



UNIVERSITY OF OXFORD
MATHEMATICAL INSTITUTE

**Random Fractals:
Traversing without Zebra Crossing**

Candidate Number

1051774

Course

MSc Mathematical Sciences

Trinity Term
April 26, 2021

Percolation Fractals

Abstract

Percolations are geometric figures obtained after repetitively removing some material from an initial set, in our case, a cuboid. Studying their properties provide a better understanding of some physical models for material, as some porous materials are modelled using percolations. In this paper, we study mathematical properties of percolations in any dimensions, and make numerical experiments on the two and three dimensional case.

After introducing the concept of dimensionality for sets, and the concept of fractal, we will narrow down our interest to a special type of fractals: the one obtained by percolation. Two competing models will be studied: the classical percolation, and the recursive scheme.

After discussing some basic properties (density, dimension, size of the central "blob"), we will look at crossings on percolations. This is the main purpose of this paper, we will derive some mathematical result in extreme cases, and use some innovative algorithms to obtain relevant numerics. Finally, we will extend techniques used before in the spatial case of starting in two dimensions to acquire some more general results.

Acknowledgment

First of all, I would like to thank my supervisor, Ben Hambly, for guiding and supporting through this project.

I would like to thank the whole administration team of Oxford, for all the help they provided despite the pandemic context.

Contents

0	Introduction	3
1	Background Theory	4
1.1	Dimensions	4
1.1.1	Intuition	4
1.1.2	Topological Dimension	5
1.1.3	Box Dimension	5
1.1.4	Hausdorff Dimension	5
1.1.5	Some Relations Between Dimensions	6
1.2	Fractals	7
1.2.1	Formal Definition	7
1.2.2	Famous Examples	7
1.2.3	Real Life Fractals	10
2	Percolation Fractals	11
2.1	Classical Percolation	11
2.2	Recursive Percolation	11
2.3	Extension to other dimensions	12
2.4	Density	13
2.5	Dimensionality	13
2.6	Blob	15
3	Percolation Crossings	20
3.1	Types of Crossings	20
3.2	Percolation Crossings	20
3.2.1	Crossings Probability	20
3.2.2	Crossings Length	21
3.3	Percolation Complement Crossings	21
3.3.1	Crossings Probability	21
3.3.2	Crossings Length	23
3.4	Link Between Crossings and Complement Crossings	23
3.5	Connectivity Properties	23
A	More on Fractals	1
A.1	Mandelbrot and Julia Sets	1
A.2	Sierpiski Fractals	1
A.3	Coastline of Islands	2
B	Finding Crossings	4
B.1	Examples	4
B.2	Algorithms	4

0 Introduction

1 Background Theory

We derive here a mathematical basis on fractals and dimensions of sets. This will be used later in our more specific study of random fractals.

1.1 Dimensions

The concept of dimension is quite intuitive from an everyday life perspective. However, the mathematical concept is more involved. From the non-mathematical world, this can be used to have better understanding of other objects arising from other fields such as DNA structure¹, or lungs[Ionescu et al.(2009)Ionescu, Oust

1.1.1 Intuition

Some objects that we are used to working with have an established dimension:

- **Empty set / Point**: dimension 0
- **Curve** (e.g.: *line*): dimension 1
- **Surface** (e.g.: *square*): dimension 2
- **Volume** (e.g.: *cube*): dimension 3

All of these usual objects have integral dimensions, making it (relatively) easy to understand.

A rule of thumb to calculate the dimension is to double (or, in general, multiply by n) the size of the object, and count the number of copies of the original object obtained. If there are N original objects, the dimension is $d = \frac{\ln(N)}{\ln(n)}$. This is so that when scaling by n , the length/area/volume of the set is multiplied by $N = n^d$.

Some objects have a more complicated dimension (in fact, a non-integral one)²:

- **Cantor Set** (see fig. 3): dimension $\log_3(2) = \frac{\ln(2)}{\ln(3)} \simeq 0.631$
- **Koch Snowflake** (see fig. 4): dimension $\log_3(4) = \frac{\ln(4)}{\ln(3)} \simeq 1.262$
- **Sierpinski Carpet** (see fig. 5): dimension $\log_2(8) = \frac{\ln(8)}{\ln(3)} \simeq 1.893$

The fractional dimensions of these objects justify creating a formal mathematical definition.

After this quick overview, 3 properties seem desirable for a definition of dimension [Pol(2009)]. For a set X ($\subset \mathbb{R}^n$, in general):

1. If X is a manifold, then the dimension should coincide with our natural preconception.
2. In some cases, X may have a fractional (i.e. non-integral) dimension.
This is needed in order to describe sets as the ones discussed above.
3. If X is countable, then X has dimension 0.

There are several definitions for dimension, satisfying different properties.

¹<https://news.mit.edu/2009/3d-genome>

²This will be discussed in more detail in 1.2.2.

1.1.2 Topological Dimension

The topological dimension is the most straightforward way to define dimension. It relies on the intuition that the boundary of a ball of dimension d should have dimension $d - 1$.

Definition 1.1 (Topological dimension). *The topological dimension $\dim_T(X)$ of a set X is defined recursively through the following:*

$$\dim_T(X) = \begin{cases} -1 & \text{if } X = \emptyset \\ d & \text{if } d = \min \{n \in \mathbb{N} \mid \forall x \in X, \exists r > 0 \text{ s.t. } \dim_T(\partial B_r(x) \cap X) \leq n - 1\} \end{cases}$$

This definition satisfies the first desired property (1.1.1:1). However, \dim_T is always an integer (this is clear from definition). Therefore, it does not satisfy the second desired property (1.1.1:2).

1.1.3 Box Dimension

The box dimension is more general than the topological one, in the sense that it allows non-integral dimensions. It relies on an intuition mentioned before: when scaled by n , a set contains N copies of the original set, then the dimension should be $\frac{\ln(N)}{\ln(n)}$.

Definition 1.2 (Box dimension). *The box dimension \dim_B of a set X is defined through the following limit:*

$$\dim_B(X) = \lim_{\varepsilon \rightarrow 0} \frac{\ln(N(\varepsilon))}{-\log(\varepsilon)}$$

Here, $N(\varepsilon)$ is the smallest number of ε -balls needed to cover X .

Note that box dimension exists only if this limit exists. In the case this limit does not exist, it is possible to define upper and lower box dimensions, taking respectively \limsup and \liminf in the definition above. (Both upper and lower box dimension with the general box dimension when it exists.)

This definition satisfies the first and second desired property (1.1.1:1,2). However, if we consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$, then $\dim_B(X) > 0$, but X is countable. Therefore, it does not satisfies the third desired property (1.1.1:3).

1.1.4 Hausdorff Dimension

The Hausdorff dimension (sometimes also called fractal dimension) is considered to be the most robust concept of dimension. The reason of this consortium is that Hausdorff dimension has a measure (the Hausdorff measure) associated to it). The definition is more involved than both the topological and the box dimensions.

Definition 1.3 (Hausdorff/fractal dimension). *The Hausdorff dimension \dim_H of a set X is defined as follows:*

$$\text{For } \varepsilon > 0, d \geq 0, H_\varepsilon^d(X) = \inf_{\substack{\mathcal{U} \text{ open cover of } X \\ U \in \mathcal{U} \implies \text{diam}(U) < \varepsilon}} \left\{ \sum_{U \in \mathcal{U}} \text{diam}(U)^d \right\}$$

A $H_\varepsilon^d(X)$ is an increasing function as $\varepsilon \rightarrow 0$, the following limit is well defined:

$$H^d(X) = \lim_{\varepsilon \rightarrow 0} H_\varepsilon^d(X)$$

This defines a measure, which is called the d -dimensional Hausdorff measure.

For a set X , $H^d(X)$ jumps from 0 to ∞ , when d varies from 0 to ∞ (this will be proved as a claim, see 1.1.5). The particular value at which this jump occurs is the Hausdorff dimension of X . Formally:

$$\dim_H(X) = \inf \{\delta \mid H^\delta(X) = 0\}$$

This definition satisfies all three of the desired property (1.1.1:1,2,3). The first two are clear from definition.

Property 1.1. *Countable sets have Hausdorff dimension 0.*

Proof. Suppose $X = \{x_n \mid n \in \mathbb{N}\}$ is countable. Let $\varepsilon > 0$, and take $\{\varepsilon_n \mid n \in \mathbb{N}\}$ such that $\sum_{n=0}^{\infty} \varepsilon_n^d < \varepsilon$. Then $\mathcal{B} = \{B(x_n, \varepsilon_n) \mid n \in \mathbb{N}\}$ is an open cover of X , so $H_\varepsilon^d(X) \leq \varepsilon$. As this is true $\forall \varepsilon > 0$, $\forall d \geq 0$, get $H^d(X) = 0$ for all $d \geq 0$. So finally, $\dim_H(X) = 0$. \square

The Hausdorff dimension is usually harder to calculate in practice.

1.1.5 Some Relations Between Dimensions

For dimension definition, there is a choice to make between more robust, but hard to calculate (Hausdorff dimension) and less robust but easier to calculate (Topological/Box dimension) definitions. It is therefore very useful to know some relationships between the three notions (it is then possible to use, for example, box dimension to give estimates for the Hausdorff dimension).

Property 1.2 (Upper bound for fractal dimension). *Box dimension is greater than or equal to Hausdorff dimension.*

Proof. For a set X (with box dimension well defined): Let $\eta > 0$, $\gamma = \dim_B(X) + \eta$ and $\delta = \dim_B(X) + 2\eta$. Then, $\exists \varepsilon > 0$ such that X can be covered by $N(\varepsilon) < \varepsilon^{-\gamma}$ ε -balls. Thus, $H_\varepsilon^\delta(X) \leq \varepsilon^{-\gamma} \varepsilon^\delta = \varepsilon^\eta$, so $H^\delta(X) = 0$. This gives $\dim_H(X) < \dim_B(X) + 2\eta \quad \forall \eta > 0$, and hence, $\dim_H(X) \leq \dim_B(X)$. \square

Thus, by calculating the box dimension, we also have an upper bound for the Hausdorff dimension.

Lemma 1.1. *If the d -dimensional Lebesgue measure is non-zero, then the Hausdorff dimension is greater than or equal to d .*

Proof. Suppose a set X is such that $\dim_H(X) < d$.

Claim. $H^d(X) < \infty \implies H^c(X) = 0 \quad \forall c > d$

Proof. As $H^d(X) < \infty$: For all $\varepsilon > 0$ there is an open cover \mathcal{U} for X such that $\sum_{U \in \mathcal{U}} \text{diam}(U)^d < \infty$ and $\text{diam}(U) < \varepsilon$. So

$$\sum_{U \in \mathcal{U}} \text{diam}(U)^c \leq \underbrace{\varepsilon^{c-d}}_{\substack{\rightarrow 0 \\ \text{as } \varepsilon \rightarrow 0}} \underbrace{\sum_{U \in \mathcal{U}} \text{diam}(U)^d}_{< \infty} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

\square

Thus, $H^d(X) = 0$. Now, the d -dimensional Hausdorff measure is a rescaling of the usual d -dimensional Lebesgue measure, so $\Lambda_d(X) = 0$ ³. This completes the proof by taking the contrapositive. \square

Thus, finding a d such that the Lebesgue measure is non-zero gives a lower bound for the Hausdorff dimension.

Property 1.3 (Lower bound for fractal dimension). *Topological dimension is less than or equal to Hausdorff dimension.*

³Writing $\Lambda_d(X)$ for the d -dimensional Lebesgue measure of X .

Proof. This follows directly from the last property, as a set X having Topological dimension d (i.e. $\dim_T(X) = d$) implies that its d -dimensional Lebesgue measure will be positive (i.e. $\Lambda_d(X) > 0$). \square

In fact, the Hausdorff dimension is bounded by the topological dimension and the box dimension, i.e. for any set X , $\dim_T(X) \leq \dim_H(X) \leq \dim_B(X)$.

1.2 Fractals

The first traces of fractals come back to the 17th century with the mathematician and philosopher Gottfried Leibniz, philosophizing about recursive self-similarity. Proper sketches of a mathematical definition for fractals only go back to Karl Weierstrass, during the 19th century. The recursive self-similarity pattern of most common fractals made their fame: images of some fractals (e.g. Julia sets, Koch snowflake, Sierpiski gasket) have become popular across the mathematics and non-mathematics world.

1.2.1 Formal Definition

There is a controversial mathematical definition of fractals (introduced by Benoît Mandelbrot):

Definition 1.4 (Fractal). *A fractal is a subset of Euclidean space with a Hausdorff dimension that strictly exceeds its topological dimension (i.e. X is a fractal if $\dim_T(X) < \dim_H(X)$).*

However, since sets with "recursive patterns" are often fractals. The "self-similarity" property is sometimes itself taken as the definition for fractals.

