# Problem Set 4 - Solutions

Q1. $x_n \to x$ in terms of metric distance $d(-,-)$:

$\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N, \ d(x_n, x) < \varepsilon$

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$x_n \to x$ in terms of norm $\|-\|$:

$\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N, \ \|x_n - x\| < \varepsilon$

Q2. a) By recurrence:

$\underline{n=0}$: $x_0 = b = b + a.0$

$\underline{n+1}$: Suppose $x_n = b + a.n$

We need $x_{n+1} = b + a.(n+1)$

$$\begin{aligned} x_{n+1} &= a + x_n \\ &= a + b + a.n \quad \text{Induction hypothesis} \\ &= b + a.(n+1) \quad \text{Factorize by } a \end{aligned}$$

✓

Thus, $x_n = b + a.n \quad \forall n \in \mathbb{N}$

b) $\underline{a > 0}$: $\lim\limits_{n \to +\infty} x_n = \lim\limits_{n \to +\infty} \underbrace{b}_{<\infty} + \underbrace{a.n}_{\to +\infty} = +\infty$

$\underline{a < 0}$: $\lim\limits_{n \to +\infty} x_n = \lim\limits_{n \to +\infty} \underbrace{b}_{<\infty} + \underbrace{a.n}_{\to -\infty} = -\infty$

$\underline{a = 0}$: $\lim\limits_{n \to +\infty} x_n = \lim\limits_{n \to +\infty} b + \underbrace{a.n}_{0} = b$

Q3. a) By recurrence:

$\underline{n=0}$: $x_0 = b = b.a^0$

$\underline{n+1}$: Suppose $x_n = b.a^n$

We need $x_{n+1} = b.a^{n+1}$

$$\begin{aligned} x_{n+1} &= a.x_n \quad \text{Induction hypothesis} \\ &= a.b.a^n = b\,a^{n+1} \end{aligned}$$

✓

Thus, $x_n = b.a^n \quad \forall n \in \mathbb{N}$

b) $\underline{b>0,\ a>1}$: $a = 1+\varepsilon,\ \varepsilon > 0$

So $a^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \varepsilon^k$

$> \binom{n}{1}.\varepsilon = n.\varepsilon \to +\infty$ as $n \to +\infty$

so $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \underbrace{b}_{>0}.\underbrace{a^n}_{\to+\infty} = +\infty$

$\underline{b<0,\ a>1}$: $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \underbrace{b}_{<0}.\underbrace{a^n}_{\to+\infty} = +\infty$

$\underline{b=0}$: $x_n = 0\ \forall n \in \mathbb{N}$ so $\lim_{n\to+\infty} x_n = 0$

$\underline{a=1}$: $x_n = b\ \forall n \in \mathbb{N}$ so $\lim_{n\to+\infty} x_n = b$

$\underline{|a|<1}$: $\frac{1}{|a|} > 1$ so $\frac{1}{|a^n|} = \frac{1}{|a|^n} = \left(\frac{1}{|a|}\right)^n \to +\infty$

Thus $\frac{1}{\frac{1}{|a^n|}} = |a^n| \to 0$, so $a^n \to 0$

Finally, $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \underbrace{b}_{\in\mathbb{R}}.\underbrace{a^n}_{\to 0} = 0$

$\underline{a=-1}$: $x_{2n} = b,\ x_{2n+1} = -b\ \ \forall n \in \mathbb{N}$

$\Rightarrow$ diverges*, unless $\underline{b=0}$.

* to nothing in particular

$\underline{a<-1}$: $x_n$ alternate signs, $|x_n| = b|a|^n \to +\infty$

$\Rightarrow$ diverges (to nothing in particular), unless $\underline{b=0}$.

Q4. a) One could conjecture that the larger the initial number is, the longer it would take to reach one

b) • $\boxed{7} \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5$
$\to 16 \to 8 \to 4 \to 2 \to 1$ (16 iterations)

• $\boxed{8} \to 4 \to 2 \to 1$ (3 iterations)

• $\boxed{15} \to 46 \to 23 \to 70 \to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10$
$\to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ (17 iterations)

• $\boxed{16} \to 8 \to 4 \to 2 \to 1$ (4 iterations)

c) No, powers of two converge very quickly to one (faster than expected).

Q5.

$$f(x) = \frac{2x^2 - 3x + 9}{4x - 5}$$

$$\begin{array}{rrr|l}
2x^2 & -3x & +9 & 4x - 5 \\
-2x^2 & +\frac{5}{2}x & & \frac{1}{2}x - \frac{1}{8} \\
\hline
 & -\frac{1}{2}x & +9 & \\
 & +\frac{1}{2}x & -\frac{5}{8} & \\
\hline
 & & \frac{67}{8} &
\end{array}$$

