

# Refresher Math Course

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### **Abstract**

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. The course will try to cover most of the prerequisites of the courses in the Master, mainly linear algebra, differential calculus, integration, and asymptotic analysis.

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# Introduction

## Presentation

- Paul Dubois
- will be teaching this refresher math course
- email (for any question), answer within 1 working day

## Course Format

### Lectures

- 8\*3h
- 1h20min lecture - 1/3h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)
- Lectures are recorded (if ever needed)
- 1st lecture ever => too fast/too slow: let me know
- May assume you know a concept/notation that you have never heard of, let me know if this happens

### Examination

- The course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be a full exercise sheet per lecture, it is advised to attempt it all (only one will be compulsory).
- Hand-in 1 exercise per lecture (i.e., 8 in total), due 2 weeks after the lecture
- Best  $(n-1)/n$  count (i.e., best 7/8 in our case), need avg  $\geq 50\%$  to pass
- In the unlikely event of not passing, will be able to do an extra work

## Questions?

# Chapter 1

## Sets & logic

### 1.1 Mathematical Objects & Notations

#### Sets

**Definition** (Sets). *Unordered list of elements.*

**Notation** (Sets).  $\in$ ,  $\{True, False\}$ ,  $\{a \mid condition\}$ ,  $\{a, b, c \dots\}$ ,  $\emptyset$

Need to be careful when defining set: some definitions are pathological.

**Remark** (Russell Paradox). *Take  $U = \{X \mid X \notin X\}$ .  $X \text{ in } U \Rightarrow U \text{ not in } U$ ,  $U \text{ is a set, so not all sets are in } U$   $X \text{ not in } U \Rightarrow X \text{ is a set}$*

**Notation** (Usual Sets).  $\mathbb{B}$ ,  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{N}^*$ ,  $\mathbb{R}^+ \dots$

#### Functions

**Definition** (Functions). *Assignment for a set to another.*

**Notation** (Function).  $f : X \rightarrow Y$ ,  $f(x) = blah$ ,  $f : x \mapsto blah$ .

**Definition** (Predicate). *Function to  $\mathbb{B}$*

**Question.** *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

#### Quantifiers

**Notation** ( $\forall$ ). *For all elements in set, e.g.:  $\forall x \in \mathbb{R}, x^2 \geq 0$ .*

**Notation** ( $\exists$ ). *There exists an element in set, e.g.:  $\exists x \in \mathbb{R} \text{ s.t. } x^2 > 1$ .*

**Notation** ( $\exists!$ ). *There exists a unique element in set, e.g.:  $\exists! x \in \mathbb{R} \text{ s.t. } x^2 \leq 0$ .*

**Definition** (Subset / Inclusion).  $X \subseteq Y$  if  $\forall x \in X, x \in Y$

**Definition** (Disjoint Sets).  $X$  and  $Y$  are disjoint if  $\forall x \in X, x \notin Y$  (or if  $\forall y \in Y, y \notin X$ ).

**Definition** (Powerset).  $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$

e.g.:  $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

**Definition** (Cartesian Product).  $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

e.g.:  $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Extension:  $X_1 \times \cdots \times X_n = \prod_{k=1}^n X_k$

## 1.2 Boolean algebra

### Basic operators

**Definition** (Conjunction).  $x \wedge y = xy$

**Definition** (Intersection).  $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

**Remark** (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

**Definition** (Disjunction).  $x \vee y = \min(x + y, 1)$

**Definition** (Union).  $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

**Definition** (Negation).  $\neg : 0, 1 \mapsto 1, 0$

**Definition** (Set minus / Complement).  $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

**Question.** *Selecting points outside a given region.*

### Basic properties

**Property** (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up  $\vee$  with addition and  $\wedge$  with multiplication.*

- *Associativity of  $\vee$ :  $x \vee (y \vee z) = (x \vee y) \vee z$*
- *Associativity of  $\wedge$ :  $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*
- *Commutativity of  $\vee$ :  $x \vee y = y \vee x$*
- *Commutativity of  $\wedge$ :  $x \wedge y = y \wedge x$*
- *Distributivity of  $\wedge$  over  $\vee$ :  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*
- *0 is identity for  $\vee$ :  $x \vee 0 = x$*
- *1 is identity for  $\wedge$ :  $x \wedge 1 = x$*
- *0 is annihilator for  $\wedge$ :  $x \wedge 0 = 0$*

