Refresher Math Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. The course will try to cover most of the prerequisites of the courses in the Master, mainly linear algebra, differential calculus, integration, and asymptotic analysis.

Contents

Introduction

Presentation

- Paul Dubois
- will be teaching this refresher math course
- email (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h
- 1h20min lecture - $^1/_3$h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)
- Lectures are recorded (if ever needed)
- 1st lecture ever => too fast/too slow: let me know
- May assume you know a concept/notation that you have never heard of, let me know if this happens

Examination

- The course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be a full exercise sheet per lecture, it is advised to attempt it all (only one will be compulsory).
- Hand-in 1 exercise per lecture (i.e., 8 in total), due 2 weeks after the lecture
- Best $^{(n-1)}/_n$ count (i.e., best $^7/_8$ in our case), need avg $\geq 50\%$ to pass
- In the unlikely event of not passing, will be able to do an extra work

Questions?

Chapter 1

Sets & logic

1.1 Mathematical Objects & Notations

Sets

Definition (Sets)**.** *Unordered list of elements.*

Notation (Sets)**.** $\in$, $\{True, False\}$, $\{a \mid condition\}$, $\{a, b, c \dots\}$, $\emptyset$

Need to be careful when defining set: some definitions are pathological.

Remark (Russell Paradox)**.** *Take* $U = \{X \mid X \notin X\}$*. X in U => U not in U, U is a set, so not all sets are in U X not in U => X is a set*

Notation (Usual Sets)**.** $\mathbb{B}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}^*$, $\mathbb{R}^+$*...*

Functions

Definition (Functions)**.** *Assignment for a set to another.*

Notation (Function)**.** $f : X \to Y$*,* $f(x) = blah$*,* $f : x \mapsto blah$*.*

Definition (Predicate)**.** *Function to* $\mathbb{B}$

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation ($\forall$)**.** *For all elements in set, e.g.:* $\forall x \in \mathbb{R}, x^2 \geq 0$*.*

Notation ($\exists$)**.** *There exists an element in set, e.g.:* $\exists x \in \mathbb{R}$ *s.t.* $x^2 > 1$*.*

Notation ($\exists!$)**.** *There exists a unique element in set, e.g.:* $\exists! x \in \mathbb{R}$ *s.t.* $x^2 \leq 0$*.*

Definition (Subset / Inclusion)**.** $X \subseteq Y$ *if* $\forall x \in X, x \in Y$

Definition (Disjoint Sets). *X and Y are disjoint if $\forall x \in X, x \notin Y$ (or if $\forall y \in Y, y \notin X$).*

Definition (Powerset). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Definition (Cartesian Product). $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$
e.g.: $\{a,b\} \times \{1,2,3\} = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

1.2 Boolean algebra

Basic operators

Definition (Conjonction). $x \wedge y = xy$

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

Definition (Disjunction). $x \vee y = \min(x+y, 1)$

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation). $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

Question. *Selecting points outside a given region.*

Basic properties

Property (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of $\vee$:* $x \vee (y \vee z) = (x \vee y) \vee z$
- *Associativity of $\wedge$:* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- *Commutativity of $\vee$:* $x \vee y = y \vee x$
- *Commutativity of $\wedge$:* $x \wedge y = y \wedge x$
- *Distributivity of $\wedge$ over $\vee$:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- 0 *is identity for $\vee$:* $x \vee 0 = x$
- 1 *is identity for $\wedge$:* $x \wedge 1 = x$
- 0 *is annihilator for $\wedge$:* $x \wedge 0 = 0$

Property (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of $\vee$:* $x \vee x = x$
- *Idempotence of $\wedge$:* $x \wedge x = x$
- *Absorption of $\vee$ over $\wedge$:* $x \vee (x \wedge y) = x \wedge y$
- *Absorption of $\wedge$ over $\vee$:* $x \wedge (x \vee y) = x \vee y$
- *Distributivity of $\vee$ over $\wedge$:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 *is annihilator for $\vee$:* $x \vee 1 = 1$

Property (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) $\square$

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property (Implication and Inclusion). *If* $\forall x \in X, P_1(x) \implies P_2(x)$, *then* $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.

