Modular Forms Modulo 2:

Governing Fields for the Hecke Algebra

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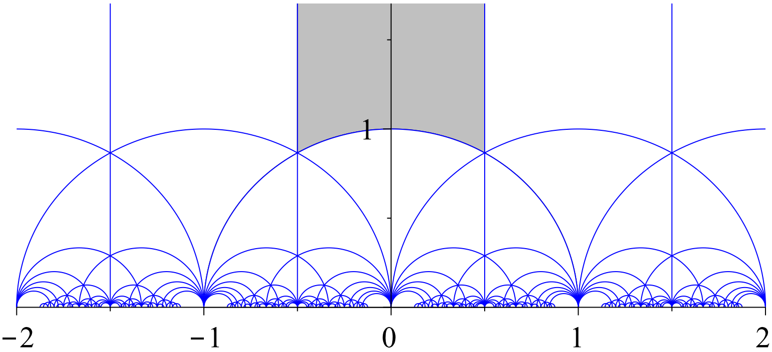
Modular Forms



Reduction Modulo 2

$$M_n = \langle \Delta, E_2, E_2 \rangle$$

$$\overline{\Delta} \rightsquigarrow \Delta$$

$$\overline{E_2} \rightsquigarrow 1$$

$$\overline{E_3} \rightsquigarrow 1$$

$$\overline{M_n} \rightsquigarrow \langle \Delta \rangle$$

Modular Forms Modulo 2

$$\mathcal{F} = \left\langle \Delta^k \mid k \text{ odd} \right\rangle = \left\langle \Delta, \Delta^3, \Delta^5, \dots \right\rangle.$$

$$\begin{aligned}
\Delta(q) &= q \prod_{n=1}^{\infty} (1 - q^n)^{24} \\
&= \sum_{n=0}^{\infty} \tau(n) q^n \\
&\equiv \sum_{m=0}^{\infty} q^{(2m+1)^2} \bmod 2
\end{aligned}$$

Hecke Operators

With

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

We define

$$\overline{T_m} | f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

Where

$$\gamma(n) = \sum_{a|(n,m),\, a \geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right)$$

Hecke Operators Modulo 2

With

$$f(q) = \sum_{n \in \mathbb{N}} c(n) q^n$$

We define

$$\overline{T_p} | f(q) = \sum_{n \in \mathbb{N}} \gamma(n) q^n$$

Where

$$\gamma(n) = \begin{cases} c(np) & \text{if } p \nmid n \\ c(np) + c(n/p) & \text{if } p \mid n \end{cases}$$

and p an odd prime.

Examples

$$T_{11}\Delta^{29}$$

[a few more examples]

The Hecke Algebra

$$A = \langle T_3, T_5, T_7, T_{11}, T_{13}, \ldots \rangle = \langle T_p \mid p \in \mathbb{P} \rangle$$

$$A = \mathbb{F}_2 [[T_3, T_5]]$$

$$T_p = \sum_{i+j \geq 1} a_{ij}(p) T_3^i T_5^j$$

Example

$$T_{19}$$

[a few examples]

Dirichlet Density Theorem: Intuition

$\boxed{\textit{Animation}}$ (see `https://pauldubois98.github.io/DirichletDensityTheoremAnimation/`)

Dirichlet Density Theorems

[to write]

Frobenius element

[define frob elements quickly]

Chebotarev Density Theorems

[to write]

Frobenian Maps

[definition]

The maps a_{ij}

$$T_p = \sum_{i+j\geq 1} a_{ij}(p)\, T_3^i\, T_5^j$$

$$a_{ij} : p \mapsto a_{ij}(p)$$

Known Governing Fields

$$M_{01} = \mathbb{Q}\left(\zeta_8\right)$$
$$M_{02} = \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}\right)$$

Potential New Governing Fields

$$M_{03} \stackrel{?}{=} \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}\right)$$

where:

$$\alpha = -\frac{3136435454775881\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} + \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656}$$

Potential New Governing Fields

$$M_{05} \stackrel{?}{=} \mathbb{Q}\left(\zeta_8, \sqrt[4]{2}, \sqrt{\alpha}, \sqrt{\beta}\right)$$

where:

$$\begin{aligned}\alpha = &- \frac{313643545477588 1\sqrt[4]{2}}{562949953421312} + \frac{4208721080340285\sqrt{2}}{2251799813685248} \\ &+ \frac{3672578267558083 \cdot \sqrt[4]{2}^3}{562949953421312} + \frac{3582104167901087}{281474976710656}\end{aligned}$$

and

$$\begin{aligned}\beta = &- \frac{8282936156772053\alpha^{\frac{13}{2}}}{1125899906842624} - \frac{1240182980093567\alpha^6}{562949953421312} \\ &- \frac{336382584949535\alpha^{\frac{9}{2}}}{2199023255552} - \frac{6445823996745319\alpha^4}{140737488355328} \\ &- \ldots\end{aligned}$$

Governing Groups

$$
\begin{aligned}
G_{01} &= D_8 & \qquad G_{10} &= D_8 \\
G_{02} &= D_8 & \qquad G_{20} &= D_8 \\
G_{03} &= D_{16} & \qquad G_{30} &= D_{16} \\
G_{04} &= D_{16} & \qquad G_{40} &= D_{16} \\
G_{05} &= D_{32} & \qquad G_{50} &= D_{32} \\
G_{06} &= D_{32} & \qquad G_{60} &= D_{32} \\
G_{07} &= D_{32} & \qquad G_{70} &= D_{32}
\end{aligned}
$$

Diagonal Governing Groups

Conjecture (Diagonal Governing Groups Conjecture)

For all $k \in \mathbb{N}^*$, there exists a field M_{0k} such that M_{0k} is a governing field for a_{0k}, and $G_{0k} = Gal(M_{0k}/\mathbb{Q})$ is dihedral.
For all $k \in \mathbb{N}^*$, there exists a field M_{k0} such that M_{k0} is a governing field for a_{k0}, and G_{k0} is dihedral.

Moreover $M_{k0} \neq M_{0k}$ in general, but $G_{k0} \cong G_{0k}$.

Questions

[add some usefull links?]