Mathematics is *not* broken.

Random Fractals

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Supervisor: Ben Hambly

Oxford University

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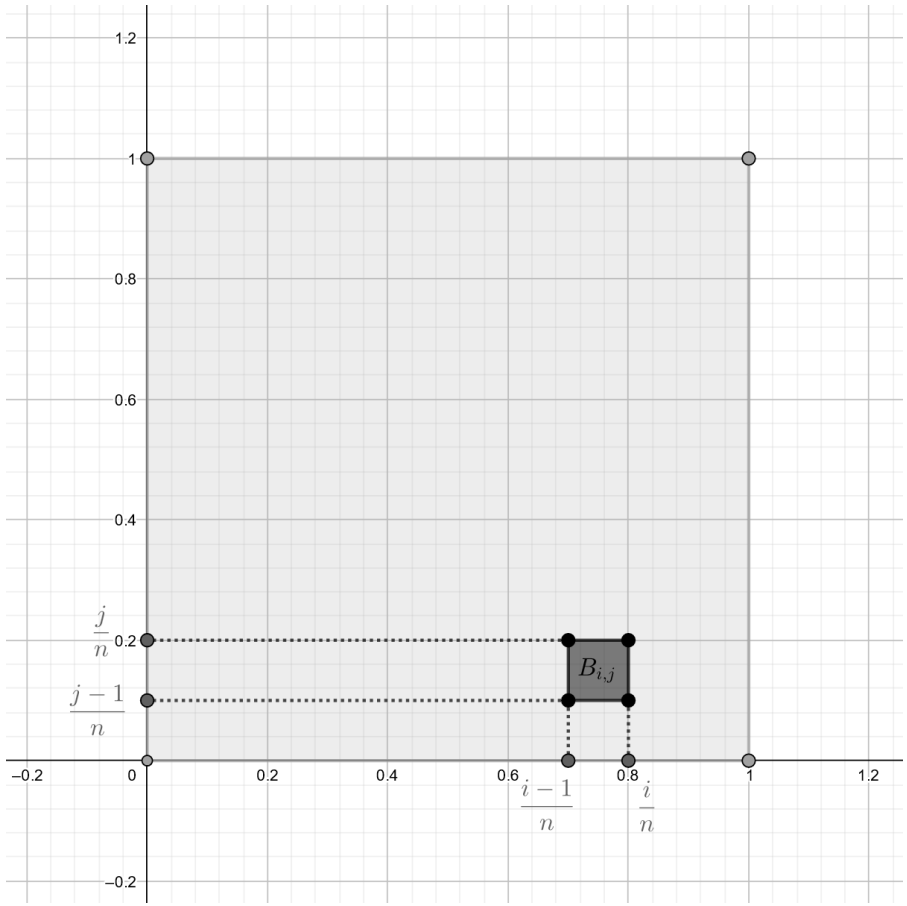
The Percolation Process

Plain: $P \sim \mathrm{Perc}(n, p, 1)$

$$B_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$



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$$B_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

$$\varepsilon_{i,j} \in \{0,1\} \text{ with } \mathbb{P}(\varepsilon_{i,j} = 1) = p \quad (\text{ i.e. } \varepsilon_{i,j} \sim \mathcal{B}(p))$$

$$P = \bigcup_{\substack{i,j \\ \varepsilon_{i,j}=1}} B_{i,j}$$

$$Z = |\{(i,j) \mid \epsilon_{i,j} = 1\}|$$

$$D = \frac{Z}{pn^2}$$

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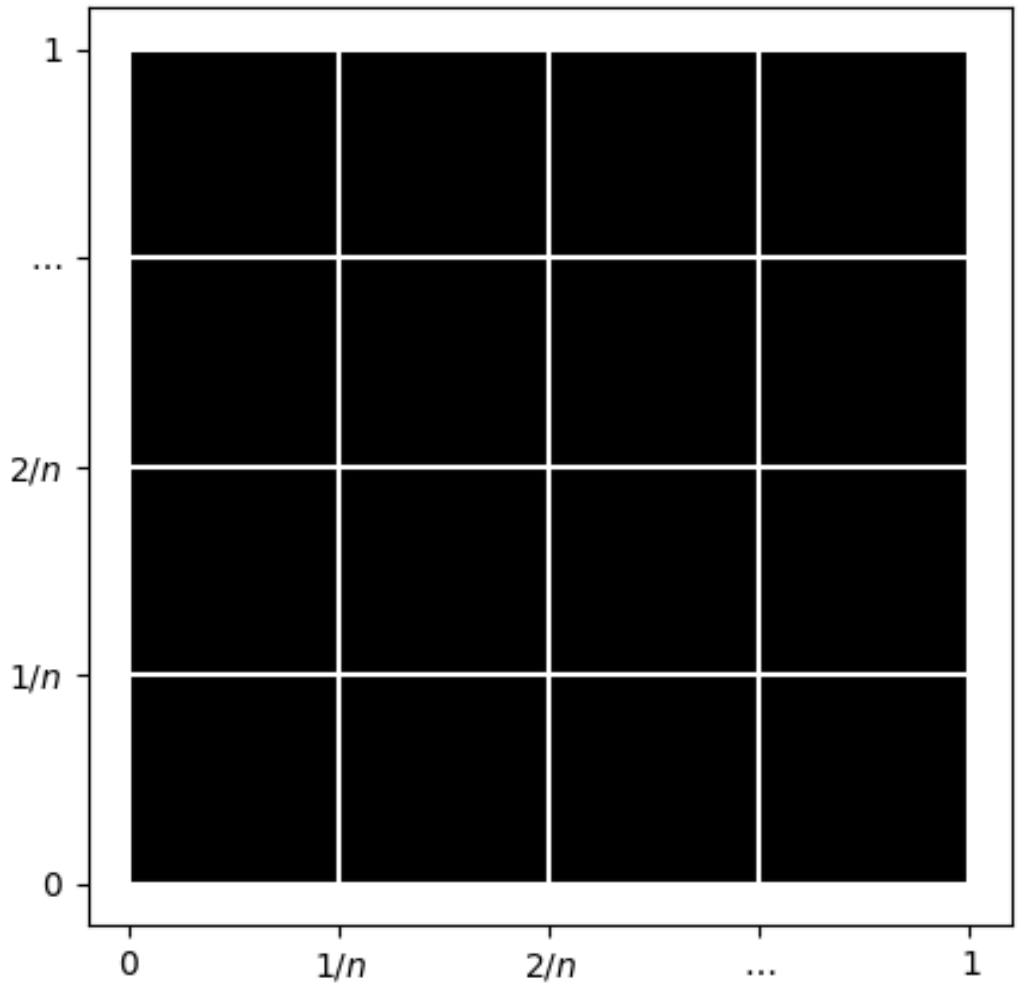

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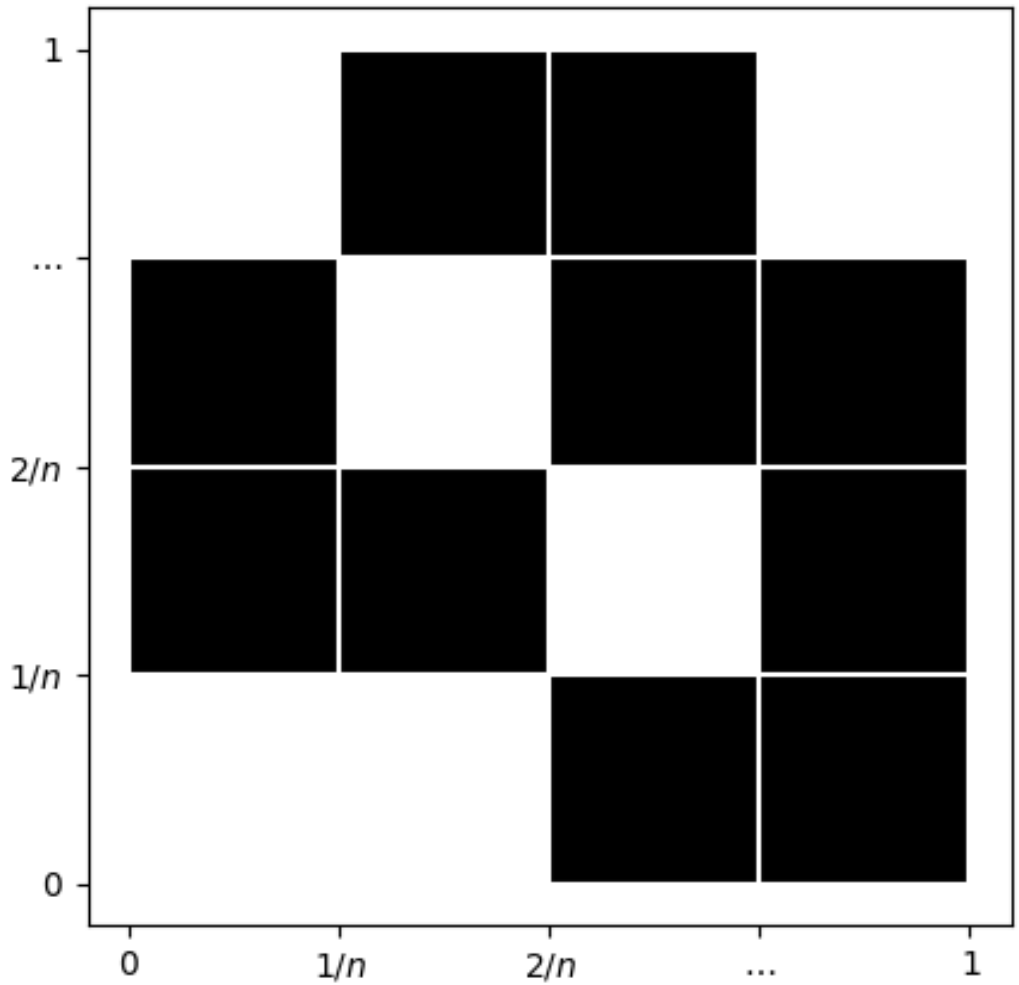
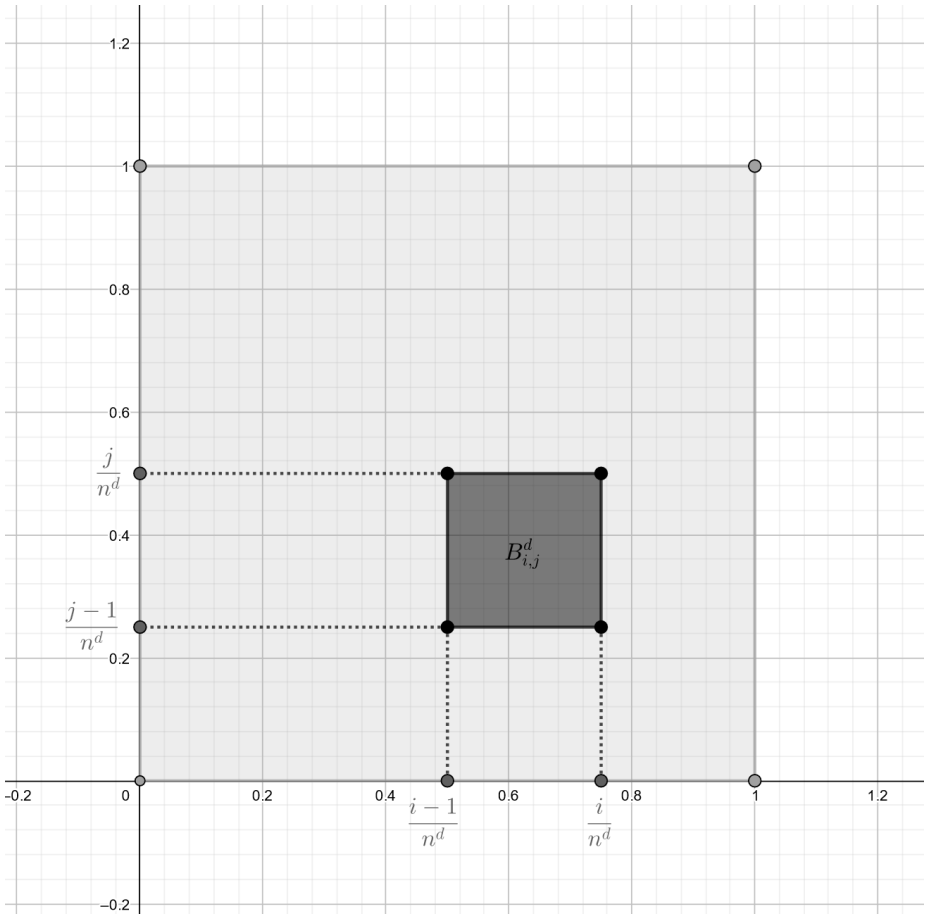


The Percolation Process

Recursive: $P_d \sim \text{Perc}(n, p, d)$

$$B^d_{i,j} = \left[\frac{i-1}{n^d}, \frac{i}{n^d}\right] \times \left[\frac{j-1}{n^d}, \frac{j}{n^d}\right]$$



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$$\varepsilon^d_{i,j} \in \{0,1\} \text{ with } \mathbb{P}\left(\varepsilon^d_{i,j} = 1\right) = p \quad (\text{ i.e. } \varepsilon^d_{i,j} \sim \mathcal{B}(p))$$

$$P_0 = [0,1]^2 \quad ; \quad P_d = P_{d-1} \bigcap \left(\bigcup_{\substack{i,j \\ \varepsilon^d_{i,j}=1}} B^d_{i,j} \right)$$

$$Z_d = \left| \left\{ (i,j) \mid \epsilon^d_{i,j} = 1 \right\} \right|$$

$$D_d = \frac{Z_d}{(pn^2)^d}$$

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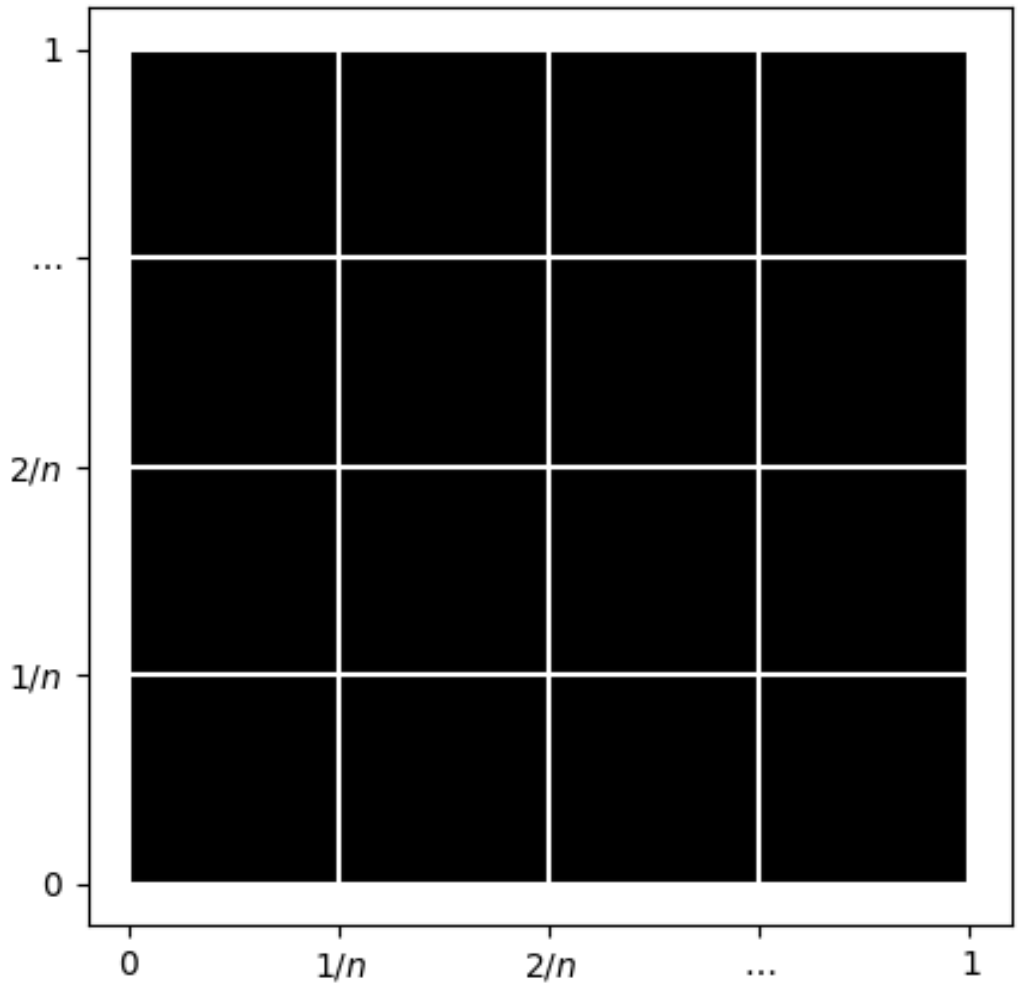

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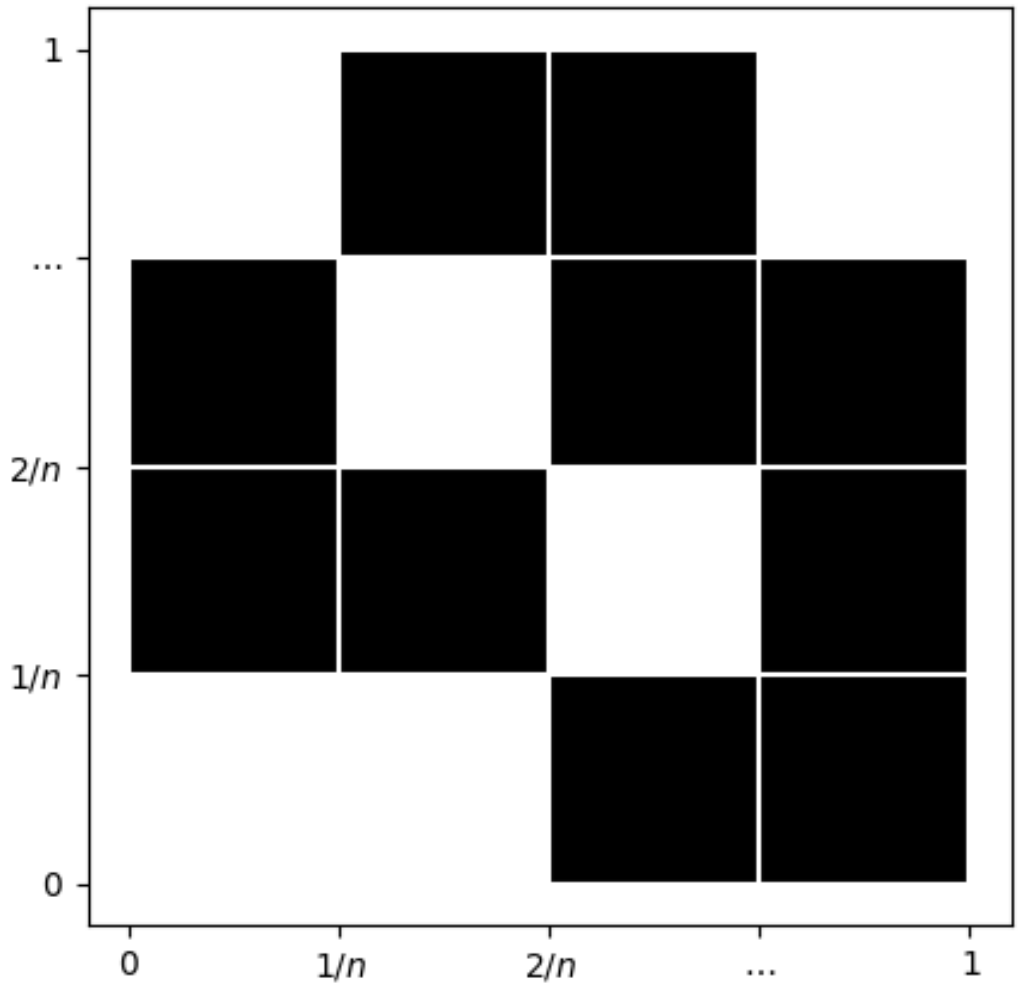

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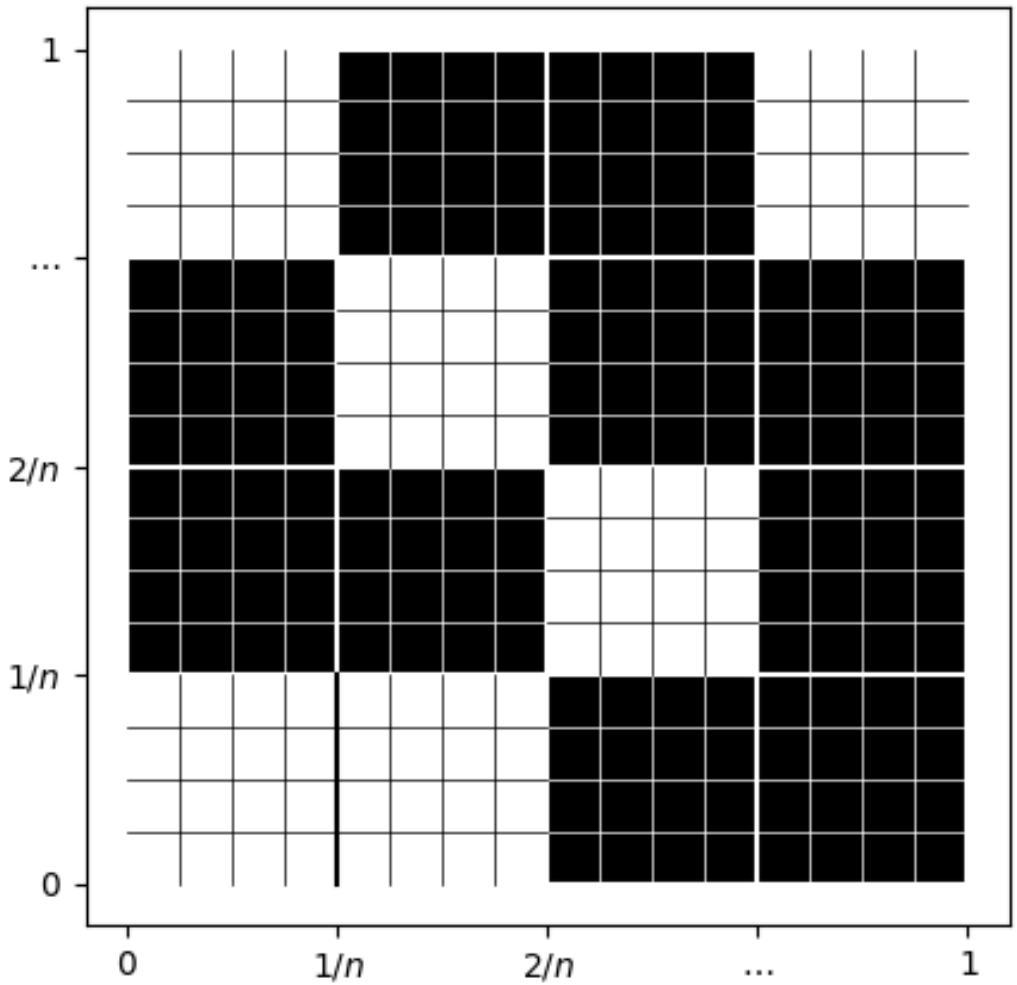

The Percolation Process

Limit: $P_\infty \sim \mathrm{Perc}\,(n, p)$

$$P_\infty = \bigcap_{d \in \mathbb{N}} P_d$$

$$D_\infty = \lim_{d \to \infty} D_d$$

The Percolation Process

Limit: $P_\infty \sim \mathrm{Perc}\,(n, p)$

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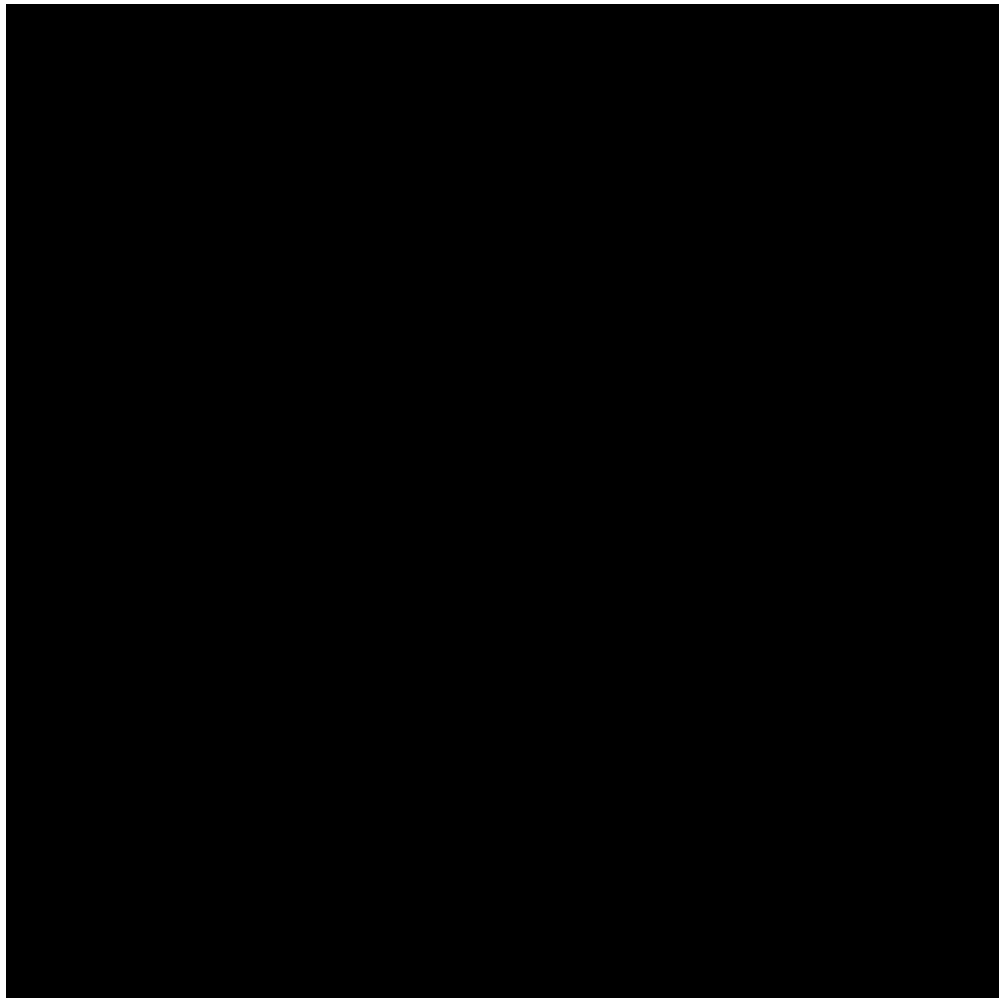
$$D_\infty > 0 \iff P_\infty \neq \emptyset$$

$$\mathbb{E}(D_\infty) = 1$$

The Percolation Process

Example for $n = 2$



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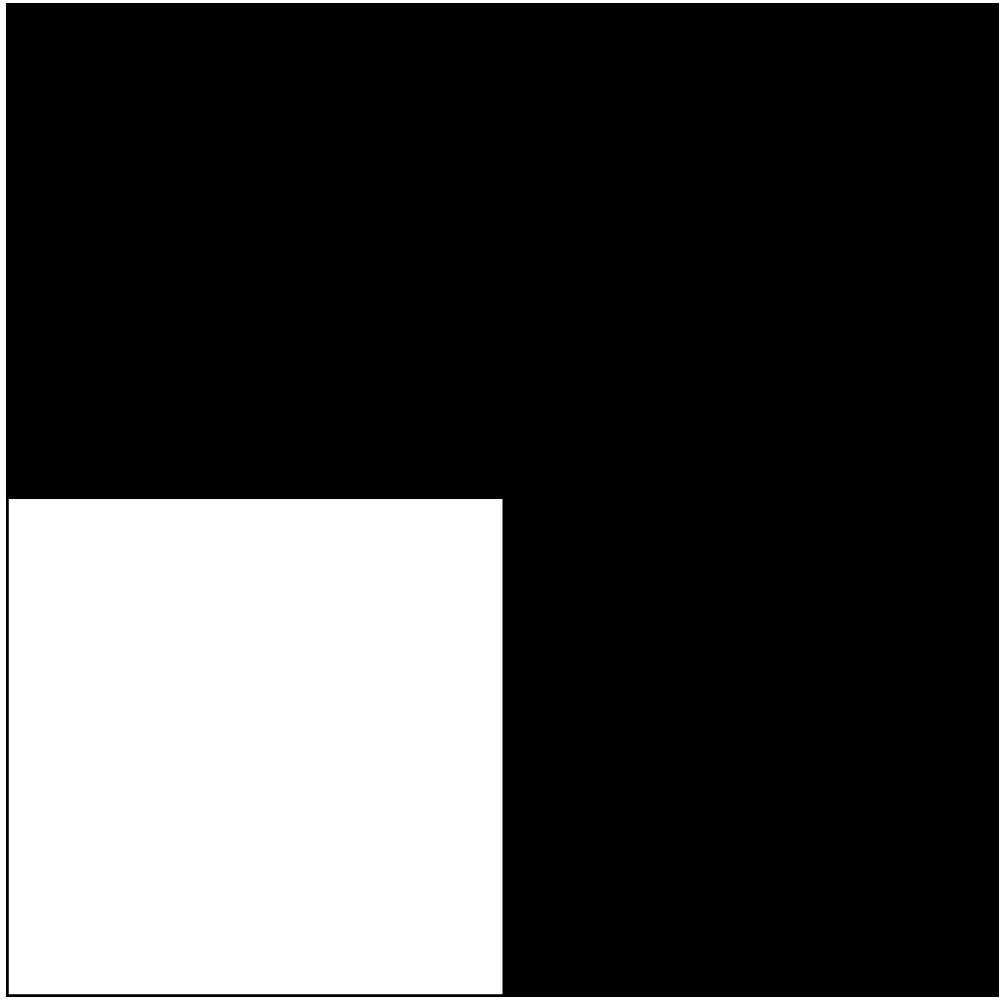

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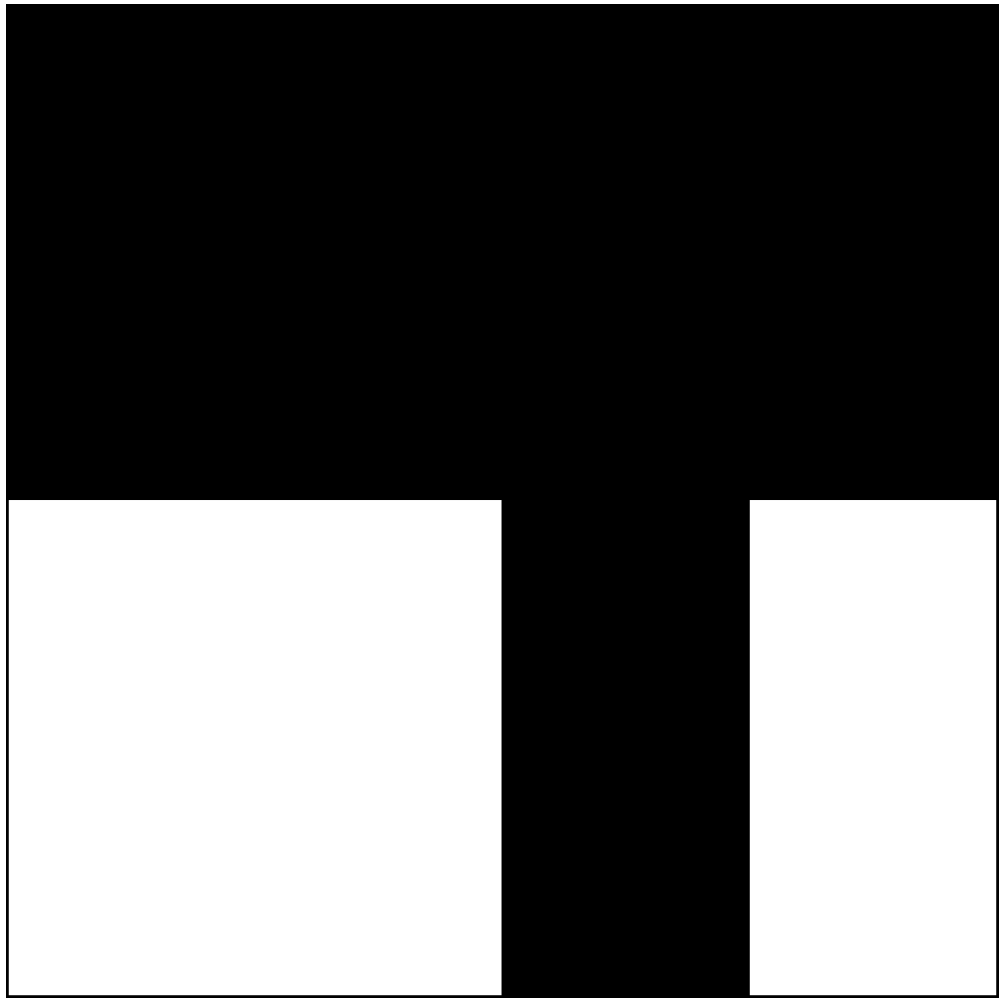

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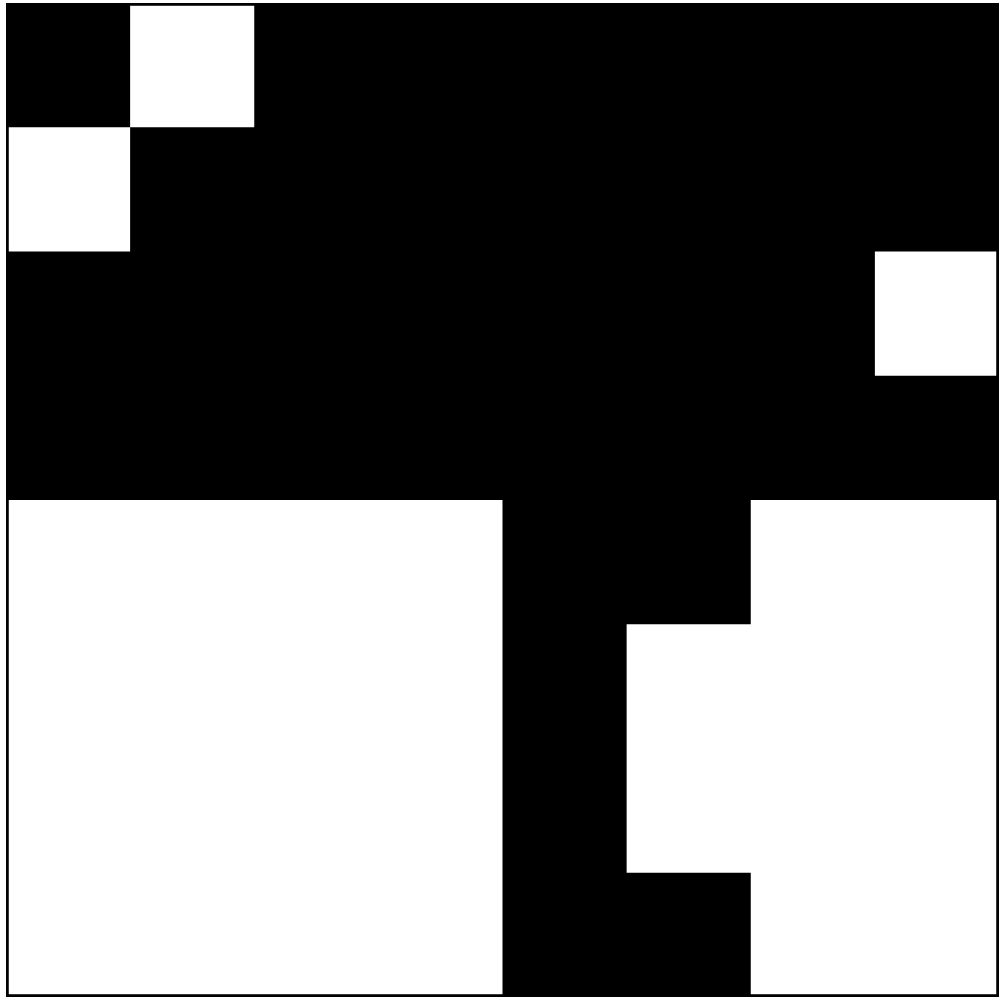

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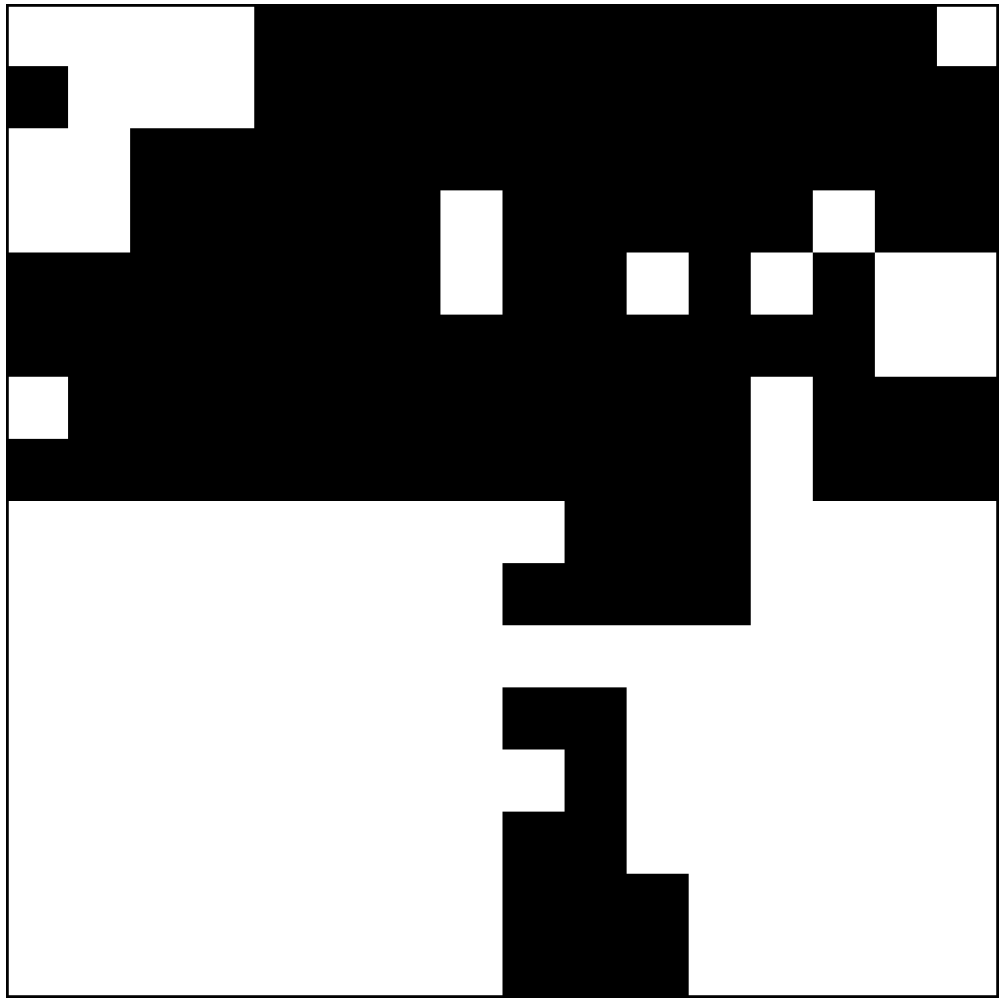

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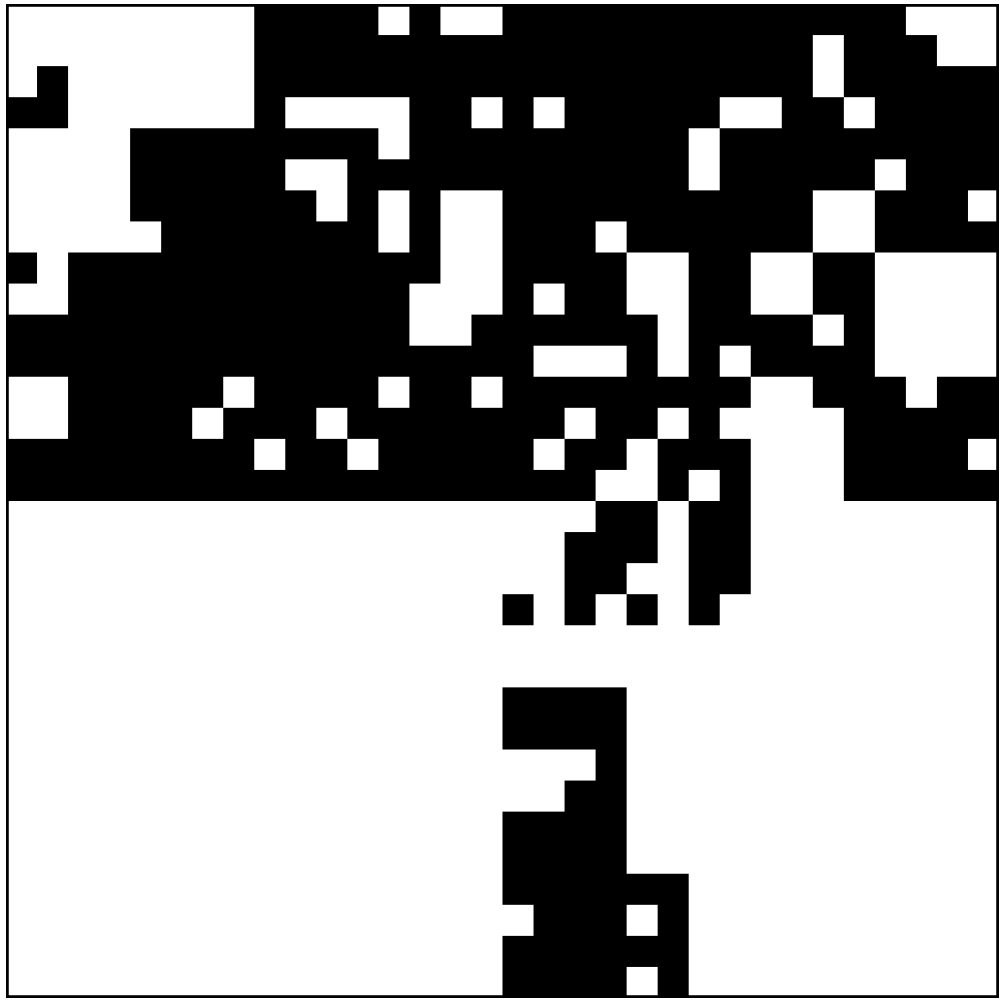

