EE 3070 Statistics

Midterm Exam Solution

April 11, 2023
10:10 $\sim$ 12:00

Note: There are **8** problems with total 100 points within **2** pages, please write your answer with detail in the answer sheet.

No credit without detail, except for question 1. No calculator. Closed books.

1. (12%) We observed $x_1, x_2, \cdots, x_{100}$ independent samples from a Gaussian random variable $N(\mu, \sigma^2)$, where μ and σ^2 are unknown. Denote $\bar{x} = \frac{\sum_{i=1}^{100} x_i}{100}$. Which of the following statements is **TRUE**?

   (a) The outcome from the estimator (for μ) $\widehat{x}_A = \frac{\sum_{i=1}^{100} x_i}{100}$ will definitely be more accurate than $\widehat{x}_B = \frac{\sum_{i=1}^{50} x_i}{50}$.
   (b) The estimator (for μ) $\widehat{x}_C = \frac{x_1+x_2}{2}$ is a valid and unbiased estimator.
   (c) The estimator (for σ^2) $\widehat{\sigma_A^2} = \frac{\sum_{i=1}^{100} (x_i - \bar{x})}{100}$ is the maximum likelihood and unbiased estimator.
   (d) Assuming σ^2 is known, the likelihood function $f(x_i\ ; \mu)$ has the property $\int f(x_i\ ; \mu)\ d\mu = 1$.
   (e) For any point in the parameter space (μ, σ^2), the likelihood function $f(x_i\ ;\ \mu, \sigma^2) \leq 1$.

   Solution: (b), (d), (e)

   Without any prior knowledge, option (d) and (e) cannot be true because the likelihood function is not a probability function. There is no guarantee that the integration of $\int f(x_i\ ; \mu)\ d\mu$ will integrate to 1 over all possible values of μ.

   Since in our case, x_i is a sample from a Gaussian random variable, and we know the integral of the PDF over the entire range of the outcomes is equal to 1.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} dx_i = 1 \iff \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(\mu-x_i)^2}{2\sigma^2}} d\mu = 1$$

   Therefore, in this case, $\int f(x_i\ ; \mu)\ d\mu$ will also equal to 1. But please be reminded that this is not always the case. The likelihood function has no such property that integrates to 1 with respect to any parameter.

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2. (12%) Assumed that discrete random variable X has a moment generating function given by

$$M_X(t) = \frac{1}{6} \cdot e^{-2t} + \frac{1}{3} \cdot e^{-t} + \frac{1}{8} \cdot e^{t} + \frac{3}{8} \cdot e^{2t}$$

Please find $P(|X| \leq 1),\ \mathrm{E}(X),\ \mathrm{Var}(X)$.

Solution:

$$M_X(t) = E(e^{tx}) = \sum_{n=0}^{\infty} e^{tx} \cdot P_X(x)$$

$$P(X=1) = \frac{1}{8},\ P(X=2) = \frac{3}{8},\ P(X=-1) = \frac{1}{3},\ P(X=-2) = \frac{1}{6}$$

$$P(|X| \leq 1) = P(X=1) + P(X=-1) = \frac{1}{8} + \frac{1}{3} = \frac{11}{24}$$

$$E(X) = \frac{5}{24},\ \ E(X^2) = \frac{21}{8},\ \ \mathrm{Var}(X) = \frac{21}{8} - (\frac{5}{24})^2 = \frac{1487}{576}$$

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3. (12%) A random variable X has the Poisson distribution given by

$$p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}, \quad \text{for } x = 0, 1, 2, \cdots$$

(a) Please derive the moment generating function of X.
(b) Please use the MGF in part (a) to find the mean and the variance of X.
Hint: Let $K(t) = \ln M_X(t) =?$

Solution:
(a) To derive $M_X(t)$, we can use the definition of the moment generating function:

$$\sum_{x=0}^{\infty} \frac{e^{-}\mu\mu^x}{x!} e^{tx} = \sum_{x=0}^{\infty} \frac{(e^t\mu)^x}{x!} e^{-\mu} = e^{-\mu} e^{e^t\mu} = e^{\mu(e^t-1)}$$

(b) To find the mean and the variance of X, we can first compute $K(t) = \ln M_X(t)$.

$$K(t) = \ln(M_X(t)) = \mu(e^t - 1)$$

Therefore, the mean and the variance of X are given by:

$$\mu = k'(t)\big|_{t=0} = \mu$$

$$\sigma^2 = k''(t)\big|_{t=0} = \mu$$

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4. (13%) Consider the mixture distribution, $\frac{7}{10} \cdot N(0,4) + \frac{3}{10} \cdot N(0,16)$. Find the kurtosis.

