



UNIVERSITY COLLEGE OF LONDON

DEPARTMENT OF MATHEMATICS

Modular Forms Modulo 2: Governing Fields for the Hecke Algebra

Author
Paul DUBOIS

Supervisor
Djordjo MILOVIC

March 12, 2020

Modular Forms Modulo 2

Abstract

Modular forms are functions over the upper half plane that play a prominent role in mathematics. Studying their properties over finite fields can yield useful information about the arithmetic of their Fourier coefficients. In this paper, we study modular forms over the smallest field of all, containing only an additive identity and a multiplicative identity, that is, \mathbb{F}_2 .

After introducing modular forms over the complex numbers, we will show how to reduce modular forms modulo 2. In this setting, modular forms may be seen as polynomials in the modular discriminant Δ or as power series in $q = \exp(2\pi iz)$. The duality between these two representations is crucial and we will use it in the development of a new computing technique: *exact computations*.

We will then concentrate on the study of prime Hecke operators T_p acting on modular forms modulo 2. These operators are nilpotent, with respect to a given modular form. The order of nilpotency is known, and we will discuss it. A Hecke operator T_p at an odd prime p acting on odd powers of Δ may also be expressed as power series in T_3 and T_5 .

The coefficient of $T_3^i T_5^j$ of this power series is denoted by $a_{ij}(p)$, and it will play a particularly important role in our study. The map $p \mapsto a_{ij}(p)$ is in fact Frobenian. We will try to discover new governing fields for these maps.

In the last section, we will discuss the computing techniques developed to achieve the various computations performed.

Acknowledgment

I would like to thank my supervisor, Djordjo Milovic for guiding and supporting me over the year. I would also like to thank Reuben Omolu, Pierre and Cécile Dubois for their time and advices.

Contents

0	Introduction	6
1	Modular Forms	7
1.1	Modular Forms of Level 1	7
1.2	Typical Modular Forms	7
1.2.1	Eisenstein Series G_k	7
1.2.2	The Modular Discriminant Δ	8
1.3	Cusp Forms	8
1.4	Dimensions of Spaces of Modular Forms	9
1.5	Fourier Expansion	10
1.5.1	Definition	10
1.5.2	Typical Modular Forms Fourier Expansion	10
1.6	A Basis for Modular Forms	10
1.7	Hecke Operators	11
2	Modular Forms Modulo Two	12
2.1	Strategy to Reduce Modulo Two	12
2.2	Integral Basis	12
2.2.1	Normalisation of Typical Modular Forms	12
2.2.2	Miller Basis	13
2.3	Basis Modulo Two	16
2.3.1	Reduced Modular Forms	16
2.3.2	Reduced Basis	17
2.4	Space of Modular Forms Modulo Two	17
2.4.1	Weights of Modular Forms Modulo Two	17
2.4.2	Powers of the Modular Discriminant Δ	18
2.4.3	The Space \mathcal{F}	19
2.4.4	Duality Between Δ and q	20
2.5	Hecke Operators Modulo Two	20
2.5.1	Reduction Modulo Two	20
2.5.2	Basic Properties	21
2.5.3	Nilpotency	23
2.5.4	Expression for $T_p \Delta^k$	24
2.5.5	Examples (for Small Powers of Δ)	24
2.5.6	Table of Hecke Operators	26
2.6	Nilpotency Order	27
2.6.1	Introduction	27
2.6.2	Code and Height of Natural Numbers	28
2.6.3	Action of T_3 and T_5	32
2.6.4	Formula for the Order of Nilpotency	33
3	Hecke Algebra	35
3.1	Definition	35
3.2	Basic Properties	35
3.3	As Generated by T_3 and T_5	36
3.4	Expansion of T_p as Series of T_3 and T_5	37

4	Frobenian Elements	39
4.1	Context	39
4.1.1	General	39
4.1.2	Residue Fields Extensions	40
4.1.3	Norms of Ideals	40
4.1.4	Galois Extension Simplifications	40
4.1.5	Unramified Prime Simplifications	40
4.2	The Frobenius Element	40
4.2.1	Definition	40
4.2.2	Examples	41
4.2.3	Behaviour in Towers of Fields	43
4.3	The Chebotarev's Density Theorem	44
4.3.1	Motivations	44
4.3.2	Notions of Density	45
4.3.3	Statement	45
4.3.4	Example	45
4.3.5	Special Case	46
4.4	The Dirichlet's Density Theorem	46
4.4.1	Statement	46
4.4.2	Link with Chebotarev	46
4.4.3	Example	46
5	Frobenian Maps and Governing Fields	47
5.1	Frobenian Maps	47
5.2	Governing Fields	47
5.2.1	Basics	47
5.2.2	Known Governing Fields	49
5.2.3	Research of Governing Fields	49
5.2.4	New Governing Fields	50
6	Numerics	53
6.1	High Performance Computations	53
6.1.1	Algorithm Optimisation	53
6.1.2	Implementation Approach	54
6.1.3	Choice of Implementation	54
6.2	Creating the library	56
6.2.1	Main Module	56
6.2.2	Sub Module	58
6.2.3	Open-Source	58
6.2.4	Official	59
6.2.5	Online Documentation	59
6.3	Finding Coefficients $a_{ij}(p)$	59
6.3.1	Strategy	59
6.3.2	Algorithm	59
6.3.3	Computations Limitations	60
6.3.4	Results	61
6.4	Finding Governing Fields	62

6.4.1	Computations Strategy	62
6.4.2	Reliability	63
6.5	Probabilistic Analysis	64
6.5.1	Motivation	64
6.5.2	Assumptions	64
6.5.3	Probability of Mistake in the Governing Fields Found	65
6.5.4	Limitation of Probabilistic Analysis	65
A	Hecke Operators	1
A.1	Primes Hecke Operators	1
A.2	Powers of Hecke Operators	2
A.3	Behaviour of Code of Integers	3
A.4	Behaviour of h on Various Scales	4
A.5	Expansions of T_p as Series of T_3 and T_5	9
B	Chebotarev Example	10
C	Speed Comparison	10
C.1	Pure Python	10
C.2	NumPy Python	11
C.3	Dense Julia	11
C.4	Sparse Python	12
C.5	Sparse Julia	12
D	ModularFormsModuloTwo.jl	13
D.1	Files Tree	13
D.2	Files details	13
D.2.1	Main Sources	13
D.2.2	Data Submodule	26
E	T_p as series of T_3 and T_5	30
E.1	$a_{ij}(p)$ Computations	30
E.2	$a_{ij}(p)$ Graphs	32
F	Governing Fields	32
F.1	Frobenian Elements Computations	32
F.2	Analysis of Extensions	34
G	Governing Fields Results	36
G.1	a_{0k}	36
G.1.1	a_{01}	36
G.1.2	a_{02}	36
G.1.3	a_{03}	36
G.1.4	a_{04}	37
G.1.5	a_{05}	37
G.1.6	a_{06}	37
G.1.7	a_{07}	37
G.1.8	a_{08}	38
G.2	a_{kk}	39

	G.2.1	a_{11}	39
G.3		a_{k0}	39
	G.3.1	a_{10}	39
	G.3.2	a_{20}	40
	G.3.3	a_{30}	40
	G.3.4	a_{40}	40
	G.3.5	a_{50}	40
	G.3.6	a_{60}	41
	G.3.7	a_{70}	41
	G.3.8	a_{80}	41

0 Introduction

Following Nicolas and Serre [2012a], we define a *modular form modulo 2 of level 1* to be a polynomial in the power series

$$\Delta(q) = \sum_{n=0}^{\infty} q^{(2n+1)^2} \in \mathbb{F}_2[[q]].$$

After we introduce the classical theory of modular forms as functions on the upper half-plane in Section 1, we will provide justification for this etymology by constructing certain suitable integral bases for the spaces of modular forms of level 1.

We will then be interested in the reduction of Hecke operators modulo 2. Our interest will be limited to Hecke operators for odd prime numbers (T_p). We will prove that this reduction is never trivial as a consequence of a new property that will be proven in this paper. The main property of Hecke operators on modular forms modulo 2 is that, for given a modular form, they are all nilpotent. Finding the order of nilpotency of a modular form is a solved problem, that we will go through.

The main object of our study is the \mathbb{F}_2 -algebra A , generated by Hecke operators T_p , p odd, acting on the \mathbb{F}_2 -space spanned by odd powers of Δ . The Hecke algebra A is isomorphic to the power series ring

$$\mathbb{F}_2[[T_3, T_5]].$$

A key feature of this presentation of A is that the coefficients of the power series corresponding to T_p are Frobenian functions of p . More precisely, for each odd prime p , let us write

$$T_p = \sum_{i+j \geq 1} a_{ij}(p) T_3^i T_5^j$$

Then, as a consequence of a theorem of Bellaïche [2012], for each fixed i and j , there is a normal extension M_{ij}/\mathbb{Q} such that the Frobenius conjugacy class of p in M_{ij}/\mathbb{Q} determines the value $a_{ij}(p) \in \mathbb{F}_2$. For instance, in Nicolas and Serre [2012a], they claim that one can take

$$M_{01} = \mathbb{Q}(\zeta_8) \quad \text{and} \quad M_{02} = \mathbb{Q}(\zeta_8, \sqrt[4]{2})$$

We carry out with various high-performance computations techniques. The structure of modular forms modulo 2 will in fact lead to a new computing technique (so-called "exact computations"). This allows us to find new results, for instance, we have computed that

$$M_{03} = \mathbb{Q} \left(\mu, \sqrt[4]{2}, \sqrt{-\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248}} + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656} \right).$$

The new data calculated will lead us to the following new conjecture: 5.2.4

Conjecture (On Diagonal Governing Groups). *For all $k \in \mathbb{N}^*$, there exists a field M_{0k} such that M_{0k} is a governing field for a_{0k} , and $G_{0k} = \text{Gal}(M_{0k}/\mathbb{Q})$ is dihedral. For all $k \in \mathbb{N}^*$, there exists a field M_{k0} such that M_{k0} is a governing field for a_{k0} , and G_{k0} is dihedral. Moreover $M_{k0} \neq M_{0k}$ in general, but $G_{k0} = G_{0k}$.*

1 Modular Forms

1.1 Modular Forms of Level 1

Let \mathbb{H} denote the *upper-half plane*, that is,

$$\mathbb{H} = \{z = x + yi \in \mathbb{C} \mid y > 0\}.$$

We say that a function $f : \mathbb{H} \rightarrow \mathbb{C}$ is *weakly modular* of *weight $2k$* if f is meromorphic and

$$f(z) = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).$$

The group $\text{SL}_2(\mathbb{Z})$ of invertible 2-by-2 matrices over \mathbb{Z} with is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

see [Conrad, 2020, p.1-2]. From this property, we can derive an alternative definition of weakly modular functions: f is weakly modular of weight $2k$ if f is meromorphic and

$$f(z + 1) = f(z) \quad \text{and} \quad f(-1/z) = z^k f(z),$$

for all $z \in \mathbb{C}$. Moreover, we define a function $f : \mathbb{H} \rightarrow \mathbb{C}$ to be *modular* of weight $2k$ if f is holomorphic and weakly modular. Lastly, we say that a function $f : \mathbb{H} \rightarrow \mathbb{C}$ is a *modular form* of weight $2k$ if it is modular and holomorphic at ∞ , that is, $f(1/z)$ is holomorphic at $z = 0$.

It is straightforward to check, using the above definition, that the set of modular forms of weight $2k$ is closed under addition and multiplication by complex scalars. More precisely:

- If f_1 and f_2 are modular forms of weight $2k$, then $f_1 + f_2 : z \mapsto f_1(z) + f_2(z)$ is modular of weight $2k$ as well.
- Similarly, if $\lambda \in \mathbb{C}$ and f is a modular form of weight $2k$, then so is $\lambda \cdot f : z \mapsto \lambda f(z)$.

Therefore, modular forms of weight $2k$ over \mathbb{C} form a vector space. We denote it M_k .

It is also possible to multiply modular forms, in which case the weights are additive: If f_1 and f_2 are modular forms of respective weights $2k_1$ and $2k_2$, then $f_1 \cdot f_2 : z \mapsto f_1(z)f_2(z)$ is modular of weight $2k_1 + 2k_2$.

We deduce that we can take powers of modular forms, and the weight is then multiplied by the exponent: if $f(z)$ is modular of weight $2k$, then $f^n(z)$ is modular of weight $2k \cdot n$ (with $n \in \mathbb{N}^1$).

1.2 Typical Modular Forms

1.2.1 Eisenstein Series G_k

The most famous class of modular forms is probably the *Eisenstein series*, usually denoted G_k . We define them as follows [Stein, 2007, Examples of Modular Forms]:

$$G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz + n)^{2k}}$$

for $k \geq 2$.

¹The set of naturals \mathbb{N} is taken to start from 0 in this paper.

It is easy to check that G_k are modular of weight $2k$ [Stein, 2007, Proposition 2.1], as:

$$G_k(z+1) = G_k(z)$$

(using $(m, n+m) \mapsto (m, n)$, an invertible map) and

$$G_k(-1/z) = z^k G_k(z)$$

(using $(m, -n) \mapsto (m, n)$, an invertible map). It is pleasant to remark that [Stein, 2007, Proposition 2.2]

$$G_k(\infty) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^{2k}} = 2\zeta(2k),$$

where $\zeta(k)$ is Riemann zeta function. The values of this function are well-known on positive even numbers, and we deduce [Lennart Rade, 2013, p.194] that:

$$G_k(\infty) = 2\zeta(2k) = \frac{(2\pi)^{2k}}{(2k)!} B_k,$$

where $B_k = (-1)^{k+1} b_{2k}$ and b_k are Bernoulli numbers.

1.2.2 The Modular Discriminant Δ

We will be interested in one main modular form in the rest of this article: the *modular discriminant* Δ . We define Δ in terms of G_k as follows [Serre, 1973, p.84]:

$$\Delta = \left(\frac{1}{(2\pi)^{12}} \right) (g_2^3 - 27g_3^2) \in M_6 \quad \text{with } g_2 = 40G_2 \text{ and } g_3 = 140G_3$$

As g_2^3 is modular of weight $4 \cdot 3 = 12$ and g_3^2 of weight $6 \cdot 2 = 12$, Δ is modular of weight 12. Multiplying by the scalar $(1/(2\pi)^{12})$ doesn't change the weight of the modular form and it will be useful later for normalization purposes.

Now, using $G_2(\infty) = 2\zeta(4) = \frac{\pi^4}{45}$ and $G_3(\infty) = 2\zeta(6) = \frac{2\pi^6}{945}$, we get

$$\Delta(\infty) = \left(\frac{1}{(2\pi)^{12}} \right) \left[\left(\frac{4\pi^4}{3} \right)^3 - \left(\frac{8\pi^6}{27} \right)^2 \right] = 0$$

so Δ has a zero at infinity.

1.3 Cusp Forms

A function $f : \mathbb{H} \rightarrow \mathbb{C}$ that is a modular form may in addition be a *cusp form*, if $f(\infty) = 0$. We will denote the *space of modular cusp forms* of weight $2k$ over \mathbb{C} by M_k^0 . This space of cusp forms of weight $2k$ is a subset of the space of modular forms of weight $2k$.

It is useful to note $G_k(\infty) = \sum_{n \in \mathbb{N}^*} \frac{2}{n^{2k}} > 2$ and in particular, $G_k(\infty) \neq 0$, so G_k are *not* cusp forms for any k . As we have shown it before, $\Delta(\infty) = 0$, so Δ is a modular cusp form of weight 12, so $\Delta \in M_6^0$. Using tools from complex analysis, we can prove that Δ has only one zero (at infinity), which has order one [Serre, 1973, p.88].

We have the following relation:

Theorem 1. [Serre, 1973, p.88]: $M_k \cong M_k^0 \oplus \mathbb{C} \cdot G_k$ for all $k \geq 2$

Proof. We let $\Phi : M_k \rightarrow \mathbb{C}$ such that if $f \in M_k$, $\Phi(f) = f(\infty)$. Now, we have $\text{Ker}(\Phi) = M_k^0$, therefore, by the 1st Isomorphism Theorem, $M_k/M_k^0 \cong \text{Im}(\Phi) \subseteq \mathbb{C}$. Note that $G_k \in M_k$, and $G_k(\infty) = \sum_{n \in \mathbb{Z}^*} \frac{1}{n^{2k}} \neq 0$, so $G_k \notin M_k^0$. As $G_k \neq 0$, $\dim(M_k/M_k^0) \geq 1$ and $\text{Im}(\Phi) = \mathbb{C}$. Thus, $G_k \in M_k \setminus M_k^0$.

Finally, we have $M_k \cong M_k^0 \oplus \mathbb{C} \cdot G_k$ if $k \geq 2$. (The above argument fails for $k < 2$ as G_k is not well defined anymore.) \square

Therefore, the dimensions of M_k and M_k^0 are closely linked.

1.4 Dimensions of Spaces of Modular Forms

The fact that multiplying two modular forms gives a function that remains modular yields that we may map one set of modular forms to another.

Theorem 2. [Serre, 1973, p.88]. We have $M_{k-6} \cong M_k^0$.

Proof. We define $\Phi : M_{k-6} \rightarrow M_k^0$ by setting,

$$\Phi(f)(z) = \Delta(z)f(z) \quad \text{for } f \in M_{k-6}.$$

This is well defined as if f has weight $2(k-6)$, $\Delta \cdot f$ has weight $2k$ since Δ has weight 12. As Δ is a cusp form, $\Delta \cdot f$ will also be a cusp form. From the definition, Φ is a linear map.

Now, if $g \in M_k^0$, we may define $\Psi : M_k^0 \rightarrow M_{k-6}$ by setting

$$\Psi(g)(z) = g(z)/\Delta(z) \quad \text{for } g \in M_k^0.$$

This is well defined as if g has weight $2k$, $\Delta \cdot f$ has weight $2k$ since Δ has weight 12. This is well defined as Δ has only one zero, at infinity, where g also has a zero (as g is a cusp form). The weights agree again as well. We remark that $\Psi = \Phi^{-1}$. So Φ is bijective, and thus an isomorphism. Finally, we have $M_{k-6} \cong M_k^0$. \square

This theorem, combined with the previous one is very powerful: it shows that there must be a pattern in the sequence of dimensions $\dim(M_k)$ and $\dim(M_k^0)$ for $k \geq 2$. We have $M_k \cong M_k^0 \oplus \mathbb{C} \cdot G_k \cong M_{k-6} \oplus \mathbb{C} \cdot G_k$, so $\dim(M_k) = \dim(M_{k-6}) + 1$ when $k \geq 2$. Thus, if we compute the dimensions of $M_0, M_1, M_2, M_3, M_4, M_5$, we can extrapolate dimensions of M_k and M_k^0 for all k .

Using complex analysis techniques again, we have:

- $\dim(M_k) = 0$ for $k < 0$
- $\dim(M_1) = 0$
- $\dim(M_0) = \dim(M_2) = \dim(M_3) = \dim(M_4) = \dim(M_5) = 1$

In the case $k = 0$, $\dim(M_0) = 1$. As $f(z) = 1$ is clearly a modular form of weight 0, $\{1\}$ is a basis for M_0 . We deduce $\dim(M_k^0) = 0$ as 1 is clearly not a cusp form. In the case $k = 1$, $\dim(M_1) = 0$, which makes $\dim(M_1^0) = 0$ automatically. (Cases $k < 0$ are similar to $k = 1$.)

Other cases may be derived directly from the relations (using induction to get general formulas), and we obtain:

Space	$k < 0$	$k \geq 0, k \equiv 1 \pmod{6}$	$k \geq 0, k \not\equiv 1 \pmod{6}$
$\dim(M_k)$	0	$\lfloor k/6 \rfloor$	$\lfloor k/6 \rfloor + 1$
$\dim(M_k^0)$	0	$\max\{0, \lfloor k/6 \rfloor - 1\}$	$\lfloor k/6 \rfloor$

Note that the maximum is taken only to avoid negative dimensions.

1.5 Fourier Expansion

1.5.1 Definition

To study modular forms, we can use Fourier expansion. As a modular form f satisfies $f(z) = f(z+1)$ for all $z \in \mathbb{C}$, we can express the Fourier expansion in terms of $q = e^{2\pi iz}$. Thus, in the case of f being a modular form of weight $2k$, a *Fourier expansion* is a representation of f as a power series of $e^{2\pi iz}$ i.e.

$$f(z) = \sum_{n \in \mathbb{Z}} a_n(f) e^{2\pi i n z}.$$

We usually denote $q = e^{2\pi iz}$ so that $q^n = e^{2\pi i n z}$ and the Fourier expansion of f become

$$f(q) = \sum_{n \in \mathbb{Z}} a_n(f) q^n.$$

When in this form, we may as well call it the *q-expansion*.

Asymptotic Notation The asymptotic behaviour of a function may be expressed in terms of another. This is done via the *big-O notation* or *asymptotic notation*. We write $f(x) = \mathcal{O}(g(x))$ as $x \mapsto a$ if there exist $\delta, M \in \mathbb{R}$ such that $|f(x)| \leq M|g(x)|$ when $0 < |x - a| \leq \delta$. For example, if $a = 0$ (which will always be the case in here), we have that if $\alpha \geq \beta$, then $x^\alpha = \mathcal{O}(x^\beta)$.

It will sometimes be useful to write the Fourier expansions only up to some coefficient. For the q -series up to m , we will write $\mathcal{O}(q^m)$. That is, if

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n,$$

then

$$f(q) = \sum_{n=0}^{m-1} c(n) q^n + \mathcal{O}(q^m).$$

1.5.2 Typical Modular Forms Fourier Expansion

Fourier Expansions of G_k The modular forms G_k have the following q -expansion [Serre, 1973, p.92]:

$$G_k(q) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where σ_d is the generalized divisor function defined by:

$$\sigma_d(n) = \sum_{m|n} m^d.$$

Fourier Expansion of Δ We also have [Serre, 1973, p.95]:

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

1.6 A Basis for Modular Forms

We established that M_k form a vector space over the complex numbers. One may ask for a basis of this vector space.

We would like to find a basis for each vector space M_k . It turns out that the modular forms G_2 and G_3 introduced before in fact generate a basis for all M_k . It is not obvious and may in fact seem

wrong at a first glance: G_2 and G_3 are modular forms of weight 4 and 6, whereas M_k in general have modular forms of weight $2k$. However, by taking combinations of G_2 and G_3 , we may obtain modular forms of any weight $2k$. It is important to remember that when multiplied, the weight of modular forms add up.

Theorem 3. [Stein, 2007, Theorem 2.17]. *The set $S = \{G_2^a G_3^b \mid a, b \in \mathbb{N}^2, 2a + 3b = k\}$ is a basis for M_k .*

Proof. Of course, the cases when $\dim(M_k) = 0$ (for $k < 0$ and $k = 1$) are trivial, as the basis is empty, and $2a + 3b = k$ has no solution for $a, b \in \mathbb{N}$. To show S is a basis, we need it to span M_k and to be linearly independent.

We start with spanning, and we proceed by induction on k , with step 6. As $\dim(M_k) = 1$ for $k = 0, 2, 3, 4, 5, 7$, and the equation $2a + 3b = k$ has exactly one solution for $a, b \in \mathbb{N}$ (namely $(a, b) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (2, 1)$), S has only one element, which must be the basis. Now, for $k > 7$, take some $a, b \in \mathbb{N}$ such that $2a + 3b = k$. Let $f \in M_k$, and $g = G_2^a G_3^b \in M_k$. We have $g(\infty) \neq 0$ as none of G_2 or G_3 is a cusp form. So there must be a complex λ such that $f - \lambda g$ is a cusp form. Then $f - \lambda g \in M_k \cong M_{k-6}^0$ and we can find a $h \in M_{k-6}^*$ such that $h \cdot \Delta = f - \lambda g$.

By induction, h must be a polynomial of G_2 and G_3 ; by definition, Δ is one as well (note that as of yet, we haven't put any restriction on powers of G_2 and G_3 , other than being positive integers). Therefore, $f = \Delta \cdot h + \lambda g$ is a polynomial of G_2 and G_3 . From the fact that $f \in M_k$ (i.e. f has weight $2k$), terms of f as a polynomial of G_2 and G_3 have the form $G_2^a G_3^b$ with $2a + 3b = k$.

We now want to show linear independence, we proceed by contradiction. Suppose there is a non-trivial linear relation of terms $G_2^a G_3^b$. We can multiply it by suitable G_2 and G_3 so that all terms have the form $2a + 3b = k \equiv 0 \pmod{12}$. Then, we can divide all terms by G_3^2 , which gives us a polynomial for which G_2^3/G_3^2 is a root. In particular, this polynomial is constant when G_2^3/G_3^2 is plugged. This contradicts the fact that the q -expansion of G_2^3/G_3^2 is not constant. \square

This set is a basis. One may even find it pleasant: given the two modular forms G_2 and G_3 , we are able to generate all the modular forms of weight $2k$ that we could think of.

1.7 Hecke Operators

We define the *Hecke operators* for a modular form f as follows [Serre, 1973, p.100]:

$$T_n f(z) = n^{2k-1} \sum_{a \geq 1, ad=n, 0 \leq b < d} d^{-2k} f\left(\frac{az+b}{d}\right)$$

with $n \in \mathbb{N}$. We can check that $T_n f$ is modular if f is (as the sum of modular forms). We may as well write $T_n f$ as a q -expansion as follows [Serre, 1973, p.100]: if $f(z) = \sum_{n \in \mathbb{Z}} c(n) q^n$, then

$$T_n f(z) = \sum_{m \in \mathbb{Z}} \gamma(m) q^m \quad \text{with} \quad \gamma(z) = \sum_{a|(n,m), a \geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right).$$

²In this paper, $0 \in \mathbb{N}$, i.e. $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$.

2 Modular Forms Modulo Two

2.1 Strategy to Reduce Modulo Two

It is not trivial, at this point, as to why and how we can reduce modulo 2 modular forms, objects that have coefficients in \mathbb{C} . In general, reduction modulo a number is only possible with whole numbers (integers). We would like to reduce modulo 2 coefficients of the Fourier series for modular forms. But at the moment, they lie in \mathbb{C} .

In fact, we will introduce a new basis for the modular forms: the so called Miller Basis. The coefficients of all the forms in this basis are integers. It is then possible to consider the space of modular forms over \mathbb{Z} instead of \mathbb{C} . Once this is done, we will reduce all the newly integral coefficients modulo 2.

In this section, we will denote all objects reduced modulo 2 with an over-line:

- The modular form f once reduced will be denoted \overline{f} ,
- The coefficients of the q -expansion c will reduce to \overline{c} ,
- The Hecke operators T_n reduced will be denoted $\overline{T_n}$.

2.2 Integral Basis

2.2.1 Normalisation of Typical Modular Forms

Normalisation of Eisenstein series G_k We first recall the formula for the q -extension of G_k and the one for $\zeta(2k)$:

$$G_k(q) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

and

$$2\zeta(2k) = \frac{(2\pi)^{2k}}{(2k)!} B_k,$$

so overall:

$$G_k(q) = \frac{(2\pi)^{2k}}{(2k)!} B_k + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

We would like to normalize this series, so that the coefficients become integers, so that we can ultimately reduce them modulo 2. Right now, the coefficients are rational.

As we want to keep the series modular with same weight, the only tool we have to normalize the series is multiplication by a constant. The normalization is a crucial point: if we multiply by 2 all coefficients of a modular form that already lie in \mathbb{Z} , the reduction modulo 2 will always give zero.

First, let's normalize the series to have particular values on some coefficients of interest. There are two justified ways to do so: normalize to have constant (q^0) coefficient set to one, and to have q^1 coefficient set to one. We will introduce both: we define E_k be such that:

$$E_k \cdot 2\zeta(2k) = G_k,$$

so that

$$E_k = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

E_k then has constant coefficient set to 1.

We also define F_k be such that:

$$F_k \cdot \left(2 \frac{(2\pi i)^{2k}}{(2k-1)!} \right) = G_k,$$

so that

$$F_k = (-1)^k \frac{B_k}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

F_k then has q -coefficient set to 1 (as $\sigma_{2k-1}(1) = 1$).

Clearly, the coefficients of this expansion remain in \mathbb{Q} at least, and we will show that for some specific k , the coefficients lie in fact in \mathbb{Z} . Both F_k and E_k are interesting, but for our purpose (reducing modulo 2), we will use E_k . Note that E_k are normalized versions of Eisenstein series G_k , but in literature, both are called Eisenstein series; see [Shrivastava, 2017, p.6] for example.

The Modular Discriminant Δ Normalized Again, we recall the formula for q extension of Δ :

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Clearly, the coefficients in expansion of Δ are integers (which we can reduce modulo 2). This is the reason why we defined Δ with the $\frac{1}{(2\pi)^{12}}$ factor in front.

2.2.2 Miller Basis

Basis with Integral Coefficients (in Fourier Series) Applying normalization $G_k \mapsto E_k$ above for $k = 2, 3$, we get:

$$\begin{aligned} E_2 &= 1 + \frac{8}{B_2} \sum_{n=1}^{\infty} \sigma_3(n) q^n & B_2 &= \frac{1}{30} \\ &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \end{aligned}$$

and

$$\begin{aligned} E_3 &= 1 - \frac{12}{B_3} \sum_{n=1}^{\infty} \sigma_5(n) q^n & B_3 &= \frac{1}{42} \\ &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \end{aligned}$$

Now, we have shown that $\{G_2^a G_3^b \mid 2a + 3b = k\}$ is a basis for modular forms of weight $2k$ over the complex (see 1.6). As $E_2 = \lambda G_2$, $\lambda \in \mathbb{C}$ and $E_3 = \mu G_3$, $\mu \in \mathbb{C}$, we have that $\{E_2^a E_3^b \mid 2a + 3b = k\}$ remains a basis for M_k over \mathbb{C} .

It is clear, from the series, that coefficients of the q -expansion of both E_2 and E_3 are all integers. Thus, so are coefficients of combinations of E_2 and E_3 . Therefore, we have found a basis for M_k such that all elements in the basis have only integral coefficients in their q -expansion.

Miller Basis for M_k^0 This is a nice result, but we can in fact do better, by forcing the first coefficients to chosen values.

Theorem 4. *For the space of modular cusp forms M_k^0 , there exists a basis $\{f_1, \dots, f_r\}$ such that:*

- $f_i \in \mathbb{Z}[[q]]$
- $a_i^j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \forall 1 \leq i, j \leq r$
 where a_i^j is the coefficient of q^j in expansion of f_i .

This is commonly called the Miller basis for M_k^0 , as it was first introduced by Victor Saul Miller [1975].

Proof. We consider separately the following cases:

- For $k < 6$, $k = 7$, we have $\dim(M_k^0) = 0$. Thus, \emptyset is a basis which satisfies the Miller basis properties.
- For $k = 6$, we have $\dim(M_k^0) = 1$. Thus, $\{\Delta\}$ is a basis which satisfies the Miller basis properties.
- For $k \geq 7$, we let $r = \dim(M_k^0) \geq 1$. We then consider the set $\{g_j \mid 1 \leq j \leq r\}$ where

$$g_j = \Delta^j E_3^{2(d-j)+b} E_2^a,$$

with $2a + 3b \leq 7$, $2a + 3b \cong k \pmod{6}$ and $d = \frac{k-(2a+3b)}{6} \in \mathbb{N}$ (with $k \geq 7$). Note that a and b are unique unless $k \cong 0 \pmod{6}$. In which case, we use by convention $a = 0$, $b = 0$. As all E_2 , E_3 , and Δ have integral coefficients, g_j will as well.

We then look at the q -series

$$\Delta(q) = q + \mathcal{O}(k^2) \implies \Delta^j(q) = q^j + \mathcal{O}(k^{j+1}).$$

As we normalized, we have

$$E_2(q) = 1 + \mathcal{O}(q) \implies E_2^\alpha(q) = 1 + \mathcal{O}(q)$$

$$\text{and } E_3(q) = 1 + \mathcal{O}(q) \implies E_3^\alpha(q) = 1 + \mathcal{O}(q).$$

This gives:

$$g_j(q) = q^j + \mathcal{O}(q^{j+1}) \quad \forall 1 \leq j \leq r.$$

Therefore, $\{g_j \mid 1 \leq j \leq r\}$ is clearly a linearly independent set. By dimension argument, it also spans M_k^0 . Therefore, it forms a basis. Moreover, in this basis: $a_i^j = \delta_{ij}$ $i \leq j$.

Now, viewing Δ -coefficients of the modular forms $\{g_j\}$ as rows of a matrix, we can perform Gaussian elimination on them. We will obtain a basis $\{f_j \mid 1 \leq j \leq r\}$ such that: $a_i^j = \delta_{ij}$ for all $1 \leq i, j \leq r$. The coefficients will remain in \mathbb{Z} .

□

Extension to all M_k We already have a basis for M_k^0 , as $\dim(M_k) = \dim(M_k^0) + 1$ (over \mathbb{C}), we just need to adjoin one element of $M_k \setminus M_k^0$ to our basis. It was shown before that $\{E_2^a E_3^b \mid 2a + 3b = k\}$ is a basis for M_k with integral coefficients (see 2.2.2). One may see from the q -expansion that $E_2^a E_3^b = 1 + \mathcal{O}(q)$ so $E_2^a E_3^b \in M_k \setminus M_k^0$. Therefore, we can just add one element of $\{E_2^a E_3^b \mid 2a + 3b = k\}$ to the Miller basis, and use Gaussian elimination again. We get a basis for M_k of the form $\{f_j \mid 0 \leq j \leq r\}$ such that in this basis: $a_i^j = \delta_{ij}$ for all $0 \leq i, j \leq r$ (with $r = \dim(M_k^0)$ i.e. $r + 1 = \dim(M_k)$).

Miller Basis Examples

Miller basis for $k = 16$ We can calculate the Miller basis for $k = 16$: $k \cong 4 \pmod{12}$ so $a = 2$ and $b = 0$; $d = 2$. We put $g_1 = \Delta^1 E_3^2 E_2^2$, so:

$$\begin{aligned} g_1(q) &= \Delta(q) E_2^2(q) E_3^2(q) \\ &= [q - 24q^2 + 252q^3 + O(q^4)] \\ &\quad \cdot [1 + 240q + 2160q^2 + 6720q^3 + O(q^4)]^2 \\ &\quad \cdot [1 - 504q - 16632q^2 + 122976q^3 + O(q^4)]^2 \\ &= q - 552q^2 - 188244q^3 + O(q^4) \end{aligned}$$

and $g_2 = \Delta^2 E_3^0 E_2^2$, so:

$$\begin{aligned} g_2(q) &= \Delta^2(q) E_2^2(q) \\ &= [q - 24q^2 + 252q^3 + O(q^4)]^2 \\ &\quad \cdot [1 + 240q + 2160q^2 + 6720q^3 + O(q^4)]^2 \\ &= q^2 + 432q^3 + O(q^4) \end{aligned}$$

Then, $f_2 = g_2$ and $f_1 = g_1 + 552g_2$, so:

$$\begin{aligned} f_1(q) &= q - 552q^2 - 188244q^3 + O(q^4) + 552 \cdot [q^2 + 432q^3 + O(q^4)] \\ &= q + 50220q^3 + O(q^4) \\ f_2(q) &= q^2 + 432q^3 + O(q^4) \end{aligned}$$

Therefore, up to $O(q^4)$, $\{f_1, f_2\} = \{q + 50220q^3 + O(q^4), q^2 + 432q^3 + O(q^4)\}$ is a basis for M_{16}^0 .

To extend this base to M_k , we adjoint a term of the form $g_0 = E_2^a E_3^b$ where $2a + 3b = 16$. We pick $g_0 = E_2^8$, so:

$$\begin{aligned} g_0(q) &= E_2^8(q) \\ &= [1 + 240q + 2160q^2 + 6720q^3 + O(q^4)]^8 \\ &= 1 + 1920q + 1630080q^2 + 803228160q^3 + O(q^4) \end{aligned}$$

Then, $f_0 = g_0 - 1920g_1 - 1630080g_2$, so:

$$\begin{aligned} f_0(q) &= g_0(q) - 1920g_1(q) - 1630080g_2(q) \\ &= [1 + 1920q + 1630080q^2 + 803228160q^3 + O(q^4)] \\ &\quad - 1920 [q + 50220q^3 + O(q^4)] \\ &\quad - 1630080 [q^2 + 432q^3 + O(q^4)] \\ &= 1 + 2611200q^3 + O(q^4) \end{aligned}$$

Therefore, up to $O(q^4)$, $\{f_0, f_1, f_2\} = \{1 + 2611200q^3 + O(q^4), q + 50220q^3 + O(q^4), q^2 + 432q^3 + O(q^4)\}$ is a basis for M_{16} .

Miller basis for $k = 92$ The calculation of this basis may be interesting by hand once; However, it is possible to automate it. The procedure that calculates such coefficients is a standard in SageMath

Contributors [2020]. Here is, up to $O(q^{10})$, the Miller basis for M_{92} :

$$\begin{aligned}
f_0 &= 1 + 3034192667130000q^8 + 137290127714549760000q^9 + O(q^{10}) \\
f_1 &= q + 91578443563200q^8 + 2651503140376278561q^9 + O(q^{10}) \\
f_2 &= q^2 + 2380310529376q^8 + 42238207588515840q^9 + O(q^{10}) \\
f_3 &= q^3 + 51682260816q^8 + 530253459731160q^9 + O(q^{10}) \\
f_4 &= q^4 + 896013480q^8 + 4882999541760q^9 + O(q^{10}) \\
f_5 &= q^5 + 11516000q^8 + 28971735750q^9 + O(q^{10}) \\
f_6 &= q^6 + 94680q^8 + 80990208q^9 + O(q^{10}) \\
f_7 &= q^7 + 312q^8 - 4860q^9 + O(q^{10})
\end{aligned}$$

2.3 Basis Modulo Two

2.3.1 Reduced Modular Forms

Now that we have a basis with integral coefficients, it makes sense to reduce forms modulo 2. For a modular form f , we denote its reduction modulo 2 by \bar{f} . It is defined as follows:

If $f(q) = \sum_{n \in \mathbb{N}} c(n)q^n$ is a modular general form, then we define its reduction \bar{f} by

$$\bar{f}(q) = \sum_{n \in \mathbb{N}} \bar{c}(n)q^n \quad \text{with } \bar{c}(n) = c(n) \bmod 2.$$

We want to reduce Miller basis modulo 2. The reason is that as we know that some coefficients are ones, the reduction will not be trivial. We will reduce separately E_2 , E_3 and Δ (which together generate the Miller basis).

E_2 reduced We have:

$$E_2(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \equiv 1 \bmod 2.$$

Therefore, the reduction modulo 2 of E_2 is just 1. We write $\overline{E_2} = 1$, so $\overline{E_2^a} = 1$, for all $a \geq 0$.

E_3 reduced We have:

$$E_3(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n \equiv 1 \bmod 2.$$

Therefore, the reduction modulo 2 of E_3 is 1 as well. We write $\overline{E_3} = 1$, so $\overline{E_3^b} = 1$, for all $b \geq 0$.

Δ reduced We defined before Δ , and we would now like to know its q -extension in the standard way. That is, an infinite sum of q^n , instead of an infinite product as we have at the moment.

We define the coefficients $\tau(n)$ to match in the equation:

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.$$

When this holds, τ is called the *Ramanujan function*. We would like an explicit formula for $\tau(n)$. More precisely, we are interested in a formula for $\tau(n) \bmod 2$. We will calculate separately the coefficients $\tau(n) \bmod 2$ for n even and odd.

Case n odd Remember $\sigma_s(n)$ as the sum of s^{th} powers of (positive) divisors of n . It is known from classical theory [Kolberg, 1962, p.8] that:

$$\tau(8n + l) \equiv a_l \sigma_{11}(8n + l) \pmod{2^{b_l}},$$

where $\gcd(l, 8) = 1$, $a_1 = 1$, $a_3 = 1217$, $a_5 = 1537$, $a_7 = 705$, $b_1 = 11$, $b_3 = 13$, $b_5 = 12$, $b_7 = 14$.

