Lecture 10. Berge's maximum theorem

Paul Delatte
delatte@usc.edu
University of Southern California

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Theorem 1 (Berge's Maximum Theorem). *Let X and Θ be topological spaces, $f: \Theta \times X \to \mathbb{R}$ a continuous function, and $C: \Theta \rightrightarrows X$ a correspondence with nonempty compact values. Define the value function*[1] $v: \Theta \to \mathbb{R}$ *by*

$$v(\theta) = \max\{f(\theta, x) : x \in C(\theta)\},$$

and the solution correspondence $S: \Theta \rightrightarrows X$ by

$$S(\theta) = \arg\max\{f(\theta, x) : x \in C(\theta)\}.$$

If the correspondence C is also continuous, then the value function v is continuous and the solution correspondence S is upper hemicontinuous with nonempty compact values.

Proof. By Lemma 2, since f and C are continuous, the value function v is continuous.

Let $\theta \in \Theta$. Since f is continuous, the projection $f_\theta: x \mapsto f(\theta, x)$ is continuous, and since $C(\theta)$ is a nonempty compact set, the extreme value theorem guarantees that the set $S(\theta)$ is nonempty and compact.

Define $D: \Theta \rightrightarrows X$ by $D(\theta) = \{x \in X : f(\theta, x) = v(\theta)\}$. Then $\mathrm{Gr}D = \{(\theta, x) \in \Theta \times X : f(\theta, x) = v(\theta)\}$ is closed since f and v are continuous and $\mathbb{R}$ is Hausdorff. Since $S = C \cap D$, the solution correspondence S is upper hemicontinuous by Lemma 3. □

Reference. Bouligand and Kuratowksi independently introduced upper and lower hemicontinuous correspondences (see p.109 in Berge (1963)). The definitions for hemicontinuity do not always match; we adopt the definitions as in Aliprantis&Border (2006) that leave out properties not conceptually related to continuity. The maximum theorem first (?) appeared in Berge (1959) for Hausdorff topological spaces. Our statement and proof of the theorem follow T.17.31. in Aliprantis&Border (2006) p.570 which adapts "Maximum theorem" in Berge (1963) p.116 to arbitrary topological spaces. Our proof of Lemma 2 mostly follows Aliprantis&Border L.17.29-30 and our proof of Lemma 3 adapts to arbitrary topological spaces the proof in Berge (1963) of T.6.7. p.112 (we slightly corrected the proof of this lemma on the Wikipedia page for the Maximum Theorem).

Lemma 2. *Let X and Θ be topological spaces, $f: \Theta \times X \to \mathbb{R}$ a function, and $C: \Theta \rightrightarrows X$ a correspondence. Define the extended value function $\bar{v}: \Theta \to \overline{\mathbb{R}}$ by*

$$\bar{v}(\theta) = \sup\{f(\theta, x) : x \in C(\theta)\}$$

[1] The value function is a well-defined function by the extreme value theorem.

with the convention that $\sup \emptyset = -\infty$. *Then:*

1. if f is lower semicontinuous and C is lower hemicontinuous, then the extended value function $\bar{v}$ is lower semicontinuous;

2. if f is upper semicontinuous and C is upper hemicontinuous with nonempty compact values, then the extended value function $\bar{v}$ is upper semicontinuous.

Proof. 1. We prove that $\{\theta \in \Theta : \bar{v}(\theta) > \alpha\}$ is open for any $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$. Suppose there is $\theta' \in \Theta$ such that $\bar{v}(\theta') > \alpha$, then $f(x', \theta') > \alpha$ for some $x' \in C(\theta')$. Since f is lower semicontinuous, then $W = \{(\theta, x) \in \Theta \times X : f(\theta, x) > \alpha\}$ is an open neighborhood of (θ', x'). Thus there are open neighborhoods U of θ' and V of x' such that $(U \times V) \cap \mathrm{Gr}(B) \subseteq W$. Define $C^l(A) = \{\theta \in \Theta : C(\theta) \cap A \neq \emptyset\}$ for any $A \subseteq X$. Since C is lower hemicontinuous and $V \cap C(\theta') \neq \emptyset$, the set $N = U \cap C^l(V)$ is a neighborhood of θ'. By definition of $C^l(V)$, for each $\theta \in N$, there is some $x \in C(\theta) \cap V$, so that $(\theta, x) \in (U \times V) \cap \mathrm{Gr}(C) \subseteq W$. Then $f(\theta, x) > \alpha$, so $\bar{v}(\theta) > \alpha$ for each $\theta \in N$. Thus $N \subseteq \{\theta \in \Theta : \bar{v}(\theta) > \alpha\}$. Therefore, $\{\theta \in \Theta : \bar{v}(\theta) > \alpha\}$ is open.

