

# Refresher Maths Course

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## Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the Master's, i.e. basic algebra/analysis and basic application.

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## 0 Introduction

Hello! welcome to this maths refresher course for DSBA 2022! This is the best course ever!

### Presentation

- Paul Dubois, PhD Student @ Centrale, end of 1st year
- Email: b00795695@essec.edu (for any question), answer within 1 working day

### Course Format

#### Lectures

- 8\*3h arranged as 1h20min lecture - 1/3h break - 1h20min lecture
- No pb class planned, but lectures will have integrated live exercises
- Interrupt if needed (but may also ask at the end of the lecture)

#### Examination

- Course is pass/fail
- Most (in fact hopefully all) of you will pass
- There will be sets of exercises (about one per lecture), it is advised to attempt it all (only the starred questions will be compulsory)
- As the goal is to learn, you will be able to resubmit exercise sets, but you will lose 10% every-time you re-submit (so that you have some incentive to try your best the 1st time)
- Best  $(n-1)/n$  count, need average  $\geq 70\%$  to pass
- In the unlikely event of not passing, you will be able to do some extra work to pass

### Questions?

# 1 Elementary Maths

Should be fairly easy, this section is just so that everyone is on the same page and use the same notation for the rest of the course. I will go fast as I assume you have already seen this before.

## 1.1 Mathematical Objects & Notations

### Sets

**Definition** (Sets). *Unordered list of elements.*

**Notation** (Sets).  $a \in A$ ,  $\{a, b, c, \dots\}$ ,  $\{e \mid \text{condition}\}$ ,  $\emptyset$

**Remark** (Russell Paradox). (*digression*)

*Need to be careful when defining set: some definitions are pathological.*

*e.g.: Take  $Y = \{x \mid x \notin x\}$ :  $Y \in Y \iff Y \notin Y$*

### Types of numbers

**Definition** (Booleans). *The set of boolean numbers  $\mathbb{B}$  is  $\{0, 1\}$  or sometimes denoted  $\{\text{False}, \text{True}\}$ .*

**Definition** (Naturals). *The set of natural numbers  $\mathbb{N}$  is  $\{0, 1, 2, 3, \dots\}$ .*

*N.B.: some countries do not count 0 as a natural number; but we are in France, so in this course, we will.*

*N.B.(bis): If we need the naturals without zero, we will use  $\mathbb{N}^*$ .*

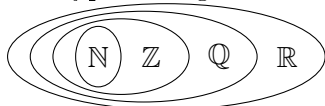
**Definition** (Integers). *The set of integral numbers  $\mathbb{Z}$  is  $\{0, 1, -1, 2, -2, 3, -3, \dots\}$ .*

*N.B.: If we need only the negative part of the integers, we will use  $\mathbb{Z}^-$ .*

**Definition** (Rationals or Quotients or Fractions). *The set of rational numbers  $\mathbb{Q}$  is  $\{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}^*\}$ .*

**Definition** (Reals). *The set of real numbers  $\mathbb{R}$  is the set of "all numbers you can think of": rational + irrationals (e.g.: roots).*

Typical diagram:



Later, we will see complex numbers. In computer science, types matter a lot.

### Functions

**Definition** (Functions). *Assignment from a set to another.*

**Notation** (Function).  $f : X \rightarrow Y$ ,  $f(x) = \text{blah}$ ,  $f : x \mapsto \text{blah}$ .

**Question.** *Which ones of these function are well-defined ?*

- $f : k \in \{0, 1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4, 5\} \mapsto 24/k \in \mathbb{N}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2\}$
- $f : k \in \{1, 2, 3, 4\} \mapsto k \in \{1, 2, 3, 4, 5\}$

## Quantifiers

**Notation** ( $\forall$ ). For all elements in set, e.g.:  $\forall x \in \mathbb{R}, x^2 \geq 0$ .

**Notation** ( $\exists$ ). There exists an element in set, e.g.:  $\exists x \in \mathbb{R}$  s.t.  $x^2 > 1$ .

**Notation** ( $\exists!$ ). There exists a unique element in set, e.g.:  $\exists! x \in \mathbb{R}$  s.t.  $x^2 \leq 0$ .

