Taylor Series Expansions

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Abstract

Taylor series expands a function about a point a as a polynomial. This is useful as an analytical tool to make approximations about the function at a point.

1 Derivation of the single variable case

Consider estimate a function $f(x)$ local to the point $f(a)$. Take the linear approximation.

$$f(x) = f(a) + f'(a)(x - a) \tag{1}$$

To improve accuracy consider a polynomial expansion

$$f(x) = \sum_{n=0}^{n=\infty} c_n (x - a)^n \tag{2}$$

The problem then becomes how to calculate the coefficients. To test a solution differentiate an expansion in a set number of terms and then differentiate.

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 \cdots \tag{3}$$

$$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 \cdots \tag{4}$$

$$f''(x) = 2c_2 + 6c_3(x - a) + 12c_4(x - a)^2 \cdots \tag{5}$$

$$f'''(x) = 6c_3 + 24(x - a) \cdots \tag{6}$$

By setting $x = a$ in each of the above we see that we isolate the coefficient that is the degree of the derivative.

$$c_0 = f(a) \tag{7}$$

$$c_1 = f'(a) \tag{8}$$

$$c_2 = \frac{f''(a)}{2!} \tag{9}$$

$$c_3 = \frac{f'''(a)}{3!} \cdots \; c_n = \frac{f^{(n)}(a)}{n!} \tag{10}$$

Using these into equation 3 we have

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 \ldots \tag{11}$$

Equation 11 can be recast in the form of equation 2.

$$f(x) = \sum_{n=0}^{n=\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n \tag{12}$$

If the series is summed to a finite value k then this only represents an approximation. We must therefore add an additional function $h_k(x)(x - a)^k$ which is a remainder that must have the form $\lim_{x \to a} h_k(x) = 0$ such that:

$$f(x) = \sum_{n=0}^{n=k} \frac{f^{(n)}(a)}{n!}(x - a)^n + h_k(x)(x - a)^k \tag{13}$$

Nothing is stated in this paper about calculating the remainder.