Problem Set 5 - Solutions

Q 1. By induction:

$\underline{n=1}$: $S_{2^1} = \sum_{k=1}^{2^1} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \geqslant \frac{1}{2}$ ✓

$\underline{n+1}$: Suppose $S_{2^n} \geqslant \frac{n}{2}$

Need $S_{2^{n+1}} \geqslant \frac{n+1}{2}$:

$$S_{2^{n+1}} = S_{2^n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}$$

Induction hypothesis & $k \leq 2^{n+1} \Rightarrow \frac{1}{k} \geqslant \frac{1}{2^{n+1}}$

$$\geqslant \frac{n}{2} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}}$$

$$\geqslant \frac{n}{2} + 2^n \frac{1}{2^{n+1}} = \frac{n}{2} + \frac{1}{2} = \frac{n+1}{2} \quad ✓$$

Thus, $S_{2^n} \geqslant \frac{n}{2}$

Now, $\frac{n}{2} \to +\infty$ as $n \to +\infty$ so $S_{2^n} \to +\infty$ as $n \to +\infty$

$\Rightarrow S_n \to +\infty$ as $n \to +\infty$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n}$ does <u>not</u> converge

Q2. a) $S = \sum_{k=0}^{n} k = 0 + 1 + 2 + \dots + n$

$= \sum_{k=0}^{n} (n-k) = n + (n-1) + \dots + 1 + 0$

b) $2S = S + S = \sum_{k=0}^{n} k + (n-k) = \sum_{k=0}^{n} n = (n+1)n$

$\Rightarrow S = \frac{n(n+1)}{2}$

c) $\sum_{k=0}^{n} x_k = \sum_{k=0}^{n} b + a.k$

$= b.(n+1) + a. \sum_{k=0}^{n} k$

$= b(n+1) + a \frac{n(n+1)}{2} = (n+1)\left[b + a.\frac{n}{2}\right]$

d) $\underline{a>0}$ / $\underline{a<0}$: $\sum_{k=0}^{\infty} x_k$ diverges to $+\infty/-\infty$.

$\underline{a=0}$: $b>0$ / $b<0$: $\sum_{k=0}^{\infty} x_k$ diverges to $+\infty/-\infty$.

$\underline{a=0,\ b=0}$: $\sum_{k=0}^{\infty} x_k = 0$ (converges)

Q3. a) $(1-\alpha).S = (1-\alpha)\sum_{k=0}^{n} \alpha^k$

$$= \sum_{k=0}^{n} \alpha^k - \sum_{k=1}^{n+1} \alpha^k$$

$$= \alpha^0 - \alpha^{n+1} = 1-\alpha^{n+1}$$

b) $S = \dfrac{1-\alpha^{n+1}}{1-\alpha}$ if $\alpha \neq 1$; $S = \sum_{k=0}^{n} 1^k = \sum_{k=0}^{n} 1 = n+1$ if $\alpha = 1$

c) $\sum_{k=0}^{n} x_k = \sum_{k=0}^{n} b.a^k$

$$= b.\left(\sum_{k=0}^{n} a^k\right)$$

$$= \begin{cases} b.\dfrac{1-a^{n+1}}{1-a} & \text{if } a \neq 1 \\ b.(n+1) & \text{if } a = 1 \end{cases}$$

d) $\underline{|a|<1}$: $a^{n+1} \to 0$ so $\sum_{k=0}^{\infty} x_k \to \dfrac{b}{1-a} \in \mathbb{R}$ $(|a|<1 \Rightarrow a \neq 1)$

& e) $\underline{a=1}$: $\sum_{k=0}^{\infty} x_k \to \begin{cases} +\infty & \text{if } b>0 \\ -\infty & \text{if } b<0 \\ 0 & \text{if } b=0 \end{cases}$

$\underline{a=-1}$: $\sum_{k=0}^{\infty} x_k$ diverges (to nothing in particular) if $\underline{b \neq 0}$,

& converges to 0 if $\underline{b=0}$.

