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Ulam's Searching Game with Two Lies

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We determine the minimal number of yes–no queries sufficient to find an unknown integer between 1 and n if at most two of the answers may be erroneous. This solves completely the problem of Ulam on searching with two lies, partially solved by Czyzowicz, Mundici, and Pelc. Their solution applied only to the case when n is a power of 2.

1. INTRODUCTION

S. M. Ulam in his "Adventures of a Mathematician" [U] raised the following question:

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half-million? and then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2(1{,}000{,}000)$ questions. Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer? One clearly needs more than n questions for guessing one of the 2^n objects because one does not know when the lie was told. This problem is not solved in general.

Several partial answers were given to this problem. The reader may find an exhaustive bibliography of the partial solutions in [Cz-M-P2]. Here we will quote only two papers, which give the best results obtained so far. A. Pelc in [P1] solved Ulam's problem in the case of one lie. His main theorem says that one can determine the number between 1 and n in k questions iff $n(k+1) \leqslant 2^k$ for even n and $n(k+1)+(k-1) \leqslant 2^k$ for odd n. J. Czyzowicz, D. Mundici, and A. Pelc in [Cz-M-P1] and [Cz-M-P2] show that if the integer n is a power of 2, say $n = 2^m$, and the Responder may lie twice, then k yes–no queries will do iff $k^2+k+2 \leqslant 2^{k-m+1}$. This result was sufficient to answer Ulam's problem for $n = 1{,}000{,}000$. It was easy to show that for 1,000,000 one has to ask at least 29 questions and, on the other hand, 29 questions were sufficient for 2^{20}, which is bigger than

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1,000,000. This method does not apply to many other integers n, when the distance to the closest power of 2 is greater.

In this paper we give a full solution to this problem. The computational difficulties in the proof and several exceptions to the theorem in its general form show that obtaining an exact solution to the problem in the case of more than two lies, is rather hopeless. Some asymptotic estimates in the case of more lies were, however, obtained by Rivest *et al.* in [R *et al.*], cf. also A. Pelc in [P2].

Ulam's game is closely connected with coding theory. We can consider a noninteractive version of the game when all questions should be asked at once. Detecting the Responder's number in such a game, which obviously is equivalent to identifying the erroneous answers, in nothing else than constructing an error-correcting code. A winning strategy in Ulam's game which allows k lies can be considered as an "interactive" counterpart of a k-error-correcting code. Thus our result yields a shortest "interactive" counterpart of a 2-error-correcting code of given size. An interactive counterpart of k-error detecting codes, the error-detecting searching games, were considered by A. Pelc in [P3].

2. Notation and Terminology

We fill follow the notation of [Cz-M-P2]. The game is played by two players: the Questioner and the Responder. The Responder chooses an integer from the set $\{1, ..., n\}$ and the Questioner has to determine this number in k queries of the form "is e in T?," where T is an arbitrary subset of the set $\{1, ..., n\}$. The Responder is allowed to lie at most twice. The Questioner wins the game iff he can determine the number in k questions. We will give the necessary and sufficient conditions on the numbers n and k for the Questioner to win the game.

At each stage, the state of the game is described by three natural numbers: (a, b, c). The first number a is the size of the truth-set: the set of those elements of the set $\{1, ..., n\}$ which satisfy all answers given so far. The second number b is the size of the one-lie-set: the set of those numbers which satisfy all but one answer. The number c is the size of the two-lies-set: the set of those numbers which satisfy all but two answers.

Following [Cz-M-P2] we define the weight and character of a state (a, b, c). The weight is defined by a formula

$$w_k(a, b, c) = F(k) \cdot a + (k+1) \cdot b + c,$$

where the function F is defined as follows

$$F(k) = \frac{k^2 + k + 2}{2}.$$

The character is defined as

$$\mathrm{ch}(a, b, c) = \min\{k : w_k(a, b, c) \leqslant 2^k\}.$$

The state is called nice iff the Questioner wins in $\mathrm{ch}(a, b, c)$ questions starting from this state.

We will consider now the queries. Let (a, b, c) be an arbitrary state and A, B, C be the truth-set, one-lie-set, and two-lies-set, respectively. Consider a query "is e in T?," where T is an arbitrary set. Let x, y, and z be cardinalities of sets $A \cap T$, $B \cap T$, and $C \cap T$, respectively. The query "is e in T?" will be denoted then as a $[x, y, z]$? query. This query yields two states: YES and NO, resulting from positive and negative answer, respectively. Then

$$\mathrm{YES} = \mathrm{YES}_{a,b,c}(x, y, z) = (x, a - x + y, b - y + z),$$

$$\mathrm{NO} = \mathrm{NO}_{a,b,c}(x, y, z) = (a - x, x + b - y, y + c - z).$$

It is easy to see that

$$w_k(a, b, c) = w_{k-1}(\mathrm{YES}) + w_{k-1}(\mathrm{NO}).$$

The state (a, b, c) is balanced iff there exists a query $[x, y, z]$? such that

$$|w_{k-1}(\mathrm{YES}) - w_{k-1}(\mathrm{NO})| \leqslant 1,$$

where $k = \mathrm{ch}(a, b, c)$. The query $[x, y, z]$? is said to balance the state (a, b, c) with the states YES and NO. We also say that the states YES and NO balance the state (a, b, c).

Note that if a query $[x, y, z]$? balances the state (a, b, c) then

$$\mathrm{ch}(\mathrm{YES}), \mathrm{ch}(\mathrm{NO}) \leqslant k - 1.$$

In particular, a state is nice iff there exists a query which balances it with nice states.

3. The Typical States

In this section we will define a large class of states, the typical states and show that most of them are nice. In the next sections we will prove the main result of the paper showing that under specific assumptions on the numbers n and k the first two queries lead to typical states.

Definition 3.1. Let $\mathrm{ch}(a, b, c) = k$. A state (a, b, c) is called typical, iff:

(1) $b \geqslant a - 1$,

(2) $c \geqslant k$.

DEFINITION 3.2. A question $[x, y, z]$? asked in a state (a, b, c) is called a splitting, iff $x=[a/2]$.

