EE 3070 Statistics

Homework #1 Solution

Due at 23:59, March 31, 2023
online submission to eeclass systems

1. At the beginning of a study of individuals, 15% were classified as heavy smokers, 30% were classified as light smokers, and 55% were classified as nonsmokers. In the five-year study, it was determined that the death rates of the heavy and light smokers were five and three times that of the nonsmokers, respectively. A randomly selected participant died over the five-year period: calculate the probability that the participant was a nonsmoker.

   Solution:
   Let the total number of participant be S and the death rate of nonsmokers be x.

$$\begin{aligned} P(\text{nonsmokers} \mid \text{total death}) &= \frac{P(\text{the nonsmokers who actually died})}{P(\text{total death})} \\ &= \frac{(0.55 \times S) \times x}{(0.55 \times S) \times x + (0.30 \times S) \times 3x + (0.15 \times S) \times 5x} \\ &= \frac{0.55}{0.55 + 0.9 + 0.75} = \frac{0.55}{2.2} = 0.25 \\ &= \frac{1}{4} \end{aligned}$$

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2. Let the space of the random variable X be $\mathcal{D} = \{x : 0 < x < 1\}$. If $D_1 = \{x : 0 < x < \frac{1}{2}\}$ and $D_2 = \{x : \frac{1}{2} \leq x < 1\}$, find $P_X(D_2)$ if $P_X(D_1) = \frac{1}{4}$.

   Solution:
   We have $P_X(D_1) = \frac{1}{4}$ and we want to find $P_X(D_2)$. Note that $\mathcal{D}$ is the space of the random variable X and we have partitioned it into two sets D_1 and D_2.
   By the law of total probability, we have:

$$P_X(D_2) = P_X(\mathcal{D}) - P_X(D_1) = 1 - \frac{1}{4} = \frac{3}{4}$$

   Therefore

$$P_X(D_2) = \frac{3}{4}$$

   which means that the probability of X taking on a value in the interval $\left[\frac{1}{2}, 1\right]$ is $\frac{3}{4}$.

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3. Let X have the pmf $p(x) = \frac{1}{3}$, $x = -1, 0, 1$. Find the pmf of $Y = X^2$.

Solution:
Since $Y = X^2, X = \{-1, 0, 1\}$, we have $Y = \{0, 1\}$. Therefore

$$P(Y = 1) = P(X = 1) + P(X = -1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$
$$P(Y = 0) = P(X = 0) = \frac{1}{3}$$

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4. Let X have the pdf $f(x) = \frac{1}{9}x^2$, $0 < x < 3$, zero elsewhere. Find the pdf of $Y = X^3$.

Solution:
For $0 < y < 27$,

$$x = y^{\frac{1}{3}}, \quad \frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}, \quad f_Y(y) = \frac{1}{3}y^{-\frac{2}{3}} \cdot \frac{1}{9}y^{\frac{2}{3}} = \frac{1}{27} \text{ or simply } f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{\frac{1}{9}x^2}{3x^2} = \frac{1}{27}$$

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5. Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere.
(a) Compute $E(X^3)$.
(b) Show that $Y = X^3$ has a uniform $(0, 1)$ distribution.
(c) Compute $E(Y)$ and compare this result with the answer obtained in part.

Solution:
(a) We can use the definition of the expected value to find $E(X^3)$:

$$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)dx = \int_0^1 x^3(3x^2)dx = 3\int_0^1 x^5 dx$$
$$= 3 \cdot \frac{1}{6} \cdot x^6 \Big|_0^1 = \frac{1}{2}$$

(b) To show that $Y = X^3$ has a uniform $(0, 1)$ distribution, we need to find the CDF of Y and show that it is equal to the CDF of a uniform distribution on $(0, 1)$. The CDF of Y is:

$$F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3}) = \int_0^{y^{1/3}} 3x^2 dx = y$$

where we can clearly see that $F_Y(y) = y$ for $0 < y < 1$ is a uniform distribution on $(0, 1)$. Therefore, we have shown that $Y = X^3$ has a uniform $(0, 1)$ distribution.

