Modular Forms Modulo 2:

Governing Fields for the Hecke Algebra

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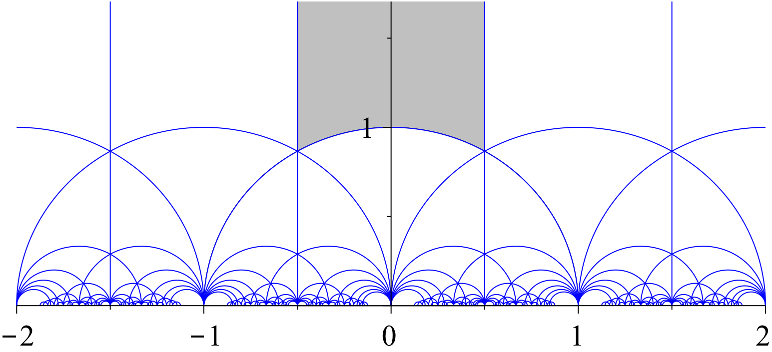
Modular Forms



Complex vector space of modular forms of weight n: M_n

For $f \in M_n$: $f(z) = \sum_{n=0}^{\infty} c(n) q^n$ with $q^n = e^{2\pi i n z}$

Reduction Modulo 2

$$\left\{ f \in M_n \mid f = \sum_{n\in\mathbb{N}} c(n)q^n \text{ s.t. } c(n) \in \mathbb{Z} \right\}$$

$$\|$$

$$\mathbb{Z}\left[\Delta, E_2, E_3\right]$$

$$\uparrow \text{(with weight n)}$$

$$\overline{\Delta} \rightsquigarrow \mathit{\Delta}$$

$$\overline{E_2} \rightsquigarrow 1$$

$$\overline{E_3} \rightsquigarrow 1$$

$$\overline{\mathbb{Z}\left[\Delta, E_2, E_3\right]} \rightsquigarrow \mathbb{F}_2\left[\mathit{\Delta}\right]$$

Modular Forms Modulo 2

$$\mathcal{F} = \left\langle \Delta^k \mid k \text{ odd} \right\rangle_{\mathbb{F}_2} = \left\langle \Delta, \Delta^3, \Delta^5, \dots \right\rangle_{\mathbb{F}_2}$$

$$\begin{aligned} \Delta(q) &= q \prod_{n=1}^{\infty} (1 - q^n)^{24} \\ &= \sum_{n=0}^{\infty} \tau(n) q^n \\ &\equiv \sum_{m=0}^{\infty} q^{(2m+1)^2} \bmod 2 \end{aligned}$$

Hecke Operators

With

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

We define

$$T_m | f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

Where

$$\gamma(n) = \sum_{a | (n,m),\, a \geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right)$$

Hecke Operators Modulo 2

With

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

We define

$$\overline{T_p} | f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

Where

$$\gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases} \quad \text{and } p \text{ an odd prime.}$$

Examples

	Δ^1	Δ^3	Δ^5	Δ^7	Δ^9	Δ^{11}	Δ^{13}
T_3	0	Δ	0	Δ^5	Δ^3	Δ^9	Δ^7
T_5	0	0	Δ	Δ^3	0	0	Δ^9
T_7	0	0	0	Δ	0	0	Δ^3
T_{11}	0	Δ	0	Δ^5	Δ^3	$\Delta+\Delta^9$	Δ^7
T_{13}	0	0	Δ	Δ^3	0	0	$\Delta+\Delta^9$
T_{17}	0	0	0	0	Δ	Δ^3	Δ^5
T_{19}	0	Δ	0	Δ^5	Δ^3	$\Delta+\Delta^9$	Δ^7

$$\begin{tikzcd}
f = \Delta^k \arrow[d, leftrightarrow] & & T_p|f = \Delta^m + \dots \arrow[d, leftrightarrow] \\
f = q^k + \dots \arrow[rr, "T_p"] & & T_p|f = q^m + \dots
\end{tikzcd}$$

