Solution of Ulam's Searching Game with Three Lies or an Optimal Adaptive Strategy for Binary Three-Error-Correcting-Codes

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List of Symbols

Symbols		Meaning
$\mathbb{N}$	$=$	$\{1, 2, \ldots\}$: set of the natural numbers
$\mathbb{N}_0$	$=$	$\{0, 1, 2, \ldots\}$: set of the natural numbers including 0
$\mathbb{R}$	:	field of the real numbers
$\mathbb{R}_0^+$	:	set of the non $-$ negative real numbers inculing 0
$\Longleftrightarrow$	:	equivalence ($a \Longleftrightarrow b$: "a if and only if b")
$\triangleq$	:	by definition ($a \triangleq b$: "a defines b")
$[a, b]$	$=$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
$(a, b]$	$=$	$\{x \in \mathbb{R} : a < x \leq b\}$
$[a, b)$	$=$	$\{x \in \mathbb{R} : a \leq x < b\}$
(a, b)	$=$	$\{x \in \mathbb{R} : a < x < b\}$
$\lvert x \rvert$	$=$	$\begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$
$\lvert S \rvert$	:	the cardinality of S
$A \cup B$	$=$	$\{x : x \in A \text{ or } x \in B\}$
$A \cap B$	$=$	$\{x : x \in A \text{ and } x \in B\}$
$A \backslash B$	$=$	$\{x : x \in A \text{ and } x \notin B\}$
$A \times B$	$=$	$\{(x, y) : x \in A \text{ and } y \in B\}$
$\bar{A}$	$=$	$\{x : x \notin A\}$
A^i	$=$	$\underbrace{A \times \ldots \times A}_{i-\text{times}}$
e	$=$	$\sum_{n=0}^{\infty} \frac{1}{n!}$ Euler′s number
log	:	logarithm to base 2
ln	:	natural logarithm
$\lceil x \rceil$	$=$	$\min\{z \in \mathbb{Z} : z \geq x\}$
$\lfloor x \rfloor$	$=$	$\max\{z \in \mathbb{Z} : z \leq x\}$
iff	:	if and only if
w.l.o.g.	:	without loss of generality

Abstract

In this paper we determine the minimal number of yes-no queries needed to find an unknown integer between 1 and N if at most three of the answers are lies. This strategy is also an optimal adaptive strategy for binary three-error-correcting codes.

1 Introduction

In 1976, Ulam [24] suggested an interesting two-person search game in his autobiography (pp. 281-282), which can be formalized as follows:

> Person 1 thinks of a number between one and one million. Person 2 is allowed to ask questions to which Person 1 is supposed to answer only yes or no. Person 2 asks for subsets of the set $\{1, \ldots, 1000000\}$. The difficulty is that Person 1 is allowed to lie l times. Now we want to know: How many questions does Person 2 have to ask in order to get the correct answer?

The problem is solved in [20] for one lie. Solution for $l = 2$ and $|\mathcal{X}| = 10^6$ can be find in [9]. Solution for $|\mathcal{X}| = 2^m$ and $l = 2$ is presented in [7] and its generalization to arbitrary $|\mathcal{X}|$ is given in [11]. The case of three lies if $N = 1000000$ is solved in [17]. In [12] it is shown how to solve Ulam's game for $l = 1, 2, 3, 4$ and $N = 1000000$. We solve Ulam's game for $l = 3$ and arbitrary N. In Section 2 we introduce some definitions and notations. Furthermore, we present some properties of the volume, which is an important auxiliary function of the game. The main idea of the strategy is as follows. If N is big enough ($N > 265$), then we can divide the game into three parts:

- The first three questions
- Typical questions
- The last 15 questions

We present the first part in Section 3. We show how to ask the first three questions in an optimal way. It turns out, that we can decide after the first three optimal questions, how many questions are necessary for Person 2 to get the correct answer. After the first three questions there exists a strategy that forms the next question until there are 15 questions left. We call these questions typical and present them in Section 4. Unfortunately, we cannot give a similar strategy for the last 15 questions. In Section 5 we present an algorithm for the last 15 questions. This algorithm was developed by Guzicki [11] to solve Ulam's problem in the case of two lies. It is easy to prove that we can also use this algorithm for three lies. Some remarks on the result of the algorithm are given in Section 7. The main theorem of the paper is presented in Section 6, where we combine all results of the previous parts. In Section 8 we present some general solution of the game for arbitrary $l \in \mathbb{N}$.
Berlekamp [5] considered 1964 the problem of transmitting messages over a noisy binary channel with noiseless feedback. In this model, sender wants to transmit a message over a noisy binary channel. Let $\mathcal{X} = \{1, \ldots, N\}$ denote the set of possible messages and let $\mathcal{Y} = \{0, 1\}$ be the binary coding alphabet. We have a passive feedback, that is the sender always knows what has been received. The codewords are elements of $\mathcal{Y}^n$, and a codeword has the following form: $\left(c_1\left(x, y^0\right), c_2\left(x, y^1\right) \ldots, c_n\left(x, y^{i-1}\right)\right)$, where $c_i : \mathcal{X} \times \mathcal{Y}^{i-1} \to \mathcal{Y}$ is a function defined for the i-th code letter, which depends on the message we want to transmit and $i-1$ bits received before. We suppose that the noise does not change more than

$l \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ bits of a codeword. One may consider each transmission as the following quiet-question-noisy-answer-game: The sender and receiver play this game against a "Devil". The sender chooses a message $x \in \mathcal{X}$. The sender and receiver have a common partition strategy depending on the received bits. Thus, the sender chooses a subset $S \subset \mathcal{X}$ by following the partition strategy. He sends either "1" if $x \in S$ or "0" if $x \notin S$ over a noisy channel. A new subset $S \subset \mathcal{X}$ is chosen depending on the previously received bits. The aim is that the receiver tries to get the message after n transmissions, and "Devil" wants to avoid this by changing at most l bits. Berlekamp [5] considered the asymptotic problem ($\frac{l}{n} \to p, n \to \infty$). Obviously, Berlekamp's quiet-question-noisy-answer-game is equivalent to Ulam's game. Many search problems, which are equivalent to a coding problem can be find in [1] and [4]. A good overview and some new results concerning channels with noiseless feedback is given in [25]. Another generalization of the problem is to use an q-partition that means an q-ary alphabet. Some ideas and results are presented in Section 9. In 1992 Spencer [22] introduced a game on a board, which is equivalent to Ulam's game. We present this game and further versions of Ulam's game in the last section.

2 Definitions and notations

We use the notations of Berlekamp [5]. In each round of the game Person 2 gets a negative vote for a subset $T \subset \mathcal{X}$. If Person 2 gets more than l negative votes for a number, then this number cannot be the searched one because Person 1 is allowed to lie at most l times. Therefore, we consider the sets

$$S_j \triangleq \{x \in \mathcal{X} : \text{Person 2 gets } (l-j) \text{ negative votes for } x\}.$$

Definition 2.1 *The vector*

$$\underline{v} = (|S_l|, |S_{l-1}|, \ldots, |S_0|) = (v_l, v_{l-1}, \ldots, v_0)$$

is referred to as a state (of the game). $\underline{v}$ *is called an k-state if k questions are left.*

Neither the states nor the dividing questions depend on a specific number, which is chosen by Person 1. Everything depends only on the cardinality of the sets S_j.

Definition 2.2 *Let* $\underline{s}$ *be an arbitrary state. The question if "$x \in S$" ($S \subset \mathcal{X}$) is introduced as a vector* $[\underline{u}] = [u_l, \ldots, u_0]$, *where* $u_i \triangleq |S_i \cap S|$. *The state* $\underline{x}$ *is reduced to the states* $\underline{y}(\triangleq YES^{\underline{s}})$ *and* $\underline{z}(\triangleq NO^{\underline{s}})$ *by the question* $[\underline{u}]$ *if there exists a* $\underline{v} \geq 0$ *such that*

1. $\underline{x} = \underline{u} + \underline{v} \triangleq (u_l + v_l, u_{l-1} + v_{l-1}, \ldots, u_0 + v_0)$,
2. $y_i = u_i + v_{i+1}$,
3. $z_i = v_i + u_{i+1}$.

Definition 2.3 *Let* $\underline{v}$ *be an arbitrary state and let* $[\underline{x}]$ *be a question. The question is called legal if*

$$0 \leq x_i \leq v_i, \; for\ all\ i = 0, \ldots, l.$$

Definition 2.4 1. *A 0-state* $\underline{x}$ *is called winning if* $\sum_{i=0}^{l} x_i \leq 1$. *Otherwise,* $\underline{x}$ *is called losing.*

2. *An k-state* $\underline{x}$ *is called winning if it can be reduced to two winning $(k-1)$ - states. Otherwise,* $\underline{x}$ *is called losing.*

3. *A winning k-state is called borderline winning if it is a losing $(k-1)$-state.*

4. *An k-state is called singlet if $\sum_{i=0}^{k} |S_i| = 1$.*

5. *An k-state is called doublet if $\sum_{i=0}^{k} |S_i| = 2$.*

Proposition 2.1 1. *A winning n-state is also a winning k-state if $k > n$.*

2. *All borderline winning 1-states have the form $(0, \ldots, 0, 2)$.*

3. *Let $\underline{x}$ be a winning k-state and let $\underline{y}$ be some state if $y_i \leq x_i$ holds for all $i \leq l$. Then $\underline{y}$ is also a winning k-state.*

Definition 2.5 *For a given state $\underline{x}$, the function*

$$V_n(\underline{x}) \triangleq \sum_{i=0}^{l} x_i \sum_{j=0}^{i} \binom{n}{j}$$

is called the n-th volume of $\underline{x}$.

The n-th volume of $\underline{x}$ can be interpreted as follows. We surround all messages at height j (Person 1 can lie j times at all these numbers) by a sphere of radius j. Spencer [22] used the fact that $\frac{1}{2^n} V_n(\underline{v}) = \sum_{i=0}^{l} Pr(X \leq i)$, where X is $B(n, \frac{1}{2})$ distributed. $B(n,p)$ is the standard binomial distribution.

Theorem (Berlekamp's Conservation of Volume [5]) *Let $\underline{x}$ be a state which can be reduced to the states $\underline{y}$ and $\underline{z}$. Then*

$$V_n(\underline{x}) = V_{n-1}(\underline{y}) + V_{n-1}(\underline{z}).$$

Theorem (Berlekamp's Volume Bound [5]) *Let $\underline{x}$ be a winning n-state. Then*

$$V_n(\underline{x}) \leq 2^n.$$

We denote by $L_l(N)$ the minimal number of questions, which Person 2 needs to find the searched number by using an optimal strategy.

Corollary 2.2 (The Hamming Bound) *Let $|\mathcal{X}| = N$ and $l \in \mathbb{N}$. Then*

$$L_l(N) = n \Longrightarrow N \leq \frac{2^n}{\sum_{j=0}^{l} \binom{n}{j}}.$$

Definition 2.6 *Let $\underline{s}$ be an arbitrary state. The number*

$$ch(\underline{s}) \triangleq \min\{k : V_k(\underline{s}) \leq 2^k\}$$

is called character of $\underline{s}$.

Hereafter, we restrict our considerations to the case $l = 3$.

Notation 2.7 *We set $F, G : \mathbb{N} \to \mathbb{N}$ with*

$$G(k) \triangleq \sum_{i=0}^{3} \binom{k}{i} = \frac{k^3 + 5k + 6}{6},$$

$$F(k) \triangleq \sum_{i=0}^{2} \binom{k}{i} = \frac{k^2 + k + 2}{2}.$$

It is easy to see that $V_k(\underline{v}) = v_3 G(k) + v_2 F(k) + v_1(k+1) + v_0$.

3 The first three crucial questions

Let $\underline{v}$ be the state of the game after the first three questions. We show that the volume of $\underline{v}$ is crucial if N is big enough in Section 5. It turns out that we can solve the game with $ch(\underline{v})$ questions by starting from the state $\underline{v}$. We construct an optimal strategy step by step. We analyze the best queries and the worst answers. In every step of the game, the Questioner reduces the current state to YES and NO states. The Responder chooses the worst state of these states. The aim of the Questioner is to reduce the character by one in every step.
Let $\underline{v}$ be a state and $\underline{x}$ be the next question. Then the question $[\underline{z}] = [v_3 - x_3, v_2 - x_2, v_1 - x_1, v_0 - x_0]$ is called the complement of the question $\underline{x}$ because it exchanges the YES-state and the NO-state. In the first two questions we only consider questions, which have the property that the j-th volume of the YES state is bigger than the j-th volume of the NO state.

Definition 3.1 *1. Let $\underline{s}$ be a state and $k \triangleq ch(\underline{s})$. The state is called balanced if there exists a query $[\underline{u}]$ such that*

$$|V_{k-1}(YES^{\underline{s}}) - V_{k-1}(NO^{\underline{s}})| \leq 1.$$

2. The state $\underline{s}$ is called nice iff the Questioner wins in $ch(\underline{s})$ questions starting from this state.

In the next three propositions we present the first three questions of the game. We denote the initial state by $\underline{s}$, the first question by $[\underline{x}]$, the state after the first question by $\underline{t}$, the second question by $[\underline{y}]$, the state after the second question by $\underline{u}$, the second question by $[\underline{z}]$ and the state after the third question by $\underline{v}$. In each proposition we prove that the questions are locally optimal, that means that the difference of the j-th volume of the YES state and the NO state is minimal. At the end it turns out that these questions are also optimal.

Proposition 3.1 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a state and let $k_1 \in \mathbb{N}$. The following questions minimize the volume difference $|V_{k_1}(YES) - V_{k_1}(NO)|$:*

1. *$[\frac{s_3}{2}, 0, 0, 0]$ if s_3 is even,*
2. *$[\frac{s_3+1}{2}, 0, 0, 0]$ if s_3 is odd.*

Furthermore, $|V_{k_1}(YES^{\underline{s}}) - V_{k_1}(NO^{\underline{s}})| = a_1\binom{k_1}{3}$, where $a_1 \in \{0, 1\}$ (the parameter a_1 is called the loss in the first question).

Proof: Let us denote a query by $[\underline{x}]$. We have to minimize $|V_{k_1}(YES) - V_{k_1}(NO)|$ that can be expressed as $|(2x_3 - s_3)\binom{k_1}{3}|$. Obviously, $x_3 = \frac{s_3}{2}$ if s_3 is even and $x_3 = \frac{s_3+1}{2}$ if s_3 is odd. Then $|V_{k_1}(YES) - V_{k_1}(NO)| \leq \binom{k_1}{3}$. □

Proposition 3.2 *Let $\underline{t} = (t_3, t_2, 0, 0)$ be a state, $k_2 \in \mathbb{N}$, $t_3 \geq 3$ and $t_2 \geq k_2 - 2$. Then there exists a question $[\underline{y}]$ minimizing the volume difference $|V_{k_2}(YES^{\underline{t}}) - V_{k_2}(NO^{\underline{t}})|$. Furthermore, $|V_{k_2}(YES^{\underline{t}}) - V_{k_2}(NO^{\underline{t}})| = a_2[k_2]\binom{k_2}{2}$, where $a_2[k_2] \in \{0, \frac{1}{3}, 1\}$ (the parameter $a_2[k_2]$ is called the loss at the second question).*

Proof: Let us denote a query by $[\underline{y}]$. We have to minimize $|V_{k_2}(YES) - V_{k_2}(NO)|$ that can be expressed as $|\binom{k_2}{2}((2y_3 - t_3)\frac{k_2-2}{3} + 2y_2 - t_2)|$. Depending on k_2 and the values of its components modular 2 we can find a question which minimizes the volume difference. This questions are presented in Appendix A. □

Proposition 3.3 *Let $\underline{u} = (u_3, u_2, u_1, 0)$ be a state, $k_3 \in \mathbb{N}$, $u_3 \geq 4$, $u_2 \geq k_3$ and $u_1 \geq k_3 - 1$. Then there exists a question $[\underline{z}]$ minimizing the volume difference $|V_{k_3}(YES^{\underline{u}}) - V_{k_3}(NO^{\underline{u}})|$. Furthermore, $|V_{k_3}(YES^{\underline{u}}) - V_{k_3}(NO^{\underline{u}})| = a_3[k_3]\binom{k_3}{2}$, where $a_3[k_3] \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1\}$ (the parameter $a_3[k_3]$ is called the loss in the third question).*

Proof: Let us denote a query by $[\underline{z}]$. We have to minimize $|V_{k_3}(YES) - V_{k_3}(NO)|$ that can be expressed as $|k_3(\frac{k_3-1}{2}((2z_3-u_3)\frac{k_3-2}{3}+2z_2-u_2)+2z_1-u_1)|$. Depending on k_3 and the values of its components modular 2 we can find a question which minimizes the volume difference. This questions are presented in Appendix B. □

It turned out to be that the first three questions only depend on k_2, k_3 and the values of its components modular 2. Thus, we write e for an even component and o for an odd one. In Table 3.1 the loss in the second question is given. The proof can be find in Appendix A.

Table 3.1

k_2 mod 6 / state	0	1	2	3	4	5
$(e,e,0,0)$	0	0	0	0	0	0
$(e,o,0,0)$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	1
$(o,e,0,0)$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	1
$(o,o,0,0)$	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	0

In Table 3.2 the loss in the third question is given. The proof can be find in Appendix B.

Table 3.2

k_3 mod 12 / state	0	1	2	3	4	5	6	7	8	9	10	11
$(e,e,e,0)$	0	0	0	0	0	0	0	0	0	0	0	0
$(e,e,o,0)$	0	1	0	$\frac{1}{3}$	0	1	0	1	0	$\frac{1}{3}$	0	1
$(e,o,e,0)$	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$(e,o,o,0)$	$\frac{1}{6}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0
$(o,e,e,0)$	0	0	0	$\frac{1}{3}$	0	0	0	1	0	0	0	1
$(o,e,o,0)$	0	1	0	0	0	1	0	0	0	$\frac{1}{3}$	0	0
$(o,o,e,0)$	$\frac{1}{6}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$(o,o,o,0)$	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1

Remark 3.4 *Let $\underline{s}$ be the initial state and let $k = ch(\underline{s})$.*

1. *We say that Person 2 follows the mini-strategy if he asks in the way defined in Propositions 3.1, A.1, B.1, where $k_1 = k - 1$, $k_2 = k - 2$ and $k_3 = k - 3$. We see in Proposition 3.7 that it is always possible if the character of the initial state is bigger than 17.*

2. *Inspecting the questions in Propositions 3.1, A.1, B.1 and the complementary ones, we get*

$$\begin{aligned}
\frac{N-1}{2} &\leq x_3 \leq \frac{N+1}{2}, \\
\frac{t_3-3}{2} &\leq y_3 \leq \frac{t_3+3}{2}, \\
\frac{t_2-k+4}{2} &\leq y_2 \leq \frac{t_2+k-4}{2}, \\
\frac{u_3-3}{2} &\leq z_3 \leq \frac{u_3+3}{2},
\end{aligned}$$

$$\begin{aligned}\frac{u_2-k+3}{2} &\leq \quad z_2 \leq \frac{u_2+k-3}{2},\\ \frac{u_1-k+4}{2} &\leq \quad z_1 \leq \frac{u_1+k-4}{2}.\end{aligned}$$

Remark 3.5 *For any strategy chosen by Person 2, we can express the loss of the k-volume difference as $a_1\binom{k}{3}$, $a_2[k]\binom{k}{2}$ and $a_3[k]k$ after the first,second and third question, respectively. The difference is always a natural number.*

Definition 3.2 *1. A question is called local optimal if its loss is minimal.*

2. The first three questions are called optimal if they can be used to get a minimal $(k-3)$-th volume of the resulting state.

