Notes on Differential Equations

DSBA Mathematics Refresher 2024

Abstract

1 First-Order Differential Equations

A first-order differential equation can often be written in the form:

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

To solve this, we use an integrating factor "IF" which is defined as:

$$\text{IF} = e^{\int P(x)\,dx}$$

Multiplying both sides of the differential equation by IF gives:

$$\text{IF} \cdot \frac{dy}{dx} + \text{IF} \cdot P(x) \cdot y = \text{IF} \cdot Q(x)$$

The left side of the equation is now the derivative of the product $\text{IF} \cdot y$:

$$\frac{d}{dx}\left[\text{IF} \cdot y\right] = \text{IF} \cdot Q(x)$$

Integrating both sides with respect to x:

$$\text{IF} \cdot y = \int \text{IF} \cdot Q(x)\,dx + C$$

Finally, solve for $y(x)$:

$$y(x) = \frac{1}{\text{IF}}\left[\int \text{IF} \cdot Q(x)\,dx + C\right]$$

Initial Conditions Consider the same first-order equation with an initial condition $y(x_0) = y_0$. After finding the general solution as in the previous section:

$$y(x) = \frac{1}{\text{IF}}\left[\int \text{IF} \cdot Q(x)\,dx + C\right]$$

Apply the initial condition to find C:

$$y(x_0) = \frac{1}{\text{IF}(x_0)}\left[\int \text{IF} Q(x)\,dx \,|_{x_0} + C\right] = y_0$$

Solve for C and conclude.

2 (Homogeneous) Second-Order Linear Differential Equations

A second-order homogeneous linear differential equation is of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

The characteristic equation associated with this differential equation is:

$$ar^2 + br + c = 0$$

The nature of the solutions depends on the discriminant $\Delta = b^2 - 4ac$:

- If $\Delta > 0$, the roots r_1 and r_2 are real and distinct. The solution is: $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$.
- If $\Delta = 0$, the roots are real and repeated $r_1 = r_2 = r$. The solution is: $y(x) = (C_1 + C_2x)e^{rx}$.
- If $\Delta < 0$, the roots are complex $r_{1,2} = \alpha \pm i\beta$. The solution is: $y(x) = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x))$.

3 Second-Order Linear Differential Equations

For the general second-order linear differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

The solution is the sum of the complementary function $y_c(x)$ (the solution to the associated homogeneous equation) and a particular solution $y_p(x)$:

$$y(x) = y_c(x) + y_p(x)$$

The method for finding $y_p(x)$ depends on the form of $g(x)$:

- If $g(x)$ is a polynomial, use the method of undetermined coefficients.
- If $g(x)$ is of an exponential or trigonometric form, use an answer of a similar form.
- For other forms of $g(x)$, the method of variation of parameters can be used.

Initial Conditions Given the general solution:

$$y(x) = y_c(x) + y_p(x)$$

And initial conditions $y(x_0) = y_0$ and $\frac{dy}{dx}\Big|_{x=x_0} = y'_0$, we determine the constants C_1 and C_2 in $y_c(x)$:

$$y(x_0) = y_c(x_0) + y_p(x_0) = y_0$$

$$\frac{dy}{dx}\Big|_{x=x_0} = y'_c(x_0) + y'_p(x_0) = y'_0$$

Solving these two equations will yield the specific values of C_1 and C_2 that satisfy the initial conditions, leading to the particular solution for the problem.