Refresher Math Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. The course will try to cover most of the prerequisites of the courses in the Master, mainly linear algebra, differential calculus, integration, and asymptotic analysis.

Contents

Introduction

Presentation

- Paul Dubois
- Email: b00795695@essec.edu (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h arranged as 1h20min lecture - $^1/_3$h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)

Examination

- The course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)
- Best $^{(n-1)}/_n$ count, need average $\geq 70\%$ to pass
- In the unlikely event of not passing, you will be able to do some extra work to pass

Questions?

Chapter 1

Sets & logic

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $\in$, $\{True, False\}$, $\{a \mid condition\}$, $\{a, b, c \dots\}$, $\emptyset$

Need to be careful when defining set: some definitions are pathological.

Remark (Russell Paradox). *Take* $Y = \{x \mid x \notin x\}$. $Y \in Y \iff Y \notin Y$

Notation (Usual Sets). $\mathbb{B}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}^*$, $\mathbb{R}^+$*...*

Functions

Definition (Functions). *Assignment for a set to another.*

Notation (Function). $f : X \to Y$, $f(x) = blah$, $f : x \mapsto blah$.

Definition (Predicate). *Function to* $\mathbb{B}$

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation ($\forall$). *For all elements in set, e.g.:* $\forall x \in \mathbb{R}, x^2 \geq 0$.

Notation ($\exists$). *There exists an element in set, e.g.:* $\exists x \in \mathbb{R}$ *s.t.* $x^2 > 1$.

Notation ($\exists!$). *There exists a unique element in set, e.g.:* $\exists! x \in \mathbb{R}$ *s.t.* $x^2 \leq 0$.

Definition (Subset / Inclusion). $X \subseteq Y$ *if* $\forall x \in X, x \in Y$

Definition (Disjoint Sets). X *and* Y *are disjoint if* $\forall x \in X, x \notin Y$ *(or if* $\forall y \in Y, y \notin X$*).*

Definition (Powerset). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Definition (Cartesian Product). $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
e.g.: $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

1.2 Boolean algebra

Basic operators

Definition (Conjonction)**.** $x \wedge y = xy$

Definition (Intersection)**.** $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection)**.** *Disjoint sets have empty intersection.*

Definition (Disjunction)**.** $x \vee y = \min(x + y, 1)$

Definition (Union)**.** $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation)**.** $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement)**.** $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

Question. *Selecting points outside a given region.*

Basic properties

Property (Boolean algebra matching ordinary algebra)**.** *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$*
- *Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*
- *Commutativity of $\vee$: $x \vee y = y \vee x$*
- *Commutativity of $\wedge$: $x \wedge y = y \wedge x$*
- *Distributivity of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*
- 0 *is identity for $\vee$: $x \vee 0 = x$*
- 1 *is identity for $\wedge$: $x \wedge 1 = x$*
- 0 *is annihilator for $\wedge$: $x \wedge 0 = 0$*

Property (Boolean algebra specific properties)**.** *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of $\vee$: $x \vee x = x$*
- *Idempotence of $\wedge$: $x \wedge x = x$*
- *Absorption of $\vee$ over $\wedge$: $x \vee (x \wedge y) = x \wedge y$*
- *Absorption of $\wedge$ over $\vee$: $x \wedge (x \vee y) = x \vee y$*
- *Distributivity of $\vee$ over $\wedge$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*
- 1 *is annihilator for $\vee$: $x \vee 1 = 1$*

Property (De Morgan Laws)**.** $\neg(x \wedge y) = \neg x \vee \neg y$ $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) $\square$

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property (Implication and Inclusion). *If* $\forall x \in X, P_1(x) \implies P_2(x)$*, then* $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.

Proof. Trivial. □

Definition (If and only if). $x \iff y$

Question. *Express in terms of and, or, not:*

- $\oplus$
- $\implies$
- $\impliedby$
- $\iff$

Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.

Negation of quantified propositions

Property (Negation of $\forall$). $\text{not}(\forall x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Property (Negation of $\exists$). $\text{not}(\exists x \in X, P(x)) = \forall x \in X, \text{not}(P(x))$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \ \ldots$ *is true ;* $\exists x \in \emptyset, \ \ldots$ *is false*

Question. *Negate the following* ($x_n \to x$)*:* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.* $\forall n > N, |x_n - x| < \epsilon$

Chapter 2

Proofs methods

2.1 Direct implication

Want to show A: may show B and $B \implies A$, or C and $C \implies B$ and $B \implies A$.

2.2 Case dis-junction

Split in cases.

E.g.: show n and n^2 have the same parity (take n odd then n even).

2.3 Contradiction

Suppose the opposite, derive a contradiction (i.e. A and $\not A$) and conclude.

E.g.: show $\sqrt{2} \notin \mathbb{Q}$ (suppose $\sqrt{2} = {}^a/_b$, WLOG $a, b \in \mathbb{N}$ co-prime).

2.4 Induction

Want to show P_n for $n \geq n_0$: show $P_n \implies P_{n+1}$ and P_{n_0}.

E.g.: show $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

2.5 Existence and Uniqueness

It is common to show existence and/or uniqueness.

E.g.: Existence and uniqueness in Euclidean division:

$$\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists!\ q \in \mathbb{Z}, r \in [0, b[\cap \mathbb{N} \text{ s.t. } a = bq + r$$

Use $q = \max\{k \in \mathbb{N} \mid bk \leq a\}$, $r = a - bq$.

Question.

- *Show that n divisible by 6 if and only if n divisible by 2 and 3.*
- *Show $\sqrt{3} \notin \mathbb{Q}$.*[1]
- *Show that $12n - 6$ is divisible by 6 for every positive integer n.*
- *Show that $2^n \geq 2n$ for all $n \in \mathbb{N}$*

[1] See `https://en.wikipedia.org/wiki/Modular_arithmetic` and use it to show $3 \mid n^2 \implies 3 \mid n$.

Chapter 3

Functions Properties

$$f : X \to Y \quad A \subseteq X, B \subseteq Y$$

Definition (Image). $f(A) = \{y \in Y \mid \exists x \in A \ s.t.\ f(x) = y\}$

Definition (Inverse Image). $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

Definition (Fiber). *Fiber of y is inverse image of $\{y\}$.*

Definition (Well definedness). $\forall x \in X, \exists! y \in Y \ s.t.\ f(x) = y$

Definition (Injectivity). $\forall x, x' \in X, x \neq x', f(x) \neq f(x')$

Definition (Surjectivity). $\forall y \in Y, \exists x \in X \ s.t.\ f(x) = y$

Definition (Bijectivity). *Injectivity plus Surjectivity:* $\forall y \in Y, \exists! x \in X \ s.t.\ f(x) = y$

Definition (Invertibility). $f^{-1} : Y \to X$ *well defined.*

Remark (Alternative Definition of Inverse). $f \circ f^{-1} = Id \mid_X$ *and* $f^{-1} \circ f = Id \mid_Y$

Remark (Invertibility and Bijectivity). f *bijective* $\iff$ f *invertible.*

Remark (Inverse is Invertible). f^{-1} *is invertible, and* $(f^{-1})^{-1} = f$.

Property (Injections between finite intervals). $m, n \in \mathbb{N}^*$, *there exists an injection* $f : [\![1; m]\!] \to [\![1; n]\!]$ *if and only if* $m \leq n$.

Proof. By induction on m, carefully checking $m \leq n$. □

Property (Bijections between finite intervals). $n, m \in \mathbb{N}^*$, *there exists a bijection* $f : [\![1; m]\!] \to [\![1; n]\!]$ *if and only if* $m = n$.

