Notes on Calculus

DSBA Mathematics Refresher 2025

Abstract

1 Derivatives: A Quick Reminder

The derivative of a function measures how the function's output value changes as the input value changes. For a function $f(x)$, the derivative, denoted by $f'(x)$ or $\frac{d}{dx}f(x)$, is defined as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Some common derivatives include:

$$\begin{aligned}
\text{Power Rule:} \quad & \frac{d}{dx}(x^n) = nx^{n-1}, \\
\text{Exponential Function:} \quad & \frac{d}{dx}(e^x) = e^x, \\
\text{Trigonometric Functions:} \quad & \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x.
\end{aligned}$$

Chain Rule The chain rule is used to differentiate composite functions. If you have a function $y = f(g(x))$, where one function is inside another, the chain rule states:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Example: If $y = \sin(3x)$, then:

$$\frac{dy}{dx} = \cos(3x) \cdot 3 = 3\cos(3x)$$

Product Rule The product rule is used to differentiate the product of two functions. If you have a function $y = u(x) \cdot v(x)$, where $u(x)$ and $v(x)$ are both functions of x, the product rule states:

$$\frac{dy}{dx} = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Example: If $y = x^2 \cdot \sin(x)$, then:

$$\frac{dy}{dx} = 2x \cdot \sin(x) + x^2 \cdot \cos(x)$$

Quotient Rule The quotient rule is used to differentiate the quotient of two functions. If you have a function $y = \frac{u(x)}{v(x)}$, where $u(x)$ and $v(x)$ are both functions of x, the quotient rule states:

$$\frac{dy}{dx} = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

Example: If $y = \frac{x^2}{\sin(x)}$, then:

$$\frac{dy}{dx} = \frac{2x \cdot \sin(x) - x^2 \cdot \cos(x)}{[\sin(x)]^2}$$

2 Integration: A Quick Reminder

Integration is the "reverse process" of differentiation. The integral of a function represents the area under the curve of the function. The indefinite integral of a function $f(x)$ is denoted by:

$$\int f(x)dx$$

Some basic integration rules include:

$$\begin{aligned}
\text{Power Rule:} \quad & \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1) \\
\text{Exponential Function:} \quad & \int e^x dx = e^x + C, \\
\text{Trigonometric Functions:} \quad & \int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C.
\end{aligned}$$

where C is the constant of integration [1] [2].

3 Integration by Parts

Integration by parts is based on the product rule for differentiation and is given by:

$$\int uv' = uv - \int u'v$$

where $u = f(x)$ and $dv = g(x)dx$. Steps:

1. Choose u and dv.
2. Differentiate u to get du.
3. Integrate dv to get v.

[1] It is very common to forget the "$+C$".

[2] When I tutor young students, I give them push-ups when they forget it... they don't usually forget more than twice.

4. Substitute into the formula.

Example:

$$\int xe^x dx$$

Let $u = x$, $dv = e^x dx$, then $du = 1 \cdot dx$ and $v = e^x$. Therefore:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

4 Integration by Substitution

Integration by substitution is used when the integral can be transformed into a simpler form by substituting $u = g(x)$. The substitution formula is:

$$\int f(g(x))\frac{dg(x)}{dx}dx = \int f(u)\,du$$

or

$$\int f(u)\frac{du}{dx}dx = \int f(u)\,du$$

Steps:

1. Choose $u = g(x)$.
2. Compute $du = g'(x)dx$.
3. Rewrite the integral in terms of u.
4. Integrate with respect to u.
5. Substitute back $u = g(x)$ if needed.

Example:

$$\int \cos(3x)dx$$

Let $u = 3x$, then $du = 3dx$ or $dx = \frac{du}{3}$. The integral becomes:

$$\int \cos(3x)dx = \frac{1}{3}\int \cos(u)\,du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(3x) + C$$