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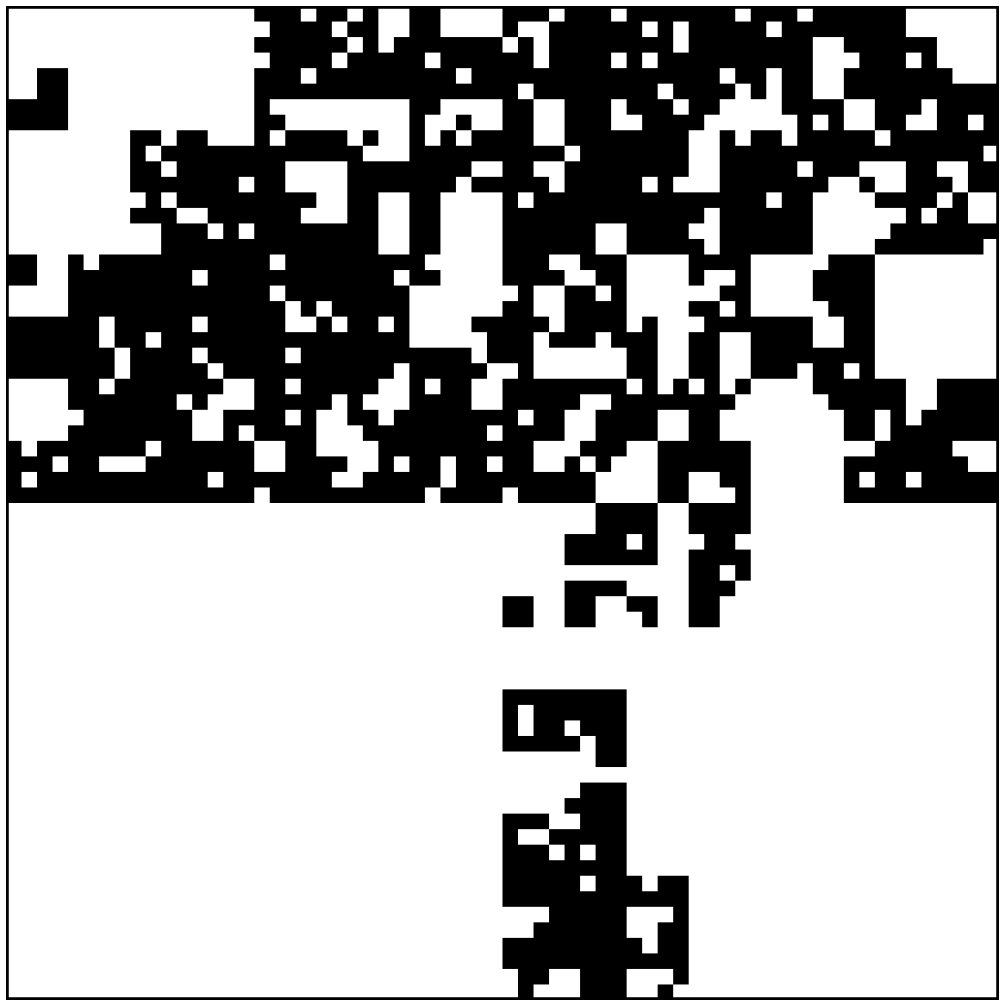

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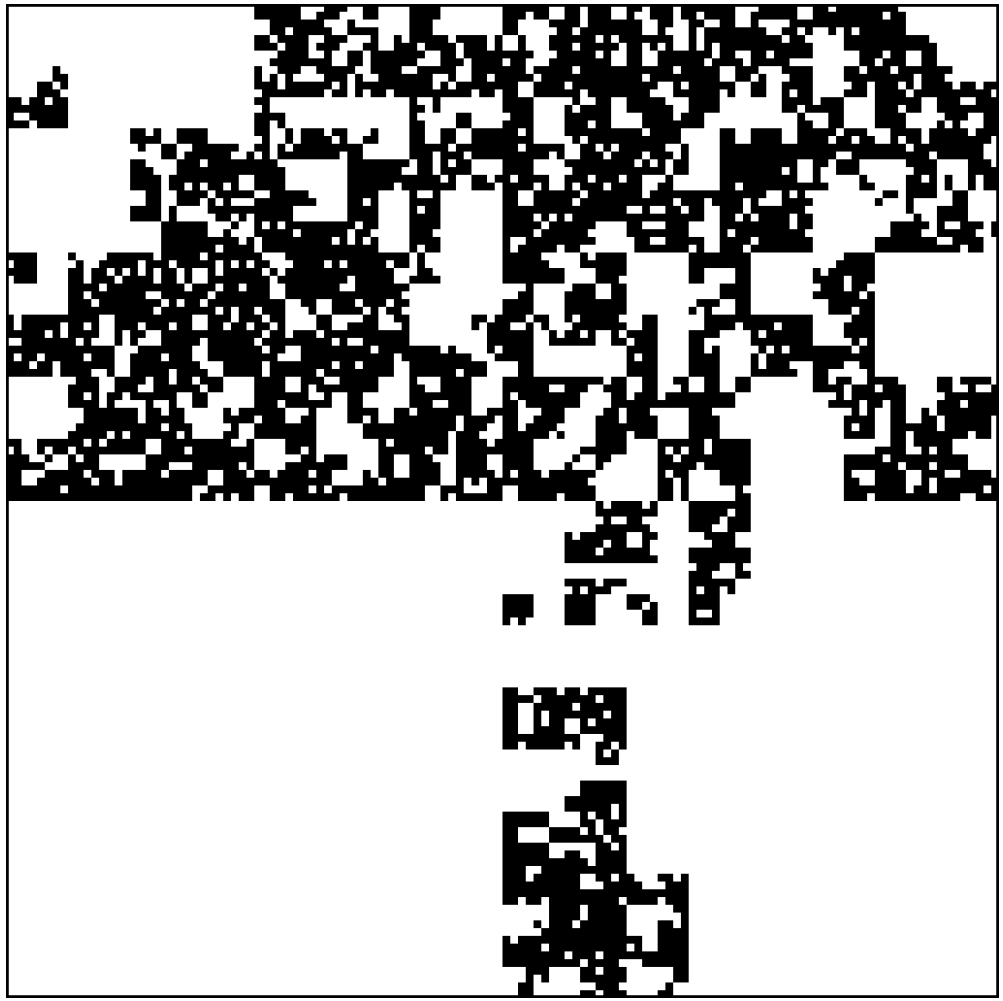

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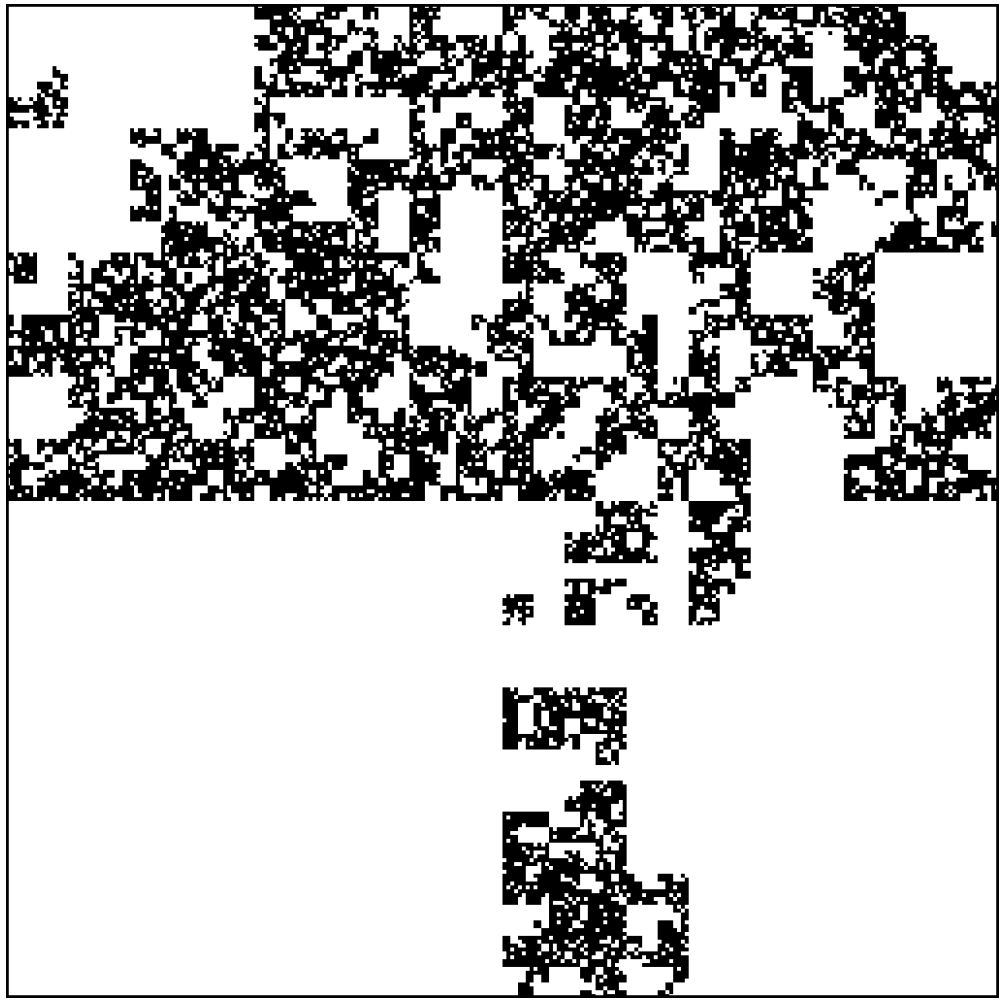

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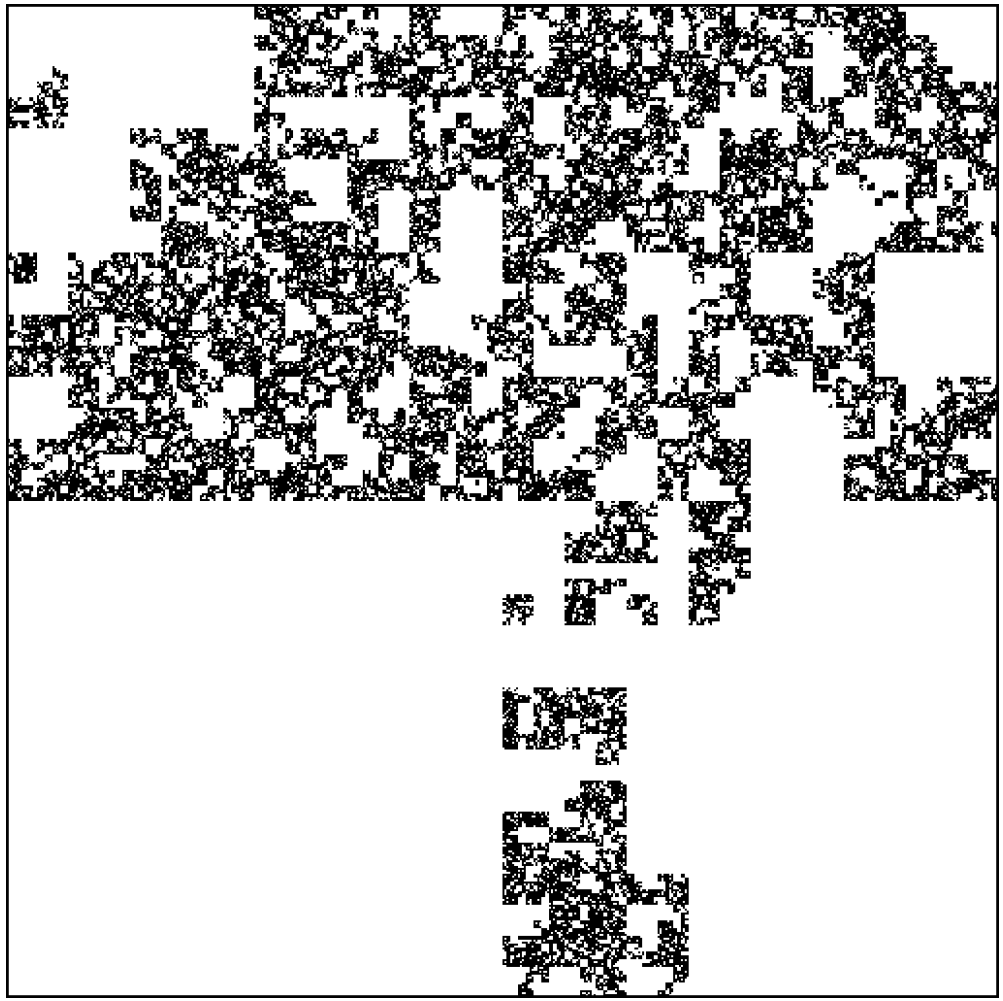
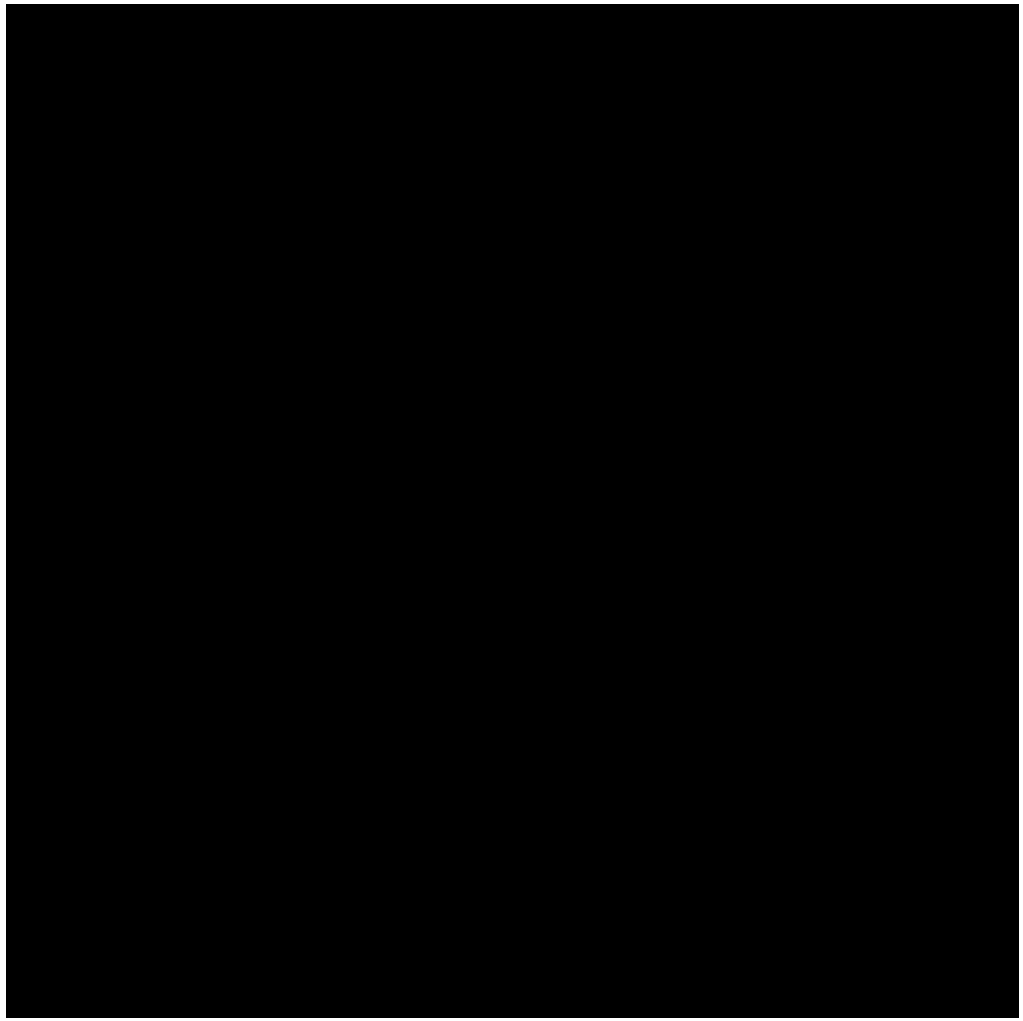


The Percolation Process

Example for $n = 3$



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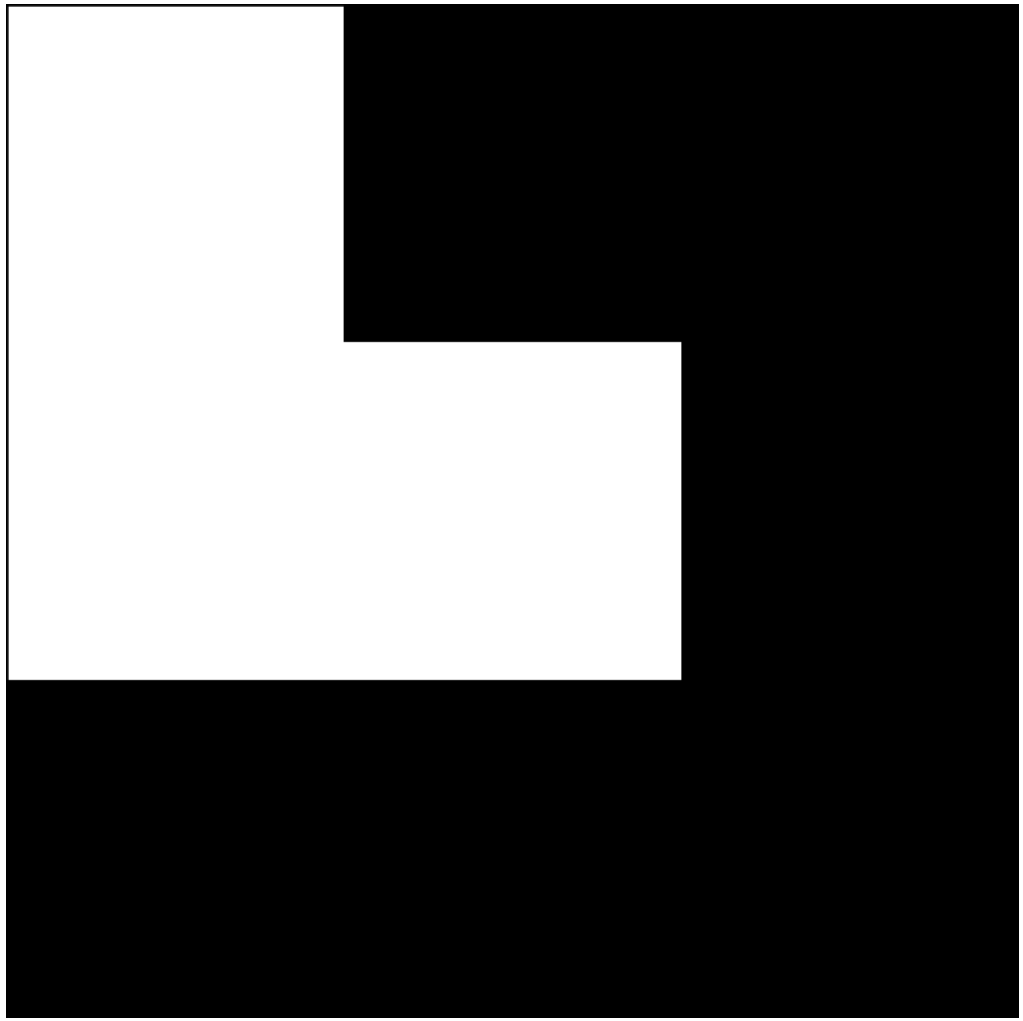

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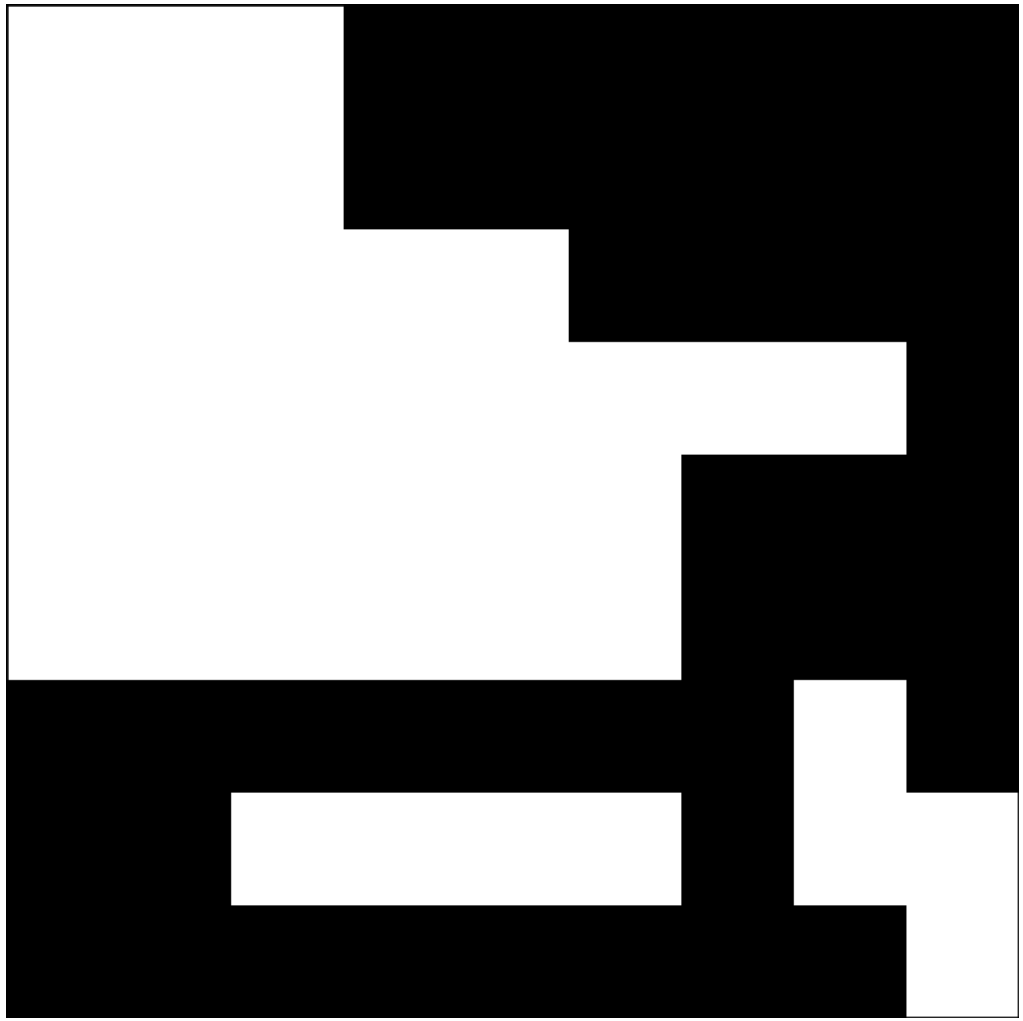

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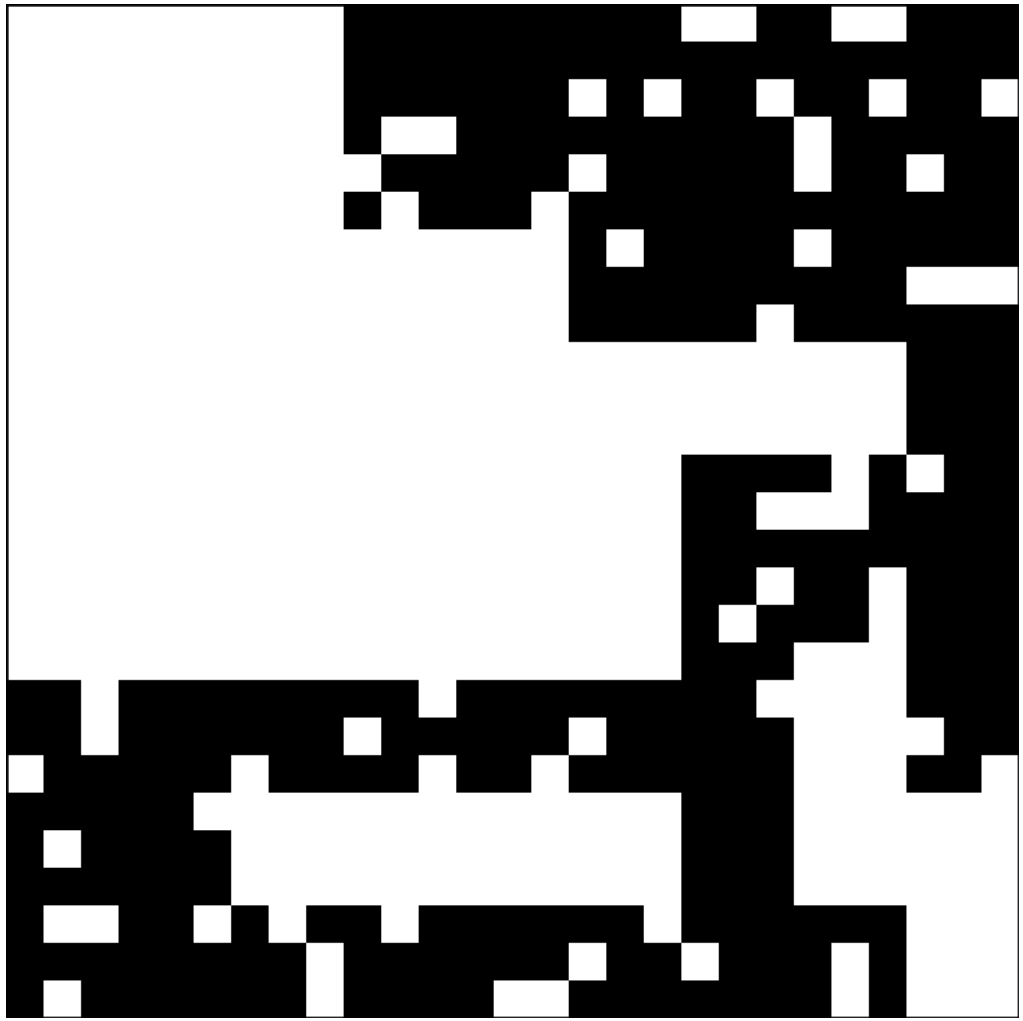

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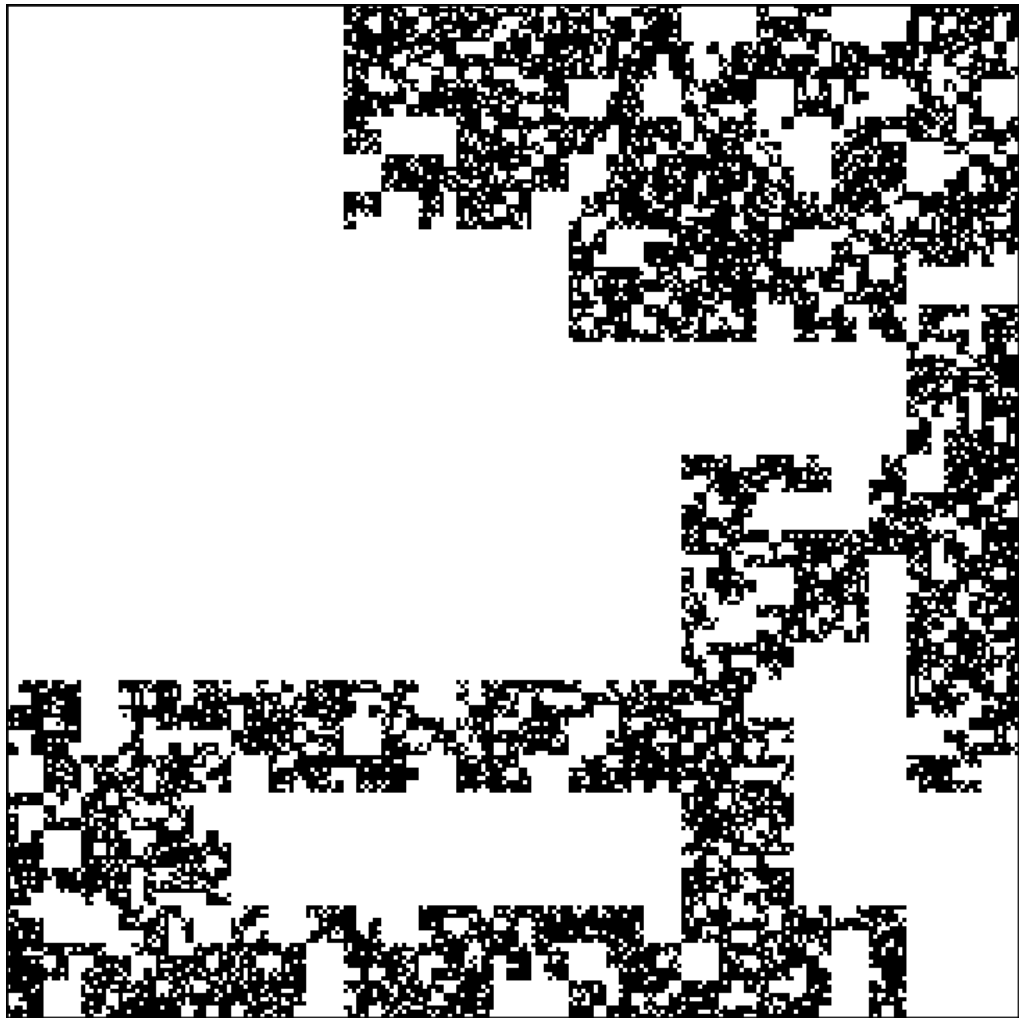

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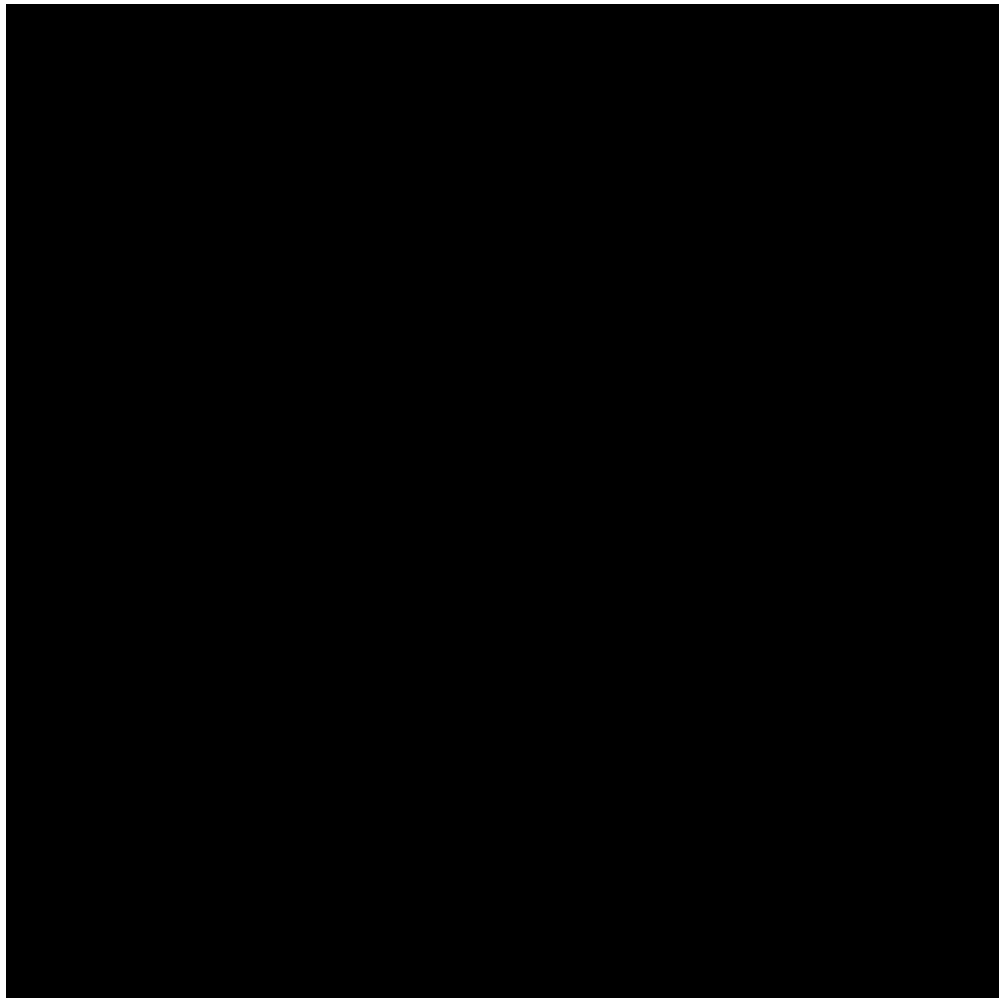
Example for $n = 5$



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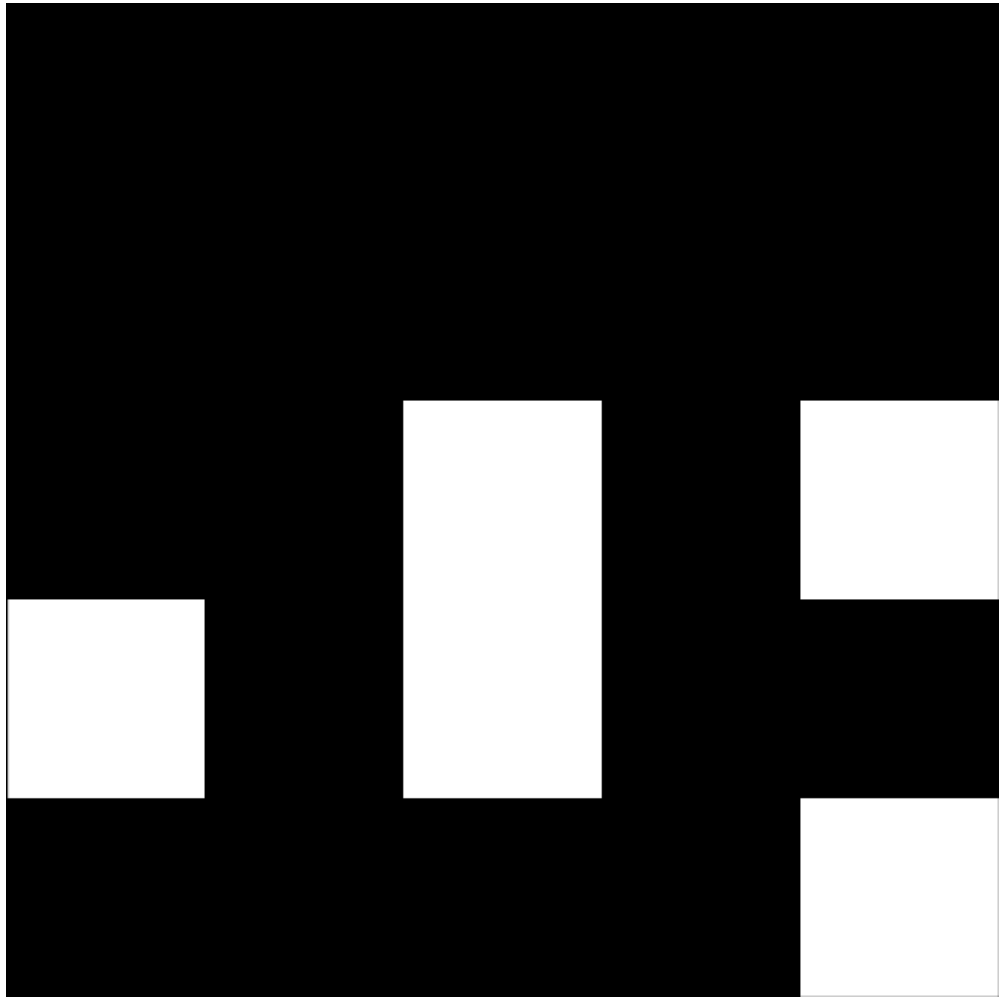

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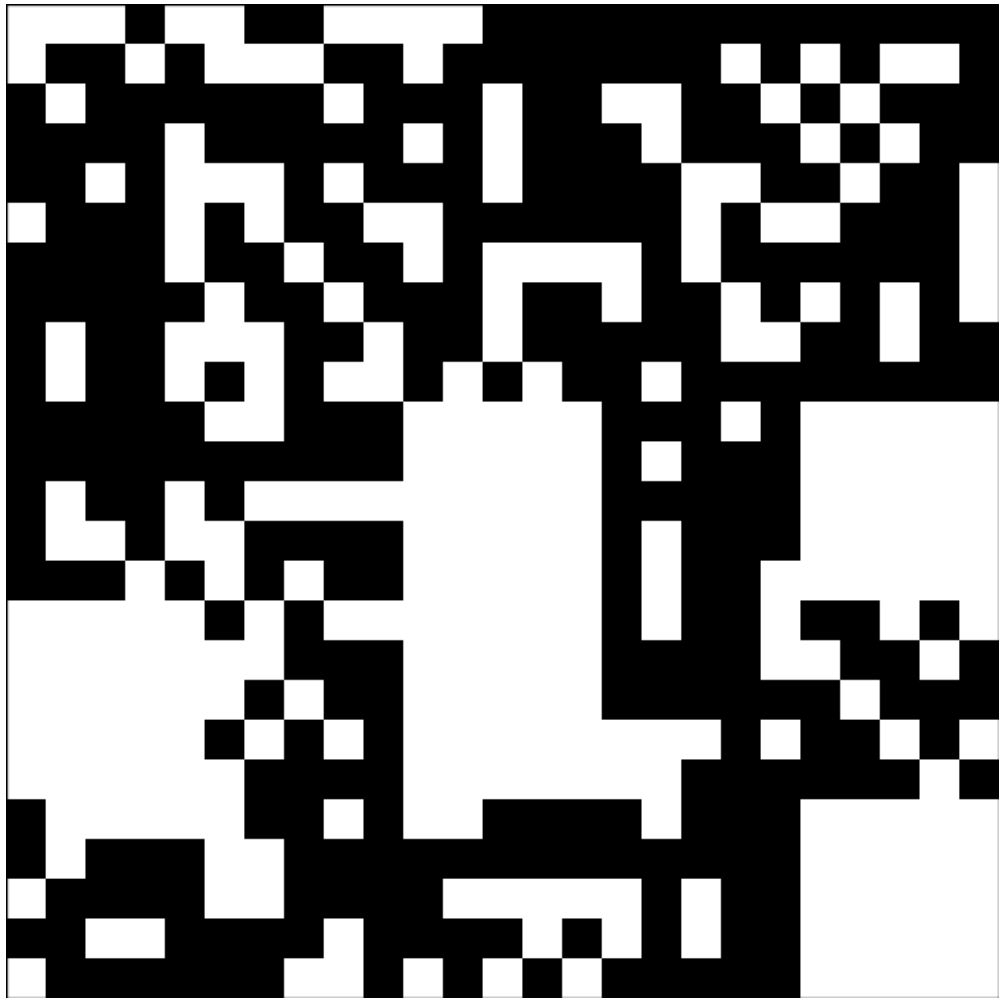

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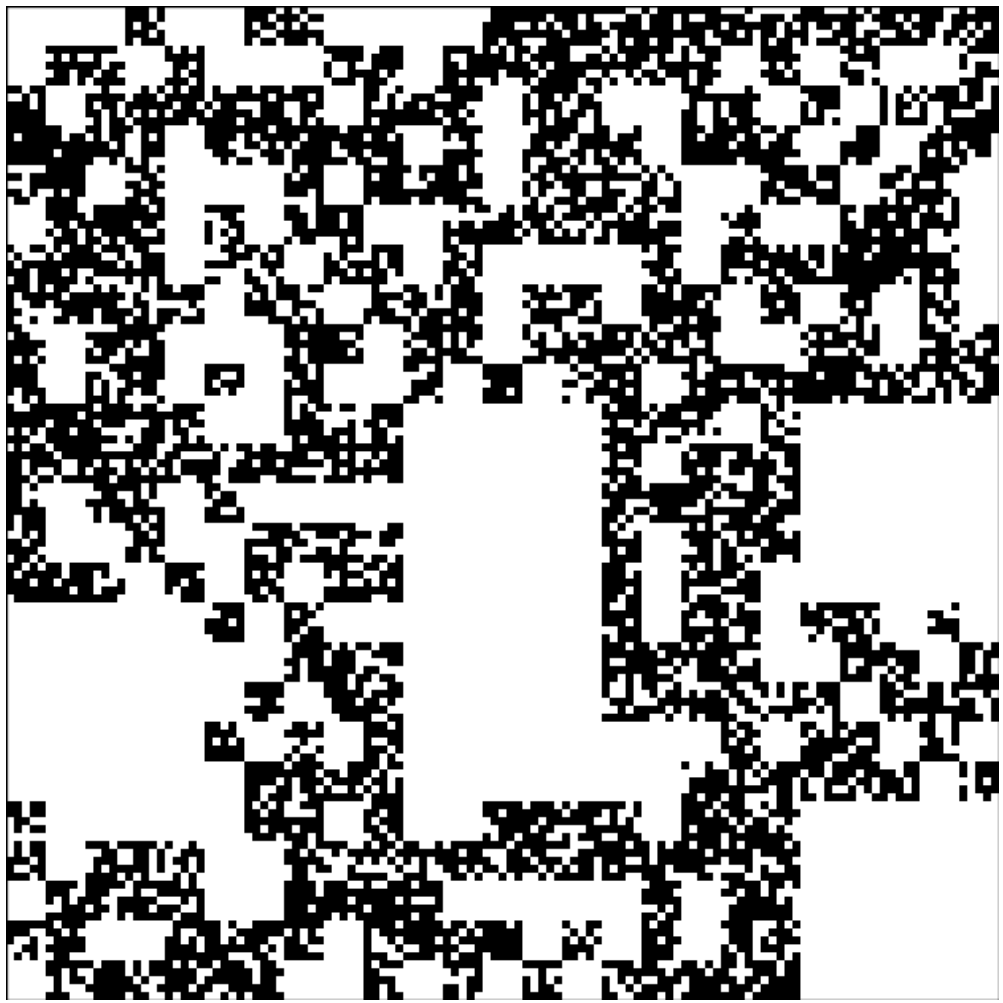

Dimensions

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...

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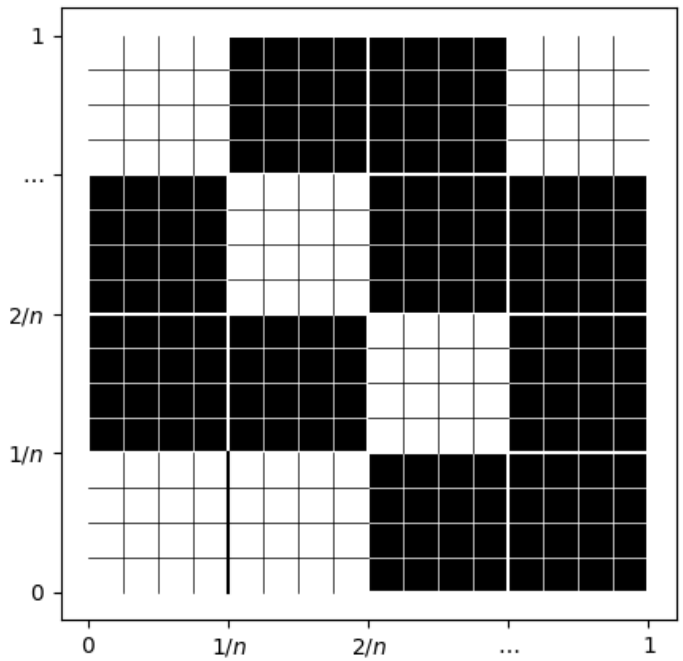
...