2. Note that, by the extreme value theorem for upper semicontinuous functions, $\bar{v}(\theta) = v(\theta) := \max\{f(\theta, x) : x \in C(\theta)\}$ for all $\theta \in \Theta$. We prove that $\{\theta \in \Theta : v(\theta) < \alpha\}$ is open for any $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$. Suppose there is θ' such that $v(\theta') < \alpha$. Define $W = \{(\theta, x) \in \Theta \times X : f(\theta, x) < \alpha\}$. For each $x \in C(\theta')$, we have $(\theta', x) \in W$. Since f is upper semicontinuity, the set W is open. Thus, for each $x \in C(\theta')$, there are an open neighborhood U_x of θ' and an open neighborhood V_x of x such that $(U_x \times V_x) \cap \mathrm{Gr}(C) \subseteq W$. The family $\{V_x : x \in C(\theta')\}$ then forms an open cover of the compact set $C(\theta')$. By compactness, we can extract a finite open subcover $\{V_{x_1}, \ldots, V_{x_n}\}$ of $C(\theta')$. Define $U = \bigcap_{i=1}^n U_{x_i}$ and $V = \bigcup_{i=1}^n V_{x_i}$. Then $(U \times V) \cap \mathrm{Gr}(C) \subseteq W$. Define $C^u(A) = \{\theta \in \Theta : C(\theta) \subseteq A\}$ for any $A \subseteq X$. By upper hemicontinuity of C, the set $N = U \cap C^u(V)$ is an open neighborhood of θ'. For each $\theta \in N$, if $x \in C(\theta)$, then $(\theta, x) \in (U \times V) \cap \mathrm{Gr}(C) \subseteq W$, so $f(\theta, x) < \alpha$, and in particular, $v(\theta) < \alpha$. Thus $N \subseteq \{\theta \in \Theta : v(\theta) < \alpha\}$. Therefore, $\{\theta \in \Theta : v(\theta) < \alpha\}$ is open. □

Lemma 3. *Let X and Θ be topological spaces, and $A\colon \Theta \rightrightarrows X$ and $B\colon \Theta \rightrightarrows X$ correspondences. If A is upper hemicontinuous with compact values and B has closed graph, then the correspondence given by $A \cap B\colon \Theta \rightrightarrows X$ defined by $(A \cap B)(\theta) = A(\theta) \cap B(\theta)$ is upper hemicontinuous.*

Proof. Let $\theta \in \Theta$. Suppose $G \subseteq X$ is an open set such that $(A \cap B)(\theta) \subseteq G$. We prove that there is a neighborhood U_θ of θ such that $(A \cap B)(U_\theta) \subseteq G$.

If $A(\theta) \subseteq G$, then the result follows by upper hemicontinuity of A.

Suppose now that $A(\theta) \nsubseteq G$. Since B is closed, for each $x \in A(\theta) \setminus G$, there is an open neighborhood $U_x \times V_x$ of (θ, x) such that $U_x \times V_x \subseteq (\Theta \times X) \setminus \mathrm{Gr}(C)$; that is, $x' \notin B(\theta')$ whenever $(\theta', x') \in U_x \times V_x$ or, equivalently, $B(U_x) \cap V_x = \emptyset$. The collection of sets $\{G\} \cup \{V_x : x \in A(\theta) \setminus G\}$ forms an open cover of the compact set $A(\theta)$. By compactness, we can extract a finite open subcover $\{G, V_{x_1}, \ldots, V_{x_n}\}$. Then, by upper hemicontinuity of A, there is a neighborhood U'_θ of θ such that $A(U'_\theta) \subseteq G \cup V_{x_1} \cup \cdots \cup V_{x_n}$. Define $U_\theta = U'_\theta \cap U_{x_1} \cap \cdots \cap U_{x_n}$. Then $A(U_\theta) \subseteq G \cup V_{x_1} \cup \cdots \cup V_{x_n}$ and $B(U_\theta) \bigcap (V_{x_1} \cup \cdots \cup V_{x_n}) = \emptyset$. Therefore $(A \cap B)(U_\theta) \subseteq G$. □

References

ALIPRANTIS, C. D., AND K. C. BORDER (2006): *Infinite dimensional analysis*. Springer.

BERGE, C. (1963): *Topological spaces*. Oliver & Boyd.