

# Random Fractals

*Student:* Paul Dubois  
*Supervisor* Ben Hambly

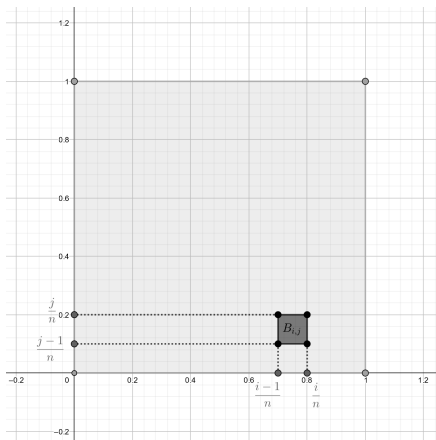
Oxford University

10th March 2021

# The Percolation Process

Plain:  $P \sim \text{Perc}(n, p, 1)$

$$B_{i,j} = \left[ \frac{i-1}{n}, \frac{i}{n} \right] \times \left[ \frac{j-1}{n}, \frac{j}{n} \right]$$



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$\varepsilon_{i,j} \in \{0, 1\}$  with  $\mathbb{P}(\varepsilon_{i,j} = 1) = p$  ( i.e.  $\varepsilon_{i,j} \sim \mathcal{B}(p)$ )

$$P = \bigcup_{\substack{i,j \\ \varepsilon_{i,j}=1}} B_{i,j}$$

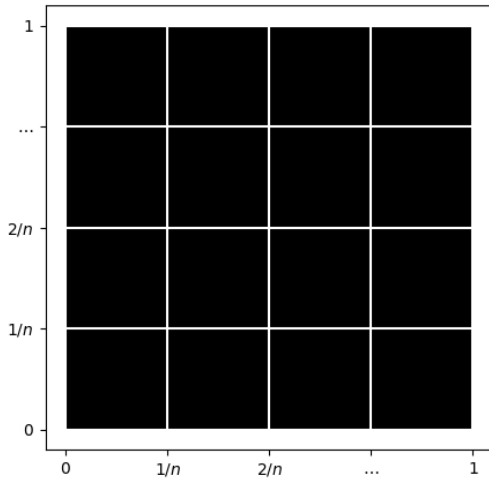
$$Z = |\{(i,j) \mid \varepsilon_{i,j} = 1\}|$$

$$D = \frac{Z}{pn^2}$$



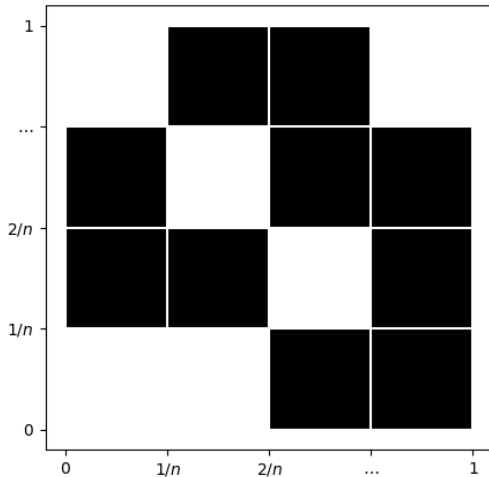
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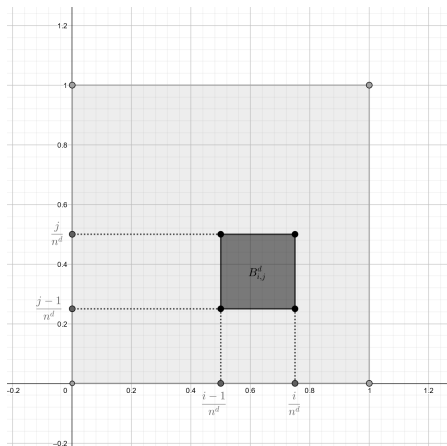
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Recursive:  $P_d \sim \text{Perc}(n, p, d)$

$$B_{i,j}^d = \left[ \frac{i-1}{n^d}, \frac{i}{n^d} \right] \times \left[ \frac{j-1}{n^d}, \frac{j}{n^d} \right]$$



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$$B_{i,j}^d = \left[ \frac{i-1}{n^d}, \frac{i}{n^d} \right] \times \left[ \frac{j-1}{n^d}, \frac{j}{n^d} \right]$$

$$\varepsilon_{i,j}^d \in \{0, 1\} \text{ with } \mathbb{P}(\varepsilon_{i,j}^d = 1) = p \quad (\text{i.e. } \varepsilon_{i,j}^d \sim \mathcal{B}(p))$$

$$P_0 = [0, 1]^2 \quad ; \quad P_d = P_{d-1} \cap \left( \bigcup_{\substack{i,j \\ \varepsilon_{i,j}^d = 1}} B_{i,j}^d \right)$$

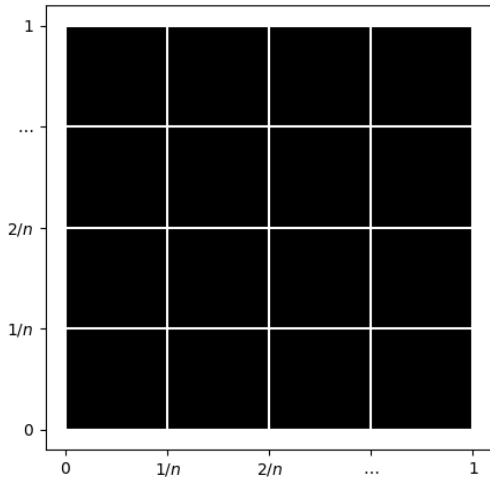
$$Z_d = \left| \left\{ (i, j) \mid \varepsilon_{i,j}^d = 1 \right\} \right|$$

$$D_d = \frac{Z_d}{(pn^2)^d}$$



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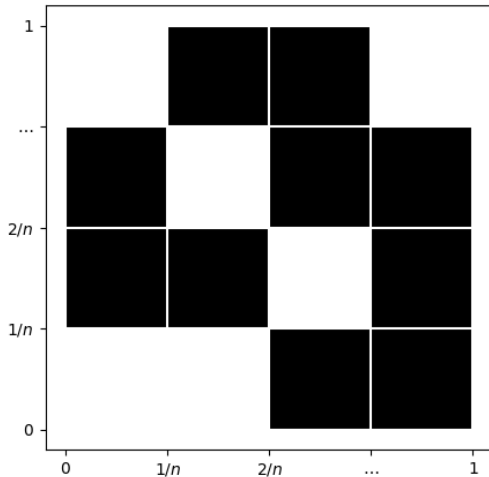
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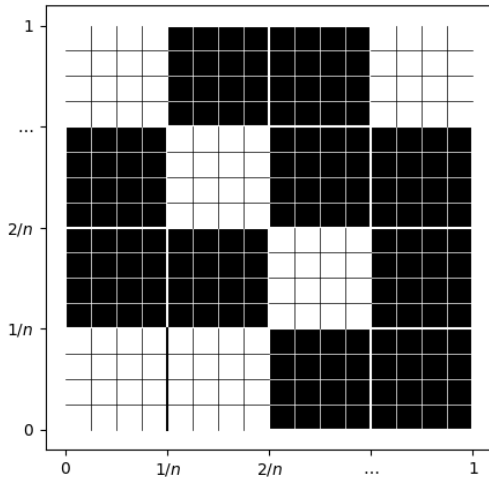
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Limit:  $P_\infty \sim \text{Perc}(n, p)$

$$P_\infty = \bigcap_{d \in \mathbb{N}} P_d$$

$$D_\infty = \lim_{d \rightarrow \infty} D_d$$



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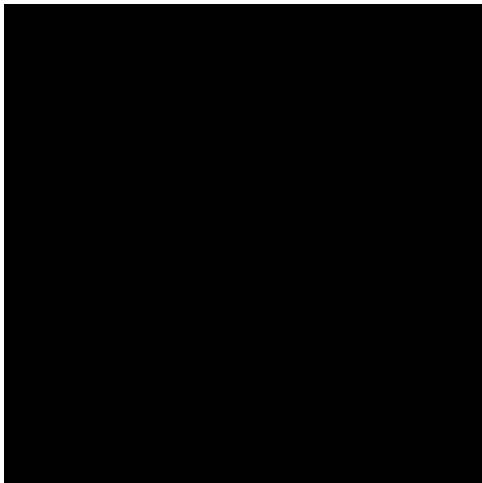
$$D_\infty > 0 \iff P_\infty \neq \emptyset$$

$$\mathbb{E}(D_\infty) = 1$$



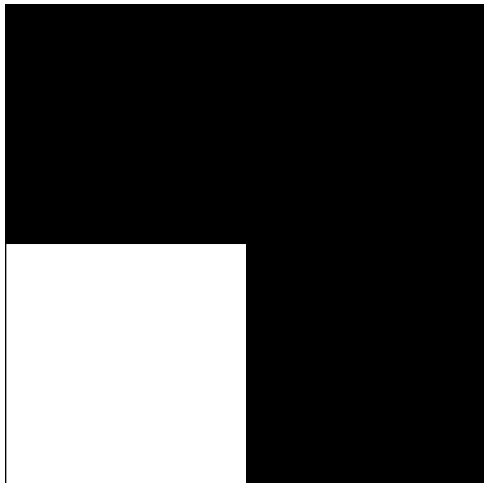
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Example for  $n = 2$



