Random Fractals

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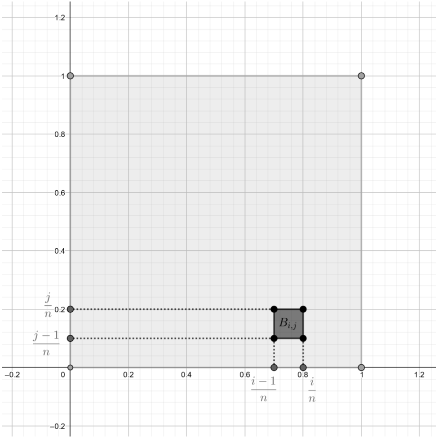
The Percolation Process

Plain: $P \sim \text{Perc}(n, p, 1)$

$$B_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$



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$$\varepsilon_{i,j} \in \{0,1\} \;\text{ with } \mathbb{P}\left(\varepsilon_{i,j} = 1\right) = p \quad (\text{ i.e. } \; \varepsilon_{i,j} \sim \mathcal{B}(p))$$

$$P = \bigcup_{\substack{i,j \\ \varepsilon_{i,j}=1}} B_{i,j}$$

$$Z = \left|\{(i,j) \mid \epsilon_{i,j} = 1\}\right|$$

$$D = \frac{Z}{pn^2}$$

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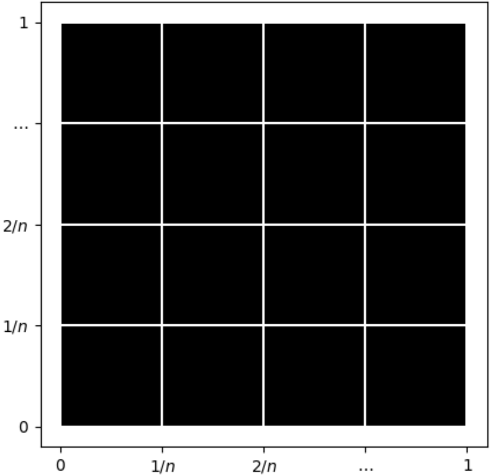

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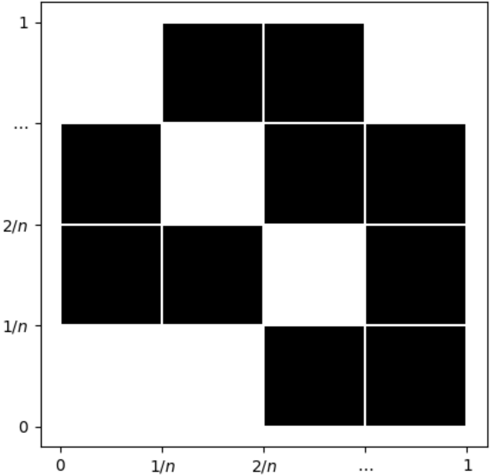
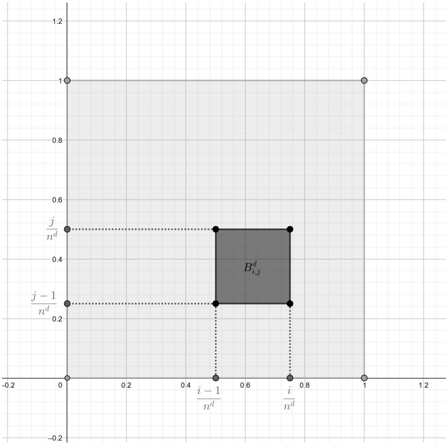


The Percolation Process

Recursive: $P_d \sim \mathrm{Perc}(n, p, d)$

$$B^d_{i,j} = \left[\frac{i-1}{n^d}, \frac{i}{n^d}\right] \times \left[\frac{j-1}{n^d}, \frac{j}{n^d}\right]$$



