

EE 3070 Statistics

Final Exam

May 30, 2023
10:10 ~ 12:00

Note: There are **6** problems with a total of 100 points on a single page. Please provide detailed answers on the answer sheets.

No credit without detail. Closed books. No calculator.

- (13%) Show that the mean \bar{X} of a random sample of size n from a distribution having pdf $f(x; \theta) = (1/\theta) \cdot e^{-(x/\theta)}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n .
- (13%) Let the joint pdf of X and Y be $f(x, y) = \frac{12}{7} \cdot x(x+y)$, $0 < x < 1$, $0 < y < 1$, zero elsewhere. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the joint pdf of U and V .
- (14%) Let X_1, X_2, \dots, X_n be a random sample from a $\Gamma(\alpha = 3, \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the mle of θ . Recall that gamma distribution is given by equation (3.3.1) as shown below:

$$f(x; \alpha, \beta) = \Gamma(\alpha, \beta) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot e^{-x/\beta} & 0 < x < \infty, \alpha > 0, \beta > 0, \Gamma(\alpha) > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

- (20%) Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_m be independent random samples from the two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$. Find the likelihood ratio Λ for testing the composite hypothesis $H_0 : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$.
- (20%) Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x; \theta) = 1/\theta$, $0 < x < \theta$, $0 < \theta < \infty$, zero elsewhere. Show that $4Y_1$ is unbiased estimators of θ .
- (20%) Consider observations $x[i]$, $i = 0 \sim N-1$ taken from the model

$$x[n] = A \cos(2\pi f_0 n + \phi) + w[n], \quad n = 0, 1, \dots, N-1.$$

The A and f_0 are known and $w[n] \sim N(0, \sigma^2)$, where σ^2 is known. Prove that the sufficient statistics for ϕ is

$$\left\{ \sum_{n=0}^{N-1} x[n] \cos(2\pi f_0 n), \sum_{n=0}^{N-1} x[n] \sin(2\pi f_0 n) \right\} \stackrel{\text{Denoted as}}{=} \{T_1(x), T_2(x)\}$$

Hint: Use the Theorem that (Theorem 7.2.1)

$$f(x[0]; \phi) \cdot f(x[2]; \phi), \dots, f(x[N-1]; \phi) = k_1[u_1(x[0], x[1], \dots, x[N-1]); \phi] \times k_2[x[0], \dots, x[N-1]].$$

Hint: $\cos(A+B) = \cos A \cos B - \sin A \sin B$