Refresher Maths Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the Master's, i.e. basic algebra/analysis and basic application.

Contents

0 Introduction

Hello! welcome to this maths refresher course for DSBA 2022! This is the best course ever!

Presentation

- Paul Dubois, PhD Student @ Centrale, end of 1st year
- Email: b00795695@essec.edu (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h arranged as 1h20min lecture - $1/3$h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)

Examination

- Course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)
- As the goal is to learn, you will be able to resubmit exercise sets, but you will lose 10% every-time you re-submit (so that you have some incentive to try your best the 1st time)
- Best $(n-1)/n$ count, need average $\geq 70\%$ to pass
- In the unlikely event of not passing, you will be able to do some extra work to pass

Questions?

1 Elementary Maths

Should be fairly easy, this section is just so that everyone is one the same page and use the same notation for the rest of the course. I will go fast as I assume you have already seen this before.

1.1 Mathematical objects

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $a \in A$, $\{a, b, c \dots\}$, $\{e \mid condition\}$

Remark (Russell Paradox). *(digression)*
Need to be careful when defining set: some definitions are pathological.
e.g.: Take $Y = \{x \mid x \notin x\}$*:* $Y \in Y \iff Y \notin Y$

Definition (Empty Set). $\emptyset = \{\}$

Definition (Power Set). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Numbers

Definition (Booleans). *The set of boolean numbers* $\mathbb{B}$ *is* $\{0, 1\}$ *or sometimes denoted* $\{False, True\}$.

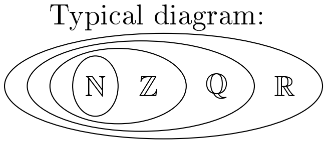
Definition (Naturals). *The set of natural numbers* $\mathbb{N}$ *is* $\{0, 1, 2, 3, \dots\}$.
N.B.: some countries do not count 0 *as a natural number; but we are in France, so in this course, we will.*
N.B.(bis): If we need the naturals without zero, we will use $\mathbb{N}^*$.

Definition (Integers). *The set of integral numbers* $\mathbb{Z}$ *is* $\{0, 1, -1, 2, -2, 3, -3, \dots\}$.
N.B.: If we need only the negative part of the integers, we will use $\mathbb{Z}^-$.

Definition (Rationals or Quotients or Fractions). *The set of natural numbers* $\mathbb{Q}$ *is* $\{{}^a\!/_b \mid a \in \mathbb{Z}, b \in \mathbb{N}^*\}$.

Definition (Reals). *The set of real numbers* $\mathbb{R}$ *is the set of "all numbers you can think of": rational + irrationals (e.g.: roots).*

Typical diagram:



Later, we will see complex numbers. In computer science, types matter a lot.

Functions

Definition (Functions). *Assignment from a set to another.*

Notation (Function). $f : X \to Y$, $f(x) = blah$, $f : x \mapsto blah$.

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$

- $f : k \in \{1,2,3,4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1,2,3,4,5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1,2,3,4\} \mapsto k \in \{1,2\}$
- $f : k \in \{1,2,3,4\} \mapsto k \in \{1,2,3,4,5\}$

Quantifiers

Notation ($\forall$)**.** *For all elements in set, e.g.:* $\forall x \in \mathbb{R}, x^2 \geq 0$.

Notation ($\exists$)**.** *There exists an element in set, e.g.:* $\exists x \in \mathbb{R}$ *s.t.* $x^2 > 1$.

Notation ($\exists!$)**.** *There exists a unique element in set, e.g.:* $\exists! x \in \mathbb{R}$ *s.t.* $x^2 \leq 0$.