Proof. Use last property & inverse. □

Property (Compositions). *Composition preserve injectivity/surjectivity/bijectivity/invertibility:*
$f : X \to Y,\ g : Y \to Z$ *injectives* $\implies g \circ f$ *is injective*
$f : X \to Y,\ g : Y \to Z$ *surjectives* $\implies g \circ f$ *is surjective*
$f : X \to Y,\ g : Y \to Z$ *bijectives/invertibles* $\implies g \circ f$ *is bijective/invertible*

Proof. Trivial. □

Property. *An injection between two sets of the same size is bijective.*

Proof. By contradiction. □

Chapter 4

Finite Cardinalities

Definition (Cardinality). *For finite sets:*
Intuitively: $|X| = n \in \mathbb{N}$ *if there are* n *elements in the set.*
Mathematically: $|X| = n \in \mathbb{N}$ *if there is a bijection between* X *and* $[\![1, n]\!]$.

Property (Cardinality of Disjoints). X, Y *disjoint sets:* $|X \cup Y| = |X| + |Y|$
Extension: $X_1, \ldots, X_n$ *pairwise disjoint sets (i.e.* $X_i \cap X_j = \emptyset\ \forall i \neq j$*):* $|\bigcup_{k=1}^{n} X_k| = \sum_{k=1}^{n} |X_k|$

Proof. Shift bijection of Y by $|Y|$; use induction. □

Property (Cardinality of Complement). $X \subseteq Y$*:* $|Y \setminus X| = |Y| - |X|$

Proof. Use previous property with X & $Y \setminus X$ disjoint. □

Property (Cardinality of Cartesian Products). X, Y *sets:* $|X \times Y| = |X| * |Y|$
Extension: $X_1, \ldots, X_n$ *sets:* $|\prod_{k=1}^{n} X_k| = \prod_{k=1}^{n} |X_k|$

Proof. $X \times \{y_k\}$ are all disjoint for $k \in [\![1, |Y|]\!]$; use induction. □

Property (Cardinality of Sub-list). X *sets:* $|\{Y \text{ list} \mid |Y| = n \text{ and } y \in Y \implies y \in X\}| = |X|^n$

Proof. Just count! □

Property (Cardinality of Ordered Subsets). X *sets:* $|\{Y \text{ ordered set} \mid |Y| = n \text{ and } y \in Y \implies y \in X\}| = \frac{|X|!}{(|x|-n)!}$

Proof. Just count! □

Property (Cardinality of Subsets). X *sets:* $|\{Y \subseteq X \mid |Y| = n\}| = \binom{|X|}{n}$

Proof. Just count! □

Property (Cardinality of Sets of Functions). $|\{f : X \to Y\}| = |Y|^{|X|}$

Proof. Just count! □

Property (Cardinality of Sets of Injections). $|\{f : X \to Y \mid f \text{ injective}\}| = \frac{|Y|!}{(|Y|-|X|)!}$

Proof. Count (without repetition). □

Property (Cardinality of Sets of Surjections). $|\{f : X \to Y \mid f \text{ surjective}\}| = \binom{|X|}{|Y|}.(|Y|!).|Y|^{(|X|-|Y|)} = \frac{|X|.|Y|^{(|X|-|Y|)}}{(|X|-|Y|)!}$

Proof. Choose $|Y|$ elements in X to map to all elements in Y, then map the rest to any thing in Y. □

Property (Cardinality of Sets of Bijections). $|\{f : X \to Y \mid f \text{ bijective}\}| = |Y|! = |X|!$

Proof. Bijection is an injection between two sets of the same size. □

Question.

- *For n students, if we record the order of people getting out of the room, how many possibilities are there?*
- *Bench for 10 people, we have 5 boys, 5 girls, how many arrangements are there such that two boys/two girls are never seated next to each others?*
- *Bench for 11 people, we have 6 boys, 5 girls, how many arrangements are there such that two boys/two girls are never seated next to each others?*

Chapter 5

Infinite Cardinalities

Definition (Alphabet). $\mathcal{A} = \{a, b, c, \dots, z\}$

To compare the size of infinite sets, we use bijections, injections:

Definition (Comparing Sets). $f : X \to Y$ *injective* $\implies |X| \leq |Y|$ $f : X \to Y$ *surjective* $\implies |X| \geq |Y|$ $f : X \to Y$ *bijective* $\implies |X| = |Y|$

Note that together with $|\,[1, n]\,| = n$, this defines cardinality.

Definition (Countable sets). *A set is countable if it has the same cardinality as the naturals (i.e. X is countable if $|X| = |\mathbb{N}|$).*

Property (Countable Union Finite). $|\mathbb{N} \cup \mathcal{A}| = |\mathbb{N}|$

Property (Countable Union Countable / Integers). $|\mathbb{Z}| = |\mathbb{N} \cup \mathbb{N}^*| = |\mathbb{N}|$

Property (Countable Union of Finites). $|X_n| < \infty\ \forall n \in \mathbb{N} \implies |\bigcup_{n \in \mathbb{N}} X_n| = |\mathbb{N}|$

Property (Countable Union of Countables / Rationals). $|\mathbb{Q}| = |\bigcup_{n \in \mathbb{N}^*} \{{}^{m}/_{n} \mid m \in \mathbb{Z}\}| = |\mathbb{N}|$

Property (Power set of Countables / Reals). $|\,[0, 1[\,| = |\mathcal{P}(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}| > |\mathbb{N}|$

Property (Bounded & Unbounded Reals). $|\,[0, 1[\,| = |\mathbb{R}|$

Property (Reals and Product of Reals). $|\,[0, 1[\,| = |\,[0, 1[^2\,|$

Question.

- *What is $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ compared to $\mathbb{N}$?*
- *What is $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ compared to $\mathbb{N}$?*
- *What is $\mathbb{R} \times \mathbb{R}$ compared to $\mathbb{R}$?*

Chapter 6

Spaces

Mathematical Space: Object based on a set with more structure.

6.1 Metric Space

A metric space is a set X together with a metric distance $d : X \times X \to \mathbb{R}^+$.
d is a metric if it satisfies the following axioms:

- Non-degenerative: $d(x, y) = 0 \iff x = y$
- Symmetric: $d(x, y) = d(y, x)$
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

Question. *Which of the following are metric spaces?*

- $X = \mathbb{N},\ d(x, y) = 1$
- $X = \mathbb{N},\ d(x, y) = 0 \ \textit{if}\ x = y,\ d(x, y) = 1\ \textit{else}$
- $X = \mathbb{R},\ d(x, y) = |x - y|$
- $X = \mathbb{R},\ d(x, y) = (x - y)^2$

6.2 Norm Space

A norm space is a set X together with a norm $|\_| : X \to \mathbb{R}^+$.
$|\_|$ is a norm if it satisfies the following axioms:

- Non-degenerative: $\mathrm{Norm}(x) = 0 \iff x = 0$
- Homogeneity: $\mathrm{Norm}(\lambda x) = |\lambda| \mathrm{Norm}(x) \qquad \lambda \in \mathbb{R}$
- Triangle inequality: $\mathrm{Norm}(x + y) \leq \mathrm{Norm}(x) + \mathrm{Norm}(y)$

Question. *Which of the following are norm spaces?*

- $X = \mathbb{R}^2,\ Norm((x, y)) = \sqrt{x^2 + y^2}$
- $X = \mathbb{R}^2,\ Norm((x, y)) = |x| + |y|$
- $X = \mathbb{R}^2,\ Norm((x, y)) = \max(|x| + |y|)$
- $X = \mathbb{R}^n,\ Norm((x_1, x_2, \ldots, x_n)) = \sqrt{x_1^2 +_2^2 + \cdots + x_n^2}$

Hint: to show triangle inequality, search for "Minkowski inequality"

Property (Norm Implies Metric)**.** *Letting* $d(x, y) = Norm(x - y)$.

