A box contains 30 balls with 10 red balls. Find the probability that the third red ball appears on the kth draw in the following two cases.
(a) Randomly draw one by one with replacement.
(b) Randomly draw one by one without replacement. CH1

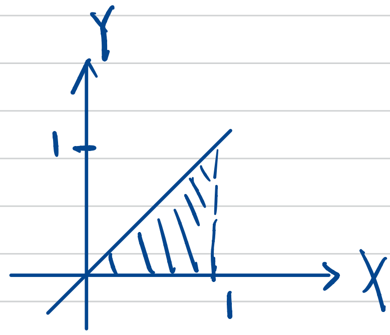
(a) $C_2^{k-1} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{k-3}$

(b) $\dfrac{C_2^{k-1} P_3^{10} P_{k-3}^{20}}{P_k^{30}}$

First a point Y is selected at random from the interval (0,1). Then another point X is selected at random from the interval $(Y,1)$. Find the probability density function of X. CH2

$0 \le Y \le X \le 1$ $f_Y(y) = 1$

$f(x|y) = \dfrac{1}{1-y} = \dfrac{f(x,y)}{f(y)} = f(x,y)$



$\int_0^x \frac{1}{1-y}\,dy = -\ln(1-y)\Big|_0^x = -\ln(1-x);\ 0<x<1$

(a) Assume that discrete r.v. X has a moment generating function given by

$M_X(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{8}e^{t} + \frac{3}{8}e^{2t}$.

Please find $\Pr(|X| \le 1)$, $E[X]$, $Var(X)$

(b) A random variable X has the Poisson distribution $p(x;\mu) = e^{-\mu}\mu^x / x!$ for $x = 0,1,2,\ldots$

(1) Please derive the moment-generating function of X.

(2) Please use the moment-generating function in (1) to find the mean and the variance of X. Hint: 令 $K(t) = \ln\{M_X(t)\} = ?$ 求 μ, σ^2

(a)

$$M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right] = \sum_{n=0}^{\infty} t^n \frac{E[X^n]}{n!} = \sum_{X=\infty}^{\infty} e^{tX} f(X) =$$

$X = 1 \Rightarrow f(X) = 1/8$

$X = 2 \Rightarrow f(X) = \frac{3}{8}$

$X = -1 \Rightarrow f(X) = 1/3$

$X = -2 \Rightarrow f(X) = 1/6$

$$P(|X| \le 1) = P(X=1) + P(X=-1)$$
$$= 1/8 + 1/3 = \frac{11}{24} \#$$

$$E[X] = 1/8 + \frac{3}{4} - 1/3 - 1/3$$
$$= \frac{7}{8} - 2/3 = \frac{21-16}{24} = \frac{5}{24}$$

$$E[X^2] = 1/8 + \frac{12}{8} + 1/3 + \frac{4}{6}$$
$$= \frac{13}{8} + \frac{6}{6}$$
$$= \frac{21}{8}$$

$$Var(X^2) = E[X^2] - E[X]^2 = \frac{21}{8} - \left(\frac{5}{24}\right)^2$$
$$= \frac{1487}{576}$$

(b)

(1)
$$\sum_{X=0}^{\infty} \frac{e^{-\mu}\mu^X}{X!} e^{tX} = \sum_{X=0} \frac{(e^t\mu)^X}{X!} e^{-\mu} = e^{-\mu} e^{e^t\mu}$$
$$= e^{\mu(e^t-1)}$$

(2)
$$K(t) = \ln(M_X(t)) = \mu(e^t - 1)$$

$$\mu = K'(t)\big|_{t=0} = \mu e^t\big|_{t=0} = \mu$$

$$\sigma^2 = K''(t)\big|_{t=0} = \mu e^t\big|_{t=0} = \mu$$

11. For two random variables X and Y, if the covariance $Cov(X,Y)=0$, please express the correlation coefficient $\rho(X+Y,X-Y)$ in terms of $Var(X)$ and $Var(Y)$.