Remark 3.6 *1. All questions stated in Propositions 3.1, 3.2 and 3.3 are local optimal. We see in Theorem 6.1 that they are also optimal.*

2. The questions stated in Proposition 3.1 are optimal because there does not exist any other local optimal question.

3. The questions stated in Proposition B.1 are optimal because they are the last of the first three questions.

Proposition 3.7 *Let $\underline{s} = (N, 0, 0, 0)$ be a state and let $k = ch(\underline{s})$. If $N > 265$ and we follow the mini-strategy, then Proposition 3.1 with $k_1 = k-1$, Proposition 3.2 with $k_2 = k-2$ and Proposition 3.3 with $k_3 = k-3$ are satisfied.*

Proof: If $N > 265$, then $ch(\underline{s}) > 18$. There is nothing to prove for Proposition 3.1. First we show:

$$\text{If } k > 18, \text{ then } N \geq 6k-7. \tag{3.1}$$

We assume that $k > 18$, but $N < 6k-7$. Then $NG(k) \leq 2^k$ and $NG(k-1) > 2^{k-1}$. However, $NG(k-1) < (6k-7)G(k-1) \leq 2^{k-1}$ if $k > 18$.

1. We show that assumptions of Proposition 3.2 are satisfied if $k_2 > 16$. Let $\underline{t}$ be the state after the first question. After this question, $t_3 \geq \frac{N-1}{2}$ and $t_2 \geq \frac{N-1}{2}$. It follows that $t_3 \geq 132$ and $t_2 \geq \frac{N-1}{2} \geq \frac{6k-8}{2} \geq 3k-4 > (k-2)-2 = k_2-2$.
2. We show that the properties of Proposition 3.3 are satisfied if $k_3 > 15$. Let $\underline{u}$ be the state after the second question. After this question $u_3 \geq \frac{N-7}{4}$, $u_2 \geq \frac{N-k+3}{2}$ and $u_1 \geq \frac{N-2k+1}{4}$. It follows that $u_3 \geq 65$ because $k > 18$, $u_2 \geq \frac{N-k}{2} \geq \frac{5k-7}{2} > (k-3) = k_3$ and $u_1 \geq \frac{N-2k+7}{4} \geq \frac{4k}{2} \geq 2k > (k-3)-1 = k_3-1$.

$\square$

Proposition 3.8 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a state, $ch(\underline{s}) = k \geq 17$ and let $\underline{v}$ be the state after the third question. If Person 2 has chosen the mini-strategy, then $ch(\underline{v}) \leq k-2$.*

Proof: Let $k \geq 9$. We assume $ch(\underline{v}) > k-2$. It follows that $V_{k-2}(\underline{v}) > 2^{k-2}$. Because of Propositions 3.1, 3.2, 3.3 and Berlekamp's Conservation of Volume we have

$$V_{k-2}(\underline{v}) \leq \frac{\frac{\frac{V_{k+1}(\underline{s})+\binom{k}{3}}{2}+\binom{k-1}{2}}{2}+k-2}{2}.$$

It holds that $V_{k+1}(\underline{s}) = V_k(\underline{s}) + s_3\sum_{i=0}^{2}\binom{k}{i} \leq 2^k + \frac{2^k\sum_{i=0}^{2}\binom{k}{i}}{\sum_{i=0}^{3}\binom{k}{i}}$. Thus, $V_{k-2}(\underline{v}) \leq 2^{k-3} + \frac{2^{k-3}\sum_{i=0}^{2}\binom{k}{i}}{\sum_{i=0}^{3}\binom{k}{i}} + \frac{\binom{k}{3}}{8} + \frac{\binom{k-1}{2}}{4} + \frac{k-2}{2}$ and $\frac{2^{k-3}\sum_{i=0}^{2}\binom{k}{i}}{\sum_{i=0}^{3}\binom{k}{i}} + \frac{\binom{k}{3}}{8} + \frac{\binom{k-1}{2}}{4} + \frac{k-2}{2} \leq 2^{k-3}$ if $k \geq 9$. We get $V_{k-2}(\underline{v}) \leq 2^{k-2}$. This is a contradiction. $\square$

Proposition 3.9 *Let $\underline{s} = (s_3, 0, 0, 0)$ a state and $ch(\underline{s}) = k \geq 7$. Let Person 2 use the mini-strategy. The best counter-strategy for Person 1 is to choose the maximal value in every step.*

Proposition 3.10 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a state, $ch(\underline{s}) = k \geq 7$ and let Person 1 play in an optimal way. Person 2 gets the smallest possible character after the third question, only if he uses local optimal questions.*

Proof: Let $\underline{v}$ be the state of the game after the third question and $e = ch(\underline{v})$. If $e = k-3$, then Person 2 cannot do better. Thus, we assume that Person 2 uses local optimal questions and $e = k-2$ ($e > k-2$ is not possible). By Propositions 3.1,Table 3.1 and 3.2, we know how much we lose in each question. Let $\underline{v}$, $\underline{s}$, k , a_1, $a_2[k-2]$ and $a_3[k-3]$, be the worst state after the first three questions, the initial state, the character of $\underline{s}$, the loss in the first, second and third question, respectively. We have to show that there is no other strategy such that $V_{k-3}(\underline{v})$ is smaller than $V_{k-3}(\underline{v})$ by using local optimal questions. By Berlekamp's Conservation of Volume we get

$$V_{k-3}(\underline{v}) = \frac{V_k(\underline{s})}{8} + \underbrace{\frac{a_1\binom{k-1}{3}}{8} + \frac{a_2[k-2]\binom{k-2}{2}}{4} + \frac{a_3[k-3](k-3)}{2}}_{=R}.$$

1. We prove: In order to get a minimal character of the state after the third question, a good strategy for Person 2 is to minimize $|V_{k_1}(YES) - V_{k_1}(NO)|$ in the first step, where $k_1 = k-1$.

   (a) s_3 is even: In this case $R \leq \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$ if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $R \geq \frac{\binom{k-1}{3}}{4}$. But $\frac{\binom{k-1}{3}}{4} > \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$ if $k > 5$.

   (b) s_3 is odd: In this case $R \leq \frac{\binom{k-1}{3}}{8} + \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$, if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $R \geq \frac{3\binom{k-1}{3}}{8}$. But $\frac{\binom{k-1}{3}}{4} > \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$ if $k > 5$.

   Therefore, we know that we have to choose a local optimal question. Let $S = \frac{a_2[k-2]\binom{k-2}{2}}{4} + \frac{a_3[k-3](k-3)}{2}$.

2. We prove: In order to get a minimal character of the state after the third question, a good strategy for Person 2 is to minimize $|V_{k_2}(YES) - V_{k_2}(NO)|$ in the second step, where $k_2 = k-2$. Let D be chosen as in the proof of Proposition A.1.

   (a) D is an odd number: In this case $S \leq \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$ if Person 2 asks a local optimal questions. If he does not ask such a question in the second step, then Person 1 can choose the answers in such a way that $S \geq \frac{3\binom{k-2}{2}}{4}$. But $\binom{k-2}{2} \geq k-3$ if $k > 2$.

   (b) D is an even number: In this case $S \leq \frac{k-3}{2}$ if Person 2 asks a local optimal questions. If he does not ask such a question in the second step, then Person 1 can choose the answers in such a way that $S \geq \frac{\binom{k-2}{2}}{2}$. But $\binom{k-2}{2} \geq k-3$ if $k > 2$.

(c) $D = \frac{z}{3}$, where z is an odd number not divisible by 3: In this case $S \leq \frac{\binom{k-2}{2}}{12} + \frac{k-3}{2}$ if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $S \geq \frac{\binom{k-2}{2}}{4}$. But $\frac{\binom{k-2}{2}}{4} > \frac{\binom{k-2}{2}}{12} + \frac{k-3}{2}$ if $k > 6$.

(d) $D = \frac{z}{3}$, where z is an even number not divisible by 3: In this case $S \leq \frac{k-3}{2}$ if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $S \geq \frac{\binom{k-2}{2}}{3}$. But $\frac{\binom{k-2}{2}}{3} > \frac{k-3}{3}$ if $k > 6$.

3. We prove: In order to get a minimal character of the state after the third question, a good strategy for Person 2 is to minimize $|V_{k_3}(YES) - V_{k_3}(NO)|$ in the third step, where $k_3 = k - 3$. This is obvious.

□

4 Typical questions

Definition 4.1 *Let $\underline{v} = (v_3, v_2, v_1, v_0)$ be a state and $ch(\underline{v}) = k$. The state is called typical if*

1. $3v_3 < v_2 + \frac{3}{2}k$,
2. $v_2 < v_1 + \frac{15}{8}k$,
3. $v_0 > k$.

Proposition 4.1 *Let $k = ch((N, 0, 0, 0)) > 18$ and Person 2 follows the mini-strategy (c.f. Remark 3.4). Then the state is typical after the first three questions.*

Proof: Let $(N, 0, 0, 0)$, $(t_3, t_2, 0, 0)$, $(u_3, u_2, u_1, 0)$ and (v_3, v_2, v_1, v_0) be the states before the first, second, third and fourth question by using the mini-strategy. We can assume, that Person 1 always has to answer the first question with yes to get the minimal or maximal value of v_3, v_2, v_1 or v_0 because Person 2 can ask in such a way that $V_j(YES^{\underline{s}}) \geq V_j(NO^{\underline{s}})$. If $V_j(YES^{\underline{s}}) = V_j(NO^{\underline{s}})$, then the two states are equal. We also have to consider the complementary questions of Propositions 3.1, A.1 and B.1. Let $[x_3, 0, 0, 0]$, $[y_3, y_2, 0, 0]$ and $[z_3, z_2, z_1, 0]$ be the first three questions and $c = ch(\underline{v})$. Since $c + 2 \leq k \leq c + 3$, we have $c > 15$.

1. We calculate the maximal value for v_3 and the minimal value for v_2 and show that $3v_3 < v_2 + \frac{3}{2}c$.

   (a) We first consider the case when Person 1 answers yes at the second and third question. We know that $v_3 = z_3$ and $v_2 = z_2 + u_3 - z_3$. With Proposition 3.3 we get $z_3 \leq \frac{u_3+3}{2}$ and $z_2 \geq \frac{u_2-k+3}{2}$. Thus, $v_3 \leq \frac{u_3+3}{2}$ and $v_2 \geq \frac{u_2-k+3}{2} + \frac{u_3-3}{2}$. Hence, $u_3 = y_3$ and $u_2 = y_2 + t_3 - y_3$. With Proposition 3.2 we get $y_3 \leq \frac{t_3+3}{2}$ and $y_2 \geq \frac{t_2-k+4}{2}$. Thus, $v_3 \leq \frac{t_3+9}{4}$ and $v_2 \geq \frac{t_2+2t_3-3k+4}{4}$. Hence, $t_2 = N - x_3$ and $t_3 = x_3$. With Proposition 3.1 we get $\frac{N-1}{2} \leq x_3 \leq \frac{N+1}{2}$. Thus, $v_3 \leq \frac{N+19}{8}$ and $v_2 \geq \frac{3N+7-6k}{8} \geq \frac{3N-11-6c}{8}$. It is easy to see that $3\frac{N+19}{8} < \frac{3N-11-6c}{8} + \frac{3}{2}c$ if $c > 11$.

   (b) Let us consider the case when Person 1 answers yes at the second and no in the third question. We know that $v_3 = y_3 - z_3$ and $v_2 = u_2 - z_2 + z_3$. With Proposition 3.3 we get $z_3 \geq \frac{u_3-3}{2}$ and $z_2 \leq \frac{u_2+k-3}{2}$. Thus, $v_3 \leq y_3 - \frac{u_3-3}{2}$ and $v_2 \geq \frac{u_2-k+u_3}{2}$. Hence, $u_3 = y_3$ and $u_2 = y_2 + t_3 - y_3$.

With Proposition 3.2 we get $\frac{t_3-3}{2} \leq y_3 \leq \frac{t_3+3}{2}$ and $y_2 \geq \frac{t_2-k+4}{2}$. Thus, $v_3 \leq \frac{t_3+9}{4}$ and $v_2 \geq \frac{t_2+2t_3-3k+4}{4}$. Hence, $t_2 = N - x_3$ and $t_3 = x_3$. With Proposition 3.1 we get $\frac{N-1}{2} \leq x_3 \leq \frac{N+1}{2}$. Thus, $v_3 \leq \frac{N+19}{8}$ and $v_2 \geq \frac{3N+7-6k}{8} \geq \frac{3N-11-6c}{8}$. It is easy to see that $3\frac{N+19}{8} < \frac{3N-11-6c}{8} + \frac{3}{2}c$ if $c > 11$.

(c) Let us consider the case when Person 1 answers no at the second and yes in the third question. We know that $v_3 = z_3$ and $v_2 = z_2 + u_3 - z_3$. With Proposition 3.3 we get $z_3 \leq \frac{u_3+3}{2}$ and $z_2 \geq \frac{u_2-k+3}{2}$. Thus, $v_3 \leq \frac{u_3+3}{2}$ and $v_2 \geq \frac{u_2-k+u_3}{2}$. Hence, $u_3 = t_3 - y_3$ and $u_2 = y_3 + t_2 - y_2$. With Proposition 3.2 we get $\frac{t_3-3}{2} \leq y_3$ and $y_2 \leq \frac{t_2+k-4}{2}$. Thus, $v_3 \leq \frac{t_3+9}{4}$ and $v_2 \geq \frac{t_2+2t_3-3k+4}{4}$. Hence, $t_2 = N - x_3$ and $t_3 = x_3$. With Proposition 3.1 we get $\frac{N-1}{2} \leq x_3 \leq \frac{N+1}{2}$. Thus, $v_3 \leq \frac{N+19}{8}$ and $v_2 \geq \frac{3N+7-6k}{8} \geq \frac{3N-11-6c}{8}$. It is easy to see that $3\frac{N+19}{8} < \frac{3N-11-6c}{8} + \frac{3}{2}c$ if $c > 11$.

(d) Let us consider the case when Person 1 answers no at the second and third question. We know that $v_3 = u_3 - z_3$ and $v_2 = u_2 - z_2 + z_3$. With Proposition 3.3 we get $z_3 \geq \frac{u_3-3}{2}$ and $z_2 \leq \frac{u_2+k-3}{2}$. Thus, $v_3 \leq \frac{u_3+3}{2}$ and $v_2 \geq \frac{u_2-k+u_3}{2}$. Hence, $u_3 = t_3 - y_3$ and $u_2 = y_3 + t_2 - y_2$. With Proposition 3.2 we get $\frac{t_3-3}{2} \leq y_3$ and $y_2 \leq \frac{t_2+k-4}{2}$. Thus, $v_3 \leq \frac{t_3+9}{4}$ and $v_2 \geq \frac{t_2+2t_3-3k+4}{4}$. Hence, $t_2 = N - x_3$ and $t_3 = x_3$. With Proposition 3.1 we get $\frac{N-1}{2} \leq x_3 \leq \frac{N+1}{2}$. Thus, $v_3 \leq \frac{N+19}{8}$ and $v_2 \geq \frac{3N+7-6k}{8} \geq \frac{3N-11-6c}{8}$. It is easy to see that $3\frac{N+19}{8} < \frac{3N-11-6c}{8} + \frac{3}{2}c$ if $c > 11$.

2. We calculate the maximal value for v_2 and the minimal value for v_1 and show then $v_2 \leq v_1 + \frac{15}{8}k$.

   (a) We first consider the case when Person 1 answers yes at the second and third question. We know that $v_2 = z_2 + u_3 - z_3$ and $v_1 = z_1 + u_2 - z_2$. With Proposition 3.3 we get $z_3 \geq \frac{u_3-3}{2}$, $z_2 \leq \frac{u_2+k-3}{2}$ and $z_1 \geq \frac{u_1-k+4}{2}$. Thus, $v_2 \leq \frac{u_2+u_3+k}{2}$ and $v_1 \geq \frac{u_1+u_2+7-2k}{2}$. Hence, $u_1 = t_2 - y_2$, $u_2 = y_2 + t_3 - y_3$ and $u_3 = y_3$. With Proposition 3.2 we get $y_3 \leq \frac{t_3+3}{2}$ and $y_2 \leq \frac{t_2+k-4}{2}$. Thus, $v_2 \leq \frac{t_2+2t_3+3k-4}{4}$ and $v_1 \geq \frac{2t_2+t_3-4k+11}{4}$. Hence, $t_2 = N - x_3$ and $t_3 = x_3$. With Proposition 3.1 we get $\frac{N-1}{2} \leq x_3 \leq \frac{N+1}{2}$. Thus, $v_2 \leq \frac{3N+6k-7}{8} \leq \frac{3N+6c+11}{8}$ and $v_1 \geq \frac{3N-8k+22}{8} \geq \frac{3N-8c-1}{8}$. It is easy to see that $\frac{3N+6c+11}{8} < \frac{3N-8c-1}{8} + \frac{15}{8}c$ if $c > 12$.

   (b) For the other cases we get the same result because of the complement questions.

3. We calculate the minimal value for v_0 and show then this value is bigger k.

   (a) We know that $v_0 = u_1 - z_1$. With Proposition 3.3 we get $z_1 \leq \frac{u_1-4+k}{2}$. Thus, $v_0 \geq \frac{u_1-k+4}{2}$. Hence, $u_1 = t_2 - y_2$. With Proposition 3.2 we get $y_2 \leq \frac{t_2+k-4}{2}$. Thus, $v_0 \geq \frac{t_2+12-3k}{4}$. Hence, $t_2 = N - x_3$. With Proposition 3.1 we get $x_3 \leq \frac{N+1}{2}$. Thus, $v_0 \geq \frac{N+23-6k}{8}$. And $\frac{N+23-6k}{8} > k > c$ if $k > 18$.

