The Cauchy Crofton Formula

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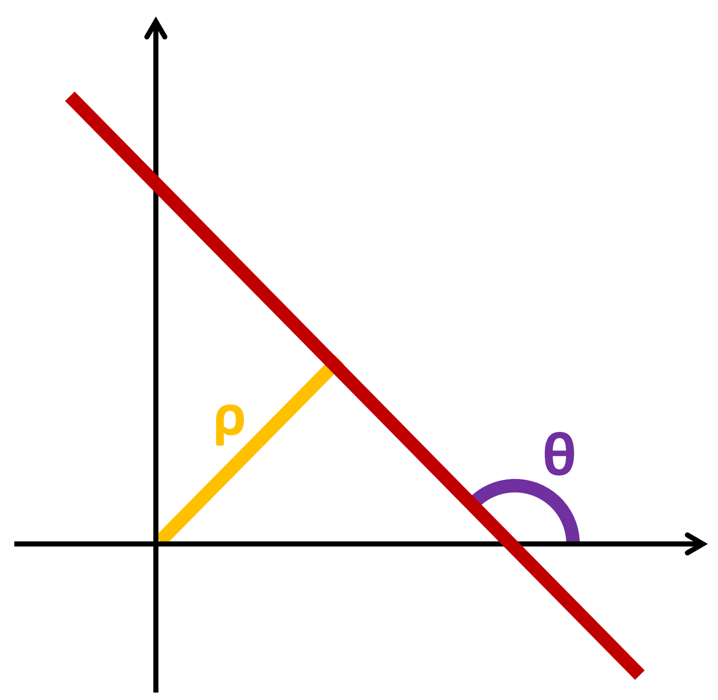
UCL

Goal: obtain the length of a curve

Suppose that you are given a curve that lies in a plane and you need to calculate its length.
We will do this by "counting" how many straight lines intersect with that curve.

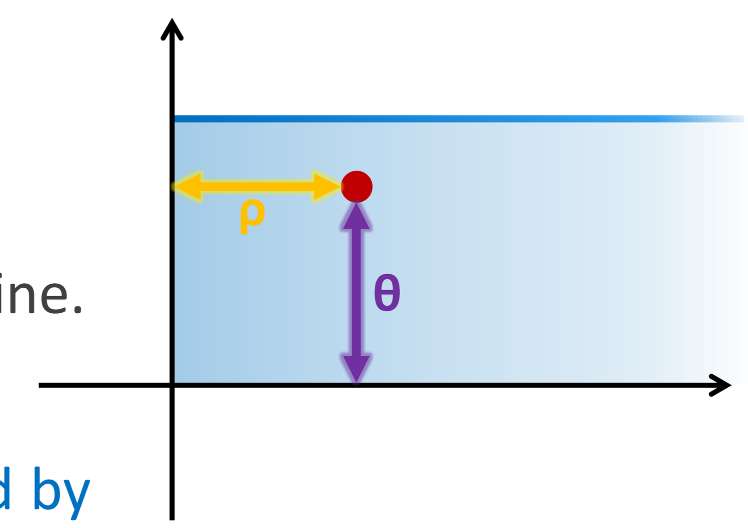
Representation of straight lines:



To characterise a line, use:
- Its **distance** to **origin** (ρ)
- Its **angle** with the **horizontal** (θ)

This gives coordinates (**ρ**, **θ**)

Now plot (**ρ**, **θ**) in another plane.
This is the new representation of the line.



Note that all the lines are represented by a point in the blue region as $0 < \theta < 2\pi$ and $0 < \rho$

Formula:

For any curve in a plane:

