Problem Set 4

Due 22$^{\text{nd}}$ September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Sequences

Question 1. *$X, Norm(\_)$ is a norm space; (x_n) is a sequence in X. Define what is meant by $x_n \to x$ in this case (remember we defined it in class for X, d a metric space, and we also saw how to define a metric space from a norm space).*

Question 2. *(Arithmetic Sequence) (x_n) is defined by iteration as follows: $x_0 = b$, $x_{n+1} = a + x_n$.*
a) Prove that the explicit formula for x_n is $x_n = b + a * n$.*
b) State and prove the behavior as $n \to +\infty$ (split in cases for different values of a or b if needed).*

Question 3. *(Geometric Sequence) (x_n) is defined by iteration as follows: $x_0 = b$, $x_{n+1} = a * x_n$.*
a) Prove that the explicit formula for x_n is $x_n = b * a^n$.*
b) State and prove the behavior as $n \to +\infty$ (split in cases for different values of a or b if needed).*

Question 4. *The Syracuse sequence is defined by the recurrence relation as follows:*

- $x_{n+1} = \frac{x_n}{2}$ *if x_n is even*
- $x_{n+1} = 3 * x_n + 1$ *if x_n is odd*

We start with a natural number (note that the sequence takes only integral values).
a) Conjecture how many iterations it takes to reach 1 starting from 7,8,15,16.
b) Calculate how many iterations it takes to reach 1 starting from 7,8,15,16.
c) Was your first intuition correct?
(You may want to use a calculator to speed up calculations)

Question 5. *(*) Let $f(x) = \frac{2x^2-3x+9}{4x-5}$*
Find the asymptote as $x \to \pm\infty$; Find the singularity of f.
Draw a graph of f to the best of your knowledge, using the above information.

Question 6. *(*) Let $x_{n+1} = \frac{1}{2}x_n + 1$, $x_0 = 0$.*
Show that the general term of this sequence is given by $x_n = 2 - \frac{2}{2^n}$.
Deduce the limit of the sequence.

Question 7. *Find a set X such that $\forall x, y \in X, d(x, y) < Diam(X)$ (i.e. $Diam(X) = \sup(\{d(x, y) \mid x, y \in X\})$ but $Diam(X) \neq \max(\{d(x, y) \mid x, y \in X\})$).*
(X, d can be a metric space of your choice, but I advise $X = \mathbb{R}$, $d(x, y) = |x - y|$ to begin with)