1.2.2 Famous Examples

Some fractals became very famous, both for their aesthetic appeal and as an example of a complex structure arising from simple rules. Fractals are one of the best-known examples of mathematical visualization and mathematical beauty.

Basin Boundaries of Complex Maps Along the most famous fractals, one finds the ones that arise from looking at the contour of sets (basin boundaries) derived from complex maps.

Contour of the Mandelbrot Set The Mandelbrot set is a subset of the complex plane defined as follows:

Definition 1.5 (Mandelbrot Set). *Let $z_0 = 0$ and $z_{n+1} = f_c(z_n)$, with $f_c(z) = z^2 + c$. Then, the Mandelbrot set $M \subseteq \mathbb{C}$ is $M = \{c \in \mathbb{C} \mid z_n \text{ does not diverge}\}$.*

The Mandelbrot set is not a fractal itself (it has dimension 2, as it contains $\{w \in \mathbb{C} \mid |w - 1| < \frac{1}{4}\}$ [3DX(2020)], and is contained in \mathbb{C} , both having dimension 2).

It is surprising that the boundary ∂M of M also has Hausdorff dimension 2. This was proved by Shishikura in 1992 [Shishikura(1992)], as a consequence of ∂M

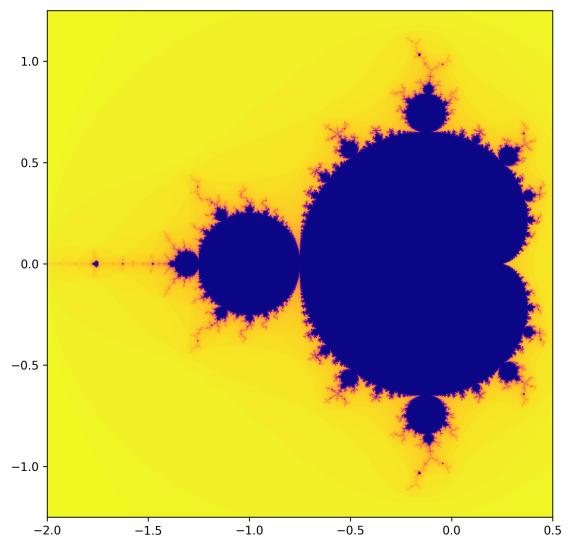


Figure 1: Mandelbrot Set Plot⁴

⁴More details on the plots in the appendix.

having a positive 2-dimensional Lebesgue measure (i.e. $\Lambda_2(\partial M) > 0$).

Despite having an integral fractional dimension, ∂M is commonly considered to be a fractal (because the integrality of fractional dimension is not obvious, and the self-similarity of the set).

Contour of Julia Sets Julia sets are also subset of the complex plane defined similarly to the Mandelbrot set:

Definition 1.6 (Julia Set). *For a complex $c \in \mathbb{C}$ constant: Let $z_0 = z$ and $z_{n+1} = f_c(z_n)$, $f_c(z) = z^2 + c$ as before. The filled Julia set about c is $K(f_c) = \{z \in \mathbb{C} \mid z_n \text{ does not diverges}\} \subseteq \mathbb{C}$. The Julia set (about c) $J(f_c)$ is the boundary of $K(f_c)$ ($J(f_c) \subseteq \mathbb{C}$).*

The dimension of $J(f_c)$ will of course depend on c . It is considered as a fractal even when the dimension is an integer. For some values of c , dimension of $J(f_c)$ is well known:

$c = 0$	$\dim_H(J(f_c)) = 1$	"Circle"
$c = \frac{1}{4}$	$\dim_H(J(f_c)) \approx 1.0812$	
$c = i$	$\dim_H(J(f_c)) \approx 1.2$	"Dendrite"
$c = -1$	$\dim_H(J(f_c)) \approx 1.2683$	
$c = -0.123 + 0.745i$	$\dim_H(J(f_c)) \approx 1.3934$	"Douady rabbit"

The Mandelbrot set and Julia sets are very closely related. In fact, Shishikura proved [Shishikura(1992)] that when c is on ∂M , then the Julia set $J(f_c)$ associated to it has (Hausdorff) dimension 2. Moreover, Heinemann and Stratmann have shown [Heinemann and Stratmann(1998)] that when the quadratic f_c is parametrized with c near the boundary of the Mandelbrot set M , the Hausdorff dimension of $J(f_c)$ is arbitrarily close to 2.

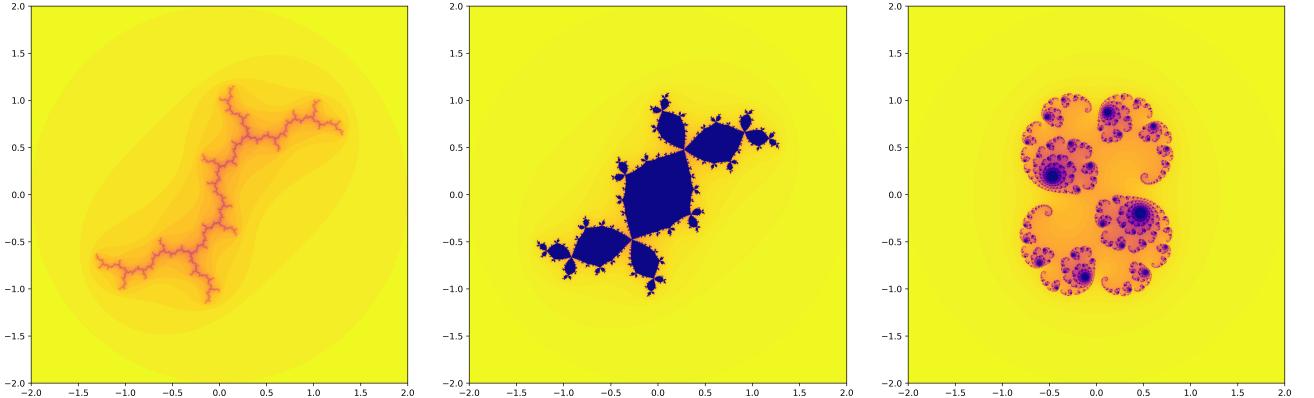


Figure 2: Popular Julia Set Plots
(left to right: "Dendrite"⁵; "Douady rabbit"⁶; other typical Julia set⁷)

Cantor Set The (middle third) Cantor set $C \subseteq [0, 1]$ is constructed via the following informal definition:

- Start with $C_0 = [0, 1]$
- Generate iteratively $C_n = \frac{C_{n-1}}{3} + \frac{2+C_{n-1}}{3}$
- Take the limit $C = \lim_{n \rightarrow \infty} C_n$



Figure 3: Cantor Set (first 6 iterations)

⁵ $c = i$

⁶ $c = -0.123 + 0.745i$

⁷ $c = 0.285 + 0.01i$

The Cantor set satisfies $C = \frac{C}{3} + \frac{2+C}{3}$, so rescaling by $\frac{1}{3}$, it contains two copies of itself. So the intuitive Hausdorff dimension for the Cantor set is $\dim_H(C) = \frac{\log(2)}{\log(3)} \approx 0.631$. This can be proved rigorously (see [Falconer(1990), p. 34-35, ex. 2.7].

Another equivalent definition for the cantor set C is "points in $[0, 1]$ with extension in base 3 composed of 0 and 2 only". That is:

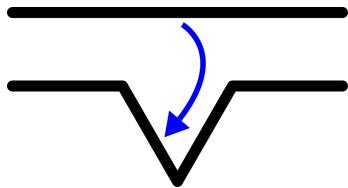
$$C = \left\{ x \in [0, 1] \mid x = \sum_{k=1}^{\infty} x_k 3^{-k}, \quad x_k \in \{0, 2\} \quad \forall k \right\}$$

From this definition, it is straightforward that the Cantor set is totally disconnected. In fact, fractals with Hausdorff dimension smaller than 1 are totally disconnected ([Falconer(1990), p. 33, prop. 2.5]).

Koch Snowflake The Koch Snowflake is an example of a fractal curve. Fractal curves are obtained by applying the same transformation recursively to each segment of the previous iteration of the curve. Their fractional dimension is in the interval $[1, 2]$ (or in $[1, 3]$ if the curve evolves in a three dimensional space).

The Koch snowflake is obtained after replacing each edge of an equilateral triangle by a Koch curve with the triangle edge as initial segment.

Now, the Koch curve is defined recursively as follows: K_n is the curve at iteration n , with segments K_n^1, \dots, K_n^m . For each $i \in \llbracket 1, m \rrbracket$, split K_n^i into 3 equal length segments, and replace the middle one with two new ones as on this diagram:



Merging back all segment gives K_{n+1} .

The Koch curve starting form a segment $[a, b]$ is the limit curve $K = \lim_{n \rightarrow \infty} K_n$ with $K_0 = [a, b]$.

At each level, the Koch curve is scaled by $1/3$, and 4 copies are created. Thus, the intuitive Hausdorff dimension for the Koch curve (which is also the dimension of the Koch snowflake) is $\dim_H(K) = \frac{\log(4)}{\log(3)} \approx 1.262$. This can be proved rigorously using similar techniques as in [Falconer(1990), p. 34-35, ex. 2.7].

Sierpiski Carpet Sierpiski carpet is a fractal constructed recursively by removing parts of its initial set.

The construction starts from a (filled) square. Then, at each step, split every square into 9 sub-square as follows:

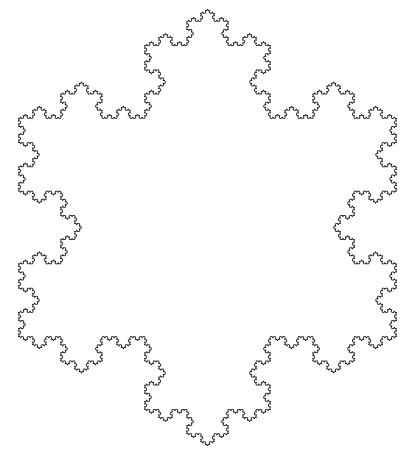
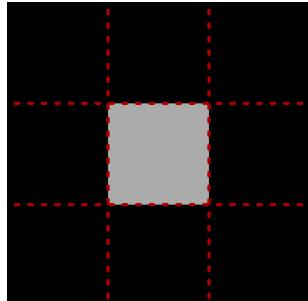


Figure 4: Koch Snowflake Curve Plot (5 iterations)

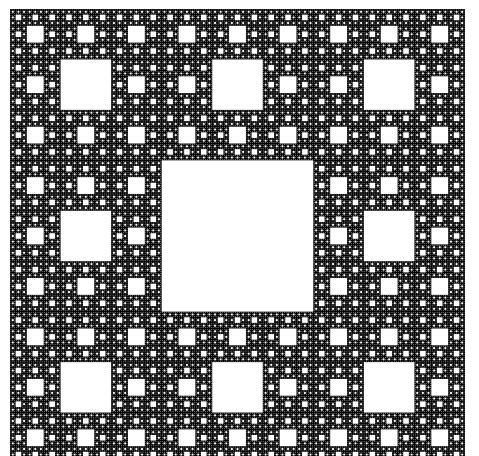


Figure 5: Sierpinski Carpet (6 steps)

Remove the central square, and repeat the operation on the remaining 8 squares.

Note that this can be seen as a version of the Cantor set starting in 2 dimensions.

The figure is made of 8 copies of itself, scaled by a factor of $1/3$. Therefore, the intuitive Hausdorff dimension for this set is $\dim_H(K) = \frac{\log(8)}{\log(3)} \approx 1.893$. Again, this can be proved rigorously using similar techniques as in [Falconer(1990), p. 34-35, ex. 2.7].

1.2.3 Real Life Fractals

Fractals are more than mathematical constructions, in fact self-similarity structures also occur in nature. The limit is never reached, because of the physical constraints, but we can view them as natural approximations of fractals. This gives a good motivation for studying fractals in a mathematical context.

Plants Some plants (not even genetically modified) have self-similarity patterns, and may therefore be considered as fractals.

Ferns Perhaps the most obvious and popular natural fractal: Ferns leaves. The leaves are self-similar, with varying pattern across variety of ferns.

Cauliflower A less obvious example: Cauliflowers' surface is a fractal. In fact, since each branch splits into about 13 branches, each about 3 times shorter, the approximate dimension of a Cauliflower's surface is $\log_3(13) \approx 2.335$.



(a) Fern Leave



(b) Cauliflower

Figure 6: Plant Fractals

Coastline of Islands When trying to measure the length of coastlines, scientists realized that the more precise their attempt was, the higher was the value of the length found. In fact, this makes sense: the approximation of a curve with shorter segments will capture more of the smaller curve details, resulting in a longer length. The approximate length found could be made really large after approximating using smaller and smaller segments (until physical boundary are reached). This yields the fact that the coastline is a fractal with dimension greater than one.

Using electronic maps, we try to estimate the fractal dimension of the coastlines for the UK islands, Iceland and Madagascar. More details on the calculations techniques used are given in the appendix (see A.3). The (approximate) coastline dimension for UK, Iceland, and Madagascar are respectively 1.24, 1.25, and 1.06.

