Consider the structure equation

$$\gamma_{11} y_{1i} + \gamma_{12} y_{2i} = \beta_1 z_{1i} + u_{1i}$$

$$\gamma_{21} y_{1i} + \gamma_{22} y_{2i} = \beta_2 z_{2i} + \beta_3 z_{3i} + u_{2i}$$

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \end{bmatrix} + \begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix}$$

"Manual" inversion:

$$y_{1i} = \frac{\beta_1}{\gamma_{11}} z_{1i} - \frac{\gamma_{12}}{\gamma_{11}} y_{2i} + \frac{u_{1i}}{\gamma_{11}} \qquad (1)$$

$$y_{2i} = \frac{\beta_2}{\gamma_{22}} z_{2i} + \frac{\beta_3}{\gamma_{22}} z_{3i} - \frac{\gamma_{21}}{\gamma_{22}} y_{1i} + \frac{u_{2i}}{\gamma_{22}} \qquad (2)$$

Replace (2) on (1) to solve for y_{1i}

$$y_{1i} = \frac{\beta_1}{\gamma_{11}} z_{1i} - \frac{\gamma_{12}}{\gamma_{22}} \beta_2 z_{2i} - \frac{\gamma_{12}}{\gamma_{22}} \beta_3 z_{3i} + \frac{\gamma_{21}\gamma_{12}}{\gamma_{11}\gamma_{22}} y_{1i} - \frac{\gamma_{12}}{\gamma_{11}} \frac{u_{2i}}{\gamma_{22}} + \frac{u_{1i}}{\gamma_{11}}$$

⊛ define $\kappa = 1 - \frac{\gamma_{21}\gamma_{12}}{\gamma_{11}\gamma_{22}}$

$$\Rightarrow \quad y_{1i} = \begin{bmatrix} \underbrace{\frac{\beta_1}{\gamma_{11}\kappa}}_{\pi_1} & \underbrace{-\frac{\gamma_{12}\beta_2}{\gamma_{22}\kappa} \quad -\frac{\gamma_{12}\beta_3}{\gamma_{22}\kappa}}_{\pi_2} \end{bmatrix} \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \end{bmatrix} + w_{1i}$$

Replace (1) on (2) to solve for y_{2i}

$$\Rightarrow \quad y_{2i} = \begin{bmatrix} \underbrace{-\frac{\gamma_{21}}{\gamma_{22}} \frac{\beta_1}{\gamma_{11}\kappa}}_{\Pi_1} & \underbrace{\frac{\beta_2}{\gamma_{22}\kappa} \quad \frac{\beta_3}{\gamma_{22}\kappa}}_{\Pi_2} \end{bmatrix} \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \end{bmatrix} + w_{2i}$$

$$\underbrace{\begin{bmatrix} 1 & \gamma_{12}/\gamma_{11} \\ \gamma_{21}/\gamma_{22} & 1 \end{bmatrix}_{2\times 2}}_{\Gamma} \underbrace{\begin{bmatrix} \pi'_1 & \pi'_2 \\ \Pi_1 & \Pi_2 \end{bmatrix}_{2\times 3}}_{\Gamma^{-1} B} = \underbrace{\begin{bmatrix} \beta_1/\gamma_{11} & 0 & 0 \\ 0 & \beta_2/\gamma_{22} & \beta_3/\gamma_{22} \end{bmatrix}_{2\times 3}}_{B}$$

- $$\pi'_1 + \frac{\gamma_{12}}{\gamma_{11}} \Pi_1 = \frac{\beta_1}{\gamma_{11}}$$

  $$\underset{1\times 2}{\pi'_2} + \frac{\gamma_{12}}{\gamma_{11}} \underset{1\times 2}{\Pi_2} = 0 \Rightarrow \frac{\gamma_{12}}{\gamma_{11}} = (\Pi_2')^{+} \pi_2'$$

- $$\frac{\gamma_{21}}{\gamma_{22}} \pi_1 + \Pi_1 = 0 \Rightarrow \frac{\gamma_{21}}{\gamma_{22}} = \frac{\Pi_1}{\pi_1}.$$

  $$\frac{\gamma_{21}}{\gamma_{22}} \pi_2 + \Pi_2 = \begin{bmatrix} \frac{\beta_2}{\gamma_{22}} & \frac{\beta_3}{\gamma_{22}} \end{bmatrix}$$

