Linear Regression with weakly dependent data

Consider the uwoe regression model

$$y_t = x_t'\beta + u_t$$

Consistency

(*) $\{x_t u_t\}$ and $\{x_t x_t'\}$ are also α-mixing of the same size, by Proposition 1.

Provided

(a) $\{(x_t', u_t)\}$ is α-mixing of any size

(b) $\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} x_t u_t = 0$

(c) $E|x_{tj}|^{2+\eta} < \Delta$ for all t and $j=1,\dots,k$ and some $\eta>0$

(d) $E|u_t|^{2+\eta} < \Delta$ for some $\eta > 0$

(e) $M_n = \frac{1}{n}\sum_{t=1}^{n} E x_t x_t'$ is uniformly positive definite over n.

Then $\hat\beta_n - \beta = o_p(1)$

proof:

We start from the moment condition

$$0 = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} E x_t u_t = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} E x_t(y_t - x_t'\beta)$$

$$\Rightarrow \beta = \left(\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} E x_t x_t'\right)^{-1}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} E x_t y_t\right)$$

Sample analogue: drop the "E"!

$$\hat\beta_n = \left(\frac{1}{n}\sum_{t=1}^{n} x_t x_t'\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} x_t y_t\right)$$

$$= \beta + \left(\frac{1}{n}\sum_{t=1}^{n} x_t x_t'\right)^{-1}\frac{1}{n}\sum_{t=1}^{n} x_t u_t$$

$$= \beta + \left(\frac{1}{n}\sum_{t=1}^{n} x_t x_t' - M_n + M_n\right)^{-1}\frac{1}{n}\sum_{t=1}^{n} x_t u_t$$

$E\|x_t x_t'\|^{1+\eta} \leq \left(E\|x_t\|^{2+2\eta} E\|x_t\|^{2+2\eta}\right)^{1/2}$ by Cauchy-Schwarz

WLLN $\{x_t x_t'\}$

$$= \beta + \left(o_p(1)\, M_n^{-1} + M_n M_n^{-1}\right)^{-1} M_n^{-1}\frac{1}{n}\sum_{t=1}^{n} x_t u_t$$

$$= \beta + \left(o_p(1)\, O(1) + I_k\right)^{-1} O(1)\,\frac{1}{n}\sum_{t=1}^{n} x_t u_t$$

$$= \beta + [I_K + o_p(1)]\, O(1) \frac{1}{n}\sum_{t=1}^{n} X_t u_t$$

$$= \beta + [I_K + o_p(1)]\, O(1) \left[\frac{1}{n}\sum_{t=1}^{n} X_t u_t - \frac{1}{n}\sum_{t=1}^{n} EX_t u_t + \frac{1}{n}\sum_{t=1}^{n} EX_t u_t \right]$$

$E\|X_t u_t\|^{1+\eta} \le \left(E|X_t|^{2+2\eta}\, E|u_t|^{2+2\eta}\right)^{1/2}$ by Cauchy-Schwarz

$$\underset{\text{WLLN } \{X_t u_t\}}{=} \beta + [I_K + O_p(1)]\, O(1) \left[o_p(1) + o(1) \right]$$

$$= \beta + I_K\, O(1)\, o_p(1) + o_p(1)\, O(1)\, o_p(1)$$

$$= \beta + o_p(1). \quad \blacksquare$$

Asymptotic Normality

⊛ Reminder: conditions for CLT in Proposition 6
(i) α-mixing size $-p/p-2$
(ii) $E\|X_{nt}\|^p \le \Delta$ for all t, n
(iii) $\Omega_n = Var\left(\frac{1}{\sqrt{n}}\sum X_{nt}\right)$ is uniformly p.d. and $p > 2$.

Provided

(a) $\{(X_t' u_t)\}$ is α-mixing of size $-p/p-2$

(b) $\frac{1}{\sqrt{n}}\sum_{t=1}^{n} EX_t u_t = o(1)$

(c) $E|X_{tj}|^{2p} \le \Delta$ for all t and $j = 1, \ldots, K$

(d) $E|u_t|^{2p} \le \Delta$ for all t

(e) $M_n = \frac{1}{n}\sum_{t=1}^{n} EX_t X_t'$ is uniformly positive definite over n

(f) $\Omega_n = Var\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t u_t\right)$ is uniformly positive definite over n

Then

$$V_n^{-1/2} \sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, I_K)$$

where $V_n = M_n^{-1} \Omega_n M_n^{-1}$.

