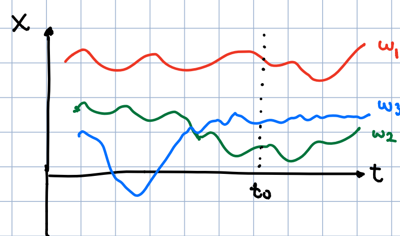
Def.- (Strict Stationarity) We say a random sequence is strictly stationary if the shift transformation is measure preserving. This implies that $\{X_t\}_1^{\infty}$ and $\{X_{t+k}\}_1^{\infty}$ have the same joint distribution for every $k>0$.

Def.- (Mean Stationary) Let $\mu_t := EX_t$ and $\gamma_{kt} := Cov(X_t, X_{t+k})$ and consider cases where in which the sequence $\{\mu_t\}_1^{\infty}$ and the array $\{\{\gamma_{mt}\}_{m=0}^{\infty}\}_{t=1}^{\infty}$ are well defined. If $\mu_t = \mu$ $\forall t$ we say the sequence is mean stationary.

Def.- (Covariance Stationary) If a mean stationary sequence has $\gamma_{mt} = \gamma_m$ then we say it's covariance stationary or just stationary.



$$\bar{X}_{N,t_0} = \frac{1}{N}\sum_{i=1}^{N} X_{t_0}(\omega_i) \quad \text{"Ensemble average"}$$

$$\bar{X}_N = \frac{1}{N}\sum_{t=1}^{N} X_t(\omega_i) \quad \text{"Time average"}$$

Mixing Processes

For two σ-fields $\mathcal{F}$ and $\mathcal{G}$, we define

$$\alpha(\mathcal{F},\mathcal{G}) = \sup_{F\in\mathcal{F},\, G\in\mathcal{G}} |P(F\cap G) - P(G)P(F)|$$

And define for $r \le s$ $\quad B_r^s = \sigma(X_r, \dots, X_s)$.

Def.- (Strong Mixing Coefficients) $\alpha(m) = \sup_j \alpha(B_{-\infty}^{j}, B_{j+m}^{\infty})$ for $m=1,2,\dots$

Def.- (Strong Mixing Process) A process $\{X_t\}$ is strong or α-mixing if $\lim_{m\to\infty} \alpha(m) = 0$.

Def.- $\alpha(m)$ is of size $-a$, $a>0$, if $\alpha(m) = O(m^{-a-\epsilon})$ for some $\epsilon>0$.

Proposition 1.- Let $\{X_t\}$ be α-mixing of size $-a$, and $Y_t = g(X_t, \dots, X_{t-h})$ where g is measurable. Then $\{Y_t\}$ is also α-mixing of size $-a$.

proof: For $r\le s$ let $B_r^s = \sigma(X_r,\dots,X_s)$ and $C_r^s = \sigma(Y_r,\dots,Y_s)$. It can be inferred that $C_{-\infty}^{j} \subset B_{-\infty}^{j}$ and $C_{j+m}^{\infty} \subset B_{j+m-h}^{\infty}$ for all j and $m\ge h$.

$$\alpha_Y(m) \underset{def}{=} \sup_j \sup_{F\in C_{-\infty}^{j},\, G\in C_{j+m}^{\infty}} |P(F\cap G) - P(G)P(F)|$$

$$\le \sup_j \sup_{F\in B_{-\infty}^{j},\, G\in B_{j+m}^{\infty}} |P(F\cap G) - P(G)P(F)|$$

$$= \alpha_X(m-h)$$

$$= O(m^{-a-\epsilon}) \quad \text{for } \epsilon>0 \text{ and } m>h.$$

Covariance Inequalities for Mixing Processes

Proposition 2.- Suppose that $|X_t| \le C_1$ and $|X_{t-h}| \le C_2$ for some $C_1, C_2 > 0$. Then

$$|\operatorname{Cov}(X_t, X_{t-h})| \le 4\, C_1 C_2\, \alpha(h).$$

proof:

