Problem Set 3

Due 22^{nd} September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Metric/Norm Spaces

Question 1. *(*) $X, Norm(\_)$ is a norm space; we define $d(x, y) = Norm(x - y)$, so that X, d is a metric space. We have seen what the ball of radius r centered at x is in a metric space (i.e. using d). Define it in terms of then norm.*

Question 2. *(*) We take $X = \mathbb{R}^2$, draw $B_1((0, 0))$ (the unit ball) for the following norms:*

- $Norm((x, y)) = |x| + |y|$
- $Norm((x, y)) = \sqrt{x^2 + y^2}$
- $Norm((x, y)) = \max(|x|, |y|)$
- $Norm((x, y)) = 0$ *if* $(x, y) = (0, 0)$, $Norm((x, y)) = 1$ *else*
- $Norm((x, y)) = 0$ *if* $(x, y) = (0, 0)$, $Norm((x, y)) = 2$ *else*

Question 3. *For which values of $p \geq 0$ is $Norm((x, y)) = \sqrt[p]{|x|^p + |y|^p}$ a norm for $\mathbb{R}^2$?*

Question 4. *Find two different norms for $\mathbb{R}^n$ (give their definitions).*

2 Sequences Convergence

Question 5. *Find the limit of the following sequences, and prove your claim carefully:*

- *(*)* $x_n = \frac{1}{n^2}$
- $x_n = \frac{1}{n^3}$
- $x_n = \frac{1}{n^k}$ *for some* $k > 0$
- *(*)* $x_n = \frac{1}{2^n}$
- $x_n = \frac{1}{3^n}$
- $x_n = \frac{1}{k^n}$ *for some* $k > 1$
- *(*)* $x_n = \frac{3n^2+2n+1}{6n^2-n+2}$
- $x_n = \frac{2n+1}{6n^2+9n-5}$
- *(hard)* $x_n = \frac{n}{2^n}$

Question 6. $x_n \to x$, $y_n \to y$; *Prove the following:*

- *(*)* $x_n + y_n \to x + y$
- $x_n * y_n \to x * y$
- *if* $y_n \neq 0$, *and* $y \neq 0$*:* $x_n/y_n \to x/y$