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$$P_0 = [0,1]^2 \quad ; \quad P_d = P_{d-1} \bigcap \left(\bigcup_{\substack{i,j \\ \varepsilon^d_{i,j}=1}} B^d_{i,j} \right)$$

$$Z_d = \left| \left\{ (i,j) \mid \epsilon^d_{i,j} = 1 \right\} \right|$$

$$D_d = \frac{Z_d}{(pn^2)^d}$$

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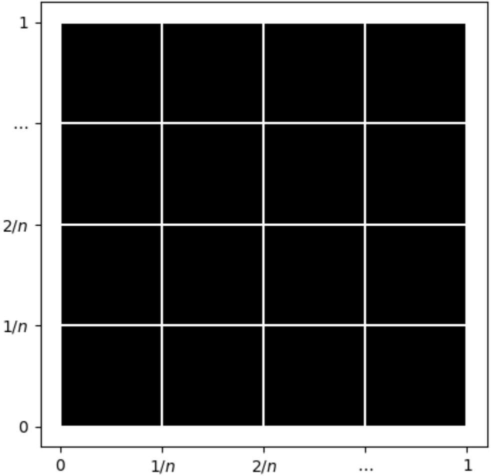

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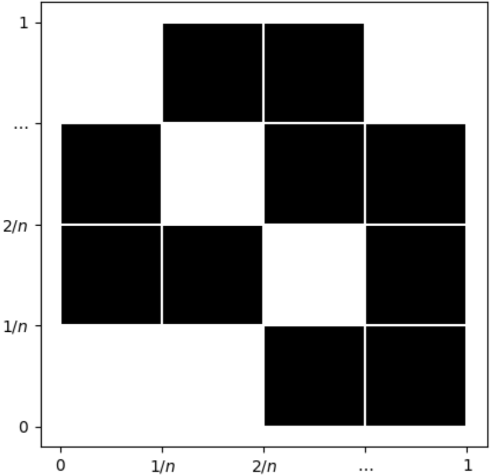

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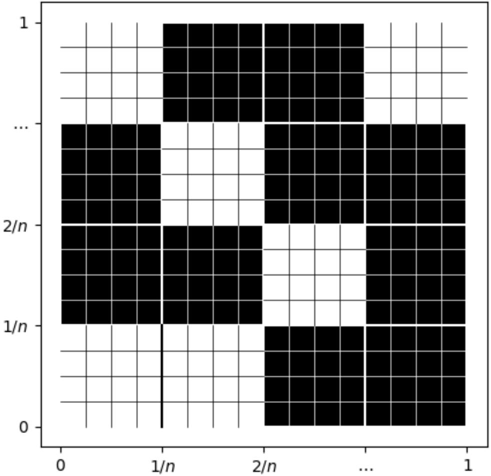

The Percolation Process

Limit: $P_\infty \sim \mathrm{Perc}(n, p)$

$$P_\infty = \bigcap_{d \in \mathbb{N}} P_d$$

$$D_\infty = \lim_{d \to \infty} D_d$$

The Percolation Process

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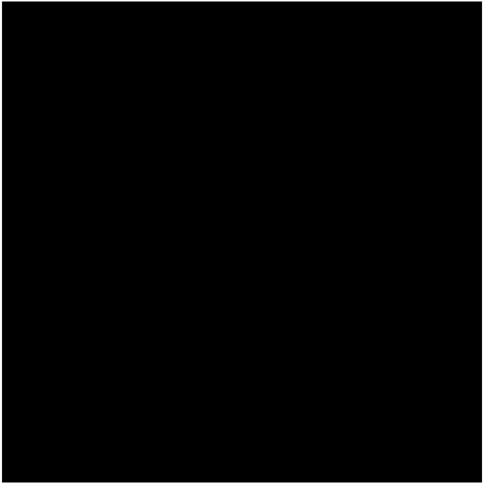
$$D_\infty > 0 \iff P_\infty \neq \emptyset$$