6.3 Inner Product Space

An inner product space is a set X together with an inner product $\langle \_, \_ \rangle : X \times X \to \mathbb{C}$. $\langle \_, \_ \rangle$ is an inner product if it satisfies the following axioms:

- Linear (in 1^{st} argument): $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \lambda \in \mathbb{C}$ and $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$
- Conjugate symmetry: $|x + y| \leq |x| + |y|$
- Positive definiteness $\langle x, x \rangle > 0 \ \forall x \neq 0$
- *(implied)* Non-degenerative: $\langle x, 0 \rangle = 0$ and $\langle 0, x \rangle = 0$
- *(implied)* Conjugate linear (in 2^{nd} argument): $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle \quad \lambda \in \mathbb{C}$ and $\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle$

Property (Inner Product implies Norm)**.** *Letting* $|x| = \sqrt{\langle x, x \rangle}$.

Property (Cauchy-Schwarz inequality)**.** $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$

Proof. Let $P(\lambda) = \langle x + \lambda y, x + \lambda y \rangle$. This polynomial is never negative, so its discriminant must be non-positive. Deduce the inequality from $\Delta \geq 0$. $\square$

Definition (Orthogonal / Normal)**.** x, y *orthogonal* $\iff x \perp y \iff \langle x, y \rangle = 0$

Property (Pythagoras Theorem)**.** $x \perp y \implies |x + y|^2 = |x|^2 + |y|^2$

Property (Parallelogram Identity)**.** $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$

Property (Polarization Identity)**.** $4 \langle x, y \rangle = |x + y|^2 - |x - y|^2 + i(|x + iy|^2 - |x - iy|^2)$

Question. *Draw ball of radius one in* $\mathbb{R}^2$ *for the following norms:* $|\_|_1$, $|\_|_2$, $|\_|_3$, $|\_|_\infty$.

6.4 Openness

Here, we work over (X, d), a metric space.

Definition (Open Ball)**.** $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$

Definition (Closed Ball)**.** $\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$

Definition (Open Set)**.** U *is open* $\iff \forall x \in U, \exists \epsilon > 0$ *s.t.* $B(x_0, \epsilon) \subseteq U$

Definition (Closed Set)**.** C *is closed* $\iff X \setminus C$ *is open*

Property. *Open balls are open.*

Proof. Use triangle inequality & draw scheme $\square$

Property. *Closed balls are closed.*

Proof. Use triangle inequality & draw scheme $\square$

Chapter 7

Limit Behaviors

7.1 Convergence & Divergence

Definition ($(x_n) \subseteq X$ converges to $x \in X$)**.** $\forall \varepsilon > 0,\ \exists N\ s.t.\ \forall n \geq N,\ d(x_n, x) < \varepsilon$

We write $x_n \xrightarrow[n \to +\infty]{} x$ or $\lim\limits_{n \to +\infty} x_n = x$. Note that convergence is defined w.r.t. a metric (or a norm/inner product, which induces a metric).

Definition ($(x_n) \subseteq \mathbb{R}$ converges to $x \in \mathbb{R}$)**.** $\forall \varepsilon > 0,\ \exists N\ s.t.\ \forall n \geq N,\ |x_n - x| < \varepsilon$

Definition ($(x_n) \subseteq \mathbb{R}$ diverges to $+\infty$)**.** $\forall M \in \mathbb{R},\ \exists N\ s.t.\ \forall n \geq N,\ x_n > M$

We write $x_n \xrightarrow[n \to \infty]{} +\infty$ or $\lim\limits_{n \to +\infty} x_n = +\infty$. Note that divergence is only defined over $\mathbb{R}$; divergence to $-\infty$ is defined similarly.

Question.

Define divergence to $-\infty$

Arithmetic sequence: find limit

Geometric sequence: find limit

Definition (Sub-sequence)**.** $\phi : \mathbb{N} \to \mathbb{N}$ *strictly increasing defines the sub-sequence* $(x_{\phi(n)})$ *of the sequence* (x_n).

Property (Convergence & Divergence of Sub-sequences)**.** $x_n \to x \implies x_{\phi(n)} \to x$
moreover, $x_n \to +\infty \implies x_{\phi(n)} \to +\infty$

Definition ($f : X \to Y$ converges to $y \in Y$ at $x \in X$)**.** $\forall \epsilon > 0,\ \exists \delta > 0\ s.t.\ d_X(x, x') < \delta \implies d_Y(y, y') < \epsilon$

Equivalent definition: $\forall x_n \to x$ as $n \to +\infty, y_n = f(x_n) \to y$; we write $\lim\limits_{x' \to x} f(x') = y$

Question.

- $\lim\limits_{x \to a} \phi(f(x))$
- $\lim\limits_{x \to a} f(x) + g(x)$
- $\lim\limits_{x \to a} f(x) * g(x)$
- $\lim\limits_{x \to a} f(x)/g(x) \qquad g(x) \neq 0$

Proof. left as exercise ☐

Property ("Determinate Forms")**.** $\frac{1}{0} = \infty,\ \frac{1}{\infty} = 0$

Property ("Indeterminate Forms"). $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 1^∞, $\infty - \infty$, 0^0, ∞^0

E.g.: $x^2 \times \frac{1}{x} \to \infty$; $x^2 \times \frac{1}{x^2} \to 1$; $x^2 \times \frac{1}{x^3} \to 0$.

Theorem 7.1.1 (Comparison Test). *Let $x_n \leq y_n\ \forall n \in \mathbb{N}$, then:*

- $x_n \to +\infty \implies y_n \to +\infty$
- $y_n \to -\infty \implies x_n \to -\infty$

Proof. from def $\square$

Theorem 7.1.2 (Sand-wish Theorem / Two Policemen Theorem). *Let $x_n \leq y_n \leq z_n\ \forall n \in \mathbb{N}; l \in \mathbb{R}$, then $x_n \to l$ and $z_n \to l \implies y_n \to l$.*

Proof. from def $\square$

Theorem 7.1.3 (Fixed Point Theorem). *$x_{n+1} = f(x_n)$ and $(x_n) \to l \implies l = f(l)$ (i.e. l is a fixed point of f).*

Proof. easy: x_n and x_{n+1} must both go to l $\square$

E.g.: $x_0 = 1$ and $x_{n+1} = \frac{x_n^2+2}{2x_n}$ give $x_n \to \sqrt{2}$.

7.2 Maximum vs Supremum

Definition (a maximum of A). *$a \in A$ and $\forall x \in A, x \leq a$*

Maximum doesn't always exists (even if A is bounded).

Definition (a supremum of A). *$\forall \epsilon > 0, \exists\, x \in A$ s.t. $a - \epsilon \leq x \leq a$*

Supremum is the "smallest upper bound". Exists if A is bounded.