   (b) For the other cases we get the same result because of the complement questions. $\square$

Proposition 4.2 *Let $\underline{v}$ be a typical state and $ch(\underline{v}) = k + 1 \geq 7$. Then the state is balanced and can be reduced to two typical states if $v_0 > \binom{k}{3} + 2k$.*

Proof: We have to consider eight cases. We show that we can always find a question $[\underline{x}]$ which balances the state $\underline{v}$ to two typical states: $YES^{\underline{v}} = \underline{y} = (x_3, x_2 + v_3 -$

$x_3, x_1+v_2-x_2, x_0+v_1-x_1)$ and $NO^{\underline{v}} = \underline{n} = (v_3-x_3, v_2-x_2+x_3, v_1-x_1+x_2, v_0-x_0+x_1)$. We use the fact that $\underline{v}$ is typical. That means that we use the inequalities: $3v_3 < v_2 + \frac{3}{2}(k+1)$, $v_2 < v_1 + 2(k+1)$, $v_0 > \binom{k}{3} + 2k$, $v_0 > 6$. In all cases it is easy to verify that all questions are legal. It holds that $ch(YES^{\underline{v}}) = ch(NO^{\underline{v}}) = k$.

1. v_3 is even, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil$. Obviously, the resulting states are balanced. We have to show that they are typical:

$$3y_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3}{4} + \frac{3k}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = y_2 + \frac{3}{2}k,$$

$$y_2 = \frac{v_2}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k+1}{4} < \frac{v_1}{2} + \frac{5k+5}{4} + \frac{v_2}{2} < y_1 + \frac{15}{8}k,$$

$$y_0 \geq \frac{v_0}{2} + \frac{v_1}{2} > k,$$

$$3n_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3}{4} + \frac{3k}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = n_2 + \frac{3}{2}k,$$

$$n_2 = \frac{v_2}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k}{4} < \frac{v_1}{2} + \frac{5k+4}{4} + \frac{v_2}{2} < n_1 + \frac{15}{8}k,$$

$$n_0 \geq \frac{v_0}{2} - 1 + \frac{v_1}{2} > k.$$

2. v_3 is odd, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}}{2} \right\rfloor$. Obviously, the resulting states are balanced. We have to show that they are typical:

$$3y_3 = 3\frac{v_3+1}{2} < \frac{v_2}{2} + \frac{5}{4} + \frac{3k}{4} < \frac{v_2}{2} + \frac{v_3-1}{2} + \frac{3}{2}k = y_2 + \frac{3}{2}k,$$

$$y_2 = \frac{v_2}{2} + \frac{v_3-1}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k}{4} < \frac{v_1}{2} + \frac{5k+4}{4} + \frac{v_2}{2} < y_1 + \frac{15}{8}k,$$

$$y_0 \geq \frac{v_0}{2} - \frac{\binom{k}{3}}{2} + \frac{v_1}{2} > \frac{\binom{k}{3}}{2} + k - \frac{\binom{k}{3}}{2} + \frac{v_1}{2} > k,$$

$$3n_3 = 3\frac{v_3-1}{2} < \frac{v_2}{2} + \frac{3}{4} + \frac{3k}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = n_2 + \frac{3}{2}k,$$

$$n_2 = \frac{v_2}{2} + \frac{v_3+1}{2} < \frac{v_1}{2} + k + \frac{v_2}{6} + \frac{k}{4} + \frac{3}{2} < \frac{v_1}{2} + \frac{v_2}{2} + \frac{5k+6}{4} = n_1 + \frac{15}{8}k,$$

$$n_0 \geq \frac{v_0}{2} - 2 - \frac{\binom{k}{3}}{2} + \frac{v_1}{2} > k.$$

3. v_3 is even, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{2}}{2} \right\rfloor$. Obviously, the resulting states are balanced. We have to show that they are typical:

$$3y_3 = 3\frac{v_3}{2} < \frac{v_2+1}{2} + \frac{1}{4} + \frac{3k}{4} < \frac{v_2+1}{2} + \frac{v_3}{2} + \frac{3}{2}k = y_2 + \frac{3}{2}k,$$

$$y_2 = \frac{v_2+1}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + \frac{5}{2} + \frac{v_2}{6} + \frac{k}{4} - \frac{1}{2} < \frac{v_1}{2} + \frac{v_2-1}{2} + \frac{5k+10}{4} < y_1 + \frac{15}{8}k,$$

$$y_0 \geq \frac{v_0}{2} - \frac{\binom{k}{2}}{2} + \frac{v_1}{2} > \frac{\binom{k}{3}}{2} + k - \frac{\binom{k}{2}}{2} + \frac{v_1}{2} > k,$$

$$3n_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3}{4} + \frac{3k}{4} < \frac{v_2 - 1}{2} + \frac{v_3}{2} + \frac{3k+1}{2} < n_2 + \frac{3}{2}k,$$

$$n_2 = \frac{v_2 - 1}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + \frac{v_2}{6} + \frac{k+1}{4} < \frac{v_1}{2} + \frac{5k+1}{4} + \frac{v_2+1}{2} < n_1 + \frac{15}{8}k,$$

$$n_0 \geq \frac{v_0}{2} - 2 - \frac{\binom{k}{2}}{2} + \frac{v_1}{2} > k.$$

4. v_3 is odd, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

5. v_3 is even, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1+1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \lfloor \frac{k}{2} \rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

6. v_3 is odd, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1-1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}-k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

7. v_3 is even, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1-1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{2}-k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

8. v_3 is odd, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1-1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}-k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical. □

Proposition 4.3 *Let $\underline{v}$ be a typical state and $ch(\underline{v}) = k+1 \geq 7$. Then the state is balanced and can be reduced to two typical states, if $v_1 > \binom{k}{2} + 2k$.*

Proof: We have to consider eight cases. We show that we can always find a question $[\underline{x}]$ which balances the state $\underline{v}$ to two typical states: $YES^{\underline{v}} = \underline{y} = (x_3, x_2 + v_3 - x_3, x_1 + v_2 - x_2, x_0 + v_1 - x_1)$ and $NO^{\underline{v}} = \underline{n} = (v_3 - x_3, v_2 - x_2 + x_3, v_1 - x_1 + x_2, v_0 - x_0 + x_1)$. In all cases it is easy to verify that all questions are legal. It holds that $ch(YES^{\underline{v}}) = ch(NO^{\underline{v}}) = k$.

1. v_3 is even, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil$. Obviously, the resulting states are balanced. We have to show that they are typical:

$$3y_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3}{4} + \frac{3k}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = y_2 + \frac{3}{2}k,$$

$$y_2 = \frac{v_2}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k+1}{4} < \frac{v_1}{2} + \frac{5k+5}{4} + \frac{v_2}{2} < y_1 + \frac{15}{8}k,$$

$$y_0 \geq \frac{v_0}{2} + \frac{v_1}{2} > k,$$

$$3n_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3}{4} + \frac{3k}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = n_2 + \frac{3}{2}k,$$

$$n_2 = \frac{v_2}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k+1}{4} < \frac{v_1}{2} + \frac{5k+5}{4} + \frac{v_2}{2} < n_1 + \frac{15}{8}k,$$

$$n_0 \geq \frac{v_0}{2} - 1 + \frac{v_1}{2} > k.$$

2. v_3 is odd, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{(k-1)(k-2)}{12} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced. We have to show that they are typical:

$$y_0 \geq v_1 - x_1 \geq \frac{v_1}{2} - \frac{(k-1)(k-2)}{12} > \frac{\binom{k}{2}}{2} + k - \frac{\binom{k}{2}}{6} > k,$$

$$n_0 \geq x_1 \geq \frac{v_1}{2} - \frac{(k-1)(k-2)}{12} > \frac{\binom{k}{2}}{2} + k - \frac{\binom{k}{2}}{6} > k.$$

3. v_3 is even, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1}{2} - \left\lceil \frac{(k-1)}{4} \right\rceil$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

4. v_3 is odd, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1}{2} - \left\lceil \frac{\binom{k}{3}-\binom{k}{2}}{2k} \right\rceil$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

5. v_3 is even, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1+1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \lfloor \frac{k}{2} \rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

6. v_3 is odd, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lceil \frac{\binom{k}{3}-k}{2k} \right\rceil$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

7. v_3 is even, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lceil \frac{\binom{k}{2}-k}{2k} \right\rceil$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

8. v_3 is odd, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lceil \frac{\binom{k}{3}-\binom{k}{2}-k}{2k} \right\rceil$ and $x_0 = \lceil \frac{v_0}{2} \rceil - \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical. □

Proposition 4.4 *Let $\underline{v}$ be a typical state and $ch(\underline{v}) = k+1 \geq 7$. Then the state is balanced and can be reduced to two typical states, if $v_2 > 4k$.*

Proof: We have to consider eight cases. We show that we can always find a question $[\underline{x}]$ which balances the state $\underline{v}$ to two typical states: $YES^{\underline{v}} = \underline{y} = (x_3, x_2 + v_3 - x_3, x_1 + v_2 - x_2, x_0 + v_1 - x_1)$ and $NO^{\underline{v}} = \underline{n} = (v_3 - x_3, v_2 - x_2 + x_3, v_1 - x_1 + x_2, v_0 - x_0 + x_1)$. If $v_2 > 4k$, then $v_1 > 2k$. In all cases it is easy to verify that all questions are legal. It holds that $ch(YES^{\underline{v}}) = ch(NO^{\underline{v}}) = k$.

1. v_3 is even, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil$. Obviously, the resulting states are balanced. We have to show that they are typical:

$$3y_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3(k+1)}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = y_2 + \frac{3}{2}k,$$

$$y_2 = \frac{v_2}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k+1}{4} < \frac{v_1}{2} + \frac{5k+5}{4} + \frac{v_2}{2} < y_1 + \frac{15k}{8},$$

$$y_0 \geq \frac{v_0}{2} + \frac{v_1}{2} > k,$$

$$3n_3 = 3\frac{v_3}{2} < \frac{v_2}{2} + \frac{3(k+1)}{4} < \frac{v_2}{2} + \frac{v_3}{2} + \frac{3}{2}k = n_2 + \frac{3}{2}k,$$

$$n_2 = \frac{v_2}{2} + \frac{v_3}{2} < \frac{v_1}{2} + k + 1 + \frac{v_2}{6} + \frac{k+1}{4} < \frac{v_1}{2} + \frac{5k+5}{4} + \frac{v_2}{2} < n_1 + \frac{15k}{8},$$

$$n_0 \geq \frac{v_0}{2} - 1 + \frac{v_1}{2} > k.$$

2. v_3 is odd, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2} - \lfloor \frac{k-2}{6} \rfloor$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}}{2k} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

3. v_3 is even, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{\binom{k}{2}}{2k} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

4. v_3 is odd, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2} - \lfloor \frac{k-5}{6} \rfloor$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}}{2k} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

5. v_3 is even, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1+1}{2}$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \lfloor \frac{k}{2} \rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

6. v_3 is odd, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2} - \lfloor \frac{k-2}{6} \rfloor$, $x_1 = \frac{v_1-1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}-k}{2k} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

7. v_3 is even, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lfloor \frac{\binom{k}{2}-k}{2k} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

8. v_3 is odd, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2} - \lfloor \frac{k-5}{6} \rfloor$, $x_1 = \frac{v_1-1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_1-v_1)\binom{k}{2}-k}{2k} \right\rfloor$ and $x_0 = \lceil \frac{v_0}{2} \rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$. Obviously, the resulting states are balanced and it is easy to verify that they are typical.

□

Proposition 4.5 *Let $\underline{v}$ be a typical state with $ch(\underline{v}) = k+1 > 15$. Then $v_2 > 4k$, $v_1 > \binom{k}{2} + 2k$ or $v_0 > \binom{k}{3} + 2k$.*

Proof: We assume the contrary. Let $v_2 \leq 4k$, $v_1 \leq \binom{k}{2} + 2k$ and $v_0 \leq \binom{k}{3} + 2k$. It holds that $3v_3 < v_2 + \frac{3}{2}k$ thus, $v_3 < 2k$. Therefore,

$$V_k(\underline{v}) = v_3 G(k) + v_2 F(k) + v_1(k+1) + v_0$$

$$< 2kG(k) + 4kF(k) + (\binom{k}{2} + 2k)(k+1) + \binom{k}{3} + 2k$$
$$= \frac{k}{6}(2k^3 + 16k^2 + 31k + 59).$$

If $k > 15$, then
$$V_k(\underline{v}) < \frac{k}{6}(2k^3 + 16k^2 + 31k + 59) < 2^k.$$

This is a contradiction to the hypothesis that $ch(\underline{v}) = k + 1$. □

5 The last questions

We show in this section that all typical states with a character bigger than or equal to 15 are nice. From the previous section we know that we can reduce a typical state to two balanced typical states if its character is bigger than 15. Thus, we have to show that all typical states with character 15 are nice (starting from this state Person 2 needs 15 questions to win the game). To show this we use the Guzicki Algorithm. This algorithm, originally described by the case of two lies, can be generalized by the case of three lies without essential efforts.

Lemma 5.1 *Let $\underline{v}$ be an k-winning state, $ch(\underline{v}) \geq 1$, and let $[\underline{x}]$ be the first question of the k-winning strategy chosen in such a way that $V_{k-1}(YES^{\underline{v}}) \geq V_{k-1}(NO^{\underline{v}})$.*

1. *If $V_k(\underline{v}) < 2^k$, then $\underline{v}' = (v_3, v_2, v_1, v_0 + 1)$ is also an k-winning state and there exists an k-winning strategy with the first question $[\underline{x}]$.*
2. *If $V_k(\underline{v}) = 2^k$, then $\underline{v}' = (v_3, v_2, v_1, v_0 + 1)$ is a $(k+1)$-winning state and there exists a $(k+1)$-winning strategy with the first question $[\underline{x}]$.*

Proof: We prove Lemma 5.1 by induction in k.

- For $k = 1$ and $k = 2$ the statement is true.
- $k - 1 \to k$: If $\underline{v}$ is k-winning, then Person 2 has a winning strategy with k questions. Let $[\underline{x}]$ be the first question of this strategy chosen in such a way that $V_{k-1}(YES^{\underline{v}}) \geq V_{k-1}(NO^{\underline{v}})$.
    1. : $V_k(\underline{v}) < 2^k$. We can reduce the state to the two $(k-1)$-winning states $YES^{\underline{v}}$ and $NO^{\underline{v}} = (n_3, n_2, n_1, n_0)$ by the question $[\underline{x}]$. By Berlekamp's Conservation of Volume, it follows that $V_{k-1}(NO^{\underline{v}}) < 2^{k-1}$. Therefore, $NO' = (n_3, n_2, n_1, n_0 + 1)$ is also $(k-1)$-winning by induction. We can reduce $(v_3, v_2, v_1, v_0 + 1)$ to the $(k-1)$-winning states $YES^{\underline{v}}$ and NO'. It follows that this state is also k-winning.
    2. : $V_k(\underline{v}) = 2^k$. We can reduce the state to the two $(k-1)$-winning states $YES^{\underline{v}}$ and $NO^{\underline{v}} = (n_3, n_2, n_1, n_0)$ by the question $[\underline{x}]$. By Berlekamp's Conservation of Volume, it follows that $V_{k-1}(NO^{\underline{v}}) = 2^{k-1}$. Therefore, $NO' = (n_3, n_2, n_1, n_0 + 1)$ is k-winning by induction. We can reduce $(v_3, v_2, v_1, v_0 + 1)$ to the $(k-1)$-winning states $YES^{\underline{v}}$ and the k-winning state NO'. It follows that this state is $k + 1$-winning. □

Corollary 5.2 *Let $\underline{v}$ be a nice state. Then $\underline{v} = (v_3, v_2, v_1, v_0 + 1)$ is also a nice state.*

Proof: If $ch(\underline{v}) = 0$ the statement can be easily checked. In the other cases it follows directly with Lemma 5.1. □

Definition 5.1 *Let $v_3, v_2, v_1 \in \mathbb{N}$. Then*

$$M(v_3, v_2, v_1) \triangleq \min\{v_0 : \textit{ the state } (v_3, v_2, v_1, v_0) \textit{ is nice}\}.$$

Lemma 5.3 *If $M(v_3, v_2, v_1) = c$ and $ch((v_3.v_2, v_1, c)) = k$, then $c = 0$ or $V_{k-1}((v_3, v_2, v_1, c)) = 2^{k-1} + 1$.*

Proof: Let $M(v_3, v_2, v_1) = c$, $ch((v_3.v_2, v_1, c)) = k$ and $c \neq 0$. We assume that $V_{k-1}(v_3, v_2, v_1, c) > 2^{k-1} + 1$. Therefore, $V_{k-1}(v_3, v_2, v_1, c-1) > 2^{k-1}$ and $ch(v_3, v_2, v_1, c-1) = k$. Thus, the state $(v_3, v_2, v_1, c-1)$ is nice. This contradicts the equation $M(v_3, v_2, v_1) = c$.

□

Definition 5.2 *Let $k = ch(v_3, v_2, v_1, c)$. Then $C(v_3, v_2, v_1, x_3, x_2, x_1) \triangleq \min\{c :$ there exists an z and a question $[x_3, x_2, x_1, z]$ which reduces the state (v_3, v_2, v_1, c) to two $(k-1)$-winning states$\}$. We set*

$$C(v_3, v_2, v_1, 0, 0, 0) = \infty,$$

$$C(v_3, v_2, v_1, v_3, v_2, v_1) = \infty.$$

Calculation of $M(v_3, v_2, v_1)$:
$M(0,0,0) = 1$,
$M(0,0,1) = 0$,
$M(0,1,0) = 0$,
$M(1,0,0) = 0$,
$M(v_3, v_2, v_1) = \min\{C(v_3, v_2, v_1, x_3, x_2, x_1) : 0 \leq x_3 \leq \lceil \frac{v_3}{2} \rceil, 0 \leq x_2 \leq \lceil \frac{v_2}{2} \rceil,$
$0 \leq x_1 \leq \lceil \frac{v_1}{2} \rceil\}$.

Calculation of $C(v_3, v_2, v_1, x_3, x_2, x_1)$:

Consider the state $\underline{v} = (v_3, v_2, v_1, 0)$ and the question $[x_3, x_2, x_1, 0]$. Furthermore, set $YES^{\underline{v}} = (y_3, y_2, y_1, y_0)$, $NO^{\underline{v}} = (n_3, n_2, n_1, n_0)$,
$c_1 = M(y_3, y_2, y_1)$,
$c_2 = M(n_3, n_2, n_1)$,
$k_1 = ch(YES^{\underline{v}})$,
$k_2 = ch(NO^{\underline{v}})$,
$k_3 = ch(y_3, y_2, y_1, c_1)$,
$k_4 = ch(n_3, n_2, n_1, c_2)$.

