Refresher Maths Course

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the Master's, i.e. basic algebra/analysis and basic application.

Contents

0 Introduction

Hello! welcome to this maths refresher course for DSBA 2022! This is the best course ever!

Presentation

- Paul Dubois, PhD Student @ Centrale, end of 1st year
- Email: b00795695@essec.edu (for any question), answer within 1 working day

Course Format

Lectures

- 8*3h arranged as 1h20min lecture - $1/3$h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)

Examination

- Course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)
- As the goal is to learn, you will be able to resubmit exercise sets, but you will lose 10% every-time you re-submit (so that you have some incentive to try your best the 1st time)
- Best $(n-1)/n$ count, need average $\geq 70\%$ to pass
- In the unlikely event of not passing, you will be able to do some extra work to pass

Questions?

1 Elementary Maths

Should be fairly easy, this section is just so that everyone is one the same page and use the same notation for the rest of the course. I will go fast as I assume you have already seen this before.

1.1 Mathematical Objects & Notations

Sets

Definition (Sets). *Unordered list of elements.*

Notation (Sets). $a \in A$, $\{a, b, c \dots\}$, $\{e \mid condition\}$, $\emptyset$

Remark (Russell Paradox). *(digression)*
Need to be careful when defining set: some definitions are pathological.
e.g.: Take $Y = \{x \mid x \notin x\}$*:* $Y \in Y \iff Y \notin Y$

Types of numbers

Definition (Booleans). *The set of boolean numbers* $\mathbb{B}$ *is* $\{0, 1\}$ *or sometimes denoted* $\{False, True\}$.

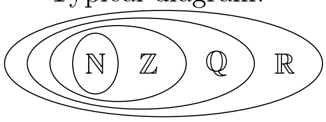
Definition (Naturals). *The set of natural numbers* $\mathbb{N}$ *is* $\{0, 1, 2, 3, \dots\}$.
N.B.: some countries do not count 0 *as a natural number; but we are in France, so in this course, we will.*
N.B.(bis): If we need the naturals without zero, we will use $\mathbb{N}^*$.

Definition (Integers). *The set of integral numbers* $\mathbb{Z}$ *is* $\{0, 1, -1, 2, -2, 3, -3, \dots\}$.
N.B.: If we need only the negative part of the integers, we will use $\mathbb{Z}^-$.

Definition (Rationals or Quotients or Fractions). *The set of natural numbers* $\mathbb{Q}$ *is* $\{{}^{a}/_{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}^*\}$.

Definition (Reals). *The set of real numbers* $\mathbb{R}$ *is the set of "all numbers you can think of": rational + irrationals (e.g.: roots).*

Typical diagram:



Later, we will see complex numbers. In computer science, types matter a lot.

Functions

Definition (Functions). *Assignment from a set to another.*

Notation (Function). $f : X \to Y$, $f(x) = blah$, $f : x \mapsto blah$.

Question. *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

Quantifiers

Notation ($\forall$). *For all elements in set, e.g.:* $\forall x \in \mathbb{R}, x^2 \geq 0$.

Notation ($\exists$). *There exists an element in set, e.g.:* $\exists x \in \mathbb{R}$ *s.t.* $x^2 > 1$.

Notation ($\exists!$). *There exists a unique element in set, e.g.:* $\exists! x \in \mathbb{R}$ *s.t.* $x^2 \leq 0$.

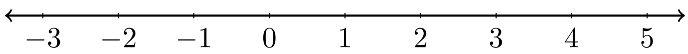
Question.
- *Express "all natural numbers are positive" with quantifiers*
- *Express* $\forall x \geq 0,\ \sqrt{x} \geq 0$ *in a sentence*