We are interested in congruence class (mod 2) of the Ramanujan function $\tau(n)$. For n odd, we deduce the following:

$$\tau(n) \equiv \sigma_{11}(n) \equiv \sum_{d|n} d^{11} \equiv \sum_{d|n} 1 \equiv \begin{cases} 1 \pmod{2} & \text{if } n \text{ is a square} \\ 0 \pmod{2} & \text{else} \end{cases}.$$

Case n even It is easy to calculate that $\tau(2) = -24 \equiv 0 \pmod{2}$.

We know $\tau(p^{n+1}) = \tau(p^n)\tau(p) - p^{11}\tau(p^{n-1})$ for all $p \in \mathbb{P}$; see [Serre, 1973, p.97]. With $p = 2$, it follows by induction that $\tau(2^k) \equiv 0 \pmod{2}$ for all $k \in \mathbb{N}$. We know as well that $\tau(nm) = \tau(n)\tau(m)$ if $\gcd(n, m) = 1$; see [Serre, 1973, p.97]. It follows that for all n even, $\tau(n) \equiv 0 \pmod{2}$.

Explicit series of the discriminant Therefore, the only non-zero coefficients (modulo 2) appears on odd squares, i.e.

$$\tau(n) \equiv \begin{cases} 1 \pmod{2} & \text{if } n = (2m+1)^2 \text{ for } m \in \mathbb{N} \\ 0 \pmod{2} & \text{else} \end{cases}.$$

Thus, we can write the power series of Δ as:

$$\Delta(q) \equiv \overline{\Delta}(q) = \sum_{m=0}^{\infty} q^{(2m+1)^2} \pmod{2}.$$

2.3.2 Reduced Basis

The Miller basis for M_k was obtained via the Gauss elimination of the set $\{\Delta^j E_3^{2(d-j)+b} E_2^a \mid 1 \leq j \leq \dim(M_k)\}$ (with some conditions on a, b, d).

But $\overline{E_2^a} = \overline{E_2}^a = 1^a = 1 \pmod{2}$ and similarly, $\overline{E_3^{2(d-j)+b}} = 1 \pmod{2}$. So once the above set is reduced modulo 2, we are left with $\{\overline{\Delta}^j \mid 1 \leq j \leq \dim(M_k)\}$. So the Miller basis just becomes the Gauss elimination of $\overline{\Delta}$ powers. This is motivation for the next section.

2.4 Space of Modular Forms Modulo Two

We would like to have a definition for this space in a similar way as M_k was used for modular forms (of weight $2k$) before reduction. By abuse of notation, we denote the reduction of Δ modulo 2 (written $\overline{\Delta}$ until here) also by Δ , that is,

$$\Delta(q) = \sum_{m=0}^{\infty} q^{(2m+1)^2} \pmod{2}.$$

2.4.1 Weights of Modular Forms Modulo Two

We just saw that the Miller basis for $\overline{M_k}$ is (the Gaussian elimination of) $\{\Delta^j \mid 1 \leq j \leq \dim(M_k)\}$. Now, if we look at this set not reduced modulo 2, we have: $\{\Delta^j \mid 1 \leq j \leq \dim(M_k)\}$. This is a set of modular forms that have different weights. However, we started with a modular form in M_k , i.e. all modular forms having weight $2k$.

We understand now that modulo 2, the weight of modular form doesn't make sense any more. This is one of the consequences of reducing modulo 2: we lose some information about the modular forms, such as the weight. From this observation, we should study all modular forms together, modulo 2 (instead of separating by weights). This is why the space of modular forms modulo 2 will be denoted \mathcal{F} , with no dependence on k .

2.4.2 Powers of the Modular Discriminant Δ

Set of Powers of the Modular Discriminant Δ As we just saw, we can treat Δ -coefficients of modular forms as entries of a matrix (each modular form represented by a row). This allows us to perform Gaussian elimination for powers Δ^k up to $\dim(\overline{M}_k)$ on the Miller basis of \overline{M}_k (modular forms of weight $2k$ reduced modulo 2).

For simplicity again, we will just take the powers of Δ to be our basis for modular forms modulo 2 (i.e. drop the Gaussian elimination process). We define the space $\mathbb{F}_2[\Delta]$ in the usual way:

$$\mathbb{F}_2[\Delta] = \left\{ \sum_{k=1}^n a_k \Delta^k \mid n \in \mathbb{N}, a_k \in \mathbb{F}_2 \right\}$$

From 2.3.1 we had:

$$\Delta(q) = \sum_{n=0}^{\infty} \tau(n) q^n = \sum_{m=0}^{\infty} q^{(2m+1)^2}.$$

Therefore, we define

$$\Delta^k(q) = \sum_{n=0}^{\infty} \tau_k(n) q^n = \left(\sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^k \pmod{2}.$$

Thus, we have $\tau(n) = \tau_1(n)$.

Proportion of zeros In fact, most of the coefficients $\tau_k(n)$ are 0 modulo 2. When $k = 1$, there is already few coefficients that are 1: only the odd squares. When raising to the k^{th} power, there are even fewer.

Conditions on non-zero coefficients We can find conditions on coefficients that may not be zero. We look at the squares modulo 8. We remark that odd squares are all 1 mod 8, and even squares are

$a =$	0	1	2	3	4	5	6	7	mod 8
$a^2 =$	0	1	4	1	0	1	4	1	mod 8

Table 1: Squares modulo 8

all 0 mod 8.

We know from previous calculations that $\Delta(q)$ only has odd powers of q . Thus, raising to the k^{th} power give terms of power n such that:

$$\begin{aligned} n &= m_1^2 + m_2^2 + m_3^2 + \cdots + m_k^2 \\ &\equiv 1 + 1 + 1 + \cdots + 1 \pmod{8} \\ &\equiv k \pmod{8} \end{aligned}$$

Therefore: if $\tau_k(n) \equiv 1 \pmod{2}$, then $n \equiv k \pmod{8}$.

Equivalently: if $n \not\equiv k \pmod{8}$ then $\tau_k(n) \equiv 0 \pmod{2}$ (by taking the contra-positive).

This means, that Δ^k may only have terms q^n such that $n \equiv k \pmod{8}$, i.e. Δ^k may only have terms of power congruent to $k \pmod{8}$. When $k = 1$, this is that Δ may only have terms of power $1 \pmod{8}$, this matches with table 1: all odd squares are $1 \pmod{8}$.

Even powers of Δ We compare $\Delta^{2k}(q)$ and $\Delta^k(q^2)$:

$$\begin{aligned}
\Delta^{2k}(q) &= \left(\sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^{2k} \\
&= \sum_{n=0}^{\infty} \#[(2m_1+1)^2 + (2m_2+1)^2 + \dots + (2m_{2k}+1)^2 = n \mid m_0, m_1, \dots, m_{2k} \in \mathbb{N}] q^n \\
&= \sum_{n \text{ even}}^{\infty} \#[(2m_1+1)^2 + (2m_2+1)^2 + \dots + (2m_k+1)^2 = n/2 \mid m_0, m_1, \dots, m_k \in \mathbb{N}] q^n \\
&= \left(\sum_{m=0}^{\infty} q^{((2m+1)^2) \cdot 2} \right)^k \\
&= \left(\sum_{m=0}^{\infty} (q^2)^{(2m+1)^2} \right)^k = \Delta^k(q^2)
\end{aligned}$$

Thus, $\Delta^{2k}(q) = \Delta^k(q^2)$. Therefore, we can write any modular form modulo 2 f as the following [Nicolas and Serre, 2012a, (3)]:

$$f = \sum_{s \geq 0} f_s^{2^s},$$

with f_s having only odd powers of Δ . We will focus on the study of the odd powers of Δ .

2.4.3 The Space \mathcal{F}

We define the *space of modular forms modulo 2* denoted \mathcal{F} to be [Nicolas and Serre, 2012a, 2.1]:

$$\mathcal{F} = \langle \Delta^k \mid k \text{ odd} \rangle = \langle \Delta, \Delta^3, \Delta^5, \Delta^7, \dots \rangle$$

That is, all finite polynomials of Δ over \mathbb{F}_2 , having only odd powers. We remark that the weight of modular forms do not appear, as it was discussed before in 2.4.1. The observations modulo 8 that we have done in 2.4.2 yields that it will be useful to denote:

$$\begin{aligned}
\mathcal{F}_1 &= \langle \Delta^k \mid k = 1 \pmod{8} \rangle = \langle \Delta, \Delta^9, \Delta^{17}, \Delta^{25}, \dots \rangle, \\
\mathcal{F}_3 &= \langle \Delta^k \mid k = 3 \pmod{8} \rangle = \langle \Delta^3, \Delta^{11}, \Delta^{19}, \Delta^{27}, \dots \rangle, \\
\mathcal{F}_5 &= \langle \Delta^k \mid k = 5 \pmod{8} \rangle = \langle \Delta^5, \Delta^{13}, \Delta^{21}, \Delta^{29}, \dots \rangle, \\
\text{and } \mathcal{F}_7 &= \langle \Delta^k \mid k = 7 \pmod{8} \rangle = \langle \Delta^7, \Delta^{15}, \Delta^{23}, \Delta^{31}, \dots \rangle.
\end{aligned}$$

Of course, we have:

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7.$$

We will also introduce (as in [Nicolas and Serre, 2012b, 2.]):

$$\mathcal{F}(n) = \langle \Delta^k \mid k \text{ odd and } k \leq 2n-1 \rangle = \langle \Delta, \Delta^3, \Delta^5, \dots, \Delta^{2n-1} \rangle.$$

2.4.4 Duality Between Δ and q

As we defined \mathcal{F} above, a modular form modulo 2 is an expression of powers Δ^k . But we had from before that $\Delta = \sum_{m=0}^{\infty} q^{(2m+1)^2} \pmod{2}$. Therefore, we can translate a modular form given as a finite polynomial of Δ into an infinite polynomial of q . Thus, there are two ways to write a modular form modulo 2.

This duality between the two definitions is what makes the study of modular forms modulo 2 so interesting: we go back and forth between an infinite series and a finite polynomial. One is easy to express, the other easy to compute. This will lead to new reasoning. In particular, there is a new technique of computation ("exact computations" 6.2.1) that uses equivalence between the two ways of writing a modular form.

2.5 Hecke Operators Modulo Two

2.5.1 Reduction Modulo Two

Definition Now that we have reduced modular forms modulo 2, we would like to study the Hecke operators on these reduced modular forms. We define *Hecke operators modulo 2* as follows:

With f a modular form modulo 2 with q -definition

$$f(q) = \sum_{n \in \mathbb{N}} c(n)q^n$$

we define

$$\overline{T_p}f(q) = \sum_{n \in \mathbb{N}} \gamma(n)q^n$$

where

$$\gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases} \quad \text{and } p \text{ an odd prime.}$$

Well-definiteness We want to check that all the definitions make sense. When we look at $T_p f$, there is a number of ways to reduce it modulo 2: $\overline{T_p}f$, $\overline{T_p}f$, $\overline{T_p}f$, $\overline{T_p}f$. Let's compare coefficients: $\overline{T_p}f$:

$$\gamma(n) = \sum_{a|(n,p), a \geq 1} a^{2k-1} c\left(\frac{np}{a^2}\right) = \begin{cases} \overline{c}(np) & \text{if } p \nmid n \\ \overline{c}(np) + \overline{c}(n/p) & \text{if } p \mid n \end{cases}$$

Divisors of (n, p) are $\{1\}$ or $\{1, p\}$ since p is prime, so the sum split in two cases, with one or two terms. We see now that looking at Hecke operators modulo 2 only for primes simplifies the sum to a computable formula.

As both 1 and p are odd, the term a^{2k-1} reduces to 1 modulo 2. We understand why Hecke operators modulo 2 aren't defined for even numbers: many terms in the summation would become zero. It would not make sense to call it a Hecke operator any more.

It also makes sense why we look at modular forms modulo 2 instead of 3 or 5: the coefficient a^{2k-1} collapse nicely modulo 2, which won't be the case modulo any other number than 2.

$\overline{T_p}f$: this is (very) similar to the case above.

$\overline{T_p}f$:

$$\gamma(n) = \begin{cases} \overline{c}(np) & \text{if } p \nmid n \\ \overline{c}(np) + \overline{c}(n/p) & \text{if } p \mid n \end{cases}$$

$\overline{T_n}f$: Again, this is (very) similar to the case above.

All reductions give in fact the same result, so it makes sense to reduce modular forms modulo 2, and still study the Hecke operators (but now only for odd primes). As this all makes sense, we will now only consider modular forms modulo 2, and we will drop the over lines for simplicity. The fact that T_p and $\overline{T_p}$ have exactly the same action on the q -expansions of modular forms is only true when p is an odd prime. This is why we will concentrate on this case.

2.5.2 Basic Properties

When reduced modulo 2, Hecke operators $\overline{T_p}$ for primes p have more properties than the general T_p . The extra properties make the study modulo two interesting.

Inherited properties From the fact that $\overline{T_p|f(q)} = \overline{T_p|f(q)}$, we get that the Hecke operators modulo 2 keep the properties they had before being reduced.

Modularity Remains From definition 1.7, a Hecke operator transforms a modular form to another. This is because from definition, $T_n f$ is a sum of modular forms (which remain modular). Therefore, Hecke operators modulo 2 will as well transform a modular form to another. This was not clear from the definition modulo 2 that we had (which was in terms of q series).

Commutativity As in general [Serre, 1973, p.101]:

$$T_n T_m = T_{mn} \text{ if } \gcd(m, n) = 1.$$

We get that:

$$\overline{T_p T_q} = \overline{T_q T_p} \quad \forall p, q \in \mathbb{P}.$$

Therefore, the Hecke operators modulo 2 commute. This, as well, was not clear from definition. It will be very convenient for future calculations.

Linearity From definition 1.7, we have that the Hecke operators are immediately linear. That is:

$$T_p|(f + g) = T_p|f + T_p|g$$

(this follows directly from definition). This property will also remain modulo 2.

Behaviour of \mathcal{F}_i Suppose $f \in \mathcal{F}_i$ ³, using 2.4.2, we have:

$$f = \sum_{m \equiv i \pmod{8}} \mu_m \Delta^m = \sum_{n \equiv i \pmod{8}} c(n) q^n$$

From the definition of Hecke operator modulo 2 (2.5.1), we have:

$$\overline{T_p}|f = \sum_{n \in \mathbb{N}} \gamma(n) q^n \quad \text{with } \gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases}$$

$c(np)$ We have $np \not\equiv i \pmod{8} \implies c(np) = 0$.

$c(n/p)$ As p is an odd prime, it is an odd number, so from 2.4.2, $p^2 \equiv 1 \pmod{8}$, so $p^{-2} \equiv 1 \pmod{8}$ as well (with p^{-2} seen mod 8).

Therefore, $np \not\equiv i \pmod{8} \implies n/p \equiv np/p^2 \equiv np \not\equiv i \pmod{8}$.

³By abuse of notation, we denote by f a modular forms modulo 2 (instead of \overline{f}).

$\gamma(n)$ We conclude that $n \equiv np^2 \not\equiv pi \pmod{8} \implies \gamma(n) = 0$

Using 2.4.2 again, we deduce that $\overline{T_p}|f \in \mathcal{F}_j$ with $j \equiv pi \pmod{8}$. Overall, we have the following:

$$f \in \mathcal{F}_i \implies \overline{T_p}|f \in \mathcal{F}_j \text{ with } j \equiv pj \pmod{8}.$$

Non-Nullity of Hecke Operator We will prove the non-nullity of Hecke Operators in two ways.

First, as a consequence of a property following the idea developed in [Ono, 2004, p.33].

Property 2.1. *If $f \in \mathcal{F}$, and $\overline{T_p}|f = 0$ for all odd primes p , then either $f = 0$ or $f = \Delta$. That is, only Δ and 0 give zero after applying any Hecke operator.*

Proof. Let's denote by $a(n)$ the coefficients of the q -expansion of f in the usual way ($f(z) = \sum_{n=0}^{\infty} a(n)q^n$, with $q = e^{2\pi iz}$). With p an odd prime, we similarly define $\overline{T_p}|f(z) = \sum_{n=0}^{\infty} \gamma(n)q^n$ with $\gamma(n) = c(np) + c(n/p)$.

1. If r simple odd:

$$p \nmid n \text{ gives } 0 = \gamma(n) = a(np), \text{ so } a(r) = 0$$

2. If r odd of power 3 or more:

$$\text{Putting } n = mp^2, \text{ we get: } 0 = \gamma(mp^2) = a(mp^3) + a(mp) = a(mp^3).$$

$$\text{Thus, } a(r) = 0$$

Thus, $a(r) \neq 0$ implies r is an odd square. Note that $0 = \gamma(np) = a(np^2) + a(n)$, so $a(1) = 1$ will implies $a(r) = 1$ for all odd squares r . In this case, $f = \Delta$.

Similarly, $a(1) = 0$ makes $a(n) = 0$ for all n . Therefore, f may only be Δ or 0. \square

An immediate consequence (by taking the contra-positive) of this property is that if $f \neq 0, \Delta$, then there exists a p such that $\overline{T_p}|f \neq 0$. Thus, for any $k > 1$, we get that there is a prime p such that $\overline{T_p}|\Delta^k \neq 0$. This means that $\overline{T_p}$ is never the null operator (so reduction modulo two doesn't become trivial).

A second way to show non-nullity of Hecke operators is by the following property, which is new to this paper:

Property 2.2. *In fact, we have $\overline{T_p}|\Delta^p = \Delta + \mathcal{O}(\Delta^9)$ for any odd primes p .*

That is, $\overline{T_p}|\Delta^p = \Delta + [\Delta^{k_1} + \Delta^{k_2} + \dots + \Delta^{k_r}]$ with $k_i \geq 9 \quad \forall 1 \leq i \leq r$ (in fact, we also have $k_i \equiv 1 \pmod{8} \quad \forall 1 \leq i \leq r$).

Proof. We denote $c(n)$ the coefficients of the q -expansion of Δ^p and $\gamma(n)$ the ones of $T_p|\Delta^p$. We recall from the definition that $\gamma(1) = c(p)$ (since $p \nmid 1$). Now, $c(p) = 1$ (in fact, p is the smallest power of q that appear in the q -expansion of Δ^p). So $\overline{T_p}|\Delta^p = \Delta + \mathcal{O}(\Delta^3)$.

But now using Behaviour of Hecke operators in \mathcal{F}_i , we have:

$$\overline{T_p}|\Delta^p = \Delta^{k_0} + \Delta^{k_1} + \Delta^{k_2} + \dots + \Delta^{k_r} \quad k_i \neq k_j \text{ if } i \neq j$$

with $k_0 \equiv k_j \pmod{8}$ for all $0 \leq j \leq r$. As $k_0 = 1$, this means that $k_j \equiv 1 \pmod{8}$. Therefore, the smallest power of Δ appearing in $\overline{T_p}|\Delta^p$ apart from 1 is 9. This gives the proposition statement. \square

Note that with this proof, we know an explicit modular form such that the Hecke operator does not vanish.

2.5.3 Nilpotency

One property of Hecke operators is that, given a modular form f , if we apply a Hecke operators enough times, the form will become zero (i.e. they are nilpotent). The strategy to show this is to prove that for any k (odd), and any prime p , we have:

$$\overline{T_p}|\Delta^k = \sum_{j < k} \mu_j \Delta^j$$

The proof of this property will be divided into two main steps:

Order of Δ doesn't increase We first want to show that:

$$\overline{T_p}|\Delta^k = \sum_{j \leq k} \mu_j \Delta^j.$$

Let f be a modular form modulo 2 of maximum degree $2n_0 - 1$ (in Δ), i.e.

$$f = \Delta^{2n_0-1} + \Delta^{2n_1-1} + \dots + \Delta^{2n_m-1},$$

with $n_0 > n_1 > \dots > n_m$. So $f \in \mathcal{F}(n_0)$. Then, there must exist a modular forms $g \in M_{6n_0}$ such that $\bar{g} = f$. For example, we can take

$$g = \Delta^{2n_0-1} + E_2^{6(n_0-n_1)} \Delta^{2n_1-1} + \dots + E_2^{6(n_0-n_m)} \Delta^{2n_m-1}$$

(it is straightforward to check that $g \in M_{6n_0}$ and $\bar{g} = f$, as it was designed for). We know that $\overline{T_p}|f = \overline{T_p}|\bar{g} = \overline{T_p}|g$. Remark as well that as $T_p : M_{6n_0} \rightarrow M_{6n_0}$, we have $T_p|g \in M_{6n_0}$, so the maximum degree of Δ that appear in $T_p|g$ is $2n_0 - 1$. Thus, the maximum degree of $\overline{\Delta}$ that appear in $\overline{T_p}|g = \overline{T_p}|f$ is $2n_0 - 1$. Therefore, the degree of f doesn't increase when applying $\overline{T_p}$ to it.

Order of Δ decrease Since T_p and $\overline{T_p}$ have exactly the same action on q -expansions of modular forms, we can interchange them as we want. By abuse of notation (again), we denote the reduction of T_p modulo 2 (usually denoted $\overline{T_p}$) by T_p as well. We have proved that

$$T_p|\Delta^k = \sum_{j \leq k} \mu_j \Delta^j.$$

We now need to show that $\mu_k = 0$, so that the maximum order of Δ in fact effectively decrease.

Let's look at $\mathcal{F}(k)$ as a vector space over \mathbb{F}_2 with basis $\{\Delta, \Delta^3, \dots, \Delta^k\}$. We may represent a modular form modulo 2 by a k -vector over \mathbb{F}_2 (note that even powers of Δ will always be zero, but we keep track of them to lighten notation). Then, as T_p are linear (see 2.5.2), we can represent each operator T_p with a matrix. Let A_p be the $(k \times k)$ -matrix (over \mathbb{F}_2) representing the action of T_p on $\mathcal{F}(k)$. Since the order of Δ doesn't increase when applying a Hecke operator, the matrix A_p should be upper-triangular, i.e.:

$$A_p = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,k} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{k,k} \end{pmatrix}.$$

We need to show that the coefficients $a_{i,i}$ are zero. We will do this by induction. Suppose we know T_p decrease the degree of Δ^j for all $j \leq k - 1$. Translating this information to the matrix, it means that $a_{j,j} = 0$ for all $j \leq k - 1$. Then, we only need to show that $a_{k,k} = 0$.

Now that we have all this information on the diagonal, it makes sense to study the trace: $\text{Tr}(A_p) = a_{k,k}$. A nice interpretation of the trace should give us an equation for $a_{k,k}$. We can interpret the trace as the sum of eigenvalues of the matrix A_p , i.e. eigenvalues of the Hecke operator T_p . Some knowledge about eigenvalues of Hecke operators has been proved already (see Hatada [1979]), we have: for p an odd prime, if λ_p is an eigenvalue of T_p , we have the congruence: $\lambda_p \equiv 1 + p \pmod{8}$. Since p is an odd (prime) number, we get: $\lambda_p \equiv 0 \pmod{2}$. As this is true for all eigenvalues of T_p , we have that the sum of eigenvalues (which corresponds to the trace of the matrix) is zero over \mathbb{F}_2 . Thus: $a_{k,k} = \text{Tr}(A_p) \equiv 0 \pmod{2}$.

Now, this is a proof by induction, but the first case really is $k = 0$, in which case, all modular forms are just 0, so all Hecke operators are obviously zero and hence nilpotent in this case. Therefore, we proved the nilpotency modulo 2 of Hecke operators T_p for all p odd primes.

2.5.4 Expression as a Sum of Powers of Δ

As the degree of a modular form doesn't increase after applying a Hecke operator, we can apply this the modular form Δ^k to get:

$$T_p|\Delta^k = \sum_{\substack{j \leq k \\ j \text{ odd}}} \mu_j \Delta^j$$

As we know, moreover, that the degree of a modular form will in fact decrease, we deduce that in fact:

$$T_p|\Delta^k = \sum_{\substack{j \leq k-2 \\ j \text{ odd}}} \mu_j \Delta^j \quad (*)$$

The observation on \mathcal{F}_i (in 2.5.2) leads us to the formula:

$$T_p|\Delta^k = \sum_{\substack{j \leq k-2 \\ j \equiv pk \pmod{8}}} \mu_j \Delta^j \quad (**)$$

(since $\Delta^k \in \mathcal{F}_i$ with $k \equiv i \pmod{8}$).

2.5.5 Examples (for Small Powers of Δ)

We will describe the behaviour of Hecke operators when applied to Δ^k with k odd, $k \leq 7$.

Δ Clearly, from (*), we have $T_p|\Delta = 0$, since the sum is empty (for any p odd prime).

Δ^3 From (*), we have $T_p|\Delta^3 = \Delta$ or 0. Moreover, (**) gives $T_p|\Delta^3 = 0$ if $1 \not\equiv 3p \pmod{8}$ i.e. if $p \not\equiv 3 \pmod{8}$. Now, if $p \equiv 3 \pmod{8}$, we may only look at the coefficient q^1 of $T_p|\Delta^3$ (if it is 1, $T_p|\Delta^3 = \Delta$ and if it is 0, $T_p|\Delta^3 = 0$, as there is no other possibilities).

From definition (in 2.5.1), we have that the coefficient of q^1 is $\gamma(1) = c(p)$ (since $p \nmid 1$) with c the q coefficients of Δ^3 . From (2.3.1), the non-zero coefficients of Δ are odd squares. So we have:

$$(\Delta(q))^3 = \left(\sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^3 = \sum_{n=0}^{\infty} \#\{m_1, m_2, m_3 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 = n\} q^n.$$

As, $c(p)$ is the p^{th} coefficient of Δ^3 , $c(p) = \#\{m_1, m_2, m_3 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 = p\} \pmod{2}$ corresponds (mod 2) to the number of ways to write p as sum of three odd squares. Now, there are a few possible cases: if $p = 3$, then $m_1 = m_2 = m_3 = 1$ is the only solution, and $c(p) = 1$ so $T_p|\Delta^3 = \Delta$. We consider separately:

- $m_1 = m_2 = m_3$: in this case, $m_1^2 + m_2^2 + m_3^2$ isn't prime, unless $p = 3$ (but we handled this case already). So there is no solution of this form.
- $m_1 \neq m_2 \neq m_3 \neq m_1$ (i.e. all distinct): in this case, if (m_1, m_2, m_3) is a solution, then so is (m_1, m_3, m_2) , (m_2, m_1, m_3) , (m_2, m_3, m_1) , (m_3, m_1, m_2) and (m_3, m_2, m_1) . Therefore, there is an even number of solutions, so the contribution to $c(p)$ of solutions of this type is 0.
- $m_1 = m_2 \neq m_3$ in this case, if (m_1, m_2, m_3) is a solution, then so is (m_1, m_3, m_2) , (m_3, m_1, m_2) . Therefore, there is an odd number of solutions, so the contribution to $c(p)$ of solutions of this type is 1 (if such kind of solution exist).

We want to know if there are solutions to $a^2 + 2b^2 = p$ with a and b odds. We look at the integers $\mathbb{Z}[\sqrt{-2}]$: if p splits, then there is a pair (a, b) of integers such that $(a + b\sqrt{-2})(a - b\sqrt{-2}) = p$, i.e. $a^2 + 2b^2 = p$. Now, p splits in $\mathbb{Z}[\sqrt{-2}]$ if $\left(\frac{-2}{p}\right) = +1$. As $p \equiv 3 \pmod{8}$, $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{-1}{p}\right) = -1$, so $\left(\frac{-2}{p}\right) = +1$ (by the first and second supplement to the law of quadratic reciprocity.) Therefore, there are two integers a and b such that $a^2 + 2b^2 = p$. Suppose a is even: then p is even, which is even (as p is an odd prime). Thus, a must be odd. Suppose b is even: then $a^2 + 2b^2 \equiv a^2 \equiv 1 \pmod{8}$, but $p \equiv 3 \pmod{8}$. Thus, b must be odd.

Therefore, if $p \equiv 3 \pmod{8}$, we found a pair of odd numbers a and b such that $a^2 + 2b^2 = p$, so $c(p) = 1$, and thus $T_p|\Delta^3 = \Delta$.

Δ^5 From (*), we have $T_p|\Delta^5 = \Delta^3$ or Δ or 0. Moreover, (**) gives:

$$\begin{array}{lll} p \equiv 7 \pmod{8} : & T_p|\Delta^5 = \Delta^3 \text{ or } 0 & \text{if } 3 \equiv 5p \pmod{8} \quad \text{i.e. } p \equiv 7 \pmod{8} \\ p \equiv 5 \pmod{8} : & T_p|\Delta^5 = \Delta \text{ or } 0 & \text{if } 1 \equiv 5p \pmod{8} \quad \text{i.e. } p \equiv 5 \pmod{8} \\ p \equiv 1 \text{ or } 3 \pmod{8} : & T_p|\Delta^5 = 0 & \text{else} \end{array}$$

$p \equiv 7 \pmod{8}$ Now, if $p \equiv 7 \pmod{8}$, we may only look at the coefficient q^3 of $T_p|\Delta^5$ (if it is 1, $T_p|\Delta^5 = \Delta^3$ and if it is 0, $T_p|\Delta^3 = 0$, as there is no other possibilities). From definition (in 2.5.1), we have that the coefficient of q^3 is $\gamma(3) = c(3p)$ (since $p \nmid 3$) with c the q coefficients of Δ^5 . From (2.3.1), the non zero coefficients of Δ are odd squares. Now, $c(3p)$ is the p^{th} coefficient of Δ^5 . We have:

$$(\Delta(q))^5 = \left(\sum_{m=0}^{\infty} q^{(2m+1)^2} \right)^5 = \sum_{n=0}^{\infty} \#\{m_1, m_2, m_3, m_4, m_5 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = n\} q^n$$

So $c(3p) = \#\{m_1, m_2, m_3, m_4, m_5 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = 3p\} \pmod{2}$ corresponds (mod 2) to the number of ways to write $3p$ as sum of five odd squares.

Now, we look at the decomposition of m_1, m_2, m_3, m_4, m_5 :

- $(1+1+1+1+1)$: in this case, there are $5! = 120$ symmetric solutions, so an even number of solution, so the contribution to $c(3p)$ is 0.
- $(1+1+1+2)$: in this case, there are $3!\binom{5}{2} = 60$ symmetric solutions, (which is even), so the contribution to $c(3p)$ is 0.
- $(1+2+2)$: in this case, there are $\binom{5}{2}\binom{3}{2} = 30$ symmetric solutions, (again, even), so the contribution to $c(3p)$ is 0.
- $(1+1+3)$: in this case, there are $2\binom{5}{3} = 20$ symmetric solutions, so the contribution to $c(3p)$ is 0.

- (1+4): in this case, there are 5 symmetric solutions, so an odd number of solution, so the contribution to $c(3p)$ is 1, if such a solution exist.
- (5): in this case, $m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = 5n$ which is never $3p$.

Now, we are looking for pairs of odd integers (a, b) such that $a^2 + 4b^2 = 3p$. that is, $a^2 + c^2 = 3p$ with $c = 2b$ so $c \equiv 2 \pmod{4}$. We look at factorization of $3p$ in $\mathbb{Z}[\sqrt{-1}]$. Now, $p \equiv 7 \pmod{8}$, so $\left(\frac{-1}{p}\right) = -1$ (by the first supplement to the law of quadratic reciprocity). Thus, p remains prime in the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$. Therefore, there is no pair of (a, b) integers solution $a^2 + 4b^2 = 3p$.

Thus, $T_p|\Delta^5 \neq \Delta^3$ for any odd prime p .

$p \equiv 5 \pmod{8}$ Now, if $p \equiv 5 \pmod{8}$, we want to know if $T_p\Delta = \Delta$ or 0. Again, we look at the coefficient q^1 of $T_p|\Delta$, which is the coefficient q^p of Δ^5 , i.e. $c(p) = \#\{m_1, m_2, m_3, m_4, m_5 \text{ odds} \mid m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 = p\} \pmod{2}$, which corresponds (mod 2) to the number of ways to write p as sum of five odd squares.

If $p = 5$, then $c(5) = 1$ clearly, and we have $T_5|\Delta^5 = \Delta$. Now, we use again the decomposition of m_1, m_2, m_3, m_4, m_5 from above: we are looking for pairs of odd integers (a, b) such that $a^2 + 4b^2 = p$. that is, $a^2 + c^2 = 3p$ with $c = 2b$ so $c \equiv 2 \pmod{4}$. We again look at factorization of p in $\mathbb{Z}[\sqrt{-1}]$. Now, $p \equiv 5 \pmod{8}$, so $\left(\frac{-1}{p}\right) = 1$ (by the first supplement to the law of quadratic reciprocity). Thus, p remains prime in the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$. Therefore, there is a pair of (a, b) integers solution $a^2 + c^2 = p$.

Suppose both a and c are odd: then $a^2 + c^2$ is even, so it may not be any odd prime p . Suppose both a and c are even: then $a^2 + c^2$ is even, so it may not be any odd prime p . So, without loss of generalities, a is odd and c is even. We can write $b = c/2$. Suppose c is even: then $a^2 + 4b^2 \equiv 1 \pmod{8}$. As $p \equiv 5 \pmod{8}$, $a^2 + 4b^2 \neq p$. So if (a, b) is a solution, then a and b are odds. Thus, $T_p|\Delta^5 = \Delta$ for any odd prime p .

Δ^7 From (*), we have $T_p|\Delta^7 = \Delta^5$ or Δ^3 or Δ or 0. In fact, we have [Nicolas and Serre, 2012a, 2.3]:

$$\begin{array}{ll} T_p|\Delta^7 = \Delta & \text{if } p \equiv 7 \pmod{16} \\ T_p|\Delta^7 = \Delta^3 & \text{if } p \equiv 5 \pmod{8} \\ T_p|\Delta^7 = \Delta^5 & \text{if } p \equiv 3 \pmod{8} \\ T_p|\Delta^7 = 0 & \text{if } p \equiv 1 \pmod{8} \text{ or } p \equiv -1 \pmod{16} \end{array}$$

2.5.6 Table of Hecke Operators

Here is a table of Hecke operators:

	Δ^1	Δ^3	Δ^5	Δ^7	Δ^9	Δ^{11}	Δ^{13}	Δ^{15}	Δ^{17}	Δ^{19}
T_3	0	Δ	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^9 + \Delta^{17}$
T_5	0	0	Δ	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7
T_7	0	0	0	Δ	0	0	Δ^3	Δ^9	0	Δ^5
T_{11}	0	Δ	0	Δ^5	Δ^3	$\Delta + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^9 + \Delta^{17}$
T_{13}	0	0	Δ	Δ^3	0	0	$\Delta + \Delta^9$	Δ^{11}	Δ^5	Δ^7
T_{17}	0	0	0	0	Δ	Δ^3	Δ^5	Δ^7	Δ	0
T_{19}	0	Δ	0	Δ^5	Δ^3	$\Delta + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta + \Delta^9 + \Delta^{17}$
T_{23}	0	0	0	Δ	0	0	Δ^3	$\Delta + \Delta^9$	0	Δ^5
T_{29}	0	0	Δ	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7
T_{31}	0	0	0	0	0	0	0	Δ	0	0
T_{37}	0	0	Δ	Δ^3	0	0	$\Delta + \Delta^9$	Δ^{11}	Δ^5	Δ^7
T_{41}	0	0	0	0	0	0	0	0	Δ	Δ^3
T_{43}	0	Δ	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^9 + \Delta^{17}$
T_{47}	0	0	0	0	0	0	0	Δ	0	0

Action of Primes Hecke Operators (primes up to 50) on Modular Forms Modulo 2 (up to Δ^{19}).

A larger table may be found in the appendix (see A.1).

It seems quite random, which makes sense since the Hecke operators depend on primes, and primes appear at random. However, it is interesting to try to find patterns and rules for this table. In the second column (of Δ^3), we get $1/4$ of the primes giving Δ (the other $3/4$ giving 0), this is a consequence of Dirichlet Density Theorem, that will be discussed later in this paper. Similarly, in the third column (of Δ^5), we get $1/4$ of the primes giving Δ . For a similar reason, in the fourth column (of Δ^7), we get $1/4$ of the primes giving Δ^3 ; $1/4$ of the primes giving Δ^5 ; $1/8$ of the primes giving Δ ; and $3/8$ of the primes giving 0.

2.6 Nilpotency Order

In this subsection, we follow the construction from Nicolas and Serre [2012a]. As we know that the Hecke operators are nilpotent, we may want to study the order of nilpotentness.

2.6.1 Introduction

Definition For a modular form modulo 2 $f \in \mathcal{F}$, we define the *nilpotentness order* to be the smallest integer $g(f)$ such that we have

$$T_{p_1} T_{p_2} \cdots T_{p_{g(f)}} |f = 0,$$

for any set of primes numbers $p_1, p_2, \dots, p_{g(f)} \in \mathbb{P}$. The primes p_i involved do not need to be distinct. Note as well that from commutativity of the Hecke operators, the order of the primes p_i doesn't matter.

By convention, we write $g(0) = -\infty$. With a slight abuse of notation, we will write $g(k)$ for $g(\Delta^k)$.

Properties

Maximum Order All Hecke operators lower by at least two the maximum degree of Δ in the Δ -expansion of a modular form modulo 2 2.5.3. We deduce that $g(f) \leq g(T_p |f) + 1$. Applied to Δ^k , we get: $g(k) \leq g(k-2) + 1$. Therefore, by induction, we have $g(k) \leq \lfloor \frac{k+1}{2} \rfloor$. This implies by the same occasion, the well definiteness of the order of nilpotentness for all modular form modulo two.

Minimum Order If the degree of f is strictly greater than 1 (i.e. $f \neq 0, \Delta$), then $g(f) \geq 2$.

We deduce this from the fact that Hecke operators are not null operators in general (2.5.2): Remember that one consequence of non nullity is that if $f \neq 0, \Delta$, then there exists an odd prime p such that $T_p|f \neq 0$. This directly implies that $g(f) > 1$ if $f \neq \Delta, 0$.

Examples We can compute a few nilpotentness "by hand" (we will see later that there is a more direct method):

- $g(0) = -\infty$
- $g(\Delta) = 1$:
 $T_p(\Delta) = 0$ as order of Δ decrease, see 2.5.3
- $g(\Delta^3) = 2$:
 $T_p|\Delta^3 = \Delta$ or 0
thus: $g(\Delta^3) = 1 + \max(g(\Delta), g(0)) = 2$
- $g(\Delta^3 + \Delta) = 2$
similarly
- $g(\Delta^5) = 2$:
 $T_p|\Delta^5 = \Delta$ or 0
thus: $g(\Delta^5) = 1 + \max(g(\Delta), g(0)) = 2$
- $g(\Delta^5 + \Delta^3 + \Delta) = g(\Delta^5 + \Delta^3) = g(\Delta^5 + \Delta) = 2$
similarly
- $g(\Delta^7) = 3$:
 $T_p(\Delta^7) = \Delta^5$ or Δ^3 or Δ or 0
thus: $g(\Delta^7) = 1 + \max(g(\Delta^5), g(\Delta^3), g(\Delta), g(0)) = 3$

2.6.2 Code and Height of Natural Numbers

Definition of n_3 , n_5 and h We consider a natural number $k \in \mathbb{N}$, and we let α_i be the digits of it's binary representation, i.e.

$$k = \sum_{i=0}^{\infty} \alpha_i 2^i \quad \text{with } \alpha_i \in \{0, 1\}.$$

We then define $n_3(k)$ and $n_5(k)$ as follows:

$$n_3(k) = \sum_{i=1}^{\infty} \alpha_{2i+1} 2^i = \sum_{i \text{ odd}} \alpha_i 2^{\frac{i-1}{2}} \quad \text{and} \quad n_5(k) = \sum_{i=1}^{\infty} \alpha_{2i+2} 2^i = \sum_{i \text{ even}} \alpha_i 2^{\frac{i-2}{2}}.$$

We also define h (the *height*) to be the sum of n_3 and n_5 , that is:

$$h(k) = n_3(k) + n_5(k).$$

The definition of n_3 and n_5 is equivalent to the following:

$n_3(k)$ Take k and write it in binary base. Ignore the units digit. Select only the digits corresponding to odd powers of 2. This forms $n_3(k)$.

