6.2.3. Given the pdf

$$f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty,$$

show that the Rao–Cramér lower bound (σ Var min) is $2/n$, where n is the size of a random sample from this Cauchy distribution. What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta}-\theta)$ if $\widehat{\theta}$ is the mle of θ?

$$L(\theta, x) = \prod_{i=1}^{n} f(x_i;\theta), \quad f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}$$

$$I(\theta) = E\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2 = E\left[\frac{\partial\left(\frac{1}{\pi(1+(x-\theta)^2)}\right)}{\partial\theta}\right]^2$$

$$= E\left[\frac{\partial}{\partial\theta}\left(-\ln\pi - \ln(1+(x-\theta)^2)\right)\right]^2$$

$$= E\left[\frac{2(x-\theta)}{1+(x-\theta)^2}\right]^2 = \int_{-\infty}^{\infty}\left(\frac{2(x-\theta)}{1+(x-\theta)^2}\right)^2 \frac{1}{\pi(1+(x-\theta)^2)}\,dx$$

$$= \int_{-\infty}^{\infty}\frac{4(x-\theta)^2}{\pi(1+(x+\theta)^2)^3}\,dx, \quad \tan t = x-\theta, \quad \sec^2 t\,dt = dx$$

$$= \int_{-\pi/2}^{\pi/2}\frac{4\tan^2 t}{\pi(1+\tan^2 t)^3}\sec^2 t\,dt$$

$$= \int_{-\pi/2}^{\pi/2}\cos^2 t\cdot\sin^2 t\,dt$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1}{\pi}\left(1-\frac{1+\cos(4t)}{2}\right)dt$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\cos(4t)\,dt, \quad u = 4t, \quad du = 4dt$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\int_{-2\pi}^{2\pi}\cos(u)\tfrac{1}{4}dt$$

$$= \tfrac{1}{2} - 0 = \tfrac{1}{2} \Rightarrow \text{RCLB} \quad \Rightarrow \frac{1}{nI(\theta)} = \frac{2}{n}$$

(b) $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{D} N\left(0, \dfrac{1}{E\left[\dfrac{\partial \ln(f(x;\theta))}{\partial\theta}\right]^2}\right)$, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{D} N(0, 2)$

依 6.2.18.

6.3.8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

(a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^{n} X_i$. Obtain the null distribution of Y.

(b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \leq 4$ or $Y \geq 17$.

(a)

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})}, \quad \Lambda \leq C \text{ where } \alpha = P_{\theta_0}(\Lambda \leq C)$$

$$f(x;\theta) = \frac{\theta^x}{x!} e^{-\theta}$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{\theta^x}{x!} e^{-\theta} = e^{-n\theta} \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$l(\theta) = \ln(L(\theta)) = -n\theta + \sum_{i=1}^{n} x_i \ln\theta - \sum_{i=1}^{n} \ln(x_i!)$$

$$\frac{\partial}{\partial\theta}(l(\theta)) = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left(-n\theta + \sum_{i=1}^{n} x_i \ln\theta - \sum_{i=1}^{n} \ln(x_i!)\right) = -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

(a)

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{e^{-n\theta_0} \dfrac{\theta_0^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}}{e^{\left\{-n\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \frac{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\right\}}} = \frac{e^{-n\theta_0} \theta_0^{\sum_{i=1}^{n} x_i}}{e^{\sum_{i=1}^{n} x_i} \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{\sum_{i=1}^{n} x_i}}$$

$$\Lambda = \frac{e^{-n\theta_0}\theta_0^{Y}}{e^{Y}\left(\frac{1}{n}Y\right)^{Y}}$$

(b)

$$\alpha = P(Y \leq 4) + P(Y \geq 17)$$

$$= P(Y \leq 4) + 1 - P(Y \leq 17)$$

$$= \sum_{x=0}^{4} e^{-10}\frac{10^x}{x!} + 1 - \sum_{x=0}^{17} e^{-10}\frac{10^x}{x!} = 0.05$$

6.4.4. The *Pareto distribution* is a frequently used model in the study of incomes and has the distribution function

$$F(x;\theta_1,\theta_2) = \begin{cases} 1-(\theta_1/x)^{\theta_2} & \theta_1 \le x \\ 0 & \text{elsewhere,} \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$. If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, find the maximum likelihood estimators of θ_1 and θ_2. (*Hint:* This exercise deals with a nonregular case.)

$$L(\theta, x) = \prod_{i=1}^{n} f(x_i;\theta) \qquad \frac{\partial}{\partial\theta_j}\ln(L(\theta)) = 0,\ j \in J$$

$$F(x;\theta_1,\theta_2) = \begin{cases} 1-\left(\frac{\theta_1}{x}\right)^{\theta_2}, & 0<\theta_1\le x,\ \theta_2>0 \\ 0, & \text{else} \end{cases}$$

$$f(x;\theta_1,\theta_2) = \begin{cases} \theta_1^{\theta_2}\theta_2 x^{-\theta_2-1}, & 0<\theta_1\le x,\ \theta_2>0 \\ 0, & \text{else} \end{cases}$$

