

EE 3070 Statistics

Final Exam Solution

May 30, 2023

10:10 ~ 12:00

Note: There are **6** problems with a total of 100 points on a single page. Please provide detailed answers on the answer sheets.

No credit without detail. Closed books. No calculator.

1. (13%) (**Exercise 7.1.1**) Show that the mean \bar{X} of a random sample of size n from a distribution having pdf $f(x; \theta) = (1/\theta) \cdot e^{-(x/\theta)}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n .

Solution:

$$\begin{aligned}\Rightarrow E(X_1) &= \int_0^\infty x \cdot f(x) dx = \int_0^\infty x \cdot \left(\frac{1}{\theta} \cdot e^{-\frac{x}{\theta}}\right) dx = \int_0^\infty u \cdot e^{-u} \cdot \theta du \quad (\text{let } u = x/\theta, \theta \cdot du = dx) \\ &= \theta \cdot \int_0^\infty u e^{-u} du = \theta \cdot (u e^{-u} \Big|_0^\infty - e^{-u} \Big|_0^\infty + 0) = \theta \cdot (0 - 0 - 0 + 1 + 0) = \theta \cdot 1 = \theta \\ \therefore E(\bar{X}) &= E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1}{n} \cdot n \cdot E(X_1) = \theta \\ \Rightarrow E(X_1^2) &= \int_0^\infty x^2 \cdot f(x) dx = \int_0^\infty x^2 \cdot \left(\frac{1}{\theta} \cdot e^{-\frac{x}{\theta}}\right) dx = \int_0^\infty \theta \cdot \left(\frac{x}{\theta}\right)^2 \cdot e^{-\frac{x}{\theta}} dx \\ &= \int_0^\infty \theta \cdot u^2 e^{-u} \cdot \theta du = \theta^2 \int_0^\infty u^2 e^{-u} du \quad (\text{let } u = x/\theta, \theta \cdot du = dx) \\ &= \theta^2 \cdot (u^2(-e^{-u}) \Big|_0^\infty - 2u e^{-u} \Big|_0^\infty + 2e^{-u} \Big|_0^\infty - 0) = \theta^2 \cdot (0 - 0 - 0 - 0 + 2 - 0 - 0) = 2 \cdot \theta^2 \\ \Rightarrow \text{Var}(X_1) &= E(X_1^2) - E(X_1)^2 = 2 \cdot \theta^2 - \theta^2 = \theta^2 \\ \therefore \text{Var}(\bar{X}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n} \cdot \text{Var}(X_1) = \frac{\theta^2}{n}\end{aligned}$$

■

2. (13%) (**Exercise 4.4.20**) Let the joint pdf of X and Y be $f(x, y) = \frac{12}{7} \cdot x(x+y)$, $0 < x < 1, 0 < y < 1$, zero elsewhere. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the joint pdf of U and V .

Solution:

(1) $U = \min(X, Y) = X$, $V = \max(X, Y) = Y$

$$\begin{aligned}f_{U,V}(u, v) &= f(x = u, y = v) \cdot \left| \frac{\frac{\partial x}{\partial u}}{\frac{\partial y}{\partial u}} \quad \frac{\frac{\partial x}{\partial v}}{\frac{\partial y}{\partial v}} \right| \\ &= \frac{12}{7} \cdot u(u+v) \cdot \left| \frac{1}{0} \quad \frac{0}{1} \right| \\ &= \frac{12}{7} \cdot u(u+v) \cdot 1 = \frac{12}{7} \cdot u(u+v)\end{aligned}$$

(2) $U = \min(X, Y) = Y, \quad V = \max(X, Y) = X$

$$\begin{aligned} f_{U,V}(u, v) &= f(x = v, y = u) \cdot \left| \frac{\frac{\partial x}{\partial u}}{\frac{\partial y}{\partial u}} \frac{\frac{\partial x}{\partial v}}{\frac{\partial y}{\partial v}} \right| \\ &= \frac{12}{7} \cdot v(v + u) \cdot \left| \frac{0}{1} \frac{1}{0} \right| \\ &= \frac{12}{7} \cdot v(v + u) \cdot |(-1)| = \frac{12}{7} \cdot v(v + u) \end{aligned}$$

$$\therefore f_{U,V}(u, v) = \begin{cases} \frac{12}{7} \cdot u(u + v), & 0 < u < v < 1, \quad u = x, \quad v = y \\ \frac{12}{7} \cdot v(v + u), & 0 < u < v < 1, \quad v = x, \quad u = y \end{cases}$$