(c) To compute $E(Y)$, we can use the definition of the expected value:

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^1 y \cdot 1 \ dy = \frac{1}{2} \cdot y^2\Big|_0^1 = \frac{1}{2}$$

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6. Suppose X_1 and X_2 have the joint pdf $f_{X_1,X_2}(x_1,x_2) = e^{-(x_1+x_2)}$, $0 < x_i < \infty, i = 1,2$, zero elsewhere.
(a) Use formula (2.2.2) to find the pdf of $Y_1 = X_1 + X_2$.
(b) Find the mgf of Y_1.
Note. formula (2.2.2) $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2)dy_2$

Solution:
(a) We have $f_{X_1,X_2}(x_1,x_2) = e^{-(x_1+x_2)}$, for $0 < x_i < \infty$, $i = 1,2$, and zero elsewhere. To find the pdf of $Y_1 = X_1 + X_2$, we use the transformation method as follows:

$$\begin{aligned}
F_{Y_1}(y_1) &= P(Y_1 \le y_1) = P(X_1 + X_2 \le y_1) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{y_1 - x_2} f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \quad \text{(by definition of joint pdf)} \\
&= \int_0^{y_1} \int_0^{y_1 - x_2} e^{-(x_1+x_2)}dx_1dx_2 \quad \text{(since the pdf is zero elsewhere)} \\
&= \int_0^{y_1} e^{-x_2} \int_0^{y_1 - x_2} e^{-x_1}dx_1dx_2 \\
&= \int_0^{y_1} e^{-x_2}(1 - e^{-(y_1 - x_2)})dx_2 \\
&= \int_0^{y_1} e^{-y_1}dx_2 \\
&= y_1e^{-y_1}, \quad \text{for } y_1 > 0,
\end{aligned}$$

Therefore, the pdf of Y_1 is given by

$$f_{Y_1}(y_1) = \frac{d}{dy_1}F_{Y_1}(y_1) = y_1e^{-y_1}, \quad \text{for} \ \ y_1 > 0$$

(b) To find the mgf of Y_1, we use the definition:

$$\begin{aligned}
M_{Y_1}(t_1) &= E(e^{t_1Y_1}) = \int_{-\infty}^{\infty} e^{t_1y_1} f_{Y_1}(y_1)\ dy_1 = \int_0^{\infty} e^{t_1y_1}y_1e^{-y_1}\ dy_1 \\
&= \int_0^{\infty} y_1 \cdot e^{-(1-t_1)y_1}\ dy_1 \ \text{(by using integration by parts} \int u\ dv = u \cdot v - \int v\ du) \\
&= \int_0^{\infty} y_1 \cdot d\left(\frac{1}{-(1-t_1)}e^{-(1-t_1)y_1}\right) \ (\because \int xe^{ax}dx = x \cdot \frac{1}{a}e^{ax} - \int \frac{1}{a} \cdot e^{ax}dx) \\
&= \left(y_1 \cdot \frac{1}{-(1-t_1)}e^{-(1-t_1)y_1}\right)\Big|_0^{\infty} - \int_0^{\infty} \frac{1}{-(1-t_1)}e^{-(1-t_1)y_1}dy_1 \\
&= \left(\infty \cdot e^{-\infty} - 0 \cdot e^0\right) - \frac{1}{-(1-t_1)} \cdot \left(\frac{1}{-(1-t_1)} \cdot e^{-(1-t_1)y_1}\Big|_0^{\infty}\right) \ (\because \frac{1}{a}\int e^{ax}dx = \frac{1}{a} \cdot \frac{1}{a}e^{ax}) \\
&= (0 - 0) - \frac{1}{(1-t_1)^2} \cdot (e^{-\infty} - e^0) = 0 - \frac{1}{(1-t_1)^2} \cdot (0 - 1) \\
&= \frac{1}{(1-t_1)^2}, \quad \text{for } t_1 < 1.
\end{aligned}$$

Therefore, the moment generating function of Y_1 is given by $M_{Y_1}(t_1) = \frac{1}{(1-t_1)^2}$, for $t_1 < 1$.

