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Random Fractals:
[subtitle]

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Random Fractals

Abstract

[blah]

Acknowledgment

First of all, I would like to thank my supervisor, Ben Hambly, for guiding and supporting through this project.

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Contents

0 Introduction

1 Background Theory

We derive here a mathematical basis for dimensionality and fractal. This will be used later in our more specific study of random fractals.

1.1 Dimensions

The concept of dimension is quite intuitive from a day-life perceptive. However, the mathematical concept is more involved. From the non-mathematical world, this can be used to have better understanding of other object arising from other fields.

1.1.1 Intuition

Some object that we are used to work with have a very commonly admitted dimension:

- **Empty set / Point**: dimension 0
- **Curve** (e.g.: *line*): dimension 1
- **Surface** (e.g.: *square*): dimension 2
- **Volume** (e.g.: *cube*): dimension 3
- **General d-dim set** (e.g.: *d-dimensional cuboid*): dimension d

All of these usual objects have integral dimensions, making it (relatively) easy to understand.

The rule of thumb to calculate the dimension is to double (or, in general, multiply by n) the size of the object, and count the number of copies of the original object obtained. If there is N original objects, the dimension is $d = \frac{\ln(N)}{\ln(n)}$. This is so that when scaling by n, the length/area/volume of the set is multiplied by $N = n^d$.

Some objects have a more complicated dimension (in fact, a non-integral one)[1]:

- **Cantor Set**: dimension $\log_3(2) = \frac{\ln(2)}{\ln(3)} \simeq 0.631$
- **Koch Snowflake**: dimension $\log_3(4) = \frac{\ln(4)}{\ln(3)} \simeq 1.262$
- **Sierpiski Carpet**: dimension $\log_2(8) = \frac{\ln(8)}{\ln(2)} \simeq 1.893$

The dimension is much less intuitive for these objects, and it justifies creating a formal mathematical definition.

After this quick overview, 3 properties seem desirable for a definition of dimension [Pol(2009)]. For a set X ($\subset \mathbb{R}^n$, in general):

1. If X is a manifold, dimension coincide with the natural preconception.
2. In some cases, X may have a fractional (i.e. non-integral) dimension.
3. If X is countable, then X has dimension 0.

There are several definition for dimension, satisfying different properties.

[1] This will be discussed in more details in 1.2.2.

1.1.2 Topological Dimension

The topological dimension is the most straightforward way to define dimension. It relies on the intuition that the boundary of a ball of dimension d should have dimension $d-1$.

Definition 1.1 (Topological dimension)**.** *The topological dimension* $\dim_T(X)$ *of a set* X *is defined recursively through the following:*

$$\dim_T(X) = \begin{cases} -1 & \textit{if} \;\; X = \emptyset \\ d & \textit{if} \;\; d = \min\{n \in \mathbb{N} \mid \forall x \in X, \; \exists r > 0 \; \textit{s.t.} \; \dim_T(\partial B_r(x) \cap X) \leq n-1\} \end{cases}$$

This definition satisfies the first desired property (1.1.1:1). However, $\dim_T$ is always an integer (this is clear from definition). Therefore, it does not satisfies the second desired property (1.1.1:2).

1.1.3 Box Dimension

The box dimension is more abstract than the topological one (def. 1.1). It relies on an intuition mentioned before: if scaled by n, a set contains N copies of the original set, then the dimension should be $\frac{\ln(N)}{\ln(n)}$.

Definition 1.2 (Box dimension)**.** *The box dimension* $\dim_B$ *of a set* X *is defined through the following limit:*

$$\dim_B(X) = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{-\log(\varepsilon)}$$

Here, $N(\varepsilon)$ *is the smallest number of* ε*-balls needed to cover* X*.*

Note that box dimension exists only if this limit exists.

This definition satisfies the first and second desired property (1.1.1:1,2). However, if we consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$, then $\dim_B(X) > 0$, but X is countable. Therefore, it does not satisfies the third desired property (1.1.1:3).

1.1.4 Hausdorff Dimension

The Hausdorff dimension (sometimes also called fractal dimension) is considered to be the most accurate of all, in particular when studying fractal, as it will be the case later in this paper. It is more involved than both the topological (def. 1.1) and the box (def. 1.2) dimensions.

Definition 1.3 (Hausdorff/fractal dimension)**.** *The Hausdorff dimension* $\dim_B$ *of a set* X *is defined as follows:*

$$H^d_\varepsilon(X) = \inf_{\substack{\mathcal{U} \text{ open cover of } X \\ U \in \mathcal{U} \implies diam(U) < \varepsilon}} \left\{ \sum_{U \in \mathcal{U}} diam(U)^d \right\}$$

$$H^d(X) = \lim_{\varepsilon \to 0} H^d_\varepsilon(X)$$

$$\dim_H(X) = \inf\{\delta \mid H^\delta(X) = 0\}$$

Note that H^d *defines the d-dimensional measure (for all* $d \geq 0$*).*

This definition satisfies all three of the desired property (1.1.1:1,2,3). The first two are clear from definition.

Property 1.1. *Countable sets have Hausdorff dimension 0.*

Proof. Suppose $X = \{x_n \mid n \in \mathbb{N}\}$ is countable. Let $\epsilon > 0$, and take $\{\epsilon_n \mid n \in \mathbb{N}\}$ such that $\sum_{n=0}^{\infty} \epsilon_n^d < \epsilon$. Then $\mathcal{B} = \{B(x_n, \epsilon_n) \mid n \in \mathbb{N}\}$ is an open cover fo X, so $H_\epsilon^d(X) \leq \epsilon$. Thus $H^d(X) = 0$ for all $d \geq 0$. So finally, $\dim_H(X) = 0$. □

However, the Hausdorff dimension is usually harder to calculate in practice.

1.1.5 Some Relations Between Dimensions

For dimension definition, there is a choice to make between highly accurate, but hard to calculate (Hausdorff dimension) and less accurate but easier to calculate (Topological/Box dimension) definitions. It is therefore very useful to know some relationships between the three notions (it is then possible to use, for example, box dimension to give estimates for the Hausdorff dimension).