The Hecke Algebra

$$
\begin{aligned}
A &= \mathbb{F}_2\,[\,T_3,\, T_5,\, T_7,\, T_{11},\, T_{13}, \ldots\,] \\
&= \mathbb{F}_2\,[\,T_p \mid p \in \mathbb{P}]
\end{aligned}
$$

$$A \cong \mathbb{F}_2\,[[\,T_3,\, T_5]]$$

$$T_p = \sum_{i+j \geq 1} a_{ij}(p)\, T_3^i\, T_5^j$$

Examples

$$x = T_3 \qquad y = T_5$$
$$\text{i.e. } x^a y^b = T_3^a T_5^b$$

$T_3 = x^1y^0 = x$

$T_5 = x^0y^1 = y$

$T_7 = x^1y^1 + x^3y^1 + x^3y^3 + x^5y^1 + x^1y^7 + x^1y^9 + x^7y^3 + x^7y^5 + x^9y^3 + x^{11}y^1 + x^3y^{11} + x^5y^9 + x^{13}y^1 + x^3y^{13} + x^5y^{11} + x^9y^7 + x^{11}y^5 + x^{13}y^3 + x^3y^{15} + x^7y^{11} + x^9y^9 + x^{13}y^5 + x^{15}y^3 + \dots$

$T_{11} = x^1y^0 + x^1y^2 + x^3y^0 + x^1y^4 + x^3y^2 + x^5y^0 + x^1y^6 + x^3y^4 + x^7y^2 + x^1y^{10} + x^3y^8 + x^7y^4 + x^9y^2 + x^{11}y^2 + x^3y^{12} + x^5y^{10} + x^7y^8 + x^{11}y^4 + x^{13}y^2 + x^9y^8 + x^{17}y^0 + \dots$

Dirichlet Density Theorem

Intuition: $\boxed{\textit{Animation}}$

(see `https://pauldubois98.github.io/DirichletDensityTheoremAnimation/`)

Theorem (Dirichlet's Density Theorem)

Let $n \in \mathbb{N}^*$, $a \in \mathbb{N}$ *such that* $\gcd(a, n) = 1$.
If $S = \{p \in \mathbb{P} \mid p \equiv a \mod n\}$, *then* S *has density* ${}^1/_{\varphi(n)}$.

Chebotarev Density Theorem

Theorem (Chebotarev Density Theorem)

With L/K an extension of Galois group $G = Gal(L/K)$.
Let C be a conjugacy class in G.
Then, the proportion of unramified primes ideals $\mathfrak{p}$ in K that have Frobenius element $Frob_{L/K}(\mathfrak{p}) = C$ is $^{|C|}/_{|G|}$.

$L = \mathbb{Q}(\zeta_n)$ and $K = \mathbb{Q}$ give Dirichlet's density theorem.

Frobenian Maps

With K a number field, P the set of primes in K. $f : P \to \Omega$ is Frobenian if there exists

- a (finite) set $S \subset P$
- a field M extending K
- a class function $\phi : \mathrm{Gal}(M/K) \to \Omega$

such that:

$$f(\mathfrak{p}) = \phi(\mathrm{Frob}_{M/K}(\mathfrak{p})) \qquad \forall \mathfrak{p} \in P \setminus S$$

The maps a_{ij}

$$T_p = \sum_{i+j\geq 1} a_{ij}(p)\, T_3^i\, T_5^j$$

$a_{ij} : p \mapsto a_{ij}(p)$ is Frobenian

M_{ij} denotes a governing field for a_{ij}

G_{ij} denotes a governing group for a_{ij}

Known identities

- $a_{10}(p) = 1 \iff p \equiv 3 \bmod 8$
- $a_{01}(p) = 1 \iff p \equiv 5 \bmod 8$
- $a_{11}(p) = 1 \iff p \equiv 7 \bmod 8$
- $$a_{20}(p) = 1 \iff \begin{array}{c} \exists a, b \in \mathbb{Z} \text{ and } b \text{ odd}, \text{ such that} \\ p = a^2 + 8b^2 \end{array}$$
- $$a_{02}(p) = 1 \iff \begin{array}{c} \exists a, b \in \mathbb{Z} \text{ and } b \text{ odd}, \text{ such that} \\ p = a^2 + 16b^2 \end{array}$$