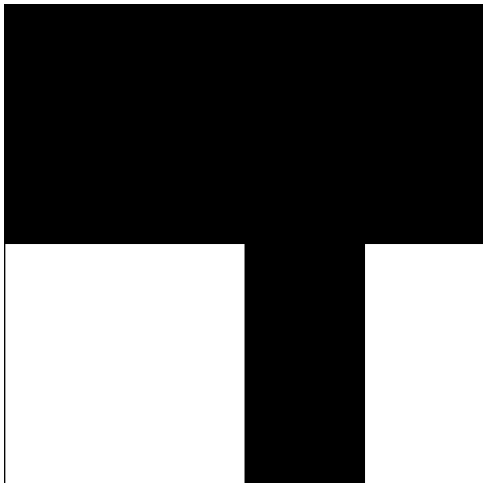
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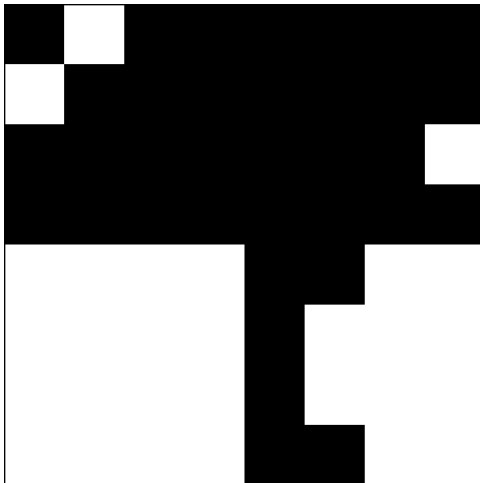
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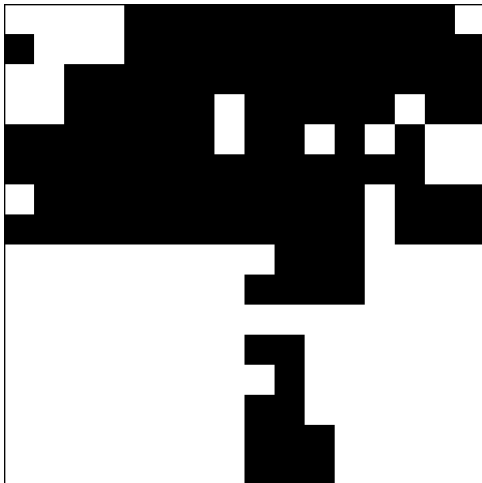
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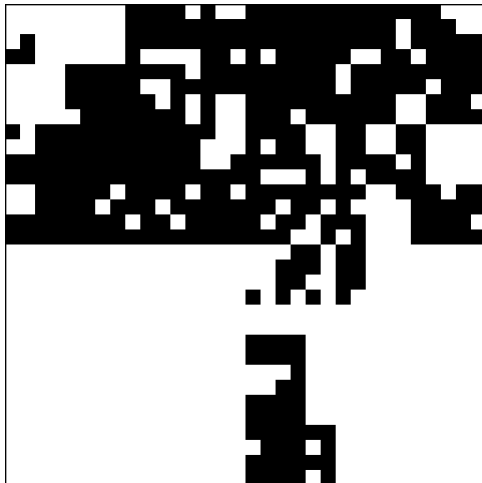
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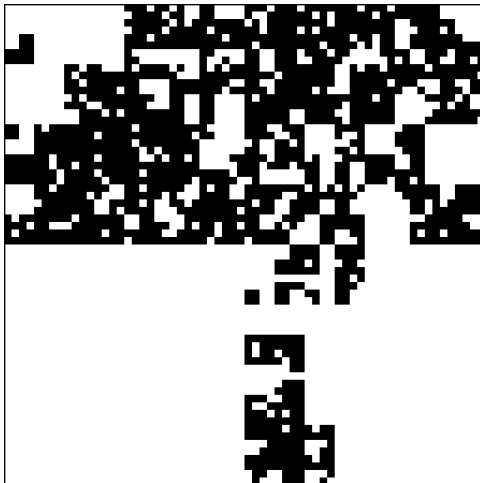
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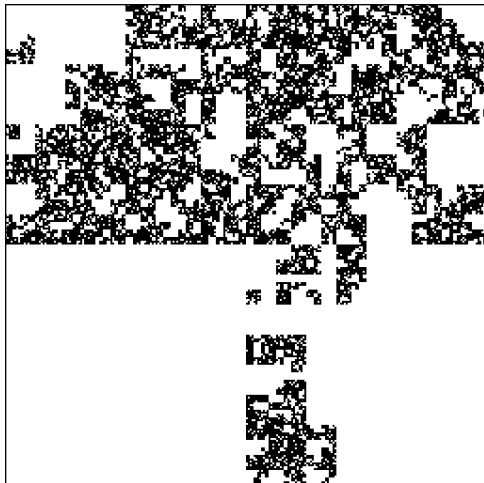
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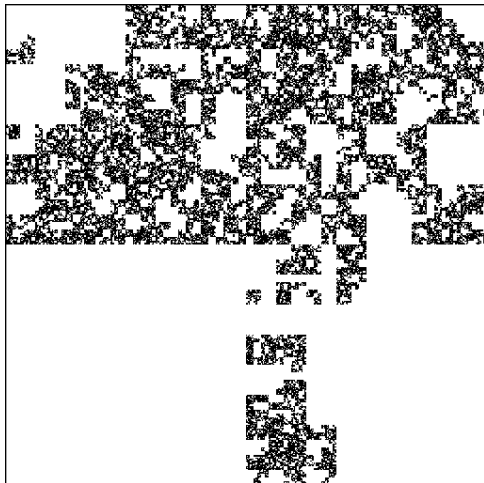
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Example for  $n = 2$



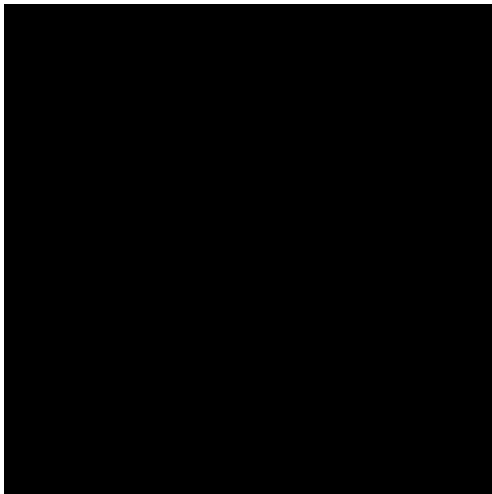
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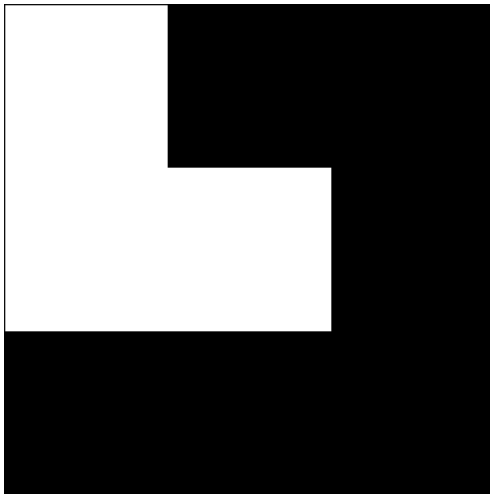
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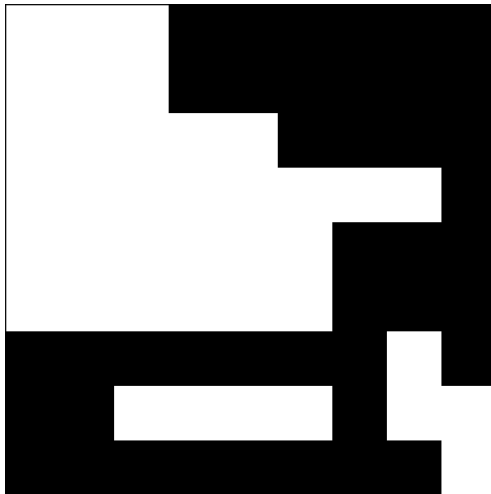
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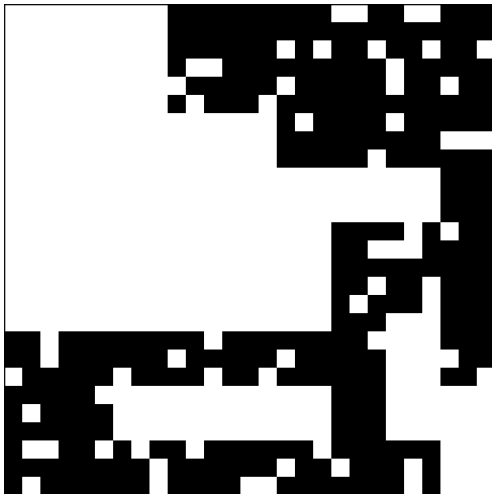
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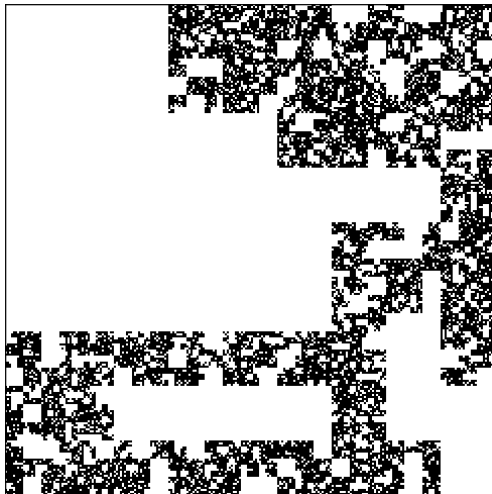
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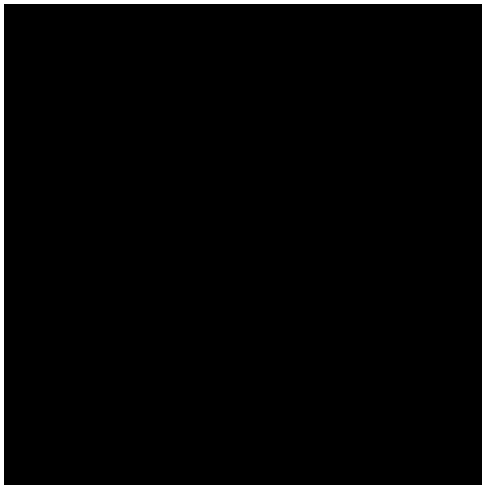
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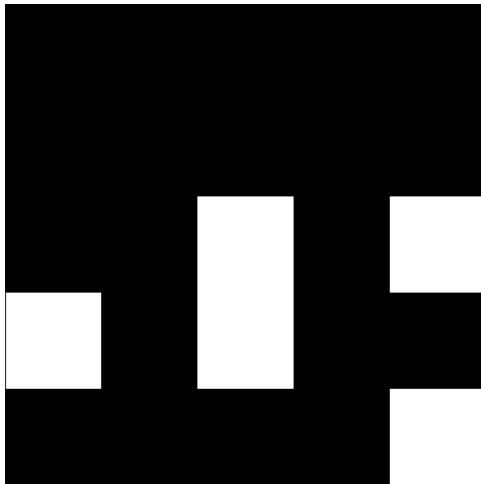
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Example for  $n = 5$



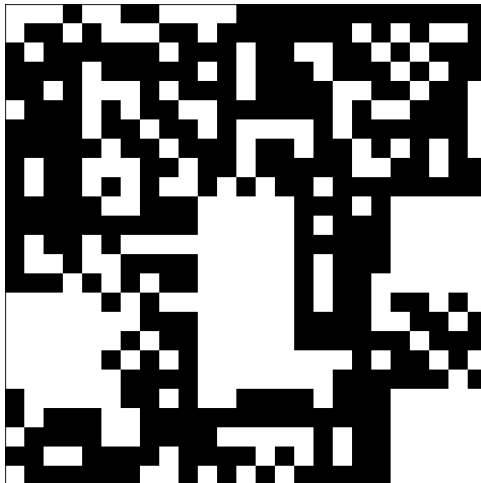
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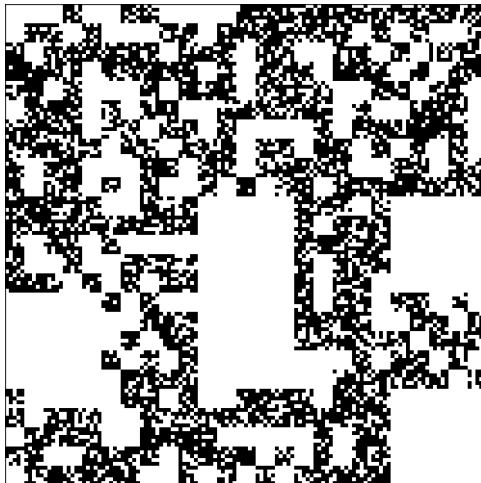
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# Dimensions

## Intuition



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**1D:** Scale by  $\lambda \iff$  Lengths multiplied by  $\lambda^1$



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**$nD$ :** Scale by  $\lambda \iff$   $n$ -Dim. Volumes multiplies by  $\lambda^n \quad \forall n \in \mathbb{N}$



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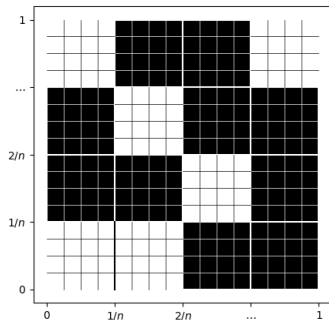
**$\alpha$ D:** Scale by  $\lambda \iff$   $n$ -Dim. Volumes multiplies by  $\lambda^\alpha \quad \forall \alpha \in \mathbb{R}^+$



# Dimensions

## Percolation dimensions

For  $P \sim \text{Perc}(n, p)$ , scaling by  $n$  gives  $pn^2$  copies of  $P$ .

