Exercise Set: Calculus

Solutions

1 - Fundamental Theorem of Calculus

$$F(b) - F(a) = \int_0^b f(t)\,dt - \int_0^a f(t)\,dt$$
$$= \cancel{\int_0^a f(t)\,dt} + \int_a^b f(t)\,dt - \cancel{\int_0^a f(t)\,dt}$$
$$= \int_a^b f(t)\,dt$$

$$\int_0^{\pi/2} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi/2} = -\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0} - \Big(-\underbrace{\cos(0)}_{1}\Big) = 1$$

$$\int_1^4 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^4 = -\frac{1}{4} - \left(-\frac{1}{1}\right) = \frac{3}{4}$$

2 - Integration Techniques

1) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + c$ since $(x^2+1)' = 2x$

(direct)

(using substitution)

$u = x^2 + 1$

$\Leftrightarrow u - 1 = x^2$

$\Rightarrow \sqrt{u-1} = x$ if $x \geq 0$

& $-\sqrt{u-1} = x$ if $x \leq 0$

if $x \geq 0$: $\dfrac{dx}{du} = \dfrac{1}{2\sqrt{u-1}}$ so $\displaystyle\int \frac{2x}{x^2+1}\,dx = \int \frac{2\sqrt{u-1}}{u} \cdot \frac{1}{2\sqrt{u-1}}\,du = \int \frac{1}{u}\,du$
$= \ln(u) + c$

if $x \leq 0$: $\dfrac{dx}{du} = -\dfrac{1}{2\sqrt{u-1}}$ so $\displaystyle\int \frac{2x}{x^2+1}\,dx = \frac{-2\sqrt{u-1}}{u}\left(-\frac{1}{2\sqrt{u-1}}\right)du = \int \frac{1}{u}\,du$
$= \ln(u) + c$

2) $\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-\sin(u)^2}} \cdot \frac{d \sin(u)}{du} du$

$= \int \frac{1}{\sqrt{\cos(u)^2}} \cos(u) du$

$= \int \frac{1}{\cos(u)} \cos(u) du = \int 1 du = u + c$

$= \arcsin(x) + c$

3) $\int \frac{1}{4+x^2} dx = \int \frac{1}{4+4u^2} \left(\frac{d 2u}{du}\right) du$

$= \int \frac{1}{4} \cdot \frac{1}{1+u^2} 2 du$

$= \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \arctan(u) + c$

$= \frac{1}{2} \arctan\left(\frac{x}{2}\right) + c$

A) $\int x \cdot \ln(x) dx = \left[\frac{1}{2} x^2 \ln(x)\right] - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx$

$= \frac{1}{2} x^2 \ln(x) - \frac{1}{2} \int x dx$

$= \frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + c$

$= \frac{1}{2} x^2 \left(\ln(x) - \frac{1}{2}\right) + c$

B) $\int x^2 e^x dx = [x^2 e^x] - \int 2x e^x dx$

$= x^2 e^x - [2x e^x] + \int 2 e^x dx$

$= x^2 e^x - 2x e^x + 2 e^x + c$

$= e^x (x^2 - 2x + 2) + c$

c) $\int x \cos(x) dx = [x \cdot \sin(x)] - \int \sin(x) dx$

$= x \cdot \sin(x) - (-\cos(x)) + c$

$= x \sin(x) + \cos(x) + C$

D) $\int e^{2x}\cos(2x)dx = \left[\frac{1}{2}e^{2x}\cos(2x)\right] - \int \frac{1}{\cancel{2}}e^{2x}\,\cancel{2}\sin(2x)\,dx$

$= \frac{1}{2}e^{2x}\cos(2x) - \left[\frac{1}{2}e^{2x}\sin(2x)\right] + \int \frac{1}{\cancel{2}}e^{2x}(-\cancel{2})\cos(2x)\,dx$

$\Leftrightarrow \quad 2\int e^{2x}\cos(2x)\,dx = \frac{1}{2}e^{2x}\cos(2x) - \frac{1}{2}e^{2x}\sin(2x) + c$

