

Vector Spaces

(A003)

Paul Valentine

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1 Introduction

A vector space is a collection of objects called **vectors** that can be scaled and added together in a linear way. It is therefore a linear space. The scaling and addition rules for a vector space form a set of axioms. Roughly speaking, the number of independent directoins is the dimension of the vector space. Two operations are defined on this space \cdot and $+$.

2 Operations \cdot and $+$

Let V be an ordered set.

$$+ : V \times V \rightarrow V \quad (1)$$

Equation 1 defines the addition rule where \times is the cartesian production [1] and V is a set. For example.

$$+ : ((x_1, y_1), (x_2, y_2)) = ((x_1 + x_2, y_1 + y_2)) \quad (2)$$

With a **field** [2] F we have.

$$\cdot : F \times V \rightarrow V \quad (3)$$

Where \cdot represents scalar multiplication. If F is a complex field then we have a complex vector space. If F is a real number field then we have a real vector space. A field is a set on which addition, subtraction, multiplication and division are defined. That it is, it is an abelian group with 0 as the additive identity and 1 as the multiplicative identity. Example:

$$\forall a \in \mathbb{R} : (a, (x_1, y_1)) = ((ax_1, ay_1)) \quad (4)$$

Examples of field are $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and \mathbb{Q} .

Coupling the above we say that a vector space V is over a field F .

3 Axioms

For the set V to form a vector space the operations $+$ and \cdot must conform to the following 8 axioms:

$$\text{Associativity of addition: } \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \quad (5)$$

$$\text{Commutativity of addition: } \hat{u} + \hat{v} = \vec{v} + \vec{u} \quad (6)$$

$$\text{Identity element of addition: } \exists \vec{0} \in V \mid \hat{v} + \vec{0} = \hat{v} \forall \hat{v} \in V \quad (7)$$

$$\text{Inverse element of addition: } \forall \vec{v} \in V \exists -\vec{v} \mid \vec{v} + -\vec{v} = \vec{0} \quad (8)$$

$$\text{Compatibility of scalar and field multiplication: } a(b\vec{v}) = (ab)\vec{v} \quad (9)$$

$$\text{Identity element of scale multiplication: } 1\vec{v} = \vec{v} \quad (10)$$

$$\text{Distributivity of scalar multiplicatoin w.r.t vector addition: } a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} \quad (11)$$

$$\text{Ditributivity of scalar multiplication w.r.t field addition: } (a + b)\vec{v} = a\vec{v} + b\vec{v} \quad (12)$$

4 Examples

4.1 Coordinates spaces

The simplest form a vector space is a field itself giving n tuples (a_1, a_2, \dots, a_n) .

4.2 Function Space

Let V be a vector space over a field F and X be any set. The set $f : X \rightarrow V$ are a vector space over F . Typically here $F, V \in \mathbb{R}$ as a common example.

$$f, g : X \rightarrow V : \forall x \in X, \forall a \in F \quad (13)$$

$$(f + g)(x) = f(x) + g(x) \quad (14)$$

$$(a \cdot f)(x) = a \cdot f(x) \quad (15)$$

If X is also a vector space the set of linear maps $X \rightarrow V$ form a vector space over F called $Hom(X, V)$. A special case is the set of linear functions $V \rightarrow F$ which form a **dual** vector space.

4.3 Integration [3]

The collectoin of integrable functions form a vector space.

$$f \mapsto \int_a^b f(x)dx \quad (16)$$

$$\int_a^b (\alpha f + \beta g)(x)dx = \int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \quad (17)$$

Since the function itself forms a vector space the integrable function takes the function to the field over which the integration occurs. Since this is $I : V \rightarrow F$ it places the integrable functions (I) in the dual vector space.

□

References

- [1] P. Valentine, *Cartesian Products*. Paul Valentine, 2020.
- [2] Wikipedia contributors, “Field (mathematics) — Wikipedia, the free encyclopedia,” 2020. [Online; accessed 19-March-2020].
- [3] Wikipedia contributors, “Integral — Wikipedia, the free encyclopedia,” 2020. [Online; accessed 19-March-2020].