7.1.1. Show that the mean $\overline{X}$ of a random sample of size n from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-(x/\theta)}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n.

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty$$

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = n \times \frac{1}{n} E[X_1] = \theta$$

$$E[X_1] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x}{\theta} e^{-\frac{x}{\theta}}\,dx$$ ，令 $u = \frac{x}{\theta}$ ，$\theta\,du = dx$

$$= \theta \int_0^\infty u e^{-u}\,du$$ ，

u	e^{-u}
1	$-e^{-u}$
0	e^{-u}

$$= \theta$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} n\, Var(X_1) = \frac{Var(X_1)}{n} = \frac{\theta^2}{n}$$

$$Var(X_1) = E[X_1^2] - E[X_1]^2 = 2\theta^2 - \theta^2 = \theta^2$$

$$E[X_1^2] = \int_0^\infty \frac{x^2}{\theta} e^{-\frac{x}{\theta}}\,dx = \theta \int_0^\infty u^2 e^{-u}\,\theta\,du$$

$+\ u^2$	e^{-u}
$-\ 2u$	$-e^{-u}$
$+\ 2$	e^{-u}
$-\ 0$	e^{u}

$$= \theta^2 \left(-u^2 e^{-u} - 2u e^{-u} + 2e^{u}\right)\Big|_0^\infty$$

$$= \theta^2 (0 - 0 + 0 + 0 - 0 + 2) = 2\theta^2$$