Property 1.2 (Upper bound for fractal dimension)**.** *Box dimension is greater than or equal to Hausdorff dimension.*

Proof. For a set X (with box dimension well defined): Let $\eta > 0$, $\gamma = dim_B(X) + \eta$ and $\delta = dim_B(X) + 2\eta$. Then, $\exists\, \epsilon > 0$ such that X can be covered by $N(\epsilon) < \epsilon^{-\gamma}$ ϵ-balls. Thus, $H_\epsilon^\delta(X) \leq \epsilon^{-\gamma}\epsilon^\delta = \epsilon^\eta$, so $H^\delta(X) = 0$. This gives $\dim_H(X) < \dim_B(X) + 2\eta \quad \forall \eta > 0$, and finally, $\dim_H(X) < \dim_B(X)$. □

Thus, by calculating the box dimension, we also have an upper bound for the Hausdorff dimension.

Lemma 1.1. *If the d-dimensional Lebesgue measure is non-zero, then the Hausdorff dimension is greater or equal to d.*

Proof. Suppose a set X is such that $\dim_H(X) < d$.

Claim. $H^d(X) < \infty \implies H^c(X) = 0 \quad \forall c > d$

Proof. As $H^d(X) < \infty$: For all $\epsilon > 0$ there is an open cover $\mathcal{U}$ for X such that $\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^d < \infty$ and $\operatorname{diam}(U) < \epsilon$. So

$$\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^c \leq \underbrace{\epsilon^{c-d}}_{\substack{\to 0 \\ \text{as } \epsilon \to 0}} \underbrace{\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^d}_{<\infty} \to 0 \quad \text{as } \epsilon \to 0$$

□

Thus, $H^d(X) = 0$. Now, the d-dimensional Hausdorff measure is a rescaling of usual d-dimensional Lebesgue measure, so $\Lambda_d(X) = 0$ [2]. This completes the proof by taking the contrapositive. □

Thus, finding a d such that the Lebesgue measure is non-zero gives a lower bound for the Hausdorff dimension.

Property 1.3 (Lower bound for fractal dimension)**.** *Topological dimension is less than or equal to Hausdorff dimension.*

Proof. This follows directly from last property. □

In fact, the Hausdorff dimension is bonded by the topological dimension and the box dimension, i.e. for any set X, $\dim_T(X) \leq \dim_H(X) \leq \dim_B(X)$.

[2] Writing $\Lambda_d(X)$ for the d-dimensional Lebesgue measure of X.

1.2 Fractals

Fractals are mathematical objects that have been studied since the 17th century. The recursive self-similarity pattern of most common fractals made their fame across the mathematics and non-mathematics world.

1.2.1 Formal Definition

The strict mathematical definition of a fractal is the following:

Definition 1.4 (Fractal)**.** *A fractal is a subset of Euclidean space with a Hausdorff dimension that strictly exceeds its topological dimension (i.e. X is a fractal if $\dim_T(X) < \dim_H(X)$).*

Sets with "recursive patterns" are often fractals (which is why this is sometimes taken a the definition for fractals).

1.2.2 Famous Examples

Some fractals became very famous, both for their aesthetic appeal and as an example of a complex structure arising from simple rules. Fractals are one of the best-known examples of mathematical visualization and mathematical beauty.

Basin Boundaries of Complex Maps Along the most famous fractals, one finds the ones that arise from looking at the contour of sets (basin boundaries) derived from complex maps.

Contour of the Mandelbrot Set The Mandelbrot set is a subset of the complex plane defined as follows:

Definition 1.5 (Mandelbrot Set)**.** *Let $z_0 = 0$ and $z_{n+1} = f_c(z_n)$, with $f_c(z) = z^2 + c$. Then, the Mandelbrot set $M \subseteq \mathbb{C}$ is $M = \{c \in \mathbb{C} \mid z_n \text{ does not diverges}\}$.*

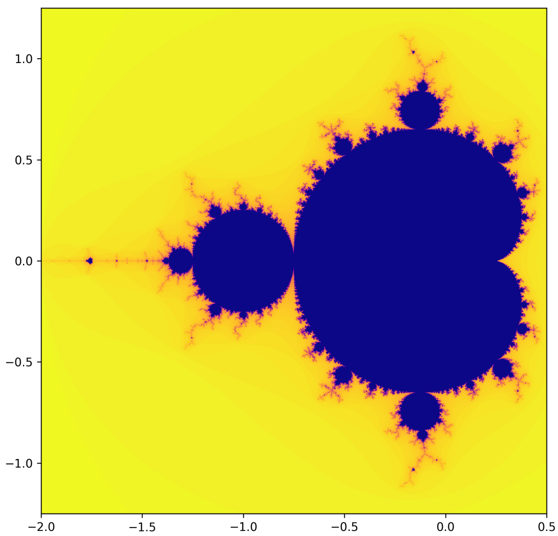


Figure 1: Mandelbrot Set Plot[3]

The Mandelbrot set is not a fractal itself (it is dimension 2, as it contains $\left\{w \in \mathbb{C} \mid |w-1| < \frac{1}{4}\right\}$ [3DX(2020)], and is contained in $\mathbb{C}$, both having dimension 2).

It is surprising that the boundary ∂M of M also has Hausdorff dimension 2. This was proved by Shishikura in 1992 [Shishikura(1992)], as a consequence of ∂M having a positive 2-dimensional Lebesgue measure (i.e. $\Lambda_2(\partial M) > 0$).

Despite having an integral fractional dimension, ∂M is commonly considered to be a fractal (because the intergality of fractional dimension is not obvious, and the self-similarity of the set).

Contour of the Julia Set The Julia set is also subset of the complex plane defined similarly to the Mandelbrot set:

[3]More details on the plots in the appendix.