$$\mathbb{E}(D_\infty) = 1$$

The Percolation Process

Example for $n = 2$



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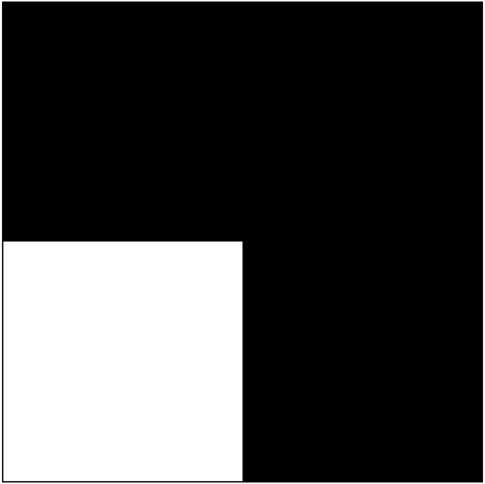

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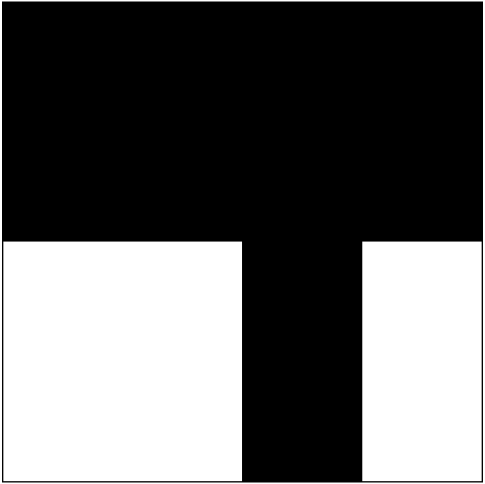

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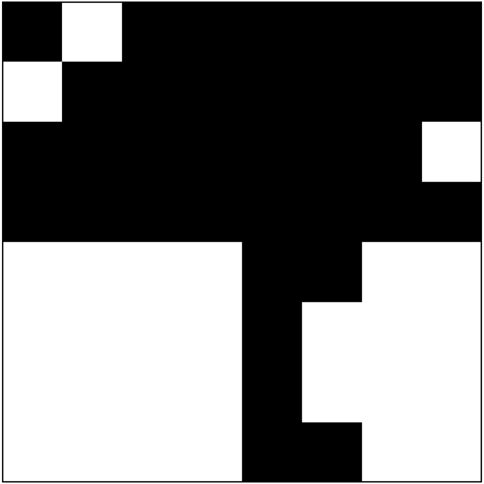

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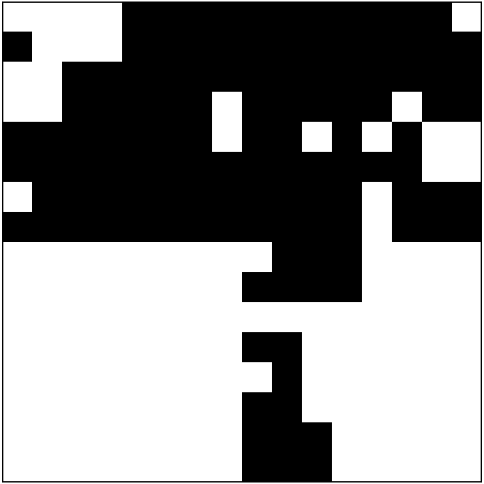

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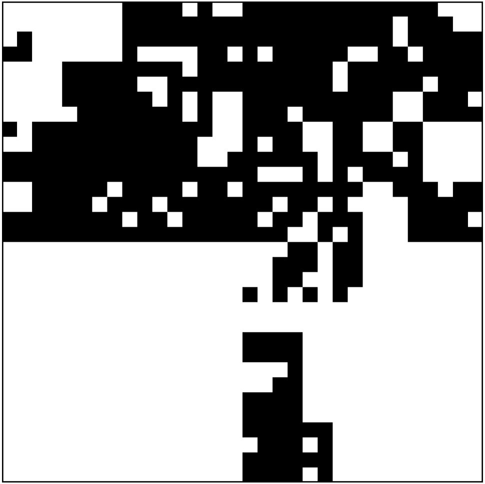

