Econ 627

Assignment 2

The due date for this assignment is Tuesday January 28.

1. Consider the following model

$$
\begin{aligned}
Y_i &= X_i'\beta + U_i, \\
EZ_iU_i &= 0, \\
\operatorname{rank}(EZ_iX_i') &= k,
\end{aligned}
$$

where β is a k-vector of unknown parameters, X_i is a k-vector of random regressors, and Z_i is an l-vector of instruments. Write $Z_i = (Z_{1i}', Z_{2i}')'$, where Z_{1i} is an l_1-sub-vector, and assume that $\operatorname{rank}(EZ_{1i}X_i') = k$.

(a) Show that the efficient GMM estimator of β based only the first l_1 moment conditions can be written as

$$\left(\sum_{i=1}^n X_iZ_i'W_n\sum_{i=1}^n Z_iX_i'\right)^{-1}\sum_{i=1}^n X_iZ_i'W_n\sum_{i=1}^n Z_iY_i,$$

where

$$W_n = \begin{pmatrix} \widehat{\Omega}_{11,n}^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

and $\widehat{\Omega}_{11,n}$ is a consistent estimator of $EU_i^2Z_{1i}Z_{1i}'$.

Solution:

$$\tilde{\beta}_n = (X'ZW_nZ'X)^{-1}X'ZW_nZ'Y$$

Using the first l_1 moment conditions imply

$$W_n = \begin{pmatrix} W^*_{l_1\times l_1} & 0_{l_1\times l_2} \\ 0_{l_2\times l_1} & 0_{l_2\times l_2} \end{pmatrix}_{l\times l}$$

Then, notice that

$$\tilde{\beta}_n = \left(X'\begin{pmatrix} \underset{n\times l_1}{Z_1} & \underset{n\times l_2}{Z_2}\end{pmatrix}\begin{pmatrix} W^* & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} Z_1' \\ Z_2'\end{pmatrix}X\right)^{-1}X'(Z_1\ Z_2)\begin{pmatrix} W^* & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} Z_1' \\ Z_2'\end{pmatrix}y$$

$$= \left(X'\begin{pmatrix} Z_1W^* & 0\end{pmatrix}\begin{pmatrix} Z_1' \\ Z_2'\end{pmatrix}X\right)^{-1}X'\begin{pmatrix} Z_1W^* & 0\end{pmatrix}\begin{pmatrix} Z_1' \\ Z_2'\end{pmatrix}y$$

$$= (X'Z_1W^*Z_1'X)^{-1}X'Z_1W^*Z_1'y$$

Therefore, the efficient matrix is $\hat{\Omega}_{11,n}^{-1}$, because the asymptotic linear representation yields a score of

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_{1i}U_i \xrightarrow{d} N(0, EU_i^2Z_{1i}Z_{1i}')$$

(b) From the result in (a), what you can say about relationship between the degree of overidentification and efficiency of GMM?

solution:

$$\text{Avar}(\overset{\overset{\ell_1 \text{ moments}}{\downarrow}}{\hat{\beta}_n}) = \left(E\, \underset{k\times\ell_1}{x_i} \underset{1\times\ell}{z_i'}\ \Omega^{-1}\ E\, z_i x_i' \right), \quad \text{let } Q := E\, z_i x_i'$$

When we do the asymptotic linear representation for $\overset{\overset{\ell \text{ moments}}{\downarrow}}{\tilde{\beta}_n}$ we get:

$$\sqrt{n}\,(\tilde{\beta}_n - \beta) = \left(\frac{X'Z}{n}\ W_n\ \frac{Z'X}{n} \right)^{-1} \frac{X'Z}{n}\ W_n\ \frac{Z'u}{\sqrt{n}}$$

$$= \left(\frac{1}{n}\sum_{i=1}^n x_i z_i'\ W_n\ \frac{1}{n}\sum_{i=1}^n z_i x_i' \right)^{-1} \frac{1}{n}\sum_{i=1}^n x_i z_i'\ W_n\ \frac{1}{\sqrt{n}}\sum_{i=1}^n z_i u_i$$

$$\xrightarrow[d]{} \mathcal{N}\left(0, (Q'WQ)^{-1} Q'W\,\Omega\, WQ\,(Q'WQ)^{-1}\right)$$