proof:

From the previous proposition we get

$$\sqrt{n}(\hat{\beta}_n - \beta) = (I_K + o_p(1))\, M_n^{-1} \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \{X_t u_t - EX_t u_t\} + o_p(1)$$

$$= M_n^{-1} \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \{X_t u_t - EX_t u_t\} + o_p(1)\, O(1) \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \{X_t u_t - EX_t u_t\} + o_p(1)$$

$$= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \underbrace{M_n^{-1}\{X_t u_t - EX_t u_t\}}_{\xi_{nt}} + o_p(1) \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \{X_t u_t - EX_t u_t\} + o_p(1)$$

$$= \underbrace{\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \xi_{n,t}}_{A} + \underbrace{o_p(1) \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \{X_t u_t - EX_t u_t\}}_{B} + o_p(1)$$

- We will deal with B first. We're interested in the process $\{X_t u_t - EX_t u_t\}$, so we check for the conditions

(i) $\{X_t u_t - EX_t u_t\}$ is a measurable function of $\{X_t', u_t\}$ so by Proposition 1 this is α-mixing of size $-p/p-2$.

(ii) $$E\| X_t u_t - EX_t u_t \|^p \underset{\text{Minkowski's Inequality}}{\leq} \{(E\|X_t u_t\|^p)^{1/p} + (E\|X_t u_t\|^p)^{1/p}\}^p$$

$$\underset{\text{Cauchy-Schwarz}}{\leq} 2^p (E\|X_t\|^{2p})^{1/2} (E\|u_t\|^{2p})^{1/2}$$

$$< \infty$$

(iii) $\text{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} [X_t u_t - EX_t u_t]\right) = \text{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_t u_t\right) = \Omega_n$ which is uniformly p.d. by assumption.

Then, by the CLT $\quad \Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \{X_t u_t - EX_t u_t\} = O_p(1)$.

We write B as:

$$o_p(1)\, \Omega_n^{-1/2} \Omega_n \Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \{X_t u_t - EX_t u_t\} = o_p(1)\, \Omega_n^{-1/2} \Omega_n\, O_p(1)$$

$$= o_p(1)\, O(1)\, O_p(1)$$

$$= o_p(1).$$

- Now we can deal with A. We're interested in the array $\{M_n^{-1}(X_t u_t - EX_t u_t)\}$, so we will check for the conditions

(i) $\{M_n^{-1}(X_t u_t - EX_t u_t)\}$ is a measurable function of $\{X_t', u_t\}$ so by Proposition 1 it's α-mixing of size $-p/p-2$.

(ii) $$E\| M_n^{-1}(X_t u_t - EX_t u_t)\| \leq \|M_n^{-1}\|\, E\| X_t - EX_t u_t\|$$

$$\leq O(1)\quad O(1)$$

$$< \infty$$

(iii) $\text{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} M_n^{-1}(X_t u_t - EX_t u_t)\right) = M_n^{-1}\Omega_n M_n^{-1}$ must be uniformly positive definite.

This requires that for arbitrary x s.t. $\|x\| = 1$

$$x' M_n^{-1} \Omega_n M_n^{-1} x > 0$$

$$
\begin{aligned}
x' M_n^{-1} \Omega_n M_n^{-1} x &= x' M_n^{-1} C_n \Delta_n C_n M_n^{-1} x \\
&= y_n' \Delta_n y_n \\
&= \sum_{i=1}^{k} e_{ni} y_{ni}^2 \\
&\geq \underline{e_n} \, \| y_n \|^2 \\
&\geq \delta \, x' M_n^{-1} C_n C_n' M_n^{-1} x \\
&= \delta \, x' M_n^{-2} x \\
&= \delta \, x' D_n P_n^{-2} D_n x \\
&= \delta \sum_{i=1}^{k} d_{ni}^{-2} w_{ni}^2 \\
&\geq \frac{\delta}{\bar{d}_n^2} \| x' D_n \| \\
&= \frac{\delta}{\bar{d}_n^2} \\
&> 0 \qquad \text{because } \; M_n \text{ uniformly p.d.}, \; \sup_n \bar{d}_n < K \text{ for some } K > 0.
\end{aligned}
$$

Then, by the CLT $\quad V_n^{-1/2} \left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} M_n^{-1} (x_t u_t - E x_t u_t) \right\} \xrightarrow{d} N(0,1).$

