EE 3070 Statistics

Final Exam

May 30, 2023
10:10 $\sim$ 12:00

Note: There are **6** problems with a total of 100 points on a single page. Please provide detailed answers on the answer sheets.

No credit without detail. Closed books. No calculator.

1. (13%) Show that the mean $\bar{X}$ of a random sample of size n from a distribution having pdf $f(x;\theta) = (1/\theta)\cdot e^{-(x/\theta)},\ 0 < x < \infty,\ 0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n.

2. (13%) Let the joint pdf of X and Y be $f(x,y) = \frac{12}{7}\cdot x(x+y), 0 < x < 1, 0 < y < 1$, zero elsewhere. Let $U = \min(X,Y)$ and $V = \max(X,Y)$. Find the joint pdf of U and V.

3. (14%) Let $X_1, X_2, \cdots, X_n$ be a random sample from a $\Gamma(\alpha = 3,\ \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the mle of θ. Recall that gamma distribution is given by equation (3.3.1) as shown below:

$$f(x;\alpha,\beta) = \Gamma(\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot e^{-x/\beta} & 0 < x < \infty,\ \ \alpha > 0,\ \beta > 0,\ \Gamma(\alpha) > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

4. (20%) Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be independent random samples from the two normal distributions $N(0,\theta_1)$ and $N(0,\theta_2)$. Find the likelihood ratio Λ for testing the composite hypothesis $H_0 : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$.

5. (20%) Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x;\theta) = 1/\theta,\ 0 < x < \theta,\ 0 < \theta < \infty$, zero elsewhere. Show that $4Y_1$ is unbiased estimators of θ.
   Hint: $f_{Y_k}(y_k) = \frac{n!}{(k-1)!\times(n-k)!} \times (F(y_k))^{k-1}(1 - F(y_k))^{n-k} \times f(y_k)$

6. (20%) Consider observations $x[i],\ i = 0 \sim N-1$ taken from the model

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n],\ n = 0, 1, \cdots, N-1.$$

The A and f_0 are known and $w[n] \sim N(0,\ \sigma^2)$, where σ^2 is known. Prove that the sufficient statistics for ϕ is

$$\left\{\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n),\ \sum_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)\right\} \overset{\text{Denoted as}}{=} \{T_1(x),\ T_2(x)\}$$

Hint: Use the Theorem that (Theorem 7.2.1)

$$f(x[0];\phi)\cdot f(x[2];\phi),\cdots, f(x[N-1];\phi) = k_1[u_1(x[0], x[1], \cdots, x[N-1]);\phi] \times k_2[x[0], \cdots, x[N-1]].$$

Hint: $\cos(A+B) = \cos A\cos B - \sin A\sin B$