Known Governing Fields

$$M_{01} = \mathbb{Q}\left(\zeta_8\right)$$

$$M_{02} = \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}\right)$$

$$M_{11} = \mathbb{Q}\left(\zeta_8, \sqrt{\zeta_8}\right) = \mathbb{Q}\left(\zeta_{16}\right)$$

$$M_{02} = \mathbb{Q}\left(\zeta_8, \sqrt{1+i}\right)$$

$$M_{01} = \mathbb{Q}\left(\zeta_8\right)$$

Note $\mathbb{Q}\left(\zeta_8\right) = \mathbb{Q}\left(i, \sqrt{2}\right)$

Potential New Governing Fields

$$M_{03} \stackrel{?}{=} \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}\right)$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^{3}}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

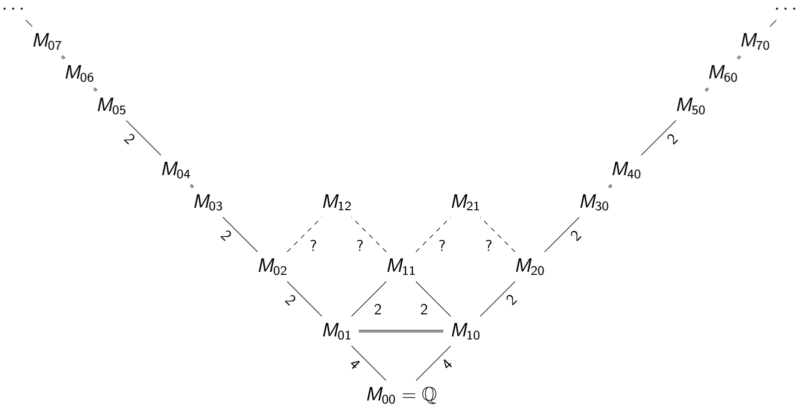
Potential New Governing Fields

$$M_{05} \stackrel{?}{=} \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}, \sqrt{\beta}\right)$$

where α remains and

$$\begin{aligned}\beta = &-\frac{8282936156772053\alpha^{\frac{13}{2}}}{1125899906842624} - \frac{1240182980093567\alpha^{6}}{562949953421312}\\ &-\frac{336382584949535\alpha^{\frac{9}{2}}}{2199023255552} - \frac{6445823996745319\alpha^{4}}{140737488355328}\\ &-\frac{4638634719581101\alpha^{\frac{5}{2}}}{35184372088832} - \frac{295472301680317\alpha^{2}}{70368744177664}\\ &-\frac{5142889464378747\sqrt[4]{2}}{140737488355328} - \frac{4198844765367981\sqrt{\alpha}}{1125899906842624}\\ &+\ldots\end{aligned}$$

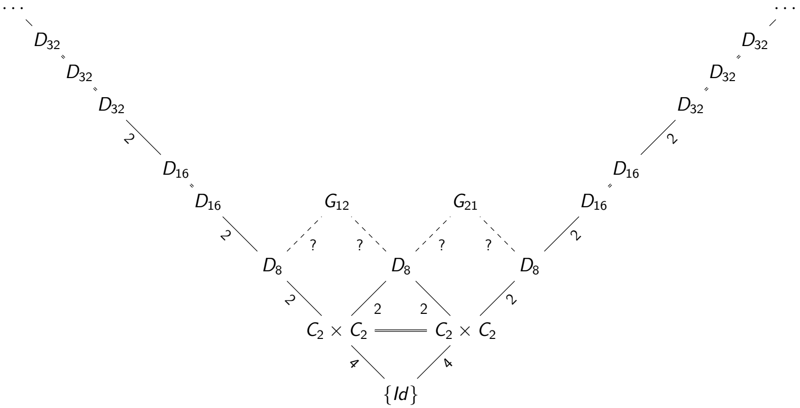
Governing Fields Extension Graph



Governing Groups

$$
\begin{aligned}
G_{01} &\cong D_4 \cong C_2 \times C_2 & \qquad G_{10} &\cong D_4 \cong C_2 \times C_2 \\
G_{02} &\cong D_8 & G_{20} &\cong D_8 \\
G_{03} &\cong D_{16} & G_{30} &\cong D_{16} \\
G_{04} &\cong D_{16} & G_{40} &\cong D_{16} \\
G_{05} &\cong D_{32} & G_{50} &\cong D_{32} \\
G_{06} &\cong D_{32} & G_{60} &\cong D_{32} \\
G_{07} &\cong D_{32} & G_{70} &\cong D_{32}
\end{aligned}
$$

Governing Groups Extension Graph



Diagonal Governing Groups

Conjecture (Diagonal Governing Groups Conjecture)

For all $k \in \mathbb{N}^*$, there exists a field M_{0k} such that M_{0k} is a governing field for a_{0k}, and $G_{0k} = Gal(M_{0k}/\mathbb{Q})$ is dihedral. For all $k \in \mathbb{N}^*$, there exists a field M_{k0} such that M_{k0} is a governing field for a_{k0}, and G_{k0} is dihedral.