Question. *Max, sup of $[0,1]$, $[0,1[$, $\mathbb{R}^+$.*

Remark. *Can define minimum & infimum similarly*

Theorem 7.2.1 (Extremum & Convergence). *$(x_n) \subseteq \mathbb{R}$ increasing:*

- *if (x_n) is upper-bounded, then $\lim\limits_{n\to+\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$*
- *else, $\lim\limits_{n\to+\infty} x_n = +\infty$*

Proof. easy (by cases) $\square$

7.3 Compactness

Definition (Diameter). *$Diam(B) = \sup\{d(x,y) \mid x, y \in B\}$*

Definition (Boundedness). *B is bounded $\iff Diam(B) < \infty$*

[proper definition: every cover has a finite sub-cover, however, in the cases we will consider (i.e. over $\mathbb{R}$ or $\mathbb{C}$):]

Definition (Compactness (over $\mathbb{R}^n$)). *C is compact over $\mathbb{R}^n \iff C$ is closed and bounded*

Property. *C is compact $\iff \forall (x_n)_{n\in\mathbb{N}} \subseteq C, (x_n)$ has a convergent subsequence*

Proof. technical, using open covers $\square$

7.4 Continuity

Definition (f continuous at x)**.** $\lim_{x' \to x} f(x') = f(x)$

Definition (f continuous on X)**.** $\forall x \in X, f$ *continuous* x

Question. *Show x^n is continuous (for all n).*

"can be plotted in a single trace/line; without lifting the pen"

7.5 Asymptotic Analysis

Definition (Asymptote)**.** *"A curve is a line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity."*
i.e. $\lim_{x \to \infty} f(x) - l(x) = 0$ *(in the case of $x \to \infty$).*

Horizontal Asymptote E.g.: $f(x) = \frac{x+1}{x}$ (asymptote is $y = 1$ as $x \to \infty$).

Vertical Asymptote E.g.: $g(x) = \frac{1}{x-2}$ (asymptote is $x = 2$ as $y \to \infty$).

Oblique Asymptote E.g.: $h(x) = \frac{3x^2+2x+1}{x}$ (asymptote is $y = 3x + 2$ as $x, y \to \infty$).

7.6 Series

Joke: I once asked someone out to "checkout some series", they went home disappointed... still don't know why.

Definition (Series)**.** *A series is a sequence (S_n) with general term x_n defined by $S_n = \sum_{k=0}^{n} x_k$. It is alternating if $x_k x_{k+1} < 0$ $\forall k \in \mathbb{N}$.*

Definition (Series Convergence)**.** *The series (S_n) converges if $(\sum_{k=0}^{n} x_k)$ converges as a sequence. The series (S_n) converges absolutely if $(\sum_{k=0}^{n} |x_k|)$ converges as a sequence.*

E.g.: $\sum_{k=1}^{n} 1$ is obviously divergent; $\sum_{k=1}^{n} \frac{1}{2^k}$ is convergent (to 1).

Property. $\sum_{k=0}^{n} a^k$ *is:*

- *Absolutely convergent for $|a| < 1$, converging to $\frac{1}{1-a}$.*
- *Divergent for $|a| \geq 1$, bounded for $a = -1$, unbounded else.*

Proof. easy (sum of geometric series) □

Property (Necessary Condition for Convergence of Series)**.** *If (S_n) converges, then $x_n \to 0$.*

Proof. trivial (by contradiction) □

However, his is **not** a sufficient condition: $\sum_{k=1}^{n} \frac{1}{k}$ is a counter-example.

Property (Criterion for Convergence of Alternating Series)**.** *If $\sum_{n \in \mathbb{N}} x_n$ is alternating, $(|x_n|)$ is decreasing, and $\lim_{n \to \infty} x_n = 0$, then $\sum_{n \in \mathbb{N}} x_n$ converges.*

Proof. WLOG, $x_{2n} > 0$ and $x_{2n-2} < 0$: $\sum_{k=0}^{2n} x_k$ is increasing, and upper bounded by $x_0 + x_1$, therefore converges; similarly, $\sum_{k=0}^{2n+1} x_k$ is decreasing, and lower bounded by x_0, therefore converges as well. $\sum_{k=0}^{2n} x_k$ and $\sum_{k=0}^{2n+1} x_k$ must have the same limit as $\sum_{k=0}^{2n+1} x_k - \sum_{k=0}^{2n} x_k = x_{2n+1} \to 0$. Thus, $\sum_{k=0}^{2n} x_k$ must be convergent. □

Property (Comparison Test for Convergence of Series)**.** $\forall n \geq n_0, 0 \leq a_n \leq b_n$*:*

- *If* $\sum_{n\in\mathbb{N}} b_n$ *converges, then* $\sum_{n\in\mathbb{N}} a_n$ *converges as well.*
- *If* $\sum_{n\in\mathbb{N}} a_n$ *diverges, then* $\sum_{n\in\mathbb{N}} b_n$ *diverges as well.*

Proof. easy by def $\square$

E.g.: $\sum_{n\geq 2} \frac{1}{n^2} \leq \sum_{n\geq 2} \frac{1}{n(n+1)} = \sum_{n\geq 2} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{2} < \infty$

Property (Integration Test for Convergence of Series)**.** $\sum_{n\in\mathbb{N}} f(n) \leq \int_{x=0} \infty f(x)$
So if $\int_{x=0} \infty f(x) < \infty$*, and* $f(x)$ *is decreasing, then* $\sum_{n\in\mathbb{N}} f(n)$ *converges.*

Proof. easy by def $\square$

E.g.: $\sum_{n\geq 2} \frac{1}{n^2} \leq \int_{x=2}^{\infty} \frac{1}{x^2} = \left[-\frac{1}{x}\right]_{x=2}^{x=\infty} = -0 + \frac{1}{2} < \infty$

Property (Ratio Test)**.** $\lim\limits_{n\to+\infty} |\frac{x_{n+1}}{x_n}| = L$

- $L < 1$*:* $\sum_{n\in\mathbb{N}} x_n$ *converges*
- $L > 1$*:* $\sum_{n\in\mathbb{N}} x_n$ *diverges*
- $L = 1$*:* $\sum_{n\in\mathbb{N}} x_n$ *is unknown*

Question. *Are these convergent series?*

- $\sum_{n=1}^{+\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$
- $\sum_{n=1}^{+\infty} \frac{n!}{5^n}$
- $\sum_{n=1}^{+\infty} \frac{n^2}{(2n-1)!}$
- $\sum_{n=1}^{+\infty} \frac{9^n}{(-2)^{n+1}n}$

Property (Root Test)**.** $\lim\limits_{n\to+\infty} \sqrt[n]{|x_n|} = L$

- $L < 1$*:* $\sum_{n\in\mathbb{N}} x_n$ *converges*
- $L > 1$*:* $\sum_{n\in\mathbb{N}} x_n$ *diverges*
- $L = 1$*:* $\sum_{n\in\mathbb{N}} x_n$ *is unknown*

Fact: $\lim\limits_{n\to+\infty} n^{\frac{1}{n}} = 1$

Question. *Are these convergent series?*

- $\sum_{n=1}^{+\infty} \frac{n^n}{3^{2n+1}}$
- $\sum_{n=1}^{+\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$
- $\sum_{n=1}^{+\infty} \frac{(-12)^n}{n}$

Chapter 8

Smoothness

Want to measure how "steep" a curve is at a pt x_0: take linear approx. from x_0 to x (take the steep of the line), and let $x \to x_0$.
Formally:

Definition. $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ *f is differentiable at x_0 if the limit $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. f is differentiable on I if the limit $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists for all $x_0 \in I$.*

Property (Differentiable implies the continuous)**.** *f differentiable at $x_0 \implies f$ continuous at x_0*

Proof. Easy □

The absolute value is continuous but not differentiable at $x = 0$.
The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere.

