**4.1.3.** Suppose the number of customers $X$ that enter a store between the hours 9:00 a.m. and 10:00 a.m. follows a Poisson distribution with parameter $\theta$. Suppose a random sample of the number of customers that enter the store between 9:00 a.m. and 10:00 a.m. for 10 days results in the values

9 7 9 15 10 13 11 7 2 12

**(a)** Determine the maximum likelihood estimate of $\theta$. Show that it is an unbiased estimator.

**(b)** Based on these data, obtain the realization of your estimator in part (a). Explain the meaning of this estimate in terms of the number of customers.

A 9.5

$X \sim P(\theta) \qquad P_X(X) = \frac{e^{-\theta}\theta^{x}}{x!} \qquad x = 0, 1, 2, 3, \ldots$

(a)

$$L(\theta) = \prod_{i=1}^{n} P(x_i;\theta) = \prod_{i=1}^{i=n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} = \frac{e^{-n\theta}\theta^{x_1+x_2+\ldots+x_n}}{x_1!\,x_2!\cdots x_n!}$$

$$\ln(L(\theta)) = \ln(e^{-n\theta}\theta^{x_1+x_2+\ldots+x_n}) - \ln(x_1!\,x_2!\ldots x_n!)$$

$$= \ln(e^{-n\theta}) + \ln(\theta^{x_1+x_2+\ldots+x_n}) - \ln(x_1!\,x_2!\ldots x_n!)$$

$$= -n\theta + \left(\sum_{i=1}^{n} x_i\right)\ln\theta - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$\frac{\partial \ln(L(\theta))}{\partial\theta} = -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = n$$

$$\Rightarrow \theta = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$\rightarrow$ ML $\quad \hat{\theta} = \bar{x}$

$$E[\bar{x}] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[x_i] = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta$$

$\because E[\hat{\theta}] = \theta \rightarrow$ unbiased estimator

(b) $\hat{\theta} = \bar{x} = \frac{1}{10}(9+7+9+15+10+13+11+7+2+12) = 9.5$

可以得知 約有 9-10 位顧客 從 9:00 - 10:00 進入商店

**4.2.1.** Let the observed value of the mean $\overline{X}$ and of the sample variance of a random sample of size 20 from a distribution that is $N(\mu, \sigma^2)$ be 81.2 and 26.5, respectively. Find respectively 90%, 95% and 99% confidence intervals for $\mu$. Note how the lengths of the confidence intervals increase as the confidence increases.

$$\boxed{\bar{x} - t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{N}} \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{N}}}$$

90%

$$\alpha = 1 - 0.9 = 0.1 \Rightarrow t_{\frac{\alpha}{2}, n-1} = \underbrace{t_{0.05, 19} = 1.729}_{\text{查表}}$$

$$81.2 - 1.729\sqrt{\frac{26.5}{20}} \leq \mu \leq 81.2 + 1.729\sqrt{\frac{26.5}{20}}$$

$$79.21 \leq \mu \leq 83.19$$

95%

$$\alpha = 1 - 0.95 = 0.05 \Rightarrow t_{0.025, 19} = 2.409$$

$$81.2 - 2.409\sqrt{\frac{26.5}{20}} \leq \mu \leq 81.2 + 2.409\sqrt{\frac{26.5}{20}}$$

$$78.43 \leq \mu \leq 83.97$$

99%

$$\alpha = 1 - 0.99 = 0.01 \Rightarrow t_{0.05, 19} = 2.861$$

$$81.2 - 2.861\sqrt{\frac{26.5}{20}} \leq \mu \leq 81.2 + 2.861\sqrt{\frac{26.5}{20}}$$

$$77.9 \leq \mu \leq 84.5$$

**4.2.9.** Let $\overline{X}$ denote the mean of a random sample of size $n$ from a distribution that has mean $\mu$ and variance $\sigma^2 = 10$. Find $n$ so that the probability is approximately 0.954 that the random interval $(\overline{X} - \frac{1}{2}, \overline{X} + \frac{1}{2})$ includes $\mu$.