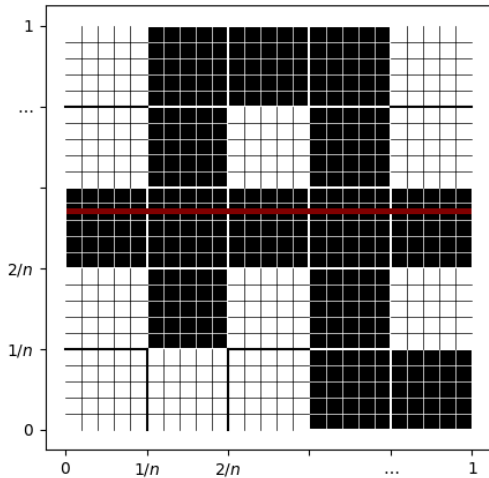


So  $\dim(P) = pn^2$ .



# Types of Crossings

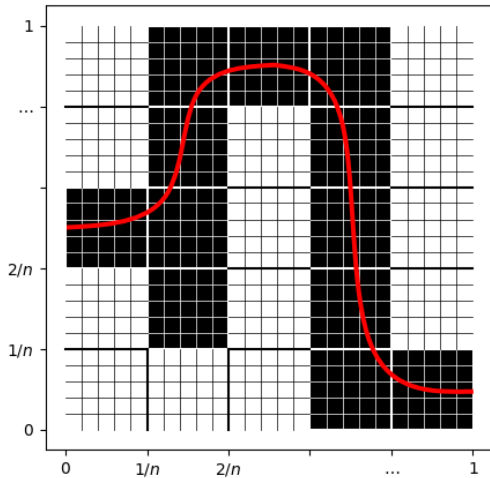
## Straight





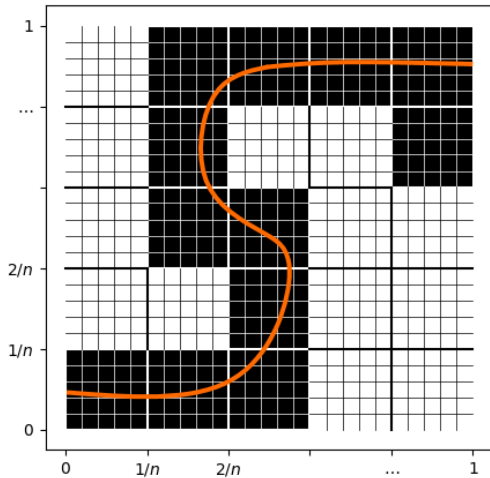
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## Semi-Straight



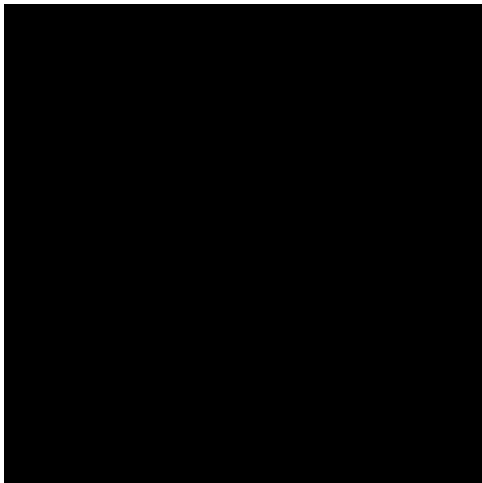
# Types of Crossings

## Non-Straight



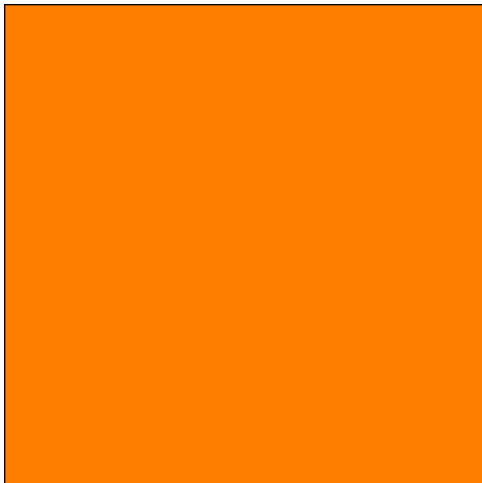
# Finding Crossings

Example for  $n = 2$



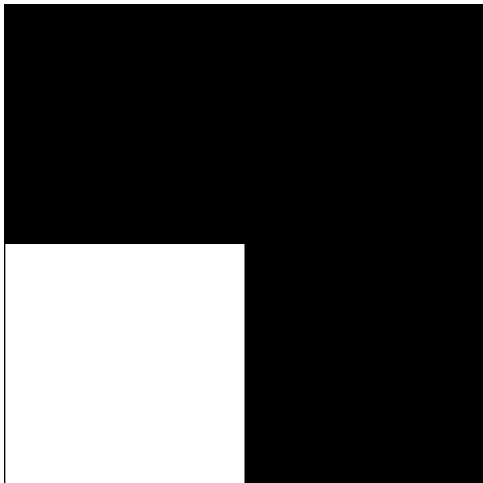
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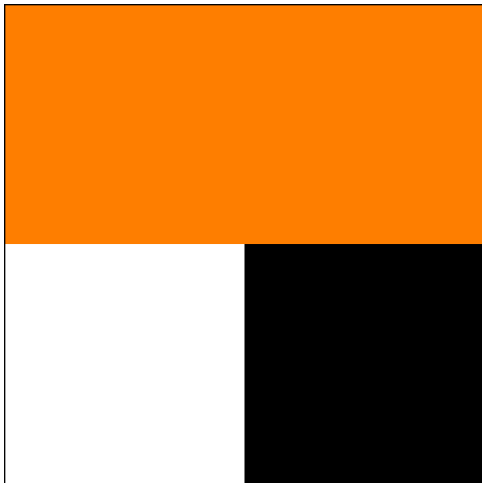
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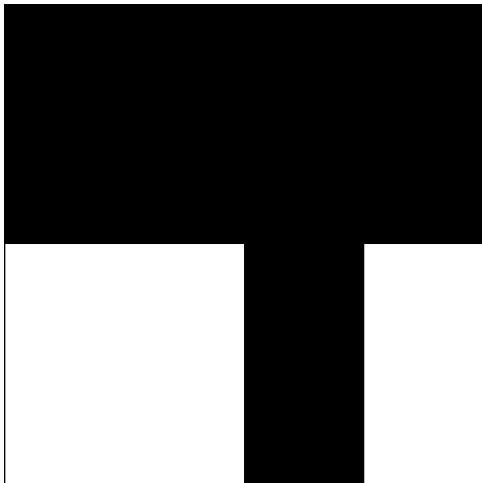
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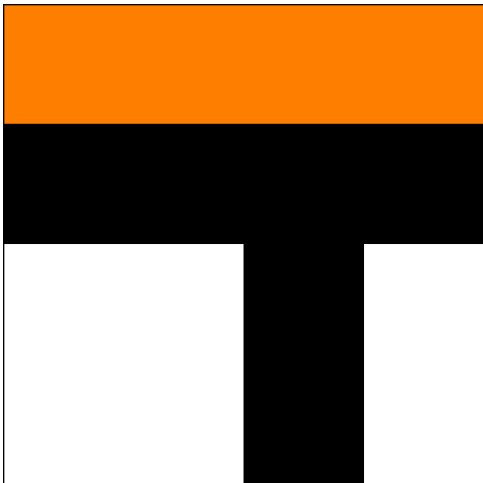
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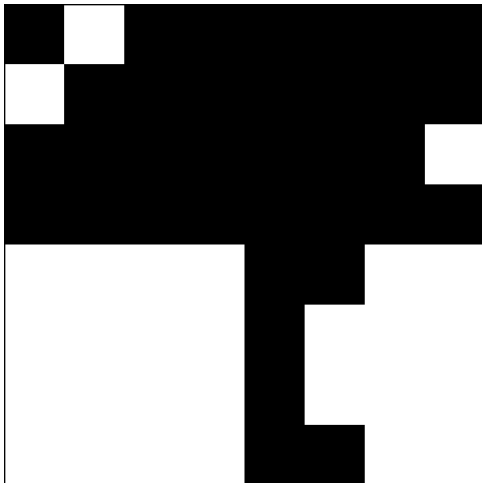
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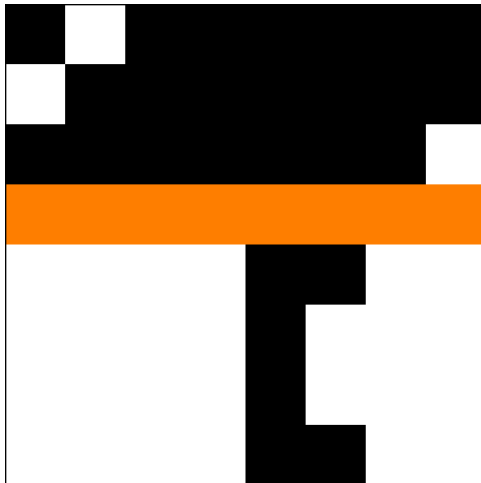
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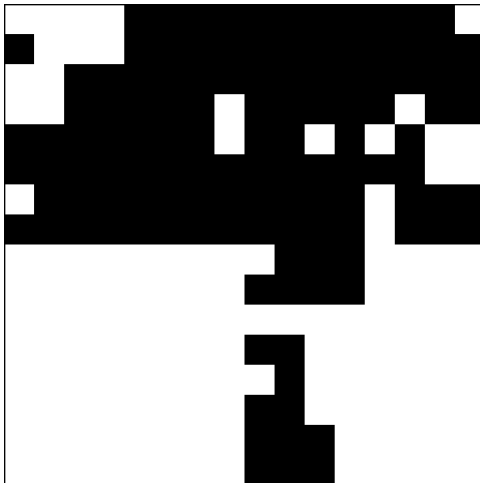
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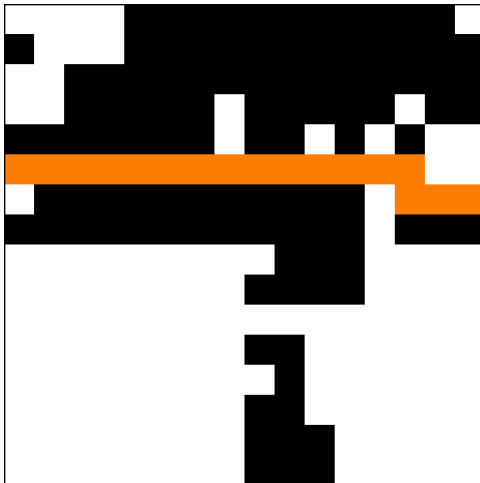
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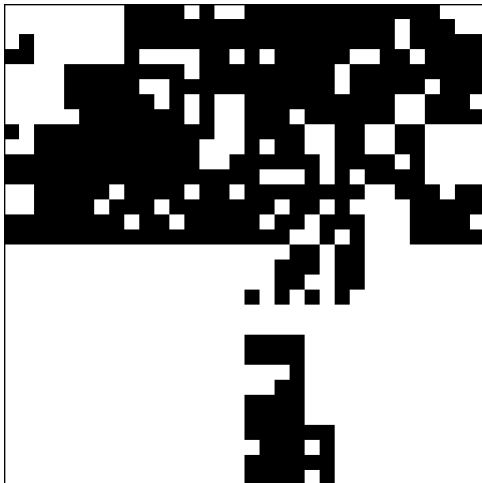
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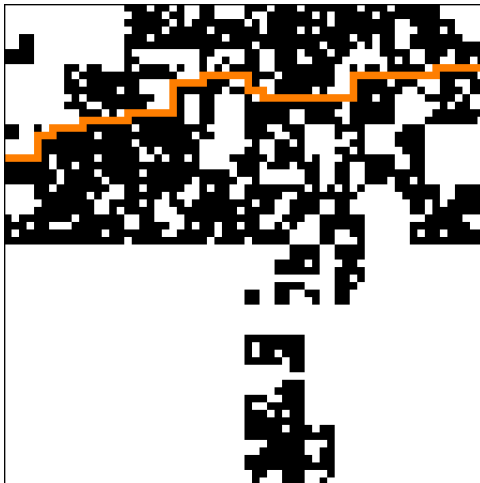
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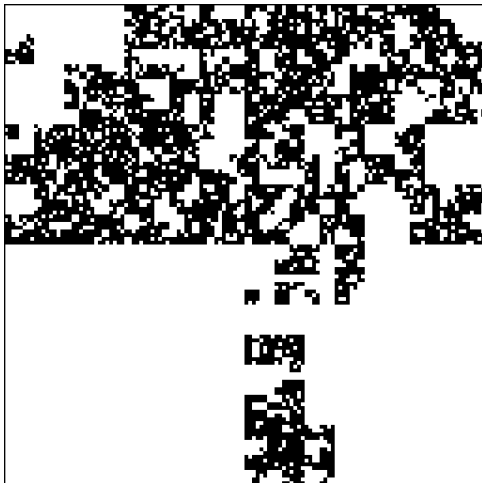
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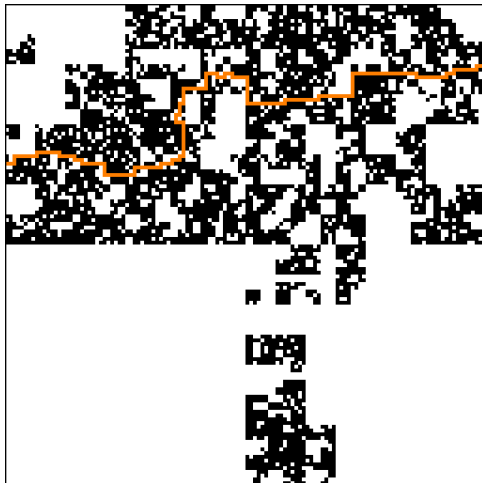
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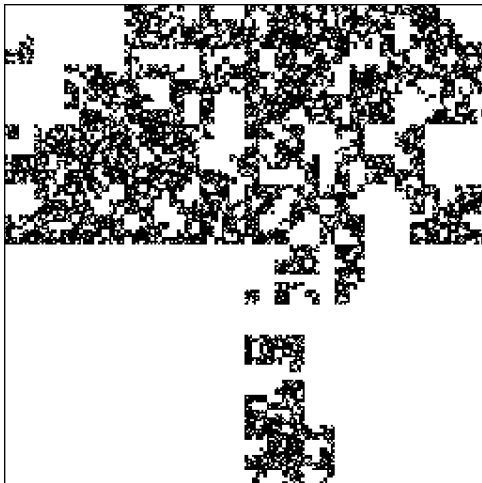
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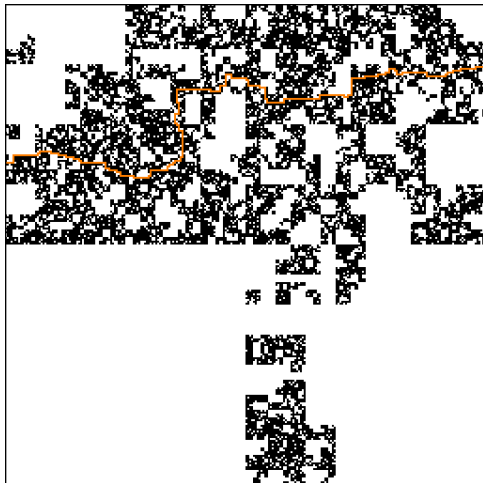
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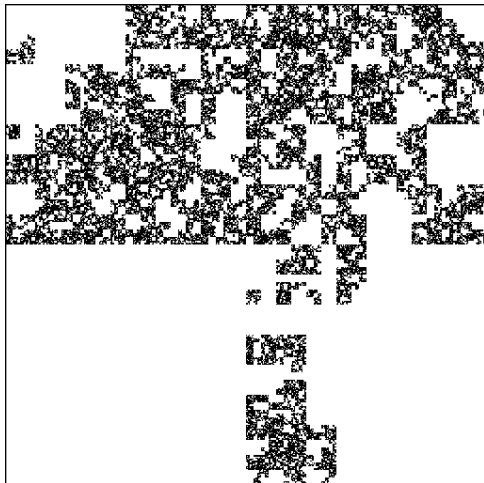
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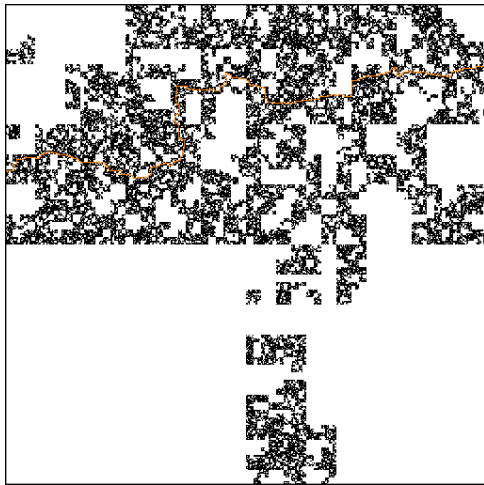
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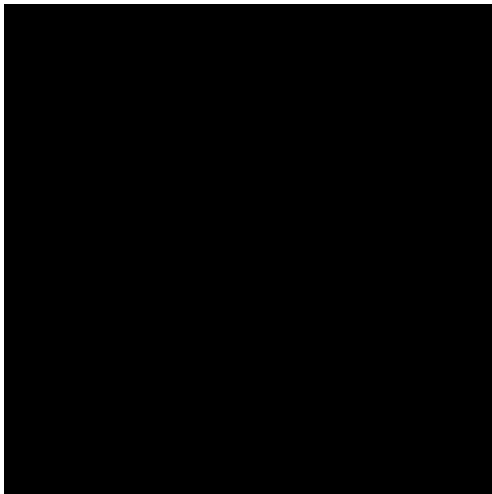
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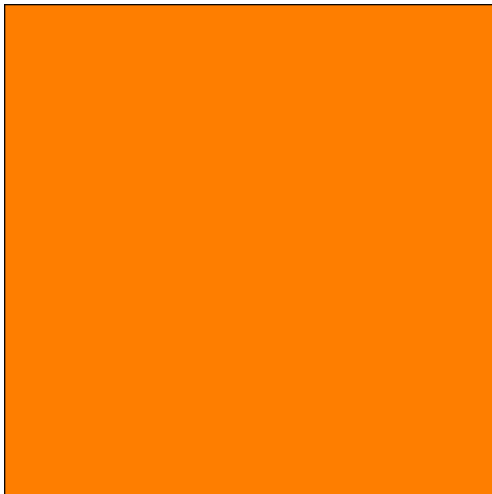
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Example for  $n = 3$