The coastlines of the UK and Iceland are much more irregular than the one of Madagascar. The approximate values found for coastlines dimensions therefore make sense.

2 Percolation Fractals

Fractals obtained by percolation will be the main object we intend to study. They are obtained after removing material from a unit initial set.

We will begin with the definition of the percolation process in the two dimensional case, as it will be our main interest (it is also the most intuitive case, since we can easily draw some pictures). We will then extend the definition to other dimensions.

There are two types of percolations: the "classical", and the "recursive", both have their interest, and we will compare the two throughout this study.

2.1 Classical Percolation

The "classical" percolation is the simplest one, as it is only composed of one filtration. The formal definition goes as follows: Split the unit square $[0, 1]^2$ into n^2 squares of side $1/n$, indexed by $1 \leq i, j \leq n$:

$$B_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n} \right] \times \left[\frac{j-1}{n}, \frac{j}{n} \right].$$

Then, each square will be selected to stay or will be thrown away, according to random variables $\epsilon_{i,j}$ following a Bernoulli distribution with probability parameter p ⁸, $\epsilon_{i,j} \sim \mathcal{B}(p)$. Finally, the classical percolation P is

$$P = \bigcup_{i,j \text{ s.t. } \epsilon_{i,j}=1} B_{i,j}.$$

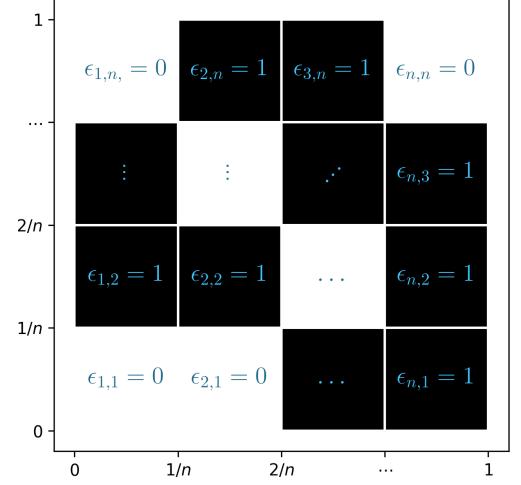


Figure 7: Classical Percolation
($n = 4, p = 0.6$)

We will adopt the notation $P \sim \text{Perc}(n, p, 1)$ for such a setup (a classical percolation of side n and probability parameter p).

In this case, it is interesting to study the behaviour as $n \rightarrow \infty$, we will write $P \sim \text{Perc}(\infty, p, 1)$ for $P = \lim_{n \rightarrow \infty} P^n$, $P^n \sim \text{Perc}(n, p, 1)$.

2.2 Recursive Percolation

The "recursive" percolation is a little more involved. Beginning with a classical percolation, we apply another filtration to each of the remaining squares, and continue recursively.

Formally, for each $1 \leq k \leq d$, split the unit square into $(n^k)^2$ squares of side $1/n^k$ indexed by $1 \leq i, j \leq n^k$

$$B_{i,j}^k = \left[\frac{i-1}{n^k}, \frac{i}{n^k} \right] \times \left[\frac{j-1}{n^k}, \frac{j}{n^k} \right].$$

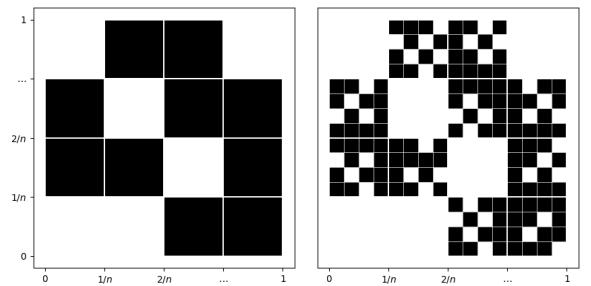


Figure 8: Recursive Percolation
($n = 4, p = 0.6, k = 1, 2$)

Again, associate to each square a Bernoulli random variable with parameter p , $\epsilon_{i,j}^k \sim \mathcal{B}(p)$. Finally, the recursive percolation P^d is defined recursively by:

$$P^k = P^{k-1} \bigcap \left(\bigcup_{i,j \text{ s.t. } \epsilon_{i,j}^k=1} B_{i,j}^k \right) \quad \forall 1 \leq k \leq d$$

⁸Writing $x \sim \mathcal{B}(p)$ for x a random variable following a Bernoulli distribution with parameter p , i.e. $\mathbb{P}(x = 1) = p$ and $\mathbb{P}(x = 0) = 1 - p$.

and

$$P^0 = [0, 1]^2.$$

We will adopt the notation $P^d \sim \text{Perc}(n, p, d)$ for such a setup (a recursive percolation of depth d , side n and probability parameter p).

In this case, it is interesting to study the behaviour as $d \rightarrow \infty$, with n fixed. We will write $P \sim \text{Perc}(n, p)$ for $P = \bigcap_{k \rightarrow \infty} P^k$, with P^k defined recursively as above.

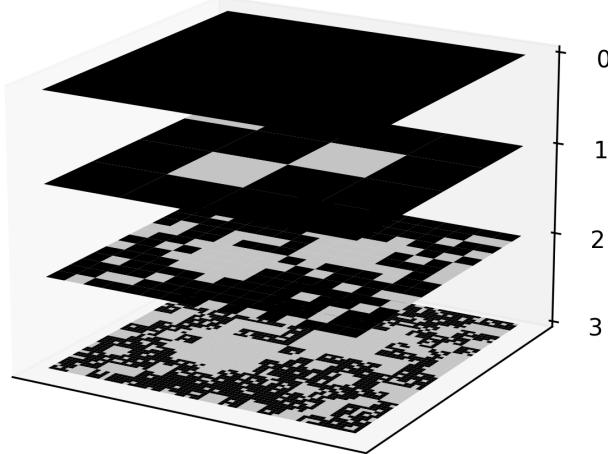


Figure 9: Recursive Percolation
($n = 4, d = 3, p = 0.8$)

2.3 Extension to other dimensions

Now that we defined percolations in two dimensions, it is straightforward to extend the definition to other dimensions.

Extension to D dimensions It suffices to add parameters in the definition to extend it to other dimensions, formally:

Classical Split the unit cuboid $[0, 1]^D$ into n^D squares of side $1/n$, indexed by $1 \leq i_1, \dots, i_D \leq n$:

$$B_{i_1, \dots, i_D} = \left[\frac{i_1 - 1}{n}, \frac{i_1}{n} \right] \times \cdots \times \left[\frac{i_D - 1}{n}, \frac{i_D}{n} \right].$$

Then, attach to each cuboid a Bernoulli random variable $\epsilon_{i_1, \dots, i_D} \sim \mathcal{B}(p)$. The classical percolation P in D dimensions is

$$P = \bigcup_{\substack{i_1, \dots, i_D \text{ s.t. } \epsilon_{i_1, \dots, i_D} = 1}} B_{i_1, \dots, i_D}.$$

We will denote this setup by $P \sim \text{Perc}^D(n, p, 1)$ (a classical percolation of side n and probability parameter p in D dimensions), and write $P \sim \text{Perc}^D(\infty, p, 1)$ for $P = \lim_{n \rightarrow \infty} P^n$, $P^n \sim \text{Perc}^D(n, p, 1)$.

Recursive For each $1 \leq k \leq d$, split the unit cuboid into $(n^k)^D$ cuboid of side $1/n^k$ indexed by $1 \leq i_1, \dots, i_D \leq n^k$

$$B_{i_1, \dots, i_D}^k = \left[\frac{i_1 - 1}{n^k}, \frac{i_1}{n^k} \right] \times \cdots \times \left[\frac{i_D - 1}{n^k}, \frac{i_D}{n^k} \right].$$

Again, attach to each cuboid a Bernoulli random variable $\epsilon_{i_1, \dots, i_D}^k \sim \mathcal{B}(p)$. Finally, the recursive percolation P^d is defined by:

$$P^k = P^{k-1} \cap \left(\bigcup_{i_1, \dots, i_D \text{ s.t. } \epsilon_{i_1, \dots, i_D}^k = 1} B_{i_1, \dots, i_D}^k \right) \quad \forall 1 \leq k \leq d$$

and

$$P^0 = [0, 1]^2.$$

We will denote this setup by $P^d \sim \text{Perc}^D(n, p, d)$ (a recursive percolation of depth d , side n and probability parameter p in D dimensions). And again, we will write $P \sim \text{Perc}^D(n, p, \infty)$ for $P = \bigcap_{k \rightarrow \infty} P^k$, with P^k as above.

In practice, we will never use more than three dimensions. First, because our world is in three dimensions, so it makes sense to stop there. In addition to that, the curse of dimensionality[Wik(2004)] stops us (calculations and notations are too heavy from a mathematical point of view, and computations are too expensive from a modelling perspective).

Restriction to 1 dimension The restriction of the percolation process to one dimension can be thought as a generalized and randomized version of the Cantor set. The Cantor set splits the interval in 3 equal parts, and keeps the two extreme ones, while the general recursive percolation process splits the interval into n equal parts, and keeps each interval with probability p .

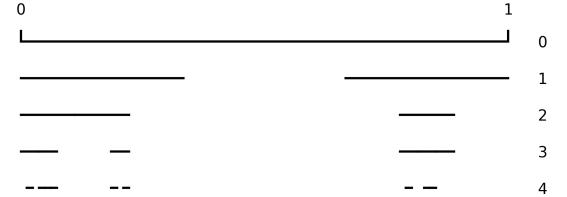


Figure 10: 1D Recursive Percolation
($n = 3, d = 4, p = 0.6$)

2.4 Density

Since the percolation process involves randomness, the density is not well defined. However, it is possible to calculate the expected density⁹.

It is straightforward to remark that the expected density of a classical percolation $P \sim \text{Perc}^D(n, p, 1)$ is p (for all D and all n). Note that this density is constant as $n \rightarrow \infty$.

Now, for a recursive percolation P such that $P \sim \text{Perc}^D(n, p, d)$, the expected density is p^d . Note that this density tends to zero as d grows to infinity (except in the trivial case $p = 1$). Thus, if $P \sim \text{Perc}^D(n, p, \infty)$, the expected density is 0.

In two dimensions ($D = 2$), the recursive percolations dimension behave as in fig. 11.

2.5 Dimensionality

Again, we can only give an expected dimension, as the process involves randomness. We ignore the cases $p = 0$ and $p = 1$ (these are trivial), and we concentrate on $p \in (0, 1)$.

We will treat separately the finite case, and the two limit cases (limit as classical percolation, and as recursive percolation).

Finite Case Take n and d finite, $P \sim \text{Perc}^D(n, p, d)$. Let Z be the number of remaining squares ($Z = |\{(i_1, \dots, i_D) \mid \epsilon_{i_1, \dots, i_D} = 1\}|$).

There are two possibilities for the dimension of P :

1. $Z = 0$: then $P = \emptyset$ and dimension of P is 0;

⁹The density of a set $X \subseteq [0, 1]^D$ is the proportion of points in the unit cuboid that are also in X .

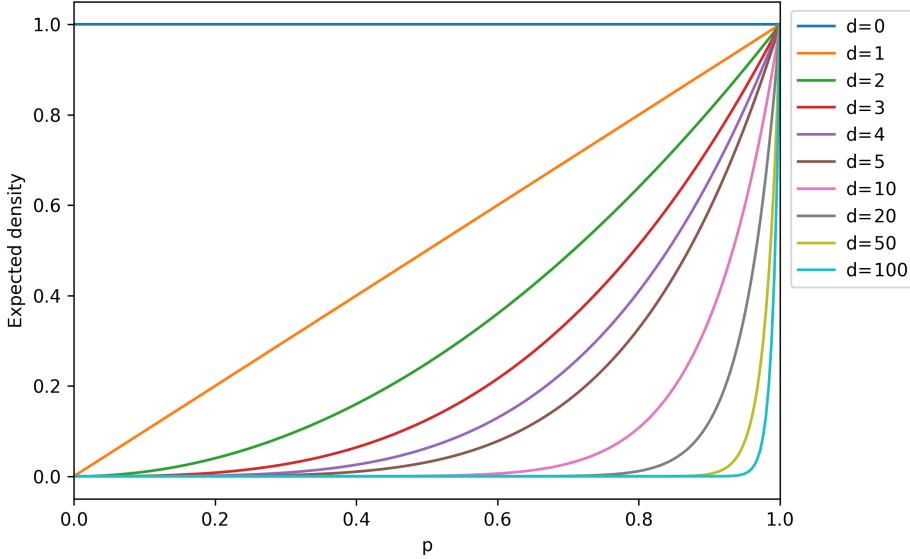


Figure 11: Density of recursive percolation fractals $P \sim \text{Perc}^2(\infty, p, d)$.

