Multiple Equation GMM

Suppose M equations are identified:

$$\underset{1\times1}{y_{1i}} = \underset{1\times k_1}{X_{1i}'}\underset{k_1\times1}{\delta_1} + \underset{1\times1}{u_{1i}}$$

$$\vdots$$

$$\underset{1\times1}{y_{Mi}} = \underset{1\times k_M}{X_{Mi}'}\underset{k_M\times1}{\delta_M} + \underset{1\times1}{u_{Mi}} \qquad \text{where} \qquad E\,\underset{\ell\times1}{Z_i}\,\underset{1\times1}{u_{ji}} = 0$$

$$\operatorname{rank}(E\,\underset{\ell\times1}{Z_i}\,\underset{1\times k_j}{X_{ji}'}) = k_j$$

In matrix form can be written as

$$\underset{n\times1}{y_1} = \underset{n\times k_1}{X_1}\,\underset{k_1\times1}{\delta_1} + \underset{n\times1}{u_1}$$

$$\vdots$$

$$\underset{n\times1}{y_M} = \underset{n\times k_M}{X_M}\,\underset{k_M\times1}{\delta_M} + \underset{n\times1}{u_M} \qquad \text{where} \qquad X_j = \begin{pmatrix} X_{j1}' \\ \vdots \\ X_{jn}' \end{pmatrix}.$$

$$\Rightarrow \quad y = \underset{nM\times K}{\begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_M \end{pmatrix}} \underset{K\times1}{\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_M \end{pmatrix}} + u$$

where $K = k_1 + \dots + k_M$

$$\underset{mn\times1}{Y} = \underset{mn\times K}{X}\,\underset{K\times1}{\delta} + \underset{mn\times1}{u}$$

The population moment conditions are

$$E\begin{pmatrix} \underset{\ell\times1}{Z_i}\,\underset{1\times1}{u_{1i}} \\ \vdots \\ \underset{\ell\times1}{Z_i}\,\underset{1\times1}{u_{Mi}} \end{pmatrix} = E(\underset{m\times1}{u_i} \otimes \underset{\ell\times1}{Z_i}) = \underset{m\ell\times1}{0}$$

The sample moment conditions are

$$\begin{pmatrix} \underset{\ell\times n}{Z'}\,\underset{n\times1}{u_1} \\ \vdots \\ \underset{\ell\times n}{Z'}\,\underset{n\times1}{u_M} \end{pmatrix} = \begin{pmatrix} Z' & & 0 \\ & \ddots & \\ 0 & & Z' \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_M \end{pmatrix}$$

$$= (I_m \otimes Z')\begin{pmatrix} u_1 \\ \vdots \\ u_M \end{pmatrix}$$

$$= \underbrace{(\underset{m\times m}{I_m} \otimes \underset{n\times\ell}{Z})'}_{m\ell\times mn}\,\underset{mn\times1}{u} = (I_m \otimes Z)'(Y - X\delta) = \underset{m\ell\times1}{0}$$

We can solve the system as with a regular GMM estimator

$$\hat{\delta_n} = \underset{d\in\mathbb{R}^K}{\operatorname{argmin}}\ (Y - Xd)'(I_m \otimes Z)\,W_n\,(I_m \otimes Z)'(Y - Xd)$$

$$= [X'(I_m \otimes Z)\,W_n\,(I_m \otimes Z)'X]^{-1}\,X'(I_m \otimes Z)\,W_n\,(I_m \otimes Z)'Y \ [(I_m \otimes Z')(Y - X\delta)]'$$

