11. Find the mean and variance of the sum $Y = \sum_{i=1}^{5} X_i$, where $X_1, ..., X_5$ are iid, having pdf $f(x) = 6x(1-x), 0 < x < 1$, zero elsewhere.

$$Y = X_1 + X_2 + X_3 + X_4 + X_5$$

$$E[Y] = E[X_1 + X_2 + X_3 + X_4 + X_5] = 5E[X] = \frac{5}{2}$$

$$E[X] = \int_0^1 x\, 6x(1-x)\, dx = \int_0^1 6x^2 - 6x^3\, dx = 2x^3 - \frac{3}{2}x^4 \Big|_0^1 = 2 - \frac{3}{2} = \frac{1}{2}$$

$$Var(Y) = Var(X_1 + X_2 + X_3 + X_4 + X_5) = 5Var(X) = 5 \times \frac{1}{20} = \frac{1}{4} \quad , \text{ iid}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

$$E[X^2] = \int_0^1 x^2\, 6x(1-x)\, dx = \int_0^1 6x^3 - 6x^4\, dx = \frac{6}{4}x^4 - \frac{6}{5}x^5 \Big|_0^1 = \frac{6}{4} - \frac{6}{5} = \frac{6}{20} = \frac{3}{10}$$

$$E[Y] = \frac{5}{2} \quad Var(Y) = \frac{1}{4}$$

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12. Let the independent random variables X_1 and X_2 have binomial distribution with parameters $n_1 = 3$, $p = \frac{2}{3}$ and $n_2 = 4$, $p = \frac{1}{2}$, respectively. Compute $P(X_1 = X_2)$.

$$X_1 \sim B(3, \tfrac{2}{3}) \qquad P_{X_1}(x_1) = C^3_{x_1} \frac{2}{3}^{x_1} \frac{1}{3}^{3-x_1}$$

$$X_2 \sim B(4, \tfrac{1}{2}) \qquad P_{X_2}(x_2) = C^4_{x_2} \frac{1}{2}^{x_2} \frac{1}{2}^{4-x_2} = C^4_{x_2} \frac{1}{2}^4$$

$$P(X_1 = X_2) = P(X_1 = X_2 = 0) + P(X_1 = X_2 = 1) + P(X_1 = X_2 = 2) + P(X_1 = X_2 = 3)$$

$$= P(X_1 = 0)P(X_2 = 0) + \dots + P(X_1 = 3)P(X_2 = 3)$$

$$= \left(1 \cdot \frac{2}{3}^0 \cdot \frac{1}{3}^3 \cdot 1 \cdot \frac{1}{2}^4\right) + \left(3 \cdot \frac{2}{3}^1 \cdot \frac{1}{3}^2 \cdot 4 \cdot \frac{1}{2}^4\right) + \left(3 \cdot \frac{2}{3}^2 \cdot \frac{1}{3}^1 \cdot 6 \cdot \frac{1}{2}^4\right)$$

$$+ \left(1 \cdot \frac{2}{3}^3 \frac{1}{3}^0 \cdot 4 \cdot \frac{1}{2}^4\right)$$

$$= \frac{1}{27} \times \frac{1}{16} + \frac{6}{27} \cdot \frac{4}{16} + \frac{12}{27} \cdot \frac{6}{16} + \frac{8}{27} \cdot \frac{4}{16}$$

$$= \frac{1 + 24 + 72 + 32}{27 \times 16} = \frac{129}{432} = \frac{43}{144}$$