Lecture 2. Von Neumann–Sion minimax theorem

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We assume all the topological spaces to be Hausdorff. For the proof of this theorem, we use the KKM lemma which was proved in lecture 1.

Theorem 1 (**Von Neumann–Sion Minimax Theorem**). *Let X be a convex compact subset of a topological vector space and Y a convex subset of a topological vector space. If $f\colon X \times Y \to \mathbb{R}$ is a function such that:*

1. *for all $x \in X$, $y \mapsto f(x, y)$ is upper semicontinuous and quasiconcave;*
2. *for all $y \in Y$, $x \mapsto f(x, y)$ is lower semicontinuous and quasiconvex;*

Then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

Reference. The result is from Sion (1958) "On general minimax theorems" which he proved using the KKM lemma and Helly's theorem. For a proofs, see Fan (1964) or Takahashi (1976) or T.II.7.1.4. in Granas&Dugundji FPT p.143 (the last two use the Fan–Browder fixed point theorem derived from KKM and we follow them). For alternative proofs using simpler arguments, see Ghouila-Houri (1966) or Komiya (1988) or Kindler (2005). See also Aubin O&E. See Simons (1995) "Minimax Theorems and Their Proofs" for a general survey of minimax theorems.

Proof. Since X is compact and $x \mapsto f(x, y)$ is lsc, $\min_{x \in X} f(x, y)$ exists. Since $x \mapsto f(x, y)$ is lsc, $x \mapsto \sup_{y \in Y} f(x, y)$ is lsc, and, similarly, $\min_{x \in X} \sup_{y \in Y} f(x, y)$ exists. Since $f(x, y) \leq \sup_{y \in Y} f(x, y)$, $\min_{x \in X} f(x, y) \leq \min_{x \in X} \sup_{y \in Y} f(x, y)$, and thus $\sup_{y \in Y} \min_{x \in X} f(x, y) \leq \min_{x \in X} \sup_{y \in Y} f(x, y)$.

Suppose for now that Y is compact. By contradiction, suppose there exists $r \in \mathbb{R}$ such that $\sup_{y \in Y} \min_{x \in X} f(x, y) < r < \min_{x \in X} \sup_{y \in Y} f(x, y)$. Define the correspondences $S\colon X \rightrightarrows Y$ and $T\colon X \rightrightarrows Y$ by $S(x) = \{y \in Y : f(x, y) > r\}$ and $T(x) = \{y \in Y : f(x, y) < r\}$. Each $T(x)$ is open by u.s.c. of $y \mapsto f(x, y)$ and each $S(x)$ is convex by the quasi-concavity of $y \mapsto f(x, y)$ and nonempty since $\sup_{y \in Y} \min_{x \in X} f(x, y) < r$. Since $S^{-1}(y) = \{x \in X : f(x, y) > r\}$ and $T^{-1}(y) = \{x \in X : f(x, y) < r\}$, we similarly have that each $S^{-1}(y)$ is open and each $T^{-1}(y)$ is convex and nonempty. Then, by Lemma 5, there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in S(x_0) \cap T(x_0)$, that is, $r < f(x_0, y_0) < r$: a contraction.

Suppose now that Y is not compact. Suppose again by contraction that there exists $r \in \mathbb{R}$ such that $\sup_{y \in Y} \min_{x \in X} f(x, y) < r < \min_{x \in X} \sup_{y \in Y} f(x, y)$. Then there exists a finite set $S \subseteq Y$ such that for any $x \in X$, there is $y \in S$ with $f(x, y) > r$. Taking $f' = f|_{X \times \mathrm{conv}(S)}$, we get $\sup_{y \in \mathrm{conv}(S)} \min_{x \in X} f'(x, y) < r < \min_{x \in X} \sup_{y \in \mathrm{conv}(S)} f'(x, y)$. This contradicts the previous result for Y compact. $\square$

Let E be a vector space and $X \subseteq E$ an arbitrary subset. Denote $\mathrm{conv}(\{x_1, \ldots, x_n\})$ the convex hull of a finite collection of points $x_1, \ldots, x_n$ in X. A correspondence $T\colon X \rightrightarrows E$ is said to be a Knaster-Kuratowski-Mazurkiewicz map (or KKM-map) if

$$\mathrm{conv}(\{x_1, \ldots, x_n\}) \subseteq \bigcup_{i=1}^{n} T(x_i)$$

for every finite subset $\{x_1, \ldots, x_n\}$ of X.

Lemma 2. *Let X be a nonempty convex subset of a vector space and $T\colon X \rightrightarrows X$. If the dual of G defined by $G^*\colon y \mapsto X - T^{-1}(y) = X - \{x \in X : y \in T(x)\}$ is not a KKM-map, then:*

1. *there exists a point $w \in X$ such that $w \in \mathrm{conv}(T(w))$;*
2. *if T has convex values, then T has a fixed point.*

Proof. Since (1.) directly implies (2.), it suffices to prove (1.). Since G^* is not a KKM-map, there exists $w \in \mathrm{conv}(\{x_1, \ldots, x_n\})$ for some $x_1, \ldots, x_n \in X$ such that $w \in X - \bigcup_{i=1}^{n} T^*(x_i) = X - \bigcup_{i=1}^{n}(X - T^{-1}(x_i)) = \bigcap_{i=1}^{n} T^{-1}(x_i)$. Therefore $x_i \in T(w)$ for each $i \in \{1, \ldots, n\}$, and so $w \in \mathrm{conv}(T(w))$. □

Reference. L.3.1.3. in Granas&Dugundji FPT p.38.

