

# Modular forms modulo 2

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## Abstract

We are interested in Modular forms modulo 2, and computing thing about it. [temporary abstract]

Key words that should appear: Modular forms; Mod 2; Duality of definitions; Governing fields; Frobenian map?; Exact computations;

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# 1 Modular forms

## 1.1 Modular forms of level 1

We will denote by  $\mathbb{H}$  the upper half plane.

We say that a complex function  $f$  on the upper half plane is weakly modular of weight  $2k$  if  $f$  is meromorphic on  $\mathbb{H}$  and

$$f(z) = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

A property of  $SL_2(\mathbb{Z})$  is that when we define

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then  $SL_2(\mathbb{Z})$  is generated by  $S$  and  $T$  (? , p.1-2).

From that property, we can derive an alternative definition of weakly modular functions:  $f$  is weakly modular of weight  $2k$  if  $f$  is meromorphic on  $\mathbb{H}$  and  $f(z+1) = f(z)$  and  $f(-1/z) = z^k f(z)$  for all  $z \in \mathbb{H}$ .

Moreover, we also define a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  to be modular of weight  $2k$  if  $f$  is holomorphic on  $\mathbb{H}$  and  $f$  is weakly modular.

Lastly, we say that a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a modular form of weight  $2k$  if  $f$  is holomorphic at  $\infty$  and  $f$  is modular.

It is easy to check, from definition, that we can add modular forms together, as well as multiply them by a complex:

- $f_1(z) + f_2(z)$  is modular of weight  $2k$  if  $f_1(z)$  and  $f_2(z)$  are modular of weight  $2k$ .
- $\lambda f(z)$  is modular of weight  $2k$  if  $f(z)$  is.

Therefore, modular forms of weight  $2k$  over  $\mathbb{C}$  form a space. We denote it  $M_k$ .

It is also possible to multiply modular forms, in which case the weights adds on: If  $f_1(z)$  &  $f_2(z)$  are modular of respective weights  $2k_1$  &  $2k_2$ , then  $f_1(z)f_2(z)$  is modular of weight  $2k_1 + 2k_2$ .

We deduce that we can take powers of modular forms, and the weight is then multiplied by the power: If  $f(z)$  is modular of weight  $2k$ , then  $f^n(z)$  is modular of weight  $2k * n$  (with  $n \in \mathbb{N}$ ).

## 1.2 Typical Modular Forms

### 1.2.1 Eisenstein series $G_k$

The most famous class of modular forms is probably the Eisenstein series, usually denoted  $G_k$ . We define them as follows(? , Examples of Modular Forms of Level 1):

$$G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz + n)^{2k}}$$

for  $k \geq 2$ .

It is easy to check that  $G_k$  are modular of weight  $2k$ (? , Proposition 2.1), as:

$$G_k(z+1) = G_k(z)$$

(using  $(m, n+m) \rightarrow (m, n)$ , an invertible map)

$$G_k(-1/z) = z^k G_k(z)$$

(using  $(m, -n) \rightarrow (m, n)$ , an invertible map).

It is pleasant to remark that (? , Proposition 2.2)

$$G_k(\infty) = \sum_{n \in \mathbb{Z}^*} \frac{1}{n^{2k}} = 2\zeta(2k)$$

. Where  $\zeta(k)$  is Riemann Zeta function. The values of this function are well known on positive even numbers, and we deduce (? , p.194):

$$G_k(\infty) = 2\zeta(2k) = \frac{(2\pi)^{2k}}{(2k)!} B_k$$

with  $B_k = (-1)^{k+1} b_{2k}$  where  $b_k$  are Bernoulli's numbers.

### 1.2.2 $\Delta$

We will be interested in one main modular form in the rest of this article:  $\Delta$ . We define  $\Delta$  in terms of  $G_k$  as follows(? , p.84):

$$\Delta = g_2^3 - 27g_3^2 \in M_6^0 \quad \text{with } g_2 = 40G_2 \text{ and } g_3 = 140G_3$$

As  $g_2^3$  is modular of weight  $4 * 3 = 12$  and  $g_3^2$  of weight  $6 * 2 = 12$ ,  $\Delta$  is modular of weight 12.

Now, using  $G_2(\infty) = 2\zeta(4) = \frac{\pi^4}{45}$  and  $G_3(\infty) = 2\zeta(6) = \frac{2\pi^4}{945}$ , we get  $\Delta(\infty) = \left(\frac{4\pi^4}{3}\right)^3 - \left(\frac{8\pi^4}{27}\right)^2 = 0$ .

### 1.3 Cusp Forms

A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  that is a modular form may in addition be a cusp form, if  $f(\infty) = 0$ . We will denote the space of modular cusp forms of weight  $2k$  over  $\mathbb{C}$  by  $M_k^0$ .

It is useful to note  $G_k(\infty) = \sum_{n \in \mathbb{N}^*} \frac{2}{n^{2k}} > 2$  and in particular,  $G_k(\infty) \neq 0$ , so  $G_k$  are *not* cusp forms for any  $k$ . As we have shown it before,  $\Delta(\infty) = 0$ , so  $\Delta$  is a modular cusp form of weight 12, so  $\Delta \in M_6^0$ . Using tools from complex analysis, we can prove that  $\Delta$  has only one zero (at infinity), which has order one(? , p.88).

We have the following relation:

**Theorem 1.**  $M_k \cong M_k^0 \oplus \mathbb{C}.G_k \quad \forall k \geq 2. (? , p.88)$

*Proof.* We let  $\Phi : M_k \rightarrow \mathbb{C}$  such that if  $f \in M_k$ ,  $\Phi(f) = f(\infty)$ .

Now, we have  $\text{Ker}(\Phi) = M_k^0$ , therefore, by the 1<sup>st</sup> Isomorphism Theorem,  $M_k/M_k^0 \cong \text{Im}(\Phi) \subseteq \mathbb{C}$ .

Note that  $G_k \in M_k$ , and  $G_k(\infty) = \sum_{n \in \mathbb{Z}^*} \frac{1}{n^{2k}} \neq 0$ , so  $G_k \notin M_k^0$ . As  $G_k \neq 0$ ,  $\dim(M_k/M_k^0) \geq 1$  and  $\text{Im}(\Phi) = \mathbb{C}$ . Thus,  $G_k \in M_k$ .

Finally, we have  $M_k \cong M_k^0 \oplus \mathbb{C}.G_k$  if  $k \geq 2$ . (The above argument fails for  $k < 2$  as  $G_k$  is not well defined any more.)  $\square$

Therefore, the dimensions of  $M_k$  and  $M_k^0$  are closely linked.