■

3. (14%) (**Exercise 6.1.1**) Let X_1, X_2, \dots, X_n be a random sample from a $\Gamma(\alpha = 3, \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the mle of θ . Recall that gamma distribution is given by equation (3.3.1) as shown below:

$$f(x; \alpha, \beta) = \Gamma(\alpha, \beta) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot e^{-x/\beta} & 0 < x < \infty, \quad \alpha > 0, \quad \beta > 0, \quad \Gamma(\alpha) > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Solution:

$$\begin{aligned} L(\theta, x) &= \prod_{i=1}^n f(x_i; \theta), \quad x \in \Omega. \quad X_1, X_2, \dots, X_n \sim \text{iid}, \quad \theta \text{ is unknown.} \quad \hat{\theta} = \max_{\theta \in \Omega} L(\theta; x) \\ f(x; \alpha, \beta) &= \Gamma(\alpha, \beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot e^{-\frac{x}{\beta}} = \frac{1}{\Gamma(3) \cdot \theta^3} \cdot x^2 \cdot e^{-\frac{x}{\theta}}, \quad 0 < x < 1. \\ \Rightarrow L(\theta) &= \prod_{i=1}^n f(x_i; 3, \theta) = \prod_{i=1}^n \frac{1}{\Gamma(3) \cdot \theta^3} \cdot x_i^2 \cdot e^{-\frac{x_i}{\theta}} = \left(\frac{1}{\Gamma(3) \cdot \theta^3} \right)^n \cdot \left(\prod_{i=1}^n x_i^2 \cdot e^{-\frac{x_i}{\theta}} \right) \\ &= \left(\frac{1}{\Gamma(3) \cdot \theta^3} \right)^n \cdot e^{-\frac{1}{\theta} \sum_{i=1}^n x_i} \cdot \prod_{i=1}^n x_i^2 \\ \Rightarrow l(\theta) &= \ln(L(\theta)) = \ln \left(\left(\frac{1}{\Gamma(3) \cdot \theta^3} \right)^n \cdot e^{-\frac{1}{\theta} \sum_{i=1}^n x_i} \cdot \prod_{i=1}^n x_i^2 \right) \\ &= -n \ln(\Gamma(3)) - 3n \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^n x_i + 2 \sum_{i=1}^n \ln(x_i) \\ \Rightarrow \frac{\partial}{\partial \theta} l(\theta) &= 0 \\ \Rightarrow \frac{\partial}{\partial \theta} \left(-n \ln(\Gamma(3)) - 3n \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^n x_i + 2 \sum_{i=1}^n \ln(x_i) \right) &= 0 \\ \Rightarrow -\frac{3n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i &= 0 \quad \rightarrow \quad \frac{3n}{\theta} = \frac{1}{\theta^2} \sum_{i=1}^n x_i \quad \rightarrow \quad 3n\theta = \sum_{i=1}^n x_i \\ \therefore \hat{\theta} &= \frac{1}{3n} \sum_{i=1}^n x_i = \frac{1}{3} \bar{x} \end{aligned}$$

■

4. (20%) (**Exercise 6.5.4**) Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_m be independent random samples from the two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$. Find the likelihood ratio Λ for testing the composite hypothesis $H_0 : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$.

Solution:

(1) $\theta_1 = \theta_2 = \theta_0$

$$\begin{aligned}
 L(\theta_0) &= \frac{1}{(2\pi\theta_0)^{\frac{m+n}{2}}} \cdot \exp \left\{ -\frac{1}{2\theta_0} \cdot \left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2 \right) \right\} \\
 l(\theta_0) &= \ln L(\theta_0) = -\frac{m+n}{2} \cdot \ln 2\pi\theta_0 - \frac{1}{2\theta_0} \cdot \left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2 \right) \\
 \frac{\partial}{\partial \theta} l(\theta_0) &= -\frac{m+n}{2} \cdot \frac{1}{2\pi\theta_0} \cdot 2\pi + \frac{1}{2\theta_0^2} \cdot \left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2 \right) = 0 \\
 \Rightarrow \frac{m+n}{2\theta_0} &= \frac{1}{2\theta_0^2} \cdot \left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2 \right) \\
 \therefore \hat{\theta}_0 &= \frac{\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2}{m+n}
 \end{aligned}$$