Property (Operations on Derivatives)**.**

- $(f+g)' = f' + g'$
- $(f * g)' = f' * g + f * g'$ *"product rule"*
- $(f/g)' = \frac{f' * g + f * g'}{g^2} f' + g' \qquad g \neq 0$ *"quotient rule"*
- $(f(g))' = f'(g) * g'$ *"chain rule"*

Proof.

- linearity of limits
- from def
- from def
- from def

□

Property (Sign of the derivative)**.** *$f' > 0$ on $I \implies f$ strictly increasing on I*

Proof. Clear graphically; mathematically, use mean value theorem. □

Can (try to) differentiate the derivative f' of f, giving $f'' = f^{(2)}$. Can then (try to) differentiate f'' giving $f''' = f^{(3)}$.

Definition. *$C[I]$ is the set of continuous functions on I.*

- *If f' exists, f is differentiable.*
- *If f'' exists, f is twice differentiable.*

- *If $f^{(k)}$ exists, f is k-times differentiable.*
- *If f' exists and is continuous, then f is continuously differentiable.*
- *If f'' exists and is continuous, then f is twice continuously differentiable.*
- *If $f^{(k)}$ exists and is continuous, then f is k-times continuously differentiable.*

$C^k[I]$ is the set of functions k-times continuously differentiable on I.

Question. *How many times are the following continuously differentiable? (on $\mathbb{R}$)*

- $f(x) = |x|$
- $f(x) = x^2$
- $f(x) = x^2$ *for* $x > 0$, $f(x) = -x^2$ *for* $x \leq 0$
- $f(x) = 2x + 1$ *for* $x > 0$, $f(x) = 3x - 1$ *for* $x \leq 0$

Chapter 9

Integration

Integral of f: area under a curve of f (draw scheme).

Proof. Take f continuous; let $A(x)$ be the area under the curve of f, from 0 to x.

Then $A(x+h) - A(x) = f(x).h + \epsilon(h)$; as $h \to 0$, $\epsilon(h) \to 0$ by continuity of f. Then get $f(x) = \frac{A(x+h)-A(x)}{h}$ as $h \to 0$. Thus, $A'(x) = f(x)$. □

Notation. *Integral from a to b of f is* $\int_a^b f(x)dx$

Theorem 9.0.1 (Fundamental Theorem of Calculus)**.** *If* $F(x) = \int_a^x f(t)dt$*, then* F *is uniformly continuous and differentiable, with derivative* $F' = f$

Corollary 9.0.1. $\int_a^b f(x)dx = F(b) - F(a)$ *where* F *is an anti-derivative of* f *(i.e.* $F' = f$[1]*)*

Integrals may also be approximated via partial sums; this is how computers calculate integrals (draw picture).

Integration is the "inverse" of differentiation: $\int f = F + C$ where $C \in \mathbb{R}$ and $F' = f$. So $\int f' = f + C$ and $\left(\int f\right)' = f$

Note that not all functions are integrable in terms of elementary functions (e.g.: $\frac{\sin(x)}{x}$). Note that not all functions are integrable in terms of area under the curve either (e.g.: $f(x) = 0$ if $x \in Q, f(x) = 1$ if $x \notin \mathbb{Q}$). However, "most" usual functions are integrable in terms of area under the curve (any continuous or monotone function is, so usually do not worry about it in applied maths).

Note that integrable does **not** imply differentiable/continuous (e.g. floor function); and differentiable does **not** imply anti-derivative exists in terms of elementary functions (e.g. $\frac{\sin(x)}{x}$). [draw diagram of implications: integrable area; integrable anti-derivative; continuous; differentiable]

[1] Anti-derivative are **not** unique (can add a constant).

Chapter 10

Elementary Functions

10.1 Enumeration

[Enumeration of the elementary function by Liouville from 1833 to 1841:]

- Polynomials function of $\mathbb{R}[x]$: 1, x, $\pi + 3.2x^2 + \frac{7}{8}x^{2021}$, ...
- Hyperbolic functions: exponential (e^x), hyperbolic sinus ($\sinh(x) = \frac{e^x - e^{-x}}{2}$), hyperbolic cosinus ($\cosh(x) = \frac{e^x - e^{-x}}{2}$), ...
- Trigonometric functions: cos, sin, tan,...
- Inverse functions of the previous functions: logarithmic functions, inverse trigonometric, ...

Closed under derivative, but not under integration:
e.g.: $\int_a^b \frac{\sin(x)}{x} dx = \mathrm{Si}(b) - \mathrm{Si}(a)$ where $\mathrm{Si}(x)$ cannot be expressed in terms of elementary functions (it is defined by the area under the curve of $\frac{\sin(x)}{x}$).

10.2 Properties

10.2.1 Polynomials Functions

Definition. $\mathbb{R}[x]$ *is the set of polynomials of x with real coefficients.*

Property. *All polynomials are continuous & differentiable, with* $(x^n)' = nx^{(n-1)}$.

Proof. x is continuous, then use algebra of continuous functions & by induction. □

Corollary 10.2.1. $(x^n)^{(k)} = \frac{n!}{(n-k)!}x^{n-k} = k!\binom{n}{k}x^{n-k}$

Degree

Definition. $P = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x] \implies \partial P = n$
by convention, $\partial P = -\infty$ *if* $P \equiv 0$

Property (Algebra of degrees).

- $\partial P + Q = \max(\partial P, \partial Q)$
- $\partial P * Q = \partial P + \partial Q$

Proof. Exercise □

Roots

Definition (Root orders). *$P \in \mathbb{R}[x]$ has a root at x if $P(x) = 0$ $P \in \mathbb{R}[x]$ has a root of order n at x if $P^{(k)}(x) = 0$ for all $k < n$ and $P^{(n)}(x) \neq 0$*

[draw graphical interpretation of roots multiplicities]

Property (Roots and Factorization). *P has a root of order k at x' iff $P(x) = (x - x')^k Q(x)$ where Q is a polynomial s.t. $\partial P = k + \partial Q$.*

Proof. definition for $\implies$ direction, Taylor formula for $\impliedby$ direction ☐

Corollary 10.2.2 (Degree and Number of roots). *If roots of P have multiplicities $k_1, k_2, k_3, \dots, k_n$, then $\sum_{i=1}^{n} k_i \leq \partial P$. "#roots $\leq$ degree"*

Proof. easy using previous property ☐

The Constant case No root(s) except for $P \equiv 0$

The Linear case One root: $P(x) = ax + b \implies x = -\frac{b}{a}$ is the only root.

The Quadratic case

$$P(x) = ax^2 + bx + c \qquad \Delta = b^2 - 4ac$$

- $\Delta > 0$: P has two roots: $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$
- $\Delta = 0$: P has one root: $x_0 = \frac{-b}{2a}$
- $\Delta > 0$: P has no root(s)

Proof. "force factorization": $ax^2 + bx + c = 0 \iff \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{c}{a} = 0$ which has solution(s) only if $\Delta \geq 0$ ☐

Taylor formula

Theorem 10.2.1 (Binomial theorem). $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$

Proof. by induction ☐

Theorem 10.2.2 (Taylor for polynomials). $P(x) = \sum_{k=0}^{+\infty} \frac{P^{(k)}(\alpha)}{p!} (x - \alpha)^k$

Proof. prove it for $P(x) = x^n$, using the binomial theorem ☐

10.2.2 Hyperbolic Functions

Exponential

Proposition. *The series $\sum_{n \in \mathbb{N}} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.*

Proof. easy ☐

Definition. *The series $\sum_{n \in \mathbb{N}} \frac{x^n}{n!}$ is called the exponential function, and denoted* $\exp(x)$ *or* e^x.