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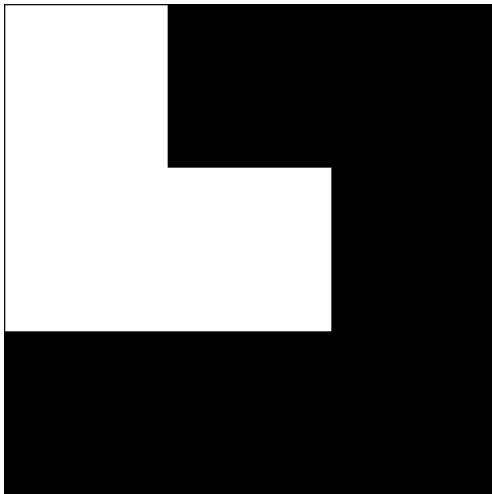
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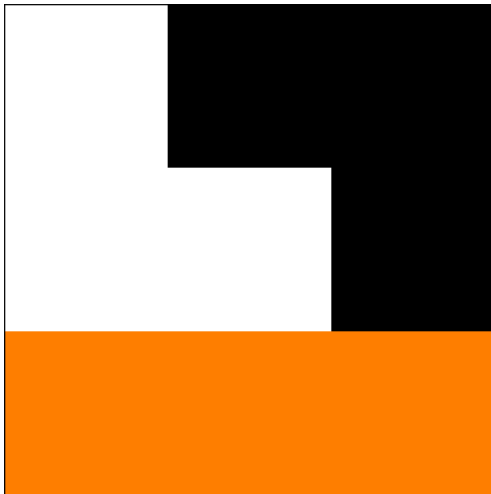
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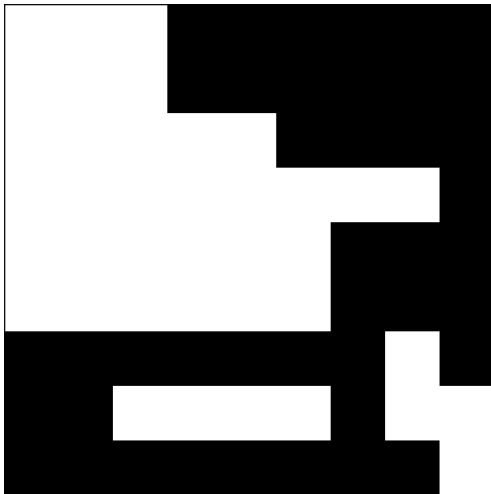
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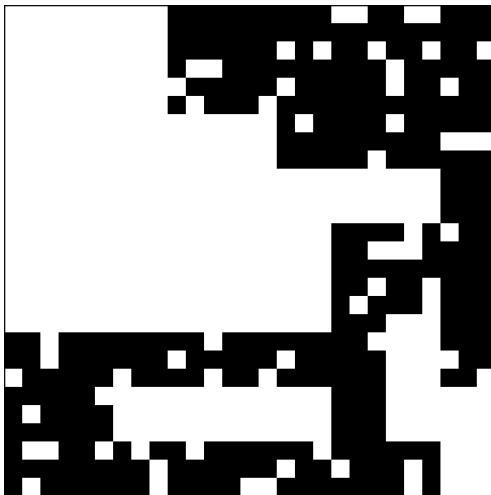
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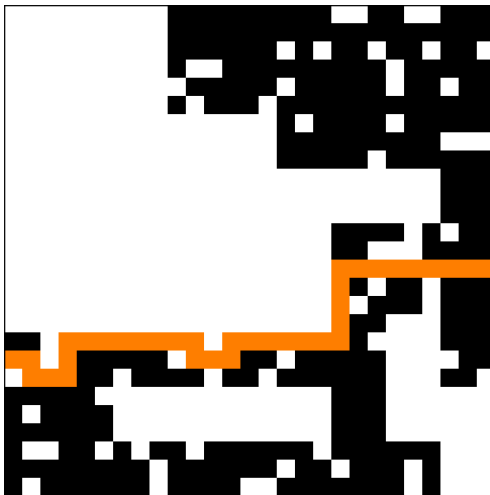
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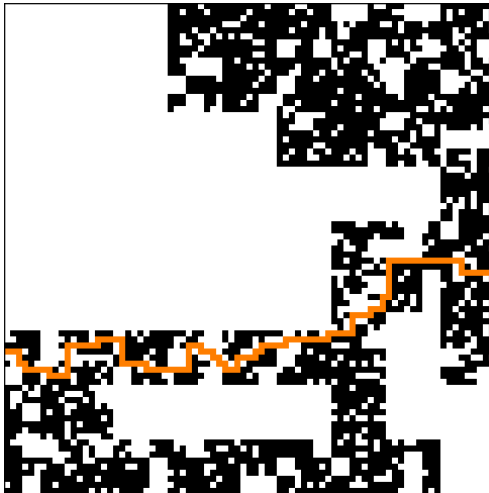
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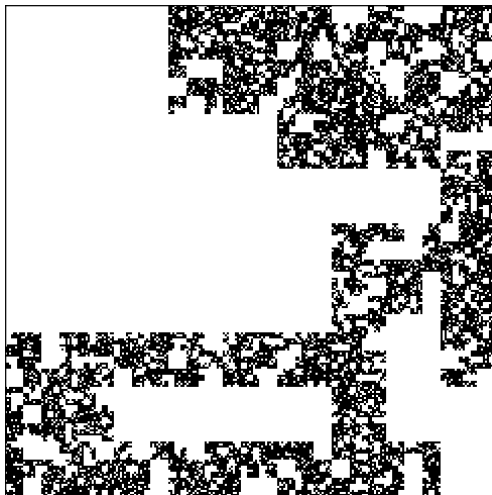
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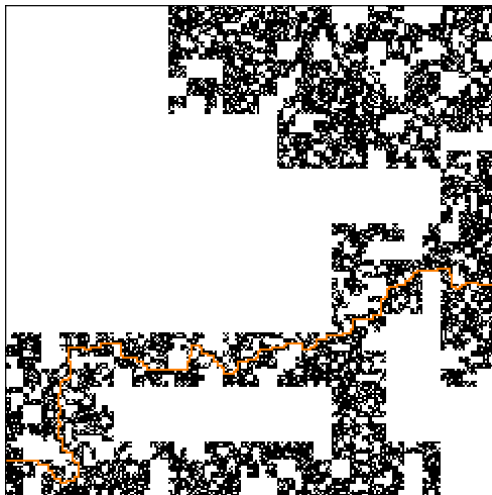
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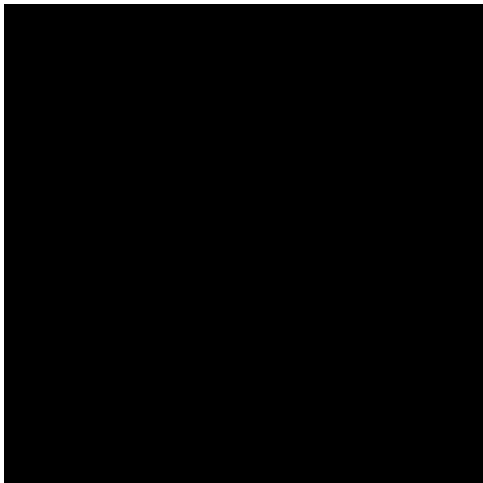
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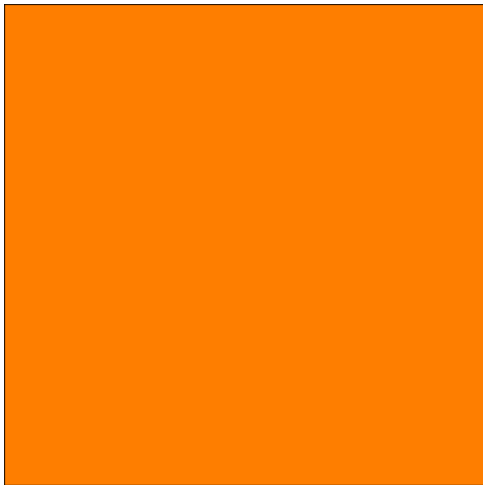
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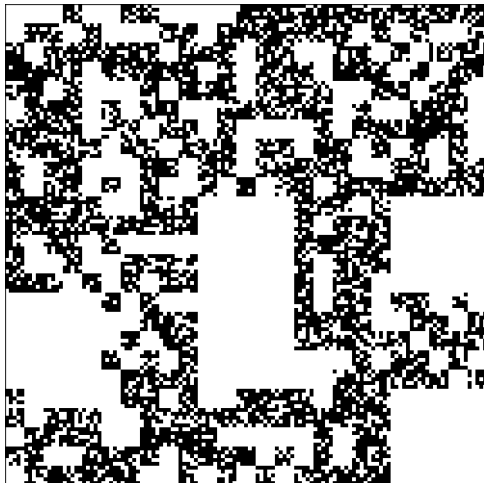
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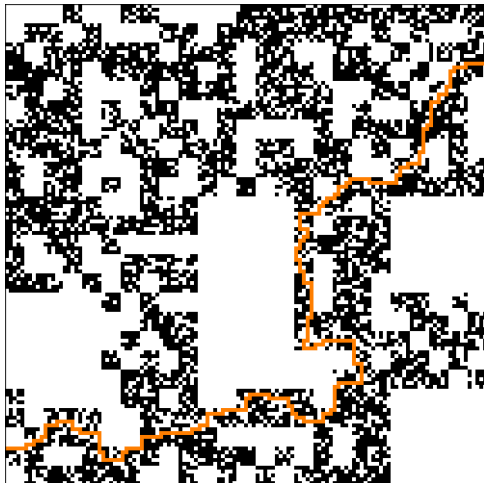
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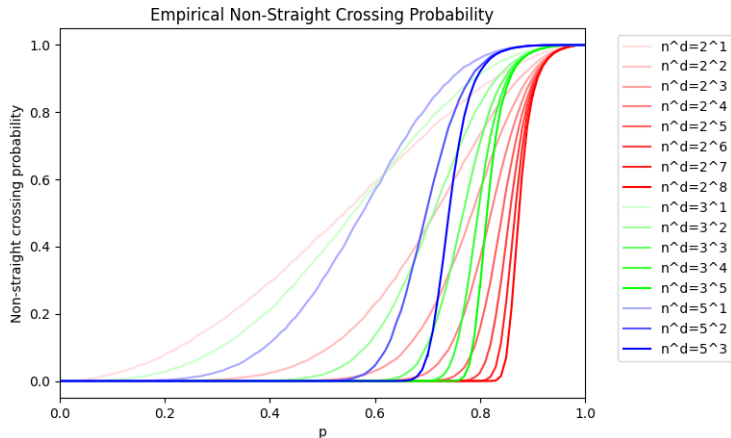
## Algorithm