$\underline{a>1}$: $\sum_{k=0}^{\infty} x_k \to \begin{cases} +\infty & \text{if } b>0 \\ -\infty & \text{if } b<0 \\ 0 & \text{if } b=0 \end{cases}$

$\underline{a<-1}$: $\sum_{k=0}^{\infty} x_k$ diverges (to nothing in particular) if $\underline{b \neq 0}$;

& converges to 0 if $\underline{b=0}$.

Q4.

- $\dfrac{(-17)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} \cdot \dfrac{4^{2n+1}(n+1)}{(-17)^n}$

$= \dfrac{-17}{4^2} \dfrac{(n+1)}{(n+2)} \to -\dfrac{17}{16} < -1$

So the series diverges by ratio test

- $\dfrac{3^{2(n+1)+1}}{(n+1)^{(n+1)}} \dfrac{n^n}{3^{2n+1}}$

$= \dfrac{3^2 . n^n}{(n+1)^{n+1}} < \dfrac{9 . n^n}{n^{n+1}} = \dfrac{9}{n} \to 0 \in]-1,1[$

so the series converges by ratio test.

- $\sqrt[n]{\dfrac{n!}{n^n}} = \dfrac{(n!)^{1/n}}{n} < \dfrac{\left(\frac{n}{2}\right)^{1/2} . n^{1/2}}{n} = \dfrac{\frac{\sqrt{n}}{\sqrt{2}}}{\sqrt{n}} = \dfrac{1}{\sqrt{2}}$

for n even

(for n odd, use similar idea)

Thus, $\lim\limits_{n \to +\infty} \sqrt[n]{\dfrac{n!}{n^n}} \in \left[0, \dfrac{1}{\sqrt{2}}\right]$, $\dfrac{1}{\sqrt{2}} < 1$ so the series converges by root test.

- $\sqrt[n]{\dfrac{n^n}{(2n-1)!}} = \dfrac{n}{[(2n-1)!]^{1/n}} < \dfrac{\cancel{n}}{\cancel{n} . [(n-1)!]^{1/n}} = \dfrac{1}{[(n-1)!]^{1/n}} \to 0$

Thus, the series converges by root test

- $\left[\left(\left|\dfrac{5n - 3n^3}{0{,}5n^3+2}\right|\right)^n\right]^{1/n} = \left|\dfrac{5n - 3n^3}{0{,}5n^3+2}\right| \to |-6| > 1$

Thus, the series diverges by root test

- $\dfrac{(-12)^{n+1}}{n+1} \cdot \dfrac{n}{(-12)^n} = \dfrac{-12 . n}{n+1} \to -12 < 1$

Thus, the series diverges by ratio test.

- $\dfrac{(-2)^{n+2} . (n+1)}{9^{n+1}} \cdot \dfrac{9^n .}{(-2)^{n+1} . n} = \dfrac{-2}{9} \dfrac{n+1}{n} \to -\dfrac{2}{9} \in [-1,1]$

Thus, the series converges by ratio test.

- $$\sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1 + \int_1^{+\infty} \frac{1}{x^3}\,dx$$

$$= 1 + \left[-\frac{1}{2x^2} \right]_1^{+\infty} = 1 + \frac{1}{2} = \frac{3}{2} < \infty$$

Thus, the series converges

Q5. $$f'(x) = \frac{(15x^2-2)(2x-7) - 2.(5x^3-2x+1)}{(2x-7)^2}$$

$$= \frac{30x^3 - 105x^2 - \cancel{4x} + 14 - 10x^3 + \cancel{4x} - 2}{(2x-7)^2}$$

$$= \frac{20x^3 - 105x^2 + 12}{(2x+7)^2}$$

$g'(x) = 2ax + b$

Q6. $$\left(\frac{x^{n+1}}{n+1}\right)' = \frac{1}{n+1}(n+1)x^n = x^n$$

So $\frac{x^{n+1}}{n+1}$ is the anti-derivative of x^n.