$\Leftrightarrow \quad \int e^{2x}\cos(2x)\,dx = \frac{1}{4}e^{2x}\left(\cos(2x) - \sin(2x)\right) + c'$

α) $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$

$3x^2 - 2x - 1 = (x-1)(3x+1)$

So $\dfrac{3x^2-2x-1}{x^3-x^2+x-1} = \dfrac{3x+1}{x^2+1}$ for $x \neq 1$

So $\displaystyle\int \frac{3x^2-2x-1}{x^3-x^2+x-1}\,dx = \int \frac{3x+1}{x^2+1}\,dx$ on $\mathbb{R}\setminus\{1\}$

$= \int \frac{3x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

$= \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + \arctan(x)$

$= \frac{3}{2}\ln(x^2+1) + \arctan(x) + c$

NB: if the numerator was $3x^2-2x+1$, then:

$(x^3-x^2+x-1)' = 3x^2-2x+1$

so $\displaystyle\int \frac{3x^2-2x+1}{x^3-x^2+x-1}\,dx = \ln(x^3-x^2+x-1) + c$

β) $\displaystyle\int_0^{+\infty} e^{-x}\,dx = \lim_{a\to+\infty}\int_0^a e^{-x}\,dx$

$= \lim_{a\to+\infty}\left[-e^{-x}\right]_0^a$

$= \lim_{a\to+\infty} \underbrace{-e^{-a}}_{\to 0} - \underbrace{(-e^{-0})}_{-1} = 1$

γ) Trapezoidal Rule with $n=3$ sub-intervals:

$\displaystyle\int_a^b f(x)\,dx \simeq (b-a)\cdot\left(\frac{1}{n}\right)\left[\frac{1}{2}f(a) + f\left(a+\left(\frac{b-a}{n}\right)\right) + f\left(a+2\frac{(b-a)}{n}\right) + \dots\right.$
$\left. + f\left(b-2\frac{(b-a)}{n}\right) + f\left(b-\left(\frac{b-a}{n}\right)\right) + \frac{1}{2}f(b)\right]$

So $\int_0^{\pi/2} \sin(x)\,dx \simeq \frac{\pi}{2} \cdot \frac{1}{3}\left[\frac{\sin(0)}{2} + \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) + \frac{\sin\left(\frac{\pi}{2}\right)}{2}\right]$

$= \frac{\pi}{6} \cdot \left[0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2}\right]$

$= \frac{\pi}{6}\left[1 + \frac{\sqrt{3}}{2}\right] \simeq 0{,}52\,[1{,}87] \simeq 0{,}97$

The real value is 1, so error is $< 3\%$!

8) Simpson Rule with $n=2$ sub intervals; let $s = \frac{b-a}{n}$

$\int_a^b f(x)\,dx = (b-a)\left(\frac{1}{n}\right)\left[\frac{1}{6} f(a) + \frac{4}{6} f\left(a+\frac{s}{2}\right) + \frac{2}{6} f(a+s) + \frac{4}{6} f\left(a+\frac{3}{2}s\right)\right.$

$+ \frac{2}{6} f(a+2s) + \dots + \frac{2}{6} f(b-2s) + \frac{4}{6} f\left(b-\frac{3}{2}s\right)$

$\left.+ \frac{2}{6} f(b-s) + \frac{4}{6} f\left(b-\frac{s}{2}\right) + \frac{1}{6} f(b)\right]$

So $\int_0^{\pi/2} \sin(x)\,dx \simeq \frac{\pi}{2} \cdot \frac{1}{2}\left[\frac{1}{6}\sin(0) + \frac{4}{6}\sin\left(\frac{\pi}{8}\right) + \frac{2}{6}\sin\left(\frac{\pi}{4}\right) + \frac{4}{6}\sin\left(\frac{3\pi}{8}\right)\right.$

$\left.+ \frac{1}{6}\sin\left(\frac{\pi}{2}\right)\right]$

$\simeq 0{,}79\,(0 + 0{,}26 + 0{,}24 + 0{,}62 + 0{,}17)$

$= 0{,}79 * 1{,}29 \simeq 1{,}02$

Error is $< 2\%$!

