Lecture 3. Tarski's fixed point theorem

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Theorem 1 (Tarski's Fixed Point Theorem). *Let $(L, \leq)$ be a complete lattice and $f: L \to L$ an increasing function, that is, $x \leq y$ implies $f(x) \leq f(y)$. Then the set $\{u \in L : f(u) = u\}$ of fixed points of f forms a complete lattice under $\leq$. In particular, f has a least fixed point $\underline{u}$ and a greatest fixed point $\bar{u}$.*

Proof. We start by showing that f has a least fixed point and a greatest fixed point. Define $H := \{x \in L : x \leq f(x)\}$ (which is nonempty since L is complete lattice) and $\bar{u} := \sup H$. Note that $\bar{u} \in L$ since L is a complete lattice. For all $x \in H$, we have $x \leq \bar{u}$, so $x \leq f(x) \leq f(\bar{u})$. Thus $f(\bar{u})$ is an upper bound for H, and so $\bar{u} \leq F(\bar{u})$. Since f is increasing, $f(\bar{u}) \leq f(f(\bar{u}))$. Thus $f(\bar{u}) \in H$, and so $f(\bar{u}) \leq \bar{u}$. This proves that $\bar{u}$ is a fixed point. If u is another fixed point, then $u \in H$, and so $u \leq \bar{u}$. Similarly, we show that $\underline{u} := \inf\{x \in L : f(x) \leq x\}$ is a least fixed point of f.

We now show that $F := \{u \in L : f(u) = u\}$ is a complete lattice under $\leq$. Let $A \subseteq F$ be nonempty and take $\bar{a} := \sup A$. Define $[\bar{a}, \sup L] := \{x \in L : \bar{a} \leq x\}$. It is directly seen that $[\bar{a}, \sup L]$ is a complete lattice. We prove that f maps $[\bar{a}, \sup L]$ into itself. If $x \in A$, then $x \leq \bar{a}$, and so $f(x) \leq f(\bar{a})$. But $f(x) = x$, so $x \leq f(\bar{a})$. That is, $f(\bar{a})$ is an upper bound of A, and so $\bar{a} \leq f(\bar{a})$. If $y \in [\bar{a}, \sup L]$, then $\bar{a} \leq y$, and so $f(\bar{a}) \leq f(y)$. Thus $\bar{a} \leq f(\bar{a}) \leq f(y)$, hence $f(y) \in [\bar{a}, \sup L]$. Denote $\hat{f}$ the restriction of f to $[\bar{a}, \sup L]$ and $\hat{F}$ the set of fixed points of $\hat{f}$ (which is nonempty since $[\bar{a}, \sup L]$ is a complete lattice). By the first part of the proof, $\underline{v} := \inf \hat{F} \in \hat{F}$, that is, $\underline{v}$ is a fixed point of $\hat{f}$, and so a fixed point of f. Since $\underline{v} \in [\bar{a}, \sup L]$, it is an upper bound of A that lies in F. If b is another upper bound of A in F, then $b \in I$, and so $b \in \hat{F}$. By definition of $\underline{v}$, we thus have $\underline{v} \leq b$. This proves that $\underline{v}$ is the supremum of A in F. A similar argument shows that A has an infimum in F. This concludes the proof. □

Reference. The result is from Tarski (1955) "A lattice-theoretical fixpoint theorem and its applications". See T.1.11. in Aliprantis&Border IDA p.17. The result is sometimes referred to as the Knaster–Tarski fixed point theorem. However, this refers to a weaker result under weaker conditions (see T.1.10. in Aliprantis&Border IDA p.16 or T.I.2.1.1. in Garnas&Dugundji FPT p.25). Another references include Topkis S&C (1998) and Davey&Priestley ILO (2002).

References

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