Problem Set 2 - Solutions

Q1. $f: A \to B$ injective ; $|A| = |B|$

f injective $\Rightarrow$ $f: A \to f(A)$ bijective (by definition)

so $|A| = |f(A)|$

Now, $f(A) \subseteq B$ & $|f(A)| = |A| = |B|$ so $f(A) = B$.

Therefore, f is bijective from A to B.

Q2. a. Method 1: Injective $\oplus$ Surjective

Injectivity: Suppose $\phi(f) = \phi(g)$

$$\Leftrightarrow (f(x_1), \dots, f(x_n)) = (g(x_1), \dots, g(x_n))$$

$$\Leftrightarrow \begin{cases} f(x_1) = g(x_1) \\ \vdots \\ f(x_n) = g(x_n) \end{cases}$$

$$\Leftrightarrow f = g$$

Thus, ϕ is injective

Surjectivity: Suppose $(y_1, \dots, y_n) \in Y^n$

let $f: X \to Y$ s.t. $f(x_i) = y_i \quad \forall 1 \leq i \leq n$

then $\phi(f) = (y_1, \dots, y_n)$

Thus, ϕ is surjective

ϕ is therefore bijective (as injective & surjective).

Method 2: Inverse function

Let $\varphi: Y^n \to Y^X$ s.t. $\varphi((y_1, \dots, y_n)) = f: X \to Y$, $x_i \mapsto y_i$

Then $\varphi \circ \phi = Id|_{Y^X}$ & $\phi \circ \varphi = Id|_{Y^n}$, so $\phi^{-1} = \varphi$

so ϕ is invertive $\Rightarrow$ ϕ is bijective

b. $|Y^X| = |Y^n|$ (using a.)

$= |Y|^n \quad (= |Y|^{|X|})$

c. $\mathfrak{S}_1 = \{ f : \{1\} \to \{1\} \mid f \text{ bijective} \} = \{ \overset{\{1\} \to \{1\}}{1 \mapsto 1} \}$

so $|\mathfrak{S}_n| = 1$

Let $f \in \mathfrak{S}_{n+1}$: then $g = f|_{[\![1,n]\!]}$ is a bijection from $[\![1,n]\!]$ to $\underbrace{[\![1,n+1]\!] \setminus \{f(n+1)\}}_{\text{(set of size } n)}$

Thus, choosing $f \in \mathfrak{S}_{n+1}$ is choosing $f(n+1) \in [\![1,n+1]\!]$ & $f|_{[\![1,n]\!]} \in \mathfrak{S}_n$

so $|\mathfrak{S}_{n+1}| = (n+1).|\mathfrak{S}_n|$

d. $f(n) = |\mathfrak{S}_n| = n!$ can be defined recursively as follows:

$f(n) = n.\, f(n-1)$ (or $f(n+1) = (n+1).\, f(n)$)

& $f(1) = 1$; $f(0) = 1$ by convention.

Q3. a. $\binom{n}{p} = |\{ Y \mid Y \subseteq [\![1,n]\!],\ |Y| = p \}|$

so $\binom{n+1}{p} = |\{ Y \mid Y \subseteq [\![1,n+1]\!],\ |Y| = p \}|$

disjoint union (LHS contains $n+1$, while RHS does not)

$= |\{ \{n+1\} \cup Y \mid Y \subseteq [\![1,n]\!],\ |Y| = p-1 \} \sqcup \{ Y \mid Y \subseteq [\![1,n]\!],\ |Y| = p \}|$

$= |\{ \{n+1\} \cup Y \mid Y \subseteq [\![1,n]\!],\ |Y| = p-1 \}| + |\{ Y \mid Y \subseteq [\![1,n]\!],\ |Y| = p \}|$

$= \binom{n}{p-1} + \binom{n}{p}$

b. By induction on $\underline{\underline{n}}$: Note: $0 \le p \le n$

<u>$n=0$</u>: $\binom{0}{p} = |\{ Y \mid Y \subseteq \emptyset,\ |Y| = p \}|$

$= |\{\emptyset\}| = 1$ (as $p=0$ is the only possibility) & $\frac{0!}{(0-p)!\,p!} = 1$ for $p=0$ ✓

<u>$n=1$</u>: $\binom{1}{p} = |\{ Y \mid Y \subseteq \{1\},\ |Y| = p \}|$

$= \begin{cases} |\{\emptyset\}| & \text{if } p=0 \\ |\{\{1\}\}| & \text{if } p=1 \end{cases}$

$= 1$ (in both cases) & $\frac{1!}{(1-p)!\,p!} = 1$ for both $p=0$ & $p=1$ ✓

<u>Induction</u>: Suppose $\binom{n}{p} = \frac{n!}{(n-p)!\,p!} \quad \forall\, 0 \le p \le n$

want $\binom{n+1}{p} = \frac{(n+1)!}{(n+1-p)!\,p!} \quad \forall\, 0 \le p \le n+1$

Case $0 \leq p \leq n$: $\binom{n+1}{p} = \binom{n}{p-1} + \binom{n}{p}$

↓ By induction hypothesis (twice)

$$= \frac{n!}{(n-(p-1))!\,(p-1)!} + \frac{n!}{(n-p)!\;p!}$$

$$= \frac{n!\,(p)}{(n+1-p)!\;p!} + \frac{n!\,(n+1-p)}{(n+1-p)!\;p!}$$

$$= \frac{n!\,(n+1-\cancel{p}+\cancel{p})}{(n+1-p)!\;p!} = \frac{(n+1)!}{(n+1-p)!\,p!}$$ ✓