Definition 1.6 (Julia Set)**.** *For a complex $c \in \mathbb{C}$ constant: Let $z_0 = z$ and $z_{n+1} = f_c(z_n)$, $f_c(z) = z^2+c$ as before. The filled Julia set is $K(f_c) = \{z \in \mathbb{C} \mid z_n$ does not diverges$\} \subseteq \mathbb{C}$. The Julia set $J(f_c) \subseteq \mathbb{C}$ is the boundary of $K(f_c)$.*

The dimension of $J(f_c)$ will of course depend on c. It is considered as a fractal even when the dimension is an integer. For some values of c, dimension of $J(f_c)$ is well known:

$c = 0$	$\dim_H(J(f_c)) = 1$	"Circle"
$c = \frac{1}{4}$	$\dim_H(J(f_c)) \approx 1.0812$	
$c = i$	$\dim_H(J(f_c)) \approx 1.2$	"Dendrite"
$c = -1$	$\dim_H(J(f_c)) \approx 1.2683$	
$c = -0.123 + 0.745i$	$\dim_H(J(f_c)) \approx 1.3934$	"Douady rabbit"

The Mandelbrot set an Julia sets are very closely related. In fact, Shishikura proved [Shishikura(1992)] that when c is on ∂M, then the Julia set $J(f_c)$ associated to it has (Hausdorff) dimension 2. Moreover, Heinemann and Stratmann have shown [Heinemann and Stratmann(1998)] that when the quadratic f_c is parametrized with c near the boundary of the Mandelbrot st M, the Hausdorff dimension of $J(f_c)$ is close to 2.

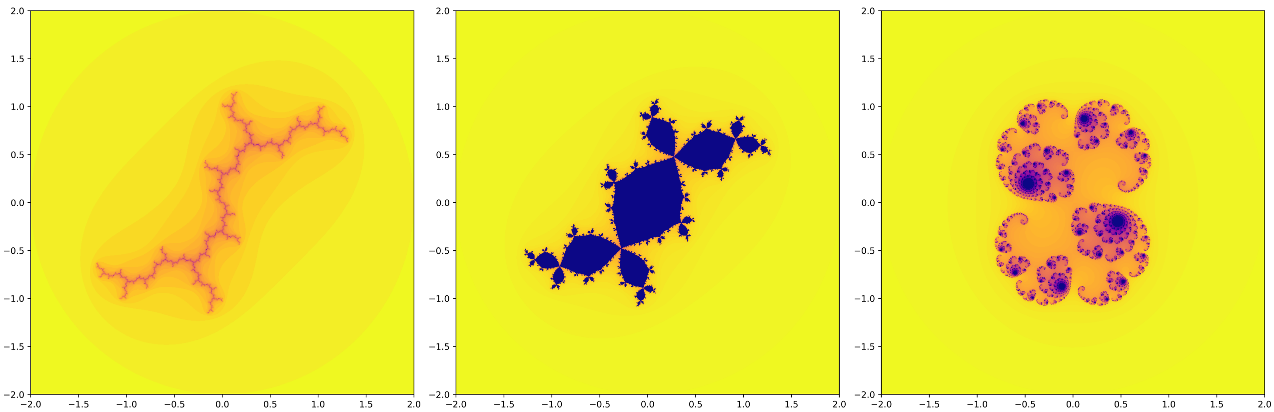


Figure 2: Popular Julia Set Plots
(left to right: "Dendrite"[4]; "Douady rabbit"[5]; other typical Julia set[6])

Cantor Set The (middle third) Cantor set $C \subseteq [0,1]$ is constructed as follows:

- Start with $C_0 = [0,1]$
- Generate iteratively $C_n = \frac{C_{n-1}}{3} + \frac{2+C_{n-1}}{3}$
- Take the limit $C = \lim_{n\to\infty} C_n$

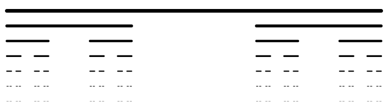


Figure 3: Cantor Set (first 6 iterations)

The Cantor set satisfies $C = \frac{C}{3} + \frac{2+C}{3}$, so rescaling by $\frac{1}{3}$, it contains two copies of itself. So the intuitive Hausdorff dimension for the Cantor set is $\dim_H(C) = \frac{\log(2)}{\log(3)} \approx 0.631$. This can be proved rigorously (see [Falconer(1990), p. 34-35, ex. 2.7].

Another equivalent definition for the cantor set C is "points in $[0,1]$ with extension in base 3 composed of 0 and 2 only". That is:

$$C = \left\{ x \in [0,1] \mid x = \sum_{k=1}^{\infty} x_k 3^{-k}, \quad x_k \in \{0,2\}\ \forall k \right\}$$

[4] $c = i$
[5] $c = -0.123 + 0.745i$
[6] $c = 0.285 + 0.01i$

From this definition, it is straightforward that the Cantor set is totally disconnected. In fact, fractals with Hausdorff dimension smaller than 1 are totally disconnected ([Falconer(1990), p. 33, prop. 2.5]).

Koch Snowflake Koch Snowflake is an example of a fractal curve. Fractal curves are obtained by applying the same transformation recursively to each segment of the previous iteration of the curve. Their fractional dimension is in the interval $[1, 2]$ (or in $[1, 3]$ if the curve evolves in a three dimensional space).

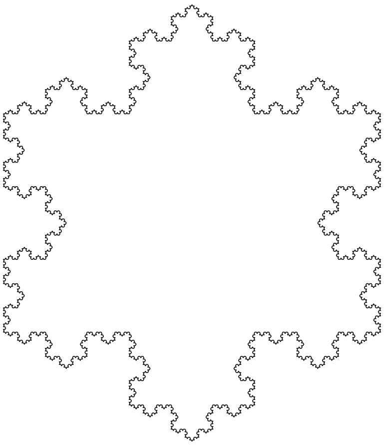
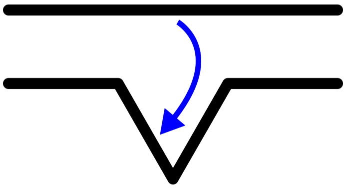


Figure 4: Koch Snowflake Curve Plot (5 iterations)

The Koch snowflake is obtained after replacing each edge of an equilateral triangle by a Koch curve with the triangle edge as initial segment.

Now, the Koch curve is defined recursively as follows: K_n is the curve at iteration n, with segments $K_n^1, \dots K_n^m$. For each $i \in [\![1, m]\!]$, split K_n^i into 3 equal length segments, and replace the middle one with two new ones as on this diagram:



Merging back all segment gives K_{n+1}.

The Koch curve starting form a segment $[a, b]$ is the limit curve $K = \lim_{n\to\infty} K_n$ with $K_0 = [a, b]$.