Putting it all together yields:

$$
V_n^{-1/2} \sqrt{n} (\hat{\beta}_n - \beta) = V_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} M_n^{-1} (x_t u_t - E x_t u_t) + V_n^{-1/2} o_p(1)
$$

$$
\begin{aligned}
V_n^{-1/2} \sqrt{n} (\hat{\beta}_n - \beta) &= V_n^{-1/2} \left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} M_n^{-1} (x_t u_t - E x_t u_t) \right\} + O(1)\, o_p(1) \\
&= V_n^{-1/2} \left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} M_n^{-1} (x_t u_t - E x_t u_t) \right\} + o_p(1) \\
&\xrightarrow{d} N(0, I_k). \qquad \blacksquare
\end{aligned}
$$

Estimation of Asymptotic Variance Matrix

Recall that $V_n = M_n^{-1} \Omega_n M_n^{-1}$ and $M_n = \frac{1}{n}\sum_{t=1}^{n} E X_t X_t'$. Then we can estimate M_n using

$$\hat{M_n} = \frac{1}{n}\sum X_t X_t' \quad \text{and hope that} \quad \hat{M_n} - M_n = o_p(1).$$

To estimate Ω_n we need $\Omega_n(h) = \frac{1}{n}\sum_{t=h+1}^{n} E\underbrace{[X_t u_t (X_{t-h} u_{t-h})']}_{\text{because they are centered at 0.}}$.

Now, our initial estimator could be

$$\tilde{\Omega}_n = \tilde{\Omega}_n(0) + \sum_{h=1}^{n-1} (\tilde{\Omega}_n(h) + \tilde{\Omega}_n(h'))$$

$$\text{where} \quad \tilde{\Omega}_n(h) = \frac{1}{n}\sum_{t=h+1}^{n} [X_t u_t (X_{t-h} u_{t-h})']$$

- Problem: we need to ensure that $(\tilde{\Omega}_n(h) + \tilde{\Omega}_n(h'))$ grow slower than n. A solution would be to allow for autocovariances to grow slower than n.

- New problem: when we truncate we can get non positive definite matrix, so we need to put weights in the sum.

The (infeasible) HAC estimator of variance is

$$\tilde{\Omega}_n = \tilde{\Omega}_n(0) + \sum_{j=1}^{m_n} w(j, m_n) (\tilde{\Omega}_n(j) + \tilde{\Omega}_n(j'))$$

Proposition HAC 1.- Suppose that for some $p > 2$ and $\Delta, \delta, C > 0$

(a) $\{(X_t', u_t)\}$ is α-mixing of size $-p/p-2$

(b) $E X_t u_t = 0$ for all t

(c) $E|X_{tj}|^{4p+\delta} \le \Delta$ for all t and all $j = 1, \dots, K$

(d) $E|u_t|^{4p+\delta} \le \Delta$ for all t

(e) $|w(j,m)| \le C$ for all j and m

(g) $\lim_{m\to\infty} w(j,m) = 1$ for all j

(h) $m_n = o(n^{1/4})$.

Then

$$\tilde{\Omega}_n - \Omega_n = o_p(1).$$

proof: By the Cramer - Wold device it suffices to show that

$$c'(\tilde{\Omega}_n - \Omega_n)c = o_p(1) \quad \text{for all } c \in \mathbb{R}^K.$$

Now, define $h_t = c' X_t u_t$ and notice that by Proposition 1 it is α-mixing of size $-p/p-2$. Then we write

$$c'(\tilde{\Omega}_n - \Omega_n)c = \underbrace{\frac{1}{n}\sum_{t=1}^{n}(h_t - Eh_t)}_{R_{n,0} = o_p(1) \text{ by LLN}} + \underbrace{2\sum_{j=1}^{m_n} w(j,m_n)\frac{1}{n}\sum(h_t h_{t-j} - Eh_t h_{t-j})}_{R_{n,1} := \text{ regular estimation error of covariances}} + \underbrace{2\sum_{j=1}^{m_n}(w(j,m_n)-1)\frac{1}{n}\sum_{t=h+1}^{n} Eh_t h_{t-j}}_{R_{n,2} := \text{ bias due to using weights}}$$

$$\underbrace{-\,2\sum_{j=m_n+1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} Eh_t h_{t-j}}_{R_{n,3} := \text{ bias due to truncation of autocovariances}}$$

