Refresher Maths Course

Paul Dubois

September 2022

Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the Master's, i.e. basic algebra/analysis and basic application.

Contents

0 Introduction

Hello! welcome to this maths refresher course for DSBA 2022! This is the best course ever!

Presentation

- Paul Dubois, PhD Student @ Centrale, end of 1st year
- Email: b00795695@essec.edu (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h arranged as 1h20min lecture - $^1/_3$h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)

Examination

- Course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)
- As the goal is to learn, you will be able to resubmit exercise sets, but you will lose 10% every-time you re-submit (so that you have some incentive to try your best the 1st time)
- Best $^{(n-1)}/_n$ count, need average $\geq 70\%$ to pass
- In the unlikely event of not passing, you will be able to do some extra work to pass

Questions?

1 Elementary Maths

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $\in$, $\{True, False\}$, $\{a \mid condition\}$, $\{a, b, c \dots\}$, $\emptyset$

Remark (Russell Paradox). *(digression)*
Need to be careful when defining set: some definitions are pathological.
e.g.: Take $Y = \{x \mid x \notin x\}$: $Y \in Y \iff Y \notin Y$

Notation (Usual Sets). $\mathbb{B}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}^*$, $\mathbb{R}^+$...

Functions

Definition (Functions). *Assignment from a set to another.*

Notation (Function). $f : X \to Y$, $f(x) = blah$, $f : x \mapsto blah$.

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation ($\forall$). *For all elements in set, e.g.:* $\forall x \in \mathbb{R}, x^2 \geq 0$.

Notation ($\exists$). *There exists an element in set, e.g.:* $\exists x \in \mathbb{R}$ *s.t.* $x^2 > 1$.

Notation ($\exists!$). *There exists a unique element in set, e.g.:* $\exists! x \in \mathbb{R}$ *s.t.* $x^2 \leq 0$.

Question.
- *Express "all natural numbers are positive" with quantifiers*
- *Express* $\forall x \geq 0, \sqrt{x} \geq 0$ *in a sentence*

Definition (Subset / Inclusion). $X \subseteq Y$ *if* $\forall x \in X, x \in Y$

Definition (Disjoint Sets). X *and* Y *are disjoint if* $\forall x \in X, x \notin Y$ *(or if* $\forall y \in Y, y \notin X$*).*

Definition (Power Set). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Definition (Cartesian Product). $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
e.g.: $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

1.2 Axioms

Here $\star$ and $\dagger$ will operations.

Definition (Associativity). *$\star$ is associative if* $\forall x, y, z,\ (x \star y) \star z = x \star (y \star z)$

Definition (Commutativity). *$\star$ is associative if* $\forall x, y,\ (x \star y) = y \star x$

Definition (Identity). $1_\star$ *is identity for* $\star$ *if* $\forall x,\ 1_\star \star x = x \star 1_\star = x$

Definition (Annihilator). $0_\star$ *is annihilator for* $\star$ *if* $\forall x,\ 0_\star \star x = x \star 0_\star = 0_\star$

Definition (Distributivity). *$\star$ is distributive over $\dagger$ if* $\forall x, y, z\ x \star (y \dagger z) = (x \star y) \dagger (x \star z)$

of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Question. *(make a table)*

- *Which of these are commutative: addition, subtraction, multiplication, division, power?*
- *Which of these are associative: addition, subtraction, multiplication, division, power?*
- *What is identity for: addition, subtraction, multiplication, division, power?*
- *What is annihilator for: addition, subtraction, multiplication, division, power?*

Question.
- *Think of an operation that is commutative, but not associative*
- *Think of an operation that is associative, but not commutative*

1.3 Boolean algebra

The reason we'll do some is because of it's application to programming, in particular to conditions ('if' blocks and 'while' loops).

