Problem Set 2

Due 14th September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Functions Properties

Question 1. *(*) Show that an injection between two sets of the same size is bijective.*

2 Finite Cardinalities

Question 2 (Counting the number of functions between two finite sets)**.** *Let X and Y be two non-empty finite sets. We want to count the number of functions in Y^X.*

a. Let us assume that $|X| = n$ with $X = \{x_1, ..., x_n\}$. Prove that the function

$$\Phi : \begin{array}{ccc} Y^X & \longrightarrow & Y^n \\ f & \longmapsto & (f(x_1), ..., f(x_n)) \end{array}$$

is a bijection.

b. Deduce the value of $|Y^X|$.

c. Let $n \in \mathbb{N}^*$. Let us consider the set $\mathfrak{S}_n \subset [\![1;n]\!]^{[\![1;n]\!]}$ containing the bijections from $[\![1;n]\!]$ to itself. Prove that the sequence $(|(\mathfrak{S}_n)|)_{n\in\mathbb{N}^*}$ is defined by the recurrence relation

$$\begin{cases} |\mathfrak{S}_1| = 1 \\ \forall n \in \mathbb{N},\ |\mathfrak{S}_{n+1}| = (n+1)|\mathfrak{S}_n| \end{cases}$$

(hint : we can use the bijections $\forall k \in [\![1;n]\!],\ g_k : [\![1;n]\!] \setminus \{k\} \to [\![1;n-1]\!]$)

d. The cardinal of $\mathfrak{S}_n$ is the factorial of n, denoted as $n!$. Write a function returning the value of the factorial for a given $n \in \mathbb{N}$ (by convention $0! = 1$).

Question 3 (Counting the number of sub-parts)**.** *(*) We study the function $(n,p) \in \mathbb{N}^2 \mapsto \binom{n}{k} \in \mathbb{N}$ the binomial coefficient, which is the number of subsets containing p elements in a set containing n elements.*

a. Prove that $\forall n, p \geq 1,\ \binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}$.

b. Deduce from this recurrence equation that

$$\forall n \in \mathbb{N},\ \forall p \leq n,\ \binom{n}{p} = \frac{n!}{(n-p)!p!}$$

c. Prove the formula

$$\forall n \in \mathbb{N}, \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

d. Derive the value of $|\mathcal{P}([\![1;n]\!])|$ (power set).

3 Infinite Cardinalities

Question 4. *(*) Find an explicit bijection between* $[a, b]$ *and* $[c, d]$

Question 5. *(*) Show that there is a bijection between* $[0, 1[$ *and* $\{0, 1\}^{\mathbb{N}}$

Question 6. *(*) Let* $\mathcal{A} = \{a, b, c, \ldots, z\}$ *be the set of letters in the alphabet. Show explicitly that* $|\mathcal{A} \cup \mathbb{N}| = |\mathbb{N}|$.

Question 7. *Let* $\mathbb{C}$ *be the set of complex numbers. Compare* $\mathbb{R}$ *and* $\mathbb{C}$.

Question 8. *Let* $\mathbb{P}$ *be the set of prime numbers. Compare* $\mathbb{P}$ *with the usual sets (in particular with* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$*).*