Proof. Trivial. $\square$

Definition (If and only if). $x \iff y$

Negation of quantified propositions

Property (Negation of $\forall$). $\text{not}(\forall x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Property (Negation of $\exists$). $\text{not}(\exists x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \ \ldots$ *is true ;* $\exists x \in \emptyset, \ \ldots$ *is false*

1.3 Python

=> use google colab'

Chapter 2

Proofs methods

2.1 Direct implication

Want to show A: may show B and $B \implies A$, or C and $C \implies B$ and $B \implies A$.

2.2 Case dis-junction

Split in cases.

E.g.: show n and n^2 have the same parity (take n odd then n even).

2.3 Contradiction

Suppose the opposite, derive a contradiction (i.e. A and $\not A$) and conclude.

E.g.: show $\sqrt{2} \notin \mathbb{Q}$ (suppose $\sqrt{2} = {}^a/_b$, WLOG $a, b \in \mathbb{N}$ co-prime).

2.4 Induction

Want to show P_n for $n \geq n_0$: show $P_n \implies P_{n+1}$ and P_{n_0}.

E.g.: show $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

2.5 Existence and Uniqueness

It is common to show existence and/or uniqueness.

E.g.: Existence and uniqueness in Euclidean division:

$$\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists!\ q \in \mathbb{Z}, r \in [0, b[\cap \mathbb{N} \text{ s.t. } a = bq + r$$

Use $q = \max\{k \in \mathbb{N} \mid bk \leq a\}$, $r = a - bq$.

Chapter 3

Functions Properties

$$f : X \to Y \quad A \subseteq X, B \subseteq Y$$

Definition (Image). $f(A) = \{y \in Y \mid \exists x \in A \ s.t. \ f(x) = y\}$

Definition (Inverse Image). $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

Definition (Fiber). *Fiber of y is inverse image of $\{y\}$.*

Definition (Well definedness). $\forall x \in X, \exists! y \in Y \ s.t. \ f(x) = y$

Definition (Injectivity). $\forall x, x' \in X, x \neq x', f(x) \neq f(x')$

Definition (Surjectivity). $\forall y \in Y, \exists x \in X \ s.t. \ f(x) = y$

Definition (Bijectivity). *Injectivity plus Surjectivity:* $\forall y \in Y, \exists! x \in X \ s.t. \ f(x) = y$

Definition (Invertibility). $f^{-1} : Y \to X$ *well defined.*

Remark (Alternative Definition of Inverse). $f \circ f^{-1} = Id\,|_X$ *and* $f^{-1} \circ f = Id\,|_Y$

Remark (Invertibility and Bijectivity). f *bijective* $\iff$ f *invertible.*

Remark (Inverse is Invertible). f^{-1} *is invertible, and* $(f^{-1})^{-1} = f$.

Property (Injections between finite intervals). $m, n \in \mathbb{N}^*$, *there exists an injection* $f : [\![1; m]\!] \to [\![1; n]\!]$ *if and only if* $m \leq n$.

Proof. By induction on m, carefully checking $m \leq n$. □

Property (Bijections between finite intervals). $n, m \in \mathbb{N}^*$, *there exists a bijection* $f : [\![1; m]\!] \to [\![1; n]\!]$ *if and only if* $m = n$.

Proof. Use last property & inverse. □

Property (Compositions). *Composition preserve injectivity/surjectivity/bijectivity/invertibility:*
$f : X \to Y, \ g : Y \to Z$ *injectives* $\implies$ $f \circ g$ *is injective*
$f : X \to Y, \ g : Y \to Z$ *surjectives* $\implies$ $f \circ g$ *is surjective*
$f : X \to Y, \ g : Y \to Z$ *bijections/invertibles* $\implies$ $f \circ g$ *is bijective/invertible*

Proof. Trivial. □

Property. *An injection between two sets of the same size is bijective.*

Proof. By contradiction. □

Chapter 4

Finite Cardinalities

Definition (Cardinality)**.** *For finite sets:*
Intuitively: $|X| = n \in \mathbb{N}$ *if there are* n *elements in the set.*
Mathematically: $|X| = n \in \mathbb{N}$ *if there is a bijection between* X *and* $[\![1, n]\!]$*.*

