

# Notes on Calculus

DSBA Mathematics Refresher 2024

## Abstract

## 1 Linear Independence Property

Given a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  in a vector space  $V$ , the set is said to be **linearly independent** if the only solution to the equation

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n = \mathbf{0}$$

is

$$c_1 = c_2 = \dots = c_n = 0.$$

In other words, none of the vectors in the set can be written as a linear combination of the others.

**Example:** Consider the vectors  $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \\ -1 \\ 2 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 5 \\ 0 \\ 1 \\ 1 \end{pmatrix}$  and  $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 3 \\ -4 \end{pmatrix}$  in

$\mathbb{R}^5$ . These vectors are linearly independent because the only solution to

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0}$$

is  $c_1 = c_2 = c_3 = 0$ .

## 2 Spanning Property

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  in a vector space  $V$  is said to **span**  $V$  if every vector in  $V$  can be expressed as a linear combination of these vectors. Mathematically, the set spans  $V$  if for every vector  $\mathbf{v} \in V$ , there exist scalars  $c_1, c_2, \dots, c_n$  such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n.$$

**Example:** In  $\mathbb{R}^2$ , the vectors  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  and  $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  span  $\mathbb{R}^2$  because any vector  $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$  can be written as

$$\mathbf{v} = x\mathbf{v}_1 + (y-x)\mathbf{v}_2 + (y-x)\mathbf{v}_2.$$

### 3 Basis

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  in a vector space  $V$  is called a **basis** for  $V$  if:

- The set is linearly independent.
- The set spans the vector space  $V$ .

The number of vectors in a basis is called the **dimension** of the vector space.

**Example:** The vectors  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  form a basis for  $\mathbb{R}^2$ . The dimension of  $\mathbb{R}^2$  is 2.

### 4 (Reduced) Row Echelon Form

A matrix is in **row echelon form** if it satisfies the following conditions:

1. All non-zero rows are above any rows of all zeros.
2. The leading entry of each non-zero row after the first occurs to the right of the leading entry of the previous row.
3. The leading entry in any non-zero row is 1 (this condition is for the **reduced** row echelon form).
4. The leading 1 is the only non-zero entry in its column (for **reduced** row echelon form).

**Example:** The matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

is in row echelon form.

### 5 Inverting a Matrix

The **inverse** of a square matrix  $A$  is denoted by  $A^{-1}$  and is defined as the matrix that satisfies

$$AA^{-1} = A^{-1}A = I,$$

where  $I$  is the identity matrix.

**Steps to find the inverse of a matrix:**

1. Form the augmented matrix  $[A|I]$ .
2. Use row operations to convert  $A$  into the identity matrix.
3. The matrix that results from the identity matrix on the left side is  $A^{-1}$  on the right side.

**Example:** For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

the augmented matrix is

$$[A|I] = \left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right)$$

after reducing to row echelon form:

$$[I|A^{-1}] = \left( \begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1.5 & -0.5 \end{array} \right)$$

hence, the inverse is

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$$

## 6 Determinant

The **determinant** of a square matrix  $A$ , denoted by  $\det(A)$ , is a scalar value that can be computed from the elements of the matrix. The determinant has important properties and applications, including:

- A matrix is invertible if and only if its determinant is non-zero.
- The determinant of a product of matrices is the product of their determinants:  $\det(AB) = \det(A)\det(B)$ .

For a  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , the determinant is calculated as:

$$\det(A) = ad - bc.$$

For a  $3 \times 3$  matrix  $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ , the determinant is calculated as:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Expanding the  $2 \times 2$  determinants (minors):

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

For higher dimension matrices, ask a computer (it's outside the scope of this course).

**Example:** For the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$\det B = 0$ , hence, the matrix is not invertible (you can try to invert it, you will encounter a problem).

While for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

$\det A = -2 \neq 0$ , hence, the matrix is invertible (and we found the inverse in the previous section).

## 7 Eigenvalues

Let  $A$  be an  $n \times n$  matrix. A scalar  $\lambda$  is called an **eigenvalue** of  $A$  if there exists a non-zero vector  $\mathbf{v} \in \mathbb{R}^n$  (or  $\mathbb{C}^n$ ) such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The vector  $\mathbf{v}$  is referred to as the corresponding **eigenvector** of  $A$  associated with  $\lambda$ .

**Finding Eigenvalues** To find the eigenvalues of a matrix  $A$ , we solve the characteristic equation:

$$\det(A - \lambda I) = 0,$$

where  $I$  is the identity matrix of the same size as  $A$ , and  $\det(\cdot)$  denotes the determinant. The solutions  $\lambda$  are the eigenvalues of  $A$ .

**Example:** Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

The characteristic equation is given by:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}\right) = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0. \end{aligned}$$

The eigenvalues are the roots of this quadratic equation,  $\lambda_1 = 5$  and  $\lambda_2 = 2$ .

## 8 Eigenvectors

Given an eigenvalue  $\lambda$  of a matrix  $A$ , the corresponding **eigenvector**  $\mathbf{v}$  is any non-zero vector that satisfies the equation:

$$A\mathbf{v} = \lambda\mathbf{v}.$$

To find the eigenvectors associated with  $\lambda$ , we solve the system:

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

This is a system of linear equations.

**Example:** For the matrix  $A$  from the previous example and  $\lambda_1 = 5$ , we solve:

$$(A - 5I)\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

One solution to this system is  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ <sup>1</sup>.

Similarly, for  $\lambda_2 = 2$ , we solve:

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0},$$

which gives  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ <sup>2</sup>.

## 9 Diagonalization

A square matrix  $A$  is said to be **diagonalizable** if there exists an invertible matrix  $P$  and a diagonal matrix  $D$  such that:

$$A = PDP^{-1}.$$

When this is possible, the diagonal elements of  $D$  are the eigenvalues of  $A$ , and the columns of  $P$  are the corresponding eigenvectors.

### Procedure for Diagonalization

1. Find the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of  $A$ .
2. For each eigenvalue  $\lambda_i$ , find the corresponding eigenvector  $\mathbf{v}_i$ .
3. Form the matrix  $P$  using the eigenvectors as columns:  $P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n)$ .

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<sup>1</sup>Any vector of the form  $\mathbf{v} = \begin{pmatrix} \nu \\ \nu \end{pmatrix}, \nu \in \mathbb{R}^*$  would work, it is common practice to set  $\nu = 1$ .

<sup>2</sup>Again, any vector of the form  $\mathbf{v} = \begin{pmatrix} -\nu \\ 2\nu \end{pmatrix}, \nu \in \mathbb{R}^*$  would work, it is common practice to set  $\nu = 1$ .

4. The matrix  $D$  is a diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} \lambda_1 & & 0 & 0 & 0 \\ & \lambda_2 & & \ddots & 0 \\ 0 & & \ddots & & 0 \\ 0 & \ddots & & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{pmatrix}.$$

5. Verify that  $A = PDP^{-1}$ .

**Example:** Consider the matrix  $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ . From our earlier work, the eigenvalues are  $\lambda_1 = 5$  and  $\lambda_2 = 2$ , with corresponding eigenvectors  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ .

Thus, we can form:

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Finally, verify that:

$$P^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \quad A = PDP^{-1} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$