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$\bar{x} - Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{N}} \leq \mu \leq \bar{X} + Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{N}}$

$P(\bar{X} - \frac{1}{2} < \mu \leq \bar{X} - \frac{1}{2}) = 0.954$

$Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{N}} = \frac{1}{2} = Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{N}}$ , $\alpha = 1 - 0.954 = 0.046$

$Z_{\frac{\alpha}{2}} = Z_{0.023} \doteqdot 1.995$ $\quad 2.056 \sqrt{\frac{10}{n}} = \frac{1}{2}$

? 1.685

$n = 10(2 \times 1.995)^2 = 159.201 \doteqdot 160$

**4.4.7.** Let $f(x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$, zero elsewhere, be the pmf of a distribution of the discrete type. Show that the pmf of the smallest observation of a random sample of size 5 from this distribution is

$$g_1(y_1) = \left(\frac{7-y_1}{6}\right)^5 - \left(\frac{6-y_1}{6}\right)^5, \quad y_1 = 1, 2, \ldots, 6,$$

zero elsewhere. Note that in this exercise the random sample is from a distribution of the discrete type. All formulas in the text were derived under the assumption that the random sample is from a distribution of the continuous type and are not applicable. Why?

$n=5$

$$\begin{aligned} P(Y_1=6) &= P(Y_1=6, Y_2=6, \ldots, Y_5=6) \\ &= P(Y=6)\,P(Y=6)\ldots P(Y=6) \\ &= f(6)^5 \\ &= \left(\tfrac{1}{6}\right)^5 = \left(\tfrac{7-6}{6}\right)^5 - \left(\tfrac{6-6}{6}\right)^5 \end{aligned}$$

$$\begin{aligned} P(Y_1=5) &= P(Y_i \geq 5,\ i=1,2,3,4,5) - P(Y_1=6) \\ &= (P(Y_i \geq 5))^5 - P(Y_1=6) \\ &= \left(\tfrac{2}{6}\right)^5 - \left(\tfrac{1}{6}\right)^5 = \left(\tfrac{7-5}{6}\right)^5 - \left(\tfrac{6-5}{6}\right)^5 \end{aligned}$$

$$P(Y_1=4) = P(Y_i > 4,\ i=1,2,3,4,5) - P(Y_1=5) - P(Y_1=6) = \left(\tfrac{7-4}{6}\right)^5 - \left(\tfrac{6-4}{6}\right)^5$$

$$P(Y_1=3) = P(Y_i > 3,\ i=1,2,3,4,5) - P(Y_1=4) - P(Y_1=5) - P(Y_1=6) = \left(\tfrac{7-3}{6}\right)^5 - \left(\tfrac{6-3}{6}\right)^5$$

$$P(Y_1=2) = P(Y_i > 2,\ i=1,2,3,4,5) - \sum_{i=3}^{6} P(Y_1=i) = \left(\tfrac{7-2}{6}\right)^5 - \left(\tfrac{6-2}{6}\right)^5$$

$$P(Y_1=1) = P(Y_i > 1,\ i=1,2,3,4,5) - \sum_{i=2}^{6} P(Y_1=i) = \left(\tfrac{7-1}{6}\right)^5 - \left(\tfrac{6-1}{6}\right)^5$$

得証 $g_1(y_1) = \left(\frac{7-y_1}{6}\right)^5 - \left(\frac{6-y_1}{6}\right)^5$, $y_1 = 1, 2, \ldots, 6$

解答

4.4.7 Since the distribution is of the discrete type, we cannot use the formulas in the book. However,

$$\begin{aligned} P(Y_1 = y_1) &= P(\text{all} \geq y_1) - P(\text{all} \geq y_1 + 1) \\ &= \left(\frac{7-y_1}{6}\right)^5 - \left(\frac{6-y_1}{6}\right)^5. \end{aligned}$$

**4.4.20.** Let the joint pdf of $X$ and $Y$ be $f(x,y) = \frac{12}{7}x(x+y)$, $0 < x < 1$, $0 < y < 1$, zero elsewhere. Let $U = \min(X,Y)$ and $V = \max(X,Y)$. Find the joint pdf of $U$ and $V$.