We will see the relation between both $\text{Avar}(\cdot)$:

$$Q'\Omega^{-1}Q - Q'WQ\,(Q'W\Omega W Q)^{-1} Q'WQ$$

Notice that Ω^{-1} is a real symmetric p.d matrix, so we can find an orthonormal eigenbasis R such that

$$\Omega^{-1} = R\,\Lambda\,R' \qquad \text{where } R = (\underset{\ell\times1}{r_1} \cdots \underset{\ell\times1}{r_\ell})_{\ell\times\ell}$$
$$= R\,\Lambda^{1/2}\,\Lambda^{1/2}\,R'$$

And consider D as a diagonal matrix:

- $RD = (r_1 \cdots r_\ell)\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_\ell \end{pmatrix} = (d_1 r_1 \ \cdots \ d_n r_\ell)$
- $DR' = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_\ell \end{pmatrix}\begin{pmatrix} r_1' \\ \vdots \\ r_\ell' \end{pmatrix} = \begin{pmatrix} d_1 r_1' \\ \vdots \\ d_\ell r_\ell' \end{pmatrix}$

Notice that $(RD)' = \begin{pmatrix} d_1 r_1' \\ \vdots \\ d_\ell r_\ell' \end{pmatrix} = DR'$.

Therefore,

$$\Omega^{-1} = \underbrace{R\,\Lambda^{1/2}}_{C'}\ \underbrace{\Lambda^{1/2}\,R'}_{C} = C'C$$

where C is invertible because R is a change of basis (independent columns) matrix.

$$Q'\Omega^{-1}Q - Q'WQ(Q'W\Omega WQ)^{-1}Q'WQ$$

$$= Q'C'\left(I - (C')^{-1}WQ\,(Q'WC^{-1}(C')^{-1}WQ)^{-1}Q'WC^{-1}\right)CQ$$

$$= Q'C'(\underbrace{I - H(H'H)^{-1}H}_{\text{projection matrix}})CQ \quad \Rightarrow \quad \text{p.s.d}$$

where $H := (C')^{-1}WQ$.

2. **(Local power of the overidentifying restrictions test)** Consider the linear IV regression model:

$$Y_i = X_i'\beta + U_i, \tag{1}$$
$$\operatorname{rank}(EZ_iX_i') = k, \tag{2}$$

where β is a k-vector of unknown parameters, X_i is a k-vector of random regressors, and Z_i is an l-vector of instruments such that $l > k$. Assume that the IVs fail locally the exogeneity condition:

$$EU_iZ_i = \frac{\delta}{\sqrt{n}}, \tag{3}$$

where $\delta \neq 0$ is an unknown l-vector of constants. Assume that data are iid, and

$$\Omega = Var(U_iZ_i) \text{ is finite and positive definite}, \tag{4}$$
$$EZ_{i,j}^4 < \infty \text{ for all } j = 1, \ldots, l, \tag{5}$$
$$EX_{i,j}^4 < \infty \text{ for all } j = 1, \ldots, k, \tag{6}$$
$$EU_i^4 < \infty. \tag{7}$$

(a) Let

$$\tilde{\beta}_n(W_n) = (X'ZW_nZ'X)^{-1}X'ZW_nZ'Y,$$

where X is the $n \times k$ matrix of regressors, Z is the $n \times l$ matrix of instruments, Y is the n-vector of dependent variables, and $W_n \to_p W$, a positive definite and symmetric $l \times l$ weight matrix. Find the probability limit of $\tilde{\beta}_n(W_n)$. Justify your answer. Is the estimator consistent?