Solution:
In our case, we have a finite and countable mixtures, which means given a finite set of PDFs $f_1(x), ..., f_n(x)$ and weights $w_1, ..., w_n$ such that $w_i \geq 0$ and $\sum w_i = 1$, the mixture distribution can be represented by writing the density function, f, as a sum (which is a convex combination):

$$f(x) = \sum_{i=1}^{n} w_i \cdot f_i(x)$$

To find the kurtosis, we first need to find the variance of this mixture distribution. In this case of a mixture of one-dimensional distributions with weights w_i, mean μ_i and variances σ_i^2, the total mean and variance will be:

$$\begin{aligned}
E(X) = \mu &= \sum_{i=1}^{n} w_i \cdot \mu_i \\
E((X-\mu)^2) &= \sigma^2 \\
&= E(X^2) - \mu^2 && \text{(standard variance reformulation)} \\
&= \left(\sum_{i=1}^{n} w_i \cdot E(X_i^2)\right) - \mu^2 \\
&= \left(\sum_{i=1}^{n} w_i \cdot (\sigma_i^2 + \mu_i^2)\right) - \mu^2 && (\because \sigma_i^2 = E(X_i^2) - \mu_i^2 \quad \therefore E(X_i^2) = \sigma_i^2 + \mu_i^2)
\end{aligned}$$

Applying the above equations to our case gives us the following results for the mean and variance.
Note that although X represents a random variable from a mixture distribution of two Gaussian distributions, we cannot assume that X is also a Gaussian variable, even though we can compute its mean and variance using a simple weighted sum. Later on, we will find out that X is truly not Gaussian.

$$\begin{aligned}
f_1(x) \sim\ & N(0,4), \quad f_2(x) \sim N(0,16), \quad f(x) = \frac{7}{10} \cdot f_1(x) + \frac{3}{10} \cdot f_2(x) \\
E(X) = \frac{7}{10} \cdot 0 + \frac{3}{10} \cdot 0 = 0, \quad & \mathrm{Var}(X) = \frac{7}{10} \cdot 4 + \frac{3}{10} \cdot 16 = \frac{76}{10} = 7.6 = \sigma_x^2
\end{aligned}$$

We know that in order to compute kurtosis, we first need to find $E(X^4)$. Let's start with deriving the formula for $E(X^4)$ in the general case, regardless of the distribution of X. Since we already know that if X is Gaussian, then $E(X^4)$ is equal to $3\sigma_x^4$.
We can find the formula for $E(X^4)$ by using the definition:

$$\begin{aligned}
E(X^4) &= \int_{-\infty}^{\infty} x^4 \cdot f_X(x)\ dx = \int_{-\infty}^{\infty} x^4 \cdot \left(\sum_{i=1}^{n} w_i \cdot f_i(x)\right) dx = \sum_{i=1}^{n} w_i \cdot (\int_{-\infty}^{\infty} x^4 \cdot f_i(x)\ dx) \\
&= \sum_{i=1}^{n} w_i \cdot E(X_i^4)
\end{aligned}$$

$$\Rightarrow E(X^4) = \frac{7}{10} \cdot E(X_1^4) + \frac{3}{10} \cdot E(X_2^4) = 264 \neq 173.28 = 3 \times 7.6^2 = 3\sigma_x^4 \quad \therefore X \text{ is not Gaussian.}$$