THEOREM 3.3. *Let a state* (a, b, c) *be typical and* $\mathrm{ch}(a, b, c)=k \geqslant 3$. *If* $b \geqslant 3k-3$, *then some splitting balances the state* (a, b, c) *with typical states.*

Proof. We consider 4 cases depending on the parity of a and b.

Case 1. $2 \mid a$. $2 \mid b$. Let $a_1=a/2$ and $b_1=b/2$. Let $c_1=[c/2]$. We ask a question $[a_1, b_1, c_1]$? Then

$$\mathrm{YES}=(a_1, a_1+b_1, b_1+c_1)$$
$$\mathrm{NO}=(a_1, a_1+b_1, b_1+c-c_1).$$

The states YES and NO obviously balance the state (a, b, c). We have to check that they are typical:

— Obviously $a_1+b_1 \geqslant a_1-1$.

— In order to have $b_1+c_1 \geqslant k-1$ and $b_1+c-c_1 \geqslant k-1$, it is enough to have $b_1 \geqslant k-1$. But $b \geqslant 3k-3=3(k-1)$. Hence $b_1 \geqslant \frac{3}{2}(k-1) \geqslant k-1$, which finishes the proof in this case.

Case 2. $2 \mid a$, $2 \nmid b$. Let $a_1=a/2$, $b_1=[b/2]$, and $c_1=[(c+k-1)/2]$. Then $b-b_1=b_1+1$. We ask a question $[a_1, b_1, c_1]$? This question is correct. Namely, $c+k-1 \leqslant c+c-1 \leqslant 2c$, hence $c_1 \leqslant c$. Then

$$\mathrm{YES}=(a_1, a_1+b_1, b_1+c_1+1),$$
$$\mathrm{NO}=(a_1, a_1+b_1+1, b_1+c-c_1).$$

Now we have

$$|w_{k-1}(\mathrm{YES})-w_{k-1}(\mathrm{NO})|=|2c_1-c-k+1|=|2c_1-(c+k-1)| \leqslant 1.$$

Thus the states YES and NO balance the state (a, b, c). We show that they are typical.

— Obviously $a_1+b_1+1 \geqslant a_1+b_1 \geqslant a_1-1$.

– It is enough to show that $b_1 \geqslant k-1$. Since $b=2b_1+1$, it is enough to show that $b \geqslant 2k-1$. But for $k \geqslant 2$ we have $b \geqslant 3k-3 \geqslant 2k-1$.

Case 3. $2 \nmid a$, $2 \mid b$. Let $B=[F(k-1)/(k-1)]$. Then $B \cdot (k-1) \leqslant F(k-1)<(B+1) \cdot (k-1)$. Let $a_1=[a/2]$, $b_1=b/2$, $c_1=[(c-t-1)/2]$, $B_1=[B/2]$, where $t=(2B_1+1) \cdot (k-1)-F(k-1)$. We ask a question $[a_1, b_1+B_1, c_1]$? We have to see that this question is correct.

— We show that $b_1+B_1 \leqslant b$. For $k \geqslant 1$ we have $b \geqslant 3k-3 \geqslant k-1$. Also for $k \geqslant 3$ we have $F(k-1) \leqslant (k-1)^2$, which implies $B \leqslant k-1$. So $b \geqslant B$, thus $b_1 \geqslant B_1$ and $b_1+B_1 \leqslant 2b_1=b$.

— We show that $c_1 \leqslant c$. We have to show that $c \geqslant -t-1$. But $c \geqslant k$ and

$$\begin{aligned}
-t-1 &= F(k-1)-(2B_1+1)\cdot(k-1)-1 \\
&\leqslant F(k-1)-B\cdot(k-1)-1 \\
&\leqslant (B+1)\cdot(k-1)-B\cdot(k-1)-1 \\
&= k-2 \leqslant k.
\end{aligned}$$

— We show that $c_1 \geqslant 0$. We have to show that $c \geqslant t+1$. Again $c \geqslant k$ and

$$\begin{aligned}
t+1 &= (2B_1+1)\cdot(k-1)-F(k-1)+1 \\
&\leqslant (B+1)\cdot(k-1)-F(k-1)+1 \\
&\leqslant (B+1)\cdot(k-1)-B\cdot(k-1)+1 \\
&= k.
\end{aligned}$$

Next we show that the states

$$\begin{aligned}
\text{YES} &= (a_1, a_1+1+b_1+B_1, b_1-B_1+c_1) \\
\text{NO} &= (a_1+1, a_1+b_1-B_1, b_1+B_1+c-c_1)
\end{aligned}$$

balance the state (a, b, c):

$$|w_{k-1}(\text{YES})-w_{k-1}(\text{NO})| = |2c_1-(c-t-1)| \leqslant 1.$$

Finally, we show that the states YES and NO are typical:

— We have $a_1+1+b_1+B_1 \geqslant a_1+b_1-B_1 \geqslant a_1 \geqslant a_1-1$.

— It is enough to show that $b_1-B_1 \geqslant k-1$. But $2B_1 \leqslant B \leqslant k-1$ and $b \geqslant 3k-3=3(k-1)$. Hence $2b_1 \geqslant 3(k-1)$ and $-2B_1 \geqslant -(k-1)$, so $2(b_1-B_1) \geqslant 2(k-1)$ and finally $b_1-B_1 \geqslant k-1$.

Case 4. $2 \nmid a$, $2 \nmid b$. Let us take the same B as in Case 3. Next let $B_1 = B-[B/2]$, $a_1=[a/2]$, $b_1=[b/2]$, $t=2B_1\cdot(k-1)-F(k-1)$, and $c_1=[(c-t-1)/2]$. We ask a question $[a_1, b_1+B_1, c_1]$?

Similarly as in Case 3 we show that this question is correct and the states

$$\begin{aligned}
\text{YES} &= (a_1, a_1+1+b_1+B_1, b_1+1-B_1+c_1) \\
\text{NO} &= (a_1+1, a_1+b_1+1-B_1, b_1+B_1+c-c_1)
\end{aligned}$$

balance the state (a, b, c).

