

Maths → Maths

Mathematics is *not* broken.

Random Fractals

Student: Paul Dubois

Supervisor: Ben Hambly

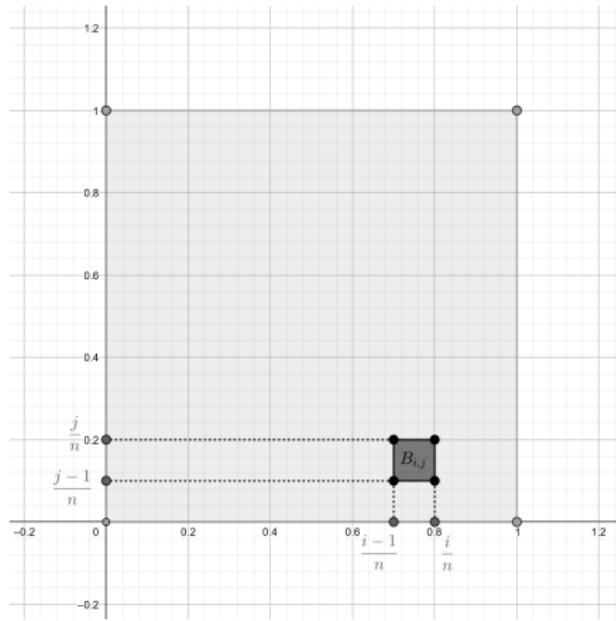
Oxford University

10th March 2021

The Percolation Process

Plain: $P \sim \text{Perc}(n, p, 1)$

$$B_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n} \right] \times \left[\frac{j-1}{n}, \frac{j}{n} \right]$$



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$\varepsilon_{i,j} \in \{0, 1\}$ with $\mathbb{P}(\varepsilon_{i,j} = 1) = p$ (i.e. $\varepsilon_{i,j} \sim \mathcal{B}(p)$)

$$P = \bigcup_{\substack{i,j \\ \varepsilon_{i,j}=1}} B_{i,j}$$

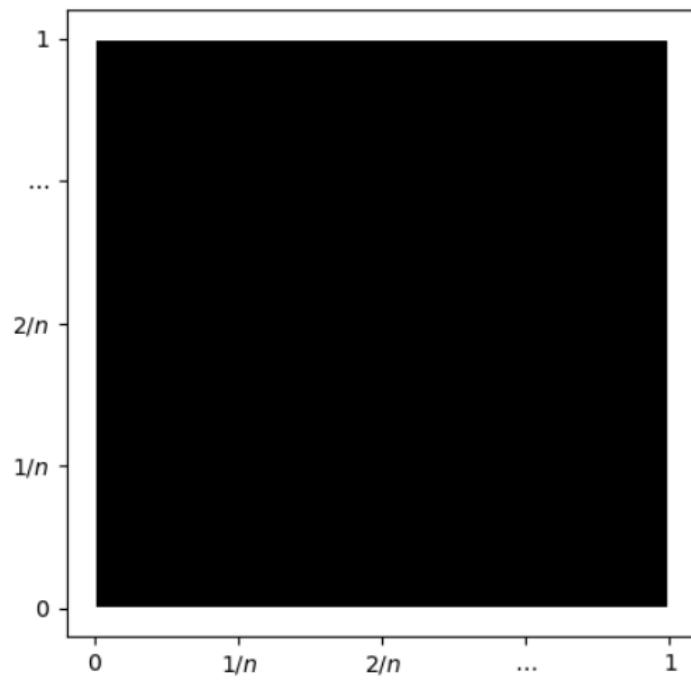
$$Z = |\{(i,j) \mid \varepsilon_{i,j} = 1\}|$$

$$D = \frac{Z}{pn^2}$$



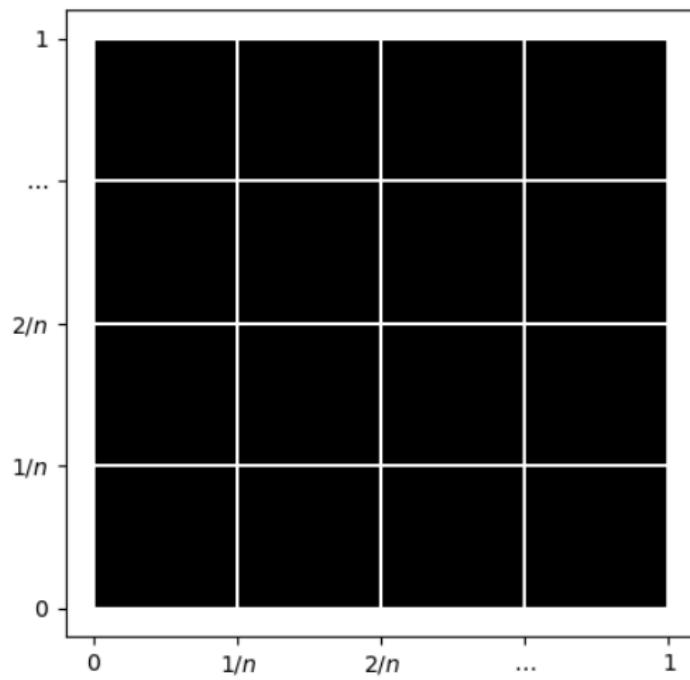
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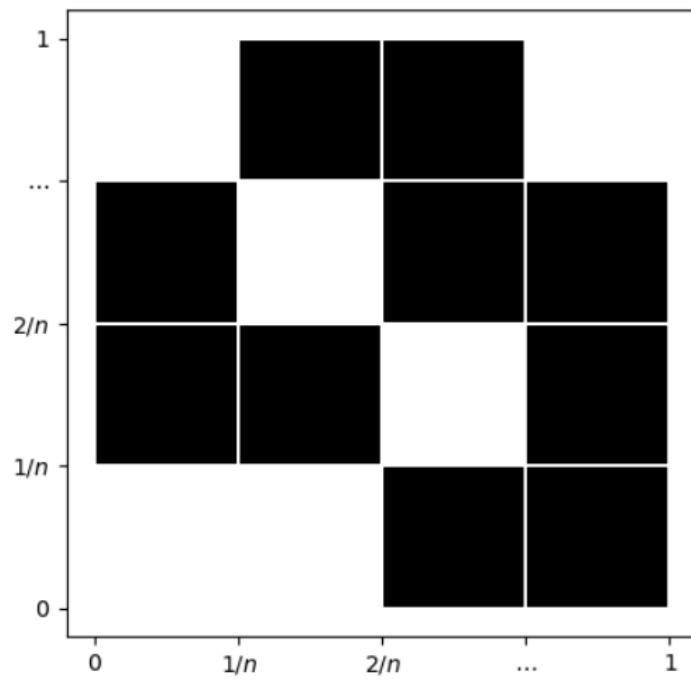
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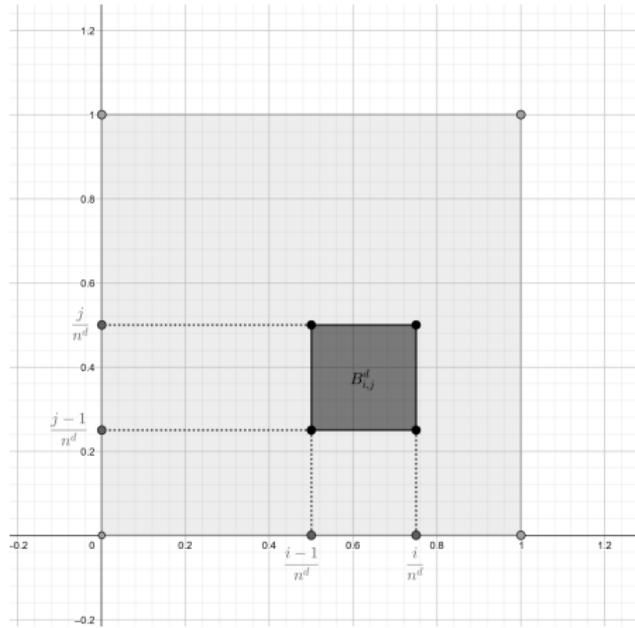
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The Percolation Process

Recursive: $P_d \sim \text{Perc}(n, p, d)$

$$B_{i,j}^d = \left[\frac{i-1}{n^d}, \frac{i}{n^d} \right] \times \left[\frac{j-1}{n^d}, \frac{j}{n^d} \right]$$



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$\varepsilon_{i,j}^d \in \{0, 1\}$ with $\mathbb{P}(\varepsilon_{i,j}^d = 1) = p$ (i.e. $\varepsilon_{i,j}^d \sim \mathcal{B}(p)$)

$$P_0 = [0, 1]^2 \quad ; \quad P_d = P_{d-1} \bigcap \left(\bigcup_{\substack{i,j \\ \varepsilon_{i,j}^d = 1}} B_{i,j}^d \right)$$

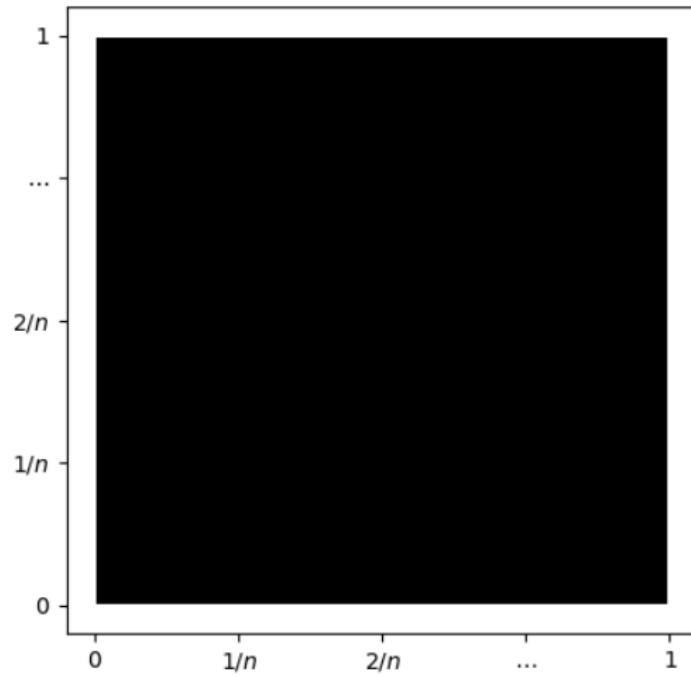
$$Z_d = \left| \left\{ (i, j) \mid \varepsilon_{i,j}^d = 1 \right\} \right|$$

$$D_d = \frac{Z_d}{(pn^2)^d}$$



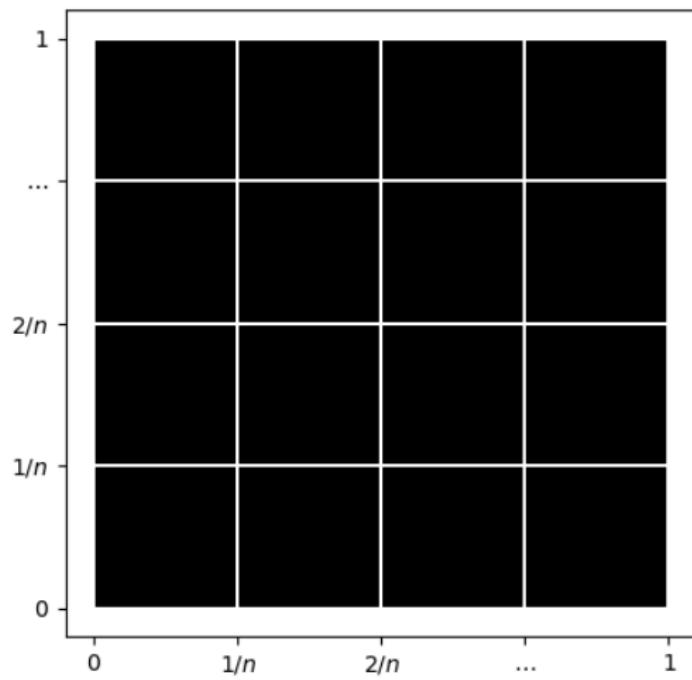
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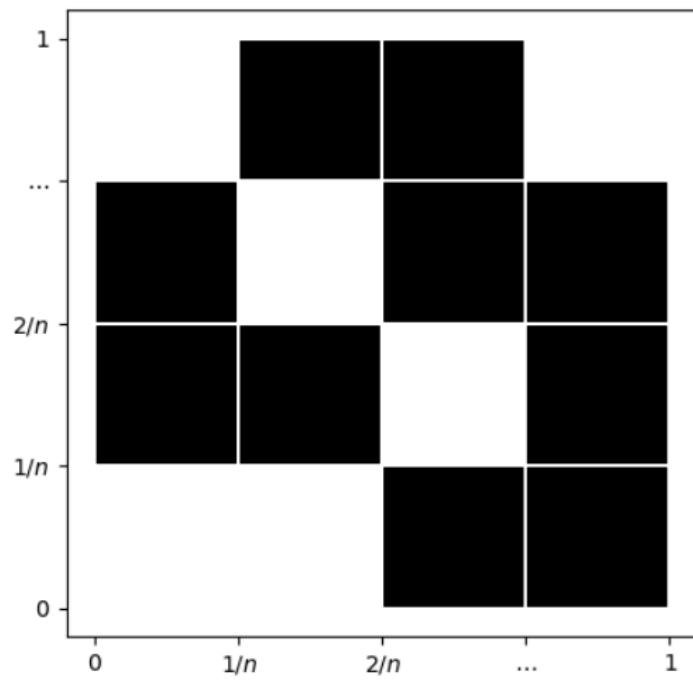
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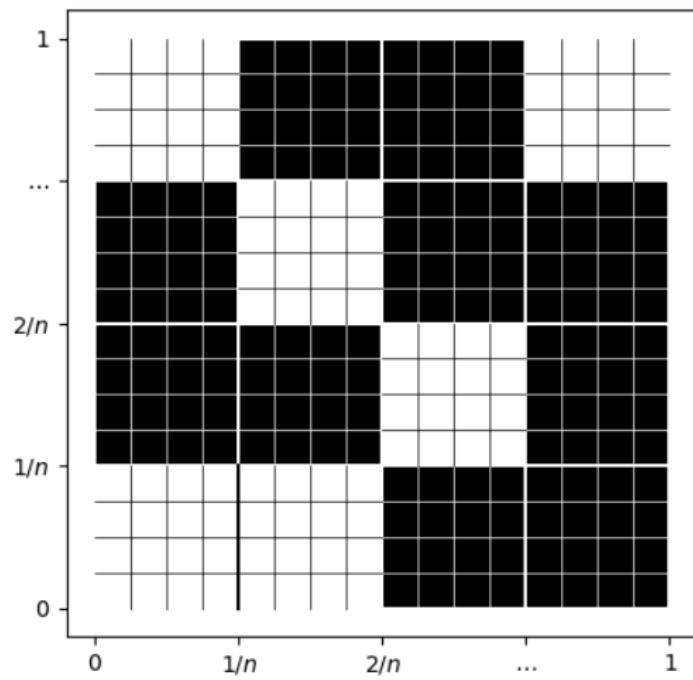
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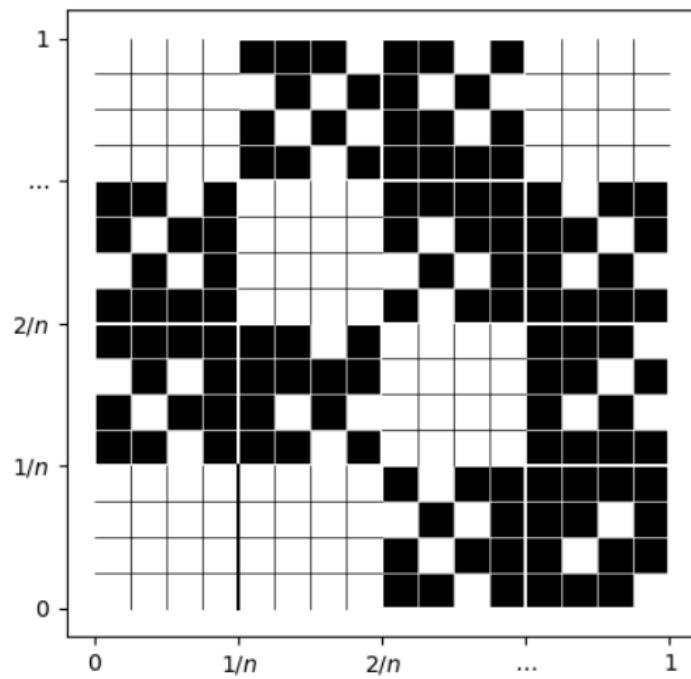
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The Percolation Process

Limit: $P_\infty \sim \text{Perc}(n, p)$

$$P_\infty = \bigcap_{d \in \mathbb{N}} P_d$$

$$D_\infty = \lim_{d \rightarrow \infty} D_d$$



The Percolation Process

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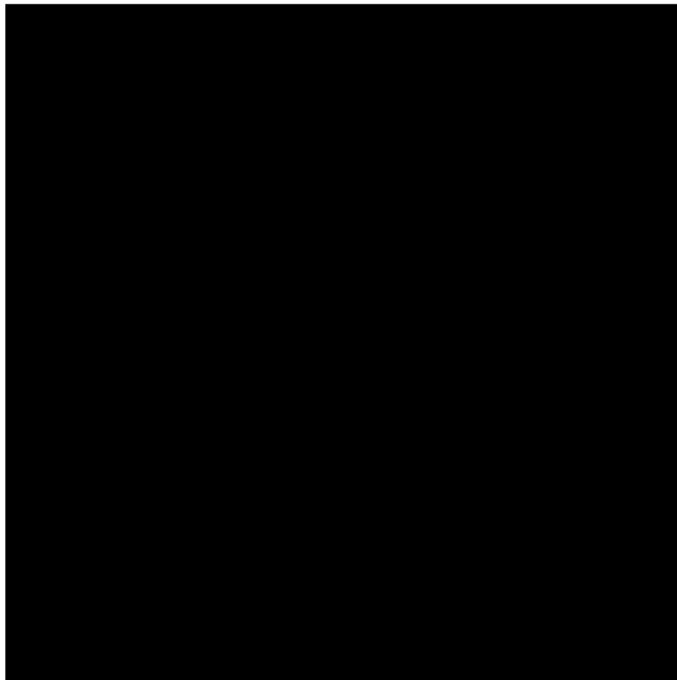
$$D_\infty > 0 \iff P_\infty \neq \emptyset$$

$$\mathbb{E}(D_\infty) = 1$$



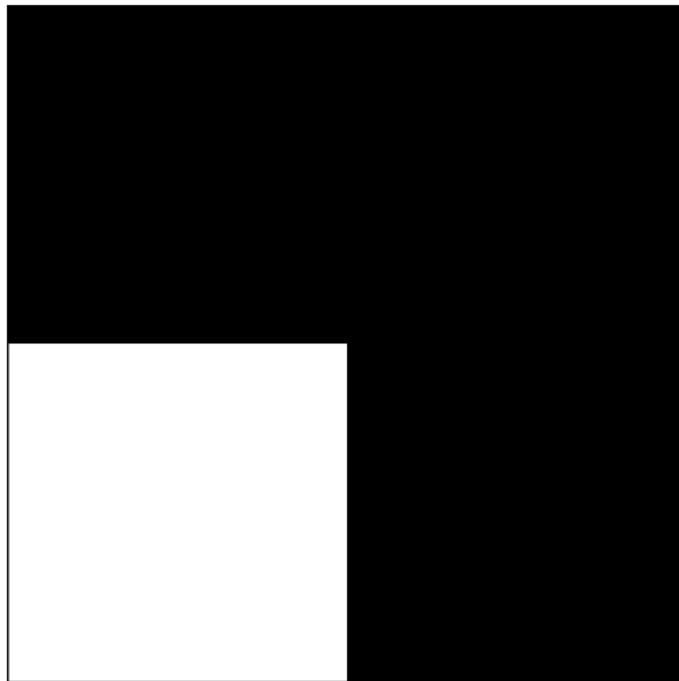
The Percolation Process

Example for $n = 2$



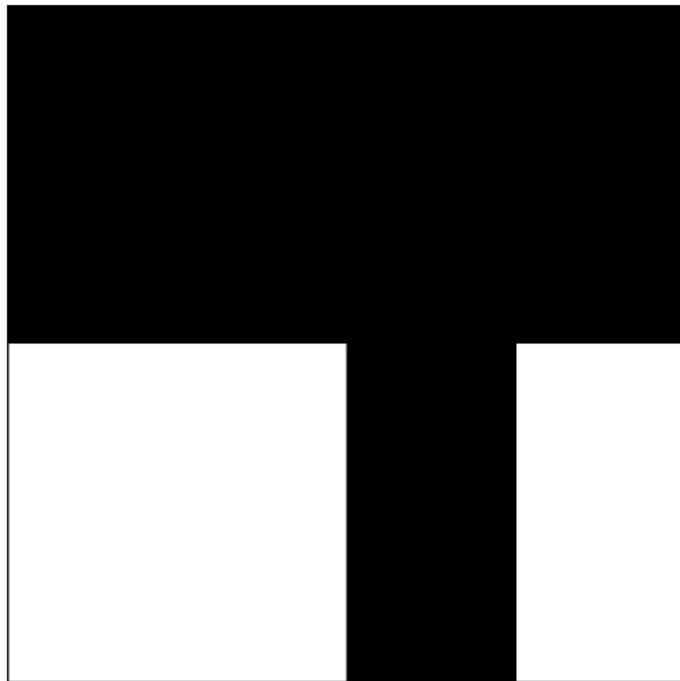
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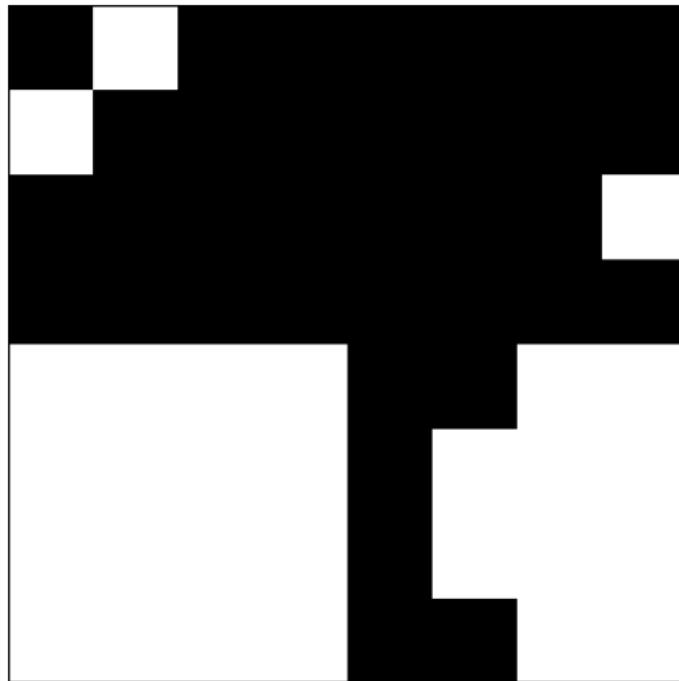
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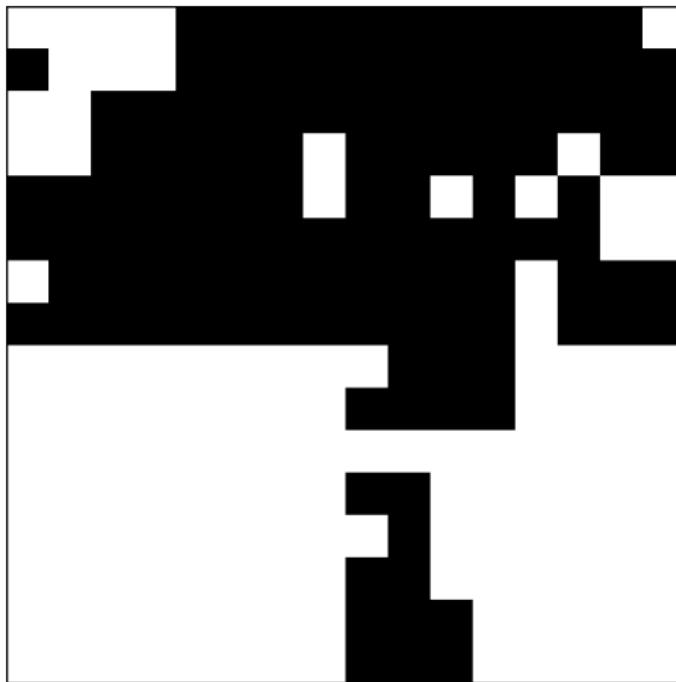
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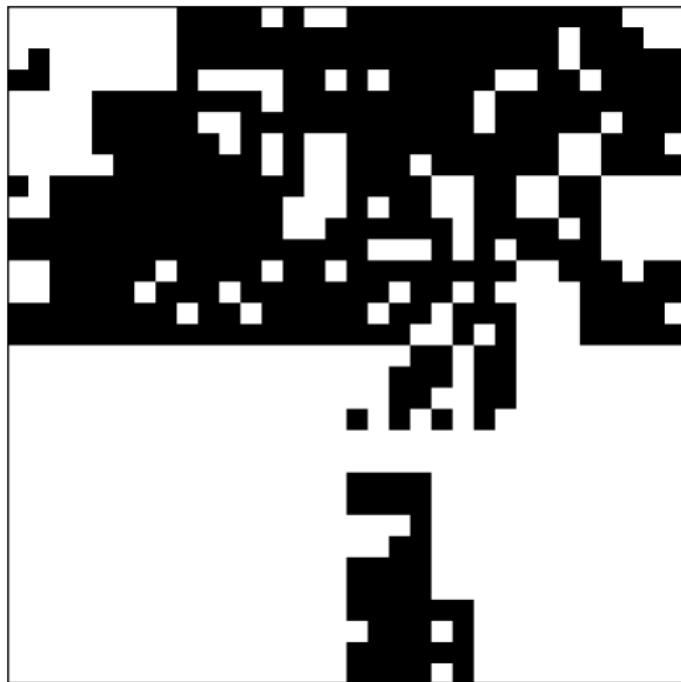
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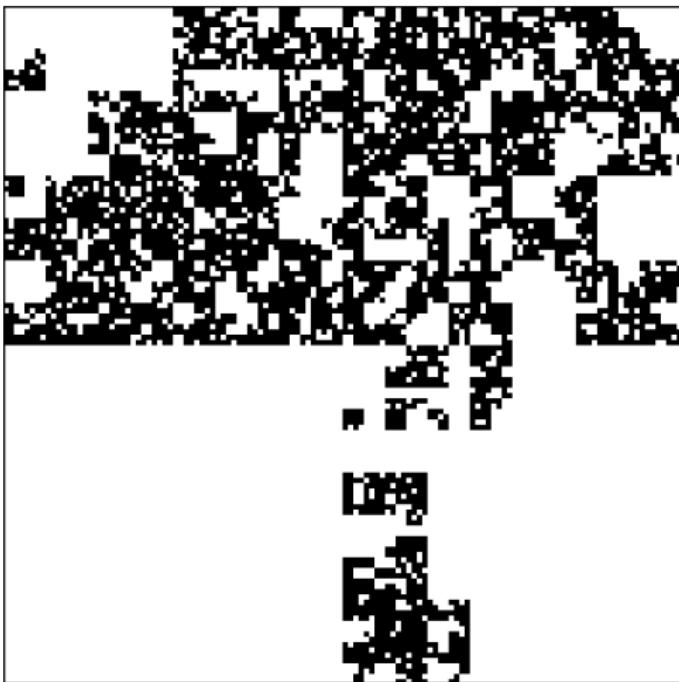
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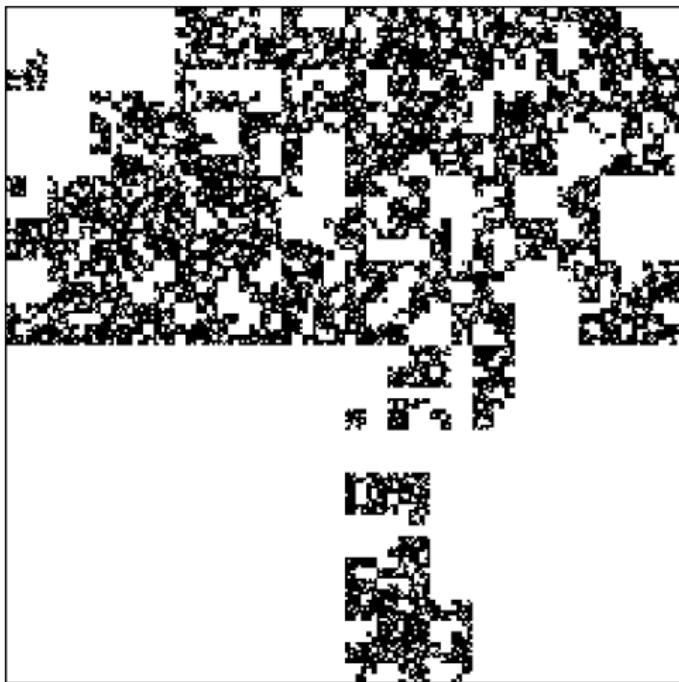
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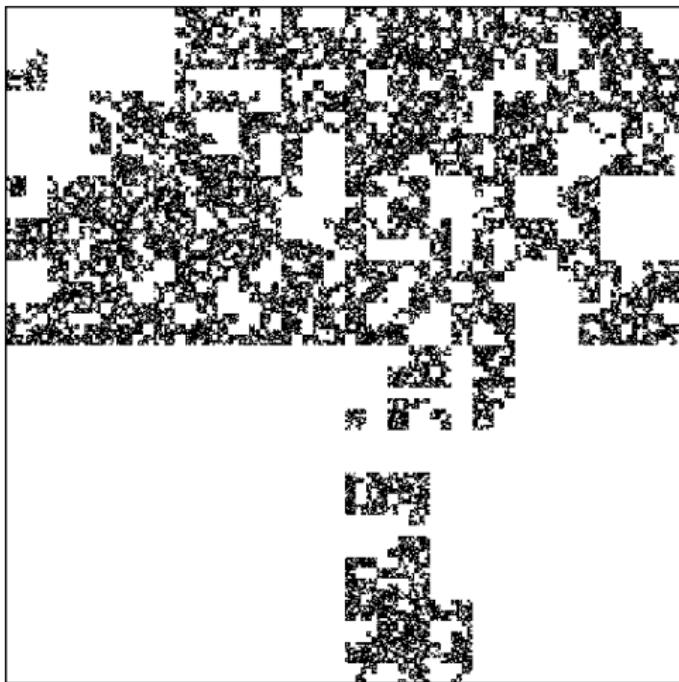
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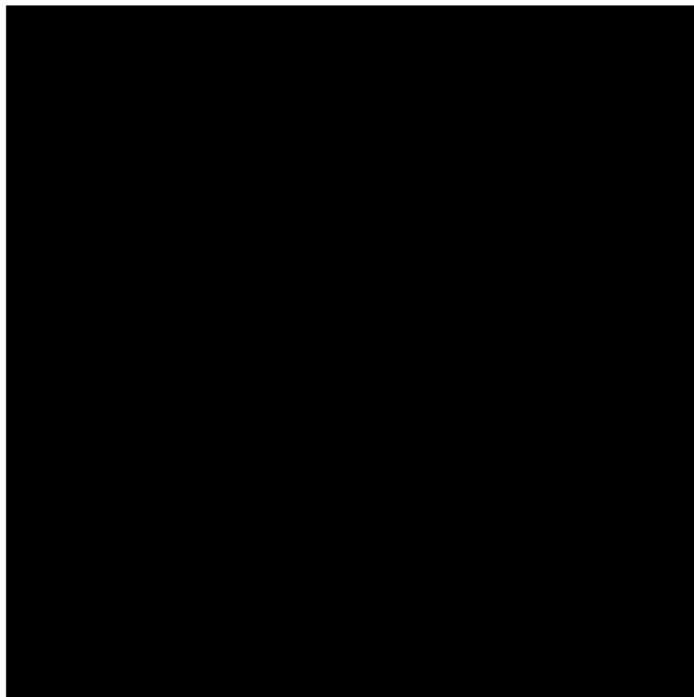
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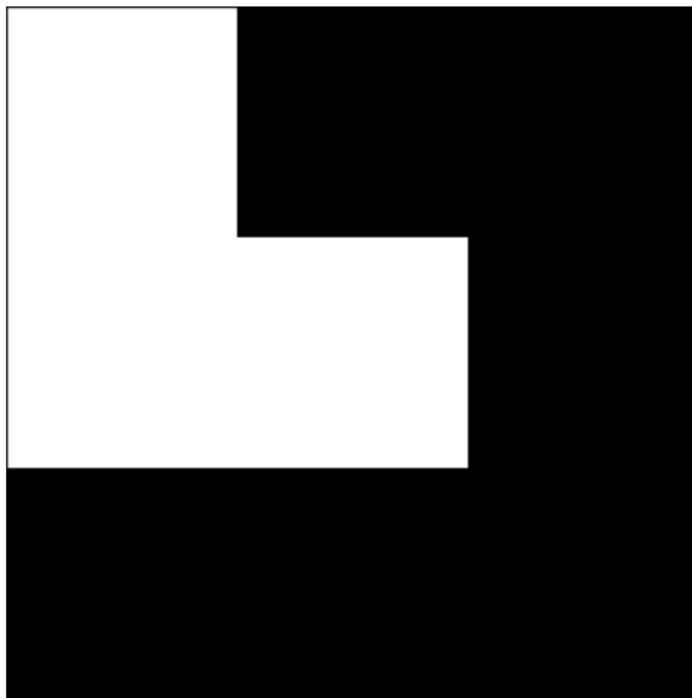
The Percolation Process

