1. (a) $Y = \beta_0 + \beta_1 X + \varepsilon$, ε, X : indep & $\varepsilon \sim N(0, \sigma^2 I)$

$\underline{\beta} = (\beta_0, \beta_1)^T$

$$\underset{\sim}{X} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

$$SE(\hat{\beta}_j)^2 = \{\sigma^2 (\underset{\sim}{X}^T \underset{\sim}{X})^{-1}\}_{jj}$$

$$\underset{\sim}{X}^T \underset{\sim}{X} = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix} = \begin{bmatrix} n & n\bar{x} \\ n\bar{x} & \sum_{i=1}^n x_i^2 \end{bmatrix}$$

$$(\underset{\sim}{X}^T \underset{\sim}{X})^{-1} = \frac{1}{\det(\underset{\sim}{X}^T \underset{\sim}{X})} \begin{bmatrix} \sum_{i=1}^n x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{bmatrix} = \frac{1}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} \begin{bmatrix} \sum_{i=1}^n x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{bmatrix}$$

$$\Rightarrow SE(\hat{\beta}_0) = \{\sigma^2 (\underset{\sim}{X}^T \underset{\sim}{X})^{-1}\}_{00}$$

$$= \frac{\sigma^2 \sum_{i=1}^n x_i^2}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} = \frac{\sigma^2}{n} \left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 - n\bar{x}^2} \right)$$

$$= \frac{\sigma^2}{n} \left(\frac{\sum x_i^2 - n\bar{x}^2}{\sum_{i=1}^n x_i^2 - n\bar{x}^2} - \frac{\sum x_i^2 - n\bar{x}^2}{\sum x_i^2 - n\bar{x}^2} + \frac{\sum x_i^2}{\sum x_i^2 - n\bar{x}^2} \right)$$

$$= \frac{\sigma^2}{n} \left(1 + \frac{n\bar{x}^2}{\sum x_i^2 - n\bar{x}^2} \right) \quad \because \sum x_i^2 - n\bar{x}^2 = \sum (x_i - \bar{x})^2$$

$$= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{\sum (x_i - \bar{x})^2} \right)$$

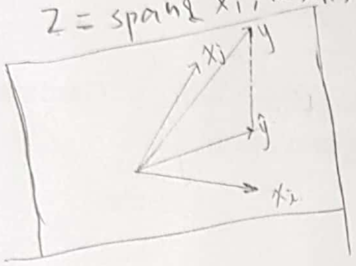
$$SE(\hat{\beta}_1) = \{\sigma^2 (\underset{\sim}{X}^T \underset{\sim}{X})^{-1}\}_{11}$$

$$= \frac{\sigma^2 n}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2}$$

$$= \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

1. (b) Show $Hz = z$, where H : project matrix & $z = Xc$, $c = (c_0, \dots c_p)^T$

$Z = \text{span}\{x_1, \dots x_n\}$



$$Hy = \hat{y} = X(X^TX)^{-1}X^Ty$$

$$HXc = X(X^TX)^{-1}X^TXc = Xc$$

2. (a)

$$\hat{\beta} = \arg\max_{\beta} p(y|X,\beta)$$

$$= \arg\max_{\beta} \prod_{i=1}^{n} \mathcal{N}(y_n | x_i^T\beta)$$

$$= \arg\max_{\beta} \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - x_i^T\beta)^2\right)$$

$$= \arg\max_{\beta} (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2\right)$$

$$= \arg\max_{\beta} -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2$$

$$= \arg\min_{\beta} \sum_{i=1}^{n}(y_i - x_i^T\beta)^2$$

$$L = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2 = \frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)$$

$$\frac{\partial L}{\partial \beta} = 0 \Rightarrow -2X^Ty + 2X^TX\beta = 0$$

$$\Rightarrow \beta = (X^TX)^{-1}X^Ty.$$

(b)

