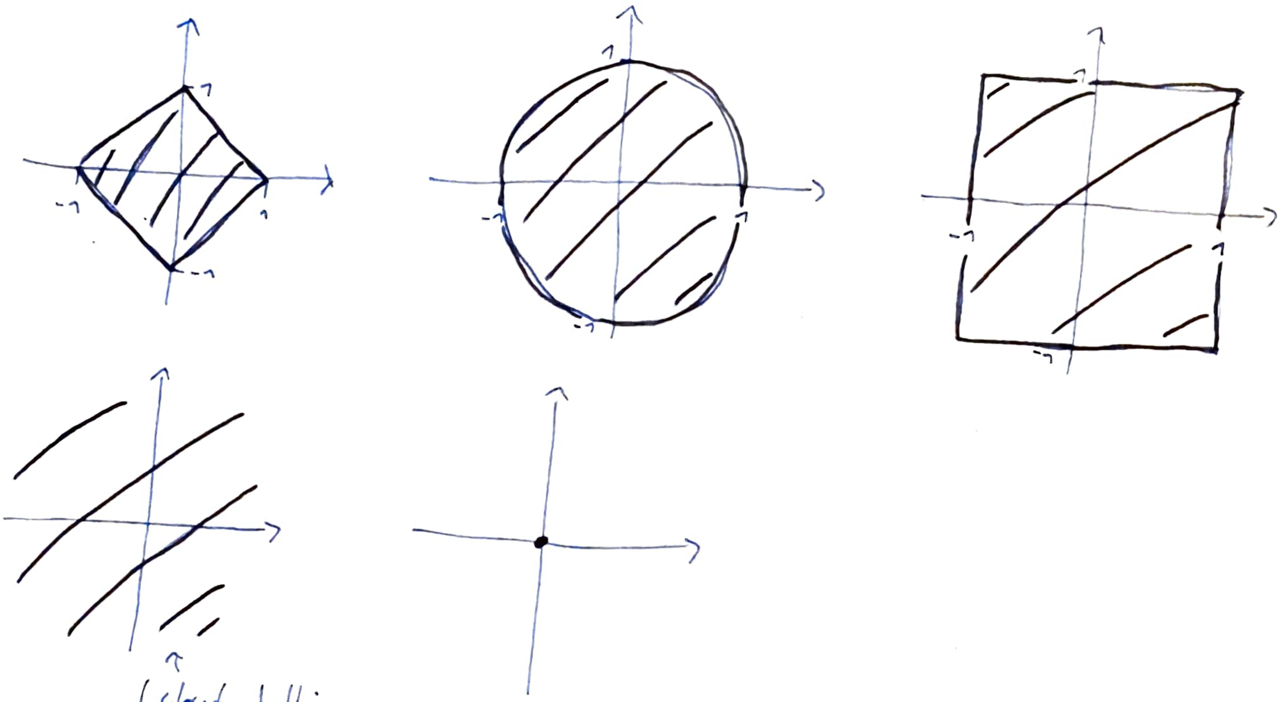
Problem Set 3 - Solutions

Q.1. $d(x,y) = \text{Norm}(x-y)$

so $B_r(x_0) = \{ x \in X \mid d(x, x_0) < r \}$

$= \{ x \in X \mid \text{Norm}(x - x_0) < r \}$

Q2.



(closed ball; open ball is just $\{(0,0)\}$)

Q3. For $p \geqslant 1$, $p < 1$ will fail triangle inequality

Q4. $\|x\|_{max} = \max(|x_1|, \ldots, |x_n|) = \max(\{ |x_k| \mid k \in [\![1, n]\!] \})$

$\|x\|_{euclidean} = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{\sum_{k=1}^{n} x_k^2}$

$\|x\|_{manhattan} = |x_1| + \ldots + |x_n| = \sum_{k=1}^{n} |x_k|$

$\|x\|_p = \sqrt[p]{|x_1|^p + \ldots + |x_n|^p} = \sqrt[p]{\sum_{k=1}^{n} |x_k|^p} \quad \forall p \geqslant 1$

Q5. • Method 1: ε, N definition

let $N = \sqrt{1/\varepsilon}$;

$\forall \varepsilon > 0, \exists N = \sqrt{\frac{1}{\varepsilon}}$ s.t. $\forall n > N$, $n > N \Rightarrow \frac{1}{n} < \frac{1}{N}$

$\frac{1}{n^2} < \frac{1}{N^2}$ $\Leftarrow$ $\frac{1}{n} > 0$ & $\frac{1}{N} > 0$ (as $n, N > 0$)

$\frac{1}{n^2} < \varepsilon$ $\Leftarrow$ $N = \sqrt{\frac{1}{\varepsilon}} \Rightarrow N^2 = \frac{1}{\varepsilon}$

so $1/n^2 \to 0$

Method 2: using algebra of limit

we know $\lim_{n\to\infty} \frac{1}{n} = 0$ (from lectures)

$$\text{so } \underbrace{\left(\lim_{n\to\infty} \frac{1}{n}\right)}_{0} \times \underbrace{\left(\lim_{n\to\infty} \frac{1}{n}\right)}_{0} = \lim_{n\to\infty} \frac{1}{n^2}$$

so $\lim_{n\to\infty} \frac{1}{n^2} = 0$

- Similar ideas, replace square root by cubic root.
- Similar ideas, replace square root by k^{th} root.
- Method 1: ε, N def.