Define the following random variable

$$\eta = \operatorname{sign}\{E(X_t \mid B_{-\infty}^{t-h}) - EX_t\}, \quad \eta \in \{-1, 0, 1\} \text{ and it's } B_{-\infty}^{t-h} \text{ measurable.}$$

⊛ Notice that we can write covariances by only demeaning one of the two random variables

$$\begin{aligned} E(X_t - EX_t)(X_{t-h} - EX_{t-h}) &= E\{X_t \cdot X_{t-h} - X_t \cdot EX_{t-h} - EX_t \cdot X_{t-h} + EX_t \cdot EX_{t-h}\} \\ &= E\, X_t \cdot X_{t-h} - EX_t \cdot EX_{t-h} \\ &= E\{X_t (X_{t-h} - EX_{t-h})\} = E\{X_{t-h}(X_t - EX_t)\}. \end{aligned}$$

Now, write

$$\begin{aligned} |\operatorname{Cov}(X_t, X_{t-h})| = |E\{X_{t-h}(X_t - EX_t)\}| &\underset{\text{LIE}}{=} |E\{X_{t-h}(E(X_t \mid B_{-\infty}^{t-h}) - EX_t)\}| \\ &\le E\{|X_{t-h}| \cdot |E(X_t \mid B_{-\infty}^{t-h}) - EX_t|\} \\ &\le C_2\, E\{\eta\, (E(X_t \mid B_{-\infty}^{t-h}) - EX_t)\} \\ &\underset{\text{LIE}}{=} C_2 (E\eta X_t - E\eta EX_t) \\ &= C_2 \operatorname{Cov}(\eta, X_t) \\ &= C_2 |\operatorname{Cov}(\eta, X_t)| \end{aligned}$$

it's non negative by construction!

Next, define the following random variable

$$\xi = \operatorname{sign}\{E(\eta \mid B_t^{\infty}) - E\eta\}, \quad \xi \in \{-1, 0, 1\} \text{ and it's } B_t^{\infty} \text{ measurable.}$$

Now, repeating the same argument

$$\begin{aligned} |\operatorname{Cov}(\eta, X_t)| = |E X_t(\eta - E\eta)| &\underset{\text{LIE}}{=} |E\{X_t (E(\eta \mid B_t^{\infty}) - E\eta)\}| \\ &\le E\{|X_t|\, |E(\eta \mid B_t^{\infty}) - E\eta|\} \\ &\le C_1\, E\{\xi (E(\eta \mid B_t^{\infty}) - E\eta)\} \\ &= C_1 (E\xi\eta - E\xi E\eta) \\ &= C_1 |E\eta\xi - E\xi E\eta|. \end{aligned}$$

Combining both results yield

$$|Cov(X_t, X_{t-h})| \leq C_1 . C_2 \; |E\xi\eta - E\xi E\eta| .$$

Define the following events $A_1 = \{\eta = 1\}$ $\quad A_{-1} = \{\eta = -1\}$ $\quad A_0 = \{\eta = 0\}$
$B_1 = \{\xi = 1\}$ $\quad B_{-1} = \{\xi = -1\}$ $\quad B_0 = \{\xi = 0\}$

- $$E\eta\xi = P(A_1 \cap B_1) . 1\times 1 + P(A_1 \cap B_{-1}) \; 1\times(-1) + \cancel{P(A_1 \cap B_0) \; 1\times 0} + P(A_{-1} \cap B_1) . 1\times(-1) + P(A_{-1} \cap B_{-1}) \; (-1)\times(-1) + \cancel{P(A_{-1} \cap B_0) \; (-1)\times 0} + \cancel{P(A_0 \cap B_1) . 0\times 1} + \cancel{P(A_0 \cap B_{-1}) \; 0\times(-1)} + \cancel{P(A_0 \cap B_{-1}) \; 0\times(-1)}$$