Recall the sample criterion function

$$\underbrace{(Y-X\delta)'(I_m\otimes Z)'}\,W_n\,(I_m\otimes Z)'(Y-X\delta)$$

$$= \begin{pmatrix} Z'(Y_1-X_1\delta_1) \\ \vdots \\ Z'(Y_m-X_M\delta_M) \end{pmatrix}' \underbrace{\begin{pmatrix} \underset{\ell\times\ell}{W_{11}} & \cdots & \underset{\ell\times\ell}{W_{1M}} \\ \vdots & \ddots & \\ \underset{\ell\times\ell}{W_{M1}} & & \underset{\ell\times\ell}{W_{MM}} \end{pmatrix}}_{m\ell\times m\ell} \begin{pmatrix} Z'(Y_1-X_1\delta_1) \\ \vdots \\ Z'(Y_m-X_M\delta_M) \end{pmatrix}$$

$$= \begin{pmatrix} \underset{1\times n}{(Y_1-X_1\delta_1)'}\,\underset{n\times\ell}{Z} \\ \vdots \\ \underset{1\times n}{(Y_m-X_m\delta_m)'}\,\underset{n\times\ell}{Z} \end{pmatrix} \begin{pmatrix} W_{11} & & W_{1m} \\ & \ddots & \\ W_{m1} & & W_{mm} \end{pmatrix} \begin{pmatrix} \underset{\ell\times n}{Z'}\,\underset{n\times1}{(Y_1-X_1\delta_1)} \\ \vdots \\ \underset{\ell\times n}{Z'}\,\underset{n\times1}{(Y_m-X_m\delta_m)} \end{pmatrix}$$

$$= (Y_1-X_1\delta_1)'ZW_{11}Z'(Y_1-X_1\delta_1) + (Y_1-X_1\delta_1)'ZW_{21}Z'(Y_1-X_1\delta_1) + \ldots + (Y_m-X_m\delta_m)'Z\,W_{mm}Z'(Y_m-X_m\delta_m)$$

⊛ If $W_{ij}=0$ for $i\neq j$ this becomes the sum of the individual criterion functions!

$$\hat{\delta}_n-\delta = \left[X'\frac{(I_m\otimes Z)}{n}W_n\frac{(I_m\otimes Z)'X}{n}\right]^{-1}\frac{X'(I_m\otimes Z)}{n}W_n\frac{(I\otimes Z)'u}{n}$$

- $$\frac{(I_m\otimes Z)'X}{n} = \frac{1}{n}\begin{pmatrix} \underset{\ell\times n}{Z'} & & 0 \\ & \ddots & \\ 0 & & \underset{\ell\times n}{Z'} \end{pmatrix}\begin{pmatrix} \underset{n\times k_1}{X_1} & & 0 \\ & \ddots & \\ 0 & & \underset{n\times k_m}{X_m} \end{pmatrix} = \begin{pmatrix} \frac{Z'X_1}{n} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{Z'X_m}{n} \end{pmatrix}$$
$$\xrightarrow{P}\begin{pmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_m \end{pmatrix} := C$$

- $$\frac{(I_m\otimes Z)'u}{n} = \frac{1}{n}\begin{pmatrix} Z'u_1 \\ \vdots \\ Z'u_m \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} z_iu_{1i} \\ \vdots \\ z_iu_{mi} \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}(u_i\otimes z_i)$$

Then $$\sqrt{n}(\hat{\delta}_n-\delta) = \left[X'\frac{(I_m\otimes Z)}{n}W_n\frac{(I_m\otimes Z)'X}{n}\right]^{-1}\frac{X'(I_m\otimes Z)}{n}W_n\frac{(I\otimes Z)'u}{\sqrt{n}}$$

- $\dfrac{(I_m \otimes Z)' u}{\sqrt{n}} = \dfrac{1}{\sqrt{n}} \sum_{i=1}^{n} \underbrace{(u_i \otimes z_i)}_{\xi_i} \xrightarrow{d} N(0, E\,\xi_i \xi_i')$

$= N(0, E[u_i u_i' \otimes z_i z_i'])$

- $E[u_i u_i' \otimes z_i z_i'] = E\left[\begin{pmatrix} u_{1i}u_{1i} & \cdots & u_{1i}u_{mi} \\ \vdots & \ddots & \\ u_{mi}u_{1i} & \cdots & u_{mi}u_{mi} \end{pmatrix} \otimes z_i z_i'\right]$

