Vector Spaces
(A003)

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1 Introduction

A vector space is a collection of objects called **vectors** that can be scaled and added together in a linear way. It is therefore a linear space. The scaling and addition rules for a vector space form a set of axioms. Roughly speaking, the number of independent directoins is the dimension of the vector space. Two operations are defined on this space $\cdot$ and $+$.

2 Operations $\cdot$ and $+$

Let V be an ordered set.

$$+ : V \times V \to V \tag{1}$$

Equation 1 defines the addition rule where $\times$ is the cartesian production [1] and V is a set. For example.

$$+ : \big((x_1, y_1), (x_2, y_2)\big) = \big((x_1 + x_2, y_1 + y_2)\big) \tag{2}$$

With a **field** [2] F we have.

$$\cdot : F \times V \to V \tag{3}$$

Where $\cdot$ represents scalar multiplication. If F is a complex field then we have a complex vector space. If F is a real number field then we have a real vector space. A field is a set on which addition, subtraction, multiplication and division are defined. That it is, it is an albelian group with 0 as the additive identity and 1 as the multiplicative identity. Example:

$$\forall a \in \mathbb{R} \cdot : \big(a, (x_1, y_1)\big) = \big((ax_1, ay_1)\big) \tag{4}$$

Examples of field are $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{N}, \mathbb{Q}$.

3 Axioms

For the set V to form a vector space the operations $+$ and $\cdot$ must conform to the following 8 axioms:

$$\text{Associativity of addition: } \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{5}$$

$$\text{Commutativity of addition: } \hat{u} + \hat{v} = \vec{v} + \vec{u} \tag{6}$$

$$\text{Identity element of addition: } \exists\, \vec{0} \in V \mid \hat{v} + \vec{0} = \hat{v} \,\forall\, \hat{v} \in V \tag{7}$$

$$\text{Inverse element of addition: } \forall\, \vec{v} \in V \,\exists\, -\vec{v} \mid \vec{v} + -\vec{v} = \vec{0} \tag{8}$$

$$\text{Compatibility of scalar and field multiplication: } a(b\vec{v}) = (ab)\vec{v} \tag{9}$$

$$\text{Identity element of scale multiplication: } 1\vec{v} = \vec{v} \tag{10}$$

$$\text{Distributivity of scalar multiplicatoin w.r.t vector addition: } a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} \tag{11}$$

$$\text{Ditributivity of scalar multiplication w.r.t field addition: } (a + b)\vec{v} = a\vec{v} + b\vec{v} \tag{12}$$

References

[1] P. Valentine, *Cartesian Products*. Paul Valentine, 2020.

[2] Wikipedia contributors, "Field (mathematics) — Wikipedia, the free encyclopedia," 2020. [Online; accessed 19-March-2020].