Refresher Maths Course

Paul Dubois

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the Master's, i.e. basic algebra/analysis and basic application.

Contents

0   Introduction

Hello! welcome to this maths refresher course for DSBA 2022! This is the best course ever!

Presentation

- Paul Dubois, PhD Student @ Centrale, end of 1st year

- Email: b00795695@essec.edu (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h arranged as 1h20min lecture - $^1/3$h break - 1h20min lecture

- No pb class planned, but lectures will have integrated live exercises

- Interrupt if needed (but may also ask at the end of the lecture)

Examination

- Course is pass/fail

- Most (in fact hopefully all) of you will pass

- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)

- As the goal is to learn, you will be able to resubmit exercise sets, but you will lose 10% every-time you re-submit (so that you have some incentive to try your best the 1st time)

- Best $^{(n-1)}/n$ count, need average $\geq 70\%$ to pass

- In the unlikely event of not passing, you will be able to do some extra work to pass

Questions?

1 Elementary Maths

Should be fairly easy, this section is just so that everyone is one the same page and use the same notation for the rest of the course. I will go fast as I assume you have already seen this before.

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $a \in A$, $\{a, b, c \dots\}$, $\{e \mid condition\}$, $\emptyset$

Remark (Russell Paradox). *(digression)*
Need to be careful when defining set: some definitions are pathological.
    e.g.: Take $Y = \{x \mid x \notin x\}$: $Y \in Y \iff Y \notin Y$

Types of numbers

Definition (Booleans). *The set of boolean numbers $\mathbb{B}$ is $\{0, 1\}$ or sometimes denoted $\{False, True\}$.*

Definition (Naturals). *The set of natural numbers $\mathbb{N}$ is $\{0, 1, 2, 3, \dots\}$.*
N.B.: some countries do not count 0 as a natural number; but we are in France, so in this course, we will.
*N.B.(bis): If we need the naturals without zero, we will use $\mathbb{N}^*$.*

Definition (Integers). *The set of integral numbers $\mathbb{Z}$ is $\{0, 1, -1, 2, -2, 3, -3, \dots\}$.*
N.B.: If we need only the negative part of the integers, we will use $\mathbb{Z}^-$.

Definition (Rationals or Quotients or Fractions). *The set of natural numbers $\mathbb{Q}$ is $\{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}^*\}$.*

Definition (Reals). *The set of real numbers $\mathbb{R}$ is the set of "all numbers you can think of": rational + irrationals (e.g.: roots).*

   Typical diagram:
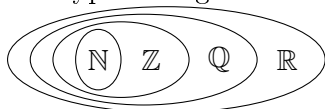
Later, we will see complex numbers. In computer science, types matter a lot.

Functions

Definition (Functions). *Assignment from a set to another.*

Notation (Function). $f : X \to Y$, $f(x) = blah$, $f : x \mapsto blah$.

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$

- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$

- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$

- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$

- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation ($\forall$). *For all elements in set, e.g.: $\forall x \in \mathbb{R}, x^2 \geq 0$.*

Notation ($\exists$). *There exists an element in set, e.g.: $\exists x \in \mathbb{R}$ s.t. $x^2 > 1$.*

Notation ($\exists!$). *There exists a unique element in set, e.g.: $\exists! x \in \mathbb{R}$ s.t. $x^2 \leq 0$.*

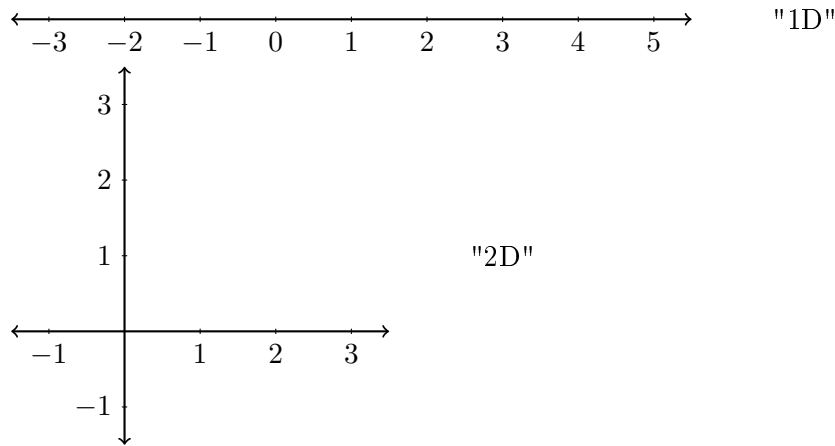**Question.** • *Express "all natural numbers are positive" with quantifiers*

• *Express $\forall x \geq 0$, $\sqrt{x} \geq 0$ in a sentence*

Definition (Subset / Inclusion). *$X \subseteq Y$ if $\forall x \in X, x \in Y$*

Definition (Disjoint Sets). *X and Y are disjoint if $\forall x \in X, x \notin Y$ (or if $\forall y \in Y, y \notin X$).*

Definition (Power Set). *$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$*
e.g.: $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Scalars vs vectors 1D vs 2D:

"1D"

"2D"

To "select" a point in 2D, we need 2 numbers, giving a coordinate. In 3D, we would need 3 numbers, in 4D, 4 numbers, etc...