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$$\rho(X+Y,X-Y) = \frac{E[(X+Y)(X-Y)] - E[(X+Y)]E[(X-Y)]}{\sqrt{Var(X+Y)\,Var(X-Y)}}$$

$$= \frac{E[X^2-Y^2] - (E[X]+E[Y])(E[X]-E[Y])}{\sqrt{(E[(X+Y)^2] - E[X+Y]^2)(E[(X-Y)^2] - E[X-Y]^2)}}$$

$$= \frac{E[X^2]-E[Y^2] - E[X]^2 + E[Y]^2}{\sqrt{(Var(X)+Var(Y))^2}}$$

$$= \frac{Var(X)-Var(Y)}{Var(X)+Var(Y)}$$

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7. Let X, Y, and Z be three independent Poisson random variables with parameters $\lambda_1, \lambda_2, \lambda_3$, respectivley. For $y = 0,1,2,\ldots,t$, calculate $E(Y \mid X+Y+Z=t)$. (10%)

$$P(Y \mid X+Y+Z=t) = \frac{P(Y, X+Y+Z=t)}{P(X+Y+Z=t)} = \frac{P(Y)P(X+Z=t-Y)}{P(X+Y+Z=t)}$$

$$= \frac{\left(\frac{e^{-\lambda_2}\lambda_2^{Y}}{Y!}\right)\frac{e^{-(\lambda_1+\lambda_3)}(\lambda_1+\lambda_3)^{t-Y}}{(t-Y)!}}{\frac{e^{-(\lambda_1+\lambda_2+\lambda_3)}(\lambda_1+\lambda_2+\lambda_3)^{t}}{t!}} = \frac{\frac{\cancel{e^{-(\lambda_1+\lambda_2+\lambda_3)}}(\lambda_1+\lambda_3)^{t-Y}\lambda_2^{Y}}{Y!(t-Y)!}}{\frac{\cancel{e^{-(\lambda_1+\lambda_2+\lambda_3)}}(\lambda_1+\lambda_2+\lambda_3)^{t}}{t!}}$$

$$= \frac{(\lambda_1+\lambda_3)^{t-Y}\lambda_2^{Y}\,t!}{(\lambda_1+\lambda_2+\lambda_3)^{t}(Y!(t-Y)!)} = \frac{t!}{Y!(t-Y)!}\,\frac{(\lambda_1+\lambda_3)^{t-Y}\lambda_2^{Y}}{(\lambda_1+\lambda_2+\lambda_3)^{t}}$$

$$= C^{t}_{Y}\left(\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\right)^{Y}\left(\frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\right)^{t-Y}$$

$$\sim B\left(t, \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\right)$$

$$E[Y \mid X+Y+Z=t] = t\left(\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\right)$$

WNJ. WX

Let $X_1, X_2, ..., X_N$ be a set of independent random variables, where each X_i is a normal random variable with mean equal to μ and variance equal to σ^2. Please derive the moment generating function of Y, where $Y = X_1 + X_2 + \cdots + X_N$ and N is a Poisson random variable with mean λ.

$r.v.'s X_i \sim N(\mu, \sigma^2)$; $r.v. N \sim P(\lambda)$; $r.v. Y = X_1 + \cdots + X_N$

$$M_Y(t) = E[e^{tY}] = E[E[e^{tY} \mid N]] = E[E[e^{t(X_1+X_2+...+X_N)} \mid N]]$$

$$E[e^{t(X_1+...+X_N)} \mid N = n]$$

$$\downarrow = E[e^{t(X_1+...+X_n)}] = (e^{\mu t + \frac{1}{2}\sigma^2 t^2})^n$$

$$= E\left[\left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)^N\right] = \sum_{n=0}^{\infty} (e^{\mu t + \frac{1}{2}\sigma^2 t^2})^{N=n} \cdot P_N(n)$$

$$= \sum_{n=0}^{\infty} e^{n\mu t + \frac{1}{2}n\sigma^2 t^2} \left(\frac{e^{-\lambda}\lambda^n}{n!}\right)$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\lambda e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)^n$$

$$= e^{-\lambda} e^{\lambda\left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)} = e^{\lambda\left[e^{\mu t + \frac{1}{2}\sigma^2 t^2} - 1\right]}$$