

Problem Set 4

Due 24nd September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Sequences

Question 1. $X, \text{Norm}(_)$ is a norm space; (x_n) is a sequence in X . Define what is meant by $x_n \rightarrow x$ in this case (remember we defined it in class for X, d a metric space, and we also saw how to define a metric space from a norm space).

Question 2. (Arithmetic Sequence) (x_n) is defined by iteration as follows: $x_0 = b$, $x_{n+1} = a + x_n$.
a*) Prove that the explicit formula for x_n is $x_n = b + a * n$.

b*) State and prove the behavior as $n \rightarrow +\infty$ (split in cases for different values of a or b if needed).

Question 3. (Geometric Sequence) (x_n) is defined by iteration as follows: $x_0 = b$, $x_{n+1} = a * x_n$.

a*) Prove that the explicit formula for x_n is $x_n = b * a^n$.

b*) State and prove the behavior as $n \rightarrow +\infty$ (split in cases for different values of a or b if needed).

Question 4. The Syracuse sequence is defined by the recurrence relation as follows:

- $x_{n+1} = \frac{x_n}{2}$ if x_n is even
- $x_{n+1} = 3 * x_n + 1$ if x_n is odd

We start with a natural number (note that the sequence takes only integral values).

a) Conjecture how many iterations it takes to reach 1 starting from 7,8,15,16.

b) Calculate how many iterations it takes to reach 1 starting from 7,8,15,16.

c) Was your first intuition correct?

(You may want to use a calculator to speed up calculations)

Question 5. (*) Let $f(x) = \frac{2x^2-3x+9}{4x-5}$

Find the asymptote as $x \rightarrow \pm\infty$; Find the singularity of f .

Draw a graph of f to the best of your knowledge, using the above information.

Question 6. (*) Let $x_{n+1} = \frac{1}{2}x_n + 1$, $x_0 = 0$.

Show that the general term of this sequence is given by $x_n = 2 - \frac{2}{2^n}$.

Deduce the limit of the sequence.

Question 7. Find a set X such that $\forall x, y \in X, d(x, y) < \text{Diam}(X)$ (i.e. $\text{Diam}(X) = \sup(\{d(x, y) \mid x, y \in X\})$ but $\text{Diam}(X) \neq \max(\{d(x, y) \mid x, y \in X\})$).

(X, d can be a metric space of your choice, but I advise $X = \mathbb{R}$, $d(x, y) = |x - y|$ to begin with)