

Refresher Math Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. The course will try to cover most of the prerequisites of the courses in the Master, mainly linear algebra, differential calculus, integration, and asymptotic analysis.

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Introduction

Presentation

- Paul Dubois
- will be teaching this refresher math course
- email (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h
- 1h20min lecture - 1/3h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)
- Lectures are recorded (if ever needed)
- 1st lecture ever => too fast/too slow: let me know
- May assume you know a concept/notation that you have never heard of, let me know if this happens

Examination

- The course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be a full exercise sheet per lecture, it is advised to attempt it all (only one will be compulsory).
- Hand-in 1 exercise per lecture (i.e., 8 in total), due 2 weeks after the lecture
- Best $(n-1)/n$ count (i.e., best 7/8 in our case), need avg $\geq 50\%$ to pass
- In the unlikely event of not passing, will be able to do an extra work

Questions?

Chapter 1

Sets & logic

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $\in, \{True, False\}, \{a \mid condition\}, \{a, b, c, \dots\}, \emptyset$

Need to be careful when defining set: some definitions are pathological.

Remark (Russell Paradox). *Take $U = \{X \mid X \notin X\}$. X in $U \Rightarrow U$ not in U , U is a set, so not all sets are in U X not in $U \Rightarrow X$ is a set*

Notation (Usual Sets). $\mathbb{B}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{N}^*, \mathbb{R}^+ \dots$

Functions

Definition (Functions). *Assignment for a set to another.*

Notation (Function). $f : X \rightarrow Y, f(x) = blah, f : x \mapsto blah$.

Definition (Predicate). *Function to \mathbb{B}*

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation (\forall). *For all elements in set, e.g.: $\forall x \in \mathbb{R}, x^2 \geq 0$.*

Notation (\exists). *There exists an element in set, e.g.: $\exists x \in \mathbb{R}$ s.t. $x^2 > 1$.*

Notation ($\exists!$). *There exists a unique element in set, e.g.: $\exists! x \in \mathbb{R}$ s.t. $x^2 \leq 0$.*

Definition (Subset / Inclusion). $X \subseteq Y$ if $\forall x \in X, x \in Y$

Definition (Powerset). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$

e.g.: $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Definition (Cartesian Product). $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

e.g.: $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Extension: $X_1 \times \dots \times X_n = \prod_{k=1}^n X_k$

1.2 Boolean algebra

Basic operators

Definition (Conjunction). $x \wedge y = xy$

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Definition (Disjunction). $x \vee y = \min(x + y, 1)$

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation). $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

Question. *Selecting points outside a given region.*

Basic properties

Property 1.1 (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up \vee with addition and \wedge with multiplication.*

- *Associativity of \vee : $x \vee (y \vee z) = (x \vee y) \vee z$*
- *Associativity of \wedge : $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*
- *Commutativity of \vee : $x \vee y = y \vee x$*
- *Commutativity of \wedge : $x \wedge y = y \wedge x$*
- *Distributivity of \wedge over \vee : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*
- *0 is identity for \vee : $x \vee 0 = x$*
- *1 is identity for \wedge : $x \wedge 1 = x$*
- *0 is annihilator for \wedge : $x \wedge 0 = 0$*

Property 1.2 (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of \vee : $x \vee x = x$*
- *Idempotence of \wedge : $x \wedge x = x$*
- *Absorption of \vee over \wedge : $x \vee (x \wedge y) = x$*
- *Absorption of \wedge over \vee : $x \wedge (x \vee y) = x$*
- *Distributivity of \vee over \wedge : $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*
- *1 is annihilator for \vee : $x \vee 1 = 1$*

Property 1.3 (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me)

□

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property 1.4 (Implication and Inclusion). *If $\forall x \in X, P_1(x) \implies P_2(x)$, then $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.*

Definition (If and only if). $x \iff y$

Negation of quantified propositions

Property 1.5 (Negation of \forall). $\text{not}(\forall x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Property 1.6 (Negation of \exists). $\text{not}(\exists x \in X, P(x)) = \forall x \in X, \text{not}(P(x))$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \dots$ is true ; $\exists x \in \emptyset, \dots$ is false

1.3 Python

=> use google colab'

Chapter 2

Proofs methods

2.0.1 Direct implication

Want to show A : may show B and $B \implies A$, or C and $C \implies B$ and $B \implies A$.

2.0.2 Case dis-junction

Split in cases.

E.g.: show n and n^2 have the same parity (take n odd then n even).

2.0.3 Contradiction

Suppose the opposite, derive a contradiction (i.e. A and $\neg A$) and conclude.

E.g.: show $\sqrt{2} \notin \mathbb{Q}$ (suppose $\sqrt{2} = a/b$, WLOG $a, b \in \mathbb{N}$ co-prime).

2.0.4 Induction

Want to show P_n for $n \geq n_0$: show $P_n \implies P_{n+1}$ and P_{n_0} .

E.g.: show $\sum_{k=0}^n k = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

2.0.5 Existence and Uniqueness

It is common to show existence and/or uniqueness.

E.g.: Existence and uniqueness in Euclidean division:

$$\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists! q \in \mathbb{Z}, r \in [0, b[\cap \mathbb{N} \text{ s.t. } a = bq + r$$

Use $q = \max\{k \in \mathbb{N} \mid bk \leq a\}$, $r = a - bq$.