**Question.** • Express "all natural numbers are positive" with quantifiers

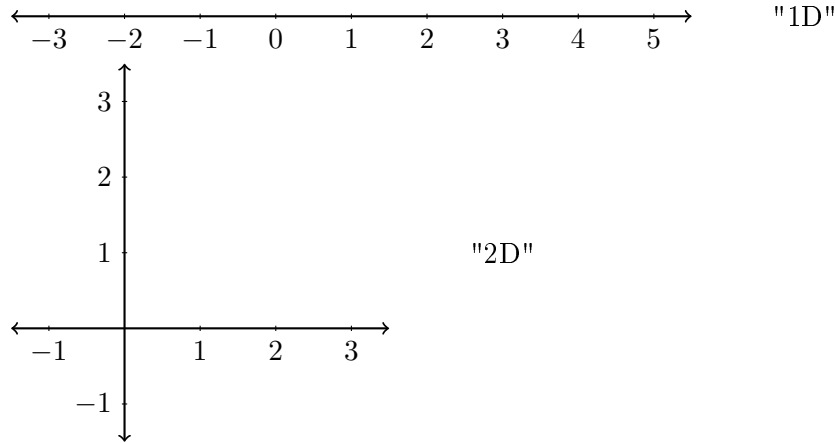
- Express  $\forall x \geq 0, \sqrt{x} \geq 0$  in a sentence

**Definition** (Subset / Inclusion).  $X \subseteq Y$  if  $\forall x \in X, x \in Y$

**Definition** (Disjoint Sets).  $X$  and  $Y$  are disjoint if  $\forall x \in X, x \notin Y$  (or if  $\forall y \in Y, y \notin X$ ).

**Definition** (Power Set).  $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$   
e.g.:  $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

**Scalars vs vectors** 1D vs 2D:



To "select" a point in 2D, we need 2 numbers, giving a coordinate. In 3D, we would need 3 numbers, in 4D, 4 numbers, etc...

**Definition** (Cartesian Product).  $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

e.g.:  $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

e.g.:  $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$

Extension:  $X_1 \times \cdots \times X_n = \prod_{k=1}^n X_k$

Moving from one point to another gives a "translation". Again, we need as many numbers as there are dimensions. Typically, we denote points horizontally, and vectors vertically.

## 1.2 Axioms

Here  $\star$  and  $\dagger$  will operations.

**Definition** (Associativity).  $\star$  is associative if  $\forall x, y, z, (x \star y) \star z = x \star (y \star z)$

**Definition** (Commutativity).  $\star$  is commutative if  $\forall x, y, (x \star y) = y \star x$

**Definition** (Identity).  $1_\star$  is identity for  $\star$  if  $\forall x, 1_\star \star x = x \star 1_\star = x$

**Definition** (Annihilator).  $0_\star$  is annihilator for  $\star$  if  $\forall x, 0_\star \star x = x \star 0_\star = 0_\star$

**Definition** (Distributivity).  $\star$  is distributive over  $\dagger$  if  $\forall x, y, z, x \star (y \dagger z) = (x \star y) \dagger (x \star z)$

of  $\wedge$  over  $\vee$ :  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

**Question.** *(make a table)*

- Which of these are commutative: addition, subtraction, multiplication, division, power?
- Which of these are associative: addition, subtraction, multiplication, division, power?
- What is identity for: addition, subtraction, multiplication, division, power?
- What is annihilator for: addition, subtraction, multiplication, division, power?

**Question.** • Think of an operation that is commutative, but not associative

- Think of an operation that is associative, but not commutative

### 1.3 Boolean algebra

The reason we'll do some is because of its application to programming, in particular to conditions ('if' blocks and 'while' loops).

#### Basic operators

**Definition** (Conjunction).  $x \wedge y = xy$

**Definition** (Intersection).  $X \cap Y = \{z \mid (z \in X) \wedge (z \in Y)\}$

**Remark** (Disjoint Sets and Intersection). *Disjoint sets have empty intersection.*

**Definition** (Disjunction).  $x \vee y = \min(x + y, 1)$

**Definition** (Union).  $X \cup Y = \{z \mid (z \in X) \vee (z \in Y)\}$

**Definition** (Negation).  $\neg : 0, 1 \mapsto 1, 0$

**Definition** (Set minus / Complement).  $X \setminus Y = \{x \in X \mid \neg(x \in Y)\}$

[Draw diagrams]

**Question.** *Selecting points outside a given region.*

#### Basic properties

**Property** (Boolean algebra matching ordinary algebra). *Same laws as ordinary algebra when one matches  $\cup$  with addition and  $\wedge$  with multiplication.*

