Refresher Math Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. The course will try to cover most of the prerequisites of the courses in the Master, mainly linear algebra, differential calculus, integration, and asymptotic analysis.

Contents

Introduction

Presentation

- Paul Dubois
- will be teaching this refresher math course
- email (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h
- 1h20min lecture - $1/3$h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)
- Lectures are recorded (if ever needed)
- 1st lecture ever => too fast/too slow: let me know
- May assume you know a concept/notation that you have never heard of, let me know if this happens

Examination

- The course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be a full exercise sheet per lecture, it is advised to attempt it all (only one will be compulsory).
- Hand-in 1 exercise per lecture (i.e., 8 in total), due 2 weeks after the lecture
- Best $(n-1)/n$ count (i.e., best $7/8$ in our case), need avg $\geq 50\%$ to pass
- In the unlikely event of not passing, will be able to do an extra work

Questions?

Chapter 1

Sets & logic

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $\in$, $\{True, False\}$, $\{a \mid condition\}$, $\{a, b, c \dots\}$, $\emptyset$

Need to be careful when defining set: some definitions are pathological.

Remark (Russell Paradox). *Take* $U = \{X \mid X \notin X\}$. *X in U => U not in U, U is a set, so not all sets are in U X not in U => X is a set*

Notation (Usual Sets). $\mathbb{B}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}^*$, $\mathbb{R}^+$...

Functions

Definition (Functions). *Assignment for a set to another.*

Notation (Function). $f : X \to Y$, $f(x) = blah$, $f : x \mapsto blah$.

Definition (Predicate). *Function to* $\mathbb{B}$

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation ($\forall$). *For all elements in set, e.g.:* $\forall x \in \mathbb{R}, x^2 \geq 0$.

Notation ($\exists$). *There exists an element in set, e.g.:* $\exists x \in \mathbb{R}$ *s.t.* $x^2 > 1$.

Notation ($\exists!$). *There exists a unique element in set, e.g.:* $\exists! x \in \mathbb{R}$ *s.t.* $x^2 \leq 0$.

Definition (Subset / Inclusion). $X \subseteq Y$ *if* $\forall x \in X, x \in Y$

Definition (Disjoint Sets). *X and Y are disjoint if $\forall x \in X, x \notin Y$ (or if $\forall y \in Y, y \notin X$).*

Definition (Powerset). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Definition (Cartesian Product). $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$
e.g.: $\{a,b\} \times \{1,2,3\} = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

1.2 Boolean algebra

Basic operators

Definition (Conjonction). $x \wedge y = xy$

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

Definition (Disjunction). $x \vee y = \min(x+y, 1)$

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation). $\neg : 0,1 \mapsto 1,0$

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

Question. *Selecting points outside a given region.*

Basic properties

Property 1.1 (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of $\vee$:* $x \vee (y \vee z) = (x \vee y) \vee z$
- *Associativity of $\wedge$:* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- *Commutativity of $\vee$:* $x \vee y = y \vee x$
- *Commutativity of $\wedge$:* $x \wedge y = y \wedge x$
- *Distributivity of $\wedge$ over $\vee$:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- 0 *is identity for $\vee$:* $x \vee 0 = x$
- 1 *is identity for $\wedge$:* $x \wedge 1 = x$
- 0 *is annihilator for $\wedge$:* $x \wedge 0 = 0$

Property 1.2 (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of $\vee$:* $x \vee x = x$
- *Idempotence of $\wedge$:* $x \wedge x = x$
- *Absorption of $\vee$ over $\wedge$:* $x \vee (x \wedge y) = x \wedge y$
- *Absorption of $\wedge$ over $\vee$:* $x \wedge (x \vee y) = x \vee y$
- *Distributivity of $\vee$ over $\wedge$:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 *is annihilator for $\vee$:* $x \vee 1 = 1$

Property 1.3 (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) $\square$

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property 1.4 (Implication and Inclusion). *If* $\forall x \in X, P_1(x) \implies P_2(x)$*, then* $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.

