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Ulam's Searching Game with Three Lies

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In this paper we determine the minimal number of yes–no queries needed to find an unknown integer between 1 and 2^m if at most three of the answers may be erroneous.

1. INTRODUCTION

S. M. Ulam in his "Adventures of a Mathematician" [U] raised the following question:

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half-million? and then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2(1000000)$. Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer? One clearly needs more than n questions for guessing one of the 2^n objects because one does not know when the lie was told. This problem is not solved in general.

Several answers to Ulam's problem and its generalizations are given in the literature.

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A complete solution for Ulam's problem with one lie was obtained by Pelc [P]. His main theorem states:

—for even n, q questions are sufficient to determine an unknown integer between 1 and n iff $n(q+1) \leq 2^q$;

—for odd n, q questions are sufficient to determine an unknown integer between 1 and n iff $n(q+1) + (q-1) \leq 2^q$.

Solutions for the problem with two lies can be found in [C1, C2], and [G]. In [C1] the authors show that 29 questions are sufficient to determine the unknown element in $\{1, \dots, 1000000\}$.

In [C2] it is shown that if the unknown element is in the set $\{1, \dots, 2^n\}$ and the Responder may lie at most twice, then q queries are sufficient to find it iff

$$2^n \cdot \frac{q^2 + q + 2}{2} \leq 2^q,$$

with $n \geq 3$. The cases $n = 1, 2$ are also considered in that paper.

In [G], more generally, it is shown that if the unknown element is in the set $\{1, \dots, n\}$, then q queries are sufficient to find it, iff

$$n \cdot \frac{q^2 + q + 2}{2} \leq 2^q.$$

In the paper the author shows that one more question is necessary only for $n = 3, 4, 5, 6, 9, 10, 11, 17, 18, 29, 30, 51, 89$.

In [N], the case of three lies was solved for the search space $\{1, 2, \dots, 1000000\}$. In that paper the authors show that 33 questions are necessary and sufficient to determine the unknown element.

In the present paper we give a solution of Ulam's problem for three lies, when the search space has size 2^m. We show that, for $m = 1$, $m = 4$, and $m \geq 6$, the unknown number can be determined in q questions, where q is the well-known Hamming's bound [V] which was given as the lower bound for Ulam's problem by [Br, R]:

$$q = \min p : 2^m \cdot \sum_{i=0}^{3} \binom{p}{i} \leq 2^p.$$

For the three exceptional cases $m = 2, 3, 5$, we prove that the minimal number of questions is given by the above value of q increased by one.

Our techniques are quite similar to those developed in [P], [C2], and [G], but as pointed out by Kleitman [K] the proofs in this case are more complicated.

2. Definitions and Notations

The game is played by two players: the Questioner and the Responder. The Responder chooses an element e in the search space $\{1, \ldots, 2^m\}$ unknown to the Questioner, who has to find it out by means of q queries of the form "does $e \in Q$?," where Q is an arbitrary subset of the search space. The Responder may lie at most three times. If the Questioner has a winning strategy to determine the hidden number in q queries independently of the Responder's answering strategy, we say that he wins the game in q questions. Our interest is focused on the minimal q, necessary to the Questioner to win the game, for a given m. Let e be an arbitrary element in the search space. Each stage of the game is associated with a *state*, i.e., a quadruple (a, b, c, d) of natural numbers, where a, b, c, and d are the numbers of elements of subsets A, B, C, and D, respectively, defined as

— $e \in A$ iff e does not falsify any of the previous answers;
— $e \in B$ iff e falsifies exactly one of the previous answers;
— $e \in C$ iff e falsifies exactly two of the previous answers;
— $e \in D$ iff e falsifies exactly three of the previous answers.

At the beginning of the game the state is given by $(2^m, 0, 0, 0)$.

Definition 1 [B]. Let (a, b, c, d) be a state when q questions remain to be asked. The *weight* of this state is

$$w_q(a, b, c, d) = a\left(\binom{q}{3}\right) + b\left(\binom{q}{2}\right) + c\left(\binom{q}{1}\right) + d\left(\binom{q}{0}\right),$$

where $\left(\binom{n}{m}\right) = \sum_{i=0}^{m} \binom{n}{i}$.

Let (a, b, c, d) be a state, and X, Y, Z, K be subsets of A, B, C, and D with cardinalities x, y, z, k, respectively. From the query "does $e \in X \cup Y \cup Z \cup K$?," in symbols $[x, y, z, k]$?, one obtains two new states $YES_{a,b,c,d}(x, y, z, k)$ and $NO_{a,b,c,d}(x, y, z, k)$, according as the answer is positive or negative, respectively. From the definition of state and weight, it follows that

$$YES_{a,b,c,d}(x, y, z, k) = (x, a - x + y, b - y + z, c - z + k),$$
$$NO_{a,b,c,d}(x, y, z, k) = (a - x, x + b - y, y + c - z, z + d - k)$$

and

$$w_j(a, b, c, d) = w_{j-1}(YES_{a,b,c,d}(x, y, z, k)) + w_{j-1}(NO_{a,b,c,d}(x, y, z, k))$$

for every $j \geq 1$.

DEFINITION 2. For any state (a, b, c, d), the number $\mathrm{ch}(a, b, c, d)$, called the *character* of this state, is defined as

$$\mathrm{ch}(a, b, c, d) = \min\{q: w_q(a, b, c, d) \leq 2^q\}.$$

DEFINITION 3. State (a, b, c, d) is called *nice* iff the Questioner wins in $\mathrm{ch}(a, b, c, d)$ questions starting from this state.

DEFINITION 4. State (a, b, c, d) is called *balanced* iff there exists a query $[x, y, z, k]?$, such that

$$|w_{q-1}(YES_{a,b,c,d}(x, y, z, k)) - w_{q-1}(NO_{a,b,c,d}(x, y, z, k))| \leq 1,$$

with $q = \mathrm{ch}(a, b, c, d)$. Any such query $[x, y, z, k]?$ is said to *balance* state (a, b, c, d).

From this definition it follows that if a query $[x, y, z, k]?$ balances the state (a, b, c, d) and $\mathrm{ch}(a, b, c, d) = q$, then

$$\mathrm{ch}(YES_{a,b,c,d}(x, y, z, k)) \leq q - 1$$

and $\mathrm{ch}(NO_{a,b,c,d}(x, y, z, k)) \leq q - 1$.

DEFINITION 5 [G]. Let $\mathrm{ch}(a, b, c, d) = q$. The state (a, b, c, d) is called *0-typical*, iff it satisfies the following three conditions:

(i) $a = 0$;

(ii) $c \geq b - 1$;

(iii) $d \geq q$.

3. MAIN RESULT

THEOREM. *Let $\mathcal{N} = \mathbb{N} \setminus \{2, 3, 5\}$. Then for each $m \in \mathcal{N}$ the state $(1, m, \binom{m}{2}, \binom{m}{3})$ is nice.*

States $m = 2$, $m = 3$, and $m = 5$ are not nice and will be considered in Section 5, where it will be shown that the minimal number of questions is given by the previous lower bound increased by one. As a corollary of the theorem and of the results of Section 5, the main result of the paper easily follows:

COROLLARY. *Let $m \in \mathcal{N}$, and $L = \min l$: $2^m((\binom{l}{3})) \leq 2^l$, then L queries are necessary and sufficient to find a number in the search space $\{1, 2, \ldots, 2^m\}$ when up to three lies are allowed. When $m = 2, 3$, or 5, the minimal number of queries is $L + 1$.*

Proof of Corollary. The case $m = 2, 3, 5$ is considered in Section 5. As for the remaining cases, note that starting from state $(2^m, 0, 0, 0)$, m consecutive questions can be asked in such a way that the state given by a positive answer is equal to the state given by a negative answer. One might, for instance, let the ith question ask "Is the ith binary digit equal to one?." After the first i questions, the state will be given by

$$\left(2^{m-i}, (i-1)2^{m-i}, \binom{i-1}{2}2^{m-i}, \binom{i-1}{3}2^{m-i}\right)$$

and the query

$$\left[2^{m-i-1}, i2^{m-i-1}, \binom{i}{2}2^{m-i-1}, \binom{i}{3}2^{m-i-1}\right]?$$

yields two equal states

$$\left(2^{m-i-1}, (i+1)2^{m-i-1}, \binom{i+1}{2}2^{m-i-1}, \binom{i+1}{3}2^{m-i-1}\right).$$

After the first m questions, the state $(1, m, \binom{m}{2}, \binom{m}{3})$ is reached. If

$$q = ch\left(1, m, \binom{m}{2}, \binom{m}{3}\right),$$

the Questioner needs $L = q + m$ questions to win the game. By definition of the character of a state, L is the least natural number l such that $2^m((\binom{l}{3})) \le 2^l$.
Hence, $L = \mathrm{ch}(2^m, 0, 0, 0)$, and from the main theorem it follows that the state $(2^m, 0, 0, 0)$ is nice. The fact that at least L questions are needed follows from the lower bound [R]. □

In the sequel we shall naturally identify the state (b, c, d) for the two-lies problem with the state $(0, b, c, d)$ for the three-lies problem.

