Lecture 4. Banach's fixed point theorem

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Theorem 1 (**Banach's Fixed Point Theorem**). *Let X be a complete (nonempty) metric space and $f\colon X \to X$ a K-Lipschitz function, that is, $d(f(x), f(y)) \le Kd(x, y)$ for all $x, y \in X$. If $K < 1$, then there is a unique $x^* \in X$ such that $f(x^*) = x^*$. Moreover, for each point $x_0 \in X$, the sequence $(f^n(x_0)_n)_{n\in\mathbb{N}}$ converges to x^* as $n \to \infty$.*

Proof. Let $K < 1$ and $f\colon X \to X$ be K-Lipschitz.

(*Uniqueness.*) Let $x, y \in X$ such that $f(x) = x$ and $f(y) = y$. Then $d(x, y) \le Kd(x, y)$, which can only happens if $d(x, y) = 0$, i.e., if $x = y$.

(*Existence.*) Pick $x_0 \in X$ arbitrarily and define the sequence (x_n) recursively by setting $x_{n+1} = f(x_n)$ for $n = 0, 1, \ldots$. For $n \ge 1$, we have $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}) \le Kd(x_n, x_{n+1})$. By recursion, we get that for all $n \in \mathbb{N}_0$, $d(x_{n+1}, x_n) \le K^n d(x_1, x_0)$. If $n < m$, then $d(x_n, x_m) \le \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \le (K^n + K^{n+1} + \cdots + K^{m-1})d(x_1, x_0) \le ((1-c)^{-1} d(x_1, x_0))K^n$. This proves that (x_n) is a Cauchy sequence. Since X is complete, $\lim_{n\to\infty} x_n = x$ for some $x \in X$. By continuity of f, $\lim_{n\to\infty} f(x_n) = f(x)$, but $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$, hence $f(x) = x$ by uniqueness of limits of sequences. □

Reference. T.9.23. in Rudin PMA p.220 or "Lipschitz's theorem" in Berge TS p.105 or T.I.1.1. in Granas&Dugundji FPT p.10. The theorem is also known as Banach's contraction principle.

References

Berge, C. (1963): *Topological spaces*. Oliver & Boyd.

Granas, A., and J. Dugundji (2003): *Fixed point theory*, vol. 14. Springer.

Rudin, W., et al. (1976): *Principles of mathematical analysis*, vol. 3. McGraw-Hill.