Problem Set 3

Due 20$^{\text{th}}$ September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Metric/Norm Spaces

Question 1. *(*)* $X, Norm(\_)$ *is a norm space; we define* $d(x,y) = Norm(x - y)$*, so that* X, d *is a metric space. We have seen what the ball of radius* r *centered at* x *is in a metric space (i.e. using* d*). Define it in terms of then norm.*

Question 2. *(*) We take* $X = \mathbb{R}^2$*, draw* $B_1((0,0))$ *(the unit ball) for the following norms:*

- $Norm((x,y)) = |x| + |y|$

- $Norm((x,y)) = \sqrt{x^2 + y^2}$

- $Norm((x,y)) = \max(|x|, |y|)$

- $Norm((x,y)) = 0$ *if* $(x,y) = (0,0)$*,* $Norm((x,y)) = 1$ *else*

- $Norm((x,y)) = 0$ *if* $(x,y) = (0,0)$*,* $Norm((x,y)) = 2$ *else*

Question 3. *For which values of* $p \geq 0$ *is* $Norm((x,y)) = \sqrt[p]{|x|^p + |y|^p}$ *a norm for* $\mathbb{R}^2$*?*

Question 4. *Find two different norms for* $\mathbb{R}^n$ *(give their definitions).*

2 Sequences Convergence

Question 5. *Find the limit of the following sequences, and prove your claim carefully:*

- *(*)* $x_n = \frac{1}{n^2}$

- $x_n = \frac{1}{n^3}$

- $x_n = \frac{1}{n^k}$ *for some* $k > 0$

- *(*)* $x_n = \frac{1}{2^n}$

- $x_n = \frac{1}{3^n}$

- $x_n = \frac{1}{k^n}$ *for some* $k > 1$

- *(*)* $x_n = \frac{3n^2 + 2n + 1}{6n^2 - n + 2}$

- $x_n = \frac{2n+1}{6n^2 + 9n - 5}$

- *(hard)* $x_n = \frac{n}{2^n}$

Question 6. $x_n \to x$, $y_n \to y$; *Prove the following:*

- *(*)* $x_n + y_n \to x + y$

- $x_n * y_n \to x * y$

- *if* $y_n \neq 0$, *and* $y \neq 0$: $x_n/y_n \to x/y$