Proposition 5.4 *Let $k = \max\{k_1, k_2, k_3, k_4\}$, $(x_3, x_2, x_1) \neq (0,0,0)$ and $(x_3, x_2, x_1) \neq (v_3, v_2, v_1)$. Then*

$$C(v_3, v_2, v_1, x_3, x_2, x_1) = \min\{c : ch(v_3, v_2, v_1, c) > k\}.$$

Proof: Let $C(v_3, v_2, v_1, x_3, x_2, x_1) = c$. First we show that $ch(v_3, v_2, v_1, c) > k$. Suppose that $ch(v_3, v_2, v_1, c) \leq k$. Then (see the definition of C), there exists a question $[(x_3, x_2, x_1, z)]$ that reduces (v_3, v_2, v_1, c) to the $(k-1)$-winning states $YES' = (y_3, y_2, y_1, y_0 + z)$ and $NO' = (n_3, n_2, n_1, n_0 + c - z)$, where $y_3, y_2, y_1, y_0, n_3, n_2, n_1, n_0$ are defined above.

- Case 1: $k = k_1$. In this case $ch(YES^{\underline{v}}) = k$. It follows that $V_{k-1}(YES') \geq V_{k-1}(YES^{\underline{v}}) > 2^{k-1}$. This is a contradiction.
- Case 2: $k = k_2$. In this case $ch(NO^{\underline{v}}) = k$. It follows that $V_{k-1}(NO') \geq V_{k-1}(NO^{\underline{v}}) > 2^{k-1}$. This is a contradiction.

- Case 3: $k = k_3$. In this case $ch(y_3, y_2, y_1, c_1) = k$, (y_3, y_2, y_1, c_1) is nice because of the definition of M and for every $c' < c_1$ it holds that (y_3, y_2, y_1, c') is not nice. It follows from Lemma 5.3 that we can distinguish between two sub-cases:

  1. $c_1 > 0$. In this sub-case $V_{k-1}(y_3, y_2, y_1, c_1) = 2^{k-1} + 1$. We get
  $$V_{k-1}(y_3, y_2, y_1, c_1 - 1) = 2^{k-1}.$$
  $(y_3, y_2, y_1, c_1 - 1)$ is nice because $(y_3, y_2, y_1, 0)$ is $(k-1)$-winning. This is a contradiction to that c_1 is minimal.
  2. $c_1 = 0$. In this sub-case $V_{k-1}(y_3, y_2, y_1, 0) > 2^{k-1}$. Thus, YES' is not $(k-1)$-winning. This is a contradiction.

- Case 4: $k = k_4$. In this case $ch(y_3, y_2, y_1, c_2) = k$, (n_3, n_2, n_1, c_2) is nice and for every $c' < c_2$ it holds that (n_3, n_2, n_1, c') is not nice. It follows from Lemma 5.3 that we can distinguish between two sub-cases:

  1. $c_2 > 0$. In this sub-case $V_{k-1}(n_3, n_2, n_1, c_2) = 2^{k-1} + 1$. We get
  $$V_{k-1}(n_3, n_2, n_1, c_2 - 1) = 2^{k-1}.$$
  $(n_3, n_2, n_1, c_2 - 1)$ is nice because $(n_3, n_2, n_1, 0)$ is $(k-1)$-winning. This is a contradiction to that c_2 is minimal.
  2. $c_2 = 0$. In this sub-case $V_{k-1}(n_3, n_2, n_1, c_2) > 2^{k-1}$. Thus, NO' is not $(k-1)$-winning. This is a contradiction.

Now we have proved that $C(v_3, v_2, v_1, x_3, x_2, x_1) \geq \min\{c : ch(v_3, v_2, v_1, c) > k\}$. Let $c = \min\{c' : ch(v_3, v_2, v_1, c') > k\}$. We show that the question $[x_3, x_2, x_1, 0]$ reduces (v_3, v_2, v_1, c) to two k-winning states.

- $c = 0$: Because of the definition of k_1 and k_2 we get $ch(YES^{\underline{v}}) \leq k$ and $ch(NO^{\underline{v}}) \leq k$. Let us consider the $YES^{\underline{v}}$ state. If $y_0 \geq c_1$, then $YES^{\underline{v}}$ is k-winning due to the definition of function M. If $y_0 < c_1$, then $k_1 \leq k_3 \leq k$. Thus, the state is k-winning. Let us consider the $NO^{\underline{v}}$ state. If $n_0 \geq c_2$, then NO is k-winning because of the definition of function M. If $y_0 < c$, then $k_2 \leq k_4 \leq k$. Thus, the state is k-winning.

- $c > 0$: In this case holds $V_k(v_3, v_2, v_1, c) > 2^k$ and $V_k(v_3, v_2, v_1, c - 1) \leq 2^k$ because c is minimal. Therefore, we get $V_k(v_3, v_2, v_1, c) = 2^k + 1$ and $V_k(v_3, v_2, v_1, 0) \leq V_k(v_3, v_2, v_1, c-1) = 2^k$. If we ask the question $[x_3, x_2, x_1, 0]$, then $YES' = YES^{\underline{v}} = (y_3, y_2, y_1, y_0)$ and $NO' = NO^{\underline{v}} = (n_3, n_2, n_1, n_0 + c)$. Obviously, $ch(YES') = ch(YES^{\underline{v}}) \leq k$. In order to show that $ch(NO') \leq k$, it suffices to show that $V_k(NO') < V_k(v_3, v_2, v_1, c)$. This condition is satisfied because $x_1(k+1) \geq x_1$, $x_2F(k) \geq x_2(k+1)$ and $x_3G(k) \geq x_3F(k)$ (G and F are defined in Notation 2.7), if $k > 0$ and $(x_3, x_2, x_1) \neq (0, 0, 0)$ (that means: one of the three terms is really bigger). It remains to show that the states YES' and NO' are $k - winning$. YES' is k-winning because $YES^{\underline{v}}$ is k-winning. Let us consider NO'. If $n_0 + c \geq c_2$, then it is k-winning by the definition of function M. If $n_0 + c < c_2$, then it is k_4-winning and thus k-winning.

□

Now we can write a computer program and with Proposition 4.5 obtain the following result:

Proposition 5.5 *Let $\underline{v}$ be a typical state and $k = ch(\underline{v})$, $\underline{v}$ is nice if $k \geq 15$.*

Proposition 5.6 *If $k \geq 15$, Person 2 needs at most $ch(k)+1$ questions to win the game.*

Proof: This follows by Proposition 5.5 and Proposition 3.8. $\square$

The Guzicki Algorithm gives also a strategy. Let (v_3, v_2, v_1, v_0) be a state, $k = ch(v_3, v_2, v_1, v_0)$, $v_0' = M(v_3, v_2, v_1)$ and $k' = ch(v_3, v_2, v_1, v_0')$. If $v_0 \leq v_0'$. Then (v_3, v_2, v_1, v_0) is a k'-winning state. We can reduce it to two $(k'-1)$-winning states by the question $[x_3, x_2, x_1, 0]$, where x_3, x_2, x_1 minimizes the function C. If $v_0 > v_0'$, then (v_3, v_2, v_1, v_0) is a k-winning state. The question $[x_3, x_2, x_1, x_0]$, which reduces the state to two $(k-1)$-winning states can be calculated with Lemma 5.1.

6 The main result

Let us put all properties together. If $N > 265$, then the character of the initial state is bigger than 18. In this case we can use the optimal mini-strategy for the first three questions and get a nice state. The analysis is given in Theorem 6.1. If $N \leq 265$, we need some more considerations.

Proposition 6.1 *Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$. Person 2 needs k questions to win the game, if $N \in M_0$.*
$M_0 = \{1, 2, 14, \ldots, 16, 22, \ldots, 28, 35, \ldots, 50, 57, \ldots, 88, 95, \ldots, 154, 158, \ldots, 265\}$.

Proof: For all $a \in M_0$, it holds that $M(a, 0, 0) = 0$. $\square$

Definition 6.1 *Let $\underline{x} = \begin{pmatrix} x_k \\ \vdots \\ x_0 \end{pmatrix}$ be a state. Then $T(\underline{x}) = \underline{t} = \begin{pmatrix} t_{k-1} \\ \vdots \\ t_0 \end{pmatrix}$ with $t_i = x_{i+1}$ is called the translation of $\underline{x}$.*

Remark 6.2 *Let $\underline{x}$ be a doublet state with $a \in S_i$ and $b \in S_j$ $(0 \leq i, j \leq l)$. Then $\underline{x}$ is a winning $(k = i + j + 1)$-state.*

The best strategy is to ask separately for each element if only two elements are left. The following property of the states is proved by Berlekamp and called Translation Bound.

Theorem (Berlekamp [5]) *Let $\underline{x}$ be a state with $\sum_{i=0}^{k} x_i \geq 3$ and $n \geq 3$. If x is a winning n-state, then $T(x)$ is a winning $(n-3)$-state.*

Corollary 6.3
1. $L_l(N) \leq k \Rightarrow L_{l-1}(N) \leq k - 3$.

2. $L_l(N) = k \Rightarrow L_{l+1}(N) \geq k + 3$.

We consider the following state-table, which was presented by Berlekamp:

Table 6.1

column / row	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	4	2	1	1	1	1	1	1	1	1	1	1	1	1
2	8	6	4	1	0	0	0	0	0	0	0	0	0	0
3	36	22	14	10	5	1	0	0	0	0	0	0	0	0
4	152	94	58	36	24	15	6	1	0	0	0	0	0	0
5	644	398	246	152	94	60	39	21	7	1	0	0	0	0
6	2728	1686	1042	644	398	246	154	99	60	28	8	1	0	0
7	11556	7142	4414	2728	1686	1042	644	400	253	159	88	36	9	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Berlekamp's state table has the following properties.

Theorem (Berlekamp [5]) *Let $A_{i,j}$ be the value in the i-th row and j-th column of Table 6.1.*

- *An $\underline{A}_{m,j}$-state is a winning $(3m-j)$-state for all $0 \leq j \leq 3m$.*
- *An $\underline{A}_{m,j}$-state can be reduced to $\underline{A}_{m,j+1}$ and $\underline{A}_{m-1,j-2}$.*
- *If $0 \leq j \leq 3 \leq i$, then holds: $A_{i,j} = 2(\frac{1+\sqrt{5}}{2})^{3i-j-2} + 2(\frac{1-\sqrt{5}}{2})^{3i-j-2}$.*

Proposition 6.4 *Let $\underline{s} = (N,0,0,0)$ and $k = ch(\underline{s})$. Person 2 needs $k+2$ questions to win the game if $N \in M_2 = \{3,5\}$.*

Proof: Guzicki proved that $L_2(3) = 8$ and $L_2(5) = 9$. It also holds that $ch(3,0,0,0) = 9$ and $ch(5,0,0,0) = 10$. By using Berlekamp's translation bound, we get $L_3(3) \geq 11 = k+2$ and $L_3(5) \geq 12 = k+2$. By using Berlekamp's state table, we know that $(4,8,36,152)$ is a 11-winning state and $(1,4,15,58,246)$ is a 12-winning state. Therefore, $L_3(3) = 11 = k+2$ and $L_3(5) = 12 = k+2$. □

Proposition 6.5 *Let $\underline{s} = (N,0,0,0)$ and $k = ch(\underline{s})$. Person 2 needs $k+1$ questions to win the game if $N \in M_1 = \{1,\ldots,165\} \backslash (M_0 \cup M_2)$, where M_0 is defined in Proposition 6.1 and M_1 is defined in Proposition 6.5.*

Proof: For all $a \in M_1$ holds $M(a,0,0) \neq 0$. Therefore, $L_3(a) \geq k+1$. It holds that $L_3(4) \leq 11 = k+1$ because $(4,8,56,152)$ is a 11-winning state. It holds that $L_3(8) \leq 12 = k+1$ because the state $(8,0,0,0)$ is reduced to $(4,4,0,0)$ with the question $[4,0,0,0]$ and $(4,4,0,0)$ is a 11-winning state because $(4,8,56,152)$ is a 11-winning state. For all $N \in M_1 \backslash \{4,8\}$ holds $L_3(N) \leq k+1$ because for each $N \in M_1 \backslash \{4,8\}$ one can always find an $N' > N$ with $ch(N,0,0,0)+1 = ch(N',0,0,0)$ and $M(N',0,0) = 0$. □

We present another possibility to get the solution for Ulam's game if $N < 266$ in Section 7. It turns out that the value of M determines the solution. Let us combine all results. We know that the first three questions have to be optimal. The first questions of Proposition 3.1 and the third questions of Proposition B.1 are obviously optimal. We have to show that the second questions of Proposition A.1 are also optimal. This holds either if the loss in the third question is equal 0 or if there does not exist any local optimal question such that the loss in the third question is equal 0.

Theorem 6.1 *Let $\underline{s} = (N,0,0,0)$ and $k = ch(\underline{s})$, Then*
$N < 266$*:*

1. *Person 2 needs $k+2$ questions to win the game if $N \in M_2$ (defined in Proposition 6.4).*
2. *Person 2 needs $k+1$ questions to win the game if $N \in M_1$ (defined in Proposition 6.5).*
3. *Person 2 needs k questions to win the game if $N \notin M_1 \cup M_2$.*

$N \geq 266$*:*

1. *Person 2 needs k questions to win the game if $N = 8m$.*
2. *Person 2 needs either k questions or $k+1$ questions to win the game, depending on conditions, concerning k and p if $N = 8m + p$.*

Proof: Let $N \geq 266$, $\underline{s}$ be the initial state, $\underline{t}$ be the state after the first question $[\underline{x}]$, $\underline{u}$ be the state after the second question $[\underline{y}]$ and $\underline{v}$ be the state after the third question $[\underline{z}]$. The character of the state after three questions of the mini-strategy is the character of the initial state minus three or two. If this character is the same as the one of the initial state minus 3, then we can win the game with $ch(\underline{s})$ questions, otherwise we need $ch(\underline{s})+1$ questions. Thus, we have to calculate if the character of the initial state is equal to the character of the state after three questions plus 3. By Propositions 3.1, Table 3.1 and 3.2 we know how much we lose in each question. Let $\underline{v}$ be the worst state after the first three questions, $\underline{s}$ the initial state, $k = ch(\underline{s})$ and a_1, $a_2[k-2]$ and $a_3[k-3]$ the loss in the first, second and third question. By Berlekamp's Conservation of Volume we get

$$V_{k-3}(\underline{v}) = \frac{V_k(\underline{s}) + a_1\binom{k-1}{3}}{8} + \frac{a_2[k-2]\binom{k-2}{2}}{4} + \frac{a_3[k-3](k-3)}{2}.$$

Thus, $ch(\underline{s}) = ch(\underline{v}) + 3$ if $V_{k-3}(\underline{v}) \leq 2^{k-3}$. We have to check

$$V_k(\underline{s}) + a_1\binom{k-1}{3} + 2a_2[k-2]\binom{k-2}{2} + 4a_3[k-3](k-3) \leq 2^k.$$

Due to the complementary questions we only have to consider the cases in which the resulting YES-state is bigger than or equal to the resulting NO-state. We have to check if the second question is optimal. The exact analysis of this can be find in Appendix C. $\square$

7 Some remarks on Guzicki's algorithm

In this section we consider the values of the function M. Let $\underline{v}$ be a typical state with $ch(\underline{v}) < 15$. Then it can happen that $M(v_3, v_2, v_1) = c > ch(v_3, v_2, v_1, c)$. In this case we can find some typical states. There exist some states (v_3, v_2, v_1, j) where $ch(\underline{v}) < j < M(v_3, v_2, v_1)$ and the states are typical but not nice. Some examples of the states are given below.

$ch = 6:$	$(0,1,5,j)$	$6 < j < 14$	(in total 3 such sequences)
$ch = 7:$	$(0,3,0,j)$	$7 < j < 42$	(in total 12 such sequences)
$ch = 8:$	$(0,5,0,j)$	$8 < j < 72$	(in total 39 such sequences)
$ch = 9:$	$(1,6,0,j)$	$9 < j < 107$	(in total 119 such sequences)
$ch = 10:$	$(3,0,12,j)$	$10 < j < 365$	(in total 479 such sequences)
$ch = 11:$	$(6,6,1,j)$	$11 < j < 243$	(in total 1132 such sequences)
$ch = 12:$	$(7,17,0,j)$	$12 < j < 661$	(in total 1638 such sequences)
$ch = 13:$	$(9,45,26,j)$	$13 < j < 287$	(in total 1801 such sequences)
$ch = 14:$	$(22,52,39,j)$	$14 < j < 128$	(in total 921 such sequences)

In total there exist 6144 sequences of typical and not nice states.

Lemma 7.1 *Let $M(v_3, v_2, v_1) \neq 0$ and $k = ch(v_3, v_2, v_1, 0)$. Then there exists an $j \in \mathbb{N}\backslash\{0\}$ such that*

$$M(v_3, v_2, v_1) = \min\{d : ch(v_3, v_2, v_1, d) > ch(v_3, v_2, v_1, 0) + (j-1)\}.$$

Proof: Let $M(v_3, v_2, v_1) = c$. Obviously, $ch(v_3, v_2, v_1, 0) = k < ch(v_3, v_2, v_1, c)$. Thus, it exists a j such that $k + j = ch(v_3, v_2, v_1, c)$. It holds $ch(v_3, v_2, v_1, c-1) = k + j - 1$ because c is minimal. The statement follows directly. $\square$

Lemma 7.2 *Let $(v_3, v_2, v_1, 0)$ be a borderline k-winning, but not a nice state and let $v_0 = M(v_3, v_2, v_1)$. Then (v_3, v_2, v_1, v_0) is a nice k-winning state.*

Proof: There exists an v_0' such that (v_3, v_2, v_1, v_0') is an k-winning state and $ch(v_3, v_2, v_1, v_0') = k$ because of Lemma 5.1. $v_0 = v_0'$ because M is minimal. $\square$

Proposition 7.3 *Let* $\underline{v} = (v_3, v_2, v_1, 0)$ *be a state,* $k = ch(v_3, v_2, v_1, 0)$ *and* $m > 0$. *Then holds*

1. $\underline{v}$ *is a borderline winning k-state* $\iff M(v_3, v_2, v_1) = 0$.
2. $\underline{v}$ *is a borderline winning* $k + m$*-state* $\iff M(v_3, v_2, v_1) = \min\{c : ch(v_3, v_2, v_1, c) > ch(v_3, v_2, v_1, 0) + (m-1)\}$.

