

Refresher Maths Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the Master's, i.e. basic algebra/analysis and basic application.

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0 Introduction

Hello! welcome to this maths refresher course for DSBA 2022! This is the best course ever!

Presentation

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- Email: b00795695@essec.edu (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h arranged as 1h20min lecture - 1/3h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)

Examination

- Course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)
- As the goal is to learn, you will be able to resubmit exercise sets, but you will lose 10% every-time you re-submit (so that you have some incentive to try your best the 1st time)
- Best $(n-1)/n$ count, need average $\geq 70\%$ to pass
- In the unlikely event of not passing, you will be able to do some extra work to pass

Questions?

1 Elementary Maths

Should be fairly easy, this section is just so that everyone is on the same page and use the same notation for the rest of the course. I will go fast as I assume you have already seen this before.

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $a \in A$, $\{a, b, c, \dots\}$, $\{e \mid \text{condition}\}$, \emptyset

Remark (Russell Paradox). (*digression*)

Need to be careful when defining set: some definitions are pathological.

e.g.: Take $Y = \{x \mid x \notin x\}$: $Y \in Y \iff Y \notin Y$

Types of numbers

Definition (Booleans). *The set of boolean numbers \mathbb{B} is $\{0, 1\}$ or sometimes denoted $\{\text{False}, \text{True}\}$.*

Definition (Naturals). *The set of natural numbers \mathbb{N} is $\{0, 1, 2, 3, \dots\}$.*

N.B.: some countries do not count 0 as a natural number; but we are in France, so in this course, we will.

N.B.(bis): If we need the naturals without zero, we will use \mathbb{N}^ .*

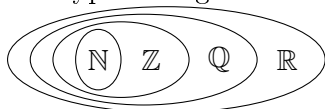
Definition (Integers). *The set of integral numbers \mathbb{Z} is $\{0, 1, -1, 2, -2, 3, -3, \dots\}$.*

N.B.: If we need only the negative part of the integers, we will use \mathbb{Z}^- .

Definition (Rationals or Quotients or Fractions). *The set of rational numbers \mathbb{Q} is $\{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}^*\}$.*

Definition (Reals). *The set of real numbers \mathbb{R} is the set of "all numbers you can think of": rational + irrationals (e.g.: roots).*

Typical diagram:



Later, we will see complex numbers. In computer science, types matter a lot.

Functions

Definition (Functions). *Assignment from a set to another.*

Notation (Function). $f : X \rightarrow Y$, $f(x) = \text{blah}$, $f : x \mapsto \text{blah}$.

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation (\forall). For all elements in set, e.g.: $\forall x \in \mathbb{R}, x^2 \geq 0$.

Notation (\exists). There exists an element in set, e.g.: $\exists x \in \mathbb{R}$ s.t. $x^2 > 1$.

Notation ($\exists!$). There exists a unique element in set, e.g.: $\exists! x \in \mathbb{R}$ s.t. $x^2 \leq 0$.

Question. • Express "all natural numbers are positive" with quantifiers

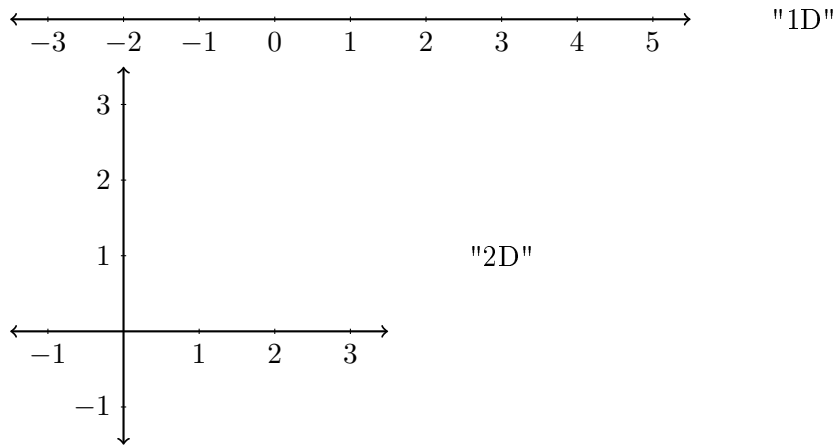
- Express $\forall x \geq 0, \sqrt{x} \geq 0$ in a sentence

Definition (Subset / Inclusion). $X \subseteq Y$ if $\forall x \in X, x \in Y$

Definition (Disjoint Sets). X and Y are disjoint if $\forall x \in X, x \notin Y$ (or if $\forall y \in Y, y \notin X$).

Definition (Power Set). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Scalars vs vectors 1D vs 2D:



To "select" a point in 2D, we need 2 numbers, giving a coordinate. In 3D, we would need 3 numbers, in 4D, 4 numbers, etc...

Definition (Cartesian Product). $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

e.g.: $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

e.g.: $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$

Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^n X_k$

Moving from one point to another gives a "translation". Again, we need as many numbers as there are dimensions. Typically, we denote points horizontally, and vectors vertically.

1.2 Axioms

Here \star and \dagger will operations.

