# Exercises Set: Calculus

DSBA Mathematics Refresher 2025

**Abstract**

Only the questions with a * are compulsory (but do all of them!).

## 1 Fundamental Theorem of Calculus

**Statement**

Let $f$ be a continuous real-valued function defined on a closed interval $[0, x]$. Let $F$ be the function defined, for all $t \in [0, x]$, by $F(x) = \int_0^x f(t)dt$.

Then $F$ is uniformly continuous on $[0, x]$ and differentiable on the open interval $(a, b)$, and $F'(x) = f(x)$ for all $x \in (a, b)$ so $F$ is an anti-derivative of $f$.

**Generalization / Corollary**

Let $f(x)$ be a continuous function on the closed interval $[a, b] \ni 0$, and let $F$ be an anti-derivative of $f$. Prove that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

**Application**

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin(x)\,dx$$

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_1^4 \frac{1}{x^2}\,dx$$

## 2 Integration Techniques

**Reminder**

$$\int f(u)\,du = \int f(g(x)) \cdot g'(x)\,dx \quad \text{or} \quad \int f(u)\,du = \int f(x) \cdot \frac{du}{dx}\,dx$$

$$\int uv' = uv - \int vu'$$

**Substitution / Change of Variable**

**Exercise 1:** Evaluate the following integral using the method of substitution:

$$\int \frac{2x}{x^2+1}\,dx$$

*Hint:* Let $u = x^2 + 1$ and then find $du$ to perform the substitution.

**Exercise 2: (*)** Evaluate the following integral using the method of substitution:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

*Hint:* Let $x = \sin(u)$ and then find $du$ to perform the substitution.

**Exercise 3:** Evaluate the following integral using a trigonometric substitution:

$$\int \frac{1}{4+x^2}\,dx$$

*Hint:* Use the substitution $u = x/2$ to simplify the integral.

**Integration by Parts**

**Exercise A:** Compute the following integral using integration by parts:

$$\int x\ln(x)\,dx$$

**Exercise B:** Find the value of the integral using integration by parts:

$$\int x^2 e^x\,dx$$

**Exercise C: (*)** Compute the following integral using integration by parts:

$$\int x\cos(x)\,dx$$

**Exercise D:** Evaluate the following integral using the method of substitution:

$$\int e^{2x}\cos(2x)\,dx$$

**Further integration techniques**

**Exercise $\alpha$:** Decomposing the fraction of the following expression:

$$\frac{3x^2-2x-1}{x^3-x^2+x-1}$$

Calculate

$$\int \frac{3x^2-2x-1}{x^3-x^2+x-1}dx$$

*Hint:* Factor the numerator and denominator.

**Exercise $\beta$: (*)** Evaluate the following improper integral by decomposition:

$$\int_0^{+\infty} e^{-x}\,dx$$

*Hint:* Evaluate the improper integral by considering the limits is $a$, and let $a$ approach infinity.

**Exercise $\gamma$:** Approximate the value of the integral

$$\int_0^{\pi/2} \sin(x)\,dx$$

using the Trapezoidal Rule with $n = 3$ sub-intervals.

**Exercise $\delta$:** Estimate the value of the integral

$$\int_0^{\pi/2} \sin(x)\,dx$$

using Simpson's Rule with $n = 2$ sub-intervals.

## 3 Applications

**Areas between curves**
Determine the area of the region enclosed by the curves $y = \sin(x)$ and $y = -\sin(x)$ over the interval $[0, \pi]$.
*Hint:* Begin by finding the points of intersection between the two curves within the given interval. Then, set up the integral to calculate the area between the curves.

**Volume of a solid of revolution (*)** (Disk Method)
Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the $y$-axis, over the interval $y \in [0, 1]$, about the $y$-axis using the disk method.
*Hint:* Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.

**Arc length of curves**
Find the arc length of the curve defined by $y = x\sqrt{x}$ over the interval $[1, 4]$.
*Hint:* Use the formula for arc length $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$ to calculate the arc length of the curve.

**Surface area of a solid of revolution**
Find the surface area of the solid generated by revolving the curve $y = x^2$ over the interval $y \in [0, 1]$ about the $y$-axis.
*Hint:* Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.