Proof:

1. This statement is true because of the definition of M.
2. (a) Let $\underline{v}$ be a borderline winning $(k+m)$-state. Then $M(v_3, v_2, v_1) \neq 0$ and $M(v_3, v_2, v_1) \geq \min\{c : ch(v_3, v_2, v_1, c) > ch(v_3, v_2, v_1, 0) + (m-1)\}$. If $M(v_3, v_2, v_1) < \min\{k : ch(v_3, v_2, v_1, k) > ch(v_3, v_2, v_1, 0) + (m-1)\}$, then we know that $(v_3, v_2, v_1, M(v_3, v_2, v_1))$ is a winning $(k+m-1)$-state. It follows that $(v_3, v_2, v_1, 0)$ is also a winning $(k+m-1)$-state. This is a contradiction. It remains to show that $M(v_3, v_2, v_1) \leq \min\{c : ch(v_3, v_2, v_1, c) > ch(v_3, v_2, v_1, 0) + (m-1)\}$. This follows by Lemma 7.2.
   (b) Let $m = M(v_3, v_2, v_1) = \min\{c : ch(v_3, v_2, v_1, c) > ch(v_3, v_2, v_1, 0) + (m-1)\}$. It follows that (v_3, v_2, v_1, m) is a borderline winning $(k+m)$-state. Thus, $(v_3, v_2, v_1, 0)$ is a winning $(k+m)$-state. It is not possible to win it with less than $k+m$ questions because of Lemma 7.2 and M is minimal.

□

It is also possible to get a solution of Ulam's game with Property 7.3.

8 Some comments on the general solution

As it is shown in [11], solution for the case of two lies depends on the character after the first two questions. Our solution for the case of three lies depends on the character after the first three questions. Spencer proved the following

Theorem (Spencer [22]) *There exists constants* $c(l)$*,* $q_0(l)$ *such that for all states* $\underline{v}$ *with* $ch(\underline{v}) \geq q_0(l)$*, the following is true: If* $v_0 > c(l)$*, then* $\underline{v}$ *is a nice state.*
Using this statement Spencer proved:

Theorem (Spencer [22]) *Let* $\underline{s} = (N, 0, \ldots, 0)$ *be the initial state,* $k = ch(\underline{s})$ *and let* $\underline{v}$ *be the state after the first l questions. Then there exists a constant* $q(l)$*, such that the following is true: If* $k > q(l)$*, then Person 2 wins the game with* $ch(\underline{v})$ *questions.*
The asymptotic problem ($\frac{l}{n} \to p, n \to \infty$) was solved by Berlekamp [5] and Zigangirov [26], who proved:

Theorem (Berlekamp [5], Zigangirov [26]) *Let l be the number of lies, n be the number of questions,* $R_0 = 0.6942$*,* $f_0 = \frac{1}{3}$*,* $R_t = 0.2965$*,* $f_t = 0.1909$ *and p be selected in such a way that* $R = 1 - h(p)$*. Then, for* $\lim_{n\to\infty} \frac{l(n)}{n} = f \leq \frac{1}{2}$

$$f = \begin{cases} f_0 - \frac{f_0}{R_0} R + o(1) & \text{if } 0 \leq R \leq R_t, \\ p + o(1) & \text{if } R_t \leq R. \end{cases}$$

For some special numbers $N_0 \in \mathbb{N}$ it is also possible to get an explicit formula $L_l(N_0) = f(l)$ by using Berlekamp's translation bound and his state table. We present some examples. The sets S_j were defined as follows:

$$S_j \triangleq \{x \in \mathcal{X} : \text{Person 2 gets } (l-j) \text{ negative votes for } x\}.$$

Obviously, this set is also defined for negative numbers. Therefore, we can define a q-position state as follows:

Definition 8.1 *Suppose that Person 2 has asked q questions. The vector*

$$\underline{v}^q = (|S_l|, |S_{l-1}|, \dots, |S_0|, \dots, |S_{l-q}|) = (v_l, v_{l-1}, \dots, v_0, \dots, v_{l-q})$$

is referred to as an q-position state.

In the same way we define a q-position question.

Proposition 8.1 *1. $L_l(2) = 1 + 2l$ for all $l \geq 0$.*

2. $L_l(4) = 7 + 3(l-2)$ for all $l \geq 2$.

3. $L_l(16) = 4 + 3l$ for all $l \geq 0$.

4. In general case: If one finds a strategy that reduces $(N_0, 0, \dots, 0)$ to an q-position state $\underline{v}$, which is less or equal $\underline{A}_{l,j_0}$ taken from Berlekamp's state table and this strategy is optimal for some $l = c_0$, then for all $l \geq c_0$

$$L_l(N_0) = f(l) = L_{c_0}(N_0) + 3(l - c_0).$$

Proof: The first statement is only a rewriting Remark 6.2. The other statements can be proved by using the first column of Berlekamp's state Table 6, Corollary 6.3 and the Hamming Bound. Further, assume that we can find a strategy (with q-position questions) which reduces the state in the worst case to an q-position state. If this vector is less than or equal to a column of the state table, then we can continue with the state table strategy. If we know that this strategy is optimal, then we can use the same strategy to solve the problem for one lie more with three additional questions. By using Corollary 6.3, we know that this is also optimal. □
The idea using a state table to solve Ulam's problem belongs to Hill and Karim [12]. The authors used another state table as compared to Table 6.1. They solved Ulam's problem for $N = 1000000$ and $l = 1, 2, 3$ by using this table. The table used does not harmonize with translation bound of Berlekamp. It is not possible to get results for fixed N and arbitrary l.

9 Some consideration of the q-ary case

In this part we consider the problem of transmitting messages over a noisy q-ary channel with noiseless feedback. This problem is closely related to a sequential search with errors. Instead of using a binary coding alphabet, we introduce $\mathcal{Y} = \{0, \dots, q-1\}$ as the q-ary coding alphabet. All other variables and functions are chosen as before. The quiet-question-noisy-answer-game is changed in the following way. The sender and receiver have a common q-partition strategy. They divide the set of possible messages to q pairwise disjoint sets $(M_0, \dots, M_{q-1})$. The sender transmits the index of the set over the noisy channel. A "Devil" wants to avoid that the receiver gets the message by changing at most l answers. Again we consider the sets

$$S_j \triangleq \{x \in \mathcal{X} : \text{Person 2 get } (l-j) \text{ negative votes for } x\}.$$

Definition 9.1 *The vector*

$$\underline{v} = (|S_l|, |S_{l-1}|, \dots, |S_0|) = (v_l, v_{l-1}, \dots, v_0)$$

is referred to a state (of the game). $\underline{v}$ is called an k-state if k questions are left.

Example 1 *Person 1 thinks of a number in $\mathcal{X} = \{1, \ldots, 9\}$. Let $q = 3$ and $l = 2$.*

$$S_2 = \{1, \ldots, 9\} \quad S_1 = \emptyset \qquad S_0 = \emptyset \qquad \underline{v} = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix}$$

$1.: M_0 = \{4,5,6\}\ M_1 = \{1,2,3\}\ M_2 = \{7,8,9\}\ Person\ 1 : 1$

$$S_2 = \{1,2,3\} \quad S_1 = \{4, \ldots, 9\} \quad S_0 = \emptyset \qquad \underline{v} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

$2.: M_0 = \{1,4,5\}\ M_1 = \{2,6,7\}\ M_2 = \{3,8,9\}\ Person\ 1 : 0$

$$S_2 = \{1\} \quad S_1 = \{2,3,4,5\} \quad S_0 = \{6,7,8,9\} \quad \underline{v} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$3.: M_0 = \{1\}\ M_1 = \{2,3,6,7\}\ M_2 = \{4,5,8,9\}\ Person\ 1 : 1$

$$S_2 = \emptyset \quad S_1 = \{1,2,3\} \quad S_0 = \{4,5,6,7\} \quad \underline{v} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$4.: M_0 = \{1,4,5\}\ M_1 = \{2,7\}\ M_2 = \{3,6\}\ Person\ 1 : 2$

$$S_2 = \emptyset \quad S_1 = \{3\} \quad S_0 = \{1,2,6\} \quad \underline{v} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$5.: M_0 = \{3\}\ M_1 = \{1,2\}\ M_2 = \{6\}\ Person\ 1 : 1$

$$S_2 = \emptyset \quad S_1 = \emptyset \quad S_0 = \{1,2,3\} \quad \underline{v} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$6.: M_0 = \{1\}\ M_1 = \{2\}\ M_2 = \{3\}\ Person\ 1 : 1$

$$S_2 = \emptyset \quad S_1 = \emptyset \quad S_0 = \{2\} \quad \underline{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Definition 9.2 *Let $\underline{v} = (|S_l|, \ldots, |S_0|)$ be an arbitrary state and let $M_0, \ldots, M_{q-1}$ be the next partition used by Person 2. We set $X_j^i = S_i \cap M_j$ and denote the question "Which subset contains the message?" by $[\underline{x}] = (x_{i,j})_{\substack{i=l\ldots 0 \\ j=0\ldots(q-1)}} = \begin{bmatrix} x_{l0} & \ldots & x_{l(q-1)} \\ \vdots & \ddots & \vdots \\ x_{00} & \ldots & x_{0(q-1)} \end{bmatrix}$, where $x_{i,j} \triangleq |X_j^i|$. $[\underline{x}]$ is called a question-matrix.*

We can formalize the question-matrices in Example 1 in the following way. 1. $\begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, 3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$, 4. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, 5. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, 6. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Definition 9.3 *The state $\underline{s}$ can be reduced to the states $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_q$ if there exists a question $[\underline{x}]$ such that $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_q$ are possible states after this question.*

Definition 9.4 *1. A 0-state $\underline{x}$ is called winning if $\sum_{i=0}^{l} x_i \leq 1$. Otherwise, it is called losing.*

2. *An k-state $\underline{x}$ is called winning if it can be reduced to q winning $(k-1)$-states. Otherwise, it is called losing.*
3. *A winning k-state is called borderline winning if it is a losing $(k-1)$-state.*
4. *An k-state is called singlet if $\sum_{i=0}^{k} |S_i| = 1$.*
5. *An k-state is called small if $\sum_{i=0}^{k} |S_i| \leq q$.*

Proposition 9.1 *1. A winning n-state is also a winning k-state if $k > n$.*

2. *All borderline winning 1-states have the form $(0, \ldots, 0, a)$, where $a \leq q$.*
3. *Let $\underline{x}$ be a winning k-state and $\underline{y}$ be some state. If $y_i \leq x_i$ for all $i \leq l$, then it is also a winning k-state.*

Definition 9.5 *Let $\underline{x}$ be an arbitrary state. Then*

$$V_n^q(\underline{x}) \triangleq \sum_{i=0}^{l} x_i \sum_{j=0}^{i} \binom{n}{j} (q-1)^j$$

is called the n-th volume of the state $\underline{x}$.

Proposition 9.2 *Let $\underline{x}$ be a state, which can be reduced to the states $\underline{a}_1, \underline{a}_2, \ldots \underline{a}_q$. Then*

$$V_n^q(\underline{x}) = \sum_{i=1}^{q} V_{n-1}^q(\underline{a}_i).$$

Proposition 9.3 *Let $\underline{x}$ be a winning n-state. Then*

$$V_n\left(\underline{x}\right) \leq q^n.$$

Let $\mathcal{X} = \{1, \ldots, N\}$ be the set of messages and l be the maximal number of lies. We denote the minimal number of questions of Person 2 to find the searched number by using an optimal strategy by $L_l^q(N)$.

Corollary 9.4 (The Hamming bound) *Let $|\mathcal{X}| = N$ and $l \in \mathbb{N}$. Then*

$$L_l^q(N) = n \Longrightarrow N \leq \frac{q^n}{\sum_{j=0}^{l} \binom{n}{j} (q-1)^j}.$$

Definition 9.6 *Let $\underline{s}$ be an arbitrary state. Its character is defined as*

$$ch(\underline{s}) \triangleq \min\{k : V_k^q(\underline{s}) \leq q^k\}.$$

Lemma 9.5 *Let $\underline{x}$ be a small state, but not a singlet state, $m \triangleq \max\{j : x_j \neq 0\}$ and* $k \triangleq \begin{cases} \max\{j : x_j \neq 0 \text{ and } j \neq m\} & \text{if } x_m = 1, \\ m & \text{otherwise.} \end{cases}$
If $z = k + m + 1$, then $\underline{x}$ is a borderline winning z-state.

Proof: Person 2 can ask for sets of cardinality 1 at every step. The statement follows directly. □

Definition 9.7 *Let* $\underline{x} = \begin{pmatrix} x_k \\ \vdots \\ x_0 \end{pmatrix}$ *be a state. Then* $T(\underline{x}) = \underline{t} = \begin{pmatrix} t_{k-1} \\ \vdots \\ t_0 \end{pmatrix}$ *with* $t_i = x_{i+1}$ *is called the translation of* $\underline{x}$.

Theorem 9.1 *Let* $\underline{x}$ *be a state. If* $\underline{x}$ *is a winning* n*-state, then* $T(\underline{x})$ *is a winning* $(n-2)$*-state.*

Proof: (Induction in n)

$\underline{n \leq q:}$ The theorem is true (see Lemma 9.5).

$\underline{n-1 \rightarrow n:}$ Let $\underline{x}$ be a winning n-state. We can reduce $\underline{x}$ to q winning $(n-1)$-states $\underline{y}^1, \ldots, \underline{y}^q$. We can use the same partition to reduce $T(\underline{x})$ to $T(\underline{y}^1), \ldots, T(\underline{y}^q)$. If there exists some $j \in \{1, \ldots, q\}$ with $\sum y_i^j > q$, then as it follows by induction, $T(\underline{y}^j)$ is a winning $(n-2)$-state. For the other states, $\sum y_i^j \leq q$. Based on Lemma 9.5 we conclude that these states are winning $(n-2)$-states. Thus, $\underline{x}$ is a winning $(n-2)$-state.

□

Corollary 9.6 *Let* $\underline{x}$ *be a winning* n*-state. Then* $T^n(\underline{x})$ *is a winning* $(n-2m)$*-state.*

Aigner and Malinowski have solved independently the q-ary case if $l = 1$. They have the following equivalent results:

Theorem (Aigner [2]) *Suppose that* N *and* q *are chosen as before and* h, r *are chosen in such a way that* $N = hq - r$ *and* $0 \leq r < q$*. Then*

$$L_1(N) = \begin{cases} \lceil \log_q N \rceil + 2 & if\ n \leq q^{q-1}, \\ \min\{k : \frac{q^k - r(k-1)(q-1)}{1+k(q-1)} \geq N\} & otherwise. \end{cases}$$

Theorem (Malinowski [16]) *Suppose that* N *and* q *chosen as before and* h, r *are chosen in such a way that* $N = h(q+1)+r$ *and* $0 \leq r < q+1$*. Let* $c = \min\{(q+1)^i : (q+1)^i \geq h\}$ *and* $r = 0$*. Then*

$$L_1(N) = 1 + \begin{cases} ch(h, qh) & if\ h \geq (q+1)^{q-1}, \\ \max\{ch(h, qh), ch(c, c(h-1))\} & otherwise. \end{cases}$$

Let $c = \min\{(q+1)^i : (q+1)^i \geq h+1\}$ *and* $r \neq 0$*. Then*

$$L_1(N) = 1 + \begin{cases} ch(h+1, qh+r-1) & if\ h \geq (q+1)^{q-1}, \\ \max\{ch(h+1, qh+r-1), ch(c, c(h-1))\} & otherwise. \end{cases}$$

10 Further versions of Ulam's game

In 1992 Spencer [22] presented another aspect of Ulam's game. He considered the following two person game. We take a board with two columns and $l+1$ rows. The rows are numbered from l to 0 and the columns by two and one. A field with some chips on it corresponds to every row. Each round of the game is played in three steps. At the first step Person 2 distributes the chips of the field on the corresponding columns. At the second step Person 1 chooses one column. All chips in this column are shifted down one row. The chips in row 0 and the selected column

are removed. At step three all chips of one row are taken on its corresponding field. Then the round is finished. The game is terminated if every chip except one is removed. The aim of Person 1 is to get the number of rounds as large as possible whereas Person 2 wants to get a small number of rounds. This game is equivalent to the Ulam's one. The j-th row corresponds to the set S_j, the number of lies corresponds to the number of rows minus one, each chip represents a number, at the end of each round the number of chips on the j-th field is equal to x_j, the distribution of the chips represent the questions and the chips in the selected row correspond to the numbers for which Person 1 in Ulam's game gives a negative vote. It is easy to modify this game to get an equivalent game for the q-ary case. We only have to play with q columns. With this model of Ulam's game we get some new interesting aspects. If we have a look at the Block Coding model and Ulam's game model, then we are only interested in some special states like $(N, 0, \dots, 0)$ and the states that can be reached from these states by using an optimal strategy. In this model, it is quite meaningful to consider arbitrary states as initial ones.
Kleitman, Meyer, Rivest, Spencer, Winklmann [13] considered Ulam's game in the continuous case. This means that Person 1 chooses a number $[0,1) \subset \mathbb{R}$. A generalized version of the continuous case is given by De Bonis, Gargano and Vaccaro in [6]. This means that Person 1 chooses a number in a Lebesgue measurable set. The authors regard the problem of group testing as a search in bi-dimensional space $[0,1) \times [0,1)$. They considered two versions of a game in which Person 1 thinks about two numbers contained in $[0,1)$. In the first version (Group Testing) Person 2 chooses subsets $S \subset [0,1)$ and asks if S contains at least one of two numbers. It the second version (Parity Testing) Person 2 chooses subsets $S \subset [0,1)$ and asks if S contains exactly one of two numbers. The main result is as follows. There exists an optimal search strategy for Person 2 (in both versions) that confines the pair (x, y) in a set whose worst-case measure is $2^{-(n+1)} \sum_{k=0}^{l} \binom{n}{k}$. The authors in [14] considered the continuous case whereby Person 1 thinks of one number. A relationship between Ulam's game and block codes without feedback was under consideration in [18] and [3]. The authors give an algorithm based on Berlekamp's volume in [15]. The authors in [23] and [10] regarded the problem from a different perspective. In their paper they considered Ulam's game, when the number of lies is a fixed fraction of the number of questions. Some aspects of Ulam's game with one lie, when we allow only questions of the form: "Is $x > a$?" are given in [21] and [2]. In [8] some results for local restriction answers of Person 1 are presented (i.e. Person 1 is not allowed to give two consecutive lies). In [19] the aim of the Questioner was extended to determining the steps where lies took place.

A The second question

In this section we present the second question of the mini-strategy.

Proposition A.1 *Let* $\underline{t} = (t_3, t_2, 0, 0)$ *be a state,* $k_2 \in \mathbb{N}$, $t_3 \geq 3$ *and* $t_2 \geq k_2 - 2$. *Then there exists a question* $[\underline{y}]$ *minimizing the volume difference* $|V_{k_2}(YES^{\underline{t}}) - V_{k_2}(NO^{\underline{t}})|$. *It can be chosen in the following way:*

1. $[\frac{t_3}{2}, \frac{t_2}{2}, 0, 0]$ *if* t_3 *is even and* t_2 *is even.*
2. $[\frac{t_3+3}{2}, \frac{t_2-1}{2} - \frac{k_2-3}{2}, 0, 0]$ *if* t_3 *is odd,* t_2 *is odd and* k_2 *is odd.*
3. $[\frac{t_3+1}{2}, \frac{t_2+1}{2} - \frac{k_2}{6}, 0, 0]$ *if* t_3 *is odd,* t_2 *is odd and* $k_2 = 6h$.
4. $[\frac{t_3+1}{2}, \frac{t_2+1}{2} - \frac{k_2-2}{6}, 0, 0]$ *if* t_3 *is odd,* t_2 *is odd and* $k_2 = 6h + 2$.
5. $[\frac{t_3-1}{2}, \frac{t_2-1}{2} + \frac{k_2+2}{6}, 0, 0]$ *if* t_3 *is odd,* t_2 *is odd and* $k_2 = 6h + 4$.