$= E\begin{bmatrix} u_{1i}^2 z_i z_i' & \cdots & u_{1i}u_{mi} z_i z_i' \\ \vdots & & \\ u_{mi}u_{1i} z_i z_i' & \cdots & u_{mi}^2 z_i z_i' \end{bmatrix}$

To have Ω finite we need that every sub-matrix is finite.
Then

- $E\|u_{ji}^2 z_i z_i'\| \le \left(E\, u_{ji}^4 \; \underbrace{E\|z_i z_i'\|^2}\right)^{1/2}$ Cauchy Schwartz

(*) Known as Frobenious Norm

$= E[\text{trace}(\underbrace{z_i z_i'}_{l\times l} \underbrace{z_i z_i'}_{l\times l})]$

$= E[\text{trace}(z_i \underbrace{z_i' z_i}_{1\times 1} z_i')]$

$= E[\underbrace{z_i' z_i}_{\text{scalar = trace}} \underbrace{\text{trace}(z_i z_i')}_{\text{trace}(A) = \text{trace}(A')}]$

$= E[\text{trace}(z_i' z_i)\ \text{trace}(z_i' z_i)]$

$= E[(\text{trace}(z_i' z_i)^{1/2})^4] = E[\|z_i\|^4]$

Then we require $E\, u_{ji}^4 < \infty$ and $E\|z_i\|^4 < \infty$.

- $E\|u_{ji} u_{si} z_i z_i'\| \le (E\, \underbrace{u_{ji}^2 u_{si}^2}\; E\|z_i z_i'\|^2)^{1/2}$

$\le (E\, u_{ji}^4 \cdot E\, u_{si}^4)^{1/2}$

so we don't require further assumptions.

When $E(u_i u_i' \otimes z_i z_i')$ is block diagonal i.e. $E u_{ji} u_{ri} z_i z_i' = 0_{\ell \times \ell}$.
then the efficient weighting matrix W_n is

$$\Omega^{-1} = \begin{bmatrix} E u_{1i}^2 z_i z_i' & & 0 \\ & \ddots & \\ 0 & & E u_{mi}^2 z_i z_i' \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (E u_{1i} z_i z_i')^{-1} & & 0 \\ & \ddots & \\ 0 & & (E u_{mi}^2 z_i z_i')^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \Omega_{11}^{-1} & & 0 \\ 0 & \ddots & \Omega_{mm}^{-1} \end{bmatrix}$$

so the criterion function will be the sum of the single equation criterion functions since W_n is block diagonal. Intuitively, there are no efficiency gains since the other equations don't provide information to the equation that we're interested.

(Testing on structural Parameters without assuming identification)

$$\underset{1\times1}{y_i} = \underset{1\times k}{X_i}{}' \underset{k\times1}{\gamma} + \underset{1\times1}{u_i}$$

$$\underset{k\times1}{X_i} = \underset{k\times \ell}{\Pi'} \underset{\ell\times1}{Z_i} + \underset{k\times1}{V_i}$$

which can be rewritten into matrix form by defining

$$\Pi = \begin{pmatrix} \underset{\ell\times1}{\pi_1} & \underset{\ell\times1}{\pi_2} & \cdots & \underset{\ell\times1}{\pi_k} \end{pmatrix}_{\ell\times k}, \quad X = \begin{pmatrix} X_1' \\ \vdots \\ X_n' \end{pmatrix}_{n\times k}$$

Structural Eq : $\underset{n\times1}{y} = \underset{n\times k}{X} \underset{k\times1}{\gamma} + \underset{n\times1}{u}$

First Stage Eq : $\underset{n\times k}{X} = \underset{n\times \ell}{Z} \underset{\ell\times k}{\Pi} + \underset{n\times k}{V}$