Question.
- *Express "all natural numbers are positive" with quantifiers*
- *Express* $\forall x \geq 0,\ \sqrt{x} \geq 0$ *in a sentence*
- *Express set inclusion (e.g.* $A \subseteq B$*) with quantifiers*
- *Express* $\forall x \in X, x \notin Y$ *in a sentence*

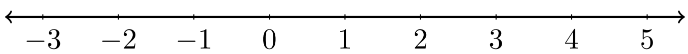
Property (Negation of $\forall$)**.** $\neg \forall x \in X, P(x) = \exists x \in X, \neg P(x)$

Property (Negation of $\exists$)**.** $\neg \exists x \in X, P(x) = \forall x \in X, \neg P(x)$

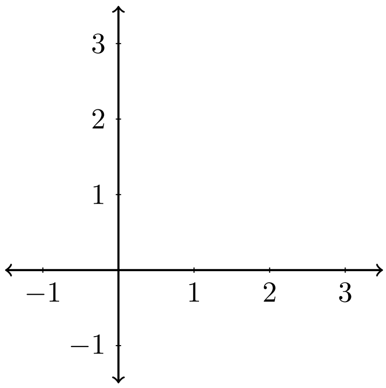
Notation (Quantifiers and the empty set)**.** $\forall x \in \emptyset, \ \ldots$ *is true ;* $\exists x \in \emptyset, \ \ldots$ *is false*

Question. *Negate the following*
- $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}$ *s.t.* $n > x$
- *(*$x_n \to x$*):* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.* $\forall n > N, |x_n - x| < \epsilon$

Vectors 1D vs 2D:



"1D"



"2D"

To "select" a point in 2D, we need 2 numbers, giving a coordinate. In 3D, we would need 3 numbers, in 4D, 4 numbers, etc...

Definition (Cartesian Product)**.** $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$
e.g.: $\{a,b\} \times \{1,2,3\} = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$
e.g.: $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

Moving from one point to another gives a "translation". Again, we need ad many numbers as there are dimensions. Typically, we denote points horizontally, and vectors vertically.

1.2 Axioms

Here $\star$ and $\dagger$ will operations.

Definition (Associativity). *$\star$ is associative if* $\forall x, y, z,\ (x \star y) \star z = x \star (y \star z)$

Definition (Commutativity). *$\star$ is associative if* $\forall x, y,\ (x \star y) = y \star x$

Definition (Identity). *$1_\star$ is identity for $\star$ if* $\forall x,\ 1_\star \star x = x \star 1_\star = x$

Definition (Annihilator). *$0_\star$ is annihilator for $\star$ if* $\forall x,\ 0_\star \star x = x \star 0_\star = 0_\star$

Definition (Distributivity). *$\star$ is distributive over $\dagger$ if* $\forall x, y, z\ x \star (y \dagger z) = (x \star y) \dagger (x \star z)$

of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Question. *(make a table)*

- *Which of these are commutative: addition, subtraction, multiplication, division, power?*
- *Which of these are associative: addition, subtraction, multiplication, division, power?*
- *What is identity for: addition, subtraction, multiplication, division, power?*
- *What is annihilator for: addition, subtraction, multiplication, division, power?*

Question.
- *Think of an operation that is commutative, but not associative*
- *Think of an operation that is associative, but not commutative*

1.3 Boolean algebra

The reason we'll do some is because of it's application to programming, in particular to conditions ('if' blocks and 'while' loops).

Boolean operators

Definition (Conjunction). $x \wedge y = xy$

Definition (Disjunction). $x \vee y = \min(x + y, 1)$

Definition (Negation). $\neg : 0, 1 \mapsto 1, 0$

Definition (Exclusive Or). $x \oplus y = x + y - 2xy$

Definition (Implication). $x \implies y$

Definition (If and only if). $x \iff y$

Question. *Express in terms of and, or, not:*

- $\oplus$
- $\implies$
- $\impliedby$
- $\iff$

Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.

