Squares and Cubes Modulo n

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Abstract. We study the asymptotics of the average number of squares (or quadratic residues) in $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$. Similar analyses are performed for cubes, square roots of 0 and 1, and cube roots of 0 and 1.

Let $\mathbb{Z}_n$ denote the ring of integers modulo n, and let $\mathbb{Z}_n^*$ denote the group (under multiplication) of integers relatively prime to n. The number of elements in $\mathbb{Z}_n^*$ is $\varphi(n)$, where φ is Euler's totient function. What is the average number of elements in $\mathbb{Z}_n^*$, given an arbitrary n? One way to answer this question is to apply the Selberg-Delange method [1, 2, 3] to the Dirichlet series

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^{s+1}} &= \prod_p\left(1+\sum_{r=1}^{\infty}\frac{\varphi(p^r)}{p^{r(s+1)}}\right)\\ &= \prod_p\left(1+\sum_{r=1}^{\infty}\frac{p-1}{p^{rs+1}}\right)\\ &= \prod_p\left(1+\frac{p-1}{p}\sum_{r=1}^{\infty}\frac{1}{p^{rs}}\right)\\ &= \prod_p\left(1+\frac{p-1}{p(p^s-1)}\right) = G(s)\cdot\zeta(s)\end{aligned}$$

where $G(s)$ is bounded in a half plane $\mathrm{Re}(s) > c$ for some $c < 1$. In fact, $G(s) = 1/\zeta(s+1)$ in this case, and hence

$$\sum_{n\leq N}\frac{\varphi(n)}{n} \sim \frac{G(1)}{\Gamma(1)}N = \frac{1}{\zeta(2)}N$$

as $N \to \infty$. It follows by partial summation that

$$\sum_{n\leq N}\varphi(n) \sim \frac{1}{2\zeta(2)}N^2 = \frac{3}{\pi^2}N^2.$$

A more elementary proof of this asymptotic formula appears in [4]. Since the Selberg-Delange method will be used throughout this paper, we choose to illustrate its application in this simple setting.

Many other questions can be asked for arbitrary n:

- What is the average number of solutions of $x^2 = 1$ in $\mathbb{Z}_n^*$?
- What is the average number of solutions of $x^3 = 1$ in $\mathbb{Z}_n^*$?
- What is the average number of solutions of $x^2 = 0$ in $\mathbb{Z}_n$?
- What is the average number of solutions of $x^3 = 0$ in $\mathbb{Z}_n$?
- What is the average number of images of the map $y \mapsto y^2$ in either $\mathbb{Z}_n$ or $\mathbb{Z}_n^*$?
- What is the average number of images of the map $y \mapsto y^3$ in either $\mathbb{Z}_n$ or $\mathbb{Z}_n^*$?

Although the answers require only straightforward use of standard techniques, they do not seem to be explicitly given in the literature. We make no claim of originality: Our purpose is only to collect results in one place and to document relevant numerical techniques.

1. Number Theory

1.1. Selberg-Delange Method. Let $F(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ be a Dirichlet series with positive coefficients and with the property that $G(s) = F(s) \cdot \zeta(s)^{-z}$ can be analytically continued and is bounded over $\mathrm{Re}(s) > c$, for some $c < 1$ and some $z \in \mathbb{C}$. Then

$$\sum_{n \le N} a(n) \sim \frac{G(1)}{\Gamma(z)} N \cdot (\ln N)^{z-1}$$

as $N \to \infty$. More terms of the asymptotic expansion are possible, as is an accurate estimate of the error, but we omit these details for brevity's sake.

A generalization of this method is required for our work involving averages over arithmetic progressions. Let χ denote the principal character modulo $k = q^m$, where q is a prime and $m \ge 1$. Here we examine

$$F_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n) a(n)}{n^s} = G_\chi(s) \cdot L_\chi(s)^z$$

where

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \left(1 - \frac{1}{k^s}\right) \zeta(s)$$

is the L-series corresponding to χ. Assuming $a(n)$ is a multiplicative function and $q \nmid \ell$, it follows that

$$\sum_{\substack{n \le N, \\ n \equiv \ell \bmod k}} a(n) \sim \frac{G_\chi(1)}{\Gamma(z)} N \cdot (\ln N)^{z-1}$$

as $N \to \infty$ and, further, that

$$G_\chi(1) = \frac{1}{\varphi(k)} \left(1 + \sum_{r=1}^{\infty} \frac{a(q^r)}{q^r}\right)^{-1} G(1).$$

In our examples, q will be either 2 or 3 and the bracketed infinite series will always collapse to a closed-form expression. Rather than directly employing the formula for $G_\chi(1)$, however, we prefer instead to deduce $F_\chi(s)$ (and hence $G_\chi(s)$) from $F(s)$ on basic principles.

1.2. Square Roots of Unity. The number $a(n)$ of solutions of $x^2 = 1$ in $\mathbb{Z}_n^*$ is [5]

$$a(n) = \begin{cases} 2^{\omega(n)-1} & \text{if } n \equiv 2, 6 \bmod 8, \\ 2^{\omega(n)} & \text{if } n \equiv 1, 3, 4, 5, 7 \bmod 8, \\ 2^{\omega(n)+1} & \text{if } n \equiv 0 \bmod 8 \end{cases}$$

where $\omega(n)$ denotes the number of distinct prime factors of n. It is well-known that [6]

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} &= \prod_p \left(1 + 2\sum_{r=1}^{\infty} \frac{1}{p^{rs}}\right) \\ &= \prod_p \left(1 + \frac{2}{p^s - 1}\right) = \frac{\zeta(s)^2}{\zeta(2s)} = G(s) \cdot \zeta(s)^2 \end{aligned}$$

and hence

$$\sum_{n \le N} 2^{\omega(n)} \sim \frac{1}{\zeta(2)} N \cdot \ln N = \frac{6}{\pi^2} N \cdot \ln N.$$

We need to generalize this asymptotic formula to arithmetic progressions $n \equiv \ell \bmod k$, where $k = 2^m$ for $m \ge 1$ and $2 \nmid \ell$. It can be shown that

$$\begin{aligned} \sum_{n \equiv \ell \bmod k} \frac{2^{\omega(n)}}{n^s} &\sim \frac{1}{\varphi(k)} \prod_{p>2} \left(1 + \frac{2}{p^s - 1}\right) \\ &= \frac{2}{k} \left(1 + \frac{2}{2^s - 1}\right)^{-1} \frac{\zeta(s)^2}{\zeta(2s)} = G_\chi(s) \cdot \zeta(s)^2 \end{aligned}$$

as $s \to 1$, and thus

$$\sum_{\substack{n\leq N,\\ n\equiv \ell \bmod k}} 2^{\omega(n)} \sim \frac{G_\chi(1)}{\Gamma(2)} N\cdot \ln N = \frac{4}{k\pi^2} N\cdot \ln N.$$