TABLE I
Case $m = 1$, after the First Question

Case	State	Ch	Query	State-YES	State-NO
$m = 1$	$(1,1,0,0)$	6	$[1,0,0,0]?$	$(1,0,1,0)$	$(0,2,0,0)$
	$(1,0,1,0)$	5	$[1,0,0,0]?$	$(1,0,0,1)$[a]	$(0,1,1,0)$
	$(0,2,0,0)$	5	$[0,1,0,0]?$	$(0,1,1,0)$	$(0,1,1,0)$
	$(0,1,1,0)$	4	$[0,1,0,0]?$	$(0,1,0,1)$[b]	$(0,0,2,0)$[c]

[a]This state is nice by Lemma 4.
[b]This state is nice by Lemma 1.
[c]This state is nice because it corresponds to the state $(2, 0)$ for the case with one lie, which is known to be nice [P].

TABLE II
Case $m = 4$, after the First Four Questions

Case	State	Ch	Query	State-YES	State-NO
$m = 4$	$(1,4,6,4)$	9	$[1,1,3,2]?$	$(1,1,6,5)$	$(0,4,4,5)$
	$(1,1,6,5)$	8	$[1,0,2,3]?$	$(1,0,3,7)$[a]	$(0,2,4,4)$
	$(0,4,4,5)$	8	$[0,2,2,3]?$	$(0,2,4,5)$	$(0,2,4,4)$
	$(0,2,4,4)$	7	$[0,1,2,2]?$	$(0,1,3,4)$	$(0,1,3,4)$
	$(0,2,4,5)$	7	$[0,1,2,3]?$	$(0,1,3,5)$	$(0,1,3,4)$
	$(0,1,3,4)$	6	$[0,1,0,4]?$	$(0,1,0,7)$[b]	$(0,0,4,0)$[c]
	$(0,1,3,5)$	6	$[0,1,0,5]?$	$(0,1,0,8)$[b]	$(0,0,4,0)$[c]

[a]This state is nice by Lemma 4.
[b]This state is nice by Lemma 1.
[c]This state is nice because it corresponds to the state $(4,0)$ for the one lie case, which is known to be nice [P].

Proof of Theorem. Searching strategies for $m = 1$ and $m = 4$ are described in Tables I and II, respectively. These strategies are optimal, as a consequence of Lemmas 1 and 4 below, together with main result of [P]. The case $m \geq 6$ will be taken care of by Lemmas 1–7. □

LEMMA 1. *The state $(0,1,0,n)$ is nice for any natural number n.*

Proof. [C2, Lemma 3]. □

LEMMA 2. *Each 0-typical state (a,b,c,d) with $\mathrm{ch}(a,b,c,d) \geq 12$ is nice.*

Proof. [G, Theorem 3.9]. □

LEMMA 3. *All 0-typical states (a,b,c,d) with $\mathrm{ch}(a,b,c,d) \leq 12$ are nice except those listed in Table VI.*

Proof. [G, Appendix]. □

LEMMA 4. *The state $(1,0,0,n)$ is nice for any natural number n.*

Proof. We may assume $n \geq 1$. Let $\mathrm{ch}(1,0,0,n) = q$. We prove the lemma by induction on n. The induction basis is trivial. Suppose that the result holds for any $m \leq n$. Then $YES_{1,0,0,n}(1,0,0,0) = (1,0,0,0)$ and $NO_{1,0,0,n}(1,0,0,0) = (0,1,0,n)$. Two cases are possible:

1. $w_{q-1}(1,0,0,0) \geq w_{q-1}(0,1,0,n)$. In this case,

$$w_{q-1}(0,1,0,n) \leq 2^{q-1}.$$

The state $(1,0,0,0)$ is a winning state for the Questioner, and the state $(0,1,0,n)$ is nice by Lemma 1. In any case, the Questioner wins with $q - 1$

more questions. It follows that the total number of questions is $\mathrm{ch}(1,0,0,n)$, and hence the state $(1,0,0,n)$ is nice.

2. $w_{q-1}(1,0,0,0) \le w_{q-1}(0,1,0,n)$. It is easy to prove that $(1,0,0,n)$ is balanced. In fact, the query $[1,0,0,x]$? yields the states $(1,0,0,x)$ and $(0,1,0,n-x)$. If $x=0$, then $w_{q-1}(1,0,0,0) \le w_{q-1}(0,1,0,n)$, if $x=n$, then $w_{q-1}(1,0,0,n) > w_{q-1}(0,1,0,0)$. Therefore, there will exist an x, with $0 \le x \le n$ such that

$$|w_{q-1}(YES_{1,0,0,n}(1,0,0,x)) - w_{q-1}(NO_{1,0,0,n}(1,0,0,x))| \le 1,$$

$$\mathrm{ch}(YES_{1,0,0,n}(1,0,0,x)) \le q-1 \quad \text{and} \quad \mathrm{ch}(NO_{1,0,0,n}(1,0,0,x)) \le q-1.$$

Moreover, the state $(1,0,0,x)$ is nice by the induction hypothesis, and the state $(0,1,0,n-x)$ is nice by Lemma 1. Therefore, the Questioner starting form $(1,0,0,n)$ wins with q questions. □

LEMMA 5. *The state* $(1,0,3,n)$ *is nice for any natural number* $n \ge 7$.

Proof. For $7 \le n \le 9$ the proof follows from Table III.

For the case $n \ge 10$, let $\mathrm{ch}(1,0,3,n) = q+1$. If we ask the question $[1,0,0,x]$? in state $(1,0,3,n)$, the resulting states are

$$YES_{1,0,3,n}(1,0,0,x) = (1,0,0,3+x)$$

and $NO_{1,0,3,n}(1,0,0,x) = (0,1,3,n-x)$. Then

$$\left|w_q(YES_{1,0,3,n}(1,0,0,x)) - w_q(NO_{1,0,3,n}(1,0,0,x))\right|$$
$$= \left|\binom{q}{3} - 3q - n + 2x\right|.$$

TABLE III
First Part of the Proof of Lemma 5

Case	State	Ch	Query	State-YES	State-NO
$n = 7$	$(1,0,3,7)$	7	$[1,0,0,2]$?	$(1,0,0,5)$[a]	$(0,1,3,5)$
	$(0,1,3,5)$	6	$[0,1,0,5]$?	$(0,1,0,8)$[b]	$(0,0,4,0)$[c]
$m = 8$	$(1,0,3,8)$	7	$[1,0,0,3]$?	$(1,0,0,6)$[a]	$(0,1,3,5)$
	$(0,1,3,5)$	6	$[0,1,0,5]$?	$(0,1,0,8)$[b]	$(0,0,4,0)$[c]
$m = 9$	$(1,0,3,9)$	7	$[1,0,0,4]$?	$(1,0,0,7)$[a]	$(0,1,3,5)$
	$(0,1,3,5)$	6	$[0,1,0,5]$?	$(0,1,0,8)$[b]	$(0,0,4,0)$[c]

[a]These states are nice by Lemma 4.
[b]These states are nice by Lemma 1.
[c]This state is nice, because it corresponds to state $(4,0)$ in the case of one lie, which is known to be nice [P].