### 1.4 Dimensions of Spaces of Modular Forms

The fact that multiplying two modular forms gives a function that remains modular yields that we may map a set of modular forms to an other.

**Theorem 2.**  $M_{k-6} \cong M_k^0. (? , p.88)$

*Proof.* We let  $\Phi : M_{k-6} \rightarrow M_k^0$  such that if  $f \in M_{k-6}$ ,  $\Phi(f)(z) = \Delta(z)f(z)$ .

This is well defined as if  $f$  has weight  $2(k-6)$ ,  $\Delta.f$  has weight  $2k$  since  $\Delta$  has weight 12. As  $\Delta$  is a cusp form,  $\Delta.f$  will also be a cusp form.

From definition,  $\Phi$  is clearly homomorphic.

Now, if  $g \in M_k^0$ , we may define  $\Psi : M_k^0 \rightarrow M_{k-6}$  such that  $\Psi(g)(z) = g(z)/\Delta(z)$

This is well defined as if  $g$  has weight  $2k$ ,  $\Delta.f$  has weight  $2k$  since  $\Delta$  has weight 12. As  $\Delta$  is a cusp form,  $\Delta.f$  will also be a cusp form.

This is well defined as  $\Delta$  has only one zero, at infinity, where  $g$  also has a zero (as  $g$  is a cusp form). The weights agree again as well.

It is then easy to remark that  $\Psi = \Phi^{-1}$ . So  $\Phi$  is bijective, and thus isomorphic.

Finally, we have  $M_{k-6} \cong M_k^0$ . □

This theorem, combined with the previous one is very powerful: it shows that there must be a pattern (of 6) in the sequence of dimensions  $\dim(M_k)$  and  $\dim(M_k^0)$  for  $k \geq 2$ . We have  $M_k \cong M_k^0 \oplus \mathbb{C}.G_k \cong M_{k-6} \oplus \mathbb{C}.G_k$ , so  $\dim(M_k) = \dim(M_{k-6}) + 1$  when  $k \geq 2$ . Thus, if we compute the dimensions of  $M_0, M_1, M_2, M_3, M_4, M_5$ , we can extrapolate dimensions of  $M_k$  and  $M_k^0$  for all  $k$ .

Using complex analysis techniques again, we have:

- $\dim(M_k) = 0 \quad k < 0$
- $\dim(M_1) = 0$
- $\dim(M_0) = \dim(M_2) = \dim(M_3) = \dim(M_4) = \dim(M_5) = 1$

In the case  $k = 0$ ,  $\dim(M_0) = 1$ . As  $f(z) = 1$  is clearly a modular form of weight 0,  $\{1\}$  is a basis for  $M_0$ . We deduce  $\dim(M_k^0) = 0$  as 1 is clearly not a cusp form. In the case  $k = 1$ ,  $\dim(M_1) = 0$ , which makes  $\dim(M_1^0) = 0$  automatically. (Cases  $k < 0$  are similar to  $k = 1$ .)

Other cases may be derived directly from the relations (using induction to get general formulas), and we obtain:

Space	$k < 0$	$k \geq 0, k \equiv 1 \pmod{6}$	$k \geq 0, k \not\equiv 1 \pmod{6}$
$\dim(M_k)$	0	$\lfloor k/6 \rfloor$	$\lfloor k/6 \rfloor + 1$
$\dim(M_k^0)$	0	$\max\{0, \lfloor k/6 \rfloor - 1\}$	$\lfloor k/6 \rfloor$

Note that the max is taken only to avoid negative dimensions.

## 1.5 Fourier Expansion

### 1.5.1 Definition

To study such function, we use Fourier Expansion. In the case of  $f$  being a modular form of weight  $2k$ , a Fourier Expansion is a representation of  $f$  as a power series of  $e^{2\pi iz}$  i.e.

$$f(z) = \sum_{n \in \mathbb{Z}} a_n(f) e^{2\pi i n z}.$$

We usually denote  $q = e^{2\pi iz}$  so that  $q^n = e^{2\pi i n z}$  and the Fourier expansion of  $f$  become

$$f(q) = \sum_{n \in \mathbb{Z}} a_n(f) q^n.$$

When in this form, we may as well call it the  $q$  expansion.

### 1.5.2 Typical Modular Forms Fourier Expansion

**Fourier Expansions of  $G_k$**  The modular forms  $G_k$  have the following  $q$  expansion(? , p.92):

$$G_k(q) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

with  $\sigma_s(n) = \sum_{d|n} d^s$ , for  $k \geq 4$ .

**Fourier Expansion of  $\Delta$**  We also have(? , p.95):

$$\Delta(q) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

## 1.6 A Basis for Modular Forms

The set of modular forms that are weight  $2k$  in fact form a vector space (we can add modular forms together, and multiply them with a constant) over the complex numbers. One may ask then a basis for this vector space.

We would like to find a basis for each set  $M_k$ . It turns out that the modular forms  $G_2$  and  $G_3$  introduced before in fact generate a basis for all  $M_k$ . It is not obvious and may in fact seems wrong at a first stage:  $G_2$  and  $G_3$  are modular forms of weight 4 and 6, whereas  $M_k$  in general have modular forms of weight  $2k$ . However, by taking combinations of  $G_2$  and  $G_3$ , we may obtain modular forms of any weight  $2k$ . It is important to remember that when multiplied, the weight of modular forms add up.

**Theorem 3.** *The set  $S = \{G_2^a G_3^b | a, b \in \mathbb{N}, 2a + 3b = k\}^1$  is a basis for  $M_k$ .(? , Theorem 2.17)*

*Proof.* Of course, the cases when  $\dim(M_k) = 0$  (for  $k < 0$  and  $k = 1$ ) are trivial, as the basis is empty, and  $2a + 3b = k$  has no solution for  $a, b \in \mathbb{N}$ .

To show  $S$  is a basis, we need it to span  $M_k$  and to be linearly independent.

We start with spanning, and we proceed by induction on  $k$ , with step 6.

As  $\dim(M_k) = 1$  for  $k = 0, 2, 3, 4, 5, 7$ , and the equation  $2a + 3b = k$  has exactly one solution for  $a, b \in \mathbb{N}$  (namely  $(a, b) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (2, 1)$ ),  $S$  has only one element, which must be the basis.

Now, for  $k > 7$ , take some  $a, b \in \mathbb{N}$  such that  $2a + 3b = k$ . Let  $f \in M_k$ , and  $g = G_2^a G_3^b \in M_k$ .  $g(\infty) \neq 0$  as none of  $G_2$  or  $G_3$  is a cusp form. So there must be a complex  $\lambda$  such that  $f - \lambda g$  is a cusp form. Then  $f - \lambda g \in M_k \cong M_{k-6}^0$  and we can find a  $h \in M_{k-6}^*$  such that  $h \cdot \Delta = f - \lambda g$ .