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7. Let the joint pdf of X and Y be given by

$$f(x,y) = \begin{cases} \dfrac{2}{(1+x+y)^3} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(a) Compute the marginal pdf of X and the conditional pdf of Y, given $X = x$.
(b) For a fixed $X = x$, compute $E(1+x+Y \big| x)$ and use the result to compute $E(Y \big| x)$.

Solution:
(a) To find the marginal pdf of X, we integrate the joint pdf over all possible values of Y:

$$\begin{aligned} f_X(x) &= \int_0^\infty f(x,y)\ dy \\ &= \int_0^\infty \frac{2}{(1+x+y)^3}\ dy \\ &= \frac{1}{(1+x)^2} \end{aligned}$$

To find the conditional pdf of Y given $X = x$, we use the conditional probability formula:

$$\begin{aligned} f_{Y|X}(y \big| x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{2}{(1+x+y)^3} \cdot \frac{(1+x)^2}{1} \\ &= \frac{2(1+x)^2}{(1+x+y)^3} \end{aligned}$$

(b) For a fixed $X = x$, the conditional expectation of $1 + x + Y$ is:

$$\begin{aligned} E(1+x+Y \big| X=x) &= E(1+x \big| X=x) + E(Y \big| X=x) \\ &= 1 + x + E(Y \big| X=x) \end{aligned}$$

To find $E(Y \big| X = x)$, we use the definition of conditional expectation:

$$\begin{aligned} E(Y \big| X=x) &= \int_0^\infty y f_{Y|X}(y|x) dy \\ &= \int_0^\infty y \cdot \frac{2(1+x)^2}{(1+x+y)^3} dy \\ &= 1 + x \end{aligned}$$

Therefore, we have:

$$\begin{aligned} E(1+x+Y|X=x) &= 1 + x + E(Y|X=x) \\ &= 1 + x + 1 + x \\ &= 2 + 2x \end{aligned}$$

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8. Let X and Y have the joint pmf $p(x,y) = \frac{1}{7}, (0,0), (1,0), (0,1), (1,1), (2,1), (1,2), (2,2)$, zero elsewhere. Find the correlation coefficient ρ.

Solution:
We first need to compute the mean and variance of X and Y as follows:

$$E(X) = \sum_{x=0}^{2} x \sum_{y=0}^{2} p(x,y) = (0+1+0+1+2+1+2) \cdot \frac{1}{7} = \frac{7}{7} = 1$$

$$E(Y) = \sum_{y=0}^{2} y \sum_{x=0}^{2} p(x,y) = (0+0+1+1+1+2+2) \cdot \frac{1}{7} = \frac{7}{7} = 1$$

$$E(XY) = \sum_{x=0,y=0}^{2} xy \sum_{x=0}^{2} p(x,y) = (0+0+0+1+2+2+4) \cdot \frac{1}{7} = \frac{9}{7}$$

$$E(X^2) = \sum_{x=0}^{2} x^2 \sum_{y=0}^{2} p(x,y) = (0^2+1^2+0^2+1^2+2^2+1^2+2^2) \cdot \frac{1}{7} = \frac{11}{7}$$

$$E(Y^2) = \sum_{y=0}^{2} y^2 \sum_{x=0}^{2} p(x,y) = (0^2+0^2+1^2+1^2+1^2+2^2+2^2) \cdot \frac{1}{7} = \frac{11}{7}$$

Now, we can compute the covariance:

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{9}{7} - 1 \cdot 1 = \frac{2}{7}$$

Finally, we can compute the standard deviations:

$$\begin{aligned}
\sigma_x &= \sqrt{E(X^2) - (E(X))^2} = \sqrt{\text{Var}(X)} \\
&= \sqrt{(\frac{11}{7}) - (1)^2} = \sqrt{\frac{11}{7} - \frac{7}{7}} \\
&= \sqrt{\frac{4}{7}} \\
\sigma_y &= \sqrt{E(Y^2) - (E(Y))^2} = \sqrt{\text{Var}(Y)} \\
&= \sqrt{(\frac{11}{7}) - (1)^2} = \sqrt{\frac{11}{7} - \frac{7}{7}} \\
&= \sqrt{\frac{4}{7}}
\end{aligned}$$

