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# Problem Set 3

Due 20th September 2021

**Abstract**

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

## 1 Metric/Norm Spaces

**Question 1.** *(\*) $X, Norm(\_)$ is a norm space; we define $d(x,y) = Norm(x-y)$, so that $X, d$ is a metric space. We have seen what the ball of radius $r$ centered at $x$ is in a metric space (i.e. using $d$). Define it in terms of then norm.*

**Question 2.** *(\*) We take $X = \mathbb{R}^2$, draw $B_1((0,0))$ (the unit ball) for the following norms:*

- $Norm((x,y)) = |x| + |y|$

- $Norm((x,y)) = \sqrt{x^2 + y^2}$

- $Norm((x,y)) = \max(|x|,|y|)$

- $Norm((x,y)) = 0$ *if* $(x,y) = (0,0)$, $Norm((x,y)) = 1$ *else*

- $Norm((x,y)) = 0$ *if* $(x,y) = (0,0)$, $Norm((x,y)) = 2$ *else*

**Question 3.** *For which values of $p \geq 0$ is $Norm((x,y)) = \sqrt[p]{|x|^p + |y|^p}$ a norm for $\mathbb{R}^2$?*

**Question 4.** *Find two different norms for $\mathbb{R}^n$ (give their definitions).*

## 2 Sequences Convergence

**Question 5.** *Find the limit of the following sequences, and prove your claim carefully:*

- *(\*) $x_n = \dfrac{1}{n^2}$*

- $x_n = \dfrac{1}{n^3}$

- $x_n = \dfrac{1}{n^k}$ *for some $k > 0$*

- *(\*) $x_n = \dfrac{1}{2^n}$*

- $x_n = \dfrac{1}{3^n}$

- $x_n = \dfrac{1}{k^n}$ *for some $k > 1$*

- *(\*) $x_n = \dfrac{3n^2 + 2n + 1}{6n^2 - n + 2}$*

- $x_n = \dfrac{2n+1}{6n^2 + 9n - 5}$

- *(hard) $x_n = \dfrac{n}{2^n}$*

**Question 6.** $x_n \to x$, $y_n \to y$; *Prove the following:*

- *(\*)* $x_n + y_n \to x + y$

- $x_n * y_n \to x * y$

- *if* $y_n \neq 0$, *and* $y \neq 0$: $x_n/y_n \to x/y$