Boolean properties

Property (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of* $\vee$*:* $x \vee (y \vee z) = (x \vee y) \vee z$
- *Associativity of* $\wedge$*:* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- *Commutativity of* $\vee$*:* $x \vee y = y \vee x$
- *Commutativity of* $\wedge$*:* $x \wedge y = y \wedge x$
- *Distributivity of* $\wedge$ *over* $\vee$*:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- 0 *is identity for* $\vee$*:* $x \vee 0 = x$
- 1 *is identity for* $\wedge$*:* $x \wedge 1 = x$
- 0 *is annihilator for* $\wedge$*:* $x \wedge 0 = 0$

Property (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of* $\vee$*:* $x \vee x = x$
- *Idempotence of* $\wedge$*:* $x \wedge x = x$
- *Absorption of* $\vee$ *over* $\wedge$*:* $x \vee (x \wedge y) = x \wedge y$
- *Absorption of* $\wedge$ *over* $\vee$*:* $x \wedge (x \vee y) = x \vee y$
- *Distributivity of* $\vee$ *over* $\wedge$*:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 *is annihilator for* $\vee$*:* $x \vee 1 = 1$

Property (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ *and* $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) $\square$

Relation with set theory

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Property (Implication and Inclusion). *If* $\forall x \in X, P_1(x) \implies P_2(x)$*, then* $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.

Property (Equivalence and Equality). *If* $\forall x \in X, P_1(x) \iff P_2(x)$*, then* $\{x \in X \mid P_1(x)\} = \{x \in X \mid P_2(x)\}$.

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

Question. *Selecting points outside a given region.*

1.4 Proofs

Proofs play a major role in mathematics. Except axioms, every statement needs a proof to be valid. There are a 4 common approaches to prove a statement:

Direct proof This techniques consists of just going straight in and aiming at the statement you want.

Property (Archimedian Property). *For any real number, there exists an integer even greater. That is $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $x < n$.*

Proof. Let $n = \lfloor x \rfloor + 1$: then $x - 1 < \lfloor x \rfloor$ so $x < \lfloor x \rfloor = n$. □

Splitting cases This techniques consists splitting the statement to prove in several easier ones.

Property. *For any $n \in \mathbb{Z}$, n and n^2 share the same parity.*

Proof. We split into 2 cases: n even and n odd:

- Suppose n is even: $n = 2k$ so $n^2 = 4k^2 = 2(2k^2)$, i.e. n^2 is even.
- Suppose n is odd: $n = 2k + 1$ so $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, i.e. n^2 is odd.

□

Contradiction This techniques consists in assuming the opposite of what you want, deriving a contradiction (both P and $\neg P$), then conclude the opposite of what was assumed.

Property. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$: then there are $a, b \in \mathbb{N}$ such that $\sqrt{2} = a/b$. Moreover, we can take a and b so that they have no common divisors. Now, we have $2b^2 = a^2$. So a^2 is even, so a is even, let $a = 2a'$. Then, $2b^2 = 4a'^2$ so $b^2 = 2a'^2$. Thus, b^2 is even, so b is even. Both a and b are divisible by 2, which contradicts them having no common divisors. Finally, we conclude our initial assumption was wrong, so $\sqrt{2} \notin \mathbb{Q}$. □

Induction This techniques is only applicable if we wish to prove a statement for all natural numbers. The idea is similar to domino: each statement will imply the next one. Mathematically, if $P(n)$ are statements ($n \in \mathbb{N}$), we give 2 proofs: First, that $P(0)$ is true. Then, that $P(n) \implies P(n+1)$.

Property. $\forall n \in \mathbb{N}$, $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

Proof. **Initialization:** for $n = 0$:

$$\sum_{k=0}^{0} k = 0 \text{ and } \frac{0(0+1)}{2} = 0$$

so clearly

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \text{ for } n = 0$$

Induction: suppose $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$, we need $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\begin{aligned}\sum_{k=0}^{n+1} k &= (n+1) + \sum_{k=0}^{n} k \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2}\end{aligned}$$

□

Be careful, BOTH initialization & induction are important.