αD: Scale by $\lambda \iff$ n-Dim. Volumes multiplies by $\lambda^\alpha \quad \forall \alpha \in \mathbb{R}^+$

Dimensions

Percolation dimensions

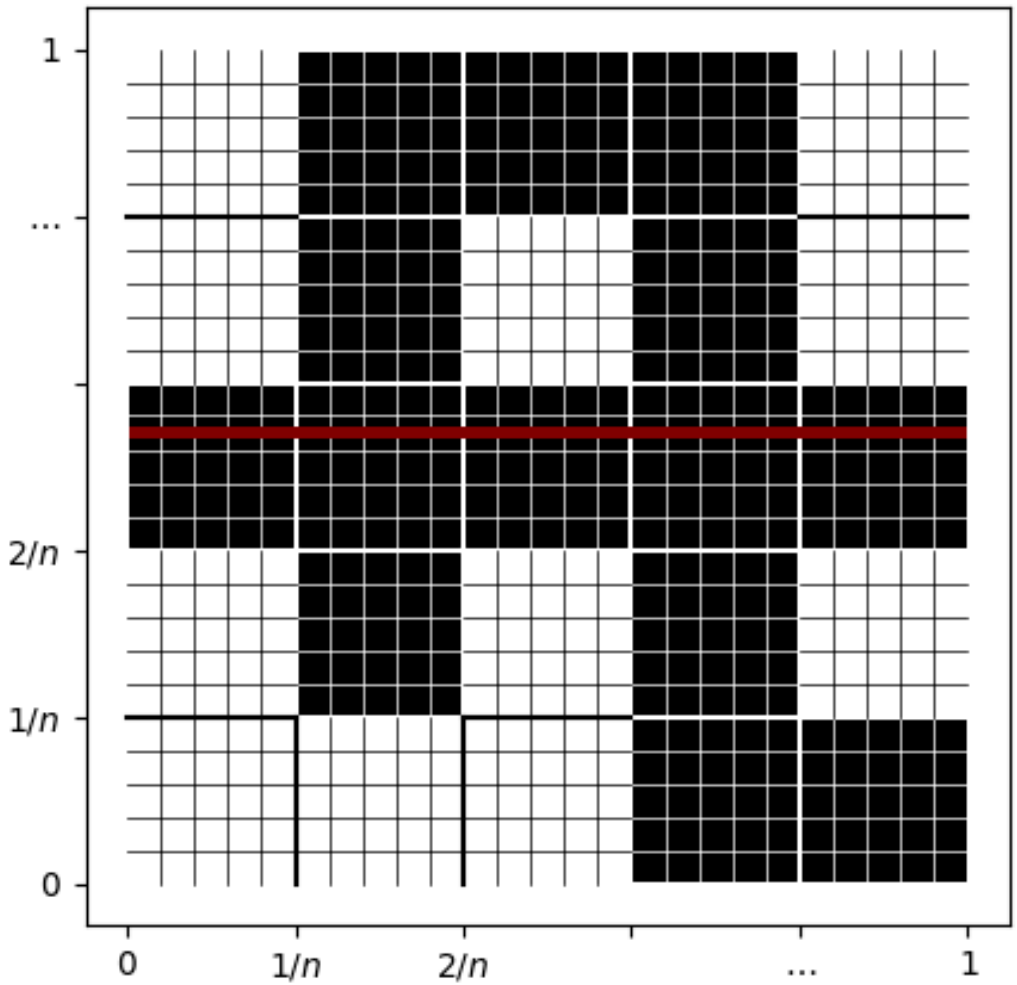
For $P \sim \mathrm{Perc}(n, p)$, scaling by n gives pn^2 copies of P.



So $\dim(P) = pn^2$.

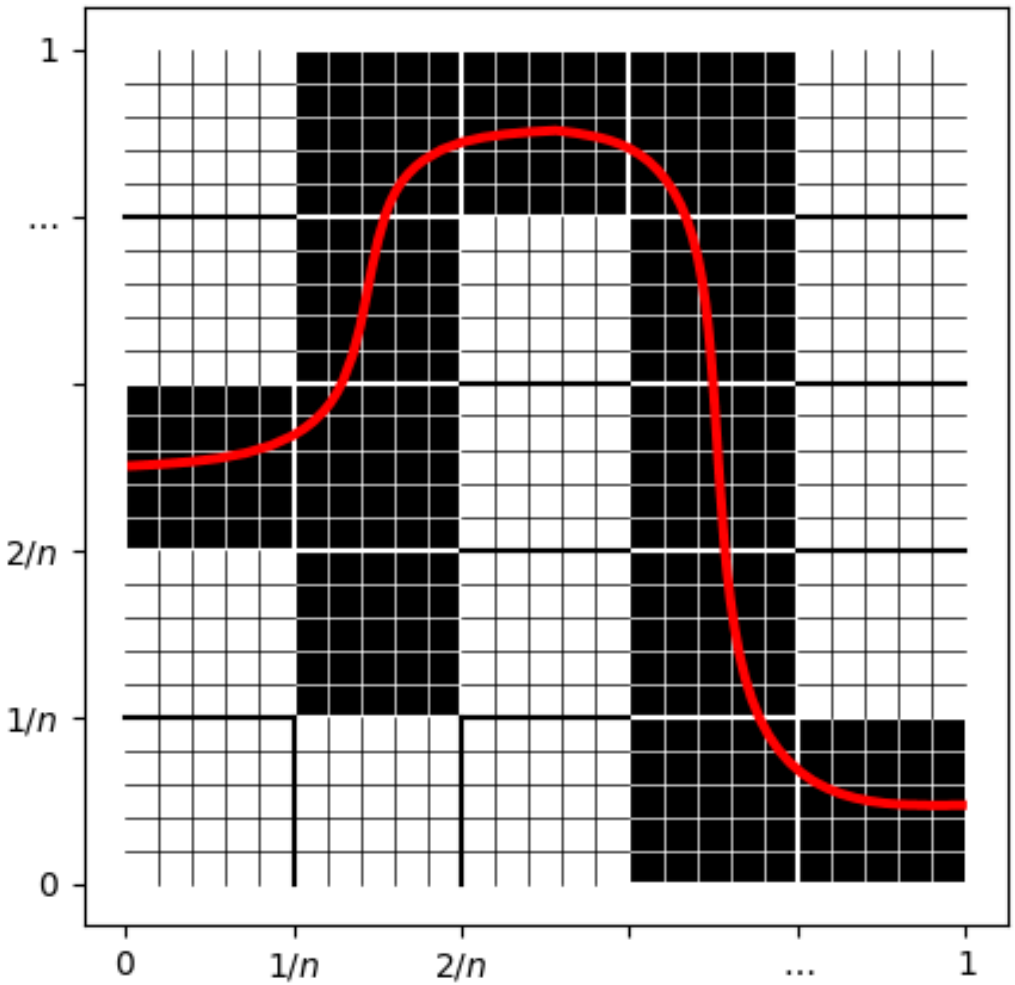
Types of Crossings

Straight



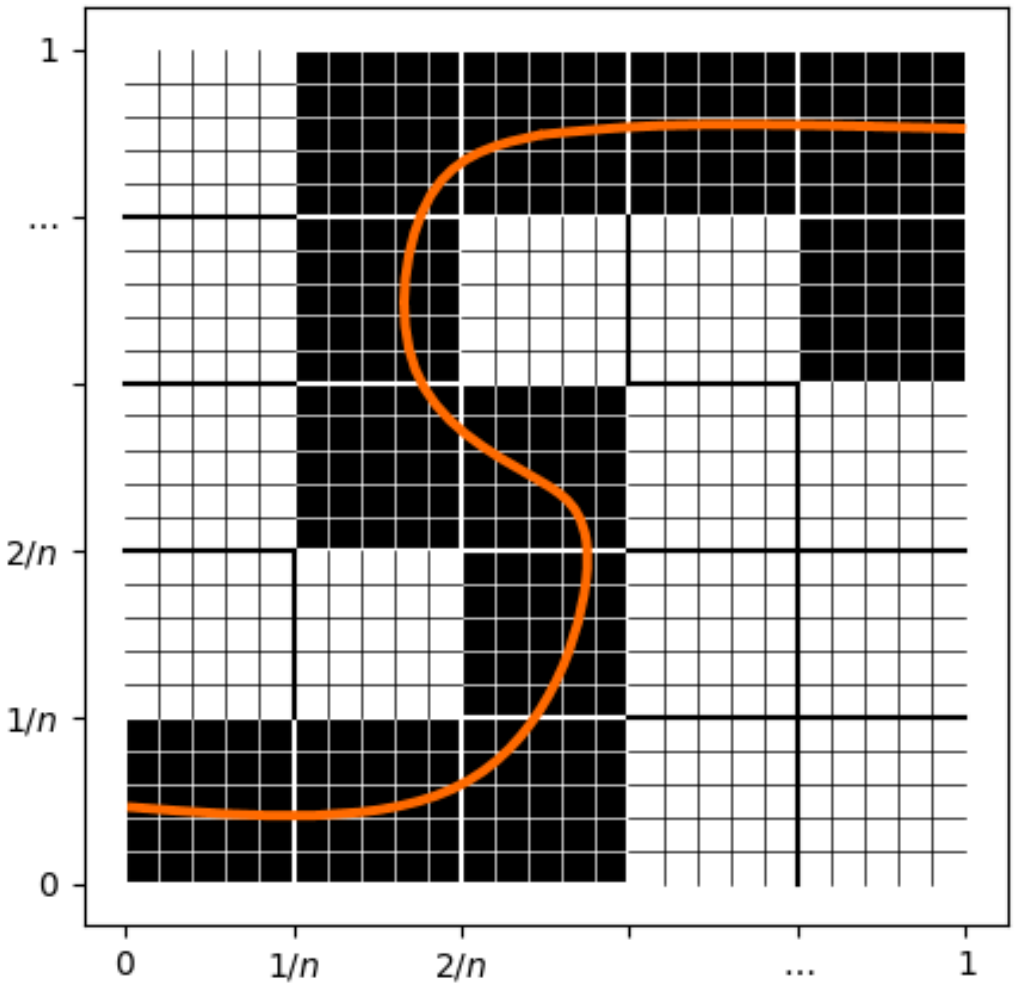
Types of Crossings

Semi-Straight



Types of Crossings

Non-Straight



Finding Crossings

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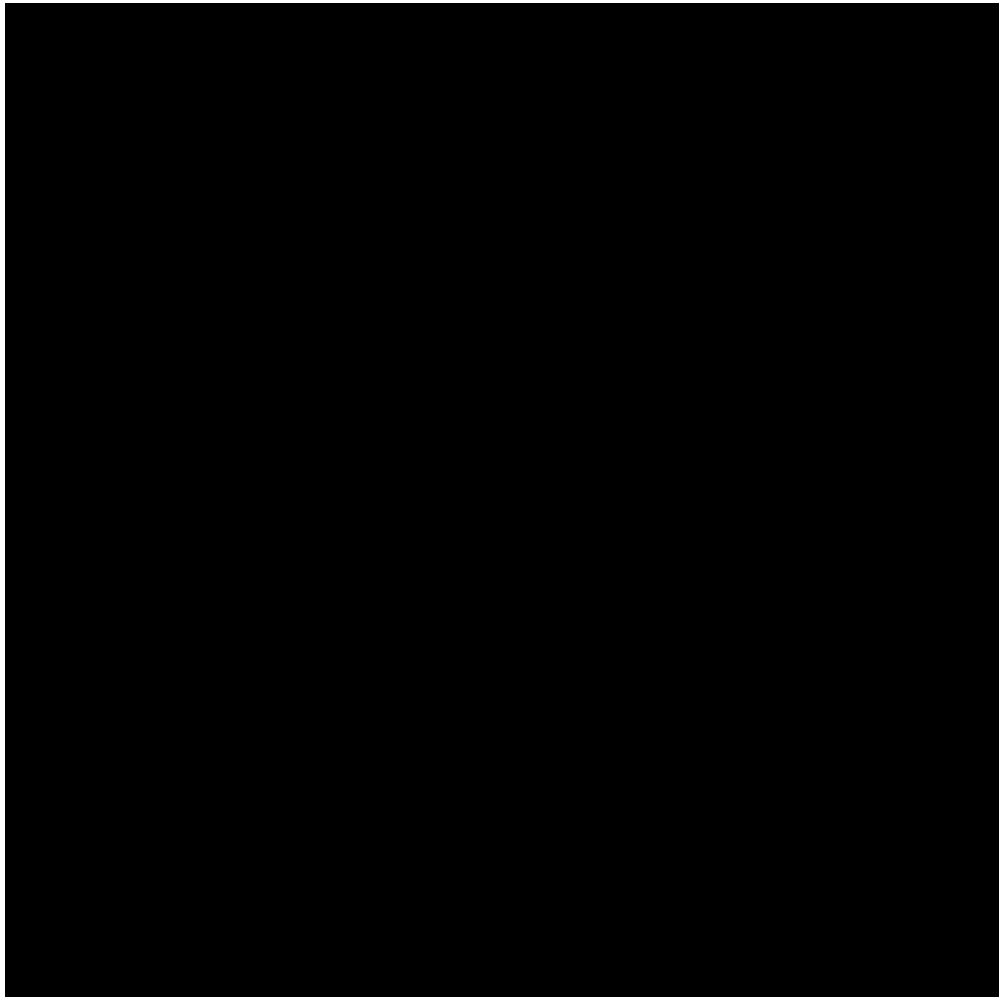

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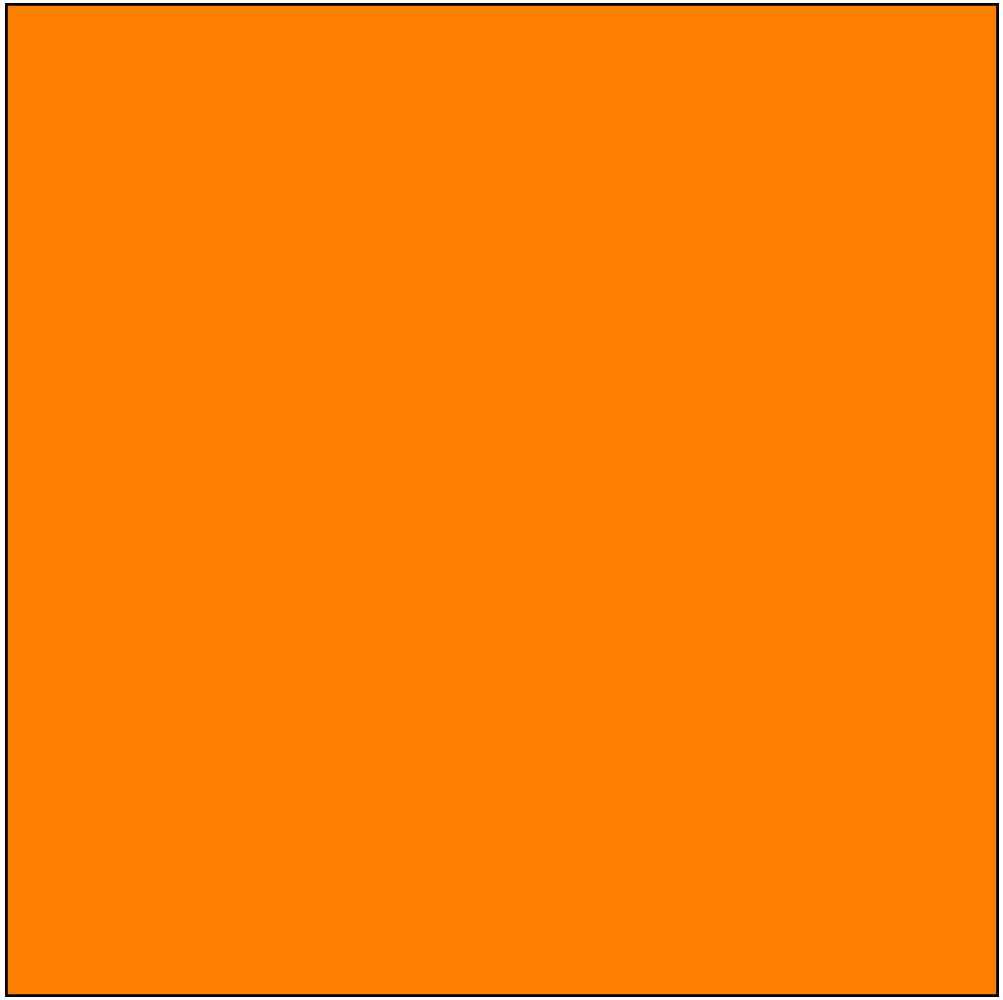

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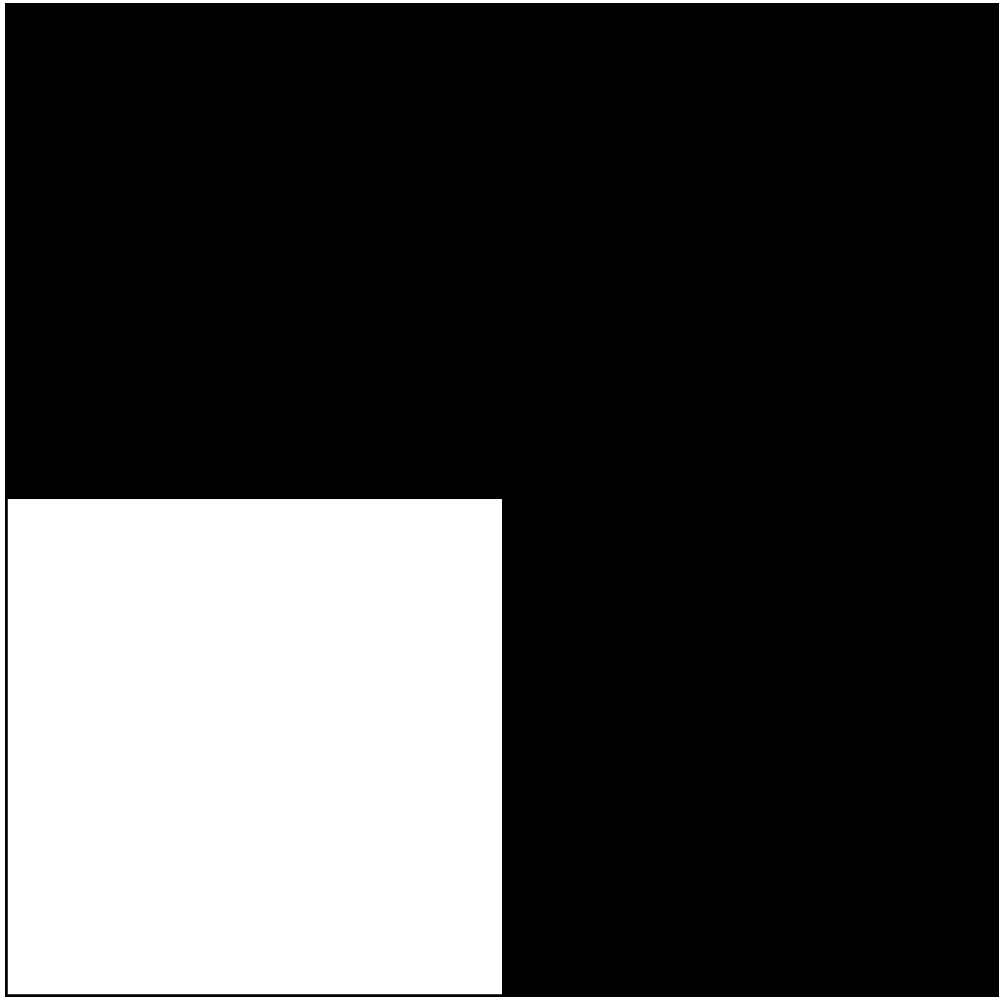

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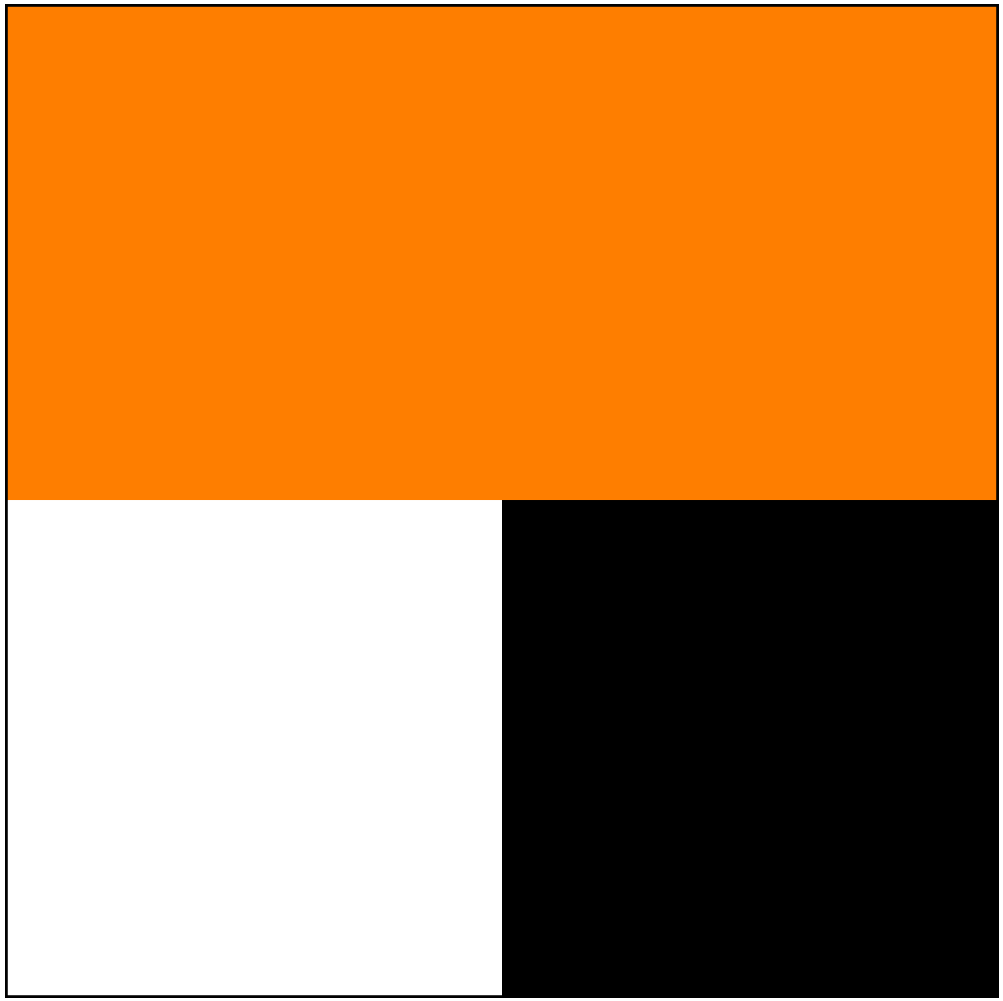

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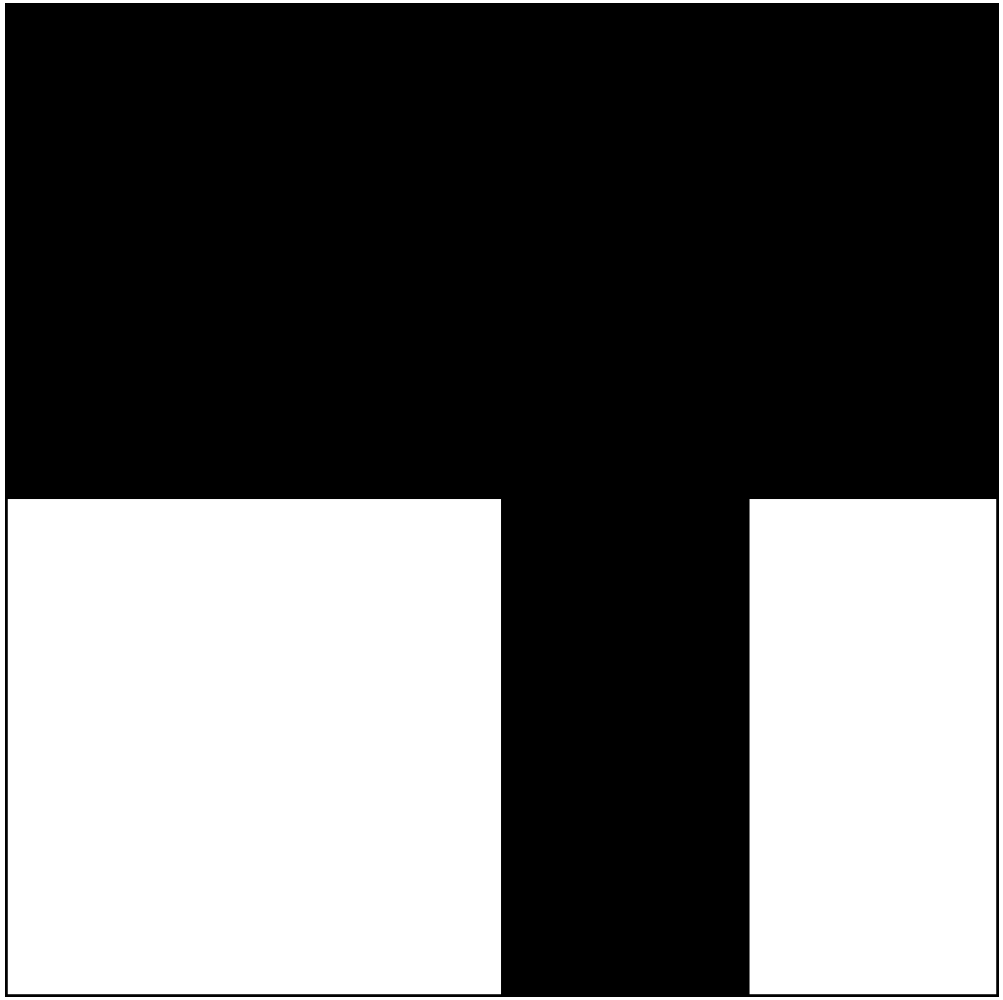

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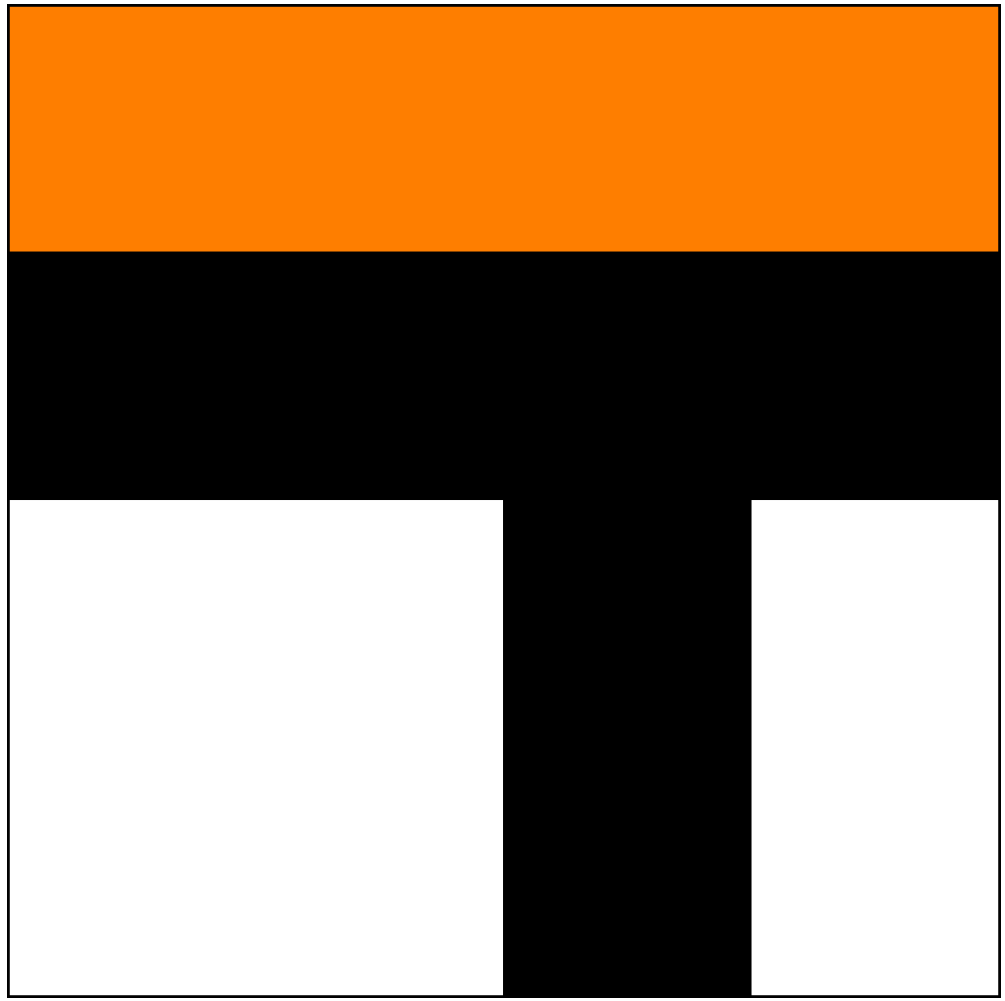

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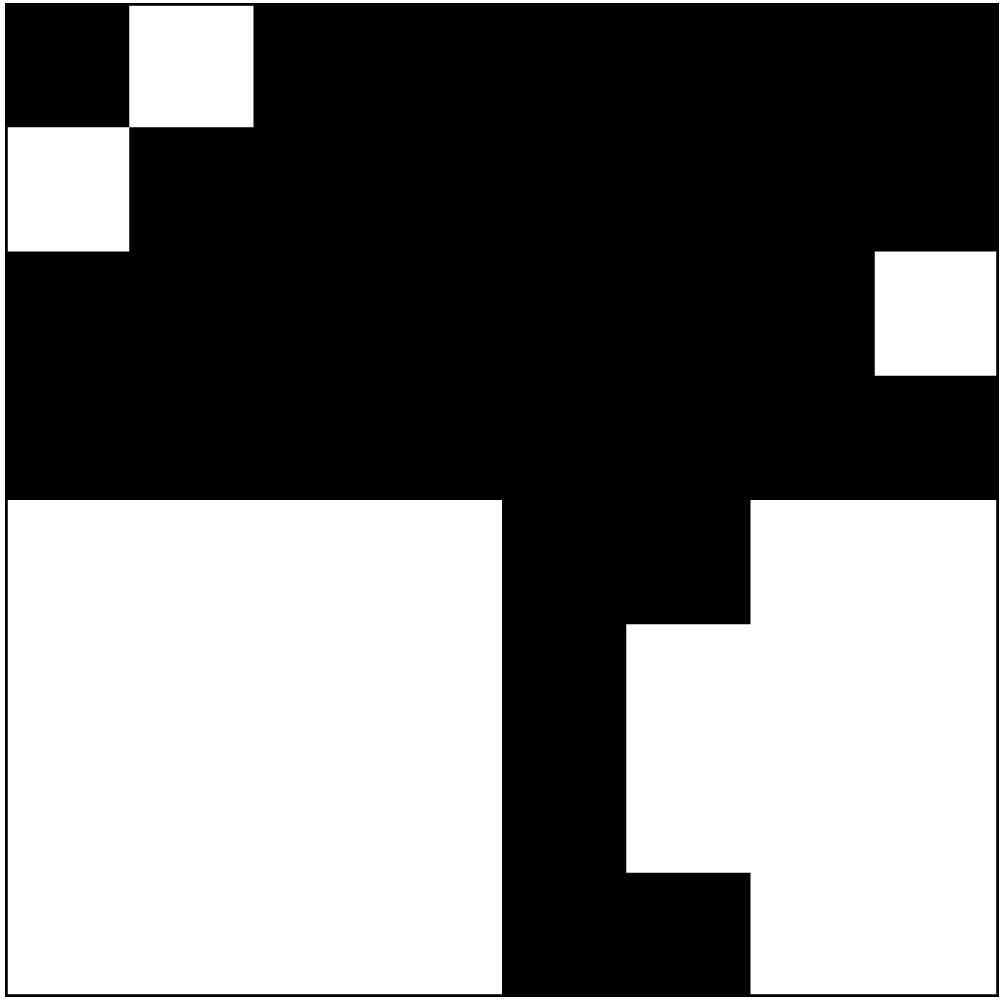

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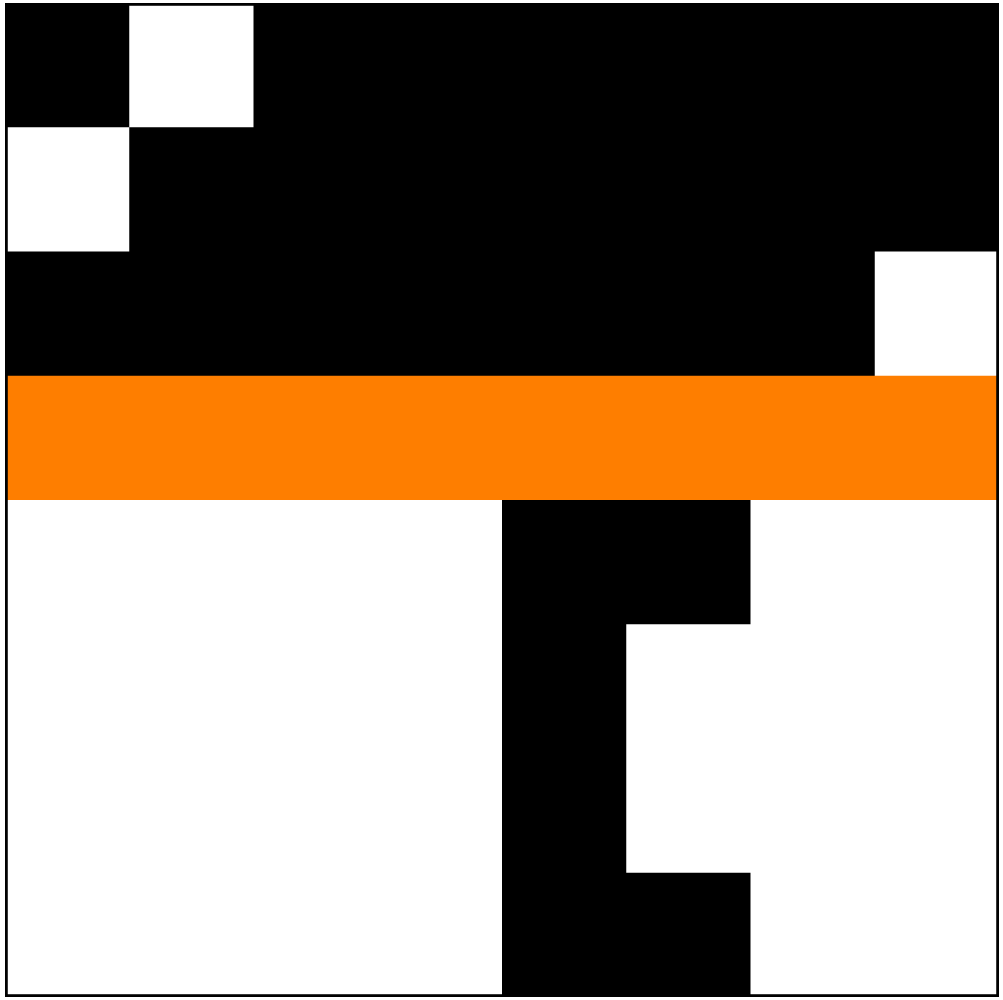

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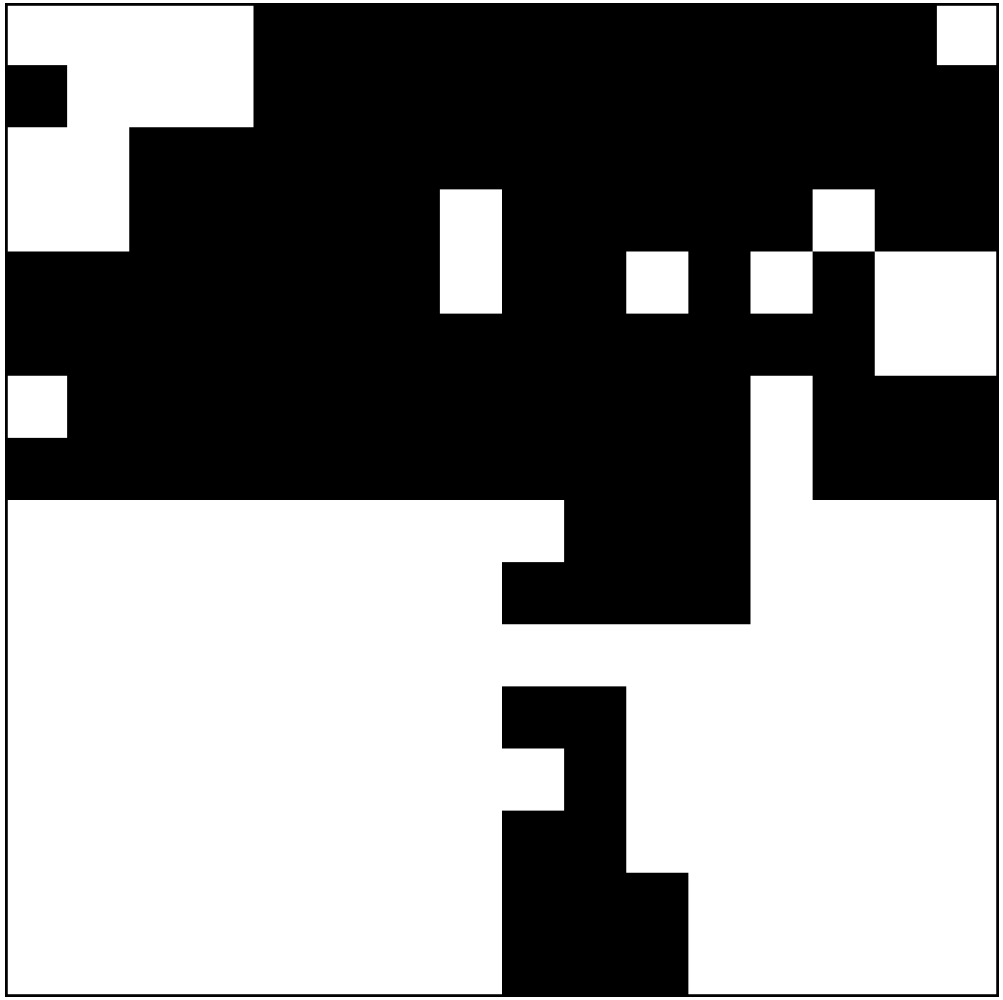

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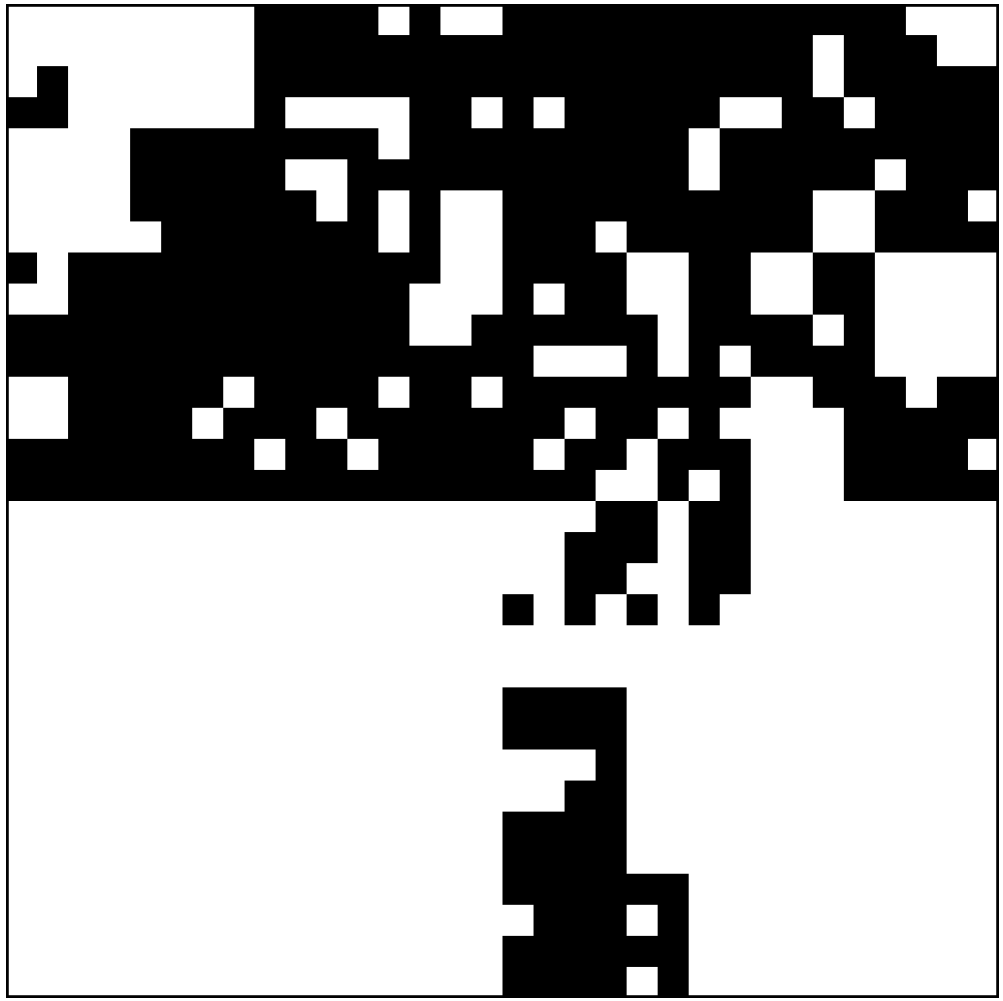

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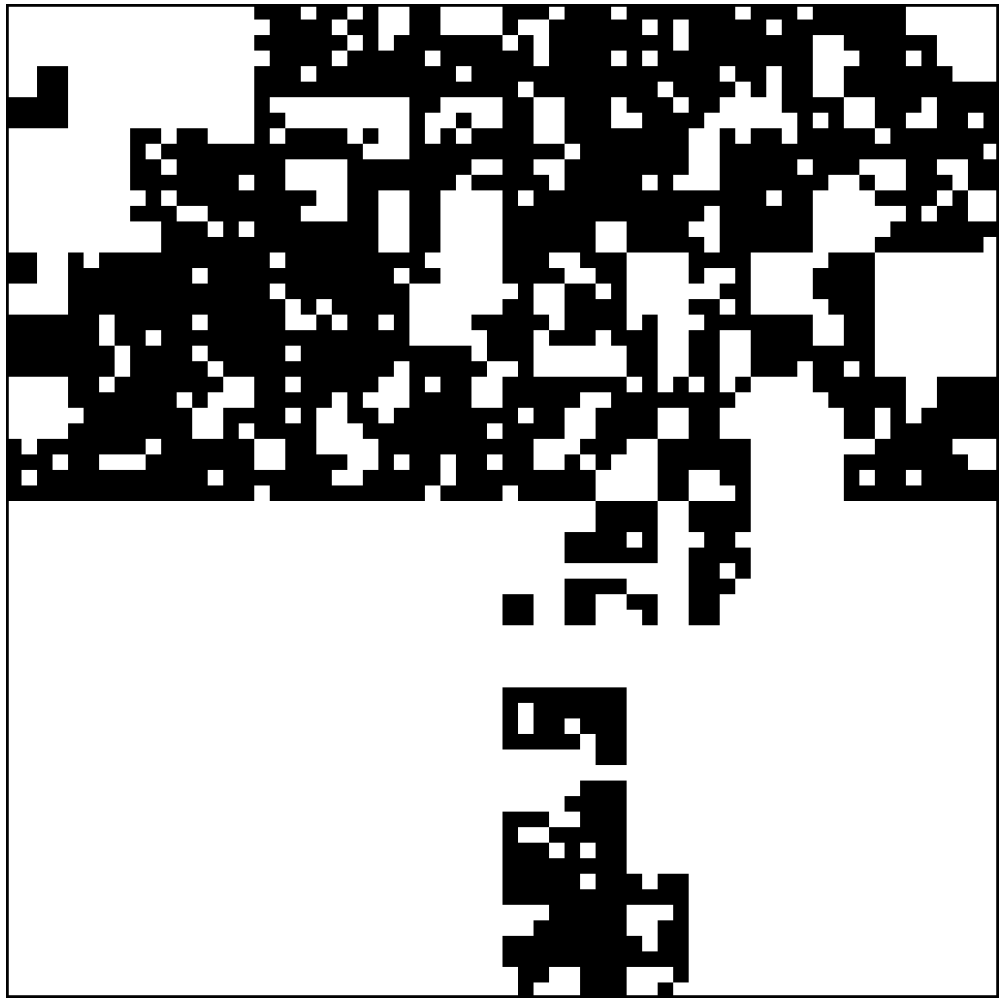