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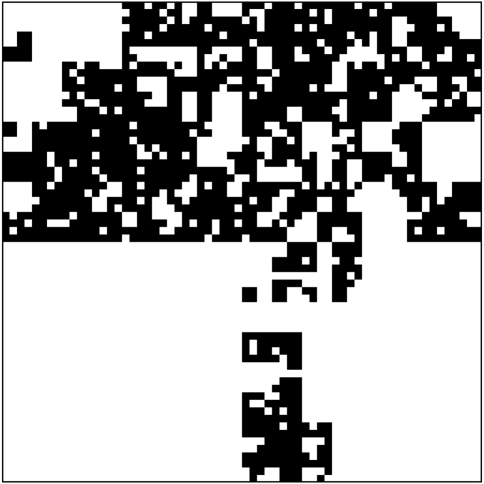

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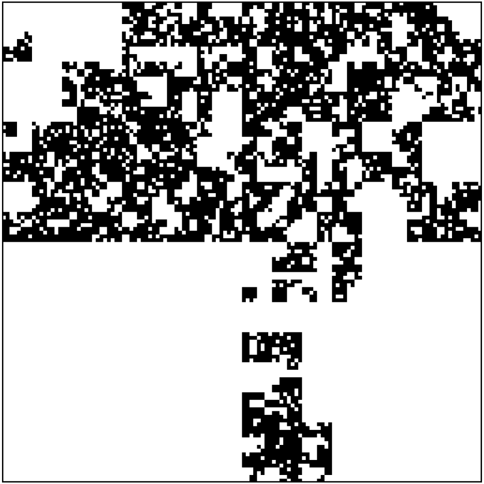

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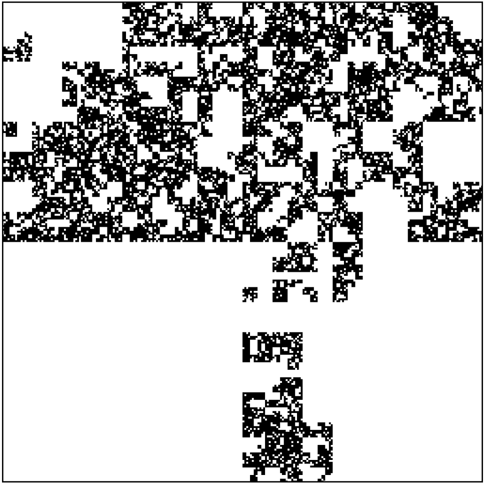

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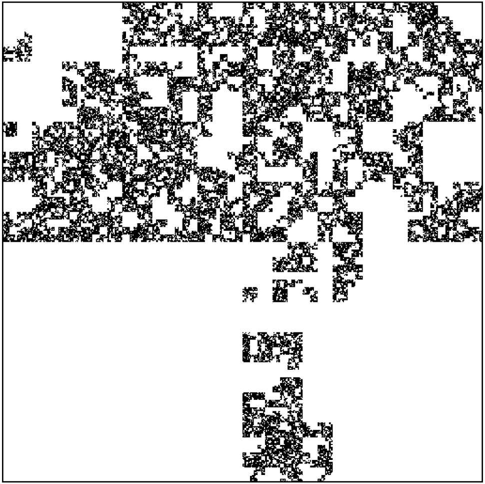