$$= P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1})$$

- $E\eta = P(A_1) - P(A_{-1})$

- $E\xi = P(B_1) - P(B_{-1})$

Hence,

$$|E\eta\xi - E\eta E\xi| = | P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) - [P(A_1) - P(A_{-1})] \times [P(B_1) - P(B_{-1})] |$$

$$= | P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) - P(A_1)P(B_1) + P(A_1)P(B_{-1}) + P(A_{-1})P(B_1) - P(A_{-1})P(B_{-1}) |$$

$$\leq 4 \sup_{A \in \mathcal{B}^{t-h}_{-\infty}, B \in \mathcal{B}^{\infty}_{t}} |P(A \cap B) - P(A)P(B)|$$

$$= 4\, \alpha(h) . \quad \blacksquare$$

<u>Proposition 3</u>.- Suppose that $E|X_{t-h}|^p < \Delta$ for some $p > 1$ and $|X_t| < C$. Then

$$|Cov(X_t, X_{t-h})| \leq 6\, C\, \Delta^{1/p} \alpha(h)^{1-1/p}$$

<u>proof</u>: Define

$$B = \left(\frac{E|X_{t-h}|^p}{\alpha(h)} \right)^{1/p}$$

$$X_{t-h}^B = X_{t-h} \; \mathbb{1}\{|X_{t-h}| < B\} \rightarrow$$ Truncation, so now this is a bounded r.v.

$$\tilde{X}_{t-h}^B = X_{t-h} - X_{t-h}^B = X_{t-h} \; \mathbb{1}\{|X_{t-h}| > B\} \rightarrow$$ Tail part

Then, write

$$|Cov(X_t, X_{t-h})| \leq |Cov(X_t, X_{t-h}^B)| + |Cov(X_t, \tilde{X}_{t-h}^B)$$

$$\underset{\text{using Proposition 2}}{\leq} 4\, C\, B\, \alpha(h) + |Cov(X_t, \tilde{X}_{t-h}^B)$$

$$= 4C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p} + \underbrace{|Cov(X_t, \tilde{X}_{t-h}^B)|}_{\text{we need to bound this new term here.}}$$

$$
\begin{aligned}
|Cov(X_t, \tilde{X}_{t-h}^B)| &= |E\; X_t\,(\tilde{X}_{t-h}^B - E\tilde{X}_{t-h}^B)| \\
&\leq E\{|X_t| \cdot |\tilde{X}_{t-h}^B - E\tilde{X}_{t-h}^B|\} \\
&\leq C\; E|\tilde{X}_{t-h}^B - E\,\tilde{X}_{t-h}^B| \\
&\leq 2C\; E|\tilde{X}_{t-h}^B| \\
&= 2C\; E\left[\,|X_{t-h}|\;\mathbb{1}\{|X_{t-h}|>B\}\right] \quad \text{exponent doesn't affect indicator} \\
&\underset{\text{Hölder's Inequality}}{\leq} 2C\,(E|X_{t-h}|^p)^{1/p}\;\left(E\;\mathbb{1}\{|X_{t-h}|>B\}^{p/(p-1)}\right)^{1-1/p} \\
&= 2C\,(E|X_{t-h}|^p)^{1/p}\;P(|X_{t-h}|>B)^{1-1/p} \\
&\underset{\text{Markov Inequality}}{\leq} 2C\,(E|X_{t-h}|^p)^{1/p}\left(\frac{E|X_{t-h}|^p}{B^p}\right)^{1-1/p} \\
&\underset{\text{replace } B}{=} 2C\,(E|X_{t-h}|^p)^{1/p}\;\alpha(h)^{1-1/p}
\end{aligned}
$$

Combining both results yield

$$|Cov(X_t, X_{t-h})| \leq 6C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p}. \blacksquare$$