Example for $n = 3$



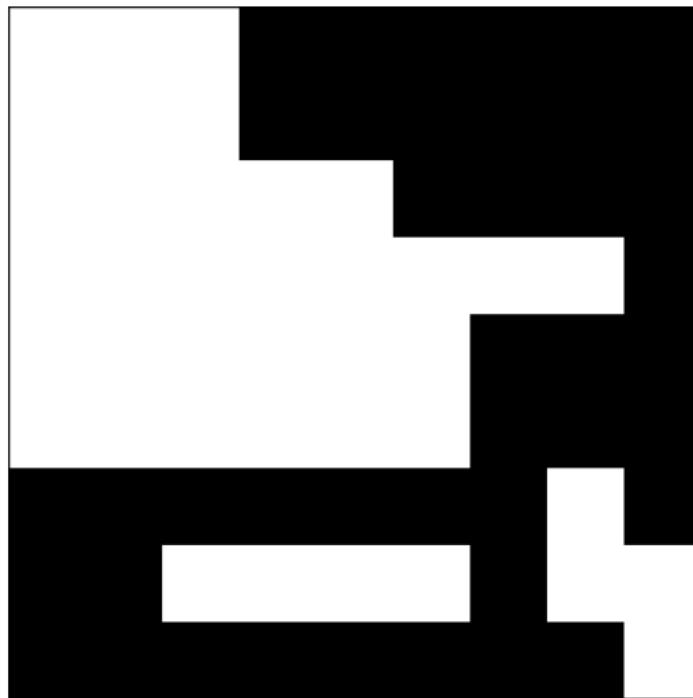
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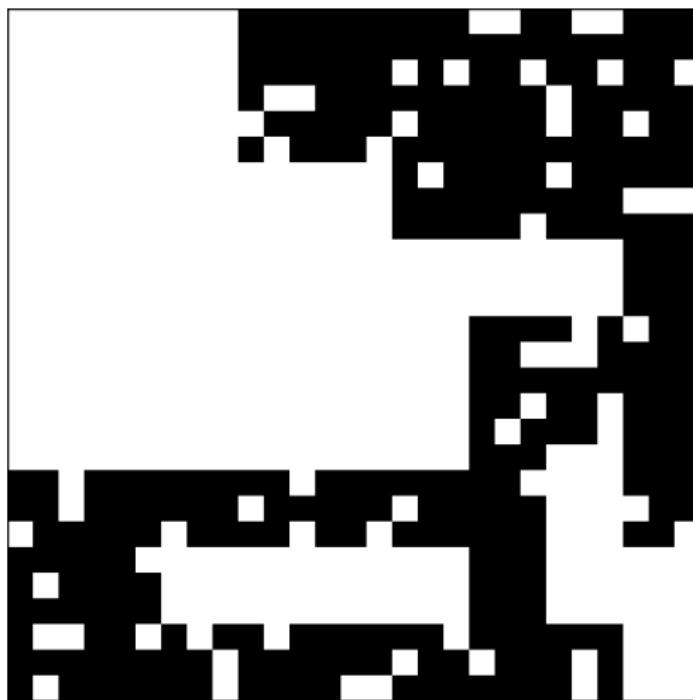
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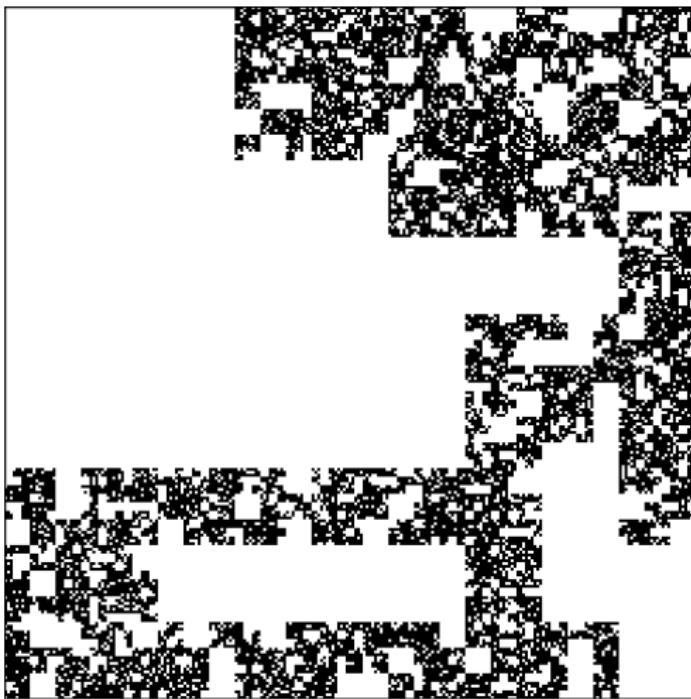
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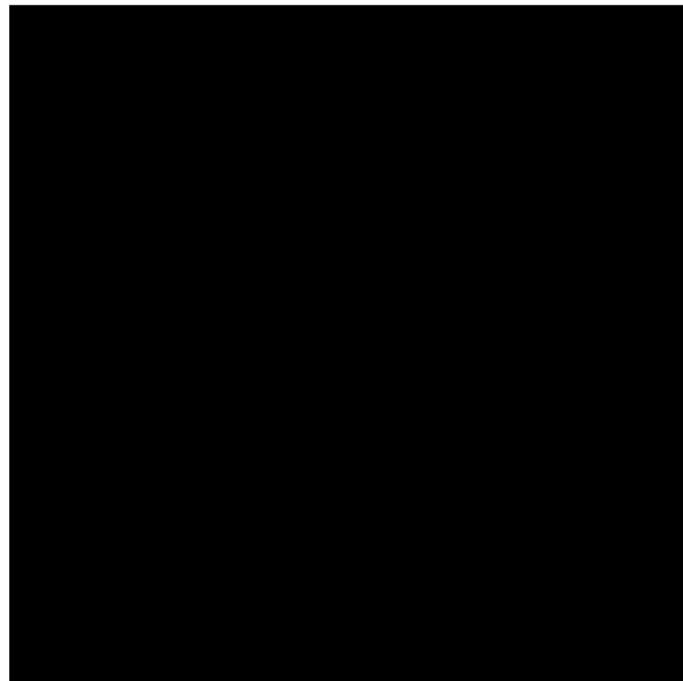
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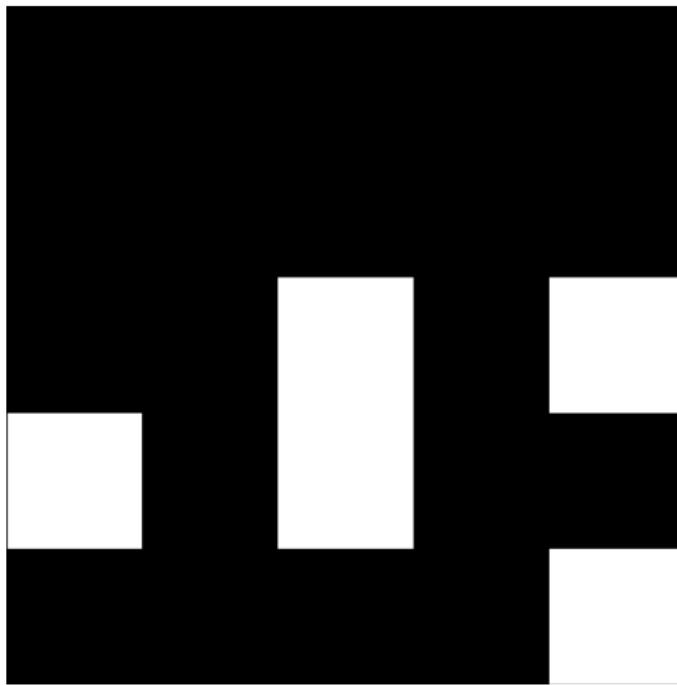
The Percolation Process

Example for $n = 5$



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Example for $n = 5$



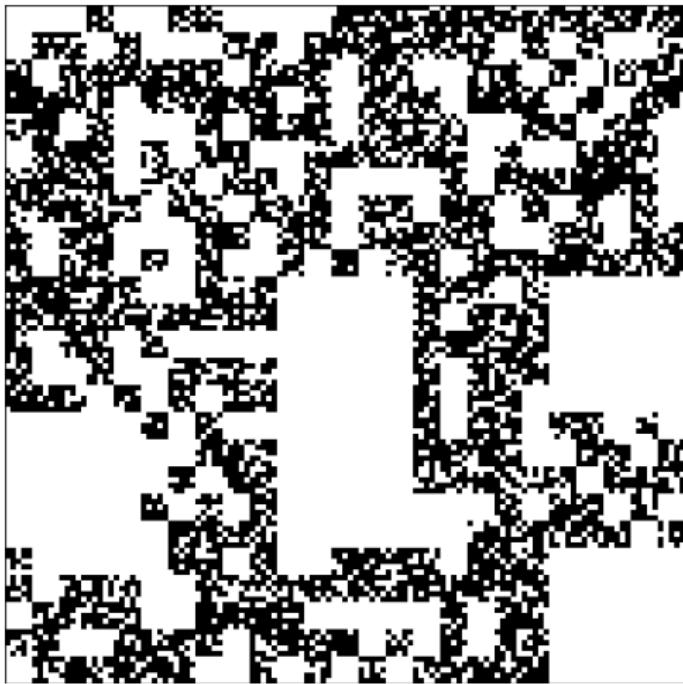
The Percolation Process

Example for $n = 5$



The Percolation Process

Example for $n = 5$



The Percolation Process

In larger dimensions

The definition extends to higher dimensions by adding components.
For computation reasons (curse of dimensionality), we only consider the 2D and 3D cases.



Dimensions

Intuition



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1D: Scale by $\lambda \iff$ Lengths multiplied by λ^1



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1D: Scale by $\lambda \iff$ Lengths multiplied by λ^1

2D: Scale by $\lambda \iff$ Areas multiplied by λ^2



Dimensions

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1D: Scale by $\lambda \iff$ Lengths multiplied by λ^1

2D: Scale by $\lambda \iff$ Areas multiplied by λ^2

3D: Scale by $\lambda \iff$ Volumes multiplied by λ^3



Dimensions

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nD: Scale by $\lambda \iff$ n-Dim. Volumes multiplies by $\lambda^n \quad \forall n \in \mathbb{N}$



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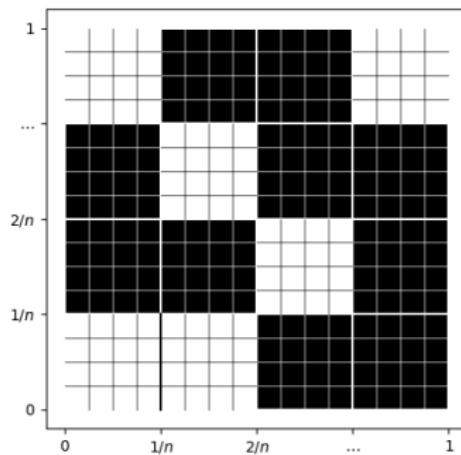
α D: Scale by $\lambda \iff$ n-Dim. Volumes multiplies by $\lambda^\alpha \quad \forall \alpha \in \mathbb{R}^+$



Dimensions

Percolation dimensions

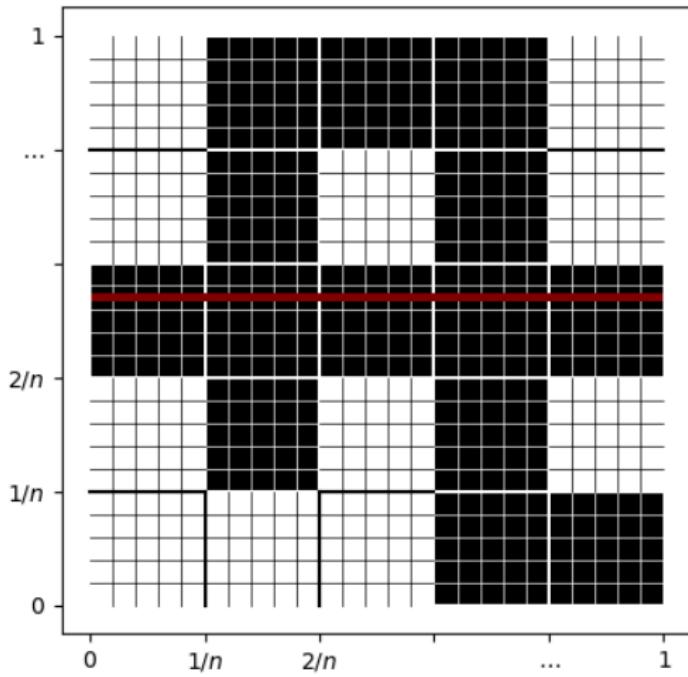
For $P \sim \text{Perc}(n, p)$, scaling by n gives pn^2 copies of P .



$$\text{So } \dim(P) = \frac{\log(pn^2)}{\log(n)} = 2 + \frac{\log(p)}{\log(n)} \leq 2.$$

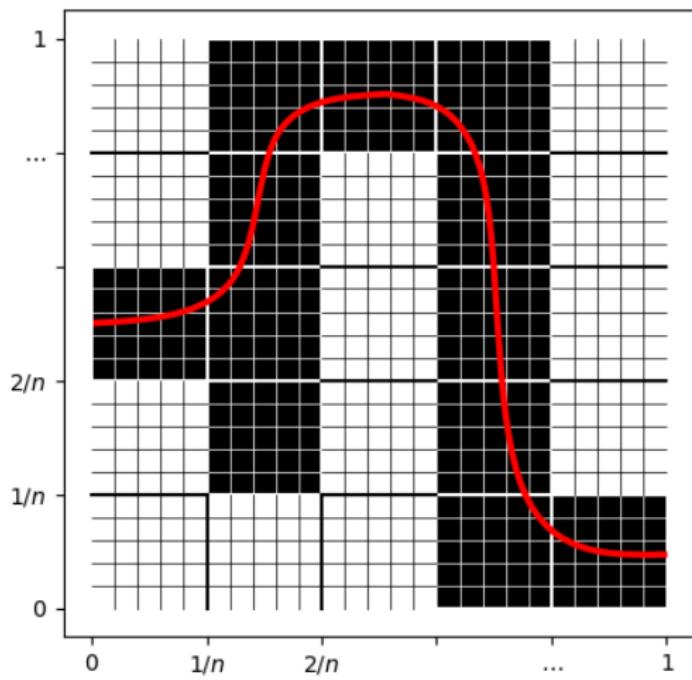
Types of Crossings

Straight



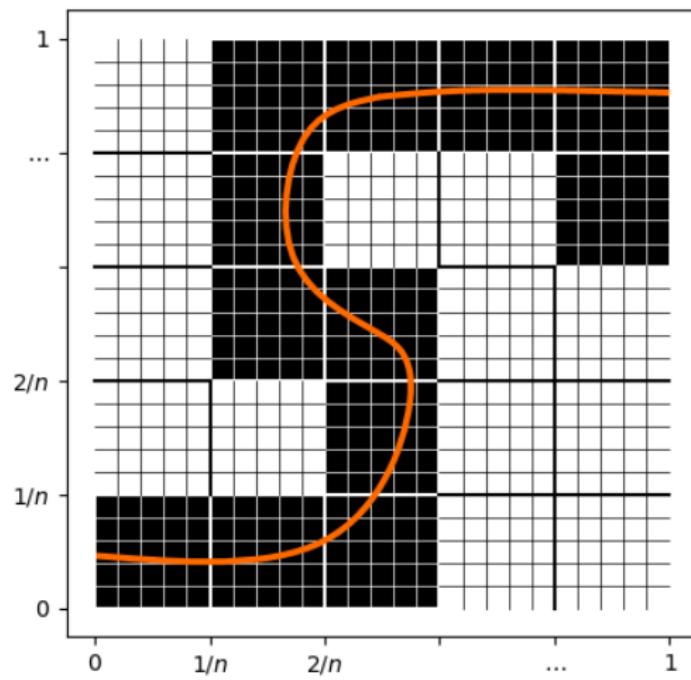
Types of Crossings

Semi-Straight



Types of Crossings

Non-Straight



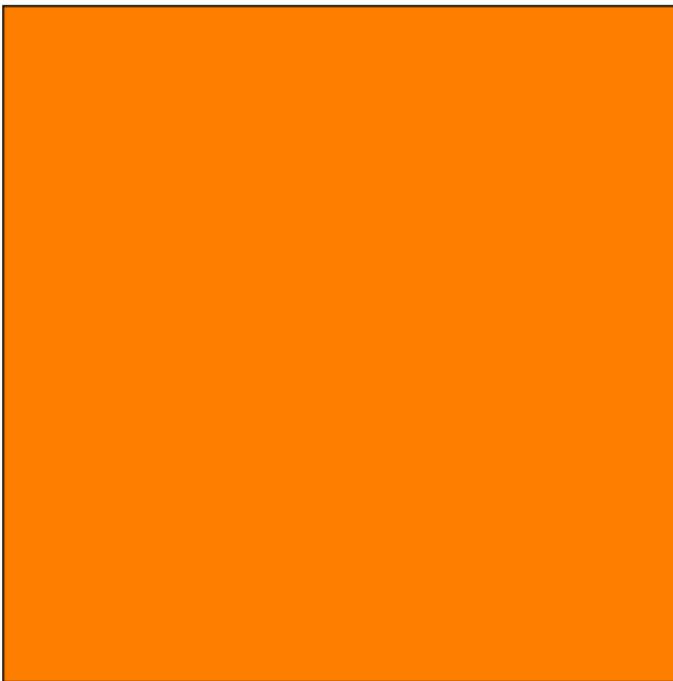
Finding Crossings

Example for $n = 2$



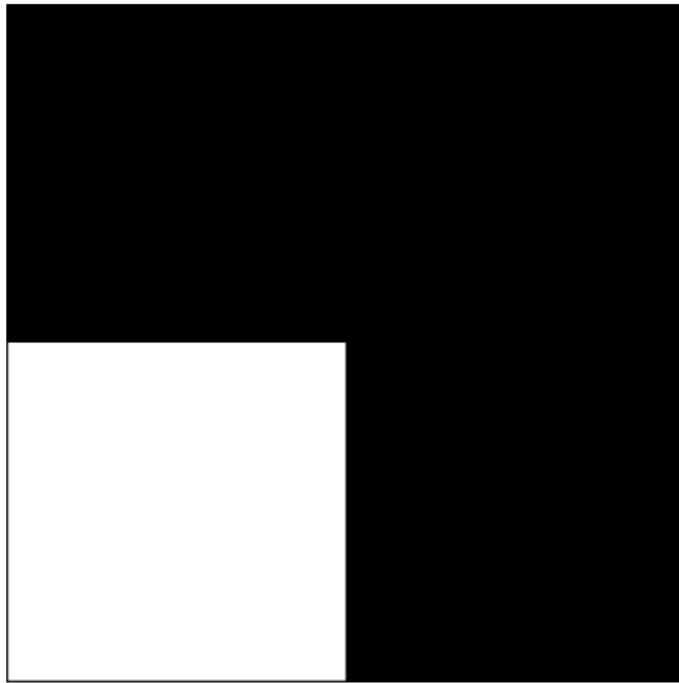
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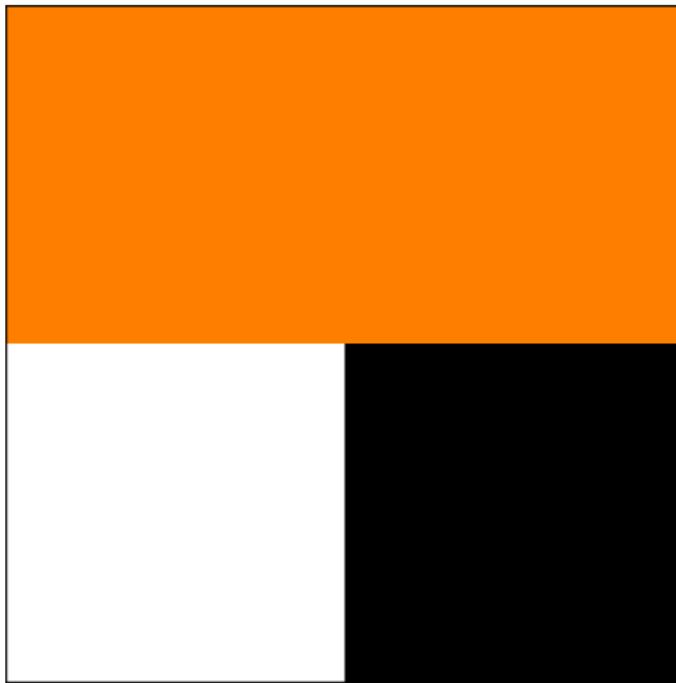
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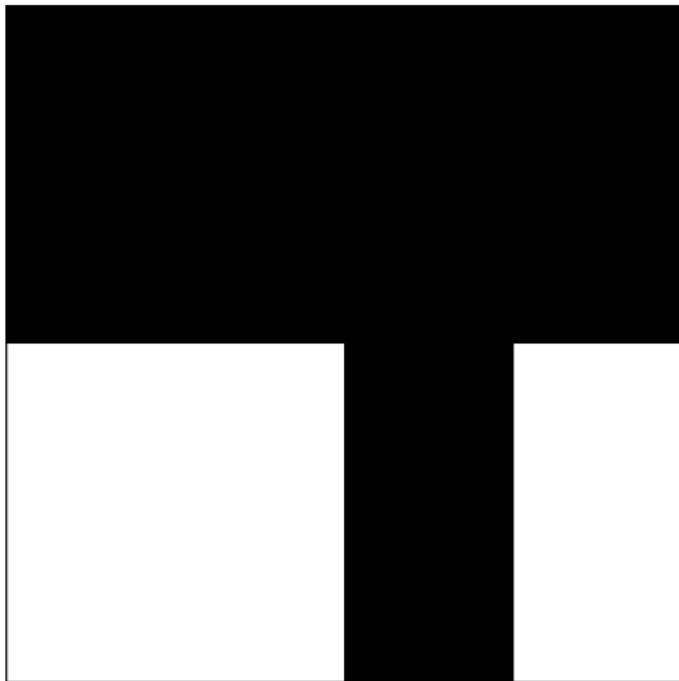
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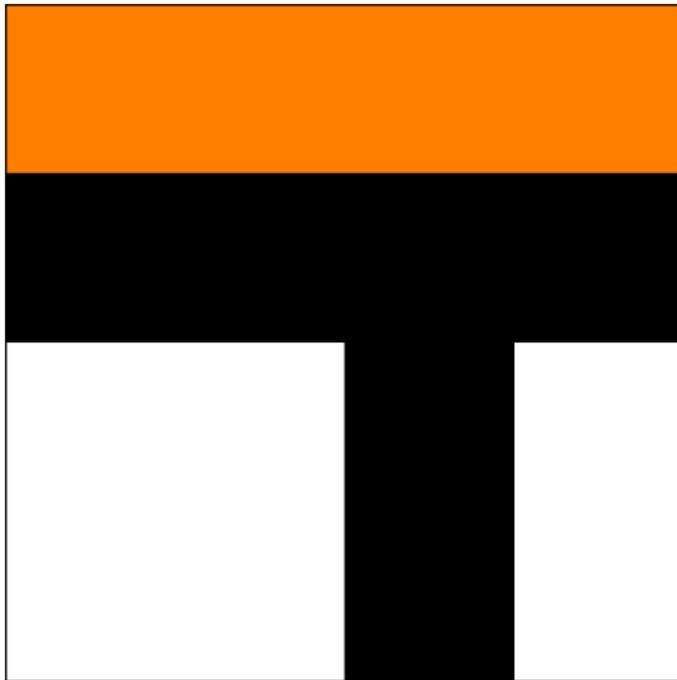
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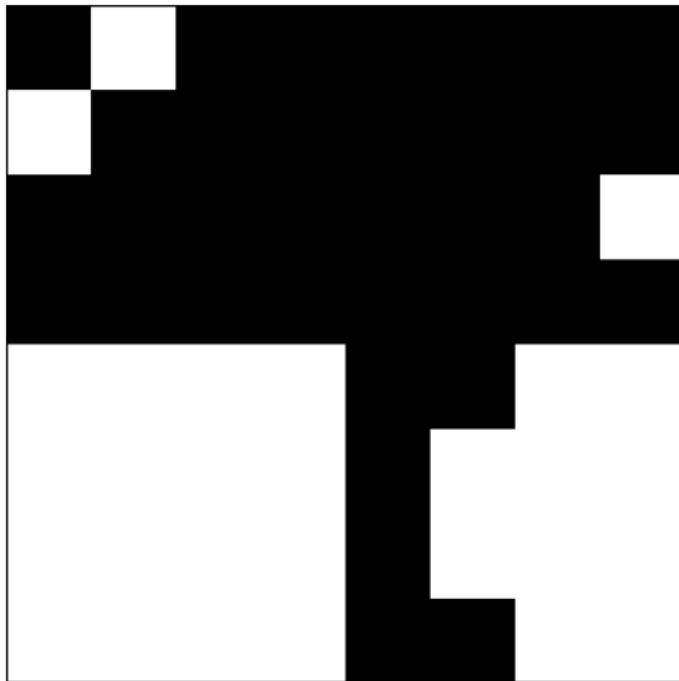
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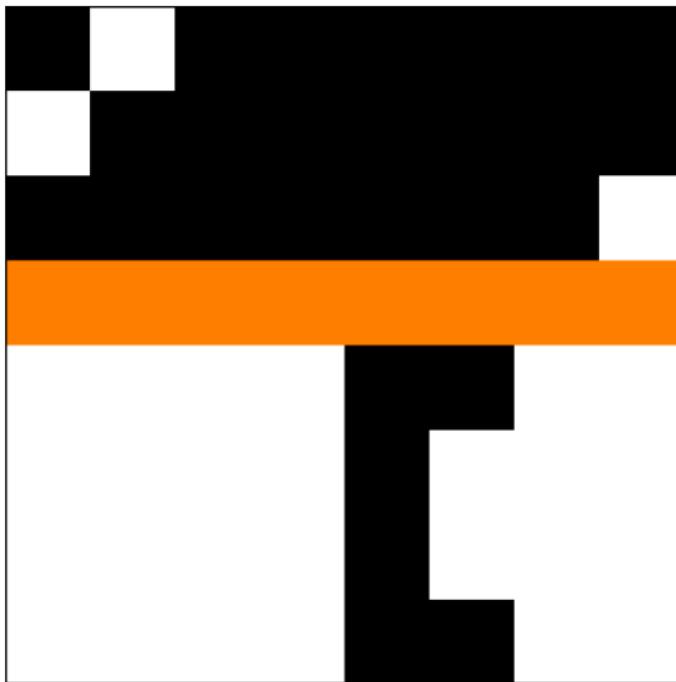
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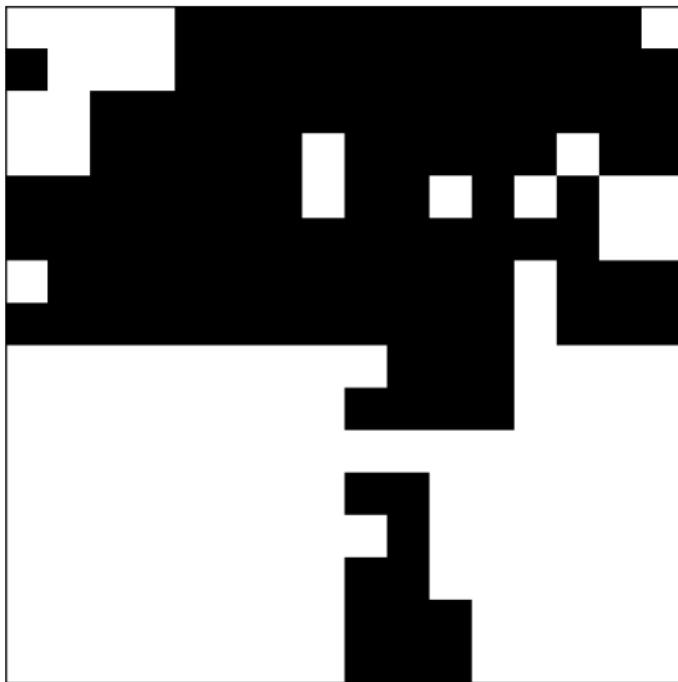
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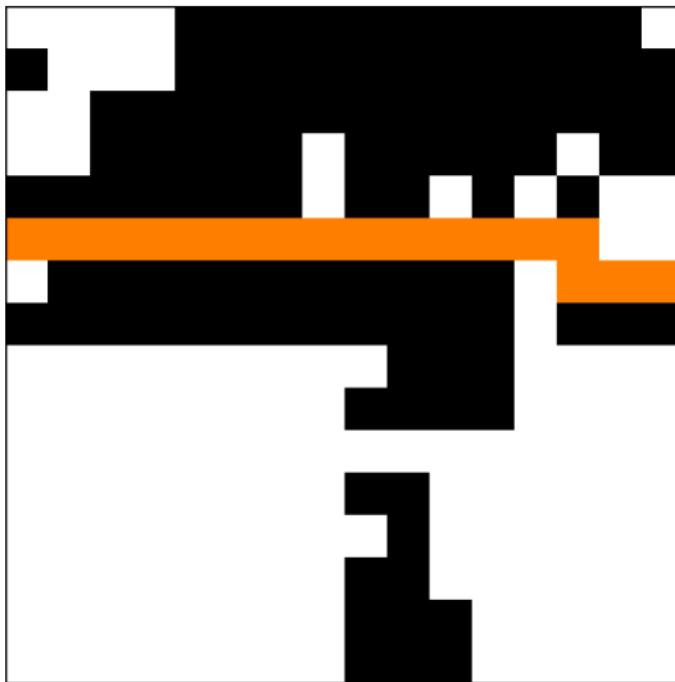
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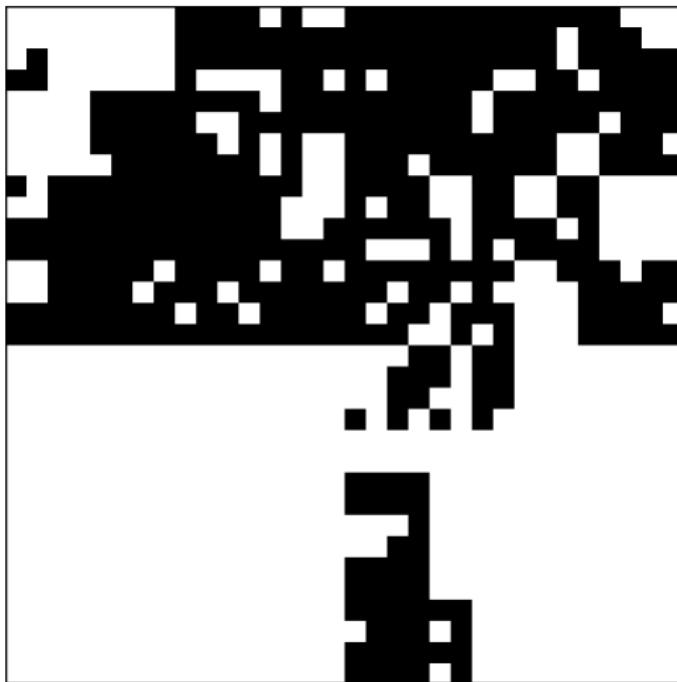
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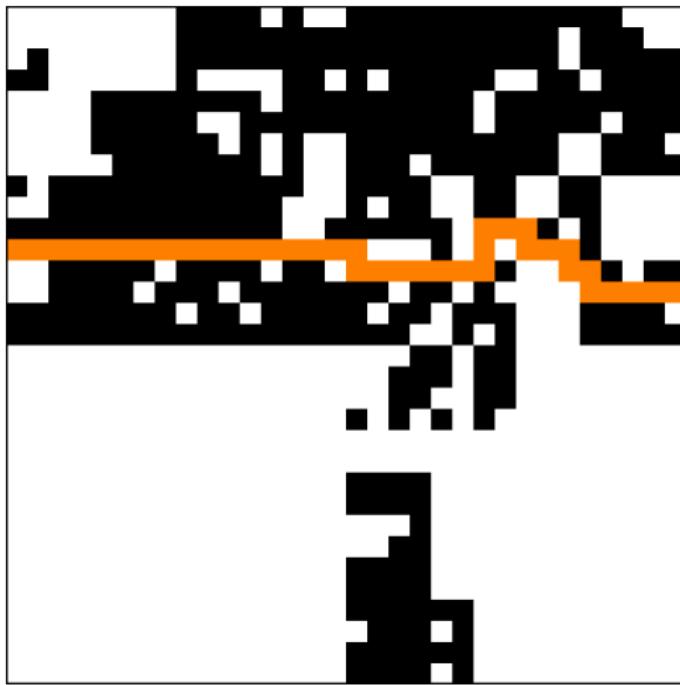
Finding Crossings