Basic operators

Definition (Conjunction). $x \wedge y = xy$

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

Definition (Disjunction). $x \vee y = \min(x + y, 1)$

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation). $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

Question. *Selecting points outside a given region.*

Basic properties

Property (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of $\vee$:* $x \vee (y \vee z) = (x \vee y) \vee z$
- *Associativity of $\wedge$:* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- *Commutativity of $\vee$:* $x \vee y = y \vee x$
- *Commutativity of $\wedge$:* $x \wedge y = y \wedge x$
- *Distributivity of $\wedge$ over $\vee$:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- 0 *is identity for $\vee$:* $x \vee 0 = x$
- 1 *is identity for $\wedge$:* $x \wedge 1 = x$
- 0 *is annihilator for $\wedge$:* $x \wedge 0 = 0$

Property (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of $\vee$:* $x \vee x = x$
- *Idempotence of $\wedge$:* $x \wedge x = x$
- *Absorption of $\vee$ over $\wedge$:* $x \vee (x \wedge y) = x \wedge y$
- *Absorption of $\wedge$ over $\vee$:* $x \wedge (x \vee y) = x \vee y$
- *Distributivity of $\vee$ over $\wedge$:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 *is annihilator for $\vee$:* $x \vee 1 = 1$

Property (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ *and* $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) $\square$

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property (Implication and Inclusion). *If* $\forall x \in X, P_1(x) \implies P_2(x)$*, then* $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.

Proof. Trivial. $\square$

Definition (If and only if). $x \iff y$

Question. *Express in terms of and, or, not:*

- $\oplus$
- $\implies$
- $\impliedby$
- $\iff$

Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.

Negation of quantified propositions

Property (Negation of $\forall$). $\mathrm{not}(\forall x \in X, P(x)) = \exists x \in X, \mathrm{not}(P(x))$

Property (Negation of $\exists$). $\mathrm{not}(\exists x \in X, P(x)) = \forall x \in X, \mathrm{not}(P(x))$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset,\ \ldots$ *is true ;* $\exists x \in \emptyset,\ \ldots$ *is false*

Question. *Negate the following*

- $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}$ *s.t.* $n > x$
- *($x_n \to x$):* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.* $\forall n > N, |x_n - x| < \epsilon$

1.4 Objects & Notations

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ - scalars vs vectors

1.5 Proofs

Example of proofs and non-proofs - direct - splitting cases - induction - contradiction

1.6 Geometry

- equations of lines/planes, etc... => vectors / scalar & equation manipulations

1.7 Sets

- min/max & sup/inf => start using for all / there exists

1.8 Integers

- prime numbers (infinite nb by Euclide) - unique factorization - finding primes between 1 and 100 => time complexity of algo?

2 Complex numbers

argand diagram

3 Sizes of infinity

[recycling house 6 pres']

4 Asymptotic analysis (limits)

- def of sequence: recursive and general form - usual sequences (arithmetic/geometric) - convergence of sequences

5 Infinite & partial sums

- sum of sequences - sum of usual (arithmetic/geometric) sequences - def of series - convergence of series

6 Functions & Inverses

finding roots & inverses

7 Usual functions

- plot & limit behaviour of: polynomials, exp, log, sin, cos, tan, sinh, cosh, tanh, arccos, arcsin, arctan

8 Differentiation

- from scratch - derivatives of usual functions - chain-law & co

9 Integration

- from scratch (area under curve, taking limit of rectangles) - antiderivative (do proof?) - integral of usual functions - integration by part? (if time!) - integration by substitution? (if time!)

10 Taylor series

- theory & practice - usual Taylor expansions - example of convergence

11 Fourier series? (if not late!)

12 Differential calculus? (if not late!)

13 Vector spaces

- def of vect sp - norm - basic propr

14 Matrices

- def - linear mapping of vect sp - inverse: def, existance (det), finding inverse - rank & kernel - eigenvalues

15 Non-linear multi-dimensional functions

- eg: cost func - partial derivatives - gradient - convexity? - optim: gradient descent

16 Regressions

- by hand - theory - non linear

17 PCA? (if time)

18 Basis of ML (perceptron)? (if time)