<u>Proposition 4</u>.- Suppose $E|X_{t-h}|^p < \Delta$ and $E|X_t|^p < \Delta$ uniformly over t and also $p>2$ for some $\Delta>0$. Then,

$$|Cov(X_t, X_{t-h})| \leq 8\,\Delta^{2/p}\,\alpha(h)^{1-2/p}.$$

<u>proof</u>: Again define $B = \left(\dfrac{E|X_{t-h}|^p}{\alpha(h)}\right)^{1/p}$.

$$
\begin{aligned}
|Cov(X_t, X_{t-h})| &\underset{\text{truncation trick}}{=} |Cov(X_t,\; X_{t-h}^B + \tilde{X}_{t-h}^B)| \\
&= |Cov(X_t, X_{t-h}^B) + Cov(X_t, \tilde{X}_{t-h}^B)| \\
&\leq |Cov(X_t, X_{t-h}^B)| + |Cov(X_t, \tilde{X}_{t-h}^B)| \\
&\underset{\text{Proposition 3}}{\leq} 6\,B\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p} + |Cov(X_t, \tilde{X}_{t-h}^B)|
\end{aligned}
$$

$$\leq 6\,\Delta^{2/p}\,\alpha(h)^{1-2/p} + \underbrace{|\mathrm{Cov}(X_t, \tilde{X}_{t-h}^{B})|}$$

replace B and use tail condition on $|X_{t-h}|$

We need to bound this term here. Notice that $|X_t|$ is not bounded like in Proposition 3.

$$|\mathrm{Cov}(X_t, \tilde{X}_{t-h}^{B})| = |E(\tilde{X}_{t-h}^{B}(X_t - EX_t))|$$

$$\leq (E|X_t - EX_t|^p)^{1/p}\,(E|\tilde{X}_{t-h}^{B}|^{\frac{p}{p-1}})^{1-1/p}$$

Hölder's inequality

$$\leq \left[(E|X_t|^p)^{1/p} + (|EX_t|^p)^{1/p}\right](E|\tilde{X}_{t-h}^{B}|^{\frac{p}{p-1}})^{1-1/p}$$

Minkowski's Inequality

$$\leq 2\,(E|X_t|^p)^{1/p}\,(E|\tilde{X}_{t-h}^{B}|^{\frac{p}{p-1}})^{1-1/p}$$

$$\leq 2\,\Delta^{1/p}\,\left(E(|X_{t-h}|^{\frac{p}{p-1}}\,\mathbb{1}\{|X_{t-h}|>B\})\right)^{1-1/p}$$

$$\leq 2\,\Delta^{1/p}\left\{(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}})^{1/\bar{p}}\,(P(|X_{t-h}|>B))^{1-1/\bar{p}}\right\}^{1-1/p}$$

Hölder's inequality

$$\leq 2\,\Delta^{1/p}\left\{(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}})^{\frac{p-1}{p}\cdot\frac{1}{\bar{p}}}\right\}\left\{\frac{E|X_{t-h}|^p}{B^p}\right\}^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}$$

Markov's Inequality

$$= 2\,\Delta^{1/p}\left\{(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}})^{\frac{p-1}{p}\cdot\frac{1}{\bar{p}}}\right.\;\alpha(h)^{\underbrace{\frac{\bar{p}-1}{\bar{p}}}\cdot\frac{p-1}{p}}\left\{\frac{E|X_{t-h}|^p}{(E|X_{t-h}|^p)}\right\}^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}$$

we want this to be $1-\frac{2}{p} = \frac{p-2}{p}$

then $\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p} = \frac{p-2}{p} \Rightarrow \boxed{\bar{p} = p-1}$

To use Hölder we need $\bar{p} > 1 \Leftrightarrow p > 2$.

$$= 2\,\Delta^{1/p}\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-\frac{2}{p}}$$