solution:

Recall

$$\tilde{\beta}_n(W_n) - \beta = \left(\frac{X'Z}{n}W_n\frac{Z'X}{n}\right)^{-1}\frac{X'Z}{n}W_n\frac{1}{n}\sum_{i=1}^n Z_iU_i$$

add and substract EZ_iU_i

$$= \Big(\underbrace{\frac{X'Z}{n}W_n\frac{Z'X}{n}}_{\to_p Q'WQ}\Big)^{-1}\underbrace{\frac{X'Z}{n}W_n}_{\to_p Q'W}\underbrace{\frac{1}{n}\sum_{i=1}^n(Z_iU_i - EZ_iU_i)}_{\to_p 0}$$

$$+ \Big(\underbrace{\frac{X'Z}{n}W_n\frac{Z'X}{n}}_{\to_p Q'WQ}\Big)^{-1}\underbrace{\frac{X'Z}{n}W_n}_{\to_p Q'W}\underbrace{\frac{\delta}{\sqrt{n}}}_{\to 0}$$

so we only require $Q'WQ$ to be invertible $\Rightarrow$ $\operatorname{rank}(Q) = k$.
Then, we conclude

$$\tilde{\beta}_n(W_n) - \beta = o_p(1).$$

(b) Derive the asymptotic distribution of $\sqrt{n}(\tilde{\beta}_n(W_n) - \beta)$. Justify your answer.
Hint: Note that the CLT cannot be applied directly to $n^{-1/2}\sum_{i=1}^{n} Z_i U_i$.

solution:

$$\sqrt{n}\,(\tilde{\beta}_n - \beta) = \left(\frac{X'Z}{n} W_n \frac{Z'X}{n}\right)^{-1} \frac{X'Z}{n} W_n \frac{1}{\sqrt{n}} \underbrace{\sum_{i=1}^{n} (Z_i U_i - E U_i Z_i)}_{\text{normal}}$$

$$+ \underbrace{\left(\frac{X'Z}{n} W_n \frac{Z'X}{n}\right)^{-1} \frac{X'Z}{n} W_n \delta}_{\text{contributes to mean, not variance}}$$

$$\xrightarrow{d} N\left((Q'WQ)^{-1} Q'W\delta,\ (Q'WQ)^{-1} Q'W\Omega WQ (Q'WQ)^{-1}\right)$$

(c) Can the asymptotic distribution in part (b) be used for the inference on β? Explain your answer.

solution:

The mean depends on δ which is unknown, so we can't use that distribution for inference.

(d) Propose a consistent estimator for Ω and show its consistency.

solution:

The score is $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_i U_i - E U_i Z_i)$ and the variance of this is

$$E\{(U_i Z_i - E U_i Z_i)(U_i Z_i - E U_i Z_i)'\}$$

$$= E\, U_i^2 Z_i Z_i' - E[E(U_i Z_i) U_i Z_i'] - E[U_i Z_i\, E U_i Z_i'] + E U_i Z_i\, E U_i Z_i'$$

$$= E\, U_i^2 Z_i Z_i' - \underbrace{E U_i Z_i}_{\frac{\delta}{\sqrt{n}}}\ \underbrace{E U_i Z_i'}_{\frac{\delta'}{\sqrt{n}}} \Rightarrow$$

should vanish with $n \to \infty$ so we propose to use the left part.

$$\hat{\Omega}_n = \frac{1}{n} \sum_{i=1}^{n} \hat{U}_i^2 Z_i Z_i', \quad \hat{U}_i = U_i + X_i'(\beta - \tilde{\beta}_n)$$

$$= \frac{1}{n} \sum_{i=1}^{n} U_i^2 Z_i Z_i' + \underbrace{\frac{1}{n} \sum_{i=1}^{n} (X_i'(\beta - \tilde{\beta}_n))^2 Z_i Z_i' - \frac{2}{n} \sum_{i=1}^{n} (X_i'(\beta - \hat{\beta}_n)) U_i Z_i Z_i'}_{\text{bound them and vanish b.c. } \beta - \hat{\beta}_n = o_p(1)}$$