The same kind of reasoning also shows that the states YES and NO are typical, which finishes the proof of the theorem. ∎

THEOREM 3.4. *Let the state* (a, b, c) *be typical and* $\mathrm{ch}(a, b, c) = k \geqslant 4$. *If* $c \geqslant k^2$, *then some splitting balances the state* (a, b, c) *with typical states.*

Proof. Again we consider four cases depending on the parity of a and b.

Case 1. $2 \mid a$, $2 \mid b$. Let $a_1 = a/2$, $b_1 = b/2$, $c = [c/2]$. We ask a question $[a_1, b_1, c_1]$?
It is obvious that the question is correct and the resulting states YES and NO balance the state (a, b, c). In order to show that the states YES and NO are typical we observe that $c \geqslant k^2 \geqslant 2(k-1)$, so $c - c_1 \geqslant c_1 \geqslant k-1$.

Case 2. $2 \mid a$, $2 \nmid b$. We follow Case 2 of the proof of Theorem 3. It is enough to show that the states YES and NO described there are typical.

— We have $c \geqslant k^2 \geqslant k-1$, so $c + k - 1 \geqslant 2(k-1)$. Hence $c_1 \geqslant k-1$.

— We have $c \geqslant k^2 \geqslant 3(k-1)$, so easily we obtain $c - c_1 \geqslant k-1$.

Case 3. $2 \nmid a$, $2 \mid b$. Let $a_1 = [a/2]$, $b_1 = b/2$, $l = (k^2 - 3k + 2)/2$, and $c_1 = [(c + l)/2]$. Then $a - a_1 = a_1 + 1$. We ask a question $[a_1, b_1, c_1]$? It is easy to see that this question is correct and the resulting states

$$\mathrm{YES} = (a_1, a_1 + b_1 + 1, b_1 + c_1)$$
$$\mathrm{NO} = (a_1 + 1, \ a_1 + b_1, b_1 + c - c_1)$$

balance the state (a_1, b_1, c_1).
We show that the states YES and NO are typical:

— $a_1 + b_1 + 1 \geqslant a_1 + b_1 \geqslant a_1 \geqslant a_1 - 1$.

— $c + l \geqslant k^2 + l \geqslant 2(k-1)$, which implies $c_1 \geqslant k-1$.

— $c \geqslant k^2 \geqslant l + 2(k-1)$, which implies $c - c_1 \geqslant k-1$.

Case 4. $2 \nmid a$, $2 \nmid b$. Let $a_1 = [a/2]$, $b_1 = [b/2]$, $l = (k^2 - 5k + 4)/2$ and $c_1 = [(c + l)/2]$. We ask a question $[a_1, b_1 + 1, c_1]$? The same kind of reasoning as before shows that this state is correct and the resulting states YES and NO are typical and balance the state (a, b, c). This proves the theorem. ∎

THEOREM 3.5. *Let the state* (a, b, c) *be typical and* $\mathrm{ch}(a, b, c) = k \geqslant 14$. *Then either* $b \geqslant 3k - 3$ *or* $c \geqslant k^2$.

Proof. Assume the contrary. Let $b < 3k - 3$ and $c < k^2$. Then $a \leqslant b + 1$, so $a \leqslant 3k - 3$. Therefore,

$$\begin{aligned} w_{k-1}(a, b, c) &= a \cdot F(k-1) + b \cdot k + c < 3k \cdot F(k-1) + 3k^2 + k^2 \\ &< 4k \cdot F(k-1) + 4k^2 \\ &= 2k(k^2 - k + 2) + 4k^2 = 2k(k^2 + k + 2). \end{aligned}$$

Now we observe that for $k \geqslant 14$ we have $k(k^2+k+2) \leqslant 2^{k-2}$, hence $w_{k-1}(a, b, c) \leqslant 2^{k-1}$. However, then $\mathrm{ch}(a, b, c) \leqslant k-1$, which contradicts the hypothesis $\mathrm{ch}(a, b, c) = k$. ▮

COROLLARY 3.6. *All typical states of character $\geqslant 14$ are balanced with typical states.*

LEMMA 3.7. *If $k \geqslant 7$ and $w_k(a, b, c) \geqslant 2^{k-1}$ then $w_{k-2}(a, b, c) > 2^{k-2}$. In particular,* $\mathrm{ch}(a, b, c) \geqslant k-1$.

Proof. We notice that

$$w_k(a, b, c) = w_{k-1}(a, b, c) + ak + b = w_{k-2}(a, b, c) + a(2k-1) + 2b.$$

Now for $k \geqslant 7$ we have

$$k^2 - 7k + 4 \geqslant 0 \qquad \text{and} \qquad k \geqslant 4,$$

which implies

$$F(k-2) - 2k + 1 \geqslant 1 \qquad \text{and} \qquad k - 3 \geqslant 1.$$

Hence

$$a \cdot (F(k-2) - 2k + 1) + b \cdot (k-3) + c \geqslant a + b + c > 0$$

which gives

$$a \cdot (2k-1) + 2b < w_{k-2}(a, b, c).$$

This obviously proves the lemma. ▮

COROLLARY 3.8. *If* $\mathrm{ch}(a, b, c) \geqslant 8$ *and the state* (a, b, c) *is balanced with the states* YES *and* NO *then* $\mathrm{ch}(\mathrm{YES}), \mathrm{ch}(\mathrm{NO}) \geqslant \mathrm{ch}(a, b, c) - 2$.

Proof. Let $\mathrm{ch}(a, b, c) = k$. From the hypothesis we have $w_{k-1}(a, b, c) > 2^{k-1}$. Hence $w_k(a, b, c) > 2^{k-1}$, so $w_{k-1}(\mathrm{YES})$, $w_{k-1}(\mathrm{NO}) \geqslant 2^{k-2}$. From the lemma we get $w_{k-3}(\mathrm{YES})$, $w_{k-3}(\mathrm{NO}) > 2^{k-3}$, thus $\mathrm{ch}(\mathrm{YES})$, $\mathrm{ch}(\mathrm{NO}) \geqslant k-2$. ▮

THEOREM 3.9. *All typical states of character $\geqslant 12$ are nice.*

Proof. The last two corollaries show that it is enough to prove that all typical states of character 12 and 13 are nice. In order to do this we will consider all typical states of character $\leqslant 13$. It will turn out that the states which are not nice are of character at most 11.