$$\iint_S d\rho\, d\theta = 2\,\ell$$

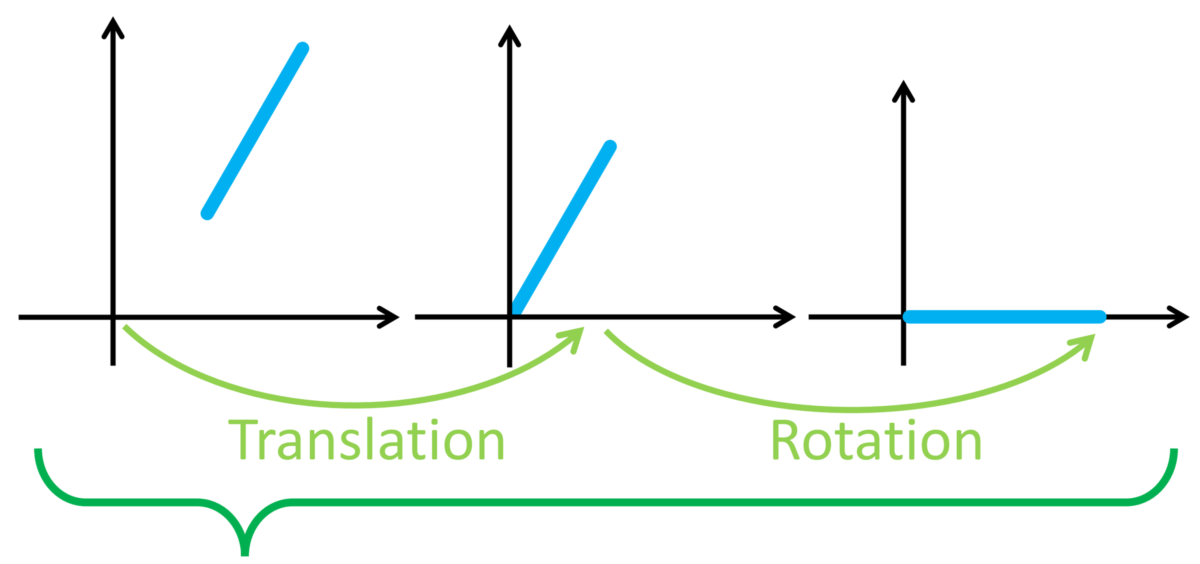
Usual Integration

Length of the curve

Where **S** is the **set** of **lines** that **intersect** the **curve**
→ Count with multiplicity
(i.e. if a line intersects twice, put it in the set S twice)

Note: As we represent the lines by points in a plane, the set S of lines that intersect the curve becomes a set of points, thus a region in a plane. We can then integrate in two dimensions over this region of the plane.

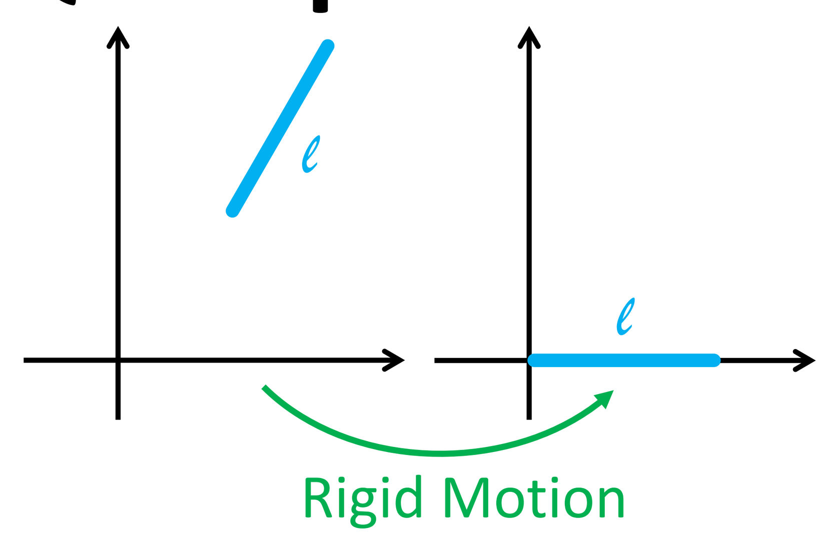
Rigid motions



One may prove that the integral in the formula remains unchanged after applying a rigid motion to our curve

