Covariance Inequalities for Mixing Processes

Proposition 2.- Suppose that $|X_t| \leq c_1$ and $|X_{t-h}| \leq c_2$ for some $c_1, c_2 > 0$. Then

$$|\mathrm{Cov}(X_t, X_{t-h})| \leq 4\, c_1 c_2\, \alpha(h).$$

proof:

Define the following random variable

$$\eta = \mathrm{sign}\{E(X_t \mid B_{-\infty}^{t-h}) - EX_t\}, \quad \eta \in \{-1, 0, 1\} \text{ and it's } B_{-\infty}^{t-h} \text{ measurable.}$$

⊛ Notice that we can write covariances by only demeaning one of the two random variables

$$\begin{aligned}
E(X_t - EX_t)(X_{t-h} - EX_{t-h}) &= E\{X_t \cdot X_{t-h} - X_t \cdot EX_{t-h} - EX_t \cdot X_{t-h} + EX_t \cdot EX_{t-h}\} \\
&= E\, X_t \cdot X_{t-h} - EX_t \cdot EX_{t-h} \\
&= E\{X_t (X_{t-h} - EX_{t-h})\} = E\{X_{t-h}(X_t - EX_t)\}.
\end{aligned}$$

Now, write

$$\begin{aligned}
|\mathrm{Cov}(X_t, X_{t-h})| = |E\{X_{t-h}(X_t - EX_t)\}| &\underset{\text{LIE}}{=} |E\{X_{t-h}(E(X_t \mid B_{-\infty}^{t-h}) - EX_t)\}| \\
&\leq E\{|X_{t-h}| \cdot |E(X_t \mid B_{-\infty}^{t-h}) - EX_t|\} \\
&\leq c_2\, E\{\eta\, (E(X_t \mid B_{-\infty}^{t-h}) - EX_t)\} \\
&\underset{\text{LIE}}{=} c_2\, (E\eta X_t - E\eta\, EX_t) \\
&= c_2\, \mathrm{Cov}(\eta, X_t) \\
&= c_2\, |\mathrm{Cov}(\eta, X_t)|
\end{aligned}$$

it's non negative by construction!

Next, define the following random variable

$$\xi = \mathrm{sign}\{E(\eta \mid B_t^{\infty}) - E\eta\}, \quad \xi \in \{-1, 0, 1\} \text{ and it's } B_t^{\infty} \text{ measurable.}$$

Now, repeating the same argument

$$\begin{aligned}
|\mathrm{Cov}(\eta, X_t)| = |E\, X_t(\eta - E\eta)| &\underset{\text{LIE}}{=} |E\{X_t\, (E(\eta \mid B_t^{\infty}) - E\eta)\}| \\
&\leq E\{|X_t|\, |E(\eta \mid B_t^{\infty}) - E\eta|\} \\
&\leq c_1\, E\{\xi\, (E(\eta \mid B_t^{\infty}) - E\eta)\} \\
&= c_1\, (E\xi\eta - E\xi\, E\eta) \\
&= c_1\, |E\eta\xi - E\xi\, E\eta|.
\end{aligned}$$

Combining both results yield

$$|Cov(X_t, X_{t-h})| \leq c_1 . c_2 \; |E \xi \eta - E\xi E\eta| .$$

Define the following events $A_1 = \{\eta = 1\}$ $A_{-1} = \{\eta = -1\}$ $A_0 = \{\eta = 0\}$
$B_1 = \{\xi = 1\}$ $B_{-1} = \{\xi = -1\}$ $B_0 = \{\xi = 0\}$

- $$\begin{aligned} E\,\eta\xi = \; & P(A_1 \cap B_1) . 1\times 1 + P(A_1 \cap B_{-1}) \; 1\times(-1) + \cancel{P(A_1 \cap B_0)} \; 1\times 0 + \\ & P(A_{-1} \cap B_1) . 1\times(-1) + P(A_{-1} \cap B_{-1}) \; (-1)\times(-1) + \cancel{P(A_{-1} \cap B_0)} \; (-1)\times 0 + \\ & \cancel{P(A_0 \cap B_1)} . 0\times 1 + \cancel{P(A_0 \cap B_{-1})} \; 0\times(-1) + \cancel{P(A_0 \cap B_{-1})} \; 0\times(-1) \\ = \; & P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) \end{aligned}$$

