3.5.2. Let X and Y have a bivariate normal distribution with parameters $\mu_1 = 3$, $\mu_2 = 1$, $\sigma_1^2 = 16$, $\sigma_2^2 = 25$, and $\rho = \frac{3}{5}$. Determine the following probabilities:

(a) $P(3 < Y < 8)$. 0.264

(b) $P(3 < Y < 8 | X = 7)$. 0.440

(c) $P(-3 < X < 3)$. 0.433

(d) $P(-3 < X < 3 | Y = -4)$. 0.643

$X \sim N(3, 16)$ $\quad$ $Y \sim N(1, 25)$

(a)

$P(3<Y<8) = P\left(\frac{3-1}{5} < \frac{Y-1}{5} < \frac{8-1}{5}\right)$

$= P\left(\frac{2}{5} < Z < \frac{7}{5}\right)$

$= \phi\left(\frac{7}{5}\right) - \phi\left(\frac{2}{5}\right) = 0.9192 - 0.6554 = 0.2638$

(b) $P(a<Y<b \mid X=c) = P\left(\frac{y-\mu_{Y|X=c}}{\sqrt{\sigma^2_{Y|X=c}}} < \frac{b-\mu_{Y|X=c}}{\sqrt{\sigma^2_{Y|X=c}}}\right) - P\left(\frac{y-\mu_{Y|X=c}}{\sqrt{\sigma^2_{Y|X=c}}} < \frac{a-\mu_{Y|X=c}}{\sqrt{\sigma^2_{Y|X=c}}}\right)$

$P(3<Y<8 \mid X=7) = P\left(Z < \frac{8-4}{4}\right) - P\left(Z < \frac{3-4}{4}\right)$

$E[Y|X] = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)$

$= 1 + \frac{3}{5}\cdot\frac{5}{4}(7-3)$

$= 1+3$

$= P(Z<1) - P(Z<-0.25)$

$= 0.8413 - 0.4013 = 0.44$

$\sigma_{Y|X=x} = \sigma_y * \sqrt{1-\rho^2}$

$= 5 \times \sqrt{1-\frac{3}{5}^2}$

$= 5 \times \frac{4}{5} = 4$

(c) $P(-3<X<3) = P\left(\frac{-3-3}{4} < \frac{X-3}{4} < \frac{3-3}{4}\right)$

$= P\left(\frac{-6}{4} < K < 0\right)$

$= \phi(0) - \phi(-1.5)$

$= 0.5 - 0.0668 = 0.4332$

(d) $P(-3<X<3 \mid Y=-4) = P\left(\frac{X-\mu_{X|Y=-4}}{\sigma_{X|Y=-4}} < \frac{-3-\mu_{X|Y=-4}}{\sigma_{X|Y=-4}}\right) - P\left(\frac{X-\mu_{X|Y=-4}}{\sigma_{X|Y=-4}} < \frac{3-\mu_{X|Y=-4}}{\sigma_{X|Y=-4}}\right)$

$= P\left(Z < \frac{3-0.6}{3.2}\right) - P\left(Z < \frac{-3-0.6}{3.2}\right)$

$= P(Z<0.75) - P(Z<-1.125)$

$= 0.7734 - 0.1303$

$= 0.6431$

3.6.3. Let T have a t-distribution with $r > 4$ degrees of freedom. Use expression (3.6.4) to determine the kurtosis of T. See Exercise 1.9.15 for the definition of kurtosis.

3.6.4 $$E[T^k] = E[W^k] \frac{2^{-k/2}\,\Gamma\left(\frac{r}{2} - \frac{k}{2}\right)}{\Gamma\left(\frac{r}{2}\right) r^{-k/2}}$$

$$K = \frac{E[(T-\mu)^4]}{\sigma^4}$$

$$= \frac{E[T^4]}{\mathrm{Var}(T)^4}$$

$$= E[W^4] \frac{2^{-2}\,\Gamma\left(\frac{r}{2}-2\right)}{\Gamma\left(\frac{r}{2}\right) r^{-2}}$$

$$= \frac{1}{4} \frac{E[W^4]\,\Gamma\left(\frac{r}{2}-2\right)}{\frac{r^2}{(r-2)^2}\,\Gamma\left(\frac{r}{2}\right) r^{-2}}$$

$$= \frac{1}{4} \frac{3\sigma^4\,\Gamma\left(\frac{r}{2}-2\right)}{\Gamma\left(\frac{r}{2}\right)} (r-2)^2$$

$$= \frac{3(r-2)^2}{4} \frac{1}{\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)}$$

$$= \frac{3(r-2)^2}{4} \frac{4}{(r-2)(r-4)}$$

$$= \frac{3(r-2)}{r-4}$$

$$K = \frac{3(r-2)}{r-4}$$

4.1.3. Suppose the number of customers X that enter a store between the hours 9:00 a.m. and 10:00 a.m. follows a Poisson distribution with parameter θ. Suppose a random sample of the number of customers that enter the store between 9:00 a.m. and 10:00 a.m. for 10 days results in the values

9 7 9 15 10 13 11 7 2 12

(a) Determine the maximum likelihood estimate of θ. Show that it is an unbiased estimator. 9.5

(b) Based on these data, obtain the realization of your estimator in part (a). Explain the meaning of this estimate in terms of the number of customers.

$$X \sim P(\theta) \qquad P_X(x) = \frac{e^{-\theta}\theta^{x}}{x!} \qquad x = 0, 1, 2, 3, \ldots$$

(a)

$$L(\theta) = \prod_{i=1}^{n} P(x_i;\theta) = \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} = \frac{e^{-n\theta}\theta^{x_1+x_2+\cdots+x_n}}{x_1!\,x_2!\cdots x_n!}$$

$$\begin{aligned}
\ln(L(\theta)) &= \ln\left(e^{-n\theta}\theta^{x_1+x_2+\cdots+x_n}\right) - \ln(x_1!\,x_2!\ldots x_n!) \\
&= \ln(e^{-n\theta}) + \ln\left(\theta^{x_1+x_2+\cdots+x_n}\right) - \ln(x_1!\,x_2!\ldots x_n!) \\
&= -n\theta + \left(\sum_{i=1}^{n} x_i\right)\ln\theta - \ln\left(\prod_{i=1}^{n} x_i!\right)
\end{aligned}$$

$$\frac{\partial \ln(L(\theta))}{\partial\theta} = -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = n$$

$$\Rightarrow \theta = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\rightarrow ML \qquad \hat{\theta} = \bar{x}$$

$$E[\bar{x}] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[x_i] = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta$$

$\because E[\hat{\theta}] = \theta \Rightarrow$ unbiased estimator

(b) $\hat{\theta} = \bar{x} = \frac{1}{10}(9+7+9+15+10+13+11+7+2+12) = 9.5$

可以得知 約有 9-10 位 顧客 從 9:00 - 10:00 進入商店