The cases $(k,\ell) = (8,1), (8,3), (8,5)$ and $(8,7)$ follow immediately. The case $(k,\ell) = (8,4)$ proceeds from the case $(k,\ell) = (2,1)$:

$$\frac{1}{N\cdot \ln N} \sum_{\substack{n\leq N,\\ n\equiv 4 \bmod 8}} 2^{\omega(n)} = \frac{1}{N\cdot \ln N} \sum_{\substack{4n\leq N,\\ n\equiv 1 \bmod 2}} 2^{\omega(n)+1} \longrightarrow \frac{1}{4}\cdot 2\cdot \frac{2}{\pi^2} = \frac{1}{\pi^2}.$$

The case $(k,\ell) = (8,2)$ proceeds from the case $(k,\ell) = (4,1)$:

$$\frac{1}{N\cdot \ln N} \sum_{\substack{n\leq N,\\ n\equiv 2 \bmod 8}} 2^{\omega(n)} = \frac{1}{N\cdot \ln N} \sum_{\substack{2n\leq N,\\ n\equiv 1 \bmod 4}} 2^{\omega(n)+1} \longrightarrow \frac{1}{2}\cdot 2\cdot \frac{1}{\pi^2} = \frac{1}{\pi^2}$$

and $(8,6)$ likewise proceeds from $(4,1)$. By everything proved thus far, we have

$$\frac{1}{N\cdot \ln N} \sum_{\substack{n\leq N,\\ n\equiv 0 \bmod 8}} 2^{\omega(n)} \longrightarrow \frac{6}{\pi^2} - 4\cdot\frac{1}{2\pi^2} - 3\cdot \frac{1}{\pi^2} = \frac{1}{\pi^2}.$$

Therefore

$$\sum_{n\leq N} a(n) \sim \left(\frac{1}{2}\cdot 2\cdot \frac{1}{\pi^2} + \left(4\cdot \frac{1}{2\pi^2} + \frac{1}{\pi^2}\right) + 2\cdot \frac{1}{\pi^2}\right) N\cdot \ln N = \frac{6}{\pi^2} N\cdot \ln N.$$

It is interesting that $(8,2)$ and $(8,6)$ balance perfectly against $(8,0)$ so that the mean value of $a(n)$ is asymptotically equivalent to the mean value of $2^{\omega(n)}$.

1.3. Cube Roots of Unity. The number $a(n)$ of solutions of $x^3 = 1$ in $\mathbb{Z}_n^*$ is [7]

$$a(n) = \begin{cases} 3^{\tilde{\omega}(n)} & \text{if } n \equiv 1,2,3,4,5,6,7,8 \bmod 9, \\ 3^{\tilde{\omega}(n)+1} & \text{if } n \equiv 0 \bmod 9 \end{cases}$$

where $\tilde{\omega}(n)$ denotes the number of distinct primes of the form $3k+1$ dividing n:

$$\tilde{\omega}(p^r) = \begin{cases} 0 & \text{if } p = 3 \text{ or } p \equiv 2 \bmod 3, \\ 1 & \text{if } p \equiv 1 \bmod 3. \end{cases}$$

First, note that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3^{\tilde{\omega}(n)}}{n^s} &= \prod_{\substack{p=3 \text{ or}\\ p\equiv 2 \bmod 3}}\left(1+\sum_{r=1}^{\infty}\frac{1}{p^{rs}}\right)\cdot\prod_{p\equiv 1 \bmod 3}\left(1+3\sum_{r=1}^{\infty}\frac{1}{p^{rs}}\right)\\ &= \prod_{\substack{p=3 \text{ or}\\ p\equiv 2 \bmod 3}}\left(1+\frac{1}{p^s-1}\right)\cdot\prod_{p\equiv 1 \bmod 3}\left(1+\frac{3}{p^s-1}\right)\\ &= \prod_{\substack{p=3 \text{ or}\\ p\equiv 2 \bmod 3}}\left(1-\frac{1}{p^s}\right)^{-1}\cdot\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-3}\left(1-\frac{1}{p^{2s}}\right)\left(1-\frac{2}{p^s(p^s+1)}\right)\\ &= \zeta(s)\cdot\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-2}\left(1-\frac{1}{p^{2s}}\right)\left(1-\frac{2}{p^s(p^s+1)}\right) = G(s)\cdot\zeta(s)^2\end{aligned}$$

and [8]

$$\lim_{s\to 1}\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-2}\cdot(s-1) = \frac{\sqrt{3}}{2\pi}\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^2}\right)^{-1};$$

hence

$$\sum_{n\le N}3^{\tilde{\omega}(n)} \sim \frac{G(1)}{\Gamma(2)}N\cdot\ln N = C\cdot N\cdot\ln N$$

where

$$C = \frac{\sqrt{3}}{2\pi}\prod_{p\equiv 1 \bmod 3}\left(1-\frac{2}{p(p+1)}\right) = \frac{\sqrt{3}}{2\pi}(0.9410349413195354517900322...).$$

We need to generalize this asymptotic formula to arithmetic progressions $n\equiv \ell \bmod k$, where $k=3^m$ for $m\ge 1$ and $3\nmid \ell$. It can be shown that

$$\begin{aligned}\sum_{n\equiv \ell \bmod k}\frac{3^{\tilde{\omega}(n)}}{n^s} &\sim \frac{1}{\varphi(k)}\prod_{p\equiv 2 \bmod 3}\left(1+\frac{1}{p^s-1}\right)\cdot\prod_{p\equiv 1 \bmod 3}\left(1+\frac{3}{p^s-1}\right)\\ &= \frac{3}{2k}\left(1+\frac{1}{3^s-1}\right)^{-1}G(s)\cdot\zeta(s)^2 = G_\chi(s)\cdot\zeta(s)^2\end{aligned}$$

as $s\to 1$, and thus

$$\sum_{\substack{n\le N,\\ n\equiv \ell \bmod k}}3^{\tilde{\omega}(n)} \sim \frac{G_\chi(1)}{\Gamma(2)}N\cdot\ln N = \frac{C}{k}N\cdot\ln N.$$

The cases $(k,\ell) = (9,1),(9,2),(9,4),(9,5),(9,7)$ and $(9,8)$ follow immediately. The case $(9,3)$ proceeds from the case $(3,1)$:

$$\frac{1}{N\cdot\ln N}\sum_{\substack{n\leq N,\\ n\equiv 3\bmod 9}} 3^{\tilde{\omega}(n)} = \frac{1}{N\cdot\ln N}\sum_{\substack{3n\leq N,\\ n\equiv 1\bmod 3}} 3^{\tilde{\omega}(n)} \longrightarrow \frac{1}{3}\cdot\frac{C}{3} = \frac{C}{9}$$

and $(9,6),(9,0)$ likewise proceed from $(3,2),(3,0)$. Therefore

$$\sum_{n\leq N} a(n) \sim \left(8\cdot\frac{C}{9} + 3\cdot\frac{C}{9}\right) N\cdot\ln N = \frac{11}{9}C\cdot N\cdot\ln N = (0.317...)N\cdot\ln N.$$

Unlike earlier, the mean value of $a(n)$ is asymptotically greater than the mean value of $3^{\tilde{\omega}(n)}$. Our estimate improves upon Cloitre [7], who gave $(0.4...)N\cdot\ln(N)$ on empirical grounds.