- ▶ 2D Crossing Algorithm:  
<https://pauldubois98.github.io/RandomFractalAlgorithmsDemo/2Dcrossing/index.html>
- ▶ 3D Crossing Algorithm:  
<https://pauldubois98.github.io/RandomFractalAlgorithmsDemo/3Dcrossing/index.html>



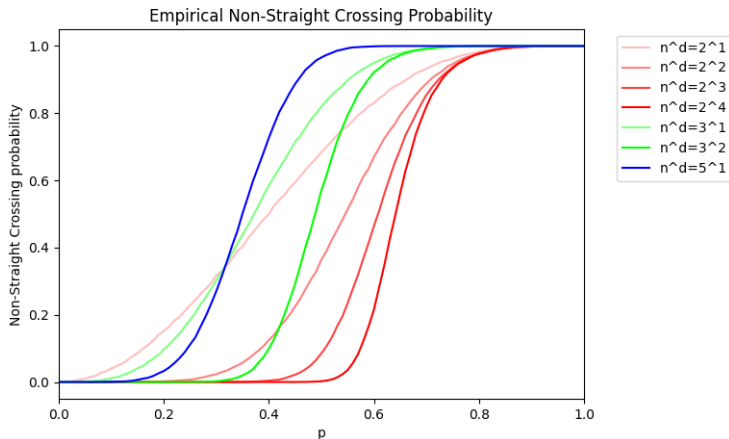
# Crossings Probability

## Non-Straight Crossing, Recursive Percolation, 2D



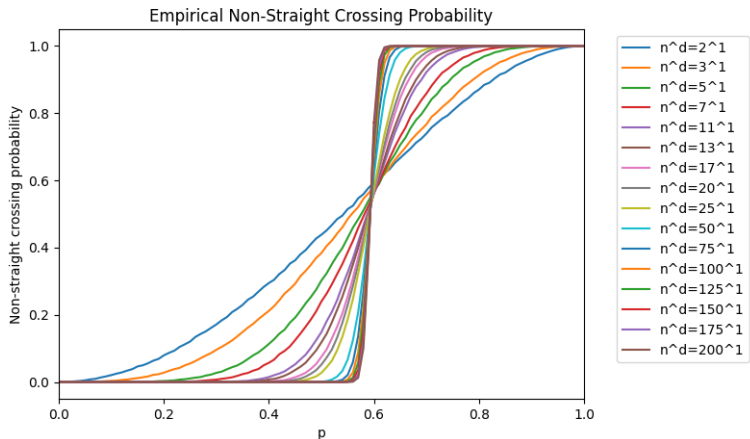
# Crossings Probability

## Non-Straight Crossing, Recursive Percolation, 3D



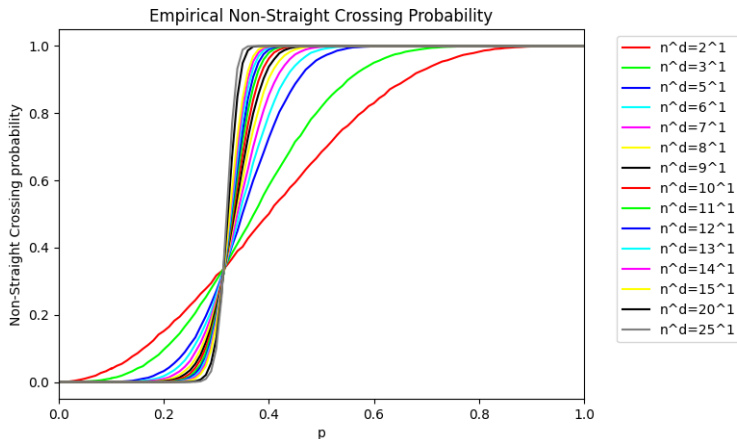
# Crossings Probability

Non-Straight Crossing, Uniform Percolation, 2D



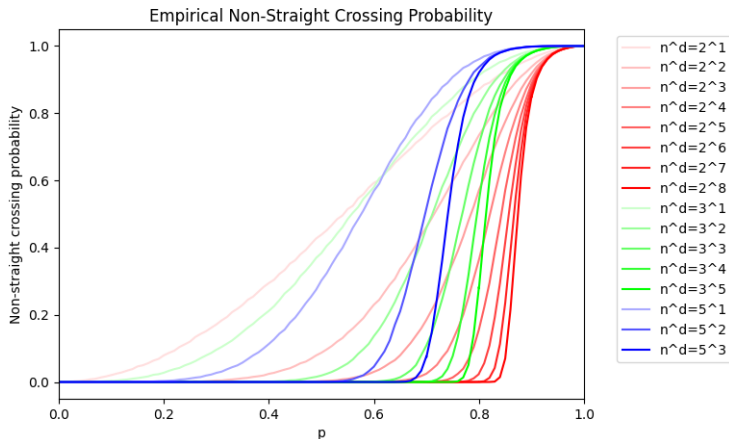
# Crossings Probability

Non-Straight Crossing, Uniform Percolation, 3D



# Crossings Probability

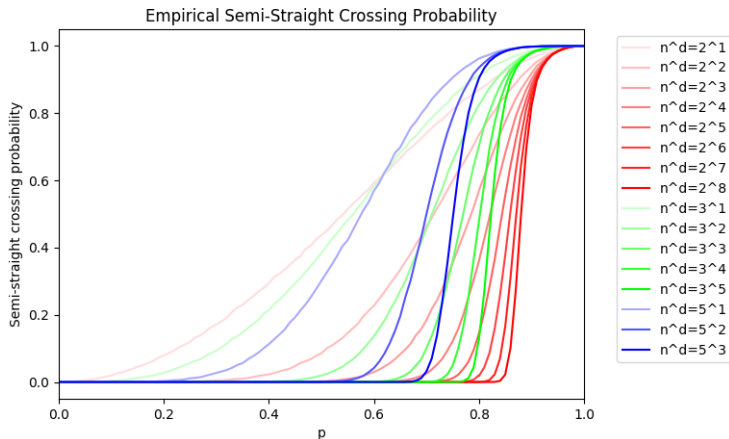
Semi-Straight Crossing, Recursive Percolation, 2D