Existence We usually show existence of an object by constructing it.

Property (Existence of Euclidean division). $\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists\ q \in \mathbb{Z}, r \in [0, b[\cap \mathbb{N}\ s.t.\ a = bq + r$

Proof. Let $q = \max\{k \in \mathbb{N} \mid bk \leq a\}$, and $r = a - bq$. One can then check that the conditions are met. □

Uniqueness To show uniqueness of an object, we usually suppose there are two, and show that they must be the same, enforcing the fact that it must be unique.

Property (Uniqueness of Euclidean division). $\forall a \in \mathbb{Z}, b \in \mathbb{N}^*, \exists!\ q \in \mathbb{Z}, r \in [0, b[\cap \mathbb{N}\ s.t.\ a = bq + r$

Proof. Suppose $a = bq_1 + r_1$ and $a = bq_2 + r_2$. Then $bq_1 + r_1 = bq_2 + r_2$, so $b(q_1 - q_2) = r_2 - r_1$. Now, $r_2 - r_1 \in]-b, b[$ and $b(q_1 - q_2) \in b * \mathbb{Z}$, so the only possibility is that they are both 0. So $r_1 = r_2$ and $q_1 = q_2$. □

A **misleading** "proof":

Property (FALSE PROPERTY). *Horses are all the same colour.*

FALSE PROOF. We will prove by induction that any set of n horses must be a set of horses all being the same colour.

Initialization: for set of 1 horse, clearly, it's true.

Induction: suppose all sets of n horses are uni-colour; we need sets of $n+1$ horses to be uni-colour.

Take $H = \{h_1, h_2, \ldots, h_n, h_{n+1}\}$ a set of $n+1$ horses. Then let $H_1 = \{h_1, \ldots, h_n, \}$ and $H_2 = \{h_2, \ldots, h_{n+1}\}$. By induction, H_1 and H_2 must be uni-colour. h_n belongs to both sets, so the colour of horses in both sets mus be the same, and H is uni-colour. □

Question.

- *Show that $12n - 6$ is divisible by 6 for every positive integer n (you do not need induction for this one).*
- *Show that n divisible by 6 if and only if n divisible by 2 and 3.*
- *Show that n^2 divisible by 3 if and only if n divisible 3.*[1]
- *Show $\sqrt{3} \notin \mathbb{Q}$.*
- *Show that $2^n \geq 2n$ for all $n \in \mathbb{N}$*

[1] You may go faster using `https://en.wikipedia.org/wiki/Modular_arithmetic`

1.5 Basic geometry

This is high-school material, so we will just warm it up with a few exercises:

Definition (Norm). *With $\vec{u}$ a vectors in N dimension:*

$$\|\vec{u}\| = \sqrt{\sum_{k=1}^{N} u_k^2}$$

NB: The distance from A to B is $\left\|\vec{AB}\right\|$.

Definition (Dot Product). *With $\vec{u}$ and $\vec{v}$ vectors:*

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{N} u_k v_k = \|\vec{u}\| \, \|\vec{v}\| \cos(angle(\vec{u}, \vec{v}))$$

In particular, if $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.
Dot product takes two vectors, and gives a scalar. It may be interesting to, similarly to scalars multiplication, have the same type of object as output of the operation. This is what the vector product was designed for.

Definition (Vector Product). *With $\vec{u}$ and $\vec{v}$ vectors in 3 dimensions:*

$$\vec{u} \times \vec{v} = \vec{w}$$

$$\textit{with } \|\vec{w}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(angle(\vec{u}, \vec{v}))$$

$$\textit{and } \vec{u}, \vec{v} \perp \vec{w}\textit{; direction given by the right-hand-rule}$$

Now, if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ *and* $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, *then:*

$$\begin{aligned} w &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= u_2 v_3 \vec{i} + u_3 v_1 \vec{j} + u_1 v_2 \vec{k} - v_1 u_2 \vec{k} - v_2 u_3 \vec{i} - v_3 u_1 \vec{j} \\ &= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \end{aligned}$$