2. $Z > 0$: then $P \supseteq \left[\frac{i_1-1}{n^k}, \frac{i_1}{n^k}\right] \times \cdots \times \left[\frac{i_D-1}{n^k}, \frac{i_D}{n^k}\right]$ for some (i_1, \dots, i_D) , so dimension of P is D .

Therefore, the expected dimension of P is:

$$\mathbb{E}(\dim_H(P)) = 0 \cdot \mathbb{P}(Z = 0) + D \cdot \mathbb{P}(Z > 0) = D \left(1 - (1-p)^{(n^D)^d}\right).$$

In practice, this is close to D , as soon as n or d grows. Note that \dim_T would only differ by the fact that $\dim_T(\emptyset) = -1$ whereas $\dim_H(\emptyset) = 0$. Box dimension \dim_B will be the same as the Hausdorff one.

Classical Percolation We are looking at $P \sim \text{Perc}^D(\infty, p, 1)$.

Heuristic Argument Intuitively, the expected dimension of this percolation should be D , since the density is positive.

Formal Calculations We show that the Hausdorff dimension is D almost surely.

Let \mathcal{U} be an open cover for P , and let V be an open subset of the unit cuboid of dimension D . Then, $\mathbb{P}(P \cap V = \emptyset) = 0$ (as V is uncountable). Thus, it is almost sure that \mathcal{U} covers $[0, 1]^D$ (otherwise, there is an open subset V of $[0, 1]^D$ not intersecting P). So $H_\epsilon^d(P) = H_\epsilon^d([0, 1]^D)$ almost surely. Therefore, $\dim_H(P) = \dim_H([0, 1]^D) = D$ almost surely.

Note that the almost sure dimension gives the expected dimension.

Recursive Percolation We are interested in $P \sim \text{Perc}^D(n, p, \infty)$.

Eventually Empty Percolation In the case of $p < 1/n^D$, the number of expected cuboid is less than one. Thus, as $d \rightarrow \infty$, the percolation will almost surely become empty. Therefore, for $p < 1/n^D$, $\dim_H(P) = 0$ almost surely.

Next, we will suppose $p \geq 1/n^D$.

Heuristically In the case of a recursive percolation, P has self-similar properties that can help finding the expected dimension. After the first percolation, each selected cuboid is another version of P scaled by $1/n$. This goes on recursively. Therefore, the intuitive dimension to expect is

$$\mathbb{E}(\dim_H(P)) = \frac{\ln(\mathbb{E}(Z_1))}{\ln(n)}.$$

Where Z_1 is the number of remaining cuboids after one filtration $Z_1 = |\{(i_1, \dots, i_D) \mid \epsilon_{i_1, \dots, i_D}^1 = 1\}|$. So $\mathbb{E}(Z_1) = pn^D$, and finally:

$$\mathbb{E}(\dim_H(P)) = \frac{\ln(pn^D)}{\ln(n)} = D + \log_n(p).$$

Formally If no randomness is involved in the process, the above argument can be made rigorous: Define a contractive map that reflects the self similarities. Then use Stefan Banach's contractive mapping fixed point theorem applied to the complete metric space of non-empty compact subsets of R^n with the Hausdorff distance.

We show the fractional dimension is $\alpha = D + \log_n(p)$ directly (this proof follows some ideas from [Falconer(1990), p. 34-35, ex. 2.7]).

From definition, $P = \bigcap_{d \rightarrow \infty} P^k$. P^k is composed of cuboids of side $1/n^k$. Let Z^k be the number of these cuboids. In expectation, P^k should have $\mathbb{E}(Z^k) = (pn^D)^k$ cuboids of side $1/n^k$ remaining. Taking P^k as a β_k -cover of P , with $\beta_k = \sqrt{D}/n^k$ ¹⁰, we have

$$\mathbb{E}(H_{\beta_k}^\alpha(P)) \leq (pn^D)^k (\sqrt{D}/n^k)^{D+\log_n(p)} \leq \sqrt{D}^\alpha.$$

Letting $k \rightarrow \infty$, we get

$$\mathbb{E}(H^\alpha(P)) \leq \sqrt{D}^\alpha < \infty.$$

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of P . For each $i \in I$, let $k \in \mathbb{N}$ be such that $\beta_{k+1} \leq \text{diam}(U_i) < \beta_k$. Then, in general, U_i may only cover fully one cuboid from P^k . Now, if $j \geq k$, then U_i intersects in expectation at most $(pn^D)^{j-k}$ cuboids of side $1/n^j$ (note $(pn^D)^{j-k} \leq \frac{(pn^D)^j}{\sqrt{D}^\alpha} \text{diam}(U_i)^\alpha$). Taking j such that $\beta_{j+1} \leq \text{diam}(U_i) \quad \text{for } i \in I$, since \mathcal{U} intersects in expectation $\mathbb{E}(Z^j) = (pn^D)^j$ cuboids of diameter $1/n^j$, we get

$$(pn^D)^j \leq \sum_{i \in I} \frac{(pn^D)^j}{\sqrt{D}^\alpha} \text{diam}(U_i)^\alpha.$$

This implies that in expectation, $\sqrt{D}^\alpha \leq H^\alpha(P)$. Thus, $\mathbb{E}(H^\alpha(P)) \geq \sqrt{D}^\alpha > 0$.

Finally, as $0 < H^\alpha(P) < \infty$, we get $\mathbb{E}(\dim_H(P)) = \alpha = D + \log_n(p)$.

Letting $W^k = \frac{Z^k}{(pn^D)^k}$, we have $W^k \rightarrow W$ a.s. when $k \rightarrow \infty$, with W a random variable such that $\mathbb{E}(W) = 1$. Use this to get almost sure result?

Dimension Plots In starting from two dimensions ($D = 2$), the recursive percolations dimension behave as in fig. 12. For plots of $D = 1, \dots, 5$ see <https://pauldubois98.github.io/PercolationFractalsStudy/percolation.html>.

2.6 Blob

To have a better understanding, we begin by studying the central "blob" of the fractal.

¹⁰So that β_k is the diameter of a cuboid of side $1/n^d$ in dimension D .

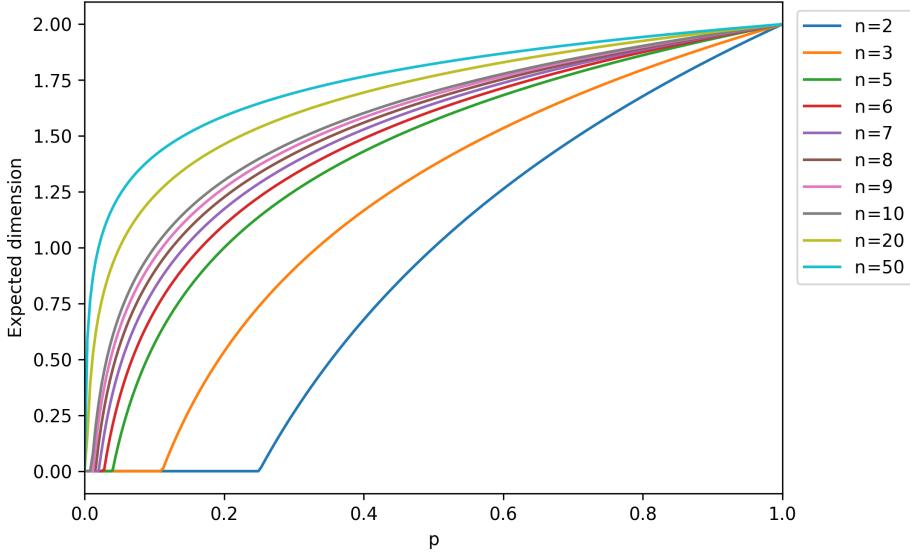


Figure 12: Dimension of recursive percolation fractals $P \sim \text{Perc}^2(n, p, \infty)$.

Definition We define the blob of a percolation P as the connected component of P that contains the point at the center of the cuboid (that is, the component that contains $(1/2, \dots, 1/2)$). Note that the blob may be empty.

In the simulations, since infinite percolation may only be approximated, we only consider cases where n is odd, so that the cuboid containing the point $(1/2, \dots, 1/2)$ is unique.

Algorithm The algorithm we use to find the blob is similar to the one used for finding crossings, and will be discussed in details in B.2.

We will use four metrics for evaluating the size and regularity of the blob.

- The size of the interior of the blob (we will use the D -dimensional Lebesgue measure).
- The size of the boundary of the blob (we will use the $(D - 1)$ -dimensional Lebesgue measure).
- The maximum (Euclidean) distance from the center (the point $(1/2, \dots, 1/2)$) to the boundary of the blob.
- The maximum step distance from the central cuboid (the cuboid containing $(1/2, \dots, 1/2)$) to cuboids at the boundary of the blob. The step distance from a cuboid A to another cuboid B is the minimal length of a cuboid-path linking A and B using only cuboids in the blob.

The interior and Euclidean distance measure how large the blob is, while the step distance and the area measure the irregularity. Figure 13 shows the four metrics on a typical percolation blob.

Simulations The study of the central blob from a mathematical theory perspective is hard. However, it is possible to get some understanding of the blob by generating percolation simulations. The following results were obtained from taking the average over 50000 percolations generated randomly, and looking at the blob of each percolation. The approximation is therefore very accurate. This was possible thanks to Oxford Mathematical Institute's Computing Services.

All plots from our simulations are available at <https://pauldubois98.github.io/PercolationFractalsStudy/blob.html>. They are all of the following form: the probability parameter p on the x-axis, the quantity we are looking at on the y-axis, and a line for each pair of parameters (n, d) , labelled with $\hat{n}d$. We considered the cases $D = 2$ and $D = 3$. For $D = 2$, the interior is measured by an area, and the

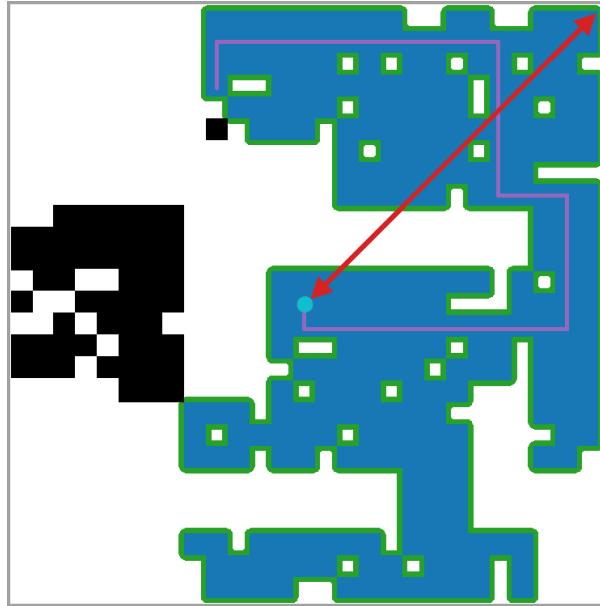


Figure 13: Blob Measures:

Blue: Blob interior.

Green: Blob boundary.

Red: Maximum center-boundary Euclidean distance.

Purple: Maximum center-boundary step distance.

Cyan: Central point $(\frac{1}{2}, \dots, \frac{1}{2})$.

boundary by a length, while for $D = 3$, the interior is a volume, and the boundary an area. We will only comment and show here the results for $D = 2$, there is not much changes for $D = 3$.

Interior For classical percolation (see fig. 14a), as $n \rightarrow \infty$, there seems to be a jump value around $p = 0.6$: for $p < 0.6$, the interior is empty, while for $p \geq 0.6$, it is non-empty and the size increase until reaching one for $p = 1$.

For recursive percolations (see fig. 14b), we observed the same phenomenon, with a shift to the right (i.e. towards $p = 1$) when d increases. For $d = 2$, the jump as $n \rightarrow \infty$ seems to occur at $p \approx 0.65$. When $d \rightarrow \infty$, it looks like the area of the blob tends to zero when $p < 1$.

In fact, this can be proved: we have shown (back in section 2.4) that the density of the percolation in this case tends to zero. Thus, the area of the percolation P tends to zero. Since the blob is a subset of P , its area must also tend to zero.