Finally, we can plug all the values into the formula for the correlation coefficient:

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_x \cdot \sigma_y} = \frac{\frac{2}{7}}{\sqrt{\frac{4}{7}} \cdot \sqrt{\frac{4}{7}}} = \frac{1}{2}$$

Therefore, the correlation coefficient between X and Y is given by

$$\rho = \frac{1}{2}$$

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9. Let $f(x_1,x_2,x_3) = e^{-(x_1+x_2+x_3)}, 0 < x_1 < \infty, 0 < x_2 < \infty, 0 < x_3 < \infty$, zero elsewhere, be the joint pdf of X_1, X_2, X_3.
(a) Compute $P(X_1 < X_2 < X_3)$ and $P(X_1 = X_2 < X_3)$.
(b) Determine the joint mgf of X_1, X_2 and X_3. Are these random variables independent?

Solution:
(a) To compute $P(X_1 < X_2 < X_3)$, we use the definition:

$$\begin{aligned}
P(X_1 < X_2 < X_3) &= \iiint_{x_1<x_2<x_3} f(x_1,x_2,x_3)\ dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} \int_0^{x_2} e^{-(x_1+x_2+x_3)}\ dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} \int_0^{x_2} e^{-x_1} \cdot e^{-(x_2+x_3)}\ dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} e^{-(x_2+x_3)} \cdot (\int_0^{x_2} e^{-x_1}\ dx_1)\ dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} e^{-(x_2+x_3)} \cdot (-e^{-x_1}\Big|_0^{x_2})\ dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} e^{-(x_2+x_3)} \cdot (1-e^{-x_2})\ dx_2dx_3 \\
&= \int_0^\infty e^{-x_3} \cdot (\int_0^{x_3} e^{-x_2}(1-e^{-x_2})\ dx_2)\ dx_3 \\
&= \int_0^\infty e^{-x_3} \cdot (\int_0^{x_3} (e^{-x_2}-e^{-2x_2})\ dx_2)\ dx_3 \\
&= \int_0^\infty e^{-x_3} \cdot (-e^{-x_2}\Big|_0^{x_3} - (-\frac{1}{2}e^{-2x_2}\Big|_0^{x_3}))\ dx_3 \\
&= \int_0^\infty e^{-x_3} \cdot (1-e^{-x_3}+\frac{1}{2}e^{-2x_3}-\frac{1}{2})\ dx_3 \\
&= \int_0^\infty \frac{1}{2}e^{-x_3} - e^{-2x_3} + \frac{1}{2}e^{-3x_3}\ dx_3 \\
&= -\frac{1}{2}e^{-x_3}\Big|_0^\infty - (-\frac{1}{2}e^{-2x_3}\Big|_0^\infty) + \frac{1}{2}\cdot(-\frac{1}{3}e^{-3x_3}\Big|_0^\infty) \\
&= -\frac{1}{2}\cdot(0-1) + \frac{1}{2}\cdot(0-1) - \frac{1}{6}\cdot(0-1) \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \\
&= \frac{1}{6}
\end{aligned}$$

To compute $P(X_1 = X_2 < X_3)$, we also use the definition:

$$\begin{aligned}
P(X_1 = X_2 < X_3) &= \iiint_{x_1=x_2<x_3} f(x_1,x_2,x_3)\ dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} \int_{x_2}^{x_2} e^{-(x_1+x_2+x_3)}\ dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} e^{-(x_2+x_3)} \cdot (\int_{x_2}^{x_2} e^{-x_1}\ dx_1)\ dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} e^{-(x_2+x_3)} \cdot 0\ dx_2dx_3 \\
&= 0
\end{aligned}$$