Definition (Subset / Inclusion). $X \subseteq Y$ *if* $\forall x \in X, x \in Y$

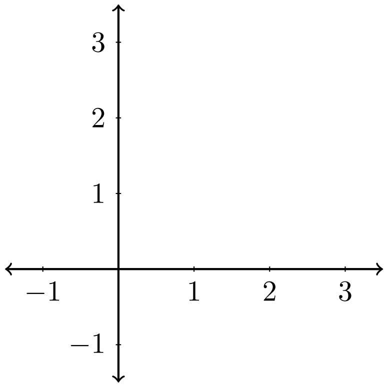
Definition (Disjoint Sets). X *and* Y *are disjoint if* $\forall x \in X, x \notin Y$ *(or if* $\forall y \in Y, y \notin X$*).*

Definition (Power Set). $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$
e.g.: $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Scalars vs vectors 1D vs 2D:



"1D"



"2D"

To "select" a point in 2D, we need 2 numbers, giving a coordinate. In 3D, we would need 3 numbers, in 4D, 4 numbers, etc...

Definition (Cartesian Product). $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
e.g.: $\{a, b\} \times \{1,2,3\} = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$
e.g.: $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$
Extension: $X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k$

Moving from one point to another gives a "translation". Again, we need ad many numbers as there are dimensions. Typically, we denote points horizontally, and vectors vertically.

1.2 Axioms

Here $\star$ and $\dagger$ will operations.

Definition (Associativity). $\star$ *is associative if* $\forall x, y, z,\ (x \star y) \star z = x \star (y \star z)$

Definition (Commutativity). $\star$ *is associative if* $\forall x, y,\ (x \star y) = y \star x$

Definition (Identity). $1_\star$ *is identity for* $\star$ *if* $\forall x,\ 1_\star \star x = x \star 1_\star = x$

Definition (Annihilator). $0_\star$ *is annihilator for* $\star$ *if* $\forall x,\ 0_\star \star x = x \star 0_\star = 0_\star$

Definition (Distributivity). $\star$ *is distributive over* $\dagger$ *if* $\forall x, y, z\ x \star (y \dagger z) = (x \star y) \dagger (x \star z)$

of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Question. *(make a table)*

- *Which of these are commutative: addition, subtraction, multiplication, division, power?*
- *Which of these are associative: addition, subtraction, multiplication, division, power?*
- *What is identity for: addition, subtraction, multiplication, division, power?*
- *What is annihilator for: addition, subtraction, multiplication, division, power?*

Question.
- *Think of an operation that is commutative, but not associative*
- *Think of an operation that is associative, but not commutative*

1.3 Boolean algebra

The reason we'll do some is because of it's application to programming, in particular to conditions ('if' blocks and 'while' loops).

Basic operators

Definition (Conjunction)**.** $x \wedge y = xy$

Definition (Intersection)**.** $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

Remark (Disjoint Sets and Intersection)**.** *Disjoint sets have empty intersection.*

Definition (Disjunction)**.** $x \vee y = \min(x + y, 1)$

Definition (Union)**.** $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

Definition (Negation)**.** $\neg : 0, 1 \mapsto 1, 0$

Definition (Set minus / Complement)**.** $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

Question. *Selecting points outside a given region.*

Basic properties

Property (Boolean algebra matching ordinary algebra)**.** *Same laws as ordinary algebra when one matches up $\vee$ with addition and $\wedge$ with multiplication.*

- *Associativity of $\vee$:* $x \vee (y \vee z) = (x \vee y) \vee z$
- *Associativity of $\wedge$:* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- *Commutativity of $\vee$:* $x \vee y = y \vee x$
- *Commutativity of $\wedge$:* $x \wedge y = y \wedge x$
- *Distributivity of $\wedge$ over $\vee$:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- 0 *is identity for $\vee$:* $x \vee 0 = x$
- 1 *is identity for $\wedge$:* $x \wedge 1 = x$
- 0 *is annihilator for $\wedge$:* $x \wedge 0 = 0$

Property (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- *Idempotence of* $\vee$*:* $x \vee x = x$
- *Idempotence of* $\wedge$*:* $x \wedge x = x$
- *Absorption of* $\vee$ *over* $\wedge$*:* $x \vee (x \wedge y) = x \wedge y$
- *Absorption of* $\wedge$ *over* $\vee$*:* $x \wedge (x \vee y) = x \vee y$
- *Distributivity of* $\vee$ *over* $\wedge$*:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 *is annihilator for* $\vee$*:* $x \vee 1 = 1$