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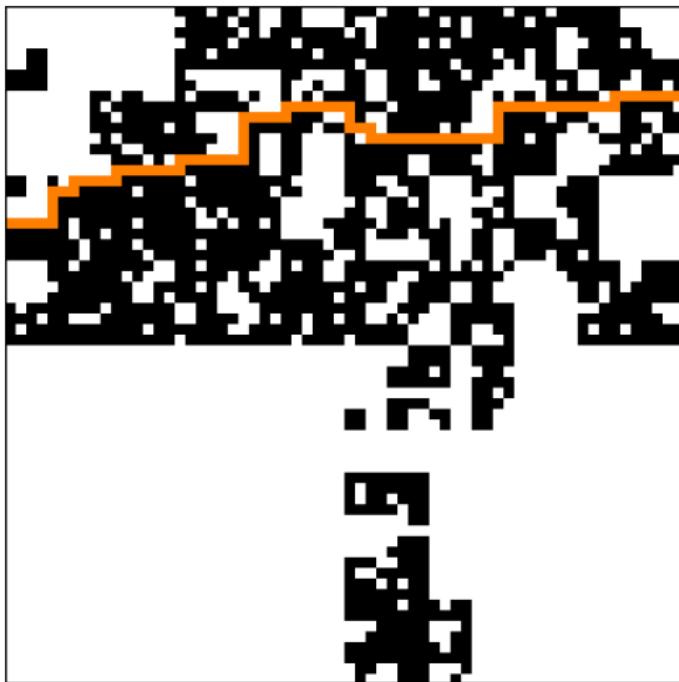
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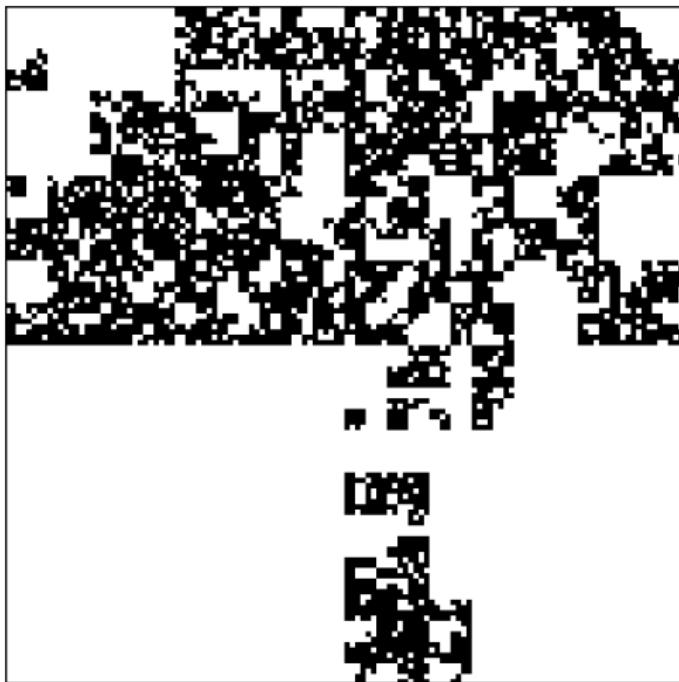
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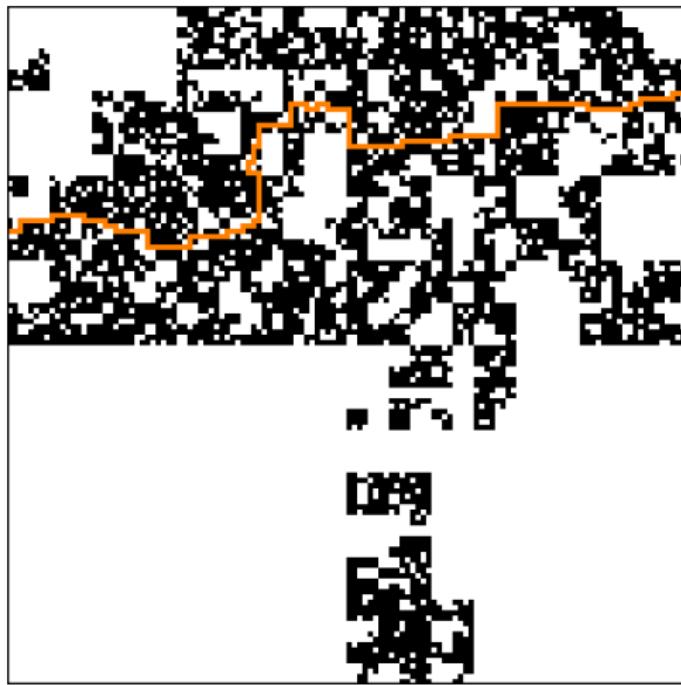
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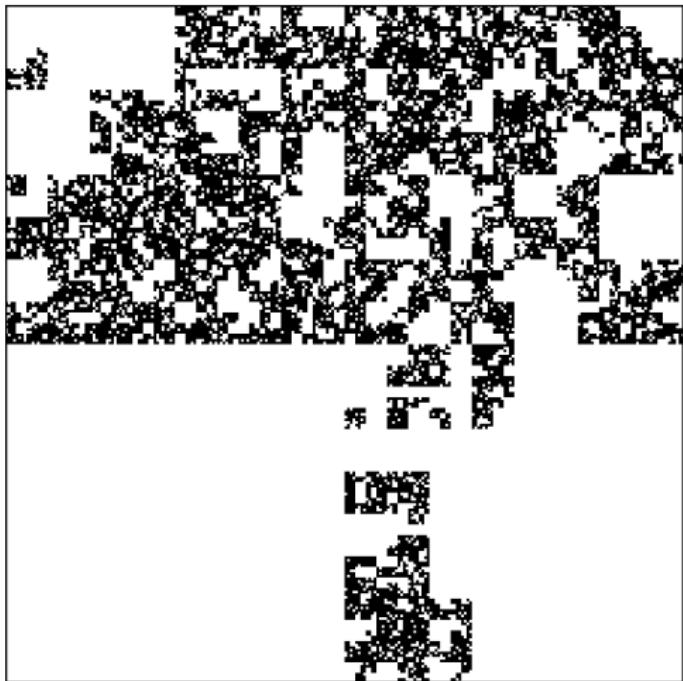
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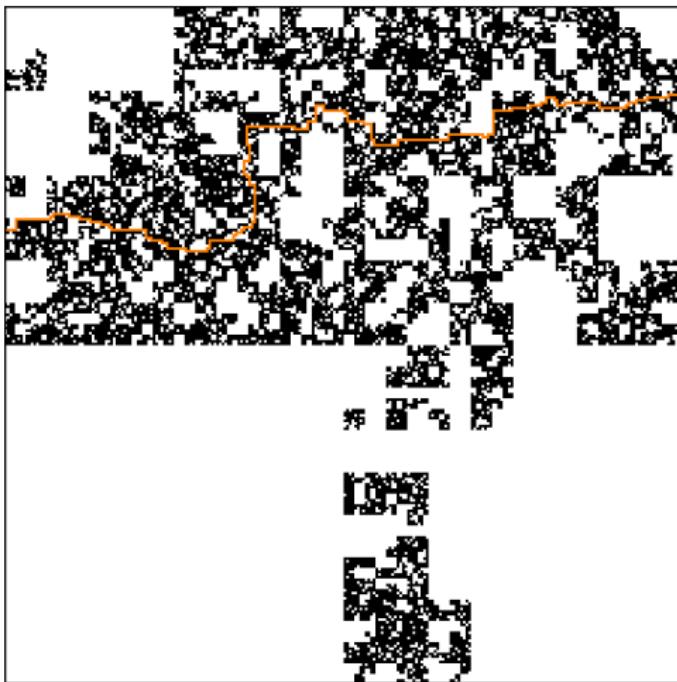
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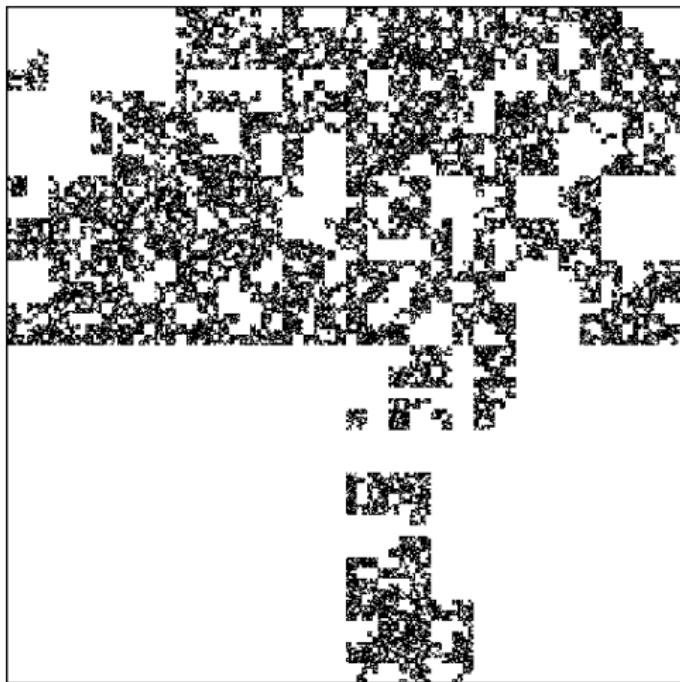
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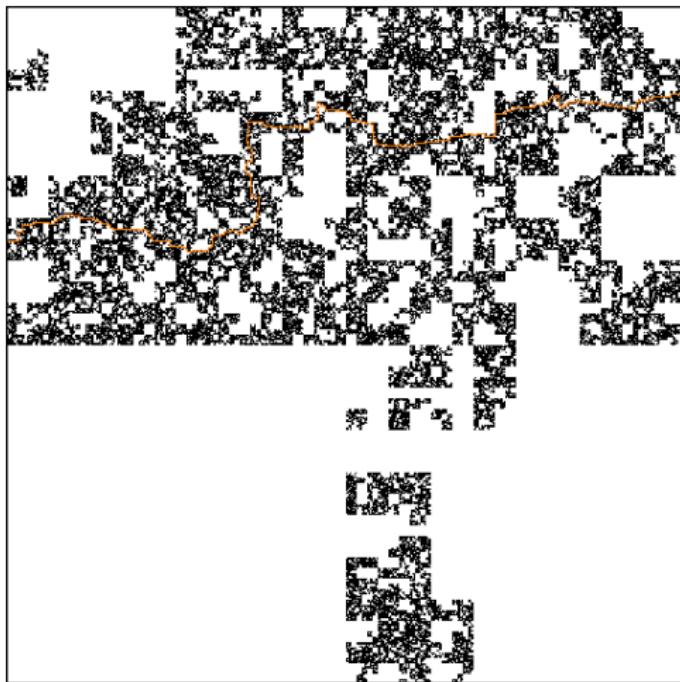
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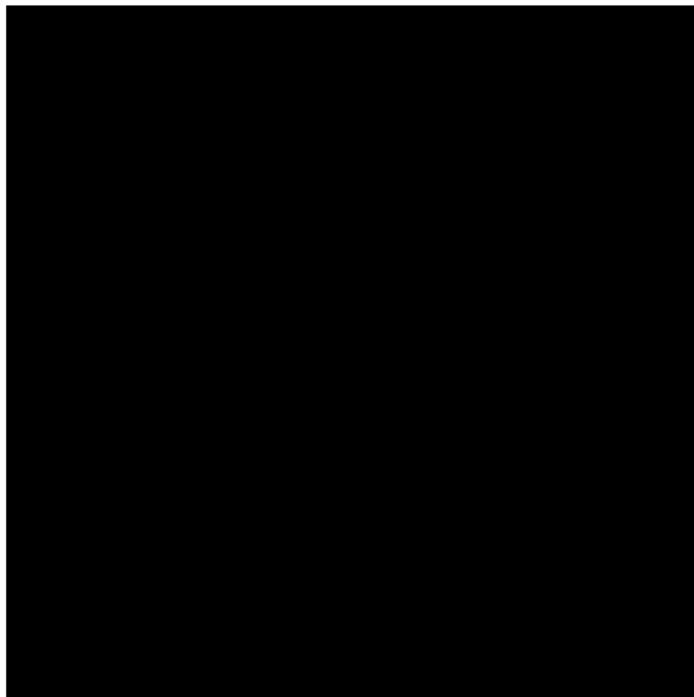
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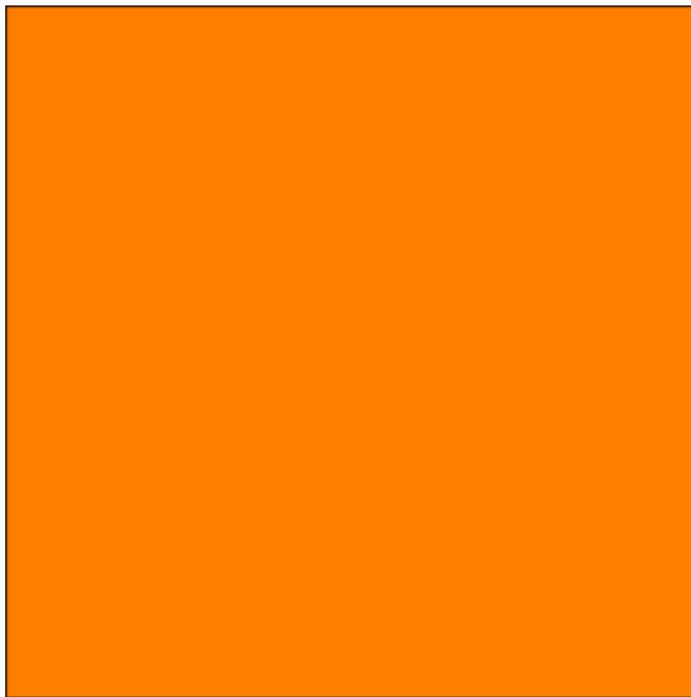
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Example for $n = 3$



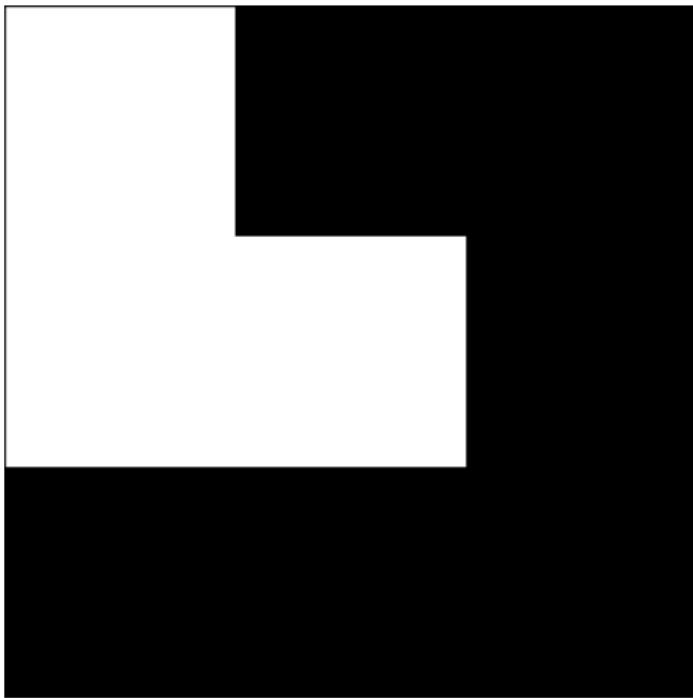
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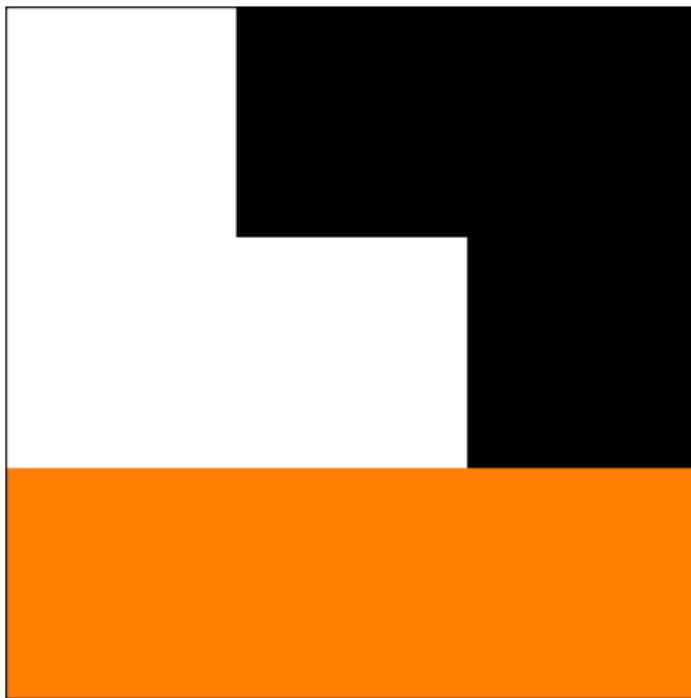
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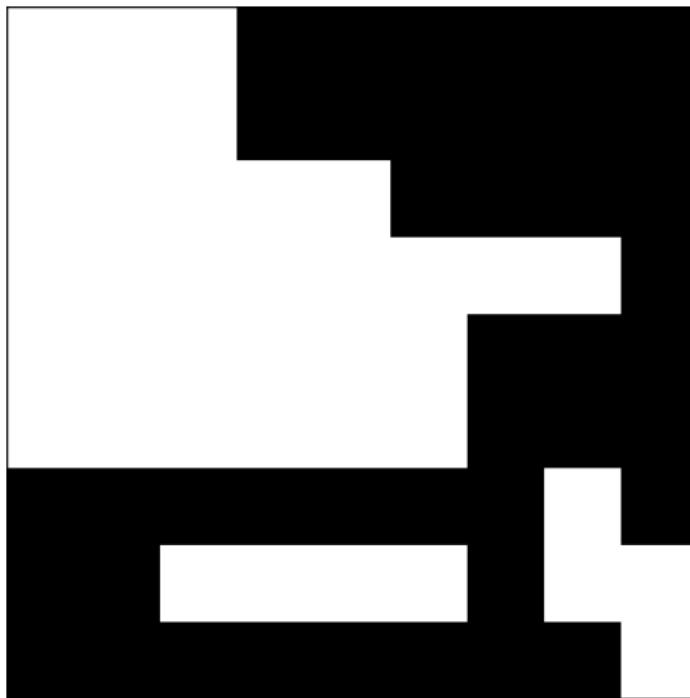
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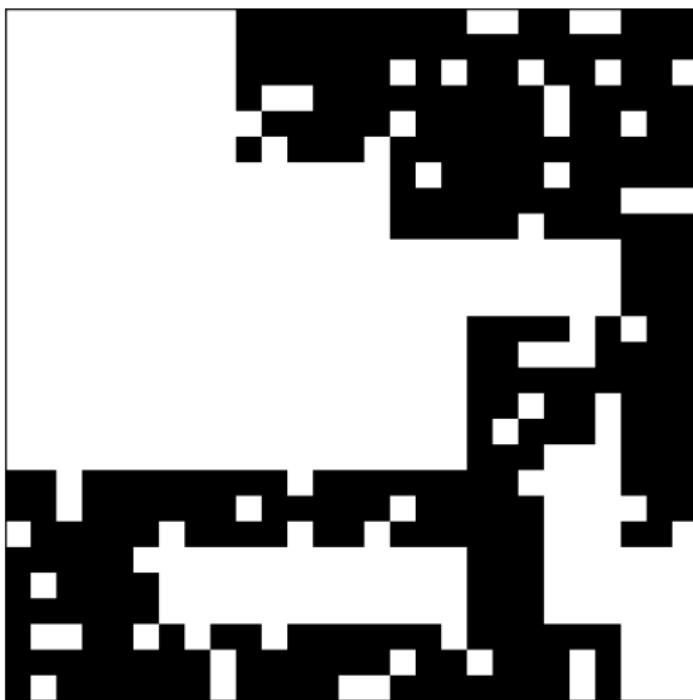
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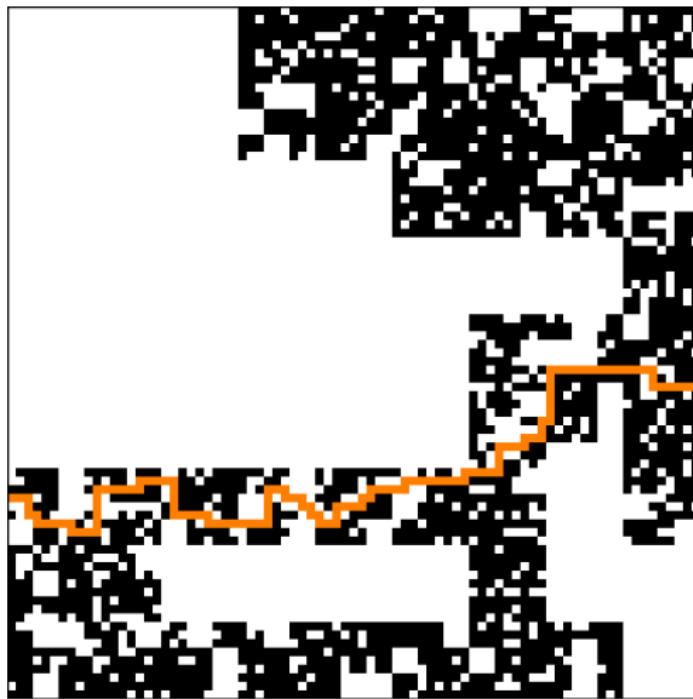
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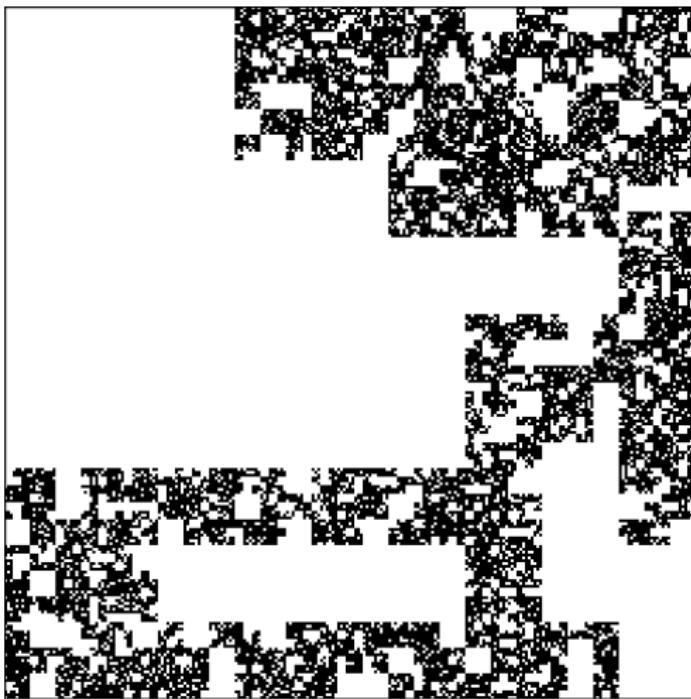
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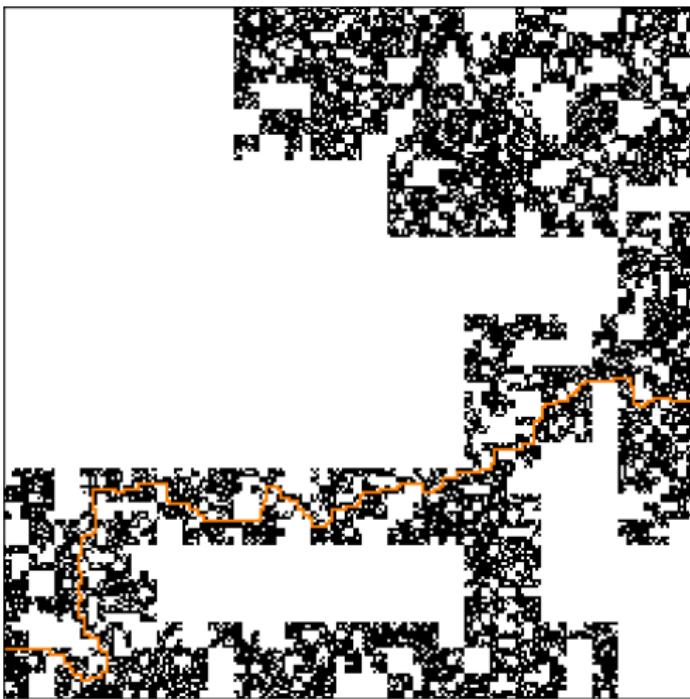
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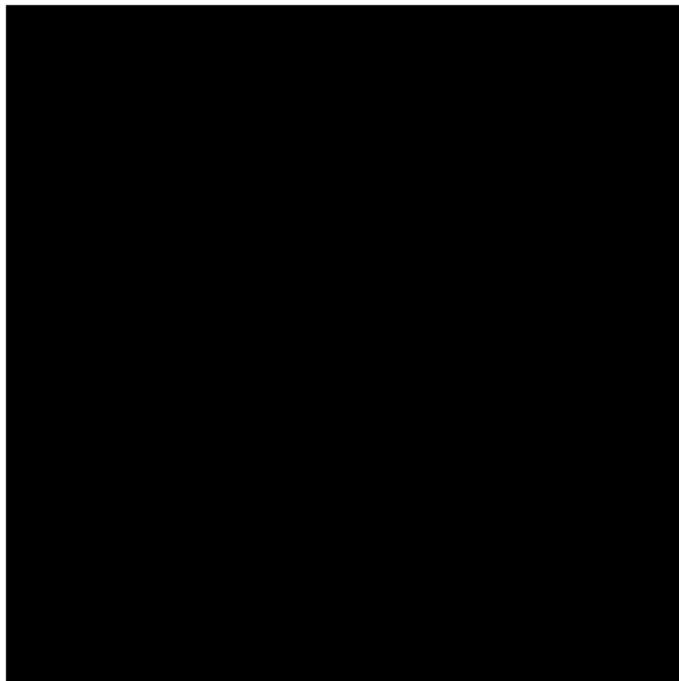
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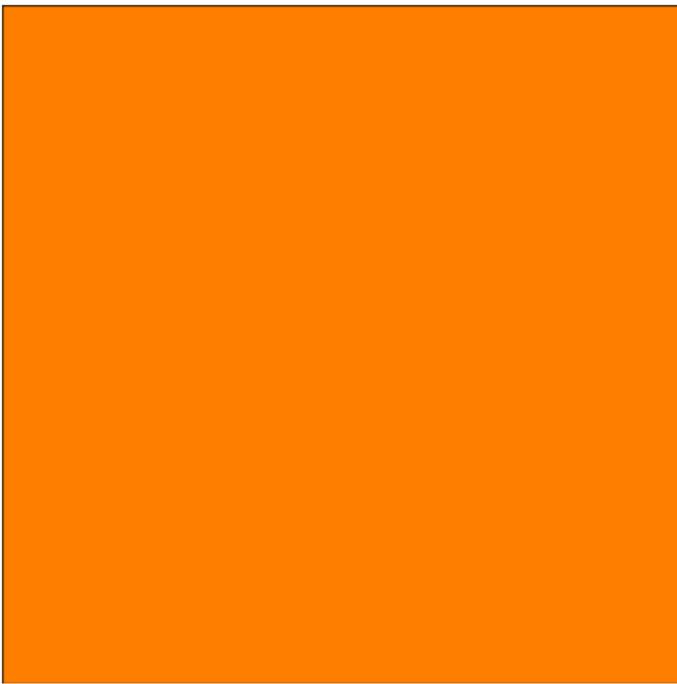
Finding Crossings

Example for $n = 5$



Finding Crossings

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Finding Crossings

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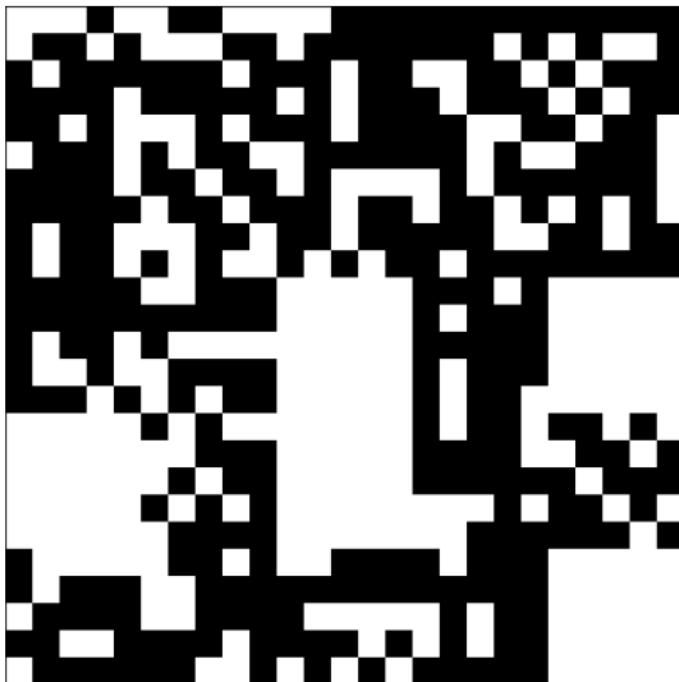
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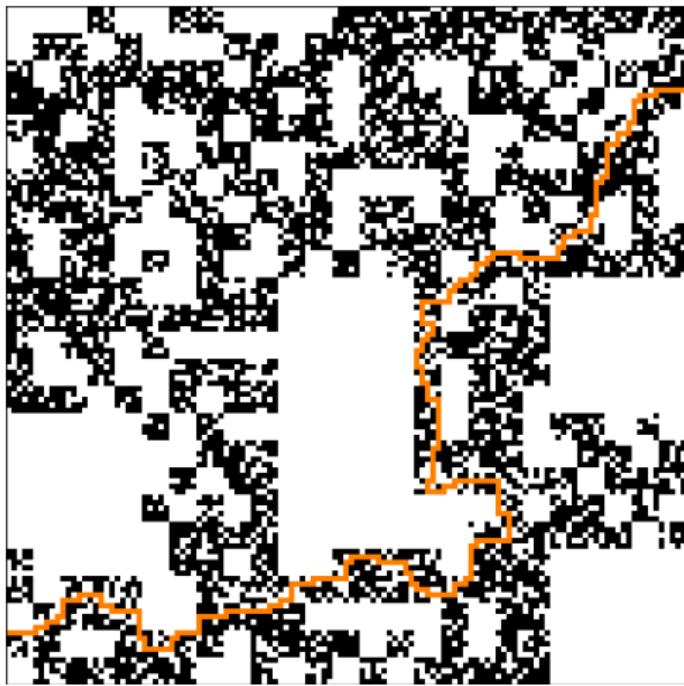
Finding Crossings

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Finding Crossings

Algorithm

- ▶ 2D Crossing Algorithm:

[https://pauldubois98.github.io/
RandomFractalAlgorithmsDemo/2Dcrossing/index.html](https://pauldubois98.github.io/RandomFractalAlgorithmsDemo/2Dcrossing/index.html)

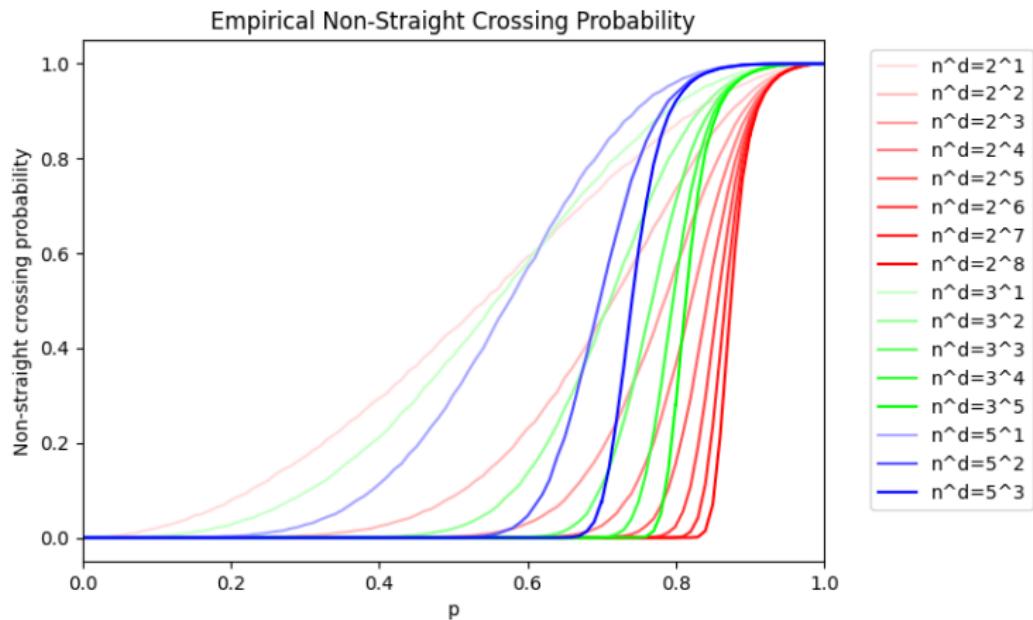
- ▶ 3D Crossing Algorithm:

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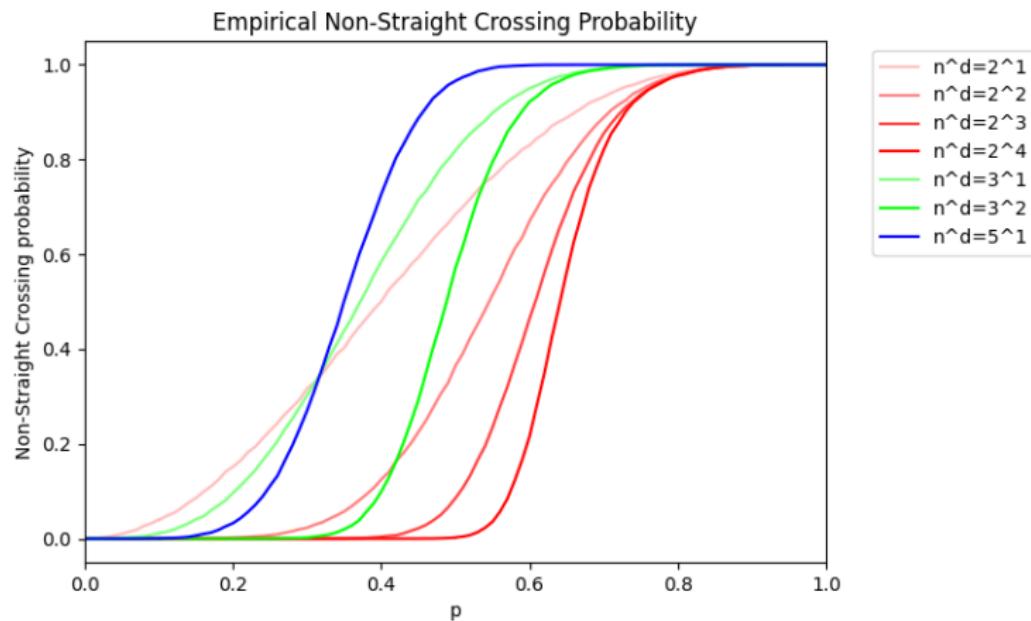
Crossings Probability

Non-Straight Crossing, Recursive Percolation, 2D



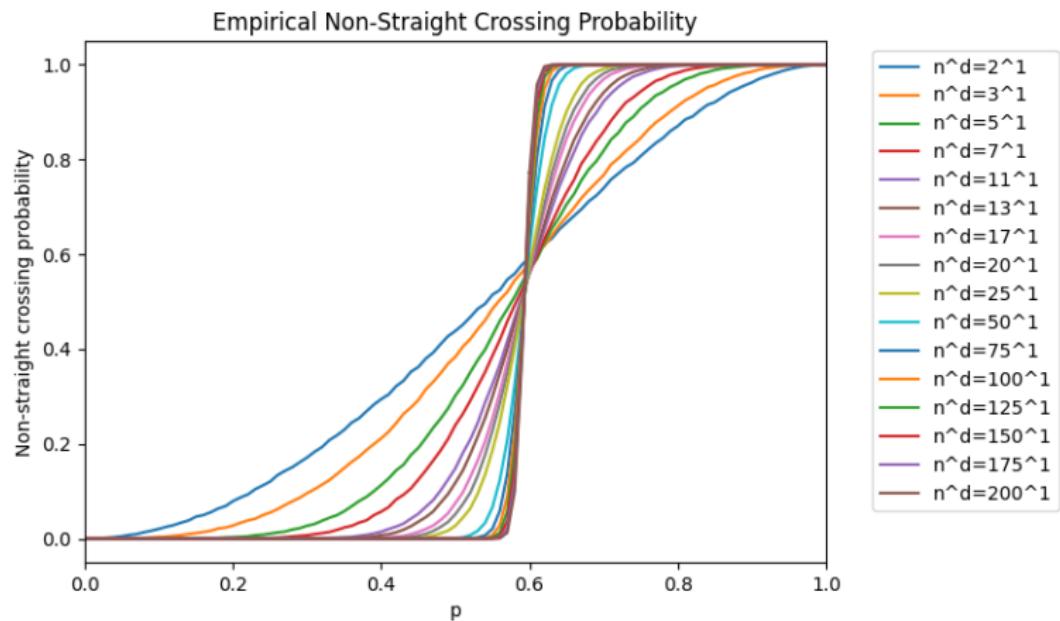
Crossings Probability

Non-Straight Crossing, Recursive Percolation, 3D



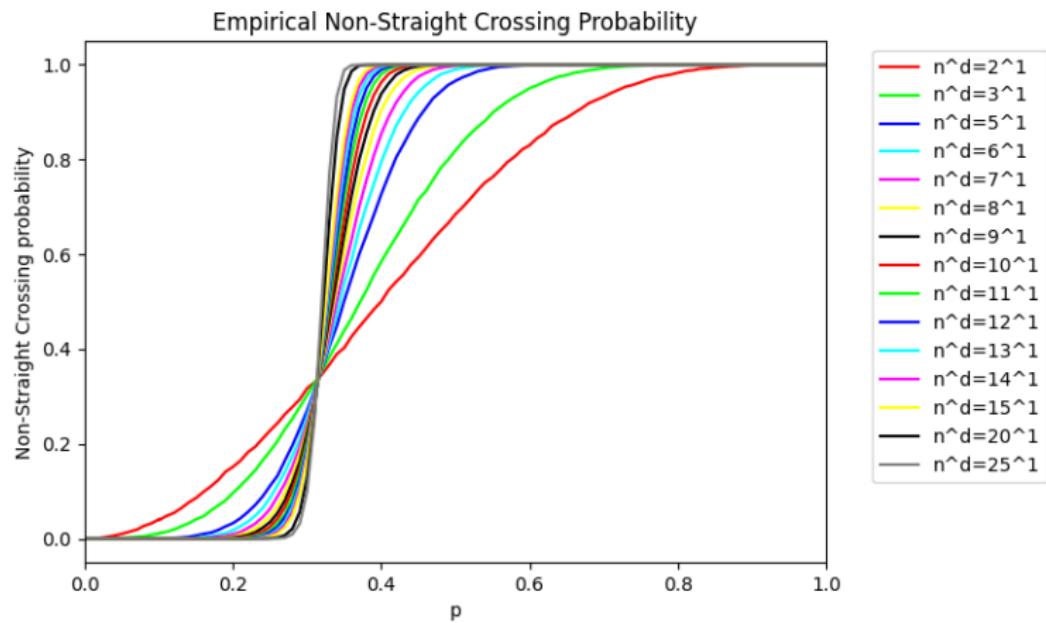
Crossings Probability

Non-Straight Crossing, Uniform Percolation, 2D



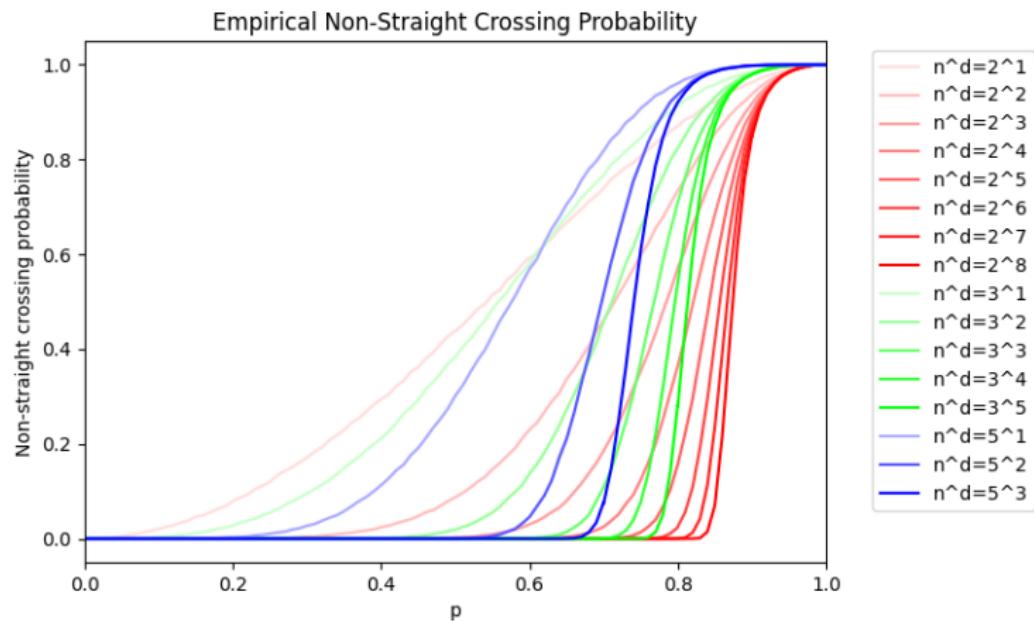
Crossings Probability

Non-Straight Crossing, Uniform Percolation, 3D



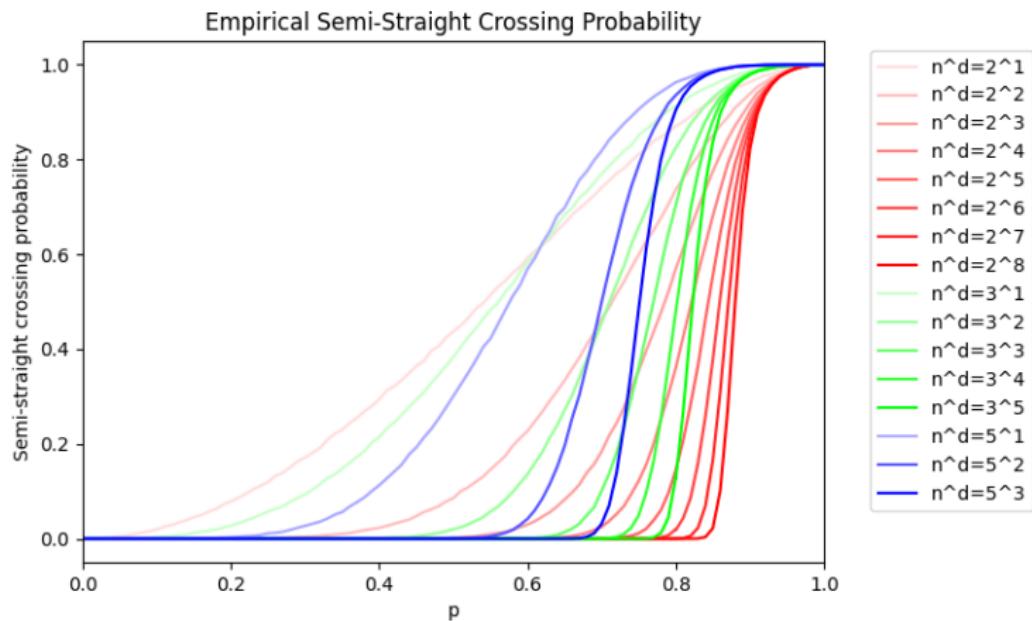
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 2D



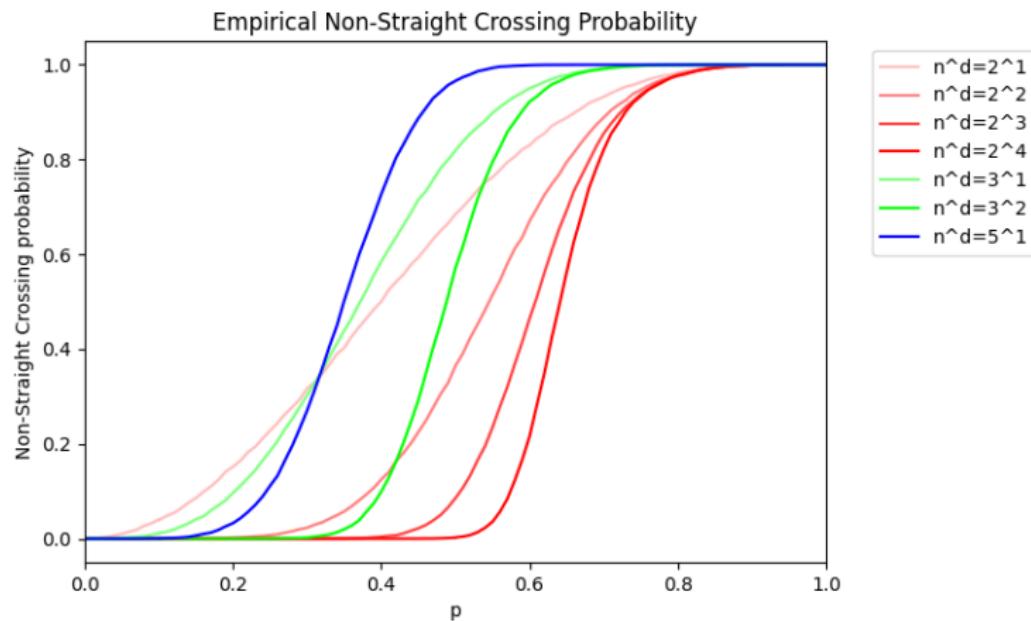
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 2D



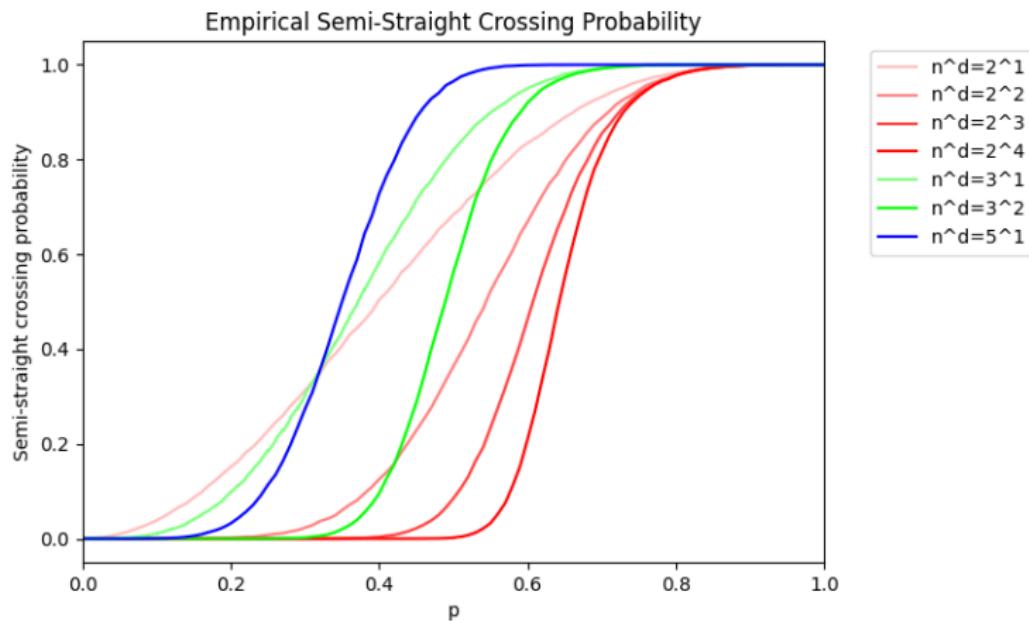
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 3D



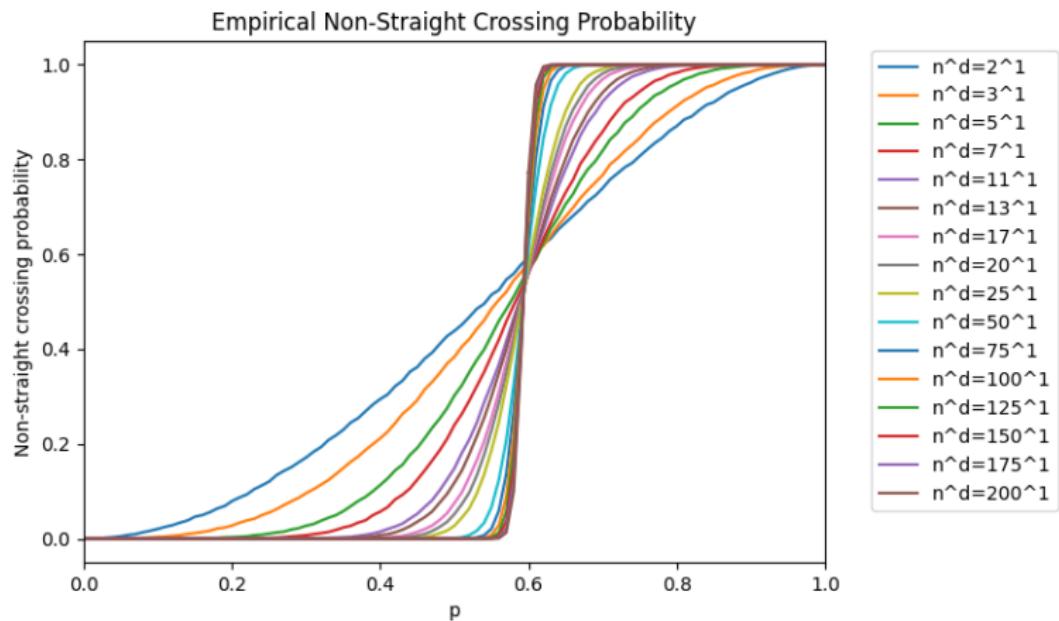
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 3D



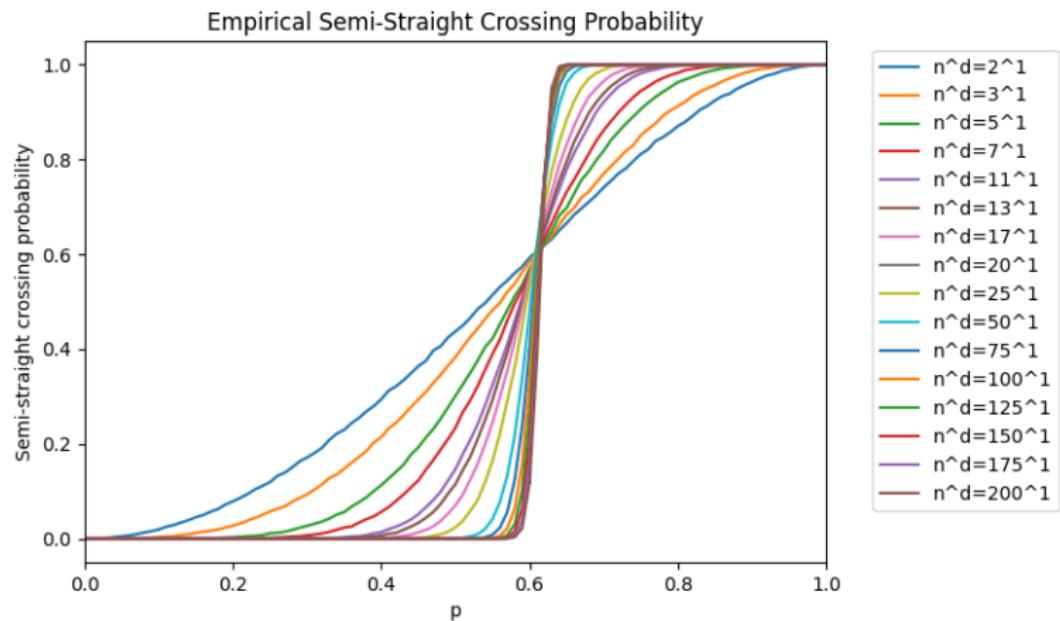
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 2D



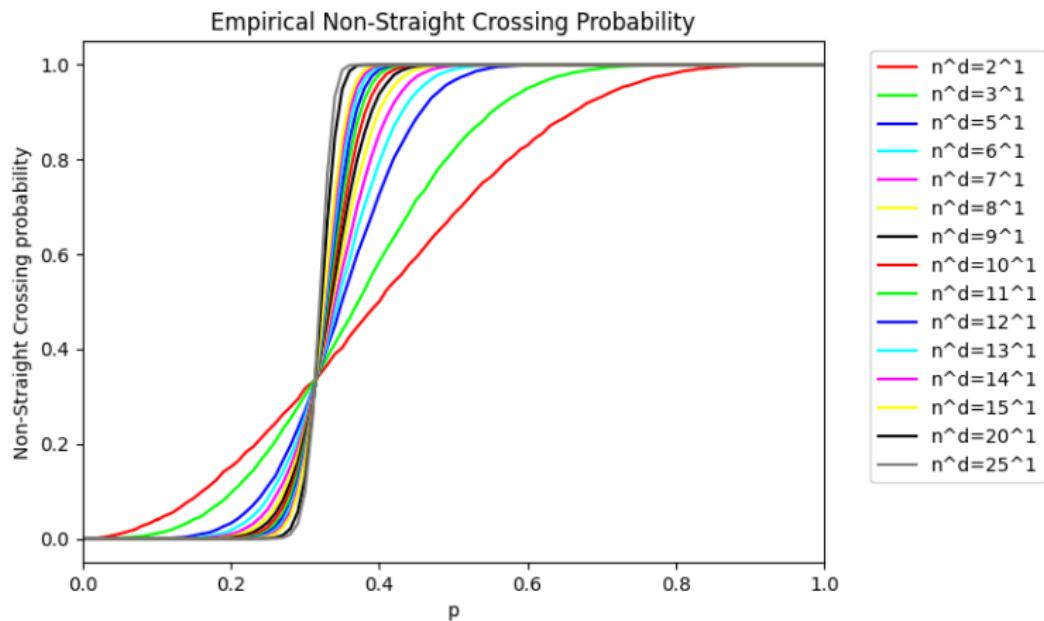
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 2D



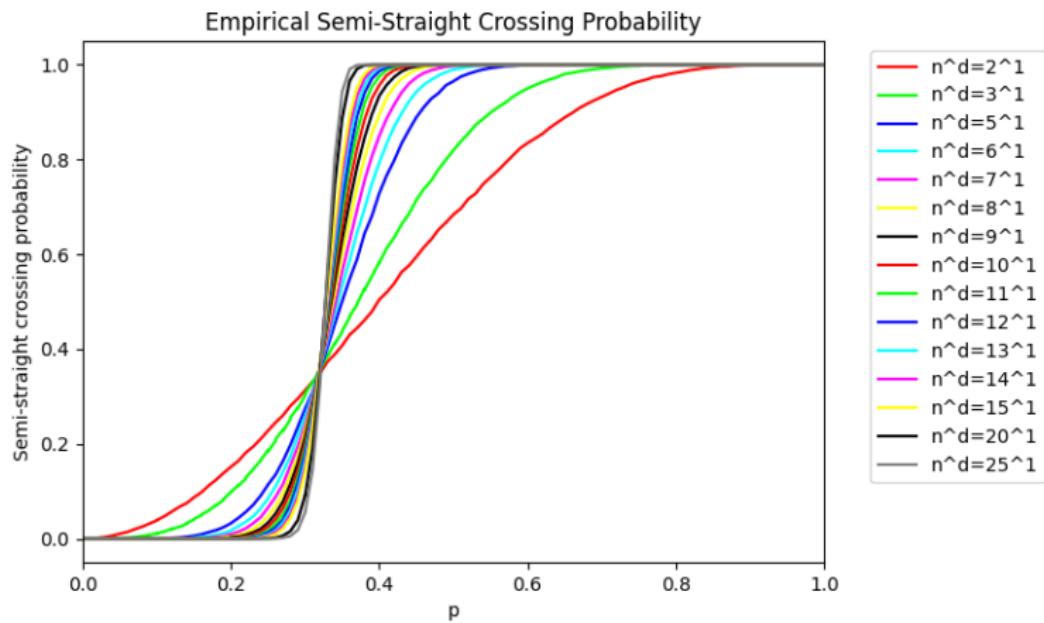
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 3D



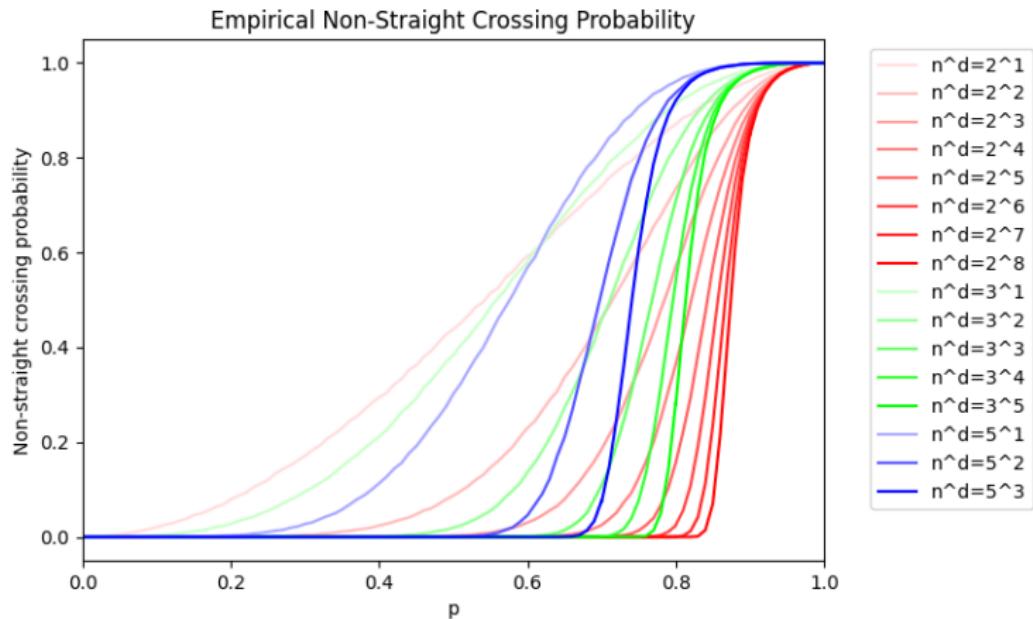
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 3D