By induction,  $h$  must be a polynomial of  $G_2$  and  $G_3$ ; by definition,  $\Delta$  is one as well (note that yet, we don't put any restriction on powers of  $G_2$  and  $G_3$ , other then being positive integers). Therefore,  $f = \Delta \cdot h + \lambda g$  is a polynomial of  $G_2$  and  $G_3$ . From the fact that  $f \in M_k$  (i.e.  $f$  has weight  $2k$ ), terms of  $f$  as a polynomial of  $G_2$  and  $G_3$  have the form  $G_2^a G_3^b$  with  $2a + 3b = k$ .

We now want to show linear independence, we proceed by contradiction.

Suppose there is a non-trivial linear relation of terms  $G_2^a G_3^b$ . We can multiply it by suitable  $G_2$  and  $G_3$  so that all terms have the form  $2a + 3b = k \equiv 0 \pmod{12}$ . Then, we can divide all terms by  $G_3^2$ , witch gives us that there is a polynomial for which  $G_2^3/G_3^2$  is a root. In particular, this polynomial is constant when  $G_2^3/G_3^2$  is plugged. This contradicts the fact that  $q$  expansion of  $G_2^3/G_3^2$  is not constant.  $\square$

This set of makes to be a basis, and one may even find it pleasant: given the two modular forms  $G_2$  and  $G_3$ , this set generates all the modular forms of weight  $2k$  that we could think of, if we only knew these two modular forms.

## 1.7 Hecke Operators

We define the Hecke operators for a modular form  $f$  as follows(? , p.100):

$$T(n)f(z) = n^{2k-1} \sum_{a \geq 1, ad=n, 0 \leq b < d} d^{-2k} f\left(\frac{az+b}{d}\right)$$

with  $n \in \mathbb{N}$ .

We can check that  $T(n)f$  is modular if  $f$  is (as the sum of modular forms).

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<sup>1</sup>The set of naturals  $\mathbb{N}$  is taken to start from 0.

We may as well write  $T(n)f$  as a Fourier Expansion of  $q = e^{2\pi iz}$  as follows(? , p.100):

$$T(n)f(z) = \sum_{m \in \mathbb{Z}} \gamma(m)q^m \quad \text{with} \quad \gamma(z) = \sum_{a|(n,m), a \geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right)$$

For modular forms  $f$  s.t.  $f(z) = \sum_{n \in \mathbb{Z}} \alpha(n)q^n$

## 2 Modular Forms Modulo Two

### 2.1 Strategy to Reduce Modulo Two

It is not trivial, at this point, why and how we can reduce modulo two modular forms, objects that have coefficients in  $\mathbb{C}$ . In general, reduction modulo a number is only possible with whole numbers (integers). We would like to reduce modulo two coefficients of the Fourier series for modular forms. But at the moment, they lie in  $\mathbb{C}$ .

In fact, we will introduce a new basis for the modular forms: the so called Miller Basis. The coefficients of all the forms in this basis are integers. It is then possible to consider the space of modular forms over  $\mathbb{Z}$  instead of  $\mathbb{C}$ . Once this is done, we will reduce all the newly integers coefficients modulo 2.

In this section, we will denote all object reduced with an over-line:  $f \rightarrow \overline{f}$ ;  $c \rightarrow \overline{c}$ ;  $T_n \rightarrow \overline{T_n}$ .

### 2.2 Integer Basis

#### 2.2.1 Normalisation of Typical Modular Forms

**Normalisation of Eisenstein series  $G_k$**  We first recall the formula for  $q$  extension of  $G_k$  and the one for  $\zeta(2k)$ :

$$G_k(q) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

and

$$2\zeta(2k) = \frac{(2\pi)^{2k}}{(2k)!} B_k$$

so overall:

$$G_k(q) = \frac{(2\pi)^{2k}}{(2k)!} B_k + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

We would like to normalize this series, so that the coefficients become integers, so that we can ultimately reduce them modulo two. Right now, coefficients are rational.

As we want to keep the series modular with same weight, the only tool we have to normalize the series is multiplication by a constant. The normalization is a crucial point: If we multiply by 2 all coefficients of a modular form that already lie in  $\mathbb{Z}$ , the reduction mod 2 will always give zero.

First, let's normalize the series to have particular values on some coefficients of interest. There are two justified ways to do so: normalize to have constant coefficient set to one, and to have  $q$  coefficient is set to one. We will introduce both: Let  $E_k$  be such that:

$$E_k \cdot 2\zeta(2k) = G_k$$

so that

$$E_k = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

$E_k$  then has constant coefficient set to one.

Let  $F_k$  be such that:

$$F_k \cdot \left( 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \right) = G_k$$

so that

$$F_k = (-1)^k \frac{B_k}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

$F_k$  then has  $q$  coefficient set to one (as  $\sigma_{2k-1}(1) = 1$ ).

Clearly, the coefficients of this expansion remain in  $\mathbb{Q}$  at least, and we will show that for some specific  $k$ , the coefficients lie in fact in  $\mathbb{Z}$ . Both  $F_k$  and  $E_k$  are interesting, but for our purpose (reducing modulo two), we will use  $E_k$ . Note that  $E_k$  are normalized versions of Eisenstein series  $G_k$ , but in literature, both are called Eisenstein series see ( ? , p.6) for example.

**Normalisation of  $\Delta$**  Again, we recall the formula for  $q$  extension of  $\Delta$ :

$$\Delta(q) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Again, we want to normalize the series so that the first coefficient is one. As there is no constant coefficients, we will normalize it so that the  $q$  coefficient becomes 1. Therefore, we define  $\Delta = (2\pi)^{12} \Delta$ , so explicitly:

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Defined that way,  $\Delta$  is known as the discriminant modular form.

Clearly, the coefficients in expansion of  $\Delta$  are integers (which we can reduce modulo two).

### 2.2.2 Miller Basis

**Basis with Integer Coefficients (in Fourier Series)** Applying normalization  $G_k \rightarrow E_k$  above for  $k = 2, 3$ , we get:

$$\begin{aligned} E_2 &= 1 + \frac{8}{B_2} \sum_{n=1}^{\infty} \sigma_3(n) q^n & B_2 &= \frac{1}{30} \\ &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \end{aligned}$$

and

$$\begin{aligned} E_3 &= 1 - \frac{12}{B_3} \sum_{n=1}^{\infty} \sigma_5(n) q^n & B_3 &= \frac{1}{42} \\ &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \end{aligned}$$

Now, we have shown that  $\{G_2^a G_3^b | 2a + 3b = k\}$  is a basis for modular forms of weight  $2k$  over the complex (see 1.6). As  $E_2 = \lambda G_2$ ,  $\lambda \in \mathbb{C}$  and  $E_3 = \mu G_3$ ,  $\mu \in \mathbb{C}$ , we have that  $\{E_2^a E_3^b | 2a + 3b = k\}$  remains a basis for  $M_k$  over  $\mathbb{C}$ .