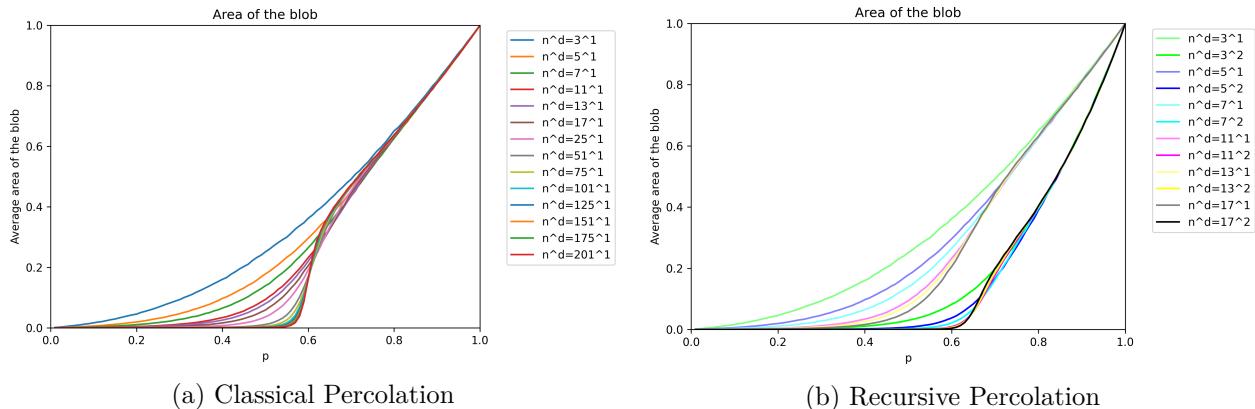


Figure 14: Blob Interior

Euclidean distance center-border The Euclidean distance from the central point to the border of the blob is also a measure of the size of the blob. Therefore, it makes sense that the Euclidean distance (plotted on fig. 15) and the interior of the blob (fig.14) behave similarly.

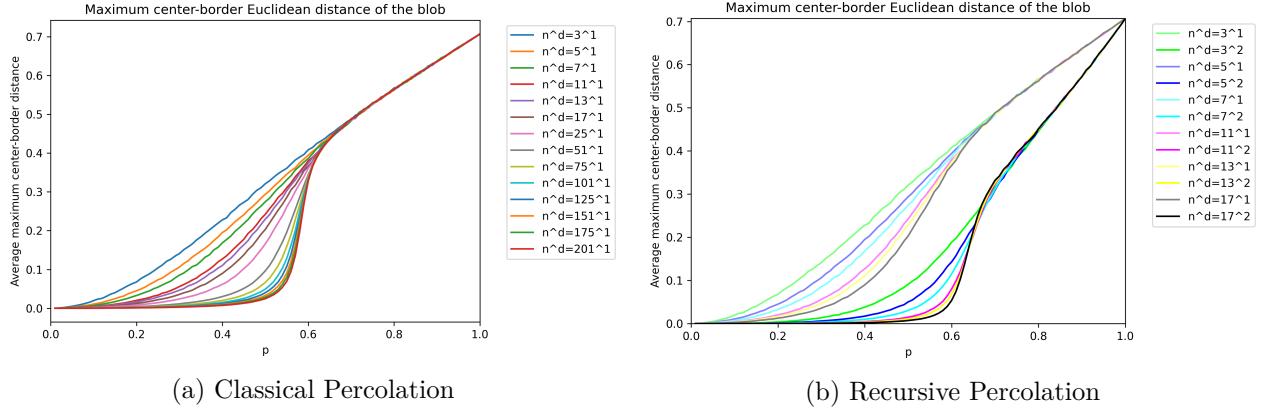


Figure 15: Blob Euclidean Distance

Boundary The size of the boundary of the blob measures how irregular the blob is. This follows the same idea as when we measure the length of islands coastlines to give a sense on how irregular the islands boundaries are (see A.3).

For regular ($d = 1$) percolations, we observe that when $p < 0.6$, the boundary length of the blob is close to zero, together with the fact that the interior area is also close to zero, this yields that the blob is nearly empty on most of the simulations. We then see a jump behaviour: for $0.6 \leq p < 1$, the length of the boundary grows rapidly as n increases.

When p is close to one, the percolation is connected. The blob and the percolation are nearly the same sets, and the removed cuboids can be assumed to be isolated. Then, the size of the boundary for $P \sim \text{Perc}^D(n, p, 1)$ is (in expectation):

$$\underbrace{(1-p)n^D}_{\text{Expected number of cuboids removed}} + \underbrace{\frac{2D}{n} \left(\frac{1}{n}\right)^{D-1}}_{\text{Boundary of a single cuboid of side } 1/n} + \underbrace{\frac{2D}{n}}_{\text{Boundary of the unit cuboid}} = 2D(1+(1-p)n) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

This behaviour is observed for $D = 2$ in 16a.

For recursive percolation (see fig. 16b), we also observe two phases (boundary close to zero for small p , followed by a rapid increase when p gets closer to one). The probability at which the behaviour split is however different, and increases as $d \rightarrow \infty$.

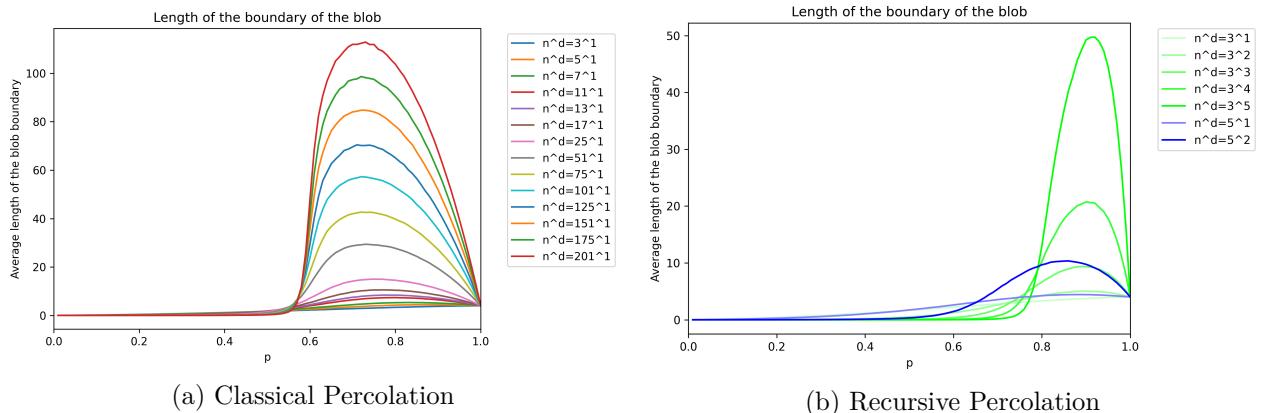


Figure 16: Blob Boundary

Step distance center-border The step distance measures a mix of the blob size and its irregularity: a large blob will need many steps to link the center to its boundary (in a similar way to Euclidean distance); and an irregular blob will need a lot of steps to link the center to the boundary, as the path will need to labyrinth round the blob's holes.

For this reason, from fig. 17a, we can extrapolate that as $n \rightarrow \infty$, there are two peaks: the first at $p \approx 0.6$ corresponding to the point where the blob stops being nearly empty most times, but the blob is still irregular, resulting in a large center to border step distance; the second at $p = 1$, when the blob is almost the full unit cuboid resulting in a step distance of 1. In between, the step distance drops as the blob is neither very large, nor very irregular.

On fig. 17b, the step distance collapses as $d \rightarrow \infty$. This is explained by the fact that the percolation density goes to zero, so the blob is empty in most cases. Thus, the step distance is nearly zero on average.

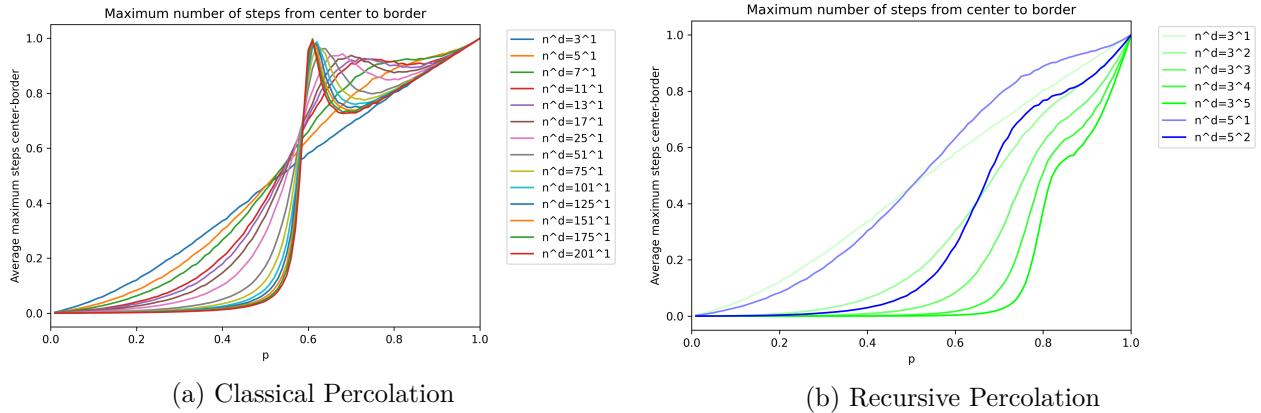


Figure 17: Blob Step Distance

3 Percolation Crossings

We are now interested in crossings from one side of the unit cuboid to the other side. There is some physical motivation for such questions: Suppose the percolation was a mathematical model for a physical material. If P has a crossing, then a fluid can pass through the material layer. However, it will be stuck if there is no crossing. The length of the crossing will tell how long it takes for the fluid to traverse the material.

The convention in mathematics is to study horizontal crossings, despite the more clear physical interpretation of vertical ones. Rotating 90° transforms vertical crossings to horizontal ones and vice-versa, so studying either is equivalent. We choose to stick to the mathematical convention.

3.1 Types of Crossings

There will be three types of crossing considered, according how the path of is allowed to behave.

Consider a percolation P on the unit cuboid $[0, 1]^D$ in D dimensions, and let $\gamma : [0, 1] \rightarrow [0, 1]^D$ be a crossing of the cuboid, i.e. $\gamma(0) \in \{0\} \times [0, 1]^{D-1}$ and $\gamma(1) \in \{1\} \times [0, 1]^{D-1}$. In the case of a finite percolation¹¹, we can view a crossing as a walk on the cuboids of side $1/n^d$.

Straight The crossing is said to be straight if $\gamma(y) = y \times (x_2, \dots, x_D)$. That is, if the path goes on a direct line from one side to the other. Only one step direction is allowed.

Semi-Straight The crossing is said to be semi-straight if $\forall 0 \leq y < z \leq 1, \gamma(y)_0 < \gamma(z)_0$ ¹². That is, if the path never goes backwards. This allows steps in $2D - 1$ direction: all the possible directions except $(-1, 0, \dots, 0)$.

Non-Straight If the crossing is neither straight or semi-straight, then it is non-straight. In the latter case, all steps directions are allowed.

The following (fig. 19) illustrate the different types of crossings in two dimensions (the easiest to picture), extension to greater dimensions is straightforward. When talking about a "crossing" in general, we will assume it may be either straight, semi-straight, or non-straight.

3.2 Percolation Crossings

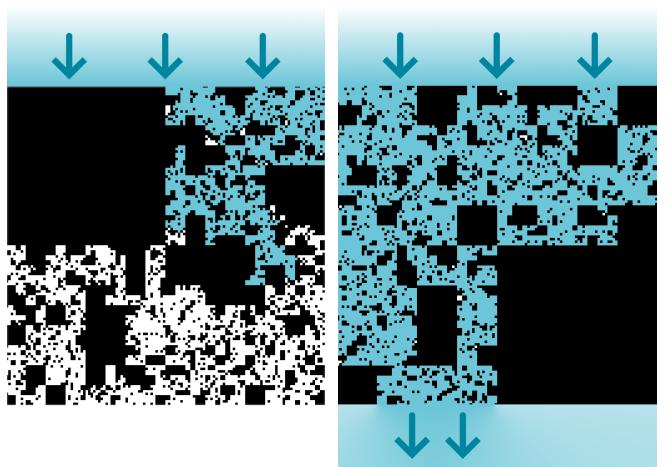
First, we are interested in crossing lying on the percolation P .

3.2.1 Crossings Probability

For a percolation $P \sim \text{Perc}^D(n, p, d)$, we look at the probability that a crossing exist.

¹¹i.e. $P \sim \text{Perc}^D(n, p, d)$ with $n, d < \infty$

¹²Writing $\gamma(y)_0$ for the first component of $\gamma(y)$.