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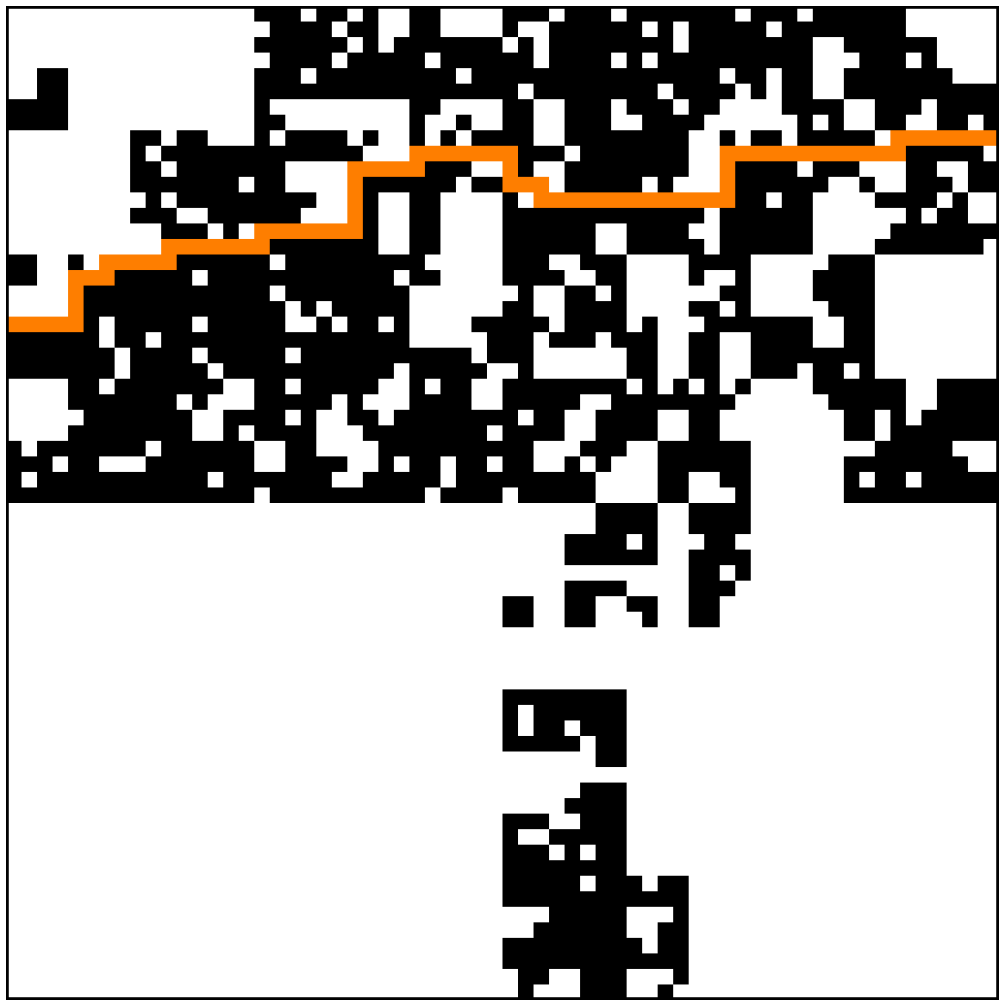

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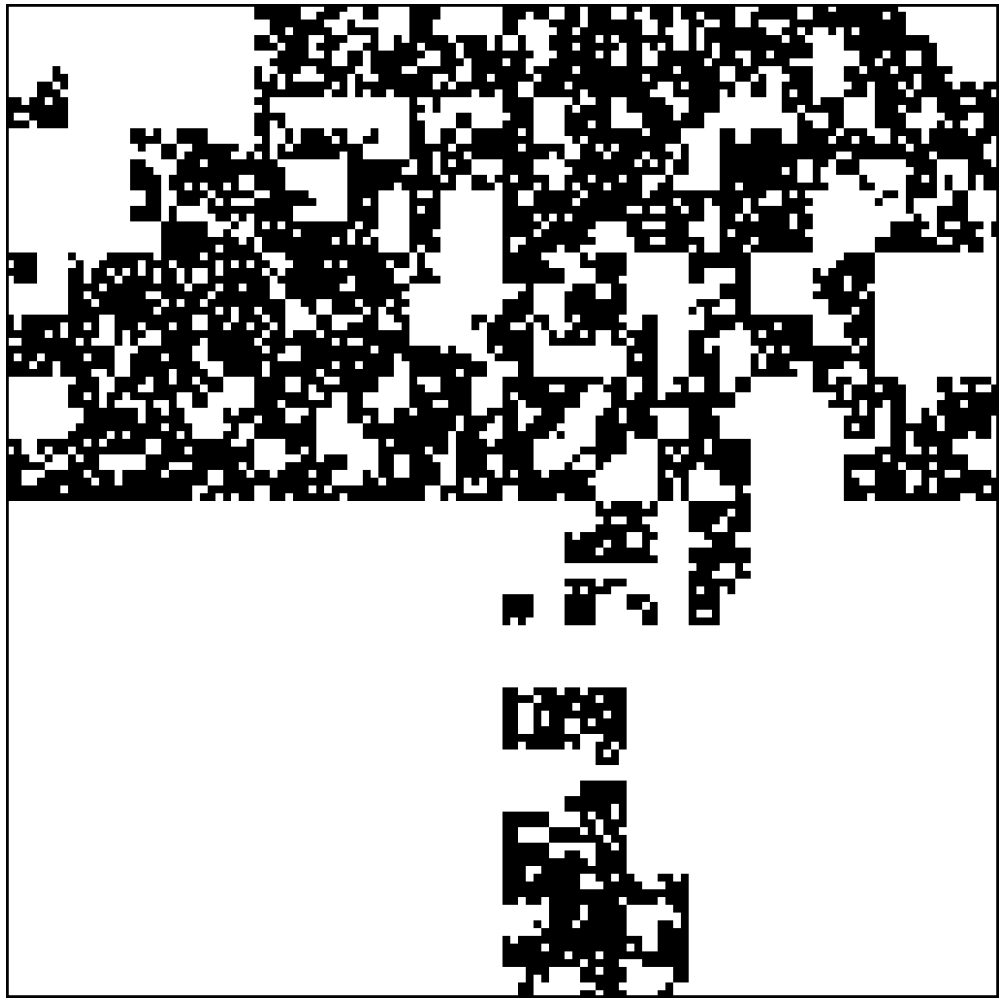

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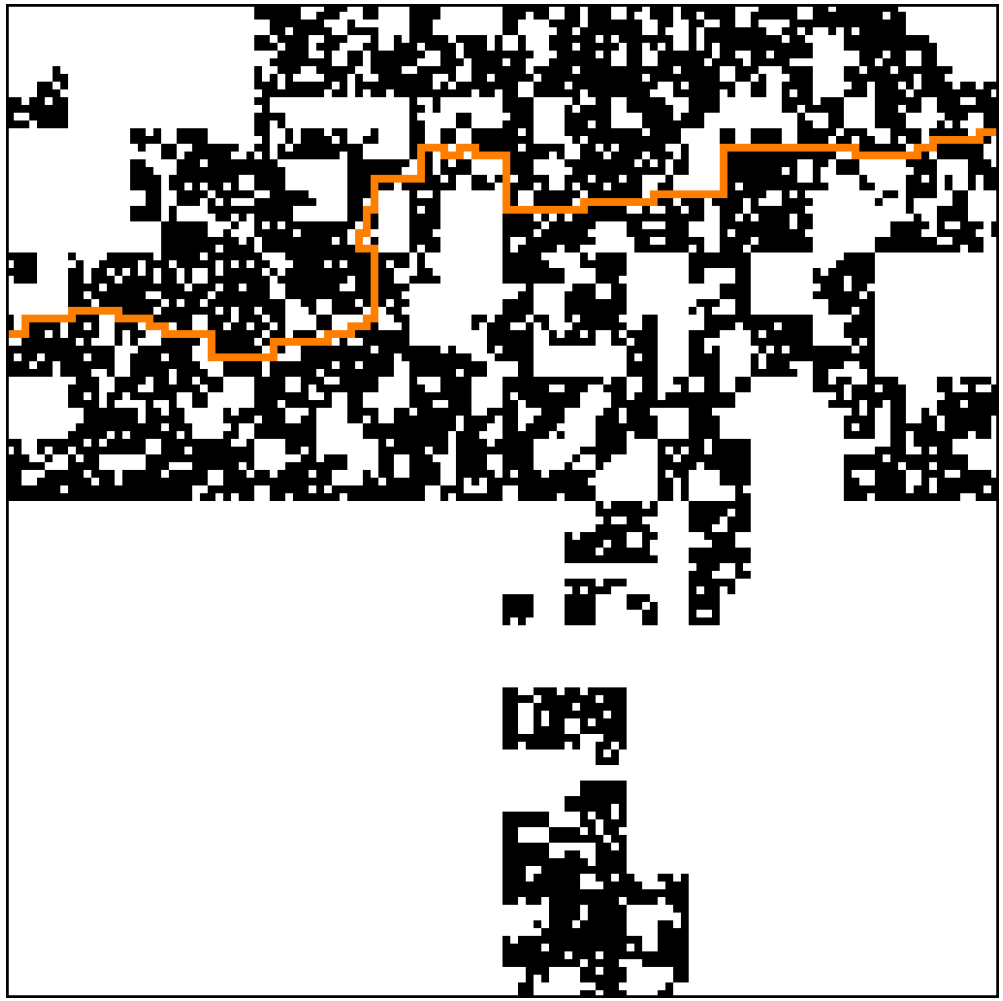

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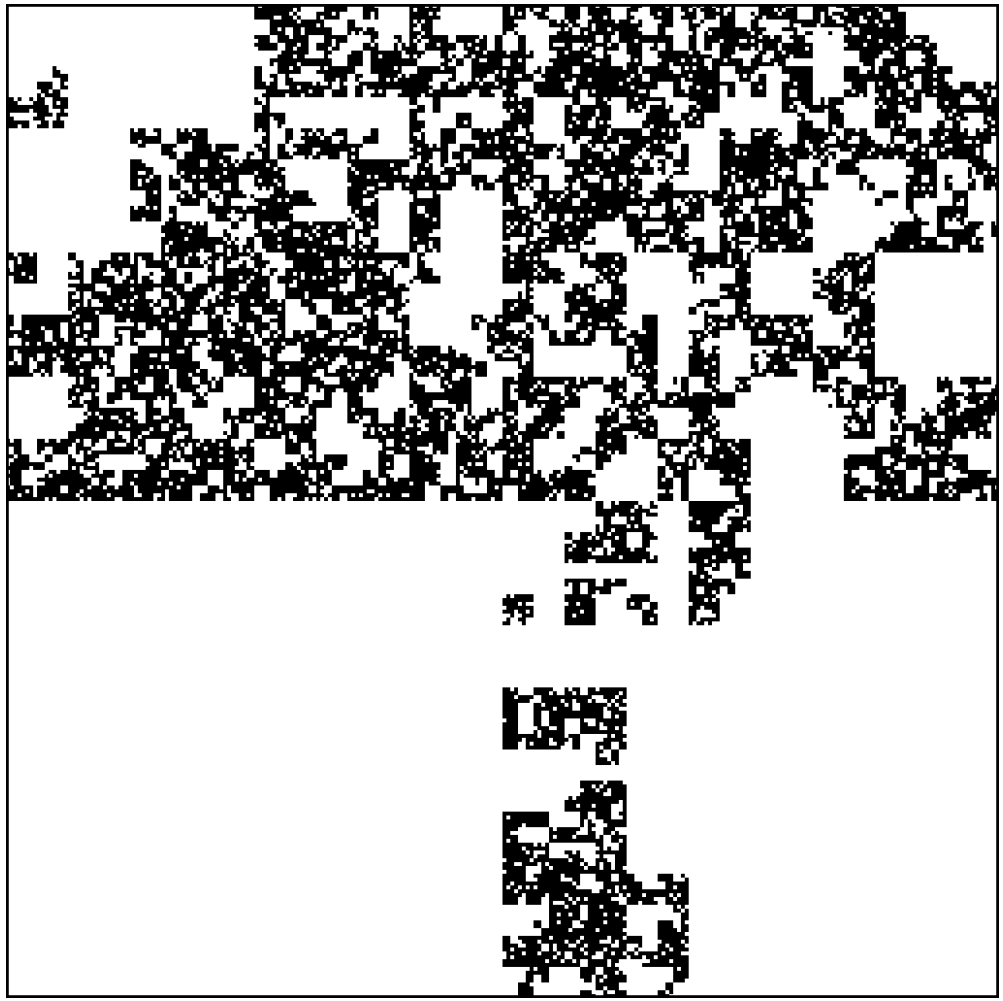

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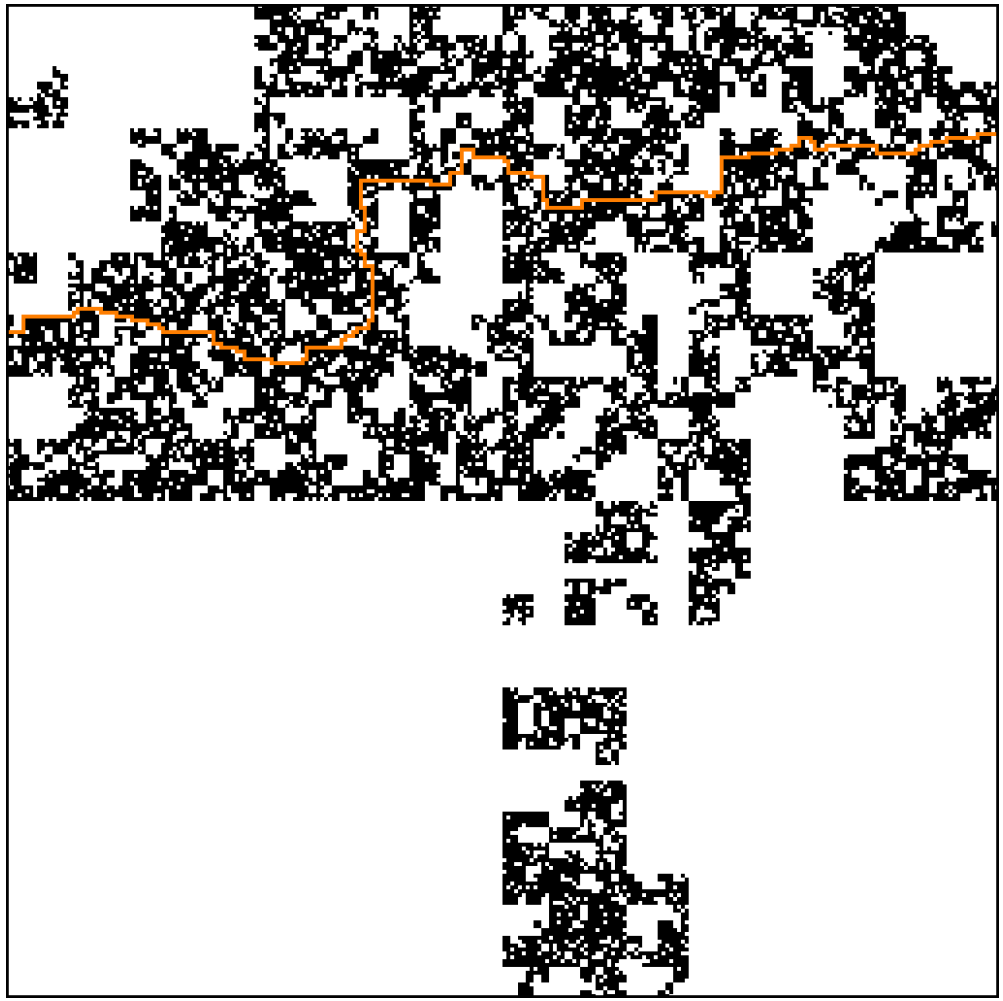
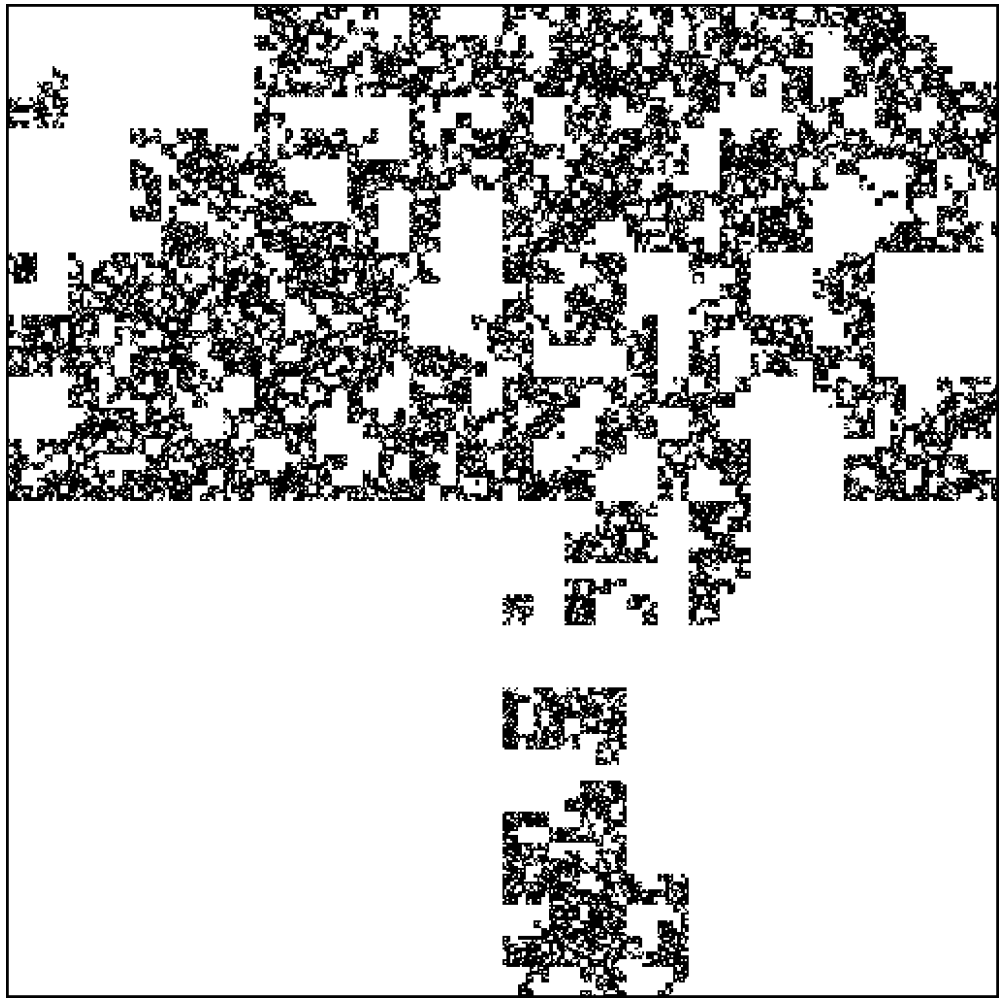