Setting $\bar{p} = p-1$

$$\leq 2\,\Delta^{1/p}\,\Delta^{1/p}\,\alpha(h)^{1-2/p}$$

$$\leq 2\,\Delta^{2/p}\,\alpha(h)^{1-2/p}.$$

Combining the two results leads us to

$$|\mathrm{Cov}(X_t, X_{t-h})| \leq 8\,\Delta^{2/p}\,\alpha(h)^{1-\frac{2}{p}}. \quad \blacksquare$$

Corollary 1.- Suppose $\{X_t\}$ is α-mixing and for some $p > 2$:

(i) $\alpha(h)$ is of size $-\frac{p}{p-2}$

(ii) $E|X_t|^p < \Delta$ for all t

Then
$$\Omega_n := Var\left(\frac{1}{\sqrt{n}} \sum X_t\right) = O(1).$$

proof:

$$\Omega_n = \frac{1}{n}\sum_{t=1}^{n} Var(X_t) + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} Cov(X_t, X_{t-h})$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} Var(X_t) + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

$$= \frac{1}{n}\sum_{t=1}^{n} \left[EX_t^2 - (EX_t)^2\right] + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} EX_t^2 + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

norm inequality
$$\leq \frac{1}{n}\sum_{t=1}^{n} (E|X_t|^p)^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

$$\leq \Delta^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

Proposition 4
$$\leq \Delta^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} 8\,\Delta^{2/p}\,\alpha(h)^{1-\frac{2}{p}}$$

$d = c + \varepsilon$ where c is the size of mixing coefficients

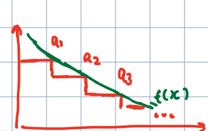
property of α coefficient
$$= \Delta^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} 8\,\Delta^{2/p}\, K\, h^{-(1-2/p)d} \quad \text{for some } K > 0.$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1}\frac{1}{n}\left(n-(h+\cancel{1})+\cancel{1}\right) h^{-d(1-2/p)}$$

$$\leq \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1}\cancel{\frac{1}{n}}\,\cancel{n}\; h^{-d(1-2/p)}$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1} h^{-\gamma} \qquad \text{where } \gamma = d(1-2/p).$$

(*) For a decreasing sequence $a_n \geq a_{n+1} \geq \dots \geq 0$ we have $\sum_{n=2}^{n} a_n \leq \int_1^n f(x)\,dx$



s.t. $f(\cdot)$ is non increasing, continous and $f(h) = a_h$.

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \left[1 + \sum_{h=2}^{n-1} h^{-\gamma} \right]$$

$$\underset{\text{using integral approximation}}{\leq} \Delta^{2/p} + 16\,\Delta^{2/p} K \left[1 + \int_1^n x^{-\gamma}\,dx \right]$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \left[1 + \frac{x^{-\gamma+1}}{1-\gamma} \Big|_1^n \right]$$

$$\underset{\substack{\text{provided} \\ \gamma \neq 1}}{=} \Delta^{2/p} + \underbrace{16\,\Delta^{2/p} K}_{\text{some constant}} \frac{n^{-\gamma+1}}{1-\gamma}$$

$$= O(n^{1-\gamma})$$

Since $\gamma \neq 1$ in order to not diverge we must have $\gamma > 1$, which requires

$$\underbrace{\delta}_{(c+\varepsilon)}(1 - 2/p) > 1$$

$(c+\varepsilon)$ where c is the size and $\varepsilon > 0$.

$$\underbrace{c\left(\frac{p-2}{p}\right)}_{\substack{\text{the smallest number} \\ \text{possible for this is } 1}} + \underbrace{\varepsilon\left(\frac{p-2}{p}\right)}_{\text{some positive number}} > 1$$

$$\Rightarrow c = \frac{p}{p-2}$$

And since the size of the α-mixing coefficients is $-c$ the size that we need is $\frac{-p}{p-2}$. ∎

⊛ What happens when $\gamma = 1$? Then $\frac{n^{1-\gamma}}{1-\gamma} \to \ln(n)$ which is divergent.