(a) No vertical crossing:
Fluid is stuck

(b) Vertical crossings:
Fluid can traverse

Figure 18: Percolations and Fluids

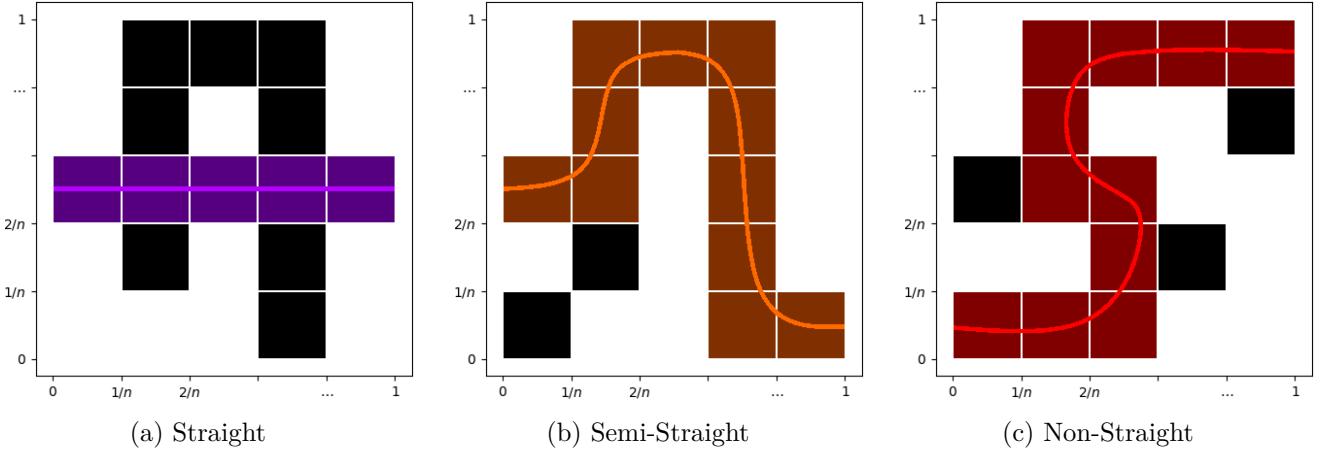


Figure 19: Crossing types studied.

Straight Crossings This case is simple enough to derive an explicit formula for both classical and recursive percolations.

Classical Percolation Take $P \sim \text{Perc}^D(\infty, p, 1)$. Then the probability that there is a straight crossing on the first line is p^n (and is the same for the n^{D-1} lines). Thus, the probability that there is no crossing on any line is $1 - (1 - p^n)^{n^{D-1}}$, i.e.

$$\mathbb{P}(P \text{ has a straight crossing}) = 1 - (1 - p^n)^{n^{D-1}} = \mathcal{O}(p^n f(n)) \text{ with } f(n) \text{ a polynomial of } n.$$

Thus, $\mathbb{P}(P \text{ has a straight crossing}) \rightarrow 0$, so it is almost sure that there is no crossing on a classical percolation P .

Recursive Percolation Now take $P \sim \text{Perc}^D(n, p, \infty)$.

Let Y^k be the number of straight crossings after k filtrations, and $E^k = \mathbb{E}(Y^k)$ be the expected number of straight crossings¹³. Clearly, $E^0 = 1$, $E^1 = p^n n$, and recursively, $E^k = p^{n^k} n E^{k-1}$. So in general, we get $E^k = n^k \prod_{i=1}^k p^{n^i}$, and $E^k \rightarrow 0$ as $k \rightarrow \infty$. Now, $\mathbb{E}(Y^k) \rightarrow 0$ together with $Y^k \geq 0$ gives $\mathbb{P}(Y^k = 0) \rightarrow 1$. So eventually, it is almost sure that a recursive percolation will have no straight crossings.

3.2.2 Crossings Length

The length of a straight crossing is always 1. However, for non-straight and semi-straight crossing, it is less clear. We study this numerically.

3.3 Percolation Complement Crossings

We turn our interest to crossings on the complement $P^C = [0, 1]^D \setminus P$ for a recursive percolation $P \sim \text{Perc}^D(n, p, \infty)$.

3.3.1 Crossings Probability

Again, we will first be interested in the probability that a crossing exists.

¹³We count two crossings as different if they go through different cuboids.

Straight Crossings There is a straight crossing on the complement P^C of the percolation if there exists a point $x \in [0, 1]^{D-1}$ such that $[0, 1] \times \{x\}$ is in P^C , i.e. do not intersect with P .

The following result is inspired from [Chayes et al.(1988)Chayes, Chayes, and Durrett, p.309 b.(1)] and [Mandelbrot(1982), p.215]. Suppose $x = (x_2, \dots, x_D)$ with $x_i \neq j_i/n^k$ for $j_i \in \llbracket 0, n^k \rrbracket$ ¹⁴. Then if $p \leq 1/n$ we have that almost surely, P has a complement crossing along $[0, 1] \times \{x\}$. This is in fact equivalent to

$$\mathbb{P}((P \cap [0, 1] \times \{x\}) = \emptyset) = 1.$$

Proof. Let $x \neq \left(\frac{j_2}{n^k}, \dots, \frac{j_D}{n^k}\right)$ for any $j_i \in \llbracket 0, n^k \rrbracket$. The number of intervals of the form $\left[\frac{j_1-1}{n^k}, \frac{j_1}{n^k}\right] \times \{x\}$ still in the percolation at depth k is a branching process in which each interval has np offspring. Thus, the branching process dies out almost surely if $np \leq 1$, i.e. $p \leq 1/n$. When the process dies, we have $[0, 1] \times \{x\} \subseteq P^C$, so there is a crossing in the complement of P . \square

Non-Straight Complement Crossings There will be a non-straight crossing in the percolation complement almost surely if $p \leq \frac{1}{2(D-1)\sqrt[n]{n}}$ ¹⁵. The following proof extends techniques from [Chayes et al.(1988)Chayes, Chayes, and Durrett, p.310 b.(2)].

Proof. This time, take $x = \left(\frac{j_2}{n^k}, \dots, \frac{j_D}{n^k}\right)$ with $j_i \in \llbracket 1, n^k - 1 \rrbracket$, $x_i = j_i/n^k$. Call a segment $\left[\frac{j_1-1}{n^k}, \frac{j_1}{n^k}\right] \times \{x\}$ vacant if any of the $2(D-1)$ cuboids adjacent at the depth k of the percolation have been removed.

Let Y^k , be the number of occupied segments of the form $\{x\} \times \left[\frac{j_1-1}{n^k}, \frac{j_1}{n^k}\right]$ at depth k . Then Y^k , is a branching process in which each the mean number of offspring is $p^{2(D-1)n}$. This branching process is almost surely eventually empty if $np^{2(D-1)} \leq 1$, i.e. if $p \leq \frac{1}{2(D-1)\sqrt[n]{n}}$.

Suppose the branching process dies at depth k . Then there are only finitely many points of the form $(j_1/n^k, x_2, \dots, x_D)$ s.t. $j_1 \in \llbracket 1, n^k - 1 \rrbracket$. It is almost sure that eventually, all the cuboids touching these points will be removed (since they are kept with a probability $p < 1$ after each filtration). Once this is the case, there is a crossing path from $(0, x_2, \dots, x_D)$ to $(1, x_2, \dots, x_D)$ that zigzags near the segment $[0, 1] \times \{x\}$ (see fig. 20 for illustrations in two and three dimensions). \square

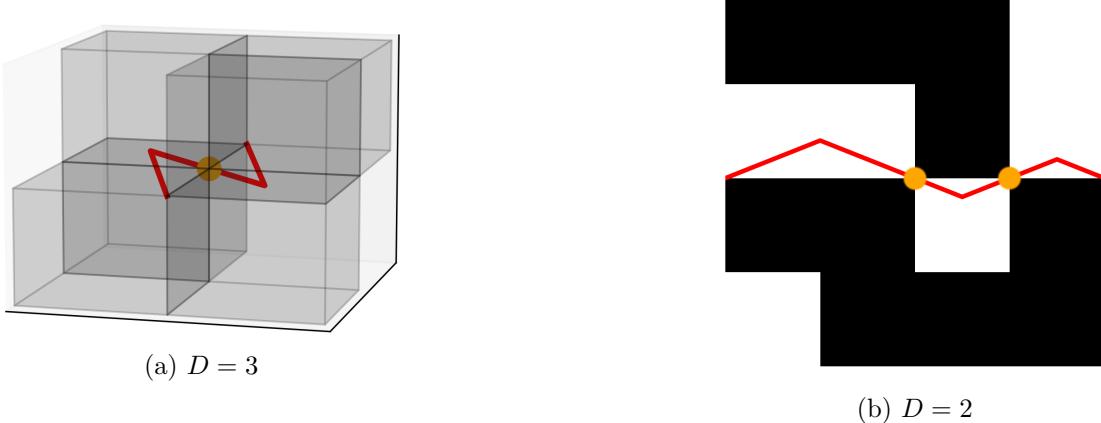


Figure 20: Eventually, the cuboids touching the orange point(s) will be removed and the red path will remain in full within the percolation complement.

¹⁴Writing $\llbracket a, b \rrbracket$ for $\mathbb{Z} \cap [0, n^d]$.

¹⁵Where $\sqrt[2(D-1)]{n}$ is the $2(D-1)^{th}$ root of n .

3.3.2 Crossings Length

3.4 Link Between Crossings and Complement Crossings

In the case of $D = 2$, vertical (respectively horizontal) crossing on P is incompatible with horizontal (respectively vertical) crossing of P^C .

Since we proved that if $p \leq 1/\sqrt{n}$, then P^C has a crossing with probability one, we get as a consequence that P have a crossing with probability zero (note that the existence probability of a vertical crossing is the same as the one of an horizontal crossing, by symmetry).

3.5 Connectivity Properties

Some properties of percolations are already known by the community. We will look at here the ones regarding connectivity of the percolation, which are related to crossings. Most results were stated in two dimensions in the literature. When possible, we will generalize the results to dimension D .

Almost Sure Disconnectedness If $p \leq 1/\sqrt{n^{D-1}}$, then the largest connected component in a percolation $P \sim \text{Perc}^D(n, p, \infty)$ is a point (almost surely).

The following proof uses techniques from [Chayes et al.(1988)Chayes, Chayes, and Durrett, p.310 b.(2)]

Proof. Similarly to last proof (see section 3.3.1), we say that a facet

$$f_{j_1, \dots, j_D}^k = \left\{ \frac{j_1}{n^k} \right\} \times \left(\prod_{i=2}^D \left[\frac{j_i - 1}{n^k}, \frac{j_i}{n^k} \right] \right)$$

is vacant is either of the two adjacent cuboids in the k^{th} filtration is.

Let Y^k be the number of occupied facets for the form f_{j_1, \dots, j_D}^k after k filtrations. Y^k is a branching process in which each element has $p^2 n^{D-1}$ offspring. Now, if $p^2 n^{D-1} \leq 1$ (i.e. $p \leq 1/\sqrt{n^{D-1}}$), then the branching process will die out with probability one. Similarly to last proof, if we wait long enough, it is almost sure that there will be a hyper-surface¹⁶ in the percolation complement P^C arbitrarily close to the hyperplane $\left\{ \frac{j_1}{n^k} \right\} \times [0, 1]^{D-1}$ (see fig. 21 for examples in two and three dimensions).

Finally, repeating the argument for $j_m \in [1, n^k]$, we get that almost surely, there are hyper-surfaces on the percolation complement arbitrarily close to the hyperplanes $\left\{ \frac{j_1}{n^k} \right\} \times [0, 1]^{D-1}$. By reflecting the argument, there are some hyper-surfaces arbitrarily close to the hyperplanes $[0, 1]^{m-1} \times \left\{ \frac{j_m}{n^k} \right\} \times [0, 1]^{D-m-1}$. Therefore, the largest connected component is a point. \square

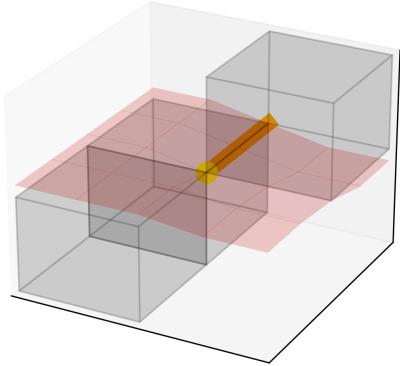
Minimal Parameter for Crossings to Appear It is interesting to study when the percolation starts having crossings. One defines (notation introduced in [Chayes et al.(1988)Chayes, Chayes, and Durrett]) p_c as the minimum value of p such that $P \sim \text{Perc}^D(n, p, \infty)$ has a non zero probability to have a crossing, i.e.

$$p_c = \inf \{p \mid \mathbb{P}(P \text{ has a crossing}) > 0\}.$$

It is reasonable to conjecture that $p_c < 1$ for all $n, D \geq 2$. In fact, this was proved in the case $D = 2$ in [Chayes et al.(1988)Chayes, Chayes, and Durrett].

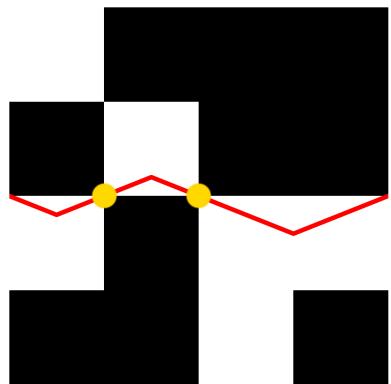
Narrowing down to $D = 2$: Chayes et al. proved in [Chayes et al.(1988)Chayes, Chayes, and Durrett] that almost surely, if $p < p_c$, the largest connected component of P is a point. For $p \geq p_c$, the

¹⁶Here, hyper-surface and hyperplane are of dimensions $D - 1$ in \mathbb{R}^D .