It is clear, from the series, that coefficients of the  $q$  expansion of both  $E_2$  and  $E_3$  are all integers. Thus, so are coefficients of combinations of  $E_2$  and  $E_3$ . Therefore, we have found a basis for  $M_k$  such that all elements in the basis have only integer coefficients in their  $q$  expansion.

**Miller Basis for  $M_k^0$**  This is a nice result, but we can in fact do better, by forcing the first coefficients to chosen values.

**Theorem 4.** *For the space of modular cusp forms  $M_k^0$ , there exists a basis  $\{f_1, \dots, f_r\}$  such that:*

- $f_i \in Z[q]$
- $a_i^j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \forall 1 \leq i, j \leq r$   
where  $a_i^j$  is the coefficient of  $q^j$  in expansion of  $f_i$ .

This is commonly called the Miller basis for  $M_k^0$ , as it was first introduced by Victor Saul Miller (? ).

*Proof.* • For  $k < 6$ ,  $k = 7$ , we have  $\dim(M_k^0) = 0$ . Thus,  $\emptyset$  is a basis which satisfies the Miller basis properties.

- For  $k = 6$ , we have  $\dim(M_k^0) = 1$ . Thus,  $\{\Delta\}$  is a basis which satisfies the Miller basis properties.



- For  $k \geq 7$ , we let  $r = \dim(M_k^0) \geq 1$ . We then consider the set

$$\{g_j | 1 \leq j \leq r\}$$

where

$$g_j = \Delta^j E_3^{2(d-j)+b} E_2^a$$

with

$$\begin{aligned} 2a + 3b &\leq 7 \text{ \& } 2a + 3b \cong k \pmod{6} \\ \& \ d = \frac{k - (2a + 3b)}{6} &\in \mathbb{N} \text{ as } k \geq 7 \end{aligned}$$

Note that  $a$  and  $b$  are unique unless  $k \cong 0 \pmod{6}$ . In witch case, we use by convention  $a = 0$ ,  $b = 0$ .

As all  $E_2$ ,  $E_3$ , and  $\Delta$  have integer coefficients,  $g_j$  will as well.

We then look at the  $q$  series:

$$\Delta(q) = q + O(k^2) \implies \Delta^j(q) = q^j + O(k^{j+1})$$

As we normalized so,

$$\begin{aligned} E_2(q) &= 1 + O(q) \implies E_2^\alpha(q) = 1 + O(q) \\ E_3(q) &= 1 + O(q) \implies E_3^\alpha(q) = 1 + O(q) \end{aligned}$$

This gives:

$$g_j(q) = q^j + O(q^{j+1}) \quad \forall 1 \leq j \leq r.$$

Therefore,  $\{g_j, | 1 \leq j \leq r\}$  is clearly a linearly independent set. By dimension argument, it also spans  $M_k^0$ . Therefore, it forms a basis. Moreover, in this basis:  $\alpha_i^j = \delta_{ij} \quad i \leq j$ .

Finally, we can use Gaussian elimination on  $\{g_j\}$  to obtain a basis  $\{f_j | 1 \leq j \leq r\}$  such that:  $\alpha_i^j = \delta_{ij} \quad \forall 1 \leq i, j \leq r$ . The coefficients will remain in  $\mathbb{Z}$  after Gaussian elimination. □

**Extension to all  $M_k$**  We already have a basis for  $M_k^0$ , as  $\dim(M_k) = \dim(M_k^0) + 1$  (over  $\mathbb{C}$ ), we just need to adjoint one element of  $M_k \setminus M_k^0$  to our basis.

It was shown before that  $\{E_2^a E_3^b | 2a + 3b = k\}$  is a basis for  $M_k$  with integer coefficients (see 2.2.2). One may see from the  $q$  expansion that  $E_2^a E_3^b = 1 + O(q)$  so  $E_2^a E_3^b \in M_k \setminus M_k^0$ .

Therefore, we can just add one element of  $\{E_2^a E_3^b | 2a + 3b = k\}$  to the Miller basis, and use Gaussian elimination again. We get a basis for  $M_k$  of the form  $\{f_j | 0 \leq j \leq r\}$  such that in this basis:  $\alpha_i^j = \delta_{ij} \quad \forall 0 \leq i, j \leq r$  (with  $r = \dim(M_k^0)$  i.e.  $r + 1 = \dim(M_k)$ ).

## Miller Basis Examples

**Miller basis for  $k = 16$**  We can calculate the Miller basis for  $k = 16$ :  $k \cong 4 \pmod{12}$  so  $a = 2$  and  $b = 0$ ;  $d = 2$ . We put  $g_1 = \Delta^1 E_3^2 E_2^2$ , so:

$$\begin{aligned} g_1(q) &= \Delta(q) E_2^2(q) E_3^2(q) \\ &= [q - 24q^2 + 252q^3 + O(q^4)] \\ &\quad * [1 + 240q + 2160q^2 + 6720q^3 + O(q^4)]^2 \\ &\quad * [1 - 504q - 16632q^2 + 122976q^3 + O(q^4)]^2 \\ &= q - 552q^2 - 188244q^3 + O(q^4) \end{aligned}$$

and  $g_2 = \Delta^2 E_3^0 E_2^2$ , so:

$$\begin{aligned} g_2(q) &= \Delta^2(q) E_2^2(q) \\ &= [q - 24q^2 + 252q^3 + O(q^4)]^2 \\ &\quad * [1 + 240q + 2160q^2 + 6720q^3 + O(q^4)]^2 \\ &= q^2 + 432q^3 + O(q^4) \end{aligned}$$

Then,  $f_2 = g_2$  and  $f_1 = g_1 + 552g_2$ , so:

$$\begin{aligned} f_1(q) &= q - 552q^2 - 188244q^3 + O(q^4) + 552 * [q^2 + 432q^3 + O(q^4)] \\ &= q + 50220q^3 + O(q^4) \\ f_2(q) &= q^2 + 432q^3 + O(q^4) \end{aligned}$$

Therefore, up to  $O(q^4)$ ,  $\{f_1, f_2\} = \{q + 50220q^3 + O(q^4), q^2 + 432q^3 + O(q^4)\}$  is a basis for  $M_{16}^0$ .