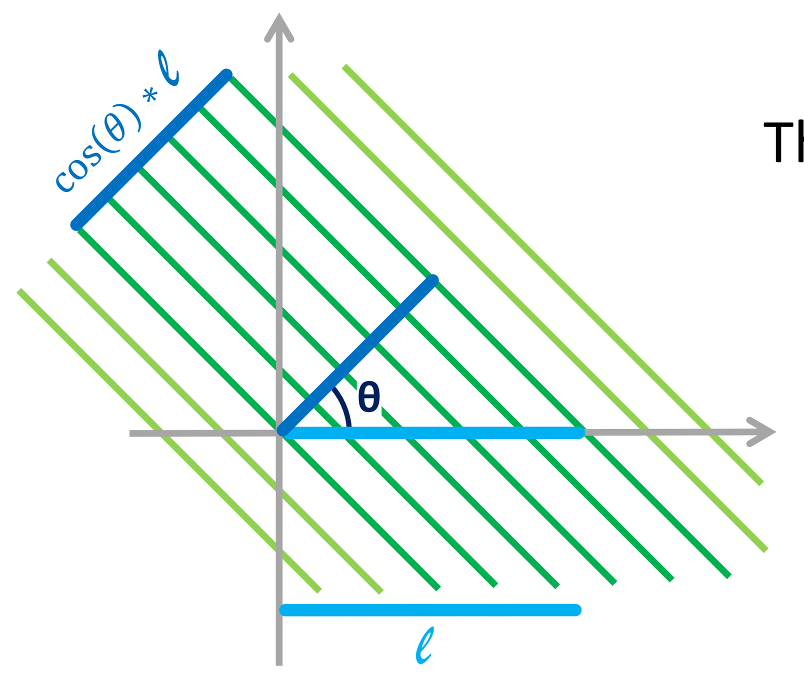
Rigid Motion: Combination of a **translation** and a **rotation**

Quick proof:



Take any line curve with length ℓ

We use a rigid motion (so the formula doesn't change) to throw the line onto the x-axis, between 0 and ℓ



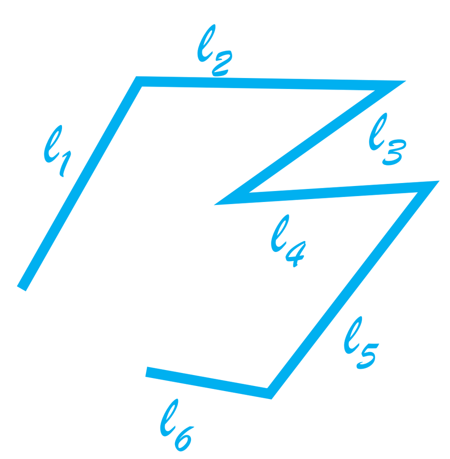
Then, the set S becomes:

$$S = \{(\rho,\theta) \mid -\pi/2 < \theta < \pi/2,\ 0 < \rho < l\,|\cos(\theta)|\}$$

Now, calculate the integral:

$$\iint_S d\rho, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{l\,|\cos(\theta)|} d\rho d\theta$$
$$= \int_{-\pi/2}^{\pi/2} l\,|\cos(\theta)|\,d\theta = 2\,l$$

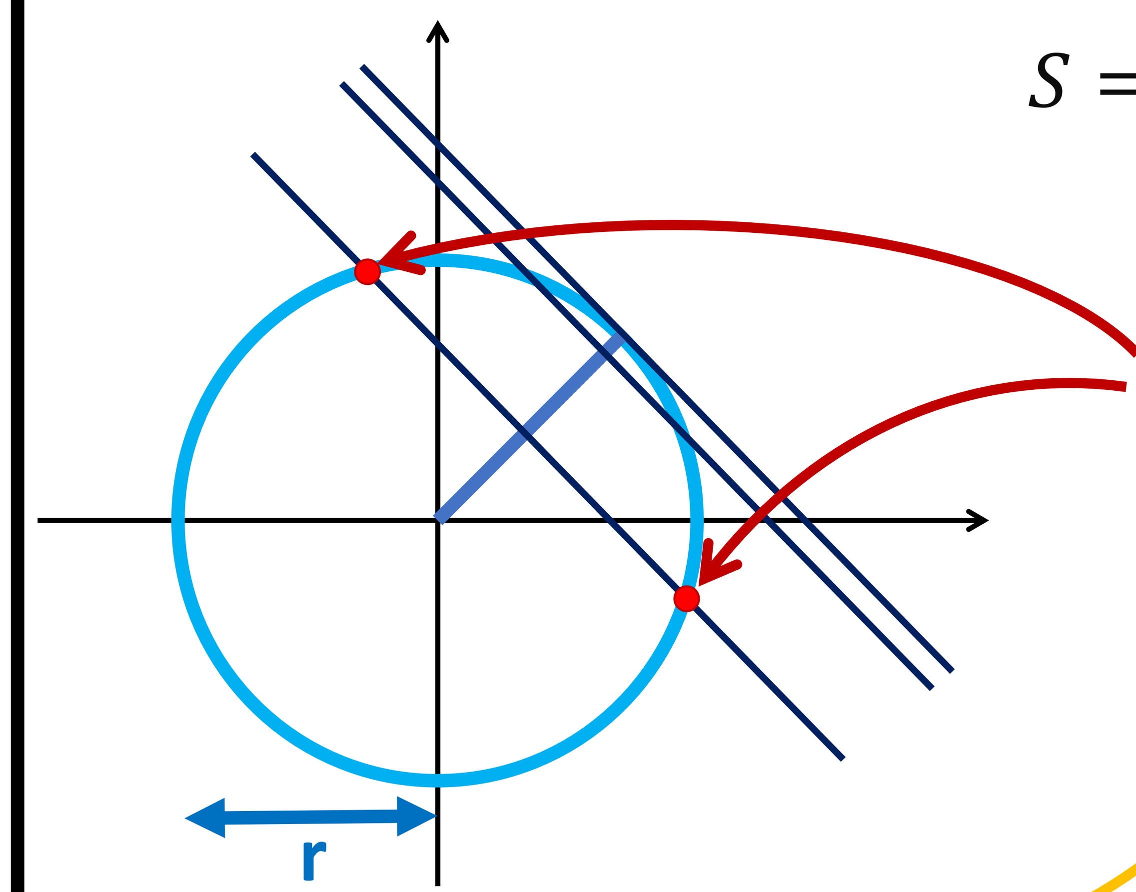
...which gives the formula!