Property (Cardinality of Disjoints)**.** X, Y *disjoint sets:* $|X \cup Y| = |X| + |Y|$
Extension: $X_1, \ldots, X_n$ *pairwise disjoint sets (i.e.* $X_i \cap X_j = \emptyset \ \forall i \neq j$*):* $|\bigcup_{k=1}^{n} X_k| = \sum_{k=1}^{n} |X_k|$

Proof. Shift bijection of Y by $|Y|$; use induction. ☐

Property (Cardinality of Complement)**.** $X \subseteq Y$*:* $|Y \setminus X| = |Y| - |X|$

Proof. Use previous property with X & $Y \setminus X$ disjoint. ☐

Property (Cardinality of Cartesian Products)**.** X, Y *sets:* $|X \times Y| = |X| * |Y|$
Extension: $X_1, \ldots, X_n$ *sets:* $|\prod_{k=1}^{n} X_k| = \prod_{k=1}^{n} |X_k|$

Proof. $X \times \{y_k\}$ are all disjoint for $k \in [\![1, |Y|]\!]$; use induction. ☐

Property (Cardinality of Sets of Functions)**.** $|\{f : X \to Y\}| = |Y|^{|X|}$

Proof. Just count! ☐

Property (Cardinality of Sets of Injections)**.** $|\{f : X \to Y \mid f \textit{ injective}\}| = \frac{|Y|!}{(|Y|-|X|)!}$

Proof. Count (without repetition). ☐

Property (Cardinality of Sets of Surjections)**.** $|\{f : X \to Y \mid f \textit{ surjective}\}| = |Y|^{|X|} - |Y| * (|Y| - 1)^{|X|}$

Proof. All functions but the non surjective ones. ☐

Property (Cardinality of Sets of Bijections)**.** $|\{f : X \to Y \mid f \textit{ bijective}\}| = |Y|! = |X|!$

Proof. Bijection is an injection between two sets of the same size. ☐

Chapter 5

Infinite Cardinalities

Definition (Alphabet). $\mathcal{A} = \{a, b, c, \ldots, z\}$

To compare the size of infinite sets, we use bijections, injections:

Definition (Comparing Sets). $f : X \to Y$ *injective* $\implies |X| \leq |Y|$ $f : X \to Y$ *surjective* $\implies |X| \geq |Y|$ $f : X \to Y$ *bijective* $\implies |X| = |Y|$

Note that together with $|\,[1, n]\,| = n$, this defines cardinality.

Definition (Countable sets). *A set is countable if it has the same cardinality as the naturals (i.e. X is countable if $|X| = |\mathbb{N}|$).*

Property (Countable Union Finite). $|\mathbb{N} \cup \mathcal{A}| = |\mathbb{N}|$

Property (Countable Union Countable / Integers). $|\mathbb{Z}| = |\mathbb{N} \cup \mathbb{N}^*| = |\mathbb{N}|$

Property (Countable Union of Finites). $|X_n| < \infty \ \forall n \in \mathbb{N} \implies |\bigcup_{n \in \mathbb{N}} X_n| = |\mathbb{N}|$

Property (Countable Union of Countables / Rationals). $|\mathbb{Q}| = |\bigcup_{n \in \mathbb{N}^*} \{{}^{m}\!/\!{}_{n} \mid m \in \mathbb{Z}\}| = |\mathbb{N}|$

Property (Power set of Countables / Reals). $|\,[0, 1[\,| = |\mathcal{P}(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}| > |\mathbb{N}|$

Chapter 6

Spaces

Mathematical Space: Object based on a set with more structure.

6.1 Metric Space

A metric space is a set X together with a metric distance $d : X \times X \to \mathbb{R}^+$.
d is a metric if it satisfies the following axioms:

- Non-degenerative: $d(x, y) = 0 \iff x = y$
- Symmetric: $d(x, y) = d(y, x)$
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

6.2 Norm Space

A norm space is a set X together with a norm $|\_| : X \to \mathbb{R}^+$.
$|\_|$ is a norm if it satisfies the following axioms:

- Non-degenerative: $|x| = 0 \iff x = 0$
- Homogeneity: $|\lambda x| = \lambda |x| \qquad \lambda \in \mathbb{R}^+$
- Triangle inequality: $|x + y| \leq |x| + |y|$

Property (Norm Implies Metric)**.** *Letting* $d(x, y) = |x - y|$.