- $R_{n,2}$: We want to use the covariance inequalities, so we need to show that $\sup_t E|h_t|^p < \infty$. To see this

$$E|h_t|^p \leq \|c\| \; E\|X_t u_t\|^p$$

$$\leq \|c\| \left(E\|X_t\|^{2p}\right)^{1/2} \left(E\|u_t\|^{2p}\right)^{1/2}$$

$$< \infty$$

Then

$$|R_{n,2}| \leq \sum_{j=1}^{m_n} |w(j,m_n) - 1| \frac{1}{n}\sum_{t=h+1}^{n} Eh_t h_{t-j}$$

$$\underset{\text{Proposition 4}}{\leq} \sum_{j=1}^{m_n} |w(j,m_n) - 1| \frac{1}{n}\sum_{t=h+1}^{n} K\,\alpha(j)^{1-2/p}$$

$$= \sum_{j=1}^{m_n} |w(j,m_n) - 1| \frac{1}{n}\sum_{t=h+1}^{n} K\, j^{\left(-\frac{p}{p-2}-\varepsilon\right)\left(\frac{p-2}{p}\right)}$$

$$= \sum_{j=1}^{m_n} |w(j,m_n) - 1| \frac{1}{n}\sum_{t=h+1}^{n} K\, j^{-1-\eta} \quad \text{for some } \eta > 0$$

$$\leq \sum_{j=1}^{m_n} |w(j,m_n) - 1| \frac{1}{n} K\, j^{-1-\eta}\,(n-(h+1)+1)$$

$$\leq K\sum_{j=1}^{\infty} |w(j,m_n) - 1|\, j^{-1-\eta}$$

$$\lim_{n\to\infty} |R_{n,2}| \leq \lim_{n\to\infty} K\sum_{j=1}^{\infty} |w(j,m_n) - 1|\, j^{-1-\eta}$$

$$= K \sum_{j=1}^{\infty} \left| \lim_{n\to\infty} w(j,m_n) - 1 \right| j^{-1-\eta}$$

Dominated Convergence Theorem

$K(C+1)\, j^{-1-\eta}$ can be the dominating function and it's integrable/summable.

$$= 0 .$$

- $R_{n,3}$: We will use the same idea as in the previous part.

$$|R_{n,3}| \le \sum_{j=m_n+1}^{n-1} \frac{1}{n} \sum_{t=1}^{n} |E h_t h_{t-j}|$$

$$\underset{\text{Using bounds computed in } R_{n,2}}{\le} K \sum_{j=m_n+1}^{n-1} j^{-1-\eta}$$

$$\le -\frac{1}{\eta} n^{-\eta} K + \frac{K}{\eta} m_n^{-\eta}$$

$$\lim_{n\to\infty} |R_{n,3}| \le \lim_{n\to\infty} -\frac{K}{\eta} n^{-\eta} + \lim_{n\to\infty} \frac{K}{\eta} m_n^{-\eta}$$

$$= 0 .$$

- $R_{n,1}$: before we deal with this I will write this term again to see why this can be difficult to check.

$$R_{1,n} := \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+1}^{n} \underbrace{(h_t h_{t-j} - E h_t h_{t-j})}$$

Call this process Z_{jt}. Moreover, notice that $Z_{jt} = g(h_t, h_{t-j})$ so by Proposition 1 it is α-mixing and $\alpha_{Z_j}(\ell) \le \alpha_h(\ell-j)$ for all $\ell = j+1, j+2, \dots$

This number must be positive!

You will see why this is important later. Mark this as ⊛.

We want to show that the object is $o_p(1)$, so we write

$$P\left(\left| \sum_{j=1}^{m_n} \underbrace{w(j,m_n)}_{\le C} \frac{1}{n} \sum_{t=j+1}^{n} Z_{jt} \right| > \varepsilon\right) \le P\left(\sum_{j=1}^{m_n} \left| \sum_{t=j+1}^{n} Z_{jt} \right| > \frac{\varepsilon \cdot n}{C} \right)$$

$$\le P\left(\left| \sum_{t=1+1}^{n} Z_{1t} \right| > \frac{\varepsilon \cdot n}{C \cdot m_n} \right) + \dots + P\left(\left| \sum_{t=m_n+1}^{n} Z_{m_n t} \right| > \frac{\varepsilon \cdot n}{C \cdot m_n} \right)$$

$$= \sum_{j=1}^{m_n} P\left(\left| \sum_{t=j+1}^{n} Z_{jt} \right| > \frac{\varepsilon \cdot n}{C \cdot m_n} \right)$$

$$\underset{\text{Markov's Inequality}}{\le} \sum_{j=1}^{m_n} \frac{C^2 m_n^2}{\varepsilon^2 n^2} E \left| \sum_{t=j+1}^{n} Z_{jt} \right|^2$$

Claim 1.- If $E|\hat{\sum} z_{jt}|^2 \leq K \cdot n \cdot (j+2)$ then $R_{1,n} = o_p(1)$.