Proposition 5.- Suppose that $\{X_t : t=1,\dots,n\}$ is an α-mixing sequence of random variables such that for some $\Delta>0$, $\delta>0$ and $\eta>0$,

$$\alpha(h) \le \Delta h^{-\delta}$$

$$E|X_t|^{1+\eta} \le \Delta \quad \text{for all } t \iff \sup_t E|X_t|^{1+\eta} \le \Delta.$$

Then,

$$\frac{1}{n}\sum_{t=1}^n X_t - \frac{1}{n}\sum_{t=1}^n EX_t \xrightarrow{p} 0$$

proof:

$$P\left(\left| \frac{1}{n}\sum_{t=1}^n X_t - \frac{1}{n}\sum_{t=1}^n EX_t \right| > \varepsilon \right) \underset{\text{Markov's Inequality}}{\le} \frac{1}{\varepsilon} \underbrace{E\left| \frac{1}{n}\sum_{t=1}^n X_t - \frac{1}{n}\sum_{t=1}^n EX_t \right|}_{\text{We need this to go to zero as } n\to\infty.}$$

Since we are not assuming second or higher moments we need to use the truncation trick.
Define

$$X_t^B = X_t \, \mathbb{1}\{|X_t| \le B\}$$

$$\tilde{X}_t^B = X_t \, \mathbb{1}\{|X_t| > B\}$$

Then we write

We want the sum to be $<\varepsilon^2$

$$E\left| \frac{1}{n}\sum_{t=1}^n X_t - \frac{1}{n}\sum_{t=1}^n EX_t \right| \le \underbrace{E\left| \frac{1}{n}\sum_{t=1}^n X_t^B - \frac{1}{n}\sum_{t=1}^n EX_t^B \right|}_{A} + \underbrace{E\left| \frac{1}{n}\sum_{t=1}^n \tilde{X}_t^B - \frac{1}{n}\sum_{t=1}^n E\tilde{X}_t^B \right|}_{B}$$

We will work with B first, to know where we must truncate

$$\begin{aligned}
E\left| \frac{1}{n}\sum_{t=1}^n \tilde{X}_t^B - \frac{1}{n}\sum_{t=1}^n E\tilde{X}_t^B \right| &\le 2\,\frac{1}{n}\sum_{t=1}^n E|\tilde{X}_t^B| \\
&\le 2 \sup_t E|\tilde{X}_t^B| \\
&= 2 \sup_t E\left[|X_t| \, \mathbb{1}\{|X_t|>B\} \right] \\
&\underset{\text{Hölder's Inequality}}{\le} 2 \sup_t \left(E|X_t|^{1+\eta} \right)^{\frac{1}{1+\eta}} \left(P(|X_t|>B) \right)^{\frac{\eta}{1+\eta}} \\
&\underset{\text{Markov's Inequality}}{\le} 2 \sup_t \left(E|X_t|^{1+\eta} \right)^{\frac{1}{1+\eta}} \left(\frac{E|X_t|^{1+\eta}}{B^\eta} \right)^{\frac{\eta}{1+\eta}} \\
&= 2 \sup_t \left(E|X_t|^{1+\eta} \right) \cdot \frac{1}{B^\eta} \\
&\le \frac{2\Delta}{B^\eta}
\end{aligned}$$

We need it to be less than $\varepsilon^2/2$ which requires $\frac{2\Delta}{B^\eta} < \varepsilon^2/2 \Rightarrow B > \left(\frac{4\Delta}{\varepsilon^2}\right)^{1/\eta}$

Now we will work with A that involve random variables bounded by the B we just defined. This means that their moments are finite. We could make use of that.