Or, if you think this is hard to understand, here's another way:
Let

$$Z_1 = X_1 + X_2, \quad Z_2 = X_2 - X_3, \quad Z_3 = X_2$$

so that we have

$$X_1 = Z_1 + Z_3, \quad X_2 = Z_3, \quad X_3 = Z_3 - Z_2$$

Then

$$\begin{aligned}
f_{Z_1,Z_2,Z_3}(z_1, z_2, z_3) &= f_{X_1,X_2,X_3}(z_1 + z_3, z_3, z_3 - z_2) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
&= e^{-(z_1+z_3+z_3+z_3-z_2)} \cdot (-0 + 0 - (-1)) \\
&= e^{-(z_1-z_2+3z_3)}, \quad -\infty < z_1, z_2 < \infty, \quad 0 < z_3 < \infty
\end{aligned}$$

Therefore

$$P(X_1 = X_2 < X_3) = P(Z_1 = 0, \ Z_2 < 0) = 0$$

(b) To find the joint moment generating function (MGF), we compute:

$$\begin{aligned}
M_X(t_1, t_2, t_3) &= E\left[e^{t_1X_1+t_2X_2+t_3X_3}\right] \\
&= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(x_1+x_2+x_3)} \cdot e^{t_1x_1+t_2x_2+t_3x_3} \ dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(1-t_1)x_1} \cdot e^{-(1-t_2)x_2} \cdot e^{-(1-t_3)x_3} \ dx_1dx_2dx_3 \\
&= (\int_0^\infty e^{-(1-t_1)x_1} \ dx_1) \cdot (\int_0^\infty e^{-(1-t_2)x_2} \ dx_2) \cdot (\int_0^\infty e^{-(1-t_3)x_3} \ dx_3) \\
&= (\frac{1}{-(1-t_1)} \cdot (0-1)) \cdot (\frac{1}{-(1-t_2)} \cdot (0-1)) \cdot (\frac{1}{-(1-t_3)} \cdot (0-1)) \\
&= \frac{1}{(1-t_1)(1-t_2)(1-t_3)}
\end{aligned}$$

Note that the MGF of X_1, X_2, and X_3 factors into the product of their individual MGFs:

$$M_{X_1}(t_1) = \frac{1}{1-t_1}, \quad M_{X_2}(t_2) = \frac{1}{1-t_2}, \quad M_{X_3}(t_3) = \frac{1}{1-t_3}$$

Therefore, X_1, X_2, and X_3 are independent.

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10. Let X_1, X_2 and X_3 be iid with common pdf $f(x) = e^{-x}, \ x > 0$, zero elsewhere. Find the joint pdf of $Y_1 = X_1$, $Y_2 = X_1 + X_2$ and $Y_3 = X_1 + X_2 + X_3$.

Solution:
We have X_1, X_2, X_3 are independent and identically distributed random variables with pdf $f(x) = e^{-x}, \ x > 0$, zero elsewhere.
(1) For $Y_1 = X_1$,

$$f_Y(y) = e^{-y}, \quad y > 0$$

(2) For $Y_2 = X_1 + X_2$,

$$\begin{aligned} M_{Y_2}(y_2) &= E[e^{ty_2}] = E[e^{t(x_1+x_2)}] = E[e^{tx_1}] \cdot E[e^{tx_2}] \\ &= \frac{1}{1-t} \cdot \frac{1}{1-t} = \frac{1}{(1-t)^2} \ \sim \ \text{Gamma}(2,\,1) \end{aligned}$$

(3) For $Y_3 = X_1 + X_2 + X_3$,

$$\begin{aligned} M_{Y_3}(y_3) &= E[e^{ty_3}] = E[e^{t(x_1+x_2+x_3)}] = E[e^{tx_1}] \cdot E[e^{tx_2}] \cdot E[e^{tx_3}] \\ &= \frac{1}{1-t} \cdot \frac{1}{1-t} \cdot \frac{1}{1-t} = \frac{1}{(1-t)^3} \ \sim \ \text{Gamma}(3,\,1) \end{aligned}$$

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11. Find the mean and variance of the sum $Y = \sum_{i=1}^{5} X_i$, where $X_1, ..., X_5$ are iid, having pdf $f(x) = 6x(1-x), 0 < x < 1$, zero elsewhere.