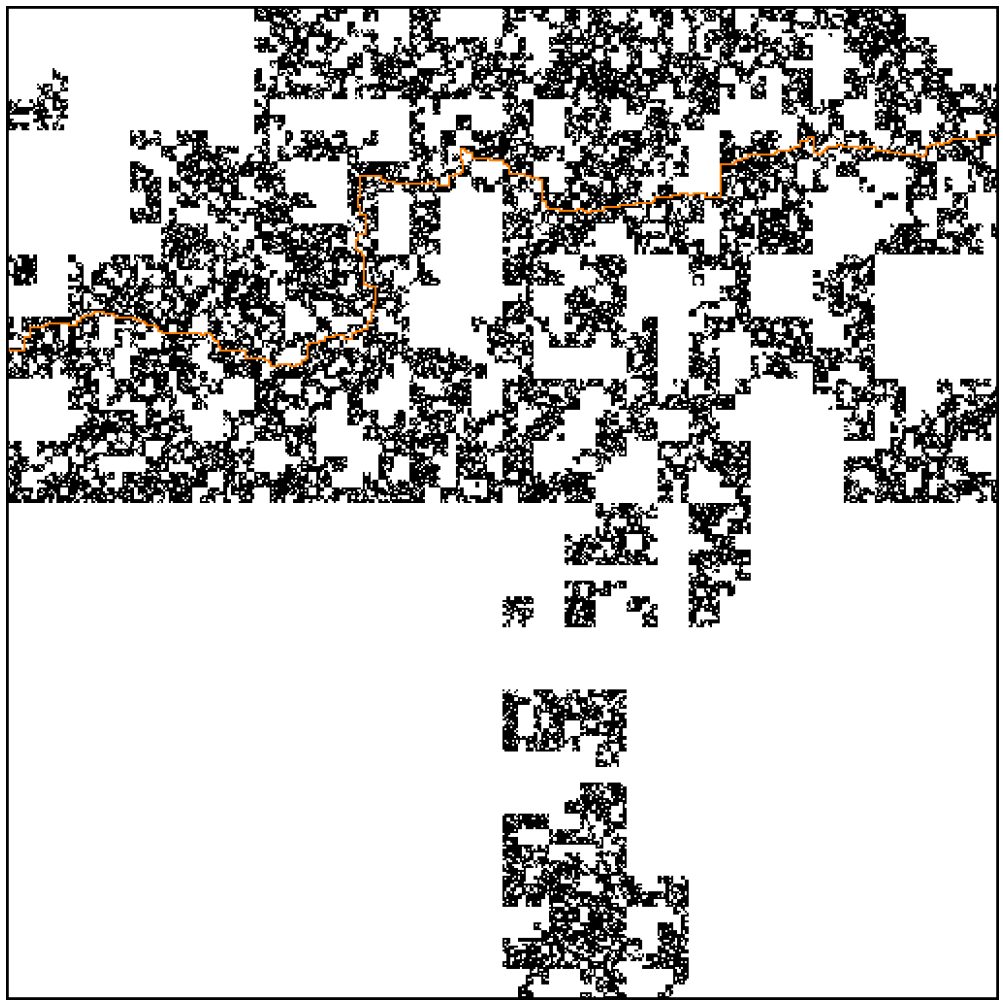
Finding Crossings

Example for $n = 2$



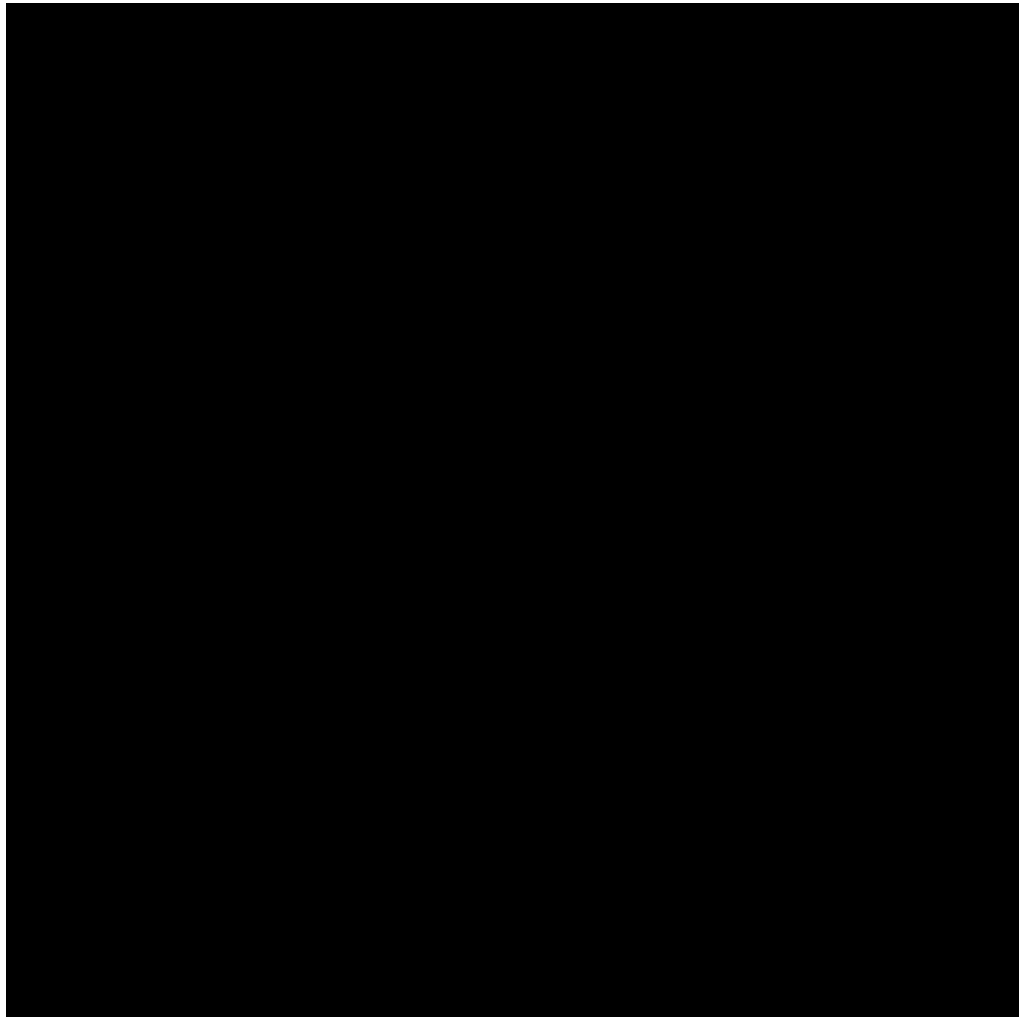
Finding Crossings

Example for $n = 2$



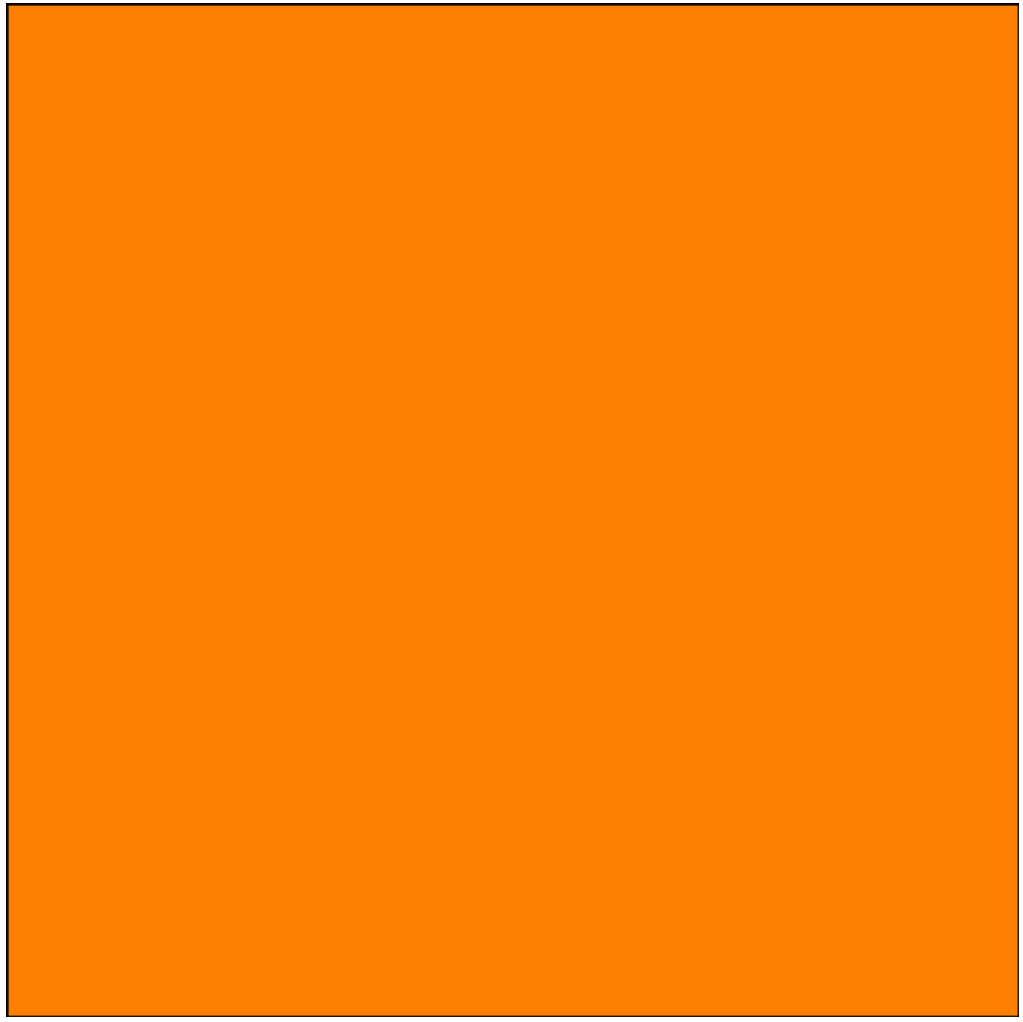
Finding Crossings

Example for $n = 3$



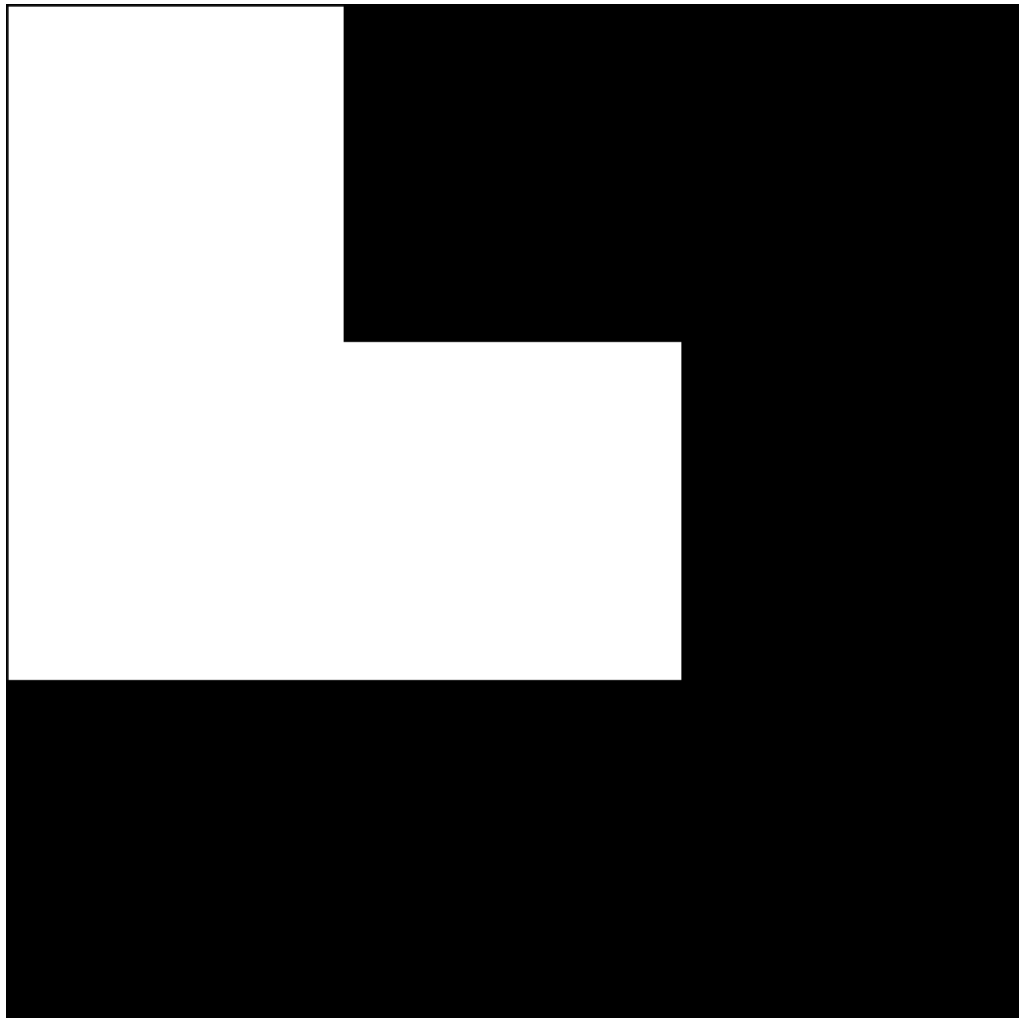
Finding Crossings

Example for $n = 3$



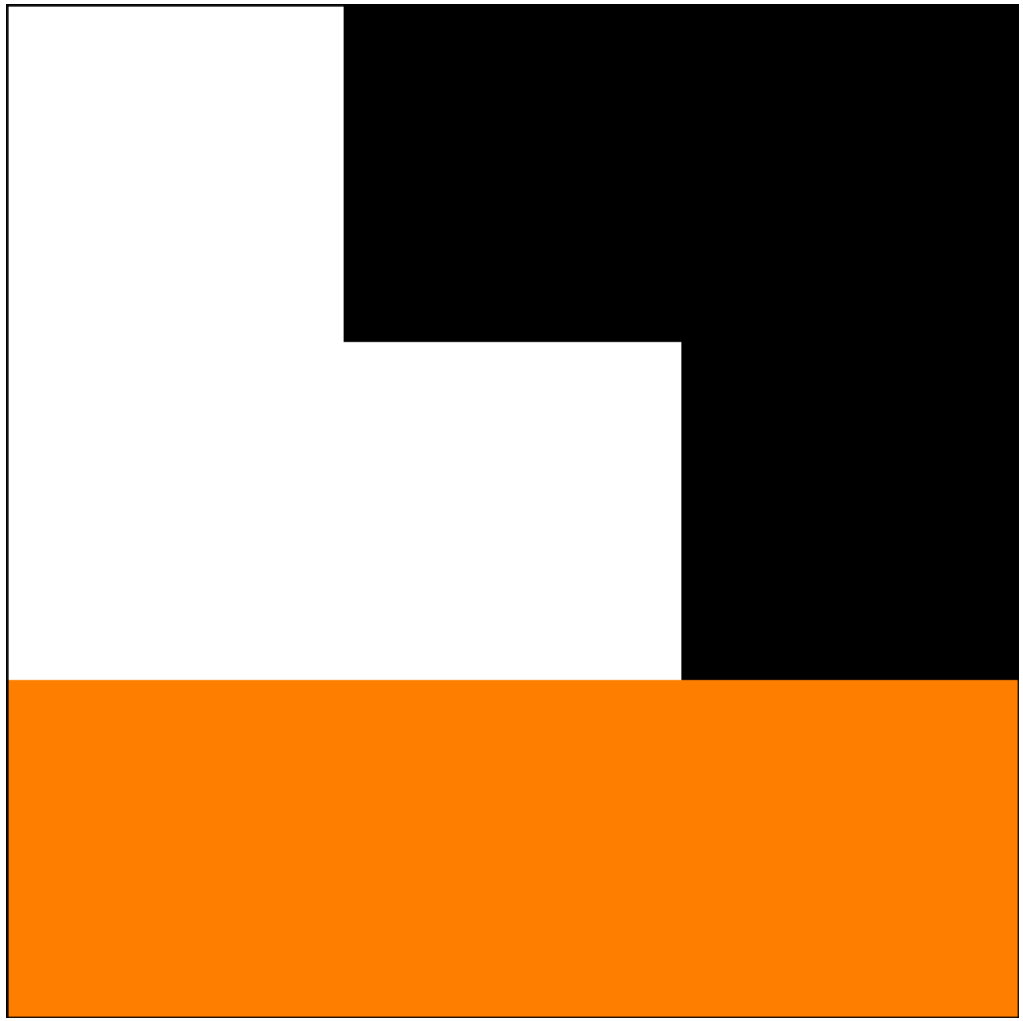
Finding Crossings

Example for $n = 3$



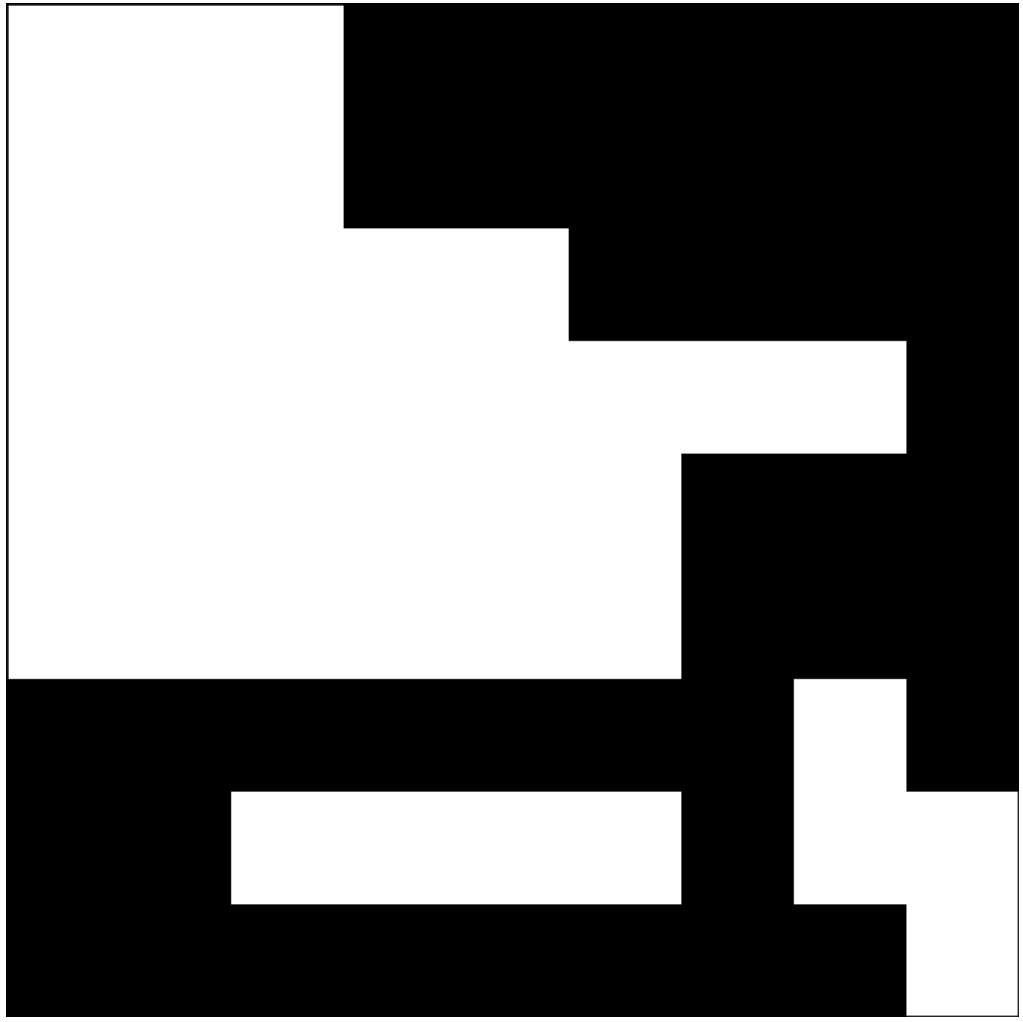
Finding Crossings

Example for $n = 3$



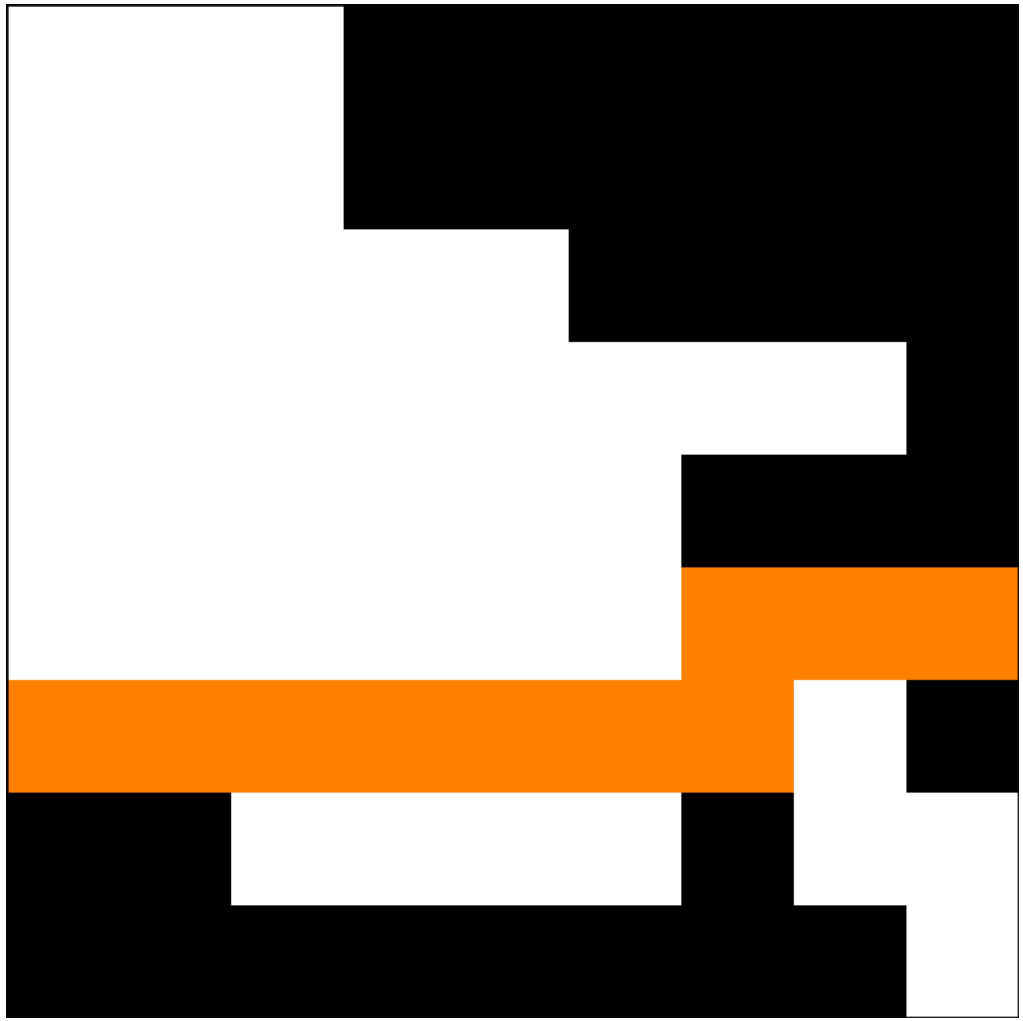
Finding Crossings

Example for $n = 3$



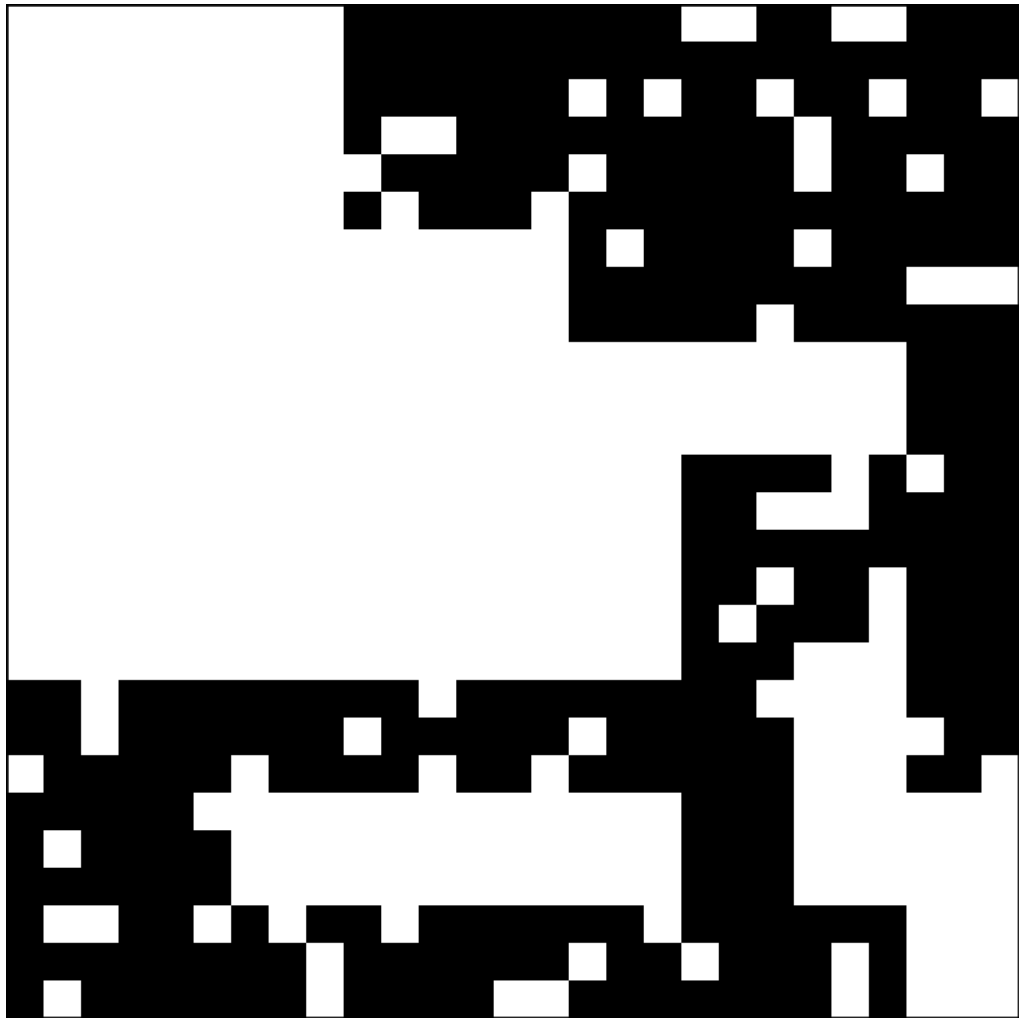
Finding Crossings

Example for $n = 3$



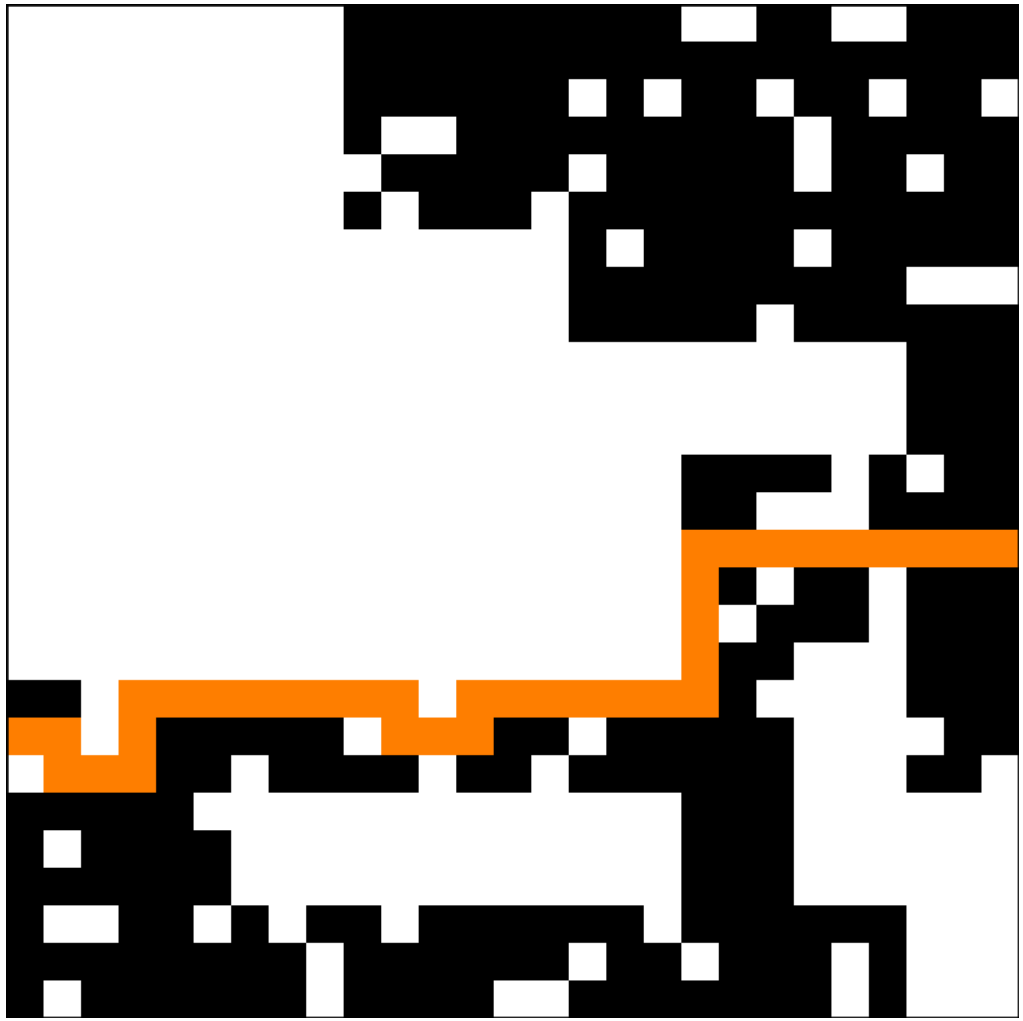
Finding Crossings

Example for $n = 3$



Finding Crossings

Example for $n = 3$

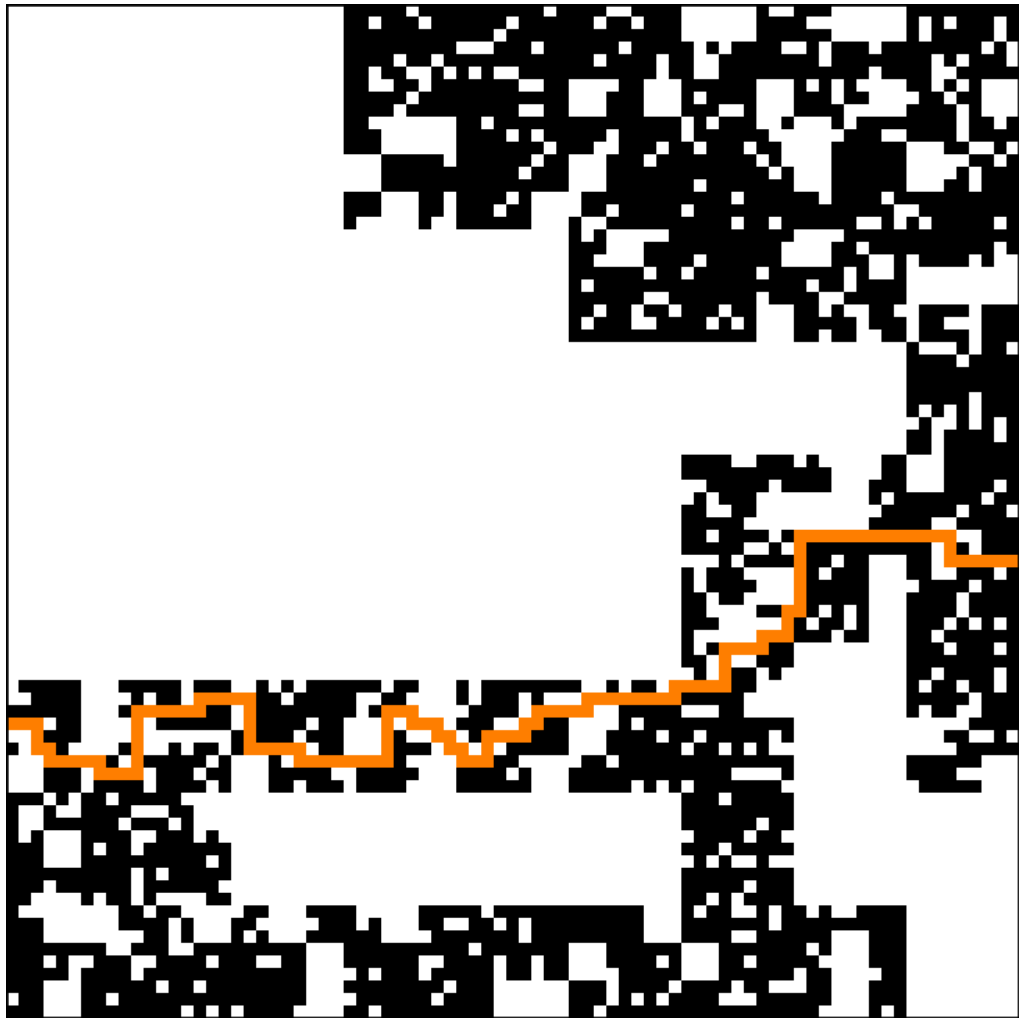


Finding Crossings

Example for $n = 3$

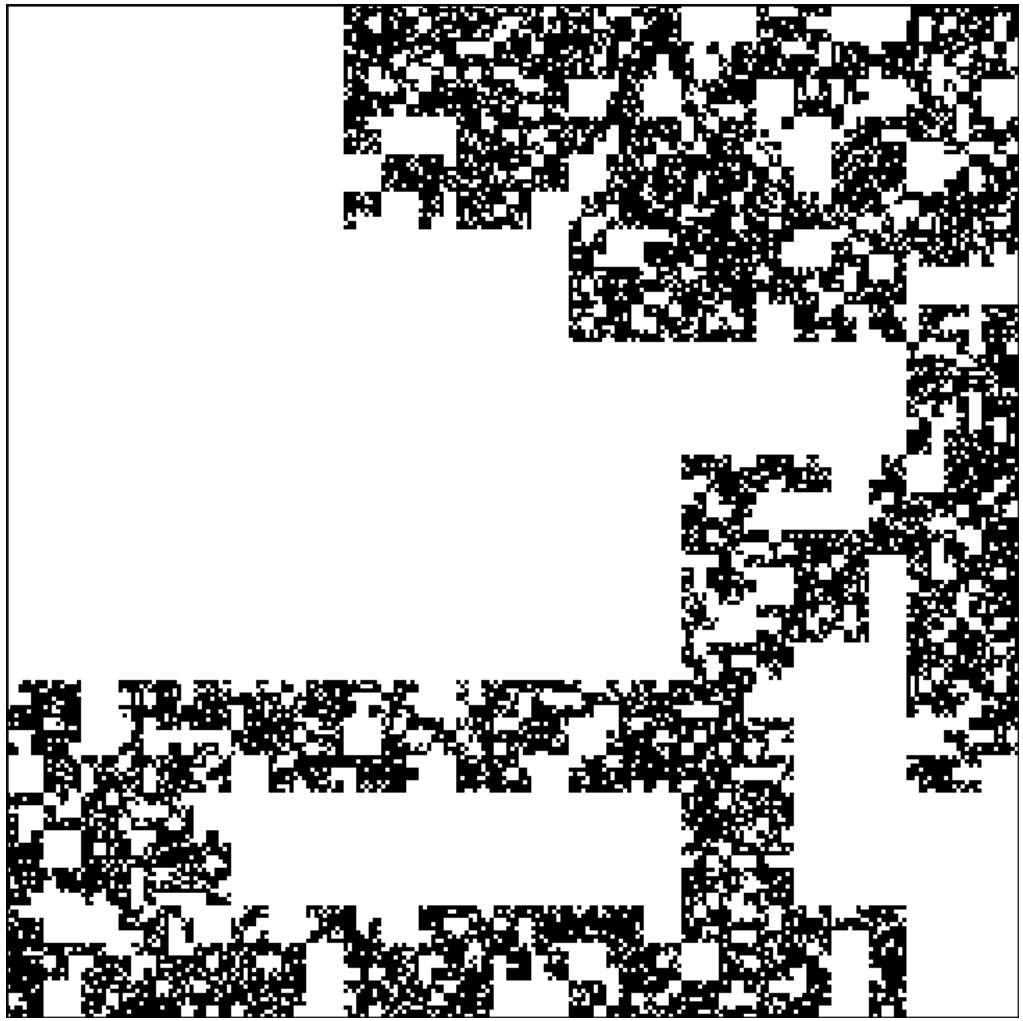
Finding Crossings

Example for $n = 3$



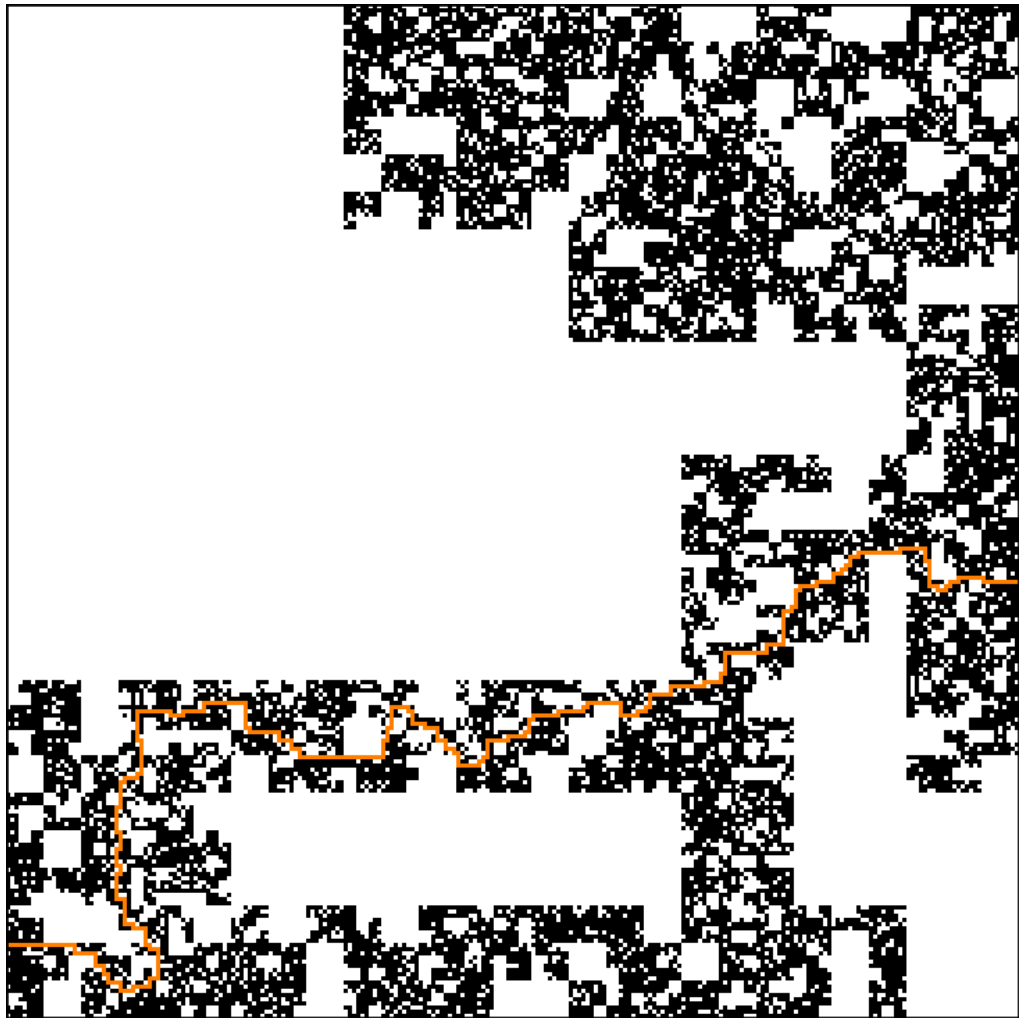
Finding Crossings

Example for $n = 3$



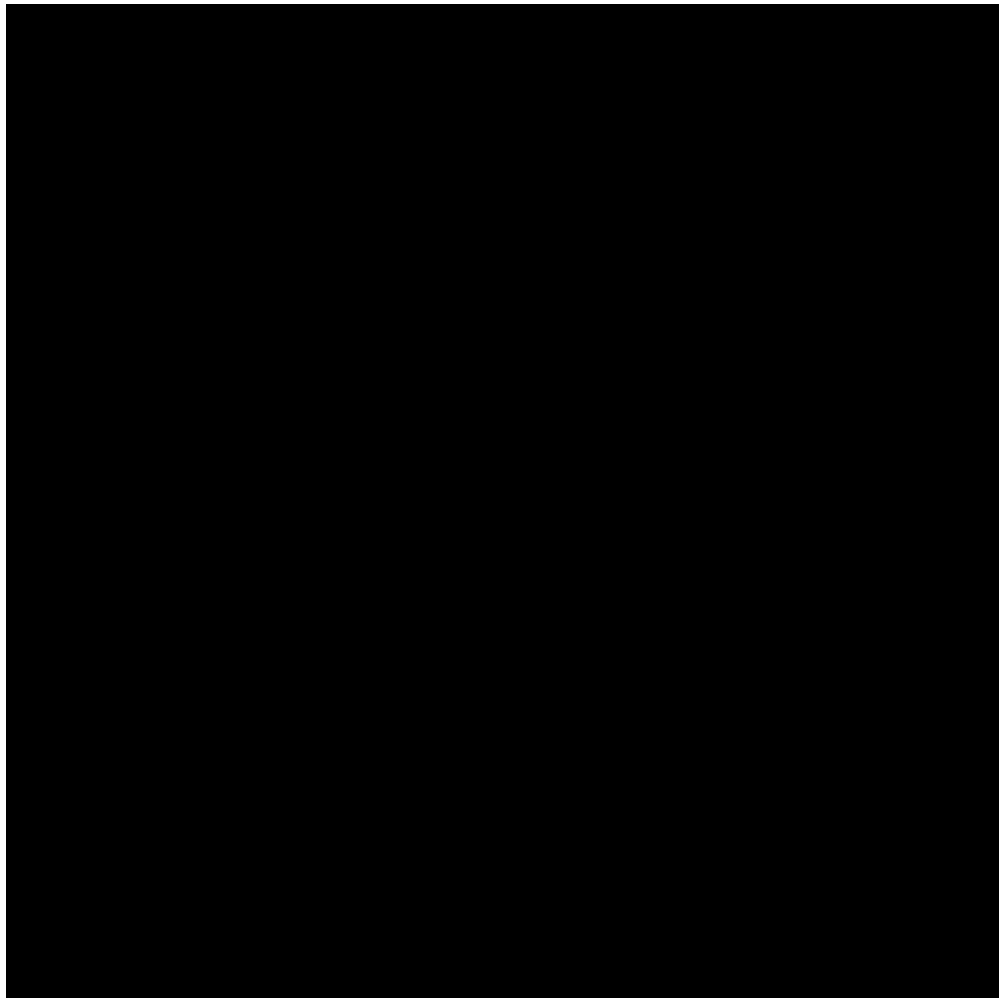
Finding Crossings

Example for $n = 3$



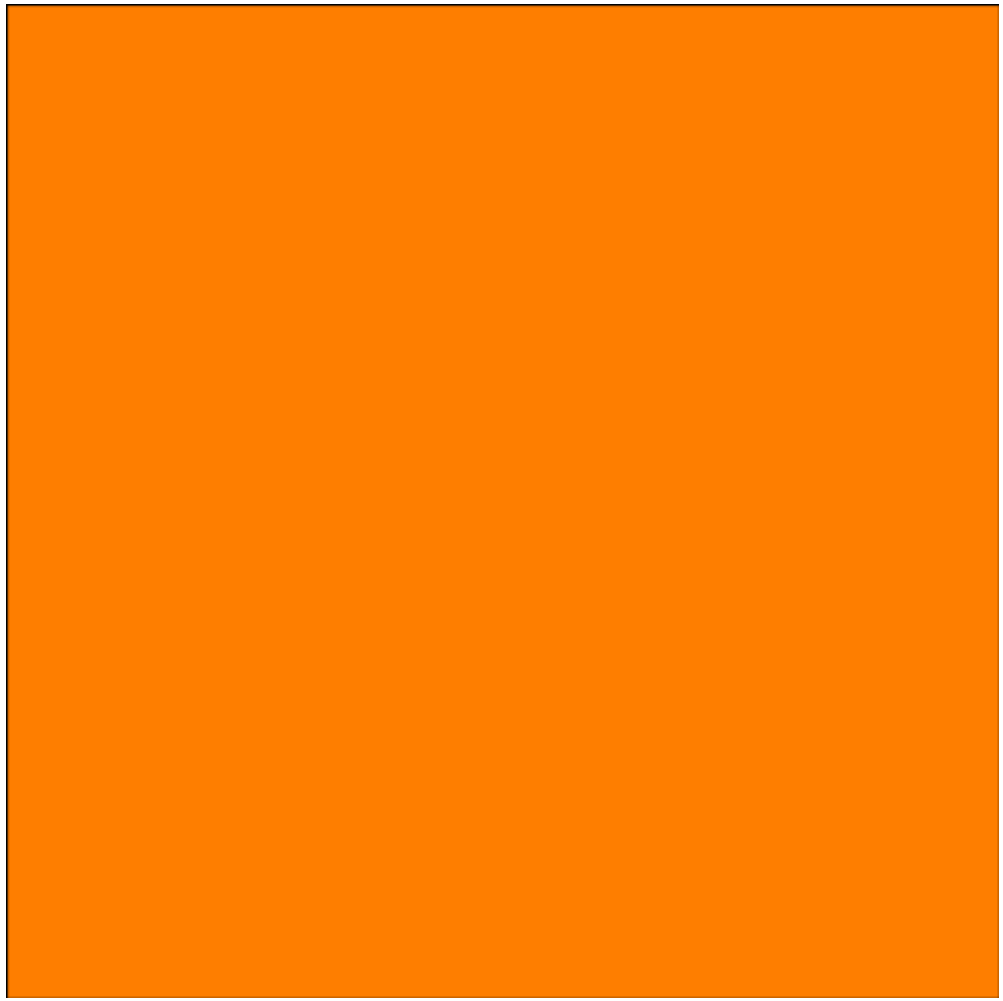
Finding Crossings

Example for $n = 5$



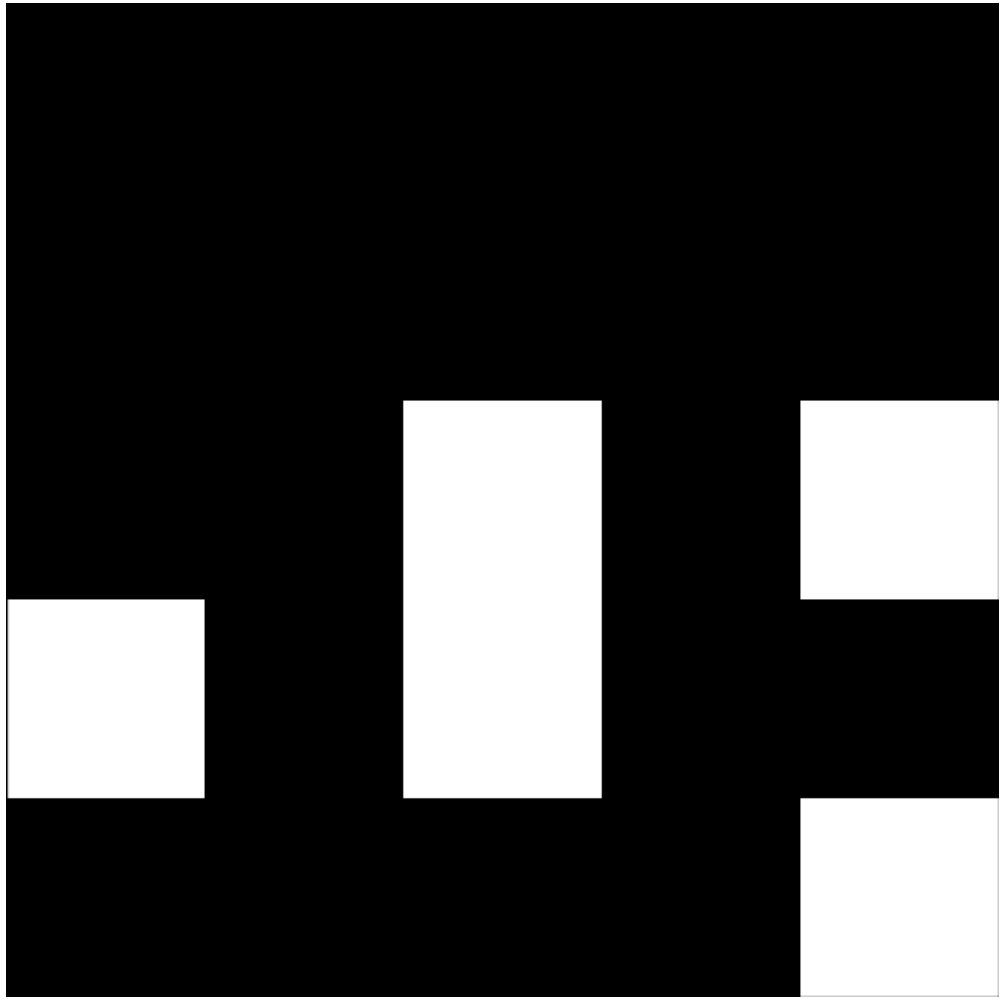
Finding Crossings

Example for $n = 5$



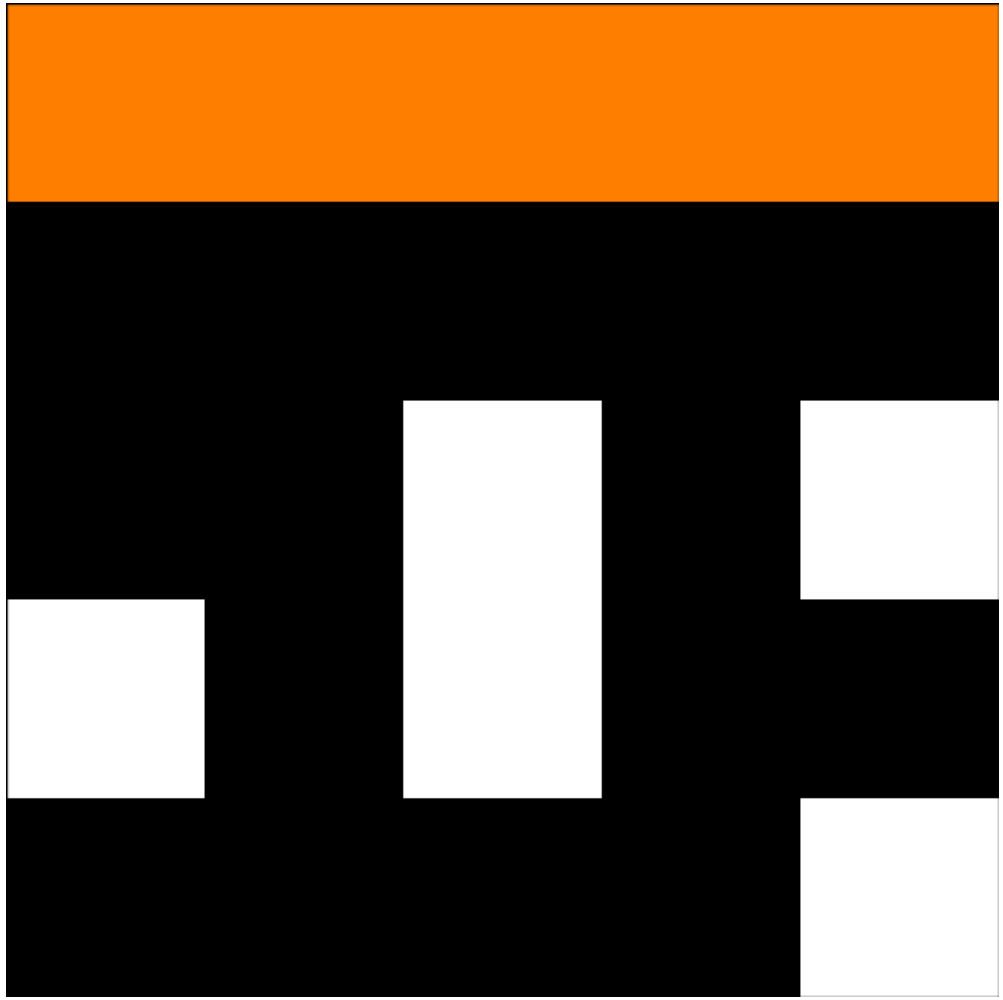
Finding Crossings

Example for $n = 5$



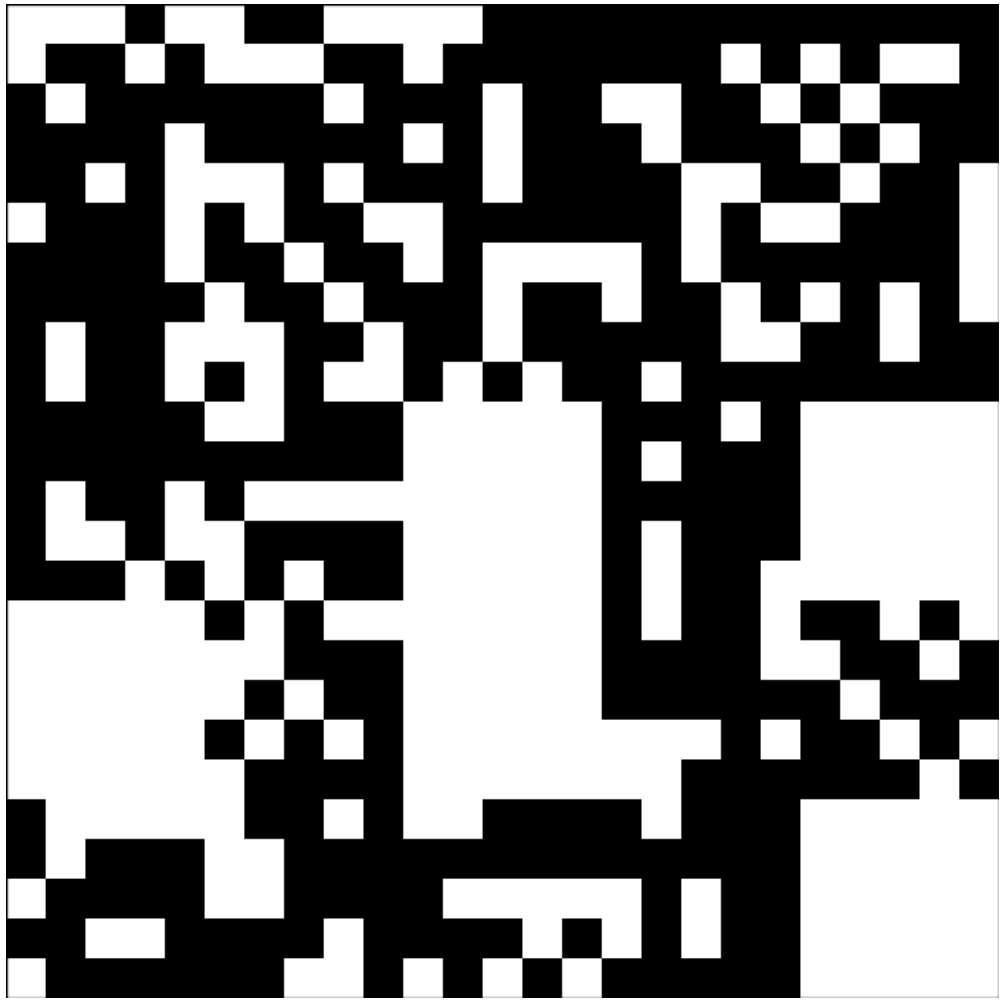
Finding Crossings

Example for $n = 5$



Finding Crossings

Example for $n = 5$

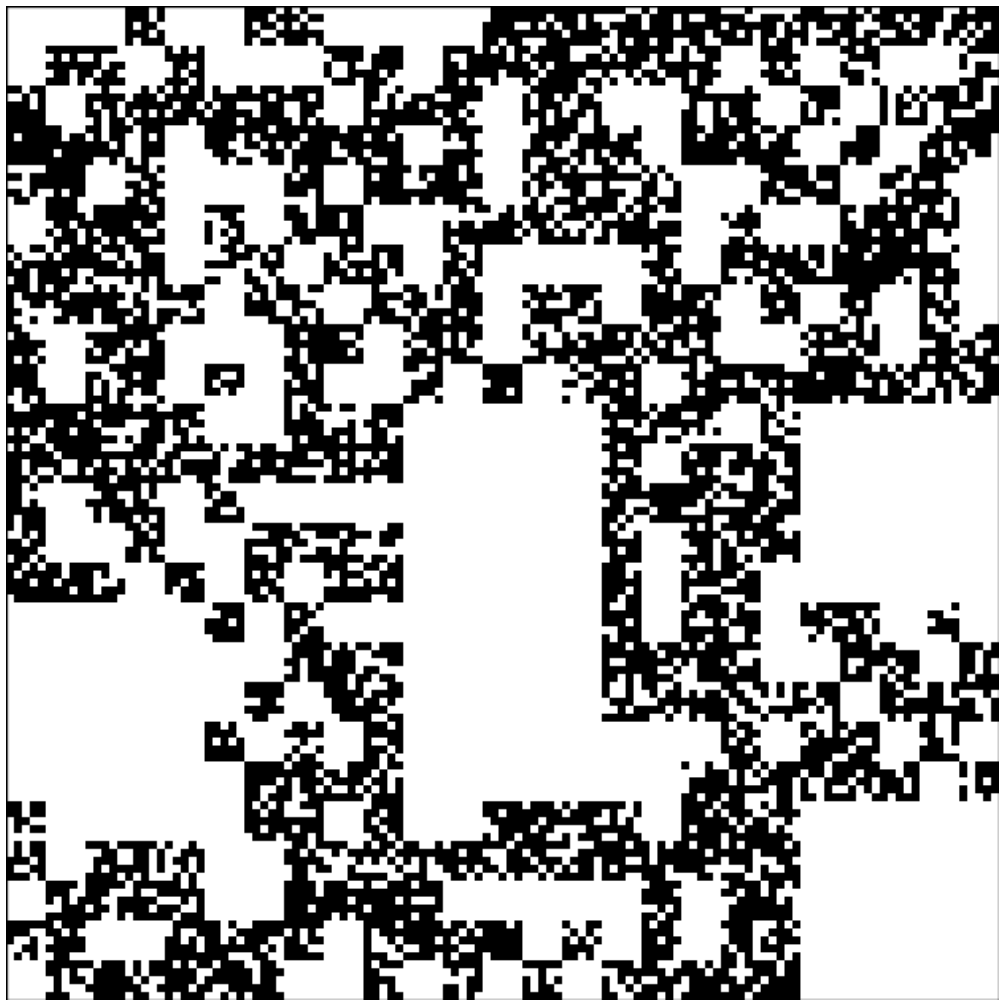


Finding Crossings

Example for $n = 5$

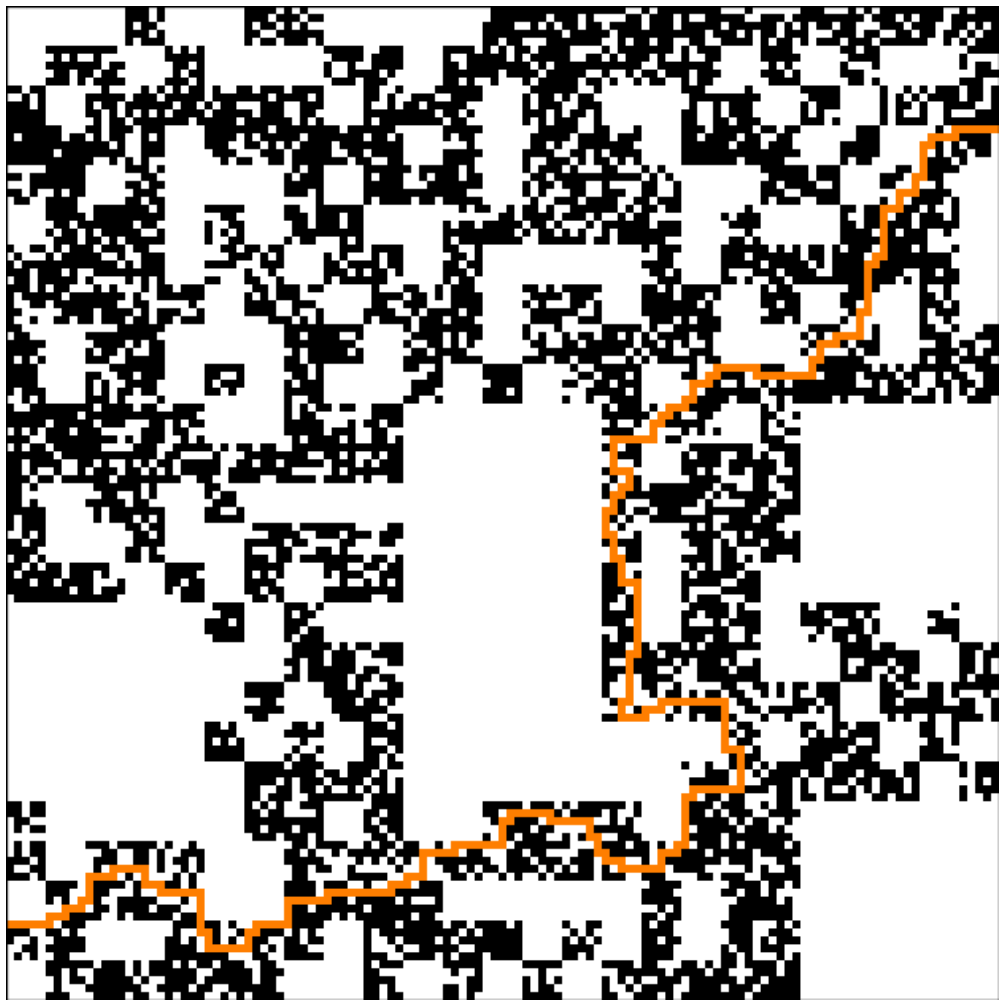
Finding Crossings

Example for $n = 5$



Finding Crossings

Example for $n = 5$



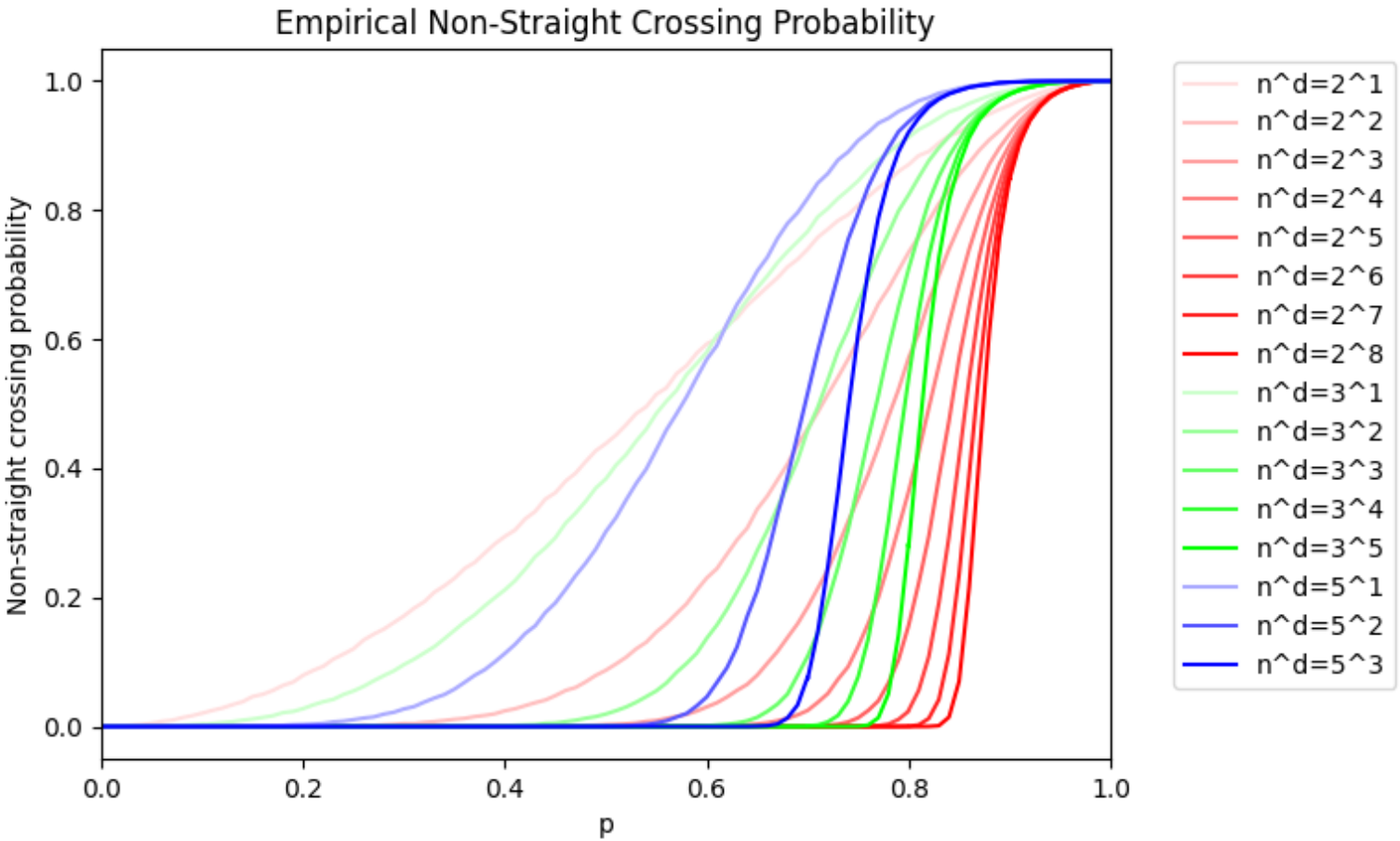
Finding Crossings

Algorithm

- 2D Crossing Algorithm:
  `https://pauldubois98.github.io/RandomFractalAlgorithmsDemo/2Dcrossing/index.html`
- 3D Crossing Algorithm:
  `https://pauldubois98.github.io/RandomFractalAlgorithmsDemo/3Dcrossing/index.html`

