HW2 4. 5. 8

6.5.4. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independent random samples from the two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$.

(a) Find the likelihood ratio Λ for testing the composite hypothesis $H_0 : \theta_1 = \theta_2$ against the composite alternative $H_1 : \theta_1 \neq \theta_2$.

(b) This Λ is a function of what F-statistic that would actually be used in this test?

$X \sim N(0, \theta_1) \qquad Y \sim N(0, \theta_2)$

(a)

$$\Lambda = \frac{n^{\frac{n}{2}} m^{\frac{m}{2}}}{(m+n)^{\frac{m+n}{2}}} \frac{\left(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2\right)^{\frac{m+n}{2}}}{\left(\sum_{i=1}^{n} X_i^2\right)^{\frac{n}{2}} \left(\sum_{j=1}^{m} Y_j^2\right)^{\frac{m}{2}}}$$

$$= \frac{n^{\frac{n}{2}} m^{\frac{m}{2}}}{(m+n)^{\frac{m+n}{2}}} \frac{\left(\frac{\sum_{i=1}^{n} X_i^2}{\sum_{j=1}^{m} Y_j^2} + 1\right)^{\frac{m+n}{2}}}{\left(\frac{\sum_{i=1}^{n} X_i^2}{\sum_{j=1}^{m} Y_j^2}\right)^{\frac{n}{2}}}$$

(b)

$$F = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\frac{1}{m}\sum_{j=1}^{m} Y_j^2}$$

The relevant F statistic $F = \dfrac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\frac{1}{m}\sum_{j=1}^{m} Y_j^2}$ #

7.1.3. Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x;\theta) = 1/\theta,\ 0 < x < \theta,\ 0 < \theta < \infty$, zero elsewhere. Show that $4Y_1$, $2Y_2$, and $\frac{4}{3}Y_3$ are all unbiased estimators of θ. Find the variance of each of these unbiased estimators.

改問其中一個

$$f_{Y_1}(y_1) = \frac{3!}{0!\,2!}\left(\frac{y_1}{\theta}\right)^{1-1}\cdot\left(1-\frac{y_1}{\theta}\right)^{3-1}\cdot\frac{1}{\theta} = \frac{3}{\theta}\left(1-\frac{y_1}{\theta}\right)^2$$

$$E[4Y_1] = 4\int_0^\theta y_1 f_{Y_1}(y_1)\,dy_1 = 4\int_0^\theta \frac{3y_1}{\theta}\left(1-\frac{y_1}{\theta}\right)^2 dy_1$$ ，令 $t = \frac{y_1}{\theta}$ ，$dt = \frac{1}{\theta}dy_1$

$$= 12\int_0^1 t(1-t)^2\,\theta\,dt$$

$$= 12\theta\cdot\frac{2}{24} = \theta$$

$$E[(4Y_1)^2] = 16E[Y_1^2] = 16\int_0^\theta y_1^2 f_{Y_1}(y_1)\,dy_1$$

$$= 48\int_0^\theta \frac{y_1^2}{\theta}\left(1-\frac{y_1}{\theta}\right)^2 dy_1$$ ，令 $\frac{y_1}{\theta} = t$ ，$dt = \frac{1}{\theta}dy_1$

$$= 48\theta^2\int_0^\theta t^2(1-t)^2\,dt$$

$$= 48\theta^2\cdot\frac{2\cdot 2}{120} = \frac{8}{5}\theta^2$$

$$\mathrm{Var}(4Y_1) = \frac{8}{5}\theta^2 - \theta^2 = \frac{3}{5}\theta^2$$

Practice 3.

6.2.3. Given the pdf

$$f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty,$$

show that the Rao–Cramér lower bound (Var min) is $2/n$, where n is the size of a random sample from this Cauchy distribution. What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta}-\theta)$ if $\widehat{\theta}$ is the mle of θ?

$$L(\theta, x) = \prod_{i=1}^{n} f(x;\theta) \quad , \quad f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}$$

$$I(\theta) = E\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2 = E\left[\frac{\partial\left(\frac{1}{\pi(1+(x-\theta)^2)}\right)}{\partial\theta}\right]^2$$

$$= E\left[\frac{\partial}{\partial\theta}\left(-\ln\pi - \ln(1+(x-\theta)^2)\right)\right]^2$$

$$= E\left[\frac{2(x-\theta)}{1+(x-\theta)^2}\right]^2 = \int_{-\infty}^{\infty}\left(\frac{2(x-\theta)}{1+(x-\theta)^2}\right)^2 \frac{1}{\pi(1+(x-\theta)^2)}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{4(x-\theta)^2}{\pi(1+(x+\theta)^2)^3}\,dx \quad , \quad \tan t = x-\theta, \quad \sec^2 t\,dt = dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{4\tan^2 t}{\pi(1+\tan^2 t)^3}\sec^2 t\,dt$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 t \cdot \sin^2 t\,dt$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\pi}\left(1 - \frac{1+\cos(4t)}{2}\right)dt$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} \cos(4t)\,dt \quad , \quad u = 4t, \quad du = 4dt$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\int_{-2\pi}^{2\pi} \cos(u)\tfrac{1}{4}dt$$

$$= \tfrac{1}{2} - 0 = \tfrac{1}{2} \Rightarrow \text{RCLB} \Rightarrow \frac{1}{nI(\theta)} = \frac{2}{n}$$

(b) $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{D} N\left(0, \frac{1}{E\left[\frac{\partial \ln(f(x;\theta))}{\partial\theta}\right]^2}\right)$, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{D} N(0, 2)$

by Thm 6.2.18.

6.3.8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

(a) Show that the likelihood ratio test of $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^n X_i$. Obtain the null distribution of Y.

(b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \leq 4$ or $Y \geq 17$.

(a)

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} \quad , \quad \Lambda \leq c \text{ where } \alpha = P_{\theta_0}(\Lambda \leq c)$$

$$f(x;\theta) = \frac{\theta^x}{x!} e^{-\theta}$$

$$L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{\theta^x}{x!} e^{-\theta} = e^{-n\theta} \frac{\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

$$l(\theta) = \ln(L(\theta)) = -n\theta + \sum_{i=1}^n x_i \ln\theta - \sum_{i=1}^n \ln(x_i!)$$

$$\frac{\partial}{\partial\theta}(l(\theta)) = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left(-n\theta + \sum_{i=1}^n x_i \ln\theta - \sum_{i=1}^n \ln(x_i!)\right) = -n + \frac{1}{\theta}\sum_{i=1}^n x_i$$

$$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^n x_i$$

(a)

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{e^{-n\theta_0} \frac{\theta_0^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}}{e^{\left\{-n\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right.} \left.\frac{\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}\right\}} = \frac{e^{-n\theta_0}\, \theta_0^{\sum_{i=1}^n x_i}}{e^{\sum_{i=1}^n x_i} \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^{\sum_{i=1}^n x_i}}$$

$$\Lambda = \frac{e^{-n\theta_0}\,\theta_0^Y}{e^Y\left(\frac{1}{n}Y\right)^Y}$$

(b)

$$\alpha = P(Y \leq 4) + P(Y \geq 17)$$

$$= P(Y \leq 4) + 1 - P(Y \leq 17)$$

$$= \sum_{x=0}^{4} e^{-10}\frac{10^x}{x!} + 1 - \sum_{x=0}^{17} e^{-10}\frac{10^x}{x!} = 0.05$$