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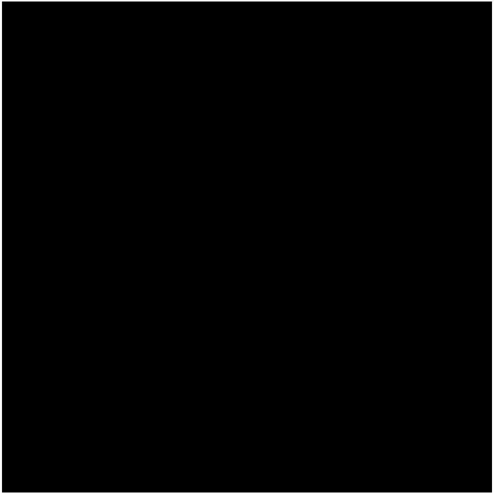
Example for $n = 3$



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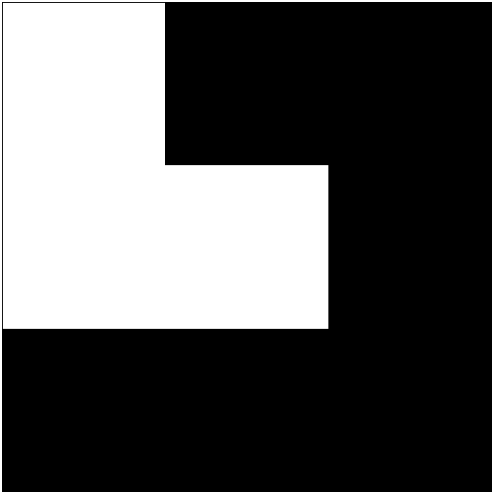

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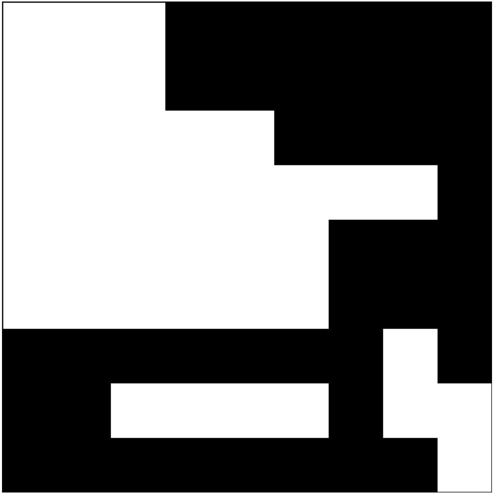

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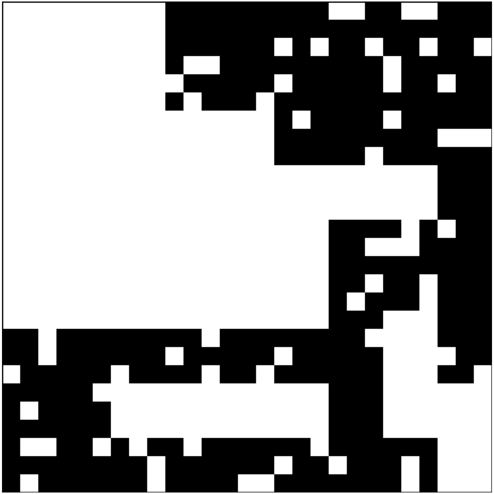

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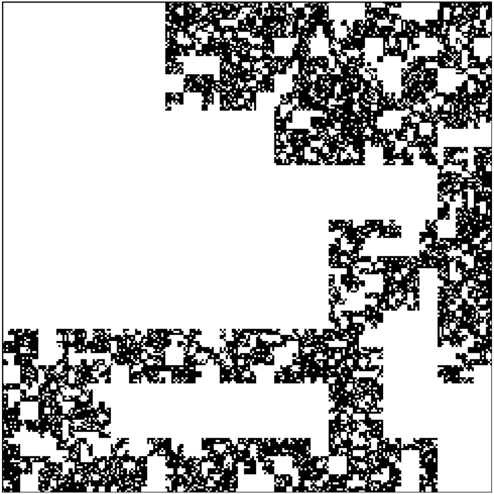