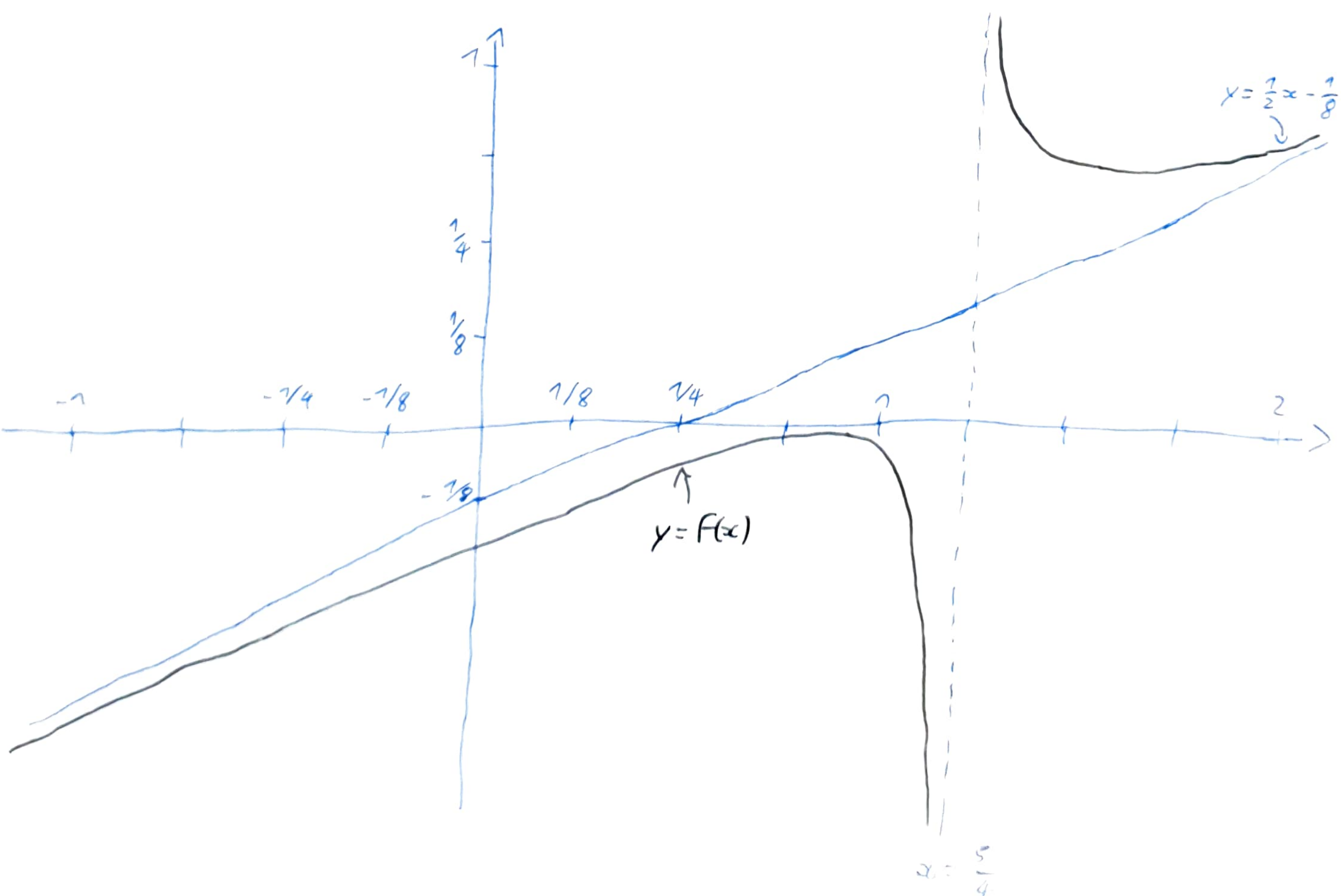
so $2x^2 - 3x + 9 = (4x-5)\left(\frac{1}{2}x - \frac{1}{8}\right) + \frac{67}{8}$

ie $f(x) = \frac{1}{2}x - \frac{1}{8} + \underbrace{\frac{67}{32x - 40}}_{\to 0 \text{ as } x \to \pm\infty}$

so $|f(x) - \ell(x)| \to 0$ as $x \to \pm\infty$ if $\ell(x) = \frac{1}{2}x - \frac{1}{8}$

The asymptote of $f$ as $x \to \pm\infty$ is oblique, of equation $y = \frac{1}{2}x - \frac{1}{8}$

At $4x - 5 = 0 \iff x = \frac{5}{4}$, $f$ is undefined & has a singularity.



Q6. By induction:

$n=0$: $x_0 = 0$

$$2 - \frac{2}{2^0} = 2 - \frac{2}{1} = 2 - 2 = 0 \quad \checkmark$$

$n+1$: Suppose $x_n = 2 - \frac{2}{2^n}$

Need $x_{n+1} = 2 - \frac{2}{2^{n+1}}$

$$
\begin{aligned}
x_{n+1} &= \frac{1}{2} x_n + 1 \\
&= \frac{1}{2}\left(2 - \frac{2}{2^n}\right) + 1 && \text{Induction Hypothesis} \\
&= 1 - \frac{2}{2^{n+1}} + 1 && \text{Distribute } \tfrac{1}{2} \text{ factor} \\
&= 2 - \frac{2}{2^{n+1}} \quad \checkmark
\end{aligned}
$$

Thus, by induction, $x_n = 2 - \frac{2}{2^n}$

Now, $\frac{2}{2^n} \to 0$ as $n \to +\infty$ (using Q 3.)

so $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} 2 - \frac{2}{2^n} = 2$

Q 7. Let $X = \{\frac{1}{n} \mid n \in \mathbb{N}^*\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Then $\text{diam}(X) = 1$ as $\forall \varepsilon > 0, \exists n \in \mathbb{N}, n > \frac{1}{\varepsilon}$ s.t. $|\underset{\substack{\uparrow \\ X}}{1} - \underset{\substack{\uparrow \\ X}}{\tfrac{1}{n}}| > 1 - \varepsilon$

& $\forall x, y \in X, |x - y| < 1$

so $\overbrace{\sup\{|x-y| \mid x, y \in X\}}^{\text{diam}(X)} = 1$

but $\max\{|x-y| \mid x,y \in X\}$ does not exist.