- *Associativity of  $\vee$ :  $x \vee (y \vee z) = (x \vee y) \vee z$*
- *Associativity of  $\wedge$ :  $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*
- *Commutativity of  $\vee$ :  $x \vee y = y \vee x$*
- *Commutativity of  $\wedge$ :  $x \wedge y = y \wedge x$*
- *Distributivity of  $\wedge$  over  $\vee$ :  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*
- *0 is identity for  $\vee$ :  $x \vee 0 = x$*
- *1 is identity for  $\wedge$ :  $x \wedge 1 = x$*
- *0 is annihilator for  $\wedge$ :  $x \wedge 0 = 0$*

**Property** (Boolean algebra specific properties). *The following laws hold in Boolean algebra, but not in ordinary algebra:*

- Idempotence of  $\vee$ :  $x \vee x = x$
- Idempotence of  $\wedge$ :  $x \wedge x = x$
- Absorption of  $\vee$  over  $\wedge$ :  $x \vee (x \wedge y) = x$
- Absorption of  $\wedge$  over  $\vee$ :  $x \wedge (x \vee y) = x$
- Distributivity of  $\vee$  over  $\wedge$ :  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- 1 is annihilator for  $\vee$ :  $x \vee 1 = 1$

**Property** (De Morgan Laws).  $\neg(x \wedge y) = \neg x \vee \neg y$  and  $\neg(x \vee y) = \neg x \wedge \neg y$

*Proof.* Truth-tables; prove De Morgan, others as exercise (or just believe me) □

### Other operators

**Definition** (Exclusive Or).  $x \oplus y$

**Definition** (Implication).  $x \implies y$

**Property** (Implication and Inclusion). *If  $\forall x \in X, P_1(x) \implies P_2(x)$ , then  $\{x \in X \mid P_1(x)\} \subset \{x \in X \mid P_2(x)\}$ .*

*Proof.* Trivial. □

**Definition** (If and only if).  $x \iff y$

**Question.** *Express in terms of and, or, not:*

- $\oplus$
- $\implies$
- $\iff$
- $\iff$

*Write 1st and 2nd digit of addition of 3 binary numbers a, b, c.*

### Negation of quantified propositions

**Property** (Negation of  $\forall$ ).  $\neg \forall x \in X, P(x) = \exists x \in X, \neg P(x)$

**Property** (Negation of  $\exists$ ).  $\neg \exists x \in X, P(x) = \forall x \in X, \neg P(x)$

**Notation** (Quantifiers and the empty set).  $\forall x \in \emptyset, \dots$  is true ;  $\exists x \in \emptyset, \dots$  is false

**Question.** *Negate the following*

- $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x$
- $(x_n \rightarrow x): \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |x_n - x| < \epsilon$

## 1.4 Proofs

Proofs play a major role in mathematics. Except axioms, every statement needs a proof to be valid. There are 4 common approaches to prove a statement:

**Direct proof** This techniques consists of just going straight in and aiming at the statement you want.

**Property** (Archimedian Property). *For any real number, there exists an integer even greater. That is  $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$  s.t.  $x < n$ .*

*Proof.* Let  $n = \lfloor x \rfloor + 1$ : then  $x - 1 < \lfloor x \rfloor$  so  $x < \lfloor x \rfloor + 1 = n$ . □

**Splitting cases** This techniques consists splitting the statement to prove in several easier ones.

**Property.** *For any  $n \in \mathbb{Z}$ ,  $n$  and  $n^2$  share the same parity.*

*Proof.* We split into 2 cases:  $n$  even and  $n$  odd:

- Suppose  $n$  is even:  $n = 2k$  so  $n^2 = 4k^2 = 2(2k^2)$ , i.e.  $n^2$  is even.
  - Suppose  $n$  is odd:  $n = 2k + 1$  so  $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ , i.e.  $n^2$  is odd.
- 

**Contradiction** This techniques consists in assuming the opposite of what you want, deriving a contradiction (both  $P$  and  $\neg P$ ), then conclude the opposite of what was assumed.