$n_5(k)$ Again, write k in binary base. Ignore the unit digit. Select only the digits corresponding to even powers of 2. This forms $n_5(k)$.

Example We look at the example $k = 91$: $91 = 1 + 2 + 8 + 16 + 64 = 2^0 + 2^1 + 2^3 + 2^4 + 2^6$

We construct $n_3(k)$ and $n_5(k)$ graphically as follows:

$$\begin{array}{c|cccccccc|c}
 k = 91 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 & \\
 \hline
 k & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1011011 = 91 \\
 n_3(k) & 1 & \boxed{0} & 1 & \boxed{1} & 0 & \boxed{1} & 1 & 011 = 3 \\
 n_5(k) & \boxed{1} & 0 & \boxed{1} & 1 & \boxed{0} & 1 & 1 & 110 = 6
 \end{array}
 \left. \vphantom{\begin{array}{c|cccccccc|c} \right\} = h(k) = 3 + 6 = 9}$$

Basic Properties

Variation Note that neither of h , n_3 , n_5 is monotone. That is, in general, it is not true that $h(n+1) \geq h(n)$, neither $n_3(n+1) \geq n_3(n)$ nor $n_5(n+1) \geq n_5(n)$.

$2k$ and $2k+1$ It is straightforward, from the definition, that n_3 and n_5 (therefore also h) are the same for an even $2m$ and the next number $2m+1$. Explicitly:

$$\left. \begin{array}{l} \text{for all } m \in \mathbb{N}: \quad n_3(2m) = n_3(2m+1) \\ \text{and} \quad n_5(2m) = n_5(2m+1) \end{array} \right\} h(2m) = h(2m+1)$$

Addition of powers of 2 Moreover, it is clear from the definition that m_3 , m_5 and h preserve addition of powers of 2, that is:

$$\begin{array}{llll}
 n_3(m_1) + n_3(m_2) & = & n_3(m_1 + m_2) & m_1 = 2^{k_1}, \quad k_1 \in \mathbb{N} \\
 n_5(m_1) + n_5(m_2) & = & n_5(m_1 + m_2) & \text{for all } m_2 = 2^{k_2}, \quad k_2 \in \mathbb{N} \\
 h(m_1) + h(m_2) & = & h(m_1 + m_2) & k_1 \neq k_2
 \end{array}$$

Powers of 2 For $m = 2^k$, $k > 0$ even:

$$n_3(m) = 0 \quad \text{and} \quad n_5(m) = 2^{\frac{k-2}{2}} = 1/2\sqrt{m} \quad \text{so} \quad h(m) = 1/2\sqrt{m}.$$

For $m = 2^k$, $k > 0$ odd:

$$n_3(m) = \sqrt{m/2} \quad \text{and} \quad n_5(m) = 0 \quad \text{so} \quad h(m) = \sqrt{m/2}.$$

Therefore, if $m = 2^k$, $m \geq 2$, we have:

$$1/2\sqrt{m} \leq h(m) \leq \sqrt{m/2}.$$

Lower Bound for h Now, combining with the property above and properties of square root, we have:

$$1/2\sqrt{m} \leq h(m) \quad \text{for all } m \in \mathbb{N}, \quad m \geq 2.$$

Upper Bound for h Let $m = 2^{k_1} + 2^{k_2} + \dots + 2^{k_r}$ with $k_1 > k_2 > \dots > k_r$.

By construction, we have:

- If k_1 is even, $n_3(m) < \sqrt{2^{k_1}} = \sqrt{m} < \sqrt{2m}$.
- If k_1 is odd, $n_3(m) < \sqrt{2^{k_1+1}} < \sqrt{2m}$.

Thus, in any case $n_3(m) < \sqrt{2m}$.

Again, by construction, we have:

- If k_1 is even, $n_5(m) < \sqrt{2^{k_1}} < \sqrt{m}$.
- If k_1 is odd, $n_5(m) < \sqrt{2^{k_1-1}} = \sqrt{m/2} < \sqrt{m}$.

Thus, in any case $n_5(m) < \sqrt{m}$.

Therefore, we have $h(m) < \sqrt{2m} + \sqrt{m} = (1 + \sqrt{2})\sqrt{m} < 5/2\sqrt{m}$.

Asymptotic Behaviour of h Note that if $m = 0, 1$, we have $h(m) = \sqrt{m} = m$. Thus, we have:

$$1/2\sqrt{m} \leq h(m) < (1 + \sqrt{2})\sqrt{m} < 5/2\sqrt{m} \quad \text{for all } m \in \mathbb{N}.$$

Using the big \mathcal{O} notation (asymptotic behaviour), this is:

$$h(m) = \mathcal{O}(\sqrt{m}).$$

Code of Natural Numbers Given a natural number $m \in \mathbb{N}$, we define its *code* as the pair $[n_3(m), n_5(m)]$.

A Representation of Odd/Even We define the maps:

$$\phi : 2\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \quad \text{such that} \quad \phi(2m) = [n_3(2m), n_5(2m)]$$

and

$$\psi : 2\mathbb{N} + 1 \rightarrow \mathbb{N} \times \mathbb{N} \quad \text{such that} \quad \psi(2m + 1) = [n_3(2m + 1), n_5(2m + 1)]$$

so that

$$\phi^{-1} : \mathbb{N} \times \mathbb{N} \rightarrow 2\mathbb{N} \quad \text{such that:}$$

$$\phi^{-1}([m_1, m_2]) = 2 \sum_{i=0}^{\infty} \alpha_i 2^{2i} + 4 \sum_{i=0}^{\infty} \alpha_i 2^{2i} \quad \text{with} \quad \begin{cases} m_1 = \sum_{i=0}^{\infty} \alpha_i 2^i \\ m_2 = \sum_{i=0}^{\infty} \beta_i 2^i \end{cases}$$

and

$$\psi^{-1} : \mathbb{N} \times \mathbb{N} \rightarrow 2\mathbb{N} + 1 \quad \text{such that:}$$

$$\psi^{-1}([m_1, m_2]) = 2 \sum_{i=0}^{\infty} \alpha_i 2^{2i} + 4 \sum_{i=0}^{\infty} \alpha_i 2^{2i} + 1 \quad \text{with} \quad \begin{cases} m_1 = \sum_{i=0}^{\infty} \alpha_i 2^i \\ m_2 = \sum_{i=0}^{\infty} \beta_i 2^i \end{cases}.$$

(i.e. squaring each power of two composing m_1 and m_2 separately.) Since ϕ (respectively ψ) has an inverse, it defines a bijection between even (respectively odd) natural numbers and their code.

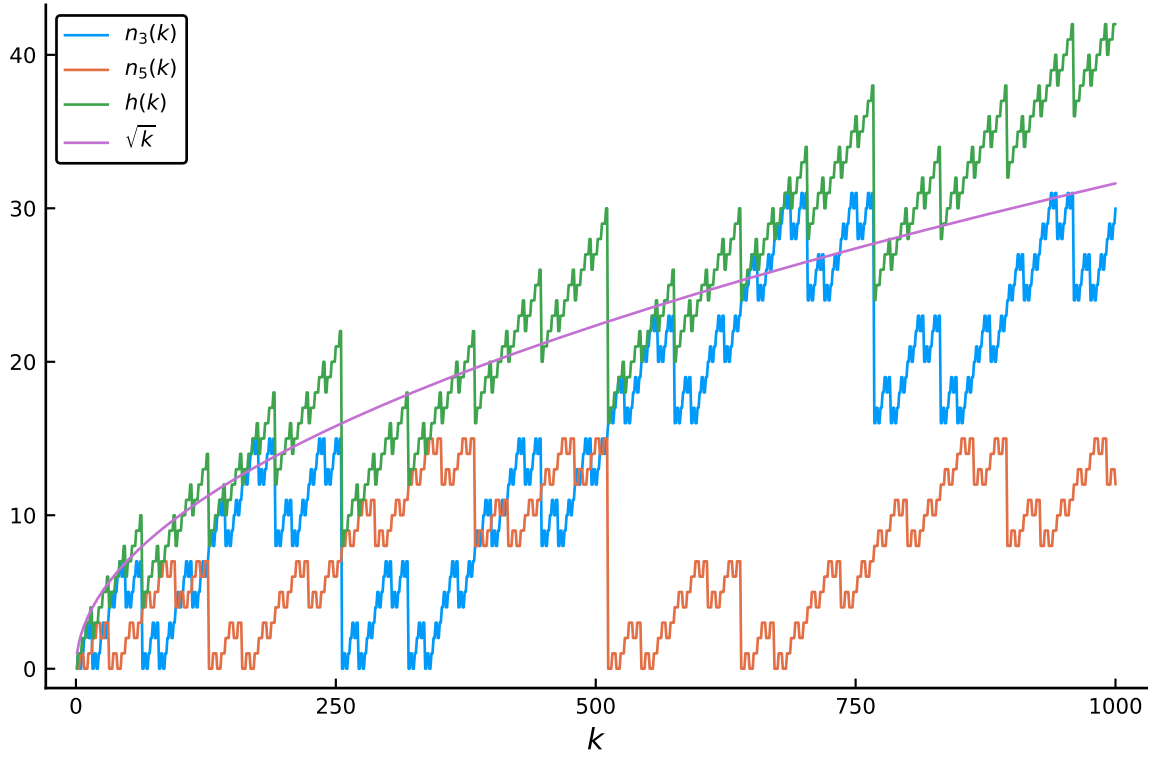
Behaviour

Height's Behaviour We observe the following behaviour:

k	0 , 1	2 , 3	4 , 5	6 , 7	8 , 9	10 , 11	12 , 13	14 , 15
Code of k	[1 , 0]	[0 , 1]	[1 , 1]	[2 , 0]	[3 , 0]	[2 , 1]	[3 , 1]	[0 , 2]
h(k)	0	1	1	2	2	3	3	4

(Larger table in A.3.)

On smaller scale, the behaviour is as follows:



(Plot with other scales in A.4.)

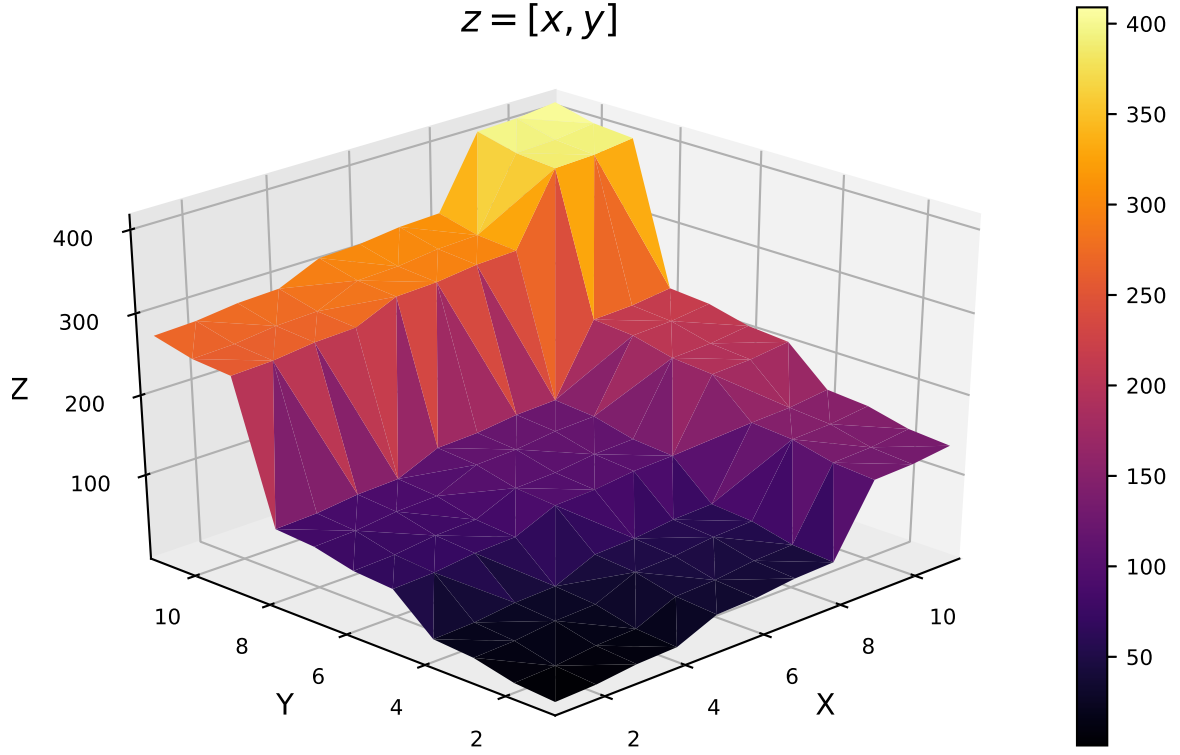
Code's Behaviour It may be interesting to represent how the code grows according to both parameters. First, we can make a small table of values:

	0	1	2	3	4	5	6	7	8	9	10
0	0	4	16	20	64	68	80	84	256	260	272
1	2	6	18	22	66	70	82	86	258	262	274
2	8	12	24	28	72	76	88	92	264	268	280
3	10	14	26	30	74	78	90	94	266	270	282
4	32	36	48	52	96	100	112	116	288	292	304
5	34	38	50	54	98	102	114	118	290	294	306
6	40	44	56	60	104	108	120	124	296	300	312
7	42	46	58	62	106	110	122	126	298	302	314
8	128	132	144	148	192	196	208	212	384	388	400
9	130	134	146	150	194	198	210	214	386	390	402
10	136	140	152	156	200	204	216	220	392	396	408

Table in of (even) integers corresponding to code $[line, column]$.

(Larger table in A.4.)

But it isn't very visual, so another way to view it is as a surface (obtained by linking with triangles the points plotted). On the grid $[0, 10]^2$, we plot the surface $z = [x, y]$ i.e. z is the integer with code $[x, y]$.



Plot of the surface with height given by the code of X and Y .
(i.e. the surface with equation $Z = [X, Y]$.)

(Other scales of plots available in A.4.)

Order Relation We define the following order relation on natural numbers: for $m, l \in \mathbb{N}$, $m \prec l$ if $h(m) < h(l)$ or $h(m) = h(l)$ and $n_5(m) \prec n_5(l)$. This relation is a total order, this is straightforward to check.

2.6.3 Action of T_3 and T_5

h on modular forms For the rest of the section, we will write a modular form $f \in \mathcal{F}$ as follows: $f = \Delta^{m_1} + \Delta^{m_2} + \dots + \Delta^{m_r}$ with $m_1 > m_2 > \dots > m_r$. In this case, m_1 is the *degree* of f , and we will denote it by ∂f . We write $\partial f = -\infty$ if $f = 0$. We define $h(f) = h(m_1)$, and $h(f) = -\infty$ if $f = 0$. Similarly, we define $n_3(f) = n_3(m_1)$ and $n_5(f) = n_5(m_1)$, and $n_3(f) = n_5(f) = -\infty$ if $f = 0$.

Action of T_3 Here, we want to give a description for the behaviour of $h(T_3|f)$ and $\partial T_3|f$.

Proposition 2.1. *Let $f \in \mathcal{F}$ be a non-zero modular form modulo 2.*

1. *In general, $h(T_3|f) \leq h(f) - 1$.*
2. *If $n_3(f) > 0$, then $h(T_3|f) = h(f) - 1$ and $\partial T_3|f$ has code $[n_3(f) - 1, n_5(f)]$.*

This proposition is stated in [Nicolas and Serre, 2012a, §4], and a complete proof is given in Gerbelli-Gauthier [2014].

Action of T_5 Similarly to the above for T_3 , we want to give a description for the behaviour of $h(T_5|f)$ and $\partial T_5|f$.

Proposition 2.2. *Let $f \in \mathcal{F}$ be a non-zero modular form modulo 2.*

1. In general, $h(T_5|f) \leq h(f) - 1$.

2. If $n_5(f) > 0$, then $h(T_5|f) = h(f) - 1$ and $\partial T_5|f$ has code $[n_3(f), n_5(f) - 1]$.

Again, this proposition is stated in [Nicolas and Serre, 2012a, §4], and a complete proof is given in Gerbelli-Gauthier [2014].

2.6.4 Formula for the Order of Nilpotency

Lower bound

Property 2.3. Let $f \in \mathcal{F}$ be a non-zero modular form modulo 2 of degree m_1 . Then we have:

$$T_3^{n_3(m_1)} T_5^{n_5(m_1)} |f = \Delta$$

Proof. We apply $n_3(m_1)$ times the proposition about T_3 above, and $n_5(m_1)$ times the proposition about T_5 , to get that the degree of $g = T_3^{n_3(m_1)} T_5^{n_5(m_1)} |f$ has code $[0, 0]$. The proposition also implies that $h(g) > -\infty$, i.e. $g \neq 0$. Therefore, $\partial g > -\infty$, so it must be odd. since it has code $[0, 0]$, $\partial g = 1$, thus $g = \Delta$. \square

Corollary 2.1. Let $f \in \mathcal{F}$ be a non-zero modular form modulo 2. We have $g(f) \geq h(f) + 1$.

Proof. This is directly implied by the proposition: as $T_3^{n_3(m_1)} T_5^{n_5(m_1)} |f = \Delta \neq 0$, we have $g(f) \geq n_3(f) + n_5(f) + 1 = h(k) + 1$. \square

Representation of \mathcal{F}

Table This means that for each k odd, there is pair $[a, b] = [n_3(k), n_5(k)] \in \mathbb{N} \times \mathbb{N}$ (which corresponds to the code of k), such that $T_3^a T_5^b | \Delta^k = \Delta$. Explicitly, $T_3^{n_3(k)} T_5^{n_5(k)} | \Delta^k = \Delta$. Thus, we can arrange all odd powers of the discriminant Δ in a table, such that applying the corresponding Hecke operators give exactly Δ^1 :

	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	T_5^5	T_5^6	T_5^7	T_5^8	T_5^9	T_5^{10}
T_3^0	Δ^1	Δ^5	Δ^{17}	Δ^{21}	Δ^{65}	Δ^{69}	Δ^{81}	Δ^{85}	Δ^{257}	Δ^{261}	Δ^{273}
T_3^1	Δ^3	Δ^7	Δ^{19}	Δ^{23}	Δ^{67}	Δ^{71}	Δ^{83}	Δ^{87}	Δ^{259}	Δ^{263}	Δ^{275}
T_3^2	Δ^9	Δ^{13}	Δ^{25}	Δ^{29}	Δ^{73}	Δ^{77}	Δ^{89}	Δ^{93}	Δ^{265}	Δ^{269}	Δ^{281}
T_3^3	Δ^{11}	Δ^{15}	Δ^{27}	Δ^{31}	Δ^{75}	Δ^{79}	Δ^{91}	Δ^{95}	Δ^{267}	Δ^{271}	Δ^{283}
T_3^4	Δ^{33}	Δ^{37}	Δ^{49}	Δ^{53}	Δ^{97}	Δ^{101}	Δ^{113}	Δ^{117}	Δ^{289}	Δ^{293}	Δ^{305}
T_3^5	Δ^{35}	Δ^{39}	Δ^{51}	Δ^{55}	Δ^{99}	Δ^{103}	Δ^{115}	Δ^{119}	Δ^{291}	Δ^{295}	Δ^{307}
T_3^6	Δ^{41}	Δ^{45}	Δ^{57}	Δ^{61}	Δ^{105}	Δ^{109}	Δ^{121}	Δ^{125}	Δ^{297}	Δ^{301}	Δ^{313}
T_3^7	Δ^{43}	Δ^{47}	Δ^{59}	Δ^{63}	Δ^{107}	Δ^{111}	Δ^{123}	Δ^{127}	Δ^{299}	Δ^{303}	Δ^{315}
T_3^8	Δ^{129}	Δ^{133}	Δ^{145}	Δ^{149}	Δ^{193}	Δ^{197}	Δ^{209}	Δ^{213}	Δ^{385}	Δ^{389}	Δ^{401}
T_3^9	Δ^{131}	Δ^{135}	Δ^{147}	Δ^{151}	Δ^{195}	Δ^{199}	Δ^{211}	Δ^{215}	Δ^{387}	Δ^{391}	Δ^{403}
T_3^{10}	Δ^{137}	Δ^{141}	Δ^{153}	Δ^{157}	Δ^{201}	Δ^{205}	Δ^{217}	Δ^{221}	Δ^{393}	Δ^{397}	Δ^{409}

Table of powers Δ^k such that the corresponding operator applied to Δ^k gives Δ^1 .
(i.e. Δ^k such that $T_3^{line} T_5^{column} | \Delta^k = \Delta$.)

(A larger table can be found in A.4.)

From the fact that codes are in bijection with odd integers, we can use this table as a basis for modular forms modulo 2.

Exact Formula We derive here an explicit formula for the order of nilpotency of a modular form modulo 2.

Theorem 5 (Order of Nilpotency of Modular Forms Modulo 2). *[Nicolas and Serre, 2012a, §5]. Let $f \in \mathcal{F}$ be a non-zero modular form modulo 2. The order of nilpotency is exactly $g(f) = h(f) + 1$.*

What remains to prove is that $g(f) \leq h(f) + 1$. This is proved in [Nicolas and Serre, 2012a, §5].

From this follows a new remark, which is useful to estimate computation times: as $g(f) = h(f) + 1$, and $h(f) = h(\partial f) = \mathcal{O}(\sqrt{\partial f})$, we have $g(f) = \mathcal{O}(\sqrt{\partial f})$, i.e. the nilpotency order of a modular form behaves asymptotically as the square root of its degree.

Corollary 2.2. *Let $f \in \mathcal{F}$ be a non-zero modular form modulo 2. If $T_3|f = T_5|f = 0$, then $f = \Delta$.*

Proof. By the previous proposition, we have both $n_3(f) = 0$ and $n_5(f) = 0$. Thus, ∂f has code $[0, 0]$. Since $f \neq 0$ and $f \in \mathcal{F}$, this means $f = \Delta$. □

3 Hecke Algebra

This section follows from Nicolas and Serre [2012b].

We recall $\mathcal{F} = \langle \Delta^k \mid k \text{ odd} \rangle$, i.e. $\mathcal{F} = \langle \Delta, \Delta^3, \Delta^5, \Delta^7, \Delta^9, \dots \rangle$.

3.1 Definition

We recall $\mathcal{F}(n) = \langle \Delta, \Delta^3, \Delta^5, \dots, \Delta^{2n-1} \rangle$ so that $\dim(\mathcal{F}(n)) = n$. We define $A(n)$ as the \mathbb{F}_2 -subalgebra of $\text{End}(\mathcal{F}(n))$ given by \mathbb{F}_2 and T_p . That is, if $\mathfrak{m}(n) = \{T_{p_1} \cdot T_{p_2} \cdots T_{p_k} \mid p_1, p_2, \dots, p_k \in \mathbb{P}, k \geq 1\}$ is a sub-vector-space of \mathcal{F} , we get $A(n) = \mathbb{F}_2 \oplus \mathfrak{m}(n)$.

Property 3.1. $\mathfrak{m}(n)$ is the only maximal ideal of $A(n)$.

Proof. Firstly, we note that $A(n)/\mathfrak{m} \cong \mathbb{F}_2$. Since \mathbb{F}_2 is a field, \mathfrak{m} must be a maximal ideal.

Now, suppose I is another (i.e. $I \neq \mathfrak{m}$) maximal ideal of $A(n)$. Then there is an operator $u \in \mathfrak{m}$ such that $(1 + u) \in I$. Since Hecke operators are nilpotent 2.5.3, there exists $n \in \mathbb{N}$ such that $u^n = 0$. By induction, $(1 + u^n) \in I$ for all $n \geq 1$. \square

Note that as Hecke operators are all nilpotent 2.5.3, the ideal $\mathfrak{m}(n)$ is itself nilpotent. In fact, from the minimum 2.6.1 and maximum 2.6.1 nilpotency order property extend to the ideal \mathfrak{m} . Let the dual of $\mathbb{F}(n)$ be $\mathcal{F}(n)^* = \{F : \mathcal{F}(n) \rightarrow \mathbb{F}_2\}$. Then $\mathbb{F}(n)^*$ is an $A(n)$ -module with operation $(u \cdot F)(f) = F(u|f)$ for $u \in A(n)$ and $F \in \mathcal{F}(n)$.

We define e_n to be the element of $\mathcal{F}(n)$ such that $e_n(\Delta) = 1$ and $e_n(\Delta^{2j+1}) = 0$ for all $1 \leq j < n$ (i.e. characteristic of Δ). Denote the q -coefficients of a modular form f by $a_m(f)$, so $f = \sum_{m \geq 0} a_m(f)q^m$. Then $e_n(f) = a_1(f)$. For an odd prime p , have $a_1(T_p|f) = a_p(f)$, so $T_p \cdot e_n(f) = a_p(f)$. By induction, this gives for odd primes p_1, p_2, \dots, p_k :

$$T_{p_1} T_{p_2} \cdots T_{p_k} \cdot e_n(f) = a_{p_1 p_2 \cdots p_k}(f).$$

To fit naturally with the above definitions, we define the Hecke algebra A as follows: as $\mathcal{F}(n) \subset \mathcal{F}(n+1)$, we can restrict elements of $A(n+1)$ to \mathcal{F} to obtain an element of $A(n)$. If we consider the map $\phi_n : A(n+1) \rightarrow A(n)$ to be the restriction to $\mathcal{F}(n)$, then ϕ_n is a homomorphism. As $A(1)$ is either identity or zero, $A(1) \cong \mathbb{F}_2$. Therefore, we have the chain:

$$\cdots \rightarrow A(n+1) \rightarrow A(n) \rightarrow A(n-1) \rightarrow \cdots \rightarrow A(2) \rightarrow A(1) \cong \mathbb{F}_2.$$

We then define the Hecke algebra A to be the projective limit of the above $A(n)$ as $n \rightarrow \infty$. Explicitly, this means

$$A = \varprojlim_{n \in \mathbb{N}} A(n) = \{T_{p_1} \cdot T_{p_2} \cdots T_{p_k} \mid p_1, p_2, \dots, p_k \in \mathbb{P}, k \geq 0\}.$$

3.2 Basic Properties

Property 3.2. Note that for a non zero modular forms $f \in \mathcal{F}(n)$, there exists an operator $u \in A(n)$ such that $e_n(u|f) = 1$.

Proof. We can write $f = q^m + \mathcal{O}(q^{m+1})$ for some m odd (as $\mathcal{F}(n)$ is generated by odd powers of Δ). Now, as m is odd, $m = p_1 p_2 \cdots p_k$ with p_i odd primes for all $1 \leq i \leq k$. Then, by the above, $T_{p_1} T_{p_2} \cdots T_{p_k} \cdot e_n(f) = a_m(f) = 1$. Letting $u = T_{p_1} T_{p_2} \cdots T_{p_k}$, we have $e_n(u|f) = u \cdot e_n(f) = 1$. \square

Property 3.3. $\mathcal{F}(n)^*$ ⁴ is free as an $A(n)$ -module, with basis e_n ; i.e., $\mathcal{F}(n)^* = A(n) \cdot e_n$.

⁴ $\mathcal{F}(n)^*$ denotes the dual of $\mathcal{F}(n)$.

Proof. By contradiction, suppose (e_n) (the $A(n)$ -module generated by e_n) isn't $\mathcal{F}(n)^*$. Then there must be a non-zero modular form $f \in \mathcal{F}(n)$ such that $u \cdot e_n(f) = 0$ for all $u \in A(n)$. This would contradict the last property. Therefore, we have $\mathcal{F}(n)^* = A(n) \cdot e_n$. \square

Corollary 3.1. *From last property, we deduce:*

- The map $\phi : A(n) \rightarrow \mathcal{F}(n)^*$ such that $\phi(u) = u \cdot e_n$ is a bijection.
- The dimension of $A(n)$ is n .
- There is a bijection $\phi : A(n) \rightarrow \mathcal{F}(n)^*$.

Proof. We prove separately:

- This follows directly from the fact that $\mathcal{F}(n)^* = A(n) \cdot e_n$.
- We have $\dim(A(n)) = \dim(\mathcal{F}(n)^*)$ by the above, and $\dim(\mathcal{F}(n)^*) = \dim(\mathcal{F}(n))$ by duality.
- Since $\mathcal{F}(n)^* \leftrightarrow A(n)$ ⁵, $\mathcal{F}(n)^{**} \leftrightarrow A(n)^*$. And as $\mathcal{F}(n)^{**} \cong \mathcal{F}(n)$, we have $\mathcal{F}(n) \leftrightarrow A(n)^*$.

\square

3.3 As Generated by T_3 and T_5

The Hecke algebra is in fact generated by powers of T_3 and T_5 , i.e., we have:

$$A = \mathbb{F}_2[[T_3, T_5]].$$

The strategy to show this splits in two parts: first, we show that $A(n) = \mathbb{F}_2[T_3, T_5]$ (with T_3 and T_5 seen in $A(n)$). Then, we show that this equation remains when taking the limit.

Property 3.4. *We have $A(n) = \mathbb{F}_2[T_3, T_5]$.*

Proof. We define $A'(n) = \mathbb{F}_2[T_3, T_5]$ as a subalgebra of $A(n)$. This is a local algebra (i.e. it has a unique maximal ideal). The idea here is similar to the one involved in A being a local algebra (see 3.1). We denote the maximal ideal of $A'(n)$ by \mathfrak{m}' .

Suppose, for contradiction, that $A' \neq A$, so $\dim(A') < n$. If $\mathcal{F}(n)^*$ (seen as a $A'(n)$ -module), was cyclic (i.e. generated by a single element), then $\dim(\mathcal{F}(n)^*) < n$. However, we know $\dim(\mathcal{F}(n)^*) = n$, so $\mathcal{F}(n)^*$ isn't cyclic. We define $V = \mathcal{F}(n)^* / \mathfrak{m}' \mathcal{F}(n)^*$. Nakayama lemma under the statement for maximal ideals implies that V has dimension > 1 .

Now we put $U = \{f \in \mathcal{F}(n) \mid a|f = 0 \ \forall a \in \mathfrak{m}'\}$, i.e. the modular forms that are zero after application of any operator in \mathfrak{m}' . We want to show that this vector space U has dimension > 1 . As vector spaces, we know: $(W_1/W_2)^* \cong W_2^0$, so:

$$\left(\frac{\mathcal{F}(n)^*}{\mathfrak{m}' \mathcal{F}(n)^*} \right)^* \cong (\mathfrak{m}' \mathcal{F}(n)^*)^0.$$

Where $(\mathfrak{m}' \mathcal{F}(n)^*)^0 = \{\tilde{f} \in \mathcal{F}(n)^{**} \mid \tilde{f}(a) = 0 \ \forall a \in \mathfrak{m}' \mathcal{F}(n)^*\}$ is the inhalator of $\mathfrak{m}' \mathcal{F}(n)^*$. We know there is an isomorphism between $\mathcal{F}(n)^{**}$ and $\mathcal{F}(n)$ ⁶. Thus, we have: $(\mathfrak{m}' \mathcal{F}(n)^*)^0 \cong \{f \in \mathcal{F}(n) \mid a|f = 0 \ \forall a \in \mathfrak{m}' \mathcal{F}(n)^*\}$. But one may recognize that this is exactly U . Therefore, $\dim(U) = \dim((\mathfrak{m}' \mathcal{F}(n)^*)^0) = \dim(V) = \dim(V^*) > 1$.

Thus, $\{0, \Delta\}$ is a subspace of U with dimension 1. As $\dim(U) > 1$, there must exist a modular form $f \in \mathcal{F}(n)$ such that: f is neither 0 or Δ , and $a|f = 0$ for all $a \in \mathfrak{m}'$. In particular, this means that $T_3|f = 0$ and $T_5|f = 0$. However, this contradicts the Order of Nilpotency of Modular Forms Modulo 2 corollary 2.6.4. Thus we have $A(n) = A'(n) = \mathbb{F}_2[T_3, T_5]$. \square

⁵ $A \leftrightarrow B$ means there exists a bijection from A to B

⁶ Just take $\phi : \mathcal{F}(n) \rightarrow \mathcal{F}(n)^{**}$ such that $f \mapsto [F \mapsto F(f)]$.

Property 3.5. We have $A = \mathbb{F}_2[[T_3, T_5]]$.

Proof. For any n , there is a homomorphism $\psi_n : \mathbb{F}_2[x, y] \rightarrow A(n)$ with $\psi_n(x) = T_3$ and $\psi_n(y) = T_5$. Therefore, we can take the limit as $n \rightarrow \infty$ to get a homomorphism $\psi : \mathbb{F}_2[[x, y]] \rightarrow A$ such that $\psi(x) = T_3$ and $\psi(y) = T_5$.

The fact that ψ is surjective follows directly from the last property.

Now we want to show that ψ is injective. Since it is already homomorphic, it suffices to show that for any element $u = \sum_{i,j} \lambda_{ij} T_3^i T_5^j$, there exist a modular form modulo 2 f such that $u|f = \sum_{i,j} \lambda_{ij} T_3^i T_5^j |f = \Delta$. Note that this sum is finite, as Hecke operators are nilpotent. If $\lambda_{00} = 1$, take $f = \Delta$. Suppose $\lambda_{00} = 0$: consider $S = \{(i, j) \in \mathbb{N}^2 \mid \lambda_{ij} = 1\}$.

Let $S_{min} = \{(m, n) \mid m+n \leq i+j \ \forall (i, j) \in S\}$ and $(a, b) \in S_{min}$ be such that $b \leq n \ \forall (m, n) \in S_{min}$. Let $k = [a, b]$ (k is the odd integer with code $[a, b]$ as in 2.6.2), and let $f = \Delta^k$. Then, by the theorem of Order Of Nilpotency 2.6.4, we have $T_3^a T_5^b |f = \Delta$. Moreover, by proposition on action of T_3 and T_5 (2.6.3 and 2.6.3), we have that $T_3^i T_5^j |f = 0$ for all $(i, j) \in S$. Thus, $u|f = \Delta$.

Therefore, ψ is an isomorphism, which completes the proof. \square

3.4 Expansion of T_p as Series of T_3 and T_5

As proved above, $A = \mathbb{F}_2[[T_3, T_5]]$, this means that each Hecke operator T_p (with p prime) may be expressed as a series of T_3 and T_5 . By comparing tables of prime Hecke operators and table of powers of Hecke operators, we can deduce the first coefficients a_{ij} in the expansion

$$T_p = \sum_{i+j \geq 1} a_{ij}(p) T_3^i T_5^j.$$

Note that the coefficients will play an important role in the next sections. We compare the two following tables:

Table of Prime Hecke Operators

	Δ^1	Δ^3	Δ^5	Δ^7	Δ^9	Δ^{11}	Δ^{13}	Δ^{15}	Δ^{17}	Δ^{19}
T_3	0	Δ	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^9 + \Delta^{17}$
T_5	0	0	Δ	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7
T_7	0	0	0	Δ	0	0	Δ^3	Δ^9	0	Δ^5
T_{11}	0	Δ	0	Δ^5	Δ^3	$\Delta + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^9 + \Delta^{17}$
T_{13}	0	0	Δ	Δ^3	0	0	$\Delta + \Delta^9$	Δ^{11}	Δ^5	Δ^7
T_{17}	0	0	0	0	Δ	Δ^3	Δ^5	Δ^7	Δ	0
T_{19}	0	Δ	0	Δ^5	Δ^3	$\Delta + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta + \Delta^9 + \Delta^{17}$

Action of Primes Hecke Operators (primes up to 20) on Modular Forms Modulo 2 (up to Δ^{19}).

A larger table may be found in the appendix (see A.1).

Table of Powers of Hecke Operators

Δ^1	T_5^0	T_5^1	T_5^2	...	Δ^3	T_5^0	T_5^1	T_5^2	...	Δ^5	T_5^0	T_5^1	T_5^2	...	Δ^7	T_5^0	T_5^1	T_5^2	...
T_3^0	Δ^1	0	0	...	T_3^0	Δ^3	0	0	...	T_3^0	Δ^5	Δ^1	0	...	T_3^0	Δ^7	Δ^3	0	...
T_3^1	0	0	0	...	T_3^1	Δ^1	0	0	...	T_3^1	0	0	0	...	T_3^1	Δ^5	Δ^1	0	...
T_3^2	0	0	0	...	T_3^2	0	0	0	...	T_3^2	0	0	0	...	T_3^2	0	0	0	...
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\ddots

Δ^9	T_5^0	T_5^1	T_5^2	T_5^3	...	Δ^{11}	T_5^0	T_5^1	T_5^2	T_5^3	...	Δ^{13}	T_5^0	T_5^1	T_5^2	T_5^3	...
T_3^0	Δ^9	0	0	0	...	T_3^0	Δ^{11}	0	0	0	...	T_3^0	Δ^{13}	Δ^9	0	0	...
T_3^1	Δ^3	0	0	0	...	T_3^1	Δ^9	0	0	0	...	T_3^1	Δ^7	Δ^3	0	0	...
T_3^2	Δ^1	0	0	0	...	T_3^2	Δ^3	0	0	0	...	T_3^2	Δ^5	Δ^1	0	0	...
T_3^3	0	0	0	0	...	T_3^3	Δ^1	0	0	0	...	T_3^3	0	0	0	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Action of Powers Hecke Operators T_3 and T_5 on Modular Forms Modulo 2 (up to Δ^{13}).

A larger table may be found in the appendix (see A.2).

Expansions Deduced The method of comparison is detailed in 6.3. Here are the obtained expansions of T_p in series of $T_3^a T_5^b$ for primes $p < 20$:

$$T_3 = T_3^1 T_5^0$$

$$T_5 = T_3^0 T_5^1$$

$$T_7 = T_3^1 T_5^1 + T_3^3 T_5^1 + T_3^3 T_5^3 + T_3^5 T_5^1 + T_3^1 T_5^7 + T_3^1 T_5^9 + T_3^7 T_5^3 + T_3^7 T_5^5 + T_3^9 T_5^3 + T_3^{11} T_5^1 + T_3^3 T_5^{11} + T_3^5 T_5^9 + T_3^{13} T_5^1 + T_3^3 T_5^{13} + T_3^5 T_5^{11} + T_3^9 T_5^7 + T_3^{11} T_5^5 + T_3^{13} T_5^3 + T_3^3 T_5^{15} + T_3^7 T_5^{11} + T_3^9 T_5^9 + T_3^{13} T_5^5 + T_3^{15} T_5^3 + \dots$$

$$T_{11} = T_3^1 T_5^0 + T_3^1 T_5^2 + T_3^3 T_5^0 + T_3^1 T_5^4 + T_3^3 T_5^2 + T_3^5 T_5^0 + T_3^1 T_5^6 + T_3^3 T_5^4 + T_3^7 T_5^2 + T_3^1 T_5^{10} + T_3^3 T_5^8 + T_3^7 T_5^4 + T_3^9 T_5^2 + T_3^{11} T_5^2 + T_3^3 T_5^{12} + T_3^5 T_5^{10} + T_3^7 T_5^8 + T_3^{11} T_5^4 + T_3^{13} T_5^2 + T_3^9 T_5^8 + T_3^{17} T_5^0 + \dots$$

$$T_{13} = T_3^0 T_5^1 + T_3^0 T_5^3 + T_3^2 T_5^1 + T_3^0 T_5^5 + T_3^4 T_5^1 + T_3^2 T_5^3 + T_3^4 T_5^3 + T_3^6 T_5^1 + T_3^0 T_5^9 + T_3^2 T_5^7 + T_3^6 T_5^3 + T_3^0 T_5^{11} + T_3^6 T_5^5 + T_3^8 T_5^3 + T_3^{10} T_5^1 + T_3^2 T_5^{11} + T_3^4 T_5^9 + T_3^6 T_5^7 + T_3^{10} T_5^3 + T_3^2 T_5^{13} + T_3^4 T_5^{11} + T_3^{14} T_5^1 + T_3^2 T_5^{15} + T_3^4 T_5^{13} + T_3^6 T_5^{11} + T_3^{12} T_5^5 + T_3^{16} T_5^1 + \dots$$

$$T_{17} = T_3^0 T_5^2 + T_3^2 T_5^0 + T_3^2 T_5^2 + T_3^0 T_5^6 + T_3^4 T_5^2 + T_3^6 T_5^0 + T_3^2 T_5^6 + T_3^4 T_5^4 + T_3^6 T_5^2 + T_3^{10} T_5^0 + T_3^2 T_5^{10} + T_3^4 T_5^8 + T_3^6 T_5^6 + T_3^{10} T_5^2 + T_3^2 T_5^{12} + T_3^6 T_5^8 + T_3^{10} T_5^4 + T_3^2 T_5^{14} + T_3^6 T_5^{10} + T_3^8 T_5^8 + T_3^{12} T_5^4 + T_3^{14} T_5^2 + T_3^4 T_5^{14} + T_3^8 T_5^{10} + T_3^{10} T_5^8 + T_3^{12} T_5^6 + T_3^{16} T_5^2 + T_3^{18} T_5^0 + \dots$$

$$T_{19} = T_3^1 T_5^0 + T_3^3 T_5^0 + T_3^1 T_5^4 + T_3^3 T_5^2 + T_3^1 T_5^6 + T_3^5 T_5^2 + T_3^3 T_5^6 + T_3^7 T_5^2 + T_3^9 T_5^0 + T_3^1 T_5^{10} + T_3^7 T_5^4 + T_3^9 T_5^2 + T_3^{11} T_5^0 + T_3^3 T_5^{12} + T_3^5 T_5^8 + T_3^{11} T_5^2 + T_3^{13} T_5^0 + T_3^3 T_5^{12} + T_3^7 T_5^8 + T_3^9 T_5^6 + T_3^{11} T_5^4 + T_3^{13} T_5^2 + T_3^3 T_5^{14} + T_3^7 T_5^{10} + T_3^{11} T_5^6 + T_3^{15} T_5^2 + T_3^{17} T_5^0 + \dots$$

Expansions for larger primes may be found in A.5.