1.4. Squares in $\mathbb{Z}_n^*$. Let $a(n)$ be as defined in section [1.2]. The number of squares, that is, the cardinality of images under the map $y\mapsto y^2$ in $\mathbb{Z}_n^*$, is [9]

$$b(n) = \frac{\varphi(n)}{a(n)} = \begin{cases} \dfrac{\varphi(n)}{2^{\omega(n)-1}} & \text{if } n\equiv 2,6\bmod 8,\\ \dfrac{\varphi(n)}{2^{\omega(n)}} & \text{if } n\equiv 1,3,4,5,7\bmod 8,\\ \dfrac{\varphi(n)}{2^{\omega(n)+1}} & \text{if } n\equiv 0\bmod 8. \end{cases}$$

First, note that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^{s+1}2^{\omega(n)}} &= \prod_p\left(1+\frac{1}{2}\sum_{r=1}^{\infty}\frac{p-1}{p^{rs+1}}\right)\\ &= \prod_p\left(1+\frac{p-1}{2p(p^s-1)}\right) = G(s)\cdot\zeta(s)^{1/2},\end{aligned}$$

hence

$$\sum_{n\leq N}\frac{\varphi(n)}{n2^{\omega(n)}} \sim \frac{G(1)}{\Gamma(1/2)}N\cdot(\ln N)^{-1/2} = C\cdot N\cdot(\ln N)^{-1/2}$$

where

$$C = \frac{1}{\sqrt{\pi}}\prod_p\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{p}\right)^{1/2} = \frac{1}{\sqrt{\pi}}(0.8121057111631225117062509...).$$

It follows by partial summation that

$$\sum_{n\le N}\frac{\varphi(n)}{2^{\omega(n)}}\sim\frac{C}{2}\cdot N^2\cdot(\ln N)^{-1/2}.$$

We need to generalize this asymptotic formula to arithmetic progressions $n\equiv\ell \bmod k$, where $k=2^m$ for $m\ge 1$ and $2\nmid\ell$. It can be shown that

$$\begin{aligned}\sum_{n\equiv\ell \bmod k}\frac{\varphi(n)}{n^{s+1}2^{\omega(n)}}&\sim\frac{1}{\varphi(k)}\prod_{p>2}\left(1+\frac{p-1}{2p(p^s-1)}\right)\\&=\frac{2}{k}\left(1+\frac{1}{4(2^s-1)}\right)^{-1}G(s)\cdot\zeta(s)^{1/2}=G_\chi(s)\cdot\zeta(s)^{1/2}\end{aligned}$$

as $s\to 1$, and thus

$$\sum_{\substack{n\le N,\\ n\equiv\ell \bmod k}}\frac{\varphi(n)}{n2^{\omega(n)}}\sim\frac{G_\chi(1)}{\Gamma(1/2)}N\cdot(\ln N)^{-1/2}=\frac{8}{5k}C\cdot N\cdot(\ln N)^{-1/2}$$

or

$$\sum_{\substack{n\le N,\\ n\equiv\ell \bmod k}}\frac{\varphi(n)}{2^{\omega(n)}}\sim\frac{4}{5k}C\cdot N^2\cdot(\ln N)^{-1/2}.$$

The cases $(k,\ell)=(8,1),(8,3),(8,5)$ and $(8,7)$ follow immediately. The case $(k,\ell)=(8,4)$ proceeds from the case $(k,\ell)=(2,1)$:

$$\frac{(\ln N)^{1/2}}{N^2}\sum_{\substack{n\le N,\\ n\equiv 4 \bmod 8}}\frac{\varphi(n)}{2^{\omega(n)}}=\frac{(\ln N)^{1/2}}{N^2}\sum_{\substack{4n\le N,\\ n\equiv 1 \bmod 2}}\frac{2\varphi(n)}{2^{\omega(n)+1}}\longrightarrow\frac{1}{16}\cdot\frac{4}{10}C=\frac{C}{40}.$$

The case $(k,\ell)=(8,2)$ proceeds from the case $(k,\ell)=(4,1)$:

$$\frac{(\ln N)^{1/2}}{N^2}\sum_{\substack{n\le N,\\ n\equiv 2 \bmod 8}}\frac{\varphi(n)}{2^{\omega(n)}}=\frac{(\ln N)^{1/2}}{N^2}\sum_{\substack{2n\le N,\\ n\equiv 1 \bmod 4}}\frac{\varphi(n)}{2^{\omega(n)+1}}\longrightarrow\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{4}{20}C=\frac{C}{40}$$

and $(8,6)$ likewise proceeds from $(4,1)$. By everything proved thus far, we have

$$\frac{(\ln N)^{1/2}}{N^2}\sum_{\substack{n\le N,\\ n\equiv 0 \bmod 8}}\frac{\varphi(n)}{2^{\omega(n)}}\longrightarrow\frac{C}{2}-4\cdot\frac{C}{10}-\frac{C}{40}-2\cdot\frac{C}{40}=\frac{C}{40}.$$

Therefore

$$\begin{aligned}\sum_{n\le N}b(n)&\sim\left(2\cdot 2\cdot\frac{C}{40}+\left(4\cdot\frac{C}{10}+\frac{C}{40}\right)+\frac{1}{2}\cdot\frac{C}{40}\right)N^2\cdot(\ln N)^{-1/2}\\&=\frac{43}{80}C\cdot N^2\cdot(\ln N)^{-1/2}=(0.246...)N^2\cdot(\ln N)^{-1/2}.\end{aligned}$$

