Exercise Set : Linear Algebra

Solutions

1 - System of Linear Equations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 2 & 3 \\ 0 & 0 & 0 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 6 & 2 & 3 \\ 0 & 0 & 0 & 10 \end{pmatrix} \qquad L_1 = L_1 - 1/3\, L_2$$

$$\to \begin{pmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} L_2 &= L_2/6 \\ L_3 &= L_3/10 \end{aligned}$$

$$\to \begin{pmatrix} 1 & 0 & 7/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} L_1 &= L_1 - 3L_3 \\ L_2 &= L_2 - 1/2\, L_3 \end{aligned}$$

Let $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

we want $A\underline{x} = \underline{u}$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow \quad \begin{aligned} L_1 &= L_1/2 \\ L_3 &= L_3/3 \end{aligned}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1/3 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

$$\downarrow \quad L_2 = L_2 - 3L_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & -3/2 & 1 & 0 \\ 0 & 1/3 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

$$\downarrow \quad L_2 = L_2 - L_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/6 & 0 & -3/2 & 1 & -1/3 \\ 0 & 1/3 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

$$\downarrow \quad L_2 = 6L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -9 & 6 & -2 \\ 0 & 1/3 & 1 & 0 & 0 & 1/3 \end{array}\right)$$

$$\downarrow \quad \begin{aligned} L_1 &= L_1 - \tfrac{1}{2} L_2 \\ L_3 &= L_3 - 1/3\, L_2 \end{aligned}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -3 & 1 \\ 0 & 1 & 0 & -9 & 6 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right) = (I_3 \mid A^{-1})$$

Therefore, $A^{-1} = \begin{pmatrix} 5 & -3 & 1 \\ 9 & 6 & -2 \\ 3 & -2 & 1 \end{pmatrix}$

and $A\underline{x} = \underline{u}$

$\Rightarrow \underbrace{A^{-1}A\underline{x}}_{\underline{x}} = A^{-1}\underline{u} = \begin{pmatrix} 5 & -3 & 1 \\ 9 & 6 & -2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 12 \\ 52 \\ 9 \end{pmatrix}$

Thus, we need $x = 11$, $y = 54$, $z = 8$.

2 - Vector Spaces

Suppose $\lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \lambda_3 \underline{v_3} = \underline{0}$ with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

Then $\begin{cases} \lambda_1 + 3\lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 + 5\lambda_3 = 0 \\ 2\lambda_3 = 0 \end{cases}$

$\Rightarrow \begin{cases} \lambda_3 = 0 & (L_3) \\ \text{so } \lambda_1 = 0 & (L_1) \\ \text{and } \lambda_2 = 0 & (L_2) \end{cases}$

Thus, $\underline{v_1}, \underline{v_2}, \underline{v_3}$ are L.I..

In fact, one way show that $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is a basis for $\mathbb{R}^3$, as they span as well.

Let $q \in \mathbb{P}_2$; $q(x) = ax^2 + bx + c$

$= \frac{a}{3}(3x^2 - 1) + \frac{a}{3} + \frac{b}{2}(2x) + c$

$= \frac{a}{3} p_3(x) + \frac{b}{2} p_2(x) + \left(c + \frac{a}{3}\right) p_1(x)$

Thus, q can be expressed as a linear sum of $\{p_1, p_2, p_3\}$, so $\{p_1, p_2, p_3\}$ span $\mathbb{P}_2$.

3 - Matrix Inverses

$$\det(A) = 2.2 - 1.0 = 4$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 \\ 0 & 1/2 \end{pmatrix}$$

$$AA^{-1} = I_2 \quad \& \quad A^{-1}A = I_2$$

$$\det(B) = 0.2.1 + 1.0.-1 + 3.1.0 - (-1).2.3 - 0.0.0 - 1.1.1$$
$$= 6 - 1 = 5$$

$$(B \,\vdots\, I_3) = \left(\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$L_1 = L_2$
$L_2 = L_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$L_1 = L_1 - 2L_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$L_3 = L_3 + L_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -5 & -2 & 1 & 1 \end{array}\right)$$

$L_3 = -L_3/5$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -6 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/5 & -1/5 & -1/5 \end{array}\right)$$

$L_1 = L_1 + 6L_3$
$L_2 = L_2 - 3L_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & -1/5 & -6/5 \\ 0 & 1 & 0 & -1/5 & 3/5 & 3/5 \\ 0 & 0 & 1 & 2/5 & -1/5 & -1/5 \end{array}\right) = (I_3 \,\vdots\, B^{-1})$$

so $$B^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 & -6 \\ -1 & 3 & 3 \\ 2 & -1 & -1 \end{pmatrix}$$

NB: It is common to put a $\frac{1}{\det(A)}$ in front of A^{-1}.

