Problem Set 1: Solutions

Q1. a. $R = [-1,3] \times [-1,3] \quad (= [-1,3]^2)$

$G = [0,1] \times [0,1]$

$B = [\tfrac{1}{2}, \tfrac{3}{2}] \times [\tfrac{1}{2}, \tfrac{3}{2}]$

b. $G \subseteq R$

$B \subseteq R$

c. $(G \cup B) \setminus (G \cap B) = \left([0,\tfrac{1}{2}] \times [0,1]\right) \cup \left([\tfrac{1}{2},1] \times [0,\tfrac{1}{2}]\right)$
$\cup \left([\tfrac{1}{2},1] \times [1,\tfrac{3}{2}]\right) \cup \left([1,\tfrac{3}{2}] \times [\tfrac{1}{2},\tfrac{3}{2}]\right)$

d. $[\tfrac{1}{2},1]^2 = G \cap B$

e. $\left([\tfrac{1}{2},\tfrac{3}{2}] \times [1,\tfrac{3}{2}]\right) \cup \left([1,\tfrac{3}{2}] \times [\tfrac{1}{2},\tfrac{3}{2}]\right) = B \setminus G$

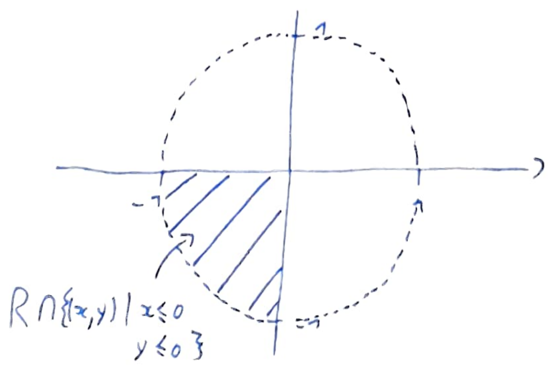
Q2. a. $R = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \}$

$G = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \leq 3,\ x-y \geq 2 \}$

$B = \{ (x,y) \in \mathbb{R}^2 \mid x \geq \tfrac{1}{2},\ y \geq \tfrac{1}{2},\ x+y \leq 2 \}$

b. $R \cap B = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1,\ x \geq \tfrac{1}{2},\ y \geq \tfrac{1}{2} \}$

c.



d.

e. $B \subseteq \mathcal{H}$, $R \not\subseteq \mathcal{H}$, $G \not\subseteq \mathcal{H}$

Q3. a. $abc + \bar{a}bc + a\bar{b}c + ab\bar{c}$

$= abc + \bar{a}bc + abc + a\bar{b}c + abc + ab\bar{c}$ ↲ $x = x + x$ (twice)

$= (a+\bar{a})bc + a(b+\bar{b})c + ab(c+\bar{c})$ ↲ $xy + xz = x(y+z)$ (3 times)

$= bc + ac + ab$ ↲ $x + \bar{x} = 1$ (3 times)

b.

a	b	c	$ab+ac+bc$	$abc + \bar{a}bc + a\bar{b}c + ab\bar{c}$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1+1+1=1	1

match!

c. $abc + ab\bar{c} + a\bar{b}cd$

$= abc + ab\bar{c} + abcd + a\bar{b}cd$ ↲ $x = x + xy$, $x = abc$, $y = d$

$= ab(c+\bar{c}) + a(b+\bar{b})cd$ ↲ Factorize (twice)

$= ab + acd$ ↲ simplify $x + \bar{x} = 1$ (twice)

Q4. twos of $a+b+c$: $abc + \bar{a}bc + a\bar{b}c + ab\bar{c} = ab + ac + bc$ (using Q3. a)

units of $a+b+c$: $abc + (a \oplus b \oplus c)$ ($\oplus$ is "exclusive or")

or $abc + a\bar{b}\bar{c} + \bar{a}b\bar{c} + \bar{a}\bar{b}c$

Q5. $\neg(\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |x_n - x| < \varepsilon)$

$= \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n > N \text{ s.t. } |x_n - x| \geq \varepsilon$

Q6. a. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } y > x$

b. $\forall n \in \mathbb{N}, n \geq 0$

c. $\nexists x \in \mathbb{R} \text{ s.t. } x^2 < 0$

d. $\exists f : \mathbb{N} \to \mathbb{Q} \text{ s.t. } f \text{ is bijective}$

Q7. a.

