

# Problem Set 5

Due 29<sup>nd</sup> September 2021

## Abstract

Only the questions with a star (\*) are compulsory for submission;  
It is however *strongly* advised to attempt all question.

## 1 Sequences

**Question 1.** (\*) Show that  $S_n = \sum_{k=1}^n \frac{1}{k}$  is such that  $S_{2^n} \geq \frac{n}{2}$ .  
Is  $\sum_{n=1}^{\infty} \frac{1}{n}$  convergent?

**Question 2.** (Sum of Arithmetic Sequence) a\*) Rewrite  $S = \sum_{k=0}^n k$  with  $k$  running from  $n$  to 0 (instead of from 0 to  $n$ ).

b\*) Using both expressions for  $S$ , express  $2S$  as a product of two terms, and deduce an expression for  $S$ .

c\*) Let  $(x_n)$  be an arithmetic sequence:  $x_0 = b$ ,  $x_{n+1} = a + x_n$  (we have shown before that the explicit formula for  $x_n$  is  $x_n = b + a * n$ ).

Find the sum of the first  $n$  terms:  $\sum_{k=0}^n x_k$

d\*) When is the series  $\sum_{n=0}^{\infty} x_n$  convergent/divergent?

**Question 3.** (Sum of Geometric Sequence) a\*) Multiply  $S = \sum_{k=0}^n \alpha^k$  by  $(1 - \alpha)$  and simplify the expression  
b\*) Deduce an expression for  $S$ .

c\*) Let  $(x_n)$  be a geometric sequence:  $x_0 = b$ ,  $x_{n+1} = a * x_n$  (we have shown before that the explicit formula for  $x_n$  is  $x_n = b * a^n$ ).

Find the sum of the first  $n$  terms:  $\sum_{k=0}^n x_k$

d\*) When is the series  $\sum_{n=0}^{\infty} x_n$  convergent/divergent?

(BONUS) e\*) When convergent (if ever), what values does the series  $\sum_{n=0}^{\infty} x_n$  converge to?

**Question 4.** Are these convergent series?

- $\sum_{n=1}^{+\infty} \frac{(-17)^n}{4^{2n+1}(n+1)}$
- $\sum_{n=1}^{+\infty} \frac{3^{2n+1}}{n^n}$
- $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$
- $\sum_{n=1}^{+\infty} \frac{n^n}{(2n-1)!}$
- $\sum_{n=1}^{+\infty} \left( \frac{5n-3n^3}{0.5n^3+2} \right)^n$
- $\sum_{n=1}^{+\infty} \frac{(-12)^n}{n}$
- $\sum_{n=1}^{+\infty} \frac{(-2)^{n+1}n}{9^n}$
- $\sum_{n=1}^{+\infty} \frac{1}{n^3}$

Fact:  $\lim_{n \rightarrow +\infty} n^{\frac{1}{n}} = 1$

**Question 5.** Calculate the derivative of the following functions:

$$f(x) = \frac{5x^3 - 2x + 1}{2x - 7} \quad g(x) = ax^2 + bx + c$$

**Question 6.** Find the anti-derivative of  $x^n$ .