DEFINITION 3.10. A state (a, b, c) is k-nice iff beginning in the state (a, b, c) the Questioner has a winning strategy in at most k questions.

Obviously a state is nice iff it is k-nice, where k is a character of it.

LEMMA 3.11. *If $w_k(a, b, c) < 2^k$ and the state (a, b, c) is k-nice then the state $(a, b, c+1)$ is k-nice as well.*

Proof. The state (a, b, c) is k-nice, so the Questioner has a winning strategy in k questions. We ask the first of these questions: $[x, y, z]$?

The resulting states are $(k-1)$-nice: it is enough to follow the winning strategy. Let us denote

$$\mathrm{YES} = (y_1, y_2, y_3),$$
$$\mathrm{NO} = (n_1, n_2, n_3).$$

Then

$$w_{k-1}(\mathrm{YES}),\ w_{k-1}(\mathrm{NO}) \leqslant 2^{k-1}.$$

Since $w_k(a, b, c) < 2^k$, one of the above inequalities is proper. Let us suppose that $w_{k-1}(\mathrm{YES}) < 2^{k-1}$. Then $w_{k-1}(y_1, y_2, y_3) < 2^{k-1}$ and the state (y_1, y_2, y_3) is $(k-1)$-nice. The inductive hypothesis shows that the state (y_1, y_2, y_3+1) is also $(k-1)$-nice. Now in the state $(a, b, c+1)$ we ask the question $[x, y, z+1]$? to obtain the states

$$\mathrm{YES}' = (y_1, y_2, y_3+1)$$
$$\mathrm{NO}' = (n_1, n_2, n_3).$$

These states are $(k-1)$-nice, so the state (a, b, c) is k-nice. If $w_{k-1}(\mathrm{NO}) < 2^{k-1}$ then it is enough to ask a question $[x, y, z]$? ∎

LEMMA 3.12. *If $w_k(a, b, c) = 2^k$ and the state (a, b, c) is k-nice then the state $(a, b, c+1)$ is $(k+1)$-nice.*

Proof. The state (a, b, c) is k-nice, so the Questioner has a winning strategy in k questions. We ask the first of these questions: $[x, y, z]$? The resulting states YES and NO are $(k-1)$-nice: we follow the winning strategy.

Next we see that $w_{k-1}(\mathrm{YES}) = w_{k-1}(\mathrm{NO}) = 2^{k-1}$. In the state $(a, b, c+1)$ we ask the question $[x, y, z+1]$? From the inductive hypothesis the resulting state YES′ is k-nice and the state NO has not changed, it is $(k-1)$-nice and thus k-nice. Therefore, the state $(a, b, c+1)$ is $(k+1)$-nice. ∎

COROLLARY 3.13. *If a state (a, b, c) is nice then the state $(a, b, c+1)$ is nice as well.*

DEFINITION 3.14. We define a function $M(a, b)$ as

$$M(a, b) = \min\{c\colon \text{the state } (a, b, c) \text{ is nice}\}.$$

We have to show that the function M is well defined. We prove that for all a and b there exists c such that the state (a, b, c) is nice. We use a double induction on a and b:

For $a=0$ we have $M(a, b) \leqslant b-1$ (see [P]).

For $a=1$ we use an induction on b.

For $b=0$ or $b=1$ we have $M(a, b)=0$ (see [Cz-M-P2]). For $b \geqslant 2$ we take c such that $c \geqslant M(1, b-1)$ and that for some k: $w_k(1, b, c) = 2^k+1$. Then $\operatorname{ch}(1, b, c) = k+1$ and it is enough to ask the question $[1, b-1, c]$?

For $a \geqslant 2$ we assume that values $M(a', b')$ are well defined for all $a' < a$ and all b'. We take a_1 such that the question $[a_1, b_1, c_1]$? is a splitting. Then $a_1, a-a_1 < a$. We take any $b_1 \leqslant b$. Next we take c such that for some k: $w_k(a, b, c) = 2^k+1$ and

$$[c/2] \geqslant \max\{M(a_1, a-a_1+b_1), M(a-a_1, a_1+b-b_1)\}.$$

Now it is enough to ask the question $[a_1, b_1, [c/2]]$?

Next we observe that if a state (a, b, c) is k-nice then the state $(a, b, c-1)$ is k-nice as well: we use the same winning strategy. It follows that if $M(a, b)=c$ and $\operatorname{ch}(a, b, c)=k$ then $\operatorname{ch}(a, b, c-1)<k$. It gives the following

COROLLARY 3.15. *If* $M(a, b)=c$ *and* $\operatorname{ch}(a, b, c)=k$ *then either* $c=0$ *or* $w_{k-1}(a, b, c) = 2^{k-1}+1$.

We are to show that the typical states of character 12 and 13 are nice. We are thus interested in states (a, b, c) such that $\operatorname{ch}(a, b, c) \leqslant 13$. Then obviously $\operatorname{ch}(a, b, 0) \leqslant 13$. From now on we will thus consider only pairs (a, b) such that $\operatorname{ch}(a, b, 0) \leqslant 13$. It is enough to show that if

$$\operatorname{ch}(a, b, 0) \leqslant 13,$$

$$M(a, b) = c,$$

$$\operatorname{ch}(a, b, c-1) = k,$$

then $k \leqslant 11$. (We recall that a state (a, b, c) is not nice iff $c < M(a, b)$).

We will compute values of the function $M(a, b)$ for all pairs (a, b) as above. We will consider $a=0, 1, 2, \ldots$ as long as $\operatorname{ch}(a, 0, 0) \leqslant 13$. For each a we will look at $b=0, 1, 2, \ldots$ and we stop when $\operatorname{ch}(a, b, 0) > 13$.

During the computation we assume that when we compute the value of

$M(a, b)$ then all values of $M(a', b')$ for $a' < a$ or $a' = a$ and $b' < b$ are already computyed and stored.

