

Problem Set 2

Due 14th September 2021

Abstract

Only the questions with a star (*) are compulsory for submission;
It is however *strongly* advised to attempt all question.

1 Functions Properties

1 **Question 1.** (*) Show that an injection between two sets of the same size is bijective.

2 Finite Cardinalities

Question 2 (Counting the number of functions between two finite sets). Let X and Y be two non-empty finite sets. We want to count the number of functions in Y^X .

a. Let us assume that $|X| = n$ with $X = \{x_1, \dots, x_n\}$. Prove that the function

$$\Phi: \begin{array}{ccc} Y^X & \longrightarrow & Y^n \\ f & \longmapsto & (f(x_1), \dots, f(x_n)) \end{array}$$

is a bijection.

b. Deduce the value of $|Y^X|$.

c. Let $n \in \mathbb{N}^*$. Let us consider the set $\mathfrak{S}_n \subset \llbracket 1; n \rrbracket^{\llbracket 1; n \rrbracket}$ containing the bijections from $\llbracket 1; n \rrbracket$ to itself. Prove that the sequence $(|\mathfrak{S}_n|)_{n \in \mathbb{N}^*}$ is defined by the recurrence relation

$$\begin{cases} |\mathfrak{S}_1| = 1 \\ \forall n \in \mathbb{N}, |\mathfrak{S}_{n+1}| = (n+1)|\mathfrak{S}_n| \end{cases}$$

(hint : we can use the bijections $\forall k \in \llbracket 1; n \rrbracket, g_k : \llbracket 1; n \rrbracket \setminus \{k\} \rightarrow \llbracket 1; n-1 \rrbracket$)

d. The cardinal of \mathfrak{S}_n is the factorial of n , denoted as $n!$. Write a function returning the value of the factorial for a given $n \in \mathbb{N}$ (by convention $0! = 1$).

Question 3 (Counting the number of sub-parts). (*) We study the function $(n, p) \in \mathbb{N}^2 \mapsto \binom{n}{p} \in \mathbb{N}$ the binomial coefficient, which is the number of subsets containing p elements in a set containing n elements.

1 a. Prove that $\forall n, p \geq 1, \binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}$.

2 b. Deduce from this recurrence equation that

$$\forall n \in \mathbb{N}, \forall p \leq n, \binom{n}{p} = \frac{n!}{(n-p)!p!}$$

2 c. Prove the formula

$$\forall n \in \mathbb{N}, \sum_{k=0}^n \binom{n}{k} = 2^n$$

1 d. Derive the value of $|\mathcal{P}(\llbracket 1; n \rrbracket)|$ (power set).

3 Infinite Cardinalities

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Question 4. (*) Find an explicit bijection between $[a, b]$ and $[c, d]$

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Question 5. (*) Show that there is a bijection between $[0, 1[$ and $\{0, 1\}^{\mathbb{N}}$

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Question 6. (*) Let $\mathcal{A} = \{a, b, c, \dots, z\}$ be the set of letters in the alphabet. Show explicitly that $|\mathcal{A} \cup \mathbb{N}| = |\mathbb{N}|$.

Question 7. Let \mathbb{C} be the set of complex numbers. Compare \mathbb{R} and \mathbb{C} .

Question 8. Let \mathbb{P} be the set of prime numbers. Compare \mathbb{P} with the usual sets (in particular with \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R}).