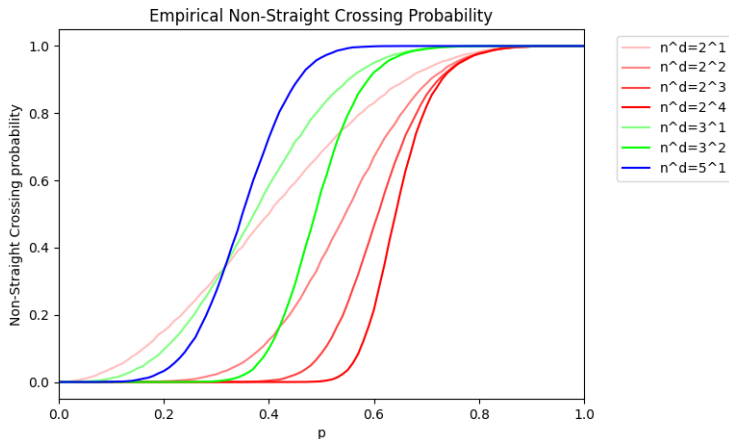
# Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 2D



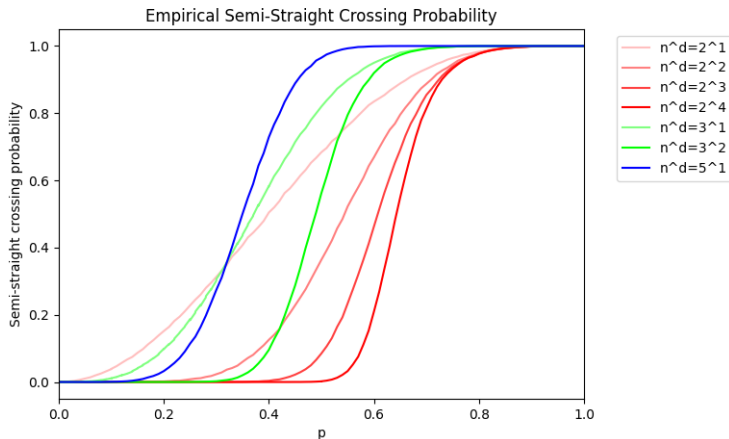
# Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 3D



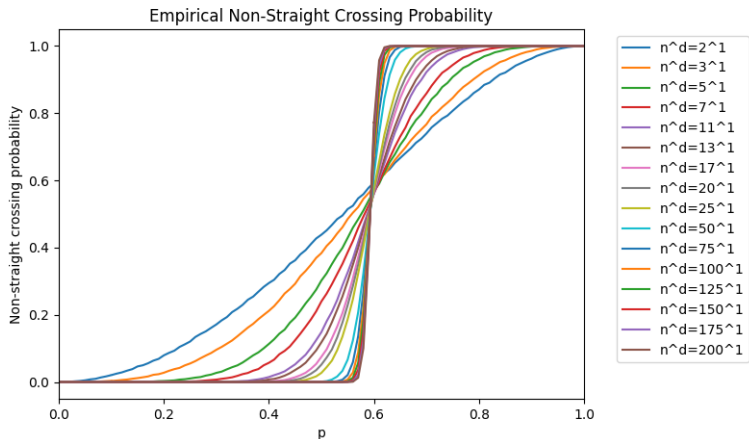
# Crossings Probability

Semi-Straight Crossing, Recursive Percolation, 3D



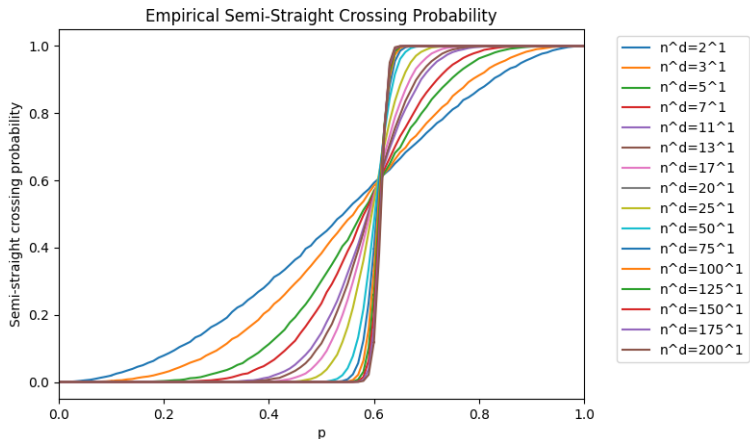
# Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 2D



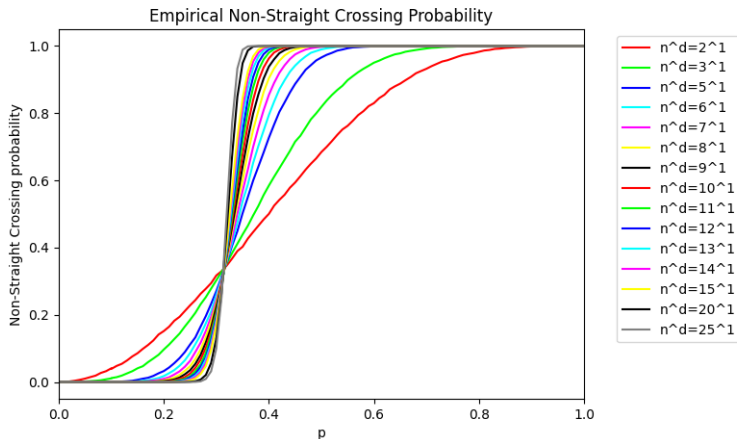
# Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 2D



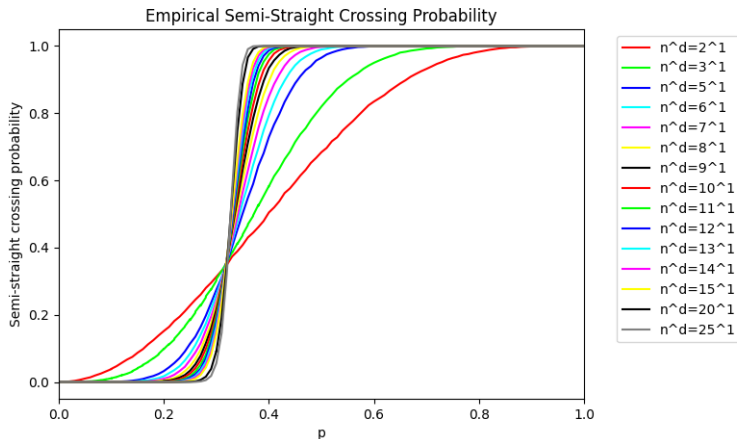
# Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 3D