If we set $x = \lfloor (n + 3q - \binom{q}{3})/2 \rfloor$ (where as usual, $\lfloor x \rfloor$ is the greatest integer $\leq x$), then

$$\left| w_q(YES_{1,0,3,n}(1,0,0,x)) - w_q(NO_{1,0,3,n}(1,0,0,x)) \right| \leq 1.$$

In order to prove that $[1,0,0,x]$? is an admissible question we have to check that $0 \leq x \leq n$. For $n = 10$, we have $q = 6$ and then

$$n + 3q - \binom{q}{3} \geq 0;$$

hence $x \geq 0$, because q, as function of n, grows slower than n. The second inequality is proved analogously. Hence the state $(1,0,3,n)$ is balanced by the query $[1,0,0,x]$?.

Let us now consider states

$$YES_{1,0,3,n}(1,0,0,x) = (1,0,0,3+x)$$

and

$$NO_{1,0,3,n}(1,0,0,x) = (0,1,3,n-x).$$

The first one is nice by Lemma 4. The second state is 0-typical. In fact, conditions (i) and (ii) of Definition 5 are clearly satisfied. Condition (iii) holds, too, since $n - x \geq q$ for $n = 10$, and n grows slower than n.

For every n such that $q \geq 12$, the state $(0,1,3,n-x)$ is nice by Lemma 2. For every n such that $q < 12$ the state is nice by Lemma 3.

Following the above search strategy, from state $(1,0,3,n)$ we obtain the states $YES_{1,0,3,n} = (1,0,0,3+x)$ and $NO_{1,0,3,n} = (0,1,3,n-x)$. Since both states are nice, then so is $(1,0,3,n)$. □

LEMMA 6. *The state* $(1, m, \binom{m}{2}, \binom{m}{3})$ *is nice, for every* m *with* $6 \leq m \leq 32$.

Proof. First note that $\mathrm{ch}(1, m, \binom{m}{2}, \binom{m}{3})$ equals 10 for $6 \leq m \leq 8$, 11 for $9 \leq m \leq 12$, 12 for $13 \leq m \leq 17$, 13 for $18 \leq m \leq 23$, and 14 for $24 \leq m \leq 32$. It suffices to prove the lemma for $m = 8, 12, 17, 23, 32$, since whenever $a_1 \leq a$, $b_1 \leq b$, $c_1 \leq c$, $d_1 \leq d$, the minimal number of questions necessary to win the game beginning from the state (a,b,c,d) trivially cannot be less than for state (a_1, b_1, c_1, d_1).

As shown in Table IV, analysis of the first three levels of the search tree is sufficient to prove the lemma. □

LEMMA 7. *The state* $(1, m, \binom{m}{2}, \binom{m}{3})$ *is nice for all* $m \geq 33$.

Proof. For $m \geq 33$ we have

$$\tfrac{9}{16}m^3 + \tfrac{15}{12}m + 1 \leq 2^{m/2},$$

TABLE IV
A Proof of Lemma 6

Case	State	Ch	Query	State-YES	State-No
$m = 8$	$(1, 8, 28, 56)$	10	$[1, 4, 10, 22]$?	$(1, 4, 14, 40)$	$(0, 5, 22, 44)^a$
	$(1, 4, 14, 40)$	9	$[1, 1, 5, 36]$?	$(1, 1, 8, 45)$	$(0, 4, 10, 9)^a$
	$(1, 1, 8, 45)$	8	$[1, 0, 2, 29]$?	$(1, 0, 3, 35)^b$	$(0, 2, 6, 18)^a$
$m = 12$	$(1, 12, 66, 220)$	11	$[1, 6, 27, 110]$?	$(1, 6, 33, 149)$	$(0, 7, 45, 137)^a$
	$(1, 6, 33, 149)$	10	$[1, 1, 13, 136]$?	$(1, 1, 18, 156)$	$(0, 6, 21, 26)^a$
	$(1, 1, 18, 156)$	9	$[1, 0, 2, 120]$?	$(1, 0, 3, 136)^b$	$(0, 2, 16, 38)^a$
$m = 17$	$(1, 17, 136, 680)$	12	$[1, 8, 60, 373]$?	$(1, 8, 69, 449)$	$(0, 10, 84, 367)^a$
	$(1, 8, 69, 449)$	11	$[1, 1, 30, 344]$?	$(1, 1, 37, 383)$	$(0, 8, 40, 135)^a$
	$(1, 1, 37, 383)$	10	$[1, 0, 2, 316]$?	$(1, 0, 3, 351)^b$	$(0, 2, 35, 69)^a$
$m = 23$	$(1, 23, 253, 1771)$	13	$[1, 11, 115, 946]$?	$(1, 11, 127, 1084)$	$(0, 13, 149, 940)^c$
	$(1, 11, 127, 1084)$	12	$[1, 1, 58, 767]$?	$(1, 1, 68, 836)$	$(0, 11, 70, 375)^c$
	$(1, 1, 68, 836)$	11	$[1, 0, 2, 700]$?	$(1, 0, 3, 766)^b$	$(0, 2, 66, 138)^a$
$m = 32$	$(1, 32, 496, 4960)$	14	$[1, 16, 232, 2545]$?	$(1, 16, 248, 2809)$	$(0, 17, 280, 2647)^c$
	$(1, 16, 248, 2809)$	13	$[1, 1, 116, 1852]$?	$(1, 1, 131, 1984)$	$(0, 16, 133, 1073)^c$
	$(1, 1, 131, 1984)$	12	$[1, 0, 2, 1635]$?	$(1, 0, 3, 1764)^b$	$(0, 2, 129, 351)^a$

[a]This state is nice by Lemma 3.
[b]This state is nice by Lemma 5.
[c]This state is nice by Lemma 2.

this inequality is equivalent to

$$\frac{(m/2)^3 + 5(m/2) + 6}{6} + m\frac{(m/2)^2 + (m/2) + 2}{2} + \binom{m}{2}\left(\frac{m}{2} + 1\right) + \binom{m}{3} \leq 2^{m/2}$$

which implies

$$\operatorname{ch}\left(1, m, \binom{m}{2}, \binom{m}{3}\right) \leq m/2.$$

Our optimal searching strategy will be a generalization of the strategies given in Table IV for the proof of Lemma 6. We show how the $(m + 1)$th, $(m + 2)$th, and $(m + 3)$th questions should be asked in order to invariably obtain nice states from state $(1, m, \binom{m}{2}, \binom{m}{3})$. We use the abbreviations $(1, b_0, c_0, d_0)$ instead of $(1, m, \binom{m}{2}, \binom{m}{3})$; using Table V, $(1, b_1, c_1, d_1)$ and $(1, 1, c_2, d_2)$ will respectively denote the states obtained after a positive answer to the first and second question; also, for each $i = 1, 2, 3$, we let states YES_i and NO_i follow from the positive or negative answer to the ith question, respectively. We let the integer q be defined by

$$q = \operatorname{ch}\left(1, m, \binom{m}{2}, \binom{m}{3}\right) - 2.$$

Depending on the values of x, y, z, our six states can be concisely written as

$$YES_1 = \left(1, \left\lfloor \frac{b_0}{2} \right\rfloor, b_0 - 2\left\lfloor \frac{b_0}{2} \right\rfloor + \left\lfloor \frac{c_0}{2} \right\rfloor, c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor + \left\lfloor \frac{b_0}{2} \right\rfloor + x\right),$$

$$NO_1 = \left(0, b_0 + 1 - \left\lfloor \frac{b_0}{2} \right\rfloor, 2\left\lfloor \frac{b_0}{2} \right\rfloor + c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor, \left\lfloor \frac{c_0}{2} \right\rfloor - \left\lfloor \frac{b_0}{2} \right\rfloor + d_0 - x\right),$$

$$YES_2 = \left(1, 1, b_1 - 1 + \left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor, c_1 - \left\lfloor \frac{c_1}{2} \right\rfloor + \left\lfloor \frac{b_1}{2} \right\rfloor + y\right),$$

