Inner Product Spaces and Normed Vector Spaces
(A004)

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Abstract

A vector space over a feild endowed with additional informatoin on how to take two vectors to the field enables a concept of length and angles to be defined in the vector space. This addition information is the inner product. This paper presents the basic concepts surrounding the inner product on a vector space. Extending from this a normed vector space is discussed representing a concepth of length.

1 Definition and Axioms

We can define an **inner product space** as a vector space V over a field F that has an inner product defined. The inner product is a map:

$$\langle \cdot, \cdot \rangle : V \times V \to F \tag{1}$$

This map must satisfy three properties $\forall v \in V$

$$\text{conjugate symmetry: } \langle a, b\rangle = \overline{\langle b, a\rangle} \tag{2}$$
$$\text{Linearity: } \alpha\langle a, b\rangle = \langle \alpha a, b\rangle,\ \ \alpha \in F \tag{3}$$
$$\langle a + b, c\rangle = \langle a, c\rangle + \langle b, c\rangle \tag{4}$$
$$\text{Positive definite: } \langle a, a\rangle > 0\ \ \forall a \in V\backslash\vec{0} \tag{5}$$

1.1 Eucledian Space

$$a, b \in \mathbb{R}^n,\ \ \langle a, b\rangle = \sum_{i=1}^{n} a_i b_i \tag{6}$$

This is a special case of an inner product in $\mathbb{R}^n$ and is called the dot product. Here $\langle a, b\rangle = a \cdot b$.

Consider a unit vector $\vec{e}_x$ along the x-axis of a Eucledian plan and $\vec{e}_y$ along the y axis. Here we know that $\vec{e}_i \cdot \vec{e}_j = \delta_i^j$. Rotate $\vec{e}_y$ to an angle θ form $\vec{e}_x$ to form a new set of basis vectors $\vec{e}_r$ and $\vec{e}_s$.

$$x = rcos(\theta) \qquad y = rsin(\theta) \tag{7}$$

Under these coordinate transformations the new basis vectors are:

$$\vec{e}_r = \frac{\partial}{\partial r} rcos(\theta)\vec{e}_x + \frac{\partial}{\partial r} rsin(\theta)\vec{e}_y = cos(\theta)\vec{e}_x + sin(\theta)\vec{e}_y \tag{8}$$
$$\vec{e}_s = \vec{e}_x \tag{9}$$

Consdier a vector V pointing along $\vec{e}_s$ and vector W pointing along $\vec{e}_r$ such that $\|V\| = V^s$ and $\|W\| = W^r$.

$$V \cdot W = V^s \vec{e}_s \cdot W^r \vec{e}_r = V^s W^r (\vec{e}_s \cdot \vec{e}_r) \tag{10}$$

From equation 8 and equation 9 $\vec{e}_s \cdot \vec{e}_r = cos(\theta)$. Applying this result to equation 10 yields:

$$V.W = \|V\|\|W\|\ cos(\theta) \tag{11}$$

Equation 11 says $V \cdot V = \|V\|^2$ and that for $V, W \neq 0$ $V \cdot W = 0 \implies V$ and W are perpendicular vectors. This definition can be carried into other inner product spaces.

2 Normed Vector Spaces

A normed vector space V over a field F is space have a nonnegative valued function, a norm, $p : V \to F$. p has the following axioms.

$$p(\vec{a} + \vec{b}) \leq p(\vec{a}) + p(\vec{b}) \tag{12}$$

$$p(\alpha\vec{a}) = |\alpha|p(\vec{a}) \tag{13}$$

$$p(\vec{a}) = 0 \implies \vec{v} = \vec{0} \tag{14}$$

If a function satisfies the above with the acception of equation 14 then it is a seminorm. The notation for a norm may be $\|\vec{a}\|$. Not all normed vector spaces are an inner product space.

2.1 Eucledian Norm

The may be referred to as the L^2 norm. L^p is a function space defined with a p-norm for a finite dimension.

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n}(x_i)^2} = \sqrt{\vec{x} \cdot \vec{x}} \tag{15}$$

2.2 Manhatton norm

This is the L^1 norm.

$$\|x\|_1 = \sum_{i=1}^{n}|x_i| \tag{16}$$

This is simply the absolute sum of the components of the vector.