$$E\left|\frac{1}{n}\sum_{t=1}^{n}(X_t^B - EX_t^B)\right| \underset{\text{Norm Inequality}}{\leq} \left\{E\left|\frac{1}{n}\sum_{t=1}^{n}(X_t^B - EX_t^B)\right|^2\right\}^{1/2}$$

$$= \left\{\frac{1}{n}\underbrace{Var\left(\frac{1}{\sqrt{n}}\sum X_t^B\right)}_{C}\right\}^{1/2}$$

Remember that

$$Var\left(\frac{1}{\sqrt{n}}\sum X_t^B\right) = \frac{1}{n}\sum_{t=1}^{n} Var(X_t^B) + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} Cov(X_t^B, X_{t-h}^B)$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} E|X_t^B|^2 + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} Cov(X_t^B, X_{t-h}^B)$$

$$\underset{\text{Proposition 2}}{\leq} B^2 + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} 4B^2\alpha(h)$$

$$= B^2 + 8B^2\sum_{h=1}^{n-1}\frac{1}{n}(n-(h+1)+1)\,\alpha(h)$$

$$\leq B^2 + 8B^2\sum_{h=1}^{n-1}\alpha(h)$$

$$\leq B^2 + 8B^2\Delta\sum_{h=1}^{n-1} h^{-\delta}$$

$$= B^2 + 8B^2\Delta\left[1 + \sum_{h=2}^{n-1} h^{-\delta}\right]$$

$$\underset{\text{integral approximation}}{\leq} B^2 + 8B^2\Delta\left[1 + \int_1^n x^{-\delta}dx\right]$$

$$\underset{\text{Provided } \delta \neq 1}{=} B^2 + 8B^2\Delta\left[1 + \frac{x^{-\delta+1}}{1-\delta}\Big|_1^n\right]$$

$$= O(n^{1-\delta})$$

Using this result yields

$$E\left|\frac{1}{n}\sum_{t=1}^{n}(X_t^B - EX_t^B)\right| \leq \left\{\frac{1}{n}O(n^{1-\delta})\right\}^{1/2}$$

$$= O(n^{-\delta/2})$$

For this to go to zero we require $\delta = c + \varepsilon > 0$, $\varepsilon > 0$ so for any size c of the mixing coefficients this condition will hold!

Putting it all together gives

$$P\left(\left| \frac{1}{n}\sum_{t=1}^{n} x_t - \frac{1}{n}\sum_{t=1}^{n} Ex_t \right| > \varepsilon \right) \leq \frac{1}{\varepsilon} E \left| \frac{1}{n}\sum_{t=1}^{n} x_t - \frac{1}{n}\sum_{t=1}^{n} Ex_t \right|$$

$$\leq \frac{1}{\varepsilon} E \left| \frac{1}{n}\sum_{t=1}^{n} x_t^B - \frac{1}{n}\sum_{t=1}^{n} Ex_t^B \right| + E \left| \frac{1}{n}\sum_{t=1}^{n} \tilde{x}_t^B - \frac{1}{n}\sum_{t=1}^{n} E\tilde{x}_t^B \right| \cdot \frac{1}{\varepsilon}$$

$$\leq \frac{1}{\varepsilon} o(1) + \varepsilon/2$$

Then by taking limits we get the desired result

$$\lim_{n\to\infty} P\left(\left| \frac{1}{n}\sum_{t=1}^{n} x_t - \frac{1}{n}\sum_{t=1}^{n} Ex_t \right| > \varepsilon \right) \leq \varepsilon/2 . \quad \blacksquare$$

<u>Lemma 1</u>: (CLT) Let $\{X_{nt}\}$ be a sequence such that $EX_{nt} = 0$ for all n, t and

(i) α coefficients are of size $\frac{-p}{p-2}$, $p > 2$

(ii) $\sup_t E|X_{nt}|^p < \Delta$ for all n

(iii) $w_n := \text{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \right) > \delta > 0$ for all n sufficiently large

Then

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{X_{nt}}{w_n^{1/2}} \xrightarrow{d} N(0,1) .$$

<u>Definition</u>: Let $\{M_n\}$ be a sequence of $k\times k$ matrices. Let e_n be the smallest eigenvalue of M_n. Then M_n is said to be uniformly positive definite if for all n sufficiently large $e_n > \delta > 0$ uniformly in n.