6.3 Inner Product Space

An inner product space is a set X together with an inner product $\langle \_, \_ \rangle : X \times X \to \mathbb{C}$.
$\langle \_, \_ \rangle$ is an inner product if it satisfies the following axioms:

- Linear (in 1$^{\text{st}}$ argument): $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \lambda \in \mathbb{C}$ and $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$
- Conjugate symmetry: $|x + y| \leq |x| + |y|$
- Positive definiteness $\langle x, x \rangle > 0 \; \forall x \neq 0$
- *(implied)* Non-degenerative: $\langle x, 0 \rangle = 0$ and $\langle 0, x \rangle = 0$
- *(implied)* Conjugate linear (in 2$^{\text{nd}}$ argument): $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle \quad \lambda \in \mathbb{C}$ and $\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle$

Property (Inner Product implies Norm). *Letting* $|x| = \sqrt{\langle x, x \rangle}$.

Definition (Orthogonal / Normal). $xyorthogonal \iff \langle x, y \rangle = 0$

Property (Pythagoras Theorem). $xyorthogonal \implies |x+y|^2 = |x|^2 + |y|^2$

Property (Parallelogram Identity). $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$

Property (Polarization Identity). $4\langle x, y \rangle = |x+y|^2 - |x-y|^2 + i(|x+iy|^2 - |x-iy|^2)$

Chapter 7

Limit Behaviors

7.1 Convergence & Divergence

Definition ($(x_n) \subseteq \mathbb{F}$ converges to $x \in \mathbb{F}$)**.** $\forall \varepsilon > 0,\ \exists N\ s.t.\ \forall n \geq N,\ d(x_n, x) < \varepsilon$

We write $x_n \xrightarrow[n\to+\infty]{} x$ or $\lim\limits_{n\to+\infty} x_n = x$. Note that convergence is defined w.r.t. a metric (or a norm/inner product, which induces a metric).

Definition ($(x_n) \subseteq \mathbb{R}$ diverges to $+\infty$)**.** $\forall M \in \mathbb{R},\ \exists N\ s.t.\ \forall n \geq N,\ x_n > M$

We write $x_n \xrightarrow[n\to\infty]{} +\infty$ or $\lim\limits_{n\to+\infty} x_n = +\infty$. Note that divergence is only defined over $\mathbb{R}$; divergence to $-\infty$ is defined similarly.

Definition (Sub-sequence)**.** $\phi : \mathbb{N} \to \mathbb{N}$ *strictly increasing defines the sub-sequence* $(x_{\phi(n)})$ *of the sequence* (x_n)*.*

Property (Convergence & Divergence of Sub-sequences)**.** $x_n \to x \implies x_{\phi(n)} \to x$ *&* $x_n \to +\infty \implies x_{\phi(n)} \to +\infty$

Definition ($f : X \to Y$ converges to $y \in Y$ at $x \in X$)**.** $\forall \epsilon > 0,\ \exists \delta > 0\ s.t.\ d_X(x, x') < \delta \implies d_Y(y, y') < \epsilon$

Equivalent definition: $\forall x_n \to x$ as $n \to +\infty, y_n = f(x_n) \to y$; we write $\lim\limits_{x'\to x} f(x') = y$

Question.

- $\lim\limits_{x\to a} \phi(f(x))$
- $\lim\limits_{x\to a} f(x) + g(x)$
- $\lim\limits_{x\to a} f(x) * g(x)$
- $\lim\limits_{x\to a} f(x)/g(x) \qquad g(x) \neq 0$

Proof. left as exercise ☐

Property ("Determinate Forms")**.** $\frac{1}{0} = \infty,\ \frac{1}{\infty} = 0$

Property ("Indeterminate Forms")**.** $\frac{0}{0},\ \frac{\infty}{\infty},\ 0 \times \infty,\ 1^\infty,\ \infty - \infty,\ 0^0,\ \infty^0$

E.g.: $x^2 \times \frac{1}{x} \to \infty$; $x^2 \times \frac{1}{x^2} \to 1$; $x^2 \times \frac{1}{x^3} \to 0$.