1.5. Cubes in $\mathbb{Z}_n^*$. Let $a(n)$ be as defined in section [1.3]. The number of cubes, that is, the cardinality of images under the map $y \mapsto y^3$ in $\mathbb{Z}_n^*$, is [10]

$$b(n) = \frac{\varphi(n)}{a(n)} = \begin{cases} \dfrac{\varphi(n)}{3^{\tilde{\omega}(n)}} & \text{if } n \equiv 1,2,3,4,5,6,7,8 \bmod 9, \\ \dfrac{\varphi(n)}{3^{\tilde{\omega}(n)+1}} & \text{if } n \equiv 0 \bmod 9. \end{cases}$$

First, note that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{s+1} 3^{\tilde{\omega}(n)}} &= \prod_{\substack{p=3 \text{ or} \\ p \equiv 2 \bmod 3}} \left(1 + \sum_{r=1}^{\infty} \frac{p-1}{p^{rs+1}}\right) \cdot \prod_{p \equiv 1 \bmod 3} \left(1 + \frac{1}{3} \sum_{r=1}^{\infty} \frac{p-1}{p^{rs+1}}\right) \\ &= \prod_{\substack{p=3 \text{ or} \\ p \equiv 2 \bmod 3}} \left(1 + \frac{p-1}{p(p^s-1)}\right) \cdot \prod_{p \equiv 1 \bmod 3} \left(1 + \frac{p-1}{3p(p^s-1)}\right) = G(s) \cdot \zeta(s)^{2/3}, \end{aligned}$$

hence

$$\sum_{n \le N} \frac{\varphi(n)}{n 3^{\tilde{\omega}(n)}} \sim \frac{G(1)}{\Gamma(2/3)} N \cdot (\ln N)^{-1/3} = C \cdot N \cdot (\ln N)^{-1/3}$$

where

$$\begin{aligned} C &= \frac{1}{\Gamma(2/3)} \prod_{\substack{p=3 \text{ or} \\ p \equiv 2 \bmod 3}} \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{2/3} \cdot \prod_{p \equiv 1 \bmod 3} \left(1 + \frac{1}{3p}\right) \left(1 - \frac{1}{p}\right)^{2/3} \\ &= \frac{1}{\Gamma(2/3)} (0.9477556177621765519078142...). \end{aligned}$$

It follows by partial summation that

$$\sum_{n \le N} \frac{\varphi(n)}{3^{\tilde{\omega}(n)}} \sim \frac{C}{2} \cdot N^2 \cdot (\ln N)^{-1/3}.$$

We need to generalize this asymptotic formula to arithmetic progressions $n \equiv \ell \bmod k$, where $k = 3^m$ for $m \ge 1$ and $3 \nmid \ell$. It can be shown that

$$\begin{aligned} \sum_{n \equiv \ell \bmod k} \frac{\varphi(n)}{n^{s+1} 3^{\tilde{\omega}(n)}} &\sim \frac{1}{\varphi(k)} \prod_{p \equiv 2 \bmod 3} \left(1 + \frac{p-1}{p(p^s-1)}\right) \cdot \prod_{p \equiv 1 \bmod 3} \left(1 + \frac{p-1}{3p(p^s-1)}\right) \\ &= \frac{3}{2k} \left(1 + \frac{2}{3(3^s-1)}\right)^{-1} G(s) \cdot \zeta(s)^{2/3} = G_\chi(s) \cdot \zeta(s)^{2/3} \end{aligned}$$

as $s \to 1$, and thus

$$\sum_{\substack{n \le N, \\ n \equiv \ell \bmod k}} \frac{\varphi(n)}{n 3^{\tilde{\omega}(n)}} \sim \frac{G_\chi(1)}{\Gamma(2/3)} N \cdot (\ln N)^{-1/3} = \frac{9}{8k} C \cdot N \cdot (\ln N)^{-1/3}$$

or

$$\sum_{\substack{n \le N, \\ n \equiv \ell \bmod k}} \frac{\varphi(n)}{3^{\tilde{\omega}(n)}} \sim \frac{9}{16k} C \cdot N^2 \cdot (\ln N)^{-1/3}.$$

The cases $(k, \ell) = (9,1), (9,2), (9,4), (9,5), (9,7)$ and $(9,8)$ follow immediately. The case $(9,3)$ proceeds from the case $(3,1)$:

$$\frac{(\ln N)^{1/3}}{N^2} \sum_{\substack{n \le N, \\ n \equiv 3 \bmod 9}} \frac{\varphi(n)}{3^{\tilde{\omega}(n)}} = \frac{(\ln N)^{1/3}}{N^2} \sum_{\substack{3n \le N, \\ n \equiv 1 \bmod 3}} \frac{2\varphi(n)}{3^{\tilde{\omega}(n)}} \longrightarrow \frac{1}{9} \cdot 2 \cdot \frac{9}{48} C = \frac{C}{24}$$

and $(9,6)$ likewise proceeds from $(3,2)$. By everything proved thus far, we have

$$\frac{(\ln N)^{1/3}}{N^2} \sum_{\substack{n \le N, \\ n \equiv 0 \bmod 9}} \frac{\varphi(n)}{3^{\tilde{\omega}(n)}} \longrightarrow \frac{C}{2} - 6 \cdot \frac{C}{16} - 2 \cdot \frac{C}{24} = \frac{C}{24}.$$

Therefore

$$\begin{aligned} \sum_{n \le N} b(n) &\sim \left(\left(6 \cdot \frac{C}{16} + 2 \cdot \frac{C}{24} \right) + \frac{1}{3} \cdot \frac{C}{24} \right) N^2 \cdot (\ln N)^{-1/3} \\ &= \frac{17}{36} C \cdot N^2 \cdot (\ln N)^{-1/3} = (0.330...) N^2 \cdot (\ln N)^{-1/3}. \end{aligned}$$

1.6. Square Roots of Nullity. The number $a(n)$ of solutions of $x^2 = 0$ in $\mathbb{Z}_n$ is a multiplicative function of n, with $a(p^r) = p^{\lfloor r/2 \rfloor}$, thus [11, 12]

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^s} + \frac{p}{p^{2s}} + \frac{p}{p^{3s}} + \frac{p^2}{p^{4s}} + \frac{p^2}{p^{5s}} + \frac{p^3}{p^{6s}} + \frac{p^3}{p^{7s}} + \cdots \right) \\ &= \prod_p \left(1 + \frac{1}{p^s} 1 + \frac{p}{p^{2s}} + \frac{1}{p^s} \frac{p}{p^{2s}} + \frac{p^2}{p^{4s}} + \frac{1}{p^s} \frac{p^2}{p^{4s}} + \frac{p^3}{p^{6s}} + \frac{1}{p^s} \frac{p^3}{p^{6s}} + \cdots \right) \\ &= \prod_p \left(\frac{1}{1 - \dfrac{p}{p^{2s}}} + \frac{1/p^s}{1 - \dfrac{p}{p^{2s}}} \right) = \prod_p \left(1 - \frac{1}{p^{2s-1}} \right)^{-1} \left(1 + \frac{1}{p^s} \right) \\ &= \frac{\zeta(2s-1)\zeta(s)}{\zeta(2s)} = G(s) \cdot \zeta(s)^2 \end{aligned}$$

and $\lim_{s\to 1}\zeta(2s-1)\cdot(s-1)=1/2$, hence

$$\sum_{n\le N} a(n)\sim\frac{G(1)}{\Gamma(2)}N\cdot\ln N=\frac{3}{\pi^2}N\cdot\ln N.$$