Then, we continued with the definition of kurtosis:

$$
\begin{aligned}
k &= \frac{E(X^4)}{\mathrm{Var}(X)^2} = \frac{\frac{7}{10}\cdot E(X_1^4) + \frac{3}{10}\cdot E(X_2^4)}{(\frac{76}{10})^2} = \frac{0.7\times 3\times 2^4 + 0.3\times 3\times 4^4}{76^2/10^2} = \frac{264\cdot 100}{5776} \\
&= \frac{1}{10}\cdot\frac{7\cdot(3\cdot\sigma_1^4)+3\cdot(3\cdot\sigma_2^4)}{76^2}\cdot 10^2 \\
&= \frac{21\cdot 2^4 + 9\cdot 4^4}{76^2}\cdot 10 = \frac{21\cdot 16 + 9\cdot 256}{5776}\cdot 10 \\
&= \frac{26400}{5776} = \frac{1650}{361} \approx 4.57063712
\end{aligned}
$$

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5. (13%) Suppose the number of customers X that enter a store between the hours 8:00 a.m. and 9:00 a.m. follows a Poisson distribution with parameter θ. Suppose a random sample of the number of customers that enter the store between 8:00 a.m. and 9:00 a.m. for 8 days results in the values: 4, 8, 12, 13, 9, 6, 5, 12.
(a) Determine the maximum likelihood estimate of θ. Show that it is an unbiased estimator.
(b) Based on these data, obtain the realization of your estimator in part **(a)**. Explain the meaning of this estimate in terms of the number of customers.

Solution:
(a) To compute $\widehat{\theta}$, we can use the definition of the ML estimator:

$$
\begin{aligned}
X \sim P(\theta) \quad P_X(x) &= \frac{e^{-\theta}\cdot\theta^x}{x!}, \quad x = 0, 1, 2, \cdots \\
L(\theta) &= \prod_{i=1}^{n} P(x_i > \theta) = \prod_{i=1}^{n}\frac{e^{-\theta}\cdot\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\cdot\theta^{x_1+x_2+\cdots+x_n}}{x_1!x_2!\cdots x_n!} \\
\Rightarrow \ln(L(\theta)) &= \ln(e^{-n\theta}\cdot\theta^{x_1+x_2+\cdots+x_n}) - \ln(x_1!x_2!\cdots x_n!) \\
&= \ln(e^{-n\theta}) + \ln(\theta^{x_1+x_2+\cdots+x_n}) + \ln(x_1!x_2!\cdots x_n!) \\
&= -n\cdot\theta + (\sum_{i=1}^{n} x_i)\cdot\ln(\theta) - \ln(\prod_{i=1}^{n} x_i!) \\
\Rightarrow \frac{\partial \ln(L(\theta))}{\partial\theta} &= -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i = 0 \\
&\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = n \\
&\Rightarrow \theta = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} \\
&\Rightarrow \text{ML estimator} \quad \widehat{\theta} = \bar{x}
\end{aligned}
$$

To show this estimator to be unbiased, we need to find its expected value of $\bar{x}$ and show that it is equal to θ. The expected value of $\bar{x}$ is:

$$
E(\bar{x}) = E(\frac{1}{n}\sum_{i=1}^{n} x_i) = \frac{1}{n}\cdot E(\sum_{i=1}^{n} x_i) = \frac{1}{n}\cdot\sum_{i=1}^{n}\theta = \frac{1}{n}\cdot n\cdot\theta = \theta
$$

Therefore, we have shown that the ML estimator $\widehat{\theta} = \bar{x}$ is unbiased.

(b)

$$\hat{\theta} = \frac{1}{8}(4+8+12+13+9+6+5+12) = \frac{69}{8} = 8.625$$

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6. (13%) Let X, Y, and Z be three independent Poisson random variables with parameters $\lambda_1, \lambda_2, \lambda_3$, respectively. For $y = 0, 1, 2, \cdots, t$, calculate $E(Y \big| X+Y+Z=t)$.