**Property** (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of  $\vee$ :  $x \vee x = x$*
- *Idempotence of  $\wedge$ :  $x \wedge x = x$*
- *Absorption of  $\vee$  over  $\wedge$ :  $x \vee (x \wedge y) = x \wedge y$*
- *Absorption of  $\wedge$  over  $\vee$ :  $x \wedge (x \vee y) = x \vee y$*
- *Distributivity of  $\vee$  over  $\wedge$ :  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*
- *1 is annihilator for  $\vee$ :  $x \vee 1 = 1$*

**Property** (De Morgan Laws).  $\neg(x \wedge y) = \neg x \vee \neg y$   $\neg(x \vee y) = \neg x \wedge \neg y$

*Proof.* Truth-tables; prove De Morgan, others as exercise (or just believe me)

□

## Other operators

**Definition** (Exclusive Or).  $x \oplus y$

**Definition** (Implication).  $x \implies y$

**Property** (Implication and Inclusion). *If  $\forall x \in X, P_1(x) \implies P_2(x)$ , then  $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$ .*

*Proof.* Trivial. □

**Definition** (If and only if).  $x \iff y$

## Negation of quantified propositions

**Property** (Negation of  $\forall$ ).  $\text{not}(\forall x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

**Property** (Negation of  $\exists$ ).  $\text{not}(\exists x \in X, P(x)) = \forall x \in X, \text{not}(P(x))$

**Notation** (Quantifiers and the empty set).  $\forall x \in \emptyset, \dots$  is true ;  $\exists x \in \emptyset, \dots$  is false

## 1.3 Python

=> use google colab'

# Chapter 2

## Proofs methods

### 2.1 Direct implication

Want to show  $A$ : may show  $B$  and  $B \implies A$ , or  $C$  and  $C \implies B$  and  $B \implies A$ .

### 2.2 Case dis-junction

Split in cases.

E.g.: show  $n$  and  $n^2$  have the same parity (take  $n$  odd then  $n$  even).

### 2.3 Contradiction

Suppose the opposite, derive a contradiction (i.e.  $A$  and  $\neg A$ ) and conclude.

E.g.: show  $\sqrt{2} \notin \mathbb{Q}$  (suppose  $\sqrt{2} = a/b$ , WLOG  $a, b \in \mathbb{N}$  co-prime).

### 2.4 Induction

Want to show  $P_n$  for  $n \geq n_0$ : show  $P_n \implies P_{n+1}$  and  $P_{n_0}$ .

E.g.: show  $\sum_{k=0}^n k = \frac{n(n+1)}{2}$  for all  $n \in \mathbb{N}$ .

### 2.5 Existence and Uniqueness

It is common to show existence and/or uniqueness.

E.g.: Existence and uniqueness in Euclidean division:

$$\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists! q \in \mathbb{Z}, r \in [0, b[ \cap \mathbb{N} \text{ s.t. } a = bq + r$$

Use  $q = \max\{k \in \mathbb{N} \mid bk \leq a\}$ ,  $r = a - bq$ .



## Chapter 3

# Functions Properties

$$f : X \rightarrow Y \quad A \subseteq X, B \subseteq Y$$

**Definition** (Image).  $f(A) = \{y \in Y \mid \exists x \in A \text{ s.t. } f(x) = y\}$

**Definition** (Inverse Image).  $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

**Definition** (Fiber). *Fiber of  $y$  is inverse image of  $\{y\}$ .*

**Definition** (Well definedness).  $\forall x \in X, \exists! y \in Y \text{ s.t. } f(x) = y$

**Definition** (Injectivity).  $\forall x, x' \in X, x \neq x', f(x) \neq f(x')$

**Definition** (Surjectivity).  $\forall y \in Y, \exists x \in X \text{ s.t. } f(x) = y$

**Definition** (Bijectivity). *Injectivity plus Surjectivity:  $\forall y \in Y, \exists! x \in X \text{ s.t. } f(x) = y$*