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n}\theta_1^{\theta_2}\theta_2 x_i^{-\theta_2-1} = \theta_1^{n\theta_2}\theta_2^{n}\prod_{i=1}^{n}x_i^{-\theta_2-1}$$

$$l(\theta_1,\theta_2) = \ln(L(\theta_1,\theta_2)) = \ln\left(\theta_1^{n\theta_2}\theta_2^{n}\prod_{i=1}^{n}x_i^{-\theta_2-1}\right) = n\theta_2\ln\theta_1 + n\ln\theta_2 + \sum_{i=1}^{n}(-\theta_2-1)\ln x_i$$

$$\begin{cases} \frac{\partial}{\partial\theta_1} l(\theta_1,\theta_2) = 0 \\ \frac{\partial}{\partial\theta_2} l(\theta_1,\theta_2) = 0 \end{cases} \Rightarrow \begin{aligned} &\frac{\partial}{\partial\theta_1}\left(n\theta_2\ln\theta_1 + \ln\theta_2 - (\theta_2+1)\sum_{i=1}^{n}\ln x_i\right) \overset{①}{=} \frac{n\theta_2}{\theta_1} \\ &\frac{\partial}{\partial\theta_2}\left(n\theta_2\ln\theta_1 + \ln\theta_2 - (\theta_2+1)\sum_{i=1}^{n}\ln x_i\right) \overset{②}{=} n\ln\theta_1 + \frac{n}{\theta_2} - \sum_{i=1}^{n}\ln(x_i) \end{aligned}$$

①. 不會等於 0，因為 $n\in N$ 不會等於 0，且 $\theta_2>0$，因此要重新找 θ_2 的 maximum likelihood estimators

$$\begin{aligned} &\hat\theta_1 = \min\{x_1, x_2, \ldots, x_n\} \\ &0 = n\ln\theta_1 + \frac{n}{\theta_2} - \sum_{i=1}^{n}\frac{\ln(x_i)}{\theta_2} \end{aligned} \Rightarrow \begin{aligned} &\hat\theta_1 = \min\{x_1,\ldots,x_n\} \\ &0 = \theta_2\left(n\ln\theta_1 - \sum_{i=1}^{n}\ln(x_i)\right)+n \end{aligned} \Rightarrow \begin{aligned} &\hat\theta_1 = \min\{x_1,\ldots,x_n\} \\ &\theta_2 = \frac{n}{\sum_{i=1}^{n}\ln(x_i) - n\ln\theta_1} \end{aligned}$$

$$\hat\theta_1 = \min\{x_1,\ldots,x_n\},\quad \hat\theta_2 = \frac{n}{\sum_{i=1}^{n}\ln(x_i) - n\ln(\min(x_1,\ldots,x_n))}$$

6.5.1. In Example 6.5.1 let $n = 10$, and let the experimental value of the random variables yield $\overline{x} = 0.6$ and $\sum_1^{10}(x_i - \overline{x})^2 = 3.6$. If the test derived in that example is used, do we accept or reject $H_0 : \theta_1 = 0$ at the 5% significance level?

$$T = \frac{\sqrt{n}\,\overline{x}}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}} = \frac{\sqrt{10} \times 0.6}{\sqrt{\frac{3.6}{9}}} = \frac{10 \times 0.6}{\sqrt{4}} = 3$$

Reject H_0 in favor of H_1 if $|T| \geq c$

$t_{\frac{\alpha}{2}, n-1}$

$|T| > t_{9, 0.025} = 2.262$

$\alpha = 0.05, \; n = 10$

6.5.5. Let X and Y be two independent random variables with respective pdfs

$$f(x;\theta_i) = \begin{cases} \left(\frac{1}{\theta_i}\right) e^{-x/\theta_i} & 0 < x < \infty, 0 < \theta_i < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, 2$. To test $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$, two independent samples of sizes n_1 and n_2, respectively, were taken from these distributions. Find the likelihood ratio Λ and show that Λ can be written as a function of a statistic having an F-distribution, under H_0.

$$\Lambda = \frac{\frac{1}{u^{n_1+n_2}} e^{-(n_1+n_2)}}{\frac{1}{\bar{x}^{n_1}\bar{y}^{n_2}} e^{-(n_1+n_2)}}$$

$$= \frac{n_1^{n_1} n_2^{n_2}}{(n_1+n_2)^{n_1+n_2}} \frac{\left(\sum_{i=1}^{n_1} X_i\right)^{n_1} \left(\sum_{j=1}^{n_2} Y_j\right)^{n_2}}{\left(\sum_{i=1}^{n_1} X_i + \sum_{j=1}^{n_2} Y_j\right)^{n_1+n_2}}$$

$$= \frac{n_1^{n_1} n_2^{n_2}}{(n_1+n_2)^{n_1+n_2}} \frac{\left(\frac{\sum_{i=1}^{n_1} X_i}{\sum_{j=1}^{n_2} Y_j}\right)^{n_1}}{\left(\frac{\sum_{i=1}^{n_1} X_i}{\sum_{j=1}^{n_2} Y_j} + 1\right)^{n_1+n_2}}$$

$$F = \frac{\frac{1}{n_1}\sum_{i=1}^{n_1} X_i}{\frac{1}{n_2}\sum_{j=1}^{n_2} Y_j}$$

7.1.1. Show that the mean $\overline{X}$ of a random sample of size n from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-(x/\theta)}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n.

