TP 5 - IMA 205

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1 PCA

Question 1

If the matrix of data X is centered, the covariance matrix is $C = \frac{1}{N-1} X_C^T X_C$ instead of

$$C = \frac{1}{N-1}(X - \bar{X})^T (X - \bar{X})$$

so it's much more easy to compute.

We also need to center the data to compute the SVD.

Question 2

Let x_p and x_q be two row-vectors representing two images, U an orthogonal matrix whose columns are the eigenvectors of X and $y_p = x_p U$, $y_q = x_q U$.

Then $x_p x_q^T = x_p (UU^T) x_q^T = (x_p U)(x_q U)^T = y_p y_q^T$.

Question 3

Question 4

Let C be the covariance matrix of X and $C = UDU^T$ its eigen decomposition. Then,

$$Cov(Y) = Cov(XU) = U^T Cov(X) U = U^T UDUU^T = D$$

2 KPCA

Question 1

As we are in a space of biger dimension, the eigenvector are no longer representable in the initial space.

3 ICA

Question 1

Let's have $X_W = D^{\frac{-1}{2}} U^T X_C = W X_C$.
Then $W^T W = U D^{\frac{-1}{2}} D^{\frac{-1}{2}} U^T = D^{-1}$
And

$$E[X_W X_W^T] = E[D^{\frac{-1}{2}} U^T X_C X_C^T U D^{\frac{-1}{2}}] = E[D^{\frac{-1}{2}} U^T \sum U D^{\frac{-1}{2}}] = E[D^{\frac{-1}{2}} D D^{\frac{-1}{2}}] = I$$

Question 2

The whitening process is simply a linear change of coordinate of the mixed data. It restores the initial "shape" of the data and then ICA must only rotate the resulting matrix. It removes any correlation in the data.

Question 3

We have $X_W = A_W S_W = A_W W_W X_W$. So $A_W = W_W^{-1}$
And, $X_C = \mathcal{W}^{-1} X_W = \mathcal{W}^{-1} W_W^{-1} S_W$
So $A_C = W^{-1} W_W^{-1}$

4 NNMF

Question 1

Let's resolve $minf(x)$ s.t $x \geq 0$

The KKT conditions are :

$x \geq 0$
$\nabla f(x) - \mu = 0$
$\mu \geq 0$
$-\mu x = 0$

That is equivalent to $x \geq 0$, $\nabla f(x) \geq 0$, $\nabla f(x) x = 0$.

Question 2

The non-negativity constraint naturally leads to sparse factors because we have always $\nabla f(x) = 0$ or $x = 0$. We are looking for a minimum where $\nabla f(x) = 0$ but if this minimum is in $x \leq 0$, the chosen x with be x=0. So the non-negativity constraint leads to a lot of x=0 : sparsity.