$$NO_2 = \left(0, b_1, 1 + c_1 - \left\lfloor \frac{c_1}{2} \right\rfloor + \left\lfloor \frac{b_1}{2} \right\rfloor, \left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor + d_1 - y\right),$$

$$YES_3 = (1, 0, 3, c_2 - 2 + z),$$

$$NO_3 = (0, 2, c_2 - 2, 2 + d_2 - z).$$

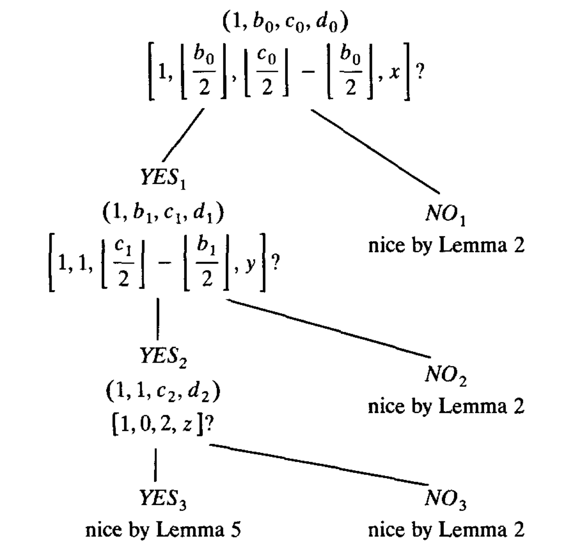
The actual values of x, y, z yielding our optimal strategies are given by

$$x = \left\lfloor \frac{d_0 + 2\left\lfloor \frac{c_0}{2} \right\rfloor - c_0 - 2\left\lfloor \frac{b_0}{2} \right\rfloor - \left(\binom{q+1}{3}\right) + \left(\binom{q+1}{2}\right)\left(b_0 + 1 - 2\left\lfloor \frac{b_0}{2} \right\rfloor\right) + (q+2)\left(c_0 + 4\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor\right)}{2} \right\rfloor,$$

$$y = \left\lfloor \frac{(b_1 - 1)\left(\binom{q}{2}\right) - \left(\binom{q}{3}\right) + (q+1)\left(c_1 + 2 - b_1 + 2\left\lfloor \frac{b_1}{2} \right\rfloor - 2\left\lfloor \frac{c_1}{2} \right\rfloor\right) + d_1 - c_1 + 2\left\lfloor \frac{c_1}{2} \right\rfloor - 2\left\lfloor \frac{b_1}{2} \right\rfloor}{2} \right\rfloor,$$

$$z = \left\lfloor \frac{d_2 + 4 - c_2 - \left(\binom{q-1}{3}\right) + 2\left(\binom{q-1}{2}\right) + q(c_2 - 5)}{2} \right\rfloor.$$

TABLE V
Sketch of Proof of Lemma 7



As a matter of fact, in Section 4 we prove the following results:

— *Sublemma* 7.1. The above values of x, y, and z are *admissible*, in the sense that $0 \le x \le d_0$, $0 \le y \le d_1$, and $0 \le z \le d_2$. In addition, for these values of x, y, and z, the three questions of Table V balance their corresponding states.

— *Sublemma* 7.2. States NO_1, NO_2, and NO_3 are nice.

— *Sublemma* 7.3. State YES_3 is nice.

From the three sublemmas, the proof of Lemma 7 immediately follows (again, see Table V). After the proof of Lemma 7, the proof of our main theorem is complete. □

Remark. In the particular case when the search space is the set $\{0, 1, \ldots, 2^{20} - 1\}$, the minimal number of queries is

$$\min q \colon \left(\binom{q}{3}\right) 2^{20} \le 2^q = 33.$$

Since $2^{19} < 10^6 < 2^{20}$, this answers Ulam's problem for the case of three lies. In [N], this result was found using a different technique.

4. Proof of Sublemmas 7.1–7.3

The proof will proceed through Claims 1–7 below. By definition, the number q is given by $\mathrm{ch}(1, b_0, c_0, d_0) = q + 2$, where b_0, c_0, and d_0 are as above. The query

$$\left[1, \left\lfloor \frac{b_0}{2} \right\rfloor, \left\lfloor \frac{c_0}{2} \right\rfloor - \left\lfloor \frac{b_0}{2} \right\rfloor, x\right]? \tag{Q1}$$

yields states

$$YES_1 = \left(1, \left\lfloor \frac{b_0}{2} \right\rfloor, b_0 - 2\left\lfloor \frac{b_0}{2} \right\rfloor + \left\lfloor \frac{c_0}{2} \right\rfloor, c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor + \left\lfloor \frac{b_0}{2} \right\rfloor + x\right),$$

$$NO_1 = \left(0, b_0 + 1 - \left\lfloor \frac{b_0}{2} \right\rfloor, 2\left\lfloor \frac{b_0}{2} \right\rfloor + c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor, \left\lfloor \frac{c_0}{2} \right\rfloor - \left\lfloor \frac{b_0}{2} \right\rfloor + d_0 - x\right).$$

An easy computation shows that

$$\begin{aligned}
&\left| w_{q+1}(YES_1) - w_{q+1}(NO_1) \right| \\
&= \Bigg| \left(\binom{q+1}{3} \right) + \left(\binom{q+1}{2} \right)\left(2\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 1\right) \\
&\quad + (q+2)\left(b_0 - 4\left\lfloor \frac{b_0}{2} \right\rfloor + 2\left\lfloor \frac{c_0}{2} \right\rfloor - c_0\right) \\
&\quad + c_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor + 2\left\lfloor \frac{b_0}{2} \right\rfloor - d_0 + 2x \Bigg|.
\end{aligned}$$

As in the previous section, defining x by

$$x = \left\lfloor \frac{\begin{array}{r} d_0 + 2\left\lfloor \frac{c_0}{2} \right\rfloor - c_0 - 2\left\lfloor \frac{b_0}{2} \right\rfloor - \left(\binom{q+1}{3} \right) + \left(\binom{q+1}{2} \right)\left(b_0 + 1 - 2\left\lfloor \frac{b_0}{2} \right\rfloor\right) \\ + (q+2)\left(c_0 + 4\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor\right) \end{array}}{2} \right\rfloor, \tag{$*$}$$

it follows that $|w_{q+1}(YES_1) - w_{q+1}(NO_1)| \leq 1$.

Claim 1. *With x as given by $(*)$, query* (Q1) *is admissible.*

Proof. We have to prove that $0 \leq x \leq d_0$. To this purpose, we first prove

$$x \geq 0. \tag{1}$$

By definition of x, to prove (1) it suffices to prove

$$d_0 + 2\left\lfloor \frac{c_0}{2} \right\rfloor - c_0 - 2\left\lfloor \frac{b_0}{2} \right\rfloor - \left(\binom{q+1}{3}\right) + \left(\binom{q+1}{2}\right)\left(b_0 + 1 - 2\left\lfloor \frac{b_0}{2} \right\rfloor\right)$$
$$+(q+2)\left(c_0 + 4\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor\right) - 1 \geq 0 \tag{1a}$$

whence, a fortiori, it suffices to prove

$$d_0 - 2 - 2\left\lfloor \frac{b_0}{2} \right\rfloor - \left(\binom{q+1}{3}\right) + \left(\binom{q+1}{2}\right) + (q+2)(b_0 - 2) \geq 0. \tag{1b}$$

Since $\left(\binom{q+1}{3}\right) - \left(\binom{q+1}{2}\right) = \binom{q+1}{3}$ and $b_0 \geq 33$ by assumption, we can write $(q+2)(b_0 - 2) \geq 2 + 2\lfloor b_0/2 \rfloor$. Therefore, (1b) is a consequence of the inequality

$$d_0 - \binom{q+1}{3} \geq 0. \tag{1c}$$

Now, for all $m \geq 33$, $\mathrm{ch}(1, m, \binom{m}{2}, \binom{m}{3}) \leq m/2$, and we have that

$$\binom{m}{3} \geq \binom{q+1}{3},$$

whence, by definition of d_0, (1c) follows. This establishes inequality (1).

