Modular forms modulo 2

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Abstract

We are interested in Modular forms modulo 2, and computing thing about it. [temporary abstract]

Key words that should appear: Modular forms; Mod 2; Duality of definitions; Governing fields; Frobenian map?; Exact computations;

Contents

1 Modular forms

1.1 Modular forms of level 1

We will denote by $\mathbb{H}$ the upper half plane.

We say that a complex function f on the upper half plane is weakly modular of weight $2k$ if f is meromorphic on $\mathbb{H}$ and

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

A property of $SL_2(\mathbb{Z})$ is that when we define

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then $SL_2(\mathbb{Z})$ is generated by S and T (**?** , p.1-2).

From that property, we can derive an alternative definition of weakly modular functions: f is weakly modular of weight $2k$ if f is meromorphic on $\mathbb{H}$ and $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$ for all $z \in \mathbb{Z}$.

Moreover, we also define a function $f : \mathbb{H} \to \mathbb{C}$ to be modular of weight $2k$ if f is holomorphic on $\mathbb{H}$ and f is weakly modular.

Lastly, we say that a function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $2k$ if f is holomorphic at ∞ and f is modular.

It is easy to check, from definition, that we can add modular forms together, as well as multiply them by a complex:

- $f_1(z) + f_2(z)$ is modular of weight $2k$ if $f_1(z)$ and $f_2(z)$ are modular of weight $2k$.
- $\lambda f(z)$ is modular of weight $2k$ if $f(z)$ is.

Therefore, modular forms of weight $2k$ over $\mathbb{C}$ form a space. We denote it M_k.

It is also possible to multiply modular forms, in which case the weights adds on: If $f_1(z)$ & $f_2(z)$ are modular of respective weights $2k_1$ & $2k_2$, then $f_1(z)f_2(z)$ is modular of weight $2k_1 + 2k_2$

We deduce that we can take powers of modular forms, and the weight is then multiplied by the power: If $f(z)$ is modular of weight $2k$, then $f^n(z)$ is modular of weight $2k * n$ (with $n \in \mathbb{N}$).

1.2 Typical Modular Forms

1.2.1 Eisenstein series G_k

The most famous class of modular forms is probably the Eisenstein series, usually denoted G_k. We define them as follows(**?** , Examples of Modular Forms of Level 1):

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}}$$

for $k \geq 2$.

It is easy to check that G_k are modular of weight $2k$(**?** , Proposition 2.1), as:

$$G_k(z+1) = G_k(z)$$

(using $(m, n+m) \to (m,n)$, an invertible map)

$$G_k(-1/z) = z^k G_k(z)$$

(using $(m,-n) \to (m,n)$, an invertible map).

It is pleasant to remark that (**?** , Proposition 2.2)

$$G_k(\infty) = \sum_{n\in\mathbb{Z}^*} \frac{1}{n^{2k}} = 2\zeta(2k)$$

. Where $\zeta(k)$ is Riemann Zeta function. The values of this function are well known on positive even numbers, and we deduce (**?** , p.194):

$$G_k(\infty) = 2\zeta(2k) = \frac{(2\pi)^{2k}}{(2k)!}B_k$$

with $B_k = (-1)^{k+1}b_{2k}$ where b_k are Bernoulli's numbers.

1.2.2 Δ

We will be interested in one main modular form in the rest of this article: Δ. We define Δ in terms of G_k as follows(**?** , p.84):

$$\Delta = g_2^3 - 27g_3^2 \in M_6^0 \qquad \text{with } g_2 = 40G_2 \text{ and } g_3 = 140G_3$$

As g_2^3 is modular of weight $4*3 = 12$ and g_3^2 of weight $6*2 = 12$, Δ is modular of weight 12.

Now, using $G_2(\infty) = 2\zeta(4) = \frac{\pi^4}{45}$ and $G_3(\infty) = 2\zeta(6) = \frac{2\pi^4}{945}$, we get $\Delta(\infty) = \left(\frac{4\pi^4}{3}\right)^3 - \left(\frac{8\pi^4}{27}\right)^2 = 0$.

1.3 Cusp Forms

A function $f : \mathbb{H} \to \mathbb{C}$ that is a modular form may in addition be a cusp form, if $f(\infty) = 0$. We will denote the space of modular cusp forms of weight $2k$ over $\mathbb{C}$ by M_k^0.

It is useful to note $G_k(\infty) = \sum_{n\in\mathbb{N}^*} \frac{2}{n^{2k}} > 2$ and in particular, $G_k(\infty) \neq 0$, so G_k are *not* cusp forms for any k. As we have shown it before, $\Delta(\infty) = 0$, so Δ is a modular cusp form of weight 12, so $\Delta \in M_6^0$. Using tools from complex analysis, we can prove that Δ has only one zero (at infinity), which has order one(**?** , p.88).

