Modular forms modulo 2

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Abstract

We are interested in Modular forms modulo 2, and computing thing about it. [temporary abstract]

Key words that should appear: Modular forms; Mod 2; Duality of definitions; Governing fields; Frobenian map?; Exact computations;

Contents

julia

```julia
function f(t::Int)
        return pi * t
end
# works?
a = "ok"
b='l'
```

python

```python
def f(t):
        return pi * t
# works?
a = "ok"
b='l'
```

1 Numerics

1.1 Finding coefficients of Hecke operators

It is important to make the program as fast as possible. Indeed, the faster the program goes, the more data it will generate (within the same amount of time). This data will be used for numerical analysis and we will also use it for interpretation. Therefore, with more data, we have more knowledge, and we can make smarter guesses.

There are two main ways to make a program faster: use a better algorithm, or use a faster implementation. A better algorithm means, for example, test factors only up to square root (in the case of primality a test). A better implementation simply means optimisation inside the computer (i.e. on operations that are made, types that are used...). We will try to optimise both.

1.1.1 Choice of implementation

As explained above, investigations on which tool will be the more suitable for the computations is an important part. Of course, the best would be to find a programming language that can already deal with modular forms modulo two. Unfortunately, this (yet) doesn't exist. There are packages that have modular forms implemented, but none with modular forms modulo two specifically. The goal of looking at modulo two is to conclude more than what we know in general. So using what has already been done in general to make computations modulo two won't give any thing interesting.

Now we realize that there is no other way than just creating a package for modular forms modulo two on our own. In fact, this is what we will do later, but before, we want to determine the tools to build this package. Modular forms modulo 2 come from maths, so it makes sense to use a high level programming language.

1.2 (Very) General Method

We want to find the coefficients a_{ij} such that

$$\sum_{i,j} a_{ij} T_3^i T_5^j = T_p$$

(with $a_{ij} \in \mathbb{F}_2$).

Let $k \geq 1$ an integer. Then there exists an integer $N(k) > 0$ such that, for all pairs of non-negative integers (i,j) with $i + j \geq N(k)$, we have $T_3^i T_5^j | \Delta^k = 0$.

This allows us to write:

$$\sum_{i+j<N(k)} a_{ij} T_3^i T_5^j | \Delta^k = T_p | \Delta^k \qquad (*)$$

Now, suppose that we want to calculate the table of the $a_{ij}(p)$ for $p \in \mathbb{P}$:

1. Take an odd power for Δ (say k, we usually start with the smallest: 1 and the increase gradually)
2. Plug Δ^k in the equation above, ie:
3. Calculate $T_3^i T_5^j|\Delta^k \forall i+j < N(k)$
4. Calculate $T_p|\Delta^k \forall i+j < N(k)$
5. Equate both sides of $(*)$, if not zero (which unfortunately happens often), use the equation to deduce $a_{ij}(p)$

[How much of the algorithm is there? too much? too little? I could develop much more on how everything is calculated: how I go back and forward between q and Δ representations of modular forms to both be efficient in calculations and catch up the error in numerical approximation, what techniques are used for speed, argue the implementation choices, describe how the code is split, etc... I could write at least pages on all of that, but is it the point of a math paper?]

A Hecke Operators

A.1 Primes Hecke Operators

T_p

A.2 Powers of Hecke Operators

$T_3^i T_5^j$

B Speed Comparison

C ModularFormsModuloTwo.jl

D Other Programs

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