- $E\,\eta = P(A_1) - P(A_{-1})$

- $E\,\xi = P(B_1) - P(B_{-1})$

Hence,
$$\begin{aligned} |E\eta\xi - E\eta E\xi| = \; & | \, P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) \\ & - [P(A_1) - P(A_{-1})] \times [P(B_1) - P(B_{-1})] \, | \\ = \; & | \, P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) \\ & - P(A_1)P(B_1) + P(A_1)P(B_{-1}) + P(A_{-1})P(B_1) - P(A_{-1})P(B_{-1}) \, | \\ \leq \; & 4 \sup_{A \in \mathcal{B}^{t-h}_{-\infty}, \, B \in \mathcal{B}^{\infty}_{t}} |P(A \cap B) - P(A)P(B)| \\ = \; & 4 \, \alpha(h) . \end{aligned}$$

<u>Proposition 3</u>.- Suppose that $E|X_{t-h}|^p < \Delta$ for some $p > 1$ and $|X_t| < C$. Then

$$|Cov(X_t, X_{t-h})| \leq 6\, C\, \Delta^{1/p} \alpha(h)^{1-1/p}$$

<u>proof</u>: Define
$$B = \left(\frac{E|X_{t-h}|^p}{\alpha(h)} \right)^{1/p}$$

$X_{t-h}^B = X_{t-h} \, \mathbb{1}\{|X_{t-h}| \leq B\}$ → Truncation, so now this is a bounded r.v.

$\tilde{X}_{t-h}^B = X_{t-h} - X_{t-h}^B = X_{t-h} \, \mathbb{1}\{|X_{t-h}| > B\}$ → Tail part

Then, write

$$\begin{aligned} |Cov(X_t, X_{t-h})| & \leq |Cov(X_t, X_{t-h}^B)| + |Cov(X_t, \tilde{X}_{t-h}^B)| \\ & \underset{\text{using Proposition 2}}{\leq} 4\, C\, B\, \alpha(h) + |Cov(X_t, \tilde{X}_{t-h}^B)| \end{aligned}$$

$$= 4C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p} + \underbrace{|\mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)|}_{\text{we need to bound this new term here.}}$$

$$\begin{aligned}
|\mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)| &= |E\, X_t(\tilde{X}_{t-h}^B - E\tilde{X}_{t-h}^B)| \\
&\le E\{|X_t| \cdot |\tilde{X}_{t-h}^B - E\tilde{X}_{t-h}^B|\} \\
&\le C\; E|\tilde{X}_{t-h}^B - E\,\tilde{X}_{t-h}^B| \\
&\le 2C\, E|\tilde{X}_{t-h}^B| \\
&= 2C\, E\left[\, |X_{t-h}|\, \mathbb{1}\{|X_{t-h}| > B\}\right] \\
&\underset{\text{Hölder's Inequality}}{\le} 2C\,(E|X_{t-h}|^p)^{1/p}\,\{E\,\mathbb{1}\{|X_{t-h}|>B\}^{p/1-p}\}^{1-1/p} \quad \text{exponent doesn't affect indicator} \\
&= 2C\,(E|X_{t-h}|^p)^{1/p}\; P(|X_{t-h}|>B)^{1-1/p} \\
&\underset{\text{Markov Inequality}}{\le} 2C\,(E|X_{t-h}|^p)^{1/p}\left(\frac{E|X_{t-h}|^p}{B^p}\right)^{1-1/p} \\
&\underset{\text{replace } B}{=} 2C\,(E|X_{t-h}|^p)^{1/p}\;\alpha(h)^{1-1/p}
\end{aligned}$$

