EE 3070 Statistics

Final Exam Solution

May 30, 2023
10:10 ~ 12:00

Note: There are **6** problems with a total of 100 points on a single page. Please provide detailed answers on the answer sheets.

No credit without detail. Closed books. No calculator.

1. (13%) **(Exercise 7.1.1)** Show that the mean $\bar{X}$ of a random sample of size n from a distribution having pdf $f(x;\theta) = (1/\theta) \cdot e^{-(x/\theta)}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n.

   Solution:

$$\Rightarrow\ E(X_1) = \int_0^\infty x \cdot f(x)\,dx = \int_0^\infty x \cdot (\frac{1}{\theta} \cdot e^{-\frac{x}{\theta}})\,dx = \int_0^\infty u \cdot e^{-u} \cdot \theta\,du \quad (\text{ let } u = x/\theta,\ \theta \cdot du = dx\ )$$

$$= \theta \cdot \int_0^\infty ue^{-u}\,du = \theta \cdot (ue^{-u}\Big|_0^\infty - e^{-u}\Big|_0^\infty + 0) = \theta \cdot (0 - 0 - 0 + 1 + 0) = \theta \cdot 1 = \theta$$

$$\therefore\ E(\bar{X}) = E(\frac{1}{n}\sum_{i=1}^n X_i) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{1}{n} \cdot n \cdot E(X_1) = \theta$$

$$\Rightarrow\ E(X_1^2) = \int_0^\infty x^2 \cdot f(x)\,dx = \int_0^\infty x^2 \cdot (\frac{1}{\theta} \cdot e^{-\frac{x}{\theta}})\,dx = \int_0^\infty \theta \cdot (\frac{x}{\theta})^2 \cdot e^{-\frac{x}{\theta}})\,dx$$

$$= \int_0^\infty \theta \cdot u^2 e^{-u} \cdot \theta\,du = \theta^2 \int_0^\infty u^2 e^{-u}\,du \quad (\text{ let } u = x/\theta,\ \theta \cdot du = dx\ )$$

$$= \theta^2 \cdot (u^2(-e^{-u})\Big|_0^\infty - 2ue^{-u}\Big|_0^\infty + 2e^{-u}\Big|_0^\infty - 0) = \theta^2 \cdot (0 - 0 - 0 - 0 + 2 - 0 - 0) = 2 \cdot \theta^2$$

$$\Rightarrow\ Var(X_1) = E(X_1^2) - E(X_1)^2 = 2 \cdot \theta^2 - \theta^2 = \theta^2$$

$$\therefore\ Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^n X_i) = \frac{1}{n^2}\sum_{i=1}^n Var(X_1) = \frac{1}{n} \cdot Var(X_1) = \frac{\theta^2}{n}$$

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2. (13%) **(Exercise 4.4.20)** Let the joint pdf of X and Y be $f(x,y) = \frac{12}{7} \cdot x(x+y), 0 < x < 1, 0 < y < 1$, zero elsewhere. Let $U = \min(X,Y)$ and $V = \max(X,Y)$. Find the joint pdf of U and V.

   Solution:

   (1) $U = \min(X,Y) = X,\ \ V = \max(X,Y) = Y$

$$f_{U,V}(u,v) = f(x = u,\ y = v) \cdot \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

$$= \frac{12}{7} \cdot u(u+v) \cdot \left| \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right|$$

$$= \frac{12}{7} \cdot u(u+v) \cdot 1 = \frac{12}{7} \cdot u(u+v)$$

(2) $U=\min(X,Y)=Y,\ \ V=\max(X,Y)=X$

$$\begin{aligned}
f_{U,V}(u,v) &= f(x=v,\ y=u)\cdot\left|\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\right| \\
&= \frac{12}{7}\cdot v(v+u)\cdot\left|\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}\right| \\
&= \frac{12}{7}\cdot v(v+u)\cdot|(-1)| = \frac{12}{7}\cdot v(v+u)
\end{aligned}$$

$$\therefore\ f_{U,V}(u,v)=\begin{cases}\frac{12}{7}\cdot u(u+v), & 0<u<v<1,\ \ u=x,\ v=y\\ \frac{12}{7}\cdot v(v+u), & 0<u<v<1,\ \ v=x,\ u=y\end{cases}$$