To extend this base to  $M_k$ , we adjoint a term of the form  $g_0 = E_2^a E_3^b$  where  $2a + 3b = 16$ . We pick  $g_0 = E_2^8$ , so:

$$\begin{aligned} g_0(q) &= E_2^8(q) \\ &= [1 + 240q + 2160q^2 + 6720q^3 + O(q^4)]^8 \\ &= 1 + 1920q + 1630080q^2 + 803228160q^3 + O(q^4) \end{aligned}$$

Then,  $f_0 = g_0 - 1920g_1 - 1630080g_2$ , so:

$$\begin{aligned} f_0(q) &= g_0(q) - 1920g_1(q) - 1630080g_2(q) \\ &= [1 + 1920q + 1630080q^2 + 803228160q^3 + O(q^4)] \\ &\quad - 1920 [q + 50220q^3 + O(q^4)] \\ &\quad - 1630080 [q^2 + 432q^3 + O(q^4)] \\ &= 1 + 2611200q^3 + O(q^4) \end{aligned}$$

Therefore, up to  $O(q^4)$ ,  $\{f_0, f_1, f_2\} = \{1 + 2611200q^3 + O(q^4), q + 50220q^3 + O(q^4), q^2 + 432q^3 + O(q^4)\}$  is a basis for  $M_{16}$ .

**Miller basis for  $k = 92$**  The calculation of this basis may be interesting by hand once; However, it is possible to automate it. The procedure that calculates such coefficients is a standard in SageMath(?). Here is, up to  $O(q^{10})$ , the Miller basis for  $M_{92}$ :

$$\begin{aligned} f_0 &= 1 + 3034192667130000 * q^8 + 137290127714549760000 * q^9 + O(q^{10}) \\ f_1 &= q + 91578443563200 * q^8 + 2651503140376278561 * q^9 + O(q^{10}) \\ f_2 &= q^2 + 2380310529376 * q^8 + 42238207588515840 * q^9 + O(q^{10}) \\ f_3 &= q^3 + 51682260816 * q^8 + 530253459731160 * q^9 + O(q^{10}) \\ f_4 &= q^4 + 896013480 * q^8 + 4882999541760 * q^9 + O(q^{10}) \\ f_5 &= q^5 + 11516000 * q^8 + 28971735750 * q^9 + O(q^{10}) \\ f_6 &= q^6 + 94680 * q^8 + 80990208 * q^9 + O(q^{10}) \\ f_7 &= q^7 + 312 * q^8 - 4860 * q^9 + O(q^{10}) \end{aligned}$$

## 2.3 Basis Modulo Two

### 2.3.1 Reduced Modular Forms

Now that we have a basis with integer coefficients, it makes sense to reduce forms modulo two. For a modular form  $f$ , we denote its reduced modulo two form  $\bar{f}$ . It is defined as follows:

If

$$f(q) = \sum_{n \in \mathbb{N}} c(n)q^n$$

then

$$\bar{f}(q) = \sum_{n \in \mathbb{N}} \bar{c}(n)q^n \quad \text{with } \bar{c}(n) = c(n) \pmod{2}.$$

We want to reduce Miller basis modulo two. The reason is that as we know that some coefficients are ones, the reduction will not be trivial. We will reduce separately  $E_2$ ,  $E_3$  and  $\Delta$  (which together generate the Miller basis).

**$E_2$  reduced** We have:

$$\bar{E}_2(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n = 1 \pmod{2}$$

Therefore, the reduction modulo two of  $E_2^a$  is just 1, for all  $a \geq 0$ .

**$E_3$  reduced** We have:

$$\bar{E}_3(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n = 1 \pmod{2}$$

Therefore, the reduction modulo two of  $E_3^b$  is just 1, for all  $b \geq 0$  as well.

**$\Delta$  reduced** We defined before  $\Delta$ , and we would now like to know its  $q$  extension in the standard way. That is, an infinite sum of  $q^n$ , instead of an infinite product as we have at the moment.

We define the coefficients  $\tau(n)$  to match in the equation:

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n$$

When this holds,  $\tau$  is called the Ramanujan function.

We would like an explicit formula for  $\tau(n)$ . More precisely, we are interested in a formula for  $\tau(n) \pmod{2}$ .

We will calculate separately the coefficients  $\tau(n) \pmod{2}$  for  $n$  even and odd.

**Case  $n$  odd** Remember  $\sigma_s(n)$  as the sum of  $s^{th}$  powers of (positive) divisors of  $n$ . It is known from classical theory ([1], p.8) that:

$$\tau(8n + l) \equiv a_l \sigma_{11}(8n + l) \pmod{2^{b_l}}$$

where  $\gcd(l, 8) = 1$ ,  $a_1 = 1$ ,  $a_3 = 1217$ ,  $a_5 = 1537$ ,  $a_7 = 705$ ,  $b_1 = 11$ ,  $b_3 = 13$ ,  $b_5 = 12$ ,  $b_7 = 14$

We are interested in congruence class  $\pmod{2}$  of the Ramanujan function  $\tau(n)$ . For  $n$  odd, we deduce the following:

$$\tau(n) \equiv \sigma_{11}(n) \equiv \sum_{d|n} d^{11} \equiv \sum_{d|n} 1 \equiv \begin{cases} 1 \pmod{2} & \text{if } n \text{ is a square} \\ 0 \pmod{2} & \text{else} \end{cases}$$

**Case  $n$  even** It is easy to calculate that  $\tau(2) = -24 \equiv 0 \pmod{2}$ .

Using  $\tau(p^{n+1}) = \tau(p^n)\tau(p) - p^{11}\tau(p^{n-1})$  ([1], p.97) with  $p = 2$ , it follows by induction that  $\tau(2^k) \equiv 0 \pmod{2} \quad \forall k \in \mathbb{N}$ .

Using  $\tau(nm) = \tau(n)\tau(m)$  ([1], p.97), it follows that for all  $n$  even,  $\tau(n) \equiv 0 \pmod{2}$ .

**Explicit series of the discriminant** Therefore, the only non-zero coefficients (modulo 2) appears on odd squares, i.e.:

$$\tau(n) \equiv \begin{cases} 1 \bmod 2 & \text{if } n = (2m+1)^2 \text{ for } m \in \mathbb{N} \\ 0 \bmod 2 & \text{else} \end{cases}$$

Thus, we can write the power series of  $\Delta$  as:

$$\Delta(q) \equiv \overline{\Delta}(q) = \sum_{m=0}^{\infty} q^{(2m+1)^2} \bmod 2$$

### 2.3.2 Reduced Basis

The Miller basis for  $M_k$  was obtained via the Gauss elimination of the set  $\{\Delta^j E_3^{2(d-j)+b} E_2^a | 1 \leq j \leq \dim(M_k)\}$  (with some conditions on  $a, b, d$ ).