Lemma 3 (Knaster–Kuratowski–Mazurkiewicz (KKM) Lemma)**.** *Let $\{x_0, x_1, \ldots, x_d\} \subseteq \mathbb{R}^{d+1}$ and $\Delta^d = \mathrm{conv}\{x_i : i \in \{0, 1, \ldots, d\}\}$ the d-simplex with vertices $\{x_0, x_1, \ldots, x_n\}$. Let $F_0, F_1, \ldots, F_d$ be closed subsets of Δ^d such that for every $I \subseteq \{0, 1, \ldots, d\}$, we have $\mathrm{conv}\{x_i : i \in I\} \subseteq \bigcup_{i \in I} F_i$. Then the intersection $\bigcap_{i=0}^{d} F_i$ is nonempty and compact.*

Proof. See Lecture 1. □

Lemma 4 (**Fan–Browder Fixed Point Theorem**)**.** *Let X be a convex compact subset of a topological vector space and $T\colon X \rightrightarrows X$ a correspondence.*

1. If T has nonempty convex values and for each $y \in X$, the set $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X, then T has a fixed point.

2. If T has open values and for each $y \in X$, the set $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is a nonempty convex subset of X, then T has a fixed point.

Proof. Since T satisfies the conditions of (2.) if and only if $T^1 := y \mapsto T^{-1}(y)$ satisfies the conditions of (1.), it suffices to prove (1.). Define the dual of T as the correspondence $T^* := y \mapsto X - T^{-1}(y)$. Since T has nonempty values, T^{-1} is surjective. We thus have $\bigcap\{T^*(y) : y \in X\} = \bigcap\{X - T^{-1}(y) : y \in X\} = X - \bigcup\{T^{-1}(y) : y \in X\} = \emptyset$. Since X is compact and T has open values, the values of T^* are compact. It follows that $\bigcap_{i=1}^{n} T^*(x_i) = \emptyset$ for some $x_1, \ldots, x_n$ in X. By the KKM lemma, it follows that T^* is not a KKM-map. Since T has convex values, it follows by Lemma 2 that T has a fixed point. □

Reference. T.7.1.2. in Granas&Dugundji FPT p.143.

Lemma 5 (Fan's Coincidence Theorem)**.** *Let X and Y be convex compact subsets of some topological vector spaces. Let $S\colon X \rightrightarrows Y$ and $T\colon X \rightrightarrows Y$ be correspondences such that: (i) S has open values and for each $y \in Y$, the set $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is a nonempty convex subset of X; (ii) T has nonempty convex values and for each $y \in Y$, the* *set* $T^{-1}(y) := \{x \in X : y \in T(x)\}$ *is open in* X*. Then there exists* $(x_0, y_0) \in X \times Y$ *such that* $y_0 \in S(x_0) \cap T(x_0)$.

Proof. Let $Z = X \times Y$ and define $H\colon Z \rightrightarrows Z$ by $H((x, y)) = T^{-1}(y) \times S(x)$. It follows that the values of H are open and for each $(x, y) \in Z$, the sets $H^{-1}(x, y) = S^{-1}(y) \times T(x)$ are nonempty and convex. By Lemma 4, H has a fixed point. That is, there exists $(x_0, y_0) \in Z$ such that $(x_0, y_0) \in T^{-1}(y_0) \times S(x_0)$. Thus $y_0 \in S(x_0) \cap T(x_0)$. □

Reference. T.7.1.3. in Granas&Dugundji FPT p.143.

References

Aubin, J.-P. (2002): *Optima and equilibria: an introduction to nonlinear analysis*, vol. 140. Springer Science & Business Media.

Fan, K. (1964): "Sur un théoreme minimax," *CR Acad. Sci. Paris*, 259, 3925–3928.

Ghouila-Houri, M. (1966): "Le théorème minimax de Sion," *Theory of Games, Engl. Univ. Press, London*, pp. 123–129.

Granas, A., and J. Dugundji (2003): *Fixed point theory*, vol. 14. Springer.

Kindler, J. (2005): "A simple proof of Sion's minimax theorem," *The American Mathematical Monthly*, 112(4), 356–358.

Komiya, H. (1988): "Elementary proof for Sion's minimax theorem," *Kodai mathematical journal*, 11(1), 5–7.

Sion, M. (1958): "On general minimax theorems.," *Pacific J. Math.*, 8(4), 171–176.

Takahashi, W. (1976): "Nonlinear variational inequalities and fixed point theorems," *Journal of the Mathematical Society of Japan*, 28(1), 168–181.