Reduced Form Eq : $\underset{n\times1}{y} = \underset{n\times \ell}{Z} \underbrace{\Pi\gamma}_{\substack{\pi \\ \ell\times1}} + \underbrace{V\gamma + u}_{W\ n\times1}$

Vectorize the first stage equation

$$\mathrm{vec}(X) = \mathrm{vec}(Z\Pi I_k) + \mathrm{vec}(V)$$

$$\begin{pmatrix} \underset{n\times1}{X_1} \\ \vdots \\ \underset{n\times1}{X_k} \end{pmatrix}_{nk\times1} = \underbrace{(I_k \otimes Z)}_{nk\times k\ell} \underbrace{\mathrm{vec}(\Pi)}_{\ell k\times1} + \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix}_{nk\times1}$$

$$\underset{nk\times1}{\tilde{X}} = \underset{nk\times k\ell}{\tilde{Z}} \underset{k\ell\times1}{\mathrm{vec}(\Pi)} + \underset{nk\times1}{\tilde{V}}$$

Then we get

- $\hat{\pi} = \pi + (Z'Z)^{-1} Z' W$

$$\sqrt{n}(\hat{\pi} - \pi) = \left(\frac{Z'Z}{n}\right)^{-1} \frac{Z'W}{\sqrt{n}} = \left(\frac{Z'Z}{n}\right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underset{\ell\times1}{Z_i} \underset{1\times1}{W_i}$$

- $\mathrm{vec}(\hat{\Pi}) = \mathrm{vec}(\Pi) + (\tilde{Z}'\tilde{Z})^{-1} \tilde{Z}' V$

$$\sqrt{n}\left(\mathrm{vec}(\hat{\Pi}) - \mathrm{vec}(\Pi)\right) = \left(\frac{\tilde{Z}'\tilde{Z}}{n}\right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\underset{k\times1}{\tilde{V}_i} \otimes \underset{\ell\times1}{Z_i}\right)$$

Consider them jointly

$$\sqrt{n}\begin{pmatrix} \hat{\pi}-\pi \\ \text{vec}(\hat{\Pi})-\text{vec}(\Pi) \end{pmatrix} = \begin{pmatrix} \left(\frac{Z'Z}{n}\right)^{-1} & \\ & \left(\frac{\tilde{Z}'\tilde{Z}}{n}\right)^{-1} \end{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underbrace{\begin{pmatrix} z_i w_i \\ \tilde{v}_i \otimes z_i \end{pmatrix}}_{\xi_i}$$

$$\xrightarrow[d]{} N\left(0, \begin{pmatrix} Q_Z^{-1} & \\ & Q_{\tilde{Z}}^{-1} \end{pmatrix} \underbrace{E\,\xi_i \xi_i'}_{\Omega} \begin{pmatrix} Q_Z^{-1} & \\ & Q_{\tilde{Z}}^{-1} \end{pmatrix}\right)$$

$$= N\left(0, \begin{pmatrix} \underset{\ell\times\ell}{Q_Z^{-1}} & \\ & \underset{k\ell\times k\ell}{Q_{\tilde{Z}}^{-1}} \end{pmatrix} \begin{pmatrix} \Omega_{11\,\ell\times\ell} & \Omega_{12\,\ell\times k\ell} \\ \underset{k\ell\times\ell}{\Omega_{21}} & \underset{k\ell\times k\ell}{\Omega_{22}} \end{pmatrix} \begin{pmatrix} \underset{\ell\times\ell}{Q_Z^{-1}} & \\ & \underset{k\ell\times k\ell}{Q_{\tilde{Z}}^{-1}} \end{pmatrix}\right)$$

$$= N\left(0, \begin{pmatrix} \Sigma_{11\,\ell\times\ell} & \Sigma_{12\,\ell\times k\ell} \\ \underset{k\ell\times\ell}{\Sigma_{21}} & \underset{k\ell\times k\ell}{\Sigma_{22}} \end{pmatrix}\right) = N(0, \Sigma)$$

Something really cool that we can do is to test $H_0: \gamma = \gamma_0$ using the equation $\Pi\gamma = \pi$ and our derived distribution.