Question. *Find an equation of the following lines in the xy-plane:*

- *vertical, crossing the x-axis at 5*
- *horizontal, crossing the y-axis at 7*
- *crossing the points* $(2, 3)$ *and* $(-3, -7)$
- *crossing the point* $(5, 1)$ *with slope* -2
- *crossing the point* $(5, 1)$ *perpendicular to the last one*
- *the mediator of* $(3, 1)$ *and* $(-9, 8)$

Question. *Find an equation of the following planes in the xyz-space:*

- *horizontal, crossing the z-axis at -8*
- *crossing the point* $(5, 1-3)$ *with normal vector* $(-2, 6, -9)$
- *crossing the points* $(2, 3, 1)$, $(-10, 4, -5)$, *and* $(-3, -7, -1)$
- *the mediator plane of* $(3, 1, 5)$ *and* $(-9, 8, -3)$

Question. *Describe (give the associated set of points) the following lines in the xyz-space:*

- *vertical, crossing the xy-plane at* $(1, 1)$
- *crossing the points* $(2, 3, -1)$ *with normal vector* $(8, 5, 9)$
- *intersection of the planes with equation* $2z + 3y - 6x = 8$ *and* $-3z + 4y - 8x = 7$

1.6 Early analysis

Definition (Minimum/Maximum)**.** *With* A *a set:* $a^* \in A$ *is the minimum (respectively maximum) of* A *if* $\forall a \in A, a \geq a^*$ *(respectively* $\forall a \in A, a \leq a^*$*).*

Definition (Lower/Upper bound)**.** *With* A *a set:* b *is a lower (respectively upper) bound of* A *if* $\forall a \in A, a \geq b$ *(respectively* $\forall a \in A, a \leq b$*).*

Definition (Infimum/Supremum)**.** *With* A *a set:* a^* *is the infimum (respectively supremum) of* A *if* a^* *is the largest lower bound (respectively the smallest upper bound).*
With quantifiers:
a^* *is the infimum of* A *if* $\forall \varepsilon > 0, \exists a \in A$ *s.t.* $a < a^* + \varepsilon$ *(and* $\forall a \in A, a \geq a^*$*).*
a^* *is the supremum of* A *if* $\forall \varepsilon > 0, \exists a \in A$ *s.t.* $a > a^* - \varepsilon$ *(and* $\forall a \in A, a \leq a^*$*).*

Infimum/supremum always exist, while minimum/maximum may not.

Question. *Find the minimum, maximum, infimum, supremum of the following sets:*

- $[0, 1]$
- $(0, 1)$
- $\{1/n \mid n \in \mathbb{N}^*\}$

1.7 Essence of number theory

Definition (Prime)**.** $p \in \mathbb{N}$ *is a prime number if it is only divisible by* 1 *and itself (p)*
The set of prime numbers is usually denotes $\mathbb{P}$*.*

Property. *Every natural number (greater than* 2*) can be decomposed into a product of prime powers, i.e.:* $\forall n \in \mathbb{N}, n \geq 2, \exists p_1, \dots, p_N, \alpha_1, \dots, \alpha_N$ *s.t.* $n = \prod_{k=1}^{N} p_k^{\alpha_k}$*.*

Property ($|\mathbb{P}| = \infty$)**.** *There are infinitely many primes.*

Proof. Suppose $\mathbb{P} = \{p_1, \dots, p_n\}$. Let $N = 1 + \prod_{k=1}^{n} p_k$; then N is not divisible by any prime from $\mathbb{P}$, but it must have a prime factorisation. Thus, there must exist a prime that isn't included in $\mathbb{P}$. This is a contradiction. □

Question. *How to check if a number is prime or not? (write pseudo code algorithm)*
Find all primes in $[0, 50]$ *(and write pseudo code algorithm to find all the primes in* $[0, 500]$*).*

2 Sizes of infinity

Little digression on sets cardinalities.