Theorem 7.1.1 (Fixed Point Theorem)**.** $x_{n+1} = f(x_n)$ *and* $(x_n) \to l \implies l = f(l)$ *(i.e. l is a fixed point of f).*

Proof. easy: x_n and x_{n+1} must both go to l ☐

E.g.: $x_0 = 1$ and $x_{n+1} = \frac{x_n^2+2}{2x_n}$ give $x_n \to \sqrt{2}$.

7.2 Maximum vs Supremum

Definition (a maximum of A)**.** *$a \in A$ and $\forall x \in A, x \leq a$*

Maximum doesn't always exists (even if A is bounded).

Definition (a supremum of A)**.** *$\forall \epsilon > 0, \exists\, x \in A$ s.t. $a - \epsilon \leq x \leq a$*

Supremum is the "smallest upper bound". Exists if A is bounded.

Question. *Max, sup of* $[0, 1]$, $[0, 1[$, $\mathbb{R}^+$.

Remark. *Can define minimum & infimum similarly*

Theorem 7.2.1 (Extremum & Convergence)**.** *$(x_n) \subseteq \mathbb{R}$ increasing:*

- *if (x_n) is upper-bounded, then* $\lim\limits_{n \to +\infty} x_n = \sup \{x_n \mid n \in \mathbb{N}\}$
- *else,* $\lim\limits_{n \to +\infty} x_n = +\infty$

Proof. easy (by cases) □

7.3 Continuity

Definition (f continuous at x)**.** $\lim\limits_{x' \to x} f(x') = f(x)$

Definition (f continuous on X)**.** *$\forall x \in X, f$ continuous x*

Question. *Show x^n is continuous (for all n).*

"can be plotted in a single trace/line; without lifting the pen" [Lipschitz-continuous??]

7.4 Asymptotic Analysis

Definition (Asymptote)**.** *"A curve is a line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity."*
i.e. $\lim\limits_{x \to \infty} f(x) - l(x) = 0$ *(in the case of $x \to \infty$).*

Horizontal Asymptote E.g.: $f(x) = \frac{x+1}{x}$ (asymptote is $y = 1$ as $x \to \infty$).

Vertical Asymptote E.g.: $g(x) = \frac{1}{x-2}$ (asymptote is $x = 2$ as $y \to \infty$).

Oblique Asymptote E.g.: $h(x) = \frac{3x^2+2x+1}{x}$ (asymptote is $y = 3x + 2$ as $x, y \to \infty$).

7.5 Series

Joke: I once asked someone out to "checkout some series", they went home disappointed... still don't know why.

Definition (Series)**.** *A series is a sequence (S_n) with general term x_n defined by $S_n = \sum_{k=0}^{n} x_k$. It is alternating if $x_k x_{k+1} < 0$ $\forall k \in \mathbb{N}$.*

Definition (Series Convergence)**.** *The series (S_n) converges if $(\sum_{k=0}^{n} x_k)$ converges as a sequence. The series (S_n) converges absolutely if $(\sum_{k=0}^{n} |x_k|)$ converges as a sequence.*

E.g.: $\sum_{k=1}^{n} 1$ is obviously divergent; $\sum_{k=1}^{n} \frac{1}{2^k}$ is convergent (to 1).

Property. *$\sum_{k=0}^{n} a^k$ is:*

- *Absolutely convergent for $|a| < 1$, converging to $\frac{1}{1-a}$.*
- *Divergent for $|a| \geq 1$, bounded for $a = -1$, unbounded else.*

Proof. easy (sum of geometric series) $\square$

Property (Necessary Condition for Convergence of Series)**.** *If (S_n) converges, then $x_n \to 0$.*

Proof. trivial (by contradiction) $\square$

However, his is **not** a sufficient condition: $\sum_{k=1}^{n} \frac{1}{k}$ is a counter-example.