4 Frobenian Elements

4.1 Context

4.1.1 General

Basic Properties Let R be a commutative ring, M and P ideals in R . We can then prove the followings:

- M is maximal $\iff R/M$ is a field
- P is prime $\iff R/P$ is an integral domain

Moreover, we have:

Property 4.1. *Maximal ideals are prime.*

Proof.

$$\begin{aligned} M \text{ maximal ideal} &\iff R/M \text{ field} \\ &\implies R/M \text{ Integral Domain} \iff M \text{ prime ideal} \end{aligned}$$

□

Definitions and Notations If L/K is a Galois extension, then we will denote it's Galois group by $\text{Gal}(L/K)$.

Let K be a number field, and \mathcal{O}_K be the corresponding ring of integers. Let \mathfrak{p} be a non-zero prime ideal in \mathcal{O}_K . Let L/K be a finite extension and again, \mathcal{O}_L be the ring of integers in L . Then we know that \mathcal{O}_L is the integral closure of \mathcal{O}_K in L . We have $\mathfrak{p}\mathcal{O}_L$ an ideal in \mathcal{O}_L . It is not a prime ideal in general, but as L/K is finite, there exists a factorization as the following:

$$\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^r \mathfrak{P}_i^{e_i}.$$

Where the integers e_i are called the ramification indexes. We also have $\mathfrak{P}_i \cap \mathcal{O}_K = \mathfrak{p}$, and we say that the ideals \mathfrak{P}_i in L extend the ideal \mathfrak{p} in K .

Then, there are three possibilities for an ideal: it may split, ramify or be inert.

Definition 4.1 (Ideal Ramifies). *We say that an ideal \mathfrak{p} ramifies in L/K if a ramification index e_i is greater than one, i.e. if $e_i > 1$ for some $1 \leq i \leq r$.*

Definition 4.2 (Ideal Splits). *We say that \mathfrak{p} splits in L/K if none of the ramification indexes e_i is greater than one, and r is at least two; i.e. if $e_i = 1 \quad \forall 1 \leq i \leq r$ and $r \geq 2$.*

Definition 4.3 (Ideal Inert). *We say that \mathfrak{p} is inert in L/K if there is only one ramification index e_1 and it is equal to one; i.e. if $e_1 = 1$ and $r = 1$.*

We know that the extension L/K is ramified in the primes that divide the discriminant. Therefore, the extension is unramified in all but finitely many prime ideals.

4.1.2 Residue Fields Extensions

The ideal \mathfrak{p} defines the *residue field* $F = \mathcal{O}_K/\mathfrak{p}$. The ideals \mathfrak{P}_i define the *residue fields* $F_i = \mathcal{O}_L/\mathfrak{P}_i$. The field F then naturally embeds to F_i (so each \mathfrak{P}_i defines a field extension). The *inertia degree* of \mathfrak{P}_i is the degree $f_i = [F_i : F] = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$ of this extension. We then observe that $[L : K] = \sum_{i=1}^r e_i f_i$. We can then specify when an ideal splits or ramifies completely:

Definition 4.4 (Ideal Splits Completely). *We say that \mathfrak{p} splits completely in L/K if all ramification indexes e_i and inertia degrees f_i are one. i.e. if $e_i = f_i = 1 \quad \forall 1 \leq i \leq r$. In this case, $r = [L : K]$.*

Definition 4.5 (Ideal Ramifies Completely). *We say that \mathfrak{p} ramifies completely in L/K if the inertia degrees f_1 is one, and r is one. i.e. if $r = 1$ and $f_1 = 1$. In this case, $e_1 = [L : K]$.*

4.1.3 Norms of Ideals

We define the *norm of an ideal* I in \mathcal{O}_K as $N(I) = |\mathcal{O}_K/I|$. If $\mathfrak{p} \subset \mathcal{O}_K$ is a prime ideal, then we can put $(p) = \mathfrak{p} \cap \mathbb{Z}$. It follows that $p\mathcal{O}_K \subset \mathfrak{p}$. \mathcal{O}_K is a free \mathbb{Z} -module of rank $[K : \mathbb{Q}] = q$, i.e. $\exists \alpha_1, \dots, \alpha_q$ s.t. $\mathcal{O}_K = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_q$. Thus, $|\mathcal{O}_K/\mathfrak{p}| \leq |\mathcal{O}_K/(p)| \leq p^q$. We have $\text{Norm}(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}| = p^m$ and $\text{Norm}_{L/\mathbb{Q}}(\mathfrak{P}_i) = \text{Norm}_{K/\mathbb{Q}}(\mathfrak{p})^{f_i}$. This implies $\text{Norm}(\mathfrak{P}_i) = |\mathcal{O}_L/\mathfrak{P}_i| = p^{mf_i}$. We also have: $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_{\text{Norm}(\mathfrak{p})}$ and $\mathcal{O}_L/\mathfrak{P}_i \cong \mathbb{F}_{\text{Norm}(\mathfrak{P}_i)}$.

4.1.4 Galois Extension Simplifications

When the extension L/K is Galois, the ramification indexes e_i are all the same ($e_i = e$), as well as the inertia degrees $f_i = f$. We then have

$$\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^r \mathfrak{P}_i^e \text{ and } [L : K] = ref.$$

The Galois group $\text{Gal}(L/K)$ is often denoted G .

We define the *decomposition group* $G_{\mathfrak{P}}$ of the ideal \mathfrak{P} to be $\{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$. It turns out that $G_{\mathfrak{P}} \cong \text{Gal}(\mathcal{O}_L/\mathfrak{P}/\mathcal{O}_K/\mathfrak{p}) \cong \text{Gal}(\mathbb{F}_{p^{mf}}/\mathbb{F}_{p^f})$. Moreover, it is a cyclic group, so $G_{\mathfrak{P}} = \langle \tilde{\sigma} \rangle$.

4.1.5 Unramified Prime Simplifications

When the ideal \mathfrak{p} is unramified, $e = 1$, so we get:

$$\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^r \mathfrak{P}_i \text{ and } [L : K] = rf.$$

4.2 The Frobenius Element

4.2.1 Definition

We can construct the *Frobenius element* (sometimes also called the *Artin symbol*, or the *Frobenius map*) that depend on the extension L/K and ideal \mathfrak{P} in \mathcal{O}_L . It is denoted $\text{Frob}_{L/K}(\mathfrak{P})$, and is the element $\sigma \in G$ such that:

$$\sigma\mathfrak{P} = \mathfrak{P} \quad \text{and} \quad \sigma(\alpha) \equiv \alpha^{\text{Norm}_{K/\mathbb{Q}}(\mathfrak{p})} \pmod{\mathfrak{P}} \quad \forall \alpha \in \mathcal{O}_L.$$

The second condition is the interesting one; while the first is only useful to make the Frobenius element unique. The second condition defines a unique element only up to conjugacy class. Most of

the time, we will consider abelian extensions, so the conjugacy classes will only have one element, and the first condition will be dropped.

We define the *Frobenius element* for \mathfrak{p} (denoted $\text{Frob}_{L/K}(\mathfrak{p})$) in a meaning full manner, to be the set

$$\{\text{Frob}_{L/K}(\mathfrak{P}) | \mathfrak{P} \text{ extending } \mathfrak{p}\} \subset G.$$

The following properties imply that $\text{Frob}_{L/K}(\mathfrak{p})$ is in fact a conjugacy class in the Galois group $\text{Gal}(L/K)$. Hence, we refer to $\text{Frob}_{L/K}(\mathfrak{p})$ as the Frobenius conjugacy class.

Property 4.2. *If $\tau \in G$, then $\text{Frob}_{L/K}(\tau\mathfrak{P}) = \tau \text{Frob}_{L/K}(\mathfrak{P}) \tau^{-1}$.*

Proof. For all $x \in \mathcal{O}_L$, we have:

$$\text{Frob}_{L/K}(\mathfrak{P})x = x^{\text{Norm}_{K/\mathbb{Q}}(\mathfrak{p})} \pmod{\mathfrak{P}}$$

But all such x may be written as $\tau^{-1}(x)$, so we have:

$$\text{Frob}_{L/K}(\mathfrak{P})\tau^{-1}(x) = (\tau^{-1}x)^{\text{Norm}_{K/\mathbb{Q}}(\mathfrak{p})} \pmod{\mathfrak{P}}.$$

Which gives:

$$\tau \text{Frob}_{L/K}(\mathfrak{P}) \tau^{-1}(x) = x^{\text{Norm}_{K/\mathbb{Q}}(\mathfrak{p})} \pmod{\mathfrak{P}}.$$

□

Property 4.3. *If \mathfrak{P}_1 and \mathfrak{P}_2 extend \mathfrak{p} , then $\text{Frob}_{L/K}(\mathfrak{P}_1)$ and $\text{Frob}_{L/K}(\mathfrak{P}_2)$ are conjugates.*

Proof. We have the following scheme:

$$\begin{array}{ccc} L & \supseteq & \mathfrak{P}_1 \quad \mathfrak{P}_2 \\ | & & \searrow \quad \swarrow \\ K & \supseteq & \mathfrak{p} \end{array}$$

There is an element $\tau \in G$ such that $\tau(\mathfrak{P}_1) = \mathfrak{P}_2$. Then using last property, we deduce that $\text{Frob}_{L/K}(\mathfrak{P}_1)$ and $\text{Frob}_{L/K}(\mathfrak{P}_2)$ are conjugates. □

Never the less, is important to notice at this point that if L/K is an abelian extension (i.e. G is abelian), then every conjugacy class in $\text{Gal}(L/K)$ is made up of only one element. In this case, we sometimes use $\text{Frob}_{L/K}(\mathfrak{p})$ to denote the Frobenius element $\text{Frob}_{L/K}(\mathfrak{P})$, where \mathfrak{P} is any prime lying above \mathfrak{p} .

4.2.2 Examples

$\mathbb{Q}[\sqrt{7}]/\mathbb{Q}$ (**quadratic field extension**) We have minimum polynomial $m(x) = x^2 - 7$, the discriminant is $\Delta = 4 \cdot 7 = 28$.

We write

$$G = \text{Gal}(\mathbb{Q}[\sqrt{7}] : \mathbb{Q}) = \langle \sigma \mid \sigma^2 = 1_G \rangle \cong C_2.$$

As C_2 is abelian, we will have no problem defining Frobenius elements.

The prime ideal (3) As $m(x) = (x+1)(x-1) \pmod{3}$, we have $(3) = (3, \sqrt{7}+1)(3, \sqrt{7}-1)$. As well, $\text{Norm}_{\mathbb{Q}[\sqrt{7}]/\mathbb{Q}}((3)) = 3$ and $\text{Norm}_{\mathbb{Q}[\sqrt{7}]/\mathbb{Q}}((3, \sqrt{7}+1)) = \text{Norm}_{\mathbb{Q}[\sqrt{7}]/\mathbb{Q}}((3, \sqrt{7}-1)) = 3$, but $\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((3)) = 3$. So we have:

$$\begin{aligned} \text{Frob}_{\mathbb{Q}[\sqrt{7}]:\mathbb{Q}}((3, \sqrt{7}+1)) : \alpha &\mapsto \alpha^{\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((3))} \pmod{(3, \sqrt{7}+1)} \\ \sqrt{7} &\mapsto (\sqrt{7})^3 \equiv \sqrt{7} \pmod{(3, \sqrt{7}+1)} \end{aligned}$$

Thus, $\text{Frob}_{\mathbb{Q}[\sqrt{7}]:\mathbb{Q}}((3, \sqrt{7}+1)) = 1_G \in G$. Similarly, $\text{Frob}_{\mathbb{Q}[\sqrt{7}]:\mathbb{Q}}((3, \sqrt{7}-1)) = 1_G \in G$.

The prime ideal (5) As $m(x)$ has no root mod 2. So $m(x)$ is irreducible mod 5 and (5) is inert in $\mathbb{Q}[\sqrt{7}]$. As well, $\text{Norm}_{\mathbb{Q}[\sqrt{7}]/\mathbb{Q}}((5)) = 5^2 = 25$ but $\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((5)) = 5$. So we have:

$$\begin{aligned} \text{Frob}_{\mathbb{Q}[\sqrt{7}]:\mathbb{Q}}((5)) : \alpha &\mapsto \alpha^{\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((5))} \pmod{(5)} \\ \sqrt{7} &\mapsto (\sqrt{7})^5 \equiv -\sqrt{7} \pmod{(5)} \end{aligned}$$

Thus, $\text{Frob}_{\mathbb{Q}[\sqrt{7}]:\mathbb{Q}}((5)) = \sigma \in G$.

$\mathbb{Q}[\zeta_n]/\mathbb{Q}$ (n^{th} **Cyclotomic Field Extensions**) We have minimum polynomial:

$$\Phi(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} \left(x - e^{2i\pi \frac{k}{n}} \right)$$

(so degree of the extension is $\varphi(n)$, where φ is Euler totient function). Discriminant of the extension is:

$$\Delta = (-1)^{\varphi(n)/2} \frac{n^{\varphi(n)}}{\prod_{p|n} p^{\varphi(n)/(p-1)}};$$

see [Washington, 1997, Proposition 2.7].

The Galois group G consist of σ_k such that $\sigma_k(\zeta_n^i) = \zeta_n^{ik}$, with $\gcd(k, n) = 1$. Note as well that G is abelian, so it is simple to calculate the Frobenius element. It is straightforward that G is naturally isomorphic to the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. Note that $\sigma \in G$ is determined by $\sigma(\zeta_n)$. Note as well that this group is abelian.

With $p \in \mathbb{P}$, a prime that is unramified in $\mathbb{Q}[\zeta_n]/\mathbb{Q}$, let P be an ideal lying above (p) . We want to look at $\text{Frob}_{\mathbb{Q}[\zeta_n]/\mathbb{Q}}(P)$. We have:

$$\begin{aligned} \text{Frob}_{\mathbb{Q}[\zeta_n]/\mathbb{Q}}(P) : \alpha &\mapsto \alpha^{\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((p))} \pmod{P} \\ \zeta_n &\mapsto \zeta_n^p \pmod{P} \end{aligned}$$

Case $\mathbb{Q}[\zeta_{10}]/\mathbb{Q}$ (10^{th} cyclotomic field extension) We denote by $\zeta_{10} = e^{\pi i/5}$ the 10^{th} root of unity. We have minimum polynomial $m(x) = x^4 - x^3 + x^2 - x + 1$ (so degree of the extension is 4), the discriminant is $\Delta = 5^3$.

We write $G = \text{Gal}(\mathbb{Q}[\zeta_{10}] : \mathbb{Q}) = \langle \sigma : \zeta_{10} \mapsto \zeta_{10}^3 \mid \sigma^4 = \text{Id} \rangle \cong C_4$.

The prime ideal (3) As $m(x)$ has no root mod 3, so (3) is inert. We have:

$$\begin{aligned} \text{Frob}_{\mathbb{Q}[\zeta_{10}]/\mathbb{Q}}((3)) : \alpha &\mapsto \alpha^{\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((3))} \pmod{(3)} \\ \zeta_{10} &\mapsto (\zeta_{10})^3 \pmod{(3)} \end{aligned}$$

Thus, $\text{Frob}_{\mathbb{Q}[\zeta_{10}]:\mathbb{Q}}((3)) = \sigma \in G$.

The prime ideal (7) As $m(x)$ has no root mod 7, so (7) is inert. We have:

$$\begin{aligned}\text{Frob}_{\mathbb{Q}[\zeta_{10}]/\mathbb{Q}}((7)) : \alpha &\mapsto \alpha^{\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((7))} \bmod (7) \\ \zeta_{10} &\mapsto (\zeta_{10})^7 \bmod (7)\end{aligned}$$

Thus, $\text{Frob}_{\mathbb{Q}[\zeta_{10}]/\mathbb{Q}}((7)) = \sigma^3 \in G$.

The prime ideal (11) As $m(x) = (x-2)(x+3)(x+4)(x+4) \bmod 11$, so (11) splits. We have:

$$\begin{aligned}\text{Frob}_{\mathbb{Q}[\zeta_{10}]/\mathbb{Q}}((11)) : \alpha &\mapsto \alpha^{\text{Norm}_{\mathbb{Q}/\mathbb{Q}}((11))} \bmod (11) \\ \zeta_{10} &\mapsto (\zeta_{10})^{11} = \zeta_{10} \bmod (11)\end{aligned}$$

Thus, $\text{Frob}_{\mathbb{Q}[\zeta_{10}]/\mathbb{Q}}((11)) = \sigma^4 = Id \in G$.

4.2.3 Behaviour in Towers of Fields

We will consider the following scheme:

$$\begin{array}{ccccc}\mathcal{O}_M \subset M & \supseteq & \mathfrak{P} \\ | & & | \\ \mathcal{O}_L \subset L & \supseteq & \mathfrak{p} \\ | & & | \\ \mathcal{O}_K \subset K & \supseteq & p\end{array}$$

In such a situation, we can define (for M/K Galois) $\text{Frob}_{M/K}(\mathfrak{P})$, $\text{Frob}_{M/K}(p)$, $\text{Frob}_{M/L}(\mathfrak{P})$, $\text{Frob}_{M/L}(\mathfrak{p})$. If, in addition, L/K is normal, we can as well define: $\text{Frob}_{L/K}(\mathfrak{p})$, and $\text{Frob}_{L/K}(p)$ (see [Janusz, 1996, p.99]). We will look at properties of these Frobenius elements (relation between each others).

Property 4.4. [Janusz, 1996, p.99]: We have

$$\text{Frob}_{M/K}(\mathfrak{P})^{f(\mathfrak{P}/\mathfrak{p})} = \text{Frob}_{M/L}(\mathfrak{P}).$$

Property 4.5.

$$\text{Frob}_{L/K}(\mathfrak{p}) = \text{Frob}_{M/K}(\mathfrak{P})|_L$$

Proof. Let $\sigma = \text{Frob}_{M/K}(\mathfrak{P}) \in \text{Gal}(M/K)$ so $\sigma : M \rightarrow M$ s.t. $\sigma|_K = Id$ and σ is an automorphism. Similarly, let $\tau = \text{Frob}_{L/K}(\mathfrak{p}) \in \text{Gal}(L/K)$ so $\tau : L \rightarrow L$ s.t. $\tau|_K = Id$ and τ is an automorphism.

As M extends L , σ being an automorphism of M makes it an automorphism of L as well. The restriction condition stays the same. \square

Property 4.6.

$$\text{Gal}(L/K) \cong \text{Gal}(M/K) / \text{Gal}(M/L)$$

Proof. Let $\sigma \in \text{Gal}(M/K)$, i.e. $\sigma : M \rightarrow M$ s.t. $\sigma|_K = Id$ and σ is an automorphism.

Let $\phi : \text{Gal}(M/K) \rightarrow \text{Gal}(L/K)$ be such that: $\phi(\sigma) = \sigma|_L$. This is well defined as an automorphism of M restricts to an automorphism of L when M extends L .

It is trivial to check that ϕ is a homomorphism.

The kernel of ϕ is clearly $\text{Gal}(M/L)$.

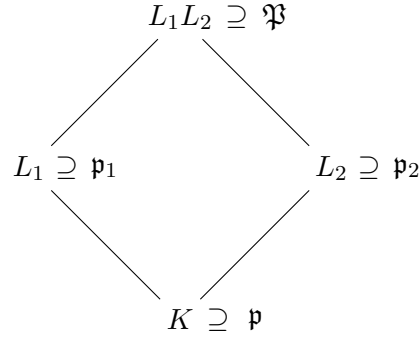
The image of ϕ is $\text{Gal}(L/K)$ as every element of $\text{Gal}(L/K)$ may be extended to $\text{Gal}(M/K)$.

Therefore, the property follows via the 1st isomorphism theorem. \square

Property 4.7. [Janusz, 1996, p.100]: We have

$$\mathfrak{p} \text{ splits completely in } L \iff \text{Frob}_{L/K}(\mathfrak{P}) = 1.$$

We will consider the following scheme:



Property 4.8. [Janusz, 1996, p.100]: We have

$$\text{Frob}_{L_1 L_2 / K}(\mathfrak{P}) = \text{Frob}_{L_1 / K}(\mathfrak{p}_1) \times \text{Frob}_{L_2 / K}(\mathfrak{p}_2).$$

Property 4.9. [Janusz, 1996, p.100]: We have

$$\mathfrak{p} \text{ splits completely in } L_1 L_2 \iff \mathfrak{p} \text{ splits completely in } L_1 \text{ and } L_2.$$

Proof. Combine the last two proposition. □

4.3 The Chebotarev's Density Theorem

4.3.1 Motivations

If we look at the distribution of primes numbers modulo a number (15 in the next example), we get a table as follows:

Table mod 15:

mod15	primes (up to 500)
0	
1	31, 61, 151, 181, 211, 241, 271, 331, 421,
2	2, 17, 47, 107, 137, 167, 197, 227, 257, 317, 347, 467,
3	3,
4	19, 79, 109, 139, 199, 229, 349, 379, 409, 439, 499,
5	5,
6	
7	7, 37, 67, 97, 127, 157, 277, 307, 337, 367, 397, 457, 487,
8	23, 53, 83, 113, 173, 233, 263, 293, 353, 383, 443,
9	
10	
11	11, 41, 71, 101, 131, 191, 251, 281, 311, 401, 431, 461, 491,
12	
13	13, 43, 73, 103, 163, 193, 223, 283, 313, 373, 433, 463,
14	29, 59, 89, 149, 179, 239, 269, 359, 389, 419, 449, 479,

It looks like there are classes of primes. We would like to characterize this repartition: that is, decide if classes are finite or infinite, and quantify the repartitions.

4.3.2 Notions of Density

As discussed previously, we are interested in subsets of \mathbb{P} (the set of primes numbers). Euler proved that there are infinitely many primes. Therefore, there are two types of subsets of \mathbb{P} : the ones that are infinite, and the finite ones. For finite sets, we can characterise the size by just counting elements. In fact, we will mainly be interested in sets that have infinitely many primes, and again, we would like a notion of size.

A suitable way would be to compare the subset with the set of all primes, and, say look at the proportions of primes included in the subset. We call this the density, there are two rigorous ways to define it:

Definition 4.6 (Natural density). *We say that $S \subseteq \mathbb{P}$ has natural density δ when:*

$$\lim_{x \rightarrow +\infty} \frac{\#\{p \in \mathbb{P}, p < x \mid p \in S\}}{\#\{p \in \mathbb{P}, p < x \mid p \in \mathbb{P}\}} = \delta$$

Definition 4.7 (Analytic density or Dirichlet density). *We say that $S \subseteq \mathbb{P}$ has analytical (or Dirichlet) density δ when:*

$$\lim_{s \rightarrow 1^+} \left(\sum_{p \in S} \frac{1}{p^s} \right) \left(\sum_{p \in \mathbb{P}} \frac{1}{p} \right)^{-1} = \delta$$

Note that the natural density may not exist. However, when both exist, the two densities are the same.

4.3.3 Statement

One of the most important results that use Frobenian maps is probably the Chebotarev density theorem.

Theorem 6 (Chebotarev Density Theorem). *With L/K an extension of Galois group $G = \text{Gal}(L/K)$. Let C be a conjugacy class in G .*

Then, the proportion of unramified primes ideals \mathfrak{p} in K that have Frobenius element $\text{Frob}_{L/K}(\mathfrak{p}) = C$ is $|C|/|G|$.

We see that Frobenius elements are in the heart of this theorem. It was proved by Nikolai Chebotarev in his thesis (Tschebotareff [1926]).

4.3.4 Example

We go through an example of Chebotarev Density Theorem for an extension of order 3. We look at K/\mathbb{Q} with $K \cong \mathbb{Q}[x]/(x^3 - 3x - 1)$ (i.e. the number field with defining polynomial $x^3 - 3x - 1$). Using SageMath⁸, we have: The discriminant of this extension is $3^4 = 81$, and the extension is Galois. We define $G = \text{Gal}(K/\mathbb{Q})$ the Galois group of the extension, and we have $G \cong C_3 = \langle \sigma \mid \sigma^3 = 1 \rangle$ since the order of the extension is 3).

Then, an unramified prime in \mathbb{Q} may remain irreducible in K/\mathbb{Q} , split in K/\mathbb{Q} . If p splits in K/\mathbb{Q} , then $\text{Frob}_{K/\mathbb{Q}}(p) = 1$ (the identity of the Galois group G). If p remains inert in K/\mathbb{Q} , then $\text{Frob}_{K/\mathbb{Q}}(p) = \sigma$ or σ^2 . As the discriminant is finite, there are finitely many primes that ramifies. Applying the Chebotarev Density theorem: one third of the primes will split in this extension, and two third will remain inert.

⁷When depending on a prime in the "lower" field, the Frobenius element is a conjugacy class to be well defined.

⁸See the code in appendix, B

4.3.5 Special Case

Here, we want to apply Chebotarev theorem in the case of a quadratic field extension. We are looking at the field extension $L/K = \mathbb{Q}[\sqrt{d}]/\mathbb{Q}$ for $d \in \mathbb{Z}$ a square-free integer. Denote by $G = \text{Gal}(\mathbb{Q}[\sqrt{d}]/\mathbb{Q}) \cong C_2$ the Galois group of this extension. This group is abelian (so all conjugacy classes are made of a single element), and for any conjugacy class C , $|C|/|G| = 1/2$. Now, for a prime p unramified, we want to calculate the Frobenius element. If p is unramified, either $p\mathcal{O}_{\mathbb{Q}[\sqrt{d}]} = R_1R_2$ or $p\mathcal{O}_{\mathbb{Q}[\sqrt{d}]} = R$.

In the first case, we have $\left(\frac{d}{p}\right) = 1$ (i.e. $\sqrt{d} \in \mathbb{F}_p$, so d is a square modulo p). In this case, $\sqrt{d}^p \equiv \sqrt{d} \pmod{p}$ so $\text{Frob}_{\mathbb{Q}[\sqrt{d}]}(p) = \{Id : \sqrt{d} \mapsto \sqrt{d}\} \in G$.

In the second case, $\left(\frac{d}{p}\right) = -1$ (i.e. $\sqrt{d} \notin \mathbb{F}_p$, so d is not a square modulo p). In this case, $\sqrt{d}^p \not\equiv \sqrt{d} \pmod{p}$ as there is no other choice, $\text{Frob}_{\mathbb{Q}[\sqrt{d}]}(p) = \{\sigma : \sqrt{d} \mapsto -\sqrt{d}\} \in G$.

Then by Chebotarev's density theorem, we have that the density of primes p such that $\left(\frac{d}{p}\right) = \pm 1$ is $1/2$ in both cases. Therefore, we have the following summary:

Primes $p \in \mathbb{P}$ such that:	Density:
$\left(\frac{d}{p}\right) = +1$	$1/2$
$\left(\frac{d}{p}\right) = 0$	0
$\left(\frac{d}{p}\right) = -1$	$1/2$

Thus, for a square free d , $\left(\frac{d}{p}\right)$ is as often $+1$ as -1 (for a prime p), and $\left(\frac{d}{p}\right) = 0$ happens only finitely many times.

4.4 The Dirichlet's Density Theorem

4.4.1 Statement

The most common application of Chebotarev density theorem is probably the Dirichlet's density theorem.

Theorem 7 (Dirichlet's Density Theorem). *Let $n \in \mathbb{N}^*$, $a \in \mathbb{N}$ such that $\gcd(a, n) = 1$. If $S = \{p \in \mathbb{P} \mid p \equiv a \pmod{n}\}$, then S has density $1/\varphi(n)$.*

4.4.2 Link with Chebotarev

This is a direct application of Chebotarev's density theorem for the field extension $\mathbb{Q}[\zeta] : \mathbb{Q}$ where ζ is the n^{th} root of unity (this is the cyclotomic field). The Galois group is abelian (it is precisely $G = \mathbb{Z}_n^\times$ and has order $\varphi(n)$). The abelian property implies that all conjugacy classes are made of a single element. Thus, for any conjugacy class C , the fraction $|C|/|G|$ is just $1/\varphi(n)$. Primes ideals in \mathbb{Q} are just primes numbers. Therefore, Chebotarev gives Dirichlet's density theorem in the particular case of cyclotomic extensions.

4.4.3 Example

Here, look at the example of Dirichlet theorem in the case $n = 15$ from the motivation subsection above (see 4.3.1). We apply the last theorem in the case of $n = 15$: $\varphi(15) = 8$. We define $S_k = \{p \in \mathbb{P} \mid p \equiv k \pmod{15}\}$. By Dirichlet density theorem, the density of S_k is $1/8$ if k and 15 are co-prime (i.e. if $k = 1, 2, 4, 7, 8, 11, 13, 14$), otherwise (if $k = 0, 3, 5, 6, 9, 10, 12$) it is 0 . This is what we could conjecture from the observations.

5 Frobenian Maps and Governing Fields

5.1 Frobenian Maps

Class functions Let G be a group, Ω a set, and $f : G \rightarrow \Omega$. We say that f is a *class function* (of G) if f is constant on conjugacy classes of G . That is, if f remains unchanged under conjugation map of G .

S -Frobenian Maps This definition is taken from [Serre, 2012, §3.3]. Let K be a number field. Let P be the set of primes ideals in K . Let $S \subseteq P$ be a subset of primes ideal of K . We say that a function $f : P \setminus S \rightarrow \Omega$ is S -Frobenian if there exists an M , extending K and a class function $\phi : \text{Gal}(M/K) \rightarrow \Omega$ such that $f = \phi \circ \text{Frob}_{M/K}()$, i.e. $f(\mathfrak{p}) = \phi \circ \text{Frob}_{M/K}(\mathfrak{p}) \quad \forall \mathfrak{p} \in P$ ⁹.

Frobenian Maps With the same setting as above, $f : P \setminus S \rightarrow \Omega$ is Frobenian if there exists a finite set $S \subset P$ such that f is S -Frobenian.

In general, we will take S to be the set of ramified primes (there are finitely many, since they divide the Discriminant, which is finite). In the case of $K = \mathbb{Q}$, the set of primes ideals becomes just the set of primes \mathbb{P} . And a map $f : \mathbb{P} \rightarrow \Omega$ is said to be *Frobenian* if there exists a field extension M/\mathbb{Q} and a class function $\phi : \text{Gal}(M/\mathbb{Q}) \rightarrow \Omega$ such that for all but finitely many (all unramified) primes $p \in \mathbb{P}$, we have $f(p) = \phi(\text{Frob}_{M/\mathbb{Q}}(p))$.

$a_{ij}(p)$ **Frobenian** We recall that for all p odd prime,

$$T_p = \sum_{i,j \geq 0} a_{ij}(p) T_3^i T_5^j \quad \text{with } a_{ij}(p) \in \mathbb{F}_2.$$

Theorem 8. [Nicolas and Serre, 2012b, §7]. For i and j fixed, the map $p \mapsto a_{ij}(p)$ is Frobenian.

That is, for all $i, j \geq 0$, there exists an extension M_{ij}/\mathbb{Q} and a class function $\phi_{ij} : \text{Gal}(M_{ij}/\mathbb{Q}) \rightarrow \mathbb{F}_2$ such that $a_{ij}(p) = \phi_{ij}(\text{Frob}_{M_{ij}/\mathbb{Q}}(p))$ for all $p \in \mathbb{P}$ unramified in M_{ij}/\mathbb{Q} .

Moreover, M_{ij}/\mathbb{Q} is a finite Galois extension, unramified for odd primes.

In such a configuration, M_{ij} are called *governing fields*.

5.2 Governing Fields

It is nice to know that the maps $p \mapsto a_{ij}(p)$ are Frobenius, but to compute $a_{ij}(p)$, we need to know explicitly the governing field (which will depend on i and j).

5.2.1 Basics

Notations We denote by M_{ij} a *governing field* of the map $p \mapsto a_{ij}(p)$. From the theorem above, we know that such a field exist. Note that such a governing fields may not be unique (and in fact we will prove next that it is never unique). We then define a *governing group* of a_{ij} to be a G_{ij} such that $G_{ij} = \text{Gal}(M_{ij}/\mathbb{Q})$ for a M_{ij} governing field. By abuse of notation, we will denote M_{ij} and call **the** governing field the first one we find. **The** governing group G_{ij} is the one that agrees with M_{ij} .

We will denote \mathbb{P}_{ij}^1 the *set of 1-primes*, i.e. $\mathbb{P}_{ij}^1 = \{p \in \mathbb{P} \mid a_{ij}(p) = 1\}$. Similarly, \mathbb{P}_{ij}^0 is the *set of 0-primes*, i.e. $\mathbb{P}_{ij}^0 = \{p \in \mathbb{P} \mid a_{ij}(p) = 0\}$. We will denote by S_{ij}^1 the *set of Frobenius 1-elements*, that is, $S_{ij}^1 = \{g \in G_{ij} \mid \exists p \in \mathbb{P} \text{ s.t. } \text{Frob}_{M_{ij}/\mathbb{Q}}(p) = g \text{ and } a_{ij}(p) = 1\}$. We define S_{ij}^0 in a similar manner to be the *set of Frobenius 0-elements*. Finally, we define C_{ij}^1 and C_{ij}^0 to be the conjugacy classes corresponding to S_{ij}^1 and S_{ij}^0 respectively.

⁹Note that $\phi(\text{Frob}_{M/K}(\mathfrak{p}))$ is well defined since ϕ is a class function, and $\text{Frob}_{M/K}(\mathfrak{p})$ is a conjugacy class of $\text{Gal}(M/K)$.

Properties

Behaviour in Extensions. Using properties of Frobenius elements, we have that if L extends M_{ij} , then L will also be a governing fields for the map $p \mapsto a_{ij}(p)$. This implies that governing fields aren't unique.

Behaviour of T_3^2 and T_5^2 We plot $a_{ij}(p)$ as a 2-dimensional table for a fixed p :

T_{19}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	T_5^5	T_5^6	T_5^7
T_3^0	0	0	0	0	0	0	0	0
T_3^1	1	0	0	0	1	0	1	0
T_3^2	0	0	0	0	0	0	0	0
T_3^3	1	0	1	0	0	0	1	0
T_3^4	0	0	0	0	0	0	0	0
T_3^5	0	0	1	0	0	0	0	0
T_3^6	0	0	0	0	0	0	0	0
T_3^7	0	0	1	0	1	0	0	0
T_3^8	0	0	0	0	0	0	0	0
T_3^9	1	0	1	0	0	0	1	0
T_3^{10}	0	0	0	0	0	0	0	0
T_3^{11}	1	0	1	0	1	0	1	0

Putting a grid like this, it seems that for T_{19} , the only $a_{ij}(19) \neq 0$ are of the form $i \equiv 1 \pmod{2}$; $j \equiv 0 \pmod{2}$ (i.e. if i is odd and j even). We find the same kind of pattern for other primes (see tables here: https://pauldubois98.github.io/HeckeOperatorsModuloTwo/a_ij_p/).

In fact, we have [Nicolas and Serre, 2012b, §7]:

- $T_p \in \mathbb{F}_2[[x^2, y^2]]$ if $p \equiv 1 \pmod{8}$
- $T_p \in x.\mathbb{F}_2[[x^2, y^2]]$ if $p \equiv 3 \pmod{8}$
- $T_p \in y.\mathbb{F}_2[[x^2, y^2]]$ if $p \equiv 5 \pmod{8}$
- $T_p \in xy.\mathbb{F}_2[[x^2, y^2]]$ if $p \equiv 7 \pmod{8}$

The argument to see this is as follows: We know from the action of Hecke operators on \mathcal{F}_i (see 2.5.2) that:

$$f \in \mathcal{F}_i \implies T_p|f \in \mathcal{F}_j \text{ with } j \equiv pi \pmod{8}.$$

Therefore, if $f \in \mathcal{F}_i$, we may remark that:

- $T_3|f \in \mathcal{F}_j$ with $j \equiv 3i \pmod{8}$
- $T_3^2|f \in \mathcal{F}_j$ with $j \equiv i \pmod{8}$
- $T_5|f \in \mathcal{F}_j$ with $j \equiv 5i \pmod{8}$
- $T_5^2|f \in \mathcal{F}_j$ with $j \equiv i \pmod{8}$
- $T_3T_5|f \in \mathcal{F}_j$ with $j \equiv 7i \pmod{8}$

Now, we have $p \in \mathbb{P}$ such that $T_p|f \in \mathcal{F}_j$ with $j \equiv pi \pmod{8}$. We write $T_p = \sum_{k+l>0} a_{kl}(p)T_3^kT_5^l$. Now, by plugging increasingly the powers of Δ , we obtain the following: $a_{kl}(p) \neq 0$ may happen only if $3^k5^l \equiv p \pmod{8}$. This splits into the following cases:

- $p \equiv 1 \pmod{8}$: Then $p \equiv 3^k 5^l \pmod{8}$ for $k \equiv 0 \pmod{2}$ and $l \equiv 0 \pmod{2}$. Thus, $T_p \in \mathbb{F}_2[[T_3, T_5]]$.
- $p \equiv 3 \pmod{8}$: Then $p \equiv 3^k 5^l \pmod{8}$ for $k \equiv 1 \pmod{2}$ and $l \equiv 0 \pmod{2}$. Thus, $T_p \in T_3 \cdot \mathbb{F}_2[[T_3, T_5]]$.
- $p \equiv 5 \pmod{8}$: Then $p \equiv 3^k 5^l \pmod{8}$ for $k \equiv 0 \pmod{2}$ and $l \equiv 1 \pmod{2}$. Thus, $T_p \in T_5 \cdot \mathbb{F}_2[[T_3, T_5]]$.
- $p \equiv 7 \pmod{8}$: Then $p \equiv 3^k 5^l \pmod{8}$ for $k \equiv 1 \pmod{2}$ and $l \equiv 1 \pmod{2}$. Thus, $T_p \in T_3 T_5 \cdot \mathbb{F}_2[[T_3, T_5]]$.