At each level, the Koch curve is scaled by $1/3$, and 4 copies are created. Thus, the intuitive Hausdorff dimension for the Koch curve (which is also the dimension of the Koch snowflake) is $\dim_H(K) = \frac{\log(4)}{\log(3)} \approx 1.262$. This can be proved rigorously using similar techniques as in [Falconer(1990), p. 34-35, ex. 2.7].

Sierpiski Carpet Sierpiski carpet is a fractal constructed recursively by removing parts of its initial set.

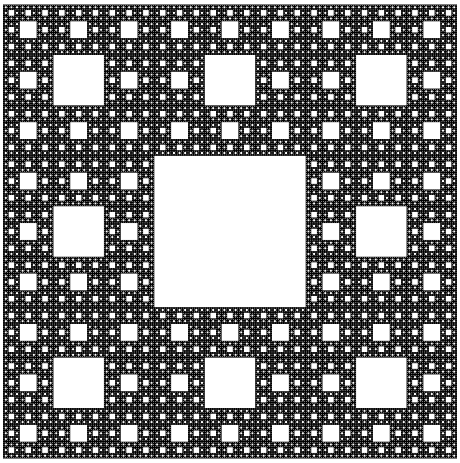
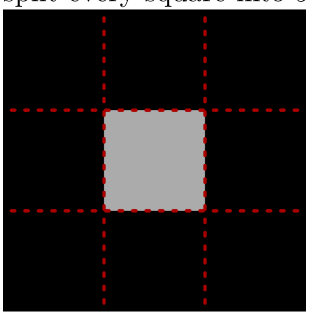


Figure 5: Sierpinski Carpet (6 steps)

The construction starts from a (filled) square. Then, at each step, split every square into 9 sub-square as follows:



Remove the central square, and repeat the operation on the remaining 8 squares.

The figure is made of 8 copies of itself, scaled by a factor of $1/3$. Therefore, the intuitive Hausdorff dimension for this set is $\dim_H(K) = \frac{\log(8)}{\log(3)} \approx 1.893$. Again, this can be proved rigorously using similar techniques as in [Falconer(1990), p. 34-35, ex. 2.7].

1.2.3 Real Life Fractals

Fractals are more than mathematical constructions, they in fact also occur in nature. Of course, the limit is never reached, because of the physical constraints of nature. However, this gives a good motivation for studying fractals mathematically.

Plants Some plants (not even genetically modified) have self-similarity patterns, and may therefore be considered as fractals.

Ferns Perhaps the most obvious and popular natural fractal: Ferns leaves. The leaves are self-similar, with varying pattern across variety of ferns.

Cauliflower A less obvious example: Cauliflowers' surface is a fractal. In fact, since each branch splits into about 13 branches, each about 3 times shorter, the approximate dimension of a Cauliflower's surface is $\log_3(13) \approx 2.335$.

(a) Fern Leave

(b) Cauliflower

Figure 6: Plants Fractals

Coastline of Islands When trying to measure the length of coastlines, scientists realized that the more precise their attempt was, the higher was the value of the length found. In fact, this makes sense: the approximation of a curve with shorter segments will capture more of the smaller curve details, resulting in a longer length. The approximate length found could be made arbitrary large after approximating using smaller and smaller segments. This yields the fact that the coastline is a fractal with dimension greater than one.

Using electronic maps, we try to estimate the fractal dimension of the coastlines for the UK islands, Iceland and Madagascar. More details on the calculations techniques used are given in the appendix (see A.3). The (approximate) coastline dimension for UK, Iceland, and Madagascar are respectively 1.24, 1.25, and 1.06.

The coastlines of the UK and Iceland are much more messy than the one of Madagascar. The approximate values found for coastlines dimensions therefore make sense.

Figure 7: United Kingdom Coastlines

2 Percolation Fractals

Percolated fractals will be the main object we intend to study. Starting from a unit square, and remove some material to obtain a fractal.

We will begin with the definition of the percolation process in the two dimensional case, as it will be our main interest (it is also the most intuitive case, as easy to picture). We will then extend the definition to other dimensions.

There are two types of percolations: the "plain", and the "recursive", both have their interest, and we will compare the two throughout this study.

2.1 Plain Percolation

The "plain" percolation is the simplest one, as it is only composed of one filtration. The formal definition goes as follows: Split the unit square $[0,1]^2$ into n^2 squares of side $1/n$, indexed by $1 \leq i, j \leq n$:

$$B_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right].$$

Then, each squares will be selected or thrown away, according to random variables $\epsilon_{i,j}$ following a Bernoulli distribution with probability parameter p [7], $\epsilon_{i,j} \sim \mathcal{B}(p)$. Finally, the plain percolation P is

$$P = \bigcup_{i,j \text{ s.t. } \epsilon_{i,j}=1} B_{i,j}.$$

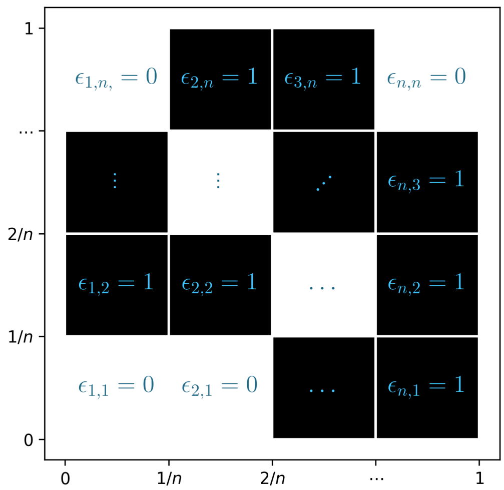


Figure 8: Plain Percolation ($n = 4, p = 0.6$)

We will adopt the notation $P \sim \text{Perc}(n, p, 1)$ for such a setup (a plain percolation of side n and probability parameter p).

In this case, it is interesting to study the behaviour as $n \to \infty$, we will write $P \sim \text{Perc}(\infty, p, 1)$ for $P = \lim_{n\to\infty} P^n$, $P^n \sim \text{Perc}(n, p, 1)$.