Property (Criterion for Convergence of Alternating Series)**.** *If $\sum_{n\in\mathbb{N}} x_n$ is alternating, $(|x_n|)$ is decreasing, and $\lim\limits_{n\to\infty} x_n = 0$, then $\sum_{n\in\mathbb{N}} x_n$ converges.*

Proof. WLOG, $x_{2n} > 0$ and $x_{2n-2} < 0$: $\sum_{k=0}^{2n} x_k$ is increasing, and upper bounded by $x_0 + x_1$, therefore converges; similarly, $\sum_{k=0}^{2n+1} x_k$ is decreasing, and lower bounded by x_0, therefore converges as well. $\sum_{k=0}^{2n} x_k$ and $\sum_{k=0}^{2n+1} x_k$ must have the same limit as $\sum_{k=0}^{2n+1} x_k - \sum_{k=0}^{2n} x_k = x_{2n+1} \to 0$. Thus, $\sum_{k=0}^{2n} x_k$ must be convergent. $\square$

Property (Comparison Test for Convergence of Series)**.** *$\forall n \geq n_0, 0 \leq a_n \leq b_n$:*

- *If $\sum_{n\in\mathbb{N}} b_n$ converges, then $\sum_{n\in\mathbb{N}} a_n$ converges as well.*
- *If $\sum_{n\in\mathbb{N}} a_n$ diverges, then $\sum_{n\in\mathbb{N}} b_n$ diverges as well.*

Proof. easy by def $\square$

E.g.: $\sum_{n\geq 2} \frac{1}{n^2} \leq \sum_{n\geq 2} \frac{1}{n(n+1)} = \sum_{n\geq 2} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{2} < \infty$

Property (Integration Test for Convergence of Series)**.** *$\sum_{n\in\mathbb{N}} f(n) \leq \int_{x=0} \infty f(x)$*
So if $\int_{x=0} \infty f(x) < \infty$, and $f(x)$ is decreasing, then $\sum_{n\in\mathbb{N}} f(n)$ converges.

Proof. easy by def $\square$

E.g.: $\sum_{n\geq 2} \frac{1}{n^2} \leq \int_{x=2}^{\infty} \frac{1}{x^2} = \left[-\frac{1}{x}\right]_{x=2}^{x=\infty} = -0 + \frac{1}{2} < \infty$

Chapter 8

Smoothness

Want to measure how "steep" a curve is at a pt x_0: take linear approx. from x_0 to x (take the steep of the line), and let $x \to x_0$.
Formally:

Definition. $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ *f is differentiable at x_0 if the limit $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. f is differentiable on I if the limit $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists for all $x_0 \in I$.*

Property (Differentiable implies the continuous)**.** *f differentiable at $x_0 \implies f$ continuous at x_0*

Proof. Easy $\square$

The absolute value is continuous but not differentiable at $x = 0$.
The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere.

Property (Operations on Derivatives)**.**

- $(f+g)' = f' + g'$
- $(f * g)' = f' * g + f * g'$ *"product rule"*
- $(f/g)' = \frac{f' * g + f * g'}{g^2} f' + g'$ $\quad g \neq 0$ *"quotient rule"*
- $(f(g))' = f'(g) * g'$ *"chain rule"*

Proof.

- linearity of limits
- from def
- from def
- from def

$\square$

Property (Sign of the derivative)**.** *$f' > 0$ on $I \implies f$ strictly increasing on I*

Proof. Clear graphically; mathematically, use mean value theorem. $\square$

Can (try to) differentiate the derivative f' of f, giving $f'' = f^{(2)}$. Can then (try to) differentiate f'' giving $f''' = f^{(3)}$.

Definition. *$C[I]$ is the set of continuous functions on I.*

- *If f' exists, f is differentiable.*
- *If f'' exists, f is twice differentiable.*

- *If* $f^{(k)}$ *exists,* f *is* k*-times differentiable.*
- *If* f' *exists and is continuous, then* f *is continuously differentiable.*
- *If* f'' *exists and is continuous, then* f *is twice continuously differentiable.*
- *If* $f^{(k)}$ *exists and is continuous, then* f *is* k*-times continuously differentiable.*

$C^k[I]$ *is the set of functions* k*-times continuously differentiable on* I*.*

Chapter 9

Integration

Integral of f: area under a curve of f (draw scheme).

Proof. Take f continuous; let $A(x)$ be the area under the curve of f, from 0 to x.