There remains to be proved

$$x \leq d_0. \tag{2}$$

This is weaker than

$$d_0 + 2\left\lfloor \frac{c_0}{2} \right\rfloor - c_0 - 2\left\lfloor \frac{b_0}{2} \right\rfloor - \left(\binom{q+1}{3}\right) + \left(\binom{q+1}{2}\right)\left(b_0 + 1 - 2\left\lfloor \frac{b_0}{2} \right\rfloor\right)$$
$$+(q+2)\left(c_0 + 4\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor\right) \leq 2d_0, \tag{2a}$$

which, in turn, follows from

$$d_0 + c_0 + 2\left\lfloor \frac{b_0}{2} \right\rfloor + \left(\binom{q+1}{3}\right) - 2\left\lfloor \frac{c_0}{2} \right\rfloor - \left(\binom{q+1}{2}\right)\left(b_0 + 1 - 2\left\lfloor \frac{b_0}{2} \right\rfloor\right)$$
$$-(q+2)\left(c_0 + 4\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor\right) \geq 0, \tag{2b}$$

and, a fortiori, from

$$d_0 + b_0 - 1 + \left(\binom{q+1}{3}\right) - 2\left(\binom{q+1}{2}\right) - (q+2)(b_0 - 1) \geq 0. \quad \text{(2c)}$$

Since $((\binom{q+1}{3})) - ((\binom{q+1}{2})) = \binom{q+1}{3}$, to prove (2c) it suffices to prove that

$$d_0 + \binom{q+1}{3} - \left(\binom{q+1}{2}\right) - (q+1)(b_0 - 1) \geq 0. \quad \text{(2d)}$$

By definition of d_0, b_0, and q, our standing assumption implies that $d_0 - (q+1)(b_0 - 1) \geq 0$ and $\binom{q+1}{3} \geq ((\binom{q+1}{2}))$. These last two inequalities are stronger than (2d), whence (2) is proved and Claim 1 is settled. □

CLAIM 2. *The state*

$$NO_1 = (0, b_0 + 1 - \lfloor b_0/2 \rfloor, 2\lfloor b_0/2 \rfloor + c_0 - \lfloor c_0/2 \rfloor, \lfloor c_0/2 \rfloor - \lfloor b_0/2 \rfloor + d_0 - x)$$

is nice.

Proof. We first prove that the state is 0-typical; i.e., it satisfies the three conditions of Definition 5 above. Condition (i) is trivially verified. Condition (ii) now has the form

$$2\left\lfloor \frac{b_0}{2} \right\rfloor + c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor \geq b_0 - \left\lfloor \frac{b_0}{2} \right\rfloor \quad (3)$$

and its proof is immediate. Condition (iii) has the form

$$\left\lfloor \frac{c_0}{2} \right\rfloor - \left\lfloor \frac{b_0}{2} \right\rfloor + d_0 - x \geq q + 1, \quad (4)$$

which is weaker than

$$\left\lfloor \frac{c_0}{2} \right\rfloor - \left\lfloor \frac{b_0}{2} \right\rfloor \geq q + 1. \quad \text{(4a)}$$

Now, inequality (4a) immediately follows from the definition of c_0 and b_0. We have proved that NO_1 is 0-typical. Since $m \geq 33$ and $\mathrm{ch}(NO_1) \geq 12$, the state is nice by Lemma 2. This settles our claim. □

We now consider state $YES_1 = (1, b_1, c_1, d_1)$. Note that,

$$\mathrm{ch}(1, b_1, c_1, d_1) = q + 1.$$

The query

(Q2) $$\left[1, 1, \left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor, y\right]?$$

yields the states

$$YES_2 = \left(1, 1, b_1 - 1 + \left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor, c_1 - \left\lfloor \frac{c_1}{2} \right\rfloor + \left\lfloor \frac{b_1}{2} \right\rfloor + y\right),$$

$$NO_2 = \left(0, b_1, 1 + c_1 - \left\lfloor \frac{c_1}{2} \right\rfloor + \left\lfloor \frac{b_1}{2} \right\rfloor, \left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor + d_1 - y\right).$$

A straightforward computation shows that

$$\begin{aligned} |w_q(YES_2) - w_q(NO_2)| = \Bigg| \left(\binom{q}{3}\right) - \left(\binom{q}{2}\right)(b_1 - 1) \\ + (q+1)\left(b_1 - 2 + 2\left\lfloor \frac{c_1}{2} \right\rfloor - 2\left\lfloor \frac{b_1}{2} \right\rfloor - c_1\right) \\ + c_1 - 2\left\lfloor \frac{c_1}{2} \right\rfloor + 2\left\lfloor \frac{b_1}{2} \right\rfloor - d_1 + 2y \Bigg|. \end{aligned}$$

As in the previous section, let y be defined by

$$y = \left\lfloor \frac{\begin{aligned} (b_1 - 1)\left(\binom{q}{2}\right) - \left(\binom{q}{3}\right) + (q+1)\left(c_1 + 2 - b_1 + 2\left\lfloor \frac{b_1}{2} \right\rfloor - 2\left\lfloor \frac{c_1}{2} \right\rfloor\right) \\ + d_1 - c_1 + 2\left\lfloor \frac{c_1}{2} \right\rfloor - 2\left\lfloor \frac{b_1}{2} \right\rfloor \end{aligned}}{2} \right\rfloor. \tag{$**$}$$

Then $|w_q(YES_2) - w_q(NO_2)| \leq 1$, whence query (Q2) yields two balanced states.

CLAIM 3. *With y as given by* ($**$), *query* (Q2) *is admissible.*

Proof. We must show that $0 \leq y \leq d_1$. We first prove

$$y \geq 0. \tag{5}$$

Proceeding as in the proof of Claim 1, from ($**$) we see that (5) is weaker than

$$\begin{aligned} (b_1 - 1)\left(\binom{q}{2}\right) - \left(\binom{q}{3}\right) + (q+1)\left(c_1 + 2 - b_1 + 2\left\lfloor \frac{b_1}{2} \right\rfloor - 2\left\lfloor \frac{c_1}{2} \right\rfloor\right) \\ + d_1 - c_1 + 2\left\lfloor \frac{c_1}{2} \right\rfloor - 2\left\lfloor \frac{b_1}{2} \right\rfloor \geq 0, \end{aligned} \tag{5a}$$

which in turn is weaker than

$$(b_1 - 1)\left(\binom{q}{2}\right) - \left(\binom{q}{3}\right) + q + d_1 - 2\lfloor b_1/2 \rfloor \geq 0. \tag{5b}$$

Since $(b_1 - 1)((\binom{q}{2})) \geq 1 + 2\lfloor b_1/2 \rfloor$, (5b) is weaker than

$$(q + 1) + d_1 - \left(\binom{q}{3}\right) \geq 0, \tag{5c}$$

i.e., weaker than

$$(q + 1) + c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor + \left\lfloor \frac{b_0}{2} \right\rfloor + x - \left(\binom{q}{3}\right) \geq 0. \tag{5d}$$

Since $((\binom{q}{3})) - (q + 1) = \binom{q}{3} + \binom{q}{2}$, to prove (5d) it suffices to prove

$$c_0 - \left\lfloor \frac{c_0}{2} \right\rfloor + \left\lfloor \frac{b_0}{2} \right\rfloor + x - \binom{q}{3} - \binom{q}{2} \geq 0, \tag{5e}$$

and, since $c_0 - \lfloor c_0/2 \rfloor \geq \binom{q}{2}$, (5e) follows from the inequality

$$\lfloor b_0/2 \rfloor + x - \binom{q}{3} \geq 0. \tag{5f}$$

An elementary computation shows that (5f) follows from the stronger

$$\begin{aligned} d_0 + 2\left\lfloor \frac{c_0}{2} \right\rfloor - c_0 - \left(\binom{q}{3}\right) + \left(\binom{q}{2}\right)\left(b_0 + 1 - 2\left\lfloor \frac{b_0}{2} \right\rfloor\right) \\ + (q + 2)\left(c_0 + 4\left\lfloor \frac{b_0}{2} \right\rfloor - b_0 - 2\left\lfloor \frac{c_0}{2} \right\rfloor\right) - 2 - 2\binom{q}{3} \geq 0. \end{aligned} \tag{5g}$$

Consider the inequality

$$d_0 - 3 - 3\binom{q}{3} + (q + 2)(b_0 - 2) \geq 0. \tag{5h}$$

This inequality certainly holds under the assumption of Lemma 7, whence, a fortiori, (5g) holds, and so does (5).