Examples We recall the expansions of T_p in series of $x^a y^b = T_3^a T_5^b$ (here, for primes $p < 152$):

$$T_3 = x^1 y^0 = x$$

$$T_5 = x^0 y^1 = y$$

$$T_7 = x^1 y^1 + x^3 y^1 + x^3 y^3 + x^5 y^1 + x^1 y^7 + x^1 y^9 + x^7 y^3 + x^7 y^5 + x^9 y^3 + x^{11} y^1 + x^3 y^{11} + x^5 y^9 + x^{13} y^1 + x^3 y^{13} + x^5 y^{11} + x^9 y^7 + x^{11} y^5 + x^{13} y^3 + x^3 y^{15} + x^7 y^{11} + x^9 y^9 + x^{13} y^5 + x^{15} y^3 + \dots$$

$$T_{11} = x^1 y^0 + x^1 y^2 + x^3 y^0 + x^1 y^4 + x^3 y^2 + x^5 y^0 + x^1 y^6 + x^3 y^4 + x^7 y^2 + x^1 y^{10} + x^3 y^8 + x^7 y^4 + x^9 y^2 + x^{11} y^2 + x^3 y^{12} + x^5 y^{10} + x^7 y^8 + x^{11} y^4 + x^{13} y^2 + x^9 y^8 + x^{17} y^0 + \dots$$

Expansions for larger primes may be found in A.5.

5.2.2 Known Governing Fields

For i, j small, we can compute explicitly the maps a_{ij} .

Known $a_{ij}(p)$ We have:

- $a_{10}(p) = 1 \iff p \equiv 3 \pmod{8}$
- $a_{01}(p) = 1 \iff p \equiv 5 \pmod{8}$
- $a_{11}(p) = 1 \iff p \equiv 7 \pmod{8}$
- $a_{20}(p) = 1 \iff \exists a, b \in \mathbb{Z} \text{ and } b \text{ odd, such that } p = a^2 + 8b^2, p \equiv 3 \pmod{8}$
- $a_{02}(p) = 1 \iff \exists a, b \in \mathbb{Z} \text{ and } b \text{ odd, such that } p = a^2 + 16b^2, p \equiv 3 \pmod{8}$

The three first ones follows directly from the examples $T_p|\Delta^3$, $T_p|\Delta^5$, $T_p|\Delta^7$ from 2.5.5. For the other two, see [Nicolas and Serre, 2012b, §7].

Corresponding Governing Fields We then have the following corresponding governing fields:

- $M_{10} = \mathbb{Q}(\zeta_8)$ with ζ_8 the 8^{th} root of unity
- $M_{01} = \mathbb{Q}(\zeta_8)$
- $M_{11} = \mathbb{Q}(\zeta_{16}) = \mathbb{Q}(\zeta_8, \sqrt{\zeta_8})$ with ζ_{16} the 16^{th} root of unity.
- $M_{10} = \mathbb{Q}(\zeta_8, \sqrt{1+i})$
- $M_{10} = \mathbb{Q}(\zeta_8, \sqrt[4]{2})$

The first three ones (M_{01} , M_{10} , and M_{11}) are consequences of Chebotarev density theorem, coupled with known values of $a_{ij}(p)$ from above. For the other two, see [Nicolas and Serre, 2012b, §7].

5.2.3 Research of Governing Fields

The only way known so far to find governing fields is via trial and error.

Check In fact, it is not easy to check if a given field is in fact a governing field. Numerically, we can check that a field is governing for a finite amount of primes. However, proving it in general is much harder. Nevertheless, we can easily check that a field is not governing: it suffices to show that two conjugate Frobenius element of primes p_1 and p_2 are not both 1-primes or both 0-primes. What we will do is to check that a field is governing for a sufficiently amount of primes, and take it as a strong evidence that it will work in general.

Guesses for Governing Fields Now we saw how to pseudo check if a field is governing. But to find governing fields, this isn't enough: we also need to have some guesses for governing fields. Some will be declined, and some will accepted (hopefully).

Reasonable Candidates It is not possible to try all fields (since there are infinitely many), we have to filter. We already know M_{02} , governing field for a_{02} . Thus, it sounds reasonable to try extensions of M_{02} as candidates for M_{03} . To create extensions, we adjoint a root to M_{02} . So the candidate for M_{03} is of the form $M_{02}(\sqrt{\alpha})$. For α , it is reasonable to take a product of fundamental unit(s) and torsion unit(s) of M_{02} . We think similarly for M_{ji} as for M_{ij} . For M_{04} , we look at extensions of M_{03} , and continue the process again for M_{05} , M_{06} and so on.

Good candidates The details of the computation can be found in 6.4, and the results (which are long to write, so we will keep the notation M_{ij} to refer to the found governing fields) can be found in G. We will discuss the details of this later. An important thing to remark is that $M_{03} = M_{03}$, and similarly, $M_{40} = M_{30}$. It turns out that $M_{05} = M_{06} = M_{07}$, and maybe (not enough data) $= M_{08}$. Similarly for M_{50} , M_{60} , M_{70} , and again maybe M_{80} .

Wrong candidates It is disappointing that the reasonable guesses for M_{11} aren't governing fields (this time, it is known for sure, as we can't prove with a computer, but we can disprove) for any a_{ij} with $i + j \leq 6$.

Limitations To test if a field extension is a governing field for a_{ij} , we check it for odd primes p up to 10^4 (i.e. $2 < p < 10^4, p \in \mathbb{P}$). Thus, there is a strong evidence that the suggested fields are in fact governing fields, but we do not provide a mathematical proof. Therefore, we may only speculate on a possible value for M_{ij} .

We could turn these calculation much stronger by proving that there is a degree for which we known a governing field exist. In this case, there won't be a finite amount of governing fields, but there would be only few of them that would be considered as good candidates.

The results aren't proved mathematically, however, we will treat pseudo governing fields as governing fields to lighten notation.

5.2.4 New Governing Fields

Here, we use the results of the various computations made. The computation methods are discussed in the next section 6.4. Full results can be found in G.

Results Samples We will discuss in here governing fields found for a_{03} and a_{30} . Details for other results may be found in G.

a_{03}

$$M_{03} = \mathbb{Q} \left(\mu, \sqrt[4]{2}, \sqrt{-\frac{3136435454775881 \sqrt[4]{2}}{562949953421312} + \frac{4208721080340285 \sqrt{2}}{2251799813685248}} \right. \\ \left. + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656} \right)$$

- $G_{03} = D_{16}$ (the dihedral group of order 16) Data used (i.e. primes $p < 10^4$):
- $|S_{03}^1| = 2$ • $\mathbb{P}_{03}^1 = 157$
- $|C_{03}^1| = 1$ • $\mathbb{P}_{03}^0 = 1071$

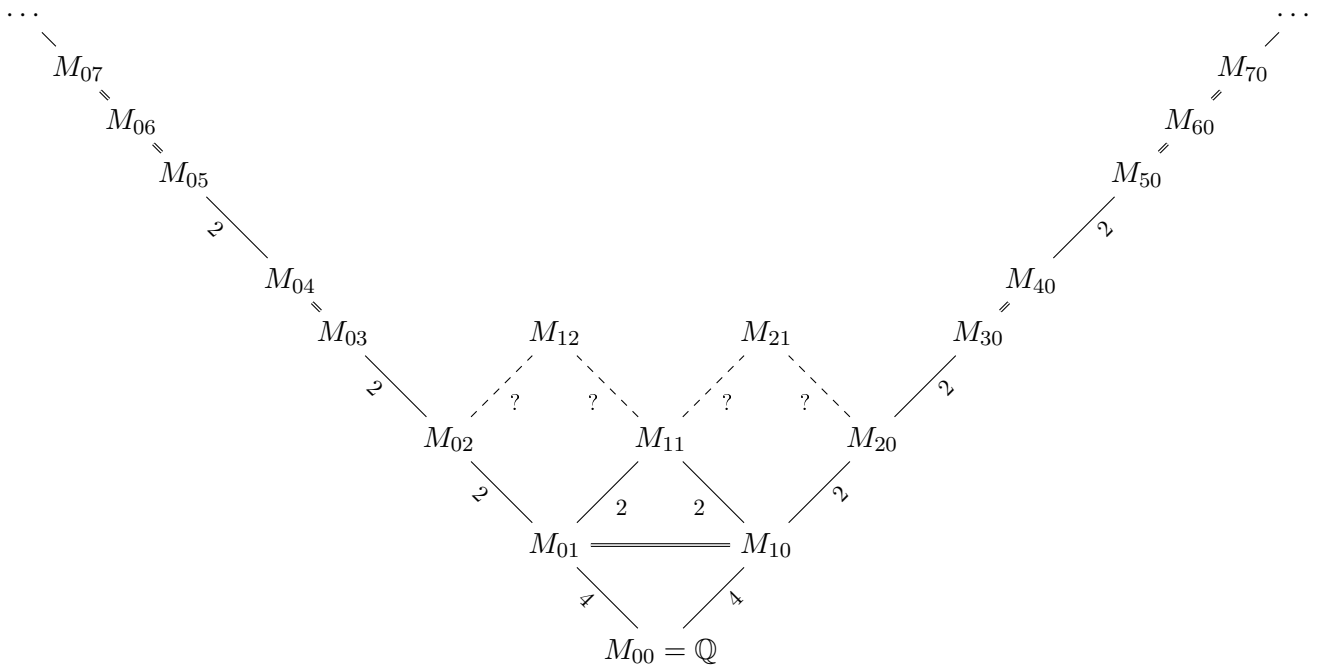
a_{30}

$$M_{30} = \mathbb{Q} \left(\mu, \sqrt{1+i}, \sqrt{-4 - \frac{65i}{16} - \frac{31(1+i)^{\frac{1}{2}}}{16} - \frac{5(1+i)^{\frac{5}{2}}}{16} - \frac{3(1+i)^3}{16}} \right. \\ \left. + \frac{(1+i)^{\frac{7}{2}}}{16} + \frac{11(1+i)^2}{16} + \frac{27(1+i)^{\frac{3}{2}}}{16} \right)$$

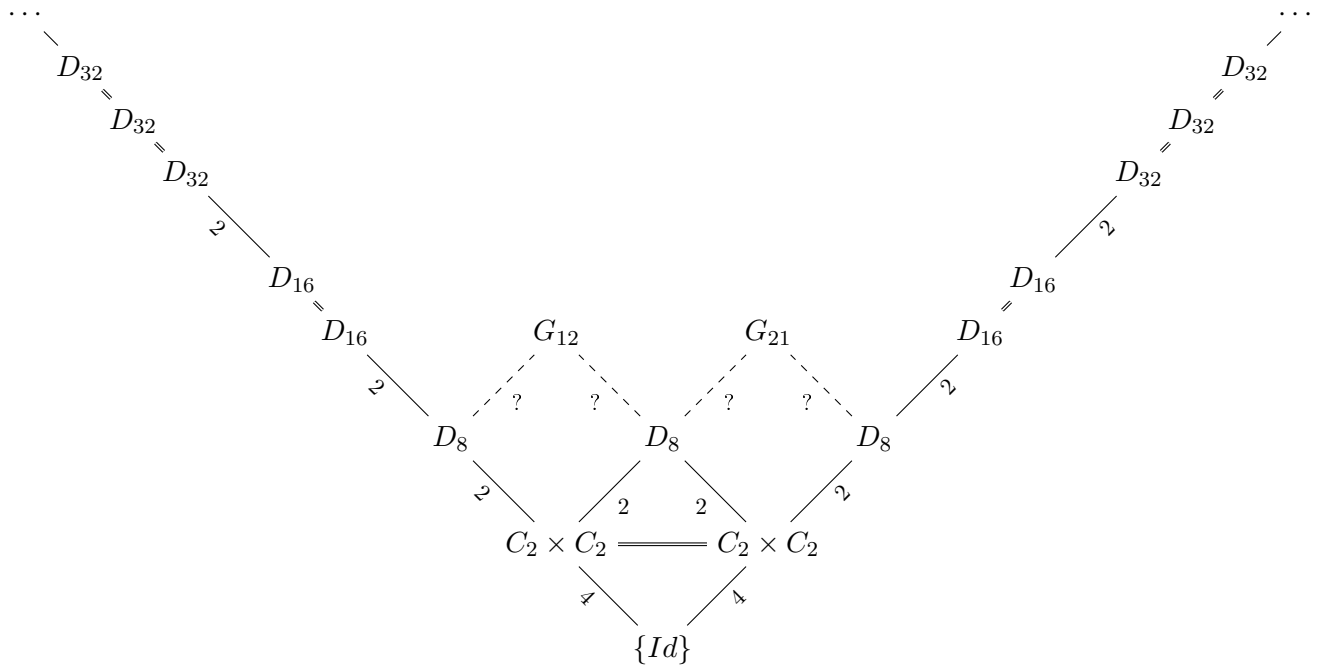
- $G_{30} = D_{16}$ (the dihedral group of order 16) Data used (i.e. primes $p < 10^4$):
- $|S_{30}^1| = 2$ • $\mathbb{P}_{30}^1 = 158$
- $|C_{30}^1| = 1$ • $\mathbb{P}_{30}^0 = 1070$

Graphs

For Fields From G , we have the following field extension diagram:



For Galois Groups We can produce the same diagram for corresponding groups:



Induced Conjecture We can remark that:

- | | |
|-----------------------------------|-----------------------------------|
| • $G_{01} = D_4 = C_2 \times C_2$ | • $G_{10} = D_4 = C_2 \times C_2$ |
| • $G_{02} = D_8$ | • $G_{20} = D_8$ |
| • $G_{03} = D_{16}$ | • $G_{30} = D_{16}$ |
| • $G_{04} = D_{16}$ | • $G_{40} = D_{16}$ |
| • $G_{05} = D_{32}$ | • $G_{50} = D_{32}$ |
| • $G_{06} = D_{32}$ | • $G_{60} = D_{32}$ |
| • $G_{07} = D_{32}$ | • $G_{70} = D_{32}$ |

This leads us to the following conjecture:

Conjecture (Diagonal Governing Groups Conjecture). *For all $k \in \mathbb{N}^*$, there exists a field M_{0k} such that M_{0k} is a governing field for a_{0k} , and $G_{0k} = \text{Gal}(M_{0k}/\mathbb{Q})$ is dihedral. For all $k \in \mathbb{N}^*$, there exists a field M_{k0} such that M_{k0} is a governing field for a_{k0} , and G_{k0} is dihedral. Moreover $M_{k0} \neq M_{0k}$ in general, but $G_{k0} = G_{0k}$.*

6 Numerics

In this section, we will discuss the computations method that allowed us to find the potential governing fields.

6.1 High Performance Computations

It is important to make the program as fast as possible. Indeed, the faster the program goes, the more data it will generate (within the same amount of time). This data will be used for numerical analysis and we will also use it for interpretation. Therefore, with more data, we have more knowledge, and we can make smarter guesses.

There are two main ways to make a program faster: use a better algorithm, or use a faster implementation. For example, in the test of primality of a number n , we may test factors only up to \sqrt{n} , instead of up to n . A better implementation simply means optimisation inside the computer (i.e. on operations that are made, types that are used...). We will try to optimise both.

6.1.1 Algorithm Optimisation

We can optimize an algorithm by optimizing (decreasing) the number of operations, or by using mathematical scheme (usually cancellations).

Optimize instructions Optimizing instructions usually comes through optimizing loops (stopping loops as soon as possible, avoiding extra loops...). For example, the following two algorithms create the same list of coefficients for the q -series of Δ .

Algorithm 1:

Require: $L \geq 1$

$f \leftarrow \text{zeros}(L)$

▷ Empty list of length L

$n=0$

while $n < L$ **do**

if $(\sqrt{n} - 1) \% 2 = 0$ **then**

$f[n] = 1$

end if

end while

Algorithm 2:

Require: $L \geq 1$

$f \leftarrow \text{zeros}(L)$

▷ Empty list of length L

$id = 1$

$i = 1$

while $id < L$ **do**

$f[id] = 1$

$i+ = 2$

$id = i^2$

end while

However, the second algorithm is significantly more efficient: the loop is faster as it only goes through odd squares instead of all numbers, and it has no condition to check. An efficient algorithm may be *harder* to understand, but is actually *better* (in terms of performance).

Mathematical ruse As we are working modulo 2, there are obviously many cancellations, which will make the calculations *faster*. It is an opportunity we shouldn't miss to make the algorithms *stronger*.

6.1.2 Implementation Approach

As explained above, investigations on which tool will be more suitable for the computation is an important part. Of course, the best would be to find a programming language that can already deal with modular forms modulo two. Unfortunately, this doesn't exist (yet). There are packages that have modular forms implemented, but none with modular forms modulo two specifically. The goal of looking at modulo two is to conclude more than what we know in general. So using what has already been done in general to make computations modulo two won't be useful.

We realize that there is no other way than just creating a package for modular forms modulo two on our own. In fact, this is what we will do later, but before, we want to determine the tools to build this package. Modular forms modulo 2 come from maths, so it makes sense to use a high level programming language. For scientific computing nowadays, there are two main open source languages: Python and Julia. Each having various packages to work with.

We will test a selection of major ones.

6.1.3 Choice of Implementation

Now it is time to wonder how to represent modular forms modulo 2. We have seen above that a modular form modulo 2 in fact have two representations: one as an infinite q -series, and one as a finite Δ -polynomial. As we want (later on) to compute Hecke operators of these forms, we will need, at some point to use the q -series representation. In fact, this will be one of the crucial points, since it is an infinite series. The way we represent infinite objects in computers, which have only a finite amount of components (memory addresses, say), is to only store informations up to a cutting point. This is equivalent (somewhat) to the asymptotic notation in mathematics. In the case of q -series of modular forms, we will store only the few first hundred/thousand/million coefficients.

This means that we will represent a modular form via its q -series, which will be stored as a list. We investigate the best ways (time wise) to do basic operations to decide what technology to use. The operations tested are creating the q -series of Δ , and squaring it (both storing coefficients up to some power `LENGTH`, the length of the list used).

There are various techniques to store lists in a computer, the main ones being continuous lists, linked lists, and sparse lists. Continuous and linked lists are both dense lists.

Dense Technique Dense storage means that we store each values of the list (next to each other, or with a link to the next). No element of the list is skipped. There are various ways to implement this technique:

Pure Python Using the Python language, this is the most elementary way to go. It represents all the q -coefficients with the default linked list python object. (code in appendix C.1).

NumPy Python NumPy is the most well known scientific computing library for Python. It interfaces with C objects to provide very fast features (such as lists). (code in appendix C.2).

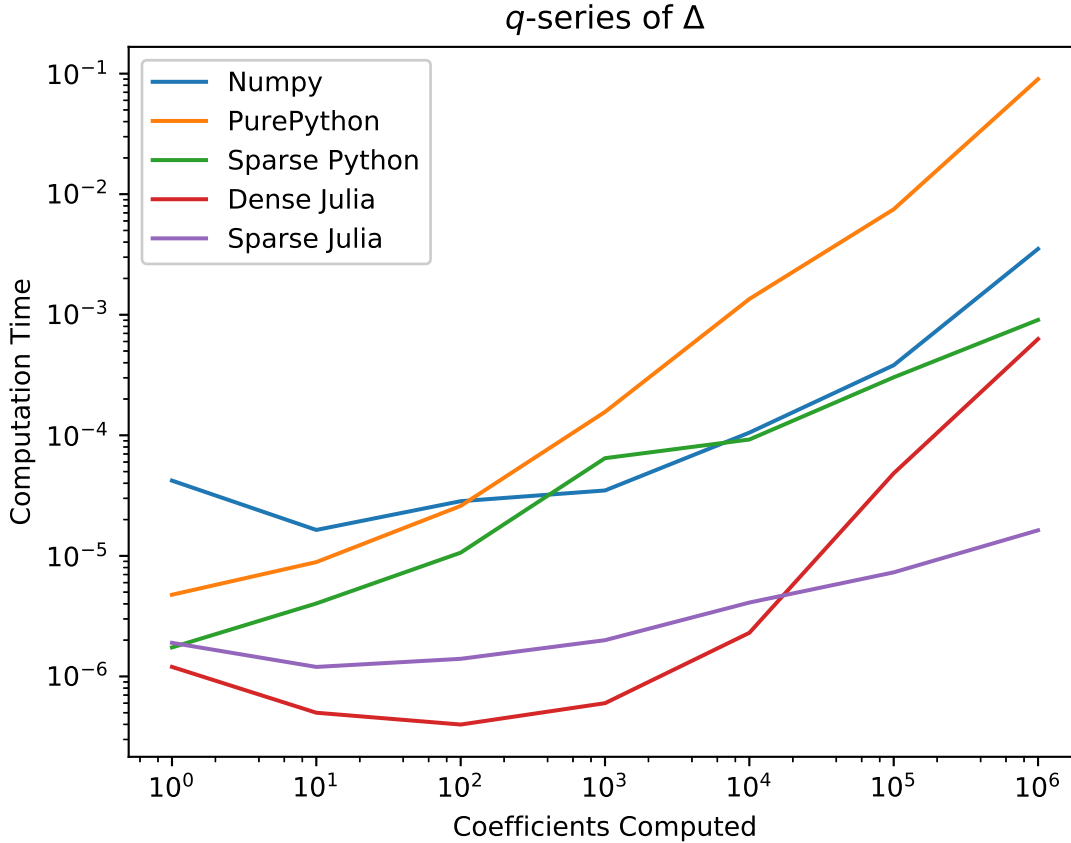
Dense Julia Julia is well-known as both high level and very fast language. Julia naturally supports lists, that we can use to represent modular forms. (code in appendix C.3).

Sparse Technique As all coefficients of the q -series are just 0 or 1, and that most of the time, they are 0, we can represent a modular forms by storing only the coefficients for which it is non-zero. This method (storing only non-zero values) is known as sparse representation. We can implement this technique in both Python and Julia:

Sparse Python We can adapt the previous code to use Python's linked lists as index of a sparse list. Note that in general, we would need a second list to store values, but there are only 0s and 1s, we can take as convention that all stored indices have value 1 and all non-store have value 0. (code in appendix C.4).

Sparse Julia Julia has a very convenient built-in sparse module. This is particularly interesting, since the built-in type already have nice methods. (code in appendix C.5).

Speed Comparison We can now compare the speed of each implementation to compute q -series. If we do that for various number of coefficients, we may obtain a graph of the following type (it is slightly dependent on the machine that execute the code, but the shape remains).



For small computations, the implementation doesn't make a big difference. However, for large computations, it seems that the sparse methods do better. It makes sense, since sparse representations are typically used for objects with more than 95% of zeros, which is the case for modular forms modulo 2.

For a more precise analysis, we now compare the speed of each implementation to compute q -series of Δ and Δ^2 ¹⁰. The following table is obtained for 10⁶ coefficients computed (i.e. up to q^{10^6}). Note

¹⁰ Δ^2 itself isn't part of our space \mathcal{F} , but it will be useful as we will compute $\Delta^{2k+1} = \Delta^{2k-1} \cdot \Delta^2$. So it makes sense to be concerned about it.

that $\mathcal{O}(q^{10^6})$ will be standard for the rest of this paper.

	Δ	Δ^2
Pure Python	0.08263147	0.26249526
NumPy Python	0.00138761	0.16163688
Dense Julia	0.000648	0.001698
Sparse Python	0.00095099	0.00134479
Sparse Julia	0.000021	0.000034

From this table, it is clear that the fastest implementation is the one using sparse lists (so called "sparse vectors") in Julia. Therefore, we will use this technique. It is nice to remark that the Pure Python implementation was 7720 times slower than the Sparse Julia one. We see here the importance of choosing the right tool to implement an algorithm.

This ratio would be even greater considering the bad algorithm presented before 6.1.1.

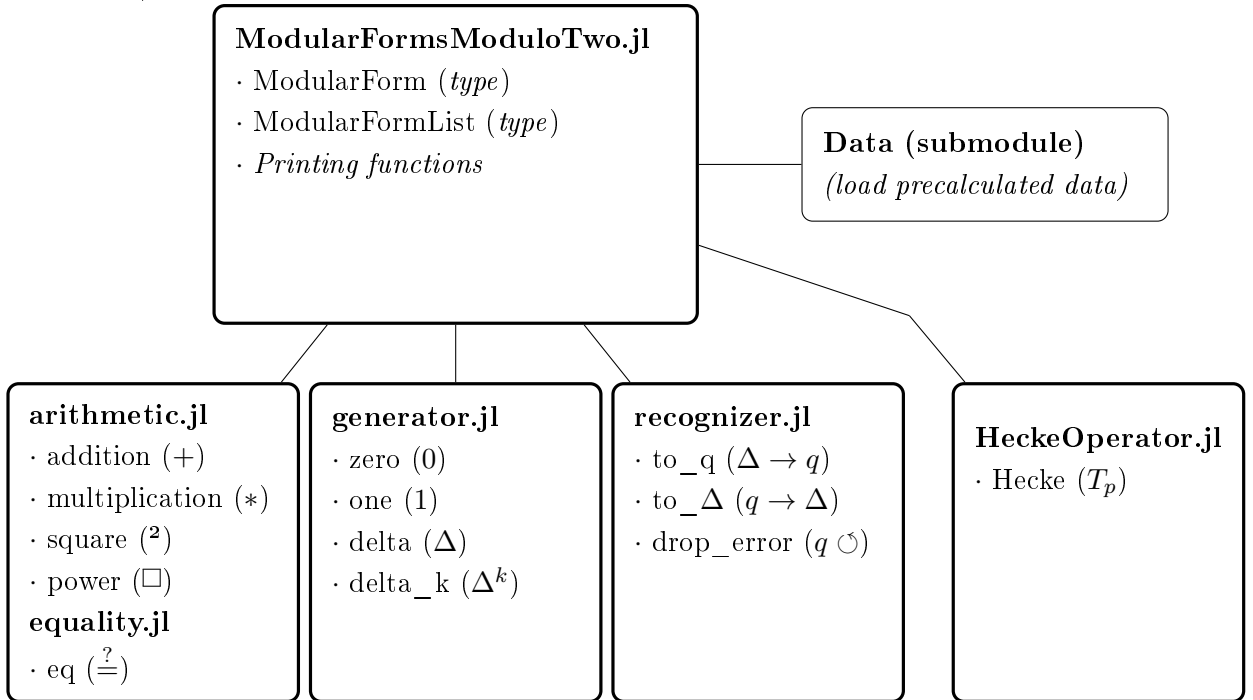
6.2 Creating the library

It is clear now that the code should be done with Julia and its Sparse objects. Now, as all the library should be created from the beginning, it is a good idea to pack all of it in a Julia module. Doing so, no code will be repeated for each small task.

6.2.1 ModularFormsModuloTwo.jl

The code will be divided in many files, for convenience.

Code Architecture The main functions are direct parts of the module, and the pre-calculated data part is written in a sub-folder, detailed next paragraph. Here is a visualization of the organisation (the "architecture"):



Major functions of the module and their repartitions into file (represented by boxes).

Files Details We will detail here the important and interesting parts of the code. For more details, the reader may refer to the source code, which is commented and documented.

Basics Operations Basic operations on modular forms modulo 2 are defined in `arithmetic.jl` (see D.2.1 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/arithmetic.jl>). All of these algorithms have been optimised as much as possible (i.e. cutting loops as soon as possible, and iterate through "ones" only, as most coefficients are zero).

Equality up to known It might be useful to compare two modular form up to some coefficients (perhaps if f_1 is known up to q^n and f_2 up to q^m , we can compare them up to $q^{\min n, m}$). This equality test is defined in `equality.jl` (see D.2.1 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/equality.jl>).

Generators of \mathcal{F} As modular forms modulo 2 are polynomials of Δ , we need to be able to generate the q -series of powers of Δ . Such functions are defined in `generator.jl`¹¹ (see D.2.1 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/generators.jl>). Here, the optimization is not trivial: To calculate Δ^{31} (say), the most trivial way is to do $\Delta \cdot \Delta \cdots \Delta$ with 30 multiplications. But what is done in fact inside the library is much faster: we calculate Δ , $\Delta^2 = \Delta \cdot \Delta$, $\Delta^4 = \Delta^2 \cdot \Delta^2$, ... until Δ^{16} , and then $\Delta^{31} = \Delta^{16} \cdot \Delta^8 \cdot \Delta^4 \cdot \Delta^2 \cdot \Delta$. Doing like this, only 8 multiplications were needed (against 30 with the naïve method). One may think that 3 was an example particularly good for this technique, but if we take another example, say 32, we would turn 31 multiplication to 5. In fact, in general, to calculate Δ^n this second method need a maximum of $\log_2(n)^2$ multiplications against $n - 1$ for the first one. So it really is a much faster algorithm.¹²

Hecke Operators The only formula we have to calculate Hecke operators, is using the q -representation of modular forms. Therefore, the Hecke operators are taking a modular forms under q -representation as input (together with a prime p). Note that when applying a Hecke operator, we loose a lot of informations on it's expansion: If f is known up to q^n , then $T_p|f$ will only be known up to $q^{n/p}$. This means that we should be careful when applying Hecke operator.

Hecke operators are implemented in the file `HeckeOperator.jl` (see D.2.1 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/HeckeOperator.jl>)

Recognise the trick? This part of the module is probably the most important, and it is the one that differ the most from any other computing technique.

We start by noting that it is easy to go from the Δ representation of a modular form to the q representation: it suffices to choose an arbitrary n , and calculate all coefficients up to q^n , using the series expansion of Δ . This step is necessary to calculate Hecke operators.

Now, one may ask if it is possible to go back from the q -representation to the Δ -representation of a modular form. In general, this is not possible. **But** if we have some assumptions on the modular form, then it may become possible. For example, if we know that the maximum degree (in terms of Δ) of f is n , and that we know it's q -coefficients up to n : then f may be written as $f = \sum_{k \leq n} \mu_k \Delta^k$, so the set $\{\Delta^k | k \leq n\}$ acts as a basis, and it is just a matter of finding the matching coefficients μ_k . This is in fact possible, and that is how the function `to_delta()` is made possible.

Now, once we have the Δ representation of a modular form, we potentially have as many q -coefficients as we want. This may seem weird or even magical, since we assumed only finitely many q -coefficients were known. In fact, we can use this fact to drop the numerical error that calculating a Hecke operator may have produced.

¹¹`generators.jl` generates Δ and then uses the power function from `arithmetic.jl`.

¹²In fact, half of the operations are squaring modular forms, which is much faster than a usual multiplication. This this algorithm is even better than stated.

Such kind of calculations are rather unusual in computer science: usually computers approximate objects, and this approximation error is never given back. But with modular forms modulo two, we can take back the approximation error.

This method will now be referred to as exact computations.

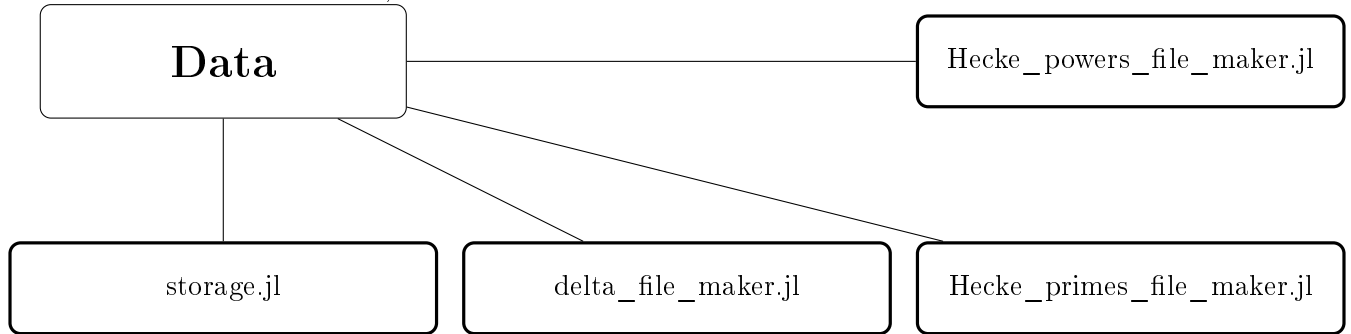
It is implemented in the file `recognizer.jl` (see D.2.1 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/recognizer.jl>).

Global The last file, `ModularFormsModuloTwo.jl` is the one that creates the link between all the small part of programs written in other files. It also defines a few general objects, such as types and printing functions. For more details, see (D.2.1 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/ModularFormsModuloTwo.jl>)

6.2.2 (Pre-calculated) Data

Data is natively part of the `ModularFormModuloTwo.jl` module, but it is treated internally as a sub-component, for better organisation.

Code Architecture This time, the architecture is much easier:



Files Details We will detail here an overview of what each file achieve. For more details, the reader may look at the source code, which is highly commented, and quite explicit in general.

- `storage.jl` (see D.2.2 or <https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/data/storage.jl>): This is the only file to be called outside of the Data sub-module.
- `delta_file_maker.jl` (see D.2.2 or https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/data/delta_file_maker.jl): The program in this file generates the q -coefficients lists for a range of powers of Δ .
- `Hecke_primes_file_maker.jl` (see D.2.2 or https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/data/Hecke_primes_file_maker.jl): The program in this file calculates the Δ -representation lists for a range of Hecke operators T_p and range of powers of Δ .
- `Hecke_powers_file_maker.jl` (see D.2.2 or https://github.com/pauldubois98/ModularFormsModuloTwo.jl/blob/master/src/data/Hecke_powers_file_maker.jl): The program in this file calculates the Δ -representation lists for a range of powers of Hecke operators T_3 and T_5 and range of powers of Δ .

6.2.3 Open-Source

This library is completely open-source. Anyone is welcome to contribute. Anyone can use it (for free).

6.2.4 Official

This module is now registered as an official Julia package. To use all the code form this library, a new user will just need to type:

```
julia> using Pkg
julia> Pkg.add("ModularFormsModuloTwo")
```

It is convenient that all the algorithms developed in this module can be used just by importing the package, which can be done in a minute.

6.2.5 Online Documentation

As it usually comes wit open-sources packages, ModularFormsModuloTwo.jl has an online documentation (see <https://pauldubois98.github.io/ModularFormsModuloTwo.jl/>).

6.3 Finding Coefficients $a_{ij}(p)$

6.3.1 Strategy

We want to find the coefficients a_{ij} such that

$$T_p = \sum_{i+j \geq 1} a_{ij}(p) T_3^i T_5^j \quad (*)$$

(with $a_{ij}(p) \in \mathbb{F}_2$). Note that this is a finite sum, since Hecke operators are nilpotent.

The strategy is as follows: we use the module we developed. It allows us to compute the (exact) Δ -representation of $T_p|\Delta^k$ and $\sum_{i+j \geq 1} a_{ij}(p) T_3^i T_5^j |\Delta^k$ for many k , p , i and j . Now, since $(*)$ should hold for all modular forms $f \in \mathcal{F}$, it has to holds, in particular, for all Δ^k (with k and odd integer). Thus, we will plug successive Δ^k and equate coefficients. This will eventually give all $a_{ij}(p)$.

6.3.2 Algorithm

As explained before, we plot Δ^k successively for a range of k . In reality, most of the terms of the sum in $*$ are zeros. This is both nice and not good: It is nice because it makes the system is easy to solve. But it also makes all coefficients in front of zeros terms being undetermined. This implies that we need to plug larger powers of Δ to fix coefficients. And bigger powers of Δ ask for heavier computations (and now we see how important it was to optimize the modular forms modulo 2 module).

As this algorithm is the heart of all computations, we give a pseudo-code simplified version:

```
Require:  $MAX_K$           ▷ Maximum power  $\Delta^k$  for which  $T_p|\Delta^k$  and  $T_3^i T_5^j |\Delta^k$  are known (computed)
Require:  $MAX_I$           ▷ Minimum  $i \in \mathbb{N}$  such that  $T_3^i |\Delta^{MAX_K} \neq 0$ 
Require:  $MAX_J$           ▷ Minimum  $j \in \mathbb{N}$  such that  $T_5^j |\Delta^{MAX_K} \neq 0$ 

for all  $p \in \mathbb{P}$ ,  $p > 2$  do                                ▷ We compute the map  $a_{ij}(p)$  for each specific odd  $p$  prime
     $a_p$  is a  $MAX_I \times MAX_J$  2-dimensional matrix          ▷  $a_p[i, j]$  will correspond to  $a_{ij}(p)$ 
    Fill  $a_p$  for known values (i.e. for  $i + j \leq 2$ )
    for  $k < MAX_K$  do
         $f = T_p|\Delta^k$ 
         $LIST_a$                                               ▷ The list of  $a_p[i, j]$  to fix using  $\Delta^k$  iteration
         $LIST_f$                                               ▷ The corresponding list of modular forms  $T_3^i T_5^j |\Delta^k$ 
        for  $0 \leq i \leq MAX_I$  do
            for  $0 \leq j \leq MAX_J$  do
```

```

    if  $a_p[i, j] = 1$  then
         $f += T_3^i T_5^j |\Delta^k$  ▷ Add  $1 * T_3^i T_5^j |\Delta^k$  to  $f$ 
    else if  $a_p[i, j] = 0$  then
        Pass ▷ Add  $0 * T_3^i T_5^j |\Delta^k$  to  $f$ 
    else ▷  $a_p[i, j]$  is unset in this case.
        Append  $a_p[i, j]$  to  $LIST_a$  ▷ We add  $a_p[i, j]$  to the list
        Append  $T_3^i T_5^j |\Delta^k$  to  $LIST_f$  ▷ We add  $T_3^i T_5^j |\Delta^k$  to the list
    end if
end for
end for
Find  $a_p[i, j]$  in  $LIST_a$  that solve that system  $f = \sum a_p[i, j] T_3^i T_5^j |\Delta^k$  ▷ using built-in linear solver, for efficiency
end for
Output  $a_p$ 
end for

```

The implementation of this algorithm (in Julia) is in E.1.

6.3.3 Computations Limitations

Here, we discuss the bounds to put in the algorithm, for a decent amount of data, and a descent amount of computation time.

Computing Hecke of Large Primes Versus Large Powers Here, we compare the difficulty to compute Hecke operators for large primes (T_p for $p \gg 2$), against the difficulty to compute Hecke operators for large powers of Hecke operators ($T_3^i T_5^j$ for $i, j \gg 1$).

Computing Hecke Operators for Large Primes When calculating a Hecke operator T_p for a modular form f , the maximum known coefficient is $q^{\frac{N}{p}}$ if f is known for coefficients up to q^N . This means that we can compute the Δ representation for $T_p|f$ only if we know it will have a degree (in terms of Δ) of maximum $\frac{N}{p}$. Thus, if we choose to compute $a_{ij}(p)$ for $p \leq P$, it means that $\partial T_p|f \leq \frac{N}{P} = K$.

Computing Hecke Operators for Large Powers Now, at a first thought, T_p isn't too pathological compared to $T_3^i T_5^j$ for large i and j , since the maximum known coefficient would be $q^{\frac{N}{3^i 5^j}}$ if f is known for coefficients up to q^N .

But in fact, we can apply a trick here: Once we know $T_3|f$ (once we computed it's q -representation), we straight calculate the Δ -representation, and then get back to the q -representation with no "lose of coefficients". This may look weird at first, since we have the q -representation of $T_3|f$, and we need it to calculate $T_3^2|f$. However, we known the q -representation of $T_3|f$ with some coefficients lost. And the fact that we go back to the Δ -representation allows us to drop the numerical error. Doing this again and again, there is no lose at all in the coefficients calculated.

This is why exact computations (this is the name we gave to this trick) is crucial for this algorithm. In fact, for powers of Hecke operators, we only need to compute the q -representations of modular forms up to $5 \partial f$, since each T_3 or T_5 will lose at most $4/5$ of the q -coefficients.

Conclusion Thus, the part which use to be the most pathological in fact becomes much nicer than the other: calculating Hecke operators for large powers is easier than for large primes.

Reasonable Bounds Here, we give limits used in the actual computations.