$$\hat{\beta} = \arg\max_{\beta} p(\beta|y,X)$$

$$= \arg\max_{\beta} \frac{p(y|X,\beta)\,p(\beta|X)}{p(y|X)}$$

$$= \arg\max_{\beta} p(y|X,\beta)\,p(\beta|X)$$

$$= \arg\max_{\beta} \log p(y|X,\beta) + \log p(\beta|X)$$

$$= \arg\max_{\beta} -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2 - \frac{n}{2}\log(2\pi r^2) - \frac{1}{2r^2}\sum_{i=1}^{n}\beta_i^2$$

$$= \arg\min_{\beta} \sum_{i=1}^{n}(y_i - x_i^T\beta)^2 + \frac{\sigma^2}{r^2}\sum_{i=1}^{n}\beta_i^2$$

$$L = \sum_{i=1}^{n}(y_i - x_i^T\beta)^2 + \frac{\sigma^2}{r^2}\sum_{i=1}^{n}\beta_i^2$$

$$= (y - X\beta)^T(y - X\beta) + \frac{\sigma^2}{r^2}\beta^T\beta$$

$$\frac{\partial L}{\partial \beta} = 0 \Rightarrow (-2X^Ty + 2X^TX\beta) + \frac{\sigma^2}{r^2}2\beta = 0$$

$$\Rightarrow \beta = \left(X^TX + \frac{\sigma^2}{r^2}I\right)^{-1}X^Ty.$$

2. (c)

$$\text{ML}: \hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} (y_i - X_i^T \beta)^2$$

$$\text{MAP}: \hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} (y_i - X_i^T \beta)^2 + \frac{\sigma^2}{\tau^2} \sum_{i=1}^{n} \beta_i^2$$

least square sol.

$\Rightarrow$ The prior distribution acts as a regularier in MAP estimation

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3. (a) $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

$x_i = 5\%, 10\%, 15\%, 20\%, 30\% \Rightarrow \bar{x} = 0.16$

$y_i = \frac{31}{200}, \frac{52}{200}, \frac{68}{200}, \frac{101}{200}, \frac{144}{200} \Rightarrow \bar{y} = 0.396$

$RSS(\hat{\beta}_0, \hat{\beta}_1) = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$

$\frac{\partial RSS}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$

$\Rightarrow \hat{\beta}_0 n \bar{x} + \hat{\beta}_1 \sum_i x_i^2 = \sum_i x_i y_i$ — ①

$\frac{\partial RSS}{\partial \hat{\beta}_0} = -2 \sum_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$

$\Rightarrow n\hat{\beta}_0 + \hat{\beta}_1 n \bar{x} = n\bar{y}$

$\Rightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ — ② $= 0.02865$

②代① $n\bar{x}\bar{y} - n\hat{\beta}_1 \bar{x}^2 + \hat{\beta}_1 \sum_i x_i^2 = \sum_i x_i y_i$

$\Rightarrow (\sum_i x_i^2 - n\bar{x})\hat{\beta}_1 = \sum_i x_i y_i - n\bar{x}\bar{y}$

$\Rightarrow \hat{\beta}_1 = \frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{\sum_i x_i^2 - n\bar{x}^2} = \frac{\sum_{i=1}^{1000} x_i y_i - 1000 \cdot 0.16 \cdot 0.396}{\sum_i x_i^2 - 1000 \cdot 0.16^2} = 2.296$

$\Rightarrow \hat{y} = 0.02865 + 2.296x$

$\Rightarrow x = 0.25 \Rightarrow \hat{y} = 0.603$

3 (b) $\ell(\beta_0,\beta_1) = \prod_{i=1}^{n} P(Y=y_i \mid X = x_i ; \beta_0,\beta_1) = \prod_{y_i=1} P(x_i;\beta_0,\beta_1) \prod_{y_i=0} (1-P(x_i;\beta_0,\beta_1))$

$$\ln \ell(\beta_0,\beta_1) = \sum_{i=1}^{n} (y_i \ln P(x_i;\beta_0,\beta_1)) + (1-y_i)\ln P(1-P(x_i;\beta_0,\beta_1))$$

$$= \sum_{i=1}^{n} \left[y_i(\beta_0+\beta_1 x_i) - \ln(1+e^{\beta_0+\beta_1 x}) \right]$$