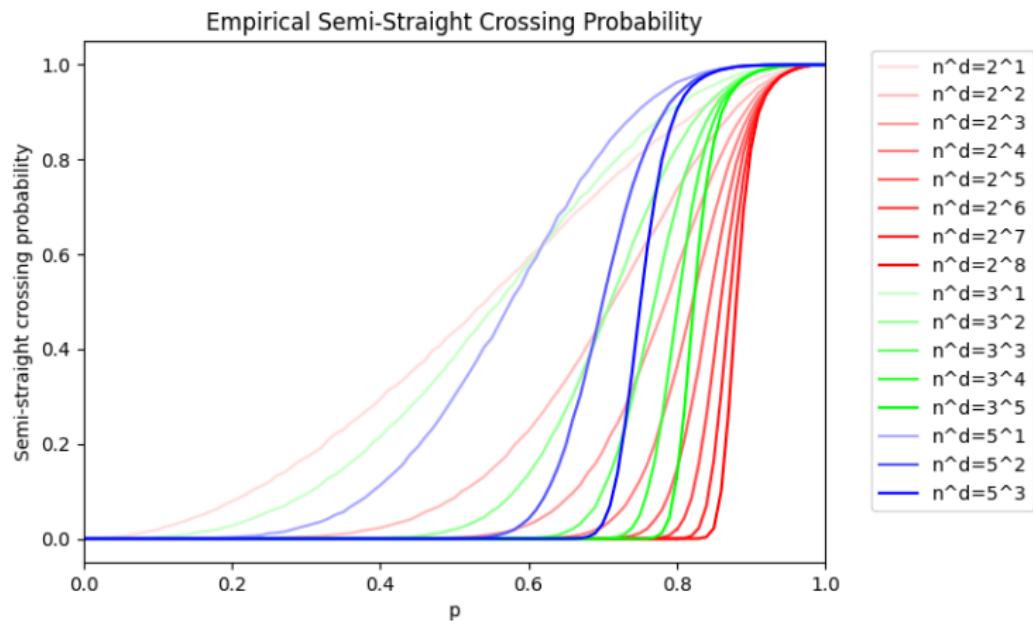
Crossings Probability

Straight Crossing, Recursive Percolation, 2D



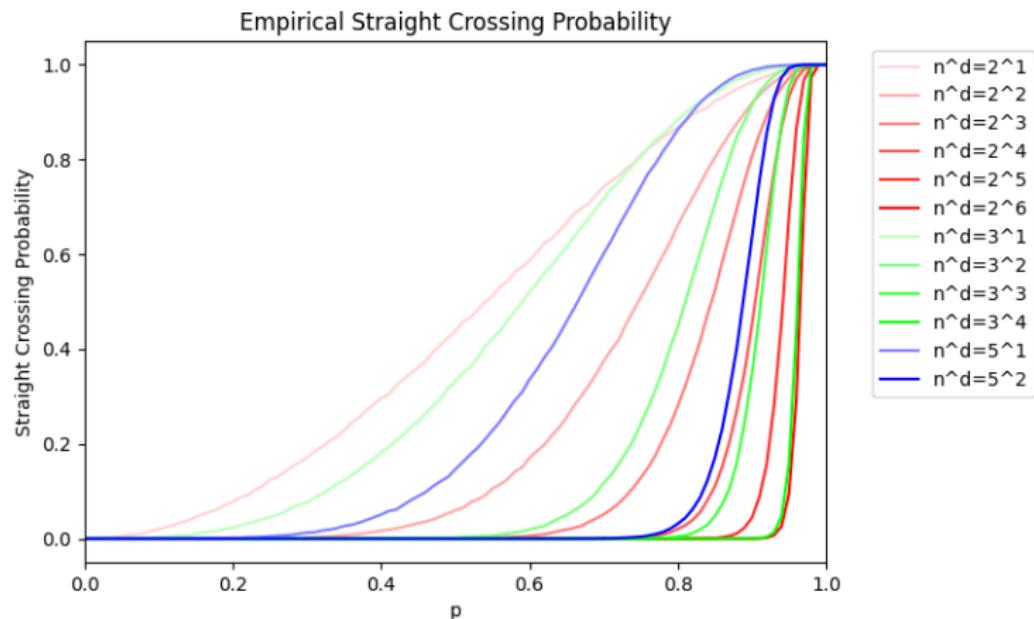
Crossings Probability

Straight Crossing, Recursive Percolation, 2D



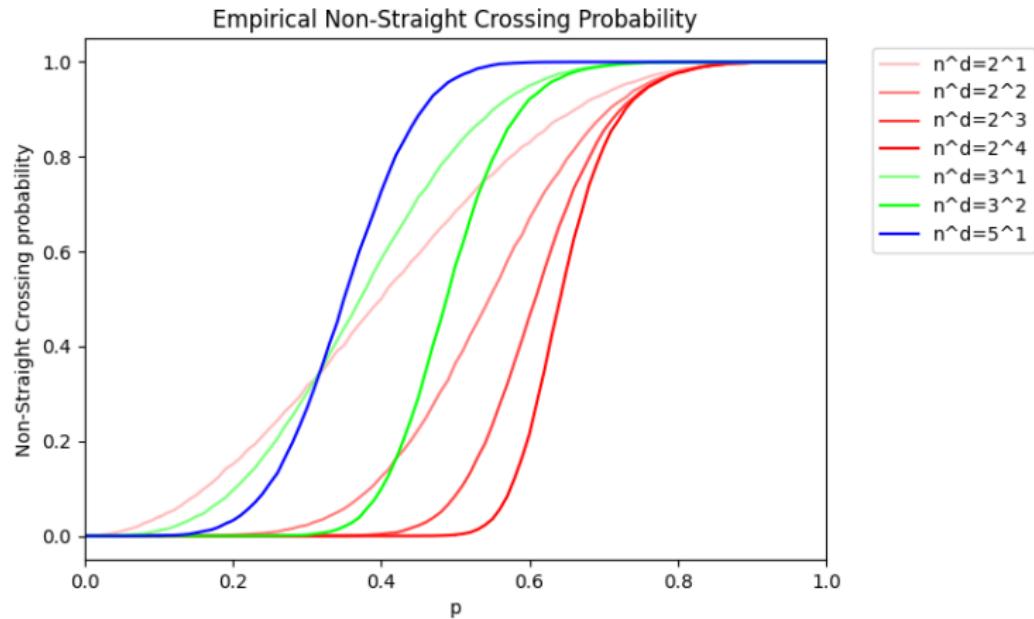
Crossings Probability

Straight Crossing, Recursive Percolation, 2D



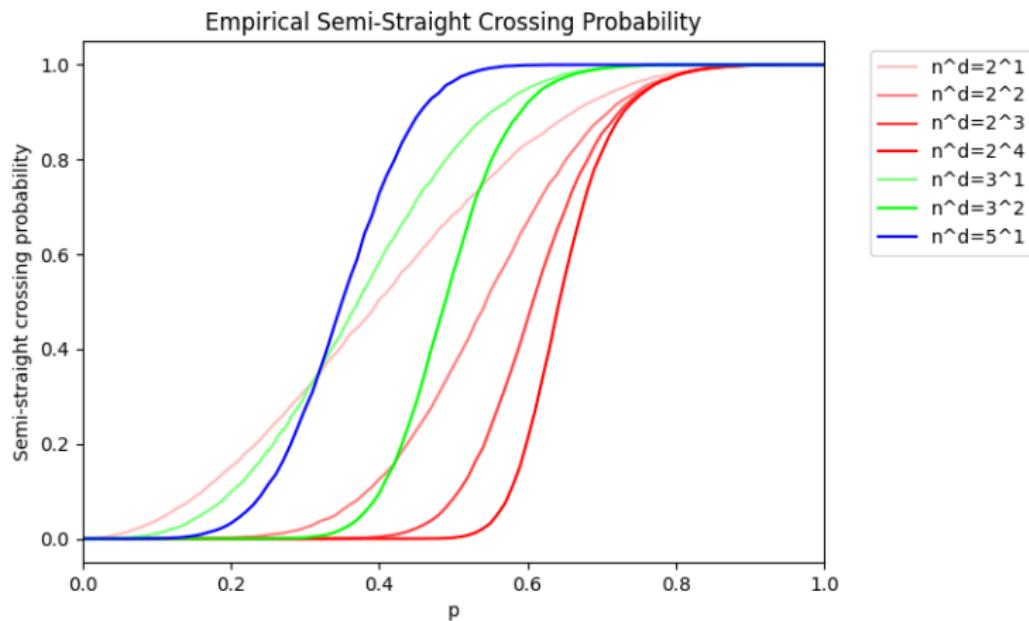
Crossings Probability

Straight Crossing, Recursive Percolation, 3D



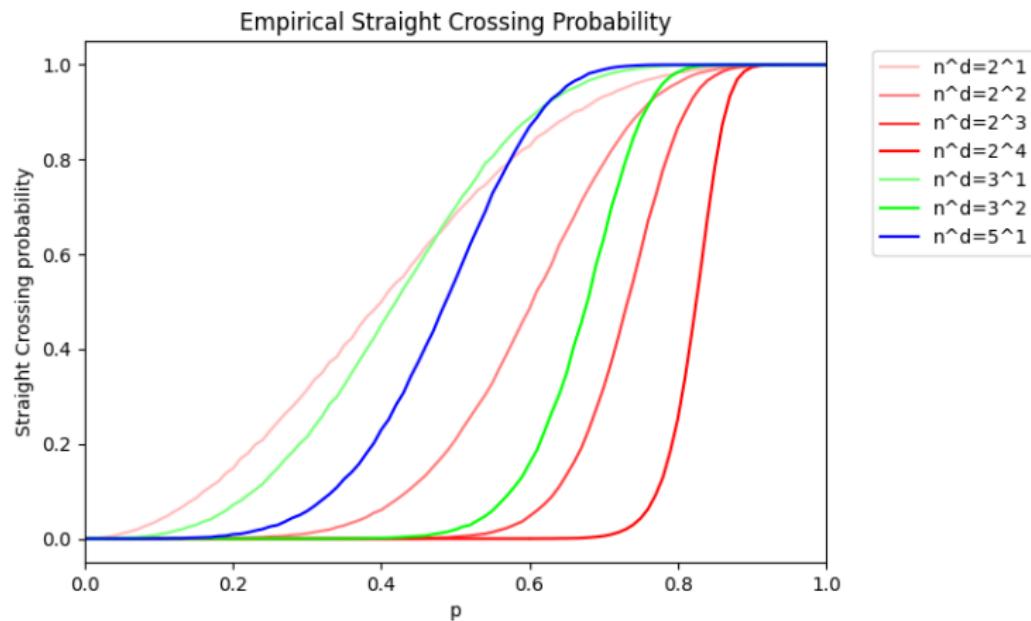
Crossings Probability

Straight Crossing, Recursive Percolation, 3D



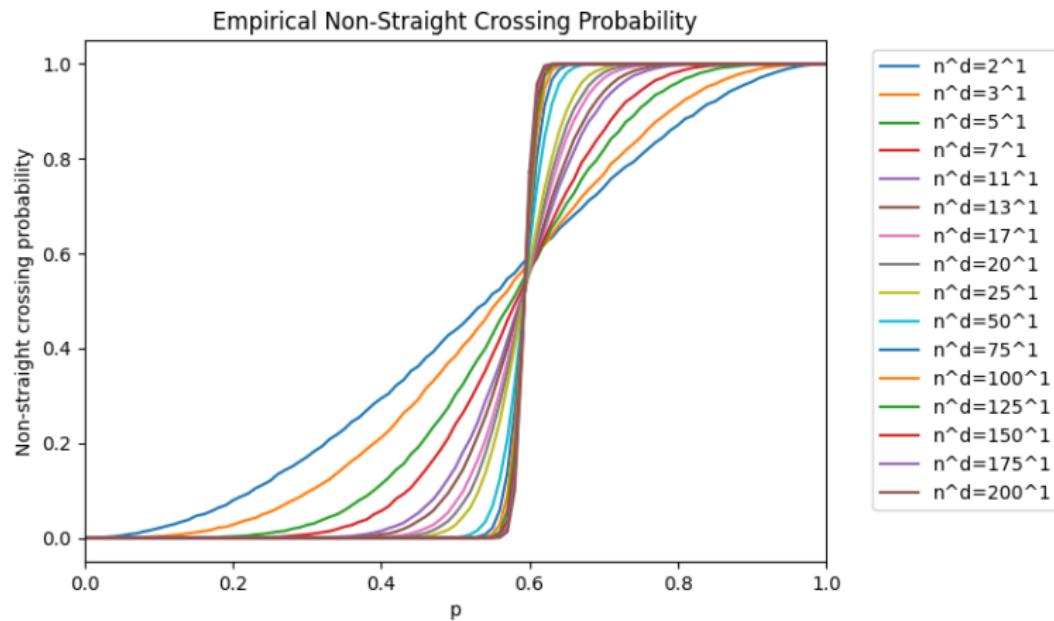
Crossings Probability

Straight Crossing, Recursive Percolation, 3D



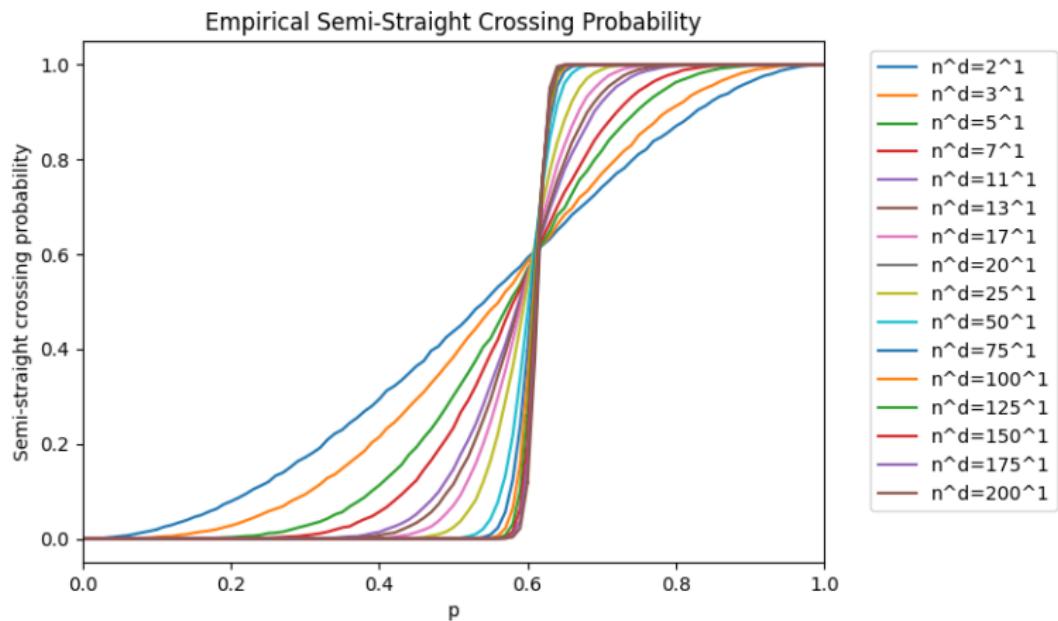
Crossings Probability

Straight Crossing, Uniform Percolation, 2D



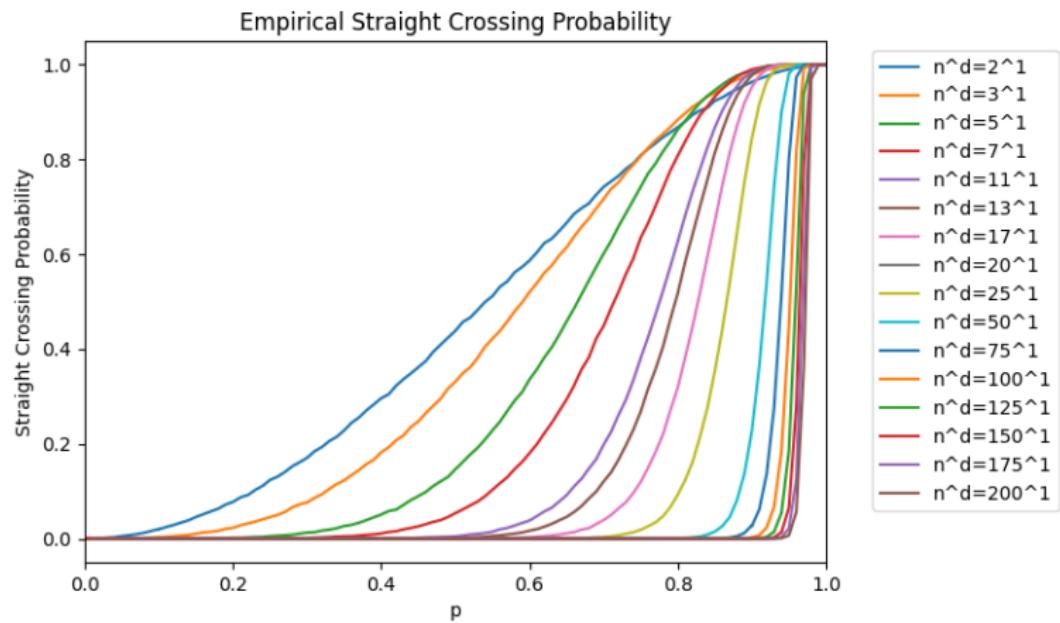
Crossings Probability

Straight Crossing, Uniform Percolation, 2D



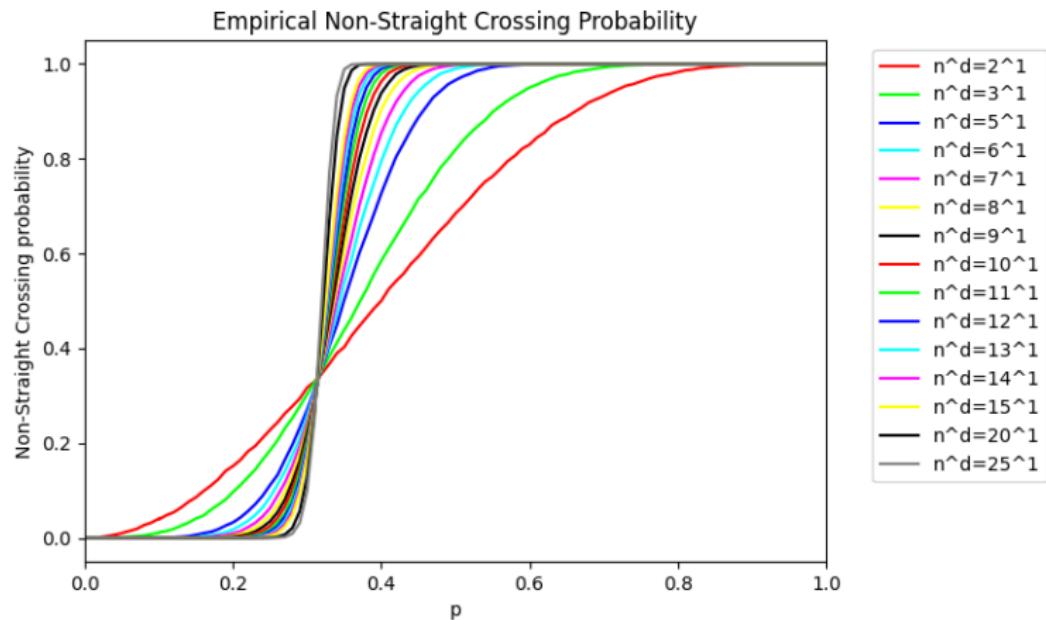
Crossings Probability

Straight Crossing, Uniform Percolation, 2D



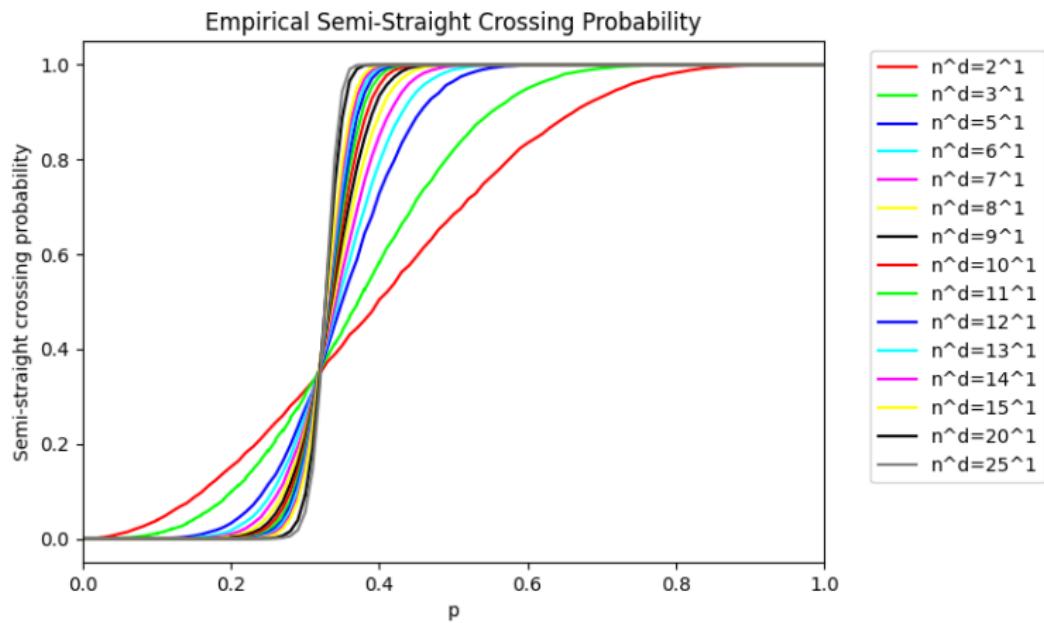
Crossings Probability

Straight Crossing, Uniform Percolation, 3D



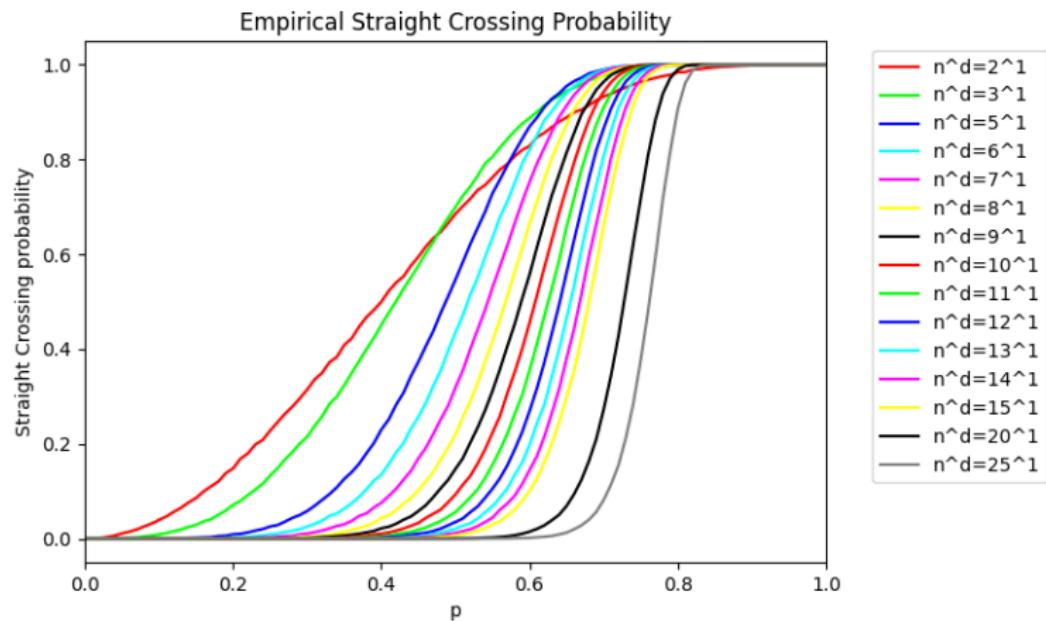
Crossings Probability

Straight Crossing, Uniform Percolation, 3D



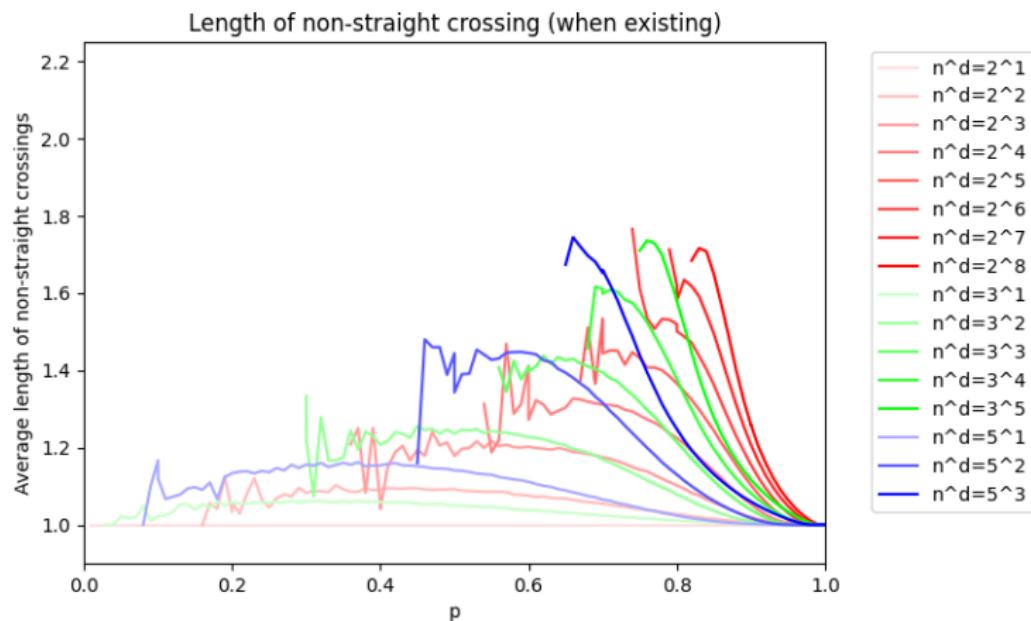
Crossings Probability

Straight Crossing, Uniform Percolation, 3D



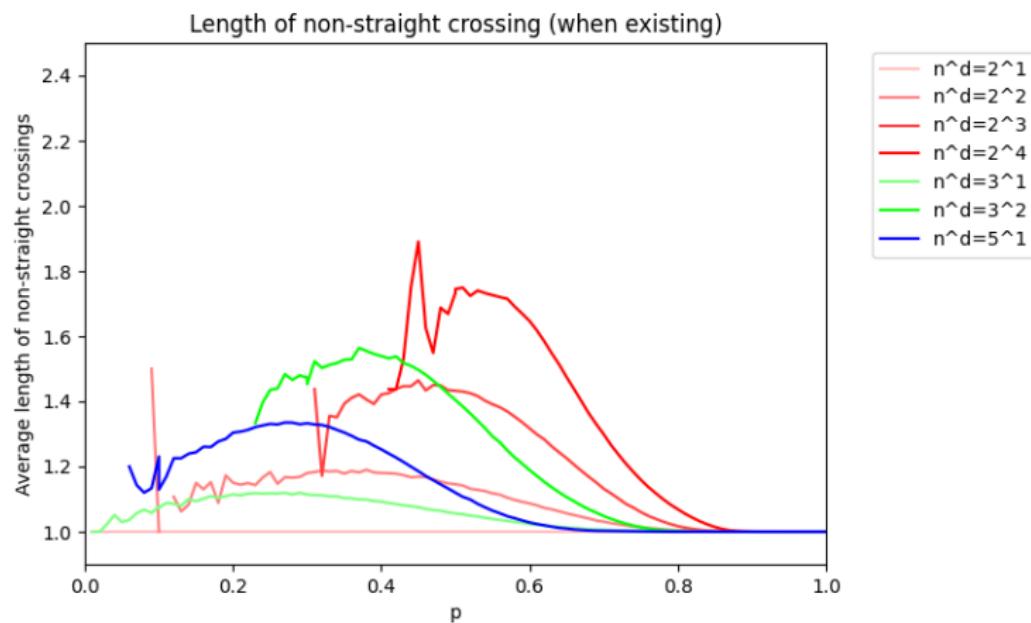
Crossings Length

Non-Straight Crossing, Recursive Percolation, 2D



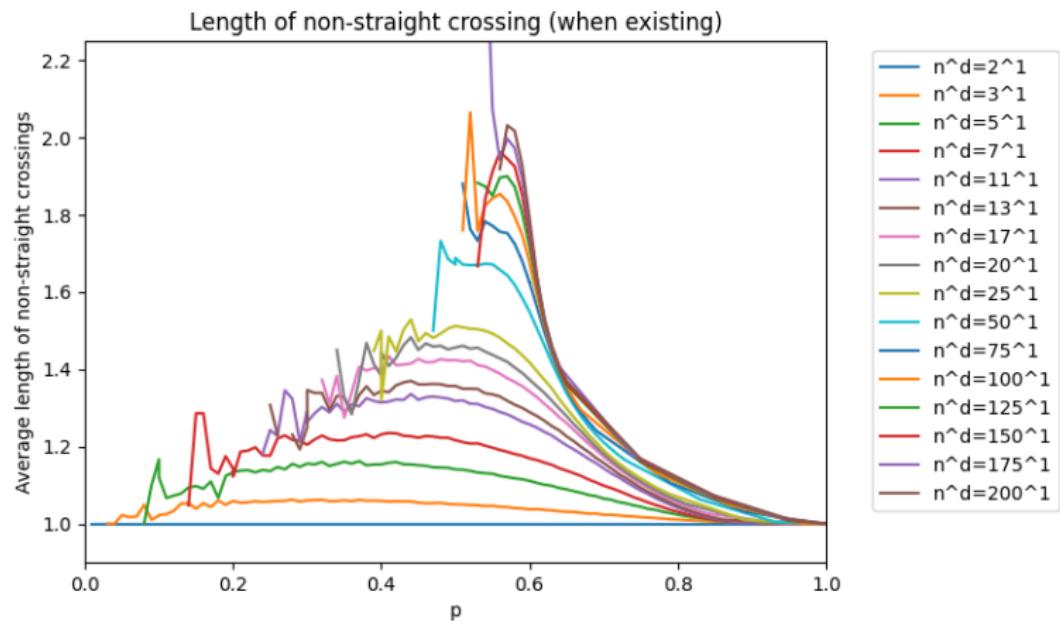
Crossings Length

Non-Straight Crossing, Recursive Percolation, 3D



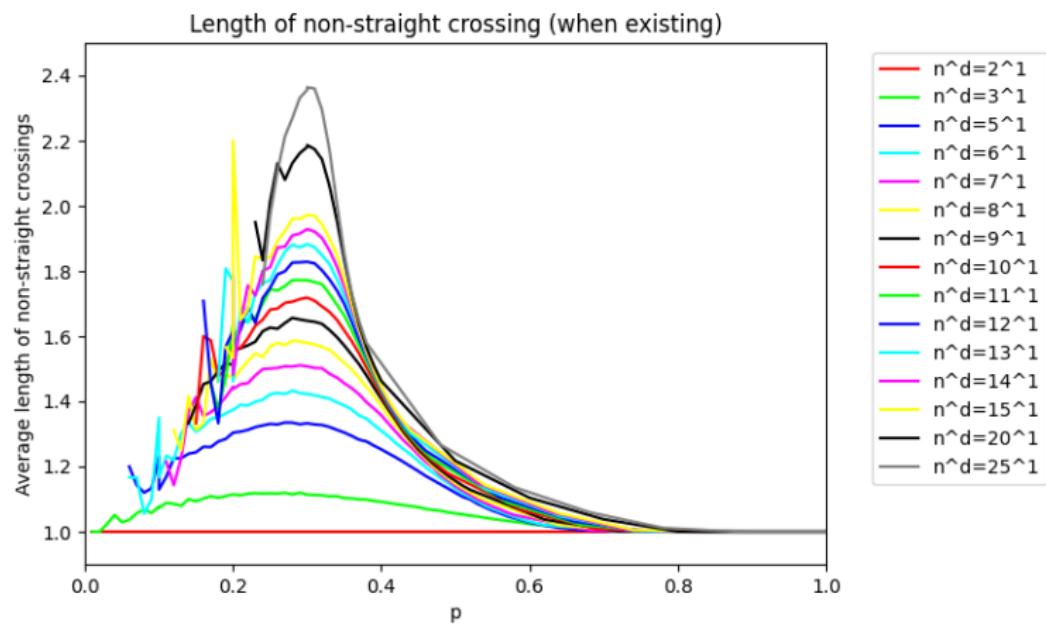
Crossings Length

Non-Straight Crossing, Uniform Percolation, 2D



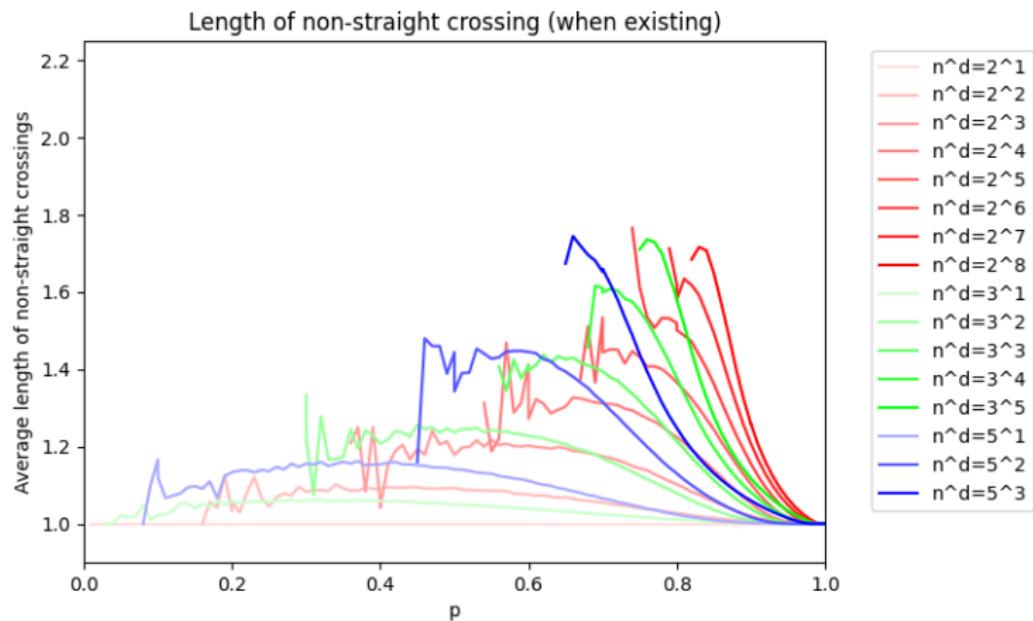
Crossings Length

Non-Straight Crossing, Uniform Percolation, 3D



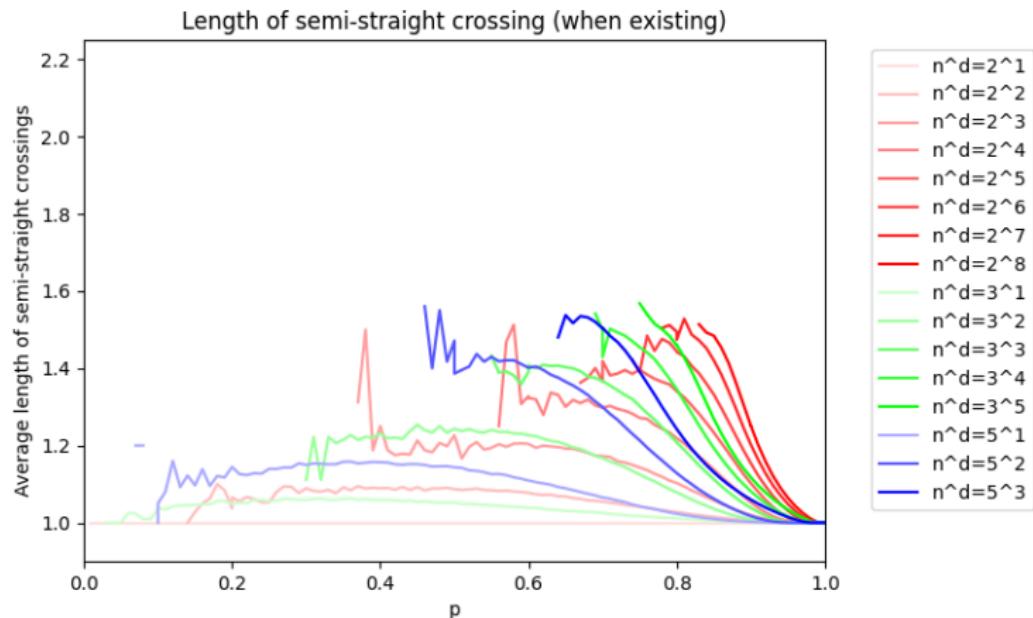
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 2D



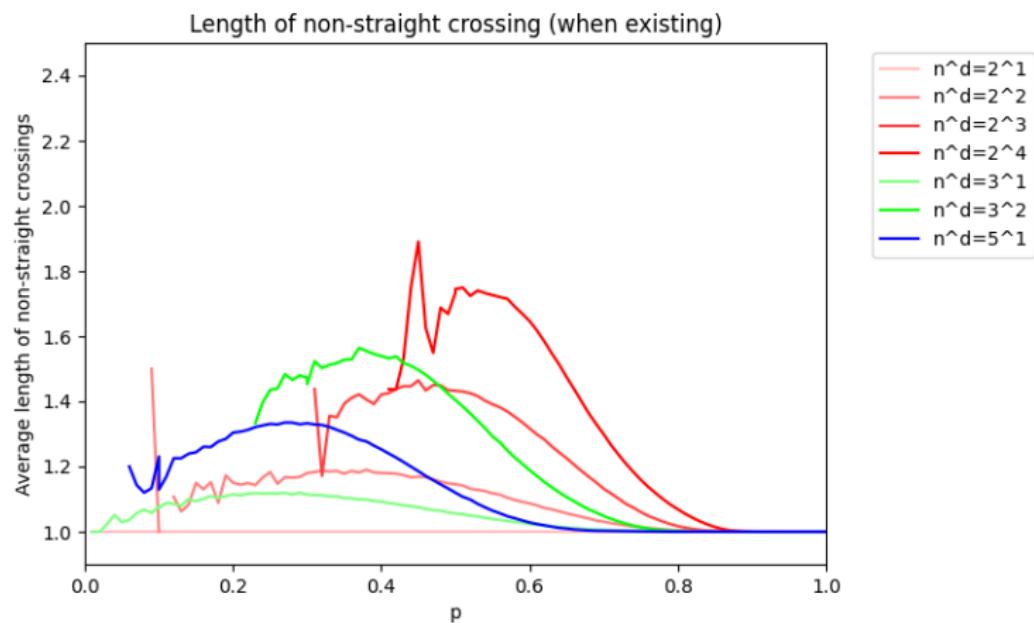
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 2D



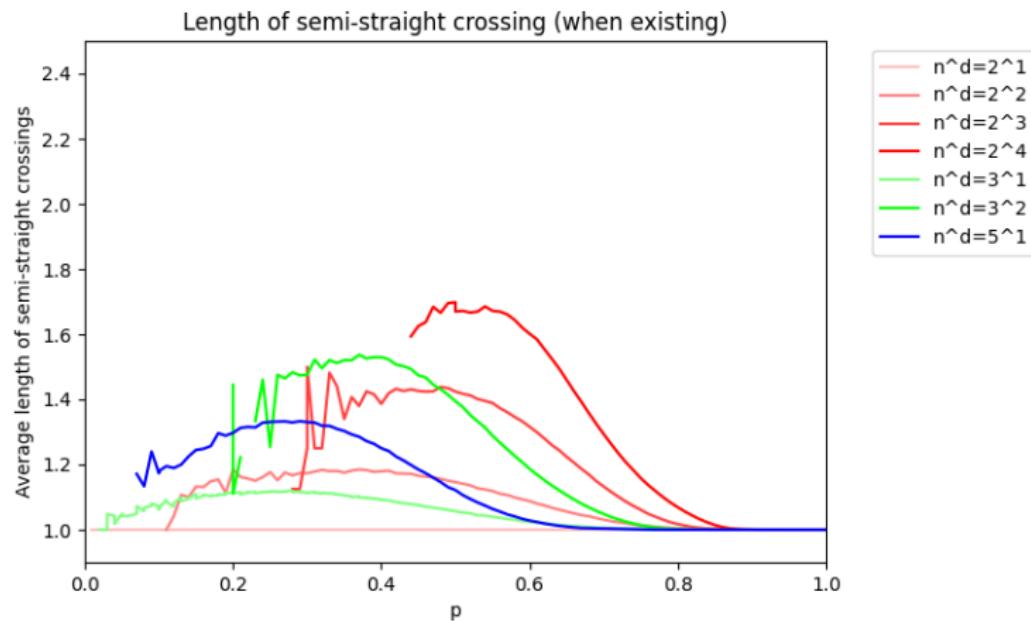
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 3D



