Modular Forms Modulo 2:

Governing Fields for the Hecke Algebra

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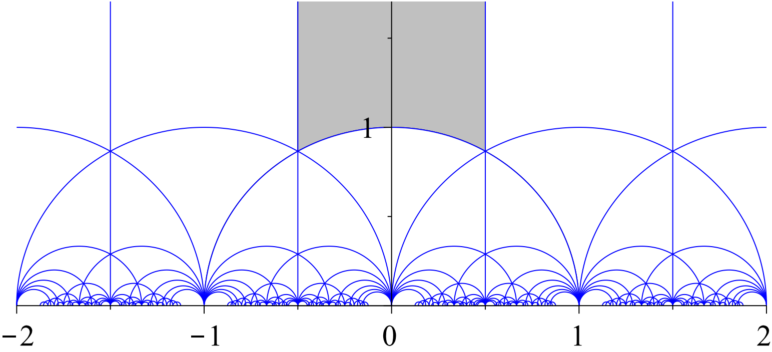
Modular Forms



Complex vector space of modular forms of weight n: M_n

For $f \in M_n$: $f(z) = \sum_{n=0}^{\infty} c(n) q^n$ with $q^n = e^{2\pi i n z}$

Reduction Modulo 2

$$\left\{ f \in M_n \mid f = \sum_{n \in \mathbb{N}} c(n) q^n \text{ s.t. } c(n) \in \mathbb{N} \right\}$$

$$||$$

$$\mathbb{F}_2 [\Delta, E_2, E_3]$$

$$\overline{\Delta} \rightsquigarrow \Delta$$

$$\overline{E_2} \rightsquigarrow 1$$

$$\overline{E_3} \rightsquigarrow 1$$

$$\overline{M_n} \rightsquigarrow \mathbb{F}_2 [\Delta]$$

Modular Forms Modulo 2

$$\mathcal{F} = \left\langle \Delta^k \mid k \text{ odd} \right\rangle_{\mathbb{F}_2} = \left\langle \Delta, \Delta^3, \Delta^5, \dots \right\rangle_{\mathbb{F}_2}$$

$$\begin{aligned} \Delta(q) &= q \prod_{n=1}^{\infty} (1 - q^n)^{24} \\ &= \sum_{n=0}^{\infty} \tau(n) q^n \\ &\equiv \sum_{m=0}^{\infty} q^{(2m+1)^2} \bmod 2 \end{aligned}$$

Hecke Operators

With

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

We define

$$\overline{T_m} | f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

Where

$$\gamma(n) = \sum_{a|(n,m),\, a \geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right)$$

Hecke Operators Modulo 2

With

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

We define

$$\overline{T_p} | f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

Where

$$\gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases} \quad \text{and } p \text{ an odd prime.}$$

Examples

	Δ^1	Δ^3	Δ^5	Δ^7	Δ^9	Δ^{11}	Δ^{13}
T_3	0	Δ	0	Δ^5	Δ^3	Δ^9	Δ^7
T_5	0	0	Δ	Δ^3	0	0	Δ^9
T_7	0	0	0	Δ	0	0	Δ^3
T_{11}	0	Δ	0	Δ^5	Δ^3	$\Delta+\Delta^9$	Δ^7
T_{13}	0	0	Δ	Δ^3	0	0	$\Delta+\Delta^9$
T_{17}	0	0	0	0	Δ	Δ^3	Δ^5
T_{19}	0	Δ	0	Δ^5	Δ^3	$\Delta+\Delta^9$	Δ^7

$$
\begin{tikzcd}
f = \Delta^k \arrow[d, leftrightarrow] & & T_p|f = \Delta^m + \dots \arrow[d, leftrightarrow] \\
f = q^k + \dots \arrow[rr, "T_p"] & & T_p|f = q^m + \dots
\end{tikzcd}
$$