(2) $\theta_1 \neq \theta_2$

$$\begin{aligned}
 L(\theta_1, \theta_2) &= (2\pi\theta_1)^{-\frac{n}{2}} \cdot \exp \left\{ -\frac{1}{2\theta_1} \sum_{i=1}^n X_i^2 \right\} \cdot (2\pi\theta_2)^{-\frac{m}{2}} \cdot \exp \left\{ -\frac{1}{2\theta_2} \sum_{j=1}^m Y_j^2 \right\} \\
 l(\theta_1, \theta_2) &= \ln L(\theta_1, \theta_2) = -\frac{n}{2} \cdot \ln 2\pi\theta_1 - \frac{1}{2\theta_1} \sum_{i=1}^n X_i^2 - \frac{m}{2} \cdot \ln 2\pi\theta_2 - \frac{1}{2\theta_2} \sum_{j=1}^m Y_j^2 \\
 \Rightarrow \frac{\partial}{\partial \theta_1} l(\theta_1, \theta_2) &= 0 \quad \rightarrow \quad -\frac{n}{2\theta_1} + \frac{1}{2\theta_1^2} \sum_{i=1}^n X_i^2 = 0 \quad \therefore \hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^n X_i^2 \\
 \Rightarrow \frac{\partial}{\partial \theta_2} l(\theta_1, \theta_2) &= 0 \quad \rightarrow \quad -\frac{m}{2\theta_2} + \frac{1}{2\theta_2^2} \sum_{j=1}^m Y_j^2 = 0 \quad \therefore \hat{\theta}_2 = \frac{1}{m} \sum_{j=1}^m Y_j^2
 \end{aligned}$$

$$(3) \Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_1^{\frac{n}{2}} \cdot \theta_2^{\frac{m}{2}}}{\hat{\theta}^{\frac{m+n}{2}}}$$

$$\begin{aligned}
 \Lambda &= \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{(2\pi\theta_0)^{-\frac{m+n}{2}} \cdot \exp \left\{ -\frac{1}{2\theta_0} (\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2) \right\}}{(2\pi\theta_1)^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2\theta_1} \sum_{i=1}^n X_i^2 \right\} \cdot (2\pi\theta_2)^{-\frac{m}{2}} \exp \left\{ -\frac{1}{2\theta_2} \sum_{j=1}^m Y_j^2 \right\}} \\
 &= \frac{(2\pi\theta_0)^{-\frac{m+n}{2}} \cdot e^{-\frac{1}{2\theta_0}(m+n)\hat{\theta}}}{(2\pi\theta_1)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_1}n\hat{\theta}_1} \cdot (2\pi\theta_2)^{-\frac{m}{2}} e^{-\frac{1}{2\theta_2}m\hat{\theta}_2}} = \frac{\theta^{-\frac{m+n}{2}} \cdot e^{-\frac{m+n}{2}}}{\theta_1^{-\frac{n}{2}} \theta_2^{-\frac{m}{2}} e^{-\frac{n}{2}} e^{-\frac{m}{2}}} = \frac{\theta_1^{\frac{n}{2}} \cdot \theta_2^{\frac{m}{2}}}{\hat{\theta}^{\frac{m+n}{2}}} \\
 &= \frac{\left(\frac{1}{n}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{m}\right)^{\frac{m}{2}} \cdot (\sum_{i=1}^n X_i^2)^{\frac{n}{2}} \cdot (\sum_{j=1}^m Y_j^2)^{\frac{m}{2}}}{\left(\frac{1}{m+n}\right)^{\frac{m+n}{2}} \cdot (\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2)^{\frac{m+n}{2}}} = \frac{(m+n)^{\frac{m+n}{2}}}{n^{\frac{n}{2}} \cdot m^{\frac{m}{2}}} \cdot \frac{(\sum_{i=1}^n X_i^2)^{\frac{n}{2}} \cdot (\sum_{j=1}^m Y_j^2)^{\frac{m}{2}}}{(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2)^{\frac{m+n}{2}}} \\
 &= \frac{(m+n)^{\frac{m+n}{2}}}{n^{\frac{n}{2}} \cdot m^{\frac{m}{2}}} \cdot \frac{\left(\frac{\sum_{i=1}^n X_i^2}{\sum_{j=1}^m Y_j^2}\right)^{\frac{n}{2}}}{\left(\frac{\sum_{i=1}^n X_i^2}{\sum_{j=1}^m Y_j^2} + 1\right)^{\frac{m+n}{2}}}
 \end{aligned}$$

■

5. (20%) (**Exercise 7.1.3**) Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x; \theta) = 1/\theta$, $0 < x < \theta$, $0 < \theta < \infty$, zero elsewhere.