**Definition** (Invertibility).  $f^{-1} : Y \rightarrow X$  well defined.

**Remark** (Alternative Definition of Inverse).  $f \circ f^{-1} = Id \mid_X$  and  $f^{-1} \circ f = Id \mid_Y$

**Remark** (Invertibility and Bijectivity).  $f$  bijective  $\iff f$  invertible.

**Remark** (Inverse is Invertible).  $f^{-1}$  is invertible, and  $(f^{-1})^{-1} = f$ .

**Property** (Injections between finite intervals).  $m, n \in \mathbb{N}^*$ , there exists an injection  $f : \llbracket 1; m \rrbracket \rightarrow \llbracket 1; n \rrbracket$  if and only if  $m \leq n$ .

*Proof.* By induction on  $m$ , carefully checking  $m \leq n$ . □

**Property** (Bijections between finite intervals).  $n, m \in \mathbb{N}^*$ , there exists a bijection  $f : \llbracket 1; m \rrbracket \rightarrow \llbracket 1; n \rrbracket$  if and only if  $m = n$ .

*Proof.* Use last property & inverse. □

**Property** (Compositions). *Composition preserve injectivity/surjectivity/bijectivity/invertibility:*

$f : X \rightarrow Y, g : Y \rightarrow Z$  injectives  $\implies f \circ g$  is injective

$f : X \rightarrow Y, g : Y \rightarrow Z$  surjectives  $\implies f \circ g$  is surjective

$f : X \rightarrow Y, g : Y \rightarrow Z$  bijections/invertibles  $\implies f \circ g$  is bijective/invertible

*Proof.* Trivial. □

**Property.** *An injection between two sets of the same size is bijective.*

*Proof.* By contradiction. □

# Chapter 4

## Finite Cardinalities

**Definition** (Cardinality). *For finite sets:*

Intuitively:  $|X| = n \in \mathbb{N}$  if there are  $n$  elements in the set.

Mathematically:  $|X| = n \in \mathbb{N}$  if there is a bijection between  $X$  and  $\llbracket 1, n \rrbracket$ .

**Property** (Cardinality of Disjoints).  $X, Y$  disjoint sets:  $|X \cup Y| = |X| + |Y|$

*Extension:*  $X_1, \dots, X_n$  pairwise disjoint sets (i.e.  $X_i \cap X_j = \emptyset \forall i \neq j$ ):  $|\bigcup_{k=1}^n X_k| = \sum_{k=1}^n |X_k|$

*Proof.* Shift bijection of  $Y$  by  $|Y|$ ; use induction. □

**Property** (Cardinality of Complement).  $X \subseteq Y$ :  $|Y \setminus X| = |Y| - |X|$

*Proof.* Use previous property with  $X$  &  $Y \setminus X$  disjoint. □

**Property** (Cardinality of Cartesian Products).  $X, Y$  sets:  $|X \times Y| = |X| * |Y|$

*Extension:*  $X_1, \dots, X_n$  sets:  $|\prod_{k=1}^n X_k| = \prod_{k=1}^n |X_k|$

*Proof.*  $X \times \{y_k\}$  are all disjoint for  $k \in \llbracket 1, |Y| \rrbracket$ ; use induction. □

**Property** (Cardinality of Sets of Functions).  $|\{f : X \rightarrow Y\}| = |Y|^{|X|}$

*Proof.* Just count! □

**Property** (Cardinality of Sets of Injections).  $|\{f : X \rightarrow Y \mid f \text{ injective}\}| = \frac{|Y|!}{(|Y|-|X|)!}$

*Proof.* Count (without repetition). □

**Property** (Cardinality of Sets of Surjections).  $|\{f : X \rightarrow Y \mid f \text{ surjective}\}| = |Y|^{|X|} - |Y| * (|Y| - 1)^{|X|}$