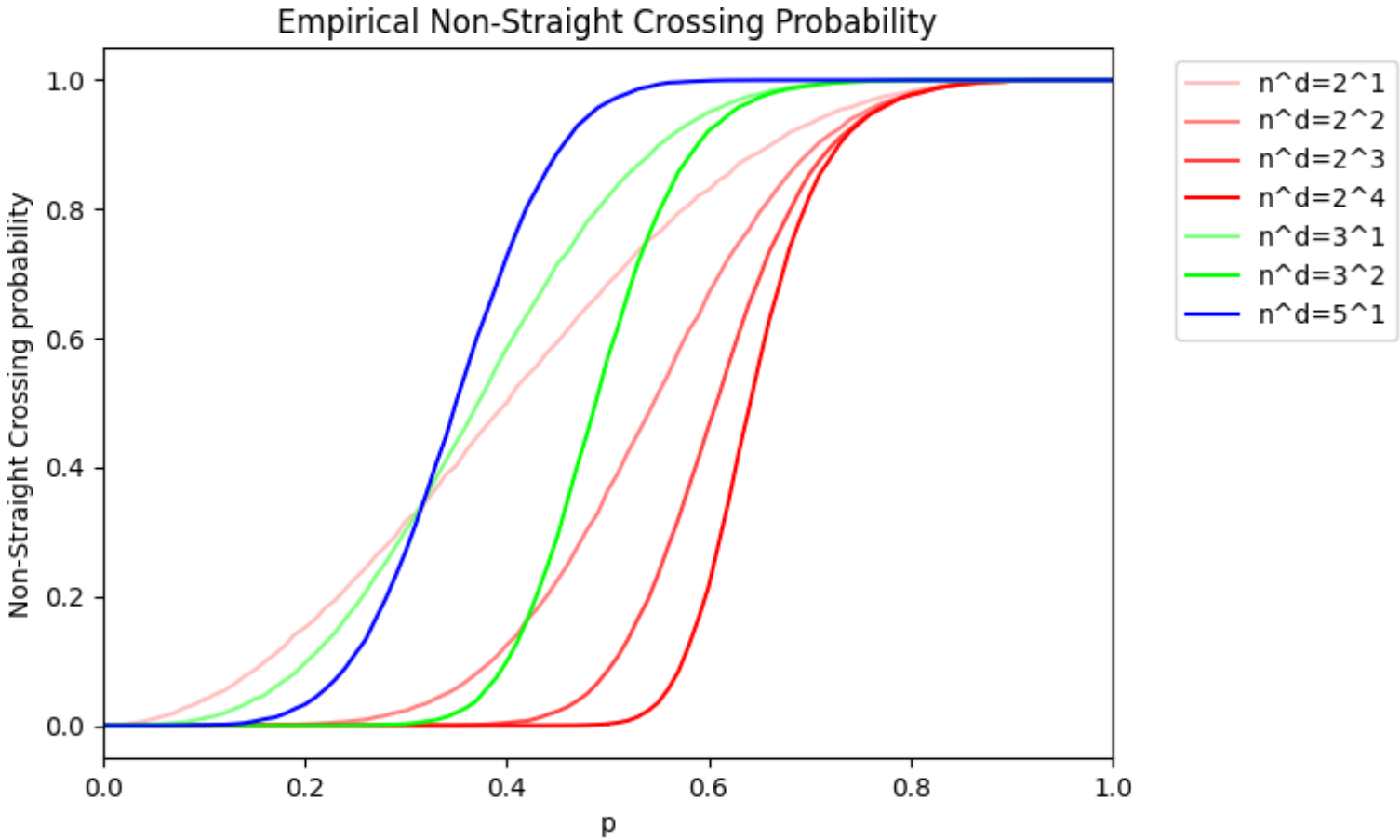
Crossings Probability

Non-Straight Crossing, Recursive Percolation, 2D



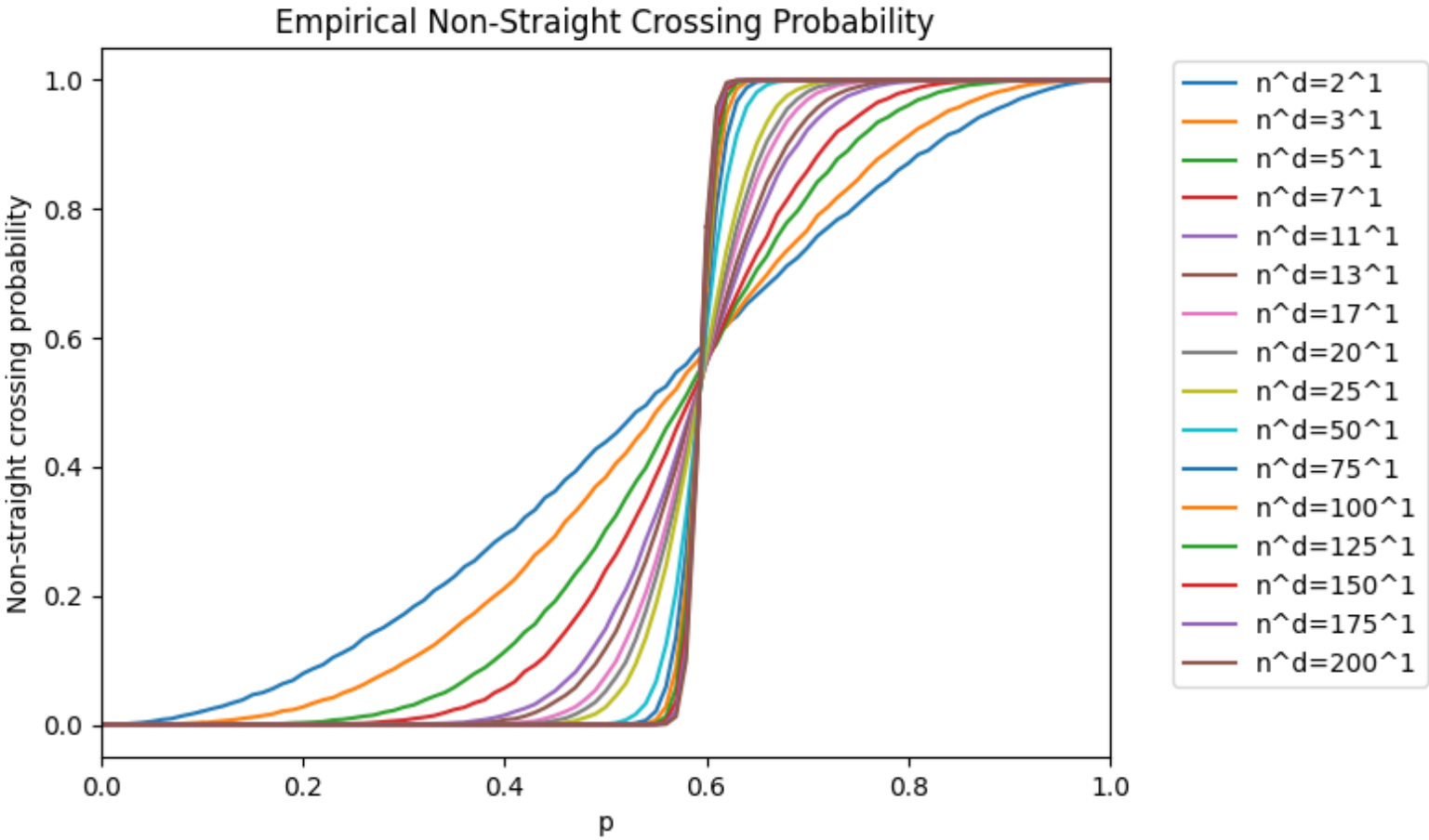
Crossings Probability

Non-Straight Crossing, Recursive Percolation, 3D



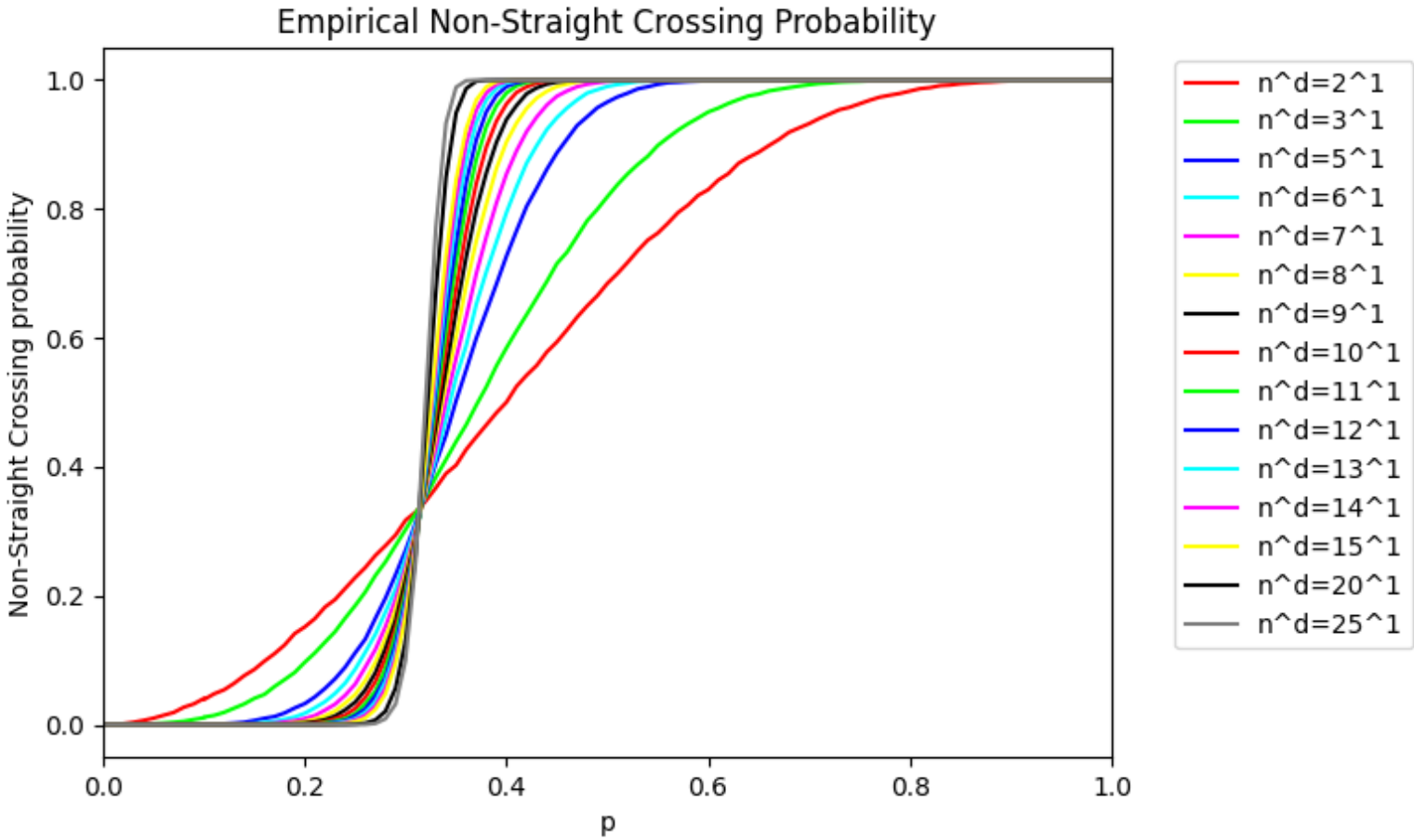
Crossings Probability

Non-Straight Crossing, Uniform Percolation, 2D



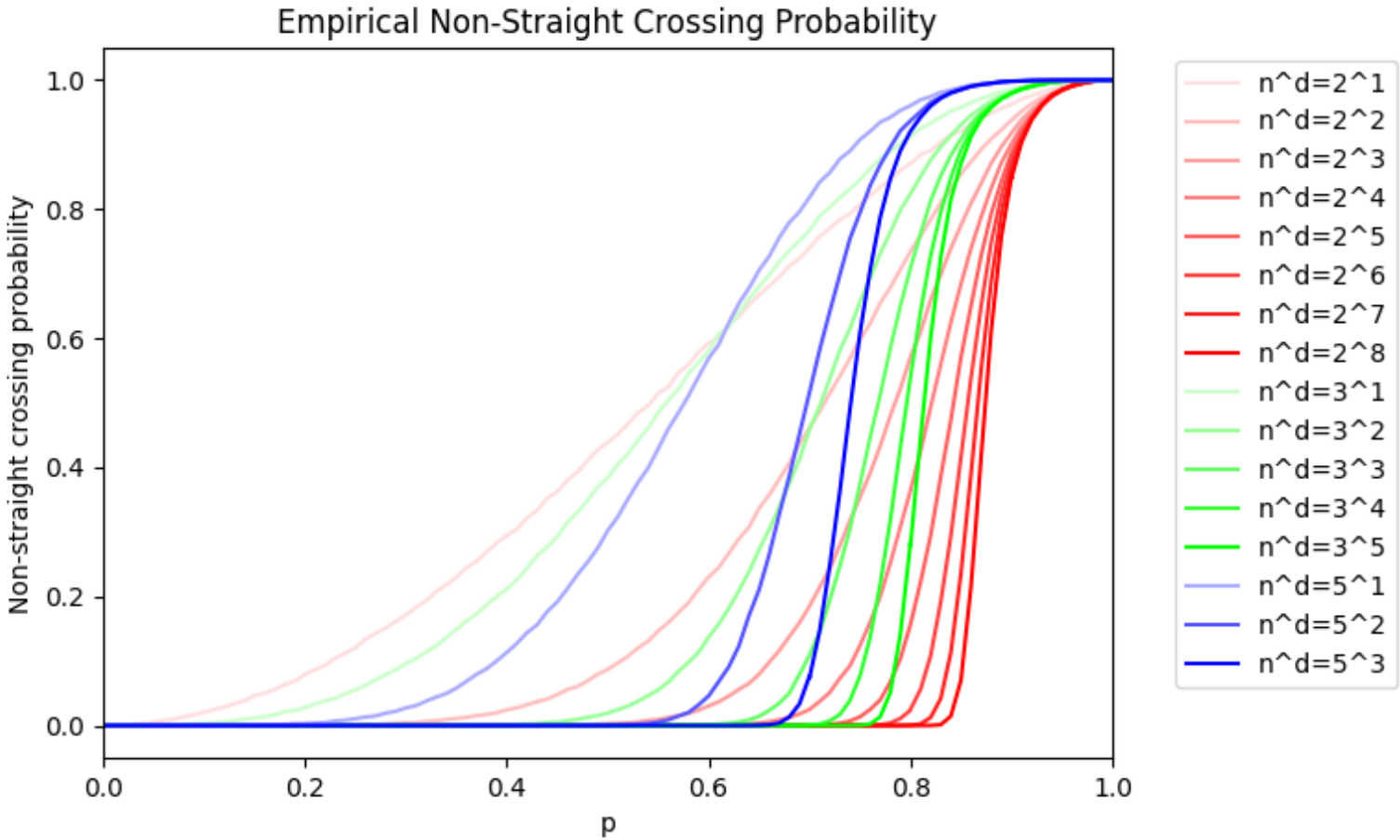
Crossings Probability

Non-Straight Crossing, Uniform Percolation, 3D



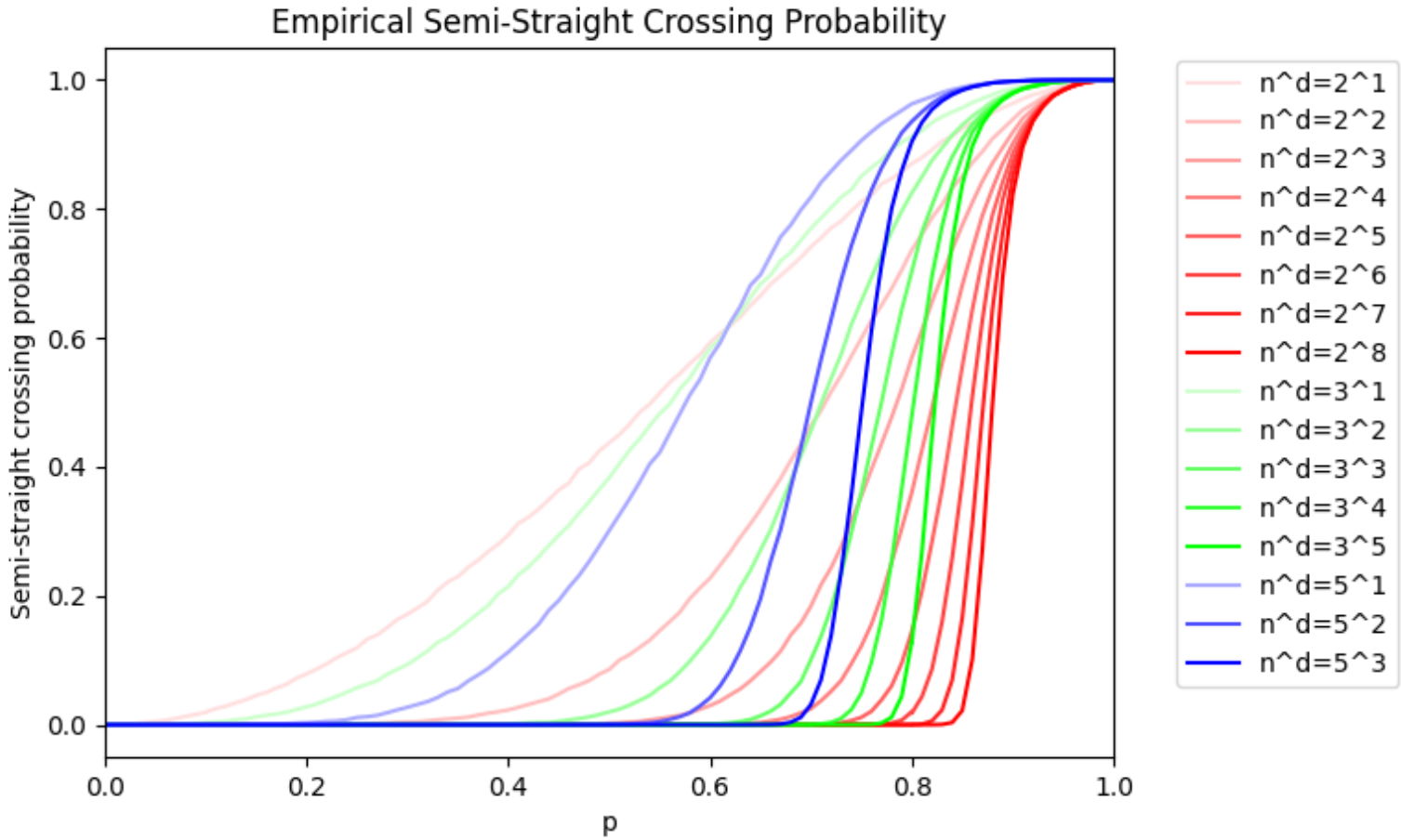
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 2D



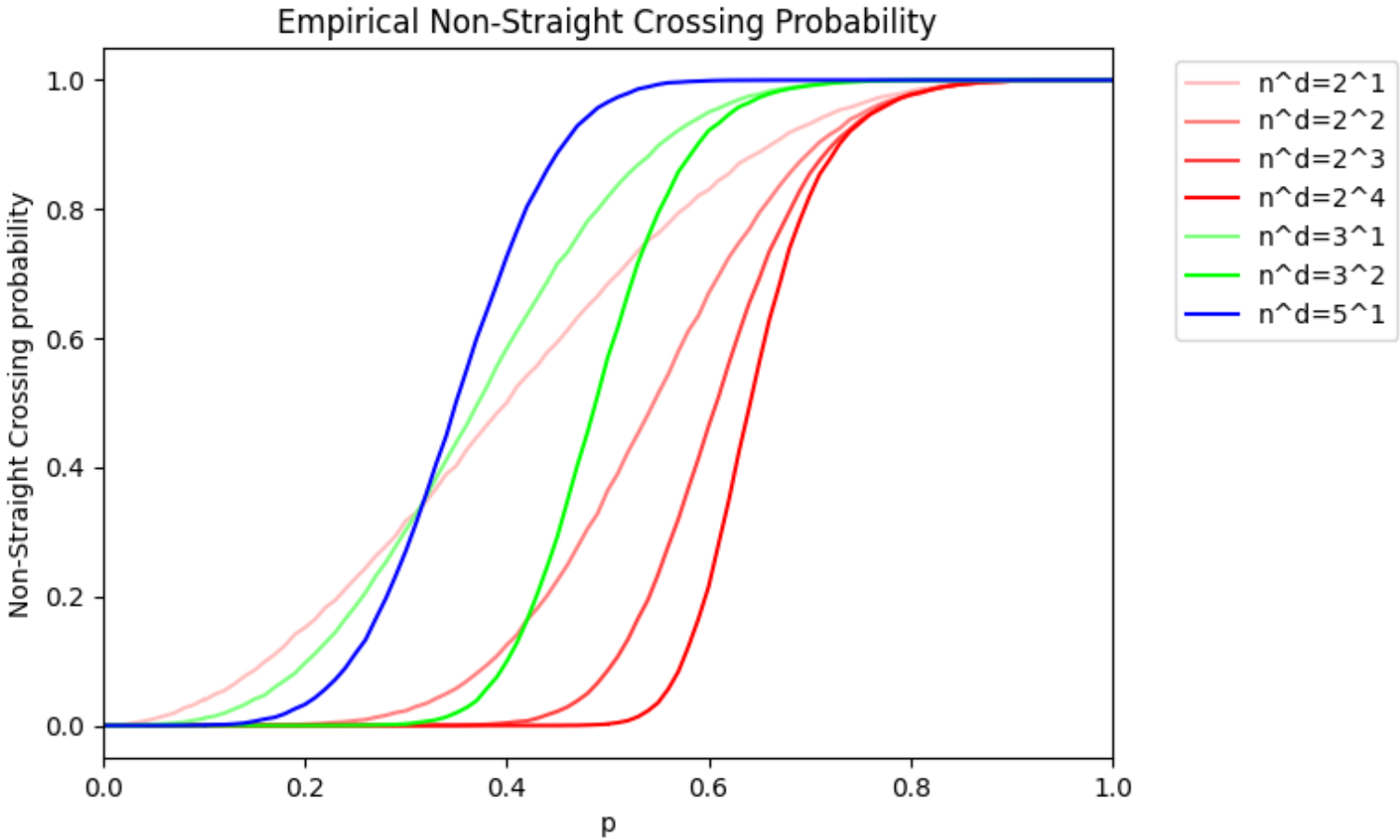
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 2D



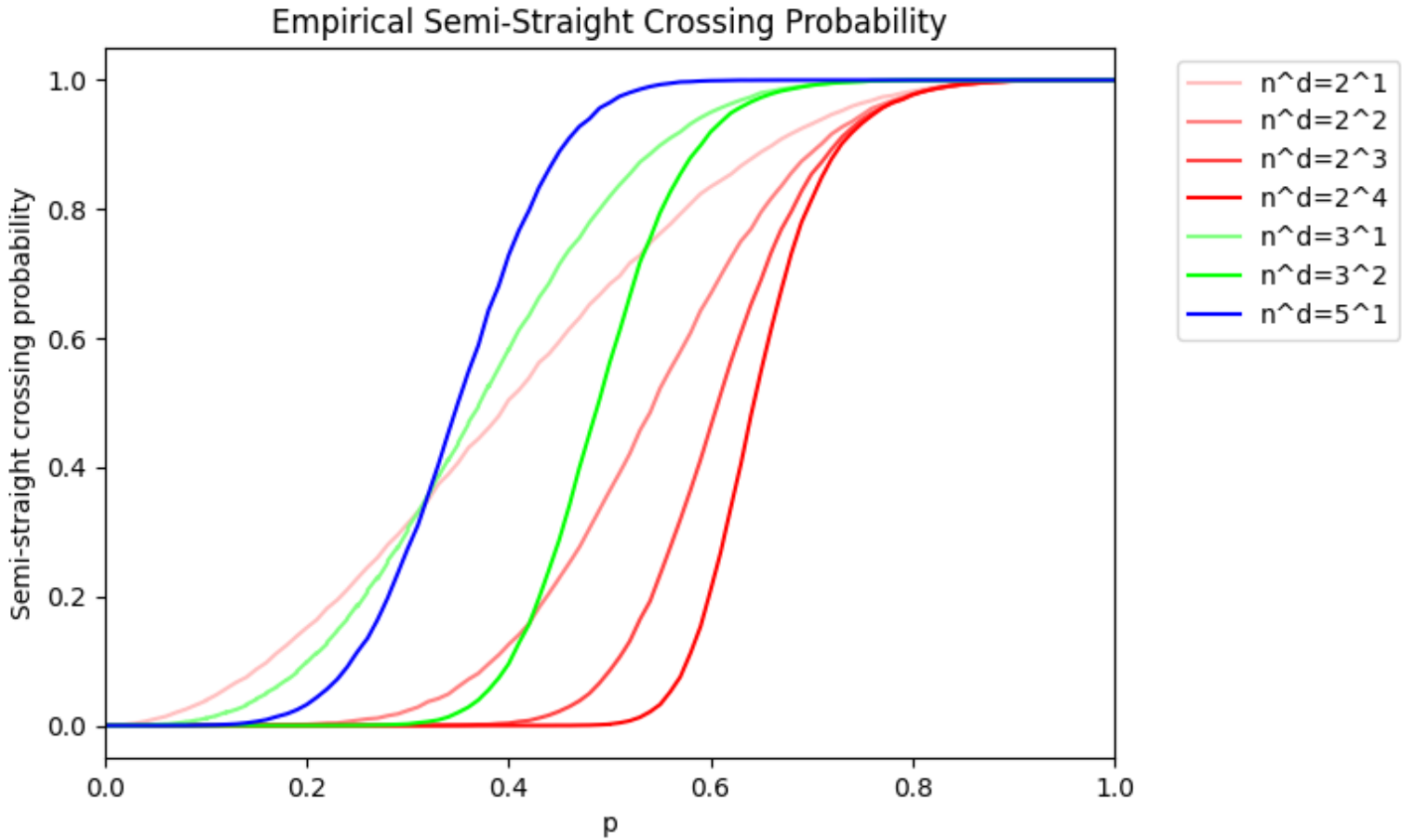
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 3D



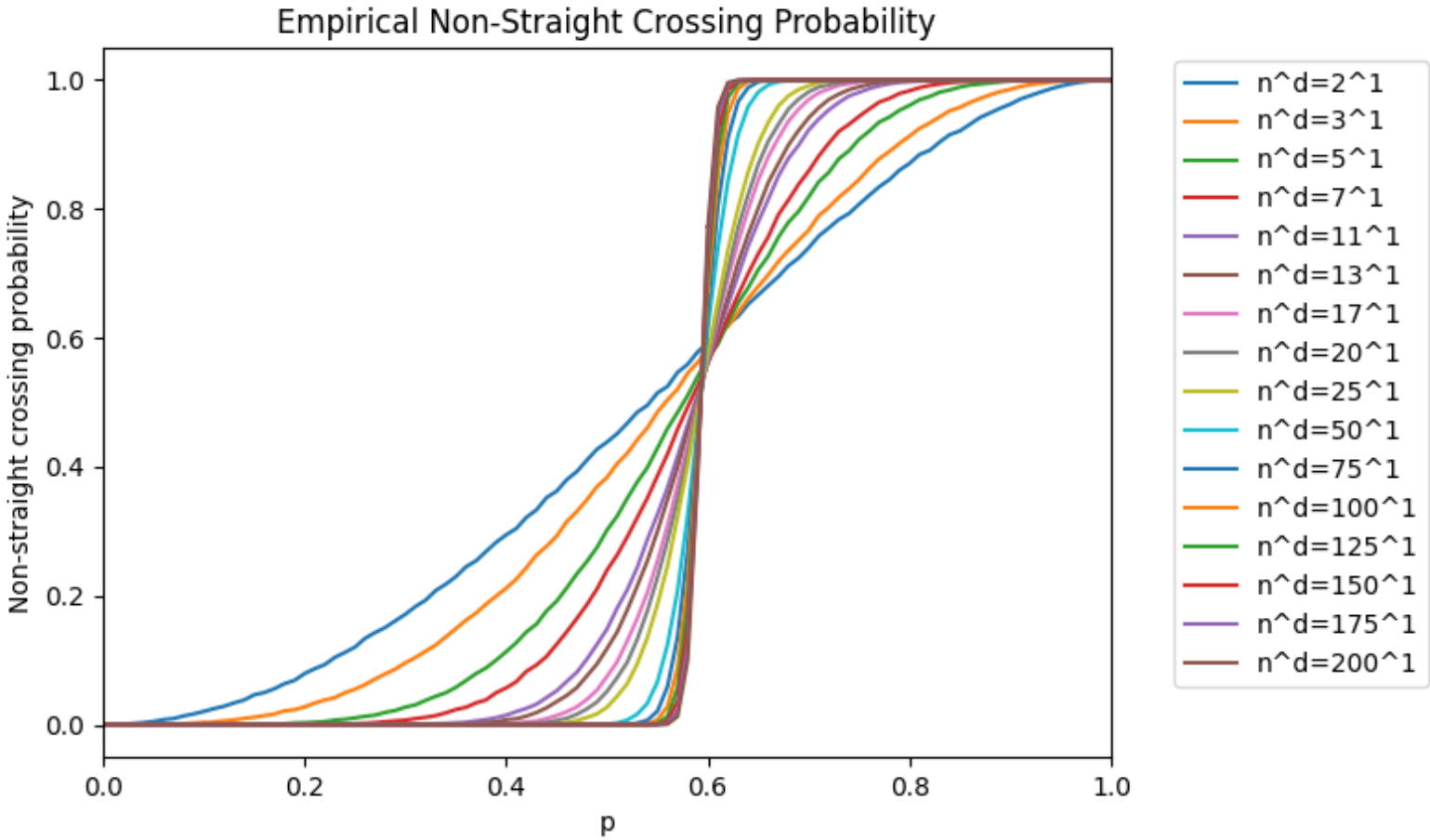
Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 3D



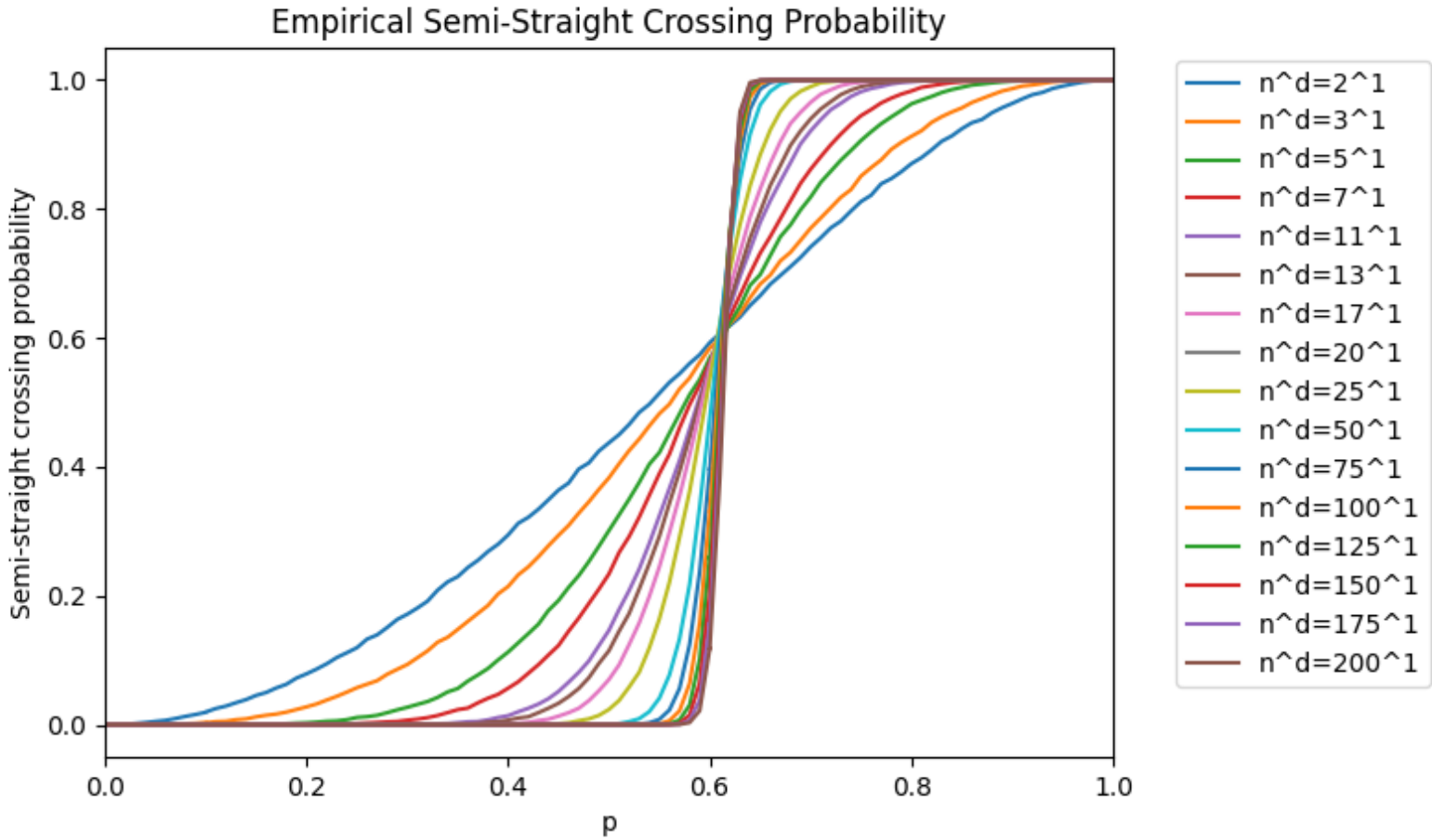
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 2D



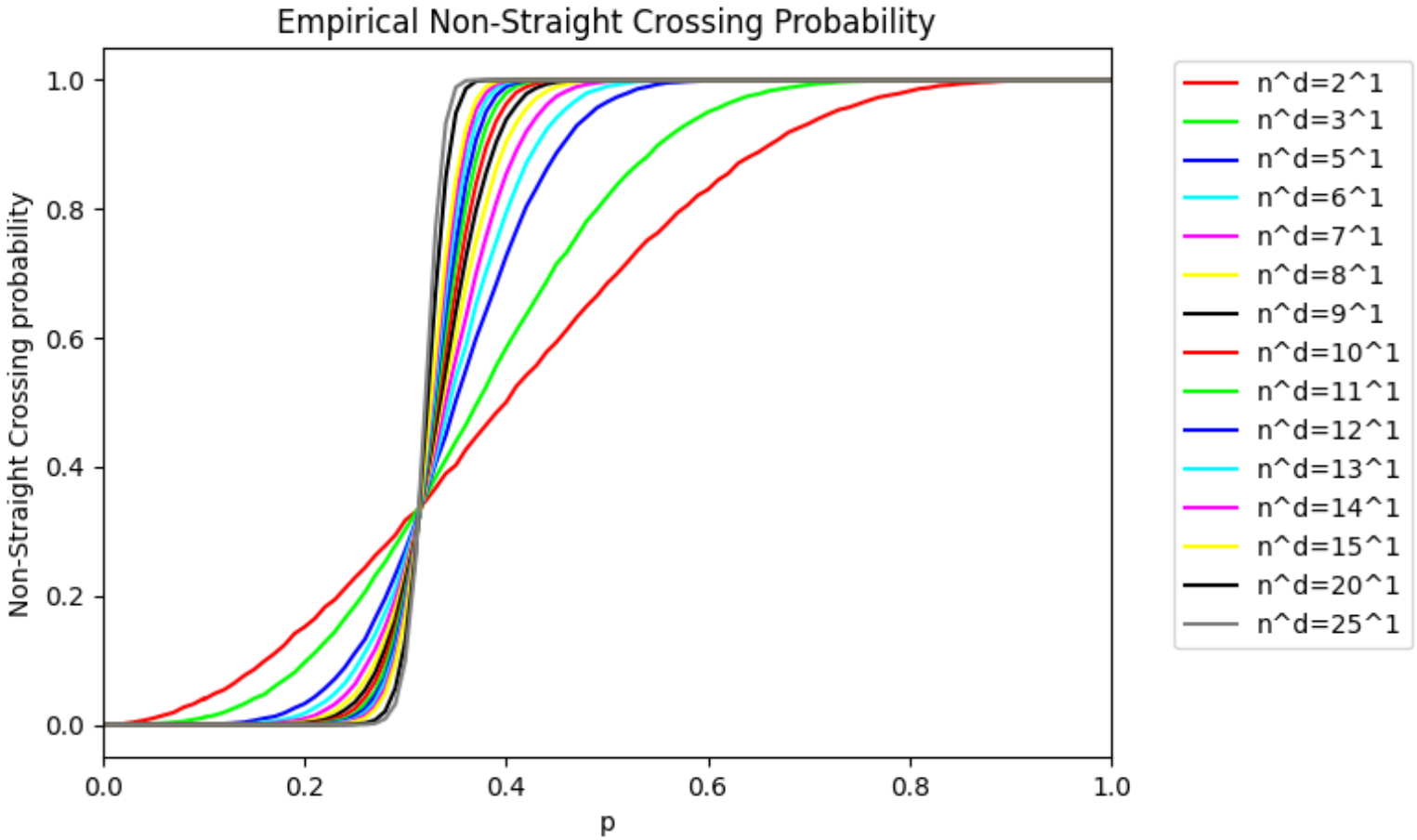
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 2D



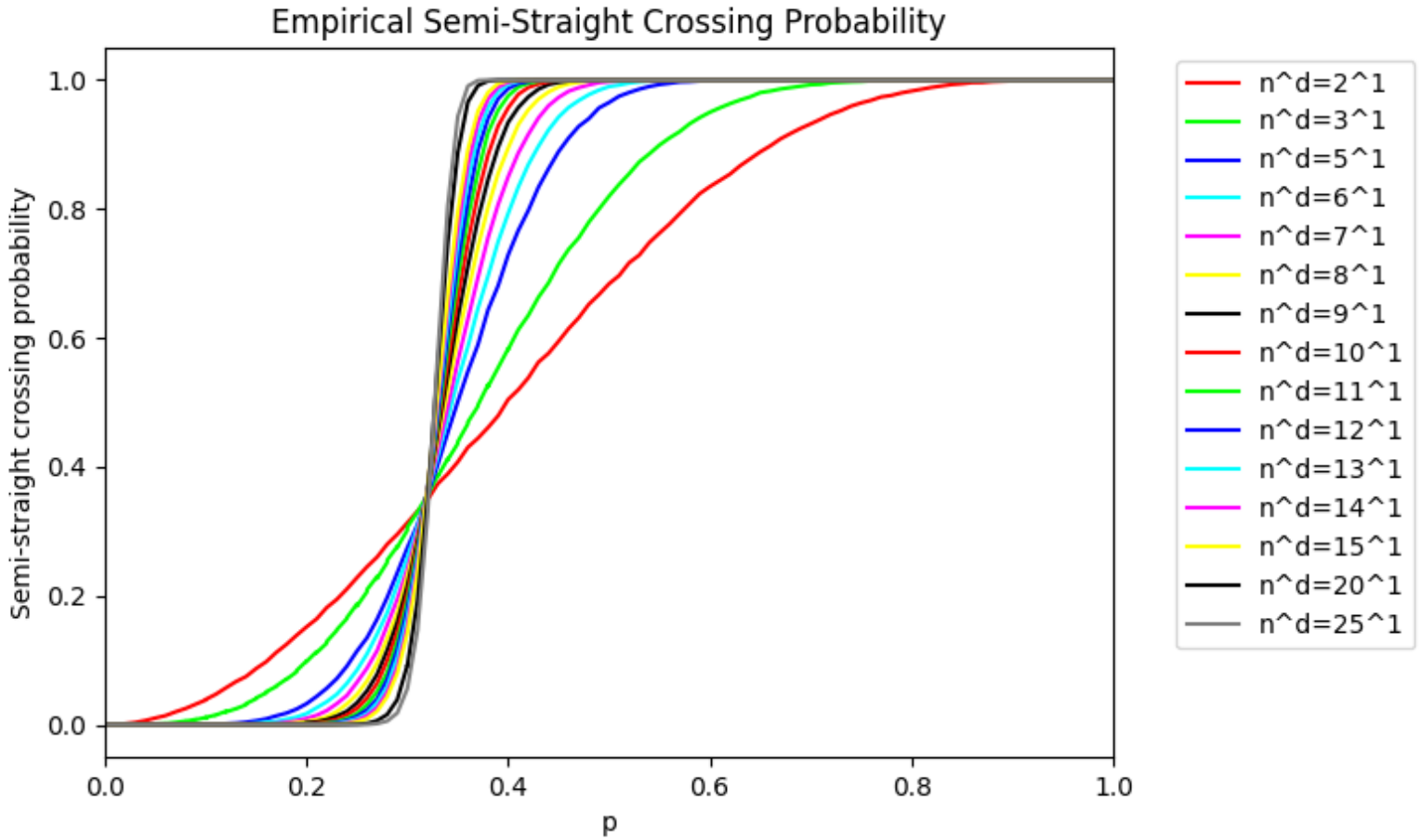
Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 3D



Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 3D

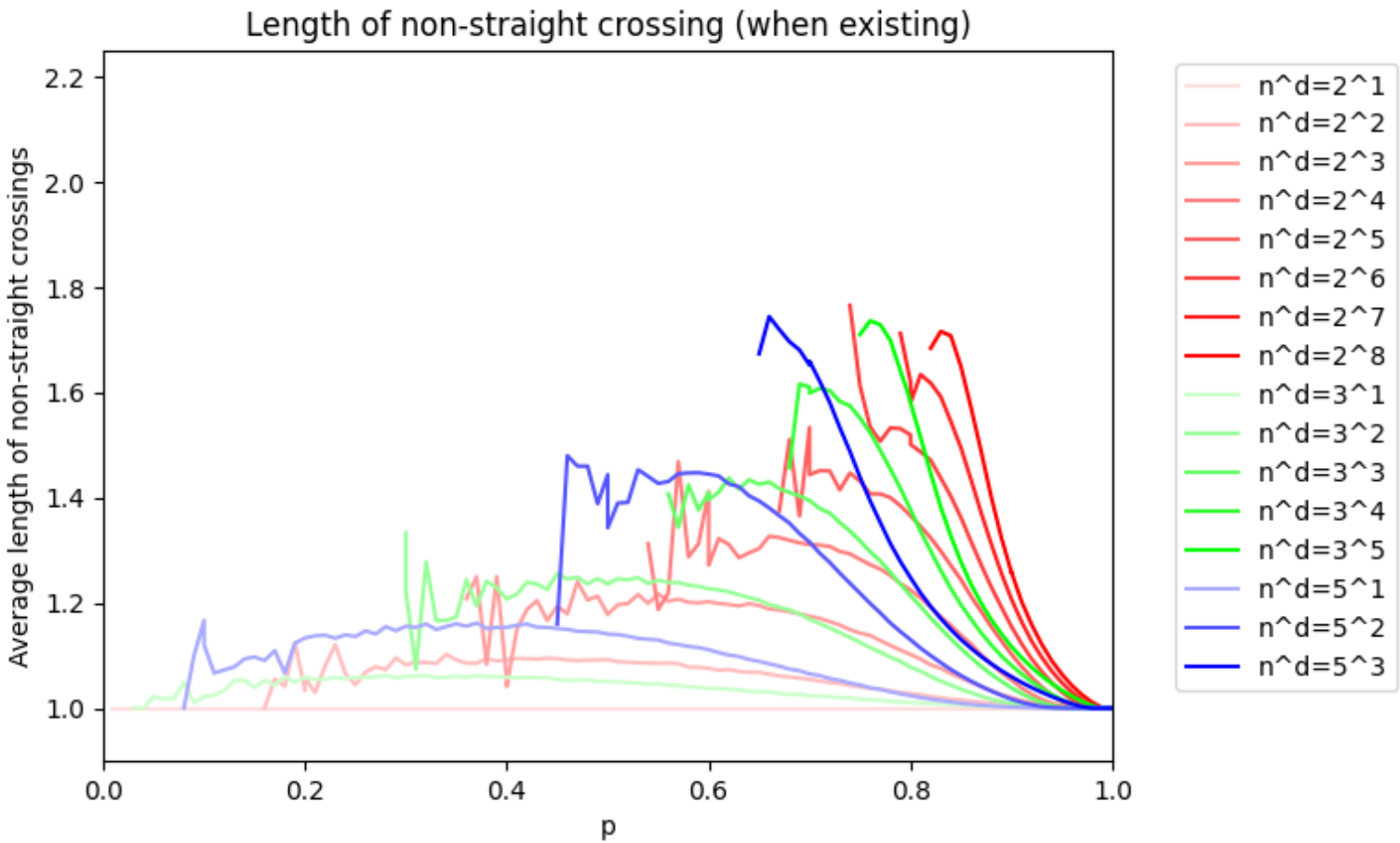


Crossings Probability

Straight Crossing

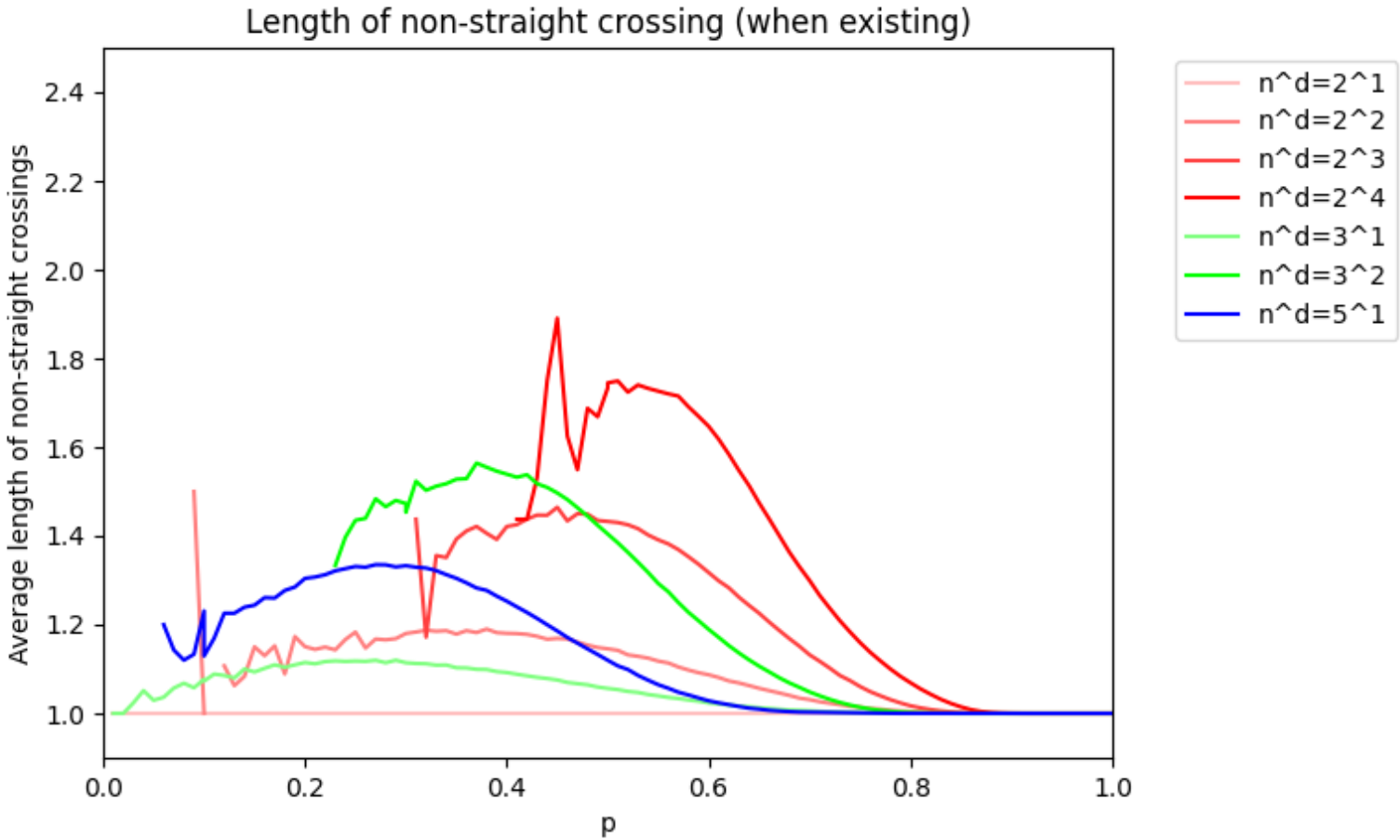
Crossings Length

Non-Straight Crossing, Recursive Percolation, 2D



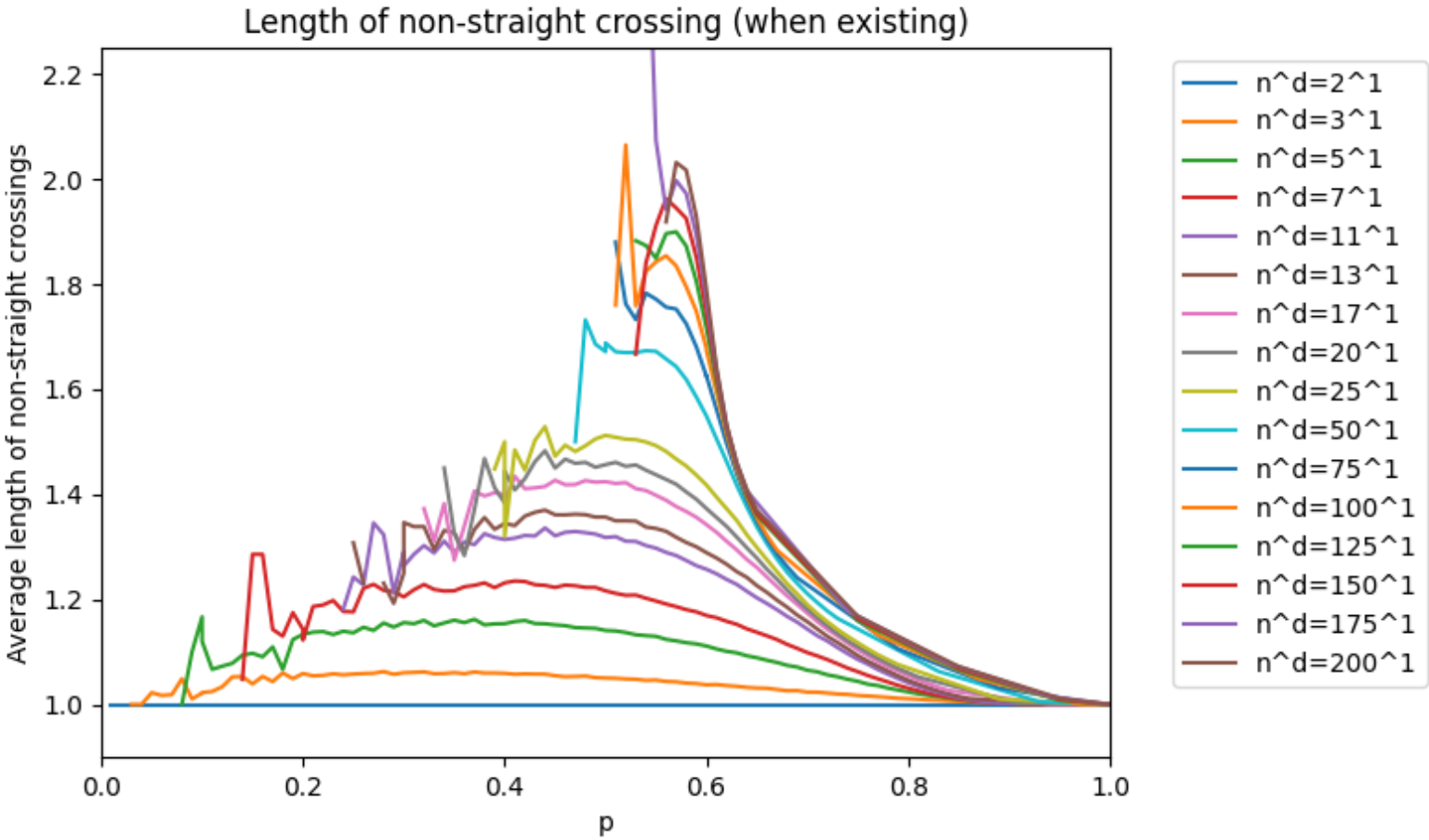
Crossings Length

Non-Straight Crossing, Recursive Percolation, 3D



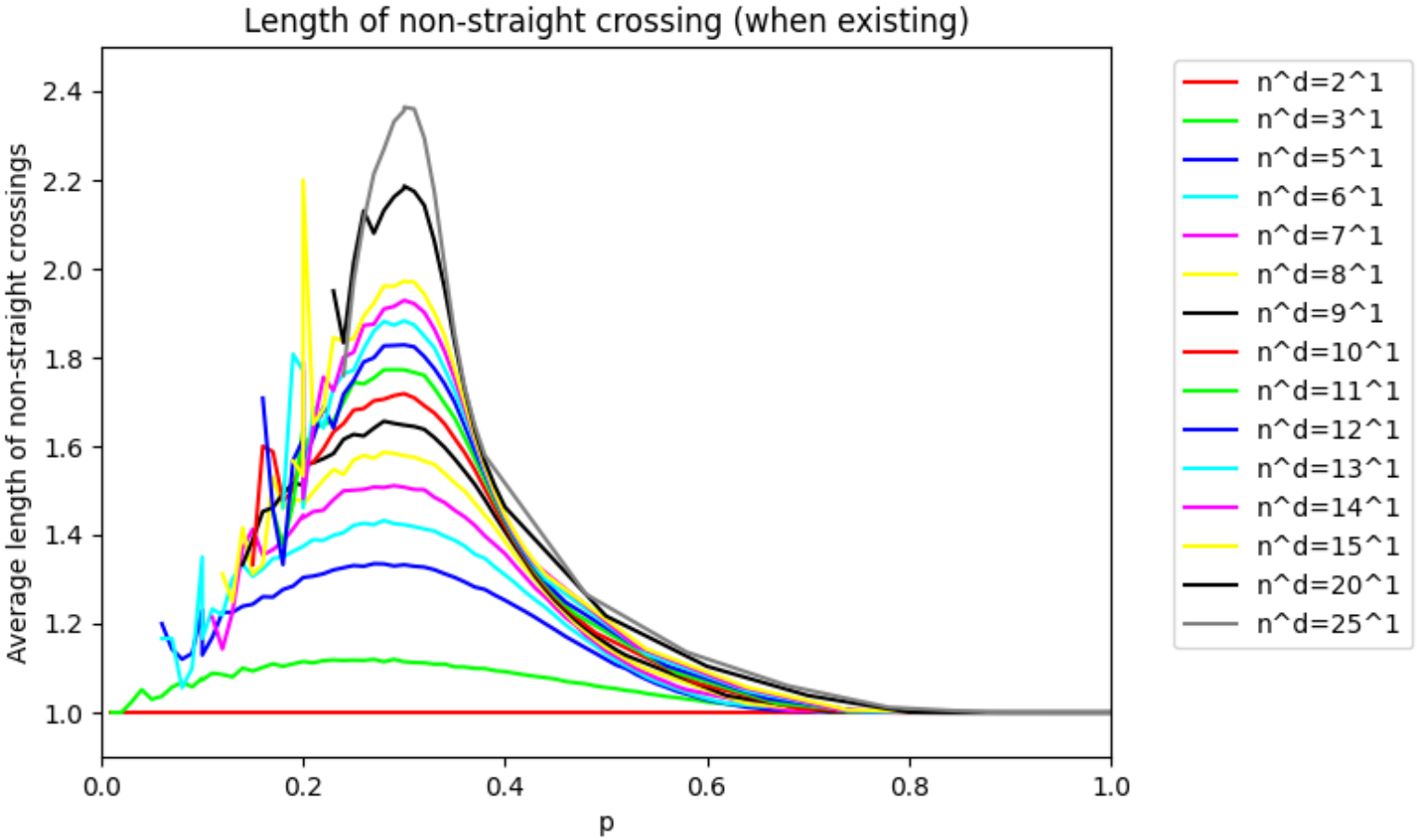
Crossings Length

Non-Straight Crossing, Uniform Percolation, 2D



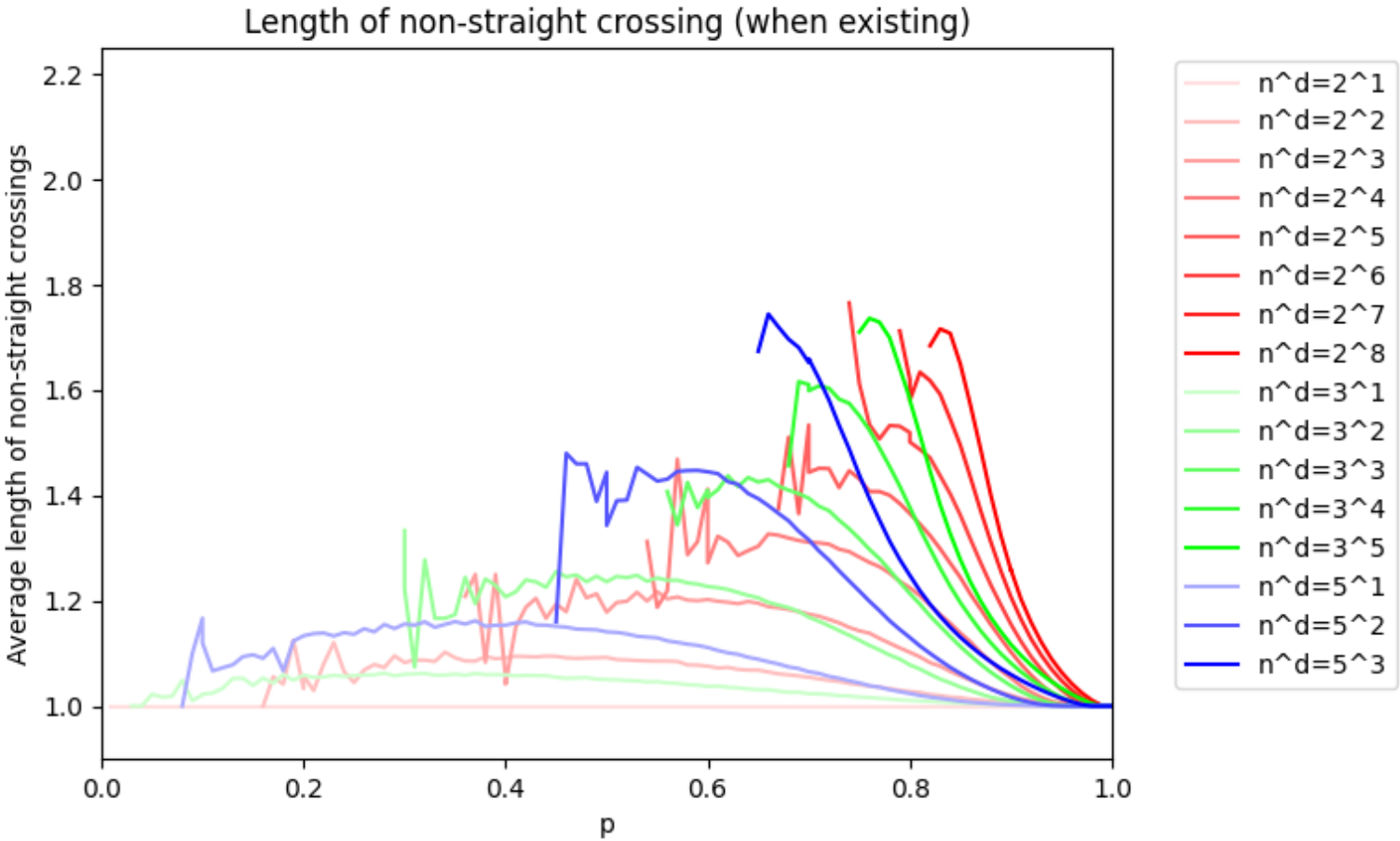
Crossings Length

Non-Straight Crossing, Uniform Percolation, 3D



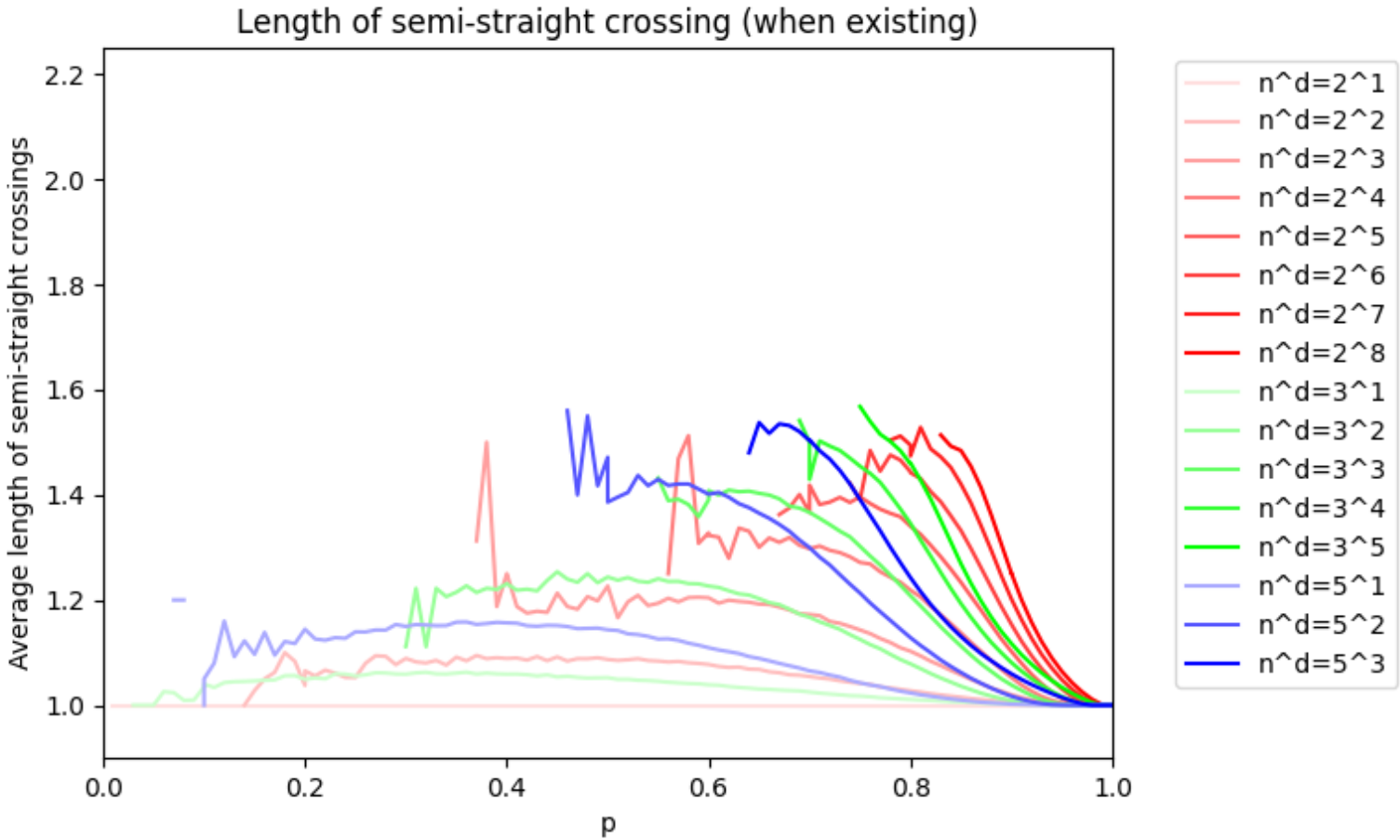
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 2D



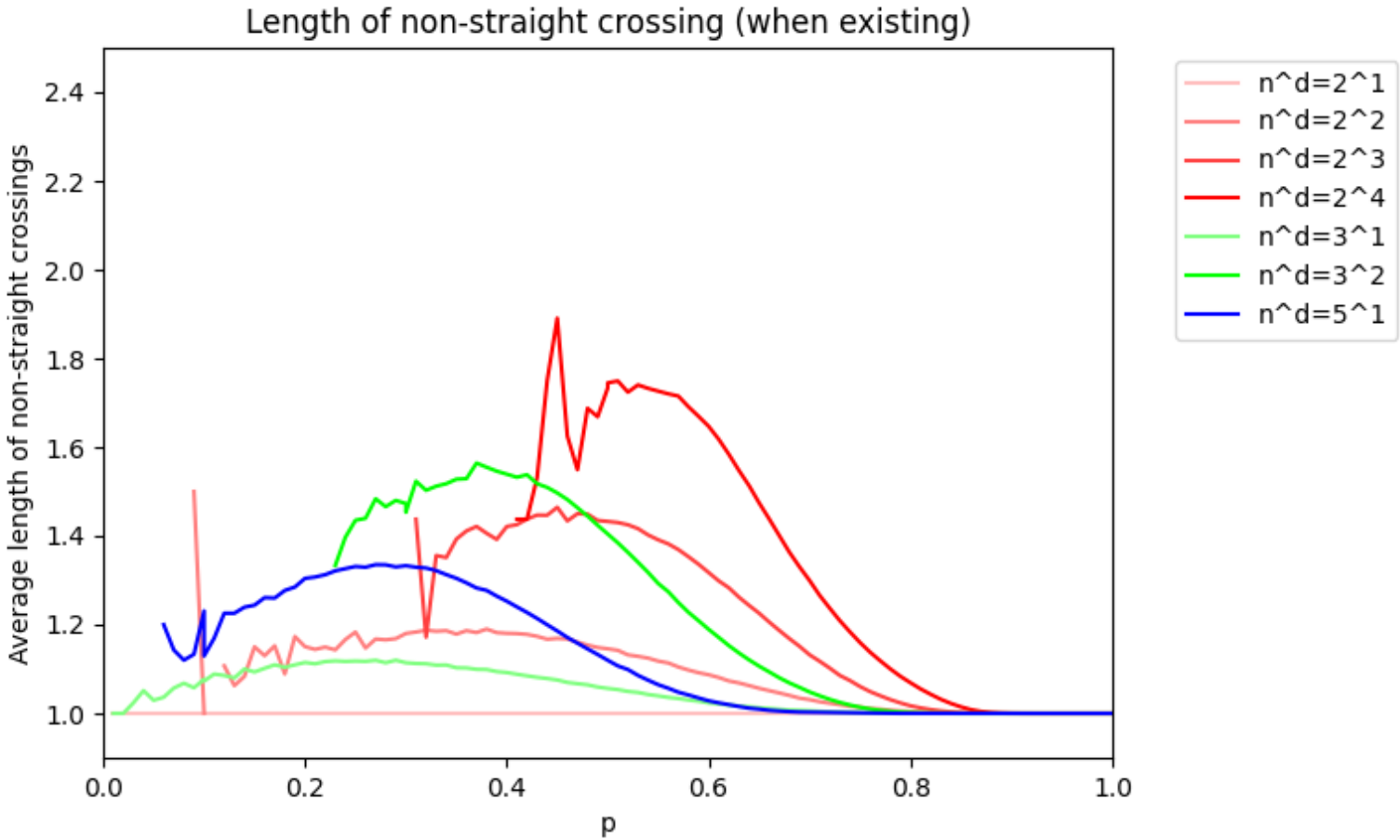
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 2D



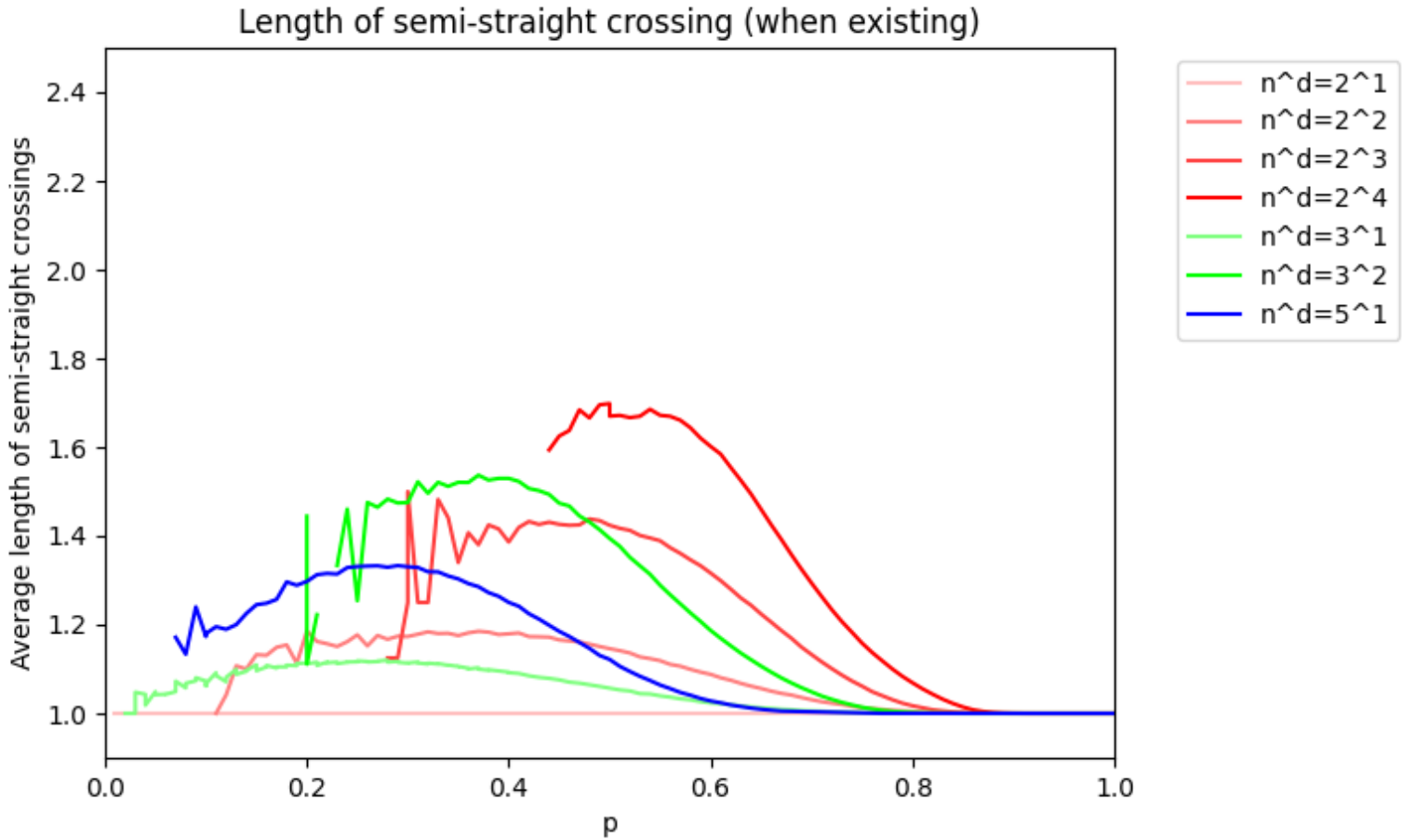
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 3D



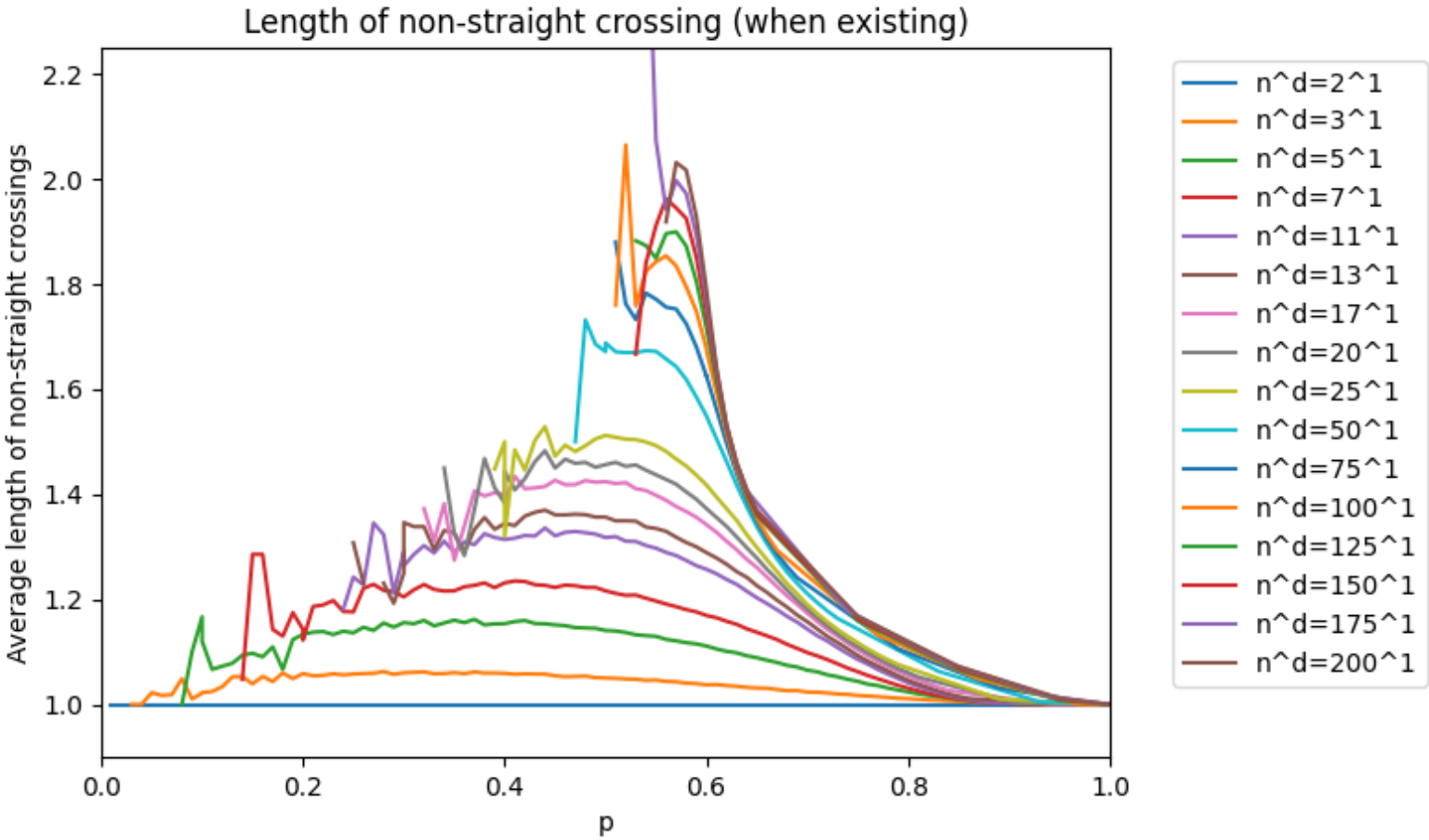
Crossings Length

Semi-Straight Crossing, Recursive Percolation, 3D



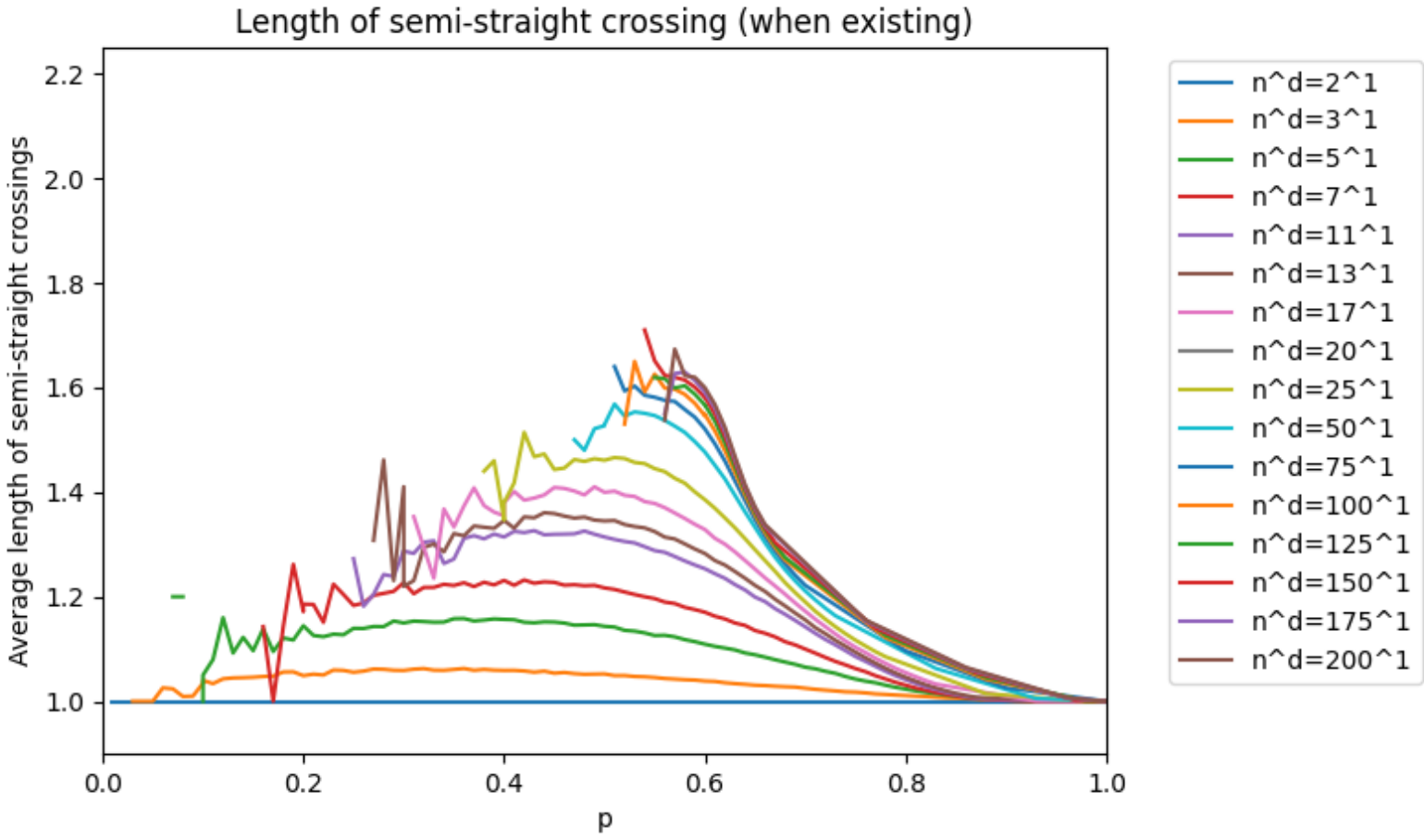
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 2D



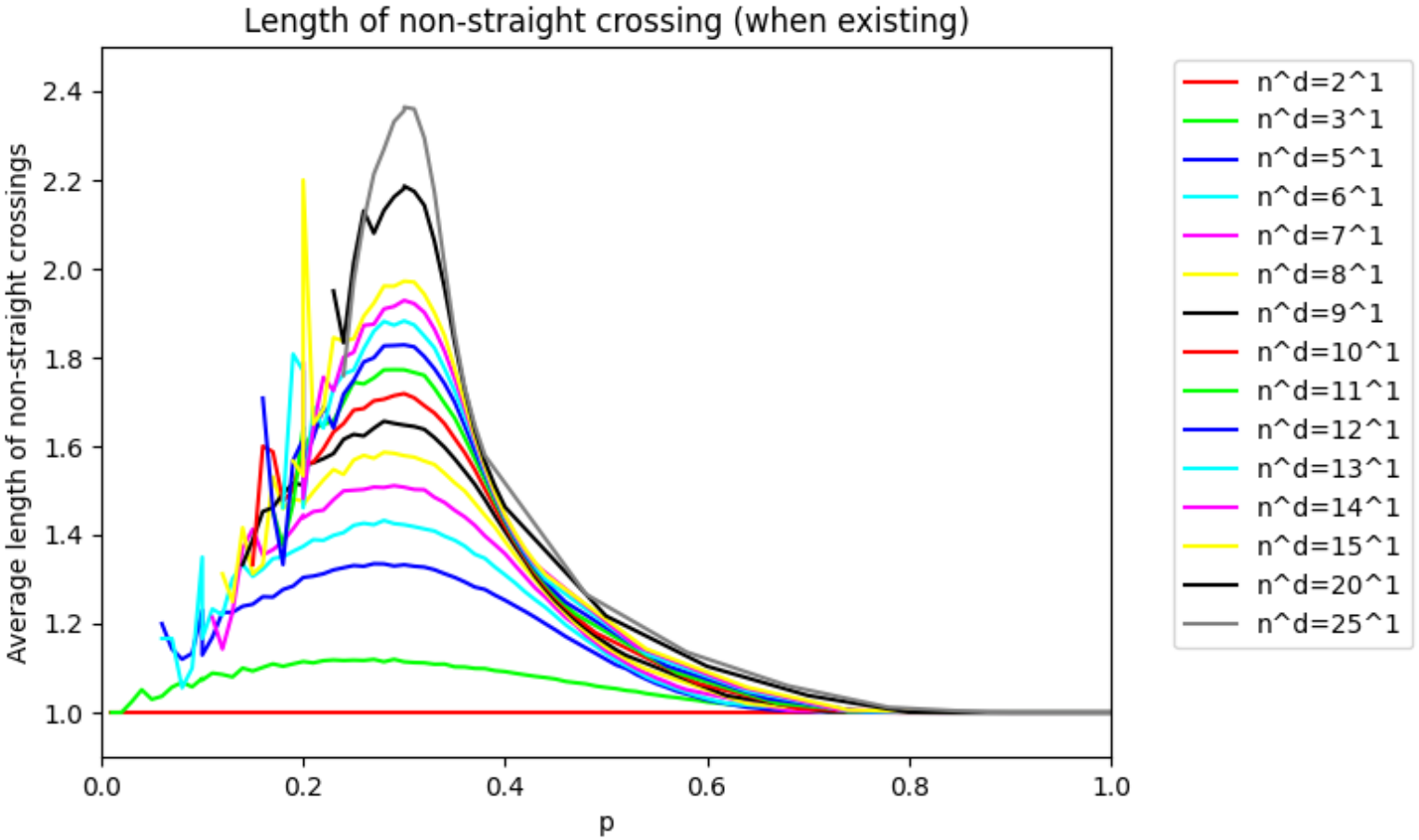
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 2D



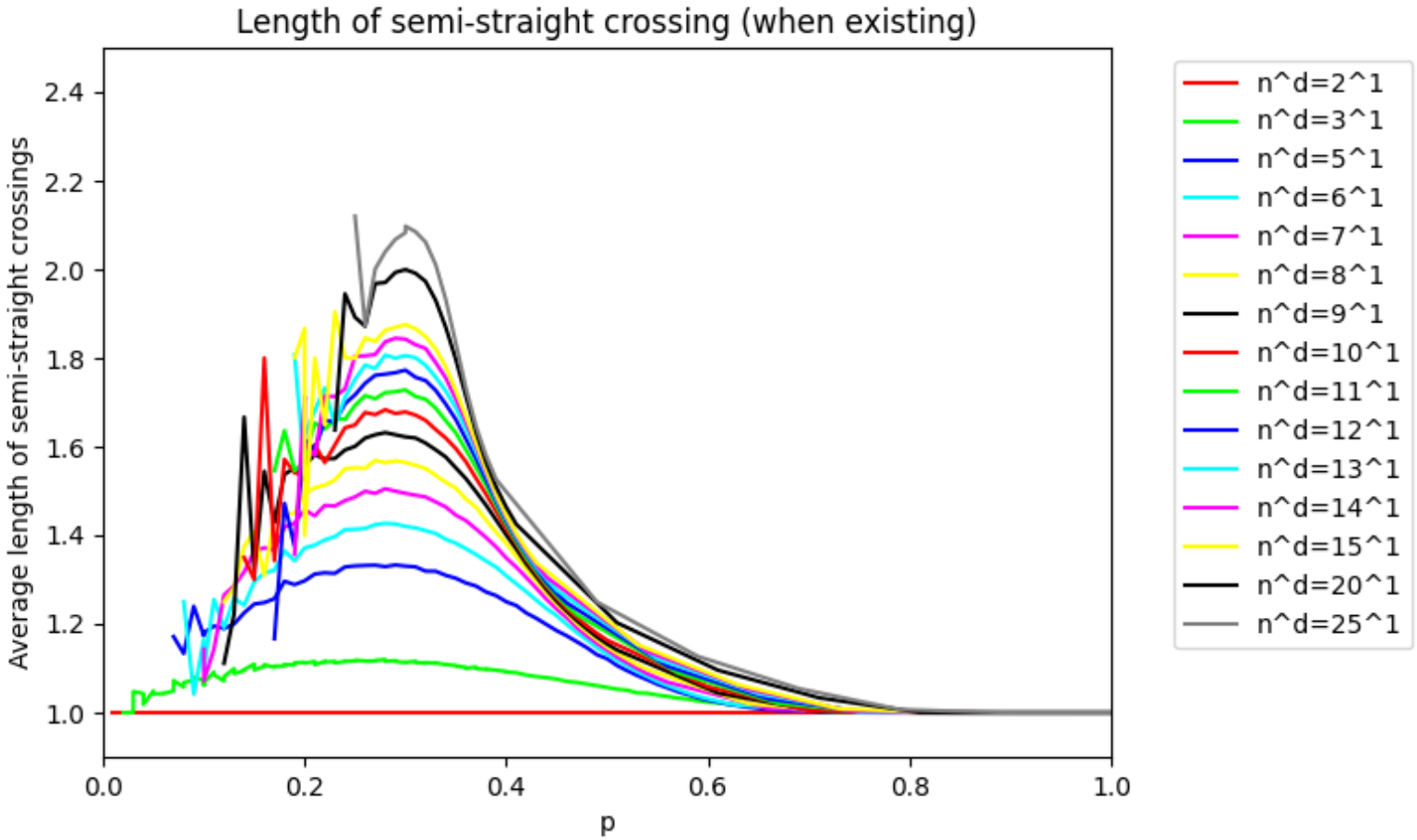
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 3D