Combining both results yield

$$|\mathrm{Cov}(X_t, X_{t-h})| \le 6C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p}. \quad \blacksquare$$

<u>Proposition 4</u>.- Suppose $E|X_{t-h}|^p < \Delta$ and $E|X_t|^p < \Delta$ uniformly over t and also $p>2$ for some $\Delta > 0$. Then,

$$|\mathrm{Cov}(X_t, X_{t-h})| \le 8\,\Delta^{2/p}\,\alpha(h)^{1-2/p}.$$

<u>proof:</u> Again define $B = \left(\dfrac{E|X_{t-h}|^p}{\alpha(h)}\right)^{1/p}$.

$$\begin{aligned}
|\mathrm{Cov}(X_t, X_{t-h})| &\underset{\text{truncation trick}}{=} |\mathrm{Cov}(X_t, X_{t-h}^B + \tilde{X}_{t-h}^B)| \\
&= |\mathrm{Cov}(X_t, X_{t-h}^B) + \mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)| \\
&\le |\mathrm{Cov}(X_t, X_{t-h}^B)| + |\mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)| \\
&\underset{\text{Proposition 3}}{\le} 6\,B\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p} + |\mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)|
\end{aligned}$$

$$\le \; 6\,\Delta^{2/p}\,\alpha(h)^{1-2/p} + \underbrace{|\mathrm{Cov}(X_t, \tilde{X}^B_{t-h})|}_{\text{We need to bound this term here.}}$$

replace B and use tail condition on $|X_{t-h}|$

Notice that $|X_t|$ is not bounded like in Proposition 3.

$$|\mathrm{Cov}(X_t, \tilde{X}^B_{t-h})| = |E(\tilde{X}^B_{t-h}(X_t - EX_t))|$$

$$\le \left(E|X_t - EX_t|^p\right)^{1/p} \left(E|\tilde{X}^B_{t-h}|^{\frac{p}{p-1}}\right)^{1-1/p}$$

Hölder's inequality

$$\le \left[(E|X_t|^p)^{1/p} + (|EX_t|^p)^{1/p}\right] \left(E|\tilde{X}^B_{t-h}|^{\frac{p}{p-1}}\right)^{1-1/p}$$

Minkowski's Inequality

$$\le 2\,(E|X_t|^p)^{1/p} \left(E|\tilde{X}^B_{t-h}|^{\frac{p}{p-1}}\right)^{1-1/p}$$

$$\le 2\,\Delta^{1/p} \left(E(|X_{t-h}|^{\frac{p}{p-1}}\,\mathbb{1}\{|X_{t-h}|>B\})\right)^{1-1/p}$$

$$\le 2\,\Delta^{1/p} \left\{ \left(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}}\right)^{1/\bar{p}} \left(P(|X_{t-h}|>B)\right)^{1-1/\bar{p}} \right\}^{1-1/p}$$

Hölder's inequality

$$\le 2\,\Delta^{1/p} \left\{ \left(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}}\right)^{\frac{p-1}{p}\cdot\frac{1}{\bar{p}}} \right\} \left\{ \frac{E|X_{t-h}|^p}{B^p} \right\}^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}$$

Markov's Inequality

$$= 2\,\Delta^{1/p} \left\{ \left(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}}\right)^{\frac{p-1}{p}\cdot\frac{1}{\bar{p}}} \right\} \alpha(h)^{\underbrace{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}} \left\{ \frac{E|X_{t-h}|^p}{(E|X_{t-h}|^p)} \right\}^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}$$

we want this to be $1 - \frac{2}{p} = \frac{p-2}{p}$

then $\frac{\bar{p}-1}{\bar{p}} \cdot \frac{p-1}{p} = \frac{p-2}{p} \Rightarrow \bar{p} = p-1$

To use Hölder we need $\bar{p} > 1 \Leftrightarrow p > 2$.