# Crossings Probability

Semi-Straight Crossing, Uniform Percolation, 3D



# Crossings Probability

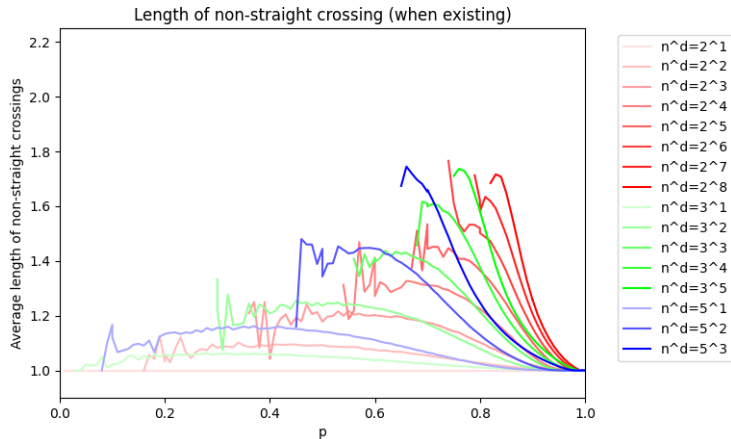
## Straight Crossing





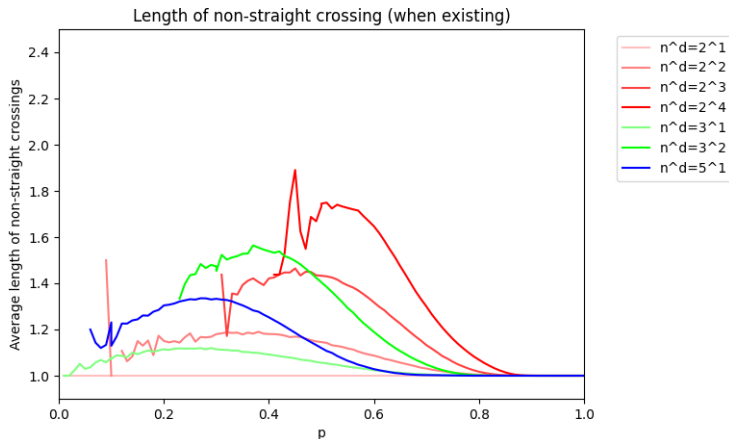
# Crossings Length

## Non-Straight Crossing, Recursive Percolation, 2D



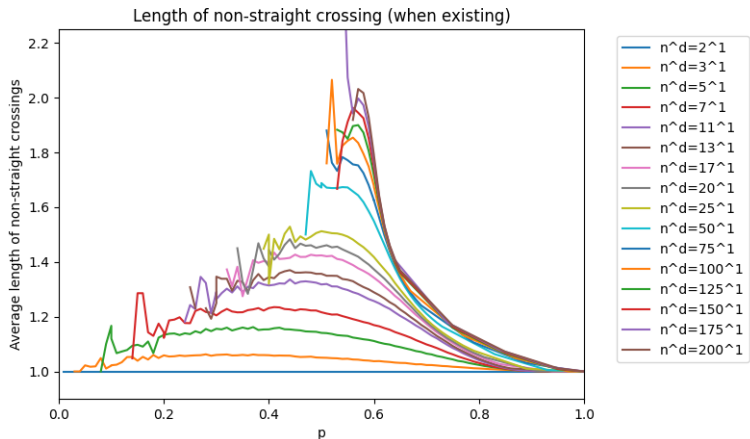
# Crossings Length

## Non-Straight Crossing, Recursive Percolation, 3D



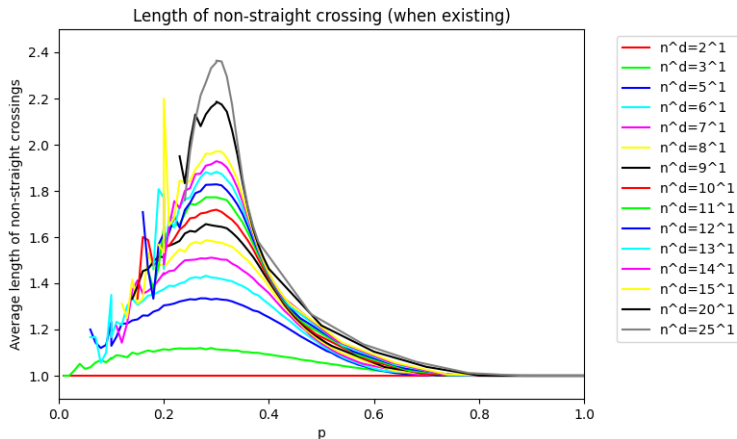
# Crossings Length

## Non-Straight Crossing, Uniform Percolation, 2D



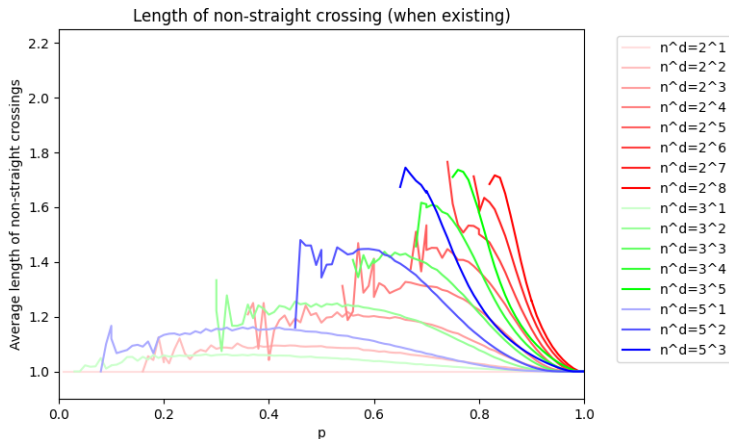
# Crossings Length

Non-Straight Crossing, Uniform Percolation, 3D



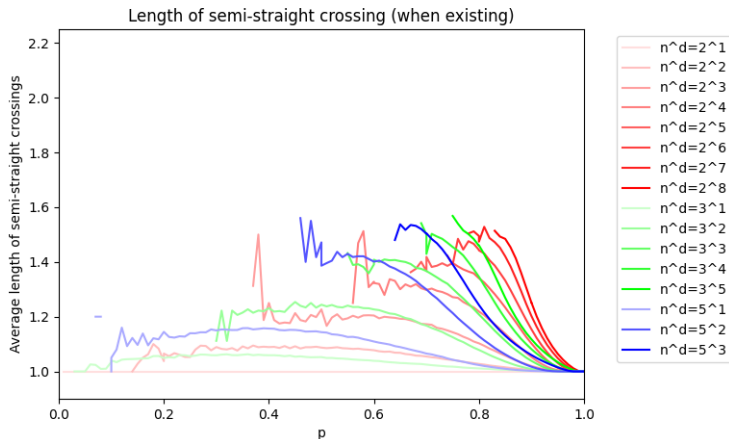
# Crossings Length

Semi-Straight Crossing, Recursive Percolation, 2D



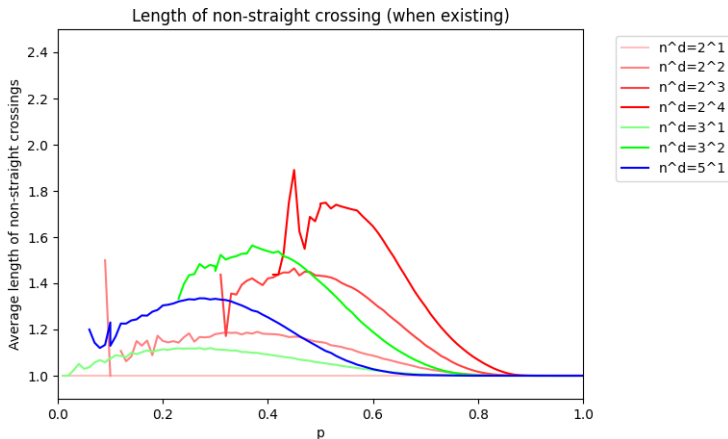
# Crossings Length

## Semi-Straight Crossing, Recursive Percolation, 2D



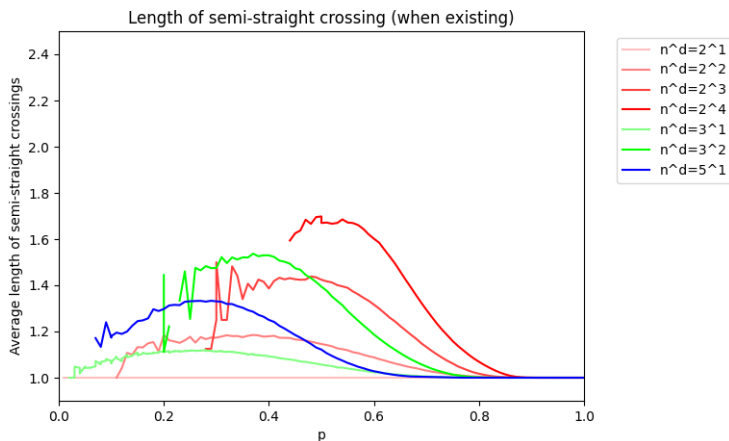
# Crossings Length

Semi-Straight Crossing, Recursive Percolation, 3D



# Crossings Length

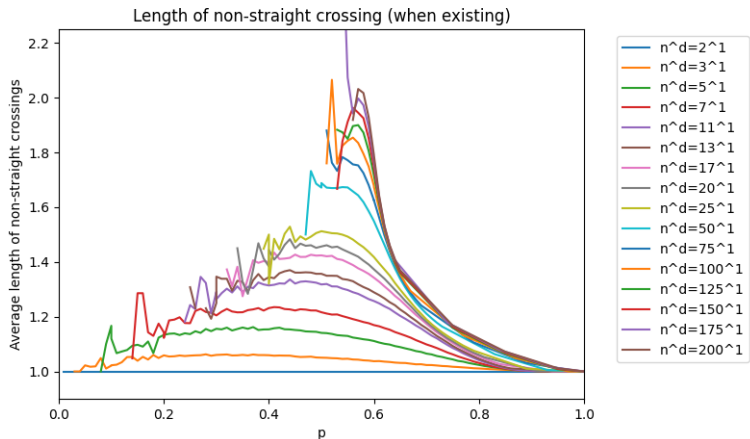
Semi-Straight Crossing, Recursive Percolation, 3D