(a) $D = 3$

Hyper-surface is a usual 3D surface.



(b) $D = 2$

Hyper-surface is just a line.

Figure 21: Eventually, the cuboids touching the orange line and yellow point(s) will be removed and the red hyper-surface will remain in full within the percolation complement.

largest connected component is almost surely not a point. Moreover, we pave \mathbb{R}^2 with a percolation $P \sim \text{Perc}^D(n, p, \infty)$ attached at each point of \mathbb{Z}^2 (and calling this P'). Then, with probability one, P' has a unique unbounded connected component. This explains the "jump" behaviour observed in section 2.6.

References

- [Ionescu et al.(2009)] Ionescu, Oustaloup, Levron, Melchior, Sabatier, and De Keyser] Clara Ionescu, Alain Oustaloup, François Levron, Pierre Melchior, Jocelyn Sabatier, and Robin De Keyser. A model of the lungs based on fractal geometrical and structural properties. *IFAC Proceedings Volumes*, 42(10):994–999, 2009. ISSN 1474-6670. doi: <https://doi.org/10.3182/20090706-3-FR-2004.00165>. URL <https://www.sciencedirect.com/science/article/pii/S1474667016387791>. 15th IFAC Symposium on System Identification.
- [Pol(2009)] Lectures on fractals and dimension theory, 2009.
- [3DX(2020)] *Mathematical Documentation of objects*, 2020.
- [Shishikura(1992)] Mitsuhiro Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *arXiv Mathematics e-prints*, art. math/9201282, April 1992.
- [Heinemann and Stratmann(1998)] Stefan M. Heinemann and Bernd O. Stratmann. Quadratic polynomials with julia sets of hausdorff dimension close to 2, 1998.
- [Falconer(1990)] Kenneth Falconer. *Fractal Geometry*. Wiley, 1990. ISBN 0978-0-471-92287-2.
- [Wik(2004)] Curse of dimensionality, 2004. URL https://en.wikipedia.org/wiki/Curse_of_dimensionality.
- [Chayes et al.(1988)] Chayes, Chayes, and Durrett] J. Chayes, L. Chayes, and R. Durrett. Connectivity properties of mandelbrot’s percolation process. *Probability Theory and Related Fields*, 77:307–324, 01 1988. doi: 10.1007/BF00319291.
- [Mandelbrot(1982)] Benoit B Mandelbrot. *The fractal geometry of nature*. Freeman, San Francisco, CA, 1982. URL <https://cds.cern.ch/record/98509>.
- [pyr(2017)] The pyromaniac game, 2017.



UNIVERSITY OF OXFORD
MATHEMATICAL INSTITUTE

Random Fractals
Appendix

Candidate Number

1051774

Course

MSc Mathematical Sciences

Trinity Term
April 26, 2021

A More on Fractals

Here will be discussed some extensions of famous fractals studied before (in section 1.2.2). This is not directly relevant for this paper, but an involved reader may be interested.

A.1 Mandelbrot and Julia Sets

Mandelbrot and Julia sets involve calculating the limit of a sequence to determine if a point is in the set or not. More precisely, one needs to determine whether the sequence diverges or not. It is impossible to do such calculations with a computer, therefore, we approximate. For each initial value z_0 , we calculate the first values of the sequence. We stipulate that the time taken by the sequence to reach $|z_n| > R$ is an indication of how fast the sequence diverges.

The plots are then made as follows: We calculate the first M values of the sequence. If $|z_n| > R$ for some $n < M$, then let $g(z_0) = n$, with n the smallest such integer. If $|z_n| \leq R$ for all $n < M$, then let $g(z_0) = M$. Finally, plot g on the area $S \subseteq \mathbb{C}$ of interest.

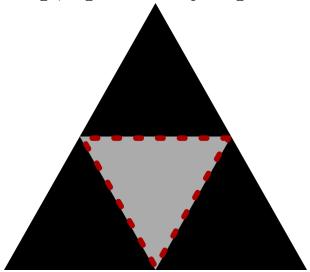
On the figures 1 and 2, the settings are $R = 2$ and $M = 100$. The plots represent the value of g through colours, with the same colour-bar as in fig. 22.

A.2 Sierpiski Fractals

Following the idea of the Sierpiski carpet, several fractals may be generated. First, using (equilateral) triangles instead of squares:

Sierpiski Triangle Sierpiski triangle is also a fractal constructed recursively by removing parts of its initial set, following a pattern similar to the one for Sierpiski carpet (although, there are equivalent definitions).

The construction starts from an equilateral triangle. Then, at each step, split every equilateral triangle into 4 sub-triangles as follows:



Remove the central triangle, and repeat the operation on the remaining 3 triangles.

The figure is made of 3 copies of itself, scaled by a factor of $\frac{1}{2}$. Therefore, the intuitive Hausdorff dimension for this set is $\dim_H(K) = \frac{\log(3)}{\log(2)} \approx 1.585$. Again, this can be proved rigorously using similar techniques as in [Falconer(1990), p. 34-35, ex. 2.7].

Sierpiski 3D Fractals As seen previously, the Sierpiski carpet and triangle live in a 2D world, but the idea of Sierpiski fractals may also be extended to 3D.

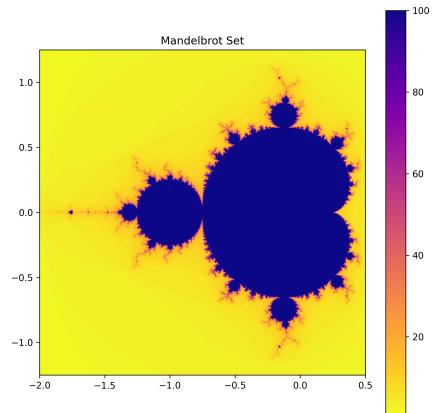


Figure 22: Mandelbrot Set

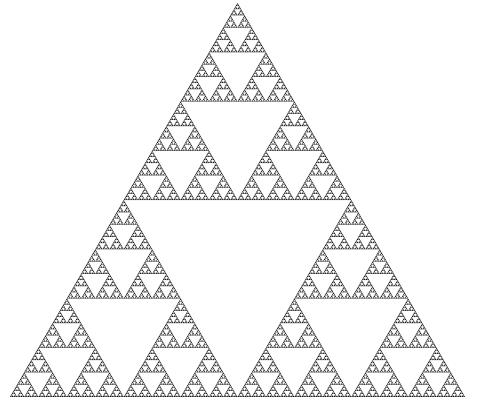


Figure 23: Sierpinski Triangle (8 steps)

Menger sponge The Menger sponge is the generalisation of Sierpiski carpet to 3 dimensions.

Start with a cube; split it into 27 identical copies scaled by $\frac{1}{3}$, and remove the central one, and the 6 cubes sharing a face with this central cube. Repeat this operation on each of the 20 remaining cubes.

From this construction arise a fractal with dimension $\frac{\log(20)}{\log(3)} \approx 2.727$.

Sierpiski tetrahedron The Sierpiski tetrahedron (or tetrix) is the analogue of Sierpiski triangle in 3 dimensions. Start with a regular tetrahedron; split it into 5 identical copies scaled by $\frac{1}{2}$, and remove the central one. Repeat this operation on each of the 4 remaining tetrahedrons.

From this construction arise a fractal with dimension $\frac{\log(4)}{\log(2)} = 2$. Note that this set is dimension exactly 2 while not being a surface.

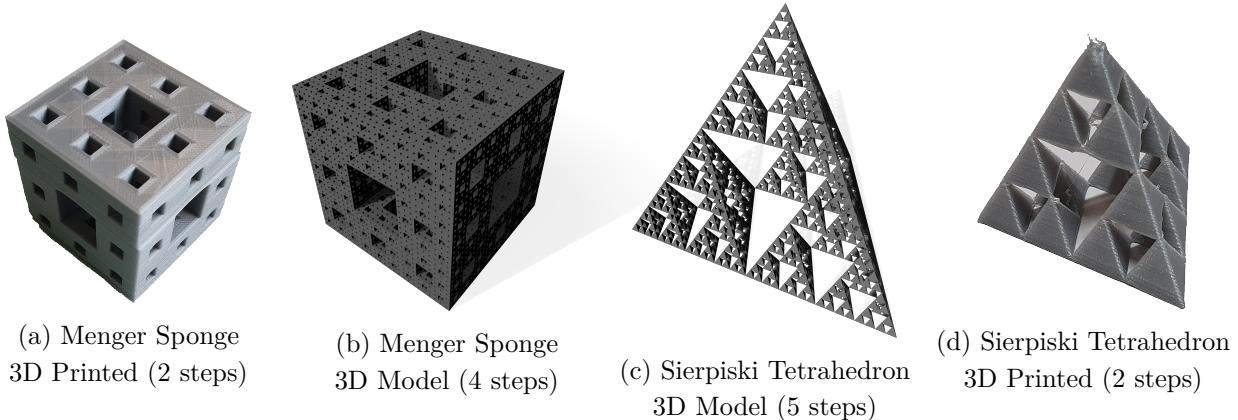


Figure 24: Sierpiski 3D Fractals

A.3 Coastline of Islands

The coastline can be measured at different scales. As the scale is refined, one measures a more detailed curve, which leads to a larger length. This refinement can be done indefinitely: from segments of 1000km (one would only capture the rough coastline shape) to segments of 1mm (one would need to go around every grain of sand), and even further considering sub-atomic scales.

Following this reasoning, the (1D) length of coastlines is infinite. It is in fact a fractal with dimension greater than one (but less than 2, as contained in a plane).

It is then possible to measure how disordered the coastline of an island is, by approximating its Hausdorff dimension. This will be done macroscopically, using maps extracted from the Google Maps service. In this paper, 3 islands will be considered: The United Kingdom, Iceland, and Madagascar. The United Kingdom and Iceland are understood as having irregular coastlines (so should have a high Hausdorff dimension), whereas Madagascar is thought to have a rather smooth coastline (Hausdorff dimension close to one).

Of course, the fractal dimension can only be an approximation. We approach the problem as follows: First, we take a high resolution map of the island, and split it into a binary colour image, black for the ground, and white for the surrounding sea (as in fig. 25). Then, we reduce the size of the image by merging pixels ("pixelizing") to obtain lower resolution images. The resolution panel will correspond to the measuring scales panel. A subset of the resulting images is shown in fig. 26.

Now, using computer vision (as a Python package), we obtain the contours of the shape in an image. It is then straightforward to measure the length of the contour (in pixels, which can be

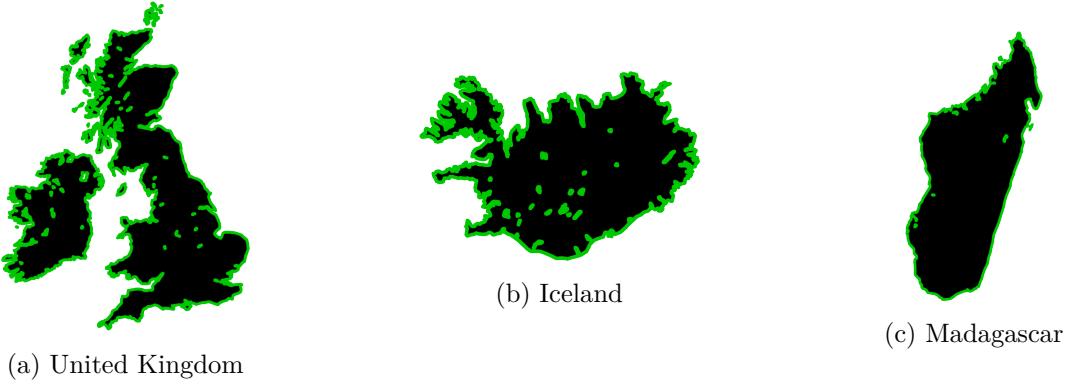


Figure 25: Coastlines (coloured in green).



Figure 26: Map scales for the UK, Iceland, and Madagascar.

converted to kilometres as we know the scale of the image). Repeating this operation for each scale of the map gives an array of coastline length with associated scales.

Plotting the values in a log-log plot shows a nearly straight line; the approximate dimension will be given by the slope of the best fitting line. We obtained the following dimensions:

United Kingdom $D \approx 1.2354$

Iceland $D \approx 1.2466$

Madagascar $D \approx 1.0600$

The fitting line is very close to the data (see fig. 27), showing that the behaviour is as expected.