$$= 2\,\Delta^{1/p} \left(E|X_{t-h}|^p\right)^{1/p} \alpha(h)^{1-\frac{2}{p}}$$

Setting $\bar{p} = p-1$

$$\le 2\,\Delta^{1/p}\,\Delta^{1/p}\,\alpha(h)^{1-2/p}$$

$$\le 2\,\Delta^{2/p}\,\alpha(h)^{1-2/p}.$$

Combining the two results leads us to

$$|\mathrm{Cov}(X_t, X_{t-h})| \le 8\,\Delta^{2/p}\,\alpha(h)^{1-\frac{2}{p}}. \qquad \blacksquare$$

Corollary 1.- Suppose $\{X_t\}$ is α-mixing and for some $p > 2$:

(i) $\alpha(h)$ is of size $-\frac{p}{p-2}$

(ii) $E|X_t|^p < \Delta$ for all t

Then
$$\Omega_n := Var\left(\frac{1}{\sqrt{n}} \sum X_t\right) = O(1).$$

proof:

$$\Omega_n = \frac{1}{n}\sum_{t=1}^{n} Var(X_t) + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} Cov(X_t, X_{t-h})$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} Var(X_t) + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

$$= \frac{1}{n}\sum_{t=1}^{n}\left[EX_t^2 - (EX_t)^2\right] + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} EX_t^2 + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

norm inequality
$$\leq \frac{1}{n}\sum_{t=1}^{n} (E|X_t|^p)^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

$$\leq \Delta^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} |Cov(X_t, X_{t-h})|$$

Proposition 4
$$\leq \Delta^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} 8\,\Delta^{2/p}\,\alpha(h)^{1-\frac{2}{p}}$$

$\delta = c + \varepsilon$ where $-c$ is the size of mixing coefficients

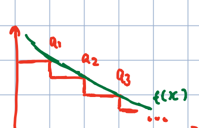
property of α coefficient
$$= \Delta^{2/p} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} 8\,\Delta^{2/p}\,K\,h^{-(1-2/p)\delta} \quad \text{for some } K > 0.$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1}\frac{1}{n}\left(n-(h+\cancel{1)+1}\right) h^{-\delta(1-2/p)}$$

$$\leq \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1}\cancel{\frac{1}{n}}\cdot\cancel{n}\; h^{-\delta(1-2/p)}$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1} h^{-\gamma} \quad \text{where } \gamma = \delta(1-2/p).$$

(*) For a decreasing sequence $a_n \geq a_{n+1} \geq \dots \geq 0$ we have $\sum_{n=2}^{n} a_n \leq \int_1^n f(x)\,dx$



s.t. $f(\cdot)$ is non increasing, continous and $f(h) = a_h$.

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \left[1 + \sum_{h=2}^{n-1} h^{-\gamma} \right]$$

using integral approximation

$$\leq \Delta^{2/p} + 16\,\Delta^{2/p} K \left[1 + \int_1^n x^{-\gamma}\, dx \right]$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \left[1 + \frac{x^{-\gamma+1}}{1-\gamma} \Big|_1^n \right]$$

provided $\gamma \neq 1$

$$= \Delta^{2/p} + \underbrace{16\,\Delta^{2/p} K}_{\text{some constant}} \frac{n^{-\gamma+1}}{1-\gamma}$$

$$= O(n^{1-\gamma})$$

Since $\gamma \neq 1$ in order to not diverge we must have $\gamma > 1$, which requires

$$\underbrace{\delta}_{(c+\varepsilon)} (1 - 2/p) > 1$$

$(c+\varepsilon)$ where c is the size and $\varepsilon > 0$.

$$\underbrace{c\left(\frac{p-2}{p}\right)}_{\text{the smallest number possible for this is 1}} + \underbrace{\varepsilon\left(\frac{p-2}{p}\right)}_{\text{some positive number}} > 1$$

$$\Rightarrow \boxed{c = \frac{p}{p-2}}$$

And since the size of the α-mixing coefficients is $-c$ the size that we need is $\frac{-p}{p-2}$. ∎

⊛ What happens when $\gamma = 1$? Then $\frac{n^{1-\gamma}}{1-\gamma} \to \ln(n)$ which is divergent.