$$= -\sum_{i=1}^{n} \ln(1+e^{-(2y_i-1)\beta^T x_i}) \triangleq J(\beta_0,\beta_1)$$

$$= -31\ln(1+e^{-(\beta_0+5\beta_1)}) - (200-31)\ln(1+e^{(\beta_0+5\beta_1)})$$
$$-52\ln(1+e^{-(\beta_0+10\beta_1)}) - (200-52)\ln(1+e^{(\beta_0+10\beta_1)})$$
$$-68\ln(1+e^{-(\beta_0+15\beta_1)}) - (200-68)\ln(1+e^{(\beta_0+15\beta_1)})$$
$$-101\ln(1+e^{-(\beta_0+20\beta_1)}) - (200-101)\ln(1+e^{(\beta_0+20\beta_1)})$$
$$-144\ln(1+e^{-(\beta_0+30\beta_1)}) - (200-144)\ln(1+e^{(\beta_0+30\beta_1)})$$

Use Gradient Descent $\beta(k+1) = \beta(k) - \eta \nabla J(\beta(k))$

3.(c)

$$p(y=1|X=x) = \frac{p(X=x|Y=1)p(y=1)}{p(X=x)} \triangleq P_1(x)$$

$$P_1(x) = \hat{\delta}_1(x) = x\frac{\hat{\mu}_1}{\hat{\sigma}^2} - \frac{\hat{\mu}_1^2}{2\hat{\sigma}^2} + \log(\hat{\pi}_1)$$

$$\pi_1 = \frac{396}{1000} \doteq 0.396$$

$$\hat{\mu}_1 = \frac{1}{n_1}\sum_{y_i=1} x_i = \frac{1}{396}(31\cdot5+52\cdot10+68\cdot15+101\cdot20+144\cdot30) = 20.290$$

$$\hat{\mu}_0 = \frac{1}{n_0}\sum_{y_i=0} x_i = \frac{1}{604}(169\cdot5+148\cdot10+132\cdot15+99\cdot20+56\cdot30) = 13.1871$$

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{k=0}^{1}\sum_{y_i=k}(x_i-\hat{\mu}_k)^2$$

$$= \frac{1}{1000-2}\Big(169\cdot(5-13.1871)^2 + 148(10-13.1871)^2 + 132(15-13.1871)^2$$
$$+99(20-13.1871)^2 + 56(30-13.1871)^2 + 31(5-20.2904)^2$$
$$+52(10-20.2904)^2 + 68(15-20.2904)^2 + 101(20-20.2904)^2$$
$$+144(30-20.2904)^2\Big)$$

$$= 62.056$$

$$P_1(x) = \frac{\pi_1 \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\hat{\mu}_1)^2\right)}{\sum_{l=0}^{1}\pi_l\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\hat{\mu}_l)^2\right)}$$

$$x=25 \Rightarrow P_1(25) = 0.628$$

4.

$$R^2 = \frac{TSS-RSS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{\sum_{j=1}^{n} y_j^2}$$

$$TSS = \sum_{j=1}^{n}(y_j - \bar{y})^2 = \sum_{j=1}^{n} y_j^2 \quad \because \bar{y} = 0$$

$$RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2$$

$\hat{\beta}_0 = 0 \quad \because \bar{x} = \bar{y} = 0$

$$= \sum_{i=1}^{n}\left(y_i - \left(\frac{\sum_{k=1}^{n} x_k y_k}{\sum_{k=1}^{n} x_k^2}\right)x_i\right)^2$$

$$Cor(X,Y) = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2}}$$

$$\Rightarrow R^2 = [Cor(X,Y)]^2$$

5. (a) If $X : U(0,1)$, then $\frac{0.65 - 0.55}{1-0} = 10\%$.

(b) $10\% \cdot 10\% = 1\%$

(c) $10\%^{100} = 10^{-100}$

(e) For $p=1$, side $= 0.1$

For $p=2$, side $= 0.1^{\frac{1}{2}} = 0.316$

For $p=100$, side $= 0.1^{\frac{1}{100}} = 0.977$

when p is too high, to use on average 10% of the observations would mean that we would need to include almost the entire range of each individual feature.