Modular Forms Modulo 2

Governing Fields for the Hecke Algebra

Oral Presentation Script

Intro

Title Slide

Hello, thank you for “coming”.

I will present my research project titled “Modular Forms Modulo 2: Governing Fields For the Hecke Algebra”.

First, we (quickly) go through reduction of modular forms modulo 2. We will also introduce Hecke operators modulo 2 and look at the structure of the Hecke algebra that they form.

Then we will turn our interest to Dirichlet and Chebotarev density theorems. This will lead to Frobenian maps.

Finally, we will link the two theories by studying governing fields (a feature related to Frobenian property) for the maps “a_{ij}” (maps linked to the structure of the Hecke Algebra).

Modular forms modulo 2

Modular Forms

Modular forms are analytic function on the upper half plane that satisfy a certain kind of equation with respect to the group action of the modular group.

The picture we usually see is the following *(show)*. Grey region is the fundamental cell, and other regions are “symmetric”.

We denote the complex vector space of modular forms modulo 2 by Mn.

We denote the coefficients of the q-series of a modular form by c(n).

What we well be interested in is modular forms MODULO 2.

Reduction Modulo 2

To reduce modulo 2, we need integers, so we will concentrate on modular forms having coefficients of q-series being integers (i.e. this set, *show*). It turns out to be generated by polynomials of the modular discriminant Delta, and Eisenstein series E2 and E3.

Once coefficients are reduced modulo 2, only Delta is not trivial. Thus, the reduction of modular forms are just polynomials of Delta over F2.

Modular Forms Modulo 2

Thus, we define F, the space of modular forms modulo 2 to be odd powers of Delta over F2. In this study, we ignore all even powers of Delta.

Note here that modular forms modulo 2 do not have “weights”, we lost this information while reducing mod 2.

Delta is defined as an infinite product. The coefficients that matches this are tau n (the Ramanujan function). Result from Kohlberg lead to Delta being only the sum of odd squares powers of q.

I would like to attract your attention here on the fact that a modular form modulo 2 has two definitions: one as a (finite) polynomial of Delta over F2, one as an (infinite) q-series. This “duality” is very specific to MF mod 2, and we will use it.

Hecke operators modulo 2

Hecke Operators

One of the interesting things we can do with modular forms is calculate Hecke operators. They are define using the q-series of modular forms as follows *(show)*.

Again, we want to reduce this modulo 2. We also want a formula not too complicated to work with. We remark that this sum has only two summands if “m” is prime. Thus, we define Hecke operators on modular forms modulo 2 as follows:

Hecke Operators Modulo 2

Which corresponds to the sum from before. We ignore the case p=2 as T2 isn’t an operator on F.

Defining Hecke operators when there are more than 2 summands do not make much sense mod 2, as the formula would be complicated, and many cancellations would happen, making the legitimacy of the process doubtable.

Examples

We can compute tables of Hecke operators as shown here *(show)*.

Here, the result is show in the finite Delta-representation, but the Hecke operators are calculated using the q-series representations of modular forms. What we in fact internally did is shown in the diagram, i.e. convert the delta polynomial of f into its q-series, calculate the Hecke operation, and then covert back the q-series of Tpf into its delta representation.

The Hecke Algebra

We now understand the behavior if Hecke operators for small primes on forms with small degree (in terms of Delta).

We create a set made of the Hecke operators: the Hecke algebra. It consists of polynomial of Hecke operators over F2. We denote it A.

It turns out that A is in fact generated by series of T3 and T5 over F2.

This means that for any Hecke operator Tp, there are some coefficients aij(p) such this *(show)* equation holds. We will be interested in these coefficients aij(p) (which are 0 or 1, as over F2).

Examples

If we denote T3 by x and T5 by y, we can write the beginning of the series expansion of Tp, for small odd primes p. Note that for T3 and T5, the series is in fact finite, but these are very particular cases.

2^{nd} part: Frobenian maps

Dirichlet Density Theorem

I would like to give some intuition before stating Dirichlet Density Theorem. *(please click the link)*

What I want to emphasis by letting you choose a number is that this theorem in fact works for any natural number n (greater than 2). Now, to have something interesting, I advise to take a number that is not prime, and for display reasons, not too large, say not greater than 20.

What we remark is that mod n, there are congruent classes which contain, apparently, finitely many primes, and some which contain infinitely many primes. Moreover, the classes of infinitely many primes seem to have all the same size.

(back to beamer)

The exact statement of Dirichlet Density Theorem is that the set S of primes a mod n with a, n coprime has density 1/phi(n).

Of course, if a and n are not coprime, the density of S will be 0.

Chebotarev Density Theorem

A more refined version of Dirichlet density theorem was proved by Chebotarev in his thesis:

It is in fact true that for an extension L/K, if C is the conjugacy class of the Galois group of L/K, then the proportion of primes ideals p such that the Frobenius element Frob L/K p is (in) C corresponds to the size of c over the size of the Galois group.

Note that if L/K is the cyclotomic field Q zeta n over the rational Q, Chebotarev density theorem gives exactly Dirichlet density theorem.

Chebotarev theorem uses Frobenius elements. Using Frobenius element, we define Frobenian maps.

Frobenian Maps

With K a number field: We say that a map f from primes P of k is Frobenian if there is:

- Subset S of P
- A field M extending K
- A Class function phi on the Galois group of M/K

Such that f(p) = phi(frob M/K (p))

In such a configuration, M is the Governing field, and G = Galois group of M/K is the governing group.

Usually, S is the set of ramified primes.

Note that if M is a governing field, any extension of M will also be a governing field. Thus, Governing fields are not unique.

Link

The maps a_{ij}

If we get back to Hecke operators, we remember that Tp can be written as a series of T3 and T5, with coefficients a_ij(p). We now look at a_ij(p) as maps as shown.

We will write M_ij and G_ij for a governing field and corresponding governing group of a_ij.

A result from Joël BELLAÏCHE is that one can find a governing field M_ij for a_ij such that M_ij/Q is normal extension, G_ij (which thus exists) is a 2-group, and M_ij is unramified outside 2 and infinity.

Known Identities

Some identities are known for small i and j. Jean-Pierre Serre and Jean-Louis Nicolas proved the following *(show)*.

From these identities, they deduced the following governing fields.

Known Governing Fields

These are all the known governing field at this point.

What we can do, with computations is, first, check these, and second, try to find new governing fields. The way we proceed is as follows: For a potential governing field, we compute Frobenian elements for many primes (primes less than 10 000) and check the if they satisfy the Frobenian property.

Potential New Governing Fields

The following fields verify the Frobenian property for all primes less than 10 000. Thus, it is very likely that they are governing fields. In the dissertation is explained a probabilistic analysis, to calculate the probability that the fields were found to satisfy Frobenian property for all these primes at random, and the probability is of order 10^-200.

It is therefore fair to assume these are proper governing fields.

Governing Fields Extension Graph

Since governing fields extend each other's, we then have the following graph. *(show)*

Governing Groups

Once we have found the governing fields, we can calculate the corresponding governing graph.

Governing Groups Extension Graph

We can again make a graph, here for the governing groups.

We remark that the groups found on the diagonal are all dihedrals. We can therefore conjecture that this property will remain.

Diagonal Governing Groups

We make the following conjecture:

On the diagonals, there exist a governing field such that the corresponding governing group is dihedral.

Moreover, their size is the same on each "level" of the diagonals.

Thank you

Thank you for your attention.