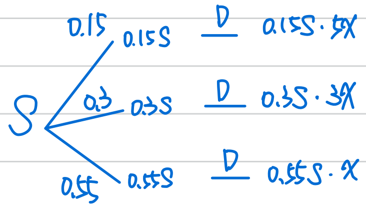
1. At the beginning of a study of individuals, 15% were classified as heavy smokers, 30% were classified as light smokers, and 55% were classified as nonsmokers. In the five-year study, it was determined that the death rates of the heavy and light smokers were five and three times that of the nonsmokers, respectively. A randomly selected participant died over the five-year period: calculate the probability that the participant was a nonsmoker.

令總人數為 S . nonsmokers 的死亡率為 x



$$P(\text{nonsmoker} \mid \text{died}) = \frac{P(\text{nonsmoker, died})}{P(\text{died})} = \frac{0.55 \cdot S \cdot x}{0.55S\cdot x + 0.9S\cdot x + 0.75S\cdot x} = \frac{1}{4}$$

3. Let X have the pmf $p(x) = \frac{1}{3}$, $x = -1, 0, 1$. Find the pmf of $Y = X^2$.

$Y = X^2$

$Y : \{1, 0\}$

$P(Y=1) = P(X=1) + P(X=-1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$P(Y=0) = P(X=0) = \frac{1}{3}$

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{\frac{1}{9}x^2}{3x^2} = \frac{1}{27}$$

5. Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere.
(a) Compute $E(X^3)$.
(b) Show that $Y = X^3$ has a uniform $(0, 1)$ distribution.
(c) Compute $E(Y)$ and compare this result with the answer obtained in part.

(a)

$$E(X^3) = \int_0^1 x^3 3x^2\, dx = 3 \cdot \frac{1}{6} x^6 \Big|_0^1 = \frac{1}{2}$$

(b)

(1) $Y = X^3$ $\quad S_Y = \{0 < y < 1\}$, $x = y^{1/3}$

$$\therefore f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{3x^2}{3x^2} = 1 \quad , 0 < y < 1$$

$\therefore$ Y is a uniform $(0,1)$

(2)

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^3 \le y) \\ &= P(X \le y^{1/3}) \\ &= \int_0^{y^{1/3}} 3x^2\, dx = x^3 \Big|_0^{y^{1/3}} = y \end{aligned}$$

(c) $E[Y] = \int_0^1 y\, dy = \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2}$

6. Suppose X_1 and X_2 have the joint pdf $f_{X_1,X_2}(x_1,x_2) = e^{-(x_1+x_2)}$, $0 < x_i < \infty, i = 1,2$, zero elsewhere.
(a) Use formula (2.2.2) to find the pdf of $Y_1 = X_1 + X_2$.
(b) Find the mgf of Y_1.
Note. formula (2.2.2) $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2)dy_2$

(a)

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(Y_1 - X_2, X_2)\, dX_2 = \int_{-\infty}^{\infty} e^{-(Y_1 - X_2 + X_2)} dX_2$$

$X_1 = Y_1 - X_2$

$$= \int_{-\infty}^{\infty} e^{-y_1} dx_2 = e^{-y_1}$$

(b)

$$M_X(t) = E(e^{ty_1}) = \int_{-\infty}^{\infty} e^{-y_1} e^{ty_1} dy_1 = \int_{-\infty}^{\infty} e^{-y_1(1-t)} dy_1$$

$$= \frac{1}{1-t} e^{-y_1} \Big|_{-\infty}^{\infty} = \frac{1}{1-t}, \quad t < 1$$

7. Let the joint pdf of X and Y be given by

$$f(x,y) = \begin{cases} \dfrac{2}{(1+x+y)^3} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(a) Compute the marginal pdf of X and the conditional pdf of Y, given $X = x$.
(b) For a fixed $X = x$, compute $E(1+x+Y|x)$ and use the result to compute $E(Y|x)$.