Focusing on the leading term

$$\frac{1}{n}\sum_{i=1}^{n}(u_i z_i)(u_i z_i)' = \frac{1}{n}\sum_{i=1}^{n}(u_i z_i - Eu_i z_i)(u_i z_i)' + Eu_i z_i \frac{1}{n}\sum_{i=1}^{n} u_i z_i'$$

$$= \underbrace{\frac{1}{n}\sum_{i=1}^{n}(u_i z_i - Eu_i z_i)(u_i z_i - Eu_i z_i)'}_{\to_p \Omega}$$

$$+ Eu_i z_i \frac{1}{n}\sum_{i=1}^{n}(\underbrace{u_i z_i}_{o_p(1)} - Eu_i z_i)'$$

$$+ \underbrace{\frac{1}{n}\sum_{i=1}^{n}(u_i z_i - Eu_i z_i)}_{o_p(1)} Eu_i z_i' + \underbrace{Eu_i z_i}_{o(1)} \underbrace{Eu_i z_i'}_{o(1)}$$

(e) Let $\hat{\Omega}_n$ be a consistent estimator of Ω, and define $\hat{\beta}_n = \tilde{\beta}_n(\hat{\Omega}_n^{-1})$. Let

$$\begin{aligned} \overline{g}_n(b) &= n^{-1}Z'(Y - Xb), \text{ and} \\ J_n(b) &= n\overline{g}_n(b)'\hat{\Omega}_n^{-1}\overline{g}_n(b). \end{aligned}$$

Also, write $\Omega^{-1} = CC'$ and $\Omega = C'^{-1}C^{-1}$. Show that

$$J_n\left(\widehat{\beta}_n\right) = n\left(C'\overline{g}_n\left(\widehat{\beta}_n\right)\right)'\left(C'\widehat{\Omega}_n C\right)^{-1}\left(C'\overline{g}_n\left(\widehat{\beta}_n\right)\right).$$

solution:

$$J_n(\hat{\beta}_n) = n\, \bar{g}_n(\hat{\beta})'\, \hat{\Omega}_n^{-1}\, \bar{g}_n(\hat{\beta})$$

$$= n\, \bar{g}_n(\hat{\beta})'\, C\, C^{-1}\, \hat{\Omega}_n^{-1}\, C'^{-1}\, C'\, \bar{g}_n(\hat{\beta})$$

$$= n\, (C'\bar{g}_n(\hat{\beta}_n))'\, (C'\hat{\Omega}_n C)^{-1}\, (C'\bar{g}_n(\hat{\beta}_n))$$

as desired.

(f) Show that $C'\overline{g}_n\left(\widehat{\beta}_n\right) = D_n C'\overline{g}_n(\beta)$, where β is the true value of the parameters, and

$$D_n = I_l - C'\left(\frac{Z'X}{n}\right)\left(\frac{X'Z}{n}\hat{\Omega}_n^{-1}\frac{Z'X}{n}\right)^{-1}\frac{X'Z}{n}\hat{\Omega}_n^{-1}C'^{-1}.$$

solution:

$$C'\bar{g}_n(\hat{\beta}_n) = C'\, n^{-1} Z'(Y - X\hat{\beta})$$

$$= C'\left[n^{-1} Z'(Y - X(X'Z\Omega_n^{-1}Z'X)^{-1}X'Z\Omega_n^{-1}Z'X\beta - X(X'Z\Omega_n^{-1}Z'X)^{-1}X'Z\Omega_n^{-1}Z'u)\right]$$