The Hecke Algebra

$$\begin{aligned} A &= \mathbb{F}_2\,[T_3,\, T_5,\, T_7,\, T_{11},\, T_{13}, \dots] \\ &= \mathbb{F}_2\,[T_p \mid p \in \mathbb{P}] \end{aligned}$$

$$A = \mathbb{F}_2\,[[T_3,\, T_5]]$$

$$T_p = \sum_{i+j\geq 1} a_{ij}(p)\, T_3^i\, T_5^j$$

Examples

$$x = T_3 \qquad y = T_5$$
$$\text{i.e. } x^a y^b = T_3^a T_5^b$$

$$T_3 = x^1y^0 = x$$

$$T_5 = x^0y^1 = y$$

$$T_7 = x^1y^1 + x^3y^1 + x^3y^3 + x^5y^1 + x^1y^7 + x^1y^9 + x^7y^3 + x^7y^5 + x^9y^3 + x^{11}y^1 + x^3y^{11} + x^5y^9 + x^{13}y^1 + x^3y^{13} + x^5y^{11} + x^9y^7 + x^{11}y^5 + x^{13}y^3 + x^3y^{15} + x^7y^{11} + x^9y^9 + x^{13}y^5 + x^{15}y^3 + \dots$$

$$T_{11} = x^1y^0 + x^1y^2 + x^3y^0 + x^1y^4 + x^3y^2 + x^5y^0 + x^1y^6 + x^3y^4 + x^7y^2 + x^1y^{10} + x^3y^8 + x^7y^4 + x^9y^2 + x^{11}y^2 + x^3y^{12} + x^5y^{10} + x^7y^8 + x^{11}y^4 + x^{13}y^2 + x^9y^8 + x^{17}y^0 + \dots$$

Dirichlet Density Theorem

Intuition: $\boxed{\textit{Animation}}$

(see `https://pauldubois98.github.io/DirichletDensityTheoremAnimation/`)

Theorem (Dirichlet's Density Theorem)

Let $n \in \mathbb{N}^*$, $a \in \mathbb{N}$ such that $\gcd(a, n) = 1$.
If $S = \{p \in \mathbb{P} \mid p \equiv a \mod n\}$, then S has density ${}^1/_{\varphi(n)}$.

Chebotarev Density Theorem

Theorem (Chebotarev Density Theorem)

With L/K an extension of Galois group $G = Gal(L/K)$.
Let C be a conjugacy class in G.
Then, the proportion of unramified primes ideals $\mathfrak{p}$ in K that have Frobenius element $Frob_{L/K}(\mathfrak{p}) = C$ is ${}^{|C|}/_{|G|}$.

$L = \mathbb{Q}(\zeta_n)$ and $K = \mathbb{Q}$ give Dirichlet's density theorem.

Frobenian Maps

With K a number field, P the set of primes in K. $f : P \to \Omega$ is Frobenian if there exists

- a set $S \subset P$
- a field M extending K
- a class function $\phi : \mathrm{Gal}(M/K) \to \Omega$

such that:

$$f(\mathfrak{p}) = \phi(\mathrm{Frob}_{M/K}(\mathfrak{p})) \qquad \forall \mathfrak{p} \in P \setminus S$$

The maps a_{ij}

$$T_p = \sum_{i+j\geq 1} a_{ij}(p)\, T_3^i\, T_5^j$$

$a_{ij} : p \mapsto a_{ij}(p)$ is Frobenian

M_{ij} denotes a governing field for a_{ij}

G_{ij} denotes a governing group for a_{ij}

Known identities

- $a_{10}(p) = 1 \iff p \equiv 3 \bmod 8$
- $a_{01}(p) = 1 \iff p \equiv 5 \bmod 8$
- $a_{11}(p) = 1 \iff p \equiv 7 \bmod 8$
- $$a_{20}(p) = 1 \iff \begin{array}{c} \exists a, b \in \mathbb{Z} \text{ and } b \text{ odd, such that} \\ p = a^2 + 8b^2, p \equiv 3 \bmod 8 \end{array}$$
- $$a_{02}(p) = 1 \iff \begin{array}{c} \exists a, b \in \mathbb{Z} \text{ and } b \text{ odd, such that} \\ p = a^2 + 16b^2, p \equiv 3 \bmod 8 \end{array}$$

Known Governing Fields

$$M_{01} = \mathbb{Q}\left(\zeta_8\right)$$

$$M_{02} = \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}\right)$$

$$M_{11} = \mathbb{Q}\left(\zeta_8, \sqrt{\zeta_8}\right) = \mathbb{Q}\left(\zeta_{16}\right)$$

$$M_{02} = \mathbb{Q}\left(\zeta_8, \sqrt{1+i}\right)$$

$$M_{01} = \mathbb{Q}\left(\zeta_8\right)$$

Note $\mathbb{Q}\left(\zeta_8\right) = \mathbb{Q}\left(i, \sqrt{2}\right)$

Potential New Governing Fields

$$M_{03} \stackrel{?}{=} \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}\right)$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^{3}}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

