Problem Set 1

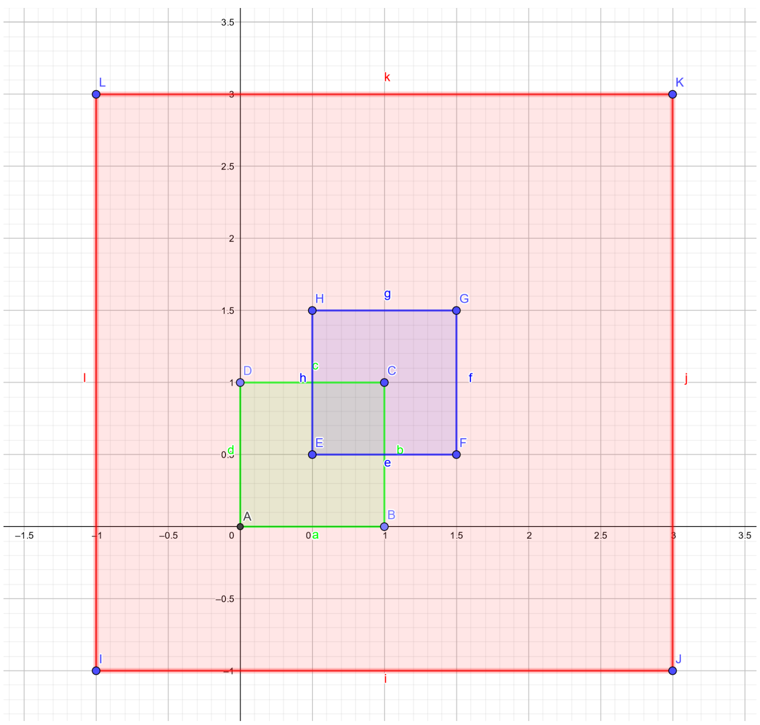
Due 9^{th} September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Sets & Logic

1.1 Sets



Question 1 (*). *We call R the set of points in the red square, B for the ones in the blue square, and G for the green one.*

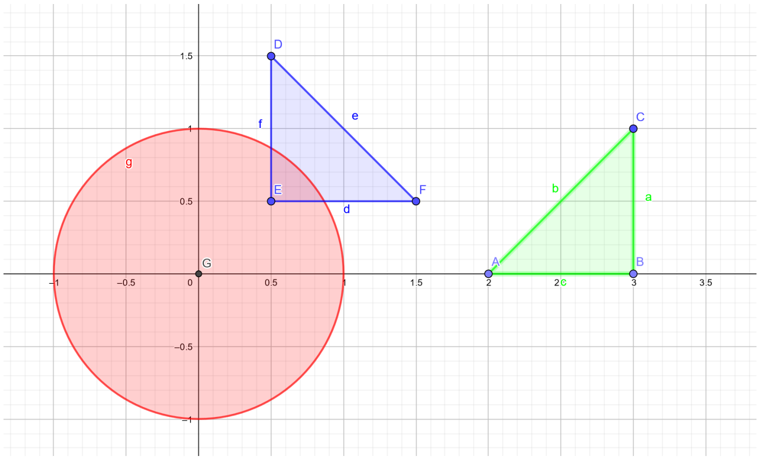
a. *Express R, G, and B in terms of Cartesian product.*[1]

b. *Give all (if any) the subsets relations between R, G, and B.*

c. *Express $G \cup B \setminus (G \cap B)$ in terms of Cartesian product.*

d. *Express $[1/2, 1]^2$ in terms of R, G, and B (using intersections, unions, ...).*

e. *Express $[1/2, 3/2] \times [1, 3/2] \cup [1, 3/2] \times [1/2, 3/2]$ in terms of R, G, and B.*



Question 2. *We call R the set of points in the red disc, B for the ones in the blue triangle, and G for the green one.*

a. *(*) Express R, G, and B using set notation with predicates (i.e. $\{object \mid condition\}$) of Cartesian product.*[2]

b. *Express $R \cap B$ using set notation.*

c. *(*) Draw $R \cap \{(x, y) \mid x \leq 0, y \leq 0\}$.*

d. *The upper half plane $\mathcal{H}$ is $\{(x, y) \mid y > 0\}$; hatch it on your figure.*

e. *Which one(s) (if any) of R, G, B is contained in $\mathcal{H}$?*

1.2 Boolean Algebra

Question 3. *Here, $a, b, c, d \in \mathbb{B}$ are boolean numbers.*

a. *(*) Show that $abc + \bar{a}bc + a\bar{b}c + ab\bar{c} = bc + ac + ab$*

b. *(*) Check that $abc + \bar{a}bc + a\bar{b}c + ab\bar{c} = bc + ac + ab$ using truth tables*

c. *Show that $abc + ab\bar{c} + a\bar{b}cd = ab + acd$*

Question 4. *Here again, $a, b, c \in \mathbb{B}$ are boolean numbers. One wants to add them, and display the result in base 2 using two LEDs (one for the units, one for the twos). The complete truth table is given below; find expressions for the units and twos.*

a	b	c	tows of $a+b+c$	units of $a+b+c$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

[1] G is in fact called the unit square in the first quadrant.
[2] R is in fact called the unit disc.

1.3 Quantified propositions

Question 5 (*). *Negate the following*[3]*:* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.* $\forall n > N, |x_n - x| < \epsilon$

Question 6. *Find the quantified notation of the following sentences:*

a. *(*) "Given a number, it always possible to find another one that is greater"*

b. *(*) "Any natural number is non-negative"*

c. *"There is no negative square"*[4]

d. *"There exists a bijection between the naturals and the set of all fractions"*

2 Modular Arithmetic

This is not needed for the rest of the course, but is nice to know Read the first two sections of `https://en.wikipedia.org/wiki/Modular_arithmetic`.

Question 7. a. *Show that* n^2 *is divisible by 3 if and only if* n *is divisible by 3.*

b. *Show that* n^2 *is divisible by 5 if and only if* n *is divisible by 5.*

c. *(harder) Show that* n^2 *is divisible by* p *if and only if* n *is divisible by* p *for any prime* p.[5]

3 Proofs Methods

Question 8. a. *(*) Show that* n *divisible by 6 if and only if* n *divisible by 2 and 3.*

b. *Show* $\sqrt{3} \notin \mathbb{Q}$.[6]

c. *Show that* $12N - 6$ *is divisible by 6 for every positive integer* n.

d. *Show that* $2^n \geq 2n$ *for all* $n \in \mathbb{N}$

4 Functions Properties

Question 9. a. *(*)* $f : X \to Y,\ g : Y \to Z$ *injectives* $\implies g \circ f$ *is injective*

b. $f : X \to Y,\ g : Y \to Z$ *surjectives* $\implies g \circ f$ *is surjective*

c. $f : X \to Y,\ g : Y \to Z$ *bijectives/invertibles* $\implies g \circ f$ *is bijective/invertible*

[3]This is in fact the definition of x_n converging to x ($x_n \to x$).
[4](in the reals)
[5]Hint: look for Euclid's lemma.
[6]See congruence on Wikipedia