Now we describe a method of computing a value of $M(a, b)$. We introduce a new function

$$\begin{aligned}\text{Min } C(a, b, x, y) = \min\{c : \exists z, k\ (k = \text{ch}(a, b, c) &\text{ and the question } [x, y, z]? \\ &\text{weakly balances the state } (a, b, c) \text{ with } (k-1)\text{-} \\ &\text{nice states i.e. } w_{k-1}(\text{YES}),\ w_{k-1}(\text{NO}) \leqslant 2^{k-1})\}.\end{aligned}$$

In this definition we assume that the question $[x, y, z]?$ is not trivial, i.e., $(x, y) \neq (0, 0)$ and $(x, y) \neq (a, b)$. For trivial questions we put

$$\text{Min } C(a, b, 0, 0) = \text{Min } C(a, b, a, b) = \infty.$$

The function Min C gives the minimal value of c such that the state (a, b, c) is nice and some question of type $[x, y, z]?$ confirms that. The function Min C is used in computing the values of $M(a, b)$:

$$\begin{aligned}&M(0, 0) = 1,\\ &M(0, 1) = 0,\\ &M(1, 0) = 0,\\ &M(a, b) = \min\{\text{Min } C(a, b, x, y) : 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\} \text{ for other pairs.}\end{aligned}$$

Next we observe that questions $[x, y, z]?$ and $[a-x, b-y, c-z]?$ are equivalent. We will thus consider only questions $[x, y, z]?$ for which $x \leqslant [a/2]$. Obviously we can stop searching when we find a pair (x, y) for which Min $C(a, b, x, y) = 0$. The algorithm will work faster when we start searching "from the center."

Finally, we will show how to compute the values of the function Min $C(a, b, x, y)$. The algorithm will have to take care of pairs $(a, b) \notin \{(0, 0), (0, 1), (1, 0)\}$. Let us thus take such a pair (a, b) and in the state $(a, b, 0)$ ask a question $[x, y, 0]?$ Then

$$\begin{aligned}&\text{YES} = (y_1, y_2, y_3) = (x, a - x + y, b - y)\\ &\text{NO} = (n_1, n_2, n_3) = (a - x, x + b - y, y).\end{aligned}$$

Next we put

$$\begin{aligned}&c_1 = M(y_1, y_2)\\ &c_2 = M(n_1, n_2)\\ &k_1 = \text{ch}(\text{YES})\\ &k_2 = \text{ch}(\text{NO})\\ &k_3 = \text{ch}(y_1, y_2, c_1)\\ &k_4 = \text{ch}(n_1, n_2, c_2)\\ &k = \max\{k_1, k_2, k_3, k_4\}.\end{aligned}$$

PROPOSITION 3.16. $\operatorname{Min} C(a, b, x, y) = \min\{c: \operatorname{ch}(a, b, c) > k\}$.

Proof. Assume first that $\operatorname{Min} C(a, b, x, y) = c$. We show that $\operatorname{ch}(a, b, c) > k$. Assume the contrary. Let $\operatorname{ch}(a, b, c) \leqslant k$. From the definition of the function $\operatorname{Min} C$ it follows that there exists a question $[x, y, z]$? which weakly balances the state (a, b, c) with $(k-1)$-nice states. Let

$$\mathrm{YES}' = (y_1, y_2, y_3 + z)$$
$$\mathrm{NO}' = (n_1, n_2, n_3 + c - z)$$

be the states generated by this question. We consider four cases.

Case 1. $k = k_1$. Then $\operatorname{ch}(\mathrm{YES}) = k$, so $w_{k-1}(y_1, y_2, y_3) > 2^{k-1}$. Thus $w_{k-1}(y_1, y_2, y_3 + z) > 2^{k-1}$ and the state YES′ connot be $(k-1)$-nice.

Case 2. $k = k_2$. Similarly, we show that the state NO′ cannot be $(k-1)$-nice.

Case 3. $k = k_3$. Then $\operatorname{ch}(y_1, y_2, c_1) = k$. From the definition of the function M we see that the state (y_1, y_2, c_1) is nice, so it is k-nice. From the minimality of c_1 the states (y_1, y_2, c') are not nice for $c' < c$.

Let us suppose that $c_1 > 0$. Then $w_{k-1}(y_1, y_2, c_1) = 2^{k-1} + 1$, so $w_{k-1}(y_1, y_2, c_1 - 1) = 2^{k-1}$. If the state $(y_1, y_2, 0)$ were $(k-1)$-nice then the state $(y_1, y_2, c_1 - 1)$ would be $(k-1)$-nice as well. Then it would be nice, contrary to the choice of c_1. Thus the state $(y_1, y_2, 0)$ is not $(k-1)$-nice. The state $\mathrm{YES}' = (y_1, y_2, y_3 + z)$ is not $(k-1)$-nice either.

Let now $c_1 = 0$. Then $\operatorname{ch}(y_1, y_2, 0) = k$, so $w_{k-1}(y_1, y_2, 0) \rangle 2^{k-1}$. The state $(y_1, y_2, 0)$ is not $(k-1)$-nice then and similarly as above the state YES′ is not $(k-1)$-nice.

Case 4. $k = k_4$. Similarly as in Case 3 we show that the state NO′ is not $(k-1)$-nice.

Now let $\operatorname{ch}(a, b, c) > k$ and let c be minimal with this property. We will show that the question $[x, y, 0]$? weakly balances the state (a, b, c) with k-nice states.

First let $c = 0$. We ask a question $[x, y, 0]$? to obtain states YES and NO. Then $\operatorname{ch}((\mathrm{YES}) = k_1 \leqslant k$ and $\operatorname{ch}(\mathrm{NO}) = k_2 \leqslant k$. Then the states YES and NO weakly balance the state (a, b, c). We show that these states are k-nice. Let us consider the state YES.

If $y_3 \geqslant c$ then from the definition of the function M it follows that the state YES is nice. Since $\operatorname{ch}(\mathrm{YES}) = k_1$ the state YES is k_1-nice. Then it is k-nice.

If $y_3 < c$ then $k_1 \leqslant k_3$. The state (y_1, y_2, c_1) is k_3-nice, so the state (y_1, y_2, y_3) is k_3-nice as well. Since $k_3 \leqslant k$, the state YES is k-nice.

Similarly, we show that the state NO is k-nice.