2.2 Recursive Percolation

The "recursive" percolation is a little more involved. Beginning with a plain percolation, we apply an other one to each of the remaining squares, and continue recursively.

Formally, for each $1 \leq k \leq d$, split the unit square into $\left(n^k\right)^D$ squares of side $1/n^k$ indexed by $1 \leq i, j \leq n^k$

$$B^k_{i,j} = \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \left[\frac{j-1}{n^k}, \frac{j}{n^k}\right].$$

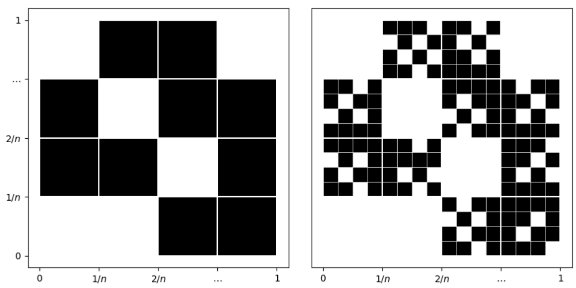


Figure 9: Recursive Percolation ($n = 4, p = 0.6, k = 1, 2$)

Again, associate to each square a Bernoulli random variable with parameter p, $\epsilon^k_{i,j} \sim \mathcal{B}(p)$. Finally, the recursive percolation P^d is defined recursively by:

$$P^k = P^{k-1} \bigcap \left(\bigcup_{i,j \text{ s.t. } \epsilon^k_{i,j}=1} B^k_{i,j} \right) \quad \forall 1 \leq k \leq d$$

[7] Writing $x \sim \mathcal{B}(p)$ for x following a Bernoulli distribution with parameter p, i.e. $\mathbb{P}(x = 1) = p$ and $\mathbb{P}(x = 0) = 1 - p$.

and

$$P^0 = [0,1]^2 .$$

We will adopt the notation $P^d \sim \text{Perc}(n, p, d)$ for such a setup (a recursive percolation of depth d, side n and probability parameter p).

In this case, it is interesting to study the behaviour as $d \to \infty$, with n fixed. We will write $P \sim \text{Perc}(n, p)$ for $P = \bigcap_{k\to\infty} P^k$, with P^k defined recursively as above.

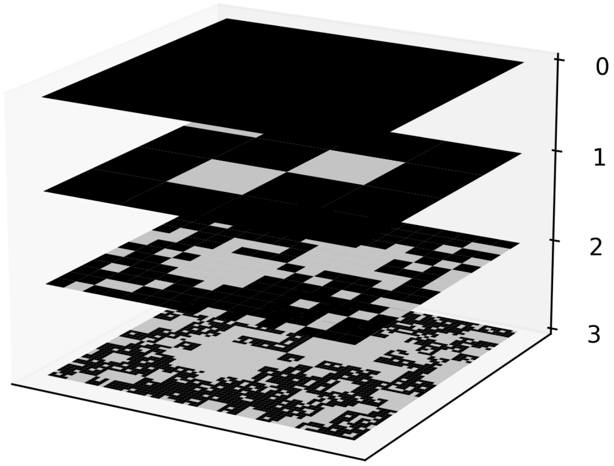


Figure 10: Recursive Percolation
$(n = 4, d = 3, p = 0.8)$

2.3 Extension to other dimensions

Now that we defined percolations in two dimensions, it is straightforward to extend the definition to other dimensions.

Extension to D dimensions It suffices to add parameters in the definition to extend it to other dimension, formally:

Plain Split the unit cuboid $[0,1]^D$ into n^D squares of side $1/n$, indexed by $1 \leq i_1, \ldots, i_D \leq n$:

$$B_{i_1,\ldots,i_D} = \left[\frac{i_1 - 1}{n}, \frac{i_1}{n}\right] \times \cdots \times \left[\frac{i_D - 1}{n}, \frac{i_D}{n}\right].$$

Then, attach to each cuboid a Bernoulli random variable $\epsilon_{i_1,\ldots,i_D} \sim \mathcal{B}(p)$. The plain percolation P in D dimension is

$$P = \bigcup_{i_1,\ldots,i_D \text{ s.t. } \epsilon_{i_1,\ldots,i_D}=1} B_{i_1,\ldots,i_D}.$$

We will denote this setup by $P \sim \text{Perc}^D(n, p, 1)$ (a plain percolation of side n and probability parameter p in D dimensions), and write $P \sim \text{Perc}^D(\infty, p, 1)$ for $P = \lim_{n\to\infty} P^n$, $P^n \sim \text{Perc}^D(n, p, 1)$.

Recursive For each $1 \leq k \leq d$, split the unit cuboid into $\left(n^k\right)^D$ cuboid of side $1/n^k$ indexed by $1 \leq i_1, \ldots, i_D \leq n^k$

$$B^k_{i_1,\ldots,i_D} = \left[\frac{i_1 - 1}{n^k}, \frac{i_1}{n^k}\right] \times \cdots \times \left[\frac{i_D - 1}{n^k}, \frac{i_D}{n^k}\right].$$

Again, attach to each cuboid a Bernoulli random variable $\epsilon^k_{i_1,\ldots,i_D} \sim \mathcal{B}(p)$. Finally, the recursive percolation P^d is defined recursively by:

$$P^k = P^{k-1} \bigcap \left(\bigcup_{i_1,\ldots,i_D \text{ s.t. } \epsilon^k_{i_1,\ldots,i_D}=1} B^k_{i_1,\ldots,i_D} \right) \quad \forall 1 \leq k \leq d$$

and

$$P^0 = [0,1]^2 .$$

We will denote this setup by $P^d \sim \mathrm{Perc}^D(n,p,d)$ (a recursive percolation of depth d, side n and probability parameter p in D dimensions). And again, we will write $P \sim \mathrm{Perc}^D(n,p,\infty)$ for $P = \bigcap_{k\to\infty} P^k$, with P^k as above.

In practice, we will never use more than three dimensions. First, our world is in three dimensions, so it makes sense to stop there. In addition to that, the curse of dimensionality stops us (calculations and notations are too heavy from the mathematical point of view, and computations are too expensive from a modelling perspective).