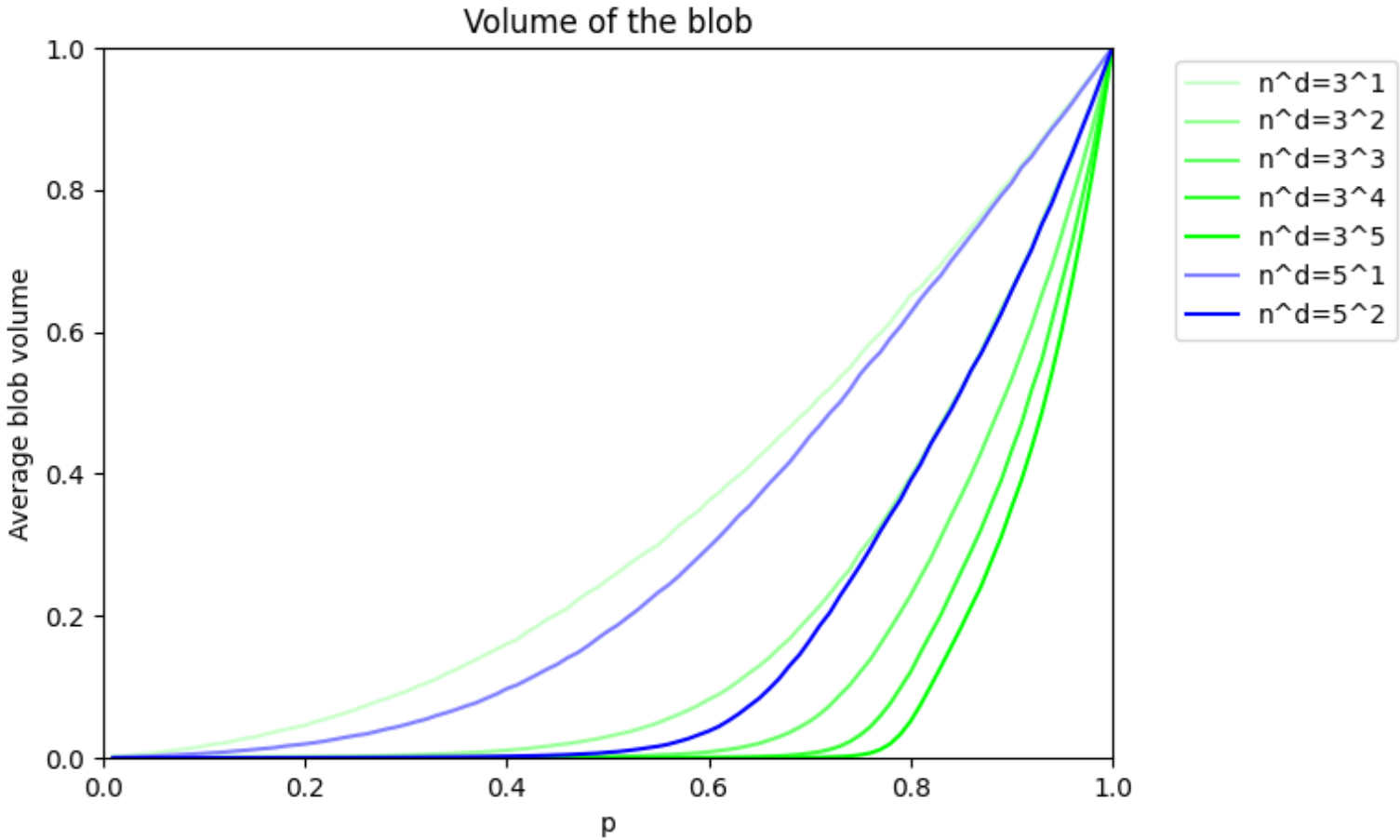
Crossings Length

Semi-Straight Crossing, Uniform Percolation, 3D



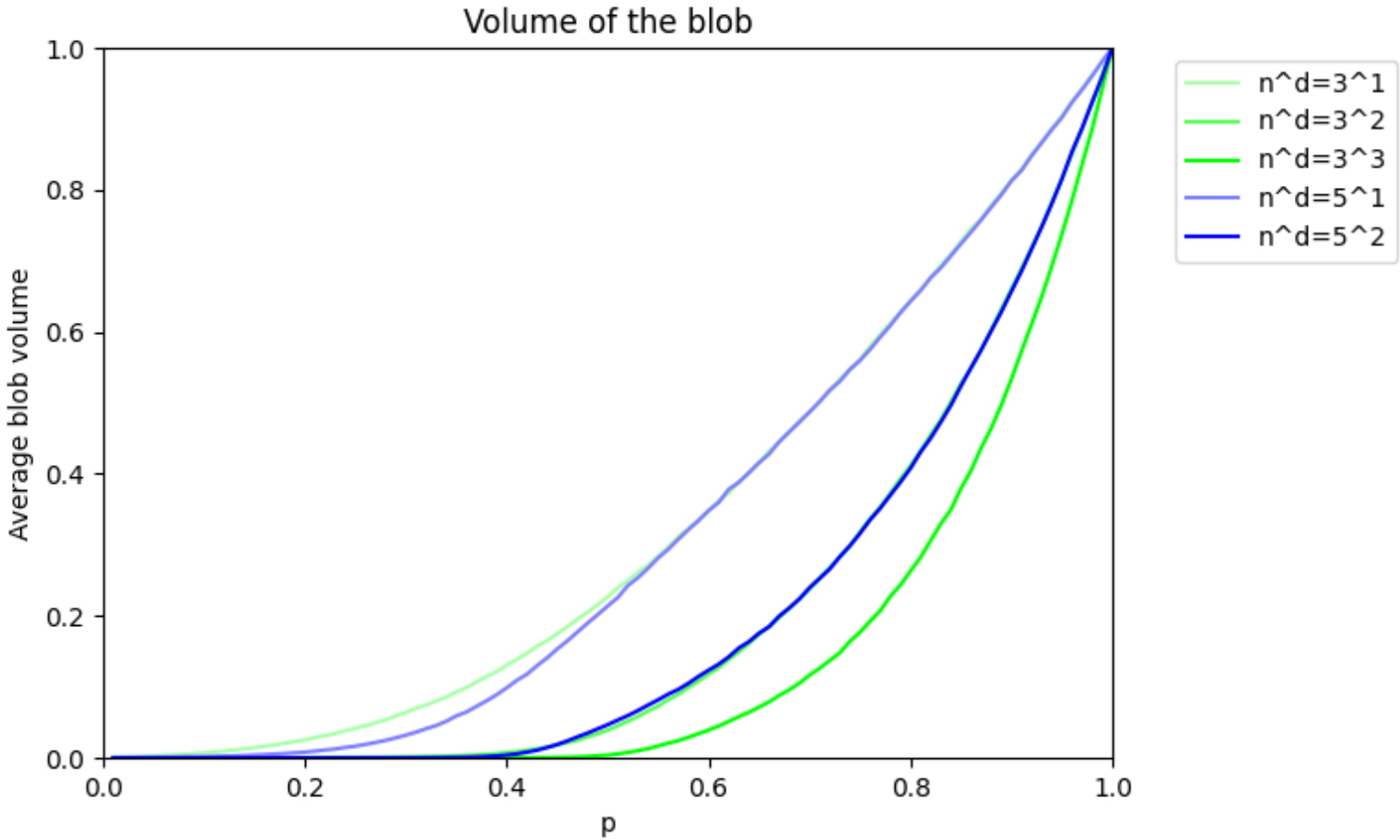
Blob

Volume, Recursive Percolation, 2D



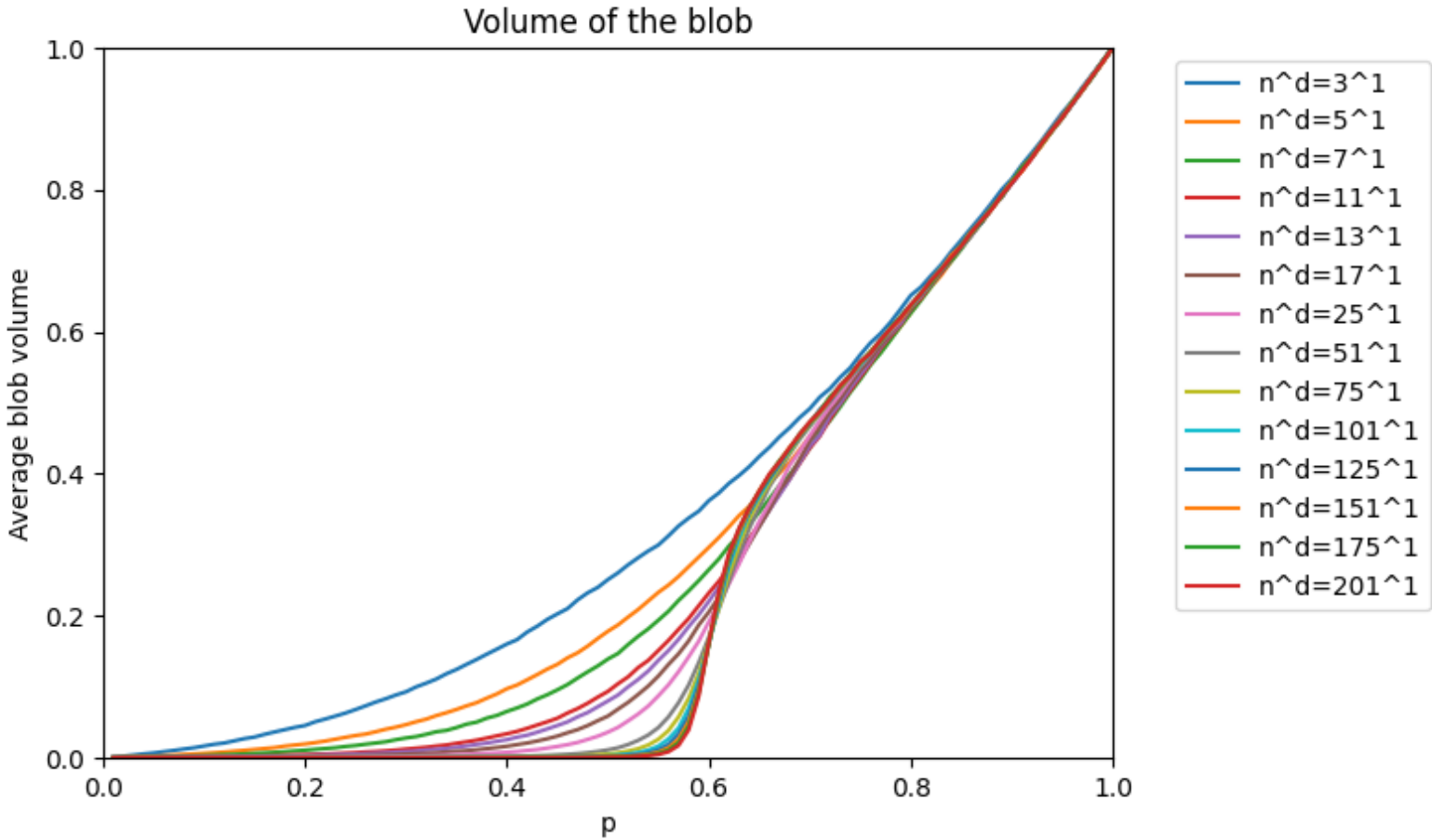
Blob

Volume, Recursive Percolation, 3D



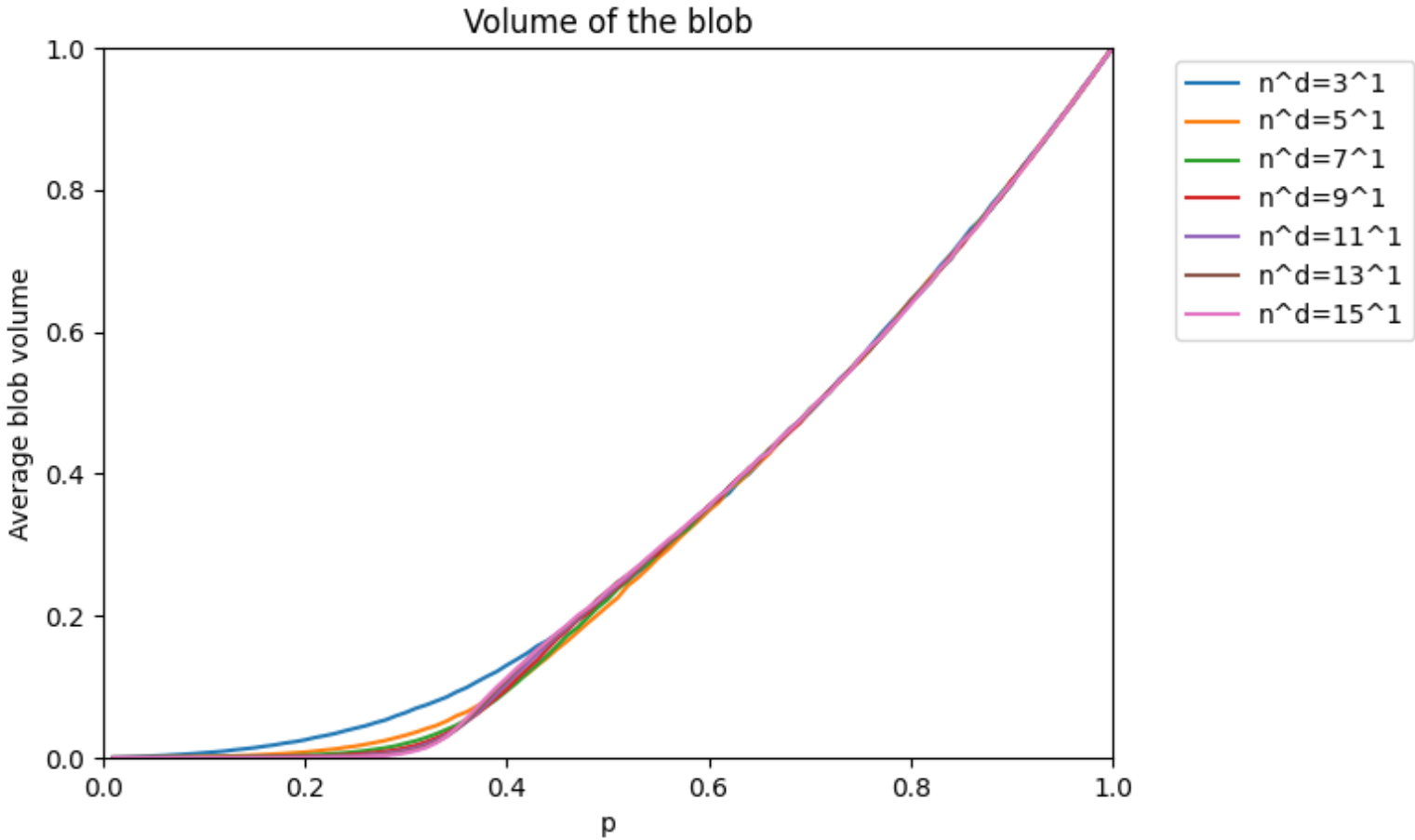
Blob

Volume, Uniform Percolation, 2D



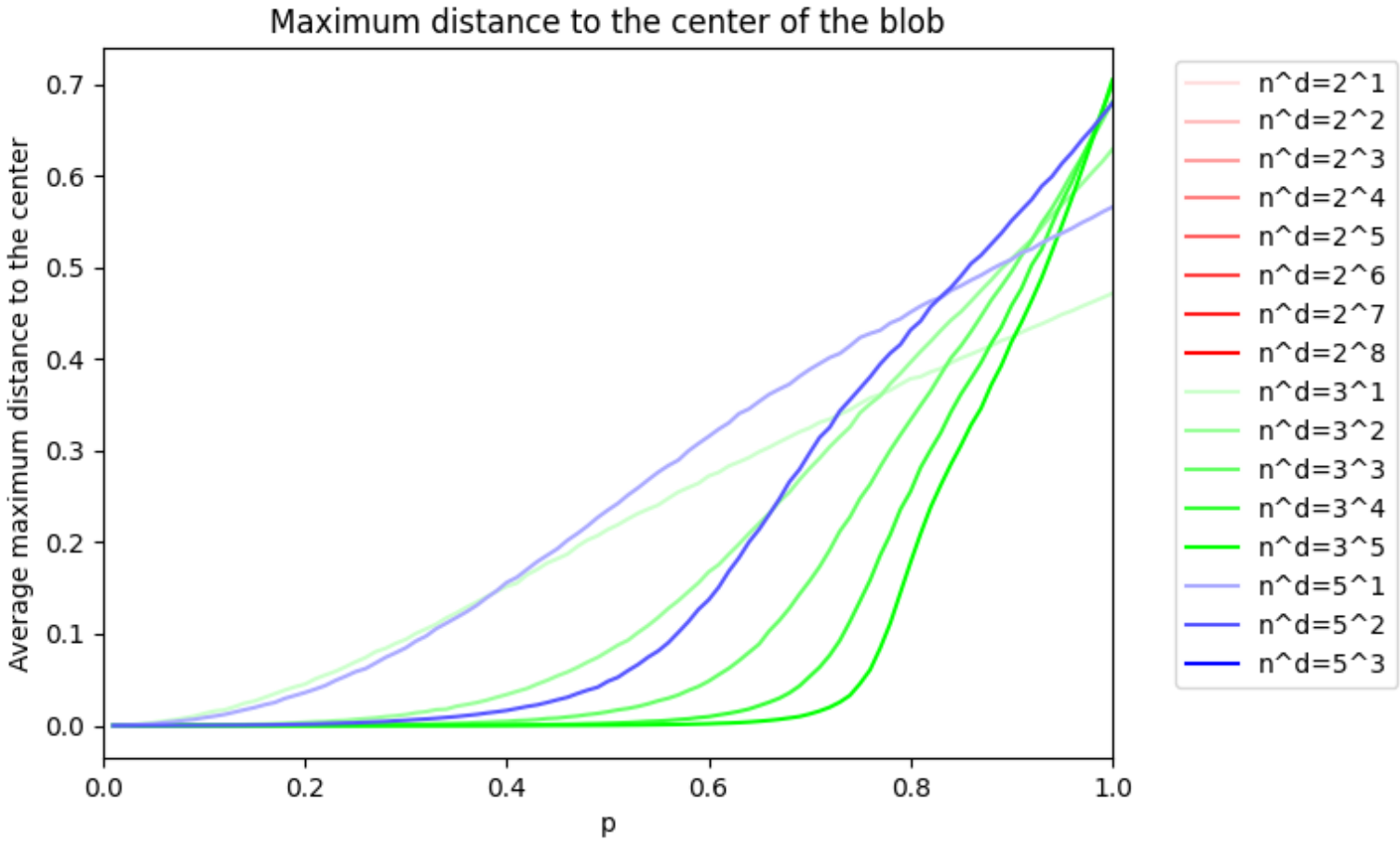
Blob

Volume, Uniform Percolation, 3D



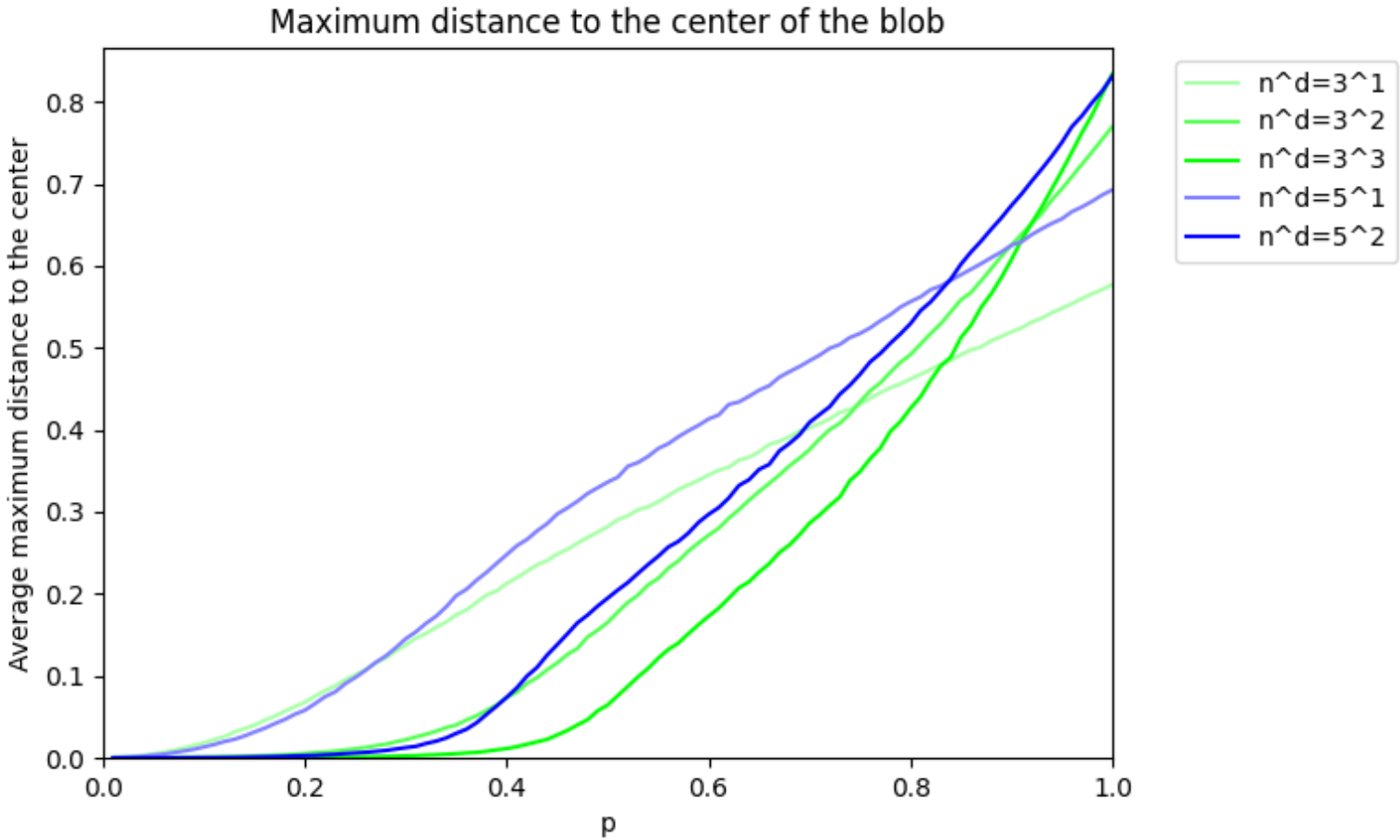
Blob

Distance to Center, Recursive Percolation, 2D



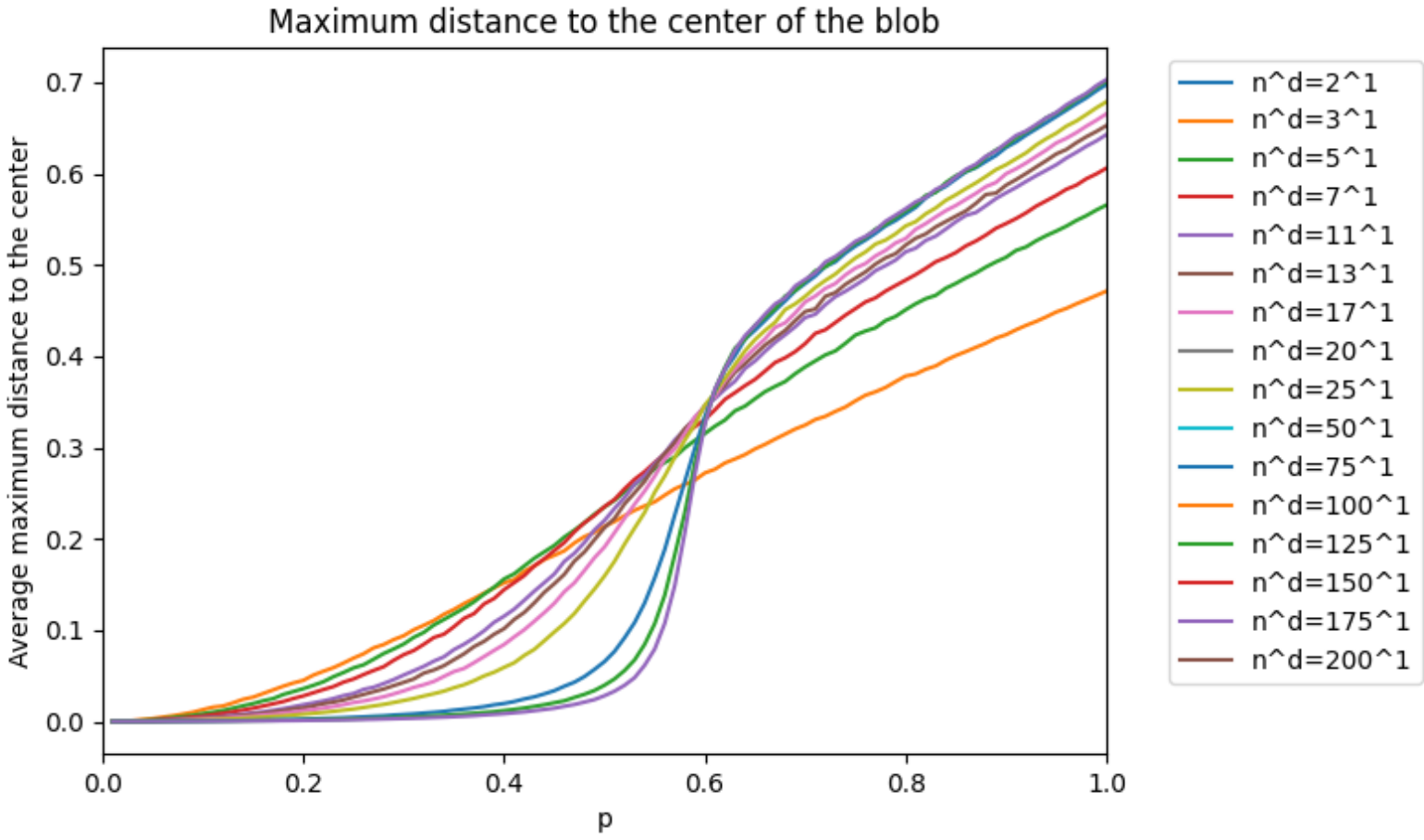
Blob

Distance to Center, Recursive Percolation, 3D



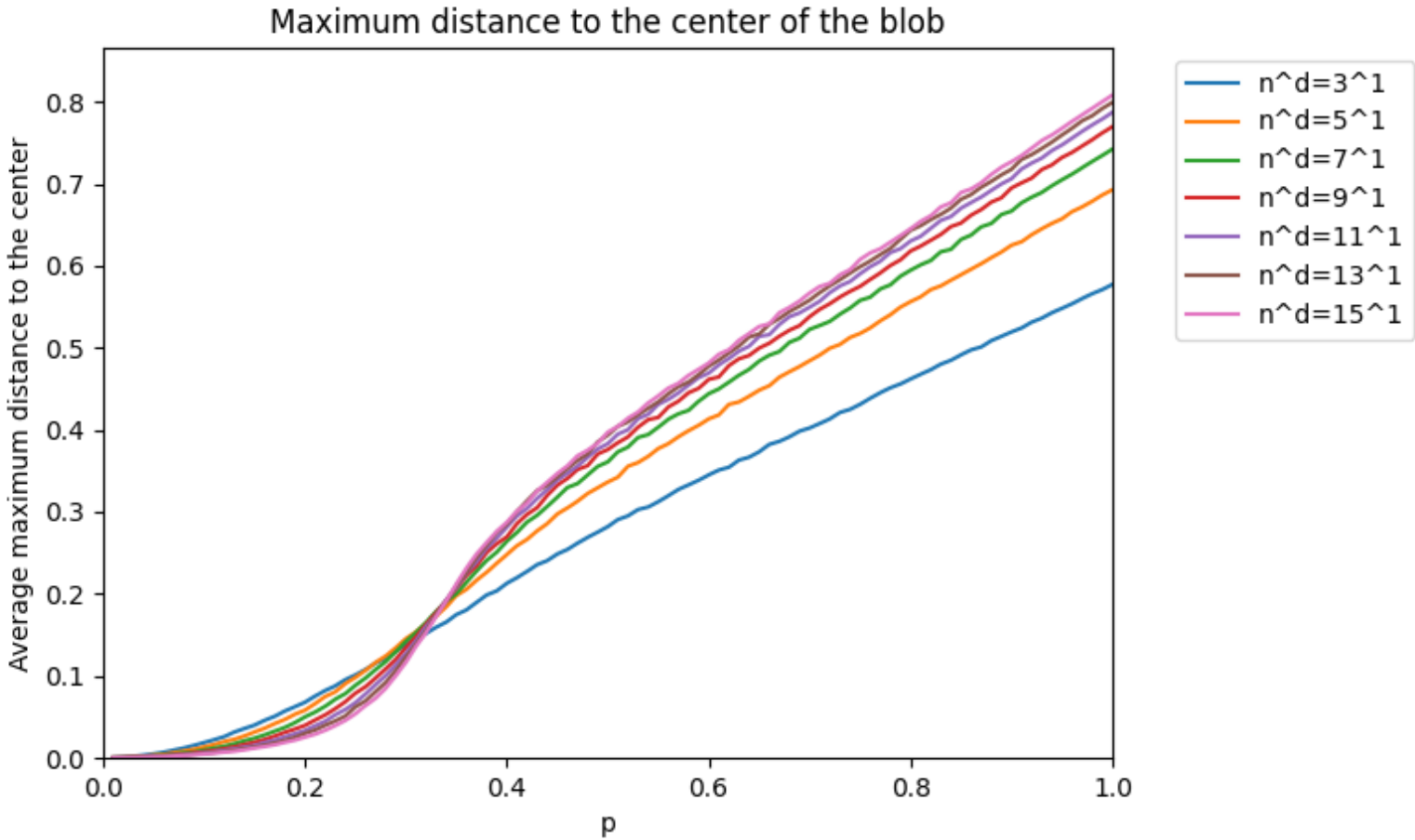
Blob

Distance to Center, Uniform Percolation, 2D



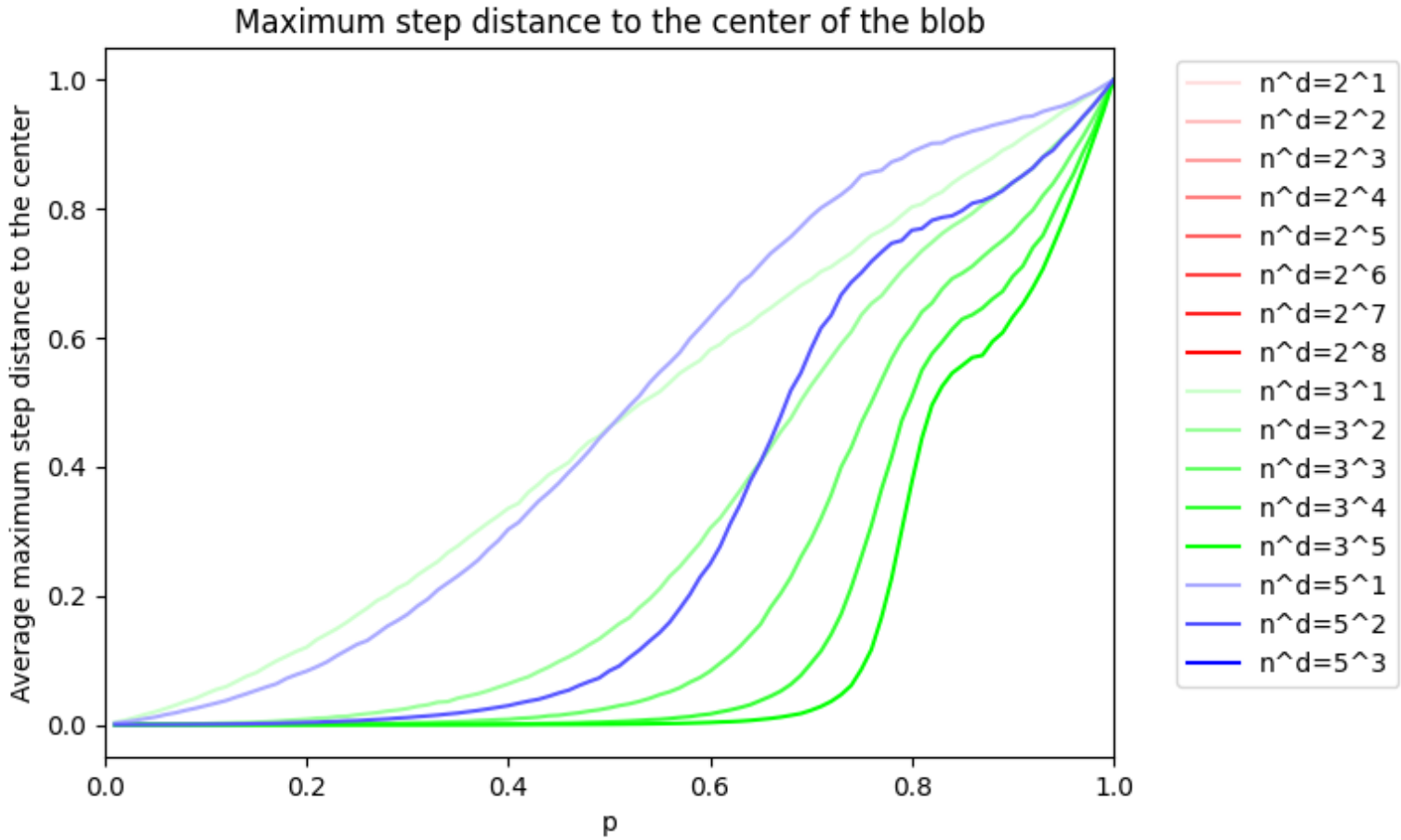
Blob

Distance to Center, Uniform Percolation, 3D



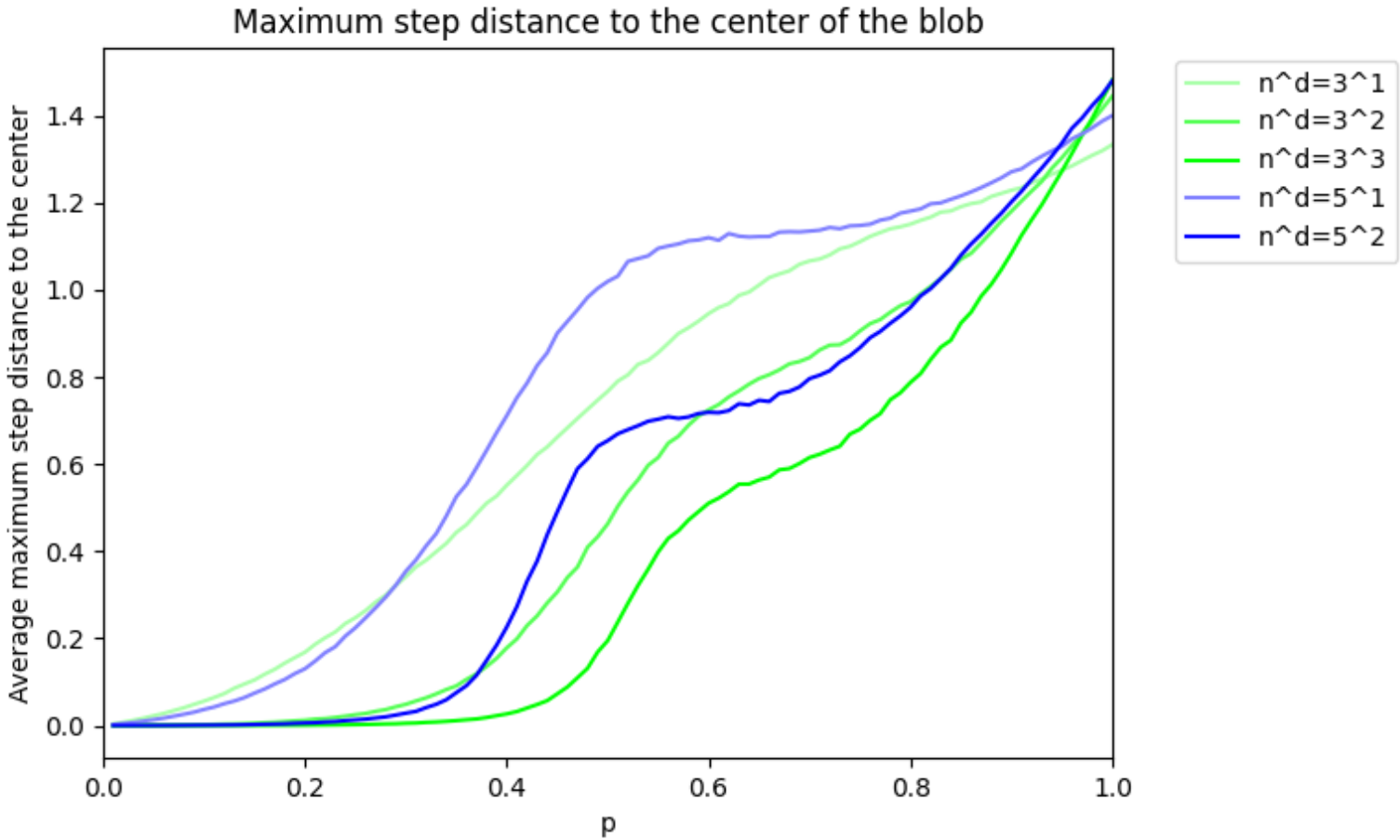
Blob

Steps to Center, Recursive Percolation, 2D



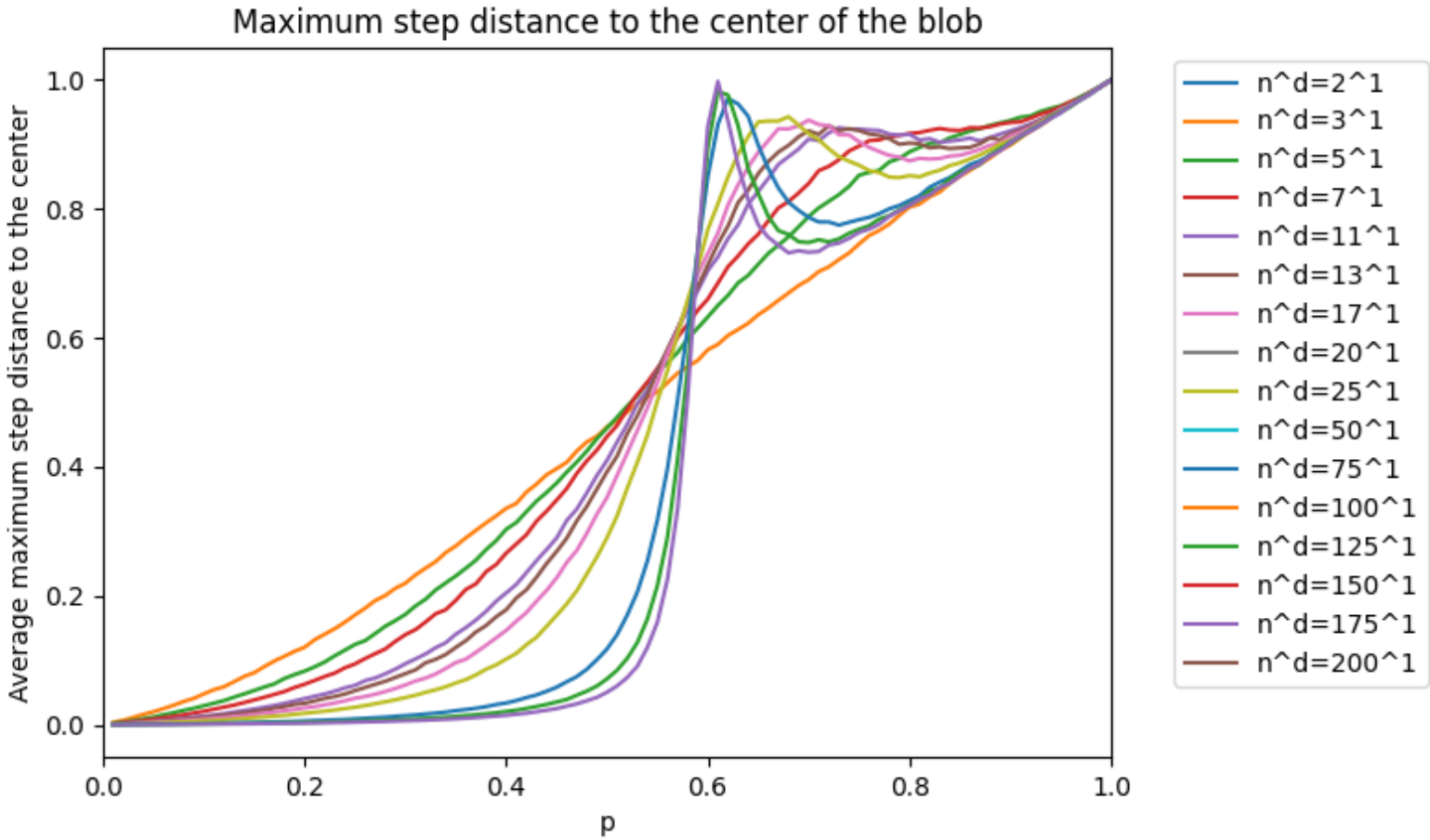
Blob

Steps to Center, Recursive Percolation, 3D



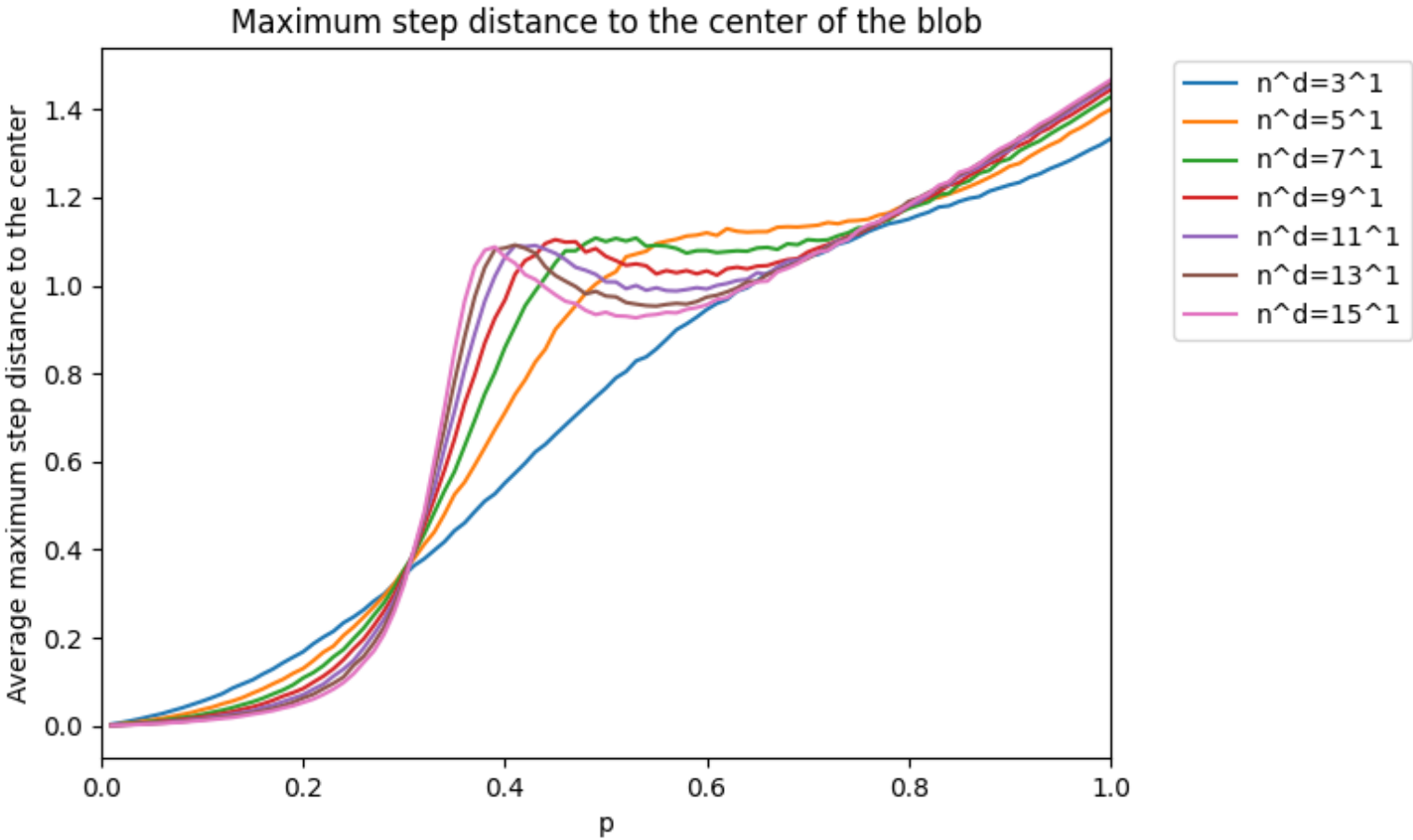
Blob

Steps to Center, Uniform Percolation, 2D



Blob

Steps to Center, Uniform Percolation, 3D



Intersection

Projection