Solution:

To compute $E(Y \big| X+Y+Z=t)$, we can use the definition of conditional probability:

$$\begin{aligned}
P(Y|X+Y+Z=t) &= \frac{P(Y,\ X+Y+Z=t)}{P(X+Y+Z=t)} = \frac{P(Y)P(X+Z=t-Y)}{P(X+Y+Z=t)} \\
&= P(Y) \cdot P(X+Z=t-Y) \cdot \frac{1}{P(X+Y+Z=t)} \\
&= \frac{e^{-\lambda_2} \cdot \lambda_2^y}{y!} \cdot \frac{e^{-(\lambda_1+\lambda_3)} \cdot (\lambda_1+\lambda_3)^{t-y}}{(t-y)!} \cdot \frac{t!}{e^{-(\lambda_1+\lambda_2+\lambda_3)} \cdot (\lambda_1+\lambda_2+\lambda_3)^t} \\
&= \frac{(\lambda_1+\lambda_3)^{t-y} \cdot \lambda_2^y \cdot t!}{(\lambda_1+\lambda_2+\lambda_3)^t \cdot y!(t-y)!} \\
&= \frac{t!}{y!(t-y)!} \cdot \frac{(\lambda_1+\lambda_3)^{t-y} \cdot \lambda_2^y}{(\lambda_1+\lambda_2+\lambda_3)^y} \\
&= C_y^t \cdot (\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3})^y \cdot (\frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+\lambda_3})^{t-y} \\
&\sim Bernoulli(t, \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3})
\end{aligned}$$

Therefore,

$$E(Y \big| X+Y+Z=t) = t \cdot \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}$$

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7. (12%) Let $X_1, X_2, ..., X_N$ be a set of independent random variables, where each X_i is a normal random variable with mean equal to μ and variance equal to σ^2. Please derive the moment generating function of Y, where $Y = X_1 + X_2 + \cdots + X_N$ and N is Poisson random variable with mean λ.

Solution:

From the problem description, we know that:

$$X_i \sim N(\mu, \sigma^2)\ ;\ N \sim Poisson(\lambda)\ ;\ Y = X_1 + \cdots + X_N$$

To derive the mgf of Y, we start with the definition:

$$
\begin{aligned}
M_Y(t) &= E(e^{ty}) = E\left(E(e^{ty}|N)\right) \\
&= E\left(E(e^{t(X_1+X_2+\cdots+X_N)|N})\right) = E\left((e^{\mu t+\frac{1}{2}\sigma^2 t^2})^N\right) \\
&= \sum_{n=0}^{\infty}(e^{\mu t+\frac{1}{2}\sigma^2 t^2})^N = n \cdot P_N(n) \\
&= \sum_{n=0}^{\infty} e^{n\mu t+\frac{1}{2}n\sigma^2 t^2} \cdot (\frac{e^{-\lambda}\lambda^n}{n!}) \\
&= e^{-\lambda} \cdot \left(\sum_{n=0}^{\infty}\frac{1}{n!} \cdot (\lambda \cdot e^{\mu t+\frac{1}{2}\sigma^2 t^2})^n\right) \\
&= e^{-\lambda} \cdot \left(e^{\lambda \cdot (e^{\mu t+\frac{1}{2}\sigma^2 t^2})}\right) = e^{\lambda(e^{\mu t+\frac{1}{2}\sigma^2 t^2}-1)}
\end{aligned}
$$

Therefore, the mgf of Y is given by

$$
M_Y(t) = \exp\{\lambda \cdot (\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\} - 1)\}
$$

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8. (13%) Consider a signal

$$
x[n] = A \cdot \cos(2\pi f_0 n + \phi) + w[n],\ n = 0, 1, \cdots, N-1.
$$

The A and f_0 are assumed known. The n is the sampling time index. The $w[n]$ is distributed as $N(0, \sigma^2)$, where σ^2 is known. The $x[n]$ is the observed data. The ϕ is the parameter to be found.
(a) Derive the likelihood function of ϕ.
(b) Derive the procedure to find the Maximum Likelihood Estimation of ϕ.

Solution:
(a) The likelihood function of ϕ is given by:

$$
P(x;\phi) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-Acos(2\pi f_0 n+\phi))}.
$$

(b) Take the first order derivative of the log likelihood function to obtain:

$$
\frac{\partial \ln\{p(x;\phi)\}}{\partial\phi} = \cdots = -\frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left(x[n]\sin(2\pi f_0 n+\phi) - \frac{A}{2}\sin(4\pi f_0 n + 2\phi)\right)
$$

Set $\frac{\partial\ ln\{p(x;\phi)\}}{\partial\ \phi} = 0$ to solve forϕ (no closed form)

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