X

1. $U = \min(X,Y) = X \qquad V = \max(X,Y) = Y$

$$f_{UV}(u,v) = f(x=u,\ y=v) \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|$$

$$= \frac{12}{7}u(u+v) \left| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right|$$

$$= \frac{12}{7}u(u+v) \cdot 1 = \frac{12}{7}u(u+v)$$

2. $U = \min(X,Y) = Y \quad , \quad V = \max(X,Y) = X$

$$f_{UV}(u,v) = f(x=v,\ y=u)\ |J|$$

$$= \frac{12}{7}v(v+u) \left| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right|$$

$$= \frac{12}{7}v(v+u) \cdot |-1| = \frac{12}{7}v(v+u)$$

$$f_{UV}(u,v) = \begin{cases} \frac{12}{7}u(u+v), & 0<u<v<1, \quad u=x,\ v=y \\ \frac{12}{7}v(v+u), & 0<u<v<1, \quad u=y\ \ v=x \end{cases}$$

**4.5.3.** Let $X$ have a pdf of the form $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, where $\theta \in \{\theta : \theta = 1, 2\}$. To test the simple hypothesis $H_0 : \theta = 1$ against the alternative simple hypothesis $H_1 : \theta = 2$, use a random sample $X_1, X_2$ of size $n = 2$ and define the critical region to be $C = \{(x_1, x_2) : \frac{3}{4} \le x_1 x_2\}$. Find the power function of the test.

$$\begin{cases} H_0 : \theta = 1 \\ H_1 : \theta = 2 \end{cases} \qquad n=2 \qquad C = \{(X_1, X_2) : \tfrac{3}{4} \le X_1 X_2\}$$

$$g(X_1, X_2) = \begin{cases} n!\,f(x_1)\,f(x_2) & , a < X_1 < X_2 < b \\ 0 & , \text{elsewhere} \end{cases}$$ 4.4.1

$$= \begin{cases} 2!\,\theta x_1^{\theta-1}\,\theta x_2^{\theta-1} & , 0 < X_1 < X_2 < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2\theta^2 (X_1 X_2)^{\theta-1} & , 0 < X_1 < X_2 < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$P\left(\tfrac{3}{4} \le X_1 X_2\right) = \int_{3/4}^{1} \int_{\frac{3}{4X_1}}^{1} 2\theta^2 x_1^{\theta-1} x_2^{\theta-1}\, dx_2\, dx_1$$

$$= 2\theta^2 \int_{3/4}^{1} x_1^{\theta-1} \left( \frac{1}{\theta} x_2^{\theta} \Big|_{\frac{3}{4x_1}}^{1} \right) dx_1$$

$$= \frac{2\theta^2}{\theta} \int_{3/4}^{1} x_1^{\theta-1} \left(1 - \left(\frac{3}{4x_1}\right)^{\theta}\right) dx_1$$

$$= \frac{2\theta^2}{\theta} \int_{3/4}^{1} x_1^{\theta-1} - \left(\frac{3}{4}\right)^{\theta} \frac{1}{x_1}\, dx_1$$

$$= 2\theta \left( \frac{1}{\theta} x_1^{\theta} - \left(\frac{3}{4}\right)^{\theta} \log x_1 \right) \Big|_{3/4}^{1}$$

$$= 2\theta \left[ \frac{1}{\theta} - \left(\frac{3}{4}\right)^{\theta} \log 1 - \frac{1}{\theta}\left(\frac{3}{4}\right)^{\theta} + \left(\frac{3}{4}\right)^{\theta} \log\left(\frac{3}{4}\right) \right)$$

$$= 2\theta \left( \frac{1}{\theta} - \frac{1}{\theta}\left(\frac{3}{4}\right)^{\theta} + \left(\frac{3}{4}\right)^{\theta} \log\left(\frac{3}{4}\right) \right)$$

$$= 2\left(1 - \left(\frac{3}{4}\right)^{\theta} + \theta\left(\frac{3}{4}\right)^{\theta} \log\left(\frac{3}{4}\right)\right), \quad \theta = 1, 2$$

解答

4.5.3 For a general $\theta$ the probability of rejecting $H_0$ is

$$\gamma(\theta) = \int_{3/4}^{1} \int_{3/4x_1}^{1} \theta^2 (x_1 x_2)^{\theta-1}\, dx_2\, dx_1 = 1 - \left(\frac{3}{4}\right)^{\theta} + \theta\left(\frac{3}{4}\right)^{\theta} \log\left(\frac{3}{4}\right)$$

$\gamma(1)$ is the significance level and $\gamma(2)$ is the power when $\theta = 2$.