Using this claim we get

$$\leq \sum_{j=1}^{m_n} \frac{c^2 m_n^2}{\varepsilon^2 n^2} K \cdot n \cdot (j+2)$$

$$= K \frac{c^2 m_n^2}{n \varepsilon^2} \sum_{j=1}^{m_n} (j+2)$$

$$\frac{1}{m_n} \sum_{j=1}^{m_n} (j+2) = \frac{\text{first} + \text{last}}{2}$$

$$= K \frac{c^2 m_n^2}{n \varepsilon^2} \left[\frac{(m_n+2)+3}{2} \right] m_n$$

$$\leq K \frac{m_n^4}{n}$$

$$= K \frac{1}{n} o(n)$$

$$= o(1).$$

To finish the proof we only need to show that the assumption for Claim 1 is true. Write

$$E|\sum^n z_{jt}|^2 \underset{\text{Variance formula}}{=} \sum_{t=j+1}^{n} E z_{jt}^2 + 2 \sum_{\ell=1}^{n-j-1} \sum_{t=\ell+j+1}^{n} E(z_{jt} z_{jt-\ell})$$

$$\leq \sum_{t=j+1}^{n} \sup_t E|h_t|^4 + 2 \sum_{\ell=1}^{n-j-1} \sum_{t=\ell+j+1}^{n} E(z_{jt} z_{jt-\ell})$$

$$\underset{\text{Cauchy Schwarz}}{\leq} \sum_{t=j+1}^{n} \left(\sup_t E|\varepsilon_t|^8 \cdot \sup_t E|c'x_t|^8 \right)^{1/2} + 2 \sum_{\ell=1}^{n-j-1} \sum_{t=\ell+j+1}^{n} E(z_{jt} z_{jt-\ell})$$

$$\leq K \cdot n + 2 \sum_{\ell=1}^{n-j-1} \sum_{t=\ell+j+1}^{n} E(z_{jt} z_{jt-\ell})$$

$$\underset{\text{split the sum}}{=} K \cdot n + \underbrace{2 \sum_{\ell=1}^{j} \sum_{t=\ell+j+1}^{n} |E z_{jt} z_{jt-\ell}|}_{\text{we cannot use mixing coefficient properties here but we can use Cauchy Schwarz, recall (*)}} + \underbrace{2 \sum_{\ell=j+1}^{n-j-1} \sum_{t=j+1+\ell}^{n} |E z_{jt} z_{jt-\ell}|}_{\text{we can use mixing coefficients properties here because } \ell \geq j+1 \text{, recall (*)}}$$

$$\underset{\text{Cauchy Schwarz + Proposition 4}}{\leq} K \cdot n + 2 \sum_{\ell=1}^{j} \sum_{t=\ell+j+1}^{n} \left(\underbrace{E|z_{jt}|^2 \, E|z_{jt-\ell}|^2}_{\text{we just showed these are bounded}} \right)^{1/2} + 2 \sum_{\ell=1}^{n-j-1} (n-\ell-j) \, \ell^{-1-\eta}$$

$$\leq Kn + K \cdot n \cdot j + 2n \sum_{\ell=1}^{n-j-\ell} \ell^{-1-\eta}$$

$$\underset{\text{By summability}}{\leq} K \cdot n + K \cdot n \cdot j + K \cdot n = K \cdot n (j+2). \quad \blacksquare$$

<u>Proposition HAC 2</u>.- Suppose that for some $p>2$ and $\Delta, \delta, C > 0$

(a) $\{(X_t', u_t)\}$ is α- mixing of size $-p/p-2$

(b) $E X_t u_t = 0$ for all t

(c) $E|X_{tj}|^{4p+\delta} \le \Delta$ for all t and all $j=1,\dots,K$

(d) $E|u_t|^{4p+\delta} \le \Delta$ for all t

(e) $|w(j,m)| \le C$ for all j and m

(g) $\lim_{m\to\infty} w(j,m) = 1$ for all j

(h) $m_n = o(n^{1/4})$.