1.7. Cube Roots of Nullity. The number $a(n)$ of solutions of $x^3=0$ in $\mathbb{Z}_n$ is a multiplicative function of n, with $a(p^r)=p^{\lfloor 2r/3\rfloor}$, thus [13]

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{a(n)}{n^s} &= \prod_p\left(1+\frac{1}{p^s}+\frac{p}{p^{2s}}+\frac{p^2}{p^{3s}}+\frac{p^2}{p^{4s}}+\frac{p^3}{p^{5s}}+\frac{p^4}{p^{6s}}+\frac{p^4}{p^{7s}}+\frac{p^5}{p^{8s}}+\frac{p^6}{p^{9s}}+\frac{p^6}{p^{10s}}+\cdots\right)\\
&= \prod_p\left(1+\tfrac{1}{p^s}1+\tfrac{p}{p^{2s}}1+\tfrac{p^2}{p^{3s}}+\tfrac{1}{p^s}\tfrac{p^2}{p^{3s}}+\tfrac{p}{p^{2s}}\tfrac{p^2}{p^{3s}}+\tfrac{p^4}{p^{6s}}+\tfrac{1}{p^s}\tfrac{p^4}{p^{6s}}+\tfrac{p}{p^{2s}}\tfrac{p^4}{p^{6s}}+\tfrac{p^6}{p^{9s}}+\tfrac{1}{p^s}\tfrac{p^6}{p^{9s}}+\cdots\right)\\
&= \prod_p\left(\frac{1}{1-\dfrac{p^2}{p^{3s}}}+\frac{1/p^s}{1-\dfrac{p^2}{p^{3s}}}+\frac{p/p^{2s}}{1-\dfrac{p^2}{p^{3s}}}\right)=\prod_p\left(1-\frac{1}{p^{3s-2}}\right)^{-1}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s-1}}\right)\\
&= \zeta(3s-2)\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s-1}}\right)=G(s)\cdot\zeta(s)^3.
\end{aligned}$$

We have

$$\begin{aligned}
\lim_{s\to 1}\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s-1}}\right)\cdot(s-1)^2 &= \lim_{s\to 1}\frac{1}{\zeta(s)\cdot 2\zeta(2s-1)}\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s-1}}\right)\\
&= \frac{1}{2}\lim_{s\to 1}\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s-1}}\right)\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{2s-1}}\right)\\
&= \frac{1}{2}\prod_p\left(1+\frac{2}{p}\right)\left(1-\frac{1}{p}\right)^2\\
&= \frac{1}{2}\prod_p\left(1+\frac{3}{p-1}\right)\left(1-\frac{1}{p}\right)^3\\
&= \frac{1}{2}\prod_p\left(1-\frac{1}{p^2}\right)\left(1-\frac{2}{p(p+1)}\right)\\
&= \frac{1}{2\zeta(2)}\prod_p\left(1-\frac{2}{p(p+1)}\right)
\end{aligned}$$

and $\lim_{s\to 1}\zeta(3s-2)\cdot(s-1)=1/3$, hence

$$\sum_{n\le N} a(n)\sim\frac{G(1)}{\Gamma(3)}N\cdot(\ln N)^2=C\cdot N\cdot(\ln N)^2$$

where [6, 13]

$$C = \frac{1}{2\pi^2}\prod_p\left(1-\frac{2}{p(p+1)}\right) = \frac{1}{12}(0.28674742843447873410789 27...).$$

1.8. Squares in $\mathbb{Z}_n$. The number $b(n)$ of images under the map $y \mapsto y^2$ in $\mathbb{Z}_n$ is a multiplicative function of n, with [14, 15, 16]

$$b(p^r) = \begin{cases} \frac{1}{3}\left(2^{r-1}+4\right) & \text{if } p = 2 \text{ and } r \equiv 0 \bmod 2, \\ \frac{1}{3}\left(2^{r-1}+5\right) & \text{if } p = 2 \text{ and } r \equiv 1 \bmod 2, \\ \frac{1}{2(p+1)}\left(p^{r+1}+p+2\right) & \text{if } p > 2 \text{ and } r \equiv 0 \bmod 2, \\ \frac{1}{2(p+1)}\left(p^{r+1}+2p+1\right) & \text{if } p > 2 \text{ and } r \equiv 1 \bmod 2 \end{cases}$$

and

$$F(s) = \sum_{n=1}^{\infty}\frac{b(n)}{n^{s+1}} = \left(1+\sum_{r=1}^{\infty}\frac{b(2^r)}{2^{r(s+1)}}\right)\cdot\prod_{p>2}\left(1+\sum_{r=1}^{\infty}\frac{b(p^r)}{p^{r(s+1)}}\right).$$

The left-hand factor in $F(s)$ simplifies to

$$\begin{aligned} &1+\frac{1}{3}\sum_{i=1}^{\infty}\frac{2^{2i-1}+4}{2^{(2i)(s+1)}}+\frac{1}{3}\sum_{j=1}^{\infty}\frac{2^{(2j-1)-1}+5}{2^{(2j-1)(s+1)}} \\ &= 1+\frac{1}{2}\left(\frac{4^{s+1}-3}{(4^{s+1}-1)(4^s-1)}+2^s\frac{2\cdot 4^{s+1}-7}{(4^{s+1}-1)(4^s-1)}\right) \\ &= \left(1+\frac{2^{2s+1}-2^{s+1}-1}{2^{s+2}(2^{2s+1}-2^{s-1}-1)}\right)\left(1-\frac{2^{s+1}+2}{2(2^{s+1}+1)(2^{s+1}-1)}\right)\left(1-\frac{1}{2^s}\right)^{-1} \end{aligned}$$

and the p^{th} right-hand factor simplifies to

$$\begin{aligned} &1+\frac{1}{2(p+1)}\sum_{i=1}^{\infty}\frac{p^{2i+1}+p+2}{p^{(2i)(s+1)}}+\frac{1}{2(p+1)}\sum_{j=1}^{\infty}\frac{p^{(2j-1)+1}+2p+1}{p^{(2j-1)(s+1)}} \\ &= 1+\frac{1}{2(p+1)}\left(\frac{p^{2s+3}+p^{2s+1}+2p^{2s}-2p-2}{(p^{2s+2}-1)(p^{2s}-1)}+p^{s+1}\frac{p^{2s+2}+2p^{2s+1}+p^{2s}-2p-2}{(p^{2s+2}-1)(p^{2s}-1)}\right) \\ &= \left(1-\frac{(p^{s+1}+2)(p-1)}{2(p^{s+1}+1)(p^{s+1}-1)}\right)\left(1-\frac{1}{p^s}\right)^{-1}. \end{aligned}$$