Restriction to 1 dimension The restriction of the percolation process to one dimension can be though as a randomized and generalized version od the Cantor set. The Cantor set splits the interval in 3 equal parts, and keep the two extreme ones, while the general recursive percolation process splits the interval into n equal parts, and keep each interval with probability p.

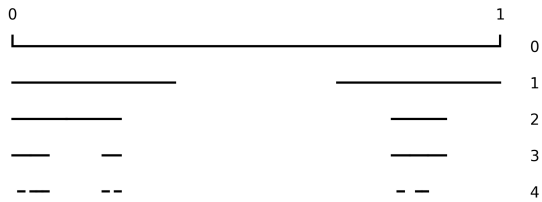


Figure 11: 1D Recursive Percolation ($n = 3, d = 4, p = 0.6$)

2.4 Density

Since the percolation process involves randomness, the density is not well defined. However, it is possible to calculate the expected density [8].

It is straightforward to remark that the expected density of a plain percolation $P \sim \mathrm{Perc}^D(n,p,1)$ is p (for all D and all n). Note that this density is constant as $n \to \infty$.

Now, for a recursive percolation P such that $P \sim \mathrm{Perc}^D(n,p,d)$, the density is p^d. Note that this density tends to zero as d grows to infinity (except in the trivial case $p = 1$). Thus, if $P \sim \mathrm{Perc}^D(n,p,\infty)$, the expected density is 0.

2.5 Dimensionality

Again, we can only give an expected dimension, as the process involves randomness. We ignore the cases $p = 0$ and $p = 1$ (these are trivial), and we concentrate on $p \in (0,1)$.

We will treat separately the finite cases, and the two limit cases (limit as plain percolation, and as recursive percolation).

Finite Cases Take n and d finite, $P \sim \mathrm{Perc}^D(n,p,d)$. Let Z be the number of remaining squares ($Z = |\{(i_1,\ldots,i_D) \mid \epsilon_{i_n,\ldots,i_D} = 1\}|$). The expected number of remaining squares is then $\mathbb{E}(Z) = p\left(n^d\right)^D$.

There are two possibilities for the dimension of P:

1. $Z = 0$: then $P = \emptyset$ and dimension of P is 0;

[8] The density of a set $X \subseteq [0,1]^D$ is the proportion of points in the unit cuboid that are also in X.

2. $Z > 0$: then $P \supseteq \left[\frac{i_1-1}{n^k}, \frac{i_1}{n^k}\right] \times \cdots \times \left[\frac{i_D-1}{n^k}, \frac{i_D}{n^k}\right]$ for some $(i_1, \ldots, i_D)$, so dimension of P is D.

Therefore, the expected dimension of P is:

$$\mathbb{E}(\dim_H(P)) = 0.\mathbb{P}(Z = 0) + D.\mathbb{P}(Z > 0) = D\left(1 - (1-p)^{\left(n^d\right)^D}\right).$$

In practice, this is close to D, as soon as n or d grows. Note that $\dim_T$ would only differ by the fact that $\dim_T(\emptyset) = -1$ whereas $\dim_H(\emptyset) = 0$. Bow dimension $\dim_B$ will be the same as the Hausdorff one.

Plain Percolation We are looking at $P \sim \mathrm{Perc}^D(\infty, p, 1)$.

Heuristic Argument Intuitively, the expected dimension of this percolation should be D, since the density is positive.

Formal Calculations We show that the Hausdorff dimension is D almost surely.

Let $\mathcal{U}$ be an open cover for P, and let V be an open subset of the unit cuboid of dimension D. Then, $\mathbb{P}(P \cap V = \emptyset) = 0$ (as V is uncountable). Thus, it is almost sure that $\mathcal{U}$ covers $[0,1]^D$ (otherwise, there is an open subset V of $[0,1]^D$ not intersecting P). So $H^d_\epsilon(P) = H^d_\epsilon\left([0,1]^D\right)$ almost surely. Therefore, $\dim_H(P) = \dim_H([0,1]^D) = D$ almost surely.

Recursive Percolation We are interested in $P \sim \mathrm{Perc}^D(n, p, \infty)$.

Heuristically In the case of a recursive percolation, P has self-similar properties that can help finding the expected dimension. After the first percolation, each selected cuboid is another version of P scaled by $1/n$. This goes on recursively. Therefore, the intuitive dimension to expect is

$$\mathbb{E}(\dim_H(P)) = \frac{\ln(\mathbb{E}(Z_1))}{\ln(n)}.$$

Where Z_1 is the number of remaining cuboids after one percolation $Z_1 = \left|\{(i_1, \ldots, i_D) \mid \epsilon^1_{i_n,\ldots,i_D} = 1\}\right|$. So $\mathbb{E}(Z_1) = pn^D$, and finally:

$$\mathbb{E}(\dim_H(P)) = \frac{\ln(pn^D)}{\ln(n)} = D + \log_n(p).$$

Formally The above argument can be made rigorous. Defining a contractive map that reflects the self similarities; Then use Stefan Banach's contractive mapping fixed point theorem applied to the complete metric space of non-empty compact subsets of R^n with the Hausdorff distance.

However, we choose to show the fractional dimension is $\alpha = D + \log_n(p)$ directly (this proof follows some ideas from [Falconer(1990), p. 34-35, ex. 2.7]).

From definition, $P = \bigcap_{d\to\infty} P^k$. P^k is composed if cuboids of side $1/n^k$. Let Z^k be the number of these cuboids. In expectation, P^k should have $\mathbb{E}(Z^k) = \left(pn^D\right)^k$ cuboids of side $1/n^k$ remaining. Taking P^k as a $\sqrt{D}/n^k$-cover (let $\beta_k = \sqrt{D}/n^k$ [9]) of P gives

$$\mathbb{E}\left(H^\alpha_{\beta_k}(P)\right) \le \left(pn^D\right)^k \left(\sqrt{D}/n^k\right)^{D+\log_n(p)} \le \sqrt{D}^\alpha.$$

Letting $k \to \infty$, get

$$\mathbb{E}\left(H^\alpha(P)\right) \le \sqrt{D}^\alpha < \infty.$$

[9] So that β_k is the diameter of a cuboid of side $1/n^d$ in dimension D.