Under H_0 we have $\Pi\gamma_0 = \pi$. Therefore,

$$\sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi}) = \sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi} - \Pi\gamma_0 + \pi)$$

$$= \sqrt{n}\left((\hat{\Pi}-\Pi)\gamma_0 - (\hat{\pi}-\pi)\right)$$

$$\circledast \quad \text{vec}(\Pi\gamma) = (\gamma' \otimes I_\ell)\,\text{vec}(\Pi)$$

$$= \sqrt{n}\left((\gamma_0' \otimes I_\ell)\,\text{vec}(\hat{\Pi}-\Pi) - (\hat{\pi}-\pi)\right)$$

$$= \left[-I_\ell \quad (\gamma_0' \otimes I_\ell)\right] \sqrt{n} \begin{pmatrix} \hat{\pi}-\pi \\ \text{vec}(\hat{\Pi}-\Pi) \end{pmatrix}$$

$$\xrightarrow[d]{} N\left(0, \left(-I_\ell \quad (\gamma_0' \otimes I_\ell)\right) \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} -I_\ell \\ (\gamma_0 \otimes I_\ell) \end{pmatrix}\right)$$

$$= N\Big(0, \underbrace{\Sigma_{11} - (\gamma_0' \otimes I_\ell)\overset{\Sigma_{12}'}{\Sigma_{21}} - \Sigma_{12}(\gamma_0 \otimes I_\ell) + (\gamma_0' \otimes I_\ell)\Sigma_{22}(\gamma_0 \otimes I_\ell)}_{\tilde{\Omega}}\Big)$$

We can get the following consistent estimator for $\tilde{\Omega}$

$$\hat{\Omega} = \hat{\Sigma}_{11} - (\gamma_0' \otimes I_\ell)\hat{\Sigma}_{21} - \hat{\Sigma}_{12}(\gamma_0 \otimes I_\ell) + (\gamma_0' \otimes I_\ell)\hat{\Sigma}_{22}(\gamma_0 \otimes I_\ell)$$

And we can construct the following Wald test using the fact that

$$\hat{\tilde{\Omega}}^{-1/2}\sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi}) \xrightarrow{d} N(0, I_\ell)$$

$$\begin{aligned}
\Rightarrow \quad W(\gamma_0) &= n(\hat{\Pi}\gamma_0 - \hat{\pi})'\hat{\tilde{\Omega}}^{-1}(\hat{\Pi}\gamma_0 - \hat{\pi}) \\
&= \sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi})'\hat{\tilde{\Omega}}^{-1/2}\hat{\tilde{\Omega}}^{-1/2}\sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi}) \\
&= [\hat{\tilde{\Omega}}^{-1/2}\sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi})]'[\hat{\tilde{\Omega}}^{-1/2}\sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi})] \\
&\xrightarrow{d} \chi^2_\ell \qquad \text{so we reject when } W(\gamma_0) > \chi^2_{\ell, 1-\alpha}.
\end{aligned}$$

Notice what happens to the asymptotic distribution under local alternatives $\gamma = \gamma_0 + \frac{\delta}{\sqrt{n}}$

$$\sqrt{n}(\hat{\Pi}\gamma - \hat{\pi}) = \sqrt{n}\left((\hat{\Pi} - \Pi)\gamma_0 - (\hat{\pi} - \pi)\right) - \Pi\delta$$

$$\xrightarrow{d} N(-\Pi\delta, \hat{\Omega})$$

Hence $W(\gamma_0) \xrightarrow{d} \chi^2_\ell(\delta'\Pi'\Pi\delta)$.