**Property.**  $\sqrt{2} \notin \mathbb{Q}$

*Proof.* Suppose  $\sqrt{2} \in \mathbb{Q}$ : then there are  $a, b \in \mathbb{N}$  such that  $\sqrt{2} = a/b$ . Moreover, we can take  $a$  and  $b$  so that they have no common divisors. Now, we have  $2b^2 = a^2$ . So  $a^2$  is even, so  $a$  is even, let  $a = 2a'$ . Then,  $2b^2 = 4a'^2$  so  $b^2 = 2a'^2$ . Thus,  $b^2$  is even, so  $b$  is even. Both  $a$  and  $b$  are divisible by 2, which contradicts them having no common divisors. Finally, we conclude our initial assumption was wrong, so  $\sqrt{2} \notin \mathbb{Q}$ . □

**Induction** This techniques is only applicable if we wish to prove a statement for all natural numbers. The idea is similar to domino: each statement will imply the next one. Mathematically, if  $P(n)$  are statements ( $n \in \mathbb{N}$ ), we give 2 proofs: First, that  $P(0)$  is true. Then, that  $P(n) \implies P(n+1)$ .

**Property.**  $\forall n \in \mathbb{N}, \sum_{k=0}^n k = \frac{n(n+1)}{2}$

*Proof.* **Initialization:** for  $n = 0$ :

$$\sum_{k=0}^0 k = 0 \text{ and } \frac{0(0+1)}{2} = 0$$

so clearly

$$\sum_{k=0}^n k = \frac{n(n+1)}{2} \text{ for } n = 0$$

**Induction:** suppose  $\sum_{k=0}^n k = \frac{n(n+1)}{2}$ , we need  $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\begin{aligned} \sum_{k=0}^{n+1} k &= (n+1) + \sum_{k=0}^n k \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \end{aligned}$$

□



Be careful, BOTH initialization & induction are important.  
 A misleading "proof":

**Property** (FALSE PROPERTY). *Horses are all the same colour.*

*FALSE PROOF.* We will prove by induction that any set of  $n$  horses must be a set of horses all being the same colour.

**Initialization:** for set of 1 horse, clearly, it's true.

**Induction:** suppose all sets of  $n$  horses are uni-colour; we need sets of  $n + 1$  horses to be uni-colour.

Take  $H = \{h_1, h_2, \dots, h_n, h_{n+1}\}$  a set of  $n + 1$  horses. Then let  $H_1 = \{h_1, \dots, h_n\}$  and  $H_2 = \{h_2, \dots, h_{n+1}\}$ . By induction,  $H_1$  and  $H_2$  must be uni-colour.  $h_n$  belongs to both sets, so the colour of horses in both sets must be the same, and  $H$  is uni-colour.  $\square$

**Question.** • Show that  $12n - 6$  is divisible by 6 for every positive integer  $n$  (you do not need induction for this one).

- Show that  $n$  divisible by 6 if and only if  $n$  divisible by 2 and 3.
- Show that  $n^2$  divisible by 3 if and only if  $n$  divisible 3.<sup>1</sup>
- Show  $\sqrt{3} \notin \mathbb{Q}$ .
- Show that  $2^n \geq 2n$  for all  $n \in \mathbb{N}$

## 1.5 Basic geometry

This is high-school material, so we will just warm it up with a few exercises:

**Definition** (Norm). With  $\vec{u}$  a vectors in  $N$  dimension:

$$\|\vec{u}\| = \sqrt{\sum_{k=1}^N u_k^2}$$

NB: The distance from  $A$  to  $B$  is  $\|\vec{AB}\|$ .

**Definition** (Dot Product). With  $\vec{u}$  and  $\vec{v}$  vectors:

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^N u_k v_k = \|\vec{u}\| \|\vec{v}\| \cos(\text{angle}(\vec{u}, \vec{v}))$$

In particular, if  $\vec{u} \perp \vec{v}$ , then  $\vec{u} \cdot \vec{v} = 0$ .

Dot product takes two vectors, and gives a scalar. It may be interesting to, similarly to scalars multiplication, have the same type of object as output of the operation. This is what the vector product was designed for.

**Definition** (Vector Product). With  $\vec{u}$  and  $\vec{v}$  vectors in 3 dimensions:

$$\vec{u} \times \vec{v} = \vec{w}$$

$$\text{with } \|\vec{w}\| = \|\vec{u}\| \|\vec{v}\| \sin(\text{angle}(\vec{u}, \vec{v}))$$

and  $\vec{u}, \vec{v} \perp \vec{w}$ ; direction given by the right-hand-rule

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<sup>1</sup>You may go faster using [https://en.wikipedia.org/wiki/Modular\\_arithmetic](https://en.wikipedia.org/wiki/Modular_arithmetic)

Now, if  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$  and  $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ , then:

$$\begin{aligned} w &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= u_2v_3\vec{i} + u_3v_1\vec{j} + u_1v_2\vec{k} - v_1u_2\vec{k} - v_2u_3\vec{i} - v_3u_1\vec{j} \\ &= \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \end{aligned}$$