$$= C'\left[n^{-1} Z'(\underbrace{Y - X\beta}_{u} - X(X'Z\Omega_n^{-1}Z'X)^{-1}X'Z\Omega_n^{-1}Z'u)\right]$$

$$= C' g_n(\beta) - C' Z'X(X'Z\Omega_n^{-1}Z'X)^{-1}X'Z\Omega_n^{-1} g_n(\beta)$$

$$= \left(I_l - C' Z'X (X'Z \Omega_n^{-1} Z'X)^{-1} X'Z \Omega_n^{-1} C'^{-1} \right) C' g_n(\beta)$$

$$= D_n C' g_n(\beta).$$

(g) Show that $D_n \to_p I_l - R\,(R'R)^{-1}\,R'$, where $R = C'Q$, and $Q = EZ_iX_i'$.

<u>solution:</u>

$$D_n = I_l - C' \frac{Z'X}{n} \left(\frac{X'Z}{n} \frac{\Omega_n^{-1}}{n} \frac{Z'X}{n} \right)^{-1} \frac{X'Z}{n} \frac{\Omega_n^{-1}}{n} C'^{-1}$$

$$\xrightarrow{p} I_l - C' Q (Q' C'C\, Q)^{-1} Q'C' \underbrace{CC'^{-1}}_{I}$$

$$\underset{p}{=} I_l - R(R'R)^{-1} R'$$

(h) Find the asymptotic distribution of $\sqrt{n} C' \bar{g}_n(\beta)$. Justify your answer.

$$\sqrt{n}\; C' \bar{g}_n(\beta) = C' \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_i z_i \pm C'\sqrt{n}\, E u_i z_i$$

$$= C' \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (u_i z_i - E u_i z_i) + C'\sqrt{n}\, E u_i z_i$$

$$= \underbrace{C' \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (u_i z_i - E u_i z_i)}_{\substack{\xrightarrow{d} \\ N(0,\; C'\Omega C) \\ = N(0,\; C' C'^{-1} C^{-1} C) \\ = N(0,\; I_l)}} + C'\delta$$

$\to$ used symmetry of Ω

$$\xrightarrow{d} N(C'\delta,\; I_l)$$

(i) Show that the asymptotic distribution of $J_n\left(\widehat{\beta}_n\right)$ is non-central χ^2. Find the non-centrality parameter and the degrees of freedom. Justify your answer.
Hints: Suppose that $\mathcal{N} \sim N(\mu, I_l)$, and let M be an $l \times l$ orthogonal projection matrix. Consider the distribution of $\mathcal{N}'M\mathcal{N}$ and how it depends on $\mu'M\mu$.

solution:

Recall that $$J_n(\hat{\beta}_n) = n\,(C'\bar{g}_n(\hat{\beta}_n))'\,(C'\hat{\Omega}_n C)^{-1}\,(C'\bar{g}_n(\hat{\beta}_n))$$

$$= (\sqrt{n}\,\underbrace{C'\bar{g}_n(\hat{\beta}_n)}_{D_n C'\bar{g}_n(\beta)})'\,(\underbrace{C'\hat{\Omega}_n C}_{\xrightarrow{p} I_l})^{-1}\,(\sqrt{n}\,\underbrace{C'\bar{g}_n(\hat{\beta}_n)}_{D_n C'\bar{g}_n(\beta)})$$

Notice that $\sqrt{n}\, D_n\, C'\bar{g}_n(\beta) \xrightarrow{d} (I_l - R(R'R)R')\,\underbrace{N(C'\delta, I_l)}_{\text{label as } \mathcal{N}}$

Therefore,

$$J_n(\hat{\beta}_n) \xrightarrow{d} [(I_l - R(R'R)R')\mathcal{N}]'\,(I_l - R(R'R)R')\mathcal{N}$$

$$\overset{d}{=} \mathcal{N}'\,\underbrace{(I_l - R(R'R)R')}_{M}\,\mathcal{N}$$ by idempotence of the proj. matrix.