Property.

- $\exp$ *is continuous*
- $\exp$ *is differentiable, and* $\exp' = \exp$
- $\exp(x + y) = \exp(x) \exp(y)$

- $\exp(-x) = 1/\exp(x)$
- $\exp(x) > 0$
- exp *is increasing on all* $\mathbb{R}$
- $\lim_{x\to+\infty} \exp(x) = +\infty$ *and* $\lim_{x\to-\infty} \exp(x) = 0$
- $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$ *"e^x grows "faster" than x^n for any n"*

Proof.
- technical (on any segment of $\mathbb{R}$, the partial sum converges uniformly to exp; partial sums are continuous, so exp is continuous on all segments of $\mathbb{R}$, thus continuous on $\mathbb{R}$)
- differentiate each term in partial sum
- from series def & binomial theorem
- corollary, ask audience
- corollary, ask audience
- corollary, ask audience
- $e^x > x$ for $x > 0$ gives limit as $x \to +\infty$, then use inverse

$\square$

[Draw exp curve]

Logarithmic

Definition. *The inverse function of* exp *is* ln *or* log*:* $\ln(x) = y$ *s.t.* $x = e^y$*. Note* $\exp : \mathbb{R} \to \mathbb{R}^{+*}$ *so* $\ln : \mathbb{R}^{+*} \to \mathbb{R}$*, so* $\exp(\ln(x)) = x\ \forall x \in \mathbb{R}^{+*}$ *and* $\ln(\exp(x)) = x\ \forall x \in \mathbb{R}$.

[Draw log curve]

Property.
- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln(x/y) = \ln(x) - \ln(y)$
- $\ln'(x) = 1/x$
- $\ln(0) = 1$
- $\lim_{x\to+\infty} \ln(x) = +\infty$ *and* $\lim_{x\to0} \ln(x) = -\infty$
- $\lim_{x\to+\infty} \frac{\ln(x)}{x^\epsilon} = 0$ *"$\ln(x)$ grows "slower" than x^ϵ for any $\epsilon > 0$"*

Proof.
- use properties of exp
- use properties of exp
- use properties of exp
- use $\exp(0) = 1$
- use limits of exp

$\square$

fun fact: cosh is the shape of a rope attached at both ends

10.2.3 Trigonometric Functions

[draw triangle def of cos, sin & tan, then graph them, observe periodicity, observe sin is cos "shifted" by pi/2; observe the location of zeros; write math def of these observations]

Property (Derivatives & Integrals of Trigonometric Functions).

- $\sin' = \cos$
- $\cos' = -\sin$
- $\tan' = 1/\cos^2$
- $\int \sin = -\cos + C$
- $\int \cos = \sin + C$
- $\int \tan = -\ln(|\cos|) + C$

Proof.

- technical
- technical
- use quotient rule
- use derivative result
- use derivative result
- technical, can be checked easily

□

Chapter 11

Complex Numbers

11.1 Introduction

Observation: $x^2 + 1 = 0$ has no solution in $\mathbb{R}$; want to extend $\mathbb{R}$ so that there is a solution.

Definition (The Complex Unit). *Let $i = \sqrt{-1}$ so that $i^2 = -1$ and $i, -i$ are two solutions of $x^2 + 1 = 0$.*

Definition (The Complex Field). $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

[draw complex plane, show $\mathbb{C}$ is in bijection with $\mathbb{R}^2$]

Proposition (Complex have all roots of all quadratics). $ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{\Delta}}{2a}$

Proof. case $\Delta < 0$ □

Operations on complex numbers: show addition, multiplication, division

Definition (Conjugate). *If $z = x + iy$, then conjugate of z is $\overline{z} = x - iy$.*

Property (Properties of Conjugate).

- $\overline{zz'} = \overline{z}\overline{z'}$
- $\overline{z + z'} = \overline{z} + \overline{z'}$
- $\overline{z^k} = \overline{z}^k$

Proof. use $z = x + iy$, $z' = x' + iy'$ □

Definition (Modulus). *If $z = x + iy$, then modulus of z is $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$.*

Property (Properties of Modulus).

- *Modulus is a metric on $\mathbb{C}$*
- $|zz'| = |z||z'|$

Proof.

- show triangle inequality (others are trivial)
- use $z = x + iy$, $z' = x' + iy'$

□

11.2 Complex Exponential

$\sum_{n \in \mathbb{N}} z_n$ converges if $\sum_{n \in \mathbb{N}} |z_n|$, so $\sum_{n \in \mathbb{N}} \frac{z^k}{k!}$ converges uniformly; we can therefore extend exponential to complex.

Note that algebra of exponential remains over the complex, and $\overline{e^z} = e^{\overline{z}}$.

11.2.1 Geometry

[Draw argand diagram: x, y; modulus, argument, conjugate]
Polar coordinates <=> Cartesian coordinates
Transformations in the complex plane: translation, scaling, rotation

11.2.2 Link with trigonometry

Property. $|\exp(i\theta)| = 1$

Proof. $|\exp(i\theta)|^2 = \exp(i\theta)\exp(-i\theta) = \exp(0) = 1$ □

Unit circle: coordinates are given by cos and sin
By proving $\theta \mapsto \exp(i\theta)$ is surjective, can show that $\exp(i\pi/2) = i$.
Have:

- $\cos(\theta) = \Re(\exp(i\theta))$
- $\sin(\theta) = \Im(\exp(i\theta))$

This gives periodicity of 2π, etc...

11.3 Complex Polynomials

Lemma 11.3.1 (Existence of the Minimum of a Polynomial). $P(z) \in \mathbb{C}[x] : \exists z_0 \in \mathbb{C}$ *s.t.* $P(z_0) = \inf\{P(z) \mid z \in \mathbb{C}\}$

Proof. Show $|P(z)| \to +\infty$ as $|z| \to +\infty$. Then $X = \{z \in \mathbb{C} \mid |P(z)| \leq \inf\{|P(z)| \mid z \in \mathbb{C}\} + 1\}$ is compact (close & bounded). By definition of infimum, there is a sequence $(z_n) \subseteq X$ such that $(P(z_n)) \to \inf\{|P(z)| \mid z \in \mathbb{C}\}$. But then there is a sub-sequence $z_{n_k} \to z \in X$. □

Theorem 11.3.1 (of d'Alembert). $P(z) \in \mathbb{C}[x] : \partial P \geq 1 \implies \exists z \in \mathbb{C}$ *s.t.* $P(z) = 0$

Proof. Suppose $\min\{|P(z)| \mid z \in \mathbb{C}\} > 0$ is reached at z_0. We can define the polynomial $Q : z \in \mathbb{C} \mapsto \frac{P(z_0+z)}{P(z_0)}$ which is such that $Q(0) = \min\{|Q(z)|, z \in \mathbb{C}\} = 1$. Let $(b_0, ..., b_p)$ be the coefficients of Q and $q = \min\{j \in [\![1;p]\!] | b_j \neq 0\}$. With these notations, $\forall z \in \mathbb{C},\ Q(z) = 1 + b_q z^q + \sum_{k=q+1}^{p} b_k z^k$.
Let $\theta = \text{Arg}(b_q)$ and $\varphi = \frac{\pi-\theta}{q}$. Then $b_q e^{iq\varphi} = -|b_q|$. So:

$$\forall r > 0, Q(re^{i\varphi}) = 1 - |b_q|r^q + \sum_{k=q+1}^{p} b_k r^k e^{ik\varphi}$$

$$|Q(re^{i\varphi})| \leq |1 - |b_q|r^q| + \sum_{k=q+1}^{p} |b_k|r^k$$

$$\forall r \in]0; |b_q|^{1/q}[,\ \ |Q(re^{i\varphi})| \leq 1 - |b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k$$

$$|Q(re^{i\varphi})| - 1 \leq -|b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k$$

$$\lim_{r\to 0} \frac{-|b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k}{r^q} = -|b_q| < 0$$