Now let $c>0$. From the hypothesis $w_k(a,b,c)>2^k$ and $w_k(a,b,c-1)\leqslant 2^k$. It follows that $w_k(a,b,c)=2^k+1$. Then

$$w_k(a,b,0)\leqslant w_k(a,b,c-1)=2^k.$$

It is easy to see that if $k\leqslant 1$ then $a+b\leqslant 1$, contrary to the choice of the pair (a,b). So $k\geqslant 2$. It follows that $F(k)-k-1\geqslant 1$. We ask a question $[x,y,0]$? Then

$$\text{YES}'=(y_1,y_2,y_3)=\text{YES},$$
$$\text{NO}'=(n_1,n_2,n_3+c).$$

Obviously $\text{ch}(\text{YES}')=\text{ch}(\text{YES})=k_1=k$. We show that $\text{ch}(\text{NO}')\leqslant k$, i.e., $w_k(\text{NO}')\leqslant 2^k$. It is enough to show that $w_k(\text{NO}')<w_k(a,b,c)$.

We recall that the value of the function $\text{Min}\, C(a,b,x,y)$ is computed only for nontrivial questions. It means that one of x and y is non-zero, i.e., $x+2y>0$. Therefore,

$$x\cdot(F(k)-k-1)+yk\geqslant x+2y>0.$$

Now it easily follows that

$$\begin{aligned}w_k(\text{NO}')&=w_k(a-x,x+b-y,y+c)\\&=(a-x)\cdot F(k)+(x+b-y)\cdot(k+1)+y+c\\&<a\cdot F(k)+b\cdot(k+1)+c\\&=w_k(a,b,c).\end{aligned}$$

This shows that the question $[x,y,0]$? weakly balances the state (a,b,c). It remains to show that the states YES$'$ and NO$'$ are k-nice.

This time we consider the state NO$'$. If $n_3+c\geqslant c_2$ then from the definition of the function M it follows that the state NO$'$ is nice. Since $\text{ch}(\text{NO}')\leqslant k$ it is also k-nice. On the other hand, if $n_3+c<c_2$ then we show as above that the state NO$'$ is k_4-nice, so it is k-nice as well. ▮

Now it is enough to run a program which follows the lines of the above algorithm and computes the values of the function $M(a,b)$. The output consisting of th list of typical states of character $\leqslant 13$ which are not nice can be found in the Appendix. We observe that among these states the ones that are not nice have character at most 11. This finishes the proof of Theorem 3.9. ▮

4. The First Question

In this section we will show that the best strategy of the Questioner should begin with a splitting as the first question. It is well known (see [R;

Cz-M-P2]) that if $w_k(a, b, c) > 2^k$ then the Questioner cannot win in k questions starting from state (a, b, c).

The first two questions should be chosen so that the biggest weight of the four resulting states be as small as possible. Let us consider a state $(m, 0, 0)$ of character $k+2$. We ask two questions obtaining four states YY, YN, NY, NN (e.g., the state YN is obtained as a result of a positive answer to the first question and a negative answer to the second). We are looking for questions such that the number

$$\max\{w_k(\mathrm{YY}), w_k(\mathrm{YN}), w_k(\mathrm{NY}), w_k(\mathrm{NN})\}$$

is smallest. Such a situation will be called the best balance.

THEOREM 4.1. *The best balance can be obtained only if the first question is a splitting.*

Proof. We will consider four cases.

Case 1. $m = 4n$. The best balance is obtained by asking $[2n, 0, 0]$? and then $[n, n, 0]$?

Case 2. $m = 4n + 1$. We ask a question $[2n - x, 0, 0]$?

If $x = 0$ then this question is a splitting. Assume then that $x \neq 0$. Then

$$\mathrm{YES} = (2n - x, 2n + 1 + x, 0),$$
$$\mathrm{NO} = (2n + 1 + x, 2n - x, 0).$$

Obviously $w_{k+1}(\mathrm{NO}) \geqslant w_{k+1}(\mathrm{YES})$.

We also consider the question $[2n, 0, 0]$? and then $[n, n, 0]$? The four resulting states will be:

$$\mathrm{YY} = (n, 2n, n+1),$$
$$\mathrm{YN} = (n, 2n+1, n),$$
$$\mathrm{NY} = (n, 2n+1, n),$$
$$\mathrm{NN} = (n+1, 2n, n).$$

It is easy to see that $w_k(\mathrm{NN})$ is the biggest weight of the four. Therefore, it is enough to show that

$$w_{k+1}(\mathrm{NO}) \geqslant 2 \cdot w_k(\mathrm{NN}).$$

An easy calculation shows that

$$w_{k+1}(\mathrm{NO}) - 2 \cdot w_k(\mathrm{NN}) = (x-1) \cdot (F(k) - 1) \geqslant 0.$$

The question $[2n+1, 0, 0]$? is equivalent to $[2n, 0, 0]$? and the question $[2n+1+x, 0, 0]$? is equivalent to $[2n-x, 0, 0]$?

Cases 3 and 4 ($m=4n+2$ and $m=4n+3$) are treated similarly as Case 2. ▮

5. The Second Question

In this section we will consider the second question in Questioner's best strategy. We have already seen that the first question should be a splitting. The resulting states will be of one of the following forms:

$$(a, a, 0),$$

$$(a+1, a, 0),$$

$$(a, a+1, 0).$$

We will assume that the character of the original state $(n, 0, 0)$ is $k+2$, i.e., the characters of the above states are at most $k+1$.

We will consider, however, the more general situation of arbitrary states $(a, b, 0)$. Our aim is to find a question which balances the state $(a, b, 0)$ in the best possible way. We will show that it is always possible to find a question which produces two states with difference of weights not exceeding k. We will also show the cases when states of equal weight are obtained and when the difference of weights is $k/2$.

We will consider four cases depending on the parity of a and b:

Case 1. $(2a, 2b, 0)$. A question $[a, b, 0]$? produces states of equal weights. We will also observe that if $b \geqslant k$ then the resulting states are typical.