Definition (Cartesian Product). *$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$*
e.g.: $\{a,b\} \times \{1,2,3\} = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$
e.g.: $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

Moving from one point to another gives a "translation". Again, we need ad many numbers as there are dimensions. Typically, we denote points horizontally, and vectors vertically.

1.2 Axioms

Here $\star$ and $\dagger$ will operations.

Definition (Associativity). *$\star$ is associative if $\forall x, y, z, \ (x \star y) \star z = x \star (y \star z)$*

Definition (Commutativity). *$\star$ is associative if $\forall x, y, \ (x \star y) = y \star x$*

Definition (Identity). *$1_\star$ is identity for $\star$ if $\forall x, \ 1_\star \star x = x \star 1_\star = x$*

Definition (Annihilator). *$0_\star$ is annihilator for $\star$ if $\forall x, \ 0_\star \star x = x \star 0_\star = 0_\star$*

Definition (Distributivity). *$\star$ is distributive over $\dagger$ if $\forall x, y, z \ x \star (y \dagger z) = (x \star y) \dagger (x \star z)$*

of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Question. *(make a table)*

- *Which of these are commutative: addition, subtraction, multiplication, division, power?*

- *Which of these are associative: addition, subtraction, multiplication, division, power?*

- *What is identity for: addition, subtraction, multiplication, division, power?*

- *What is annihilator for: addition, subtraction, multiplication, division, power?*

Question. • *Think of an operation that is commutative, but not associative*

- *Think of an operation that is associative, but not commutative*

1.3   Boolean algebra

The reason we'll do some is because of it's application to programming, in particular to conditions ('if' blocks and 'while' loops).

Basic operators

Definition (Conjunction). $x \wedge y = xy$

Definition (Intersection). $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

Definition (Disjunction). $x \vee y = \min(x + y, 1)$

Definition (Union). $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation). $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement). $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

Question. *Selecting points outside a given region.*

Basic properties

Property (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$*

- *Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*

- *Commutativity of $\vee$: $x \vee y = y \vee x$*

- *Commutativity of $\wedge$: $x \wedge y = y \wedge x$*

- *Distributivity of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*

- *0 is identity for $\vee$: $x \vee 0 = x$*

- *1 is identity for $\wedge$: $x \wedge 1 = x$*

- *0 is annihilator for $\wedge$: $x \wedge 0 = 0$*

Property (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of $\vee$: $x \vee x = x$*

- *Idempotence of $\wedge$: $x \wedge x = x$*

- *Absorption of $\vee$ over $\wedge$: $x \vee (x \wedge y) = x \wedge y$*

- *Absorption of $\wedge$ over $\vee$: $x \wedge (x \vee y) = x \vee y$*

- *Distributivity of $\vee$ over $\wedge$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*

- *1 is annihilator for $\vee$: $x \vee 1 = 1$*

Property (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ *and* $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) □

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property (Implication and Inclusion). *If $\forall x \in X, P_1(x) \implies P_2(x)$, then $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.*

Proof. Trivial. □

Definition (If and only if). $x \iff y$

Question. *Express in terms of and, or, not:*

- $\oplus$

- $\implies$

- $\impliedby$

- $\iff$

Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.

Negation of quantified propositions

Property (Negation of $\forall$). $\neg \forall x \in X, P(x) = \exists x \in X, \neg P(x)$

Property (Negation of $\exists$). $\neg \exists x \in X, P(x) = \forall x \in X, \neg P(x)$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \ldots$ *is true ;* $\exists x \in \emptyset, \ldots$ *is false*

Question. *Negate the following*

- $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ *s.t. $n > x$*

- *($x_n \to x$): $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N, |x_n - x| < \epsilon$*

1.4 Proofs

Proofs play a major role in mathematics. Except axioms, every statement needs a proof to be valid. There are a 4 common approaches to prove a statement:

Direct proof   This techniques consists of just going straight in and aiming at the statement you want.

Property (Archimedian Property). *For any real number, there exists an integer even greater. That is $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $x < n$.*

Proof. Let $n = \lfloor x \rfloor + 1$: then $x - 1 < \lfloor x \rfloor$ so $x < \lfloor x \rfloor = n$. □

Splitting cases   This techniques consists splitting the statement to prove in several easier ones.

Property. *For any $n \in \mathbb{Z}$, n and n^2 share the same parity.*

Proof. We split into 2 cases: n even and n odd:

- Suppose n is even: $n = 2k$ so $n^2 = 4k^2 = 2(2k^2)$, i.e. n^2 is even.