6. $[\frac{t_3+3}{2}, \frac{t_2}{2} - \frac{k_2-2}{2}, 0, 0]$ *if* t_3 *is odd,* t_2 *is even and* k_2 *is even.*

7. $[\frac{t_3-1}{2}, \frac{t_2}{2} + \frac{k_2-1}{6}, 0, 0]$ *if* t_3 *is odd,* t_2 *is even and* $k_2 = 6h+1$.

8. $[\frac{t_3+1}{2}, \frac{t_2}{2} - \frac{k_2-3}{6}, 0, 0]$ *if* t_3 *is odd,* t_2 *is even and* $k_2 = 6h+3$.

9. $[\frac{t_3+1}{2}, \frac{t_2}{2} - \frac{k_2-5}{6}, 0, 0]$ *if* t_3 *is odd,* t_2 *is even and* $k_2 = 6h+5$.

10. $[\frac{t_3-2}{2}, \frac{t_2-1}{2} + \frac{k_2}{3}, 0, 0]$ *if* t_3 *is even,* t_2 *is odd and* $k_2 = 6h$.

11. $[\frac{t_3+2}{2}, \frac{t_2+1}{2} - \frac{k_2-1}{3}, 0, 0]$ *if* t_3 *is even,* t_2 *is odd and* $k_2 = 6h+1$.

12. $[\frac{t_3}{2}, \frac{t_2+1}{2}, 0, 0]$ *if* t_3 *is even,* t_2 *is odd and* $k_2 = 6h+2$.

13. $[\frac{t_3-2}{2}, \frac{t_2-1}{2} + \frac{k_2}{3}, 0, 0]$ *if* t_3 *is even,* t_2 *is odd and* $k_2 = 6h+3$.

14. $[\frac{t_3+2}{2}, \frac{t_2+1}{2} - \frac{k_2-1}{3}, 0, 0]$ *if* t_3 *is even,* t_2 *is odd and* $k_2 = 6h+4$.

15. $[\frac{t_3}{2}, \frac{t_2+1}{2}, 0, 0]$ *if* t_3 *is even,* t_2 *is odd and* $k_2 = 6h+5$.

Furthermore, $|V_{k_2}(YES^{\underline{t}}) - V_{k_2}(NO^{\underline{t}})| = a_2[k_2]\binom{k_2}{2}$, *where* $a_2[k_2] \in \{0, \frac{1}{3}, 1\}$

Proof: Let us denote a query by $[\underline{y}]$. We have to minimize $|V_{k_2}(YES^{\underline{t}}) - V_{k_2}(NO^{\underline{t}})| = |(2y_3 - t_3)\binom{k_2}{3} + (2y_2 - t_2)\binom{k_2}{2}| = |\binom{k_2}{2}((2y_3 - t_3)\frac{k_2-2}{3} + 2y_2 - t_2)|$. We set $D = (2y_3 - t_3)\frac{k_2-2}{3} + 2y_2 - t_2$.

1. t_3 even and t_2 even: In this case we can get a minimal value for $D (= 0)$ if we choose $y_3 = \frac{t_3}{2}$ and $y_2 = \frac{t_2}{2}$.

2. t_3 odd, t_2 odd and k_2 is odd: In this case we can get a minimal value for $D (= 0)$ if we choose $y_3 = \frac{t_3+3}{2}$ and $y_2 = \frac{t_2-1}{2} - \frac{k_2-3}{2}$.

3. t_3 odd, t_2 odd and $k_2 = 6h$: In this case $2y_3 - t_3$ and $2y_2 - t_2$ are odd numbers and $k_2 - 2$ is an even number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3+1}{2}$ and $y_2 = \frac{t_2+1}{2} - \frac{k_2}{6}$.

4. t_3 odd, t_2 odd and $k_2 = 6h+2$: In this case $2y_3 - t_3$ and $2y_2 - t_2$ are odd numbers and $k_2 - 2$ is an even number divisible by 3. Thus, $D \geq 1$. $D = 1$ if we choose $y_3 = \frac{t_3+1}{2}$ and $y_2 = \frac{t_2+1}{2} - \frac{k_2-2}{6}$.

5. t_3 odd, t_2 odd and $k_2 = 6h+4$: In this case $2y_3 - t_3$ and $2y_2 - t_2$ are odd numbers and $k_2 - 2$ is an even number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3-1}{2}$ and $y_2 = \frac{t_2-1}{2} + \frac{k_2+2}{6}$.

6. t_3 odd, t_2 even and k_2 is even: In this case we can get a minimal value for $D (= 0)$ if we choose $y_3 = \frac{t_3+3}{2}$ and $y_2 = \frac{t_2}{2} - \frac{k_2-2}{2}$.

7. t_3 odd, t_2 even and $k_2 = 6h+1$: In this case $2y_3 - t_3$ is an odd number and $2y_2 - t_2$ is an even number and $k_2 - 2$ is an odd number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3-1}{2}$ and $y_2 = \frac{t_2}{2} + \frac{k_2-1}{6}$.

8. t_3 odd, t_2 even and $k_2 = 6h+3$: In this case $2y_3 - t_3$ is an odd number and $2y_2 - t_2$ is an even number and $k_2 - 2$ is an odd number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3+1}{2}$ and $y_2 = \frac{t_2}{2} - \frac{k_2-3}{6}$.

9. t_3 odd, t_2 even and $k_2 = 6h+5$: In this case $2y_3 - t_3$ is an odd number and $2y_2 - t_2$ is an even number and $k_2 - 2$ is an odd number divisible by 3. Thus, $D \geq 1$. $D = 1$ if we choose $y_3 = \frac{t_3+1}{2}$ and $y_2 = \frac{t_2}{2} - \frac{k_2-5}{6}$.

10. t_3 even, t_2 odd and $k_2 = 6h$: In this case $2y_3 - t_3$ is an even number and $2y_2 - t_2$ is an odd number and $k_2 - 2$ is an even number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3-2}{2}$ and $y_2 = \frac{t_2-1}{2} + \frac{k_2}{3}$.

11. t_3 even, t_2 odd and $k_2 = 6h+1$: In this case $2y_3 - t_3$ is an even number and $2y_2 - t_2$ is an odd number and $k_2 - 2$ is an odd number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3+2}{2}$ and $y_2 = \frac{t_2+1}{2} - \frac{k_2-1}{3}$.

12. t_3 even, t_2 odd and $k_2 = 6h+2$: In this case $2y_3 - t_3$ is an even number and $2y_2 - t_2$ is an odd number and $k_2 - 2$ is an even number divisible by 3. Thus, $D \geq 1$. $D = 1$ if we choose $y_3 = \frac{t_3}{2}$ and $y_2 = \frac{t_2+1}{2}$.

13. t_3 even, t_2 odd and $k_2 = 6h+3$: In this case $2y_3 - t_3$ is an even number and $2y_2 - t_2$ is an odd number and $k_2 - 2$ is an odd number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3-2}{2}$ and $y_2 = \frac{t_2-1}{2} + \frac{k_2}{3}$.

14. t_3 even, t_2 odd and $k_2 = 6h+4$: In this case $2y_3 - t_3$ is an even number and $2y_2 - t_2$ is an odd number and $k_2 - 2$ is an even number not divisible by 3. Thus, $D \geq \frac{o}{3}$ where o is an odd number. $D = \frac{1}{3}$ if we choose $y_3 = \frac{t_3+2}{2}$ and $y_2 = \frac{t_2+1}{2} - \frac{k_2-1}{3}$.

15. t_3 even, t_2 odd and $k_2 = 6h+5$: In this case $2y_3 - t_3$ is an even number and $2y_2 - t_2$ is an odd number and $k_2 - 2$ is an odd number divisible by 3. Thus, $D \geq 1$. $D = 1$ if we choose $y_3 = \frac{t_3}{2}$ and $y_2 = \frac{t_2+1}{2}$.

$\square$

B The third question

In this section we present the third question of the mini-strategy.

Proposition B.1 *Let $\underline{u} = (u_3, u_2, u_1, 0)$ be a state, $k_3 \in \mathbb{N}$, $u_3 \geq 4$, $u_2 \geq k_3$ and $u_1 \geq k_3 - 1$. Then there exists a question $[\underline{z}]$ minimizing the volume difference $|V_{k_3}(YES^{\underline{u}}) - V_{k_3}(NO^{\underline{u}})|$. It can be chosen in the following way:*

1. $[\frac{u_3}{2}, \frac{u_2}{2}, \frac{u_1}{2}, 0]$ *if u_3 is even, u_2 is even and u_1 is even.*

2. $[\frac{u_3}{2}, 1 + \frac{u_2}{2}, \frac{u_1+1-k_3}{2}, 0]$ *if u_3 is even, u_2 is even, u_1 is odd and k_3 is even.*

3. $[\frac{u_3}{2}, \frac{u_2}{2}, \frac{u_1+1}{2}, 0]$ *if u_3 is even, u_2 is even, u_1 is odd and $k_3 = 6h+1$.*

4. $[\frac{u_3+2}{2}, \frac{u_2}{2} - \frac{k_3-3}{3}, \frac{u_1-1}{2} - \frac{k_3-3}{6}, 0]$ *if u_3 is even, u_2 is even, u_1 is odd and $k_3 = 6h+3$.*

5. $[\frac{u_3}{2}, \frac{u_2}{2}, \frac{u_1+1}{2}, 0]$ *if u_3 is even, u_2 is even, u_1 is odd and $k_3 = 6h+5$.*

6. $[\frac{u_3+3}{2}, \frac{u_2+2-k_3}{2} + 1, \frac{u_1}{2} - \frac{k_3-1}{2}, 0]$ *if u_3 is odd, u_2 is odd, u_1 is even and k_3 is odd.*

7. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3}{6}, \frac{u_1}{2} - \frac{k_3}{12}, 0]$ *if u_3 is odd, u_2 is odd, u_1 is even and $k_3 = 12h$.*

8. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3-2}{6}, \frac{u_1}{2} - \frac{k_3-2}{4}, 0]$ *if u_3 is odd, u_2 is odd, u_1 is even and $k_3 = 12h+2$.*

9. $[\frac{u_3-1}{2}, \frac{u_2+1}{2} + \frac{k_3-4}{6}, \frac{u_1}{2} - \frac{k_3-4}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h+4$.

10. $[\frac{u_3+1}{2}, \frac{u_2-1}{2} - \frac{k_3}{6}, \frac{u_1}{2} + \frac{5k_3-6}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h+6$.

11. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3-2}{6}, \frac{u_1}{2} - \frac{k_3}{4}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h+8$.

12. $[\frac{u_3-1}{2}, \frac{u_2+1}{2} + \frac{k_3-4}{6}, \frac{u_1}{2} - \frac{k_3+2}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h+10$.

13. $[\frac{u_3+1}{2}, \frac{u_2-1}{2} - \frac{k_3}{6}, \frac{u_1+1}{2} + \frac{5k_3-12}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h$.

14. $[\frac{u_3+3}{2}, \frac{u_2+2-k_3}{2}, \frac{u_1+2}{2}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+1$.

15. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3-2}{6}, \frac{u_1-1}{2} - \frac{k_3-2}{4}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+2$.

16. $[\frac{u_3+1}{2}, \frac{u_2-1}{2} - \frac{k_3-3}{6}, \frac{u_1+1}{2} - \frac{k_3-3}{6}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+3$.

17. $[\frac{u_3-1}{2}, \frac{u_2+1}{2} + \frac{k_3-4}{6}, \frac{u_1-1}{2} - \frac{k_3-4}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+4$.

18. $[\frac{u_3+3}{2}, \frac{u_2+2-k_3}{2}, \frac{u_1+1}{2}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+5$.

19. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3}{6}, \frac{u_1-1}{2} - \frac{k_3-6}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+6$.

20. $[\frac{u_3+3}{2}, \frac{u_2+2-k_3}{2}, \frac{u_1+1}{2}, 0]$ *if* u_3 *is odd ,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+7$.

21. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3-2}{6}, \frac{u_1-1}{2} - \frac{k_3-4}{4}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+8$.

22. $[\frac{u_3+1}{2}, \frac{u_2+1}{2} - \frac{k_3-3}{6}, \frac{u_1+1}{2} - \frac{k_3}{3}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+9$.

23. $[\frac{u_3-1}{2}, \frac{u_2+1}{2} + \frac{k_3-4}{6}, \frac{u_1-1}{2} - \frac{k_3-10}{12}, 0]$ *if* u_3 *is odd,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+10$.

24. $[\frac{u_3+3}{2}, \frac{u_2+2-k_3}{2}, \frac{u_1+1}{2}, 0]$ *if* u_3 *is odd ,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h+11$.

25. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3-2}{2}, \frac{u_1}{2}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* k_3 *is even.*

26. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-1}{6}, \frac{u_1}{2} - \frac{k_3-1}{12}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* $k_3 = 12h+1$.

27. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-3}{6}, \frac{u_1}{2} - \frac{k_3-3}{12}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* $k_3 = 12h+3$.

28. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3-3}{2}, \frac{u_1}{2} - \frac{k_3-1}{4}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* $k_3 = 12h+5$.

29. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-1}{6}, \frac{u_1}{2} - \frac{k_3-7}{12}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* $k_3 = 12h+7$.

30. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3-3}{2}, \frac{u_1}{2} - \frac{k_3-1}{4}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* $k_3 = 12h+9$.

31. $[\frac{u_3+3}{2}, \frac{u_2-2}{2} - \frac{k_3-5}{2}, \frac{u_1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is even and* $k_3 = 12h + 11$.

32. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3}{2}, \frac{u_1-1}{2} + \frac{k_3}{2}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h$.

33. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-1}{6}, \frac{u_1-1}{2} + \frac{k_3-1}{12}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 1$.

34. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3}{2}, \frac{u_1-1}{2} + \frac{k_3}{2}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h+2$.

35. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3-3}{2}, \frac{u_1-1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 3$.

36. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-4}{6}, \frac{u_1-1}{2} - \frac{k_3-4}{6}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 4$.

37. $[\frac{u_3+3}{2}, \frac{u_2-2}{2} - \frac{k_3-5}{2}, \frac{u_1-1}{2} - \frac{k_3-5}{4}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 5$.

38. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3}{2}, \frac{u_1-1}{2} + \frac{k_3}{2}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h+6$.

39. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-1}{6}, \frac{u_1+1}{2} + \frac{k_3-7}{12}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 7$.

40. $[\frac{u_3+3}{2}, \frac{u_2}{2} - \frac{k_3-4}{2}, \frac{u_1+1}{2} - \frac{k_3}{2}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 8$.

41. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-3}{6}, \frac{u_1-1}{2} - \frac{k_3-9}{12}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 9$.

42. $[\frac{u_3+1}{2}, \frac{u_2}{2} - \frac{k_3-4}{6}, \frac{u_1+1}{2} - \frac{k_3-2}{6}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 10$.

43. $[\frac{u_3+3}{2}, \frac{u_2-2}{2} - \frac{k_3-5}{2}, \frac{u_1-1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is odd,* u_2 *is even,* u_1 *is odd and* $k_3 = 12h + 11$.

44. $[\frac{u_3+2}{2}, \frac{u_2+1}{2} - \frac{k_3}{3}, \frac{u_1}{2} + \frac{k_3}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h$.

45. $[\frac{u_3+2}{2}, \frac{u_2+1}{2} - \frac{k_3-1}{3}, \frac{u_1}{2} + \frac{k_3-1}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 1$.

46. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1}{2} - \frac{k_3-2}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 2$.

47. $[\frac{u_3+2}{2}, \frac{u_2-1}{2} - \frac{k_3-3}{3}, \frac{u_1}{2} + \frac{k_3-3}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 3$.

48. $[\frac{u_3+2}{2}, \frac{u_2-1}{2} - \frac{k_3-4}{3}, \frac{u_1}{2} - \frac{k_3-4}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 4$.

49. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1}{2} - \frac{k_3-1}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 5$.

50. $[\frac{u_3+2}{2}, \frac{u_2+1}{2} - \frac{k_3-3}{3}, \frac{u_1}{2} - \frac{5k_3-6}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 6$.

51. $[\frac{u_3+2}{2}, \frac{u_2+1}{2} - \frac{k_3-1}{3}, \frac{u_1}{2} - \frac{k_3-7}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 7$.

52. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1}{2} - \frac{k_3}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 8$.

53. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1}{2} - \frac{k_3-1}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 9$.

54. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1}{2} - \frac{k_3-2}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 10$.

55. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is even and* $k_3 = 12h + 11$.

56. $[\frac{u_3+2}{2}, \frac{u_2+1}{2} - \frac{2k_3-6}{6}, \frac{u_1+1}{2} - \frac{5k_3}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h$.

57. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-1}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 1$.

58. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-2}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 2$.

59. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 3$.

60. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-4}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 4$.

61. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-5}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 5$.

62. $[\frac{u_3+2}{2}, \frac{u_2-1}{2} - \frac{k_3-3}{3}, \frac{u_1+1}{2} + \frac{k_3-6}{12}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 6$.

63. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 7$.

64. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-4}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 8$.

65. $[\frac{u_3+2}{2}, \frac{u_2-1}{2} - \frac{k_3-3}{3}, \frac{u_1+1}{2} - \frac{k_3-9}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 9$.

66. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-2}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 10$.

67. $[\frac{u_3}{2}, \frac{u_2+1}{2}, \frac{u_1-1}{2} - \frac{k_3-3}{4}, 0]$ *if* u_3 *is even,* u_2 *is odd,* u_1 *is odd and* $k_3 = 12h + 11$.

Furthermore, $|V_{k_3}(YES^{\underline{u}}) - V_{k_3}(NO^{\underline{u}})| = a_3[k_3]k_3$ *where* $a_3[k_3] \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1\}$.

