EE 3070 Statistics

Midterm Exam

April 11, 2023
10:10 $\sim$ 12:00

Note: There are **8** problems with total 100 points within **2** pages, please write your answer with detail in the answer sheet.
No credit without detail, except for question 1. No calculator. Closed books.

1. (12%) We observed $x_1, x_2, \cdots, x_{100}$ independent samples from a Gaussian random variable $N(\mu, \sigma^2)$, where μ and σ^2 are unknown. Denote $\bar{x} = \frac{\sum_{i=1}^{100} x_i}{100}$. Which of the following statements is **TRUE**?
   (a) The outcome from the estimator (for μ) $\widehat{x}_A = \frac{\sum_{i=1}^{100} x_i}{100}$ will definitely be more accurate than $\widehat{x}_B = \frac{\sum_{i=1}^{50} x_i}{50}$.
   (b) The estimator (for μ) $\widehat{x}_C = \frac{x_1+x_2}{2}$ is a valid and unbiased estimator.
   (c) The estimator (for σ^2) $\widehat{\sigma_A^2} = \frac{\sum_{i=1}^{100}(x_i-\bar{x})}{100}$ is the maximum likelihood and unbiased estimator.
   (d) Assuming σ^2 is known, the likelihood function $f(x_i\,;\mu)$ has the property $\int f(x_i\,;\mu)\,d\mu = 1$.
   (e) For any point in the parameter space (μ, σ^2), the likelihood function $f(x_i\,;\ \mu, \sigma^2) \le 1$.

2. (12%) Assumed that discrete random variable X has a moment generating function given by
$$M_X(t) = \frac{1}{6} \cdot e^{-2t} + \frac{1}{3} \cdot e^{-t} + \frac{1}{8} \cdot e^{t} + \frac{3}{8} \cdot e^{2t}$$
   Please find $P(|X| \le 1)$, $\mathrm{E}[X]$, $\mathrm{Var}(X)$.

3. (12%) A random variable X has the Poisson distribution given by
$$p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}, \quad \text{for } x = 0, 1, 2, \cdots$$
   (a) Please derive the moment generating function of X.
   (b) Please use the MGF in part (a) to find the mean and the variance of X.
   Hint: Let $K(t) = \ln M_X(t) = ?$

4. (13%) Consider the mixture distribution, $\frac{7}{10} \cdot N(0,4) + \frac{3}{10} \cdot N(0,16)$. Find the kurtosis.

5. (13 %) Suppose the number of customers X that enter a store between the hours 8:00 a.m. and 9:00 a.m. follows a Poisson distribution with parameter θ. Suppose a random sample of the number of customers that enter the store between 8:00 a.m. and 9:00 a.m. for 8 days results in the values: 4, 8, 12, 13, 9, 6, 5, 12.
   (a) Determine the maximum likelihood estimate of θ. Show that it is an unbiased estimator.
   (b) Based on these data, obtain the realization of your estimator in part **(a)**. Explain the meaning of this estimate in terms of the number of customers.

6. (13%) Let X, Y, and Z be three independent Poisson random variables with parameters $\lambda_1, \lambda_2, \lambda_3$, respectively. For $y = 0, 1, 2, \cdots, t$, calculate $E(Y \big| X + Y + Z = t)$.

7. (12%) Let $X_1, X_2, ..., X_N$ be a set of independent random variables, where each X_i is a normal random variable with mean equal to μ and variance equal to σ^2. Please derive the moment generating function of Y, where $Y = X_1 + X_2 + \cdots + X_N$ and N is Poisson random variable with mean λ.

8. (13%) Consider a signal

$$x[n] = A \cdot \cos(2\pi f_0 n + \phi) + w[n], \; n = 0, 1, \cdots, N-1.$$

The A and f_0 are assumed known. The n is the sampling time index. The $w[n]$ is distributed as $N(0, \sigma^2)$, where σ^2 is known. The $x[n]$ is the observed data. The ϕ is the parameter to be found.
(a) Derive the likelihood function of ϕ.
(b) Derive the procedure to find the Maximum Likelihood Estimation of ϕ.