$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k 1^{n-k}$ (Binomial th.)

$\geq \binom{n}{1} = n$

so $2^n \geq n \Rightarrow \frac{1}{2^n} \leq \frac{1}{n}$ (could also show this by induction)

$\forall \varepsilon > 0, \exists N = \frac{1}{\varepsilon}$ st $\forall n > N$, $\frac{1}{2^n} \leq \frac{1}{n} < \frac{1}{N} = \varepsilon$

so $1/2^n \to 0$

Method 2: Algebra of limits

$2^n \geq n$ (shown by induction in Problem Set 1)

so $\lim_{n\to\infty} n = +\infty \Rightarrow \lim_{n\to\infty} 2^n = +\infty$ $\leftarrow$ can also be show from M, N def.

Thus, $\lim_{n\to\infty} \frac{1}{2^n} = 0$

- Similar ideas, using $3^n \geq 2^n$
- Similar ideas, using $k^n \geq (k-1).n$, $(k-1) > 0$

- $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3n^2+2n+1}{6n^2-n+2}$

$= \lim_{n\to\infty} \frac{3+2/n+1/n^2}{6-1/n+2/n^2} = \frac{3}{6} = \frac{1}{2}$

- $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n+1}{6n^2+9n-5}$

$= \lim_{n\to\infty} \frac{2/n+1/n^2}{6+9/n-5/n^2} = \frac{0}{6} = 0$

- <u>Claim</u>: $\forall n \geq 10,\ 2^n > n^2$

Proof by induction:

- <u>$n=10$</u>: $2^{10} = 1024 > 100 = 10^2$
- Supp. $2^n > n^2$

want $2^{n+1} > (n+1)^2 = n^2+2n+1$

$2^{n+1} = 2 \cdot 2^n$
$> n^2 + n^2$
$> n^2 + 10n$ $\Leftarrow n \geq 10$
$> n^2 + 2n + 1$ $\Leftarrow 8n \geq 8 > 1$ as $n \geq 10$
$> (n+1)^2$

$\square$

So $2^n > n^2 \quad \forall n \geq 10$

So $0 < \frac{n}{2^n} < \frac{n}{n^2} = \frac{1}{n} \to 0$

so $\frac{n}{2^n} \to 0$

Q6. • $\forall \varepsilon > 0$, $\exists N_x$ st $\forall n > N_x$, $|x_n - x| < \varepsilon/2$

$\exists N_y$ st $\forall n > N_y$, $|y_n - y| < \varepsilon/2$

Thus, $\exists N = \max(N_x, N_y)$ st $\forall n > N$, $|(x_n+y_n)-(x+y)| \underset{\text{Triangle inequ.}}{\leq} |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

- $|x_n y_n - xy| = |(x_n - x) y_n + x(y_n - y)|$
$\leq |(x_n - x) y_n| + |x(y_n - y)|$

← idea, try to find a formal solution using this ☺

$|x_n - x| \to 0$; y_n is bdd (as $y_n \to y$), so $|(x_n - x) \cdot y| \to 0$ (process is similar)

$(y_n - y) \to 0$ so $|x(y_n - y)| \to 0$

Thus, $|x_n y_n - xy| \to 0$ so $x_n y_n \to xy$

$\square$

- $$\lim_{n\to\infty}\left|\frac{x_n}{y_n}-\frac{x}{y}\right| = \lim_{n\to\infty}\left|\frac{x_n y - x y_n}{y_n y}\right|$$

← idea again, find formal solution on your own ☺

$$= \lim_{n\to\infty}\left|\frac{-x_n y_n + xy + x_n y - x y_n}{y_n y}\right|$$

as $(x_n y_n - xy) \to 0$

$$= \lim_{n\to\infty}\left|\frac{-x(y_n-y) - x_n(y_n-y)}{y_n y}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{(x+x_n)}{y_n y}(y_n-y)\right| = 0$$

as $(y_n - y) \to 0$

$$\left.\begin{array}{l} x+x_n \to 2x \in \mathbb{R} \\ y_n y \to y^2 \in \mathbb{R} \end{array}\right\} -\frac{x+x_n}{x_n y} \to \frac{2x}{y} \in \mathbb{R}$$