Proof. Trivial. $\square$

Definition (If and only if). $x \iff y$

Negation of quantified propositions

Property 1.5 (Negation of $\forall$). $\text{not}(\forall x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Property 1.6 (Negation of $\exists$). $\text{not}(\exists x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \ \dots$ *is true ;* $\exists x \in \emptyset, \ \dots$ *is false*

1.3 Python

=> use google colab'

Chapter 2

Proofs methods

2.0.1 Direct implication

Want to show A: may show B and $B \implies A$, or C and $C \implies B$ and $B \implies A$.

2.0.2 Case dis-junction

Split in cases.

E.g.: show n and n^2 have the same parity (take n odd then n even).

2.0.3 Contradiction

Suppose the opposite, derive a contradiction (i.e. A and $\not A$) and conclude.

E.g.: show $\sqrt{2} \notin \mathbb{Q}$ (suppose $\sqrt{2} = {}^a/_b$, WLOG $a, b \in \mathbb{N}$ co-prime).

2.0.4 Induction

Want to show P_n for $n \geq n_0$: show $P_n \implies P_{n+1}$ and P_{n_0}.

E.g.: show $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

2.0.5 Existence and Uniqueness

It is common to show existence and/or uniqueness.

E.g.: Existence and uniqueness in Euclidean division:

$$\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists!\ q \in \mathbb{Z}, r \in [0, b[\cap \mathbb{N} \text{ s.t. } a = bq + r$$

Use $q = \max\{k \in \mathbb{N} \mid bk \leq a\}$, $r = a - bq$.

Chapter 3

Functions Properties

$$f : X \to Y \quad A \subseteq X, B \subseteq Y$$

Definition (Image)**.** $f(A) = \{y \in Y \mid \exists x \in A \ s.t. \ f(x) = y\}$

Definition (Inverse Image)**.** $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

Definition (Fiber)**.** *Fiber of* y *is inverse image of* $\{y\}$*.*

Definition (Well definedness)**.** $\forall x \in X, \exists! y \in Y \ s.t. \ f(x) = y$

Definition (Injectivity)**.** $\forall x, x' \in X, x \neq x', f(x) \neq f(x')$

Definition (Surjectivity)**.** $\forall y \in Y, \exists x \in X \ s.t. \ f(x) = y$

Definition (Bijectivity)**.** *Injectivity plus Surjectivity:* $\forall y \in Y, \exists! x \in X \ s.t. \ f(x) = y$

Definition (Invertibility)**.** $f^{-1} : Y \to X$ *well defined.*

Remark (Alternative Definition of Inverse)**.** $f \circ f^{-1} = Id\mid_X$ *and* $f^{-1} \circ f = Id\mid_Y$

Remark (Invertibility and Bijectivity)**.** f *bijective* $\iff$ f *invertible.*

Remark (Inverse is Invertible)**.** f^{-1} *is invertible, and* $(f^{-1})^{-1} = f$*.*

Property 3.1 (Injections between finite intervals)**.** $m, n \in \mathbb{N}^*$*, there exists an injection* $f : [\![1; m]\!] \to [\![1; n]\!]$ *if and only if* $m \leq n$*.*

Proof. By induction on m, carefully checking $m \leq n$. □

Property 3.2 (Bijections between finite intervals)**.** $n, m \in \mathbb{N}^*$*, there exists a bijection* $f : [\![1; m]\!] \to [\![1; n]\!]$ *if and only if* $m = n$*.*

Proof. Use last property & inverse. □

Property 3.3 (Compositions)**.** *Composition preserve injectivity/surjectivity/bijectivity/invertibility:*
$f : X \to Y, \ g : Y \to Z$ *injectives* $\implies f \circ g$ *is injective*
$f : X \to Y, \ g : Y \to Z$ *surjectives* $\implies f \circ g$ *is surjective*
$f : X \to Y, \ g : Y \to Z$ *bijections/invertibles* $\implies f \circ g$ *is bijective/invertible*

Proof. Trivial. □

Property 3.4. *An injection between two sets of the same size is bijective.*

Proof. By contradiction. □

Chapter 4

Finite Cardinalities

Definition (Cardinality). *For finite sets:*
Intuitively: $|X| = n \in \mathbb{N}$ *if there are* n *elements in the set.*
Mathematically: $|X| = n \in \mathbb{N}$ *if there is a bijection between* X *and* $[\![1, n]\!]$.