Solution:
Since $X_1, ..., X_5$ are iid, we have $\mathrm{E}(X_i) = \int_{-\infty}^{\infty} x \cdot f(x)\ dx$ and $\mathrm{Var}(X_i) = \mathrm{E}(X_i^2) - (\mathrm{E}(X_i))^2$. Thus, we have:

$$\begin{aligned} \mathrm{E}(X) &= \int_0^1 x \cdot 6x(1-x)dx = 6 \cdot \int_0^1 x^2 - x^3 dx \\ &= 6 \cdot (\frac{1}{3}x^3\Big|_0^1 - \frac{1}{4}x^4\Big|_0^1) = 6 \cdot (\frac{4}{12} - \frac{3}{12}) = \frac{6}{12} \\ &= \frac{1}{2} \\ Y &= X_1 + X_2 + X_3 + X_4 + X_5 \\ \mathrm{E}(Y) &= \mathrm{E}\left(X_1 + X_2 + X_3 + X_4 + X_5\right) = 5 \cdot \mathrm{E}(X) = 5 \cdot \frac{1}{2} \\ &= \frac{5}{2} \end{aligned}$$

Using independence, we can find the variance of Y as:

$$\begin{aligned} \mathrm{E}(X^2) &= \int_0^1 x^2 \cdot 6x(1-x)\ dx = 6 \cdot \int_0^1 x^3 - x^4 dx \\ &= 6 \cdot (\frac{1}{4}x^4\Big|_0^1 - \frac{1}{5}x^5\Big|_0^1) = 6 \cdot (\frac{5}{20} - \frac{4}{20}) \\ &= \frac{3}{10} \\ \mathrm{Var}(X) &= E(X^2) - E(X)^2 = \frac{3}{10} - (\frac{1}{2})^2 = \frac{6}{20} - \frac{5}{20} = \frac{1}{20} \\ \mathrm{Var}(Y) &= \mathrm{Var}\left(X_1 + X_2 + X_3 + X_4 + X_5\right) = 5 \cdot \mathrm{Var}(X) = 5 \cdot \frac{1}{20} \\ &= \frac{1}{4} \end{aligned}$$

Therefore, the mean and variance of Y are $\frac{5}{2}$ and $\frac{1}{4}$, respectively.

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12. Let the independent random variables X_1 and X_2 have binomial distribution with parameters $n_1 = 3$, $p = \frac{2}{3}$ and $n_2 = 4$, $p = \frac{1}{2}$, respectively. Compute $P(X_1 = X_2)$.

Solution:
Since X_1 and X_2 are independent, we have

$$
\begin{aligned}
X_1 &\sim B(3, \frac{2}{3}) \qquad P_{X_1}(x_1) = C^3_{x_1}(\frac{2}{3})^{x_1} \cdot (\frac{1}{3})^{3-x_1} \\
X_2 &\sim B(4, \frac{1}{2}) \qquad P_{X_2}(x_2) = C^4_{x_2}(\frac{1}{2})^{x_2} \cdot (\frac{1}{2})^{4-x_2} = C^4_{x_2}(\frac{1}{2})^4
\end{aligned}
$$