It seems that using q series capped at q^{10^6} is reasonable (i.e. computations will be a few days long). We would like to compute $a_{ij}(p)$ for small i, j and $p \leq 10^4$.

So the setup (in the same notation as above) is $N = 10^6$, $P = 10^4$, so $K = 10^2$. Then the algorithm will compute as much $a_{ij}(p)$ as possible, for each $p \leq P$. This maximum can in fact be calculated implicitly: $a_{ij}(p)$ will be computed if and only if $T_3^i T_5^j | f$ is non-zero for one of the forms f plugged. As we plug powers of Δ up to K , this means $a_{ij}(p)$ will be computed for all $p \leq P$ if and only if $k \leq K$, where k is the odd integer with code $[i, j]$.

6.3.4 Results

Here, we will give results for the prime $p = 19$. There are many links in this section, if the reader is interested in all the data calculated (it won't fit in this paper).

Expansions of T_p We have the following extension:

$$T_{19} = T_3^1 T_5^0 + T_3^3 T_5^0 + T_3^1 T_5^4 + T_3^3 T_5^2 + T_3^1 T_5^6 + T_3^5 T_5^2 + T_3^3 T_5^6 + T_3^7 T_5^2 + T_3^9 T_5^0 + \dots$$

Writing $x = T_3$ and $y = T_5$, this is:

$$T_{19} = x^1 y^0 + x^3 y^0 + x^1 y^4 + x^3 y^2 + x^1 y^6 + x^5 y^2 + x^3 y^6 + x^7 y^2 + x^9 y^0 + x^1 y^{10} + x^7 y^4 + x^9 y^2 + x^{11} y^0 + \dots$$

For expansions of other primes, please visit this web site:

https://pauldubois98.github.io/HeckeOperatorsModuloTwo/T_p_extensions/.

For $p = 19$, we can also look at $a_{ij}(p)$ as an infinite 2-dimensional table:

T_{19}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	T_5^5	T_5^6	T_5^7	T_5^8	T_5^9	T_5^{10}	T_5^{11}	T_5^{12}	T_5^{13}	T_5^{14}	T_5^{15}
T_3^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_3^1	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0
T_3^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T_3^3	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0
T_3^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
T_3^5	0	0	1	0	0	0	0	0	1	0	0	0	0	0		
T_3^6	0	0	0	0	0	0	0	0	0	0	0	0	0			
T_3^7	0	0	1	0	1	0	0	0	1	0	1	0				
T_3^8	0	0	0	0	0	0	0	0	0	0	0					
T_3^9	1	0	1	0	0	0	1	0	0	0						
T_3^{10}	0	0	0	0	0	0	0	0	0							
T_3^{11}	1	0	1	0	1	0	1	0								
T_3^{12}	0	0	0	0	0	0	0									
T_3^{13}	1	0	1	0	0	0										
T_3^{14}	0	0	0	0	0											
T_3^{15}	0	0	1	0												
T_3^{16}	0	0	0													
T_3^{17}	1	0														
T_3^{18}	0															

Here, blanks are coefficients not computed.

For tables of other primes, please visit this web site:

https://pauldubois98.github.io/HeckeOperatorsModuloTwo/a_ij_p/.

Now, we will later be interested in the map $p \mapsto a_{ij}(p)$, so it makes sense for each pair (i, j) , to list the *1-primes* (i.e. the set $\{p \in \mathbb{P} \mid a_{ij}(p) = 1\}$) or the *0-primes*. This is what we do in the following table:

a_{ij}	$\{p \in \mathbb{P} \mid a_{ij}(p) = 1\}$
a_{01}	5, 13, 29, 37, 53, 61, 101, 109, 149, 157, 173, 181, 197, 229, 269, 277, 293, 317, 349, ...
a_{10}	3, 11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251, 283, 307, 331, 347, ...
a_{02}	17, 41, 97, 137, 193, 241, 313, 401, 409, 433, 449, 457, 521, 569, 641, 673, 761, 769, ...
a_{11}	7, 23, 31, 47, 71, 79, 103, 127, 151, 167, 191, 199, 223, 239, 263, 271, 311, 359, 367, ...
a_{20}	17, 73, 89, 97, 193, 233, 241, 281, 401, 433, 449, 601, 617, 641, 673, 769, 929, 937, ...
a_{03}	13, 29, 37, 101, 149, 173, 181, 317, 349, 389, 557, 661, 677, 709, 733, 757, 773, 997, ...
a_{12}	11, 59, 67, 83, 107, 131, 179, 211, 251, 331, 347, 379, 419, 499, 523, 547, 587, 619, ...
a_{21}	13, 37, 53, 61, 101, 157, 173, 277, 373, 389, 509, 541, 557, 677, 701, 709, 773, 797, ...
a_{30}	11, 19, 67, 107, 131, 283, 307, 331, 419, 443, 467, 523, 547, 563, 571, 587, 619, 643, ...
a_{04}	73, 89, 113, 233, 353, 577, 593, 601, 937, 1153, 1201, 1289, 1433, 1601, 1609, 1721, ...
a_{13}	23, 31, 71, 79, 103, 127, 151, 191, 223, 239, 263, 359, 431, 463, 479, 503, 631, 647, ...
a_{22}	17, 41, 73, 89, 113, 233, 241, 257, 313, 353, 401, 409, 433, 457, 601, 761, 809, 937, ...
a_{31}	7, 79, 167, 199, 239, 311, 383, 431, 439, 463, 487, 599, 607, 719, 727, 743, 751, 823, ...
a_{40}	41, 113, 257, 313, 337, 409, 457, 577, 761, 809, 881, 1129, 1249, 1553, 1657, 1889, ...
a_{05}	13, 53, 61, 101, 109, 157, 173, 197, 269, 317, 349, 389, 421, 461, 613, 653, 661, 701, ...
a_{14}	11, 19, 67, 107, 131, 163, 179, 211, 227, 251, 283, 307, 331, 347, 419, 491, 643, 811, ...
a_{23}	29, 37, 53, 61, 101, 149, 157, 173, 197, 269, 293, 389, 397, 421, 541, 557, 613, 653, ...
a_{32}	11, 19, 43, 83, 107, 131, 163, 211, 251, 347, 379, 419, 443, 467, 491, 523, 563, 571, ...
a_{41}	13, 53, 101, 149, 157, 173, 181, 229, 317, 373, 397, 421, 461, 613, 661, 701, 709, ...
a_{50}	11, 43, 59, 67, 83, 139, 163, 251, 283, 419, 467, 499, 547, 587, 619, 643, 659, 811, ...
...	

Table of 1-primes

Again, for complete tables, please visit this web site:

https://pauldubois98.github.io/HeckeOperatorsModuloTwo/a_ij_p/a_ij_p-1.html.

Note that tables for *0-primes* (i.e. the set $\{p \in \mathbb{P} \mid a_{ij}(p) = 0\}$) may be found here:

https://pauldubois98.github.io/HeckeOperatorsModuloTwo/a_ij_p/a_ij_p-0.html.

6.4 Finding Governing Fields

6.4.1 Computations Strategy

As it is usually the case with computations, this algorithm will not give a proof that the considered field is a governing field. What we will do, is to check that for sufficiently many primes, the field we consider is a governing field.

Algorithm

Outline Consider the field L extending \mathbb{Q} , suppose we suspect it to be a governing field for a_{ij} . We want to check that there exists a subset $S \subseteq G = \text{Gal}(L/\mathbb{Q})$ that is stable under conjugacy, such that $\text{Frob}_{L/\mathbb{Q}}(p) \in S$ if and only if $a_{ij}(p) = 1$. That is $\text{Frob}_{L/\mathbb{Q}}(p) \in S$ if $a_{ij}(p) = 1$ and $\text{Frob}_{L/\mathbb{Q}}(p) \notin S$ if $a_{ij}(p) = 0$.

Pseudo-code Here is a pseudo-code version of the algorithm used:

```

 $MAX_P \in \mathbb{N}$ 
 $i, j \in \mathbb{N}$  given
 $L$  a given number field, suspected to be a governing field for  $a_{ij}$ .
 $G = \text{Gal}(L/\mathbb{Q})$ 
 $Primes - 1$  an empty set.
for  $p \in \mathbb{P}, p < MAX_P, a_{ij}(p) = 1$  do
  if  $\text{Frob}_{L/\mathbb{Q}}(p) \in Primes - 0$  then
    Add  $\text{Frob}_{L/\mathbb{Q}}(p)$  to the set  $Primes - 1$ 
  end if
end for
for  $p \in \mathbb{P}, p < MAX_P, a_{ij}(p) = 0$  do
  if  $\exists F_q \in Primes - 0$  s.t.  $\text{Frob}_{L/\mathbb{Q}}(p) \sim F_q$  then
    REJECT!
  end if
end for
 $L$  is a Governing field for  $a_{ij}$ .

```

▷ Accept as not rejected yet.

The implementation of this algorithm (in Python) is in F.2. Note that the hardest part to program is in fact the transition between computer's representation of mathematical objects, and the human readable versions. This part isn't mathematically interesting, so has been hidden in this paper. The interested reader may find complete programs on GitHub (see <https://github.com/pauldubois98/HeckeOperatorsModuloTwo/tree/master/GoverningFields>).

Implementation Strategy For algebraic computations (such as computing Frobenius elements), many libraries already exist. However, many (such as SageMath) are limited in terms of degree of field extensions. This is why we will use a very powerful and respected library: PARI GP. PARI GP may be used through C, but for simplicity, we will use the the GP language that PARI GP developers suggest.

It isn't easy to connect the Julia computations for the maps a_{ij} and the PARI GP language. So we will export results from both sides in text files, and analyse it with Python. Note here that the analysis is just checking if whether or not, the field is a governing field. All the "difficult" computations are made in very efficient languages (Julia and the highly optimized PARI GP library). So it is fine, for convenience, to use a slower language as Python for the last step, which isn't computationally fast.

6.4.2 Reliability

We want the program elaborated to be reliable. Since we are mixing 3 different languages (PARI GP, Python, and Julia), it isn't easy to check that no error was made when coding.

Checking on Known Governing Fields This is the reason why we first check that the fields $\mathbb{Q}(\zeta_8, \sqrt[4]{2})$, $\mathbb{Q}(\zeta_8, \sqrt{1+i})$, and $\mathbb{Q}(\zeta_{16})$ are validated by the program as governing fields for a_{02} , a_{20} , and a_{11} . Even if maths is an exact science (so we don't need to check again some result found before), it is still nice to see that the theory is consistent.

Unsurprisingly, we in fact find results expected form [Nicolas and Serre, 2012b, §7].

New Governing Fields We apply the exact same method as before, and find very good candidates governing fields for: a_{03} , a_{04} , a_{05} , a_{06} , a_{07} , a_{30} , a_{40} , a_{50} , a_{60} , and a_{70} . We also have some less reliable

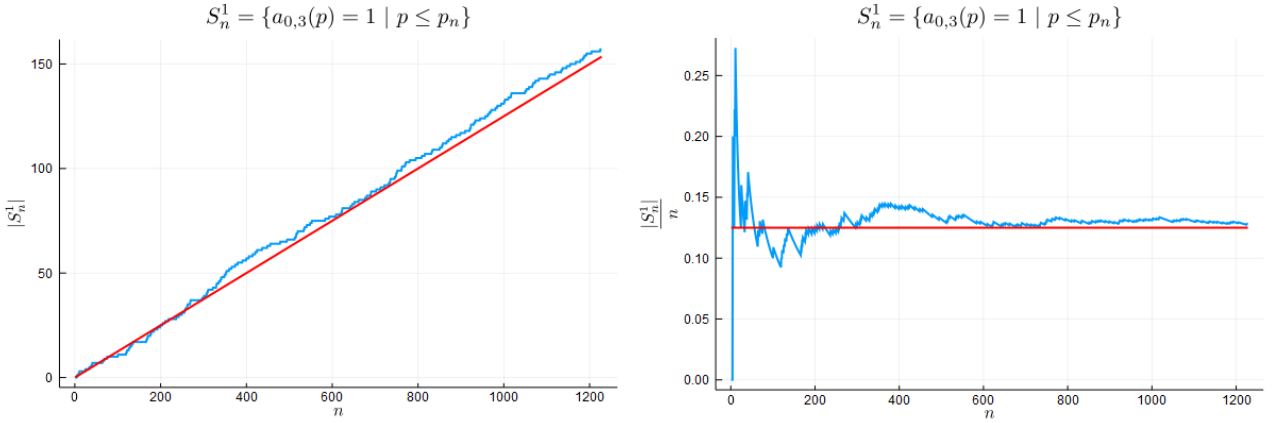
results ¹³ for a_{08} and a_{80} . All the details can be found in G.

6.5 Probabilistic Analysis

In here, we want to estimate the probability that the fields found are wrong. To do this, we calculate the probability that the matching found happens at random.

6.5.1 Motivation

We can plot the number of 1-primes (that is, $\{p < p_n \mid a_{ij}(p) = 1\}$) as a function of the number of primes n . We obtain a plot as follows (here, we take the example of a_{03} , others may be found at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/behaviour_a_ij_p/):



It seems like there is a ratio $r \in (0, 1)$ such that

$$\frac{|\{p \leq p_n \mid a_{03}(p) = 1\}|}{|\{p \leq p_n\}|} \rightarrow r.$$

In fact, since we know that the map a_{ij} is Frobenian, there exists a field M_{ij} such that $a_{ij}(p)$ is determined by the conjugacy class of $\text{Frob}_{M_{ij}/\mathbb{Q}}(p)$. Now, by Chebotarev's Density Theorem, we know the density of conjugacy classes: for a conjugacy class $C \subseteq G_{ij} = \text{Gal}(M_{ij}/\mathbb{Q})$, there will be, asymptotically, a proportion of $|C|/|G|$ of the primes p such that $\text{Frob}_{M_{ij}/\mathbb{Q}}(p) \in C$. Therefore, the ratio r we conjectured does in fact exist.

6.5.2 Assumptions

We consider primes $p < 10^4$ (the ones for which we have computational results). We have a set of values $a_{ij}(p)$ that is fixed. As we said, the asymptotic density of primes p such that $\text{Frob}_{M_{ij}/\mathbb{Q}}(p) \in C$ in G_{ij} is $|C|/|G|$. We will consider the assignment of $\text{Frob}_{M_{ij}}(p)$ to a conjugacy class of G as a random variable. We assume that $\text{Frob}_{M_{ij}}(p)$ is assigned to a conjugacy class C with probability $|C|/|G|$.

With $S \subseteq G_{ij}$ stable under conjugacy, we define

$$b_S(p) = \begin{cases} 1 & \text{if } \text{Frob}_{M_{ij}/\mathbb{Q}}(p) \in S \\ 0 & \text{else} \end{cases}.$$

Using this, we make b_S a Bernoulli random variable with probability $p = |S|/|G|$.

¹³In the sense of no coding error.

6.5.3 Probability of Mistake in the Governing Fields Found

General a_{ij} case we want to know the following probability:

$$P_{ij} = P(\exists S \subseteq G_{ij} \text{ stable under conjugacy} \mid b_S(p) = a_{ij}(p) \quad \forall p < 10^4).$$

That is, the probability that the field M_{ij} was found to match all the calculated coefficients at random.

Now, let $S \subseteq G$ be stable under conjugacy. Then $P(b_S(p) = a_{ij}(p) \quad \forall p < 10^4)$ is determined $|S|$. For a_{ij} , let $p_0 = |\{p \in \mathbb{P}_{ij}^1 \mid p < 10^4\}|$ and $p_1 = |\{p \in \mathbb{P}_{ij}^1 \mid p < 10^4\}|$. So we have:

$$P(b_S(p) = a_{ij}(p) \quad \forall p < 10^4) = P(b_S(p) = 1)^{p_1} P(b_S(p) = 0)^{p_0} = \left(\frac{|S|}{|G|}\right)^{p_1} \left(\frac{|G| - |S|}{|G|}\right)^{p_0}$$

This is the probability that a particular set S is such that b_S matches a_{ij} for primes $p < 10^4$. Now, we want to know the probability that the field extension produced this phenomenon. Therefore, we have to sum this probability over all possible sets $S \subseteq G$ that are stable under conjugation.

Case a_{03} When calculating for $i = 0, j = 3$ so, we find: $p_0 = 157, p_1 = 1071$ and $G_{03} = D_{16}$.

$$P(\exists S \subseteq G_{03} \text{ stable under conjugacy} \mid b_S(p) = a_{03}(p) \quad \forall p < 10^4) \approx 5.0998 \cdot 10^{-204}.$$

Therefore, it is very unlikely that the governing field that we have found is in fact matching the coefficients randomly. It really is a strong evidence that the governing fields found are correct. However, this isn't a proof.

The data for all other cases is in the appendix G.

Cases a_{08} and a_{80} For these two cases, we stopped computations at primes $p < 10^3$ (because it was too long to continue). Thus, we have less data, so the probability that the governing fields found are due to random should be greater than for the other governing fields found. And it is effectively the case (see G for more details).

6.5.4 Limitation of Probabilistic Analysis

This analysis is interesting, however, one may be septic on the step where we take the assignment of a Frobenius element to it's conjugacy class as a binary random variable (being in the set S , or not). We know the Bernoulli random variables very well, comparing the Frobenian element conjugacy class to a random variable would need more thought. We have shown that the expectations are the same (as we calibrated the Bernoulli random variable for it). However, it would be interesting for example, in further study, to see if the variance is the same.



UNIVERSITY COLLEGE OF LONDON

DEPARTMENT OF MATHEMATICS

Modular Forms Modulo 2

Appendix

Author

Paul DUBOIS

Supervisor

Djordjo MILOVIC

March 12, 2020

A Hecke Operators

A.1 Primes Hecke Operators

	Δ^1	Δ^3	Δ^5	Δ^7	Δ^9	Δ^{11}	Δ^{13}	Δ^{15}	Δ^{17}	Δ^{19}	Δ^{21}
T_3	0	Δ^1	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_5	0	0	Δ^1	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7	$\Delta^9 + \Delta^{17}$
T_7	0	0	0	Δ^1	0	0	Δ^3	Δ^9	0	Δ^5	Δ^3
T_{11}	0	Δ^1	0	Δ^5	Δ^3	$\Delta^1 + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_{13}	0	0	Δ^1	Δ^3	0	0	$\Delta^1 + \Delta^9$	Δ^{11}	Δ^5	Δ^7	$\Delta^9 + \Delta^{17}$
T_{17}	0	0	0	0	Δ^1	Δ^3	Δ^5	Δ^7	Δ^1	0	0
T_{19}	0	Δ^1	0	Δ^5	Δ^3	$\Delta^1 + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^1 + \Delta^9 + \Delta^{17}$	Δ^7
T_{23}	0	0	0	Δ^1	0	0	Δ^3	$\Delta^1 + \Delta^9$	0	Δ^5	Δ^3
T_{29}	0	0	Δ^1	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7	$\Delta^1 + \Delta^9 + \Delta^{17}$
T_{31}	0	0	0	0	0	0	0	Δ^1	0	0	0
T_{37}	0	0	Δ^1	Δ^3	0	0	$\Delta^1 + \Delta^9$	Δ^{11}	Δ^5	Δ^7	$\Delta^9 + \Delta^{17}$
T_{41}	0	0	0	0	0	0	0	0	Δ^1	Δ^3	Δ^5
T_{43}	0	Δ^1	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_{47}	0	0	0	0	0	0	0	Δ^1	0	0	0
T_{53}	0	0	Δ^1	Δ^3	0	0	$\Delta^1 + \Delta^9$	Δ^{11}	Δ^5	Δ^7	$\Delta^1 + \Delta^9 + \Delta^{17}$
T_{59}	0	Δ^1	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^1 + \Delta^9 + \Delta^{17}$	Δ^7
T_{61}	0	0	Δ^1	Δ^3	0	0	$\Delta^1 + \Delta^9$	Δ^{11}	Δ^5	Δ^7	$\Delta^1 + \Delta^9 + \Delta^{17}$
T_{67}	0	Δ^1	0	Δ^5	Δ^3	$\Delta^1 + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_{71}	0	0	0	Δ^1	0	0	Δ^3	$\Delta^1 + \Delta^9$	0	Δ^5	Δ^3
T_{73}	0	0	0	0	Δ^1	Δ^3	Δ^5	Δ^7	0	Δ^3	Δ^5
T_{79}	0	0	0	0	0	0	0	0	0	0	0
T_{83}	0	Δ^1	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^1 + \Delta^9 + \Delta^{17}$	Δ^7
T_{89}	0	0	0	0	Δ^1	Δ^3	Δ^5	Δ^7	0	Δ^3	Δ^5
T_{97}	0	0	0	0	Δ^1	Δ^3	Δ^5	Δ^7	Δ^1	0	0
T_{101}	0	0	Δ^1	Δ^3	0	0	$\Delta^1 + \Delta^9$	Δ^{11}	Δ^5	Δ^7	$\Delta^9 + \Delta^{17}$
T_{103}	0	0	0	Δ^1	0	0	Δ^3	$\Delta^1 + \Delta^9$	0	Δ^5	Δ^3
T_{107}	0	Δ^1	0	Δ^5	Δ^3	$\Delta^1 + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_{109}	0	0	Δ^1	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7	$\Delta^9 + \Delta^{17}$
T_{113}	0	0	0	0	0	0	0	0	0	0	0
T_{127}	0	0	0	0	0	0	0	Δ^1	0	0	0
T_{131}	0	Δ^1	0	Δ^5	Δ^3	$\Delta^1 + \Delta^9$	Δ^7	Δ^{13}	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_{137}	0	0	0	0	0	0	0	0	Δ^1	Δ^3	Δ^5
T_{139}	0	Δ^1	0	Δ^5	Δ^3	Δ^9	Δ^7	$\Delta^5 + \Delta^{13}$	0	$\Delta^9 + \Delta^{17}$	Δ^7
T_{149}	0	0	Δ^1	Δ^3	0	0	Δ^9	$\Delta^3 + \Delta^{11}$	Δ^5	Δ^7	$\Delta^1 + \Delta^9 + \Delta^{17}$

Action of Primes Hecke Operators (primes up to 150) on Modular Forms Modulo 2 (up to Δ^{21}).

A larger table may be found online at https://pauldubois98.github.io/ModularFormsModuloTwo.jl/tables/Hecke_primes_table.html.

A.2 Powers of Hecke Operators

Δ^1	T_5^0	T_5^1	T_5^2	...	Δ^3	T_5^0	T_5^1	T_5^2	...	Δ^5	T_5^0	T_5^1	T_5^2	...	Δ^7	T_5^0	T_5^1	T_5^2	...
T_3^0	Δ^1	0	0	...	T_3^0	Δ^3	0	0	...	T_3^0	Δ^5	Δ^1	0	...	T_3^0	Δ^7	Δ^3	0	...
T_3^1	0	0	0	...	T_3^1	Δ^1	0	0	...	T_3^1	0	0	0	...	T_3^1	Δ^5	Δ^1	0	...
T_3^2	0	0	0	...	T_3^2	0	0	0	...	T_3^2	0	0	0	...	T_3^2	0	0	0	...
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\ddots
Δ^9	T_5^0	T_5^1	T_5^2	T_5^3	...	Δ^{11}	T_5^0	T_5^1	T_5^2	T_5^3	...	Δ^{13}	T_5^0	T_5^1	T_5^2	T_5^3	...		
T_3^0	Δ^9	0	0	0	...	T_3^0	Δ^{11}	0	0	0	...	T_3^0	Δ^{13}	Δ^9	0	0	...		
T_3^1	Δ^3	0	0	0	...	T_3^1	Δ^9	0	0	0	...	T_3^1	Δ^7	Δ^3	0	0	...		
T_3^2	Δ^1	0	0	0	...	T_3^2	Δ^3	0	0	0	...	T_3^2	Δ^5	Δ^1	0	0	...		
T_3^3	0	0	0	0	...	T_3^3	Δ^1	0	0	0	...	T_3^3	0	0	0	0	...		
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	
Δ^{15}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...	Δ^{17}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...						
T_3^0	Δ^{15}	$\Delta^3 + \Delta^{11}$	0	0	0	...	T_3^0	Δ^{17}	Δ^5	Δ^1	0	0	...						
T_3^1	$\Delta^5 + \Delta^{13}$	$\Delta^1 + \Delta^9$	0	0	0	...	T_3^1	0	0	0	0	0	...						
T_3^2	Δ^7	Δ^3	0	0	0	...	T_3^2	0	0	0	0	0	...						
T_3^3	Δ^5	Δ^1	0	0	0	...	T_3^3	0	0	0	0	0	...						
T_3^4	0	0	0	0	0	...	T_3^4	0	0	0	0	0	...						
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots						
Δ^{19}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...	Δ^{21}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...						
T_3^0	Δ^{19}	Δ^7	Δ^3	0	0	...	T_3^0	Δ^{21}	$\Delta^9 + \Delta^{17}$	Δ^5	Δ^1	0	...						
T_3^1	$\Delta^9 + \Delta^{17}$	Δ^5	Δ^1	0	0	...	T_3^1	Δ^7	Δ^3	0	0	0	...						
T_3^2	Δ^3	0	0	0	0	...	T_3^2	Δ^5	Δ^1	0	0	0	...						
T_3^3	Δ^1	0	0	0	0	...	T_3^3	0	0	0	0	0	...						
T_3^4	0	0	0	0	0	...	T_3^4	0	0	0	0	0	...						
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots						
Δ^{23}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...	Δ^{25}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...						
T_3^0	Δ^{23}	$\Delta^{11} + \Delta^{19}$	Δ^7	Δ^3	0	...	T_3^0	Δ^{25}	$\Delta^5 + \Delta^{13}$	$\Delta^1 + \Delta^9$	0	0	...						
T_3^1	$\Delta^{13} + \Delta^{21}$	Δ^{17}	Δ^5	Δ^1	0	...	T_3^1	$\Delta^{11} + \Delta^{19}$	Δ^7	Δ^3	0	0	...						
T_3^2	0	0	0	0	0	...	T_3^2	Δ^{17}	Δ^5	Δ^1	0	0	...						
T_3^3	0	0	0	0	0	...	T_3^3	0	0	0	0	0	...						
T_3^4	0	0	0	0	0	...	T_3^4	0	0	0	0	0	...						
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots						
Δ^{27}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...	Δ^{29}	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	...						
T_3^0	Δ^{27}	$\Delta^7 + \Delta^{15}$	Δ^{11}	0	0	...	T_3^0	Δ^{29}	$\Delta^{17} + \Delta^{25}$	Δ^{13}	Δ^9	0	...						
T_3^1	$\Delta^9 + \Delta^{17} + \Delta^{25}$	Δ^{13}	Δ^9	0	0	...	T_3^1	Δ^{23}	$\Delta^{11} + \Delta^{19}$	Δ^7	Δ^3	0	...						
T_3^2	$\Delta^3 + \Delta^{11} + \Delta^{19}$	Δ^7	Δ^3	0	0	...	T_3^2	$\Delta^{13} + \Delta^{21}$	Δ^{17}	Δ^5	Δ^1	0	...						
T_3^3	$\Delta^1 + \Delta^{17}$	Δ^5	Δ^1	0	0	...	T_3^3	0	0	0	0	0	...						
T_3^4	0	0	0	0	0	...	T_3^4	0	0	0	0	0	...						
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots						

Action of Powers Hecke Operators T_3 and T_5 on Modular Forms Modulo 2 (up to Δ^{31}).

A larger table may be found online at https://pauldubois98.github.io/ModularFormsModuloTwo.jl/tables/Hecke_powers_table.html.

A.3 Behaviour of Code of Integers

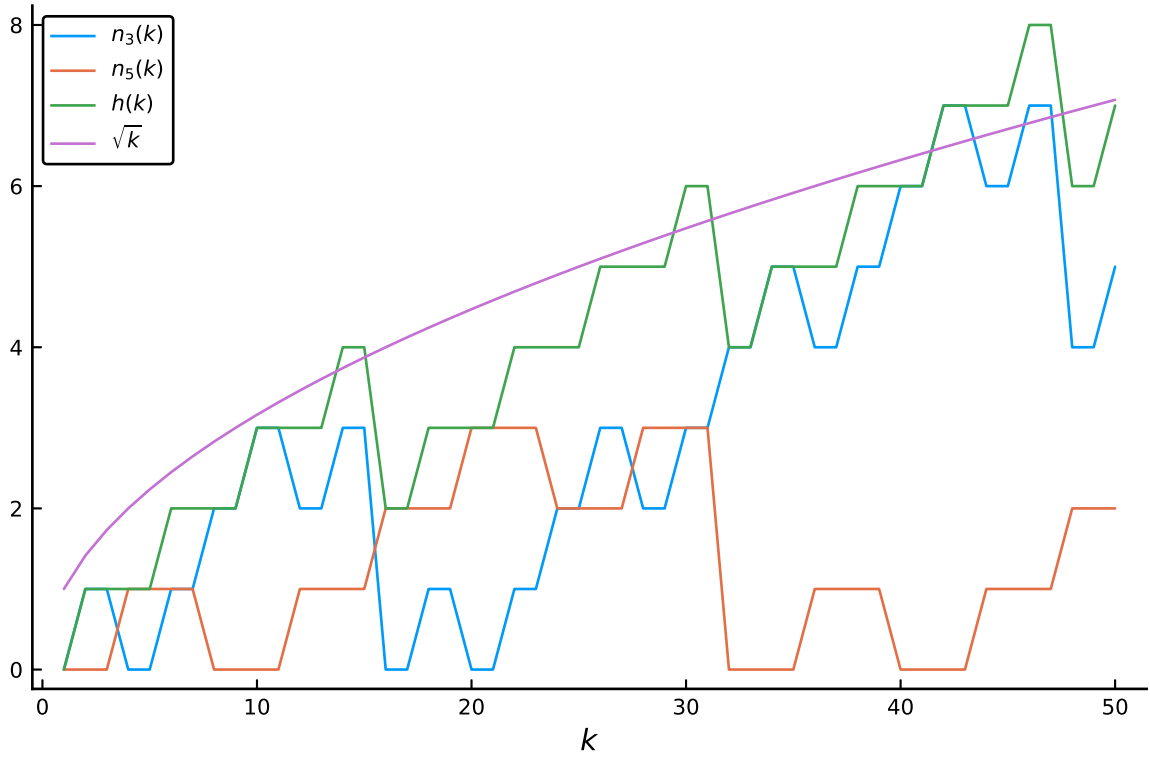
Code of Integers up to 150 Codes, as a function of integers.

k	code of k	h(k)	k	code of k	h(k)	k	code of k	h(k)
0 , 1	[0 , 0]	0	50 , 51	[5 , 2]	7	100 , 101	[4 , 5]	9
2 , 3	[1 , 0]	1	52 , 53	[4 , 3]	7	102 , 103	[5 , 5]	10
4 , 5	[0 , 1]	1	54 , 55	[5 , 3]	8	104 , 105	[6 , 4]	10
6 , 7	[1 , 1]	2	56 , 57	[6 , 2]	8	106 , 107	[7 , 4]	11
8 , 9	[2 , 0]	2	58 , 59	[7 , 2]	9	108 , 109	[6 , 5]	11
10 , 11	[3 , 0]	3	60 , 61	[6 , 3]	9	110 , 111	[7 , 5]	12
12 , 13	[2 , 1]	3	62 , 63	[7 , 3]	10	112 , 113	[4 , 6]	10
14 , 15	[3 , 1]	4	64 , 65	[0 , 4]	4	114 , 115	[5 , 6]	11
16 , 17	[0 , 2]	2	66 , 67	[1 , 4]	5	116 , 117	[4 , 7]	11
18 , 19	[1 , 2]	3	68 , 69	[0 , 5]	5	118 , 119	[5 , 7]	12
20 , 21	[0 , 3]	3	70 , 71	[1 , 5]	6	120 , 121	[6 , 6]	12
22 , 23	[1 , 3]	4	72 , 73	[2 , 4]	6	122 , 123	[7 , 6]	13
24 , 25	[2 , 2]	4	74 , 75	[3 , 4]	7	124 , 125	[6 , 7]	13
26 , 27	[3 , 2]	5	76 , 77	[2 , 5]	7	126 , 127	[7 , 7]	14
28 , 29	[2 , 3]	5	78 , 79	[3 , 5]	8	128 , 129	[8 , 0]	8
30 , 31	[3 , 3]	6	80 , 81	[0 , 6]	6	130 , 131	[9 , 0]	9
32 , 33	[4 , 0]	4	82 , 83	[1 , 6]	7	132 , 133	[8 , 1]	9
34 , 35	[5 , 0]	5	84 , 85	[0 , 7]	7	134 , 135	[9 , 1]	10
36 , 37	[4 , 1]	5	86 , 87	[1 , 7]	8	136 , 137	[10 , 0]	10
38 , 39	[5 , 1]	6	88 , 89	[2 , 6]	8	138 , 139	[11 , 0]	11
40 , 41	[6 , 0]	6	90 , 91	[3 , 6]	9	140 , 141	[10 , 1]	11
42 , 43	[7 , 0]	7	92 , 93	[2 , 7]	9	142 , 143	[11 , 1]	12
44 , 45	[6 , 1]	7	94 , 95	[3 , 7]	10	144 , 145	[8 , 2]	10
46 , 47	[7 , 1]	8	96 , 97	[4 , 4]	8	146 , 147	[9 , 2]	11
48 , 49	[4 , 2]	6	98 , 99	[5 , 4]	9	148 , 149	[8 , 3]	11

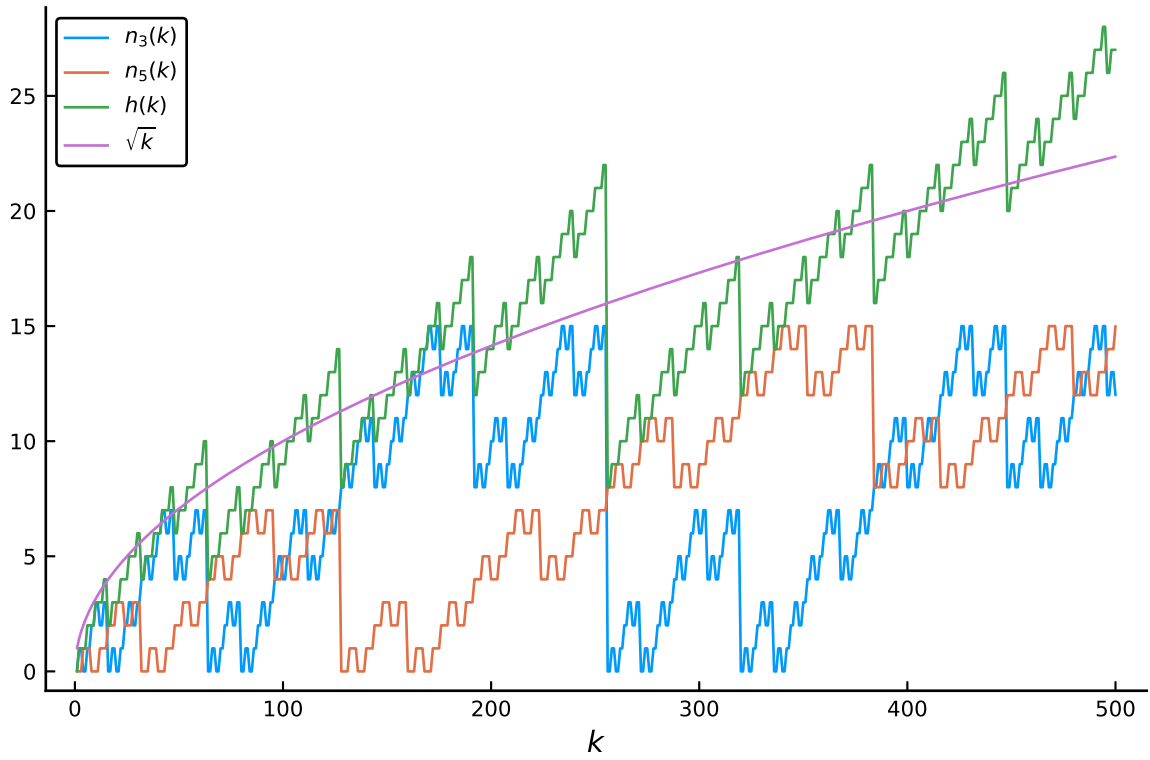
A larger table may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/int_to_code/.

A.4 Behaviour of h on Various Scales

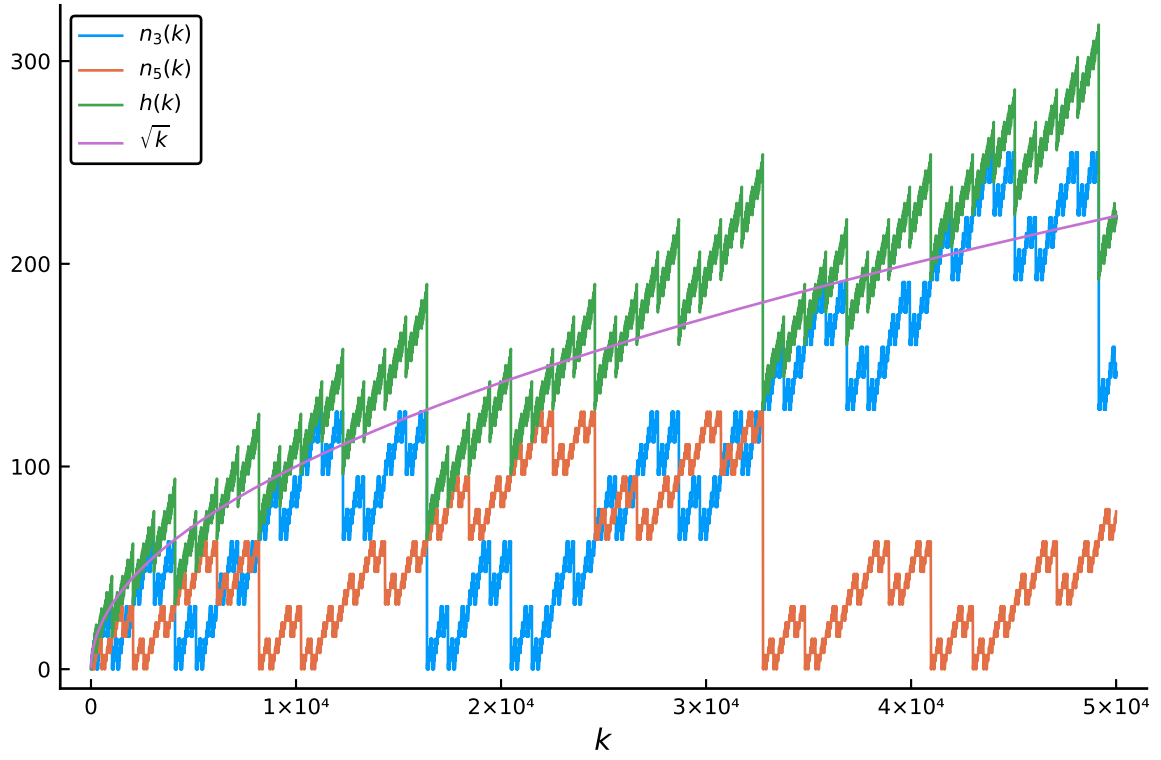
Range 0 to $5 * 10^1$



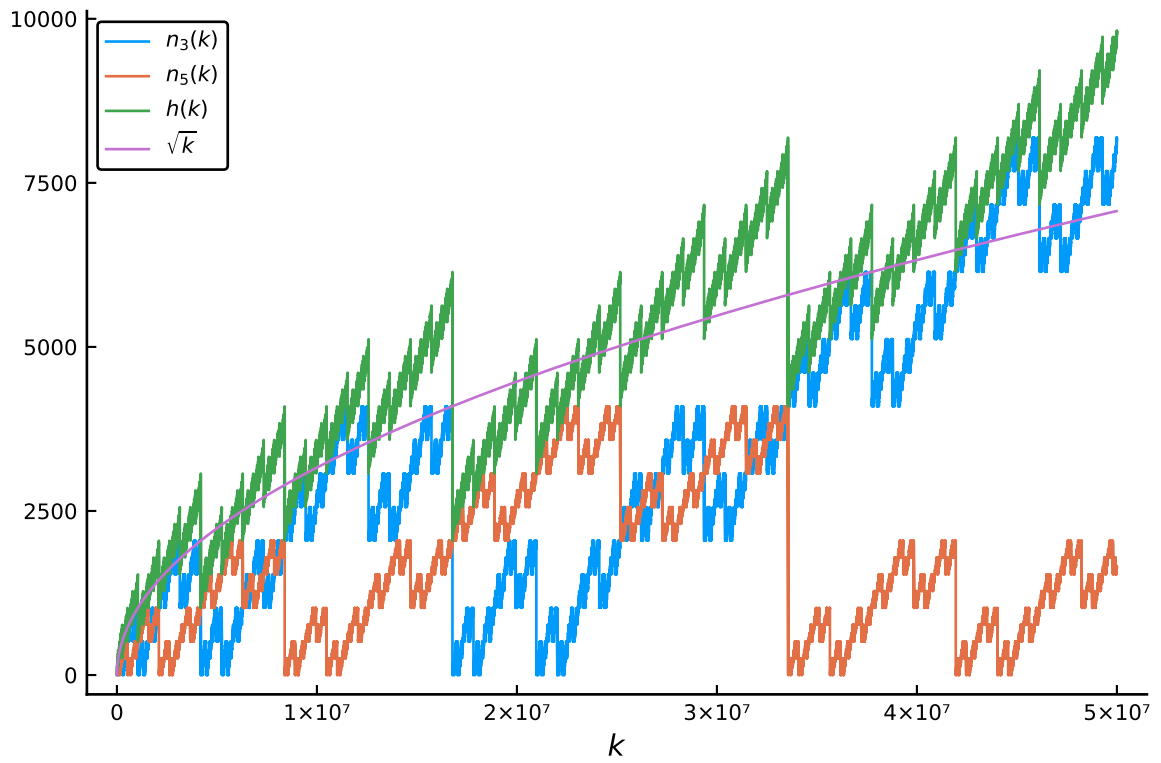
Range 0 to $5 * 10^2$



Range 0 to 5×10^4



Range 0 to 5×10^7



Other pictures (including animated ones) may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/behaviour_h/.