(a) 求 $f(y|x) =$

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{2}{(1+x+y)^3}}{1/(1+x)^2} = \frac{2(1+x)^2}{(1+x+y)^3}, \quad 0<x, y<\infty$$

$$f_X(x) = \int_0^\infty \frac{2}{(1+x+y)^3}\,dy = \frac{1}{(1+x)^2}$$

(b) $E(1+x+Y|x) = 1+x+E[Y|x] = 1+x+1+x = 2+2x$

$$E[Y|X=x] = \int_0^\infty y\, f(y|x)\,dy = \int_0^\infty \frac{2y(1+x)^2}{(1+x+y)^3}\,dy, \quad 令\ u = 1+x+y$$

$$= \int_{1+x}^\infty \frac{2(1+x)^2}{u^3}(u-x-1)\,du$$

$$= \int_{1+x}^\infty \left(\frac{2(1+x)^2}{u^2} - \frac{2x(1+x)^2}{u^3} - \frac{2(1+x)^2}{u^3}\right) du$$

$$= 2(1+x)^2 \int_{1+x}^\infty \left(\frac{1}{u^2} - \frac{x}{u^3} - \frac{1}{u^3}\right) du$$

$$= 2(1+x)^2 \left(\frac{-1}{u} - \frac{-x}{2u^2} - \frac{-1}{2u^2}\right)\Bigg|_{1+x}^\infty = 2(1+x)^2\left(\frac{1}{1+x} - \frac{x}{2(1+x)^2} - \frac{1}{2(1+x)^2}\right)$$

$$= 2(1+x) - x - 1 = 1+x$$

8. Let X and Y have the joint pmf $p(x, y) = \frac{1}{7}, (0,0), (1,0), (0,1), (1,1), (2,1), (1,2), (2,2)$, zero elsewhere. Find the correlation coefficient ρ.

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\frac{2}{7}}{4/7} = 1/2$$

$$P(x,y) = \begin{cases} \frac{1}{7}, & (0,0)(1,0)(0,1)(1,1)(2,1)(1,2)(2,2) \\ 0, & \text{else} \end{cases}$$

$$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$
$$= E[XY] - E[X]E[Y] = \frac{9}{7} - 1 = \frac{2}{7}$$

$$\sigma_X \sigma_Y = \sqrt{\frac{4}{7}} \cdot \sqrt{\frac{4}{7}} = \frac{4}{7}$$

$$E[XY] = 0 + 0 + 0 + \frac{1}{7} + \frac{2}{7} + \frac{2}{7} + \frac{4}{7} = \frac{9}{7}$$

$$P_X(x) = \begin{cases} P(X=0, Y=0) + P(X=0, Y=1) = \frac{2}{7}, & X=0 \\ P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2) = \frac{3}{7}, & X=1 \\ P(X=2, Y=1) + P(X=2, Y=2) = \frac{2}{7}, & X=2 \end{cases}$$

$$E[X] = 0 \cdot \frac{2}{7} + 1 \cdot \frac{3}{7} + 2 \cdot \frac{2}{7} = \frac{7}{7} = 1$$

$$E[X^2] = 0 \cdot \frac{2}{7} + 1 \cdot \frac{3}{7} + 4 \cdot \frac{2}{7} = \frac{11}{7}$$

$$\sigma_X^2 = E[X^2] - E[X]^2 = \frac{11}{7} - 1 = \frac{4}{7}$$

$$P_Y(y) = \begin{cases} P(X=0, Y=0) + P(X=1, Y=0) = \frac{2}{7}, & Y=0 \\ P(X=0, Y=1) + P(X=1, Y=1) + P(X=2, Y=1) = \frac{3}{7}, & Y=1 \\ P(X=1, Y=2) + P(X=2, Y=2) = \frac{2}{7}, & Y=2 \end{cases}$$

$$E[Y] = 0 \cdot \frac{2}{7} + 1 \cdot \frac{3}{7} + 2 \cdot \frac{2}{7} = 1$$

$$E[Y^2] = 0 \cdot \frac{2}{7} + 1 \cdot \frac{3}{7} + 4 \cdot \frac{2}{7} = \frac{11}{7}$$

$$\sigma_Y^2 = E[Y^2] - E[Y]^2 = \frac{11}{7} - 1 = \frac{4}{7}$$