But  $\overline{E_2^a} = \overline{E_2}^a = 1^a = 1 \bmod 2$  and similarly,  $\overline{E_3^{2(d-j)+b}} = 1 \bmod 2$ . So once the above set is reduced modulo two, we are left with  $\{\overline{\Delta}^j | 1 \leq j \leq \dim(M_k)\}$ . So the Miller basis just becomes the Gauss elimination of  $\overline{\Delta}$  powers.

This is what motivates the next section.

## 2.4 Space of Modular Forms Modulo Two

We would like to have a definition for this space in a similar way as  $M_k$  was used for modular forms (of weight  $2k$ ) before reduction.

### 2.4.1 Weights of Modular Forms Modulo Two

We just saw that the Miller basis for  $\overline{M_k}$  is (the Gaussian elimination of)  $\{\overline{\Delta}^j | 1 \leq j \leq \dim(\overline{M_k})\}$ .

Now, if we look at this set not reduced modulo two, we have:  $\{\Delta^j | 1 \leq j \leq \dim(M_k)\}$ . This is a set of modular forms that have different weights. However, we started with a modular forms in  $M_k$ , i.e. all modular forms having weight  $2k$ .

We understand now that modulo two, the weight of modular form doesn't make sense any more. This is one of the consequences of reducing modulo two: we lose some informations about the modular forms, such as the weight.

From this observation, we should study all modular forms together, modulo two (instead of separating by weights). This is why the space of modular forms modulo two will be denoted  $\mathcal{F}$ , with no dependence on  $k$ .

### 2.4.2 Powers of $\Delta$

**Set of Powers of  $\Delta$**  As we just saw, the Gaussian elimination of powers  $\overline{\Delta}^k$  up to  $\dim(\overline{M_k})$  form the Miller basis of  $\overline{M_k}$  (modular forms of weight  $2k$  reduced modulo two).

To lighten the notation, we will now write  $\Delta$  instead of  $\overline{\Delta}$ , and consider everything modulo two. For simplicity again, we will just take the powers of  $\Delta$  to be our basis for modular forms modulo two (i.e. drop the Gaussian elimination process).

We define the space  $\mathbb{F}_2[\Delta]$  in the usual way:

$$\mathbb{F}_2[\Delta] = \left\{ \sum_{k=1}^n a_k \Delta^k \mid n \in \mathbb{N}, a_k \in \mathbb{F}_2 \right\}$$

From 2.3.1 we had:

$$\Delta(q) = \sum_{n=0}^{\infty} \tau(n) q^n = \sum_{m=0}^{\infty} q^{(2m+1)^2}$$

Therefore, we define

$$\Delta^k(q) = \sum_{n=0}^{\infty} \tau_k(n) q^n = \left( \sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^k \pmod{2}$$

Thus, we have  $\tau(n) = \tau_1(n)$ .

**Proportion of zeros** In fact, most of the coefficients  $\tau_k(n)$  are 0 modulo two.

When  $k = 1$ , there is already few coefficients that are ones: only the odd squares. When raising to the  $k^{th}$  power, there are even "less".

**Conditions on non-zero coefficients** We can find conditions on coefficients that may not be zero.

We observe: We remark that odd squares are all  $1 \pmod{8}$ , and even squares are all  $0 \pmod{8}$ .

$a =$	0	1	2	3	4	5	6	7	$\pmod{8}$
$a^2 =$	0	1	4	1	0	1	4	1	$\pmod{8}$

Table 1: Squares modulo 8

We know from previous calculations that  $\Delta(q)$  only has odd powers of  $q$ . Thus, raising to the  $k^{th}$  power give terms of power  $n$  such that:

$$\begin{aligned} n &= m_1^2 + m_2^2 + m_3^2 + \dots + m_k^2 \\ &\equiv 1 + 1 + 1 + \dots + 1 \pmod{8} \\ &\equiv k \pmod{8} \end{aligned}$$

Therefore:  $\tau_k(n) \equiv 1 \pmod{2} \implies n \equiv k \pmod{8}$

Equivalently:  $n \not\equiv k \pmod{8} \implies \tau_k(n) \equiv 0 \pmod{2}$  (by taking the contra-positive)

This means, that  $\Delta^k$  may only have terms  $q^n$  such that  $n \equiv k \pmod{8}$ , i.e.  $\Delta^k$  may only have terms of power congruent to  $k \pmod{8}$ . When  $k = 1$ , this is that  $\Delta$  may only have terms of power  $1 \pmod{8}$ , this matches with table 1: all odd squares are  $1 \pmod{8}$ .

**Even powers of  $\Delta$**  We compare  $\Delta^{2k}(q)$  and  $\Delta^k(q^2)$ :

$$\begin{aligned} \Delta^{2k}(q) &= \left( \sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^{2k} \\ &= \sum_{n=0}^{\infty} \#[(2m_1+1)^2 + (2m_2+1)^2 + \dots + (2m_{2k}+1)^2 = n \mid m_0, m_1, \dots, m_{2k} \in \mathbb{N}] q^n \\ &= \sum_{n \text{ even}}^{\infty} \#[(2m_1+1)^2 + (2m_2+1)^2 + \dots + (2m_k+1)^2 = n/2 \mid m_0, m_1, \dots, m_k \in \mathbb{N}] q^n \\ &= \left( \sum_{m=0}^{\infty} q^{((2m+1)^2)*2} \right)^k \\ &= \left( \sum_{m=0}^{\infty} (q^2)^{(2m+1)^2} \right)^k = \Delta^k(q^2) \end{aligned}$$

Thus,  $\Delta^{2k}(q) = \Delta^k(q^2)$ . Therefore, we can write any modular form modulo two  $f$  as the following:

$$f = \sum_{s \geq 0} f_s^{2^s} \quad \text{with } f_s \text{ having only odd powers of } \Delta$$

(? , (3)) So it is sufficient to study only the odd powers of  $\Delta$ .