3 - Applications

Area between curves

$y = \sin(x)$ & $y = -\sin(x)$ intercept at 0 and π.

Hence, we just need $\int_0^{\pi} \sin(x) - (-\sin x)\,dx = 2\int_0^{\pi} \sin(x)\,dx$

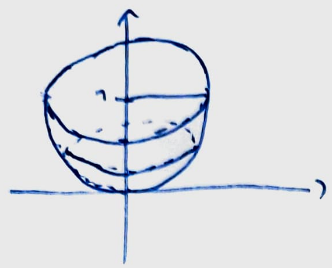
$= 2\left[-\cos(x)\right]_0^{\pi}$

$= 2\left[-\underbrace{\cos(\pi)}_{-1} + \underbrace{\cos(0)}_{1}\right] = 4$

The area is 4.

Volume of revolution

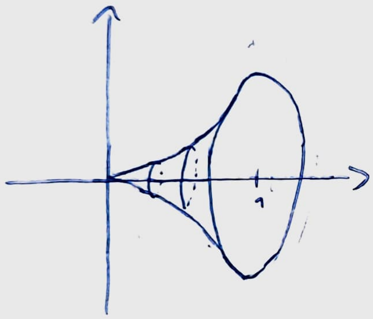
$y = x^2$, $y \in [0,1]$, revolve about y-axis:



At y, the radius of the circle is $\sqrt{y}$

so $$\begin{aligned} V &= \int_0^1 \pi . r(y)^2 \, dy \\ &= \int_0^1 \pi (\sqrt{y})^2 \, dy \\ &= \int_0^1 \pi . y \, dy \\ &= \left[\tfrac{1}{2} y^2 \pi \right]_0^1 = \pi/2 \end{aligned}$$

$y = x^2$, $x \in [0,1]$, revolve about x-axis:



At x, re radius of the circle is x^2

so $$\begin{aligned} V &= \int_0^1 \pi . r(x)^2 \, dx \\ &= \int_0^1 \pi . (x^2)^2 \, dx = \int_0^1 \pi x^5 \, dx \\ &= \pi \left[\tfrac{1}{5} x^5 \right]_0^1 = \pi/5 \end{aligned}$$

Arclength of curves: $f(x) = x\sqrt{x} = x^{3/2}$ so $f'(x) = \frac{3}{2} x^{1/2} = \frac{3}{2}\sqrt{x}$

$$\begin{aligned} \mathcal{L} &= \int_1^4 \sqrt{1 + \left(\tfrac{3}{2}\sqrt{x}\right)^2} \, dx \\ &= \int_1^4 \sqrt{1 + \tfrac{9}{4} x} \, dx \\ &= \left[\tfrac{2}{3} \left(1 + \tfrac{9}{4} x\right)^{3/2} \right]_1^4 \\ &= \tfrac{2}{3} \left(1 + \tfrac{9}{4} 4\right)^{3/2} - \tfrac{2}{3} \left(1 + \tfrac{9}{4}\right)^{3/2} \\ &= \tfrac{2}{3} \left(10 . \sqrt{10} - \tfrac{13}{4} \tfrac{\sqrt{13}}{2} \right) = \frac{20\sqrt{10}}{3} - \frac{13\sqrt{13}}{12} \end{aligned}$$

Surface of revolution

$y = x^2$ over $y \in [0,1]$ about y-axis:

$$\mathcal{A} = \int_0^1 2 \cdot \pi \cdot r(y)\, dy$$
$$= \int_0^1 2\pi \sqrt{y}\, dy$$
$$= 2\pi \left[\frac{2}{3} y^{3/2} \right]_0^1 = \frac{4\pi}{3}$$

$y = x^2$ over $x \in [0,1]$ about x-axis:

$$\mathcal{A} = \int_0^1 2\pi\, x^2\, dx$$
$$= 2\pi \left[\frac{1}{3} x^3 \right]_0^1 = \frac{2\pi}{3}$$