Notes on Calculus

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Abstract

1 Linear Independence Property

Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V, the set is said to be **linearly independent** if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

is

$$c_1 = c_2 = \cdots = c_n = 0.$$

In other words, none of the vectors in the set can be written as a linear combination of the others.

Example: Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 5 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 3 \\ -4 \end{pmatrix}$ in $\mathbb{R}^5$. These vectors are linearly independent because the only solution to

$$c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{0}$$

is $c_1 = c_2 = c_3 = 0$.

2 Spanning Property

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is said to **span** V if every vector in V can be expressed as a linear combination of these vectors. Mathematically, the set spans V if for every vector $\mathbf{v} \in V$, there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Example: In $\mathbb{R}^2$, the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ span $\mathbb{R}^2$ because any vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ can be written as

$$\mathbf{v} = x\mathbf{v}_1 + (y - x)\mathbf{v}_2 + (y - x)\mathbf{v}_2.$$

3 Basis

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a **basis** for V if:

- The set is linearly independent.
- The set spans the vector space V.

The number of vectors in a basis is called the **dimension** of the vector space.

Example: The vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$. The dimension of $\mathbb{R}^2$ is 2.

4 (Reduced) Row Echelon Form

A matrix is in **row echelon form** if it satisfies the following conditions:

1. All non-zero rows are above any rows of all zeros.
2. The leading entry of each non-zero row after the first occurs to the right of the leading entry of the previous row.
3. The leading entry in any non-zero row is 1 (this condition is for the **reduced** row echelon form).
4. The leading 1 is the only non-zero entry in its column (for **reduced** row echelon form).

Example: The matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

is in row echelon form.

5 Inverting a Matrix

The **inverse** of a square matrix A is denoted by A^{-1} and is defined as the matrix that satisfies

$$AA^{-1} = A^{-1}A = I,$$

where I is the identity matrix.

Steps to find the inverse of a matrix:

1. Form the augmented matrix $[A|I]$.
2. Use row operations to convert A into the identity matrix.
3. The matrix that results from the identity matrix on the left side is A^{-1} on the right side.

Example: For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

the augmented matrix is

$$[A|I] = \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{pmatrix}$$

after reducing to row echelon form:

$$[I|A^{-1}] = \begin{pmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & 1.5 & -0.5 \end{pmatrix}$$

hence, the inverse is

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$$

6 Determinant

The **determinant** of a square matrix A, denoted by $\det(A)$, is a scalar value that can be computed from the elements of the matrix. The determinant has important properties and applications, including:

- A matrix is invertible if and only if its determinant is non-zero.
- The determinant of a product of matrices is the product of their determinants: $\det(AB) = \det(A)\det(B)$.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is calculated as:

$$\det(A) = ad - bc.$$

For a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the determinant is calculated as:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Expanding the 2×2 determinants (minors):

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

For higher dimension matrices, ask a computer (it's outside the scope of this course).

Example: For the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$\det B = 0$, hence, the matrix is not invertible (you can try to invert it, you will encounter a problem).

While for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

$\det A = -2 \neq 0$, hence, the matrix is invertible (and we found the inverse in the previous section).