**Question.** Find an equation of the following lines in the  $xy$ -plane:

- vertical, crossing the  $x$ -axis at 5
- horizontal, crossing the  $y$ -axis at 7
- crossing the points  $(2, 3)$  and  $(-3, -7)$
- crossing the point  $(5, 1)$  with slope  $-2$
- crossing the point  $(5, 1)$  perpendicular to the last one
- the mediator of  $(3, 1)$  and  $(-9, 8)$

**Question.** Find an equation of the following planes in the  $xyz$ -space:

- horizontal, crossing the  $z$ -axis at  $-8$
- crossing the point  $(5, 1 - 3)$  with normal vector  $(-2, 6, -9)$
- crossing the points  $(2, 3, 1)$ ,  $(-10, 4, -5)$ , and  $(-3, -7, -1)$
- the mediator plane of  $(3, 1, 5)$  and  $(-9, 8, -3)$

**Question.** Describe (give the associated set of points) the following lines in the  $xyz$ -space:

- vertical, crossing the  $xy$ -plane at  $(1, 1)$
- crossing the points  $(2, 3, -1)$  with normal vector  $(8, 5, 9)$
- intersection of the planes with equation  $2z + 3y - 6x = 8$  and  $-3z + 4y - 8x = 7$

## 1.6 Sets extremums

**Definition** (Minimum/Maximum). With  $A$  a set:  $a^* \in A$  is the minimum (respectively maximum) of  $A$  if  $\forall a \in A, a \geq a^*$  (respectively  $\forall a \in A, a \leq a^*$ ).

**Definition** (Lower/Upper bound). With  $A$  a set:  $b$  is a lower (respectively upper) bound of  $A$  if  $\forall a \in A, a \geq b$  (respectively  $\forall a \in A, a \leq b$ ).

**Definition** (Infimum/Supremum). With  $A$  a set:  $a^*$  is the infimum (respectively supremum) of  $A$  if  $a^*$  is the largest lower bound (respectively the smallest upper bound).

With quantifiers:

$a^*$  is the infimum of  $A$  if  $\forall \varepsilon > 0, \exists a \in A$  s.t.  $a < a^* + \varepsilon$  (and  $\forall a \in A, a \geq a^*$ ).

$a^*$  is the supremum of  $A$  if  $\forall \varepsilon > 0, \exists a \in A$  s.t.  $a > a^* - \varepsilon$  (and  $\forall a \in A, a \leq a^*$ ).

Infimum/supremum always exist, while minimum/maximum may not.

**Question.** Find the minimum, maximum, infimum, supremum of the following sets:

- $[0, 1]$
- $(0, 1)$
- $\{1/n \mid n \in \mathbb{N}^*\}$

## 1.7 Integers

- prime numbers (infinite nb by Euclide) - unique factorization - finding primes between 1 and 100 => time complexity of algo?

## **2 Complex numbers**

argand diagram

## **3 Sizes of infinity**

[recycling house 6 pres']

## **4 Asymptotic analysis (limits)**

- def of sequence: recursive and general form - usual sequences (arithmetic/geometric) - convergence of sequences

## **5 Infinite & partial sums**

- sum of sequences - sum of usual (arithmetic/geometric) sequences - def of series - convergence of series

## **6 Functions & Inverses**

finding roots & inverses

## **7 Usual functions**

- plot & limit behaviour of: polynomials, exp, log, sin, cos, tan, sinh, cosh, tanh, arccos, arcsin, arctan

## **8 Differentiation**

- from scratch - derivatives of usual functions - chain-law & co

## **9 Integration**

- from scratch (area under curve, taking limit of rectangles) - antiderivative (do proof?) - integral of usual functions - integration by part? (if time!) - integration by substitution? (if time!)

## **10 Taylor series**

- theory & practice - usual Taylor expansions - example of convergence

## **11 Fourier series? (if not late!)**

## **12 Differential calculus? (if not late!)**

## **13 Vector spaces**

- def of vect sp - norm - basic propr

## **14 Matrices**

- def - linear mapping of vect sp - inverse: def, existence (det), finding inverse - rank & kernel - eigenvalues

## **15 Non-linear multi-dimensional functions**

- eg: cost func - partial derivatives - gradient - convexity? - optim: gradient descent

## **16 Regressions**

- by hand - theory - non linear

## **17 PCA? (if time)**

## **18 Basis of ML (perceptron)? (if time)**