For any **polygonal line**, just apply the formula above for **each component** (that is straight), and **sum** them : $\ell = \sum_{i=1}^{n} \ell_i$

For any **regular curve**, take its **limit** as small pieces of straight lines

Check for a circle:



$$S = \{(\rho,\theta) \mid \rho \le r,\ \theta \in (0, 2\pi)\}$$

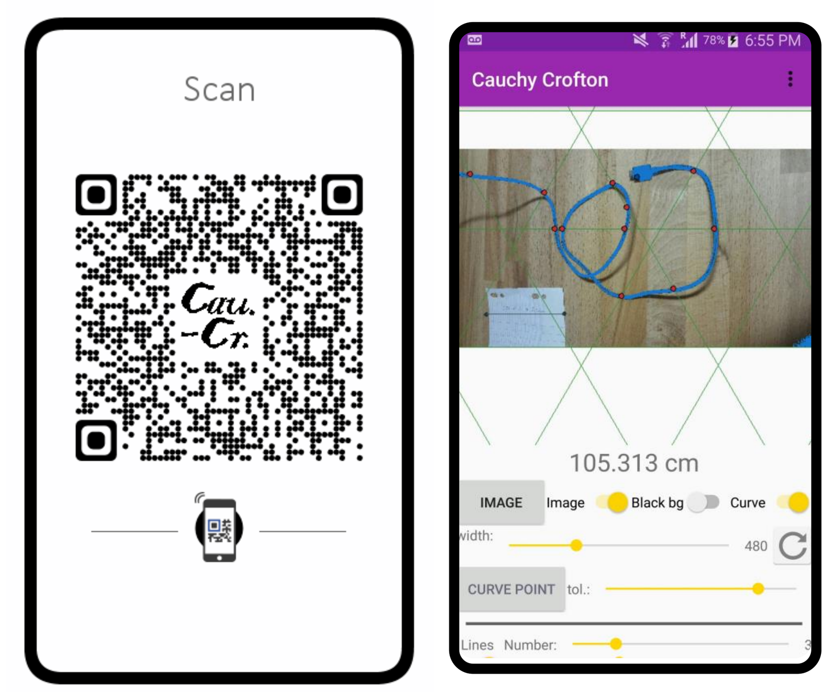
2 intersections, so count twice in the integral

$$\iint_S d\rho\, d\theta = \int_0^{2\pi}\int_0^{r} 2\, d\rho d\theta = 2\pi r.2 = 2\ell$$

Finally, obtain: $\boldsymbol{\ell = 2\pi r}$ Did you expect this result?

Uses of this formula:

This is not that useful for the case of a circle, as we already know the length (although, we can use it to derive the formula).
Nevertheless, remember that you may use the formula on any regular curve in a plane, which makes it very powerful.
Also, as the formula involves an integral, we can use all the tools we already know about integrals. In particular, it is quite easy to make approximations.



Try this formula in real life: this app enables you to apply the formula to a picture.

Scan the QR code
Or search "Cauchy Crofton"

(Only available on Play Store)

This poster summarises a second-year project undertaken in June 2018. Reference: M. do Carmo, Differential Geometry of Curves and Surfaces