Link for the slides:
`https://drive.google.com/file/d/1_Yb8hH4OKMac3xvwTG6WLkYiQooIaSOV/view?usp=sharing`

Summary:
Although $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, it is in fact the case that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$.

3 Complex numbers

3.1 Introduction

Observation: $x^2 + 1 = 0$ has no solution in $\mathbb{R}$; want to extend $\mathbb{R}$ so that there is a solution.

Definition (The Complex Unit). *Let $i = \sqrt{-1}$ so that $i^2 = -1$ and $i, -i$ are two solutions of $x^2 + 1 = 0$.*

Definition (The Complex Field). $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

[draw complex plane, show $\mathbb{C}$ is in bijection with $\mathbb{R}^2$]

Proposition (Complex have all roots of all quadratics). $ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{\Delta}}{2a}$

Proof. case $\Delta < 0$ □

Operations on complex numbers: show addition, multiplication, division

Definition (Conjugate). *If $z = x + iy$, then conjugate of z is $\overline{z} = x - iy$.*

Property (Properties of Conjugate).

- $\overline{zz'} = \overline{z}\overline{z'}$
- $\overline{z + z'} = \overline{z} + \overline{z'}$
- $\overline{z^k} = \overline{z}^k$

Proof. use $z = x + iy$, $z' = x' + iy'$ □

Definition (Modulus). *If $z = x + iy$, then modulus of z is $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$.*

Property (Properties of Modulus).

- *Modulus is a metric on $\mathbb{C}$*
- $|zz'| = |z||z'|$

Proof.

- show triangle inequality (others are trivial)
- use $z = x + iy$, $z' = x' + iy'$

□

3.2 Complex Exponential

$\sum_{n \in \mathbb{N}} z_n$ converges if $\sum_{n \in \mathbb{N}} |z_n|$, so $\sum_{n \in \mathbb{N}} \frac{z^k}{k!}$ converges uniformly; we can therefore extend exponential to complex.

Note that algebra of exponential remains over the complex, and $\overline{e^z} = e^{\overline{z}}$.

3.2.1 Geometry

[Draw argand diagram: x, y; modulus, argument, conjugate]

Polar coordinates <=> Cartesian coordinates

Transformations in the complex plane: translation, scaling, rotation

3.2.2 Link with trigonometry

Property. $|\exp(i\theta)| = 1$

Proof. $|\exp(i\theta)|^2 = \exp(i\theta)\exp(-i\theta) = \exp(0) = 1$ $\square$

Unit circle: coordinates are given by cos and sin
By proving $\theta \mapsto \exp(i\theta)$ is surjective, can show that $\exp(i\pi/2) = i$.
Have:

- $\cos(\theta) = \Re(\exp(i\theta))$
- $\sin(\theta) = \Im(\exp(i\theta))$

This gives periodicity of 2π, etc...

3.3 Complex Polynomials

Lemma 3.1 (Existence of the Minimum of a Polynomial)**.** $P(z) \in \mathbb{C}[x] : \exists z_0 \in \mathbb{C}$ *s.t.* $P(z_0) = \inf\{P(z) \mid z \in \mathbb{C}\}$

Proof. Show $|P(z)| \to +\infty$ as $|z| \to +\infty$. Then $X = \{z \in \mathbb{C} \mid |P(z)| \leq \inf\{|P(z)| \mid z \in \mathbb{C}\} + 1\}$ is compact (close & bounded). By definition of infimum, there is a sequence $(z_n) \subseteq X$ such that $(P(z_n)) \to \inf\{|P(z)| \mid z \in \mathbb{C}\}$. But then there is a sub-sequence $z_{n_k} \to z \in X$. $\square$