*Proof.* All functions but the non surjective ones. □

**Property** (Cardinality of Sets of Bijections).  $|\{f : X \rightarrow Y \mid f \text{ bijective}\}| = |Y|! = |X|!$

*Proof.* Bijection is an injection between two sets of the same size. □

## Chapter 5

# Infinite Cardinalities

**Definition** (Alphabet).  $\mathcal{A} = \{a, b, c, \dots, z\}$

To compare the size of infinite sets, we use bijections, injections:

**Definition** (Comparing Sets).  $f : X \rightarrow Y$  *injective*  $\implies |X| \leq |Y|$   $f : X \rightarrow Y$  *surjective*  $\implies |X| \geq |Y|$   $f : X \rightarrow Y$  *bijective*  $\implies |X| = |Y|$

Note that together with  $|[1, n]| = n$ , this defines cardinality.

**Definition** (Countable sets). *A set is countable if it has the same cardinality as the naturals (i.e.  $X$  is countable if  $|X| = |\mathbb{N}|$ ).*

**Property** (Countable Union Finite).  $|\mathbb{N} \cup \mathcal{A}| = |\mathbb{N}|$

**Property** (Countable Union Countable / Integers).  $|\mathbb{Z}| = |\mathbb{N} \cup \mathbb{N}^*| = |\mathbb{N}|$

**Property** (Countable Union of Finites).  $|X_n| < \infty \ \forall n \in \mathbb{N} \implies |\bigcup_{n \in \mathbb{N}} X_n| = |\mathbb{N}|$

**Property** (Countable Union of Countables / Rationals).  $|\mathbb{Q}| = |\bigcup_{n \in \mathbb{N}^*} \{m/n \mid m \in \mathbb{Z}\}| = |\mathbb{N}|$

**Property** (Power set of Countables / Reals).  $|[0, 1[| = |\mathcal{P}(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}| > |\mathbb{N}|$

# Chapter 6

## Spaces

Mathematical Space: Object based on a set with more structure.

### 6.1 Metric Space

A metric space is a set  $X$  together with a metric distance  $d : X \times X \rightarrow \mathbb{R}^+$ .  
 $d$  is a metric if it satisfies the following axioms:

- Non-degenerative:  $d(x, y) = 0 \iff x = y$
- Symmetric:  $d(x, y) = d(y, x)$
- Triangle inequality:  $d(x, z) \leq d(x, y) + d(y, z)$

### 6.2 Norm Space

A norm space is a set  $X$  together with a norm  $|\_| : X \rightarrow \mathbb{R}^+$ .  
 $|\_|$  is a norm if it satisfies the following axioms:

- Non-degenerative:  $|x| = 0 \iff x = 0$
- Homogeneity:  $|\lambda x| = \lambda |x| \quad \lambda \in \mathbb{R}^+$
- Triangle inequality:  $|x + y| \leq |x| + |y|$

**Property** (Norm Implies Metric). *Letting  $d(x, y) = |x - y|$ .*

### 6.3 Inner Product Space

An inner product space is a set  $X$  together with an inner product  $\langle \_, \_ \rangle : X \times X \rightarrow \mathbb{C}$ .  
 $\langle \_, \_ \rangle$  is an inner product if it satisfies the following axioms:

- Linear (in 1<sup>st</sup> argument):  $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \lambda \in \mathbb{C}$  and  $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$
- Conjugate symmetry:  $|x + y| \leq |x| + |y|$
- Positive definiteness  $\langle x, x \rangle > 0 \ \forall x \neq 0$
- (*implied*) Non-degenerative:  $\langle x, 0 \rangle = 0$  and  $\langle 0, x \rangle = 0$
- (*implied*) Conjugate linear (in 2<sup>nd</sup> argument):  $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle \quad \lambda \in \mathbb{C}$  and  $\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle$

**Property** (Inner Product implies Norm). *Letting  $|x| = \sqrt{\langle x, x \rangle}$ .*

**Property** (Pythagoras Theorem).  $\langle x, y \rangle = 0 \implies |x + y|^2 = |x|^2 + |y|^2$

**Property** (Parallelogram Identity).  $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$

**Property** (Polarization Identity).  $4 \langle x, y \rangle = |x + y|^2 - |x - y|^2 + i(|x + iy|^2 - |x - iy|^2)$