Then
$$\hat{\Omega}_n - \Omega_n = o_p(1).$$

where $\hat{\Omega}_n = \hat{\Omega}_n(0) + \sum_{j=1}^{m_n} w(j,m_n)\,(\hat{\Omega}_n(j) + \hat{\Omega}_n(j))$ is the feasible estimator.

$$\hat{\Omega}_n(j) = \frac{1}{n}\sum_{t=j+1}^{n} (X_t \hat{u}_t)(X_{t-j}\hat{u}_{t-j})'$$

<u>proof</u>:

$$\hat{\Omega}_n - \Omega_n = \underbrace{\hat{\Omega}_n - \tilde{\Omega}_n}_{\text{we just need to show that this is } o_p(1).} + \underbrace{\tilde{\Omega}_n - \Omega_n}_{o_p(1) \text{ by Proposition HAC 1}}$$

Recall that $\hat{u}_t = u_t - X_t'(\hat{\beta}_n - \beta)$. We now write

$$\hat{\Omega}_n - \tilde{\Omega}_n = \left(\frac{1}{n}\sum_{t=1}^{n}\hat{u}_t^2 X_t X_t' + \sum_{j=1}^{m_n} w(j,m_n)\frac{1}{n}\sum \hat{u}_t \hat{u}_{t-j}\,(X_t X_{t-j}' + X_{t-j}X_t')\right)$$
$$-$$
$$\left(\frac{1}{n}\sum_{t=1}^{n} u_t^2 X_t X_t' + \sum_{j=1}^{m_n} w(j,m_n)\frac{1}{n}\sum u_t u_{t-j}\,(X_t X_{t-j}' + X_{t-j}X_t')\right)$$

$$= -2\underbrace{\frac{1}{n}\sum_{t=1}^{n}\left((\hat{\beta}_n-\beta)'X_t u_t\right)X_t X_t'}_{B_{n,1}} + \underbrace{\frac{1}{n}\sum_{t=1}^{n}\left((\hat{\beta}_n-\beta)'X_t\right)^2 X_t X_t'}_{B_{n,2}}$$
$$-\underbrace{\sum_{j=1}^{m_n} w(j,m_n)\frac{1}{n}\sum_{t=j+1}^{n}\left((\hat{\beta}_n-\beta)'X_t u_{t-j}\right)(X_t X_{t-j}' + X_{t-j}X_t')}_{B_{n,3}}$$
$$-\underbrace{\sum_{j=1}^{m_n} w(j,m_n)\frac{1}{n}\sum_{t=j+1}^{n}\left((\hat{\beta}_n-\beta)'X_{t-j} u_t\right)(X_t X_{t-j}' + X_{t-j}X_t')}_{B_{n,4}}$$

$$+ \underbrace{\sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+1}^{n} \left((\hat\beta_n - \beta)' x_t \right) \left((\hat\beta_n - \beta)' x_{t-j} \right) \left(x_t x_{t-j}' + x_{t-j} x_t' \right)}_{B_{n,5}}$$

Then by the cramer wold device we will work with $c' (\hat\Omega_n - \tilde\Omega_n) c$ for $c \in \mathbb{R}^K$. We will work with each term separately.

- $B_{n,1}$:

$$|c' B_{n,1} c| = \left| \frac{1}{n} \sum^{n} \left((\hat\beta_n - \beta)' x_t u_t \right) c' x_t x_t' c \right|$$

$$\leq \| \hat\beta_n - \beta \| \frac{1}{n} \sum_{t=1}^{n} \|x_t\|^3 \|c\|^2 |u_t|$$

using $\|c\| = O(1)$

$$= o_p(1)\, O(1) \left[\frac{1}{n} \sum_{t=1}^{n} \|x_t\|^3 |u_t| - \frac{1}{n} \sum_{t=1}^{n} E\|x_t\|^3 |u_t| + \frac{1}{n} \sum_{t=1}^{n} E\|x_t\|^3 |u_t| \right]$$

$$= o_p(1)\, O(1) \left[\frac{1}{n} \sum_{t=1}^{n} E\|x_t\|^3 |u_t| + o_p(1) \right]$$