Property 4.1 (Cardinality of Disjoints). X, Y *disjoint sets:* $|X \cup Y| = |X| + |Y|$
Extension: $X_1, \dots, X_n$ *pairwise disjoint sets (i.e.* $X_i \cap X_j = \emptyset\ \forall i \neq j$*):* $|\bigcup_{k=1}^{n} X_k| = \sum_{k=1}^{n} |X_k|$

Proof. Shift bijection of Y by $|Y|$; use induction. □

Property 4.2 (Cardinality of Complement). $X \subseteq Y$*:* $|Y \setminus X| = |Y| - |X|$

Proof. Use previous property with X & $Y \setminus X$ disjoint. □

Property 4.3 (Cardinality of Cartesian Products). X, Y *sets:* $|X \times Y| = |X| * |Y|$
Extension: $X_1, \dots, X_n$ *sets:* $|\prod_{k=1}^{n} X_k| = \prod_{k=1}^{n} |X_k|$

Proof. $X \times \{y_k\}$ are all disjoint for $k \in [\![1, |Y|]\!]$; use induction. □

Property 4.4 (Cardinality of Sets of Functions). $|\{f : X \to Y\}| = |Y|^{|X|}$

Proof. Just count! □

Property 4.5 (Cardinality of Sets of Injections). $|\{f : X \to Y \mid f \text{ injective}\}| = \frac{|Y|!}{(|Y|-|X|)!}$

Proof. Count (without repetition). □

Property 4.6 (Cardinality of Sets of Surjections). $|\{f : X \to Y \mid f \text{ surjective}\}| = |Y|^{|X|} - |Y| * (|Y| - 1)^{|X|}$

Proof. All functions but the non surjective ones. □

Property 4.7 (Cardinality of Sets of Bijections). $|\{f : X \to Y \mid f \text{ bijective}\}| = |Y|! = |X|!$

Proof. Bijection is an injection between two sets of the same size. □

Chapter 5

Infinite Cardinalities

Definition (Alphabet). $\mathcal{A} = \{a, b, c, \dots, z\}$

To compare the size of infinite sets, we use bijections, injections:

Definition (Comparing Sets). $f : X \to Y$ *injective* $\implies |X| \leq |Y|$ $f : X \to Y$ *surjective* $\implies |X| \geq |Y|$ $f : X \to Y$ *bijective* $\implies |X| = |Y|$

Note that together with $|\,[1, n]\,| = n$, this defines cardinality.

Definition (Countable sets). *A set is countable if it has the same cardinality as the naturals (i.e. X is countable if $|X| = |\mathbb{N}|$).*

Property 5.1 (Countable Union Finite). $|\mathbb{N} \cup \mathcal{A}| = |\mathbb{N}|$

Property 5.2 (Countable Union Countable / Integers). $|\mathbb{Z}| = |\mathbb{N} \cup \mathbb{N}^*| = |\mathbb{N}|$

Property 5.3 (Countable Union of Finites). $|X_n| < \infty\ \forall n \in \mathbb{N} \implies |\bigcup_{n \in \mathbb{N}} X_n| = |\mathbb{N}|$

Property 5.4 (Countable Union of Countables / Rationals). $|\mathbb{Q}| = |\bigcup_{n \in \mathbb{N}^*} \{m/n \mid m \in \mathbb{Z}\}| = |\mathbb{N}|$

Property 5.5 (Power set of Countables / Reals). $|\,[0, 1[\,| = |\mathcal{P}(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}| > |\mathbb{N}|$