$$
\begin{aligned}
P(X_1 = X_2) &= P(X_1 = X_2 = 0) + P(X_1 = X_2 = 1) + P(X_1 = X_2 = 2) + P(X_1 = X_2 = 3) \\
&= P(X_1 = 0) \cdot P(X_2 = 0) + P(X_1 = 1) \cdot P(X_2 = 1) \\
&\quad + P(X_1 = 2) \cdot P(X_2 = 2) + P(X_1 = 3) \cdot P(X_2 = 3) \\
&= \left(1 \cdot (\frac{2}{3})^0 \cdot (\frac{1}{3})^3 \cdot 1 \cdot (\frac{1}{2})^4\right) + \left(3 \cdot (\frac{2}{3})^1 \cdot (\frac{1}{3})^2 \cdot 4 \cdot (\frac{1}{2})^4\right) \\
&\quad + \left(3 \cdot (\frac{2}{3})^2 \cdot (\frac{1}{3})^1 \cdot 6 \cdot (\frac{1}{2})^4\right) + \left(1 \cdot (\frac{2}{3})^3 \cdot (\frac{1}{3})^0 \cdot 4 \cdot (\frac{1}{2})^4\right) \\
&= (\frac{1}{27} \cdot \frac{1}{16}) + (\frac{6}{27} \cdot \frac{4}{16}) + (\frac{12}{27} \cdot \frac{6}{16}) + (\frac{8}{27} \cdot \frac{4}{16}) = \frac{1 + 24 + 72 + 32}{27 \cdot 16} = \frac{129}{432} \\
&= \frac{43}{144}
\end{aligned}
$$

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13. Let X have a Poisson distribution. If $P(X = 1) = P(X = 3)$, find the mode of the distribution.

Solution:

$$\frac{e^{-\mu}\mu}{1!} = \frac{e^{-\mu}\mu^3}{3!} \text{ requires } \mu^2 = 6 \text{ and } \mu = \sqrt{6}.$$

$$\text{Since } \frac{e^{-\sqrt{6}}(\sqrt{6})^2}{2!} = 3 \cdot e^{-\sqrt{6}} > \frac{e^{-\sqrt{6}} \cdot \sqrt{6}}{1!},\ x = 2 \text{ is the mode.}$$

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14. If X is $N(1, 4)$, compute the probability $P(1 < X^2 < 9)$

Solution:
Since X follows a normal distribution with mean $\mu = 1$ and variance $\sigma^2 = 4$, we know that $Z = \frac{X-\mu}{\sigma}$ follows a standard normal distribution $N(0, 1)$. Therefore, we can transform the inequality $1 < X^2 < 9$ into $-2 < Z < 2$ as follows:

$$1 < X^2 < 9 \Rightarrow -3 < X < -1 \text{ or } 1 < X < 3 \Rightarrow -\frac{3-1}{2} < \frac{X-\mu}{\sigma} < \frac{3-1}{2} \Rightarrow -1 < Z < 1$$

Using a standard normal table, we can find $P(-1 < Z < 1)$ to be approximately 0.477. Therefore, we have:

$$P(1 < X^2 < 9) = P(-1 < Z < 1) \approx 0.477$$

■

15. Let the random variable X have a distribution that is $N(\mu, \sigma^2)$.
(a) Does the random variable $Y = X^2$ also have a normal distribution?
(b) Would the random variable $Y = aX + b$, a and b nonzero constants have a normal distribution?
Hint: In each case, first determine $P(Y \leq y)$.

Solution:
(a) No, the random variable $Y = X^2$ does not have a normal distribution.

To determine whether the random variable $Y = X^2$ also has a normal distribution, we need to find the distribution of Y.
First, we note that if $X \sim N(\mu, \sigma^2)$, then X has a symmetric bell-shaped distribution. When we square X, we remove the negative values of X since the square of any real number is non-negative. Therefore, the distribution of $Y = X^2$ is not symmetric anymore.
To find the distribution of Y, we can use the cumulative distribution function (CDF) method. Let $y > 0$ be a fixed value. Then we have:

$$\begin{aligned} P(Y \leq y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \Phi\left(\frac{\sqrt{y}-\mu}{\sigma}\right) - \Phi\left(\frac{-\sqrt{y}-\mu}{\sigma}\right), \end{aligned}$$

where Φ is the standard normal CDF.
We can differentiate this expression with respect to y to find the PDF of Y:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} P(Y \leq y) \\ &= \frac{1}{2\sqrt{y}\sigma}\left[\phi\left(\frac{\sqrt{y}-\mu}{\sigma}\right) + \phi\left(\frac{-\sqrt{y}-\mu}{\sigma}\right)\right], \end{aligned}$$