Show that $4Y_1$ is unbiased estimators of θ .

Solution:

By using equation (4.4.2) $g_k(y_k) = \frac{n!}{(k-1)! \times (n-k)!} \times (F(y_k))^{k-1} (1 - F(y_k))^{n-k} \times f(y_k)$

$$\begin{aligned} \Rightarrow f_{Y_1}(y_1) &= \frac{3!}{(1-1)! \times (3-1)!} \times \left(\frac{y_1}{\theta}\right)^{1-1} \left(1 - \frac{y_1}{\theta}\right)^{3-1} \times \frac{1}{\theta} = \frac{3}{\theta} \cdot \left(1 - \frac{y_1}{\theta}\right)^2 \\ \Rightarrow E(4y_1) &= 4 \int_0^\theta y_1 \cdot f_{Y_1}(y_1) dy_1 = 4 \int_0^\theta y_1 \cdot \left(\frac{3}{\theta} \cdot \left(1 - \frac{y_1}{\theta}\right)^2\right) dy_1 \\ &= 12 \int_0^1 u \cdot (1-u)^2 \theta du = 12 \cdot \theta \cdot \left(\frac{1}{2}u^2 - \frac{2}{3}u^3 + \frac{1}{4}u^4\right) \Big|_0^1 \quad (\text{let } u = \frac{y_1}{\theta}, \theta \cdot du = dy_1) \\ &= 12 \cdot \theta \cdot \left(\frac{6}{12} - \frac{8}{12} + \frac{3}{12}\right) = 12 \cdot \theta \cdot \frac{1}{12} = \theta \end{aligned}$$

■

6. (20%) (**Chapter 7.2**) Consider observations $x[i]$, $i = 0 \sim N-1$ taken from the model

$$x[n] = A \cos(2\pi f_0 n + \phi) + w[n], \quad n = 0, 1, \dots, N-1.$$

The A and f_0 are known and $w[n] \sim N(0, \sigma^2)$, where σ^2 is known. Prove that the sufficient statistics for ϕ is

$$\left\{ \sum_{n=0}^{N-1} x[n] \cos(2\pi f_0 n), \sum_{n=0}^{N-1} x[n] \sin(2\pi f_0 n) \right\} \text{ Denoted as } \{T_1(x), T_2(x)\}$$

Hint: Use the Theorem that (Theorem 7.2.1)

$$f(x[0]; \phi) \cdot f(x[2]; \phi), \dots, f(x[N-1]; \phi) = k_1[u_1(x[0], x[1], \dots, x[N-1]); \phi] \times k_2[x[0], \dots, x[N-1]].$$

Hint: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Solution:

$$P(x; \phi) = f(x[0]; \phi) \cdots f(x[N-1]; \phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} \cdot \exp\left\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A \cdot \cos(2\pi f_0 n + \phi))^2\right\}$$

The exponent can be expanded as below

$$\begin{aligned} \Rightarrow \sum_{n=0}^{N-1} x^2[n] - 2A \sum_{n=0}^{N-1} x[n] \cos(2\pi f_0 n + \phi) + \sum_{n=0}^{N-1} A^2 \cos^2(2\pi f_0 n + \phi) \\ = \sum_{n=0}^{N-1} x^2[n] - 2A \sum_{n=0}^{N-1} x[n] \cos(2\pi f_0 n) \cdot \cos(\phi) + 2A \sum_{n=0}^{N-1} x[n] \sin(2\pi f_0 n) \cdot \sin(\phi) + \sum_{n=0}^{N-1} A^2 \cos^2(2\pi f_0 n + \phi) \end{aligned}$$

Individual samples' pdf multiply together can be decomposed as the multiplication result of two parts. The first part includes both the sufficient statistics and parameters. The second part consists only of coefficients, which do not involve any parameters.

$$\begin{aligned} \therefore P(x; \phi) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \cdot \exp\left\{-\frac{1}{2\sigma^2} \cdot \left(\sum_{n=0}^{N-1} A^2 \cos^2(2\pi f_0 n + \phi) - 2A \cdot T_1(x) \cdot \cos(\phi) \right.\right. \\ &\quad \left.\left. + 2A \cdot T_2(x) \cdot \sin(\phi)\right)\right\} \cdot \exp\left\{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n]\right\} \\ &= k_1[T_1(x), T_2(x), \phi] \times k_2[x] \end{aligned}$$

■