$n \equiv$	0	1	2	[3]
$n^2 \equiv$	0	1	1	[3]

so $3 | n^2 \Rightarrow n^2 \equiv 0\ [3]$

$\Rightarrow n \equiv 0\ [3] \Rightarrow 3 | n$

b.

$n \equiv$	0	1	2	3	4	[5]
$n^2 \equiv$	0	1	4	4	1	[5]

so $5 | n^2 \Rightarrow n^2 \equiv 0\ [5]$

$\Rightarrow n \equiv 0\ [5] \Rightarrow 5 | n$

c. let $n \in \mathbb{N}$, with prime decomposition as follows:

$n = \prod_{i=1}^{k} p_i^{\alpha_i}$ where $p_i \in \mathbb{P}$, $\alpha_i \in \mathbb{N}^*$, $p_i \neq p_j\ \forall i \neq j$

so $n^2 = \prod_{i=1}^{k} p_i^{2\alpha_i}$

Now, if $p \in \mathbb{P}$ s.t. $p | n^2 \Rightarrow p = p_i$ for some $i \in [\![1, k]\!]$

but then $p | n$ as well

Q8. a. $6 | n \Leftrightarrow 2 | n\ \&\ 3 | n$;

$(\Rightarrow)$ $6 | n \Rightarrow n = 6k \Rightarrow n = 2.3.k = 2.k' = 3.k'' \Rightarrow 2 | n\ \&\ 3 | n$

where $k' = 3k$, $k'' = 2k$

$(\Leftarrow)$ $2 | n\ \&\ 3 | n \Rightarrow n = 2k = 3\tilde{k}$

$2 \nmid 3$ so $2 | \tilde{k} \Rightarrow \tilde{k} = 2\tilde{\tilde{k}}$ so $n = 3.2.\tilde{\tilde{k}} = 6\tilde{\tilde{k}}$

$\Rightarrow 6 | n$

b. supp. $\sqrt{3} = \frac{a}{b}$, $a \in \mathbb{N}\ \&\ b \in \mathbb{N}^*$, $\gcd(a, b = 1)$

$\Rightarrow \sqrt{3}\, b = a$

$\Rightarrow 3b^2 = a^2$

so $3 | a^2 \Rightarrow 3 | a$ (by Q7.a.)

so $a = 3\tilde{a}$

$3b^2 = 3^2 \tilde{a}^2 = 9\tilde{a}^2$

$\Rightarrow b^2 = 3\tilde{a}^2$

so $3 | b^2 \Rightarrow 3 | b$ (by Q7.a.)

Thus, $3 | a, b \Rightarrow \gcd(a, b) \neq 1$ # $\square$

c. $12n-6 = 6\underbrace{(2n-1)}_{\in \mathbb{N}}$ so $12n-6$ is divisible by 6.

d. $\underline{n=0}$: $2^0=1$, $2.0=0$

$1>0 \Rightarrow 2^0 \geqslant 2.0$

$\underline{n=1}$: $2^1=2$, $2.1=2$

so $2^1 \geqslant 2.1$

$\underline{\text{Induction}}$: Supp. $2^n \geqslant 2.n$

want $2^{n+1} \geqslant 2(n+1)$

$$
\begin{aligned}
2^{n+1} &= 2.2^n \geqslant 2.2n = 2n+2n \\
&\geqslant 2n+2.1 \qquad \swarrow \; n \geqslant 1 \\
&= 2(n+1) \qquad \checkmark
\end{aligned}
$$

Thus, $2^n \geqslant 2.n \quad \forall n \in \mathbb{N}$

Q9. a. Supp. $g\circ f(x) = g\circ f(x')$

$\Rightarrow f(x) = f(x')$ $\swarrow$ as g is inj.

$\Rightarrow x = x'$ $\swarrow$ as f is inj.

Thus, $g\circ f$ is inj.

b. $\left.\begin{array}{l} \forall z \in Z,\ \exists y \in Y \text{ s.t. } g(y)=z \\ \forall y \in Y,\ \exists x \in X \text{ s.t. } f(x)=y \end{array}\right\}$ $\forall z \in Z, \exists x \in X$ s.t. $g\circ f(x)=z$

Thus, $f\circ g$ is surj.

c. $\left.\begin{array}{l} f,g \text{ inj.} \Rightarrow f\circ g \text{ inj.} \\ f,g \text{ surj.} \Rightarrow f\circ g \text{ surj.} \end{array}\right\}$ f,g bij. $\Rightarrow f\circ g$ bij.