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3. (14%) **(Exercise 6.1.1)** Let $X_1,X_2,\cdots,X_n$ be a random sample from a $\Gamma(\alpha=3,\ \beta=\theta)$ distribution, $0<\theta<\infty$. Determine the mle of θ. Recall that gamma distribution is given by equation (3.3.1) as shown below:

$$f(x;\alpha,\beta)=\Gamma(\alpha,\beta)=\begin{cases}\dfrac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot e^{-x/\beta} & 0<x<\infty,\ \ \alpha>0,\ \beta>0,\ \Gamma(\alpha)>0\\ 0 & \text{elsewhere.}\end{cases}$$

Solution:

$$L(\theta,x)=\prod_{i=1}^{n}f(x_i;\theta),\quad x\in\Omega.\qquad X_1,X_2,\cdots,X_n\sim\text{iid},\quad \theta\text{ is unknown.}\qquad \hat{\theta}=\max_{\theta\in\Omega}L(\theta;x)$$

$$\begin{aligned}
f(x;\alpha,\beta)=\Gamma(\alpha,\beta) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot e^{-\frac{x}{\beta}}=\frac{1}{\Gamma(3)\cdot\theta^3}\cdot x^2\cdot e^{-\frac{x}{\theta}},\quad 0<x<1.\\
\Rightarrow\ \ L(\theta) &= \prod_{i=1}^{n}f(x_i\ ;\ 3,\theta)=\prod_{i=1}^{n}\frac{1}{\Gamma(3)\cdot\theta^3}\cdot x_i^2\cdot e^{-\frac{x_i}{\theta}}=(\frac{1}{\Gamma(3)\cdot\theta^3})^n\cdot(\prod_{i=1}^{n}x_i^2\cdot e^{-\frac{x_i}{\theta}})\\
&= (\frac{1}{\Gamma(3)\cdot\theta^3})^n\cdot e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i}\cdot\prod_{i=1}^{n}x_i^2\\
\Rightarrow\ \ l(\theta) &= \ln(L(\theta))=\ln\left((\frac{1}{\Gamma(3)\cdot\theta^3})^n\cdot e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i}\cdot\prod_{i=1}^{n}x_i^2\right)\\
&= -n\ln(\Gamma(3))-3n\ln(\theta)-\frac{1}{\theta}\sum_{i=1}^{n}x_i+2\sum_{i=1}^{n}\ln(x_i)
\end{aligned}$$

$$\Rightarrow\ \ \frac{\partial}{\partial\theta}l(\theta)=0$$

$$\Rightarrow\ \ \frac{\partial}{\partial\theta}\left(-n\ln(\Gamma(3))-3n\ln(\theta)-\frac{1}{\theta}\sum_{i=1}^{n}x_i+2\sum_{i=1}^{n}\ln(x_i)\right)=0$$

$$\Rightarrow -\frac{3n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i=0\ \ \rightarrow\ \ \frac{3n}{\theta}=\frac{1}{\theta^2}\sum_{i=1}^{n}x_i\ \ \rightarrow\ \ 3n\theta=\sum_{i=1}^{n}x_i$$

$$\therefore\ \ \hat{\theta}=\frac{1}{3n}\sum_{i=1}^{n}x_i=\frac{1}{3}\bar{x}$$

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4. (20%) **(Exercise 6.5.4)** Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be independent random samples from the two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$. Find the likelihood ratio Λ for testing the composite hypothesis $H_0 : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$.

Solution:

(1) $\theta_1 = \theta_2 = \theta_0$

$$L(\theta_0) = \frac{1}{(2\pi\theta_0)^{\frac{m+n}{2}}} \cdot \exp\{-\frac{1}{2\theta_0} \cdot (\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2)\}$$

$$l(\theta_0) = \ln L(\theta_0) = -\frac{m+n}{2} \cdot \ln 2\pi\theta_0 - \frac{1}{2\theta_0} \cdot (\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2)$$

$$\frac{\partial}{\partial\theta} l(\theta_0) = -\frac{m+n}{2} \cdot \frac{1}{2\pi\theta_0} \cdot 2\pi + \frac{1}{2\theta_0^2} \cdot (\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2) = 0$$

$$\Rightarrow \frac{m+n}{2\theta_0} = \frac{1}{2\theta_0^2} \cdot (\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2)$$

$$\therefore \hat{\theta}_0 = \frac{\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2}{m+n}$$