We have

$$F(s) = \zeta(s)\left(1+\frac{2^{2s+1}-2^{s+1}-1}{2^{s+2}(2^{2s+1}-2^{s-1}-1)}\right)\prod_p\left(1-\frac{(p^{s+1}+2)(p-1)}{2(p^{s+1}+1)(p^{s+1}-1)}\right) = G(s)\cdot\zeta(s)^{1/2}$$

and hence

$$\sum_{n\leq N}\frac{b(n)}{n} \sim \frac{G(1)}{\Gamma(1/2)}N\cdot(\ln N)^{-1/2} = C\cdot N\cdot(\ln N)^{-1/2}$$

where

$$\begin{aligned} C &= \frac{17}{16}\frac{1}{\sqrt{\pi}}\prod_p\left(1-\frac{p^2+2}{2(p^2+1)(p+1)}\right)\left(1-\frac{1}{p}\right)^{-1/2} \\ &= \frac{17}{16}\frac{1}{\sqrt{\pi}}(1.2569136102101885959492115...). \end{aligned}$$

It follows by partial summation that

$$\sum_{n\leq N}b(n) \sim \frac{C}{2}\cdot N^2\cdot(\ln N)^{-1/2} = (0.376...)N^2\cdot(\ln N)^{-1/2}.$$

1.9. Cubes in $\mathbb{Z}_n$. The number $b(n)$ of images under the map $y\mapsto y^3$ in $\mathbb{Z}_n$ is a multiplicative function of n, with

$$b(p^r) = \begin{cases} \frac{1}{13}\left(3^{r+1}+10\right) & \text{if } p=3 \text{ and } r\equiv 0 \bmod 3, \\ \frac{1}{13}\left(3^{r+1}+30\right) & \text{if } p=3 \text{ and } r\equiv 1 \bmod 3, \\ \frac{1}{13}\left(3^{r+1}+12\right) & \text{if } p=3 \text{ and } r\equiv 2 \bmod 3, \\ \frac{1}{p^2+p+1}\left(p^{r+2}+p+1\right) & \text{if } p\equiv 2 \bmod 3 \text{ and } r\equiv 0 \bmod 3, \\ \frac{1}{p^2+p+1}\left(p^{r+2}+p^2+p\right) & \text{if } p\equiv 2 \bmod 3 \text{ and } r\equiv 1 \bmod 3, \\ \frac{1}{p^2+p+1}\left(p^{r+2}+p^2+1\right) & \text{if } p\equiv 2 \bmod 3 \text{ and } r\equiv 2 \bmod 3, \\ \frac{1}{3(p^2+p+1)}\left(p^{r+2}+2p^2+3p+3\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } r\equiv 0 \bmod 3, \\ \frac{1}{3(p^2+p+1)}\left(p^{r+2}+3p^2+3p+2\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } r\equiv 1 \bmod 3, \\ \frac{1}{3(p^2+p+1)}\left(p^{r+2}+3p^2+2p+3\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } r\equiv 2 \bmod 3 \end{cases}$$

and

$$F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+1}} = \left(1 + \sum_{r=1}^{\infty} \frac{b(3^r)}{3^{r(s+1)}}\right) \cdot \prod_{\substack{p\equiv 2 \\ \bmod 3}} \left(1 + \sum_{r=1}^{\infty} \frac{b(p^r)}{p^{r(s+1)}}\right) \cdot \prod_{\substack{p\equiv 1 \\ \bmod 3}} \left(1 + \sum_{r=1}^{\infty} \frac{b(p^r)}{p^{r(s+1)}}\right).$$

The expressions for $b(p^r)$ follow from a conjecture by Wilson [17]; a proof for the case $p = 2$, $r \equiv 0 \bmod 3$ was given by Wilmer & Schirokauer [18]. The left-hand factor in $F(s)$ simplifies to

$$\begin{aligned}
&1 + \frac{1}{13}\left(\sum_{i=1}^{\infty} \frac{3^{3i+1}+10}{3^{(3i)(s+1)}} + \sum_{j=1}^{\infty} \frac{3^{(3j-2)+1}+30}{3^{(3j-2)(s+1)}} + \sum_{k=1}^{\infty} \frac{3^{(3k-1)+1}+12}{3^{(3k-1)(s+1)}}\right) \\
&= 1 + \left(\frac{7\cdot 27^s - 1}{(27^{s+1}-1)(27^s-1)} + 3^{2s+1}\frac{9\cdot 27^s - 7}{(27^{s+1}-1)(27^s-1)} + 3^{s+1}\frac{3\cdot 27^s - 1}{(27^{s+1}-1)(27^s-1)}\right) \\
&= \left(1 - \frac{2\left(3^{s+2}+1\right)}{(3^{s+1}+3^{(s+1)/2}+1)(3^{s+1}-3^{(s+1)/2}+1)(3^{s+1}-1)}\right)\left(1 - \frac{1}{3^s}\right)^{-1}.
\end{aligned}$$

The p^{th} right-hand factor simplifies to

$$\begin{aligned}
&1 + \frac{1}{p^2+p+1}\left(\sum_{i=1}^{\infty} \frac{p^{3i+2}+p+1}{p^{(3i)(s+1)}} + \sum_{j=1}^{\infty} \frac{p^{(3j-2)+2}+p^2+p}{p^{(3j-2)(s+1)}} + \sum_{k=1}^{\infty} \frac{p^{(3k-1)+2}+p^2+1}{p^{(3k-1)(s+1)}}\right) \\
&= 1 + \frac{1}{p^2+p+1}\left(\frac{p^{3s+5}+p^{3s+1}+p^{3s}-p^2-p-1}{(p^{3s+3}-1)(p^{3s}-1)}\right. \\
&\quad + p^{2s+2}\frac{p^{3s+3}+p^{3s+2}+p^{3s+1}-p^2-p-1}{(p^{3s+3}-1)(p^{3s}-1)} \\
&\quad \left. + p^{s+1}\frac{p^{3s+4}+p^{3s+2}+p^{3s}-p^2-p-1}{(p^{3s+3}-1)(p^{3s}-1)}\right) \\
&= \left(1 - \frac{(p^{s+1}+1)(p-1)}{(p^{s+1}+p^{(s+1)/2}+1)(p^{s+1}-p^{(s+1)/2}+1)(p^{s+1}-1)}\right)\left(1 - \frac{1}{p^s}\right)^{-1}
\end{aligned}$$