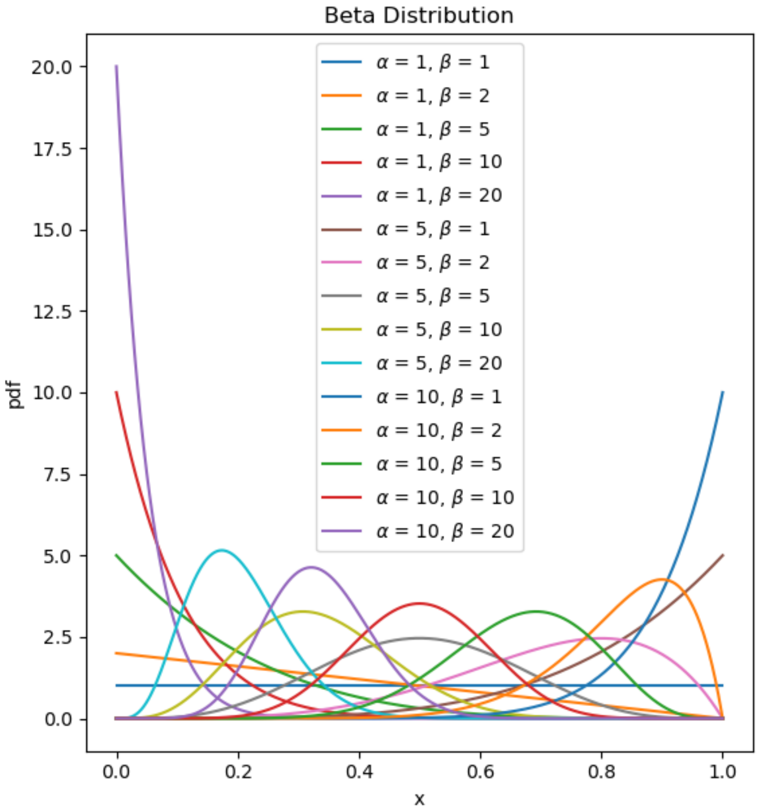
where ϕ is the standard normal PDF.
We see that the PDF of Y depends on the PDF of X, which is normal, and on the square root of Y. Therefore, $Y = X^2$ does not have a normal distribution.
Note that if X had a standard normal distribution, then $Y = X^2$ would have a chi-squared distribution with one degree of freedom.
(b)
Let $Y = aX + b$. To determine the distribution of Y, we can use the following formula for the expected value and variance of Y:
$E(Y) = E(aX + b) = aE(X) + b = a\mu + b$
$Var(Y) = Var(aX + b) = a^2Var(X) = a^2\sigma^2$
Therefore, Y is also a normal distribution, since it is a linear transformation of the normal random variable X.
More specifically, Y has a normal distribution with mean $a\mu + b$ and variance $a^2\sigma^2$.

■

16. (The original question 14.) Using the computer, obtain plots of beta pdfs for $\alpha = 1, 5, 10$ and $\beta = 1, 2, 5, 10, 20$.

Solution:

■




```python
# -*- coding: utf-8 -*-
"""
Created on Mon Mar 29 17:33:24 2023

@author: Paul
"""

import numpy as np
import matplotlib.pyplot as plt

def plot_beta_dist():
    alpha = [1, 5, 10]
    beta_values = [1, 2, 5, 10, 20]
    x = np.linspace(0, 1, 1000)

    for a in alpha:
        for b in beta_values:
            y = x**(a-1) * (1-x)**(b-1)
            y /= np.trapz(y, x)
            plt.plot(x, y, label=r'$\alpha$ = {}, $\beta$ = {}'.format(a, b))

    plt.title('Beta Distribution')
    plt.xlabel('x')
    plt.ylabel('pdf')
    plt.legend()
    plt.show()


if __name__ == "__main__":
    plot_beta_dist()
```

Figure 1: Beta distribution and code implementation