### 2.4.3 The Space $\mathcal{F}$

We define the space of modular forms modulo two denoted  $\mathcal{F}$  to be ([2], 2.1):

$$\mathcal{F} = \langle \Delta^k | k \text{ odd} \rangle = \langle \Delta, \Delta^3, \Delta^5, \Delta^7, \dots \rangle$$

That is, all finite polynomials of  $\Delta$  over  $\mathbb{F}_2$ , having only odd powers. We remark that the weight of modular forms do not appear, as it was discussed before in 2.4.1. The observations modulo 8 that we have done in 2.4.2 yields that it will be useful to denote:

$$\mathcal{F}_1 = \langle \Delta^k | k \equiv 1 \pmod{8} \rangle = \langle \Delta, \Delta^9, \Delta^{17}, \Delta^{25}, \dots \rangle$$

$$\mathcal{F}_3 = \langle \Delta^k | k \equiv 3 \pmod{8} \rangle = \langle \Delta^3, \Delta^{11}, \Delta^{19}, \Delta^{27}, \dots \rangle$$

$$\mathcal{F}_5 = \langle \Delta^k | k \equiv 5 \pmod{8} \rangle = \langle \Delta^5, \Delta^{13}, \Delta^{21}, \Delta^{29}, \dots \rangle$$

$$\mathcal{F}_7 = \langle \Delta^k | k \equiv 7 \pmod{8} \rangle = \langle \Delta^7, \Delta^{15}, \Delta^{23}, \Delta^{31}, \dots \rangle$$

Of course, we have:

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7$$

We will also introduce (as in ([2], 2.)):

$$\mathcal{F}(n) = \langle \Delta^k | k \text{ odd and } k \leq 2n-1 \rangle = \langle \Delta, \Delta^3, \Delta^5, \dots, \Delta^{2n-1} \rangle$$

This matches specifically  $\overline{M_{12(2n-1)}} = \mathcal{F}(n)$ .

### 2.4.4 Duality $\Delta - q$

As we defined  $\mathcal{F}$  above, a modular form modulo two is an expression of powers  $\Delta^k$ . But we had from before that  $\Delta = \sum_{m=0}^{\infty} q^{(2m+1)^2} \pmod{2}$ . Therefore, we can translate a modular form given as a finite polynomial of  $\Delta$  into an infinite polynomial of  $q$ . Thus, there are two ways to write a modular form modulo two.

This duality between the two definitions is what makes the study of modular forms modulo two so interesting: we go back and forth between an infinite series and a finite polynomial. One is easy to express, the other easy to compute. This will lead to new reasoning. In particular, there is a new technique of computation ("exact computations") that uses equivalence between the two ways of writing a modular form.

## 2.5 Hecke Operators Modulo Two

### 2.5.1 Reduction Modulo Two

**Definition** Now that we have reduced modular forms modulo two, we would like to study the Hecke operators on these reduced modular forms. We define Hecke operators modulo two as follows:

With  $f$  a modular form modulo two with  $q$  definition

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

we define

$$\overline{T_p} f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

where

$$\gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases} \quad \& \text{ } p \text{ an odd prime}$$

**Well-definiteness** We want to check that all the definitions make sense. When we look at  $T_p|f$ , there is a number of ways to reduce it modulo two:  $\overline{T_p|f}$ ,  $\overline{T_p|f}$ ,  $\overline{T_p|f}$ ,  $\overline{T_p|f}$ .

Let's compare coefficients:

$\overline{T_p|f}$ :

$$\gamma(n) = \sum_{a|(n,p), a \geq 1} a^{2k-1} c\left(\frac{np}{a^2}\right) = \begin{cases} \overline{c(np)} & \text{if } p \nmid n \\ \overline{c(np)} + \overline{c(n/p)} & \text{if } p \mid n \end{cases}$$

Divisors of  $(n, p)$  are  $\{1\}$  or  $\{1, p\}$  since  $p$  is prime, so the sum split in two cases, with one or two terms. We see now that looking at Hecke operators modulo two only for primes simplifies the sum to a computable formula.

As both 1 and  $p$  are odd, the term  $a^{2k-1}$  reduces to 1 modulo two. We understand why Hecke operators modulo two isn't defined for even numbers: many terms in the summation would become zero. It would not make sense to call it a Hecke operator any more.

It also makes sense why we look at modular forms modulo two and not say three or five: the coefficient  $a^{2k-1}$  collapse nicely modulo two, which won't be the case modulo an other number than two.

$\overline{T_p|f}$ : This is (very) similar to the case before.

$\overline{T_p|f}$ :

$$\gamma(n) = \begin{cases} \overline{c(np)} & \text{if } p \nmid n \\ \overline{c(np)} + \overline{c(n/p)} & \text{if } p \mid n \end{cases}$$

$\overline{T_n|f}$ : Again, this is (very) similar to the case before.

All reductions give in fact the same result, so it makes sense to reduce modular forms modulo two, and still study the Hecke operators (but now only for odd primes). As this all makes sense, we will now write only consider modular forms modulo two, and we will drop the over lines for simplicity.

### 2.5.2 Properties

What makes Hecke operators interesting, is that they have many properties.

**Inherited properties** From the fact that  $\overline{T_p|f}(q) = \overline{T_p|f(q)}$ , we get that the Hecke operators modulo two keep all the properties they had before being reduced.

**Modularity Remains** From definition 1.7, a Hecke operators transform a modular form to an other. This is because from definition,  $T(n)f$  is a sum of modular forms (which remain modular). Therefore, Hecke operators modulo two will as well transform a modular form to an other. This was not clear from the definition modulo two that we had (which was in terms of  $q$  series).

**Commutativity** As in general(? , p.101):

$$T(n)T(m) = T(mn) \quad \text{if } \gcd(m, n) = 1$$

We get that:

$$T_p T_q = T_q T_p \quad \forall p, q \in \mathbb{P}$$

Therefore, the Hecke operators modulo two commute. This, as well, was not clear from definition. It will be very convenient for future calculations.

**Nilpotent** The properties of Hecke operators is that, given a modular form  $f$ , if we apply a Hecke operators enough times, the form will become zero (i.e. they are nilpotent).

**Order of  $\Delta$  doesn't increase** From definition 1.7, a Hecke operators takes a modular form of weight  $2k$  to an other modular form of weight  $2k$ .

Take a modular form modulo two  $\bar{f}$  with degree  $k$  (in terms of  $\Delta$ ). Now we want to know the maximum degree (again in terms of  $\Delta$ ) of  $T_p|\bar{f}$ . Let  $n$  be the smallest integer such that  $\bar{f} \in \overline{M_{12(2n-1)}}$ .

We know  $T_p|\bar{f} = \overline{T(p)f}$  and  $\bar{f} \in \mathcal{F}(n) = \overline{M_{12(2n-1)}}$  so  $f \in M_{12(2n-1)}$ . This implies that  $T(p)f \in M_{12(2n-1)}$  so  $T_p|\bar{f} = \overline{T(p)f} \in \overline{M_{12(2n-1)}} = \mathcal{F}(n)$ .

Therefore, the maximum degree (in terms of  $\Delta$ ) of  $T_p|\bar{f}$  is  $k$  as well. Thus, the degree of  $\bar{f}$  doesn't increase after applying a Hecke operator.