One may check that $B^{-1}B = I_3$ and $BB^{-1} = I_3$

4 - Eigenvalues & Eigenvectors

$$ch_A(\lambda) = 0 \iff \det(A - \lambda I_2) = 0$$
$$\iff (3-\lambda)^2 - 1 = 0$$
$$\iff 9 - 6\lambda + \lambda^2 - 1 = 0$$
$$\iff \lambda^2 - 6\lambda + 8 = 0$$

$$\Delta = 6^2 - 4.8.1 = 36 - 32 = 4$$

$$\lambda = \frac{6 \pm 2}{2} = \frac{8}{2} \text{ or } \frac{4}{2} = 4 \text{ or } 2$$

$$A\underline{x} = 2\underline{x} \Rightarrow \begin{cases} 3x + y = 2x \\ x + 3y = 2y \end{cases} \Rightarrow x + y = 0$$

Setting $x = 1 \Rightarrow y = -1$; $\underline{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$A\underline{x} = 4\underline{x} \Rightarrow \begin{cases} 3x + y = 4x \\ x + 3y = 4y \end{cases} \Rightarrow x = y$$

Setting $x = 1 \Rightarrow y = 1$; $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvalue 2 has eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

——— 4 ——— $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$ch_B(\lambda) = 0 \iff \det\begin{pmatrix} 1-\lambda & -1 \\ 0 & 1-\lambda \end{pmatrix} = 0$$
$$\iff (1-\lambda)(1-\lambda) - 0.(-1) = 0$$
$$\iff (1-\lambda)^2 = 0 \quad \text{so} \quad \lambda = 1 \quad \text{with multiplicity } 2.$$

$$B\underline{x} = 1.\underline{x} \iff \begin{cases} x - y = x \\ y = y \end{cases} \iff y = 0 \quad \text{letting } x = 1 ; \ \underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For generalized eigenvector, we solve $(B - \lambda I)\underline{x} = \underline{x}_0$ instead of $(B - \lambda I)\underline{x} = \underline{0}$

$$(B - \lambda I)\underline{x} = \underline{x}_1$$
$$\iff \begin{cases} 0.x - y = 1 \\ 0.x + 0.y = 0 \end{cases} \Rightarrow y = -1 \quad \text{letting } x = 0 ; \ \underline{x}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$\det(C - \lambda I) = 0$

$\Leftrightarrow (1-\lambda)(2-\lambda)(3-\lambda) + 2.0.0 + 0\,0.1 - 0.2\,0 - 1.0.1 - 3.0\,2 = 0$

$\Leftrightarrow \lambda = 1, 2, 3$

$C.\underline{x} = 1.\underline{x} \Leftrightarrow \begin{cases} x + 2y = x \\ 2y = y \\ y + 3z = z \end{cases} \Rightarrow \begin{matrix} y = 0 \\ z = 0 \end{matrix}$ letting $x = 1$; $\underline{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$C.\underline{x} = 2\underline{x} \Leftrightarrow \begin{cases} x + 2y = 2x \\ 2y = 2y \\ y + 3z = 2z \end{cases} \Rightarrow \begin{matrix} x = 2y \\ y + z = 0 \end{matrix}$ letting $x = 2$: $y = 1, z = -1$

so $\underline{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$C.\underline{x} = 3.\underline{x} \Leftrightarrow \begin{cases} x + 2y = 3x \\ 2y = 3y \\ y + 3z = 3z \end{cases} \Rightarrow \begin{matrix} y = 0 \\ x = 0 \end{matrix}$ letting $z = 1$: $\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

5 - Diagonalization

$A = P.D.P^{-1}$ with $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\uparrow$ eigen values $\uparrow$ eigen vectors

$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (either use formula for 2x2 matrix or inverse it by hand with $(P \,|\, I)$)

One may check that $P.D.P^{-1}$ is indeed A.

B is not diagonalizable as it does not have linearly independent eigenvectors.

$C = P\,D\,P^{-1}$ with $P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P^{-1} =$

One may find $P^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

6 - Orthogonal Vectors

$\underline{u} \cdot \underline{v} = 2 \cdot 1 + (-1) \cdot 2 + 0 \cdot 1 = 0 \Rightarrow \underline{u} \perp \underline{v}$

$\underline{u} \cdot \underline{w} = 2 \cdot 0 + (-1) \cdot 1 + 0 \cdot (-2) = -1 \Rightarrow \underline{u} \not\perp \underline{w}$

$\underline{v} \cdot \underline{w} = 1 \cdot 0 + 2 \cdot 1 + 1 \cdot (-2) = 0 \Rightarrow \underline{v} \perp \underline{w}$

$\|\underline{v}_1\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$

$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{\sqrt{5}}{5} \underline{v}_1 = \begin{pmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{pmatrix}$

$\tilde{\underline{u}}_2 = \underline{v}_2 - (\underline{v}_2 \cdot \underline{u}_1)\,\underline{u}_1$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{2\sqrt{5}}{5} \begin{pmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \end{pmatrix}$

$= \begin{pmatrix} -2/5 \\ 1/5 \end{pmatrix}$

$\underline{u}_2 = \frac{\tilde{\underline{u}}_2}{\|\tilde{\underline{u}}_2\|} = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{pmatrix}$

7 - Norms

$$\|\underline{v}+\underline{w}\|_1 = \left\| \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\| = 3$$

$$3 \leq 6+7 \quad \checkmark$$

$$\|\underline{v}\|_1 = 6 \quad , \|\underline{w}\| = 7$$

$$\underline{e} = \begin{pmatrix} -0.1 \\ 0.05 \\ 0.2 \end{pmatrix}$$

1) $\|\underline{e}\|_1 = 0,35$

2) $\|\underline{e}\| = 0,0525$

3) L_1 is manhattan distance, L_2 is Euclidean
depending on the context, one may be better than the other in terms of interpretability.