(2) $\theta_1 \neq \theta_2$

$$L(\theta_1, \theta_2) = (2\pi\theta_1)^{-\frac{n}{2}} \cdot \exp\{-\frac{1}{2\theta_1} \sum_{i=1}^{n} X_i^2\} \cdot (2\pi\theta_2)^{-\frac{m}{2}} \cdot \exp\{-\frac{1}{2\theta_2} \sum_{j=1}^{m} Y_j^2\}$$

$$l(\theta_1, \theta_2) = \ln L(\theta_1, \theta_2) = -\frac{n}{2} \cdot \ln 2\pi\theta_1 - \frac{1}{2\theta_1} \sum_{i=1}^{n} X_i^2 - \frac{m}{2} \cdot \ln 2\pi\theta_2 - \frac{1}{2\theta_2} \sum_{j=1}^{m} Y_j^2$$

$$\Rightarrow \frac{\partial}{\partial\theta_1} l(\theta_1, \theta_2) = 0 \quad \rightarrow -\frac{n}{2\theta_1} + \frac{1}{2\theta_1^2} \sum_{i=1}^{n} X_i^2 = 0 \quad \therefore \hat{\theta_1} = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

$$\Rightarrow \frac{\partial}{\partial\theta_2} l(\theta_1, \theta_2) = 0 \quad \rightarrow -\frac{m}{2\theta_2} + \frac{1}{2\theta_2^2} \sum_{j=1}^{m} Y_j^2 = 0 \quad \therefore \hat{\theta_2} = \frac{1}{m} \sum_{j=1}^{m} Y_j^2$$

(3) $\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_1^{\frac{n}{2}} \cdot \theta_2^{\frac{m}{2}}}{\theta^{\frac{m+n}{2}}}$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{(2\pi\theta_0)^{-\frac{m+n}{2}} \cdot \exp\{-\frac{1}{2\theta_0}(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2)\}}{(2\pi\theta_1)^{-\frac{n}{2}} \exp\{-\frac{1}{2\theta_1}\sum_{i=1}^{n} X_i^2\} \cdot (2\pi\theta_2)^{-\frac{m}{2}} \exp\{-\frac{1}{2\theta_2}\sum_{j=1}^{m} Y_j^2)\}}$$

$$= \frac{(2\pi\theta_0)^{-\frac{m+n}{2}} \cdot e^{-\frac{1}{2\theta_0}(m+n)\hat{\theta}}}{(2\pi\theta_1)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_1} m\theta_2} \cdot (2\pi\theta_2)^{-\frac{m}{2}} e^{-\frac{1}{2\theta_2} n\theta_1}} = \frac{\theta^{-\frac{m+n}{2}} \cdot e^{-\frac{m+n}{2}}}{\theta_1^{-\frac{n}{2}} \theta_2^{-\frac{m}{2}} e^{-\frac{n}{2}} e^{-\frac{m}{2}}} = \frac{\theta_1^{\frac{n}{2}} \cdot \theta_2^{\frac{m}{2}}}{\theta^{\frac{m+n}{2}}}$$

$$= \frac{(\frac{1}{n})^{\frac{n}{2}} \cdot (\frac{1}{m})^{\frac{m}{2}} \cdot (\sum_{i=1}^{n} X_i^2)^{\frac{n}{2}} \cdot (\sum_{j=1}^{m} Y_j^2)^{\frac{m}{2}}}{(\frac{1}{m+n})^{\frac{m+n}{2}} \cdot (\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j)^{\frac{m+n}{2}}} = \frac{(m+n)^{\frac{m+n}{2}}}{n^{\frac{n}{2}} \cdot m^{\frac{m}{2}}} \cdot \frac{(\sum_{i=1}^{n} X_i^2)^{\frac{n}{2}} \cdot (\sum_{j=1}^{m} Y_j^2)^{\frac{m}{2}}}{(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j)^{\frac{m+n}{2}}}$$

$$= \frac{(m+n)^{\frac{m+n}{2}}}{n^{\frac{n}{2}} \cdot m^{\frac{m}{2}}} \cdot \frac{(\frac{\sum_{i=1}^{n} X_i^2}{\sum_{j=1}^{m} Y_j^2})^{\frac{n}{2}}}{(\frac{\sum_{i=1}^{n} X_i^2}{\sum_{j=1}^{m} Y_j^2} + 1)^{\frac{m+n}{2}}}$$

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5. (20%) **(Exercise 7.1.3)** Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x; \theta) = 1/\theta,\ 0 < x < \theta,\ 0 < \theta < \infty$, zero elsewhere.

