Vector Spaces
(A003)

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Abstract

Vector spaces appear in many forms of mathematics. Indeed $\mathbb{R}$ forms a vector space. Vectors are elements of a vector space. Understanding what defines a vector space can lead to some suprising catagorisations of vectos that are not seen in undergraduate mathematics. This paper provides an first touch introduction to the what defines a vector space and some comment elements of a vector space that are encounted in everyday mathematics and physics.

1 Introduction

A vector space is a collection of objects called **vectors** that can be scaled and added together in a linear way. It is therefore a linear space. The scaling and addition rules for a vector space form a set of axioms. Roughly speaking, the number of independent elements is the dimension of the vector space. Two operations are defined on this space $\cdot$ and $+$. This paper describes the basics of vectors spaces.

2 Operations $\cdot$ and $+$

Let V be an ordered set.

$$+ : V \times V \to V \tag{1}$$

Equation 1 defines the addition rule where $\times$ is the cartesian production [1] and V is a set. For example.

$$+ : \big((x_1, y_1), (x_2, y_2)\big) = \big((x_1 + x_2, y_1 + y_2)\big) \tag{2}$$

With a **field** [2] F we have.

$$\cdot : F \times V \to V \tag{3}$$

Where $\cdot$ represents scalar multiplication. If F is a complex field then we have a complex vector space. If F is a real number field then we have a real vector space. A field is a set on which addition, subtraction, multiplication and division are defined. That it is, it is an albelian group with 0 as the additive identity and 1 as the multiplicative identity. Example:

$$\forall a \in \mathbb{R} \cdot : \big(a, (x_1, y_1)\big) = \big((ax_1, ay_1)\big) \tag{4}$$

Examples of fields are $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{Q}$.

Coupling the above we say that a vector space V is over a field F.

3 Axioms

For the set V to form a vector space the operations $+$ and $\cdot$ must conform to the following 8 axioms:

$$\text{Associativity of addition: } \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{5}$$
$$\text{Commutativity of addition: } \hat{u} + \hat{v} = \vec{v} + \vec{u} \tag{6}$$
$$\text{Identity element of addition: } \exists\, \vec{0} \in V \mid \hat{v} + \vec{0} = \hat{v}\, \forall\, \hat{v} \in V \tag{7}$$
$$\text{Inverse element of addition: } \forall\, \vec{v} \in V\, \exists\, -\vec{v} \mid \vec{v} + -\vec{v} = \vec{0} \tag{8}$$
$$\text{Compatibility of scalar and field multiplication: } a(b\vec{v}) = (ab)\vec{v} \tag{9}$$
$$\text{Identity element of scale multiplication: } 1\vec{v} = \vec{v} \tag{10}$$
$$\text{Distributivity of scalar multiplicatoin w.r.t vector addition: } a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} \tag{11}$$
$$\text{Ditributivity of scalar multiplication w.r.t field addition: } (a + b)\vec{v} = a\vec{v} + b\vec{v} \tag{12}$$

4 Examples

4.1 Coordinates spaces

The simplest form a vector space is a field itself giving n tuples $(a_1, a_2, \ldots, a_n)$.

4.2 Function Space

Let V be a vector space over a field F and X be any set. Then for $f : X \to V$, the set of f forms a vector space over F. Typically here $F, V \in \mathbb{R}$ as a common example.

$$f, g : X \to V : \forall x \in X, \forall a \in F \tag{13}$$
$$(f + g)(x) = f(x) + g(x) \tag{14}$$
$$(a \cdot f)(x) = a \cdot f(x) \tag{15}$$

If X is also a vector space the set of linear maps $X \to V$ form a vector space over F called $Hom(X, V)$. A special case is the set of linear functions $V \to F$ which form a **dual** vector space.

4.3 Integration [3]

The collectoin of integrable functions form a vector space from the above.

$$f \mapsto \int_a^b f(x)dx \tag{16}$$
$$\int_a^b (\alpha f + \beta g)(x)dx = \int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \tag{17}$$

Since the function itself forms a vector space $f(x)dx$ takes the function to the field over which the integration occurs. Since this is $I : V \to F$ it places $f(x)dx$ (I) in the dual vector space.

5 Basis Vectors

A set of vectors $B \subseteq V$ (Vector space) over a field F is a linearly independent set that spans V [4]. Spans means is able to describe all elements of V. Practically these means that all other elements of V can be written as linear function of the elements in B.

$$\text{Linear} \implies \forall b \in B, \forall a \in F, \sum_{i=1}^{n} a_i b_i = 0 \implies a_i = 0 \tag{18}$$
$$\text{Spans} \implies \forall v \in V\, \exists\, a \in F, b \in B \mid v = \sum_{i=1}^{n} a_i b_i \tag{19}$$

As an example, for a vector space of $\mathbb{R}^2$ over field $\mathbb{R}$ the basis vectors are $(1, 0)$ and $(0, 1)$.

$\square$

References

[1] P. Valentine, *Cartesian Products*. Paul Valentine, 2020.

[2] Wikipedia contributors, "Field (mathematics) — Wikipedia, the free encyclopedia," 2020. [Online; accessed 19-March-2020].

[3] Wikipedia contributors, "Integral — Wikipedia, the free encyclopedia," 2020. [Online; accessed 19-March-2020].

[4] Wikipedia contributors, "Basis (linear algebra) — Wikipedia, the free encyclopedia," 2020. [Online; accessed 20-March-2020].