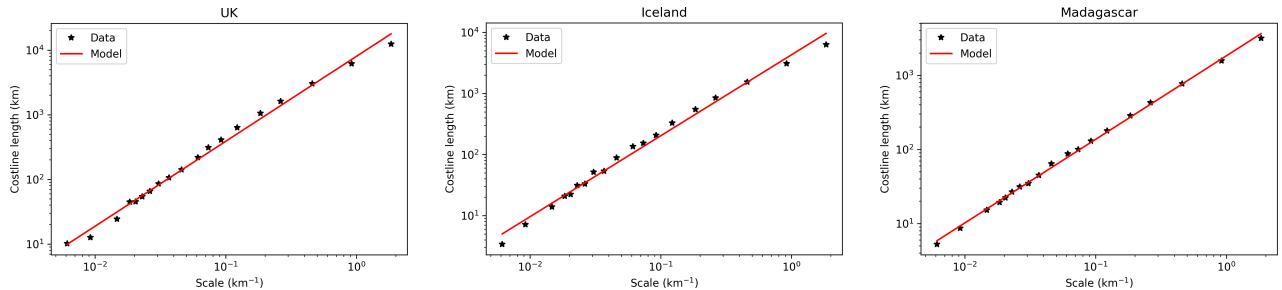


Figure 27: Islands Dimension Regression

The full code for this study can be found here: <https://colab.research.google.com/drive/1qo8S8oxqcLsw9UFrq5wTuVyxVA48uwFF?usp=sharing>.

B Finding Crossings

Percolations are complicated objects. Even for finite percolations, say one with side n , depth d in D dimensions, here are $2^{(n^D)^d}$ possibilities. This grows extremely rapidly, and makes theoretical results hard to derive.

Therefore, we choose to study these object further with numerical simulations.

B.1 Examples

To see how complicated finding crossing on percolation becomes when d grows, we look at some examples in two dimensions ($D = 2$), with sides $n = 2, 3, 5$ (see fig. 28, 29, 30).

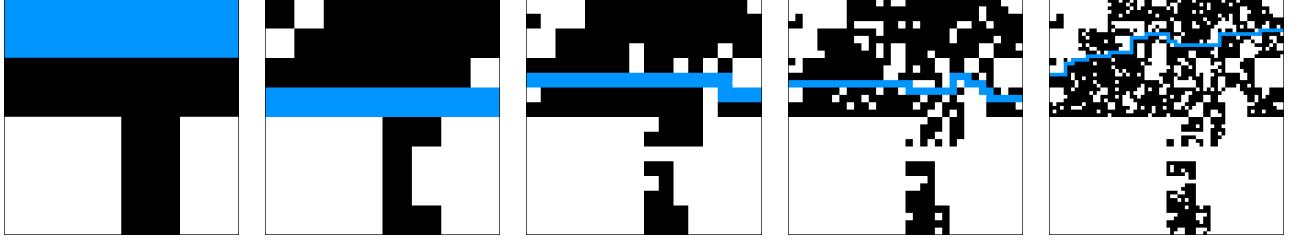


Figure 28: Crossings for a percolation with side $n = 2$, at depths $k = 2, 3, 4, 5, 6$.

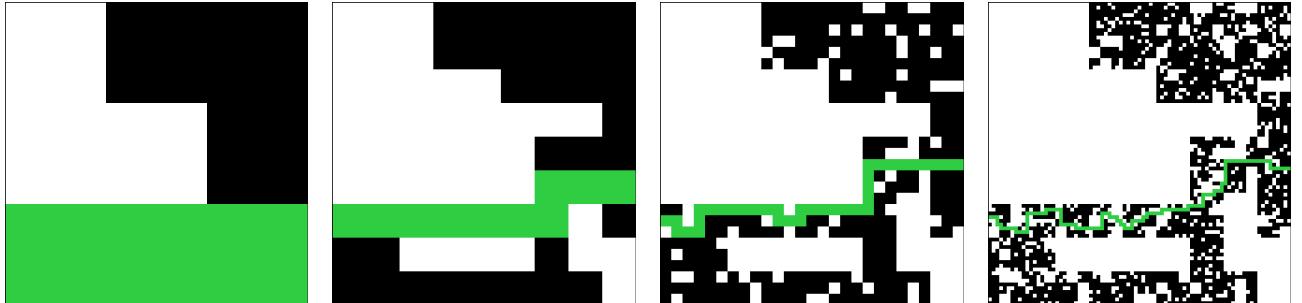


Figure 29: Crossings for a percolation with side $n = 3$, at depths $k = 1, 2, 3, 4$.

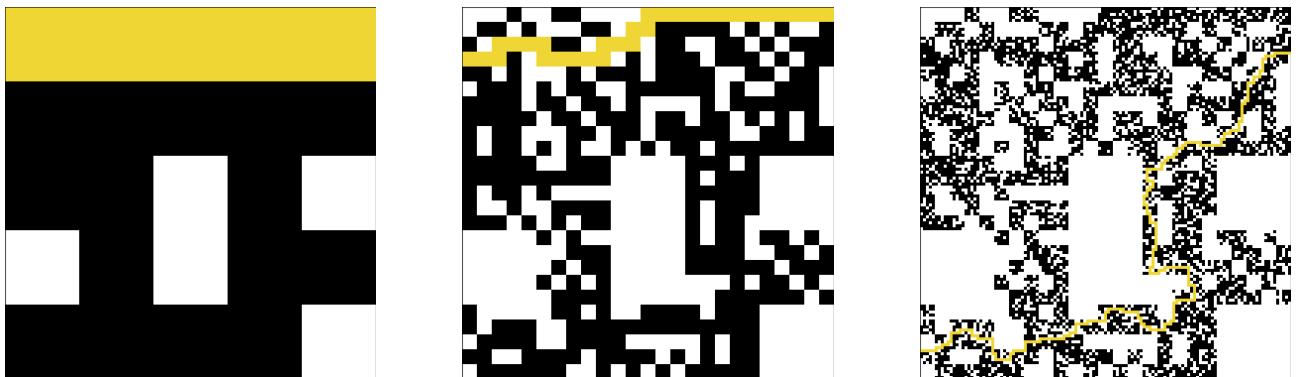


Figure 30: Crossings for a percolation with side $n = 5$, at depths $k = 1, 2, 3$.

B.2 Algorithms

Percolations As (recursive) percolations are generated through a recursive process, it is logical (and elegant) to use a recursive function to generate them computationally. We will represent a (finite)

percolation $P \sim \text{Perc}^D(n, p, d)$ with a binary D -dimensional array of length n^d in each dimensions. Note that D is a super-parameter.

This algorithm is given in pseudocode in algorithm 1. Julia implementations^{17 18} (used later in simulations for empirical estimations) can be found on the GitHub repository¹⁹.

Algorithm 1 Percolation generation algorithm

```

1: Fills the  $[i_1, i_1 + n^k] \times \dots \times [i_D, i_D + n^k]$  sub-array of  $P$ .
2: procedure PERCOLATIONFILL $^D(P, i_1, \dots, i_D, n, p, k)$ 
3:   if  $k = 0$  then
4:      $P[i_1, \dots, i_D] \leftarrow \text{random}()$ 20  $< p$                                  $\triangleright$  stop recursion
5:   else
6:     for  $0 \leq j_1, \dots, j_D < n$  do
7:       if  $\text{random}() < p$  then
8:         PERCOLATIONFILL( $P, i_1 + (j_1 n^k), \dots, i_D + (j_D n^k), n, p, k - 1$ )       $\triangleright$  recursion
9:       else
10:         $P[i_1 : i_1 + n^k, \dots, i_D : i_D + n^k] \leftarrow \text{zeros}^D(n^k, \dots, n^k)$        $\triangleright$  void cells
11:       end if
12:     end for
13:   end if
14: end procedure
15: Generates a percolation  $P \sim \text{Perc}^D(n, p, d)$ .
16: procedure PERCOLATION $^D(n, p, d)$ 
17:    $P \leftarrow \text{init}^D(n^d, \dots, n^d)$ 
18:   PERCOLATIONFILL( $P, i_1, \dots, i_D, n, p, d$ )                                      $\triangleright$  fill  $P$  recursively
19:   return  $P$ 
20: end procedure

```

Crossings Given a percolation, it is not clear how to proceed on the task of determining if a crossing exists.

Our approach of the problem is rather unusual.

We simulate a "fire" propagating from one side of the cuboid, and check if it reaches the other side.²² If it does, then a crossing exists, otherwise, no crossing exists. The rules of propagation for the fire (materialized by active cells²³) are as follows:

1. If a cell is active at time t , then the non-empty adjacent cells become active at time $t + 1$.
2. If a cell is active at time t , it deactivates and can no longer be active on times $\tilde{t} > t$.

At the beginning ($t = 0$), we set all the non-empty cells from one side of the unit cuboid as active cells. As soon as a cell from the opposite side of the unit cuboid is active, a path has been found (the

¹⁷2D percolation Julia implementation: https://github.com/pauldubois98/PercolationFractalsStudy/blob/main/fractal_percolation2D.jl

¹⁸3D percolation Julia implementation: https://github.com/pauldubois98/PercolationFractalsStudy/blob/main/fractal_percolation3D.jl

¹⁹See <https://github.com/pauldubois98/PercolationFractalsStudy>.

²⁰**random()** generates a random number $x \sim \mathcal{U}(0, 1)$ ²¹.

²¹Writing $x \sim \mathcal{U}(a, b)$ for x a random variable with uniform distribution on $[a, b]$.

²²This is inspired by simulations of forest fires from the game "The Pyromaniac Game" [pyr(2017)].

²³We call a "cell" one of the D dimensional cuboids of side $1/n^d$

time t even gives the length of the crossing), and this is the shortest crossing. If at some time t , no cell is active and no crossing was found before (i.e. the blaze turns off before reaching the other side), then no crossing exist.

Note that this technique tells us if there is a crossing, and what is the minimal length. However, it does not give the crossing path explicitly. To find it, one needs to save the set of active cells at each time, and propagate backwards.

This algorithm 2 is a pseudocode implementation of this process. Julia implementations^{[24](#) [25](#)} (used for empirical probability approximation) can be found again on the GitHub repository^{[26](#)}.

Visualisation of this algorithm:

- In 2D: <https://pauldubois98.github.io/PercolationFractalsAlgorithmsDemo/2Dcrossing/index.html>
- In 3D: <https://pauldubois98.github.io/PercolationFractalsAlgorithmsDemo/3Dcrossing/index.html>

By restricting the possible steps (not allowing activation to propagate backwards), this algorithm may be adapted to find semi-straight crossings (the above would give non-straight crossings). It can also be used for straight crossings, but in this case, it is faster to sum the percolation array on the first dimension: a straight crossing exists if one of the sums gives n^d .

^{[24](#)}2D crossing Julia implementation: <https://github.com/pauldubois98/PercolationFractalsStudy/blob/main/crossings2D.jl>

^{[25](#)}3D crossing Julia implementation: <https://github.com/pauldubois98/PercolationFractalsStudy/blob/main/crossings3D.jl>

^{[26](#)}See <https://github.com/pauldubois98/PercolationFractalsStudy>.

Algorithm 2 Crossing finding algorithm

1: Perform one step propagation on A , with respect to P

2: **procedure** NEIGHBORS $^D(A, P, n, d)$

3: $B \leftarrow \text{zeros}^D(n^d, \dots, n^d)$ ▷ newly active cells

4: **for** $1 \leq i_1, \dots, i_D < n^d$ **do**

5: **if** $A[i_1, \dots, i_D] = 2$ **then**

6: $B[i_1, \dots, i_D] = 2$ ▷ remain inactive

7: **end if**

8: **if** $A[i_1, \dots, i_D] = 1$ **then**

9: $B[i_1, \dots, i_D] = 2$ ▷ deactivate cell

10: **for** $1 \leq j \leq D$ **do**

11: **if** $P[i_1, \dots, i_j - 1, \dots, i_D]$ and $A[i_1, \dots, i_j - 1, \dots, i_D] = 0$ **then**

12: $B[i_1, \dots, i_j - 1, \dots, i_D] \leftarrow 1$ ▷ activate cell

13: **end if**

14: **if** $P[i_1, \dots, i_j + 1, \dots, i_D]$ and $A[i_1, \dots, i_j + 1, \dots, i_D] = 0$ **then**

15: $B[i_1, \dots, i_j + 1, \dots, i_D] \leftarrow 1$ ▷ activate cell

16: **end if**

17: **end for**

18: **end if**

19: **end for**

20: **return** B

21: **end procedure**

22: Returns the length of the shortest crossing on percolation $P \sim \text{Perc}^D(n, p, d)$; returns 0 if none exist.

23: **procedure** CROSSING $^D(P, n, d)$

24: $A \leftarrow \text{zeros}^D(n^d, \dots, n^d)$ ▷ active cells

25: $A[0, :, \dots, :] \leftarrow P[0, :, \dots, :]$ ▷ initialize active cells

26: $t \leftarrow 0$

27: **while** **any**($A = 1$) and **any**($A[n^d, :, \dots, :] = 1$) **do**

28: $A \leftarrow \text{NEIGHBORS}(P, A)$ ▷ propagate one step

29: $t \leftarrow t + 1$

30: **end while**

31: **if** **any**($A[n^d, :, \dots, :] = 1$) **then**

32: **return** t

33: **else**

34: **return** 0

35: **end if**

36: **end procedure**