when $p \equiv 2 \bmod 3$ and

$$1 + \tfrac{1}{3(p^2+p+1)} \left(\sum_{i=1}^{\infty} \tfrac{p^{3i+2}+2p^2+3p+3}{p^{(3i)(s+1)}} + \sum_{j=1}^{\infty} \tfrac{p^{(3j-2)+2}+3p^2+3p+2}{p^{(3j-2)(s+1)}} + \sum_{k=1}^{\infty} \tfrac{p^{(3k-1)+2}+3p^2+2p+3}{p^{(3k-1)(s+1)}} \right)$$

$$= 1 + \frac{1}{3(p^2+p+1)} \left(\frac{p^{3s+5}+2p^{3s+2}+3p^{3s+1}+3p^{3s}-3p^2-3p-3}{(p^{3s+3}-1)(p^{3s}-1)} \right.$$

$$+ p^{2s+2} \frac{p^{3s+3}+3p^{3s+2}+3p^{3s+1}+2p^{3s}-3p^2-3p-3}{(p^{3s+3}-1)(p^{3s}-1)}$$

$$\left. + p^{s+1} \frac{p^{3s+4}+3p^{3s+2}+2p^{3s+1}+3p^{3s}-3p^2-3p-3}{(p^{3s+3}-1)(p^{3s}-1)} \right)$$

$$= \left(1 - \frac{(2p^{2s+2}+3p^{s+1}+3)(p-1)}{3(p^{s+1}+p^{(s+1)/2}+1)(p^{s+1}-p^{(s+1)/2}+1)(p^{s+1}-1)} \right) \left(1 - \frac{1}{p^s} \right)^{-1}$$

when $p \equiv 1 \bmod 3$. We have

$$F(s) = \zeta(s) \left(1 - \frac{2\,(3^{s+2}+1)}{(3^{s+1}+3^{(s+1)/2}+1)(3^{s+1}-3^{(s+1)/2}+1)(3^{s+1}-1)} \right)$$

$$\cdot \prod_{\substack{p\equiv 2 \\ \bmod 3}} \left(1 - \frac{(p^{s+1}+1)(p-1)}{(p^{s+1}+p^{(s+1)/2}+1)(p^{s+1}-p^{(s+1)/2}+1)(p^{s+1}-1)} \right)$$

$$\cdot \prod_{\substack{p\equiv 1 \\ \bmod 3}} \left(1 - \frac{(2p^{2s+2}+3p^{s+1}+3)(p-1)}{3(p^{s+1}+p^{(s+1)/2}+1)(p^{s+1}-p^{(s+1)/2}+1)(p^{s+1}-1)} \right)$$

$$= G(s) \cdot \zeta(s)^{2/3}$$

and hence

$$\sum_{n \le N} \frac{b(n)}{n} \sim \frac{G(1)}{\Gamma(2/3)} N \cdot (\ln N)^{-1/3} = C \cdot N \cdot (\ln N)^{-1/3}$$

where

$$C = \frac{12}{13} \frac{1}{\Gamma(2/3)} \left(1 - \frac{1}{3} \right)^{-1/3}$$

$$\cdot \prod_{\substack{p\equiv 2 \\ \bmod 3}} \left(1 - \frac{p^2+1}{(p^2+p+1)(p^2-p+1)(p+1)} \right) \left(1 - \frac{1}{p} \right)^{-1/3}$$

$$\cdot \prod_{\substack{p\equiv 1 \\ \bmod 3}} \left(1 - \frac{2p^4+3p^2+3}{3(p^2+p+1)(p^2-p+1)(p+1)} \right) \left(1 - \frac{1}{p} \right)^{-1/3}$$

$$= \frac{12}{13} \frac{1}{\Gamma(2/3)} (1.4225831466986636811460982...).$$

It follows by partial summation that

$$\sum_{n\le N} b(n) \sim \frac{C}{2}\cdot N^2\cdot(\ln N)^{-1/3} = (0.484...)N^2\cdot(\ln N)^{-1/3}.$$

We emphasize that this result is only conjectural.

1.10. Other Problems. The power of the Selberg-Delange method is evident (many deeper applications occur elsewhere in the literature). We merely mention that the number $a(n)$ of solutions of $x^2 = -1$ in $\mathbb{Z}_n^*$ satisfies

$$\sum_{n\le N} a(n) \sim \frac{3}{2\pi}N;$$

in particular, $x^2 = -1$ has asymptotically far fewer solutions than $x^2 = 1$. Such asymmetry does not occur for $x^3 = \pm 1$ (just replace x by $-x$). See other modular polynomial equations at [19] and the enumeration of weakly primitive Dirichlet characters at [20, 21].

A more difficult exercise concerns the number $b(n)$ of elements of $\mathbb{Z}_n$ that are *both* squares and cubes. If $w = z^6$, then clearly $w = (z^3)^2 = (z^2)^3$. Conversely, if $w = u^2 = v^3$, then $(uv^{-1})^6 = (u^2)^3(v^3)^{-2} = w^3w^{-2} = w$. Hence $b(n)$ is the same as the number of sixth-powers in $\mathbb{Z}_n$. Wilson's conjecture again provides expressions for $b(p^r)$, which in turn give formulas for $F(s)$ and $G(s)$. The details of this and other higher-power problems are left to someone else [22].

2. Numerical Techniques

2.1. Prime Products. Here is a method for evaluating constants of the form

$$C = \prod_{p \equiv \ell \bmod k} f(p)$$

to high precision, where the product is taken over all primes of the form $p = mk + \ell$. Suppose that the function $\ln f$ has asymptotic expansion

$$\ln f(p) = \frac{c_2}{p^{s_2}} + \frac{c_3}{p^{s_3}} + \cdots + \frac{c_n}{p^{s_n}} + \cdots$$

as $p \to \infty$, where (c_n, s_n) are real numbers and $1 < s_2 < ... < s_n < ....$ (Often $s_n = n$ occurs.) Define the $(k, \ell)^{\text{th}}$ *prime zeta function*

$$P_{k,\ell}(s) = \sum_{p \equiv \ell \bmod k} \frac{1}{p^s}$$

for $\mathrm{Re}(s) > 1$; it follows that

$$\ln C = \sum_{p \equiv \ell \bmod k} \ln f(p) = \sum_{n \geq 2} c_n P_{k,\ell}(s_n).$$

Let $p_{k,\ell}$ denote the smallest prime of the form $mk + \ell$; clearly $P_{k,\ell}(n) \sim 1/p_{k,\ell}^n$ as $n \to \infty$. Consequently, if the coefficients c_n are uniformly bounded, the convergence of the sum is fast (geometric). It hence remains to accurately compute the values $P_{k,\ell}(s_n)$.