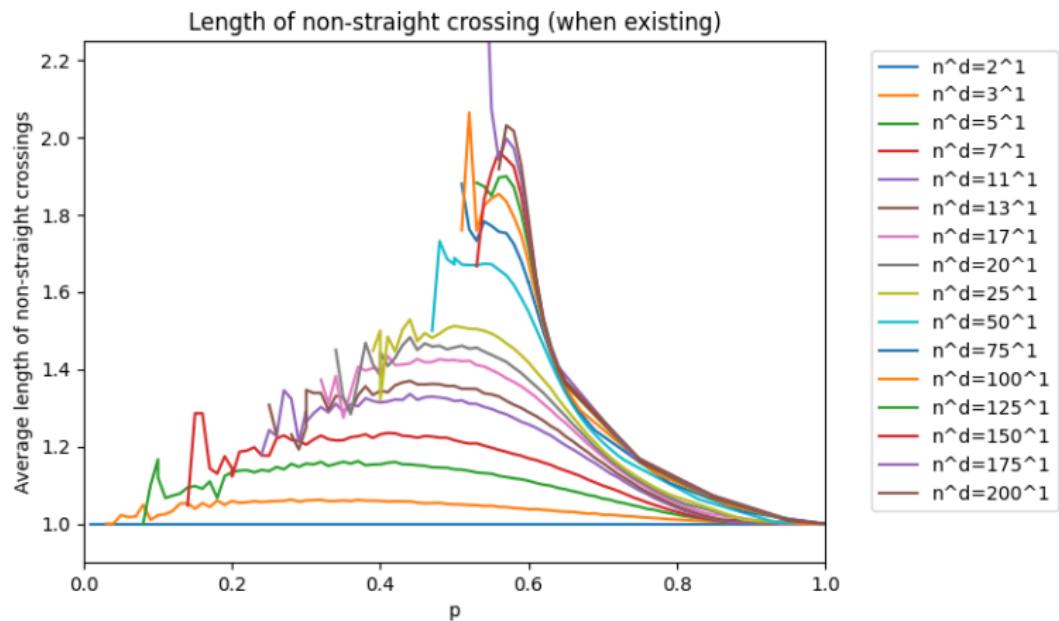
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 3D



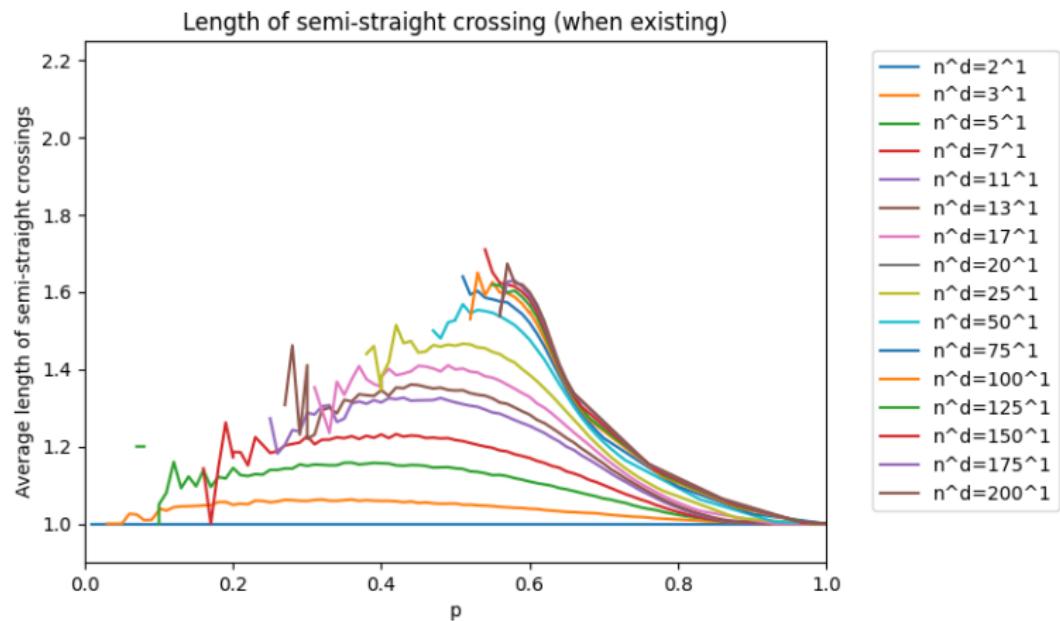
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 2D



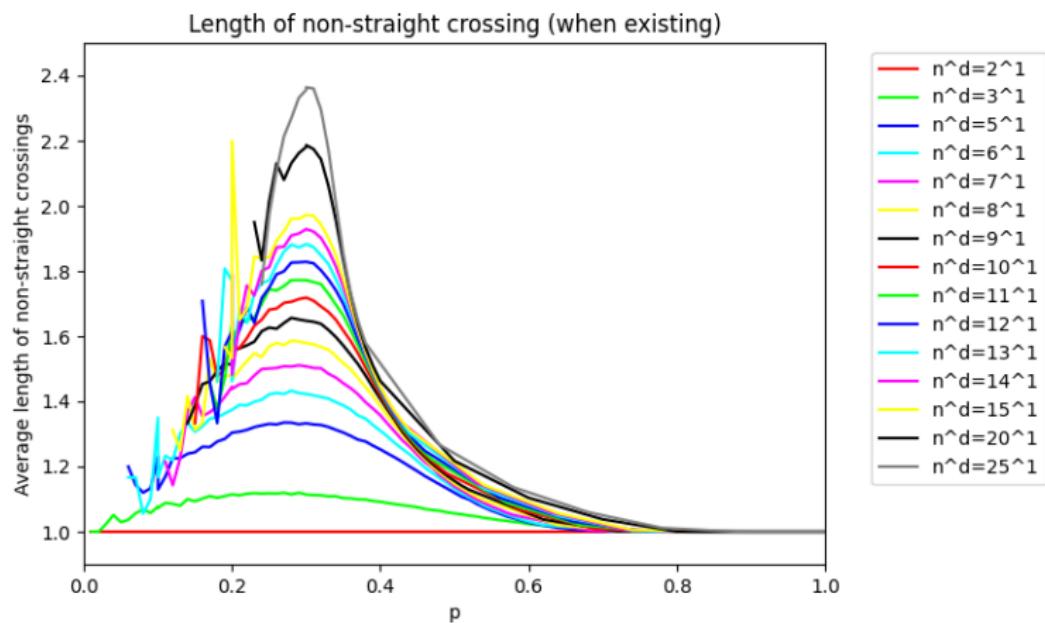
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 2D



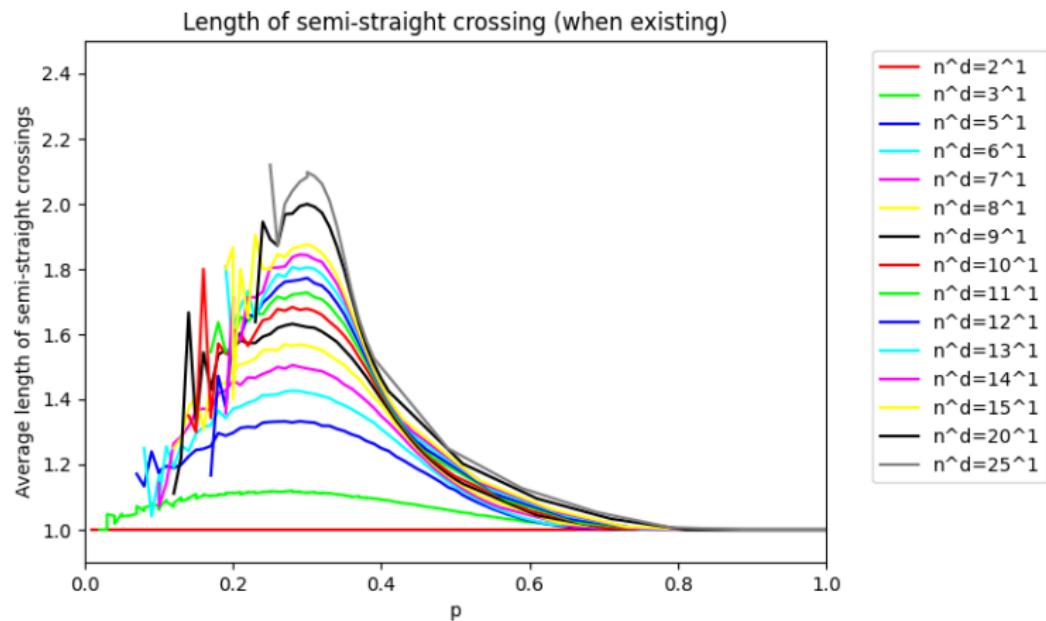
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 3D



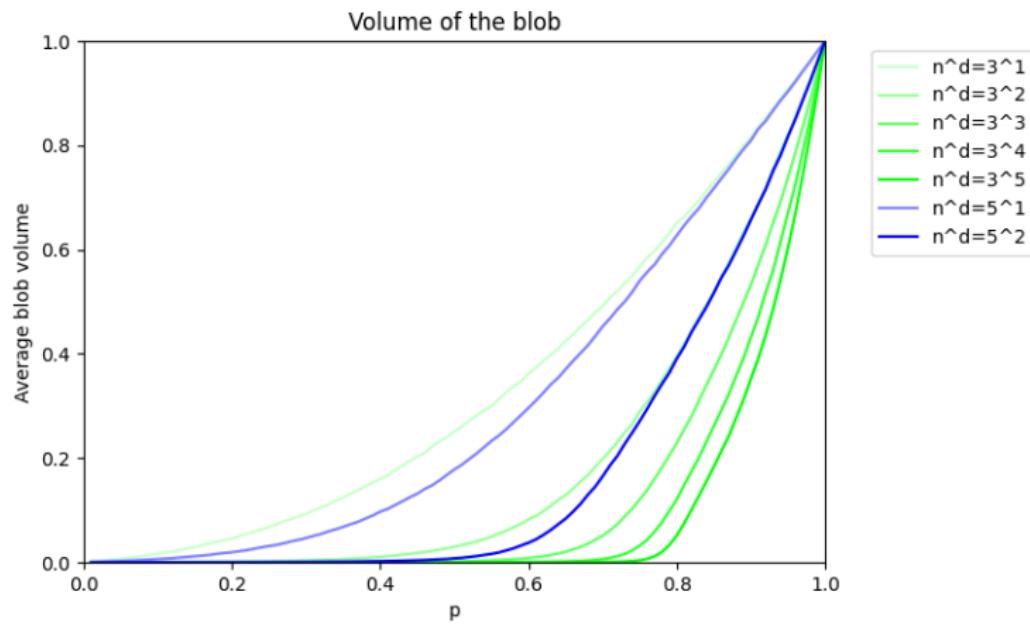
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 3D