<u>Proposition 6</u>: Let $\{X_{nt}\}$ be an α-mixing sequence of random vectors such that $EX_{nt} = 0$ for all n, t and for some $p > 2$ and $\Delta > 0$,

(*) Recall $X_{nt} = \begin{pmatrix} X_{nt1} \\ \vdots \\ X_{ntk} \end{pmatrix}_{k\times 1}$

(i) α is of size $\frac{-p}{p-2}$

(ii) $E|X_{nt}|^p \leq \Delta$ for all t, n

(iii) $\Omega_n = \text{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \right)$ is uniformly positive definite.

Then

$$\Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \xrightarrow{d} N(0, I_k)$$

Proof: Let $\lambda \in \mathbb{R}^k$ such that $\|\lambda\| = 1$. Then by the Cramér-Wold device we want to show that $\lambda' \Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \xrightarrow{d} N(0,1)$ using Lemma 1.

We need to check for the conditions in Lemma 1

- $E(\lambda' \Omega_n^{-1/2} X_{nt}) = \lambda' \Omega_n^{-1/2} E(X_{nt}) \Omega_n^{-1/2} \lambda = 0$ for all n,t.

- Define the process $\{Y_{nt}\} := \{\lambda' \Omega_n^{-1/2} X_{nt}\} = g(X_{nt1}, \cdots, X_{ntk})$ where g is measurable. Then by Proposition 1 $\{Y_{nt}\}$ is α-mixing of the same size $\frac{-p}{p-2}$.

- $$\sup_t E|\lambda' \Omega_n^{-1/2} X_{nt}|^p = \sup_t E\left|\sum_{j=1}^{k} c_{jn} X_{ntj}\right|^p$$
$$= \sup_t \left(E\left|\sum_{j=1}^{k} c_{jn} X_{ntj}\right|^p\right)^{1/p \cdot p}$$
$$\underset{\text{Minkowski's Inequality}}{\leq} \sup_t \left\{\sum_{j=1}^{k} |c_{jn}| (E|X_{ntj}|^p)^{1/p}\right\}^p$$
$$\leq \Delta \left(\sum_{j=1}^{k} |c_{jn}|\right)^p$$
$$\underset{\text{Norm inequality}}{\leq} \Delta \left(\sum_{j=1}^{k} |c_{jn}|^2\right)^{p/2}$$
$$= \Delta (\lambda' \Omega_n^{-1} \lambda)^{p/2}$$
$$\underset{\text{spectral decomposition}}{=} \Delta (\underbrace{\lambda' C_n}_{d_n'} \Lambda_n^{-1} \underbrace{C_n' \lambda}_{d_n})^{p/2} \quad \text{where } d_n' d_n = 1 \text{ by construction}$$
$$= \Delta (d_n' \Lambda_n^{-1} d_n)^{p/2}$$
$$\underset{e_n^{-1}\text{ is the largest eigenvalue of } \Omega_n^{-1}}{\leq} \Delta \left(e_n^{-1} \sum_{i=1}^{n} d_{ni}^2\right)^{p/2}$$
$$< \Delta \delta^{-p/2}$$
$$< \infty.$$

- $$\mathrm{Var}\left(\lambda' \Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt}\right) = \lambda' \Omega_n^{-1/2} \mathrm{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt}\right) \Omega_n^{-1/2} \lambda$$
$$= \lambda' \Omega_n^{-1/2} \Omega_n \Omega_n^{-1/2} \lambda = 1$$

Therefore, by Lemma 1 the desired result holds and the proof is complete. ∎