There remains to be proved

$$y \leq d_1. \tag{6}$$

For this purpose, it suffices to prove

$$(b_1 - 1)\left(\left(\begin{matrix} q \\ 2 \end{matrix}\right)\right) - \left(\left(\begin{matrix} q \\ 3 \end{matrix}\right)\right) + (q+1)\left(c_1 + 2 - b_1 + 2\left\lfloor \frac{b_1}{2} \right\rfloor - 2\left\lfloor \frac{c_1}{2} \right\rfloor\right)$$
$$+ d_1 - c_1 + 2\left\lfloor \frac{c_1}{2} \right\rfloor - 2\left\lfloor \frac{b_1}{2} \right\rfloor \le 2d_1, \tag{6a}$$

which in turn is weaker than

$$d_1 + c_1 + \left(\left(\begin{matrix} q \\ 3 \end{matrix}\right)\right) + 2\left\lfloor \frac{b_1}{2} \right\rfloor - 2\left\lfloor \frac{c_1}{2} \right\rfloor - (b_1 - 1)\left(\left(\begin{matrix} q \\ 2 \end{matrix}\right)\right)$$
$$-(q+1)\left(c_1 + 2 - b_1 + 2\left\lfloor \frac{b_1}{2} \right\rfloor - 2\left\lfloor \frac{c_1}{2} \right\rfloor\right) \ge 0, \tag{6b}$$

and, a fortiori, weaker than

$$d_1 + (b_1 - 1) + \left(\left(\begin{matrix} q \\ 3 \end{matrix}\right)\right) - b_1\left(\left(\begin{matrix} q \\ 2 \end{matrix}\right)\right) + \left(\left(\begin{matrix} q \\ 2 \end{matrix}\right)\right) - 3(q+1) \ge 0. \tag{6c}$$

Since $((\begin{smallmatrix} q \\ 2 \end{smallmatrix})) \ge 3(q+1)$, (6c) follows from the stronger inequality,

$$d_1 + (b_1 - 1) + \left(\left(\begin{matrix} q \\ 3 \end{matrix}\right)\right) - b_1\left(\left(\begin{matrix} q \\ 2 \end{matrix}\right)\right) \ge 0. \tag{6d}$$

Now, (6d) holds under the assumed values of m, whence (6) follows, as required to settle our claim. □

CLAIM 4. *The state*

$$NO_2 = \left(0, b_1, 1 + c_1 - \lfloor c_1/2 \rfloor + \lfloor b_1/2 \rfloor, \lfloor c_1/2 \rfloor - \lfloor b_1/2 \rfloor + d_1 - y\right)$$

is nice.

Proof. We first prove that the state is 0-typical, i.e., it satisfies the three conditions of Definition 5 above. Condition (i) is trivially verified. Condition (ii) has the form

$$c_1 - \left\lfloor \frac{c_1}{2} \right\rfloor + \left\lfloor \frac{b_1}{2} \right\rfloor + 1 \ge b_1 - 1, \tag{7}$$

which is weaker than

$$\left\lfloor \frac{c_1}{2} \right\rfloor + 2 \ge \left\lfloor \frac{b_1}{2} \right\rfloor + 1. \tag{7a}$$

Now, inequality (7a) immediately follows from the definition of c_1 and b_1.

Condition (iii) now has the form

$$\left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor + d_1 - y \geq q, \tag{8}$$

which is weaker than

$$\left\lfloor \frac{c_1}{2} \right\rfloor - \left\lfloor \frac{b_1}{2} \right\rfloor \geq q. \tag{8a}$$

Now, inequality (8a) immediately follows from definition of c_1 and b_1. We have proved that NO_2 is 0-typical.

Since $m \geq 33$ and $\mathrm{ch}(NO_2) \geq 12$, the state is nice by Lemma 2, and this concludes the proof of this claim. □

Having proved Claim 4, we consider state $YES_2 = (1, 1, c_2, d_2)$. Note that $\mathrm{ch}(1, 1, c_2, d_2) = q$. Let query (Q3) be defined by

(Q3) $$[1, 0, 3, z]?.$$

Then starting from state YES_2, (Q3) yields states

$$YES_3 = (1, 0, 3, c_2 - 2 + z)$$

and $NO_3 = (0, 2, c_2 - 2, 2 + d_2 - z)$. An easy calculation shows that

$$\begin{aligned} &\left| w_{q-1}(YES_3) - w_{q-1}(NO_3) \right| \\ &\quad = \left| \left(\binom{q-1}{3} \right) - 4 - c_2 - q(c_2 - 5) - 2\left(\binom{q-1}{2} \right) - d_2 + 2z \right|. \end{aligned}$$

As in the previous section, let z defined by

$$z = \left\lfloor \frac{d_2 + 4 - c_2 - \left(\binom{q-1}{3} \right) + 2\left(\binom{q-1}{2} \right) + q(c_2 - 5)}{2} \right\rfloor. \tag{$***$}$$

Then $|w_{q-1}(YES_3) - w_{q-1}(NO_3)| \leq 1$, whence (Q3) yields two balanced states.

CLAIM 5. *With z as given by $(***)$, query* (Q3) *is admissible.*

Proof. We have to prove that $0 \leq z \leq d_2$. To this purpose, we first prove

$$z \geq 0. \tag{9}$$

By definition of z, to prove (9) it suffices to prove

$$d_2 - 4 - c_2 - \left(\!\binom{q-1}{3}\!\right) + 2\left(\!\binom{q-1}{2}\!\right) + q(c_2 - 5) \geq 0, \quad \text{(9a)}$$

and, since $q(c_2 - 5) \geq c_2$, (9a) is implied by

$$d_2 - 4 - \binom{q-1}{3} + \left(\!\binom{q-1}{2}\!\right) \geq 0. \quad \text{(9b)}$$

By definition of d_2, the inequality (9b) holds and so (9) is proved.

There remains to be proved

$$z \leq d_2. \quad \text{(10)}$$

This is weaker than

$$d_2 + 4 - c_2 - \left(\!\binom{q-1}{3}\!\right) + 2\left(\!\binom{q-1}{2}\!\right) + q(c_2 - 5) \leq 2d_2 \quad \text{(10a)}$$

which, in turn, follows from

$$d_2 + c_2 + \binom{q-1}{3} - \left(\!\binom{q-1}{2}\!\right) - q(c_2 - 5) \geq 0. \quad \text{(10b)}$$

Consider the inequality $\binom{q-1}{3} \geq \left(\!\binom{q-1}{2}\!\right)$; this certainly holds under the assumptions of Lemma 7, whence (10b) is implied by

$$d_2 + c_2 - q(c_2 - 5) \geq 0. \quad \text{(10c)}$$

By definition of d_2, c_2, q, and under the assumptions of Lemma 7, this inequality holds, whence (10) is proved, and Claim 5 is settled. □

CLAIM 6. *The state* $YES_3 = (1, 0, 3, c_2 - 2 + z)$ *is nice.*

Proof. In order to prove the claim it is only necessary to show that

$$c_2 - 2 + z \geq 7; \quad \text{(11)}$$

then the proof of the claim will follow from Lemma 5. By definition of c_2 (11) holds for $z = 0$ as well, whence, a fortiori, (11) holds, and the claim is proved. □

CLAIM 7. *The state* $NO_3 = (0, 2, c_2 - 2, 2 + d_2 - z)$ *is nice.*

Proof. We first show that state NO_3 is 0-typical. Conditions (ii) and (iii) of Definition 5 are now

$$c_2 - 2 \geq 1 \quad \text{(12)}$$

and

$$2 + d_2 - z \geq q - 1. \tag{13}$$

By definition of c_2, d_2, and z and by the assumptions of Lemma 7, the inequalities (12) and (13) hold, whence the state NO_3 is 0-typical, and as a consequence of Lemma 2 it is nice too. □

Having thus proved Claims 1–7, the proof of Sublemmas 7.1–7.3 is complete, whence so is the proof of Lemma 7.