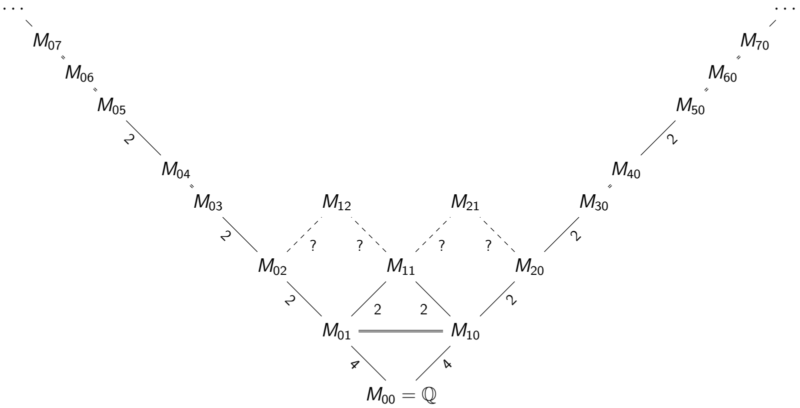
Potential New Governing Fields

$$M_{05} \stackrel{?}{=} \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}, \sqrt{\beta}\right)$$

where α remains and

$$\begin{aligned}\beta = &- \frac{8282936156772053\alpha^{\frac{13}{2}}}{1125899906842624} - \frac{1240182980093567\alpha^{6}}{562949953421312} \\ &- \frac{336382584949535\alpha^{\frac{9}{2}}}{2199023255552} - \frac{6445823996745319\alpha^{4}}{140737488355328} \\ &- \frac{4638634719581101\alpha^{\frac{5}{2}}}{35184372088832} - \frac{2954723016803317\alpha^{2}}{70368744177664} \\ &- \frac{5142889464378747\sqrt[4]{2}}{140737488355328} - \frac{4198844765367981\sqrt{\alpha}}{1125899906842624} \\ &+ \dots\end{aligned}$$

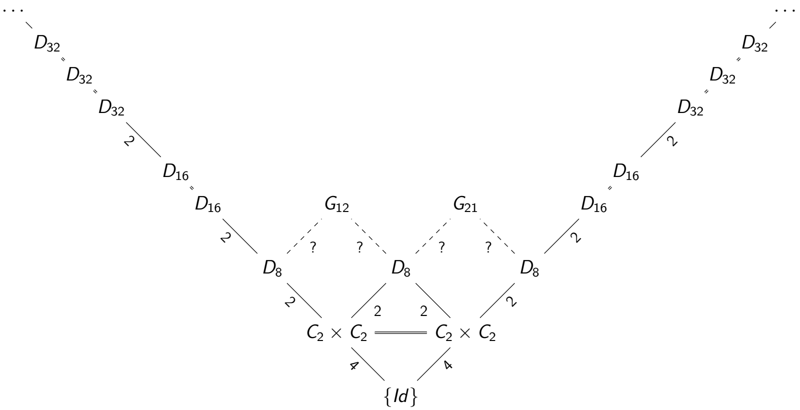
Governing Fields Extension Graph



Governing Groups

$$
\begin{aligned}
G_{01} &\cong D_4 \cong C_2 \times C_2 & \quad G_{10} &\cong D_4 \cong C_2 \times C_2 \\
G_{02} &\cong D_8 & G_{20} &\cong D_8 \\
G_{03} &\cong D_{16} & G_{30} &\cong D_{16} \\
G_{04} &\cong D_{16} & G_{40} &\cong D_{16} \\
G_{05} &\cong D_{32} & G_{50} &\cong D_{32} \\
G_{06} &\cong D_{32} & G_{60} &\cong D_{32} \\
G_{07} &\cong D_{32} & G_{70} &\cong D_{32}
\end{aligned}
$$

Governing Groups Extension Graph



Diagonal Governing Groups

Conjecture (Diagonal Governing Groups Conjecture)

For all $k \in \mathbb{N}^*$, there exists a field M_{0k} such that M_{0k} is a governing field for a_{0k}, and $G_{0k} = Gal(M_{0k}/\mathbb{Q})$ is dihedral.
For all $k \in \mathbb{N}^*$, there exists a field M_{k0} such that M_{k0} is a governing field for a_{k0}, and G_{k0} is dihedral.

Moreover $M_{k0} \neq M_{0k}$ in general, but $G_{k0} \cong G_{0k}$.

Thank you