So there exists r_1 such that $0 < r_1 < |b_q|^{-1/q}$ such that $\forall r < r_1,\ \frac{-|b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k}{r^q} < 0$, so $|Q(re^{i\varphi})| < 1$, which is a contradiction. □

Corollary 11.3.1. *Let $P \in \mathbb{C}[X]$ such that* $\deg(P) \geq 1$. *Let $z_1, ..., z_m$ be the roots of P of multiplicities $\alpha_1, ..., \alpha_m$. Then we have that $\alpha_1 + ... + \alpha_m = \deg(P)$ and there exists $\lambda \in \mathbb{C}^*$ such that*

$$\forall z \in \mathbb{C},\ P(z) = \lambda \prod_{k=1}^{m} (z - z_k)^{\alpha_k}$$

Chapter 12

Vector Spaces

12.1 Axioms

Definition (Vector Space)**.** *$\mathbb{F}$ a field (usually $\mathbb{R}$ or $\mathbb{C}$). V is a vector field over $\mathbb{F}$ if:*

- $\forall v, w \in V : v + w \in V$ *"V has addition"*
- $\forall v \in V, k \in \mathbb{F} : k.v \in V$ *"V has multiplication by a scalar"*

Such that:

- $\forall v \in V, k, l \in \mathbb{F} : (kl).v = k.(l.v)$
- $\forall v \in V, k, l \in \mathbb{F} : (k + l).v = k.v + l.v$
- $\forall v, w \in V, k \in \mathbb{F} : k.(v + w) = k.v + k.w$
- $\forall v \in V : 1.v = v$
- $\forall v \in V : 0.v = \boldsymbol{0}$
- $\forall k \in \mathbb{F} : k.\boldsymbol{0} = \boldsymbol{0}$

Example.

- 0 *over* $\mathbb{R}$
- $\mathbb{R}$ *over* $\mathbb{R}$
- $\mathbb{C}$ *over* $\mathbb{R}$
- $\mathbb{C}$ *over* $\mathbb{C}$
- $\mathbb{R}^n$ *over* $\mathbb{R}$
- $\mathbb{R}[x]$ *over* $\mathbb{R}$ *(real polynomials)*
- $\mathbb{R}^{\mathbb{N}}$ *over* $\mathbb{R}$ *(real sequences)*
- $\mathbb{R}^{\mathbb{R}}$ *over* $\mathbb{R}$ *(real functions)*
- $V_1 \times V_2$ *if* V_1 *and* V_2 *are vector spaces over the same field* $\mathbb{F}$

Definition (Vector Sub-space)**.** *$W \subseteq V$ is a vector subspace of V if it is a vector space on its own. Need to check:*

- *Closed under addition:* $w, w' \in W \implies w + w' \in W$
- *Multiplication by a scalar:* $w \in W, k \in \mathbb{F} \implies k.w \in W$

- *Contains the null vector:* $\mathbf{0} \in W$

Example.
- 0 *is a vector sub-space of* $\mathbb{R}$
- $\mathbb{R}$ *is a vector sub-space of* $\mathbb{C}$ *over* $\mathbb{R}$
- $\mathbb{R}_d[x]$ *is a vector sub-space of* $\mathbb{R}[x]$ *over* $\mathbb{R}$ *(real polynomials of degree d)*
- $C[\mathbb{R}]$ *is a vector sub-space of* $\mathbb{R}^{\mathbb{R}}$ *over* $\mathbb{R}$ *(real functions)*

Proposition. W_1, W_2 *subspaces of* $V \implies W_1 \cap W_2$ *subspace of* V

Proof. in problem set □

Property (Direct product of vector spaces are vector spaces)**.** V_1, V_2 *vector spaces* $\implies V_1 \times V_2$ *is a vector space*
$(v_1, v_2), (v_1', v_2') \in V_1 \times V_2, k \in \mathbb{F}$:

- $(v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2')$
- $k.(v_1, v_2) = (k.v_1, k.v_2)$

Definition (Linear combination)**.** $A \subset V$ *vector space: x is a linear combination of A iff* $\exists k \in \mathbb{N}$ *s.t.* $\exists k_1, \ldots, k_n \in \mathbb{F}, x_1, \ldots, x_n \in A$ *s.t.* $x = \sum_{i=1}^{n} k_i.x_i$

Definition (Span)**.** $A \subset V$ *vector space: Span*(A) *is the set of all vector that can be expressed as a linear combination of A, i.e. Span*$(A) = \{\sum_{i=1}^{n} k_i.x_i \mid n \in \mathbb{N}, k_i \in \mathbb{F}, x_i \in A\}$

Example.
- $Span(\{\mathbf{1}\}) = \mathbb{R}$
- $Span(\{\mathbf{0}\}) = \{\mathbf{0}\}$
- $Span(\{\mathbf{1}, \mathbf{i}\}) = \mathbb{C}$

Property. *Span*(A) *is the smallest vector space containing* A

Proof. any smaller set containing A is not closed under addition/multiplication by scalar □

Exercise. *Which of these can be seen as vector spaces?*

- $\mathbb{R}^{\mathbb{N}}$
- $\{(x, y) \mid x^2 + y^2 \leq 1\}$ *"unit disk"*
- $\{(x, y) \mid x + y = 0\}$
- $\{(x, y) \mid x + y = 1\}$
- $\{f \in C[a, b] \mid f(a) = f(b)\}$

12.2 Dimension & Basis

Definition (Linear Independence)**.** $x_1, \ldots, x_n$ *are linearly independent (LI) if* $\forall k_1, \ldots, k_n \in \mathbb{F}, \sum_{i=1}^{n} k_i.x_i = 0 \implies \forall 1 \leq i \leq n, k_i = 0$

Example.
- $(1, 0, 0)$ *&* $(0, 1, 0)$ *in* $\mathbb{R}^3$
- 1 *&* i *in* $\mathbb{C}$
- $(1, 0, 1)$, $(5, 0, 1)$ *&* $(1, 3, 0)$ *in* $\mathbb{R}^3$

Definition (Basis of a Vector Space)**.** *V has basis $B \subset V$ if Span$(B) = V$ and B is LI.*

Example.
- $\{(1,0,0),(0,1,0),(0,0,1)\}$ *in* $\mathbb{R}^3$
- $\{1, i\}$ *in* $\mathbb{C}$

The LI property tends to make the basis "small", while the spanning property tends to make it "large".
A basis is a largest LI set, or a smaller spanning set.