Moreover $M_{k0} \neq M_{0k}$ in general, but $G_{k0} \cong G_{0k}$.

Thank you

Extra Slides and Links

Computations Results

Dirichlet Density Theorem Animation

Governing Fields Found (Complete List)

Plots of Codes

Wikipedia Page on Modular Forms Modulo 2

Modularity

$$\mathbb{H} = \{z = x + yi \in \mathbb{C} |\ y > 0\}.$$

We say that a function $f : \mathbb{H} \to \mathbb{C}$ is *weakly modular* of *weight* $2k$ if f is meromorphic and

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The group $\mathrm{SL}_2(\mathbb{Z})$ of invertible 2-by-2 matrices over $\mathbb{Z}$ with is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

$\mathrm{SL}_2(\mathbb{Z})$ is called the modular group.

Eisenstein Series

Eisenstein Series:

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}} \text{ for } k \geq 2$$

Normalized Eisenstein Series:

$$E_k \cdot 2\zeta(2k) = G_k,$$

Discriminant Delta (before normalization):

$$\Delta = \left(\frac{1}{(2\pi)^{12}}\right)(g_2^3 - 27g_3^2) \in M_6$$

with $g_2 = 40G_2$ and $g_3 = 140G_3$

Subspaces of Modular Forms Modulo 2

$$\mathcal{F}_1 = \left\langle \Delta^k \mid k = 1 \bmod 8 \right\rangle = \left\langle \Delta^1, \Delta^9, \Delta^{17}, \Delta^{25}, \dots \right\rangle$$

$$\mathcal{F}_3 = \left\langle \Delta^k \mid k = 3 \bmod 8 \right\rangle = \left\langle \Delta^3, \Delta^{11}, \Delta^{19}, \Delta^{27}, \dots \right\rangle$$

$$\mathcal{F}_5 = \left\langle \Delta^k \mid k = 5 \bmod 8 \right\rangle = \left\langle \Delta^5, \Delta^{13}, \Delta^{21}, \Delta^{29}, \dots \right\rangle$$

$$\mathcal{F}_7 = \left\langle \Delta^k \mid k = 7 \bmod 8 \right\rangle = \left\langle \Delta^7, \Delta^{15}, \Delta^{23}, \Delta^{31}, \dots \right\rangle$$

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_7$$

$$f \in \mathcal{F}_i \implies T_p|f \in \mathcal{F}_j \quad \text{with } j \equiv pj \bmod 8$$

Frobenius Element

With L/K a Normal extension, and ideal $\mathfrak{P}$ in $\mathcal{O}_L$.
$\mathrm{Frob}_{L/K}(\mathfrak{P})$, and is *the* element $\sigma \in \mathrm{Gal}(L/K)$ such that:

$$\sigma\mathfrak{P} = \mathfrak{P} \quad \text{and} \quad \sigma(\alpha) \equiv \alpha^{\mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{p})} \bmod \mathfrak{P} \quad \forall \alpha \in \mathcal{O}_L.$$

Densities

Natural density: We say that $S \subseteq \mathbb{P}$ has natural density δ when:

$$\lim_{x \to +\infty} \frac{\#\{p \in \mathbb{P}, p < x \mid p \in S\}}{\#\{p \in \mathbb{P}, p < x \mid p \in \mathbb{P}\}} = \delta$$

Analytic density or Dirichlet density: We say that $S \subseteq \mathbb{P}$ has analytical (or Dirichlet) density δ when:

$$\lim_{s \to 1^+} \left(\sum_{p \in S} \frac{1}{p^s} \right) \left(\sum_{p \in \mathbb{P}} \frac{1}{p} \right)^{-1} = \delta$$