Consider the eigendecomposition of $M = H\Lambda H'$ where $H = [h_1 \cdots h_l]$ and $H'H = I_l$. Then consider the following dot product

$h_j'\mathcal{N}$ ($1\times l$, $l\times 1$)

- $E[h_j'\mathcal{N}\mathcal{N}^T h_j] = h_j' E\mathcal{N}\mathcal{N}^T h_j$
$= h_j' I_l h_j = h_j'h_j = 1$ because they're orthonormal.

- $E[h_j'\mathcal{N}] = h_j' C'\delta = (Ch_j)'\delta$

Then

$$H'\mathcal{N} = \begin{bmatrix} h_1' \\ \vdots \\ h_l' \end{bmatrix}\mathcal{N} \sim N\left(\begin{bmatrix} (Ch_1)'\delta \\ \vdots \\ (Ch_l)'\delta \end{bmatrix}, I_l\right) \equiv N\left(\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_l \end{bmatrix}, I_l\right)$$

Finally, this implies that

$\mathcal{N}'M\mathcal{N} = (H'\mathcal{N})'\Lambda(H'\mathcal{N})$ is the sum of $l-k$ independent normal random variables with unit variance and means given by μ_j.

The non centrality parameter is

$$\sum_{j=1}^{l}\mu_j^2\lambda_j = (H'C'\delta)'\Lambda(H'C'\delta) = \delta'C(I_l - R(R'R)^{-1}R')C'\delta$$

$$= \delta'(CC' - CC'Q(Q'CC'Q)^{-1}Q'CC')\delta$$

$$= \delta' \left(\Omega^{-1} - \Omega^{-1} Q (Q'\Omega^{-1}Q)^{-1} Q'\Omega^{-1} \right) \delta$$

(j) Suppose that $\delta = Q\pi$, where Q is defined in (g) and $\pi \neq 0$ is some k-vector. Show that in this case the overidentifying restrictions test has only *trivial* power: the probability of rejecting H_0: $EZ_i(Y_i - X_i'\beta) = 0$ against H_1: $EZ_i(Y_i - X_i'\beta) = Q\pi/\sqrt{n}$ is equal to α (the same as the size of test).

solution:

If $\delta = Q\pi$ then the non centrality parameter becomes

$$\pi' (Q'\Omega^{-1}Q - Q'\Omega^{-1}Q (Q'\Omega^{-1}Q^{-1}) Q'\Omega^{-1}Q) \pi = 0$$

Therefore,

$$P\left(J_n(\hat{\beta}_n) > \chi^2_{\ell-k,\,1-\alpha} \right) \to \alpha$$

so we say that this test has trivial power under such δ's.

(k) To explain the result in (j), show that $EZ_i(Y_i - X_i'\beta) = Q\pi/\sqrt{n}$ can be re-written as $EZ_i(Y_i - X_i'\beta^*) = 0$ for some $\beta^* \neq \beta$ and find β^*. Hence, even though $EZ_i(Y_i - X_i'\beta) \neq 0$, the following more general null hypothesis holds in this case:

$$H_0 : EZ_i(Y_i - X_i'\beta^*) = 0 \text{ for some } \beta^* \in \mathbb{R}^k,$$

where β^* can be different from the "true" β.

solution:

$$E Z_i u_i = \frac{Q\pi}{\sqrt{n}} = E Z_i X_i' \frac{\pi}{\sqrt{n}}$$

$$\Rightarrow \left(E Z_i u_i - E Z_i X_i' \frac{\pi}{\sqrt{n}} \right) = 0$$

$$E Z_i \left(Y_i - X_i'\beta - X_i' \frac{\pi}{\sqrt{n}} \right) = 0$$

$$E Z_i (Y_i - X_i' \beta^*) = 0$$

with $\beta^* = \beta + \frac{\pi}{\sqrt{n}}$ so the restriction holds but for parameters that differ from the true value.