Property (De Morgan Laws). $\neg(x \wedge y) = \neg x \vee \neg y$ *and* $\neg(x \vee y) = \neg x \wedge \neg y$

Proof. Truth-tables; prove De Morgan, others as exercise (or just believe me) $\square$

Other operators

Definition (Exclusive Or). $x \oplus y$

Definition (Implication). $x \implies y$

Property (Implication and Inclusion). *If* $\forall x \in X, P_1(x) \implies P_2(x)$*, then* $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$.

Proof. Trivial. $\square$

Definition (If and only if). $x \iff y$

Question. *Express in terms of and, or, not:*

- $\oplus$
- $\implies$
- $\impliedby$
- $\iff$

Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.

Negation of quantified propositions

Property (Negation of $\forall$). $\neg\forall x \in X, P(x) = \exists x \in X, \neg P(x)$

Property (Negation of $\exists$). $\neg\exists x \in X, P(x) = \forall x \in X, \neg P(x)$

Notation (Quantifiers and the empty set). $\forall x \in \emptyset, \ \ldots$ *is true ;* $\exists x \in \emptyset, \ \ldots$ *is false*

Question. *Negate the following*

- $\forall x \in \mathbb{R}, \ \exists n \in \mathbb{N}$ *s.t.* $n > x$
- *(*$x_n \to x$*):* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.* $\forall n > N, |x_n - x| < \epsilon$

1.4 Proofs

Proofs play a major role in mathematics. Except axioms, every statement needs a proof to be valid. There are a 4 common approaches to prove a statement:

Direct proof This techniques consists of just going straight in and aiming at the statement you want.

Property (Archimedian Property)**.** *For any real number, there exists an integer even greater. That is* $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ *s.t.* $x < n$.

Proof. Let $n = \lfloor x \rfloor + 1$: then $x - 1 < \lfloor x \rfloor$ so $x < \lfloor x \rfloor = n$. □

Splitting cases This techniques consists splitting the statement to prove in several easier ones.

Property. *For any* $n \in \mathbb{Z}$, n *and* n^2 *share the same parity.*

Proof. We split into 2 cases: n even and n odd:

- Suppose n is even: $n = 2k$ so $n^2 = 4k^2 = 2(2k^2)$, i.e. n^2 is even.
- Suppose n is odd: $n = 2k+1$ so $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, i.e. n^2 is odd.

□

Contradiction This techniques consists in assuming the opposite of what you want, deriving a contradiction (both P and $\neg P$), then conclude the opposite of what was assumed.

Property. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$: then there are $a, b \in \mathbb{N}$ such that $\sqrt{2} = a/b$. Moreover, we can take a and b so that they have no common divisors. Now, we have $2b^2 = a^2$. So a^2 is even, so a is even, let $a = 2a'$. Then, $2b^2 = 4a'^2$ so $b^2 = 2a'^2$. Thus, b^2 is even, so b is even. Both a and b are divisible by 2, which contradicts them having no common divisors. Finally, we conclude our initial assumption was wrong, so $\sqrt{2} \notin \mathbb{Q}$. □

Induction This techniques is only applicable if we wish to prove a statement for all natural numbers. The idea is similar to domino: each statement will imply the next one. Mathematically, if $P(n)$ are statements ($n \in \mathbb{N}$), we give 2 proofs: First, that $P(0)$ is true. Then, that $P(n) \implies P(n+1)$.

Property. $\forall n \in \mathbb{N},\ \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

Proof. **Initialization:** for $n = 0$:

$$\sum_{k=0}^{0} k = 0 \text{ and } \frac{0(0+1)}{2} = 0$$

so clearly

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \text{ for } n = 0$$

Induction: suppose $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$, we need $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\begin{aligned}\sum_{k=0}^{n+1} k &= (n+1) + \sum_{k=0}^{n} k \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2}\end{aligned}$$

□

Be careful, BOTH initialization & induction are important.
A misleading "proof":

Property (FALSE PROPERTY). *Horses are all the same colour.*

FALSE PROOF. We will prove by induction that any set of n horses must be a set of horses all being the same colour.

