Exercise Set: Binary Classification

Solutions

1 - Lagrangian multiplier technique

1.1 Unconstrained optimization

$$f(x,y) = 2(x-3)^2 + 4(y+1)^2 - 2$$

Hence, $(x^*, y^*) = (3, -1)$

(or) $\nabla f = \begin{pmatrix} 4x - 12 \\ 8y + 8 \end{pmatrix}$

$\nabla f = 0 \iff \begin{cases} x = 12/4 = 3 \\ y = -8/8 = -1 \end{cases}$

1.2 (Equality) constrained optimization

$$y = \frac{2}{5} - \frac{3}{5}x$$

$$f(x) = 2x^2 - 12x + 4\left(\frac{2}{5} - \frac{3}{5}x\right)^2 + 8\left(\frac{2}{5} - \frac{3}{5}x\right) + 20$$

$$f'(x) = 4x - 12 + 8\left(\frac{2}{5} - \frac{3}{5}x\right)\left(-\frac{3}{5}\right) - \frac{24}{5}$$

$$= \frac{172}{25}x - \frac{468}{25}$$

$f'(x) = 0 \iff x = \frac{117}{43}$ & $y = -\frac{53}{43}$

1.3 Lagrange multiplier

$$\mathcal{L}(x, y, \lambda) = 2x^2 - 12x + 4y^2 + 8y + 20 - 3\lambda x - 3\lambda y + 2\lambda$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} = 0 &\iff 4x - 12 - 3\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 &\iff 8y + 8 - 5\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 &\iff -3x - 3y + 2 = 0 \end{aligned}\right\} \quad x = \frac{117}{43}, \; y = -\frac{53}{43}, \; \lambda = -\frac{16}{43}$$

1.4 (Inequality) Constrained Optimization

$f(x) = (x-1)^2 - 1$

Globally: $x^* = 1$ but $3x^* = 3 \not\leq 2$

On constraint: $3x^* = 2 \Rightarrow x^* = 2/3$

$f(x) = (x+1)^2 - 1$

Globally: $x^* = -1$ and $3x^* = -3 \leq 2$

1.5 Lagrange multiplier & slack variable

$$\mathcal{L}(x, \lambda, s) = x^2 - 2x + \lambda(2 - 3x - s^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2 - 3\lambda$$

$$\frac{\partial \mathcal{L}}{\partial s} = 2\lambda s \quad \Rightarrow \quad \lambda = 0 \text{ or } s = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2 - 3x - s^2$$

$\lambda = 0 \Rightarrow x = 1$ & $s^2 = 2 - 3 = -1$ #

$s = 0 \Rightarrow x = 2/3$ & $\lambda = -2/9$

hence, $x^* = \frac{2}{3}$

$$\mathcal{L}(x, \lambda, s) = x^2 + 2x + \lambda(2 - 3x - s^2)$$

$\lambda = 0$ or $s = 0$

$\lambda = 0 \Rightarrow x = -1 \Rightarrow s^2 = 5$ so $s = \pm\sqrt{5}$

$s = 0 \Rightarrow x = 2/3 \Rightarrow \lambda = \frac{10}{9}$

$f\left(\frac{2}{3}\right) > f(-1)$ so $x^* = -1$