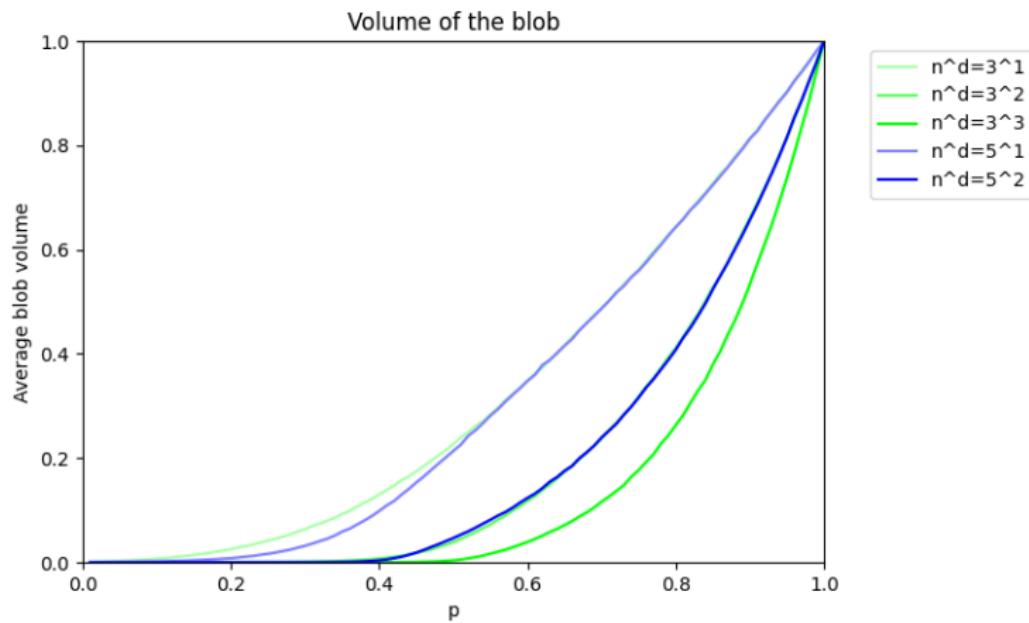
Blob

Volume, Recursive Percolation, 2D



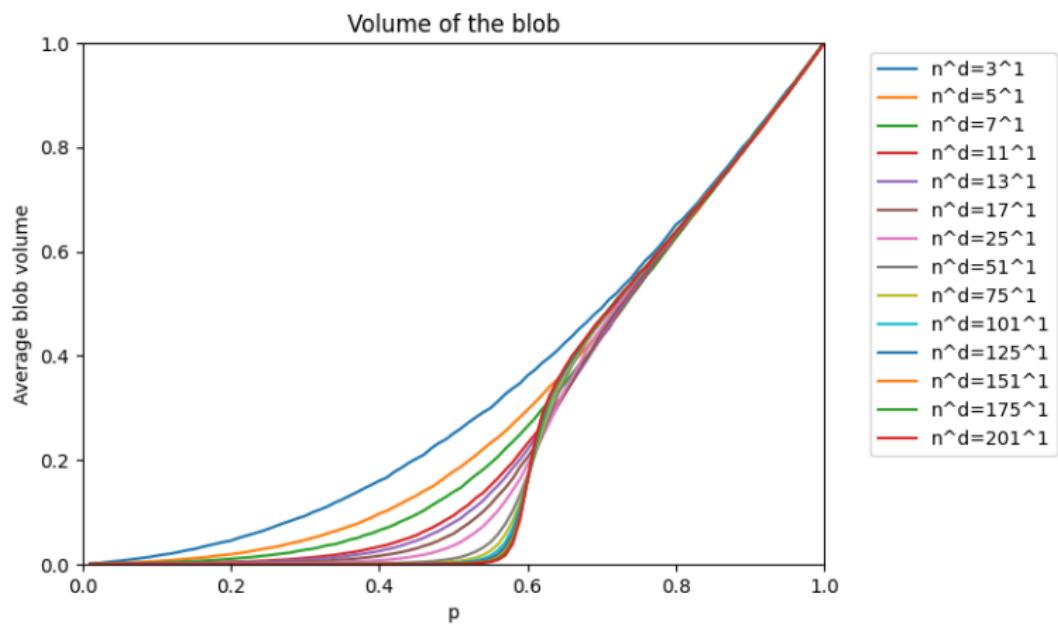
Blob

Volume, Recursive Percolation, 3D



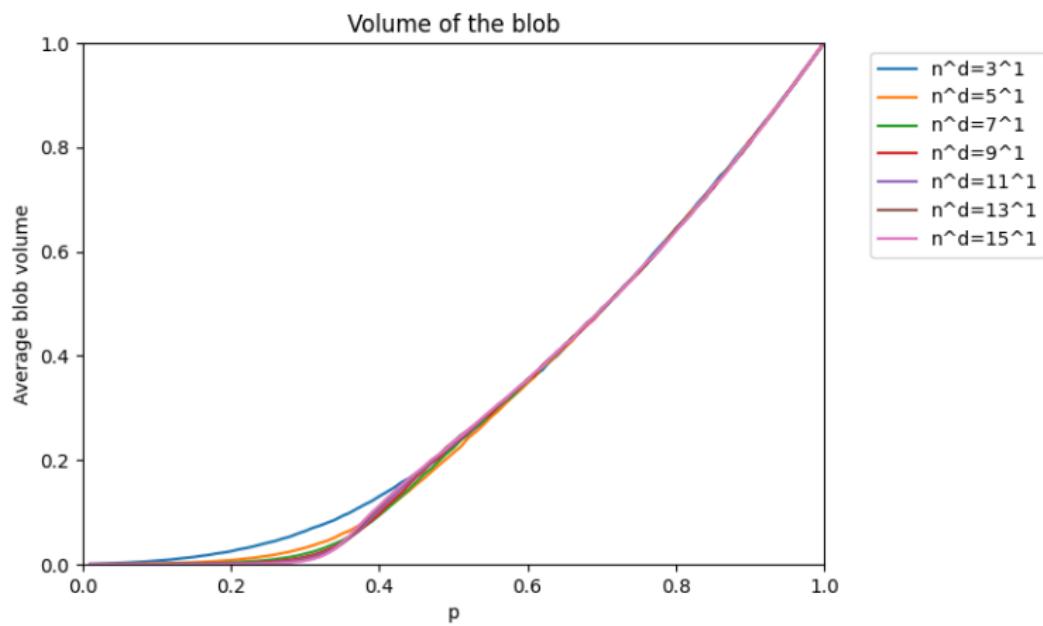
Blob

Volume, Uniform Percolation, 2D



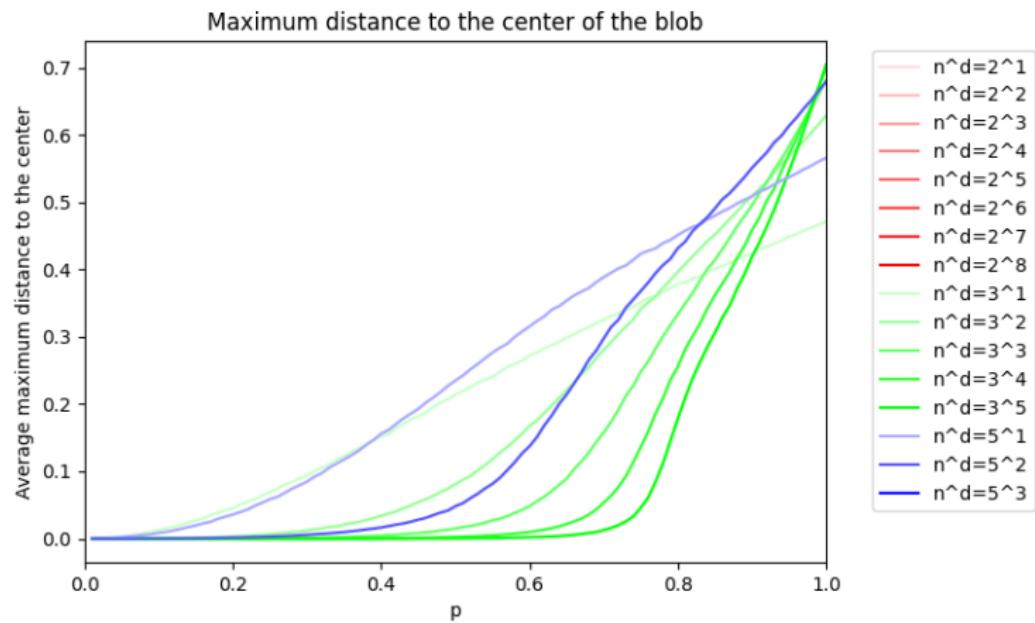
Blob

Volume, Uniform Percolation, 3D



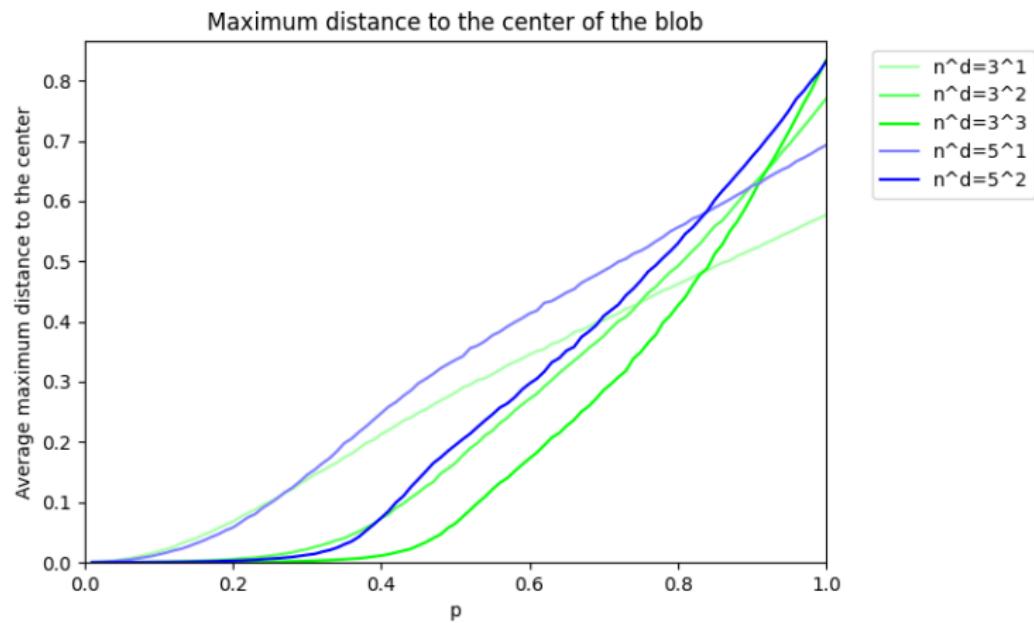
Blob

Distance to Center, Recursive Percolation, 2D



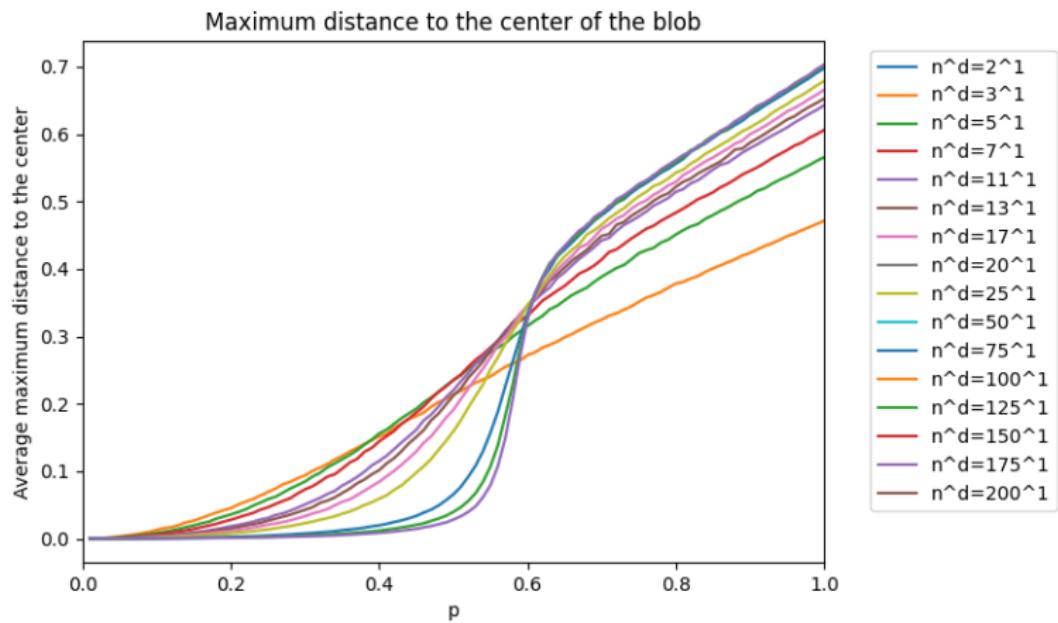
Blob

Distance to Center, Recursive Percolation, 3D



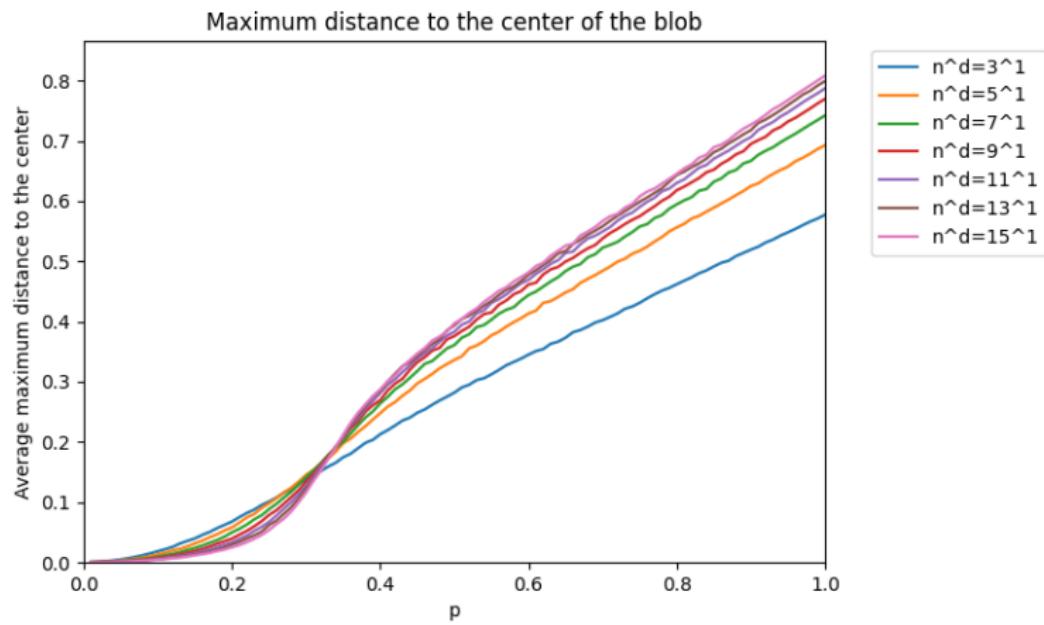
Blob

Distance to Center, Uniform Percolation, 2D



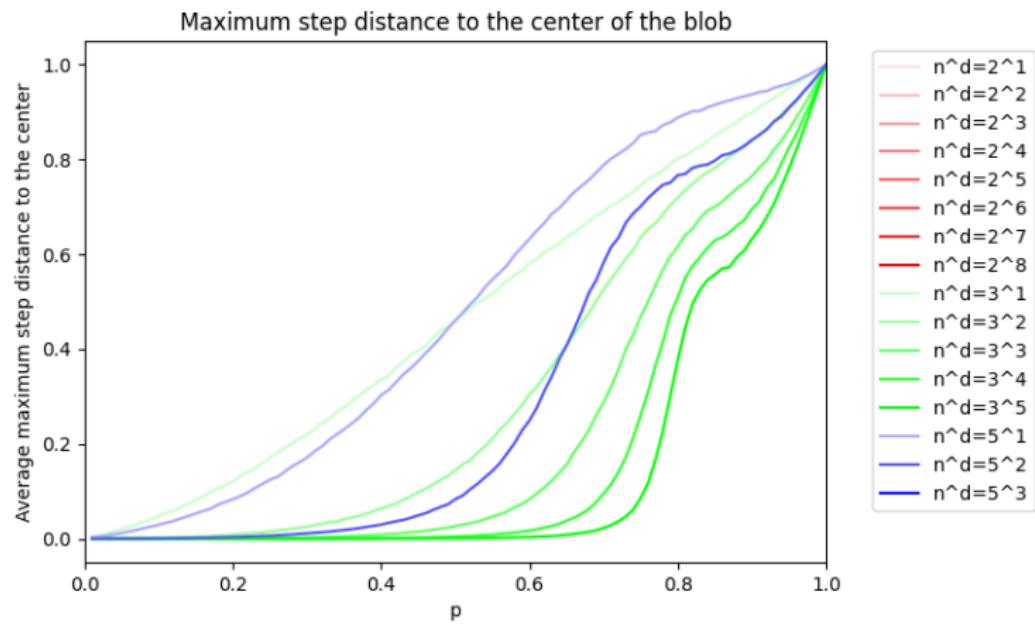
Blob

Distance to Center, Uniform Percolation, 3D



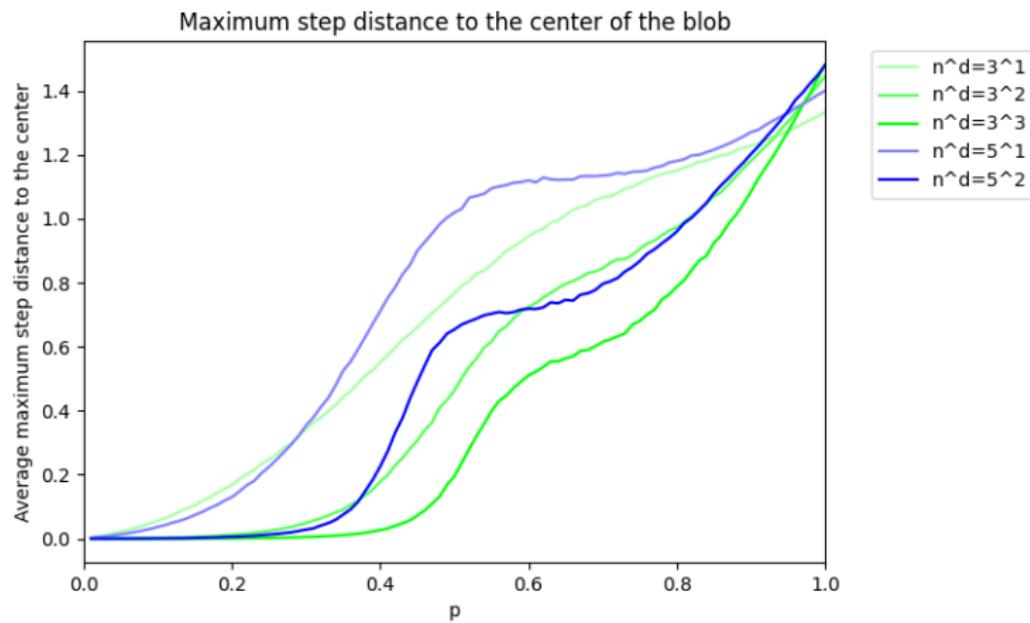
Blob

Steps to Center, Recursive Percolation, 2D



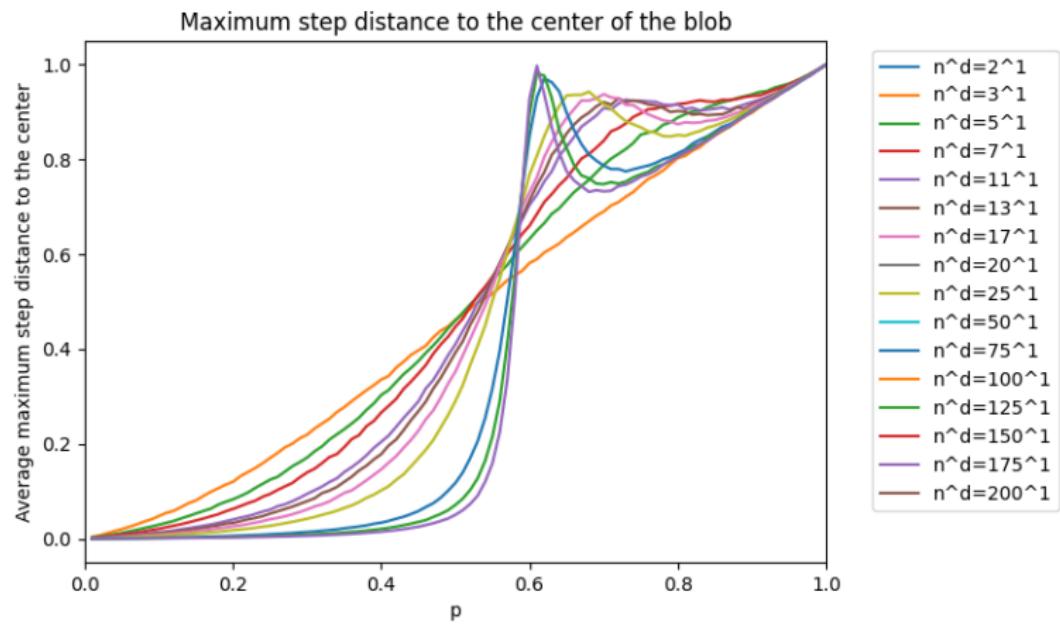
Blob

Steps to Center, Recursive Percolation, 3D



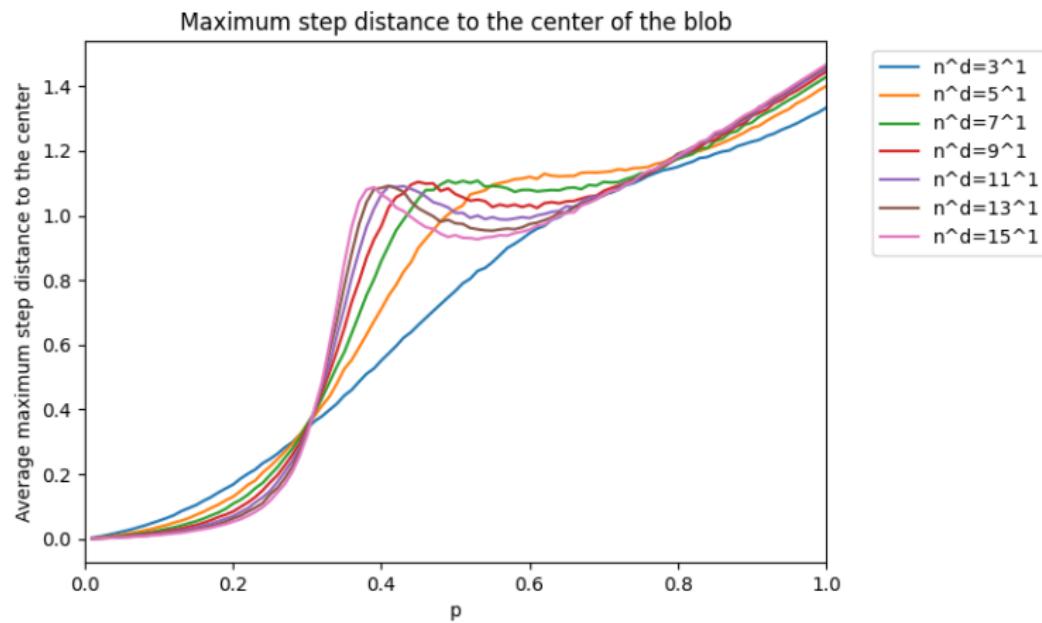
Blob

Steps to Center, Uniform Percolation, 2D



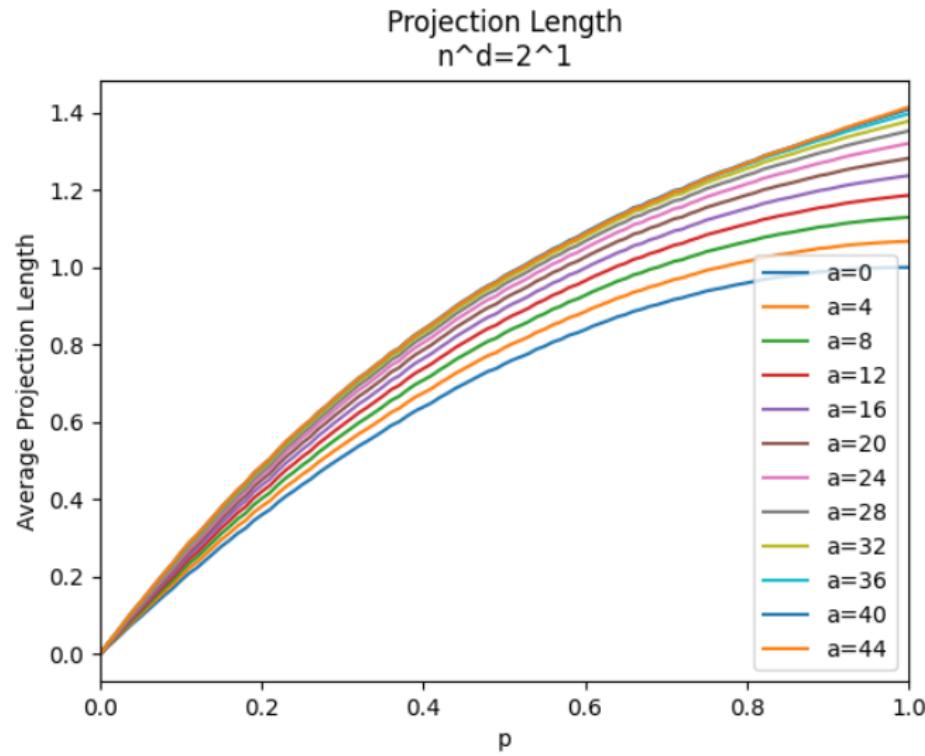
Blob

Steps to Center, Uniform Percolation, 3D



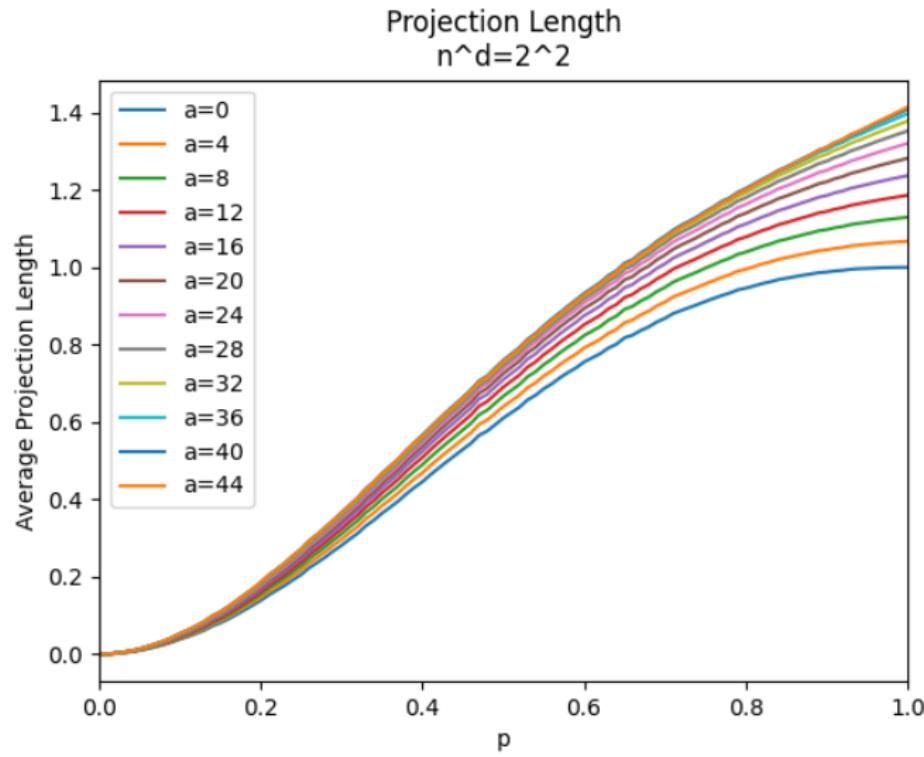
Projection

$n = 2$, Recursive Percolation, 2D



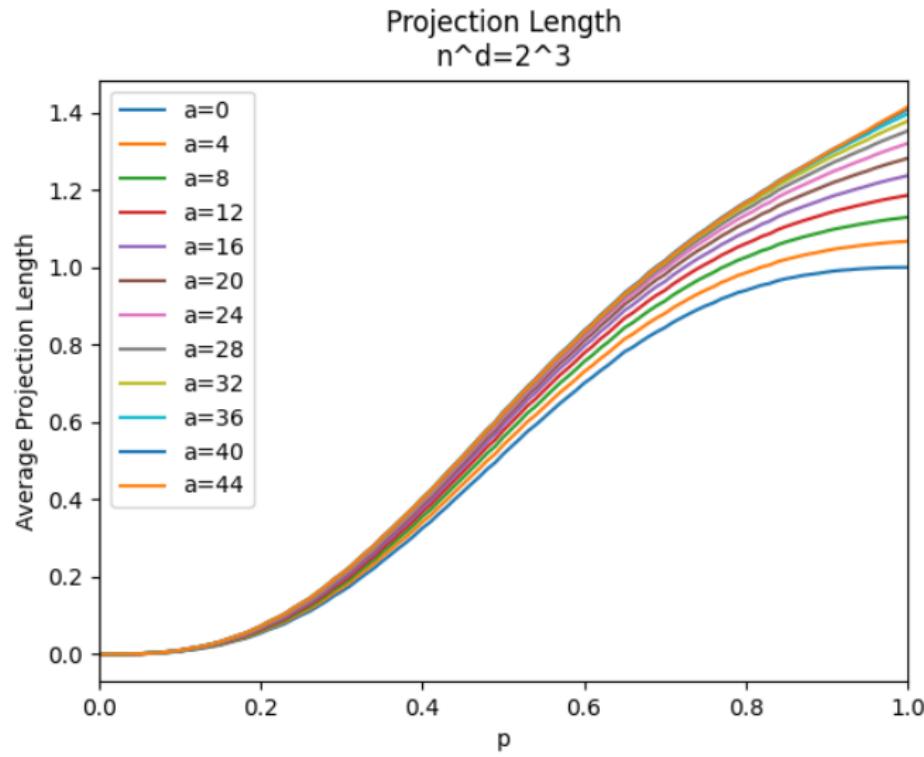
Projection

$n = 2$, Recursive Percolation, 2D



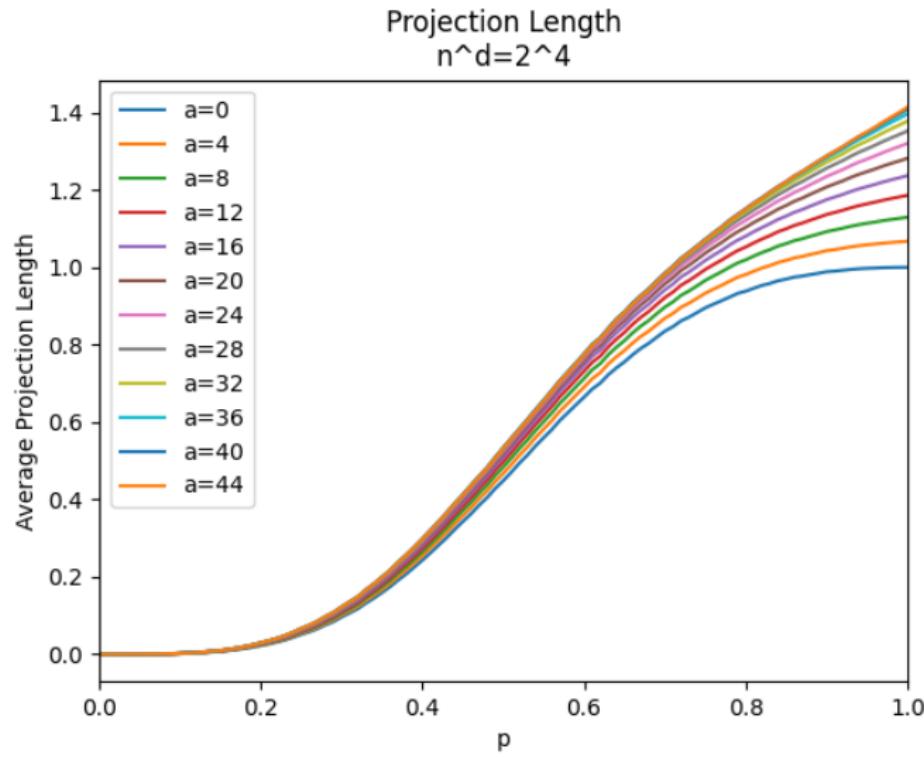
Projection

$n = 2$, Recursive Percolation, 2D



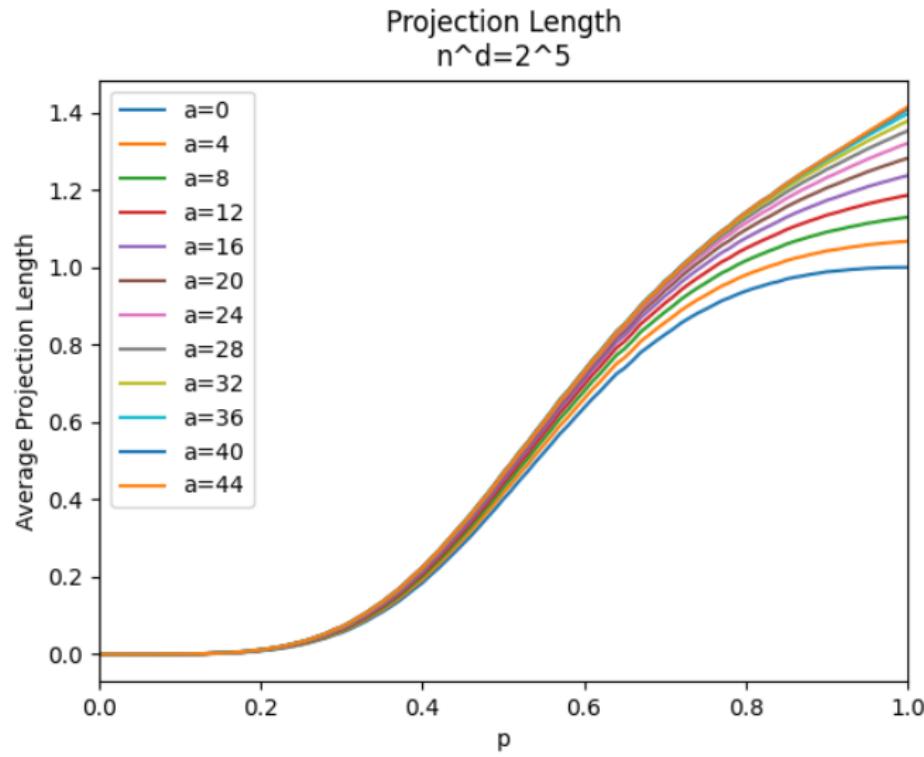
Projection

$n = 2$, Recursive Percolation, 2D



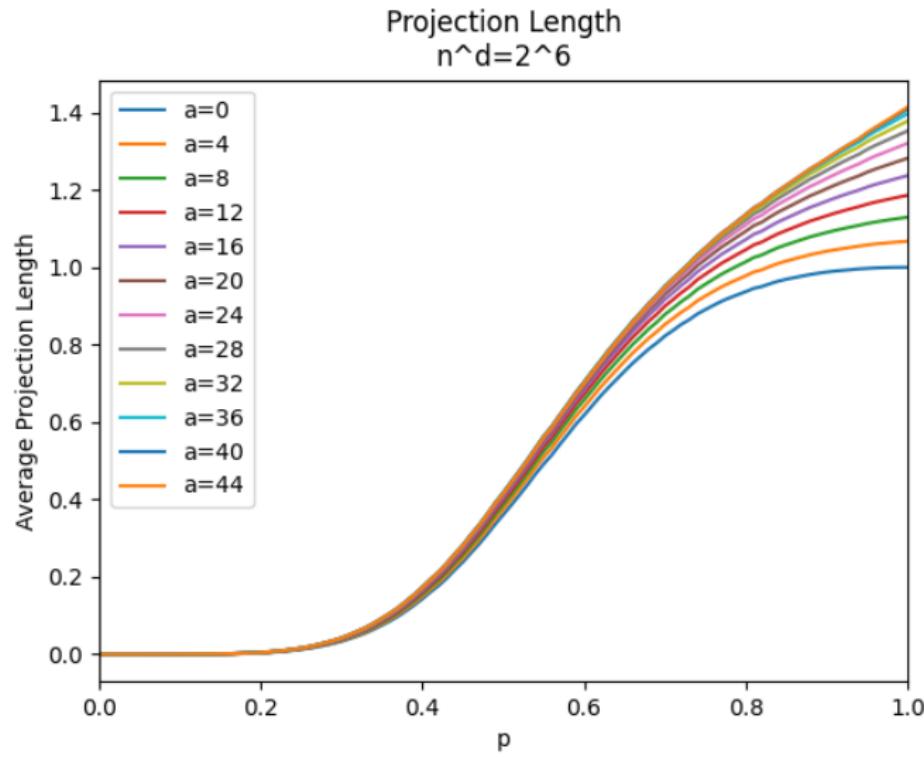
Projection

$n = 2$, Recursive Percolation, 2D



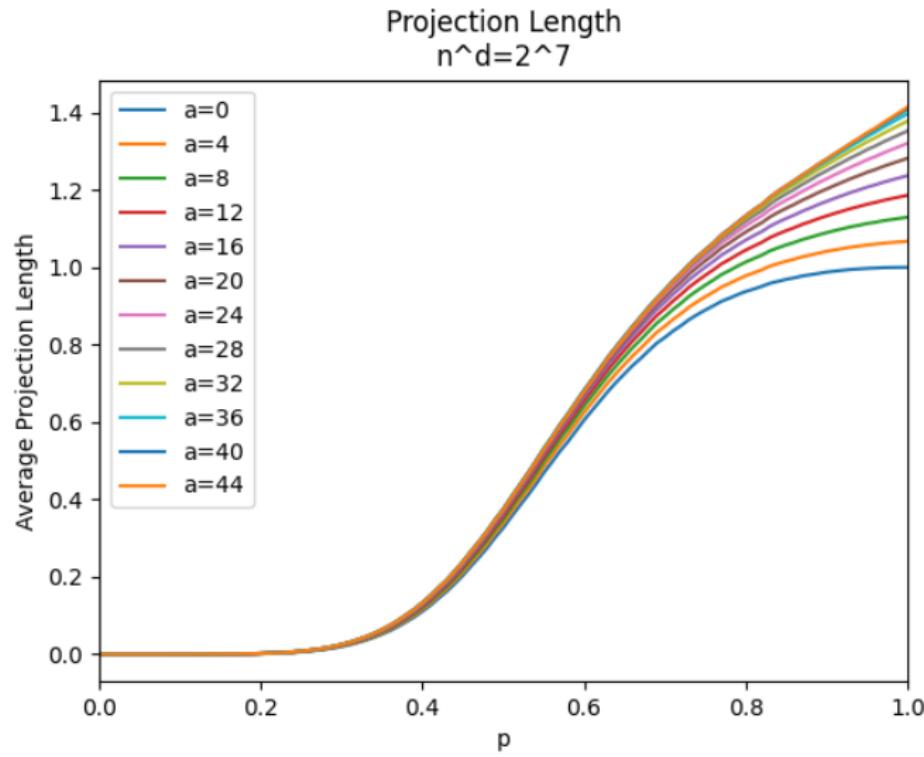
Projection

$n = 2$, Recursive Percolation, 2D



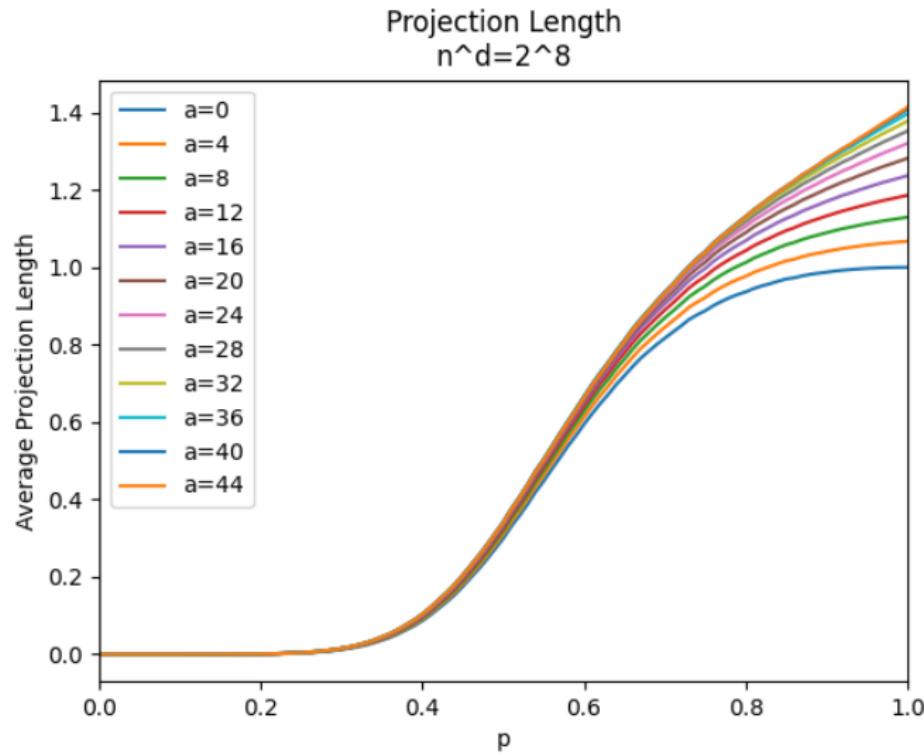
Projection

$n = 2$, Recursive Percolation, 2D



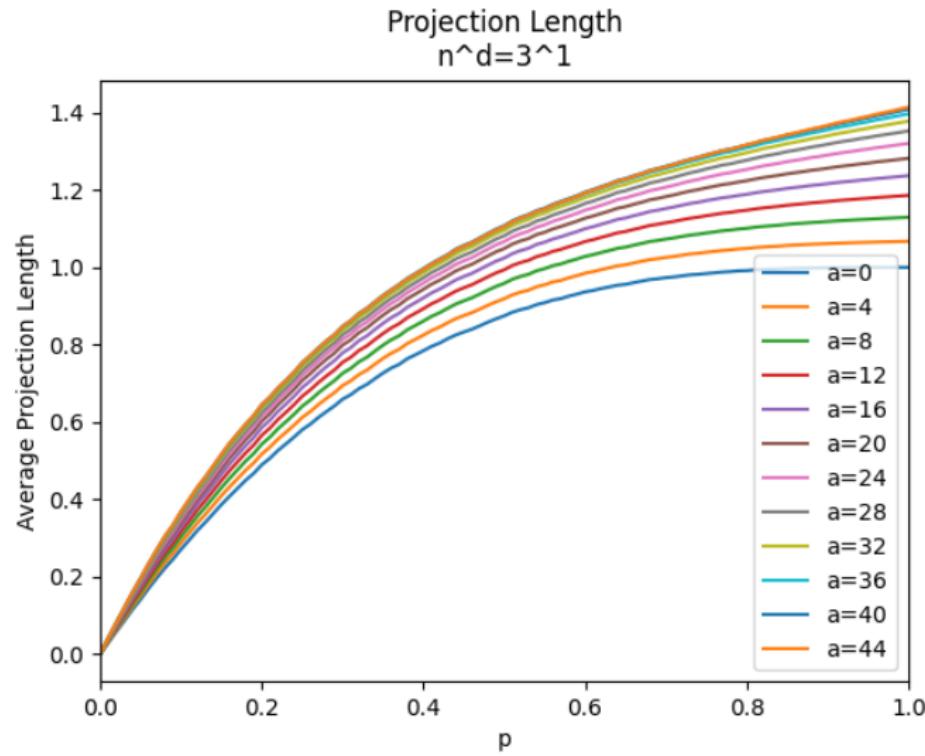
Projection