Case 2. $(2a, 2b+1, 0)$. We consider a question $[x, y, 0]$? Then

$$\mathrm{YES} = (x, 2a-x+y, 2b+1-y),$$
$$\mathrm{NO} = (2a-x, x+2b+1-y, y).$$

An easy calculation shows that

$$w_k(\mathrm{YES}) - w_k(\mathrm{NO}) = k \cdot ((x-a) \cdot (k-1) - 2 \cdot (b-y) - 1).$$

Therefore,

$$k \mid w_k(\mathrm{YES}) - w_k(\mathrm{NO}).$$

There are two possibilities:

Subcase 2.1. $2 \nmid k$. Then $2 \mid k-1$ and the number $(x-a) \cdot (k-1) - 2 \cdot (b-y) - 1$ is odd. It follows that the absolute value of the difference

$w_k(\text{YES}) - w_k(\text{NO})$ is at least k. The best way to balance this state in this case is by asking question $[a, b, 0]$? and obtaining states

$$(a, a+b, b+1) \qquad \text{and} \qquad (a, a+b+1, b).$$

Then

$$w_k(a, a+b+1, b) = w_k(a, a+b, b+1) + k.$$

Subcase 2.2. $2 \mid k$. We ask a question $[x, y, 0]$?, where $x = a+1$ and $y = b - (k-2)/2$. It is easy to see that we obtain two states of equal weights.

In both subcases the question asked is a splitting and the resulting states are typical if $b - (k-2)/2 \geqslant k$, i.e., when $b \geqslant (3k-2)/2$.

Case 3. $(2a+1, 2b, 0)$. We ask a question $[x, y, 0]$? and obtain states

$$\text{YES} = (x, 2a+1-x+y, 2b-y),$$
$$\text{NO} = (2a+1-x, x+2b-y, y).$$

As above,

$$w_k(\text{YES}) - w_k(\text{NO})$$
$$= k \cdot (k-1) \cdot (x-a) + 2k \cdot (y-b) - \frac{k \cdot (k-1)}{2}.$$

Now there are four subcases:

Subcase 3.1. $k = 4l$. In this case

$$w_k(\text{YES}) - w_k(\text{NO}) = k \cdot \left((k-1) \cdot (x-a) + 2 \cdot (y-b) - \frac{k-1}{2} \right)$$

which shows that this difference is not zero. Namely, the expression in brackets is not an integer. On the other hand, the difference of weights is divisible by $2l$. We can thus obtain the best possible balance asking a question $[x, y, 0]$?, where $x = a$ and $y = b + l$. Then we obtain states

$$(a, a+1+b+l, b-l) \qquad \text{and} \qquad (a+1, a+b-l, b+l)$$

such that

$$w_k(a, a+1+b+l, b-l) = w_k(a+1, a+b-l, b+l) + k/2.$$

The resulting states are typical if $b - l \geqslant k$. In particular, they are typical if $b \geqslant (3k-2)/2$.

Subcase 3.2. $k=4l+2$. As above we show that the difference of weights is not zero and is divisible by $k/2$. The best possible balance is obtained asking a question $[a, b+l, 0]$? and as above the difference of weights of the resulting states is $k/2$. Under the same assumption as above the resulting states are typical.

Subcase 3.3. $k=4l+1$. We ask a question $[a, b+l, 0]$? to obtain typical states of equal weights under the same assumptions.

Subcase 2.4. $k=4l+3$. It is easy to see that in this case the best possible balance is obtained with a question $[a, b+l+1, 0]$? and the difference of weights of resulting states is k. Again the resulting states are typical under the above assumptions.

Case 4. $(2a+1, 2b+1, 0)$. Similarly as in the above cases we consider a question $[x, y, 0]$? and the difference of weights of the resulting states is in this case

$$k \cdot (k-1) \cdot (x-a) + k \cdot (2y-2b-1) - \frac{k^2-k}{2}.$$

Again there are four subcases.

Subcase 4.1. $k=4l$. The best possible balance is obtained with $x=a$ and $y=b+l$. The difference of weights will be $k/2$.

Subcase 4.2. $k=4l+2$. In this case we let $x=a$ and $y=b+l+1$ to get the same result as in the above subcase.

Subcase 4.3. $k=4l+1$. Now $x=a$ and $y=b+l+1$ give two states with the difference of weights k.

Subcase 4.4. $k=4l+3$. Two states of equal weights are obtained with $x=a$ and $y=b+l+1$. In the above four subcases the resulting states are typical under the assumption $b \geqslant (3k-2)/2$.

6. The Main Theorem

We will formulate now the main result of the paper.

THEOREM 6.1. *Let m be an integer such that* $\mathrm{ch}(m, 0, 0) = k+2 \geqslant 14$. *Then there exists a winning strategy for the Questioner in the game of $k+2$ questions iff the following conditions hold*:

Case 1. $m=4n$. *Then* $m \cdot F(k+2) \leqslant 2^{k+2}$.

Case 2. $m=4n+1$. *Then*:

if $k=4l$ *then* $n \cdot F(k)+(2n+l+1) \cdot (k+1)+n-l \leqslant 2^k$,
if $k=4l+1$ *then* $(2n+1) \cdot F(k+1)+2n \cdot (k+2) \leqslant 2^{k+1}$,
if $k=4l+2$ *then* $(n+1) \cdot F(k)+(2n-l) \cdot (k+1)+n+l \leqslant 2^k$,
if $k=4l+3$ *then* $n \cdot F(k)+(2n+l+2) \cdot (k+1)+n-l-1 \leqslant 2^k$.

Case 3. $m=4n+2$. *Then*:

if $k=4l$ *then* $(n+1) \cdot F(k)+(2n+1-l) \cdot (k+1)+n+l \leqslant 2^k$,
if $k=4l+3$ *then* $m \cdot F(k+2) \leqslant 2^{k+2}$,
otherwise $n \cdot F(k)+(2n+l+2) \cdot (k+1)+n-l \leqslant 2^k$.

Case 4. $m=4n+3$. *Then*:

if $k=2l$ *then* $(2n+2) \cdot F(k+1)+(2n+1) \cdot (k+2) \leqslant 2^{k+1}$,
if $k=2l+1$ *then* $(n+1) \cdot F(k)+(2n+2) \cdot (k+1)+n \leqslant 2^k$.