Then $A(x+h) - A(x) = f(x).h + \epsilon(h)$; as $h \to 0$, $\epsilon(h) \to 0$ by continuity of f. Then get $f(x) = \frac{A(x+h)-A(x)}{h}$ as $h \to 0$. Thus, $A'(x) = f(x)$. □

Notation. *Integral from a to b of f is $\int_a^b f(x)dx$*

Theorem 9.0.1 (Fundamental Theorem of Calculus)**.** *If $F(x) = \int_a^x f(t)dt$, then F is uniformly continuous and differentiable, with derivative $F' = f$*

Corollary 9.0.1. *$\int_a^b f(x)dx = F(b) - F(a)$ where F is an anti-derivative of f (i.e. $F' = f$[1])*

Integrals may also be approximated via partial sums; this is how computers calculate integrals (draw picture).

Integration is the "inverse" of differentiation: $\int f = F + C$ where $C \in \mathbb{R}$ and $F' = f$. So $\int f' = f + C$ and $\left(\int f\right)' = f$

Note that not all functions are integrable in terms of elementary functions (e.g.: $\frac{\sin(x)}{x}$). Note that not all functions are integrable in terms of area under the curve either (e.g.: $f(x) = 0$ if $x \in Q, f(x) = 1$ if $x \notin \mathbb{Q}$). However, "most" usual functions are integrable in terms of area under the curve (any continuous or monotone function is, so usually do not worry about it in applied maths).

Note that integrable does **not** imply differentiable/continuous (e.g. floor function); and differentiable does **not** imply anti-derivative exists in terms of elementary functions (e.g. $\frac{\sin(x)}{x}$). [draw diagram of implications: integrable area; integrable anti-derivative; continuous; differentiable]

[1] Anti-derivative are **not** unique (can add a constant).

Chapter 10

Elementary Functions

10.1 Enumeration

[Enumeration of the elementary function by Liouville from 1833 to 1841:]

- Polynomials function of $\mathbb{R}[x]$: 1, x, $\pi + 3.2x^2 + \frac{7}{8}x^{2021}$, ...
- Hyperbolic functions: exponential (e^x), hyperbolic sinus ($\sinh(x) = \frac{e^x - e^{-x}}{2}$), hyperbolic cosinus ($\cosh(x) = \frac{e^x - e^{-x}}{2}$), ...
- Trigonometric functions: cos, sin, tan,...
- Inverse functions of the previous functions: logarithmic functions, inverse trigonometric, ...

Closed under derivative, but not under integration:
e.g.: $\int_a^b \frac{\sin(x)}{x} dx = \mathrm{Si}(b) - \mathrm{Si}(a)$ where $\mathrm{Si}(x)$ cannot be expressed in terms of elementary functions (it is defined by the area under the curve of $\frac{\sin(x)}{x}$).

10.2 Properties

10.2.1 Polynomials Functions

Definition. $\mathbb{R}[x]$ *is the set of polynomials of* x *with real coefficients.*

Property. *All polynomials are continuous & differentiable, with* $(x^n)' = nx^{(n-1)}$.

Proof. x is continuous, then use algebra of continuous functions & by induction. □

Corollary 10.2.1. $(x^n)^{(k)} = \frac{n!}{(n-p)!} x^{n-p} = p!\binom{n}{p} x^{n-p}$

Degree

Definition. $P = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x] \implies \partial P = n$
by convention, $\partial P = -\infty$ *if* $P \equiv 0$

Property (Algebra of degrees)**.**

- $\partial P + Q = \max(\partial P, \partial Q)$
- $\partial P * Q = \partial P + \partial Q$

Proof. Exercise □

Roots

Definition (Root orders)**.** *$P \in \mathbb{R}[x]$ has a root at x if $P(x) = 0$ $P \in \mathbb{R}[x]$ has a root of order n at x if $P^{(k)}(x) = 0$ for all $k < n$ and $P^{(n)}(x) \neq 0$*

[draw graphical interpretation of roots multiplicities]

Property (Roots and Factorization)**.** *P has a root of order k at x' iff $P(x) = (x - x')^k Q(x)$ where Q is a polynomial s.t. $\partial P = k + \partial Q$.*

Proof. definition for $\implies$ direction, Taylor formula for $\impliedby$ direction ☐

Corollary 10.2.2 (Degree and Number of roots)**.** *If roots of P have multiplicities $k_1, k_2, k_3, \ldots, k_n$, then $\sum_{i=1}^n k_i \leq \partial P$. "#roots $\leq$ degree"*

Proof. easy using previous property ☐

The Constant case No root(s) except for $P \equiv 0$

The Linear case One root: $P(x) = ax + b \implies x = -\frac{b}{a}$ is the only root.