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of P. For each $i \in I$, let $k \in \mathbb{N}$ be such that $\beta_{k+1} \leq \operatorname{diam}(U_i) < \beta_k$. Then, in general, U_i may only cover one cuboid from P^k. As generating P is a random process, it may happen that a U_i covers more than one, but this is not the case in general (as it involves a very particular selection configuration), and will not be consistently the case as we take $k \to \infty$. Now, if $j \geq k$, then U_i intersects in expectation at most $\left(pn^D\right)^{j-k} \leq \frac{\left(pn^D\right)^j}{\sqrt{D}^\alpha} \operatorname{diam}(U_i)^\alpha$ cuboids of side $1/n^j$. Taking j such that $\beta_{j+1} \leq \operatorname{diam}(U_i) \quad i \in I$, since $\mathcal{U}$ intersect in expectation $\mathbb{E}(Z^j) = \left(pn^D\right)^j$ cuboids of diameter $1/n^j$, we get

$$\left(pn^D\right)^j \leq \sum_{i \in I} \frac{\left(pn^D\right)^j}{\sqrt{D}^\alpha} \operatorname{diam}(U_i)^\alpha.$$

This implies that in expectation, $\sqrt{D}^\alpha \leq H^\alpha(P)$. Thus, $\mathbb{E}\left(H^\alpha(P)\right) \geq \sqrt{D}^\alpha > 0$.

Finally, as $0 < H^\alpha(P) < \infty$, get $\mathbb{E}(\dim_H(P)) = \alpha = D + \log_n(p)$.

2.6 Blob

To have a better understanding, we begin by studying the central "blob" of the fractal.

Definition We define the blob of a percolation P as the connected component of P that contains the point at the centre of the cuboid (that is, the component that contains $(1/2, \ldots, 1/2)$). Note that the blob may be empty.

In the simulations, since infinite percolation may only be approximated, we only consider cases where n is odd, so that the cuboid containing the point $(1/2, \ldots, 1/2)$ is uniquely well defined.

Algorithm The algorithm we use to find the blob is the following:

2.6.1 Manhattan (step) distance

2.6.2 Euclidean distance

2.6.3 Area

2.6.4 Volume

2.7 Random Walks

3 Percolation Crossings

3.1 Types of Crossings

Straight

Semi-Straight

Non-Straight

3.2 Finding Crossings

Examples

Algorithms

3.3 Crossings Probability

3.4 Crossings Length

4 Projections

4.1 2D to 1D

4.2 3D to 2D

4.3 3D to 1D

4.4 Dimension of the (non void) Universe

References

[Pol(2009)] Lectures on fractals and dimension theory, 2009.

[3DX(2020)] *Mathematical Documentation of objects*, 2020.

[Shishikura(1992)] Mitsuhiro Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *arXiv Mathematics e-prints*, art. math/9201282, April 1992.

[Heinemann and Stratmann(1998)] Stefan M. Heinemann and Bernd O. Stratmann. Quadratic polynomials with julia sets of hausdorff dimension close to 2, 1998.

[Falconer(1990)] Kenneth Falconer. *Fractal Geometry.* Wiley, 1990. ISBN 0978-0-471-92287-2.

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Random Fractals

Appendix

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A More on Fractals

Here will be discussed some extensions of famous fractals studied before (in 1.2.2). This is not directly relevant for this paper, but an involved reader may be interested.

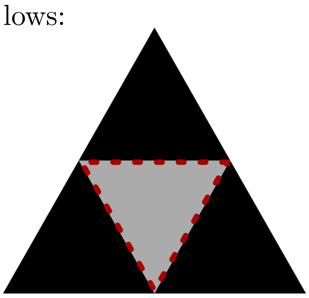
A.1 Mandelbrot and Julia Sets

A.2 Sierpiski Fractals

Following the idea of Sierpiski carpet, several othe fractals may be generated. First, using (equilateral) triangles instead of squares:

Sierpiski Triangle Sierpiski triangle is also a fractal constructed recursively by removing parts of its initial set, following a pattern similar to the one for Sierpiski carpet (although, there are equivalent definitions).

The construction is starts from an equilateral triangle. Then, at each step, split every equilateral triangle into 4 sub-triangles as follows:



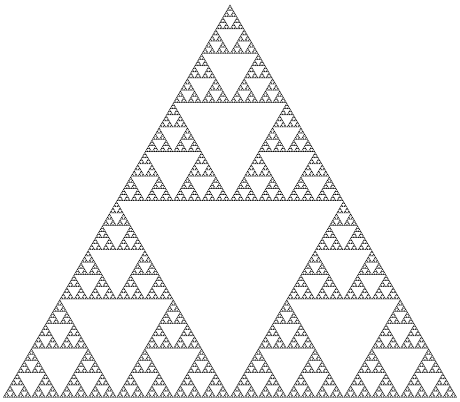


Figure 12: Sierpinski Triangle (8 steps)

Remove the central triangle, and repeat the operation on the remaining 3 triangles.

The figure is made of 3 copies of itself, scaled by a factor of $\frac{1}{2}$. Therefore, the intuitive Hausdorff dimension for this set is $\dim_H(K) = \frac{log(3)}{log(2)} \approx 1.585$. Again, this can be proved rigorously using similar techniques as in [Falconer(1990), p. 34-35, ex. 2.7].

Sierpiski 3D Fractals Seen previously, Sierpiski carpet and triangle live in a 2D world, but the idea of Sierpiski fractal may also be extended to 3D.

Menger sponge The Menger sponge is the generalisation of Sierpiski carpet to 3 dimensions.

Start with a cube; split it into 27 identical copies scaled by $\frac{1}{3}$, and remove the central one, and the 6 cubes sharing a face with the central cube. Repeat this operation on each of the 20 remaining cubes.

From this construction arise a fractal with dimension $\frac{log(20)}{log(3)} \approx 2.727$.