Definition (Associativity). \star is associative if $\forall x, y, z, (x \star y) \star z = x \star (y \star z)$

Definition (Commutativity). \star is commutative if $\forall x, y, (x \star y) = y \star x$

Definition (Identity). 1_\star is identity for \star if $\forall x, 1_\star \star x = x \star 1_\star = x$

Definition (Annihilator). 0_\star is annihilator for \star if $\forall x, 0_\star \star x = x \star 0_\star = 0_\star$

Definition (Distributivity). \star is distributive over \dagger if $\forall x, y, z, x \star (y \dagger z) = (x \star y) \dagger (x \star z)$

of \wedge over \vee : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Question. *(make a table)*

- Which of these are commutative: addition, subtraction, multiplication, division, power?
- Which of these are associative: addition, subtraction, multiplication, division, power?
- What is identity for: addition, subtraction, multiplication, division, power?
- What is annihilator for: addition, subtraction, multiplication, division, power?

Question. • Think of an operation that is commutative, but not associative

- Think of an operation that is associative, but not commutative

1.3 Boolean algebra

The reason we'll do some is because of its application to programming, in particular to conditions ('if' blocks and 'while' loops).

Basic operators

Definition (Conjunction). $x \wedge y = xy$

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

Definition (Disjunction). $x \vee y = \min(x + y, 1)$

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation). $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

Question. *Selecting points outside a given region.*

Basic properties

Property (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches \cup with addition and \wedge with multiplication.*

- *Associativity of \vee : $x \vee (y \vee z) = (x \vee y) \vee z$*
- *Associativity of \wedge : $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*
- *Commutativity of \vee : $x \vee y = y \vee x$*
- *Commutativity of \wedge : $x \wedge y = y \wedge x$*
- *Distributivity of \wedge over \vee : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*
- *0 is identity for \vee : $x \vee 0 = x$*
- *1 is identity for \wedge : $x \wedge 1 = x$*
- *0 is annihilator for \wedge : $x \wedge 0 = 0$*

Property (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- Idempotence of \vee : $x \vee x = x$
- Idempotence of \wedge : $x \wedge x = x$
- Absorption of \vee over \wedge : $x \vee (x \wedge y) = x$
- Absorption of \wedge over \vee : $x \wedge (x \vee y) = x$
- Distributivity of \vee over \wedge : $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 is annihilator for \vee : $x \vee 1 = 1$

Property (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) □

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property (Implication and Inclusion). *If $\forall x \in X, P_1(x) \implies P_2(x)$, then $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.*

Proof. Trivial. □

Definition (If and only if). $x \iff y$

Question. *Express in terms of and, or, not:*

- \oplus
- \implies
- \iff
- \iff

Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.

Negation of quantified propositions

Property (Negation of \forall). $\text{not}(\forall x \in X, P(x)) = \exists x \in X, \text{not}(P(x))$

Property (Negation of \exists). $\text{not}(\exists x \in X, P(x)) = \forall x \in X, \text{not}(P(x))$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \dots$ is true ; $\exists x \in \emptyset, \dots$ is false

Question. *Negate the following*

- $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x$
- $(x_n \rightarrow x): \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |x_n - x| < \epsilon$

1.4 Objects & Notations

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ - scalars vs vectors

1.5 Proofs

Example of proofs and non-proofs - direct - splitting cases - induction - contradiction

1.6 Geometry

- equations of lines/planes, etc... \Rightarrow vectors / scalar & equation manipulations

1.7 Sets

- min/max & sup/inf \Rightarrow start using for all / there exists

1.8 Integers

- prime numbers (infinite nb by Euclide) - unique factorization - finding primes between 1 and 100 \Rightarrow time complexity of algo?

2 Complex numbers

argand diagram

3 Sizes of infinity

[recycling house 6 pres']

4 Asymptotic analysis (limits)

- def of sequence: recursive and general form - usual sequences (arithmetic/geometric) - convergence of sequences

5 Infinite & partial sums

- sum of sequences - sum of usual (arithmetic/geometric) sequences - def of series - convergence of series

6 Functions & Inverses

finding roots & inverses

7 Usual functions

- plot & limit behaviour of: polynomials, exp, log, sin, cos, tan, sinh, cosh, tanh, arccos, arcsin, arctan

8 Differentiation

- from scratch - derivatives of usual functions - chain-law & co

9 Integration

- from scratch (area under curve, taking limit of rectangles) - antiderivative (do proof?) - integral of usual functions - integration by part? (if time!) - integration by substitution? (if time!)

10 Taylor series

- theory & practice - usual Taylor expansions - example of convergence

11 Fourier series? (if not late!)

12 Differential calculus? (if not late!)

13 Vector spaces

- def of vect sp - norm - basic propr

14 Matrices

- def - linear mapping of vect sp - inverse: def, existence (det), finding inverse - rank & kernel - eigenvalues

15 Non-linear multi-dimensional functions

- eg: cost func - partial derivatives - gradient - convexity? - optim: gradient descent

16 Regressions

- by hand - theory - non linear

17 PCA? (if time)

18 Basis of ML (perceptron)? (if time)