Proof. An easy induction argument shows that if the integer m satisfies the hypotheses of the theorem, then the states $(a, b, 0)$ obtained after the first splitting satisfy the condition $b \geqslant (3k-2)/2$, sufficient for the next question to lead to typical states. Therefore we can apply the results of the previous section. ∎

COROLLARY 6.2. *Let* n *be an integer and* $\mathrm{ch}(n, 0,, 0)=k$. *Then the Questioner has a winning strategy in the game of* $k+1$ *questions.*

Proof. Follows directly from the proof of the Theorem 6.1. ∎

The question arises about the number of questions in the best Questioner's strategy in the games with the character less than 14 at the beginning, i.e., for the numbers n less than 90. The computer caılculations of the function $M(a, b)$ of Section 3 show that except for the numbers 3, 4, 5, 6, 9, 10, 11, 17, 18, 29, 30, 51, 89, all other numbers are nice, i.e., the best strategy requires $\mathrm{ch}(n, 0, 0)$ questions. The exceptional numbers require one more question.

Our proof shows also an algorithm of asking questions for states of character $\geqslant 14$, the states of smaller character should be treated individually. One should compare the relative simplicity of the case of one lie with much bigger complexity of the proof of Theorem 6.1. The main reason is the existence of the exceptions, one of them—the number 4—already found in [Cz-M-P2]. In fact, the exceptional role of the state

$(1, 2, 1)$ which caused the state $(4, 0, 0)$ to be "nasty" was also the reason for other states to be "nasty." Several states could not be balanced because all questions led to the nasty state $(1, 2, 1)$, others led to these states and so on. The nasty states created new ones, the last nasty state being $(25, 30, c)$.

It would be interesting to see if the same situation arises in other search games. Recently A. Pelc used the method of considering typical states (which he calls normal) to analyse a search of a counterfeit coin game [P3]. His result that every normal state is nice in one-lie game does not prejudge whether in two-lies game similar exceptions would appear, though A. Pelc rather expects serious technical difficulties in the proof.

Appendix

The list of typical states of character $\leqslant 13$ which are not nice:

ch = 6	$(1, 5, c)$	for	$6 \leqslant c \leqslant 7$
	$(2, 1, c)$	for	$6 \leqslant c \leqslant 13$
	$(2, 2, 6)$		
ch = 7	$(2, 7, c)$	for	$7 \leqslant c \leqslant 14$
	$(3, 2, c)$	for	$7 \leqslant c \leqslant 25$
	$(3, 3, c)$	for	$7 \leqslant c \leqslant 17$
	$(3, 4, c)$	for	$7 \leqslant c \leqslant 9$
ch = 8	$(3, 15, c)$	for	$8 \leqslant c \leqslant 10$
	$(4, 9, c)$	for	$8 \leqslant c \leqslant 27$
	$(4, 10, c)$	for	$8 \leqslant c \leqslant 18$
	$(4, 11, c)$	for	$8 \leqslant c \leqslant 9$
	$(5, 4, c)$	for	$8 \leqslant c \leqslant 35$
	$(5, 5, c)$	for	$8 \leqslant c \leqslant 26$
	$(5, 6, c)$	for	$8 \leqslant c \leqslant 17$
	$(5, 7, 8)$		
ch = 9	$(7, 16, c)$	for	$9 \leqslant c \leqslant 30$
	$(7, 17, c)$	for	$9 \leqslant c \leqslant 20$
	$(7, 18, c)$	for	$9 \leqslant c \leqslant 10$
	$(8, 9, c)$	for	$9 \leqslant c \leqslant 54$
	$(8, 10, c)$	for	$9 \leqslant c \leqslant 44$
	$(8, 11, c)$	for	$9 \leqslant c \leqslant 34$
	$(8, 12, c)$	for	$9 \leqslant c \leqslant 24$
	$(8, 13, c)$	for	$9 \leqslant c \leqslant 14$
	$(9, 8, c)$	for	$9 \leqslant c \leqslant 18$

ch = 10	(13, 25, c)	for	$10 \leqslant c \leqslant 21$
	(13, 26, 10)		
	(14, 17, c)	for	$10 \leqslant c \leqslant 53$
	(14, 18, c)	for	$10 \leqslant c \leqslant 42$
	(14, 19, c)	for	$10 \leqslant c \leqslant 31$
	(14, 20, c)	for	$10 \leqslant c \leqslant 20$
	(15, 14, c)	for	$10 \leqslant c \leqslant 30$
	(15, 15, c)	for	$10 \leqslant c \leqslant 19$
ch = 11	(25, 30, c)	for	$11 \leqslant c \leqslant 13$

REFERENCES

[B] E. R. BERLEKAMP, Block coding for the binary symmetric channel with noiseless, delayless feedback, *in* "Error-Correcting Codes," pp. 61–85, Wiley, New York, 1968.

[Cz-M-P1] J. CZYZOWICZ, D. MUNDICI, AND A. PELC, Solution of Ulam's problem on binary search with two lies, *J. Combin. Theory Ser. A* **49**, No. 2 (1988), 384–388.

[Cz-M-P2] J. CZYZOWICZ, D. MUNDICI, AND A. PELC, Ulam's searching game with lies, *J. Combin. Theory Ser. A*, in press.

[P1] A. PELC, Solution of Ulam's problem on searching with a lie, *J. Combin. Theory Ser. A* **44**, No. 1 (1987), 129–140.

[P2] A. PELC, Searching with known error probability, *Theoret. Comput. Sci.* **63** (1989), 185–202.

[P3] A. PELC, Detecting errors in searching games, *J. Combin. Theory Ser. A* **51**, No. 1 (1989), 43–54.

[P4] A. PELC, Detecting a counterfeit coin with unreliable weighings, *Ars Combin.* **27** (1989), 181–192.

[R *et al.*] R. L. RIVEST *et al.*, Coping with errors in binary search procedures, *J. Comput. System Sci.* **20** (1980), 396–404.

[S] J. SPENCER, Guess a number—With lying, *Math. Mag.* **57**, No. 2 (1984), 105–108.

[U] S. M. ULAM, "Adventures of a Mathematicial," Scribner, New York, 1976.