$n = 2$, Recursive Percolation, 2D



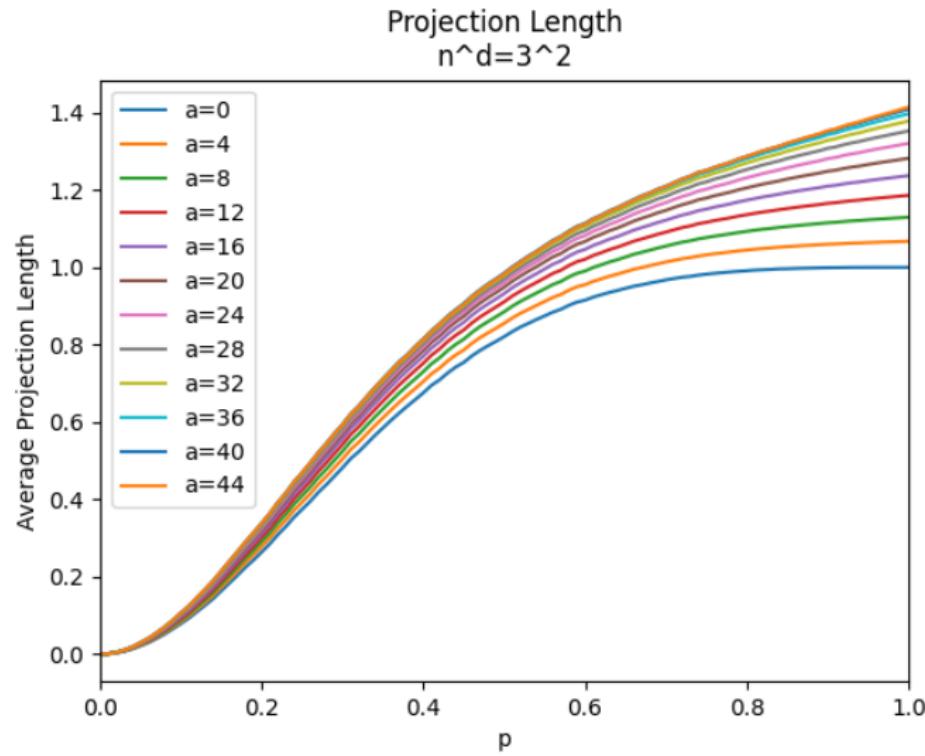
Projection

$n = 3$, Recursive Percolation, 2D



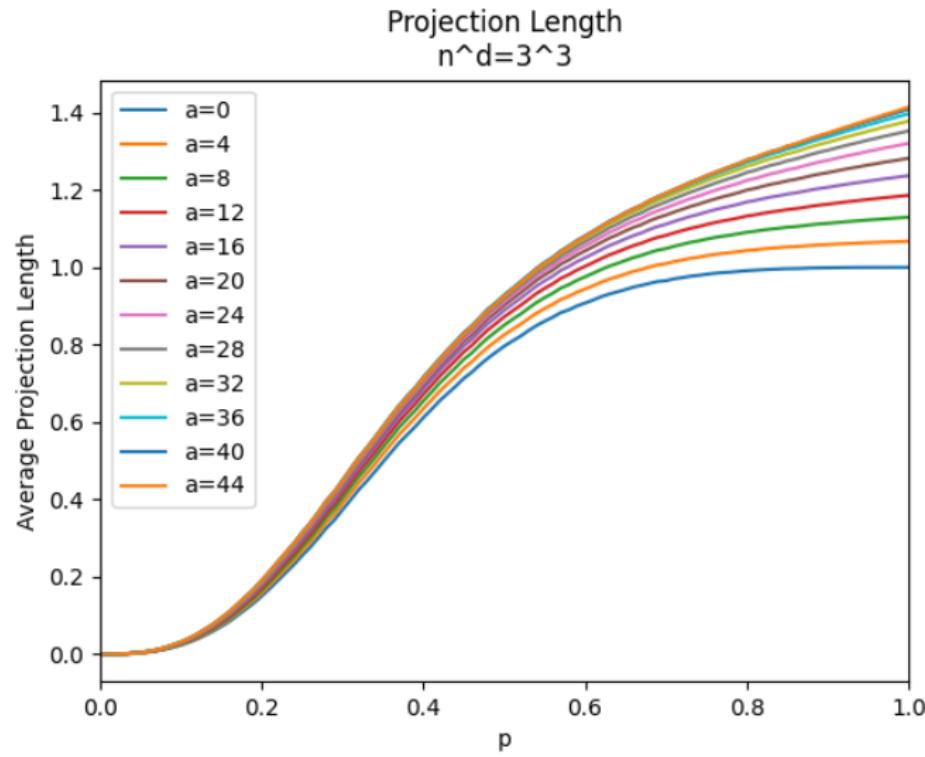
Projection

$n = 3$, Recursive Percolation, 2D



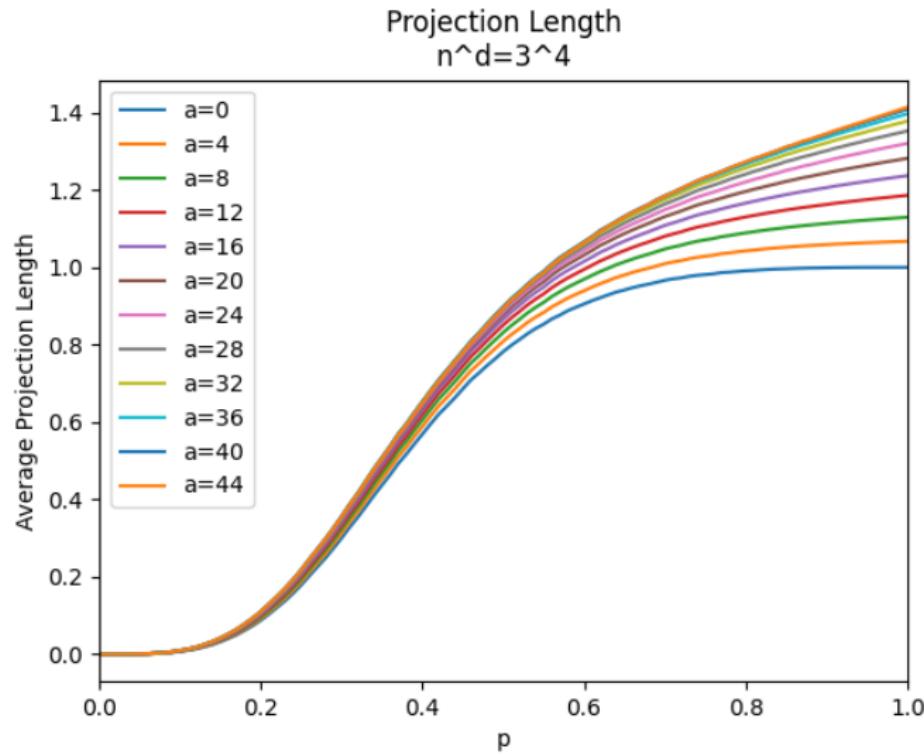
Projection

$n = 3$, Recursive Percolation, 2D



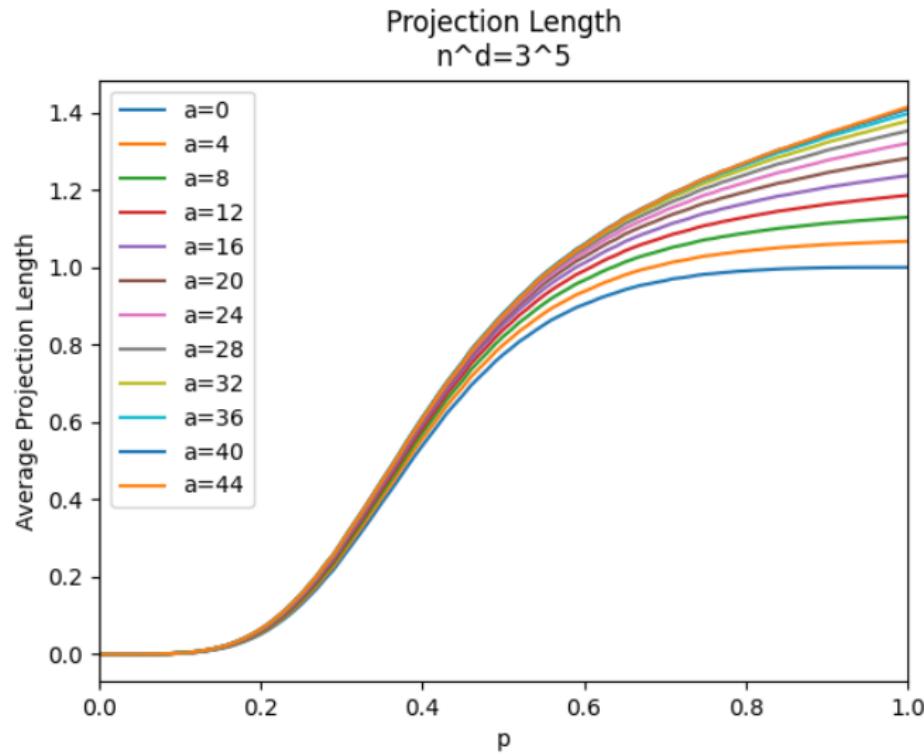
Projection

$n = 3$, Recursive Percolation, 2D



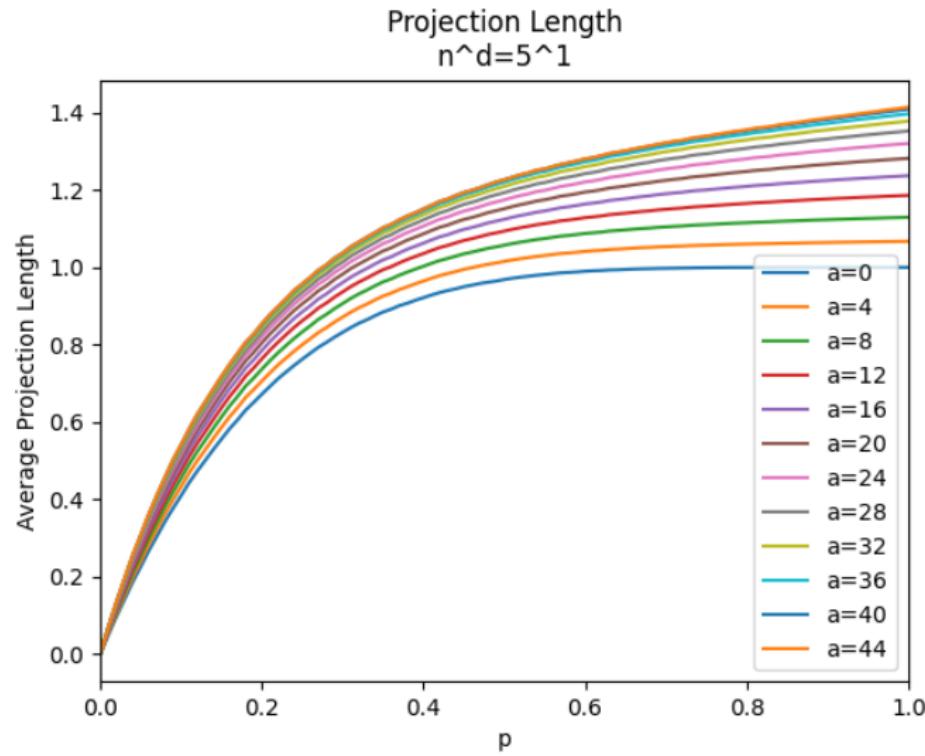
Projection

$n = 3$, Recursive Percolation, 2D



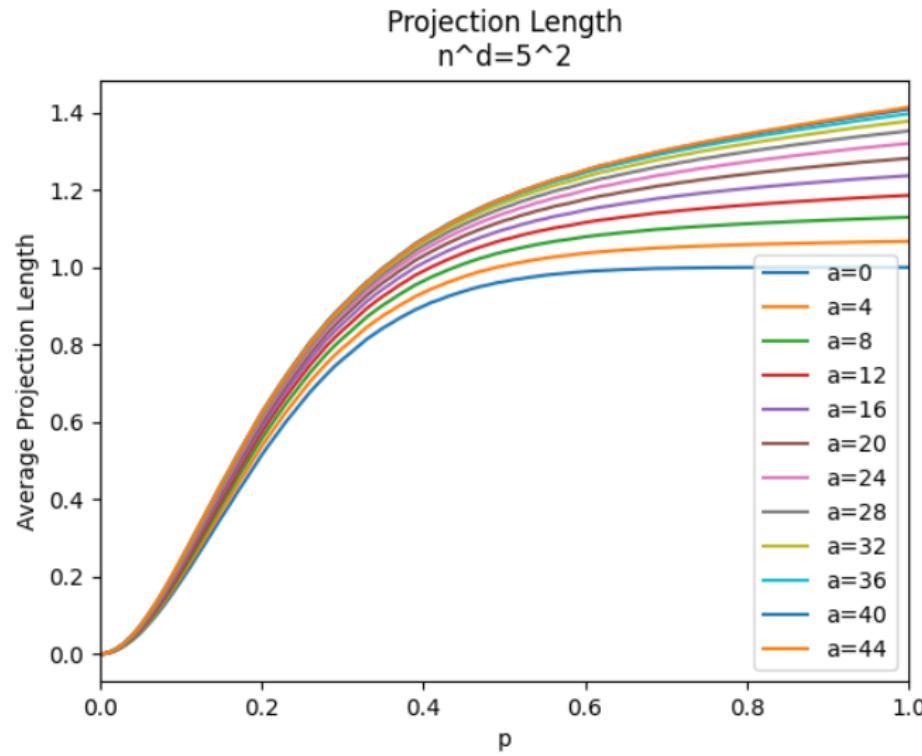
Projection

$n = 5$, Recursive Percolation, 2D



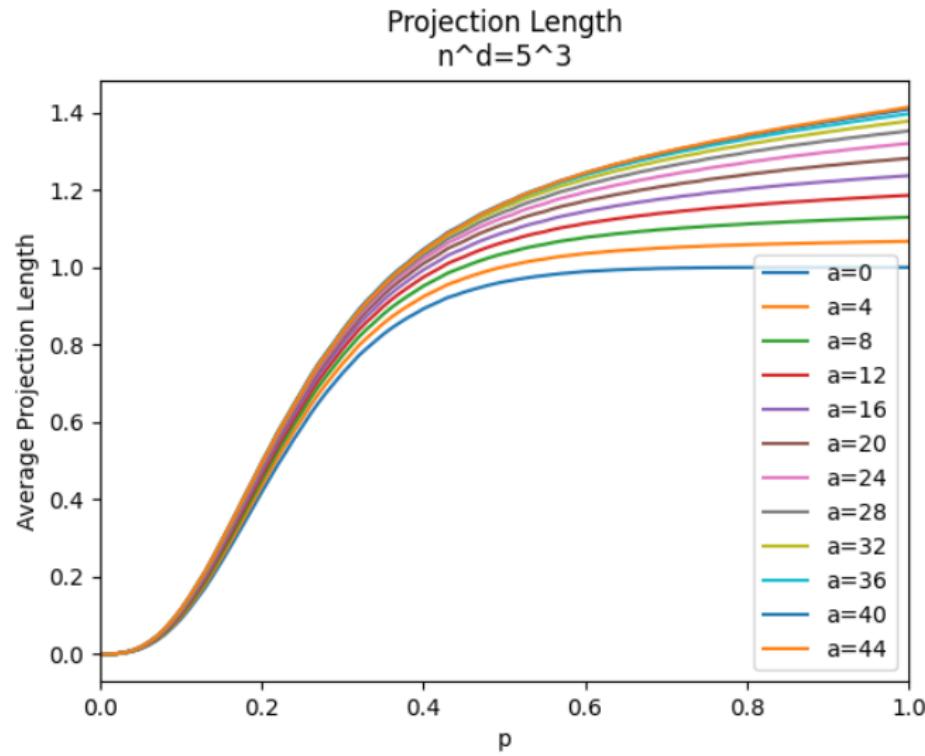
Projection

$n = 5$, Recursive Percolation, 2D



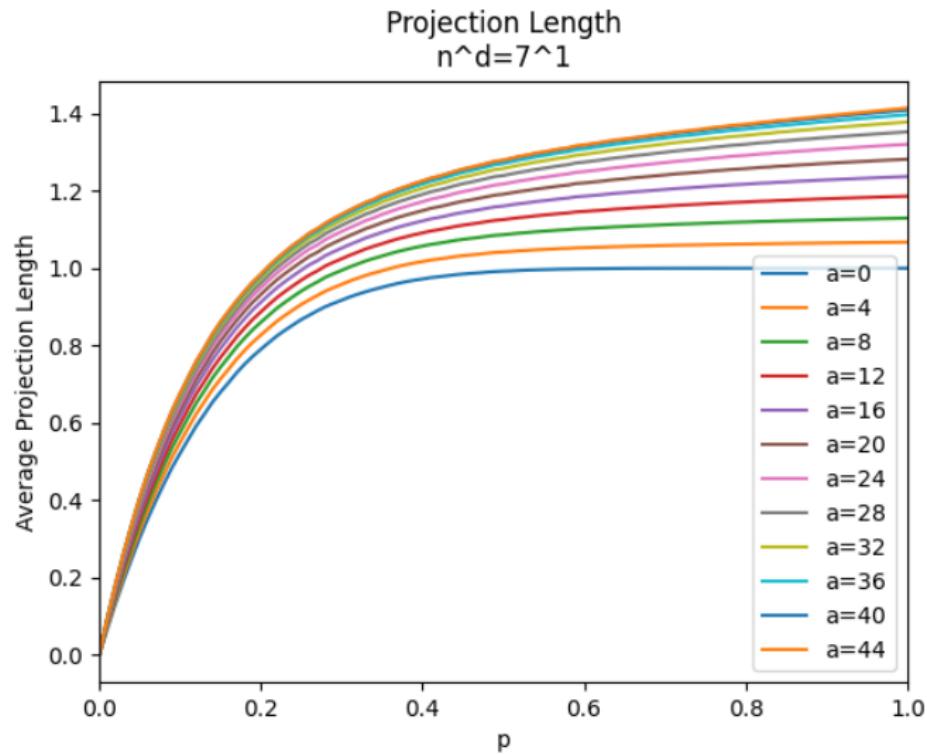
Projection

$n = 5$, Recursive Percolation, 2D



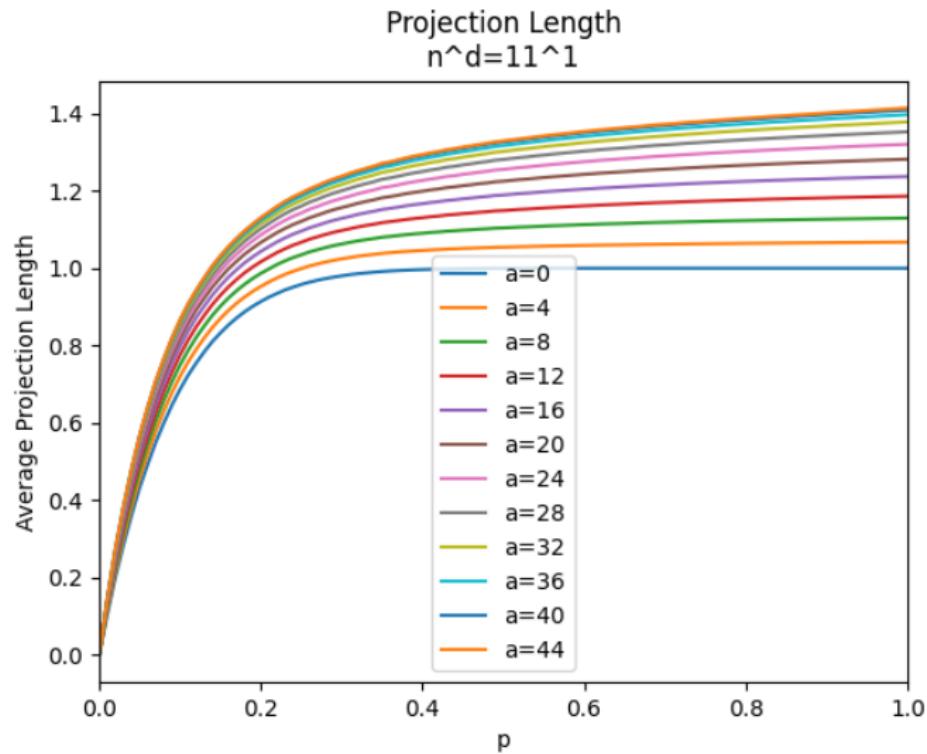
Projection

Uniform Percolation, 2D



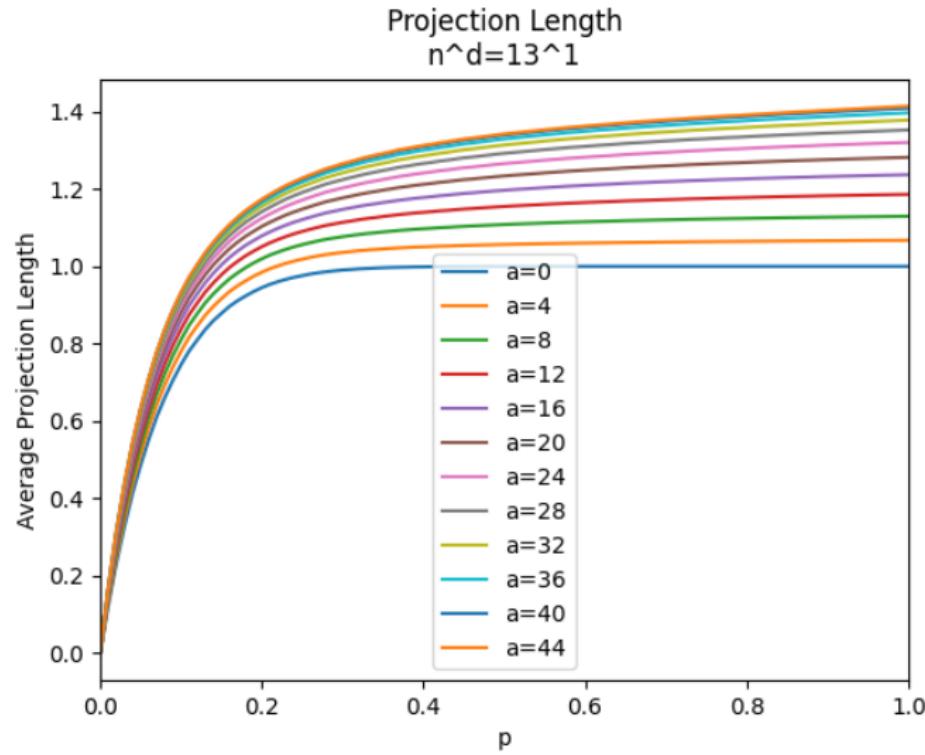
Projection

Uniform Percolation, 2D



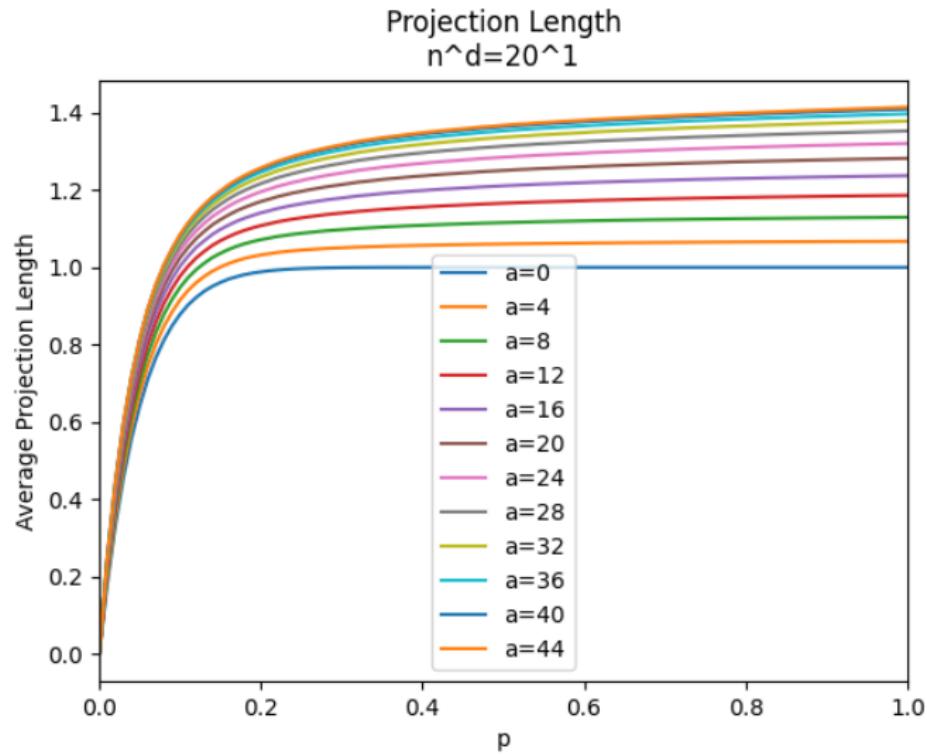
Projection

Uniform Percolation, 2D



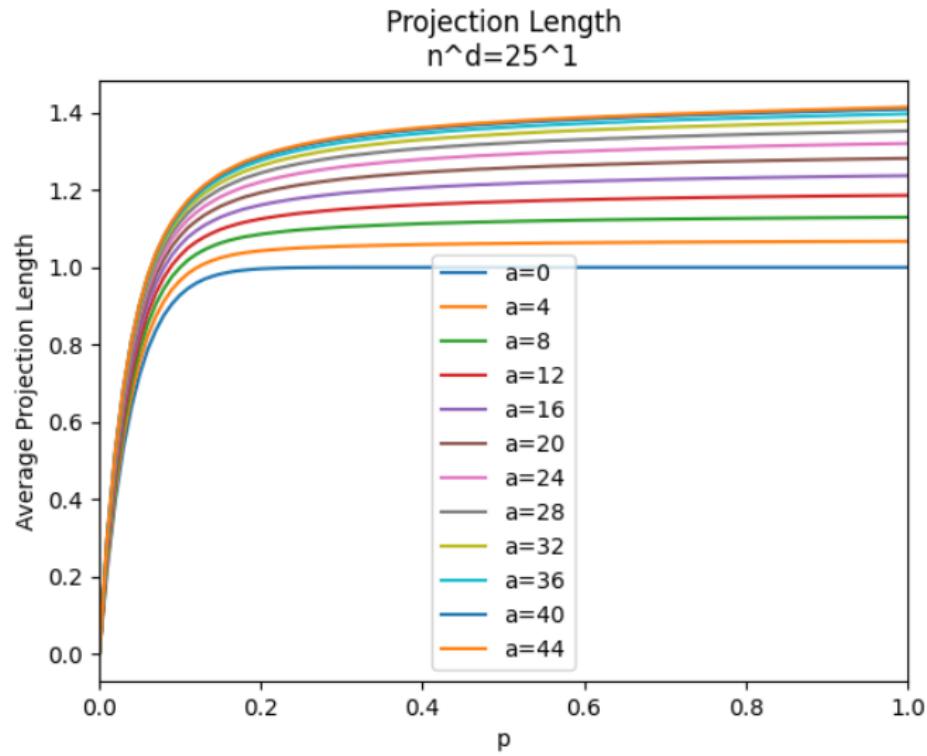
Projection

Uniform Percolation, 2D



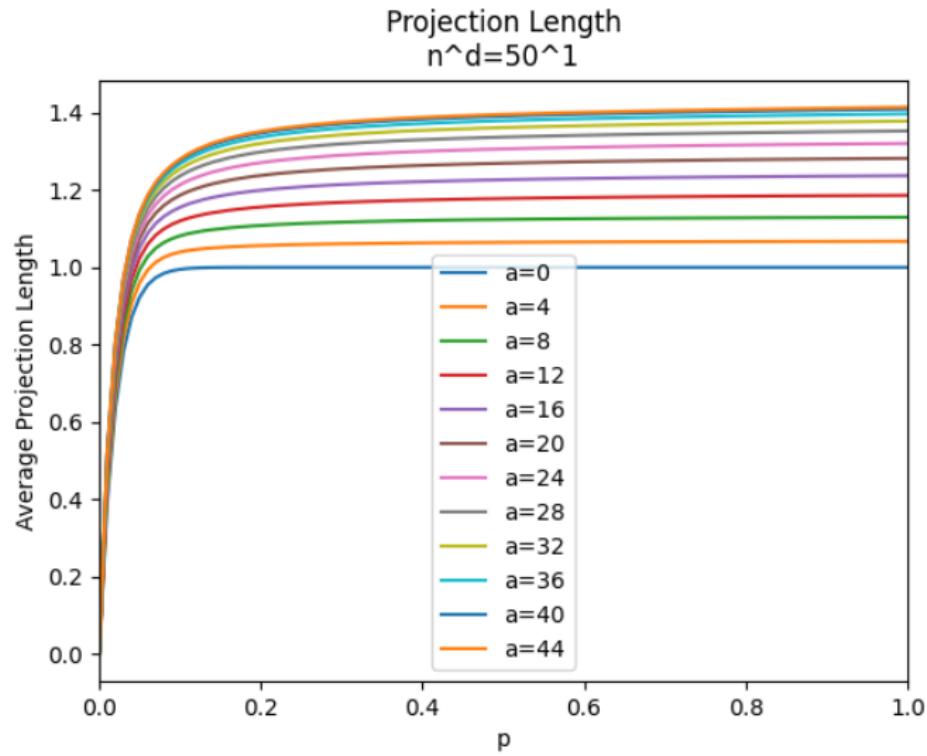
Projection

Uniform Percolation, 2D



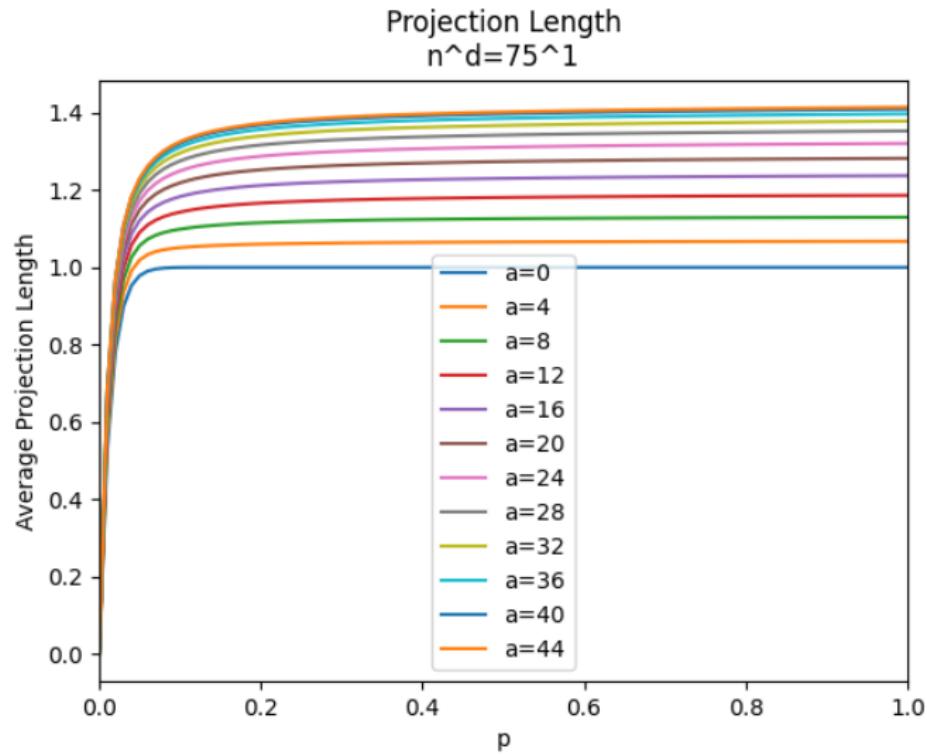
Projection

Uniform Percolation, 2D



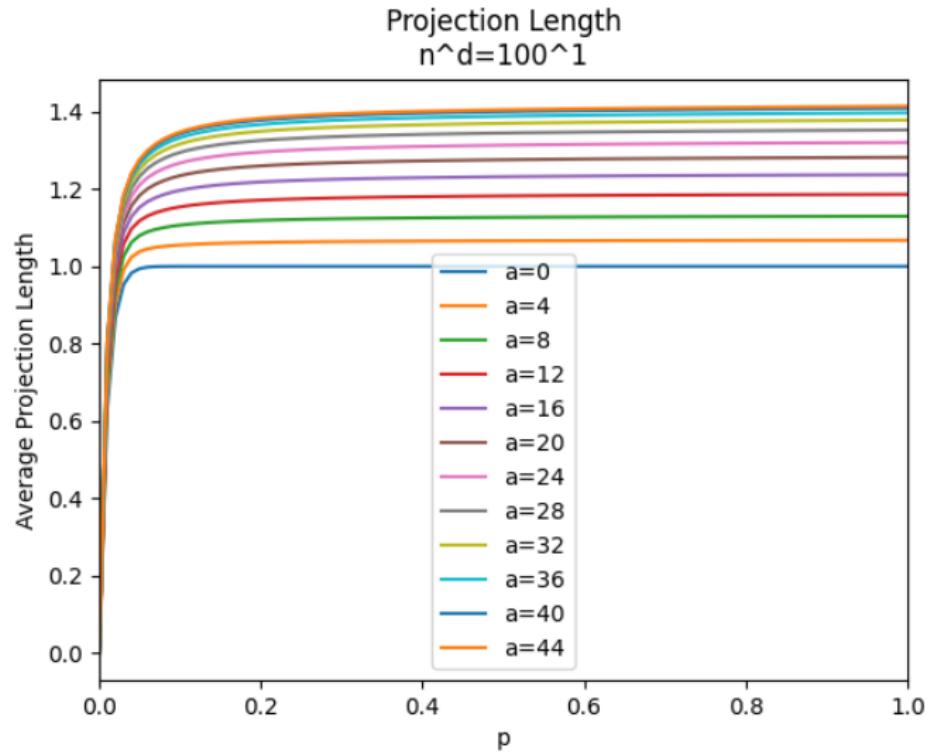
Projection

Uniform Percolation, 2D



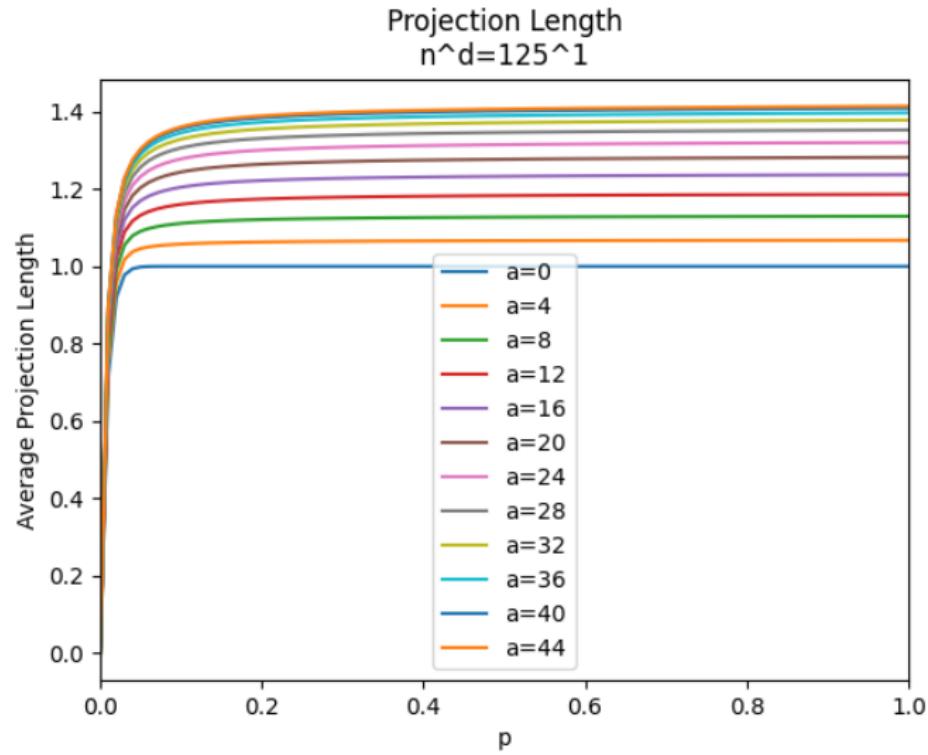
Projection

Uniform Percolation, 2D



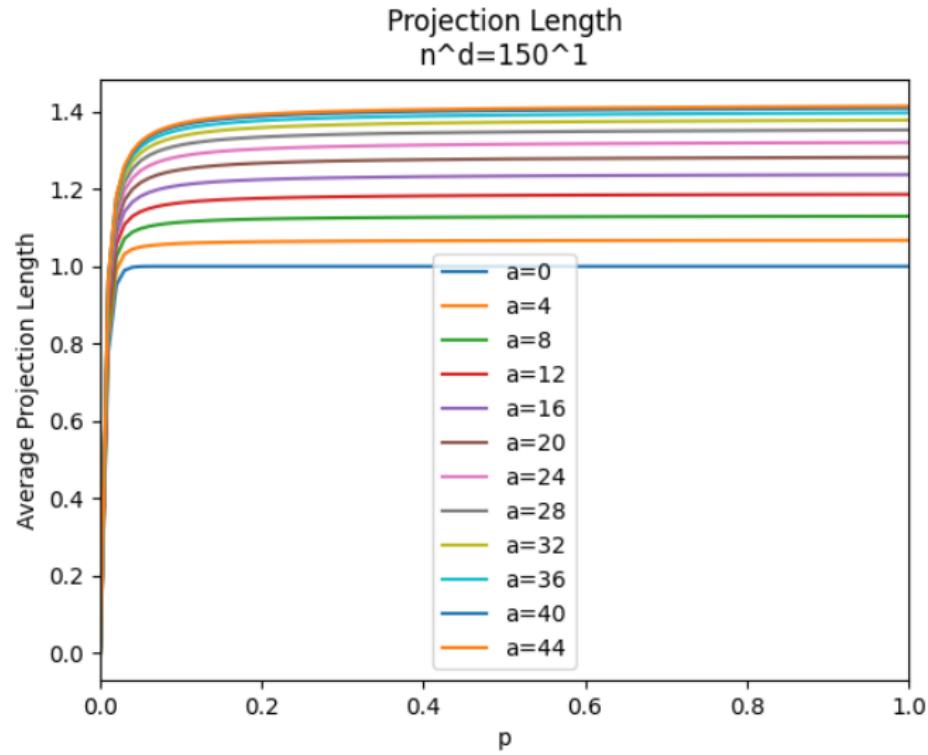
Projection

Uniform Percolation, 2D



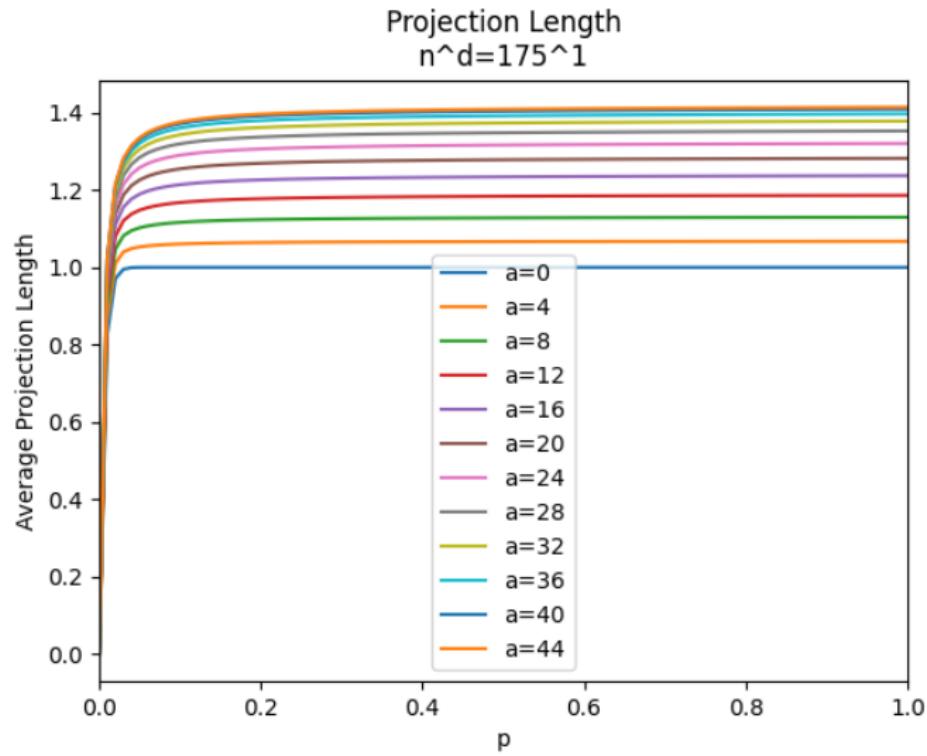
Projection

Uniform Percolation, 2D



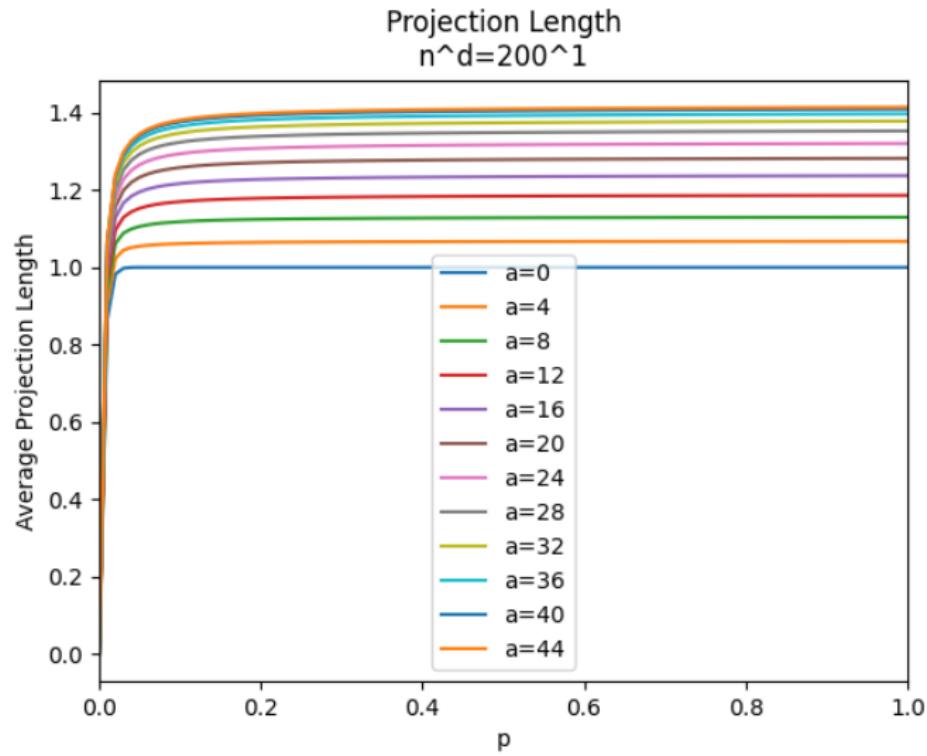
Projection

Uniform Percolation, 2D

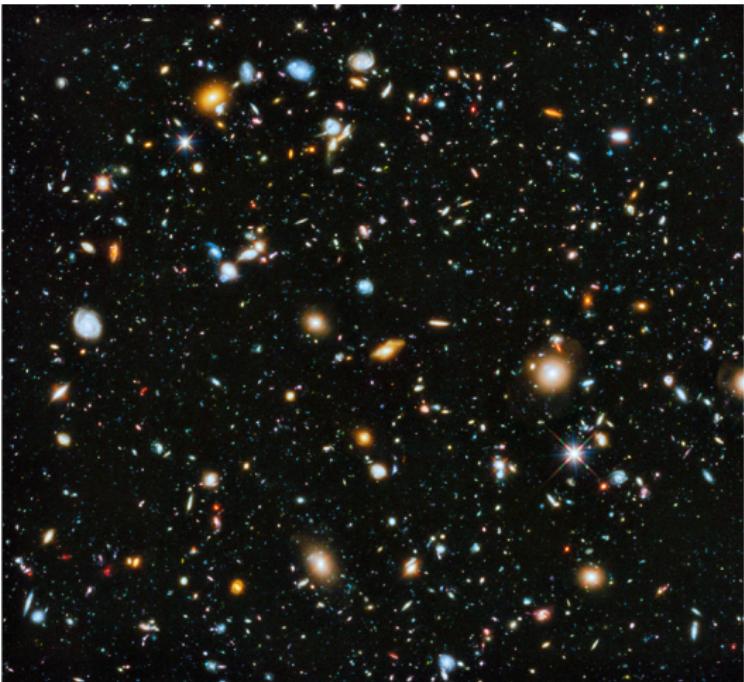


Projection

Uniform Percolation, 2D



Application to physics



"Hubble Ultra Deep Field"