We have the following relation:

Theorem 1. *$M_k \cong M_k^0 \oplus \mathbb{C}.G_k \qquad \forall k \geq 2$.(**?** , p.88)*

Proof. We let $\Phi : M_k \to \mathbb{C}$ such that if $f \in M_k$, $\Phi(f) = f(\infty)$.

Now, we have $\mathrm{Ker}(\Phi) = M_k^0$, therefore, by the 1st Isomorphism Theorem, $M_k/M_k^0 \cong \mathrm{Im}(\Phi) \subseteq \mathbb{C}$.

Note that $G_k \in M_k$, and $G_k(\infty) = \sum_{n\in\mathbb{Z}^*} \frac{1}{n^{2k}} \neq 0$, so $G_k \notin M_k^0$. As $G_k \neq 0$, $\dim(M_k/M_k^0) \geq 1$ and $\mathrm{Im}(\Phi) = \mathbb{C}$. Thus, $G_k \in M_k$
M_k^0

Finally, we have $M_k \cong M_k^0 \oplus \mathbb{C}.G_k$ if $k \geq 2$. (The above argument fails for $k < 2$ as G_k is not well defined any more.) □

Therefore, the dimensions of M_k and M_k^0 are closely linked.

1.4 Dimensions of Spaces of Modular Forms

The fact that multiplying two modular forms gives a function that remains modular yields that we may map a set of modular forms to an other.

Theorem 2. *$M_{k-6} \cong M_k^0$.(**?** , p.88)*

Proof. We let $\Phi : M_{k-6} \to M_k^0$ such that if $f \in M_k$, $\Phi(f)(z) = \Delta(z)f(z)$.

This is well defined as if f has weight $2(k-6)$, $\Delta.f$ has weight $2k$ since Δ has weight 12. As Δ is a cusp from, $\Delta.f$ will also be a cusp form.

From definition, Φ is clearly homomorphic.

Now, if $g \in M_k^0$, we may define $\Psi : M_k^0 \to M_{k-6}$ such that $\Psi(g)(z) = g(z)/\Delta(z)$

This is well defined as if g has weight $2k$, $\Delta.f$ has weight $2k$ since Δ has weight 12. As Δ is a cusp from, $\Delta.f$ will also be a cusp form.

This is well defined as Δ has only one zero, at infinity, where g also has a zero (as g is a cusp form). The weights agree again as well.

It is then easy to remark that $\Psi = \Phi^{-1}$. So Φ is bijective, and thus isomorphic.

Finally, we have $M_{k-6} \cong M_k^0$. $\square$

This theorem, combined with the previous one is very powerful: it shows that there must be a pattern (of 6) in the sequence of dimensions $\dim(M_k)$ and $\dim(M_k^0)$ for $k \geq 2$. We have $M_k \cong M_k^0 \oplus \mathbb{C}.G_k \cong M_{k-6} \oplus \mathbb{C}.G_k$, so $\dim(M_k) = \dim(M_{k-6}) + 1$ when $k \geq 2$. Thus, if we compute the dimensions of M_0, M_1, M_2, M_3, M_4, M_5, we can extrapolate dimensions of M_k and M_k^0 for all k.

Using complex analysis techniques again, we have:

- $\dim(M_k) = 0 \quad k < 0$
- $\dim(M_1) = 0$
- $\dim(M_0) = \dim(M_2) = \dim(M_3) = \dim(M_4) = \dim(M_5) = 1$

In the case $k = 0$, $\dim(M_0) = 1$. As $f(z) = 1$ is clearly a modular from of weight 0, $\{1\}$ is a basis for M_0. We deduce $\dim(M_k^0) = 0$ as 1 is clearly not a cusp form. In the case $k = 1$, $\dim(M_1) = 0$, which makes $\dim(M_1^0) = 0$ automatically. (Cases $k < 0$ are similar to $k = 1$.)

Other cases may be derived directly from the relations (using induction to get general formulas), and we obtain:

Space	$k < 0$	$k \geq 0,\ k \equiv 1 \mod 6$	$k \geq 0,\ k \not\equiv 1 \mod 6$
$\dim(M_k)$	0	$\lfloor k/6 \rfloor$	$\lfloor k/6 \rfloor + 1$
$\dim(M_k^0)$	0	$\max\{0, \lfloor k/6 \rfloor - 1\}$	$\lfloor k/6 \rfloor$

Note that the max is taken only to avoid negative dimensions.

1.5 Fourier Expansion

1.5.1 Definition

To study such function, we use Fourier Expansion. In the case of f being a modular form of weight $2k$, a Fourier Expansion is a representation of f as a power series of $e^{2\pi i n z}$ i.e.

$$f(z) = \sum_{n \in \mathbb{Z}} a_n(f) e^{2\pi i n z}.$$

We usually denote $q = e^{2\pi i z}$ so that $q^n = e^{2\pi i n z}$ and the Fourier expansion of f become

$$f(q) = \sum_{n \in \mathbb{Z}} a_n(f) q^n.$$

When in this form, we may as well call it the q expansion.