Question 2 2011 midterm

Consider the following structural equation:

$$y_i = Y_i'\gamma + u_i, \tag{8}$$

where y_i is the "dependent" endogenous variable, Y_i is the $m \times 1$ vector of endogenous regressors, and γ is the $m \times 1$ vector of unknown structural parameters. Let Z_i denote the $l \times 1$ vector of excluded exogenous variables (there are no included exogenous regressors). The reduced form equations are

$$\begin{aligned} y_i &= \pi Z_i + v_i, \\ Y_i &= \Pi Z_i + V_i, \end{aligned} \tag{9}$$

where π is the $1 \times l$ vector of reduced form coefficients, and Π is the $m \times l$ matrix of the reduced form coefficients. Assume that $m < l$. Assume also that the structural equation is identified and therefore

$$\gamma = (\Pi')^+ \pi', \tag{10}$$

where A^+ denotes the Moore-Penrose (MP) inverse of a matrix A.

(a) (10 points) Derive the asymptotic distribution of the MP estimator of γ:

$$\hat{\gamma}^+ = \left(\hat{\Pi}'\right)^+ \hat{\pi}', \tag{11}$$

where $\hat{\pi}$ and $\hat{\Pi}$ denote the OLS estimators of π and Π respectively. Justify any additional assumptions you have to make.

Notice that $l > m$ and we assume γ is identified. Hence Π' has full column rank:

$$\underbrace{(\Pi')}_{l \times m}{}^{+} = (\underbrace{\Pi}_{m \times l} \underbrace{\Pi'}_{l \times m})^{-1} \underbrace{\Pi}_{m \times l}$$

Write

$$\hat{\gamma} = (\hat{\Pi}\hat{\Pi}')^{-1}\hat{\Pi}\hat{\pi}' \quad \text{where}$$

$$\hat{\pi} = (Z'Z)^{-1}Z'y \quad , \quad \hat{\Pi} = (Z'Z)^{-1}Z'Y$$

so that

$$\hat{\gamma}^{+} = \left(Y' Z (Z'Z)^{-1} (Z'Z)^{-1} Z' Y \right)^{-1} Y' Z (Z'Z)^{-1} (Z'Z)^{-1} Z' y$$

Under (i) $E \|Z_i\|^2 < \infty$

(ii) $E Z_i Z_i'$ p.d. (bounded away from zero $\Rightarrow$ invertible)

(iii) iid data for WLLN

Then, we consider 3 objects

- $$\frac{Z'Y}{n} = \frac{Z'Z}{n} \Pi' + \frac{Z'V}{n}$$
$$= E Z_i Z_i' \Pi' + o_p(1)$$

- $$\frac{Z'Z}{n} = E Z_i Z_i' + o_p(1)$$

- $$\frac{Z'y}{n} = \frac{Z'Z}{n} \pi + \frac{Z'u}{n}$$
$$= E Z_i Z_i' \pi + \frac{Z'u}{n} + o_p(1)$$

$$\left(\neq \xrightarrow[p]{} E Z_i Z_i' \pi + \frac{Z'u}{n} + o_p(1) \right)$$

Then

$$\hat{\gamma}^{+} = \left(\Pi \, E Z_i Z_i' (E Z_i Z_i')^{-1} (E Z_i Z_i')^{-1} E Z_i Z_i' \Pi' \right)^{-1} \Pi \, E Z_i Z_i' (E Z_i Z_i')^{-1} (E Z_i Z_i')^{-1} \times$$
$$\left\{ E Z_i Z_i' \pi + \frac{Z'u}{n} \right\} + o_p(1)$$

$$= (\Pi \Pi')^{-1} \Pi \Pi' + o_p(1)$$

$$= \gamma + o_p(1)$$

renormalize rate of convergence

$$\sqrt{n}(\hat{\gamma}^+ - \gamma) = (\Pi \Pi')^{-1} \Pi (E z_i z_i')^{-1} \underbrace{\frac{Z'u}{\sqrt{n}}}_{\xrightarrow{d} N(0,\ E u_i^2 z_i z_i')} + o_p(1)$$

$$\xrightarrow{d} N\left(0,\ (\Pi\Pi')^{-1} \Pi (E z_i z_i')^{-1} (E u_i^2 z_i z_i') (E z_i z_i')^{-1} \Pi' (\Pi \Pi')^{-1}\right)$$