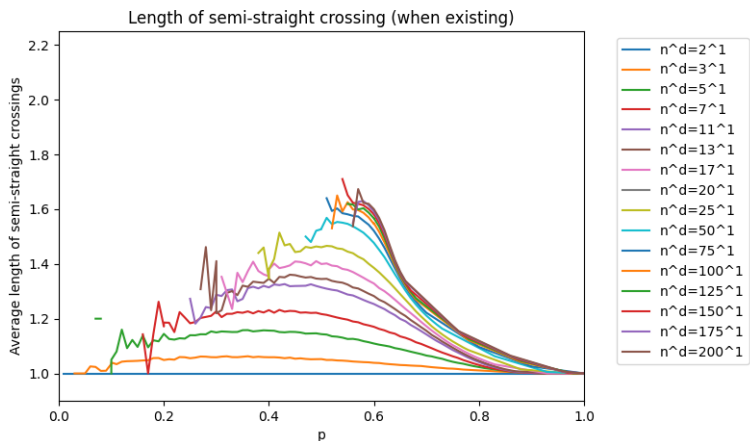
# Crossings Length

Semi-Straight Crossing, Uniform Percolation, 2D



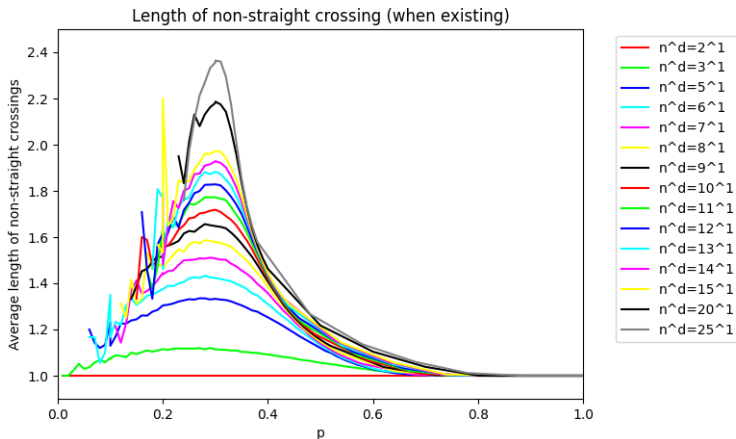
# Crossings Length

Semi-Straight Crossing, Uniform Percolation, 2D



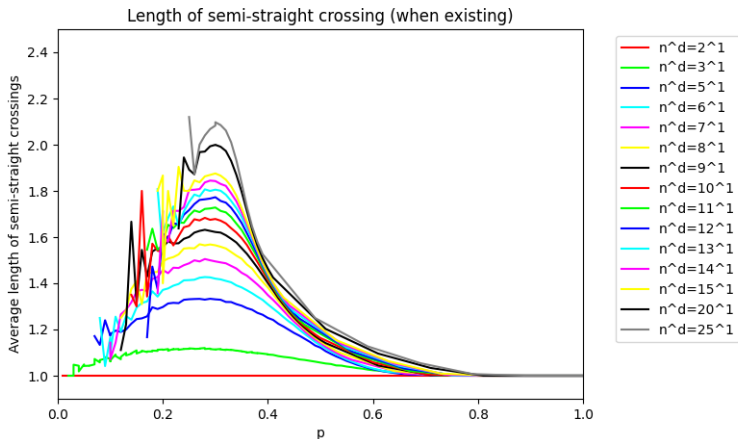
# Crossings Length

Semi-Straight Crossing, Uniform Percolation, 3D



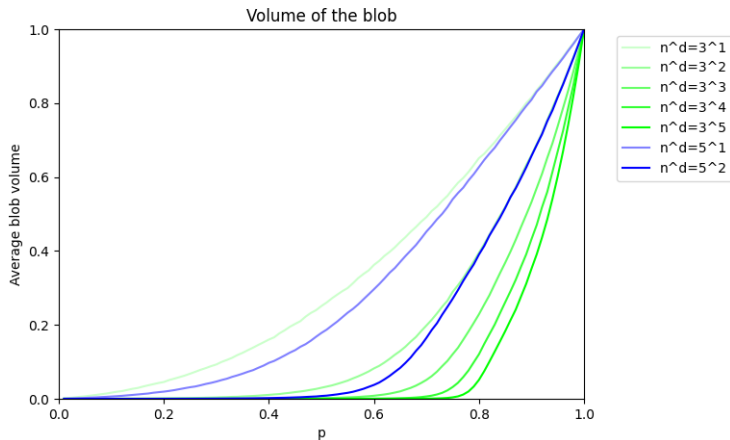
# Crossings Length

Semi-Straight Crossing, Uniform Percolation, 3D



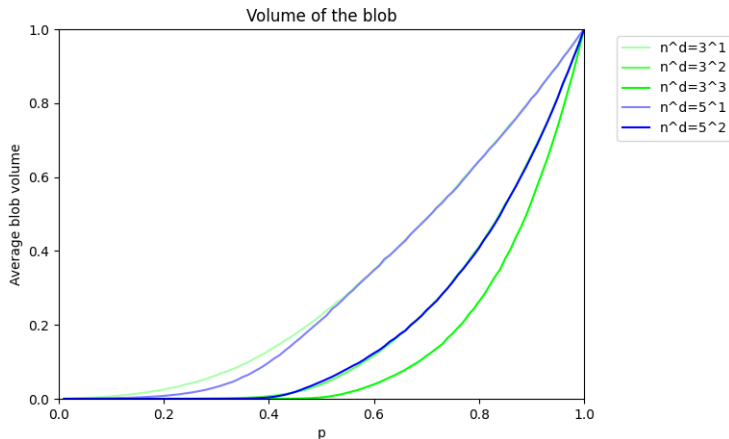
# Blob

## Volume, Recursive Percolation, 2D



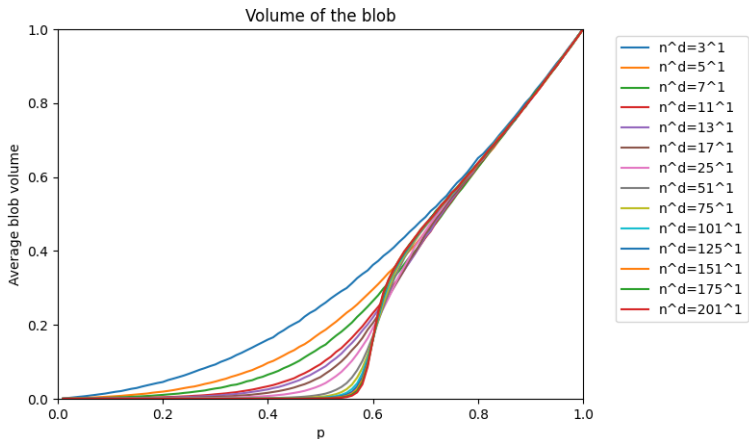
# Blob

## Volume, Recursive Percolation, 3D



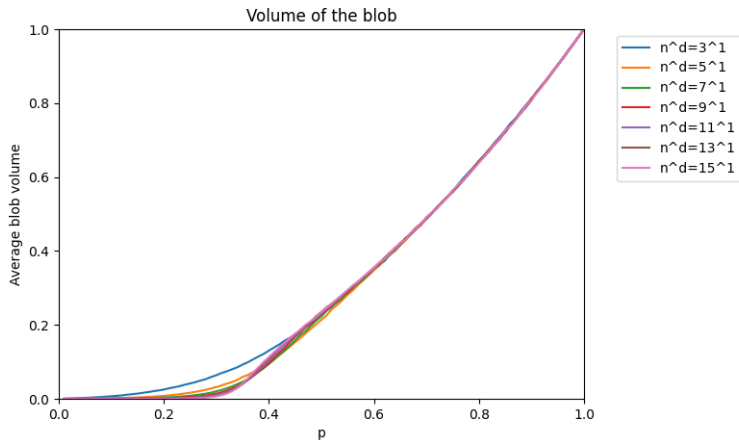
# Blob

## Volume, Uniform Percolation, 2D



# Blob

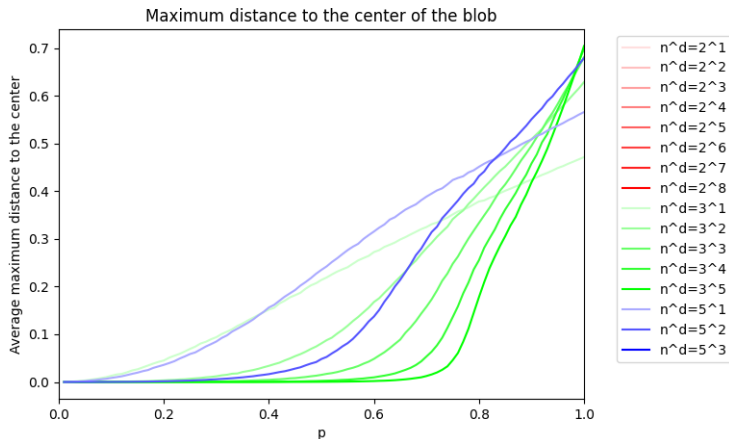
## Volume, Uniform Percolation, 3D





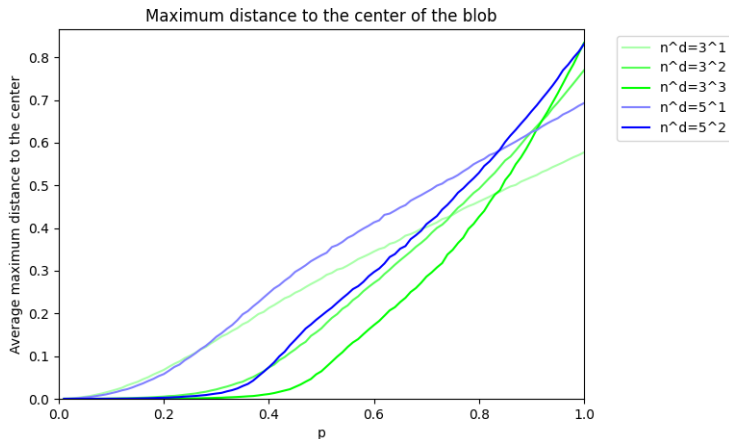
# Blob

## Distance to Center, Recursive Percolation, 2D



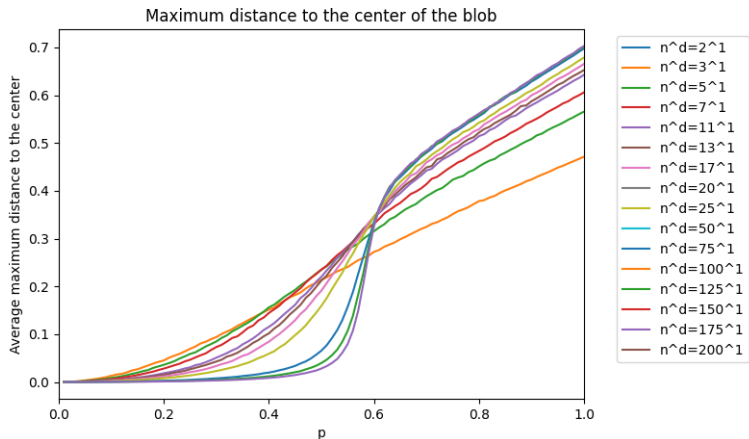
# Blob

## Distance to Center, Recursive Percolation, 3D



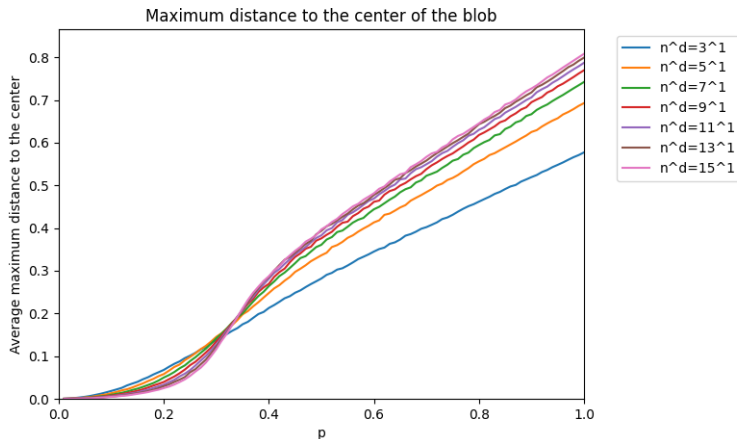
# Blob

## Distance to Center, Uniform Percolation, 2D



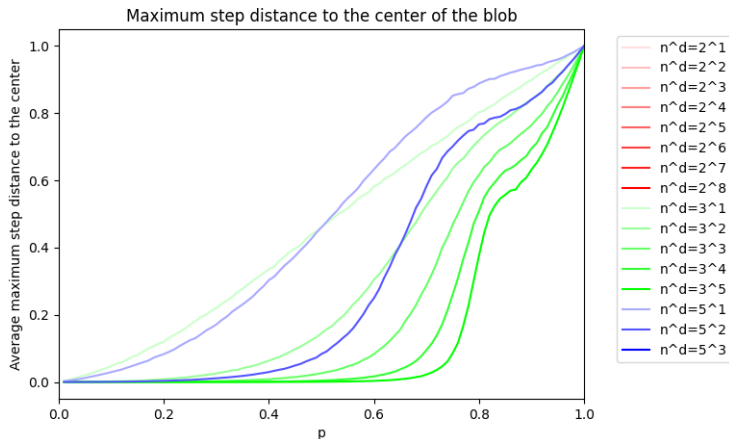
# Blob

## Distance to Center, Uniform Percolation, 3D



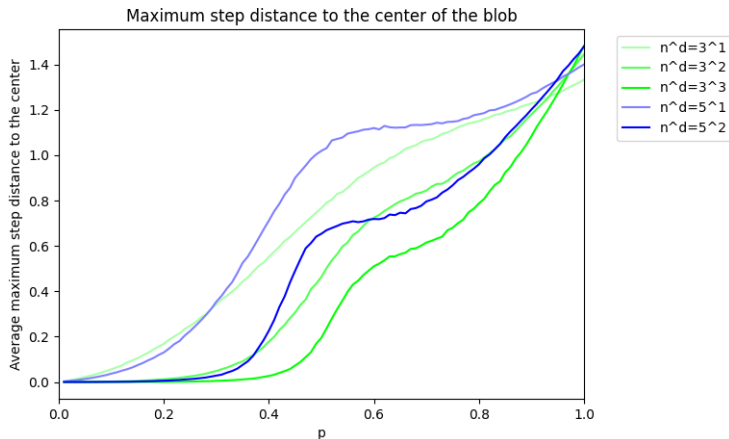
# Blob

## Steps to Center, Recursive Percolation, 2D



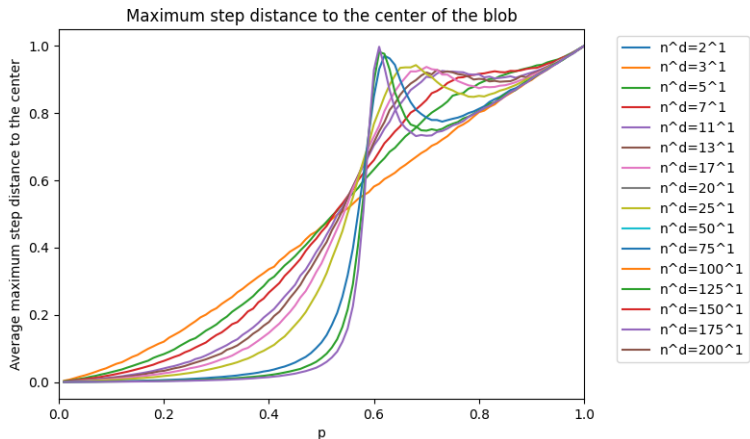
# Blob

## Steps to Center, Recursive Percolation, 3D



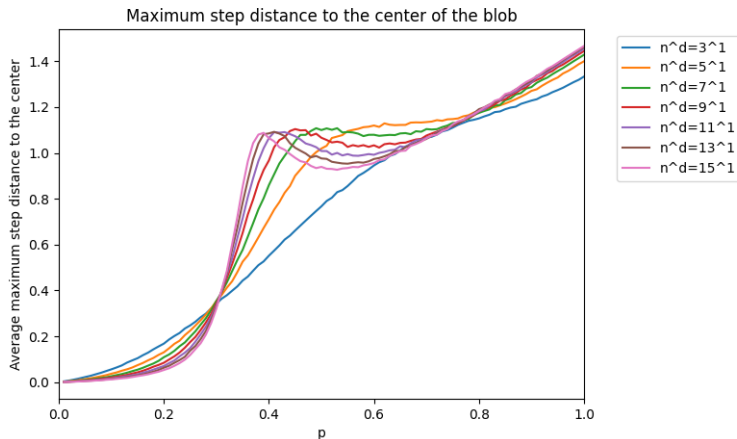
# Blob

## Steps to Center, Uniform Percolation, 2D



# Blob

## Steps to Center, Uniform Percolation, 3D





# Intersection



# Projection