1.5.2 Typical Modular Forms Fourier Expansion

Fourier Expansions of G_k The modular forms G_k have the following q expansion(**?** , p.92):

$$G_k(q) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

with $\sigma_s(n) = \sum_{d|n} d^s$, for $k \geq 4$.

Fourier Expansion of Δ We also have(**?** , p.95):

$$\Delta(q) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

1.6 A Basis for Modular Forms

The set of modular forms that are weight $2k$ in fact form a vector space (we can add modular forms together, and multiply them with a constant) over the complex numbers. One may ask then a basis for this vector space.

We would like to find a basis for each set M_k. It turns out that the modular forms G_2 and G_3 introduced before in fact generate a basis for all M_k. It is not obvious and may in fact seems wrong at a first stage: G_2 and G_3 are modular forms of weight 4 and 6, whereas M_k in general have modular forms of weight $2k$. However, by taking combinations of G_2 and G_3, we may obtain modular forms of any weight $2k$. It is important to remember that when multiplied, the weight of modular forms add up.

Theorem 3. *The set $S = \{G_2^a G_3^b | a, b \in \mathbb{N}, 2a + 3b = k\}$[1] is a basis for M_k.(**?** , Theorem 2.17)*

Proof. Of course, the cases when $\dim(M_k) = 0$ (for $k < 0$ and $k = 1$) are trivial, as the basis is empty, and $2a + 3b = k$ has no solution for $a, b \in \mathbb{N}$.

To show S is a basis, we need it to span M_k and to be linearly independent.

We start with spanning, and we proceed by induction on k, with step 6.

As $\dim(M_k) = 1$ for $k = 0, 2, 3, 4, 5, 7$, and the equation $2a + 3b = k$ has exactly one solution for $a, b \in \mathbb{N}$ (namely $(a, b) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (2, 1)$), S has only one element, which must be the basis.

Now, for $k > 7$, take some $a, b \in \mathbb{N}$ such that $2a + 3b = k$. Let $f \in M_k$, and $g = G_2^a G_3^b \in M_k$. $g(\infty) \neq 0$ as none of G_2 or G_3 is a cusp form. So there must be a complex λ such that $f - \lambda g$ is a cusp form. Then $f - \lambda g \in M_k \cong M_{k-6}^0$ and we can find a $h \in M_{k-6}^*$ such that $h.\Delta = f - \lambda g$.

By induction, h must be a polynomial of G_2 and G_3; by definition, Δ is one as well (note that yet, we don't put any restriction on powers of G_2 and G_3, other then being positive integers). Therefore, $f = \Delta.h + \lambda g$ is a polynomial of G_2 and G_3. From the fact that $f \in M_k$ (i.e. f has weight $2k$), terms of f as a polynomial of G_2 and G_3 have the from $G_2^a G_3^b$ with $2a + 3b = k$.

We now want to show linear independence, we proceed by contradiction.

Suppose there is a non-trivial linear relation of terms $G_2^a G_3^b$. We can multiply it by suitable G_2 and G_3 so that all terms have the form $2a + 3b = k \equiv 0 \mod 12$. Then, we can divide all terms by G_3^2, witch gives us that there is a polynomial for which ${G_2^3}/{G_3^2}$ is a root. In particular, this polynomial is constant when ${G_2^3}/{G_3^2}$ is plugged. This contradicts the fact that q expansion of ${G_2^3}/{G_3^2}$ is not constant. ☐

This set of makes to be a basis, and one may even find it pleasant: given the two modular forms G_2 and G_3, this set generates all the modular forms of weight $2k$ that we could think of, if we only knew these two modular forms.

1.7 Hecke Operators

We define the Hecke operators for a modular form f as follows(**?** , p.100):

$$T(n)f(z) = n^{2k-1} \sum_{a \geq 1,\, ad=n,\, 0 \leq b < d} d^{-2k} f\left(\frac{az+b}{d}\right)$$

with $n \in \mathbb{N}$.

We can check that $T(n)f$ is modular if f is (as the sum of modular forms).

[1] The set of naturals $\mathbb{N}$ is taken to start from 0.

We may as well write $T(n)f$ as a Fourier Expansion of $q = e^{2\pi iz}$ as follows(**?** , p.100):

$$T(n)f(z) = \sum_{m\in\mathbb{Z}} \gamma(m)q^m \quad \text{with} \quad \gamma(z) = \sum_{a|(n,m),\, a\geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right)$$

$$\text{For modular forms } f \text{ s.t. } f(z) = \sum_{n\in\mathbb{Z}} \alpha(n)q^n$$

References