- Suppose n is odd: $n = 2k + 1$ so $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, i.e. n^2 is odd.

□

Contradiction   This techniques consists in assuming the opposite of what you want, deriving a contradiction (both P and $\neg P$), then conclude the opposite of what was assumed.

Property. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$: then there are $a, b \in \mathbb{N}$ such that $\sqrt{2} = a/b$. Moreover, we can take a and b so that they have no common divisors. Now, we have $2b^2 = a^2$. So a^2 is even, so a is even, let $a = 2a'$. Then, $2b^2 = 4a'^2$ so $b^2 = 2a'^2$. Thus, b^2 is even, so b is even. Both a and b are divisible by 2, which contradicts them having no common divisors. Finally, we conclude our initial assumption was wrong, so $\sqrt{2} \notin \mathbb{Q}$. □

Induction   This techniques is only applicable if we wish to prove a statement for all natural numbers. The idea is similar to domino: each statement will imply the next one. Mathematically, if $P(n)$ are statements ($n \in \mathbb{N}$), we give 2 proofs: First, that $P(0)$ is true. Then, that $P(n) \implies P(n + 1)$.

Property. $\forall n \in \mathbb{N}, \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

Proof. **Initialization:** for $n = 0$:

$$\sum_{k=0}^{0} k = 0 \text{ and } \frac{0(0+1)}{2} = 0$$

so clearly

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \text{ for } n = 0$$

Induction: suppose $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$, we need $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{k=0}^{n} k$$
$$= (n+1) + \frac{n(n+1)}{2}$$
$$= \frac{(n+2)(n+1)}{2}$$

□

Be careful, BOTH initialization & induction are important.

A misleading "proof":

Property (FALSE PROPERTY). *Horses are all the same colour.*

FALSE PROOF. We will prove by induction that any set of n horses must be a set of horses all being the same colour.

Initialization: for set of 1 horse, clearly, it's true.

Induction: suppose all sets of n horses are uni-colour; we need sets of $n+1$ horses to be uni-colour.

Take $H = \{h_1, h_2, \ldots, h_n, h_{n+1}\}$ a set of $n+1$ horses. Then let $H_1 = \{h_1, \ldots, h_n, \}$ and $H_2 = \{h_2, \ldots, h_{n+1}\}$. By induction, H_1 and H_2 must be uni-colour. h_n belongs to both sets, so the colour of horses in both sets mus be the same, and H is uni-colour. $\qquad\square$

Question.
- *Show that $12n - 6$ is divisible by 6 for every positive integer n (you do not need induction for this one).*

- *Show that n divisible by 6 if and only if n divisible by 2 and 3.*

- *Show that n^2 divisible by 3 if and only if n divisible 3.*[1]

- *Show $\sqrt{3} \notin \mathbb{Q}$.*

- *Show that $2^n \geq 2n$ for all $n \in \mathbb{N}$*

1.5 Geometry

- equations of lines/planes, etc... => vectors / scalar & equation manipulations

1.6 Sets

- min/max & sup/inf => start using for all / there exists

1.7 Integers

- prime numbers (infinite nb by Euclide) - unique factorization - finding primes between 1 and 100 => time complexity of algo?

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[1]You may go faster using `https://en.wikipedia.org/wiki/Modular_arithmetic`

2   Complex numbers

argand diagram

3   Sizes of infinity

[recycling house 6 pres']

4   Asymptotic analysis (limits)

- def of sequence: recursive and general form - usual sequences (arithmetic/geometric) - convergence of sequences

5   Infinite & partial sums

- sum of sequences - sum of usual (arithmetic/geometric) sequences - def of series - convergence of series

6   Functions & Inverses

finding roots & inverses

7   Usual functions

- plot & limit behaviour of: polynomials, exp, log, sin, cos, tan, sinh, cosh, tanh, arccos, arcsin, arctan

8   Differentiation

- from scratch - derivatives of usual functions - chain-law & co

9   Integration

- from scratch (area under curve, taking limit of rectangles) - antiderivative (do proof?) - integral of usual functions - integration by part? (if time!) - integration by substitution? (if time!)

10   Taylor series

- theory & practice - usual Taylor expansions - example of convergence

11   Fourier series? (if not late!)

12   Differential calculus? (if not late!)

13   Vector spaces

- def of vect sp - norm - basic propr

14 Matrices

- def - linear mapping of vect sp - inverse: def, existance (det), finding inverse - rank & kernel - eigenvalues

15 Non-linear multi-dimensional functions

- eg: cost func - partial derivatives - gradient - convexity? - optim: gradient descent

16 Regressions

- by hand - theory - non linear

17 PCA? (if time)

18 Basis of ML (perceptron)? (if time)