(b) (10 points) Is $\hat{\gamma}^+$ an efficient estimator under heteroskedastic or homoskedastic errors? Explain.

$$\hat{\gamma}^+ = (Y' Z \underbrace{(Z'Z)^{-1}(Z'Z)^{-1}}_{\text{This is the weighting matrix}} Z'Y)^{-1} Y'Z (Z'Z)^{-1} (Z'Z)^{-1} Z'y$$

$$\left(\frac{Z'Z}{n}\right)^{-1} \left(\frac{Z'Z}{n}\right)^{-1} \xrightarrow{p} (E z_i z_i')^{-1} (E z_i z_i')^{-1}$$

Under heterosk this should be $(E u_i^2 z_i z_i')^{-1}$

Under homosk this should be $(E z_i z_i')^{-1}$.

Therefore, the MP inverse is not efficient.

1. Consider the following system of equations: Midterm 2017

$$
\begin{aligned}
y_{1i} &= X_{1i}'\delta_1 + u_{1i}, \\
y_{2i} &= X_{2i}'\delta_2 + u_{2i},
\end{aligned}
$$

where X_{1i} and X_{2i} are the k_1- and k_2-vectors of endogenous regressors respectively, and $\delta_j \in \mathbb{R}^{k_j}$, $j = 1, 2$. Let Z_i be the l-vector of instruments:

$$EZ_i u_{ji} = 0 \text{ for } j = 1, 2.$$

Assume that $l \geq k_j$, $j = 1, 2$, and that data are iid. Consider the following GMM estimator of $(\delta_1', \delta_2')'$,

$$
\begin{aligned}
&\begin{pmatrix} \widehat{\delta}_1(W_n) \\ \widehat{\delta}_2(W_n) \end{pmatrix} \\
= &\left[\begin{pmatrix} \sum_{i=1}^n X_{1i}Z_i' & 0 \\ 0 & \sum_{i=1}^n X_{2i}Z_i' \end{pmatrix}\begin{pmatrix} W_{n,11} & W_{n,12} \\ W_{n,21} & W_{n,22} \end{pmatrix}\begin{pmatrix} \sum_{i=1}^n Z_iX_{1i}' & 0 \\ 0 & \sum_{i=1}^n Z_iX_{2i}' \end{pmatrix}\right]^{-1} \\
&\times \begin{pmatrix} \sum_{i=1}^n X_{1i}Z_i' & 0 \\ 0 & \sum_{i=1}^n X_{2i}Z_i' \end{pmatrix}\begin{pmatrix} W_{n,11} & W_{n,12} \\ W_{n,21} & W_{n,22} \end{pmatrix}\begin{pmatrix} \sum_{i=1}^n Z_iy_{1i} \\ \sum_{i=1}^n Z_iy_{2i} \end{pmatrix},
\end{aligned}
$$

where

$$W_n = \begin{pmatrix} W_{n,11} & W_{n,12} \\ W_{n,21} & W_{n,22} \end{pmatrix}$$

is some positive definite symmetric matrix (possibly random and data-dependent).

(a) **(10 points)** Show consistency of $\hat{\delta}_1(W_n)$ and $\hat{\delta}_2(W_n)$. Carefully justify all additional assumptions you have to make.

Assume

(i) $W_n \xrightarrow{p} W_{2l \times 2l}$

(ii) $E\|X_{1i}\|^2 < \infty$, $E\|X_{2i}\|^2 < \infty$, $E\|Z_i\|^2 < \infty$ $\Rightarrow$ $EZ_iX_{1i}' = O(1)$, $EZ_iX_{2i}' = O(1)$

$$\delta = \left[\begin{pmatrix} Q_1' + o_p(1) & 0 \\ 0 & Q_2' + o_p(1) \end{pmatrix}\left(W + o_p(1)\right)\begin{pmatrix} Q_1 + o_p(1) & 0 \\ 0 & Q_2 + o_p(1) \end{pmatrix}\right]^{-1}$$