Finally, if $\delta \in N(\Pi)$ then the test will have trivial power:

$$\Pr_{\delta \in N(\Pi)}(W(\gamma_0) > \chi^2_{\ell, 1-\alpha}) \rightarrow \alpha, \text{ as } n \rightarrow \infty.$$

2. **(Testing hypotheses on structural parameters without assuming identification)** Consider the following system of equations:

$$
\begin{aligned}
y &= Y\gamma + u, \\
Y &= Z\Pi + V, \\
y &= Z\pi + v,
\end{aligned}
$$

where γ is the $m \times 1$ vector of structural coefficients on the endogenous regressor Y, Π is the $l \times m$ matrix of coefficients in the reduced-form equation for the endogenous regressors Y, and π is the $l \times 1$ vector of coefficients in the reduced-form equation for the dependent variably y. Recall that the structural and reduced-form parameters are related by

$$\Pi\gamma = \pi. \tag{1}$$

We assume, however, that the rank identification condition might fail, i.e. it is possible that $\operatorname{rank}(\Pi) \leq m$, and therefore consistent estimation of γ might be impossible. Let $\hat{\Pi}$ and $\hat{\pi}$ be some consistent and asymptotically normal estimators of Π and π respectively:

$$\sqrt{n}\begin{pmatrix} \hat{\pi} - \pi \\ \operatorname{vec}(\hat{\Pi} - \Pi) \end{pmatrix} \to_d N\left(0, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}\right), \tag{2}$$

where vec is the vectorization operator: for a matrix $A = \begin{bmatrix} a_1 & \ldots & a_m \end{bmatrix}$, where a_j are

$l \times 1$ vectors for $j = 1, \ldots, m$, we have:

$$\operatorname{vec}(A) = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}.$$

Assume that the asymptotic variance-covariance matrix is positive definite, and that there are estimators $\hat{\Sigma}_{11}$, $\hat{\Sigma}_{12}$, and $\hat{\Sigma}_{22}$, which consistently estimate Σ_{11}, Σ_{12}, and Σ_{22} respectively. We are interested in testing $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$, where $\gamma_0 \in \mathbb{R}^m$ is a *known* vector of constants.

(a) **(10 points)** Show that under H_0 (when $\gamma = \gamma_0$), $\sqrt{n}(\hat{\Pi}\gamma_0 - \hat{\pi}) \to_d N(0, \Omega_0)$, where Ω_0 is an $l \times l$ asymptotic variance-covariance matrix. Find the expression for Ω_0 in terms of γ_0 and Σ's. Hints: Use equations (1) and (2), and the fact that $\text{vec}(\Pi\gamma) = (\gamma' \otimes I_l)\text{vec}(\Pi)$, where $A \otimes B$ denotes the Kronecker product of A and B.

(b) **(5 points)** Given your result in (a), suggest $\hat{\Omega}_0$, a consistent estimator for Ω_0 that may depend on γ_0 and the estimators for Σ's.

(c) **(5 points)** Define a statistic $W(\gamma_0) = n(\hat{\Pi}\gamma_0 - \hat{\pi})'\hat{\Omega}_0^{-1}(\hat{\Pi}\gamma_0 - \hat{\pi})$. Using $W(\gamma_0)$, propose an asymptotic size α test of $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$. When should the econometrician reject H_0?

(d) **(10 points)** Suppose that the following local alternative hypothesis is true: $H_1 : \gamma = \gamma_0 + \delta/\sqrt{n}$, where $\delta \in \mathbb{R}^l$ is a vector of unknown constants. Find the asymptotic distribution of $W(\gamma_0)$ in this case.

(e) **(5 points)** Suppose that $H_1 : \gamma = \gamma_0 + \delta/\sqrt{n}$ is true. Against which alternatives (δ's) the test you proposed in (c) has asymptotic power greater than α? Under what condition the test will have an asymptotic power greater than α for any m-vector $\delta \neq 0$?