9. Let $f(x_1, x_2, x_3) = e^{-(x_1+x_2+x_3)}, 0 < x_1 < \infty, 0 < x_2 < \infty, 0 < x_3 < \infty$, zero elsewhere, be the joint pdf of X_1, X_2, X_3.
(a) Compute $P(X_1 < X_2 < X_3)$ and $P(X_1 = X_2 < X_3)$.
(b) Determine the joint mgf of X_1, X_2 and X_3. Are these random variables independent?

(a) $P(X_1<X_2<X_3) = \int_0^\infty \int_0^{x_3} \int_0^{x_2} e^{-(x_1+x_2+x_3)} \, dx_1 \, dx_2 \, dx_3$

$$= \int_0^\infty e^{-x_3} \int_0^{x_3} e^{-x_2} \left(-e^{-x_1} \Big|_0^{x_2} \right) dx_2 \, dx_3$$

$$= \int_0^\infty e^{-x_3} \int_0^{x_3} e^{-x_2} (-e^{-x_2} + 1) \, dx_2 \, dx_3$$

$$= \int_0^\infty e^{-x_3} \int_0^{x_3} e^{-x_2} - e^{-2x_2} \, dx_2 \, dx_3$$

$$= \int_0^\infty e^{-x_3} \left(-e^{-x_2} - \frac{-1}{2} e^{-2x_2} \Big|_0^{x_3} \right) dx_3$$

$$= \int_0^\infty e^{-x_3} \left(-e^{-x_3} + \frac{1}{2} e^{-2x_3} + 1 - \frac{1}{2} \right) dx_3$$

$$= \int_0^\infty \frac{1}{2} e^{-x_3} - e^{-2x_3} + \frac{1}{2} e^{-3x_3} \, dx_3$$

$$= \frac{-1}{2} e^{-x_3} + \frac{1}{2} e^{-2x_3} - \frac{1}{6} e^{-3x_3} \Big|_0^\infty$$

$$= 0 + 0 - 0 + \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

$P(X_1 = X_2 < X_3) = ?$

(b) 小題是对的.

10. Let X_1, X_2 and X_3 be iid with common pdf $f(x) = e^{-x}, x > 0$, zero elsewhere. Find the joint pdf of $Y_1 = X_1$, $Y_2 = X_1 + X_2$ and $Y_3 = X_1 + X_2 + X_3$.

$X \sim E(1) \quad M_X(t) = \frac{1}{1-t}, \; 1 > t$

(1) $Y_1 = X_1 \quad f_Y(y) = e^{-y_1}, \; y > 0$

(2) $Y_2 = X_1 + X_2$

$M_{Y_2}(y_2) = E[e^{tY_2}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} \cdot e^{tX_2}] = E[e^{tX_1}]E[e^{tX_2}]$

$= \frac{1}{1-t} \cdot \frac{1}{1-t} = \frac{1}{(1-t)^2} \sim Gamma(2, 1)$

(3)

$Y_3 = X_1 + X_2 + X_3$

$M_{Y_3}(y_3) = E[e^{tY_3}] = E[e^{tX_1}]E[e^{tX_2}]E[e^{tX_3}] = \frac{1}{(1-t)^3} \sim Gamma(3, 1)$