Theorem 3.1 (of d'Alembert)**.** $P(z) \in \mathbb{C}[x] : \partial P \geq 1 \implies \exists z \in \mathbb{C}$ *s.t.* $P(z) = 0$

Proof. Suppose $\min\{|P(z)| \mid z \in \mathbb{C}\} > 0$ is reached at z_0. We can define the polynomial $Q : z \in \mathbb{C} \mapsto \frac{P(z_0+z)}{P(z_0)}$ which is such that $Q(0) = \min\{|Q(z)|, z \in \mathbb{C}\} = 1$. Let $(b_0, ..., b_p)$ be the coefficients of Q and $q = \min\{j \in [\![1;p]\!] | b_j \neq 0\}$. With these notations, $\forall z \in \mathbb{C},\ Q(z) = 1 + b_q z^q + \sum_{k=q+1}^{p} b_k z^k$.
Let $\theta = \operatorname{Arg}(b_q)$ and $\varphi = \frac{\pi-\theta}{q}$. Then $b_q e^{iq\varphi} = -|b_q|$. So:

$$\forall r > 0, Q(re^{i\varphi}) = 1 - |b_q|r^q + \sum_{k=q+1}^{p} b_k r^k e^{ik\varphi}$$

$$|Q(re^{i\varphi})| \leq |1 - |b_q|r^q| + \sum_{k=q+1}^{p} |b_k|r^k$$

$$\forall r \in]0; |b_q|^{1/q}[,\ \ |Q(re^{i\varphi})| \leq 1 - |b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k$$

$$|Q(re^{i\varphi})| - 1 \leq -|b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k$$

$$\lim_{r\to 0} \frac{-|b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k}{r^q} = -|b_q| < 0$$

So there exists r_1 such that $0 < r_1 < |b_q|^{-1/q}$ such that $\forall r < r_1,\ \frac{-|b_q|r^q + \sum_{k=q+1}^{p} |b_k|r^k}{r^q} < 0$, so $|Q(re^{i\varphi})| < 1$, which is a contradiction. $\square$

Corollary 3.1. *Let* $P \in \mathbb{C}[X]$ *such that* $\deg(P) \geq 1$. *Let* $z_1, ..., z_m$ *be the roots of* P *of multiplicities* $\alpha_1, ..., \alpha_m$. *Then we have that* $\alpha_1 + ... + \alpha_m = \deg(P)$ *and there exists* $\lambda \in \mathbb{C}^*$ *such that*

$$\forall z \in \mathbb{C},\ P(z) = \lambda \prod_{k=1}^{m} (z - z_k)^{\alpha_k}$$

4 Asymptotic analysis (limits)

- def of sequence: recursive and general form - usual sequences (arithmetic/geometric) - convergence of sequences

5 Infinite & partial sums

- sum of sequences - sum of usual (arithmetic/geometric) sequences - def of series - convergence of series

6 Functions & Inverses

finding roots & inverses

7 Usual functions

- plot & limit behaviour of: polynomials, exp, log, sin, cos, tan, sinh, cosh, tanh, arccos, arcsin, arctan

8 Differentiation

- from scratch - derivatives of usual functions - chain-law & co

9 Integration

- from scratch (area under curve, taking limit of rectangles) - antiderivative (do proof?) - integral of usual functions - integration by part? (if time!) - integration by substitution? (if time!)

10 Taylor series

- theory & practice - usual Taylor expansions - example of convergence

11 Fourier series? (if not late!)

12 Differential calculus? (if not late!)

13 Vector spaces

- def of vect sp - norm - basic propr

14 Matrices

- def - linear mapping of vect sp - inverse: def, existance (det), finding inverse - rank & kernel - eigenvalues

15 Non-linear multi-dimensional functions

- eg: cost func - partial derivatives - gradient - convexity? - optim: gradient descent

16 Regressions

- by hand - theory - non linear

17 PCA? (if time)

18 Basis of ML (perceptron)? (if time)