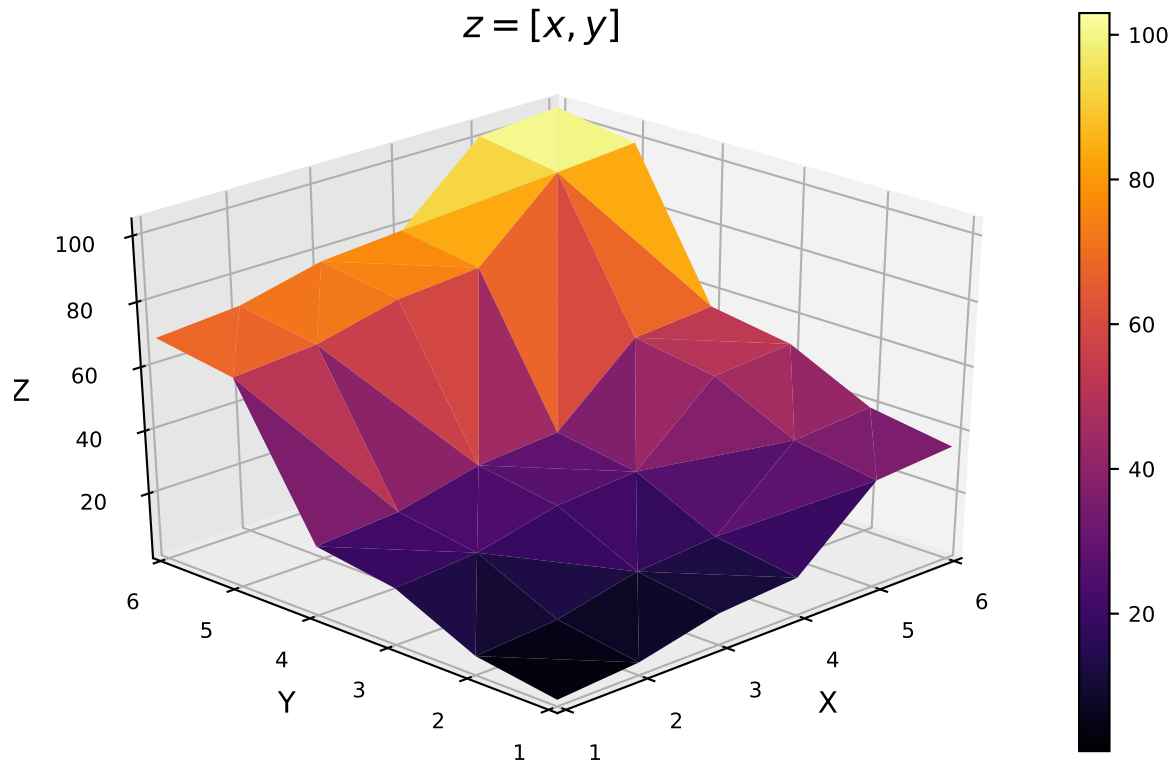
Integers with Small Code

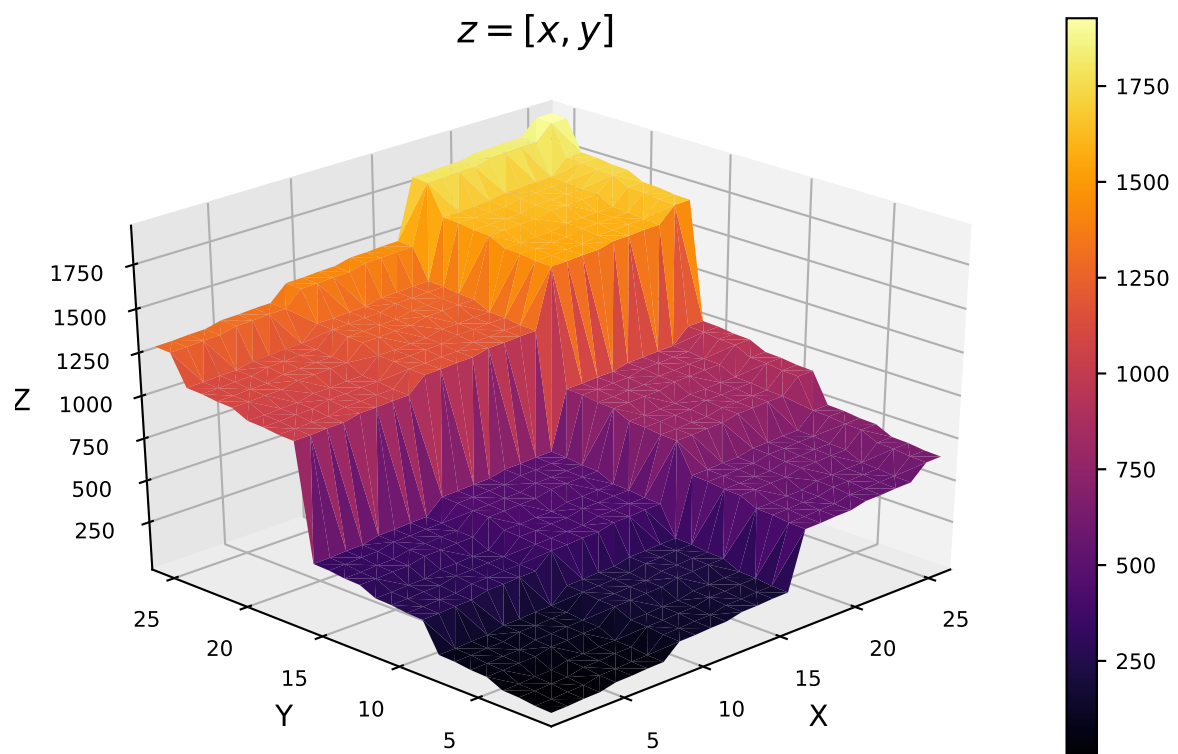
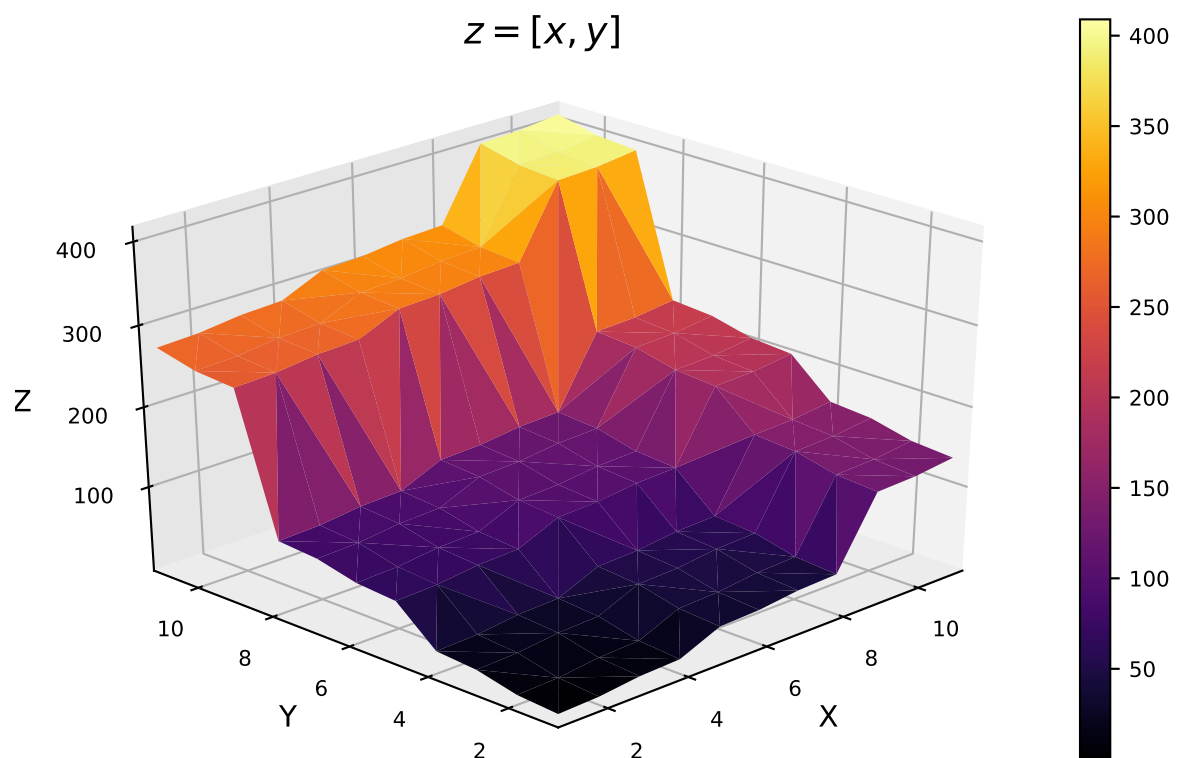
Table As the codes of integers are in bijection with even numbers (or odd numbers), we can also plot even (or odd) integers as function of their code.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	4	16	20	64	68	80	84	256	260	272	276	320	324	336	340
1	2	6	18	22	66	70	82	86	258	262	274	278	322	326	338	342
2	8	12	24	28	72	76	88	92	264	268	280	284	328	332	344	348
3	10	14	26	30	74	78	90	94	266	270	282	286	330	334	346	350
4	32	36	48	52	96	100	112	116	288	292	304	308	352	356	368	372
5	34	38	50	54	98	102	114	118	290	294	306	310	354	358	370	374
6	40	44	56	60	104	108	120	124	296	300	312	316	360	364	376	380
7	42	46	58	62	106	110	122	126	298	302	314	318	362	366	378	382
8	128	132	144	148	192	196	208	212	384	388	400	404	448	452	464	468
9	130	134	146	150	194	198	210	214	386	390	402	406	450	454	466	470
10	136	140	152	156	200	204	216	220	392	396	408	412	456	460	472	476

A larger table may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/code_to_even/. Note that the same table may be produced for odd integers, find it online at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/code_to_odd/.

Plots We plot the surface $z = [x, y]$ i.e. z is the integer with code $[x, y]$.





Other pictures (including animated ones) may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/plot_code_to_int/.

Powers of T_3 and T_5 giving Δ^1 Here, we look at the power of Δ^k such that when $T_3^a T_5^b$ is applied to it, we have $T_3^a T_5^b | \Delta^k = \Delta^1$.

	T_5^0	T_5^1	T_5^2	T_5^3	T_5^4	T_5^5	T_5^6	T_5^7	T_5^8	T_5^9	T_5^{10}	T_5^{11}	T_5^{12}	T_5^{13}	T_5^{14}
T_3^0	Δ^1	Δ^5	Δ^{17}	Δ^{21}	Δ^{65}	Δ^{69}	Δ^{81}	Δ^{85}	Δ^{257}	Δ^{261}	Δ^{273}	Δ^{277}	Δ^{321}	Δ^{325}	Δ^{337}
T_3^1	Δ^3	Δ^7	Δ^{19}	Δ^{23}	Δ^{67}	Δ^{71}	Δ^{83}	Δ^{87}	Δ^{259}	Δ^{263}	Δ^{275}	Δ^{279}	Δ^{323}	Δ^{327}	Δ^{339}
T_3^2	Δ^9	Δ^{13}	Δ^{25}	Δ^{29}	Δ^{73}	Δ^{77}	Δ^{89}	Δ^{93}	Δ^{265}	Δ^{269}	Δ^{281}	Δ^{285}	Δ^{329}	Δ^{333}	Δ^{345}
T_3^3	Δ^{11}	Δ^{15}	Δ^{27}	Δ^{31}	Δ^{75}	Δ^{79}	Δ^{91}	Δ^{95}	Δ^{267}	Δ^{271}	Δ^{283}	Δ^{287}	Δ^{331}	Δ^{335}	Δ^{347}
T_3^4	Δ^{33}	Δ^{37}	Δ^{49}	Δ^{53}	Δ^{97}	Δ^{101}	Δ^{113}	Δ^{117}	Δ^{289}	Δ^{293}	Δ^{305}	Δ^{309}	Δ^{353}	Δ^{357}	Δ^{369}
T_3^5	Δ^{35}	Δ^{39}	Δ^{51}	Δ^{55}	Δ^{99}	Δ^{103}	Δ^{115}	Δ^{119}	Δ^{291}	Δ^{295}	Δ^{307}	Δ^{311}	Δ^{355}	Δ^{359}	Δ^{371}
T_3^6	Δ^{41}	Δ^{45}	Δ^{57}	Δ^{61}	Δ^{105}	Δ^{109}	Δ^{121}	Δ^{125}	Δ^{297}	Δ^{301}	Δ^{313}	Δ^{317}	Δ^{361}	Δ^{365}	Δ^{377}
T_3^7	Δ^{43}	Δ^{47}	Δ^{59}	Δ^{63}	Δ^{107}	Δ^{111}	Δ^{123}	Δ^{127}	Δ^{299}	Δ^{303}	Δ^{315}	Δ^{319}	Δ^{363}	Δ^{367}	Δ^{379}
T_3^8	Δ^{129}	Δ^{133}	Δ^{145}	Δ^{149}	Δ^{193}	Δ^{197}	Δ^{209}	Δ^{213}	Δ^{385}	Δ^{389}	Δ^{401}	Δ^{405}	Δ^{449}	Δ^{453}	Δ^{465}
T_3^9	Δ^{131}	Δ^{135}	Δ^{147}	Δ^{151}	Δ^{195}	Δ^{199}	Δ^{211}	Δ^{215}	Δ^{387}	Δ^{391}	Δ^{403}	Δ^{407}	Δ^{451}	Δ^{455}	Δ^{467}
T_3^{10}	Δ^{137}	Δ^{141}	Δ^{153}	Δ^{157}	Δ^{201}	Δ^{205}	Δ^{217}	Δ^{221}	Δ^{393}	Δ^{397}	Δ^{409}	Δ^{413}	Δ^{457}	Δ^{461}	Δ^{473}
T_3^{11}	Δ^{139}	Δ^{143}	Δ^{155}	Δ^{159}	Δ^{203}	Δ^{207}	Δ^{219}	Δ^{223}	Δ^{395}	Δ^{399}	Δ^{411}	Δ^{415}	Δ^{459}	Δ^{463}	Δ^{475}
T_3^{12}	Δ^{161}	Δ^{165}	Δ^{177}	Δ^{181}	Δ^{225}	Δ^{229}	Δ^{241}	Δ^{245}	Δ^{417}	Δ^{421}	Δ^{433}	Δ^{437}	Δ^{481}	Δ^{485}	Δ^{497}
T_3^{13}	Δ^{163}	Δ^{167}	Δ^{179}	Δ^{183}	Δ^{227}	Δ^{231}	Δ^{243}	Δ^{247}	Δ^{419}	Δ^{423}	Δ^{435}	Δ^{439}	Δ^{483}	Δ^{487}	Δ^{499}
T_3^{14}	Δ^{169}	Δ^{173}	Δ^{185}	Δ^{189}	Δ^{233}	Δ^{237}	Δ^{249}	Δ^{253}	Δ^{425}	Δ^{429}	Δ^{441}	Δ^{445}	Δ^{489}	Δ^{493}	Δ^{505}
T_3^{15}	Δ^{171}	Δ^{175}	Δ^{187}	Δ^{191}	Δ^{235}	Δ^{239}	Δ^{251}	Δ^{255}	Δ^{427}	Δ^{431}	Δ^{443}	Δ^{447}	Δ^{491}	Δ^{495}	Δ^{507}
T_3^{16}	Δ^{513}	Δ^{517}	Δ^{529}	Δ^{533}	Δ^{577}	Δ^{581}	Δ^{593}	Δ^{597}	Δ^{769}	Δ^{773}	Δ^{785}	Δ^{789}	Δ^{833}	Δ^{837}	Δ^{849}
T_3^{17}	Δ^{515}	Δ^{519}	Δ^{531}	Δ^{535}	Δ^{579}	Δ^{583}	Δ^{595}	Δ^{599}	Δ^{771}	Δ^{775}	Δ^{787}	Δ^{791}	Δ^{835}	Δ^{839}	Δ^{851}
T_3^{18}	Δ^{521}	Δ^{525}	Δ^{537}	Δ^{541}	Δ^{585}	Δ^{589}	Δ^{601}	Δ^{605}	Δ^{777}	Δ^{781}	Δ^{793}	Δ^{797}	Δ^{841}	Δ^{845}	Δ^{857}
T_3^{19}	Δ^{523}	Δ^{527}	Δ^{539}	Δ^{543}	Δ^{587}	Δ^{591}	Δ^{603}	Δ^{607}	Δ^{779}	Δ^{783}	Δ^{795}	Δ^{799}	Δ^{843}	Δ^{847}	Δ^{859}
T_3^{20}	Δ^{545}	Δ^{549}	Δ^{561}	Δ^{565}	Δ^{609}	Δ^{613}	Δ^{625}	Δ^{629}	Δ^{801}	Δ^{805}	Δ^{817}	Δ^{821}	Δ^{865}	Δ^{869}	Δ^{881}
T_3^{21}	Δ^{547}	Δ^{551}	Δ^{563}	Δ^{567}	Δ^{611}	Δ^{615}	Δ^{627}	Δ^{631}	Δ^{803}	Δ^{807}	Δ^{819}	Δ^{823}	Δ^{867}	Δ^{871}	Δ^{883}
T_3^{22}	Δ^{553}	Δ^{557}	Δ^{569}	Δ^{573}	Δ^{617}	Δ^{621}	Δ^{633}	Δ^{637}	Δ^{809}	Δ^{813}	Δ^{825}	Δ^{829}	Δ^{873}	Δ^{877}	Δ^{889}
T_3^{23}	Δ^{555}	Δ^{559}	Δ^{571}	Δ^{575}	Δ^{619}	Δ^{623}	Δ^{635}	Δ^{639}	Δ^{811}	Δ^{815}	Δ^{827}	Δ^{831}	Δ^{875}	Δ^{879}	Δ^{891}
T_3^{24}	Δ^{641}	Δ^{645}	Δ^{657}	Δ^{661}	Δ^{705}	Δ^{709}	Δ^{721}	Δ^{725}	Δ^{897}	Δ^{901}	Δ^{913}	Δ^{917}	Δ^{961}	Δ^{965}	Δ^{977}
T_3^{25}	Δ^{643}	Δ^{647}	Δ^{659}	Δ^{663}	Δ^{707}	Δ^{711}	Δ^{723}	Δ^{727}	Δ^{899}	Δ^{903}	Δ^{915}	Δ^{919}	Δ^{963}	Δ^{967}	Δ^{979}
T_3^{26}	Δ^{649}	Δ^{653}	Δ^{665}	Δ^{669}	Δ^{713}	Δ^{717}	Δ^{729}	Δ^{733}	Δ^{905}	Δ^{909}	Δ^{921}	Δ^{925}	Δ^{969}	Δ^{973}	Δ^{985}
T_3^{27}	Δ^{651}	Δ^{655}	Δ^{667}	Δ^{671}	Δ^{715}	Δ^{719}	Δ^{731}	Δ^{735}	Δ^{907}	Δ^{911}	Δ^{923}	Δ^{927}	Δ^{971}	Δ^{975}	Δ^{987}
T_3^{28}	Δ^{673}	Δ^{677}	Δ^{689}	Δ^{693}	Δ^{737}	Δ^{741}	Δ^{753}	Δ^{757}	Δ^{929}	Δ^{933}	Δ^{945}	Δ^{949}	Δ^{993}	Δ^{997}	Δ^{1009}
T_3^{29}	Δ^{675}	Δ^{679}	Δ^{691}	Δ^{695}	Δ^{739}	Δ^{743}	Δ^{755}	Δ^{759}	Δ^{931}	Δ^{935}	Δ^{947}	Δ^{951}	Δ^{995}	Δ^{999}	Δ^{1011}
T_3^{30}	Δ^{681}	Δ^{685}	Δ^{697}	Δ^{701}	Δ^{745}	Δ^{749}	Δ^{761}	Δ^{765}	Δ^{937}	Δ^{941}	Δ^{953}	Δ^{957}	Δ^{1001}	Δ^{1005}	Δ^{1017}

A larger table may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTwo/T3T5_powers_to_delta/.

A.5 Expansions of T_p as Series of T_3 and T_5

Using T_3 and T_5 Here are expansions of T_p in series of $T_3^a T_5^b$ for primes $p < 15$:

$$T_3 = T_3^1 T_5^0$$

$$T_5 = T_3^0 T_5^1$$

$$T_7 = T_3^1 T_5^1 + T_3^3 T_5^1 + T_3^5 T_5^1 + T_3^7 T_5^1 + T_3^9 T_5^1 + T_3^{11} T_5^1 + T_3^{13} T_5^1 + T_3^{15} T_5^1 + T_3^{17} T_5^1 + T_3^{19} T_5^1 + T_3^{21} T_5^1 + T_3^{23} T_5^1 + \dots$$

$$T_{11} = T_3^1 T_5^0 + T_3^3 T_5^0 + T_3^5 T_5^0 + T_3^7 T_5^0 + T_3^9 T_5^0 + T_3^{11} T_5^0 + T_3^{13} T_5^0 + T_3^{15} T_5^0 + T_3^{17} T_5^0 + T_3^{19} T_5^0 + T_3^{21} T_5^0 + T_3^{23} T_5^0 + \dots$$

$$T_{13} = T_3^0 T_5^1 + T_3^2 T_5^1 + T_3^4 T_5^1 + T_3^6 T_5^1 + T_3^8 T_5^1 + T_3^{10} T_5^1 + T_3^{12} T_5^1 + T_3^{14} T_5^1 + T_3^{16} T_5^1 + T_3^{18} T_5^1 + T_3^{20} T_5^1 + T_3^{22} T_5^1 + \dots$$

Expansions for larger primes may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTw/T_p_extensions/T_p_expansions_T3T5.html.

Using x and y For readability reasons, we usually use $x = T_3$ and $y = T_5$. Here are expansions of T_p in series of $T_3^a T_5^b = x^a y^b$ for primes $p < 50$:

$$T_3 = x^1 y^0$$

$$T_5 = x^0 y^1$$

$$T_7 = x^1 y^1 + x^3 y^1 + x^5 y^1 + x^7 y^1 + x^9 y^1 + x^{11} y^1 + x^{13} y^1 + x^{15} y^1 + x^{17} y^1 + x^{19} y^1 + x^{21} y^1 + x^{23} y^1 + \dots$$

$$T_{11} = x^1 y^0 + x^3 y^0 + x^5 y^0 + x^7 y^0 + x^9 y^0 + x^{11} y^0 + x^{13} y^0 + x^{15} y^0 + x^{17} y^0 + x^{19} y^0 + x^{21} y^0 + x^{23} y^0 + \dots$$

$$T_{13} = x^0 y^1 + x^2 y^1 + x^4 y^1 + x^6 y^1 + x^8 y^1 + x^{10} y^1 + x^{12} y^1 + x^{14} y^1 + x^{16} y^1 + x^{18} y^1 + x^{20} y^1 + x^{22} y^1 + \dots$$

$$T_{17} = x^0 y^2 + x^2 y^2 + x^4 y^2 + x^6 y^2 + x^8 y^2 + x^{10} y^2 + x^{12} y^2 + x^{14} y^2 + x^{16} y^2 + x^{18} y^2 + x^{20} y^2 + x^{22} y^2 + \dots$$

$$T_{19} = x^1 y^0 + x^3 y^0 + x^5 y^0 + x^7 y^0 + x^9 y^0 + x^{11} y^0 + x^{13} y^0 + x^{15} y^0 + x^{17} y^0 + x^{19} y^0 + x^{21} y^0 + x^{23} y^0 + \dots$$

$$T_{23} = x^1 y^1 + x^3 y^1 + x^5 y^1 + x^7 y^1 + x^9 y^1 + x^{11} y^1 + x^{13} y^1 + x^{15} y^1 + x^{17} y^1 + x^{19} y^1 + x^{21} y^1 + x^{23} y^1 + \dots$$

$$T_{29} = x^0 y^1 + x^2 y^1 + x^4 y^1 + x^6 y^1 + x^8 y^1 + x^{10} y^1 + x^{12} y^1 + x^{14} y^1 + x^{16} y^1 + x^{18} y^1 + x^{20} y^1 + x^{22} y^1 + \dots$$

$$T_{31} = x^1 y^1 + x^3 y^1 + x^5 y^1 + x^7 y^1 + x^9 y^1 + x^{11} y^1 + x^{13} y^1 + x^{15} y^1 + x^{17} y^1 + x^{19} y^1 + x^{21} y^1 + x^{23} y^1 + \dots$$

$$T_{37} = x^0 y^1 + x^2 y^1 + x^4 y^1 + x^6 y^1 + x^8 y^1 + x^{10} y^1 + x^{12} y^1 + x^{14} y^1 + x^{16} y^1 + x^{18} y^1 + x^{20} y^1 + x^{22} y^1 + \dots$$

$$T_{41} = x^0 y^2 + x^2 y^2 + x^4 y^2 + x^6 y^2 + x^8 y^2 + x^{10} y^2 + x^{12} y^2 + x^{14} y^2 + x^{16} y^2 + x^{18} y^2 + x^{20} y^2 + x^{22} y^2 + \dots$$

$$T_{43} = x^1 y^0 + x^3 y^0 + x^5 y^0 + x^7 y^0 + x^9 y^0 + x^{11} y^0 + x^{13} y^0 + x^{15} y^0 + x^{17} y^0 + x^{19} y^0 + x^{21} y^0 + x^{23} y^0 + \dots$$

$$T_{47} = x^1 y^1 + x^3 y^1 + x^5 y^1 + x^7 y^1 + x^9 y^1 + x^{11} y^1 + x^{13} y^1 + x^{15} y^1 + x^{17} y^1 + x^{19} y^1 + x^{21} y^1 + x^{23} y^1 + \dots$$

Expansions for larger primes may be found online at https://pauldubois98.github.io/HeckeOperatorsModuloTw/T_p_extensions/T_p_expansions_xy.html.

B Chebotarev Example

```
m = 0
f = x^3 + m*x^2 + (m-3)*x - 1
K = NumberField(f, 'a')

print("Defining polynomial:")
print(f)
> x^3 - 3*x - 1
print("Discriminant:")
print(K.discriminant())
> 81
print("Extension Degree:")
print(K.degree())
> 3
print("Galois extension:")
print(K.is_galois())
> True
print("Basis:")
True
print(K.maximal_order().basis())
> [1, a, a^2]
```

C Speed Comparison

C.1 Pure Python

```
def delta(LENGTH):
    f=[0 for i in range(LENGTH)]
    indice=1
    i=1
    while indice<LENGTH:
        f[indice] = 1
        i+=2
        indice = i**2
    return f

def square(f):
    f_sq = [0 for i in range(len(f))]
    i = 0
    while 2*i < len(f):
        if f[i]:
            f_sq[2*i] = 1
        i += 1
    return f_sq
```

C.2 NumPy Python

```
import numpy as np

def delta(LENGTH):
    f=np.zeros(LENGTH, dtype=np.int8)
    indice=1
    i=1
    while indice<LENGTH:
        f[indice] = 1
        i+=2
        indice = i**2
    return f

def square(f):
    f_sq=np.zeros(len(f), dtype=np.int8)
    i = 0
    while 2*i < len(f):
        if f[i]:
            f_sq[2*i] = 1
        i += 1
    return f_sq
```

C.3 Dense Julia

```
function delta(LENGTH)
    f = zeros{Int8, LENGTH}
    indice = 2
    i = 1
    while indice < LENGTH
        f[indice] = 1
        i += 2
        indice = i^2 + 1
    end
    return f
end

function square(f)
    f_sq = zeros{Int8, length(f)}
    i = 1
    while 2 * i - 1 < length(f)
        if f[i] == 1
            f_sq[2 * i - 1] = 1
        end
        i += 1
    end
    return f_sq
end
```

```
end
```

C.4 Sparse Python

```
def delta(LENGTH):
    f = []
    indice = 1
    i = 1
    while indice < LENGTH:
        f.append(indice)
        i += 2
        indice = i**2
    return (f, LENGTH)

def square(form):
    f_sq = []
    f = form[0]
    for n in f:
        if 2*n-1 <= form[1]:
            f_sq.append(2*n-1)
    return (f_sq, form[1])
```

C.5 Sparse Julia

```
using SparseArrays: SparseVector, spzeros
```

```
function delta(LENGTH)
    f = spzeros{Int8}(LENGTH)
    indice::Int = 2
    i::Int = 1
    while indice <= f.n
        f[indice] = Int8(1)
        i += 2
        indice = i^2 + 1
    end
    return f
end
```

```
function square(f)
    f_sq = spzeros{Int8}(f.n)
    for n in f.nzind
        if 2n - 1 <= f_sq.n
            f_sq[2n - 1] = 1
        end
    end
    return f_sq
end
```


end

D ModularFormsModuloTwo.jl

D.1 Module structure

```
ModularFormsModuloTwo.jl (module)
├── docs
│   └── (automatic documentation)
├── examples
│   ├── Hecke_primes_table.jl
│   └── Hecke_powers_table.j
└── src
    ├── data
    │   ├── storage.jl
    │   ├── delta_file_maker.jl
    │   ├── Hecke_primes_file_maker.jl
    │   ├── Hecke_powers_file_maker.jl
    │   └── data_files (.JLD2)
    ├── arithmetic.jl
    ├── equality.jl
    ├── generators.jl
    ├── HeckeOperator.jl
    ├── ModularFormsModuloTwo.jl
    └── recognizer.jl
```

Full module on GitHub (<https://github.com/pauldubois98/ModularFormsModuloTwo.jl>). The official documentation (<https://pauldubois98.github.io/ModularFormsModuloTwo.jl/>) has more details on how to use.

D.2 Files details

D.2.1 Main Sources

ModularFormsModuloTwo.jl

```
module ModularFormsModuloTwo
    """
    A standard module for computations on modular forms modulo two.
    """

    using SparseArrays: SparseVector, spzeros, dropzeros!, sparse
    import Base: +, *, ^

    """
    We can represent a modular forms mod 2 by it's coefficients as a polynomial in q
    ↪ or D.
    The routines in this file are made for q-series.
```

```

Modular forms modulo 2 have coefficients in q-series being 0 most of the times,
↪ and 1 otherwise.
Thus, we will represent them as sparse 1-dimensional arrays (sparse vectors) of
↪ type SparseVector{Int8,Int}.
"""
ModularForm = SparseVector{Int8,Int}
ModularFormOrNothing = Union{ModularForm, Nothing}
ModularFormList = Array{SparseVector{Int8,Int}, 1}
"""
Lists of Modular Forms will be useful for storage.
"""
ModularFormOrNothingList = Array{ModularFormOrNothing, 1}

"""
disp(f[, maxi])

Display details of f, a modular forms mod 2.

Displays what type of data the object id, up to which coefficient is the form
↪ known.
Then displays the first few coefficients. Coefficients are displayed until maxi
↪ (50 by default).

# Example
```julia-repl
[f is a modular form mod 2]
julia> disp(f)
MF mod 2 (coef to 100) - 01000000010000000000000001000000000000000000000000001...
...
"""
function disp(f::ModularForm, maxi::Int = 50)
 print("MF mod 2 (coef to " * string(f.n) * ") - ")
 for i = 1:min(maxi, f.n)
 print(f[i])
 end
 if maxi < f.n
 println("...")
 else
 println()
 end
end

function brackets(k::Int, brackets_level::Int=1)::String
 if brackets_level==0 # never put brackets
 return string(k)
 elseif brackets_level==1 # put brackets if more than one digit

```

```

 if k>9
 return "{"*string(k)*"}"
 else
 return string(k)
 end
 else # always put brackets
 return "{"*string(k)*"}"
 end
end
end

function delta_repr(f::ModularFormOrNothing, Delta_symbol::String="\\Delta",
 ↪ brackets_level::Int=1, math_mode::Bool=true)::String
 if f===nothing
 return "error"
 end
 if length(f.nzind)==0
 return "0"
 else
 k = f.nzind[1]
 s = Delta_symbol*"^"*brackets(k-1, brackets_level)
 for k in f.nzind[2:end]
 s *= " + "*Delta_symbol*"^"*brackets(k-1, brackets_level)
 end
 if math_mode
 return "\\$"*s* "\\$"
 else
 return s
 end
 end
end
end

```

```

include("arithmetic.jl")
include("equality.jl")
include("generators.jl")
include("HeckeOperator.jl")
include("data/storage_text.jl")
include("data/storage.jl")
include("recognizer.jl")

```

end

## arithmetic.jl

"""

We can represent a modular forms mod 2 by it's coefficients as a polynomial in  $q$  or  $\leftrightarrow D$ .

The routines in this file are made for  $q$ -series.

"""

*### ARITHMETIC OPERATIONS of modular forms modulo 2*

"""

$+(f1, f2)$

Compute the addition of two modular forms (with mathematical accuracy).

# Example

```julia-repl

[f1 & f2 are modular forms mod 2]

julia> f1+f2

1000-element SparseVector{Int8,Int64} with 27 stored entries:

[...]

```

"""

**function**  $+(f1::\text{ModularForm}, f2::\text{ModularForm})::\text{ModularForm}$

$m = \min(f1.n, f2.n)$

$f = \text{truncate}(f1, m)$

**for**  $n$  **in**  $f2.nzind$

**if**  $n \leq m$

$f[n] = 1 - f[n]$

**end**

**end**

**return**  $\text{dropzeros!}(f)$

**end**

"""

$*(f1, f2)$

Compute the multiplication of two modular forms (with mathematical accuracy).

# Example

```julia-repl

[f1 & f2 are modular forms mod 2]

julia> f1*f2

1000-element SparseVector{Int8,Int64} with 86 stored entries:

```

[...]
...
"""
function *(f1::ModularForm, f2::ModularForm)::ModularForm
    n::Int = min(f1.n, f2.n)
    f = spzeros(Int8, n)
    for n in f1.nzind
        for m in f2.nzind
            if n+m-1 <= f.n
                f[n+m-1]=1-f[n+m-1]
            else
                break
            end
        end
    end
    return dropzeros!(f)
end

"""
    sq(f)

```

Compute the square of a modular form (with mathematical accuracy).

This is a much more efficient method then computing the square with multiplication.
 sq(f) is (much) more efficient then f*f, time wise and memory wise.

```

# Example
```julia-repl
[f is a modular form mod 2]
julia> @time f*f
 0.169466 seconds (37 allocations: 1.127 MiB)

julia> @time sq(f)
 0.000020 seconds (23 allocations: 9.875 KiB)
...
"""
function sq(f::ModularForm)::ModularForm
 f_sq = spzeros(Int8, f.n)
 for n in f.nzind
 if 2n-1 <= f_sq.n
 f_sq[2n-1] = 1
 end
 end
 return f_sq
end

"""

```

```
^(f, k)
```

Compute  $f^k$  (with mathematical accuracy).

```
Example
```

```
```julia-repl
```

```
[f is a modular form mod 2]
```

```
julia> f^5
```

```
1000-element SparseVector{Int8,Int64} with 75 stored entries:
```

```
[...]
```

```
```
```

```
"""
```

```
function ^(f::ModularForm, k::Int)::ModularForm
```

```
 # we use binary decomposition of k for efficiency
```

```
 f_pow=one(f.n)
```

```
 while k != 0
```

```
 if k&1 != 0
```

```
 f_pow *= f
```

```
 end
```

```
 f = sq(f)
```

```
 k >>= 1
```

```
 end
```

```
 return f_pow
```

```
end
```

**equality.jl**

```
"""
```

Routines to check equality of modular forms modulo two up to known coefficients.

Also defines useful truncations methods.

```
"""
```

```
TRUNCATION of modular forms modulo 2
```

```
"""
```

```
 truncate(f, LENGTH)
```

Truncate  $f$  to the  $LENGTH$  first coefficients with no error.

```
Example
```

```
```julia-repl
```

```
[f is a modular form mod 2]
```

```
julia> f
```

```
1000-element SparseVector{Int8,Int64} with 16 stored entries:
```

```

[...]

julia> truncate(f, 100)
100-element SparseVector{Int8,Int64} with 5 stored entries:
 [...]
...
"""
function truncate(f::ModularForm, LENGTH::Int=10^3)::ModularForm
    if f.n>LENGTH
        return f[1:LENGTH]
    else
        return f
    end
end

"""
    truncate(f1, f2[, LENGTH])

Truncate f1 and f2 to LENGTH first coefficients with no error.
Truncate to min length of f1 & f2 if LENGTH = -1.

# Example
```julia-repl
[f1 & f2 are modular forms mod 2]
julia> disp(f1)
MF mod 2 (coef to 1000) - 01000000010000000000000001000000000000000000000000000001...
julia> disp(f2)
MF mod 2 (coef to 100) - 01000000010000000000000001000000000000000000000000000001...

julia> f1, f2 = truncate(f1,f2)

julia> disp(f1)
MF mod 2 (coef to 100) - 01000000010000000000000001000000000000000000000000000001...
julia> disp(f2)
MF mod 2 (coef to 100) - 01000000010000000000000001000000000000000000000000000001...
...
"""
function truncate(f1::ModularForm, f2::ModularForm,
 ↪ LENGTH::Int=-1)::Tuple{ModularForm,ModularForm}
 if LENGTH == -1
 if f1.n>f2.n
 return (f1[1:f2.n], f2)
 elseif f1.n==f2.n
 return (f1, f2)
 else
 return (f1, f2[1:f1.n])
 end
 end
end

```

```

 else
 n=min(LENGTH, f1.n, f2.n)
 return (f1[1:n], f2[1:n])
 end
end
end

```

### EQUALITY up to size

"""

```
 eq(f1, f2)
```

Up to maximum coefficient known for both f1 and f2, tell equality.

"""

```
function eq(f1::ModularForm, f2::ModularForm)::Bool
```

```
 f1, f2 = truncate(f1, f2)
```

```
 return f1==f2
```

```
end
```

## generators.jl

"""

This file contains routines to generate standard modular forms mod 2.

Essentially, this is 0 form, 1 form, and powers of D.

"""

"""

```
 zero([LENGTH])
```

Create a zero form of length LENGTH

# Example

```
```julia-repl
```

```
julia> zero()
```

```
1000-element SparseVector{Int8,Int64} with 0 stored entries
```

```
julia> zero(1)
```

```
1-element SparseVector{Int8,Int64} with 0 stored entries
```

```
```
```

"""

```
function zero(LENGTH::Int=10^3)::ModularForm
```

```
 return spzeros{Int8, LENGTH}
```

```
end
```

"""



```
one ([LENGTH])
```

```
Create a one form of length LENGTH
```

## # Example

```
```julia-repl
```

```
julia> one()
```

1000-element SparseVector{Int8,Int64} with 1 stored entry:

$$[1] = 1$$

```
julia> one(1)
```

1-element SparseVector{Int8,Int64} with 1 stored entry:

$$[1] = 1$$

— — —

● ● ●

```
function one(LENGTH::Int=10^3)::ModularForm
```

```
f = spzeros(Int8, LENGTH)
```

$$f[1] = 1$$

```
return f
```

end

● ● ●

delta([LENGTH])

Create the standard D form, with coefficients up to LENGTH

=> as a D-series!