2.2. Prime Zeta Functions. Let $\mathrm{Re}(s) > 1$. The classical prime zeta function $P(s) = P_{1,0}(s)$ can be related to the classical zeta function by Euler's famous product:

$$\ln \zeta(s) = -\sum_{p} \ln\left(1 - \frac{1}{p^s}\right) = \sum_{p} \sum_{n \geq 1} \frac{1}{np^{ns}} = \sum_{n \geq 1} \frac{P(ns)}{n}.$$

Applying the Möbius inversion formula, we obtain [23, 24]

$$P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln \zeta(ns).$$

Since $\ln \zeta(ns) \sim 2^{-ns}$ as $n \to \infty$, only a few terms in this series are required to compute an accurate value of $P(s)$. Also $P(s) \sim -\ln(s-1)$ as $s \to 1^{+}$. These facts are useful in computing constants of the form $\prod_p f(p)$.

For constants of the form $\prod_{p\equiv\ell \bmod 3} f(p)$, we need $P_{3,1}(s)$ and $P_{3,2}(s)$. To achieve this, it is necessary to introduce the two characters modulo 3:

$$\chi_0(n)=\begin{cases}1 & \text{if } n\equiv 1 \bmod 3,\\ 1 & \text{if } n\equiv 2 \bmod 3,\\ 0 & \text{if } n\equiv 0 \bmod 3,\end{cases}\qquad \chi_1(n)=\begin{cases}1 & \text{if } n\equiv 1 \bmod 3,\\ -1 & \text{if } n\equiv 2 \bmod 3,\\ 0 & \text{if } n\equiv 0 \bmod 3,\end{cases}$$

and their associated Dirichlet L-series:

$$L_j(s)=L_{\chi_j}(s)=\sum_{n=1}^{\infty}\frac{\chi_j(n)}{n^s}=\frac{1}{3^s}\left(\chi_j(1)\zeta\left(s,\tfrac{1}{3}\right)+\chi_j(2)\zeta\left(s,\tfrac{2}{3}\right)\right),\qquad j=0,1$$

where $\zeta(s,a)$ is the Hurwitz zeta-function. By well-known acceleration procedures, series of this nature can be evaluated to many decimal places.

From the Euler product expressions

$$L_0(s)=\prod_p\left(1-\frac{\chi_0(p)}{p^s}\right)^{-1}=\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{p\equiv 2 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-1},$$

$$L_1(s)=\prod_p\left(1-\frac{\chi_1(p)}{p^s}\right)^{-1}=\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{p\equiv 2 \bmod 3}\left(1+\frac{1}{p^s}\right)^{-1},$$

we obtain

$$\frac{1}{2}\ln\left(\frac{L_0(s)}{L_1(s)}\right)=\frac{1}{2}\sum_{p\equiv 2 \bmod 3}\ln\left(\frac{1+p^{-s}}{1-p^{-s}}\right)=\sum_{n=0}^{\infty}\frac{P_{3,2}((2n+1)s)}{2n+1},$$

$$\frac{1}{2}\ln\left(\frac{L_0(s)L_1(s)}{L_0(2s)}\right)=\frac{1}{2}\sum_{p\equiv 1 \bmod 3}\ln\left(\frac{1+p^{-s}}{1-p^{-s}}\right)=\sum_{n=0}^{\infty}\frac{P_{3,1}((2n+1)s)}{2n+1}$$

and, again, by Möbius inversion,

$$P_{3,2}(s)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\mu(2n+1)}{2n+1}\ln\left(\frac{L_0((2n+1)s)}{L_1((2n+1)s)}\right),$$

$$P_{3,1}(s)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\mu(2n+1)}{2n+1}\ln\left(\frac{L_0((2n+1)s)L_1((2n+1)s)}{L_0((4n+2)s)}\right).$$

Of course, $L_0(s)=\zeta(s)(1-1/3^s)$ and $L_1(s)=1-1/2^s+1/4^s-1/5^s+\cdots$. Also, $P_{3,1}(s)\sim-\frac{1}{2}\ln(s-1)$ and $P_{3,2}(s)\sim-\frac{1}{2}\ln(s-1)$ as $s\to 1^+$.

Similar techniques involving characters modulo k can be used to compute constants of the form $\prod_{p\equiv\ell \bmod k} f(p)$, but for brevity's sake we do not discuss these here.

2.3. A Simple Example. Let us compute the constant

$$C = \prod_{p\equiv 1 \bmod 3}\left(1-\frac{2}{p(p+1)}\right) = \prod_{p\equiv 1 \bmod 3}\left(\frac{(p-1)(p+2)}{(p+1)p}\right)$$

that appears in section [1.3]. It is easy to establish that

$$\ln C = \sum_{p\equiv 1 \bmod 3}\left(\ln\left(\frac{p-1}{p+1}\right)+\ln\left(1+\frac{2}{p}\right)\right) = \sum_{n\geq 2}\frac{c_n}{n}P_{3,1}(n)$$

where $c_n = 2^n - 2$ when n is odd and $c_n = -2^n$ when n is even. Since $c_n = O(2^n)$ and $P_{3,1}(n) = O(7^{-n})$, it is more efficient to compute directly the product up to a certain cutoff p_c. For example, if we take $p_c = 31$, we find

$$C = \frac{3247695}{3430336}\prod_{\substack{p\equiv 1 \bmod 3,\\ p>31}}\left(1-\frac{2}{p(p+1)}\right)$$

and consequently

$$\ln C = \ln\frac{3247695}{3430336} + \sum_{n=2}^{\infty}\frac{c_n}{n}\left(P_{3,1}(n)-\frac{1}{7^n}-\frac{1}{13^n}-\frac{1}{19^n}-\frac{1}{31^n}\right)$$

enjoys much faster convergence because $P_{3,1}(n) - \frac{1}{7^n} - \frac{1}{13^n} - \frac{1}{19^n} - \frac{1}{31^n} = O(37^{-n})$. The first few terms of this series produce

n	C
2	0.94(0943837952389629219520 6...)
3	0.94103(87732177050567463275...)
4	0.941034(8096648041499806620...)
5	0.94103494(70255355752383278...)
10	0.94103494131953(43277214763...)
15	0.9410349413195354517 90(3566...)

and only 15 terms are necessary to obtain 20 correct decimal places.

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