Hölder inequality $p=4$

$$\leq o_p(1)\, O(1) \left[\frac{1}{n} \sum_{t=1}^{n} \left(E|u_t|^4\right)^{1/4} \left(E\|x_t\|^4\right)^{3/4} + o_p(1) \right]$$

$$= o_p(1)\, O(1) \left[O(1) + o_p(1) \right]$$

$$= o_p(1).$$

- $B_{n,2}$:

$$|c' B_{n,2} c| = \left| \frac{1}{n} \sum_{t=1}^{n} \left((\hat\beta_n - \beta)' x_t \right)^2 c' x_t x_t' c \right|$$

$$\leq \|c\|^2 \|\hat\beta_n - \beta\|^2 \left[\frac{1}{n} \sum_{t=1}^{n} \|x_t\|^4 - \frac{1}{n} \sum_{t=1}^{n} E\|x_t\|^4 + \frac{1}{n} \sum_{t=1}^{n} E\|x_t\|^4 \right]$$

$$= O(1)\, o_p(1) \left[o_p(1) + O(1) \right]$$

$$= o_p(1).$$

- $B_{n,3}$:

$$|c' B_{n,3} c| \leq 2 \|c\|^2 \|\hat\beta_n - \beta\| \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+1}^{n} \left(|u_{t-j}| \cdot \|x_t\|^2 \cdot \|x_{t-j}\| \right)$$

$$= o_p(1) \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+}^{n} \{ \underbrace{|u_{t-j}| \cdot \|x_t\|^2 \|x_{t-j}\| - E|u_{t-j}| \cdot \|x_t\|^2 \|x_{t-j}\|}_{\text{Define as } Z_{j,t} \text{ and notice that it's } \alpha\text{-mixing of size } \frac{-p}{p-2}} + E|u_{t-j}| \|x_t\|^2 \|x_{t-j}\| \}$$

$B_{n,1}$ in Proposition HAC 1 + Cauchy Schwarz

$$\leq o_p(1) \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+}^{n} \{ o_p(1) + (E|u_{t-j}|^2 \|x_{t-j}\|^2 E\|x_t\|^4)^{1/2} \}$$

Cauchy Schwarz

$$\leq o_p(1) \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+}^{n} \{ o_p(1) + ((E|u_{t-j}|^4 E\|x_{t-j}\|^4)^{1/2} E\|x_t\|^4)^{1/2} \}$$

$$\leq O_p(1) \quad O(m_n)$$

we need to use $\|\hat{\beta}_n - \beta\|$

$$\leq \|V_n\|^{1/2} \|V_n^{-1/2} \sqrt{n} (\hat{\beta}_n - \beta)\| \quad \frac{O(m_n)}{\sqrt{n}}$$

$$\leq O(1) \quad O_p(1) \quad \frac{o(n^{1/4})}{n^{1/2}}$$

$$= K \quad O_p(1) \quad o(n^{-1/4})$$

$$= o_p(1).$$

- $B_{n,4}$:

$$|c' B_{n,4} c| \leq 2\|c\|^2 \|\hat{\beta}_n - \beta\| \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+1}^{n} |u_{t-j}| \|x_t\| \|x_{t-j}\|^2$$

$$= o_p(1) \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+}^{n} \{ \underbrace{|u_{t-j}| \cdot \|x_t\| \cdot \|x_{t-j}\|^2 - E|u_{t-j}| \cdot \|x_t\| \|x_{t-j}\|^2}_{\text{Define as } Z_{j,t} \text{ and notice that it's } \alpha\text{-mixing of size } \frac{-p}{p-2}} + E|u_{t-j}| \|x_t\| \|x_{t-j}\|^2 \}$$

$$= o_p(1).$$

By same steps as $B_{n,3}$

- $B_{n,5}$:

$$|c' B_{n,5} c| \leq 2\|c\|^2 \|\hat{\beta}_n - \beta\|^2 \sum_{j=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+1}^{n} \|x_t\|^2 \|x_{t-j}\|^2$$

$$= o_p(1) \sum_{=1}^{m_n} w(j,m_n) \frac{1}{n} \sum_{t=j+}^{n} \{ \underbrace{\|x_t\| \cdot \|x_{t-j}\|^2 - E\|x_t\|^2 \|x_{t-j}\|^2}_{\text{Define as } Z_{j,t} \text{ and notice that it's } \alpha\text{-mixing of size } \frac{-p}{p-2}} + E\|x_t\|^2 \|x_{t-j}\|^2 \}$$

$$= o_p(1).$$

By same steps as $B_{n,3}$

■