Initialization: for set of 1 horse, clearly, it's true.

Induction: suppose all sets of n horses are uni-colour; we need sets of $n+1$ horses to be uni-colour.

Take $H = \{h_1, h_2, \ldots, h_n, h_{n+1}\}$ a set of $n+1$ horses. Then let $H_1 = \{h_1, \ldots, h_n, \}$ and $H_2 = \{h_2, \ldots, h_{n+1}\}$. By induction, H_1 and H_2 must be uni-colour. h_n belongs to both sets, so the colour of horses in both sets mus be the same, and H is uni-colour. ☐

Question.
- *Show that $12n - 6$ is divisible by 6 for every positive integer n (you do not need induction for this one).*
- *Show that n divisible by 6 if and only if n divisible by 2 and 3.*
- *Show that n^2 divisible by 3 if and only if n divisible 3.*[1]
- *Show $\sqrt{3} \notin \mathbb{Q}$.*
- *Show that $2^n \geq 2n$ for all $n \in \mathbb{N}$*

1.5 Basic geometry

This is high-school material, so we will just warm it up with a few exercises:

Question. *Find an equation of the following lines in the xy-plane:*
- *vertical, crossing the x-axis at 5*
- *horizontal, crossing the y-axis at 7*
- *crossing the points $(2,3)$ and $(-3,-7)$*
- *crossing the point $(5,1)$ with slope -2*
- *crossing the point $(5,1)$ perpendicular to the last one*

Question. *Find an equation of the following planes in the xyz-space:*
- *horizontal, crossing the z-axis at -8*
- *crossing the point $(5, 1-3)$ with normal vector $(-2,6,-9)$*
- *crossing the points $(2,3,1)$, $(-10,4,-5)$, and $(-3,-7,-1)$*

Question. *Describe (give the associated set of points) the following lines in the xyz-space:*
- *vertical, crossing the xy-plane at $(1,1)$*
- *crossing the points $(2,3,-1)$ with normal vector $(8,5,9)$*
- *intersection of the planes with equation $2z+3y-6x=8$ and $-3z+4y-8x=7$*

[1] You may go faster using `https://en.wikipedia.org/wiki/Modular_arithmetic`

1.6 Sets

- min/max & sup/inf => start using for all / there exists

1.7 Integers

- prime numbers (infinite nb by Euclide) - unique factorization - finding primes between 1 and 100 => time complexity of algo?

2 Complex numbers

argand diagram

3 Sizes of infinity

[recycling house 6 pres']

4 Asymptotic analysis (limits)

- def of sequence: recursive and general form - usual sequences (arithmetic/geometric) - convergence of sequences

5 Infinite & partial sums

- sum of sequences - sum of usual (arithmetic/geometric) sequences - def of series - convergence of series

6 Functions & Inverses

finding roots & inverses

7 Usual functions

- plot & limit behaviour of: polynomials, exp, log, sin, cos, tan, sinh, cosh, tanh, arccos, arcsin, arctan

8 Differentiation

- from scratch - derivatives of usual functions - chain-law & co

9 Integration

- from scratch (area under curve, taking limit of rectangles) - antiderivative (do proof?) - integral of usual functions - integration by part? (if time!) - integration by substitution? (if time!)

10 Taylor series

- theory & practice - usual Taylor expansions - example of convergence

11 Fourier series? (if not late!)

12 Differential calculus? (if not late!)

13 Vector spaces

- def of vect sp - norm - basic propr

14 Matrices

- def - linear mapping of vect sp - inverse: def, existance (det), finding inverse - rank & kernel - eigenvalues

15 Non-linear multi-dimensional functions

- eg: cost func - partial derivatives - gradient - convexity? - optim: gradient descent

16 Regressions

- by hand - theory - non linear

17 PCA? (if time)

18 Basis of ML (perceptron)? (if time)