The Quadratic case

$$P(x) = ax^2 + bx + c \qquad \Delta = b^2 - 4ac$$

- $\Delta > 0$: P has two roots: $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$
- $\Delta = 0$: P has one root: $x_0 = \frac{-b}{2a}$
- $\Delta > 0$: P has no root(s)

Proof. "force factorization": $ax^2 + bx + c = 0 \iff \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{c}{a} = 0$ which has solution(s) only if $\Delta \geq 0$ ☐

Taylor formula

Theorem 10.2.1 (Binomial theorem)**.** $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^n y^{n-k}$

Proof. by induction ☐

Theorem 10.2.2 (Taylor for polynomials)**.** $P(x) = \sum_{k=0}^{+\infty} \frac{P^{(k)}(\alpha)}{p!}(x - \alpha)^k$

Proof. prove it for $P(x) = x^n$, using the binomial theorem ☐

10.2.2 Hyperbolic Functions

Exponential

Proposition. *The series $\sum_{n \in \mathbb{N}} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.*

Proof. easy ☐

Definition. *The series $\sum_{n \in \mathbb{N}} \frac{x^n}{n!}$ is called the exponential function, and denoted* $\exp(x)$ *or* e^x.

Property.
- $\exp$ *is continuous*
- $\exp$ *is differentiable, and* $\exp' = \exp$
- $\exp(x + y) = \exp(x)\exp(y)$

- $\exp(-x) = 1/\exp(x)$
- $\exp(x) > 0$
- $\exp$ *is increasing on all* $\mathbb{R}$
- $\lim_{x \to +\infty} \exp(x) = +\infty$ *and* $\lim_{x \to -\infty} \exp(x) = 0$
- $\lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty$ *"e^x grows "faster" than x^n for any n"*

Proof.
- technical (on any segment of $\mathbb{R}$, the partial sum converges uniformly to exp; partial sums are continuous, so exp is continuous on all segments of $\mathbb{R}$, thus continuous on $\mathbb{R}$)
- differentiate each term in partial sum
- from series def & binomial theorem
- corollary, ask audience
- corollary, ask audience
- corollary, ask audience
- $e^x > x$ for $x > 0$ gives limit as $x \to +\infty$, then use inverse

□

[Draw exp curve]

Logarithmic

Definition. *The inverse function of* exp *is* ln *or* log*:* $\ln(x) = y$ *s.t.* $x = e^y$. *Note* $\exp : \mathbb{R} \to \mathbb{R}^{+*}$ *so* $\ln : \mathbb{R}^{+*} \to \mathbb{R}$, *so* $\exp(\ln(x)) = x \ \forall x \in \mathbb{R}^{+*}$ *and* $\ln(\exp(x)) = x \ \forall x \in \mathbb{R}$.

[Draw log curve]

Property.
- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln(x/y) = \ln(x) - \ln(y)$
- $\ln'(x) = 1/x$
- $\ln(0) = 1$
- $\lim_{x \to +\infty} \ln(x) = +\infty$ *and* $\lim_{x \to 0} \ln(x) = -\infty$
- $\lim_{x \to +\infty} \frac{\ln(x)}{x^\epsilon} = 0$ *"$\ln(x)$ grows "slower" than x^ϵ for any $\epsilon > 0$"*

Proof.
- use properties of exp
- use properties of exp
- use properties of exp
- use $\exp(0) = 1$
- use limits of exp

□

fun fact: cosh is the shape of a rope attached at both ends

10.2.3 Trigonometric Functions

[draw triangle def of cos, sin & tan, then graph them, observe periodicity, observe sin is cos "shifted" by pi/2; observe the location of zeros; write math def of these observations]

Property (Derivatives & Integrals of Trigonometric Functions).

- $\sin' = \cos$
- $\cos' = -\sin$
- $\tan' = 1/\cos^2$
- $\int \sin = -\cos + C$
- $\int \cos = \sin + C$
- $\int \tan = -\ln(|\cos|) + C$

Proof.

- technical
- technical
- use quotient rule
- use derivative result
- use derivative result
- technical, can be checked easily

□