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty$$

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = n \times \frac{1}{n} E[X_1] = \theta \quad \#$$

$$E[X_1] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x}{\theta} e^{-\frac{x}{\theta}}\,dx, \quad 令\ u = \frac{x}{\theta},\ \theta\,du = dx$$

$$= \theta \int_0^\infty u e^{-u}\,du \qquad \begin{array}{cc} u & e^{-u} \\ 1 & -e^{-u} \\ 0 & e^{-u} \end{array}$$

$$= \theta$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} n\,Var(X_1) = \frac{Var(X_1)}{n} = \frac{\theta^2}{n} \quad \#$$

$$Var(X_1) = E[X_1^2] - E[X_1]^2 = 2\theta^2 - \theta^2 = \theta^2$$

$$\begin{array}{cc} +\ u^2 & e^{-u} \\ -\ 2u & -e^{-u} \\ +\ 2 & e^{-u} \\ -\ 0 & e^{u} \end{array}$$

$$E[X_1^2] = \int_0^\infty \frac{x^2}{\theta} e^{-\frac{x}{\theta}}\,dx = \theta \int_0^\infty u^2 e^{-u}\,\theta\,du$$

$$= \theta^2 \left(-u^2 e^{-u} - 2u e^{-u} + 2e^{u}\right)\Big|_0^\infty$$

$$= \theta^2 (0 - 0 + 0 + 0 - 0 + 2) = 2\theta^2$$

7.2.1. Let $X_1, X_2, \ldots, X_n$ be iid $N(0,\theta)$, $0 < \theta < \infty$. Show that $\sum_1^n X_i^2$ is a sufficient statistic for θ.

$T = \sum_{i=1}^{n} X_i^2$

$$u_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$$

$$f(x_1, \ldots, x_n; \theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \cdot e^{-\frac{x_1^2 + \ldots + x_n^2}{2\theta}}$$

$$\left(\frac{1}{\sqrt{\theta}}\right)^n = \theta^{-n/2} = e^{-\frac{n}{2}\ln\theta}$$

$$\Rightarrow f(x_1, \ldots, x_n; \theta) = \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^n}_{= k_2(x_1, \ldots, x_n)} \cdot e^{\underbrace{-\frac{x_1^2 + \ldots + x_n^2}{2\theta} - \frac{n}{2}\ln(\theta)}_{= k_1(u_1(x_1, \ldots, x_n); \theta)}}$$

$$k_1(u_1(x_1, \ldots, x_n); \theta) = e^{-\frac{x_1^2 + \ldots + x_n^2}{2\theta} - \frac{n}{2}\ln\theta}$$

and

$$k_2(x_1, \ldots, x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n$$

根據 Neyman factorization theorm $T = \sum_{i=1}^{n} X_i^2$ is a sufficient statistic for θ

7.2.6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a beta distribution with parameters $\alpha = \theta$ and $\beta = 5$. Show that the product $X_1 X_2 \cdots X_n$ is a sufficient statistic for θ.

$$f_{X_i}(x_i) = \frac{\Gamma(\theta+5)}{\Gamma(\theta)\cdot\Gamma(5)} \cdot x_i^{\theta-1} \cdot (1-x_i)^{5-1}, \quad x \in (0,1)$$

$$u_1(x_1, \ldots, x_n) = x_1 \cdot, \ldots, \cdot x_n$$

$$f(x_1, \ldots, x_n; \theta) = \left(\frac{\Gamma(\theta+5)}{\Gamma(\theta)\Gamma(5)}\right)^n \cdot (x_1 \cdot \ldots \cdot x_n)^{\theta-1} \cdot [(1-x_1) \cdot, \ldots, \cdot (1-x_n)]^4$$

$$= \underbrace{[(1-x_1) \cdot, \ldots, \cdot (1-x_n)]^4}_{= k_2(x_1, \ldots, x_n)} \underbrace{\left(\frac{\Gamma(\theta+5)}{\Gamma(\theta)\Gamma(5)}\right)^n \cdot (x_1 \cdot \ldots\, x_n)^{\theta-1}}_{= k_1(u_1(x_1, \ldots, x_n); \theta)}$$

$$k_1(u_1(x_1, \ldots, x_n); \theta) = \left(\frac{\Gamma(\theta+5)}{\Gamma(\theta)\Gamma(5)}\right)^n \cdot (x_1 \cdot \ldots\, x_n)^{\theta-1}$$

$$k_2(x_1, \ldots, x_n) = [(1-x_1) \cdot, \ldots, \cdot (1-x_n)]^4$$