Case $p = n+1$: $\binom{n+1}{p} = \left|\{ Y \mid Y \subseteq [\![1, n+1]\!],\ |Y| = n+1 \}\right|$

$$= \left|\{ [\![1, n+1]\!] \}\right| = 1$$

$$\frac{(n+1)!}{(n+1-p)!\;p!} = \frac{(n+1)!}{0!\,(n+1)!} = 1$$ ✓

So by induction, $\binom{n}{p} = \dfrac{n!}{(n-p)!\,p!}$

c. <u>Method 1</u>: By induction

<u>$n=0$</u>: $\sum_{k=0}^{n} \binom{n}{k} = \binom{0}{0} = 1$

$2^n = 2^0 = 1$ ✓

<u>$n=1$</u>: $\sum_{k=0}^{n} \binom{n}{k} = \binom{1}{0} + \binom{1}{1} = 1+1 = 2$

$2^n = 2^1 = 2$ ✓

<u>Induction</u>: Suppose $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Want $\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^{n+1}$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n+1} \binom{n}{k} + \binom{n}{k-1} \qquad \text{(by a))}$$

$$= \sum_{k=0}^{n} \binom{n}{k} + \binom{n}{n+1} + \binom{n}{-1} + \sum_{k'=0}^{n} \binom{n}{k'}$$ ← relabel $k' = k-1$

↓ by induction hypothesis (twice)

$$= 2^n + 0 + 0 + 2^n$$

$$= 2.2^n = 2^{n+1}$$ ✓

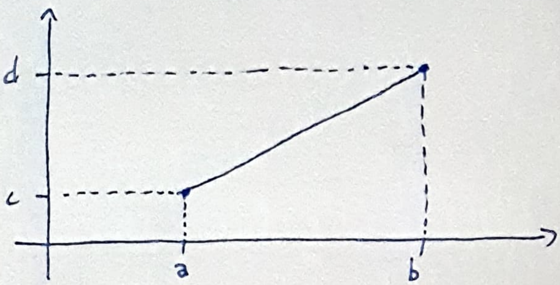
So, by induction, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Method 2: Using Binomial theorem

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

d.
$$\begin{aligned}|\mathcal{P}([\![1,n]\!])| &= |\{ Y \mid Y \subseteq [\![1,n]\!] \}| \\ &= \left| \bigsqcup_{k=0}^{n} \{ Y \mid Y \subseteq [\![1,n]\!] , |Y| = k \} \right| \\ &= \sum_{k=0}^{n} |\{ Y \mid Y \subseteq [\![1,n]\!] , |Y| = k \}| \\ &= \sum_{k=0}^{n} \binom{n}{k} = 2^n\end{aligned}$$

Q4.



$$f : [a,b] \to [c,d]$$
$$x \mapsto c + (x-a)\frac{d-c}{b-a} \quad \text{is bijective}$$

(you may check it yourself!)

Q5.
$$f : \{0,1\}^{\mathbb{N}} \longrightarrow [0,1[$$
$$(a_1, a_2, a_3, \dots) \mapsto 0,a_1 a_2 a_3 \cdots$$

binary sequence ($a_k \in \{0,1\} \ \forall k \in \mathbb{N}^*$)

binary extension ($0,a_1 a_2 a_3 \dots \in [0,1[$)

Q6.
$$f : \mathcal{A} \cup \mathbb{N} \to \mathbb{N}$$
$$\mathcal{A} \ni \left\{ \begin{array}{ccc} a & \mapsto & 0 \\ b & \mapsto & 1 \\ c & \mapsto & 2 \\ & \dots & \\ z & \mapsto & 25 \end{array} \right\} \in \mathbb{N}$$
$$\mathbb{N} \ni n \mapsto n+26 \in \mathbb{N}$$

f is bijective (can be proved finding the inverse function)

so $|\mathcal{A} \cup \mathbb{N}| = |\mathbb{N}|$

Q7. $|\mathbb{R}| = |\mathbb{C}|$

$f: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto z$ s.t. if $\ldots x_{-2}\, x_{-1}\, x_0, x_1\, x_2\, x_3 \ldots = x$ ← decimal extension

$\ldots y_{-2}\, y_{-1}\, y_0, y_1\, y_2\, y_3 \ldots = y$

then $z = \ldots y_{-2}\, x_{-2}\, y_{-1}\, x_{-1}\, y_0\, x_0, y_1\, x_1\, y_2\, x_2\, y_3\, x_3 \ldots$

f is bijective (can be shown finding inverse function)

so $|\mathbb{R}^2| = |\mathbb{R}|$

Now, $g: \mathbb{R}^2 \to \mathbb{C}$

$(x,y) \mapsto x+iy$ is bijective (again, can be shown finding the inverse function).

So $|\mathbb{R}^2| = |\mathbb{C}|$

Thus, $|\mathbb{R}| = |\mathbb{R}^2| = |\mathbb{C}|$

Q8. Using Euclid theorem on prime numbers (may be found online, is relatively easy to understand), there are infinitely many prime numbers. So we can label the prime numbers in increasing order:

$$\underset{\substack{\shortparallel \\ 2}}{p_0} < \underset{\substack{\shortparallel \\ 3}}{p_1} < \underset{\substack{\shortparallel \\ 5}}{p_2} < \underset{\substack{\shortparallel \\ 7}}{p_3} < \ldots < p_n < p_{n+1} < \ldots$$

Then, $f: \mathbb{N} \to \mathbb{P}$, $k \mapsto p_k$ is bijective, so $|\mathbb{P}| = |\mathbb{N}|$

Now, $|\mathbb{N}| = |\mathbb{Z}| \Rightarrow |\mathbb{P}| = |\mathbb{Z}|$

$|\mathbb{N}| = |\mathbb{Q}| \Rightarrow |\mathbb{P}| = |\mathbb{Q}|$

$|\mathbb{N}| < |\mathbb{R}| \Rightarrow |\mathbb{P}| = |\mathbb{R}|$