4 **6.5.4.** Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independent random samples from the two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$.

(a) Find the likelihood ratio Λ for testing the composite hypothesis $H_0 : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$.

(b) This Λ is a function of what F-statistic that would actually be used in this test?

$X \sim N(0, \theta_1) \qquad Y \sim N(0, \theta_2) \qquad \Lambda = \dfrac{L(\hat{\omega})}{L(\hat{\Omega})}$

(a) $\theta_1 = \theta_2 = \theta_0$ 5%

$$L(\theta_0) = \frac{1}{(2\pi\theta_0)^{\frac{m+n}{2}}} e^{-\frac{1}{2\theta_0}\left(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2\right)}$$

$$l(\theta_0) = -\frac{m+n}{2}\ln(2\pi\theta_0) + \frac{-1}{2\theta_0}\left(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2\right)$$

$$\frac{\partial}{\partial\theta} l(\theta_0) = -\frac{m+n}{2} \times \frac{1}{2\pi\theta_0} \times 2\pi + \frac{1}{2\theta_0^2}\left(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2\right) = 0 \qquad \frac{m+n}{2\theta_0} = \frac{1}{2\theta_0^2}\left(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2\right)$$

$$\hat{\theta}_0 = \frac{\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2}{m+n}$$

$\theta_1 \neq \theta_2$ 5%

$$L(\theta_1, \theta_2) = (2\pi\theta_1)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_1}\sum_{i=1}^{n} x_i^2} \cdot (2\pi\theta_2)^{-\frac{m}{2}} e^{-\frac{1}{2\theta_2}\sum_{j=1}^{m} y_j^2}$$

$$l(\theta_1, \theta_2) = -\frac{n}{2}\ln(2\pi\theta_1) + \left(-\frac{1}{2\theta_1}\sum_{i=1}^{n} x_i^2\right) + \left(\frac{-m}{2}\right)\ln(2\pi\theta_2) + \left(\frac{-1}{2\theta_2}\sum_{j=1}^{m} y_j^2\right)$$

$$\frac{\partial l(\theta_1, \theta_2)}{\partial\theta_1} = 0 \Rightarrow \frac{-n}{2\theta_1} + \frac{1}{2\theta_1^2}\sum_{i=1}^{n} x_i^2 = 0 \qquad \hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial l(\theta_1, \theta_2)}{\partial\theta_2} = 0 \Rightarrow \frac{-m}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{j=1}^{m} y_j^2 = 0 \qquad \hat{\theta}_2 = \frac{1}{m}\sum_{j=1}^{m} x_j^2$$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{(2\pi\theta_0)^{-\frac{m+n}{2}} e^{-\frac{1}{2\theta_0}\underbrace{\left(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} Y_j^2\right)}_{(m+n)\hat{\theta}}}}{(2\pi\theta_1)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_1}\underbrace{\sum_{i=1}^{n} x_i^2}_{m\theta_2}} (2\pi\theta_2)^{-\frac{m}{2}} e^{-\frac{1}{2\theta_2}\underbrace{\sum_{j=1}^{m} y_j^2}_{n\theta_1}}} = \frac{\theta^{-\frac{m+n}{2}} \cancel{e^{-\left(\frac{m+n}{2}\right)}}}{\theta_1^{-\frac{n}{2}} \theta_2^{-\frac{m}{2}} \cancel{e^{-\frac{n}{2}} e^{-\frac{m}{2}}}}$$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_1^{\frac{n}{2}} \theta_2^{\frac{m}{2}}}{\theta^{\frac{m+n}{2}}} \Rightarrow \frac{\left(\frac{1}{n}\right)^{\frac{n}{2}} \left(\frac{1}{m}\right)^{\frac{m}{2}} \left(\sum_{i=1}^{n} X_i^2\right)^{-\frac{n}{2}} \left(\sum_{j=1}^{m} Y_j^2\right)^{-\frac{m}{2}}}{\left(\frac{1}{m+n}\right)^{\frac{m+n}{2}} \left(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2\right)^{-\frac{m+n}{2}}}$$

$$\Lambda = \frac{(m+n)^{\frac{m+n}{2}}}{n^{\frac{n}{2}} m^{\frac{m}{2}}} \frac{\left(\sum_{i=1}^{n} X_i^2\right)^{\frac{n}{2}} \left(\sum_{j=1}^{m} Y_j^2\right)^{\frac{m}{2}}}{\left(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2\right)^{\frac{m+n}{2}}}$$

(b) $F = \dfrac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\frac{1}{m}\sum_{j=1}^{m} Y_j^2}$ The relevant F statistic $F = \dfrac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\frac{1}{m}\sum_{j=1}^{m} Y_j^2}$ #