Exercise Set: Differential Equations

Solutions

1- 1^{st} Order Differential Equations

Basics

Exercise 0: $y' = 3x^2 - 2x + 1$

$y = x^3 - x^2 + x + c, \quad c \in \mathbb{R}$

Exercise 1: $y' = 5y$

let $y = \alpha e^{\lambda x} \Rightarrow y' = \alpha \lambda e^{\lambda x} = \lambda y$

so need $\lambda = 5$

$y = \alpha e^{5x} \quad \alpha \in \mathbb{R}$

Separable

Exercise 2: $\frac{dy}{dx} = \frac{x}{y}$

$\Leftrightarrow y\,dy = x\,dx$

$\Rightarrow \int y\,dy = \int x\,dx$

$\Leftrightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c/2 \quad c \in \mathbb{R}$

$\Leftrightarrow y = \pm\sqrt{x^2 + c}$

$y' = \pm\frac{2x}{2\sqrt{x^2+c}} = \pm\frac{x}{\sqrt{x^2+c}} = \frac{x}{y}$

Exercise 3: $\frac{dy}{dx} = y' = 2x^2 e^y$

$\Leftrightarrow e^{-y}\,dy = 2x^2\,dx$

$\Rightarrow \int e^{-y}\,dy = \int 2x^2\,dx$

$\Leftrightarrow -e^{-y} = \frac{2}{3}x^3 + c' \quad c' \in \mathbb{R}$

$\Leftrightarrow \ln(e^{-y}) = \ln(-\frac{2}{3}x^3 + c) \quad c \in \mathbb{R}$

$\Leftrightarrow y = -\ln(-\frac{2}{3}x^3 + c)$

Integrating Factor

Exercise 4: $y' + 2y = 4x$

$IF = e^{\int 2\,dx} = e^{2x}$

$\Rightarrow y' e^{2x} + 2y e^{2x} = e^{2x} \cdot 4x$

$\Leftrightarrow (y e^{2x})' = e^{2x} \cdot 4x$

$\Leftrightarrow y \cdot e^{2x} = \int e^{2x} \cdot 4x\,dx$

$\Leftrightarrow y \cdot e^{2x} = e^{2x} \cdot 2x - e^{2x} + c$

$\Leftrightarrow y = 2x - 1 + c e^{-2x}$

$y' = 2 + c(-2) e^{-2x}$

so $y' + 2y = 4x$ ✓

Exercise 5: $y' - \frac{1}{x} y = x^3$

$IF = e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x}$

$y' \frac{1}{x} - \frac{1}{x^2} y = x^3 \cdot \frac{1}{x} = x^2$

$\Rightarrow y \frac{1}{x} = \int x^2\,dx$

$\Leftrightarrow y \frac{1}{x} = \frac{1}{3} x^3 + c$

$\Leftrightarrow y = \frac{1}{3} x^4 + cx$

$y' = \frac{4}{3} x^3 + c$

so $y' - \frac{1}{x} y = x^3$ ✓

2 - 2^{nd} Order Differential Equations

Basics

Exercise 0 :

$$y'' = e^x + 4\sin(2x) - 5x$$

$$y' = e^x - 2\cos(2x) - \frac{5}{2}x^2 + c \qquad c \in \mathbb{R}$$

$$y = e^x - \sin(2x) - \frac{5}{6}x^3 + cx + c' \qquad c' \in \mathbb{R}$$

Exercise 1 : $y'' + 2y + y = 0$

$y = e^{rx} \Rightarrow r^2 e^{rx} + 2r e^{rx} + e^{rx} = 0$

$\Leftrightarrow r^2 + 2r + 1 = 0$

$\Delta = 4 - 4 = 0$

$r = \frac{-2}{2} = -1$

So $y = Ae^{-x}$ works. Because $\Delta = 0$, we also try:

$y = xe^{rx} \Rightarrow y' = rxe^{rx} + e^{rx}$

& $y'' = r^2 x e^{rx} + 2re^{rx}$

$\Rightarrow xe^{rx}(1 + 2r + r^2) + e^{rx}(2 + 2r) = 0$

$\Rightarrow r = -1$

So $y = Bxe^{-x}$ is also a solution

So in general, $y = Ae^{-x} + Bxe^{-x}$, $A, B \in \mathbb{R}$

Exercise 2 : $y'' + 4y = 0$

$y = e^{\lambda x} \Rightarrow \lambda^2 + 4 = 0$

$\Rightarrow \lambda = \pm 2i$

So $y = Ae^{2ix} + Be^{-2ix}$

$= A(\cos(2x) + i\sin(2x)) + B(\cos(2x) - i\sin(2x))$

let $C = A + B$, $D = (A - B)i$:

$y = C\cos(2x) + D\sin(2x)$

Separable

Exercise 3: $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

let $z = y' = \frac{dy}{dx}$, then,

$$z' = z^2 \iff \frac{dz}{dx} = z^2$$

$$\iff \frac{dz}{z^2} = dx$$

$$\Rightarrow \int \frac{dz}{z^2} = \int dx$$

$$\iff -\frac{1}{z} = x + c \qquad c \in \mathbb{R}$$

$$\iff z = -\frac{1}{x+c}$$

$$\iff \frac{dy}{dx} = -\frac{1}{x+c}$$

$$\iff dy = -\frac{1}{x+c}\, dx$$

$$\Rightarrow \int dy = -\int \frac{1}{x+c}\, dx$$

$$\iff y = -\ln(x+c) + c' \qquad c' \in \mathbb{R}$$

$$\Rightarrow y' = -\frac{1}{x+c}$$

$$\Rightarrow y'' = \frac{1}{(x+c)^2}$$

So $y'' = (y')^2$ ✓

Non-Homogeneous

Exercise 4: $y'' + 4y' + 4y = 6x^2 + 10x + 2$

$$r^2 + 4r + 4 = 0$$

$$\Delta = 16 - 4.4 = 0 \Rightarrow r = -\frac{4}{2} = -2$$

$$y_{hom} = Ae^{-2x} + Bxe^{-2x}$$

let
$$\left.\begin{array}{l} y = ax^2 + bx + c \\ y' = 2ax + b \\ y'' = 2a \end{array}\right\} \quad y'' + 4y' + 4y = 4ax^2 + (4b + 8a)x + 4c + 4b + 2a$$

want $y'' + 4y' + 4y = 6x^2 + 10x + 2 \quad \forall x$

so
$$\begin{cases} 4a = 6 \\ 4b + 8a = 10 \\ 4c + 4b + 2a = 2 \end{cases}$$

$$\Rightarrow a = 3/2, \quad b = -\frac{1}{2}, \quad c = 1/4$$

So $y_{part} = \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{4}$

$$y = y_{hom} + y_{part} = Ae^{-2x} + Bxe^{-2x} + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{4}$$

Exercise 5: $y'' - 2y' + 5y = 3e^{2t}$

$$y_{hom} = Ae^{(1+2i)t} + Be^{(1-2i)t} \qquad A, B \in \mathbb{R}$$
$$= e^t(C\cos(2t) + D\sin(2t)) \qquad C = A+B, \ D = (A-B)i$$

let $y = de^{2t}$: $y'' - 2y' + 5y = e^{2t}(4d - 2.2d + 5d)$
$$= e^{2t}.5d$$

let $d = 3/5$ works, so $y_{part} = \frac{3}{5}e^{2t}$

Thus, $y = e^t(C.\cos(2t) + D\sin(2t)) + \frac{3}{5}e^{2t}$

Boundary Conditions

Exercise 6: $y'' + 2y' - 3y = 0$

$y = Ae^{x} + Be^{3x}$ $\qquad A, B \in \mathbb{R}$

$y(0) = 1$ so $A + B = 1$

$y(2) = 5$ so $Ae^{2} + Be^{-6} = 5$

[...] so $A = \frac{5 - e^{-6}}{e^{2} - e^{-6}}$ & $B = \frac{e^{2} - 5}{e^{2} - e^{-6}}$

Finally, $y = \frac{5 - e^{-6}}{e^{2} - e^{-6}} e^{x} + \frac{e^{2} - 5}{e^{2} - e^{-6}} e^{-3x}$

Exercise 7: $y'' + 4y = 12x$

$y_{hom} = C\cos(2x) + D\sin(2x)$ $\qquad C, D \in \mathbb{R}$

$y_{part} = 3x$

so $y = C\cos(2x) + D\sin(2x) + 3x$

$y(0) = 0 \Rightarrow C = 0$

$y'(0) = 0 \Rightarrow -2C + 2D + 3 = 0 \Rightarrow D = -3/2$

so, Finally, $y = -\frac{3}{2}\sin(2x) + 3x$