**4.6.2.** Consider the power function $\gamma(\mu)$ and its derivative $\gamma'(\mu)$ given by (4.6.5) and (4.6.6). Show that $\gamma'(\mu)$ is strictly negative for $\mu < \mu_0$ and strictly positive for $\mu > \mu_0$.

X

$$\gamma(\mu) = P_\mu\left(\bar{X} \le \mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) + P_\mu\left(\bar{X} \ge \mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \Phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} - z_{\frac{\alpha}{2}}\right) + 1 - \Phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_{\frac{\alpha}{2}}\right)$$

$$\gamma'(\mu) = \frac{\sqrt{n}}{\sigma}\left[\phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_{\frac{\alpha}{2}}\right) - \phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} - z_{\frac{\alpha}{2}}\right)\right]$$

$\gamma'(\mu)$ is strictly negative for $\mu < \mu_0$

is strictly positive for $\mu > \mu_0$

解答.

4.6.2 Suppose $\mu > \mu_0$. Then

$$\left|\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_{\alpha/2}\right| < \left|\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} - z_{\alpha/2}\right|.$$

Hence,

$$\phi\left(\left|\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_{\alpha/2}\right|\right) > \phi\left(\left|\frac{\sqrt{n}(\mu_0-\mu)}{\sigma} - z_{\alpha/2}\right|\right).$$

Because $\phi(t)$ is symmetric about 0, $\phi(t) = \phi(|t|)$. This observation plus the last inequality shows that $\gamma'(\mu)$ is increasing, (for $\mu > \mu_0$). Likewise for $\mu < \mu_0$, $\gamma'(\mu)$ is decreasing.

**6.1.1.** Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\Gamma(\alpha = 3, \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the mle of $\theta$.

$\bar{X}/3$

$$L(\theta, x) = \prod_{i=1}^{n} f(x_i ; \theta) \quad , \theta \in \Omega$$

$X_1, X_2, \ldots X_n \sim$ iid , $\theta$ is unknown

$$\hat{\theta} = \max_{\theta \in \Omega} L(\theta ; x)$$

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}}$$

$$= \frac{1}{\Gamma(3)\ \theta^3} x^2 e^{-\frac{x}{\theta}} \quad , 0 < x < \infty$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i ; 3, \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\Gamma(3)\ \theta^3} x_i^2 e^{-\frac{x_i}{\theta}}$$

$$= \left(\frac{1}{\Gamma(3)\theta^3}\right)^n \prod_{i=1}^{n} x_i^2 e^{-\frac{x_i}{\theta}}$$

$$= \left(\frac{1}{\Gamma(3)\ \theta^3}\right)^n e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^2$$

$$\hat{\theta} = \frac{1}{3n}\sum_{i=1}^{n} x_i = \frac{\bar{x}}{3}$$

$$l(\theta) = \ln(L(\theta))$$

$$= \ln\left( \left(\frac{1}{\Gamma(3)\theta^3}\right)^n e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^2 \right)$$

$$= -n\ln(\Gamma(3)) - 3n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial}{\partial\theta} l(\theta) = 0$$

$$\frac{\partial}{\partial\theta}\left(-n\ln(\Gamma(3)) - 3n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n} \ln(x_i)\right) = 0$$

$$-\frac{3n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0$$

$$\frac{3n}{\theta} = \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$$

$$3n\theta = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{\theta} = \frac{1}{3n}\sum_{i=1}^{n} x_i = \frac{\bar{x}}{3}$$