Example

```
```julia-repl
```

```
julia> disp(delta())
```

MF mod 2 (coef to 1000) - 0100000001000000000000001000000000000000000001...

```
julia> disp(delta(10^6))
```

[illegible]

— — —

● ● ●

```
function delta(LENGTH::Int=10^3)::ModularForm
```

```
f = spzeros(Int8, LENGTH)
```

```
indice::Int = 2
```

```
i :: Int = 1
```

```
while indice <= f.n
```

```
f[indice] = Int8(1)
```

```
i += 2
```

$$\text{indice} = i^{2+1}$$

end

```
return f
```

end

```

"""
 delta_k(k[, LENGTH])

Create the standard D^k form, with coefficients up to LENGTH
=> as a q-series!

Example
```julia-repl
julia> disp(delta_k(0))
MF mod 2 (coef to 1000) - 10000000000000000000000000000000000000000...

julia> disp(delta_k(1))
MF mod 2 (coef to 1000) - 010000000100000000000000100000000000000001...

julia> disp(delta_k(2))
MF mod 2 (coef to 1000) - 00100000000000000010000000000000000000000...

julia> disp(delta_k(3))
MF mod 2 (coef to 1000) - 0001000000010000000100000000000000000001000000...

julia> disp(delta_k(5))
MF mod 2 (coef to 1000) - 0000010000000100000000000000010000000100000010000...
```

function delta_k(k::Int, LENGTH::Int=10^3)::ModularForm
 if k==0
 return one(LENGTH)
 elseif k==1
 return delta(LENGTH)
 elseif k==2
 return sq(delta(LENGTH))
 else
 return delta(LENGTH)^k
 end
end

Delta(k[, LENGTH])

Create the standard D form, with coefficients up to LENGTH
=> as a D-series!

Example
```julia-repl

```


recognizer.jl

Note that for speed purposes, no checks of any kind are made.
The user should make sure that the form may be written in terms of form from the
→ precalculated variable.

```
to_D(f, precalculated)
```

Compute the D-series representation of f (using precalculated).

[illegible]

```
julia> df = to_delta(f, precalculated)
100-element SparseArrays.SparseVector{Int8, Int64} with 2 stored entries:
 [2 ] = 1
 [4 ] = 1

julia> disp(df)
MF mod 2 (coef to 100) - 0101000000000000000000000000000000000000...
^^^
```

■■■■■

```

function to_D(mf::ModularForm, precalculated::ModularFormOrNothingList,
↳ LENGTH::Int=length(precalculated))::ModularForm
    f = deepcopy(mf)
    k = 1
    df = zero(LENGTH)
    for k in 1:2:LENGTH
        if f[k+1]==1
            #there is D^k if f
            df[k+1] = 1
            f += precalculated[k+1]
        else
            #there is no D^k if f
            #pass
        end
        k += 1
    end
    return df
end
to_delta = to_D

```

■■■■■

```
drop_error(f, precalculated, LENGTH)
```

Drops the numerical error that f might have (as long as this error isn't too large).

Example

```
```julia-repl
```

```
julia> precalculated = loadFormListBinary(10^2, 10^6)
```

```
julia> f = delta(10^6) + delta_k(3, 10^6)
```

```
julia> disp(f)
```

MF mod 2 (coef to 1000000) - 01010000010100000001000001000000000000000001000001...

```
julia> T11f = MFmod2.Hecke(11, f)
```

```
julia> disp(T11f)
```

MF mod 2 (coef to 90909) - 01000000010000000000000010000000000000000000000000001...

```
julia> T11f_exact = drop_error(T11f, precalculated)
```

```
julia> disp(T11f_exact)
```

MF mod 2 (coef to 1000000) - 0100000001000000000000000100000000000000000000001...

— — —

● ● ● ● ● ●

```
function drop_error(f::ModularForm, precalculated::ModularFormOrNothingList,
```

```
↪ LENGTH::Int=length(precalculated[2]))::ModularForm
```

```
df = to_D(f, precalculated)
```

```
f = to_q(df, precalculated)
```

```
return f
```

end

### D.2.2 Data Submodule

storage.jl

```
using JLD2, FileIO
```

● ● ●

Load the modular forms lists from binary .jdl2 files, using this module's standard

→ naming system.

■■ ■■ ■■

● ● ●

```
loadFormListBinary(MAXI, LENGTH)
```

Loads the list of q-coefficients of D powers form file.

|| || ||

```
function loadFormListBinary(MAXI::Int, LENGTH::Int)
```

```
standard naming
```

```
file_name = "delta_q-"+"maxi"*string(MAXI)*"-"+"length"*string(LENGTH)*".jdl2"
```

```
load
```

```
return load(joinpath(@__DIR__, file_name), "list")
```

```

end

"""
 loadHeckePrimesListBinary(MAX_PRIME, MAX_DELTA)

Loads the list of D-coefficients of prime Hecke operators applied to powers of D.
"""

function loadHeckePrimesListBinary(MAX_PRIME::Int, MAX_DELTA::Int)
 # standard naming
 file_name =
 ↪ "Hecke_primes-"*"max_prime"*string(MAX_PRIME)*"-*"max_delta"*string(MAX_DELTA)*".jdl2"
 # load
 return load(joinpath(@__DIR__, file_name), "list")
end

"""
 loadHeckePowersListBinary(MAX_POWER, MAX_DELTA)

Loads the list of D-coefficients of powers of Hecke operators applied to powers of D.
"""

function loadHeckePowersListBinary(MAX_POWER::Int, MAX_DELTA::Int)
 # standard naming
 file_name =
 ↪ "Hecke_powers-"*"max_power"*string(MAX_POWER)*"-*"max_delta"*string(MAX_DELTA)*".jdl2"
 # load
 return load(joinpath(@__DIR__, file_name), "list")
end

```

### delta\_file\_maker.jl

```

include("../ModularFormsModuloTwo.jl")
using .ModularFormsModuloTwo
MFmod2 = ModularFormsModuloTwo
using JLD2, FileIO

parameters
LENGTH = 10^6
MAXI = 10^3

list
list = MFmod2.ModularFormOrNothingList(nothing, MAXI)

D^2
d2 = MFmod2.delta_k(2, LENGTH)
1st iteration
println("Calculating: ", "D^1")
d = MFmod2.delta(LENGTH)

```

```

list[2] = d
k = 1
main loop
while k < MAXI-2
 global k, d, d2
 k += 2
 println("Calculating: ", "D^"*string(k))
 d *= d2
 list[k+1] = d
end

final saving (standard naming)
@save joinpath(@__DIR__,
 ↪ "delta_q-"*"maxi"*string(MAXI)*"-"*"length"*string(LENGTH)*".jld12") list

Hecke_primes_file_maker.jl

include("../ModularFormsModuloTwo.jl")
using ModularFormsModuloTwo
MFmod2 = ModularFormsModuloTwo
using JLD2, FileIO
using Primes

parameters
MAXI = 10^2
LENGTH = 10^6

binary read
precalculated = MFmod2.loadFormListBinary(MAXI, LENGTH)

parameters
MAX_DELTA = length(precalculated)
MAX_PRIME = 10^4

list
list = Array{Union{MFmod2.ModularFormOrNothingList, Nothing}, 1}(nothing, MAX_PRIME)

for p in Primes.primes(3, MAX_PRIME) #avoid prime 2
 print("Calculating: T", p, "| ")
 l = MFmod2.ModularFormOrNothingList(nothing, MAX_DELTA)
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")
 f = MFmod2.delta_k(k, LENGTH)
 end
end

```



```

 Tf = MFmod2.Hecke(p, f)
 dTf = MFmod2.to_delta(Tf, precalculated)
 l[k+1] = dTf
 end
 println()
 list[p] = 1
end

final saving (standard naming)
@save joinpath(@__DIR__,
 ↪ "Hecke_primes-"*"max_prime"*string(MAX_PRIME)*"-"*"max_delta"*string(MAX_DELTA)*".jdl2")
 ↪ list

Hecke_powers_file_maker.jl

include("../ModularFormsModuloTwo.jl")
using .ModularFormsModuloTwo
MFmod2 = ModularFormsModuloTwo
using JLD2, FileIO
using Primes

parameters
MAXI = 10^3
LENGTH = 10^6

binary read
precalculated = MFmod2.loadFormListBinary(MAXI, LENGTH)

parameters
MAX_DELTA = length(precalculated)
MAX_POWER = 20

list
list = Array{Union{MFmod2.ModularFormOrNothingList, Nothing}, 2}(nothing, MAX_POWER,
 ↪ MAX_POWER)

for i in 1:MAX_POWER
 for j in 1:MAX_POWER
 print("Calculating: T3^", i-1, "T5^", j-1, "| ")
 l = MFmod2.ModularFormOrNothingList(nothing, MAX_DELTA)
 # use previous calculations
 if i == 1
 if j == 1
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")

```

```

 df = MFmod2.Delta_k(k, MAX_DELTA)
 l[k+1] = df
 end
else
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")
 df = list[i,j-1][k+1]
 f = MFmod2.to_q(df, precalculated)
 T5f = MFmod2.Hecke(5, f)
 dT5f = MFmod2.to_delta(T5f, precalculated)
 l[k+1] = dT5f
 end
end
else
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")
 df = list[i-1,j][k+1]
 f = MFmod2.to_q(df, precalculated)
 T3f = MFmod2.Hecke(3, f)
 dT3f = MFmod2.to_delta(T3f, precalculated)
 l[k+1] = dT3f
 end
end
println()
save
list[i,j] = l
end
end

final saving (standard naming)
@save joinpath(@__DIR__,
 ↪ "Hecke_powers-"*max_power*string(MAX_POWER)*"-"*max_delta*string(MAX_DELTA)*".jld12")
 ↪ list

```

## E $T_p$ as series of $T_3$ and $T_5$

### E.1 $a_{ij}(p)$ Computations

```

include("../ModularFormsModuloTwo.jl")
using .ModularFormsModuloTwo
MFmod2 = ModularFormsModuloTwo
using JLD2, FileIO
using Primes

parameters

```

```

MAXI = 10^3
LENGTH = 10^6

binary read
precalculated = MFmod2.loadFormListBinary(MAXI, LENGTH)

parameters
MAX_DELTA = length(precalculated)
MAX_POWER = 20

list
list = Array{Union{MFmod2.ModularFormOrNothingList, Nothing}, 2}(nothing, MAX_POWER,
↪ MAX_POWER)

for i in 1:MAX_POWER
 for j in 1:MAX_POWER
 print("Calculating: T3^", i-1, "T5^", j-1, "| ")
 l = MFmod2.ModularFormOrNothingList(nothing, MAX_DELTA)
 # use previous calculations
 if i == 1
 if j == 1
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")
 df = MFmod2.Delta_k(k, MAX_DELTA)
 l[k+1] = df
 end
 else
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")
 df = list[i,j-1][k+1]
 f = MFmod2.to_q(df, precalculated)
 T5f = MFmod2.Hecke(5, f)
 dT5f = MFmod2.to_delta(T5f, precalculated)
 l[k+1] = dT5f
 end
 end
 else
 for k in 1:2:MAX_DELTA-2
 print("D^", k, " ")
 df = list[i-1,j][k+1]
 f = MFmod2.to_q(df, precalculated)
 T3f = MFmod2.Hecke(3, f)
 dT3f = MFmod2.to_delta(T3f, precalculated)
 l[k+1] = dT3f
 end
 end
 end
end

```

```

 end
 println()
 # save
 list[i,j] = 1
 end
end
end

```

```

final saving (standard naming)
@save joinpath(@__DIR__,
 ↪ "Hecke_powers-"*"max_power"*string(MAX_POWER)*"-"*"max_delta"*string(MAX_DELTA)*".jdl2")
 ↪ list

```

Find this program file with context on GitHub. Note that it in fact uses the library described above, and some other peices of code that are not important mathematically (all can be found on GitHub in this folder: [https://github.com/pauldubois98/HeckeOperatorsModuloTwo/tree/master/a\\_ijComptation](https://github.com/pauldubois98/HeckeOperatorsModuloTwo/tree/master/a_ijComptation)).

## E.2 $a_{ij}(p)$ Graphs

# F Governing Fields

## F.1 Frobenian Elements Computations

We provide here an example for the case of extensions of the governing field of  $a_{02}$ . It also has been a little bit simplified. Others are very similar, and may be found here: <https://github.com/pauldubois98/HeckeOperatorsModuloTwo/tree/master/GoverningFields/guessGoverningFields>

```

P = z^4+1;
Q8 = nfinit(P);
M0 = rnfinit(Q8, y^2-(z+z^7), 1);
M1 = nfinit(M0);
M = bnfinit(M1[1]);

/* generator for the torsion units */
mu = M.tu[2];
/* fundamental units */
v1 = M.fu[1];
v2 = M.fu[2];
v3 = M.fu[3];

forvec(X=[[0,1],[0,1],[0,1],[0,1]], {
 if(X==[0,0,0,0], {
 next()
 });

 /* element to extend */
 alpha = mu^X[1]*v1^X[2]*v2^X[3]*v3^X[4];

```

```

/* extension */
L0 = rnfinit(M1, x^2-alpha, 0);
L = nfinit(L0);
G = galoisinit(L);

/* *** avoid non galois ext *** */
if(G==0, {
 print(" not Galois");
 next()
});

/* open file */
filename = Str("a_02_ext_" X[1] X[2] X[3] X[4] ".txt");
file = fileopen(filename, "w");
print(filename);
filewrite(file, "Extension");
filewrite(file, "#3");
filewrite(file, "\\mu");
filewrite(file, "\\sqrt[4]{2}");
filewrite(file, "sqrt(alpha)");

/* first group identification */
filewrite(file, "Galois identify");
filewrite(file, galoisidentify(G));
filewrite(file, "\\n");

/* second conjugacy classes */
filewrite(file, "Galois conjug classes");
filewrite(file, galoisconjclasses(G));
filewrite(file, "\\n");

/* forbenius elemnts of primes >2 */
forprime(p=3, b=10150, pr = idealprimedec(L,p)[1]; idealfrob =
 ↪ idealfrobenius(L,G,pr); filewrite(file, p); filewrite(file, idealfrob));

/*pr = idealprimedec(L,3)[1];
idealfrob = idealfrobenius(L,G,pr);
print("idealfrob: ", idealfrob);*/

fileclose(file)

})

```

## F.2 Analysis of Extensions

Here, we provide the piece of code that is interesting, which checks if the field is a governing field (it follows 6.4.1). But we in fact have to loop through the possible fields and the possibilities for  $a_{ij}$ . General codes may be found here: <https://github.com/pauldubois98/HeckeOperatorsModuloTwo/tree/master/GoverningFields/guessGoverningFields/Common>

```
DATA
primes1, primes0, primes_undefined = a_ij(i,j)

#init frob lists
frobenius0 = list()
frobenius1 = list()

#fill 1-primes frob elements
for p in primes1:
 #load element
 F_p = frobenius_of_prime(p, file)
 # we add it to the list of 0-primes frob
 found=False
 for F in frobenius1:
 if same_conjugacy_classes(F_p, F, conjugacy_classes):
 found=True
 break
 if found:
 pass
 else:
 frobenius1.append(F_p)

#fill 0-primes frob elements
for p in primes0:
 #load element
 F_p = frobenius_of_prime(p, file)
 # check it is not conjugate to a frob of a 1-prime
 found=False
 for F in frobenius1:
 if same_conjugacy_classes(F_p, F, conjugacy_classes):
 found=True
 break
 if found:
 # if it is, there is a problem for this prime
 print(p, ": wrong conjugacy class")
 else:
 # if not, we add it to the list of 0-primes frob
 found=False
 for F in frobenius0:
 if same_conjugacy_classes(F_p, F, conjugacy_classes):
 found=True
```

```
 break
 if found:
 pass
 else:
 frobenius0.append(F_p)

print("Frobenius classes:")
print(frobenius0)
print(frobenius1)
```

## G Governing Fields Results

The same results are available online at <https://pauldubois98.github.io/HeckeOperatorsModuloTwo/GoverningFields/>. There are full names for extensions online, for size reason, we will use some variables  $\alpha$  and  $\beta$  in here.

We will write  $\stackrel{?}{=}$  for the fields with strong evidence of being governing fields, and  $=$  for the ones we are sure.

### G.1 $a_{0k}$

Here we look at the maps  $a_{0k}$ .

#### G.1.1 $a_{01}$

Governing field:

$$M_{01} = \mathbb{Q}(\zeta_8)$$

- $G_{01} = C_2 \times C_2$
- $|S_{01}^1| = 1$
- $|C_{01}^1| = 1$
- $P_{01} \approx 2.297 \cdot 10^{-303}$
- $|\{p \in \mathbb{P} \mid a_{01}(p) = 0, p < 10^4\}| = 914$   
Ratio:  $914/1228 \approx 3/4$
- $|\{p \in \mathbb{P} \mid a_{01}(p) = 1, p < 10^4\}| = 314$   
Ratio:  $314/1228 \approx 1/4$

#### G.1.2 $a_{02}$

Governing field:

$$M_{02} = \mathbb{Q}(\zeta_8, \sqrt[4]{2})$$

- $G_{02} = D_8$
- $|S_{02}^1| = 1$
- $|C_{02}^1| = 1$
- $P_{02} \approx 4.285 \cdot 10^{-200}$
- $|\{p \in \mathbb{P} \mid a_{02}(p) = 0, p < 10^4\}| = 1076$   
Ratio:  $1076/1228 \approx 7/8$
- $|\{p \in \mathbb{P} \mid a_{02}(p) = 1, p < 10^4\}| = 152$   
Ratio:  $152/1228 \approx 1/8$

#### G.1.3 $a_{03}$

Governing field:

$$M_{03} \stackrel{?}{=} \mathbb{Q}(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha})$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

- $G_{03} = D_{16}$
- $|S_{03}^1| = 2$
- $|C_{03}^1| = 1$
- $P_{03} \approx 5.099 \cdot 10^{-204}$
- $|\{p \in \mathbb{P} \mid a_{03}(p) = 0, p < 10^4\}| = 1071$   
Ratio:  $1071/1228 \approx 14/16$
- $|\{p \in \mathbb{P} \mid a_{03}(p) = 1, p < 10^4\}| = 157$   
Ratio:  $157/1228 \approx 2/16$



#### G.1.4 $a_{04}$

Same governing field as  $a_{03}$ , thus also same Galois group ( $M_{04} \stackrel{?}{=} M_{03}$  and  $G_{04} \stackrel{?}{=} G_{03}$ ).

- $G_{04} = D_{16}$
- $|S_{04}^1| = 1$
- $|C_{04}^1| = 1$
- $P_{04} \approx 1.064 \cdot 10^{-120}$
- $|\{p \in \mathbb{P} \mid a_{04}(p) = 0, p < 10^4\}| = 1155$   
Ratio:  $1155/1228 \approx 15/16$
- $|\{p \in \mathbb{P} \mid a_{04}(p) = 1, p < 10^4\}| = 73$   
Ratio:  $73/1228 \approx 1/16$

#### G.1.5 $a_{05}$

Governing field:

$$\mathbb{Q}(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}, \sqrt{\beta})$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

and

$$\begin{aligned} \beta = & -\frac{8282936156772053\alpha^{\frac{13}{2}}}{1125899906842624} - \frac{1240182980093567\alpha^6}{562949953421312} - \frac{336382584949535\alpha^{\frac{9}{2}}}{2199023255552} \\ & - \frac{6445823996745319\alpha^4}{140737488355328} - \frac{4638634719581101\alpha^{\frac{5}{2}}}{35184372088832} - \frac{2954723016803317\alpha^2}{70368744177664} \\ & - \frac{5142889464378747\sqrt[4]{2}}{140737488355328} - \frac{4198844765367981\sqrt{\alpha}}{1125899906842624} + \frac{3450571136356681\sqrt{2}}{281474976710656} \\ & + \frac{6022015868546055 \cdot \sqrt[4]{2}^3}{140737488355328} + \frac{5763554133419461}{70368744177664} + \frac{7633450872164841\alpha^{\frac{3}{2}}}{281474976710656} \\ & + \frac{615248862392953\alpha^3}{8796093022208} + \frac{8240373942248553\alpha^{\frac{7}{2}}}{35184372088832} + \frac{8030384673908857\alpha^5}{562949953421312} \\ & + \frac{4981425151744809\alpha^7}{18014398509481984} + \frac{1676680829315919\alpha^{\frac{11}{2}}}{35184372088832} + \frac{8299866982438859\alpha^{\frac{15}{2}}}{9007199254740992} \end{aligned}$$

- $G_{265} = D_{32}$
- $|S_{265}^1| = 4$
- $|C_{265}^1| = 2$
- $P_{265} \approx 7.652 \cdot 10^{-205}$
- $|\{p \in \mathbb{P} \mid a_{265}(p) = 0, p < 10^4\}| = 1069$   
Ratio:  $1069/1228 \approx 28/32$
- $|\{p \in \mathbb{P} \mid a_{265}(p) = 1, p < 10^4\}| = 159$   
Ratio:  $159/1228 \approx 4/32$

#### G.1.6 $a_{06}$

Same governing field as  $a_{05}$ , thus also same Galois group ( $M_{06} \stackrel{?}{=} M_{05}$  and  $G_{06} \stackrel{?}{=} G_{05}$ ).

- $G_{06} = D_{32}$
- $|S_{06}^1| = 2$
- $|C_{06}^1| = 1$
- $P_{06} \approx 1.668 \cdot 10^{-129}$
- $|\{p \in \mathbb{P} \mid a_{06}(p) = 0, p < 10^4\}| = 1147$   
Ratio:  $1147/1228 \approx 30/32$
- $|\{p \in \mathbb{P} \mid a_{06}(p) = 1, p < 10^4\}| = 81$   
Ratio:  $81/1228 \approx 2/32$

#### G.1.7 $a_{07}$

Same governing field as  $a_{05}$ , thus also same Galois group ( $M_{07} \stackrel{?}{=} M_{05}$  and  $G_{07} \stackrel{?}{=} G_{05}$ ).

- $G_{07} = D_{32}$
- $|S_{07}^1| = 2$
- $|C_{07}^1| = 1$
- $P_{07} \approx 5.612 \cdot 10^{-126}$
- $|\{p \in \mathbb{P} \mid a_{07}(p) = 0, p < 10^4\}| = 1150$   
Ratio:  $1150/1228 \approx 30/32$
- $|\{p \in \mathbb{P} \mid a_{07}(p) = 1, p < 10^4\}| = 78$   
Ratio:  $78/1228 \approx 2/32$

### G.1.8 $a_{08}$

For  $a_{08}$ , it is a little bit special: We found 3 governing fields. This is possible, since governing fields aren't unique. Note, for this case, we did computation only up to  $10^3$ , which doesn't give a result as strong as for other  $a_{ij}$  discussed.

**First possibility** Same governing field as  $a_{05}$ , thus also same Galois group ( $M_{08} \stackrel{?}{=} M_{05}$  and  $G_{08} \stackrel{?}{=} G_{05}$ ).

- $G_{08} = D_{32}$
- $|S_{08}^1| = 1$
- $|C_{08}^1| = 1$
- $P_{08} \approx 5.800 \cdot 10^{-10}$
- $|\{p \in \mathbb{P} \mid a_{08}(p) = 0, p < 10^4\}| = 162$   
Ratio:  $162/167 \approx 31/32$
- $|\{p \in \mathbb{P} \mid a_{08}(p) = 1, p < 10^4\}| = 5$   
Ratio:  $5/167 \approx 1/32$

**Second possibility** Governing field:

$$\mathbb{Q}(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}, \sqrt{\beta})$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

and

$$\begin{aligned} \beta = & -\frac{105347359704573\alpha^7}{9007199254740992} - \frac{6998074608946635\alpha^5}{9007199254740992} - \frac{5920145374826629\alpha^3}{1125899906842624} \\ & - \frac{5641391423669567\sqrt[4]{2}}{281474976710656} + \frac{1892518063068637\sqrt{2}}{281474976710656} + \frac{6605731837972057 \cdot \sqrt[4]{2}^3}{281474976710656} \\ & + \frac{6318487279300541}{140737488355328} + \frac{4398252267665923\alpha^2}{2251799813685248} + \frac{6781736280981887\alpha^4}{2251799813685248} \\ & + \frac{2182195308166155\alpha^6}{18014398509481984} \end{aligned}$$

- $G_{268} = (C_8 \times C_2) : C_2$
- $|S_{268}^1| = 2$
- $|C_{268}^1| = 2$
- $P_{268} \approx 1.050 \cdot 10^{-09}$
- $|\{p \in \mathbb{P} \mid a_{268}(p) = 0, p < 10^4\}| = 162$   
Ratio:  $162/167 \approx 31/32$
- $|\{p \in \mathbb{P} \mid a_{268}(p) = 1, p < 10^4\}| = 5$   
Ratio:  $5/167 \approx 1/32$

**Third possibility** Governing field:

$$\mathbb{Q}(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}, \sqrt{\beta})$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

and

$$\begin{aligned} \beta = & -\frac{8834128592369193\alpha^{\frac{15}{2}}}{18014398509481984} - \frac{5764005538121637\alpha^7}{18014398509481984} - \frac{6525657542771441\alpha^{\frac{11}{2}}}{281474976710656} \\ & - \frac{8271178638947881\alpha^5}{562949953421312} - \frac{6081987607676345\alpha^{\frac{7}{2}}}{70368744177664} - \frac{6598671728459543\alpha^3}{140737488355328} \\ & - \frac{1208437681968305\alpha^{\frac{3}{2}}}{140737488355328} - \frac{183920253122025}{4398046511104} - \frac{6079543723614321 \cdot \sqrt[4]{2}^3}{281474976710656} \\ & - \frac{6967068354798893\sqrt{2}}{1125899906842624} - \frac{1068523219860669\sqrt{\alpha}}{2251799813685248} + \frac{1298004773107437\sqrt[4]{2}}{70368744177664} \\ & + \frac{1958567319150421\alpha^2}{281474976710656} + \frac{463099233299539\alpha^{\frac{5}{2}}}{17592186044416} + \frac{1458843417828589\alpha^4}{35184372088832} \\ & + \frac{4742512389557653\alpha^{\frac{9}{2}}}{70368744177664} + \frac{5197763479709557\alpha^6}{2251799813685248} + \frac{507219318934741\alpha^{\frac{13}{2}}}{140737488355328} \end{aligned}$$

- $G_{268} = QD_{32}$
- $|S_{268}^1| = 2$
- $|C_{268}^1| = 2$
- $P_{268} \approx 5.800 \cdot 10^{-10}$
- $|\{p \in \mathbb{P} \mid a_{268}(p) = 0, p < 10^4\}| = 162$   
Ratio:  $162/167 \approx 31/32$
- $|\{p \in \mathbb{P} \mid a_{268}(p) = 1, p < 10^4\}| = 5$   
Ratio:  $5/167 \approx 1/32$

## G.2 $a_{kk}$

Here we look at the maps  $a_{kk}$ .

### G.2.1 $a_{11}$

Governing field:

$$M_{11} = \mathbb{Q}(\zeta_{16})$$

- $G_{11} = C_4 \times C_2$
- $|S_{11}^1| = 2$
- $|C_{11}^1| = 2$
- $P_{11} \approx 1.172 \cdot 10^{-299}$
- $|\{p \in \mathbb{P} \mid a_{11}(p) = 0, p < 10^4\}| = 920$   
Ratio:  $920/1228 \approx 6/8$
- $|\{p \in \mathbb{P} \mid a_{11}(p) = 1, p < 10^4\}| = 308$   
Ratio:  $308/1228 \approx 2/8$

## G.3 $a_{k0}$

Here we look at the maps  $a_{k0}$ .

### G.3.1 $a_{10}$

Governing field:

$$M_{10} = \mathbb{Q}(\zeta_8)$$

- $G_{10} = C_2 \times C_2$

- $|S_{10}^1| = 1$

- $|C_{10}^1| = 1$

- $P_{10} \approx 6.202 \cdot 10^{-302}$

- $|\{p \in \mathbb{P} \mid a_{10}(p) = 0, p < 10^4\}| = 917$   
Ratio:  $917/1228 \approx 3/4$

- $|\{p \in \mathbb{P} \mid a_{10}(p) = 1, p < 10^4\}| = 311$   
Ratio:  $311/1228 \approx 1/4$

### G.3.2 $a_{20}$

Governing field:

$$M_{20} = \mathbb{Q}(\zeta_8, \sqrt{1+i})$$

- $G_{20} = D_8$

- $|S_{20}^1| = 1$

- $|C_{20}^1| = 1$

- $P_{20} \approx 1.469 \cdot 10^{-197}$

- $|\{p \in \mathbb{P} \mid a_{20}(p) = 0, p < 10^4\}| = 1079$   
Ratio:  $1079/1228 \approx 7/8$

- $|\{p \in \mathbb{P} \mid a_{20}(p) = 1, p < 10^4\}| = 149$   
Ratio:  $149/1228 \approx 1/8$

### G.3.3 $a_{30}$

Governing field:

$$M_{30} \stackrel{?}{=} \mathbb{Q}(\zeta_8, \sqrt{1+i}, \sqrt{\alpha})$$

where:

$$\alpha = -4 - \frac{65i}{16} - \frac{31\sqrt{1+i}}{16} - \frac{5(1+i)^{\frac{5}{2}}}{16} - \frac{3(1+i)^3}{16} + \frac{(1+i)^{\frac{7}{2}}}{16} + \frac{11(1+i)^2}{16} + \frac{27(1+i)^{\frac{3}{2}}}{16}$$

- $G_{30} = D_{16}$

- $|S_{30}^1| = 2$

- $|C_{30}^1| = 1$

- $P_{30} \approx 7.285 \cdot 10^{-205}$

- $|\{p \in \mathbb{P} \mid a_{30}(p) = 0, p < 10^4\}| = 1070$   
Ratio:  $1070/1228 \approx 14/16$

- $|\{p \in \mathbb{P} \mid a_{30}(p) = 1, p < 10^4\}| = 158$   
Ratio:  $158/1228 \approx 2/16$

### G.3.4 $a_{40}$

Same governing field as  $a_{30}$ , thus also same Galois group ( $M_{40} \stackrel{?}{=} M_{30}$  and  $G_{40} \stackrel{?}{=} G_{30}$ ).

- $G_{40} = D_{16}$

- $|S_{40}^1| = 1$

- $|C_{40}^1| = 1$

- $P_{40} \approx 1.401 \cdot 10^{-126}$

- $|\{p \in \mathbb{P} \mid a_{40}(p) = 0, p < 10^4\}| = 1150$   
Ratio:  $1150/1228 \approx 15/16$

- $|\{p \in \mathbb{P} \mid a_{40}(p) = 1, p < 10^4\}| = 78$   
Ratio:  $78/1228 \approx 1/16$

### G.3.5 $a_{50}$

Governing field:

$$M_{50} \stackrel{?}{=} \mathbb{Q}(\zeta_8, \sqrt{1+i}, \sqrt{\alpha}, \sqrt{\beta})$$

where:

$$\alpha = -4 - \frac{65i}{16} - \frac{31\sqrt{1+i}}{16} - \frac{5(1+i)^{\frac{5}{2}}}{16} - \frac{3(1+i)^3}{16} + \frac{(1+i)^{\frac{7}{2}}}{16} + \frac{11(1+i)^2}{16} + \frac{27(1+i)^{\frac{3}{2}}}{16}$$

and

$$\beta = \frac{\alpha^{\frac{11}{2}}}{4} - \frac{7\alpha^{\frac{7}{2}}}{4} - \sqrt{\alpha} + \frac{7\alpha^{\frac{3}{2}}}{4} - \frac{\alpha^{\frac{15}{2}}}{4}$$

- $G_{50} = D_{32}$
- $|S_{50}^1| = 4$
- $|C_{50}^1| = 2$
- $P_{50} \approx 1.242 \cdot 10^{-200}$
- $|\{p \in \mathbb{P} \mid a_{50}(p) = 0, p < 10^4\}| = 1074$   
Ratio:  $1074/1228 \approx 28/32$
- $|\{p \in \mathbb{P} \mid a_{50}(p) = 1, p < 10^4\}| = 154$   
Ratio:  $154/1228 \approx 4/32$

### G.3.6 $a_{60}$

Same governing field as  $a_{50}$ , thus also same Galois group ( $M_{60} \stackrel{?}{=} M_{50}$  and  $G_{60} \stackrel{?}{=} G_{50}$ ).

- $G_{60} = D_{32}$
- $|S_{60}^1| = 2$
- $|C_{60}^1| = 1$
- $P_{60} \approx 1.892 \cdot 10^{-122}$
- $|\{p \in \mathbb{P} \mid a_{60}(p) = 0, p < 10^4\}| = 1153$   
Ratio:  $1153/1228 \approx 30/32$
- $|\{p \in \mathbb{P} \mid a_{60}(p) = 1, p < 10^4\}| = 75$   
Ratio:  $75/1228 \approx 2/32$

### G.3.7 $a_{70}$

Same governing field as  $a_{50}$ , thus also same Galois group ( $M_{70} \stackrel{?}{=} M_{50}$  and  $G_{70} \stackrel{?}{=} G_{50}$ ).

- $G_{70} = D_{32}$
- $|S_{70}^1| = 2$
- $|C_{70}^1| = 1$
- $P_{70} \approx 1.892 \cdot 10^{-122}$
- $|\{p \in \mathbb{P} \mid a_{70}(p) = 0, p < 10^4\}| = 1153$   
Ratio:  $1153/1228 \approx 30/32$
- $|\{p \in \mathbb{P} \mid a_{70}(p) = 1, p < 10^4\}| = 75$   
Ratio:  $75/1228 \approx 2/32$

### G.3.8 $a_{80}$

Same governing field as  $a_{50}$ , thus also same Galois group ( $M_{80} \stackrel{?}{=} M_{50}$  and  $G_{80} \stackrel{?}{=} G_{50}$ ).

- $G_{80} = D_{32}$
- $|S_{80}^1| = 1$
- $|C_{80}^1| = 1$
- $P_{80} \approx 1.420 \cdot 10^{-08}$
- $|\{p \in \mathbb{P} \mid a_{80}(p) = 0, p < 10^4\}| = 163$   
Ratio:  $163/167 \approx 31/32$
- $|\{p \in \mathbb{P} \mid a_{80}(p) = 1, p < 10^4\}| = 4$   
Ratio:  $4/167 \approx 1/32$

## References

- Jean-Louis Nicolas and Jean-Pierre Serre. Formes modulaires modulo 2: l'ordre de nilpotence des opérateurs de Hecke. *C. R. Math. Acad. Sci. Paris*, 350(7-8):343–348, 2012a. ISSN 1631-073X. doi: 10.1016/j.crma.2012.03.013. URL <https://doi.org/10.1016/j.crma.2012.03.013>.
- Joël Bellaïche. Une représentation galoisienne universelle attachée aux formes modulaires modulo 2. *C. R. Math. Acad. Sci. Paris*, 350(9-10):443–448, 2012. ISSN 1631-073X. doi: 10.1016/j.crma.2012.04.012. URL <https://doi.org/10.1016/j.crma.2012.04.012>.
- Keith Conrad.  $SL_2(\mathbb{Z})$ . [Online], 2020. URL [https://kconrad.math.uconn.edu/blurbs/grouptheory/SL\(2,Z\).pdf](https://kconrad.math.uconn.edu/blurbs/grouptheory/SL(2,Z).pdf). Available from [https://kconrad.math.uconn.edu/blurbs/grouptheory/SL\(2,Z\).pdf](https://kconrad.math.uconn.edu/blurbs/grouptheory/SL(2,Z).pdf).
- William Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. ISBN 978-0-8218-3960-7; 0-8218-3960-8. doi: 10.1090/gsm/079. URL <https://doi.org/10.1090/gsm/079>. With an appendix by Paul E. Gunnells.
- Bertil Westergren Lennart Rade. *Mathematics Handbook for Science and Engineering*. Springer Science and Business Media, 2013, 2013. 5th edition, illustrated.
- J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.
- Sagar Shrivastava. Introduction to modular forms, 2017.
- Victor Saul Miller. DIOPHANTINE AND P-ADIC ANALYSIS OF ELLIPTIC CURVES AND MODULAR FORMS, 1975. URL [http://gateway.proquest.com/openurl?url\\_ver=Z39.88-2004&rft\\_val\\_fmt=info:ofi/fmt:kev:mtx:dissertation&res\\_dat=xri:pqdiss&rft\\_dat=xri:pqdiss:0295447](http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:0295447). Thesis (Ph.D.)—Harvard University.
- SageMath Contributors. Sagemath. Software tool, 2020. URL <https://www.sagemath.org/>.
- O. Kolberg. Congruences for Ramanujan's function  $\tau(n)$ . *Arbok Univ. Bergen Mat.-Natur. Ser.*, 1962 (11), 1962. ISSN 0522-9189.
- Jean-Louis Nicolas and Jean-Pierre Serre. Formes modulaires modulo 2: structure de l'algèbre de Hecke. *C. R. Math. Acad. Sci. Paris*, 350(9-10):449–454, 2012b. ISSN 1631-073X. doi: 10.1016/j.crma.2012.03.019. URL <https://doi.org/10.1016/j.crma.2012.03.019>.
- Ken Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004. ISBN 0-8218-3368-5.
- Kazuyuki Hatada. Eigenvalues of Hecke operators on  $SL(2, \mathbb{Z})$ . *Math. Ann.*, 239(1):75–96, 1979. ISSN 0025-5831. doi: 10.1007/BF01420494. URL <https://doi.org/10.1007/BF01420494>.
- Mathilde Gerbelli-Gauthier. Modular forms and galois representations mod  $p$ , and the nilpotent action of hecke operators mod 2, 2014. URL <http://www.math.mcgill.ca/darmon/theses/gerbelli-gauthier/mathilde-gg.pdf>.

- Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997. ISBN 0-387-94762-0. doi: 10.1007/978-1-4612-1934-7. URL <https://doi.org/10.1007/978-1-4612-1934-7>.
- Gerald J. Janusz. *Algebraic number fields*, volume 7 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 1996. ISBN 0-8218-0429-4.
- N. Tschebotareff. Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören. *Math. Ann.*, 95(1):191–228, 1926. ISSN 0025-5831. doi: 10.1007/BF01206606. URL <https://doi.org/10.1007/BF01206606>.
- Jean-Pierre Serre. *Lectures on  $N_X(p)$* , volume 11 of *Chapman & Hall/CRC Research Notes in Mathematics*. CRC Press, Boca Raton, FL, 2012. ISBN 978-1-4665-0192-8.