5. The Three Exceptional Cases

This section is devoted to the proof of the following

Proposition. *For $m = 2$, $m = 3$, and $m = 5$, the state $(1, m, \binom{m}{2}, \binom{m}{3})$ is not nice. Indeed, the minimal number of queries is given by* $\mathrm{ch}(1, m, \binom{m}{2}, \binom{m}{3}) + 1$.

Since queries $[x, y, z, k]$? and $[a - x, b - y, c - z, d - k]$? asked in the state (a, b, c, d) yield the same states, we can restrict our investigation to queries $[x, y, z, k]$? with $x \leq a/2$.

Claim 1. *Let $S_1 = (a_1, b_1, c_1, d_1)$ and $S_2 = (a_2, b_2, c_2, d_2)$ be two states with $\mathrm{ch}(a_1, b_1, c_1, d_1) = \mathrm{ch}(a_2, b_2, c_2, d_2)$ and $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$. If S_1 is not nice then S_2 is not nice.*

Proof. Trivial. □

In this case we say that the state S_2 *includes* the state S_1.

Claim 2. *The state $(0, 0, 3, 0)$ is not nice.*

Proof. This state corresponds to the state $(3, 0)$ for the case of one lie. We have that $\mathrm{ch}(0, 0, 3, 0) = 4$, but in [P] it was proved that starting from the state $(3, 0)$, the Questioner needs five queries to win the game. □

Claim 3. *States $(0, 1, 2, 0)$ and $(0, 1, 2, 1)$ are not nice.*

Proof. Since $(0, 1, 2, 1)$ includes $(0, 1, 2, 0)$, by Claim 1 it is only necessary to prove that state $(0, 1, 2, 0)$ is not nice. In this case $\mathrm{ch}(0, 1, 2, 0) = 5$. If we ask query $[0, 1, x, 0]$? with $0 \leq x \leq 2$, the resulting states are:

$$YES_{0,1,2,0}(0, 1, x, 0) = (0, 1, x, 2 - x) \quad \text{and}$$
$$NO_{0,1,2,0}(0, 1, x, 0) = (0, 0, 3 - x, x).$$

Two cases are possible:

1. $x \geq 1$, then $w_4(0,1,x,2-x) = 11 + 5x + 2 - x = 13 + 4x > 2^4$. Then no query $[0,1,x,0]$? with $x \geq 1$ can balance state $(0,1,2,0)$.

2. $x = 0$, in this case we obtain the state $NO_{0,1,2,0}(0,1,0,0) = (0,0,3,0)$, which is not nice by Claim 2.

Then state $(0,1,2,0)$ is not nice. □

CLAIM 4. *States* $(0,2,1,0)$ *and* $(0,2,1,1)$ *are not nice.*

Proof. Since $(0,2,1,1)$ includes $(0,2,1,0)$, by Claim 1 it is sufficient to prove that state $(0,2,1,0)$ is not nice. In this case $\mathrm{ch}(0,2,1,0) = 6$. Asking query $[0,x,y,0]$? with $1 \leq x \leq 2$ and $0 \leq y \leq 1$, the resulting states are:

$$YES_{0,2,1,0}(0,x,y,0) = (0,x,2-x+y,1-y) \qquad \text{and}$$
$$NO_{0,2,1,0}(0,x,y,0) = (0,2-x,x+1-y,y).$$

Two cases are possible:

1. $x = 2$, in this case: $w_5(0,x,2-x+y,1-y) = w_5(0,2,y,1-y) = 32 + 6y + 1 - y = 33 + 5y > 2^5$, then the query $[0,x,y,0]$? with $x = 2$ does not balance the state $(0,2,1,0)$.

2. $x = 1$, in this case

• for $y = 0$ we obtain the state $NO_{0,2,1,0}(0,1,0,0) = (0,1,2,0)$, which is not nice by Claim 4.

• for $y = 1$ we obtain $YES_{0,2,1,0}(0,1,1,0) = (0,1,2,0)$, which is not nice by Claim 3.

Then the state $(0,2,1,0)$ is not nice. □

CLAIM 5. *The states* $(0,3,0,0)$, $(0,5,0,0)$, $(0,3,0,1)$, *and* $(0,3,1,0)$ *are not nice.*

Proof. The states $(0,3,0,0)$ and $(0,5,0,0)$ respectively correspond to states $(3,0,0)$ and $(5,0,0)$ for the case of two lies; in [G], it has been shown that these states are not nice. (See Table VI.) The states $(0,3,0,1)$ and $(0,3,1,0)$ are not nice because they include the state $(0,3,0,0)$. □

CLAIM 6. *The state* $(1,0,2,0)$ *is not nice.*

Proof. In this case $\mathrm{ch}(1,0,2,0) = 6$. The query $[1,0,x,0]$? with $0 \leq x \leq 2$ yields the states $YES_{1,0,2,0}(1,0,x,0) = (1,0,x,2-x)$ and

TABLE VI
States That are 0-Typical But Not Nice [G, Appendix]

Character	State	Values of d	Character	State	Values of d
ch = 6	$(0,1,5,d)$	$6 \le d \le 7$	ch = 9	$(0,7,16,d)$	$9 \le d \le 30$
	$(0,2,1,d)$	$6 \le d \le 13$		$(0,7,17,d)$	$9 \le d \le 20$
	$(0,2,2,6)$			$(0,7,18,d)$	$9 \le d \le 10$
				$(0,8,9,d)$	$9 \le d \le 54$
				$(0,8,10,d)$	$9 \le d \le 44$
ch = 7	$(0,2,7,d)$	$7 \le d \le 14$		$(0,8,11,d)$	$9 \le d \le 34$
	$(0,3,2,d)$	$7 \le d \le 25$		$(0,8,12,d)$	$9 \le d \le 24$
	$(0,3,3,d)$	$7 \le d \le 17$		$(0,8,13,d)$	$9 \le d \le 14$
	$(0,3,4,d)$	$7 \le d \le 9$		$(0,9,8,d)$	$9 \le d \le 18$
			ch = 10	$(0,13,25,d)$	$10 \le d \le 21$
				$(0,13,26,10)$	
ch = 8	$(0,3,15,d)$	$8 \le d \le 10$		$(0,14,17,d)$	$10 \le d \le 53$
	$(0,4,9,d)$	$8 \le d \le 27$		$(0,14,18,d)$	$10 \le d \le 42$
	$(0,4,10,d)$	$8 \le d \le 18$		$(0,14,19,d)$	$10 \le d \le 31$
	$(0,4,11,d)$	$8 \le d \le 9$		$(0,14,20,d)$	$10 \le d \le 20$
	$(0,5,4,d)$	$8 \le d \le 35$		$(0,15,14,d)$	$10 \le d \le 30$
	$(0,5,5,d)$	$8 \le d \le 26$		$(0,15,15,d)$	$10 \le d \le 19$
	$(0,5,6,d)$	$8 \le d \le 17$			
	$(0,5,7,8)$		ch = 11	$(0,25,30,d)$	$11 \le d \le 13$

$NO_{1,0,2,0}(1,0,x,0) = (0,1,2-x,x)$. There are two possible cases:

1. If $x \ge 1$ then $w_5(1,0,x,2-x) = 26+6+2-x = 28+5x \ge 2^5$; hence no query $[1,0,x,0]$? can balance state $(1,0,2,0)$.