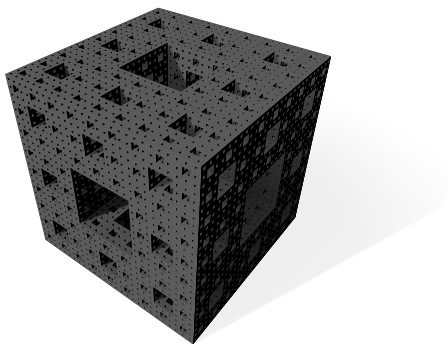


Figure 13: Menger Sponge (4 steps)

Sierpiski tetrahedron The Sierpiski tetrahedron (or tetrix) is the analogue of Sierpiski triangle in 3 dimensions. Start with a regular tetrahedron; split it into 5 identical copies scaled by $\frac{1}{2}$, and remove the central one. Repeat this operation on each of the 4 remaining tetrahedrons.

From this construction arise a fractal with dimension $\frac{log(4)}{log(2)} = 2$. Note that this set is dimension exactly 2 while not being a surface.

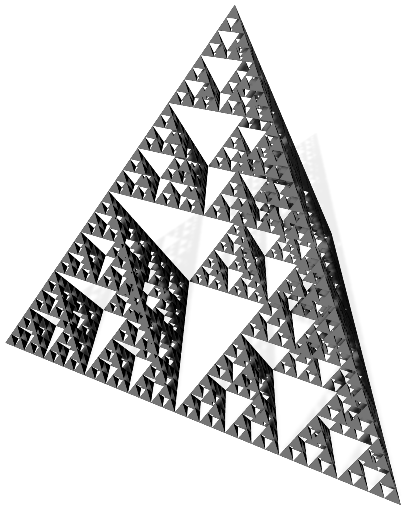


Figure 14: Sierpiski Tetrahedron (5 steps)

A.3 Coastline of Islands

The coastline can be measured at different scales. As the scale is refined, one measures a more detailed curve, which leads to a larger length. This refinement can be done indefinitely: from segments of $1000km$ (one would only capture the rough coastline shape) to segments of $1mm$ (one would need to go around every grain of sand), and even further considering sub-atomic scales.

Following this reasoning, the (1D) length of coastlines is infinite. It is in fact a fractal with dimension greater than one (but less than 2, as contain in a plane).

It is then possible to measure how disordered the coastline of an island is, by approximating its Hausdorff dimension. This will be done macroscopically, using maps extracted from the Google Maps service. In this paper, 3 islands will be considered: The United Kingdom, Iceland, and Madagascar. The United Kingdom and Iceland are understood as having messy coastlines (so should have a high Hausdorff dimension), whereas Madagascar is though to have a rather smooth coastline (Hausdorff dimension close to one).

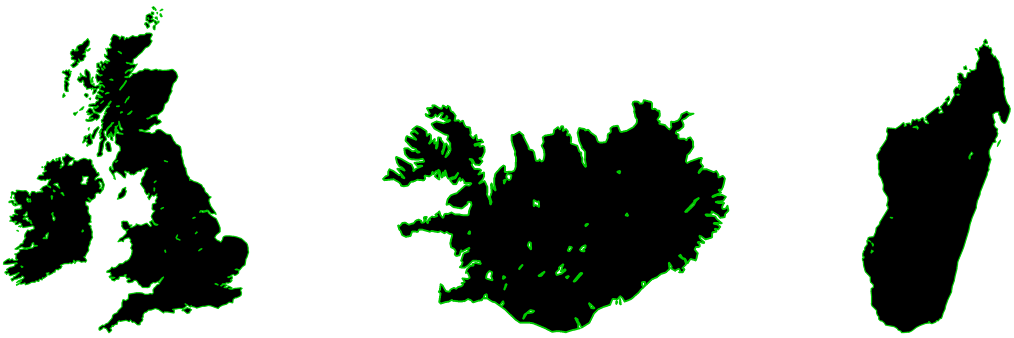


Figure 15: Coastlines (coloured in green) of the UK, Iceland, and Madagascar.

Of course, fractal dimension can only be an approximation. We approach the problem as follows: First, we take a high resolution map of the island, and split it into a binary colour image, black for the ground, and white for the surrounding sea (as in fig. 15). Then, we reduce the size of the image by merging pixels ("pixelizing") to obtain lower resolution images. The resolution panel will correspond to the measuring scales panel. A subset of the resulting images is shown in fig. 16.

Now, using computer vision (as a Python package), we obtain the contours of the shape in an image. It is then straightforward to measure the length of the contour (in pixels, which can be converted to kilometres as we know the scale of the image). Repeating this operation for each scale of the map gives an array of coastline length with associated scales.

Plotting the values in a log-log plot show a nearly straight line; the approximate dimension will be given by the slope of the best fitting line. We obtained the following dimensions:

United Kingdom	$D \approx 1.2354$
Iceland	$D \approx 1.2466$
Madagascar	$D \approx 1.0600$

The fitting line is very close to the data (see fig. 17), showing that the behaviour is as expected.

The full code for this study can be found here: `https://colab.research.google.com/drive/1qo8S8oxqcLsw9UFrq5wTuVyxVA48uwFF?usp=sharing`.

Figure 16: Map scales for the UK, Iceland, and Madagascar.

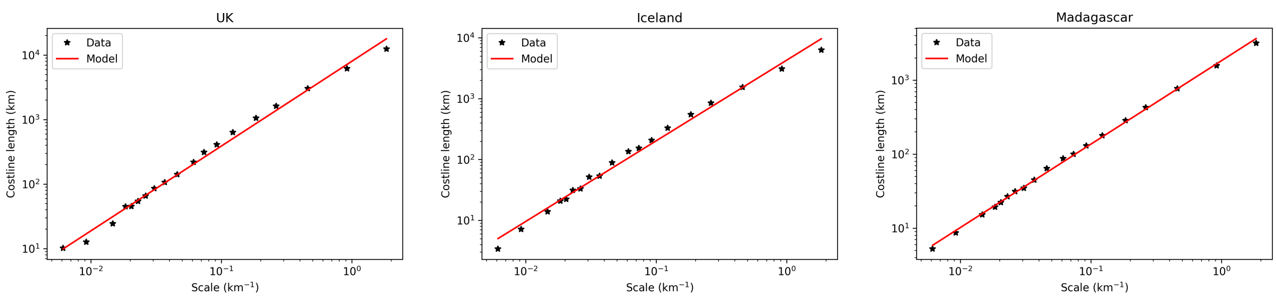


Figure 17: Islands Dimension Regression

B Plots

C Codes