Property (Basis always exist)**.** *Any vector space V has a basis*

Proof. Either remove from a spanning set (e.g. V spans itself), or (in finite dimensions), add elements to LI set until the set spans all of V. ☐

Property (All Basis have the Same Number of Elements)**.** *If B and B' are both basis of V (i.e. span V and are LI), then $|B| = |B'|$*

Proof. Technical, omitted ☐

Definition (Dimension)**.** *The dimension of a vector space is the cardinal of a basis (note it ay be finite of infinite)*

Example.
- $\mathbb{R}^3$ *has dimension 3 (finite)*
- $\mathbb{R}[x]$ *has infinite dimension*
- $\dim_{\mathbb{R}}(\mathbb{C}) = 2$
- $\dim_{\mathbb{C}}(\mathbb{C}) = 1$
- $\dim(\mathbb{R}^n) = n$
- $\dim(\{\boldsymbol{0}\}) = n$

Property (Algebra of Dimensions)**.**
- $\dim(V \times V') = \dim(V) + \dim(V')$
- $\dim(V + V') = \dim(V) + \dim(V') - \dim(V \cap V')$

12.3 Linear Maps

Here, V and W are vector spaces over $\mathbb{F}$.

Definition (Linear Map)**.** *$f : V \to W$ is linear if:*

- $f(v + v') = f(v) + f(v') \forall v, v' \in V$
- $f(k.v) = k.f(v) \forall v \in V, k \in \mathbb{F}$

Definition. *With $f : V \to W$:*

- *f is an endomorphism if $V = W$*
- *f is an isomorphism if f is bijective*
- *f is an automorphism if $V = W$ and f is bijective*
- *f is a linear form if $W = \mathbb{F}$*

Example.
- $\mathbb{R} \ni x \mapsto ax \in \mathbb{R}$ *"classic linear map"*

- $\mathbb{R}[x] \ni P \mapsto P' \in \mathbb{R}[x]$ *"derivative"*
- $\mathbb{R}^{\mathbb{R}} \ni f \mapsto f(0) \in \mathbb{R}$ *"evaluation at 0"*

Property. f *linear* $\implies f(0) = 0$

Proof. exercise ☐

Property. $f : V \to W$ *linear: Im*(f) *is a subspace of* W*, and* $f^{-1}(W')$ *is a subspace of* V *if* W' *is a subspace of* W

Proof. exercise ☐

Definition (Kernel)**.** *Ker*$(f) = f^{-1}(\{0\})$

Property (Injectivity for Linear Functions)**.** f *linear:* f *is injective iff Ker*$(f) = \{0\}$

Proof. prove both directions, easy from def ☐

Property (Composition of Linear Maps)**.** $f : V_1 \to V_2, g : V_2 \to V_3$ *linear maps* $\implies f \circ g : V_1 \to V_3$ *is a linear map*

Proof. easy from definition ☐

Property (Inverse of a Linear Map)**.** $f : V_1 \to V_2$ *linear map with inverse* $\implies f^{-1} : V_2 \to V_1$ *is a linear map*

Proof. easy from definition ☐

Property. $f : V_1 \to V_2$ *isomorphism*, B *basis for* $V_1 \implies f(B)$ *is a basis for* V_2

Proof. Use injectivity of f to show $f(B)$ is LI, and surjectivity of f to show $f(B)$ spans V_2. ☐

Notation (Isomorphic Spaces)**.** $V_1 \equiv V_2 \iff \exists f : V_1 \to V_2$ *isomorphic*

Property. *A linear map is fully determined by its action on a basis.*

Proof. using linearity ☐

Property (Dimension of Isomorphic Spaces)**.** $V_1 \equiv V_2 \iff \dim(V_1) = \dim(V_2)$ *(in finite dimensions)*

Proof. $\implies$ use isomorphism to transform basis; $\impliedby$ basis are of the same size, create a linear isomorphism from one basis to the other ☐

Definition (Rank of a Linear Map)**.** *Rank*$(f) = \dim($*Im*$(f))$

Definition (Nullity of a Linear Map)**.** *Nullity*$(f) = \dim($*Ker*$(f))$

Theorem 12.3.1 (Kernel-Rank/Nullity-Rank)**.** $f : V \to W$ *linear: Rank*(f) + *Nullity*$(f) = \dim(V)$

Proof. Let V_0 a subspace such that $V_0 \oplus \mathrm{Ker}(f) = V$. Then $g : x \in V_0 \to f(x) \in \mathrm{Im}(f)$ is an isomorphism. g is clearly injective, since $\mathrm{Ker}(g) = \mathrm{Ker}(f) \cap V_0 = \{0\}$ And g is surjective since for any $y \in \mathrm{Im}(f)$, there exists $x \in V$ such that $f(x) = y$ and x is uniquely decomposed into $x_1 + x_2$ with $x_1 \in V_0$ and $x_2 \in \mathrm{Ker}(f)$, so $f(x_1 + x_2) = f(x_1) = g(x_1) = y$.

☐

12.4 Link with matrices

Representation of a finite-dimensional linear map by a matrix
$f : V_1 \to V_2$, $B_1 = \{x_1, \ldots, x_n\}, B_2 = \{y_1, \ldots, y_m\}$ basis for V_1, V_2 respectively. As f is linear, there are coefficients $\{a_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ such that: $f(x_j) = \sum_{i=1}^{m} a_{ij} y_i$ Then, we put

$$M_{B_1,B_2}(f) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

A matrix can always be interpreted as the representation of a linear map. This interpretation is used extensively in the proofs of linear algebra. But this interpretation may be useless in some other contexts, e.g. when the matrix is just used to store data.

Exercise. *Represent in the canonical basis of $\mathbb{R}^2$ the linear map $r_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates any vector by an angle of α anticlockwise.*

The spaces of matrices and linear maps are isomorphic
There is a one-to-one correspondence between $n \times m$ matrices and linear maps from a vector space of dimension m to a vector space of dimension n.

Multiplication of a column vector by a matrix [Draw it & Explain]

Multiplication of two matrices [Draw it & Explain]

Note: multiplication of matrices is **not** commutative, and is defined only for matching dimensions.

Example of non-commutativity:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

Associativity of the matrix product

- $(\lambda A)B = \lambda(AB) = A(\lambda B)$
- $(AB)C = A(BC)$

Relation between the composition of linear maps and the multiplication of matrices
$f : V_1 \to V_2, g : V_2 \to V_3$, B_1, B_2, B_3 basis for V_1, V_2, V_3 respectively.

$$M_{B_1,B_3}(f \circ g) = M_{B_1,B_2}(f) M_{B_2,B_3}(g)$$

The identity matrix

Definition (Identity Matrix)**.** *I_n is the $n \times n$ matrix with all entries being 0 except on the diagonal, where entries are 1.*
e.g.:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Property (Multiplication by the Identity Matrix)**.** *A a $n \times n$ matrix:*
$AI_n = I_n A = A$

Inverse of a square matrix

Definition. *A a $n \times n$ matrix:*
B is the inverse of A if $AB = I_n = BA$; it is then denoted A^{-1}

Question. *What is the inverse of:*

- I_n
- R_α
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Why searching the inverse of a matrix ? Common problem: find x such that $Ax = b$. This is not easy by itself, but can be solved easily if we know A^{-1}: $x = A^{-1}b$.

Relation between inverse function and inverse matrix:

Property. *$f : V_1 \to V_2$ with inverse $f^{-1} : V_2 \to V_1$: $M_{B_2,B_1}(f^{-1}) = M_{B_1,B_2}(f)^{-1}$*

How to compute the inverse of a matrix in practice?

- 2×2 matrices: explicit formula
- Gauss-Jordan elimination
- LU decomposition

In general, inverting a matrix takes $\mathcal{O}(n^3)$ operations for an $n \times n$ matrix.