**Order of  $\Delta$  decrease** (? , 2.3) ==> need a proof!!! [I am stuck]

**Behaviour of  $\mathcal{F}_i$**  Suppose  $f \in \mathcal{F}_i$ , using 2.4.2, we have:

$$f = \sum_{m \equiv i \pmod 8} \mu_m \Delta^m = \sum_{n \equiv i \pmod 8} c(n) q^n$$

From the definition of Hecke operator modulo two (2.5.1), we have:

$$T_p|f = \sum_{n \in \mathbb{N}} \gamma(n) q^n \quad \text{with } \gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases}$$

$c(np)$ : We have  $np \not\equiv i \pmod 8 \implies c(np) = 0$ .

$c(n/p)$ : As  $p$  is an odd prime, it is an odd number, so from 2.4.2,  $p^2 \equiv 1 \pmod 8$ , so  $p^{-2} \equiv 1 \pmod 8$  as well (with  $p^{-2}$  seen mod 8).

Therefore,  $np \not\equiv i \pmod 8 \implies n/p \equiv np/p^2 \equiv np \not\equiv i \pmod 8$ .

$\gamma(n)$ : We conclude that  $n \equiv np^2 \not\equiv pi \pmod 8 \implies \gamma(n) = 0$

Using 2.4.2 again, we deduce that  $T_p|f \in \mathcal{F}_j$  with  $j \equiv pi \pmod 8$ .

Overall, we have the following:

$$f \in \mathcal{F}_i \implies T_p|f \in \mathcal{F}_j \text{ with } j \equiv pi \pmod 8$$

**Application to  $\Delta^k$**  As the degree of a modular form doesn't increase after applying a Hecke operator, we can apply this the modular form  $\Delta^k$  to get:

$$T_p|\Delta^k = \sum_{\substack{j \leq k \\ j \text{ odd}}} \mu_j \Delta^j$$

As we know, moreover, that the degree of a modular form will in fact decrease, we deduce that in fact:

$$T_p|\Delta^k = \sum_{\substack{j \leq k-2 \\ j \text{ odd}}} \mu_j \Delta^j \quad (*)$$

The last observation (on  $\mathcal{F}_i$ ), leads us to the formula:

$$T_p|\Delta^k = \sum_{\substack{j \leq k-2 \\ j \equiv pk \pmod 8}} \mu_j \Delta^j \quad (**)$$

(since  $\Delta^k \in \mathcal{F}_i$  with  $k \equiv i \pmod 8$ )

### 2.5.3 Examples (for Small Powers of $\Delta$ )

We will describe the behaviour of Hecke operators when applied to  $\Delta^k$  with  $k$  odd,  $k \leq 7$ .



$\Delta$  Clearly, from (\*), we have  $T_p|\Delta = 0$ , since the sum is empty (for any  $p$  odd prime).

$\Delta^3$  From (\*), we have  $T_p|\Delta^3 = \Delta$  or 0.

Moreover, (\*\*) gives  $T_p|\Delta^3 = 0$  if  $1 \not\equiv 3p \pmod{8}$  i.e. if  $p \not\equiv 3 \pmod{8}$ .

Now, if  $p \equiv 3 \pmod{8}$ , we may only look at the coefficient  $q^1$  of  $T_p|\Delta^3$  (if it is 1,  $T_p|\Delta^3 = \Delta$  and if it is 0,  $T_p|\Delta^3 = 0$ , as there is no other possibilities).

From definition (in 2.5.1), we have that the coefficient of  $q^1$  is  $\gamma(1) = c(p)$  (since  $p \nmid 1$ ) with  $c$  the  $q$  coefficients of  $\Delta^3$ .

From (2.3.1), the none zero coefficients of  $\Delta$  are odd squares.

Now,  $c(p)$  is the  $p^{th}$  coefficient of  $\Delta^3$ . We have:

$$(\Delta(q))^3 = \left( \sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^3 = \sum_{n=0}^{\infty} \#\{m_1, m_2, m_3 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 = n\} q^n$$

So  $c(p) = \#\{m_1, m_2, m_3 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 = p\} \pmod{2}$  corresponds  $(\pmod{2})$  to the number of ways to write  $p$  as sum of three odd squares.

Need in fact  $m_1 = m_2 \neq m_3$ , but then?? [I am stuck]

$\Delta^5$  From (\*), we have  $T_p|\Delta^5 = \Delta^3$  or  $\Delta$  or 0.

Moreover, (\*\*) gives:

$$\begin{array}{llll} p \equiv 7 \pmod{8} : & T_p|\Delta^5 = \Delta^3 \text{ or } 0 & \text{if } 3 \equiv 5p \pmod{8} & \text{i.e. } p \equiv 7 \pmod{8} \\ p \equiv 5 \pmod{8} : & T_p|\Delta^5 = \Delta \text{ or } 0 & \text{if } 1 \equiv 5p \pmod{8} & \text{i.e. } p \equiv 5 \pmod{8} \\ p \equiv 1 \text{ or } 3 \pmod{8} : & T_p|\Delta^5 = 0 & \text{else} & \end{array}$$

Now, if  $p \equiv 7 \pmod{8}$ , we may only look at the coefficient  $q^3$  of  $T_p|\Delta^5$  (if it is 1,  $T_p|\Delta^5 = \Delta^3$  and if it is 0,  $T_p|\Delta^5 = 0$ , as there is no other possibilities).

From definition (in 2.5.1), we have that the coefficient of  $q^3$  is  $\gamma(3) = c(3p)$  (since  $p \nmid 3$ ) with  $c$  the  $q$  coefficients of  $\Delta^5$ .

From (2.3.1), the none zero coefficients of  $\Delta$  are odd squares. Now,  $c(3p)$  is the  $p^{th}$  coefficient of  $\Delta^5$ . We have:

$$(\Delta(q))^5 = \left( \sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^5 = \sum_{n=0}^{\infty} \#\{m_1, m_2, m_3, m_4, m_5 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = n\} q^n$$

So  $c(3p) = \#\{m_1, m_2, m_3, m_4, m_5 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = 3p\} \pmod{2}$  corresponds  $(\pmod{2})$  to the number of ways to write  $3p$  as sum of five odd squares.

When looked  $\pmod{8}$ ,  $m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 \equiv 5 \pmod{8}$ . We can check,  $p \equiv 7 \pmod{8}$  so  $3p \equiv 5 \pmod{8}$ .

Need in fact  $m_1 = m_2 \neq m_3$ , but then?? [I am stuck]

$\Delta^7$

#### 2.5.4 Nilpotent Order

As we know that the Hecke operators are nilpotent, we may want to study the order of nilpotentness.

##### Definition

##### Properties

**Example** We will look at  $\Delta^{95}$ ...

References