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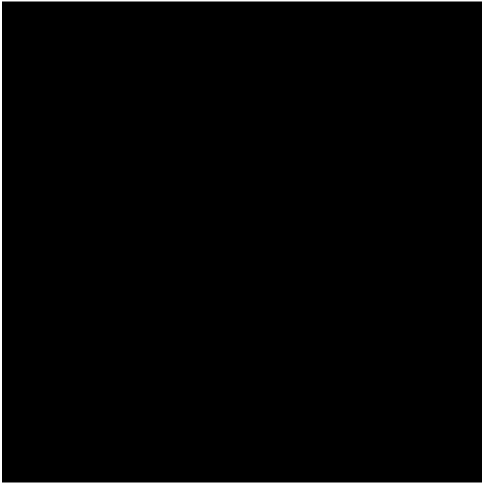
Example for $n = 5$



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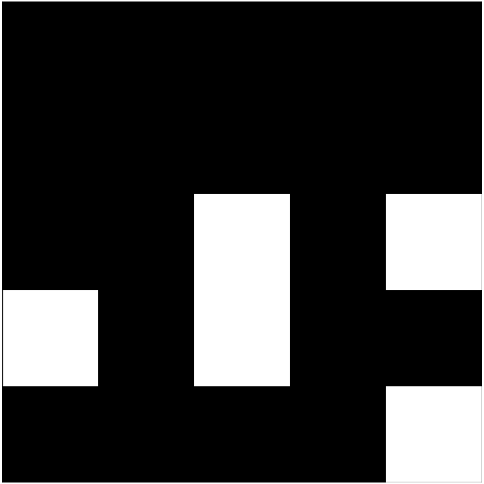

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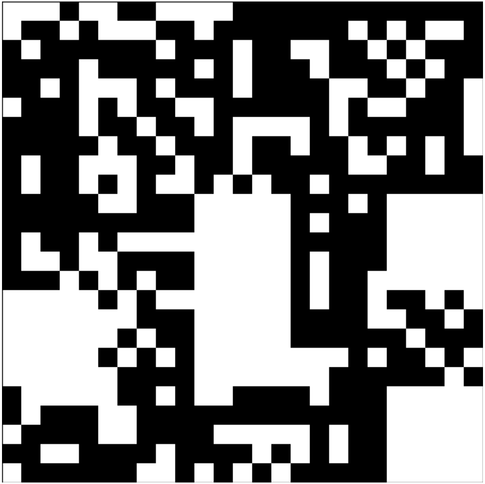

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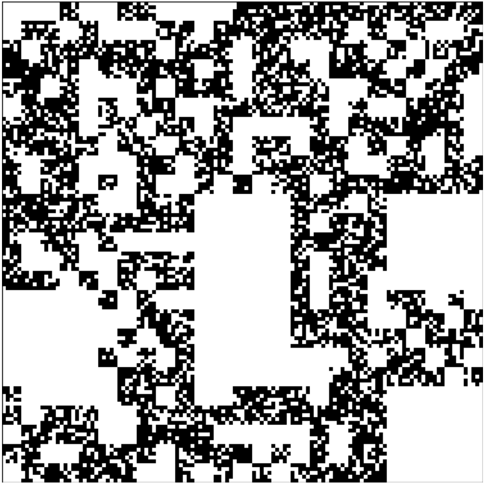

Dimensions

Intuition

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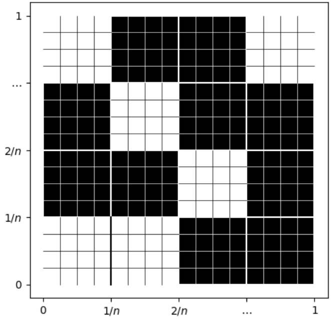
αD: Scale by $\lambda \iff$ n-Dim. Volumes multiplies by $\lambda^\alpha \quad \forall \alpha \in \mathbb{R}^+$

Dimensions

Percolation dimensions

For $P \sim \text{Perc}\,(n, p)$, scaling by n gives pn^2 copies of P.



So $\dim(P) = pn^2$.

Crossings

straight

Blob