Show that $4Y_1$ is unbiased estimators of θ.

Solution:
By using equation (4.4.2) $g_k(y_k) = \frac{n!}{(k-1)!\times(n-k)!} \times (F(y_k))^{k-1}(1-F(y_k))^{n-k} \times f(y_k)$

$$\Rightarrow\ f_{Y_1}(y_1) = \frac{3!}{(1-1)!\times(3-1)!} \times (\frac{y_1}{\theta})^{1-1}(1-\frac{y_1}{\theta})^{3-1} \times \frac{1}{\theta} = \frac{3}{\theta}\cdot(1-\frac{y_1}{\theta})^2$$

$$\Rightarrow\ E\left(4y_1\right) = 4\int_0^\theta y_1 \cdot f_{Y_1}(y_1)\, dy_1 = 4\int_0^\theta y_1 \cdot (\frac{3}{\theta}\cdot(1-\frac{y_1}{\theta})^2)\, dy_1$$

$$= 12\int_0^1 u\cdot(1-u)^2\,\theta\, du = 12\cdot\theta\cdot(\frac{1}{2}u^2 - \frac{2}{3}u^3 + \frac{1}{4}u^4)\Big|_0^1 \quad (\text{let } u = \frac{y_1}{\theta},\ \theta\cdot du = dy_1)$$

$$= 12\cdot\theta\cdot(\frac{6}{12} - \frac{8}{12} + \frac{3}{12}) = 12\cdot\theta\cdot\frac{1}{12} = \theta$$

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6. (20%) **(Chapter 7.2)** Consider observations $x[i],\ i = 0 \sim N-1$ taken from the model

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n],\ n = 0, 1, \cdots, N-1.$$

The A and f_0 are known and $w[n] \sim N(0,\ \sigma^2)$, where σ^2 is known. Prove that the sufficient statistics for ϕ is

$$\left\{\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n),\ \sum_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)\right\} \overset{\text{Denoted as}}{=} \{T_1(x),\ T_2(x)\}$$

Hint: Use the Theorem that (Theorem 7.2.1)

$$f(x[0];\phi)\cdot f(x[2];\phi), \cdots, f(x[N-1];\phi) = k_1[u_1(x[0], x[1], \cdots, x[N-1]);\phi] \times k_2[x[0], \cdots, x[N-1]].$$

Hint: $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Solution:

$$P(x;\phi) = f(x[0];\phi)\cdots f(x[N-1];\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}}\cdot\exp\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A\cdot\cos(2\pi f_0 n + \phi))^2\}$$

The exponent can be expanded as below

$$\Rightarrow \sum_{n=0}^{N-1} x^2[n] - 2A\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n + \phi) + \sum_{n=0}^{N-1} A^2\cos^2(2\pi f_0 n + \phi)$$

$$= \sum_{n=0}^{N-1} x^2[n] - 2A\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)\cdot\cos(\phi) + 2A\sum_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)\cdot\sin(\phi) + \sum_{n=0}^{N-1} A^2\cos^2(2\pi f_0 n + \phi)$$

Individual samples' pdf multiply together can be decomposed as the multiplication result of two parts. The first part includes both the sufficient statistics and parameters. The second part consists only of coefficients, which do not involve any parameters.

$$\therefore\ P(x;\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}}\cdot\exp\{-\frac{1}{2\sigma^2}\cdot(\sum_{n=0}^{N-1} A^2\cos^2(2\pi f_0 n + \phi) - 2A\cdot T_1(x)\cdot\cos(\phi)$$

$$+ 2A\cdot T_2(x)\cdot\sin(\phi))\}\cdot\exp\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\}$$

$$= k_1[T_1(x),\ T_2(x),\ \phi] \times k_2[x]$$

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