Proof: Let us denote a query by $[\underline{z}]$. We have to minimize $|V_{k_3}(YES^{\underline{u}}) - V_{k_3}(NO^{\underline{u}})| = |(2z_3 - u_3)\binom{k_3}{3} + (2z_2 - u_2)\binom{k_3}{2} + (2z_1 - u_1)k_3| = |k_3(\frac{k_3-1}{2}((2z_3 - u_3)\frac{k_3-2}{3} + 2z_2 - u_2) + 2z_1 - u_1)|$. We set $D = (2z_3 - u_3)\frac{k_3-2}{3} + 2z_2 - u_2$ and $E = |\frac{k_3-1}{2}((2z_3 - u_3)\frac{k_3-2}{3} + 2z_2 - u_2) + 2z_1 - u_1|$. We abbreviate an odd integer as o and an even integer as e.

1. u_3, u_2 and u_1 even: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2}{2}$ and $z_1 = \frac{u_1}{2}$

2. u_3, u_2 even, u_1 odd and k_3 is even: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2}{2} + 1$ and $z_1 = \frac{u_1+1-k_3}{2}$.

3. u_3, u_2 even, u_1 odd and $k_3 = 6h + 1$: In this case $\frac{k_3-1}{2}$ is an integer divisible by 3 and $D = \frac{e}{3}$. It follows that $\frac{k_3-1}{2}D$ is an even integer. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2}{2}$ and $z_1 = \frac{u_1+1}{2}$.

4. u_3, u_2 even, u_1 odd and $k_3 = 6h + 3$: In this case $\frac{k_3-1}{2}$ is an integer and $D = \frac{e}{3}$. Thus, $E \geq \frac{o}{3}$. $E = \frac{1}{3}$, if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-3}{3}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-3}{6}$.

5. u_3, u_2 even, u_1 odd and $k_3 = 6h + 5$: In this case $\frac{k_3-1}{2}$ is an integer and D is an even number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2}{2}$ and $z_1 = \frac{u_1+1}{2}$.

6. u_3, u_2 odd, u_1 even and k_3 is odd: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2+2-k_3}{2} + 1$ and $z_1 = \frac{u_1}{2} - \frac{k_3-1}{2}$.

7. u_3, u_2 odd, u_1 even and $k_3 = 12h$: In this case $D = \frac{o}{3}$. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3}{6}$ and $z_1 = \frac{u_1}{2} - \frac{k_3}{12}$.

8. u_3, u_2 odd, u_1 even and $k_3 = 12h + 2$: In this case $\frac{k_3-2}{3}$ is an even integer. Thus, D is an odd integer and $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-2}{6}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-2}{4}$.

9. u_3, u_2 odd, u_1 even and $k_3 = 12h + 4$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3-1}{2}$, $z_2 = \frac{u_2+1}{2} + \frac{k_3-4}{6}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-4}{12}$.

10. u_3, u_2 odd, u_1 even and $k_3 = 12h + 6$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is not divisible by 2 or 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3}{6}$ and $z_1 = \frac{u_1}{2} + \frac{5k_3-6}{12}$.

11. u_3, u_2 odd, u_1 even and $k_3 = 12h + 8$: In this case $\frac{k_3-2}{3}$ is an even integer. Thus, D is an odd integer and $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-2}{6}$ and $z_1 = \frac{u_1}{2} - \frac{k_3}{4}$.

12. u_3, u_2 odd, u_1 even and $k_3 = 12h + 10$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3-1}{2}$, $z_2 = \frac{u_2+1}{2} + \frac{k_3-4}{6}$ and $z_1 = \frac{u_1}{2} - \frac{k_3+2}{12}$.

13. u_3, u_2, u_1 odd and $k_3 = 12h$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is odd and not divisible by 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3}{6}$ and $z_1 = \frac{u_1+1}{2} + \frac{5k_3-12}{12}$.

14. u_3, u_2, u_1 odd and $k_3 = 12h+1$: In this case $D = \frac{e}{3}$ and k_3-1 is divisible by 12. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2+2-k_3}{2}$ and $z_1 = \frac{u_1+2}{2}$.

15. u_3, u_2, u_1 odd and $k_3 = 12h+2$: In this case $D = 1$ and k_3-1 is not divisible by 2. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-2}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-2}{4}$.

16. u_3, u_2, u_1 odd and $k_3 = 12h + 3$: In this case $D = \frac{e}{3}$ and $\frac{k_3-1}{2}$ is an odd number. Thus, $E \geq \frac{1}{3}$. $E = \frac{1}{3}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3-3}{6}$ and $z_1 = \frac{u_1+1}{2} - \frac{k_3-3}{6}$.

17. u_3, u_2, u_1 odd and $k_3 = 12h + 4$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is odd and divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3-1}{2}$, $z_2 = \frac{u_2+1}{2} + \frac{k_3-4}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-4}{12}$.

18. u_3, u_2, u_1 odd and $k_3 = 12h+5$: In this case D is an even number and k_3-1 is an even number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2+2-k_3}{2}$ and $z_1 = \frac{u_1+1}{2}$.

19. u_3, u_2, u_1 odd and $k_3 = 12h + 6$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number not divisible by 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-6}{12}$.

20. u_3, u_2, u_1 odd and $k_3 = 12h+7$: In this case $D = \frac{e}{3}$ and k_3-1 is divisible by 6. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2+2-k_3}{2}$ and $z_1 = \frac{u_1+1}{2}$.

21. u_3, u_2, u_1 odd and $k_3 = 12h + 8$: In this case D is an odd number and $k_3 - 1$ is an odd number. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-2}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-4}{4}$.

22. u_3, u_2, u_1 odd and $k_3 = 12h+9$: In this case $D = \frac{e}{3}$ and $\frac{k_3-1}{2}$ is an even number. Thus, $E \geq \frac{1}{3}$. $E = \frac{1}{3}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-3}{6}$ and $z_1 = \frac{u_1+1}{2} - \frac{k_3}{3}$.

23. u_3, u_2, u_1 odd and $k_3 = 12h+10$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3-1}{2}$, $z_2 = \frac{u_2+1}{2} + \frac{k_3-4}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-10}{12}$.

24. u_3, u_2, u_1 odd and $k_3 = 12h+11$: In this case D is an even number and $\frac{k_3-1}{2}$ is an even number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2+2-k_3}{2}$ and $z_1 = \frac{u_1+1}{2}$.

25. u_3 odd, u_2, u_1 even and k_3 is even: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-2}{2}$ and $z_1 = \frac{u_1}{2}$

26. u_3 odd, u_2, u_1 even and $k_3 = 12h+1$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-1}{6}$ and $z_1 = \frac{u_1}{2} + \frac{k_3-1}{12}$

27. u_3 odd, u_2, u_1 even and $k_3 = 12h+3$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an odd number. Thus, $E \geq \frac{1}{3}$. $E = \frac{1}{3}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-3}{6}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-3}{12}$.

28. u_3 odd, u_2, u_1 even and $k_3 = 12h+5$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-3}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-1}{4}$

29. u_3 odd, u_2, u_1 even and $k_3 = 12h+7$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an odd number divisible by 3. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-1}{6}$ and $z_1 = \frac{u_1}{2} + \frac{k_3-7}{12}$.

30. u_3 odd, u_2, u_1 even and $k_3 = 12h+9$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-3}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-1}{4}$

31. u_3 odd, u_2, u_1 even and $k_3 = 12h+11$: In this case D and $\frac{k_3-1}{2}$ are odd numbers. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2-2}{2} - \frac{k_3-5}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-3}{4}$.

32. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3}{2}$ and $z_1 = \frac{u_1-1}{2} + \frac{k_3}{2}$

33. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+1$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}D$ is an even number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-1}{6}$ and $z_1 = \frac{u_1-1}{2} + \frac{k_3-1}{12}$.

34. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+2$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3}{2}$ and $z_1 = \frac{u_1-1}{2} + \frac{k_3}{2}$

35. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+3$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-3}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-3}{4}$.

36. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+4$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-4}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-4}{6}$.

37. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+5$: In this case D is an odd number and $\frac{k_3-1}{2}D$ is an even number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2-2}{2} - \frac{k_3-5}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-5}{4}$.

38. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+6$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3}{2}$ and $z_1 = \frac{u_1-1}{2} + \frac{k_3}{2}$

39. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+7$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-1}{6}$ and $z_1 = \frac{u_1+1}{2} + \frac{k_3-7}{12}$.

40. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+8$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-4}{2}$ and $z_1 = \frac{u_1+1}{2} - \frac{k_3}{2}$.

41. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+9$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an even number. Thus, $E \geq \frac{1}{3}$. $E = \frac{1}{3}$ if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-3}{6}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-9}{12}$.

42. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+10$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+1}{2}$, $z_2 = \frac{u_2}{2} - \frac{k_3-4}{6}$ and $z_1 = \frac{u_1+1}{2} - \frac{k_3+2}{6}$.

43. u_3 odd, u_2 even, u_1 odd and $k_3 = 12h+11$: In this case we can get a minimal value for E(=0) if we choose we choose $z_3 = \frac{u_3+3}{2}$, $z_2 = \frac{u_2-2}{2} - \frac{k_3-5}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-3}{4}$.

44. u_3 even, u_2 odd, u_1 even and $k_3 = 12h$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number not divisible by 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3}{3}$ and $z_1 = \frac{u_1}{2} + \frac{k_3}{12}$.

45. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+1$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-1}{3}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-1}{12}$.

46. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+2$: In this case D is an odd number and $k_3 - 1$ is an odd number. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-2}{4}$.

47. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+3$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an odd number not divisible by 3. Thus, $E \geq \frac{1}{3}$. $E = \frac{1}{3}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3-3}{3}$ and $z_1 = \frac{u_1}{2} + \frac{k_3-3}{12}$.

48. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+4$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3-4}{3}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-4}{12}$.

49. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+5$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-1}{4}$.

50. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+6$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number not divisible by 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-3}{3}$ and $z_1 = \frac{u_1}{2} - \frac{5k_3-6}{12}$.

51. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+7$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}D$ is an odd number divisible by 3. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{k_3-1}{3}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-7}{12}$.

52. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+8$: In this case D is an odd number and $k_3 - 1$ is an odd number. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3}{4}$.

53. u_3 even, u_2 odd, u_1 even and $k_3 = 12h+9$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-1}{4}$.

54. u_3 even, u_2 odd, u_1 even and $k_3 = 12h + 10$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-2}{4}$.

55. u_3 even, u_2 odd, u_1 even and $k_3 = 12h + 11$: In this case D is an odd number and $\frac{k_3-1}{2}$ is an odd number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1}{2} - \frac{k_3-3}{4}$.

56. u_3 even, u_2, u_1 odd and $k_3 = 12h$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number not divisible by 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2+1}{2} - \frac{2k_3-6}{6}$ and $z_1 = \frac{u_1+1}{2} - \frac{5k_3}{12}$.

57. u_3 even, u_2, u_1 odd and $k_3 = 12h + 1$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an even number divisible by 3. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-1}{4}$.

58. u_3 even, u_2, u_1 odd and $k_3 = 12h + 2$: In this case D is an odd number and $k_3 - 1$ is an odd number. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-2}{4}$.

59. u_3 even, u_2, u_1 odd and $k_3 = 12h + 3$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-3}{4}$.

60. u_3 even, u_2, u_1 odd and $k_3 = 12h + 4$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an odd number divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-4}{4}$.

61. u_3 even, u_2, u_1 odd and $k_3 = 12h + 5$: In this case D is an an odd number and $\frac{k_3-1}{2}$ is an even number. Thus, $E \geq 1$. $E = 1$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-5}{4}$.

62. u_3 even, u_2, u_1 odd and $k_3 = 12h + 6$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number not divisible by 3. Thus, $E \geq \frac{1}{6}$. $E = \frac{1}{6}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3-3}{3}$ and $z_1 = \frac{u_1+1}{2} + \frac{k_3-6}{12}$.

63. u_3 even, u_2, u_1 odd and $k_3 = 12h + 7$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-3}{4}$.

64. u_3 even, u_2, u_1 odd and $k_3 = 12h + 8$: In this case D is an odd number and $k_3 - 1$ is an odd number. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-4}{4}$.

65. u_3 even, u_2, u_1 odd and $k_3 = 12h + 9$: In this case $D = \frac{o}{3}$ and $\frac{k_3-1}{2}$ is an even number not divisible by 3. Thus, $E \geq \frac{1}{3}$. $E = \frac{1}{3}$ if we choose $z_3 = \frac{u_3+2}{2}$, $z_2 = \frac{u_2-1}{2} - \frac{k_3-3}{3}$ and $z_1 = \frac{u_1+1}{2} + \frac{k_3-9}{12}$.

66. u_3 even, u_2, u_1 odd and $k_3 = 12h + 10$: In this case $D = \frac{o}{3}$ and $k_3 - 1$ is an odd number divisible by 3. Thus, $E \geq \frac{1}{2}$. $E = \frac{1}{2}$ if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-2}{4}$.

67. u_3 even, u_2, u_1 odd and $k_3 = 12h + 11$: In this case we can get a minimal value for E(=0) if we choose $z_3 = \frac{u_3}{2}$, $z_2 = \frac{u_2+1}{2}$ and $z_1 = \frac{u_1-1}{2} - \frac{k_3-3}{4}$.

$\square$

C All conditions of the main theorem

In this section we present the extended version of Theorem 6.1.

Theorem C.1 *Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$. Then*
$N < 266$*:*

1. *Person 2 needs $k+2$ questions to win the game if $N \in M_2$ (defined in Proposition 6.4).*
2. *Person 2 needs $k+1$ questions to win the game if* $N \in M_1$.
3. *Person 2 needs k questions to win the game if $N \notin M_1 \cup M_2$ (defined in Proposition 6.5).*

$N \geq 266$*: Let $N = 8m + p$ Person 2 needs either k questions or $k+1$ questions to win the game, depending on conditions, concerning k and p.*

1. *Person 2 needs k questions to win the game if $N = 8m$.*
2. *Let $N = 8m+1$, Person 2 needs k questions to win the game if*
   (a) $(8m+1)G(k) + \binom{k-1}{3} \leq 2^k$ *and* $k = 12h$,
   (b) $(8m+1)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+1$,
   (c) $(8m+1)G(k) + \binom{k-1}{3} + 4(k-3) \leq 2^k$ *and* $k = 12h+2$.
   (d) $(8m+1)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+3$,
   (e) $(8m+1)G(k) + \binom{k-1}{3} \leq 2^k$ *and* $k = 12h+4$,
   (f) $(8m+1)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+5$,
   (g) $(8m+1)G(k) + \binom{k-1}{3} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h+6$,
   (h) $(8m+1)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+7$,
   (i) $(8m+1)G(k) + \binom{k-1}{3} \leq 2^k$ *and* $k = 12h+8$,
   (j) $(8m+1)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+9$,
   (k) $(8m+1)G(k) + \binom{k-1}{3} + 4(k-3) \leq 2^k$ *and* $k = 12h+10$,
   (l) $(8m+1)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+11$.
3. *Let $N = 8m+2$, Person 2 needs k questions to win the game if*
   (a) $(8m+2)G(k) + \frac{2}{3}\binom{k-2}{2} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h$,
   (b) $(8m+2)G(k) + 2(k-3) \leq 2^k$ *and* $k = 12h+1$,
   (c) $(8m+2)G(k) + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+2$,
   (d) $(8m+2)G(k) + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+3$,
   (e) $(8m+2)G(k) + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+4$,
   (f) $(8m+2)G(k) + 2(k-3) \leq 2^k$ *and* $k = 12h+5$,
   (g) $(8m+2)G(k) + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+6$,
   (h) $(8m+2)G(k) \leq 2^k$ *and* $k = 12h+7$,
   (i) $(8m+2)G(k) + \frac{2}{3}\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+8$,
   (j) $(8m+2)G(k) + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+9$,
   (k) $(8m+2)G(k) + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+10$,

(l) $(8m+2)G(k) \leq 2^k$ *and* $k = 12h+11$.

4. *Let* $N = 8m+3$, *Person 2 needs* k *questions to win the game if*

    (a) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h$,
    (b) $(8m+3)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+1$,
    (c) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+2$,
    (d) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+3$,
    (e) $(8m+3)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+4$,
    (f) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+5$,
    (g) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h+6$,
    (h) $(8m+3)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+7$,
    (i) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+8$,
    (j) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+9$,
    (k) $(8m+3)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+10$,
    (l) $(8m+3)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+11$.

5. *Let* $N = 8m+4$, *Person 2 needs* k *questions to win the game if*

    (a) $(8m+4)G(k) + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h$,
    (b) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+1$,
    (c) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+2$,
    (d) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+3$,
    (e) $(8m+4)G(k) + 4(k-3) \leq 2^k$ *and* $k = 12h+4$,
    (f) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+5$,
    (g) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+6$,
    (h) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+7$,
    (i) $(8m+4)G(k) + 4(k-3) \leq 2^k$ *and* $k = 12h+8$,
    (j) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+9$,
    (k) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+10$,
    (l) $(8m+4)G(k) \leq 2^k$ *and* $k = 12h+11$.

6. *Let* $N = 8m+5$, *Person 2 needs* k *questions to win the game if*

    (a) $(8m+5)G(k) + \binom{k-1}{3} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h$,
    (b) $(8m+5)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+1$,
    (c) $(8m+5)G(k) + \binom{k-1}{3} + 4(k-3) \leq 2^k$ *and* $k = 12h+2$,
    (d) $(8m+5)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+3$,
    (e) $(8m+5)G(k) + \binom{k-1}{3} + 4(k-3) \leq 2^k$ *and* $k = 12h+4$,
    (f) $(8m+5)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+5$,
    (g) $(8m+5)G(k) + \binom{k-1}{3} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h+6$,
    (h) $(8m+5)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+7$,
    (i) $(8m+5)G(k) + \binom{k-1}{3} + 4(k-3) \leq 2^k$ *and* $k = 12h+8$,

(j) $(8m+5)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+9$,

(k) $(8m+5)G(k) + \binom{k-1}{3} + 4(k-3) \leq 2^k$ *and* $k = 12h+10$,

(l) $(8m+5)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+11$.

7. *Let* $N = 8m+6$, *Person 2 needs* k *questions to win the game if*

(a) $(8m+6)G(k) + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h$,

(b) $(8m+6)G(k) + 2(k-3) \leq 2^k$ *and* $k = 12h+1$,

(c) $(8m+6)G(k) + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+2$,

(d) $(8m+6)G(k) \leq 2^k$ *and* $k = 12h+3$,

(e) $(8m+6)G(k) + 2\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+4$,

(f) $(8m+6)G(k) + 2(k-3) \leq 2^k$ *and* $k = 12h+5$,

(g) $(8m+6)G(k) + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+6$,

(h) $(8m+6)G(k) \leq 2^k$ *and* $k = 12h+7$,

(i) $(8m+6)G(k) + \frac{2}{3}\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+8$,

(j) $(8m+6)G(k) + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+9$,

(k) $(8m+6)G(k) + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+10$,

(l) $(8m+6)G(k) \leq 2^k$ *and* $k = 12h+11$.