$$\left[\begin{pmatrix} Q_1' + o_p(1) & 0 \\ 0 & Q_2' + o_p(1) \end{pmatrix}\left(W + o_p(1)\right)\right]\begin{Bmatrix} Q_1 + \frac{Z'u_1}{n} + o_p(1) \\ Q_2 + \frac{Z'u_2}{n} + o_p(1) \end{Bmatrix}$$

$$= \delta + \begin{pmatrix} Q_1' W_{11} Q_1 & Q_1 W_{12} Q_2 \\ Q_2' W_{21} Q_1 & Q_2' W_{22} Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_0' & 0 \\ 0 & Q_2' \end{pmatrix} W \begin{pmatrix} 0 \\ 0 \end{pmatrix} + o_p(1)$$

$$= \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + o_p(1) .$$

(b) **(15 points)** Show joint asymptotic normality of $\hat{\delta}_1(W_n)$ and $\hat{\delta}_2(W_n)$. Find their asymptotic variance-covariance matrix. Carefully justify all additional assumptions you have to make.

Change the rate of convergence to get influence function

$$\sqrt{n}(\hat{\delta}-\delta) = \begin{pmatrix} Q_1' W_{11} Q_1 & Q_1 W_{12} Q_2 \\ Q_2' W_{21} Q_1 & Q_2' W_{22} Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_0' & 0 \\ 0 & Q_2' \end{pmatrix} W \begin{pmatrix} \frac{Z'u_1}{\sqrt{n}} \\ \frac{Z'u_2}{\sqrt{n}} \end{pmatrix} + o_p(1)$$

$$= \begin{pmatrix} Q_1' W_{11} Q_1 & Q_1 W_{12} Q_2 \\ Q_2' W_{21} Q_1 & Q_2' W_{22} Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_0' & 0 \\ 0 & Q_2' \end{pmatrix} W \begin{pmatrix} \frac{1}{\sqrt{n}} \sum z_i u_{1i} \\ \frac{1}{\sqrt{n}} \sum z_i u_{2i} \end{pmatrix} + o_p(1)$$

$$= \begin{pmatrix} Q_1' W_{11} Q_1 & Q_1 W_{12} Q_2 \\ Q_2' W_{21} Q_1 & Q_2' W_{22} Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_0' & 0 \\ 0 & Q_2' \end{pmatrix} W \frac{1}{\sqrt{n}} \sum u_i \otimes z_i + o_p(1)$$

where

$$\frac{1}{\sqrt{n}} \sum u_i \otimes z_i \xrightarrow{d} N(0, E u_i u_i' \otimes z_i z_i')$$

$$= N\left(0, \underbrace{\begin{pmatrix} E u_{1i}^2 z_i z_i' & E u_{1i} u_{2i} z_i z_i' \\ E u_{1i} u_{2i} z_i z_i' & E u_{2i}^2 z_i z_i' \end{pmatrix}}_{\Omega}\right)$$

(c) **(15 points)** What choice of W_n (possibly random) does result in efficient in the system sense GMM estimators of δ_1 and δ_2? What is their asymptotic variance-covariance matrix?

$$W_n = \hat{\Omega}_n^{-1}.$$

Using such matrix yields

$$V^* = \begin{pmatrix} Q_1' \Omega_{11} Q_1 & Q_1 \Omega_{12} Q_2 \\ Q_2' \Omega_{21} Q_1 & Q_2 \Omega_{22} Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_1' & 0 \\ 0 & Q_2' \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} Q_1' \Omega_{11} Q_1 & Q_1 \Omega_{12} Q_2 \\ Q_2' \Omega_{21} Q_1 & Q_2 \Omega_{22} Q_2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} Q_1' \Omega_{11} Q_1 & Q_1 \Omega_{12} Q_2 \\ Q_2' \Omega_{21} Q_1 & Q_2 \Omega_{21} Q_2 \end{pmatrix}^{-1}.$$

(d) **(10 points)** What choice of W_n corresponds to the efficient equation-by-equation GMM estimators of δ_1 and δ_2?

set

$$W_n = \begin{pmatrix} \hat{\Omega}_{11}^{-1} & 0 \\ 0 & \hat{\Omega}_{22}^{-1} \end{pmatrix}$$

(e) **(15 points)** Under what conditions the the efficient equation-by-equation GMM estimators in (d) are efficient in the system estimation sense? Explain.

If the Ω matrix is block diagonal.