2. For $x = 0$ we obtain the state $NO_{1,0,2,0}(1,0,0,0) = (0,1,2,0)$, which is not nice by Claim 3.

Hence state $(1,0,2,0)$ is not nice. □

CLAIM 7. *States* $(1,1,1,0)$ *and* $(1,1,1,1)$ *are not nice.*

Proof. Since $(1,1,1,1)$ includes $(1,1,1,0)$, by Claim 1 it is sufficient to prove that state $(1,1,1,0)$ is not nice. In this case $\mathrm{ch}(1,1,1,0) = 7$. The query $[1,x,y,0]$? with $0 \le x \le 1$ and $0 \le y \le 1$ yields the states

$$YES_{1,1,1,0}(1,x,y,0) = (1,x,1-x+y,1-y) \qquad \text{and}$$

$$NO_{1,1,1,0}(1,x,y,0) = (0,2-x,x+1-y,y).$$

Two cases are possible:

1. If $x = 1$ then $w_6(1, x, 1 - x + y, 1 - y) = w_6(1, 1, y, 1 - y) = 42 + 22 + 7y + 1 - y = 65 + 6y \geq 2^6$; hence no query can balance state $(1, 1, 1, 0)$.

2. If $x = 0$ then for $y = 0$ we obtain the state $NO_{1,1,1,0}(1, 0, 0, 0) = (0, 2, 1, 0)$, which is not nice by Claim 4; on the other hand, for $y = 1$, we obtain the state $YES_{1,1,1,0}(1, 0, 1, 0) = (1, 0, 2, 0)$, which is not nice by Claim 6.

Then state $(1, 1, 1, 0)$ is not nice. □

CLAIM 8. *The state* $(1, 2, 0, 1)$ *is not nice.*

Proof. In this case we have that $\mathrm{ch}(1, 2, 0, 1) = 7$. If we ask the query $[1, x, 0, y]$?, with $0 \leq x \leq 2$ and $0 \leq y \leq 1$, we obtain the states

$$YES_{1,2,0,1}(1, x, 0, y) = (1, x, 2 - x, y) \quad \text{and}$$
$$NO_{1,2,0,1}(1, x, 0, y) = (0, 3 - x, x, 1 - y).$$

There are three possible cases:

1. If $x = 0$, then $w_6(0, 3, 0, 1 - y) = 66 + 1 - y > 2^6$, and hence no query can balance state $(1, 2, 0, 1)$.

2. If $x = 1$ then $YES_{1,2,0,1}(1, 1, 0, y) = (1, 1, 1, y)$, which is not nice by Claim 7.

3. If $x = 2$ then $NO_{1,2,0,1}(1, 2, 0, y) = (0, 1, 2, 1 - y)$, which is not nice by Claim 3.

Then state $(1, 2, 0, 1)$ is not nice. □

CLAIM 9. *The states* $(1, m, \binom{m}{2}, \binom{m}{3})$ *with* $m = 2$ *and* $m = 3$ *are not nice.*

Proof. The states are respectively given by $(1, 2, 1, 0)$ and $(1, 3, 3, 1)$. Since $(1, 3, 3, 1)$ includes $(1, 2, 1, 0)$, by Claim 1 it is sufficient to prove that state $(1, 2, 1, 0)$ is not nice. In this case $\mathrm{ch}(1, 2, 1, 0) = 8$. The query $[1, x, y, 0]$?, with $0 \leq x \leq 2$ and $0 \leq y \leq 1$, yields two states $YES_{1,2,1,0}(1, x, y, 0) = (1, x, 2 - x + y, 1 - y)$ and $NO_{1,2,1,0}(1, x, y, 0) = (0, 3 - x, x + 1 - y, y)$. There are three possible cases:

1. If $x = 0$ then $NO_{1,2,1,0}(1, 0, y, 0) = (0, 3, 1 - y, y)$; this state cannot be nice; as a matter of fact, for $y = 0$ we obtain state $(0, 3, 1, 0)$, and for $y = 1$ state $(0, 3, 0, 1)$, neither being nice by Claim 5.

2. If $x = 1$ two cases are possible:

- if $y = 0$ then $YES_{1,2,1,0}(1, 1, 0, 0) = (1, 1, 1, 1)$, which is not nice by Claim 7;

• if $y = 1$ then $NO_{1,2,1,0}(1,1,1,0) = (0,2,1,1)$, which is not nice by Claim 4.

3. If $x = 2$ two cases are possible:

• if $y = 0$ then $YES_{1,2,1,0}(1,2,0,0) = (1,2,0,1)$, which is not nice by Claim 8;

• if $y = 1$ then $NO_{1,2,1,0}(1,2,1,0) = (0,1,2,1)$, which is not nice by Claim 3.

Our analysis of state $(1,2,1,0)$ is now complete, and the claim is settled. □

CLAIM 10. *The state* $(1, m, \binom{m}{2}, \binom{m}{3})$ *with* $m = 5$ *is not nice.*

Proof. In this case the state is $(1,5,10,10)$ and $\mathrm{ch}(1,5,10,10) = 9$. The query $[1, x, y, z]$? with $0 \le x \le 5$, $0 \le y \le 10$, and $0 \le z \le 10$ yields two states

$$YES_{1,5,10,10}(1, x, y, z) = (1, x, 5 - x + y, 10 - y + z),$$

$$NO_{1,5,10,10}(1, x, y, z) = (0, 6 - x, x + 10 - y, y + 10 - z).$$

Three cases are possible:

1. If $0 \le x \le 1$ then $NO_{1,5,10,10}(1,0,y,z) = (0, 6 - x, x + 10 - y, y + 10 - z)$. In this case, $(0,5,0,0)$ will be included in every possible state of the form $(0, 6 - x, x + 10 - y, y + 10 - z)$, whence the latter cannot be nice, since by Claim 5, $(0,5,0,0)$ is not nice.

2. If $2 \le x \le 3$ then $YES_{1,5,10,10}(1, x, y, z) = (1, x, 5 - x + y, 10 - y + z)$. For every x and y, this state is not nice, because it includes state $(1,2,1,0)$, which is not nice by Claim 9.

3. If $4 \le x \le 5$ then $w_8(1, x, 5 - x + y, 10 - y + z) = 93 + 37x + 9(5 - x + y) + 10 - y + z = 148 + 28x + 8y + z > 2^8$, for every x and y with $0 \le y$, $z \le 10$. We conclude that no query can balance the state. □

Having thus settled Claim 10, we have completed the proof of the proposition stated at the beginning of this section. □

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REFERENCES

[B] E. R. BERLEKAMP, Block coding for the binary symmetric channel with noiseless, delayless feedback, *in* "Error-Correcting Codes," pp. 61–85, Wiley, New York, 1968.

[BR] T. A. BRYLAWSKI, The Mathematics of Watergate (An analysis of a two-party game), University of North Carolina, 1978.

[C1] J. CZYZOWICZ, D. MUNDICI, AND A. PELC, Solution of Ulam's problem on binary search with two lies, *J. Combin. Theory Ser. A* **49** (1988), 384–388.

[C2] J. CZYZOWICZ, D. MUNDICI, AND A. PELC, Ulam's searching game with lies, *J. Combin. Theory Ser. A* **52** (1989), 62–76.

[G] W. GUZICKI, Ulam's searching game with two lies, *J. Combin. Theory Ser. A* **54** (1990), 1–19.

[K] D. J. KLEITMAN, review of "Ulam's Searching Game with Lies," by J. Czyzowicz, D. Mundici, and A. Pelc [C2], *Math. Rev.* Review Number 94A Communication, pp. 6831–6832, 1990.

[N] A. NEGRO, AND M. SERENO, Solution of Ulam's problem on binary search with three lies, *J. Combin. Theory Ser. A* **59** (1992), 149–154.

[P] A. PELC, Solution of Ulam's problem on searching with lie, *J. Combin. Theory Ser. A* **44** (1987), 129–140.

[R] R. L. RIVEST, A. R. MEYER, D. J. KLEITMAN, K. WINKLMANN, AND J. SPENCER, Coping with errors in binary search procedures, *J. Comput. System Sci.* **20** (1980), 396–406.

[U] S. M. ULAM, "Adventures of a Mathematician," p. 281, Scribner's, New York, 1976.

[V] J. H. VAN LINT, "Introduction to Coding Theory," Springer-Verlag, New York/Berlin, 1982.