8. *Let* $N = 8m+7$, *Person 2 needs* k *questions to win the game if*

(a) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h$,

(b) $(8m+7)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+1$,

(c) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+2$,

(d) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+3$,

(e) $(8m+7)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+4$,

(f) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+5$,

(g) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{4}{3}(k-3) \leq 2^k$ *and* $k = 12h+6$,

(h) $(8m+7)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+7$,

(i) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} \leq 2^k$ *and* $k = 12h+8$,

(j) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + \frac{2}{3}(k-3) \leq 2^k$ *and* $k = 12h+9$,

(k) $(8m+7)G(k) + \binom{k-1}{3} + 2\binom{k-2}{2} + 4(k-3) \leq 2^k$ *and* $k = 12h+10$,

(l) $(8m+7)G(k) + \binom{k-1}{3} + \frac{2}{3}\binom{k-2}{2} + 2(k-3) \leq 2^k$ *and* $k = 12h+11$.

Proof: Let $N \geq 266$, $\underline{s}$ be the initial state, $\underline{t}$ be the state after the first question $[\underline{x}]$, $\underline{u}$ be the state after the second question $[\underline{y}]$ and $\underline{v}$ be the state after the third question $[\underline{z}]$. If the character of $\underline{v}$ is the same as the one of the initial state minus 3, then we can win the game with $ch(\underline{s})$ questions, otherwise we need $ch(\underline{s})+1$ questions. Thus, we have to calculate if the character of the initial state is equal to the character of the state after three questions plus 3. By Propositions 3.1 A.1 and B.1 we know how much we lose in each question. Let $\underline{v}$ be the worst state after the first three questions, $\underline{s}$ the initial state, $k = ch(\underline{s})$ and a_1, $a_2[k-2]$ and $a_3[k-3]$ the loss in the first, second and third question. By Berlekamp's Conservation of Volume we get

$$V_{k-3}(\underline{v}) = \frac{V_k(\underline{s}) + a_1\binom{k-1}{3}}{8} + \frac{a_2[k-2]\binom{k-2}{2}}{4} + \frac{a_3[k-3](k-3)}{2}$$

Thus, $ch(\underline{s}) = ch(\underline{v})$ if $V_{k-3}(\underline{v}) \leq 2^{k-3}$. We have to check:

$$V_k(\underline{s}) + a_1\binom{k-1}{3} + 2a_2[k-2]\binom{k-2}{2} + 4a_3[k-3](k-3) \leq 2^k$$

Due to the complementary questions we only have to consider the cases in which the resulting YES-state is bigger than or equal to the resulting NO-state. For some formal reason we set $L = (2y_3 - t_3)(k-4) + 6y_2 - 3t_2$. Then $3a_2[k-2] = L$.

1. $N = 8m$: In this case we do not lose anything during the first three question. Therefore, we can win the game with $ch(\underline{s})$ questions.

2. $N = 8m+1$: In the first question $a_1 = 1$. Let $\underline{t}$ be the state after the first question. In the worst case t_3 is odd and t_2 is even. The loss at the second and third question depends on the character of the state:

   (a) $k = 12h$: In this case $ch(\underline{s}) - 2 = 12h + 10$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. Person 1 can choose between the YES-state and the NO-state. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (b) $k = 12h+1$: In this case $ch(\underline{s}) - 2 = 12h+11$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is odd. With this it follows that y_3 has to be even and y_2 has to be odd or y_3 has to be odd and y_2 has to be even to get $a_3[k-3] = 0$ because $u_2 = y_2 + t_3 - y_3$ has to be even. Then holds $L \bmod 4 = 1$. This is a contradiction to $3a_2[k-2] = 3$.

   (c) $k = 12h+2$: In this case $ch(\underline{s}) - 2 = 12h$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$,$(o,e,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1$ is odd. But this is not possible.

   (d) $k = 12h+3$: In this case $ch(\underline{s}) - 2 = 12h+1$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is odd. Thus, we know $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is even and y_2 is odd. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

   (e) $k = 12h+4$: In this case $ch(\underline{s}) - 2 = 12h+2$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (f) $k = 12h+5$: In this case $ch(\underline{s}) - 2 = 12h+3$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (g) $k = 12h+6$: In this case $ch(\underline{s}) - 2 = 12h+4$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$, $(e,o,o,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1$ is odd. But this is not possible.

(h) $k = 12h + 7$: In this case $ch(\underline{s}) - 2 = 12h + 5$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(i) $k = 12h + 8$: In this case $ch(\underline{s}) - 2 = 12h + 6$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(j) $k = 12h + 9$: In this case $ch(\underline{s}) - 2 = 12h + 7$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(k) $k = 12h + 10$: In this case $ch(\underline{s}) - 2 = 12h + 8$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, o, 0)$,$(o, e, o, 0)$ or $(o, o, e, 0)$ and $u_3 + u_2 + u_1 = o$. This is not possible.

(l) $k = 12h + 11$: In this case $ch(\underline{s}) - 2 = 12h + 9$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0), (e, e, o, 0), (o, e, e, 0) or (o, e, o, 0)$ and $u_3 + u_2 + u_1 = o$. Thus, we know $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is even and y_2 is odd. But this is a contradiction to $(2y_3 - t_3)\frac{k-4}{3} + 2y_2 - t_2 = \frac{1}{3}$. Then holds $L \bmod 4 = 1$. This is a contradiction to $3a_2[k-2] = 3$.

3. $N = 8m + 2$: In the first question $a_1 = 0$. Let $\underline{t}$ be the state after the first question. In the worst case t_3 is odd and t_2 is odd. The loss at the second and third question depends on the character of the state:

   (a) $k = 12h$: In this case $ch(\underline{s}) - 2 = 12h + 10$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0), (e, o, e, 0), (o, e, e, 0) or (o, o, e, 0)$. It follows that $u_1 = t_2 - y_2$ has to be even. Thus, y_2 has to be odd. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

   (b) $k = 12h + 1$: In this case $ch(\underline{s}) - 2 = 12h + 11$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, e, o, 0)$,$(o, e, e, 0)$ or $(o, e, o, 0)$ and $u_3 + u_2 + u_1$ is even. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is odd and y_2 is even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

   (c) $k = 12h + 2$: In this case $ch(\underline{s}) - 2 = 12h$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (d) $k = 12h + 3$: In this case $ch(\underline{s}) - 2 = 12h + 1$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to obtain $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, e, o, 0)$,$(o, e, e, 0)$ or $(o, e, o, 0)$ and $u_3 + u_2 + u_1$ is even. Thus, we know $YES_2^t = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is even and y_2 is odd.

(e) $k = 12h+4$: In this case $ch(\underline{s}) - 2 = 12h+2$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(f) $k = 12h+5$: In this case $ch(\underline{s}) - 2 = 12h+3$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is even. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is odd and y_2 is even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

(g) $k = 12h+6$: In this case $ch(\underline{s}) - 2 = 12h+4$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(h) $k = 12h+7$: In this case $ch(\underline{s}) - 2 = 12h+5$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(i) $k = 12h+8$: In this case $ch(\underline{s}) - 2 = 12h+6$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to obtain $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$,$(o,e,e,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1$ is even. Thus, we know $u_1 = t_2 - y_2$ has to be even. It follows, that y_2 has to be odd. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

(j) $k = 12h+9$: In this case $ch(\underline{s}) - 2 = 12h+7$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is even. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is odd and y_2 is even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

(k) $k = 12h+10$: In this case $ch(\underline{s}) - 2 = 12h+8$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(l) $k = 12h+11$: In this case $ch(\underline{s}) - 2 = 12h+9$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

4. $N = 8m + 3$: In the first question $a_1 = 1$. Let $\underline{t}$ be the state after the first question. In the worst case t_3 is even and t_2 is odd. The loss at the second and third question depends on the character of the state:

(a) $k = 12h$: In this case $ch(\underline{s}) - 2 = 12h + 10$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$, $(o,e,e,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1$ is odd. It follows that $u_1 = y_2 - t_2$ has to be even. Thus, y_2 has to be odd. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

(b) $k = 12h+1$: In this case $ch(\underline{s})-2 = 12h+11$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(c) $k = 12h+2$: In this case $ch(\underline{s})-2 = 12h$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$, $(o,e,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1 = o$. This is not possible.

(d) $k = 12h+3$: In this case $ch(\underline{s})-2 = 12h+1$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3+u_2+u_1$ is odd. Thus, we know $u_2 = y_2+t_3-y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even.

(e) $k = 12h+4$: In this case $ch(\underline{s})-2 = 12h+2$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(f) $k = 12h+5$: In this case $ch(\underline{s})-2 = 12h+3$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3+u_2+u_1$ is odd. It follows that $u_2 = y_2+t_3-y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even.

(g) $k = 12h+6$: In this case $ch(\underline{s})-2 = 12h+4$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1 = o$. This is not possible.

(h) $k = 12h+7$: In this case $ch(\underline{s})-2 = 12h+5$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(i) $k = 12h+8$: In this case $ch(\underline{s})-2 = 12h+6$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$, $(o,e,e,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1$ is odd. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^t = y_2$ has to be even.

(j) $k = 12h+9$: In this case $ch(\underline{s})-2 = 12h+7$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(k) $k = 12h+10$: In this case $ch(\underline{s})-2 = 12h+8$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$, $(o,e,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1 = o$. This is not possible.

(l) $k = 12h + 11$: In this case $ch(\underline{s}) - 2 = 12h + 9$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, o, 0)$,$(o, e, o, 0)$ or $(o, o, e, 0)$ and $u_3 + u_2 + u_1$ is odd. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

5. $N = 8m + 4$: In the first question $a_1 = 0$ and at the second question $a_2[k-2] = 0$. The loss in the third question depends on the character of the state:

   (a) $k = 12h$ By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, e, 0)$,(o, e, e) or $(o, o, e, 0)$ and $u_3 + u_2 + u_1$ is even. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^t = y_2$ has to be even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

   (b) $k = 12h + 1$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (c) $k = 12h + 2$: By Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (d) $k = 12h + 3$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (e) $k = 12h + 4$: By Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, e, 0)$,(o, e, e) or $(o, o, e, 0)$ and $u_3 + u_2 + u_1$ is even. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^t = y_2$ has to be even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

   (f) $k = 12h + 5$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (g) $k = 12h + 6$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (h) $k = 12h + 7$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (i) $k = 12h + 8$: By Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, e, 0)$, $(o, e, e, 0)$ or $(o, o, e, 0)$ and $u_3 + u_2 + u_1$ is even. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^t = y_2$ has to be even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

   (j) $k = 12h + 9$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(k) $k = 12h+10$: By Table 3.2 we get $a_3[k-3] \in \{0,1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(l) $k = 12h + 11$: By Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

6. $N = 8m + 5$: In the first question $a_1 = 1$. Let $\underline{t}$ the state after the first question. In the worst case t_3 is odd and t_2 is even. The loss at the second and third question depends on the character of the state:

   (a) $k = 12h$: In this case $ch(\underline{s}) - 2 = 12h + 10$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$,$(o,e,e,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1 = o$. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^{\underline{t}} = y_2$ has to be even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

   (b) $k = 12h+1$: In this case $ch(\underline{s})-2 = 12h+11$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (c) $k = 12h + 2$: In this case $ch(\underline{s}) - 2 = 12h$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0,1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$,$(o,e,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1 = o$. This is not possible.

   (d) $k = 12h+3$: In this case $ch(\underline{s})-2 = 12h+1$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3+u_2+u_1 = o$. Thus, we know $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is even and y_2 is odd. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

   (e) $k = 12h+4$: In this case $ch(\underline{s})-2 = 12h+2$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0,1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$,$(o,e,e,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1$ is odd. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^{\underline{t}} = y_2$ has to be even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

   (f) $k = 12h+5$: In this case $ch(\underline{s})-2 = 12h+3$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

   (g) $k = 12h+6$: In this case $ch(\underline{s})-2 = 12h+4$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$, $(e,o,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1$ is odd. But this is not possible.

(h) $k = 12h + 7$: In this case $ch(\underline{s}) - 2 = 12h + 5$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(i) $k = 12h + 8$: In this case $ch(\underline{s}) - 2 = 12h + 6$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$,$(o,e,e,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1$ is odd. The YES-state and the NO-state after the second question have equal weight. Person 1 can choose between them. Thus, both have to be in the form mentioned above. It holds that $NO_1^{\underline{t}} = y_2$ has to be even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

(j) $k = 12h + 9$: In this case $ch(\underline{s}) - 2 = 12h + 7$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(k) $k = 12h + 10$: In this case $ch(\underline{s}) - 2 = 12h + 8$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$, $(o,e,o,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1 = o$. This is not possible.

(l) $k = 12h + 11$: In this case $ch(\underline{s}) - 2 = 12h + 9$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is odd. Thus, we know $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is even and y_2 is odd. But this is a contradiction to $(2y_3 - t_3)\frac{k-4}{3} + 2y_2 - t_2 = \frac{1}{3}$. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

7. $N = 8m + 6$: In the first question $a_1 = 0$. Let $\underline{t}$ be the state after the first question. In the worst case t_3 is odd and t_2 is odd. The loss at the second and third question depends on the character of the state:

(a) $k = 12h$: In this case $ch(\underline{s}) - 2 = 12h + 10$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(b) $k = 12h + 1$: In this case $ch(\underline{s}) - 2 = 12h + 11$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is even. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is odd and y_2 is even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

(c) $k = 12h + 2$: In this case $ch(\underline{s}) - 2 = 12h$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(d) $k = 12h + 3$: In this case $ch(\underline{s}) - 2 = 12h + 1$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(e) $k = 12h+4$: In this case $ch(\underline{s}) - 2 = 12h+2$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, e, 0)$,$(o, e, e, 0)$ or $(o, o, e, 0)$ and $u_3 + u_2 + u_1$ is even. Thus, we know $u_1 = t_2 - y_2$ has to be even. It follows, that y_2 has to be odd. Then holds $L \bmod 4 = 1$. This is a contradiction to $3a_2[k-2] = 3$.

(f) $k = 12h+5$: In this case $ch(\underline{s}) - 2 = 12h+3$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, e, o, 0)$,$(o, e, e, 0)$ or $(o, e, o, 0)$ and $u_3 + u_2 + u_1$ is even. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is odd and y_2 is even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

(g) $k = 12h+6$: In this case $ch(\underline{s}) - 2 = 12h+4$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(h) $k = 12h+7$: In this case $ch(\underline{s}) - 2 = 12h+5$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(i) $k = 12h+8$: In this case $ch(\underline{s}) - 2 = 12h+6$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, o, e, 0)$,$(o, e, e, 0)$ or $(o, o, e, 0)$ and $u_3 + u_2 + u_1$ is even. Thus, we know $u_1 = t_2 - y_2$ has to be even. It follows, that y_2 has to be odd. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

(j) $k = 12h+9$: In this case $ch(\underline{s}) - 2 = 12h+7$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e, e, e, 0)$,$(e, e, o, 0)$,$(o, e, e, 0)$ or $(o, e, o, 0)$ and $u_3 + u_2 + u_1$ is even. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is even or if y_3 is odd and y_2 is even. Then holds $L \bmod 4 = 2$. This is a contradiction to $3a_2[k-2] = 0$.

(k) $k = 12h+10$: In this case $ch(\underline{s}) - 2 = 12h+8$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(l) $k = 12h+11$: In this case $ch(\underline{s}) - 2 = 12h+9$ it follows that $a_2[k-2] = 0$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

8. $N = 8m + 7$: In the first question $a_1 = 1$. In the first question $a_1 XS = 1$. Let $\underline{t}$ be the state after the first question. In the worst case t_3 is even and t_2 is odd. The loss at the second and third question depends on the character of the state:

(a) $k = 12h$ In this case $ch(\underline{s}) - 2 = 12h+10$ it follows that $a_2[k-2] = \frac{1}{3}$ and in the worst case Person 1 answers yes. By Table 3.2 we get $a_3[k-3] \in$

$\{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2 and Person 1 answers yes then $a_3[k-3] = 0$.

(b) $k = 12h+1$: In this case $ch(\underline{s}) - 2 = 12h+11$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(c) $k = 12h+2$: In this case $ch(\underline{s}) - 2 = 12h$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state $\underline{u}$ is equal to $(e,e,e,0)$,$(e,o,o,0)$,$(o,e,o,0)$ or $(o,o,e,0)$. But we know $u_3 + u_2 + u_1 = o$. This is not possible.

(d) $k = 12h+3$: In this case $ch(\underline{s}) - 2 = 12h+1$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state $\underline{u}$ is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is odd. Thus, we know $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

(e) $k = 12h+4$: In this case $ch(\underline{s}) - 2 = 12h+2$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,e,0)$,$(o,e,e,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1$ is odd. Thus, we know $u_1 = t_2 - y_2$ has to be even. It follows that y_2 has to be odd. Then holds $L \bmod 4 = 1$. This is a contradiction to $3a_2[k-2] = 3$.

(f) $k = 12h+5$: In this case $ch(\underline{s}) - 2 = 12h+3$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3 + u_2 + u_1$ is odd. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

(g) $k = 12h+6$: In this case $ch(\underline{s}) - 2 = 12h+4$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{3}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{3}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$, $(e,o,o,0)$ or $(o,o,e,0)$ and $u_3 + u_2 + u_1$ is odd. But this is not possible.

(h) $k = 12h+7$: In this case $ch(\underline{s}) - 2 = 12h+5$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(i) $k = 12h+8$: In this case $ch(\underline{q}s) - 2 = 12h+6$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, 1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 0$.

(j) $k = 12h+9$: In this case $ch(\underline{s}) - 2 = 12h+7$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{6}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{6}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3+u_2+u_1$ is odd. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

(k) $k = 12h+10$: In this case $ch(\underline{s})-2 = 12h+8$ it follows that $a_2[k-2] = 1$ and by Table 3.2 we get $a_3[k-3] \in \{0,1\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = 1$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,o,o,0)$, $(o,e,o,0)$ or $(o,o,e,0)$ and $u_3+u_2+u_1 = o$. This is not possible.

(l) $k = 12h+11$: In this case $ch(\underline{s})-2 = 12h+9$ it follows that $a_2[k-2] = \frac{1}{3}$ and by Table 3.2 we get $a_3[k-3] \in \{0, \frac{1}{2}\}$. If we ask the second question suggested by Proposition 3.2, then $a_3[k-3] = \frac{1}{2}$. In the worst case it is not possible to get $a_3[k-3] = 0$ because this happens only if the state is equal to $(e,e,e,0)$,$(e,e,o,0)$,$(o,e,e,0)$ or $(o,e,o,0)$ and $u_3+u_2+u_1$ is odd. It follows that $u_2 = y_2 + t_3 - y_3$ has to be even. This happens only if y_3 is odd and y_2 is odd or if y_3 is even and y_2 is even. Then holds $L \bmod 4 = 3$. This is a contradiction to $3a_2[k-2] = 1$.

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