Equality constraint

$f(x, y) = x - y$

$g(x, y) = x^2 + y^2 - 1 \quad = 0$

KKT conditions :

On a donc $L(x, y, \lambda) = x - y + \lambda x^2 + \lambda y^2 - \lambda$

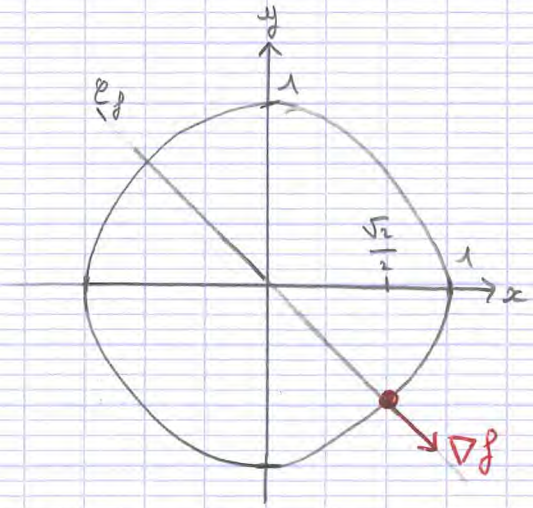
$\nabla_x L = 0 \iff 1 + 2\lambda x = 0 \iff x = -\frac{1}{2\lambda}$

$\nabla_y L = 0 \iff -1 + 2\lambda y = 0 \iff y = \frac{1}{2\lambda}$

et $x^2 + y^2 - 1 = 0$

Donc $2 \times \frac{1}{4\lambda^2} = 1 \qquad \lambda^2 = \frac{1}{2} \qquad \lambda = -\frac{1}{\sqrt{2}}$

cad $\boxed{x^* = \frac{\sqrt{2}}{2} \qquad y^* = -\frac{\sqrt{2}}{2}}$



Inequality constraint

$f(x, y) = xy$

$g(x, y) = 4 - x^2 - y^2 \geq 0$

$\mathcal{L}(x, y, \lambda) = xy + 4\lambda - \lambda x^2 - \lambda y^2$

$\nabla_x \mathcal{L} = 0 \iff y - 2\lambda x = 0 \quad (\Leftrightarrow) \; x = \frac{y}{2\lambda} \quad \lambda \neq 0$

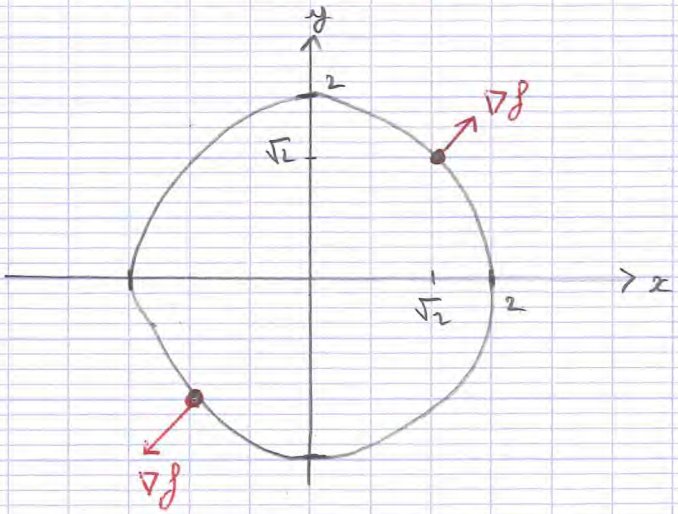
$\nabla_y \mathcal{L} = 0 \quad (\Leftrightarrow) \ldots \qquad y = \frac{x}{2\lambda}$

Donc $\lambda = \frac{1}{2}$ et $x = y$ (car $\lambda \geq 0$)

$\lambda g(x) = 0$ donc $g(x) = 0$ $(\lambda \neq 0)$

donc $4 - 2x^2 = 0$

et $\boxed{x^* = y^* = \pm\sqrt{2}}$



3) $f(x,y) = \ln(x) + y \qquad x > 0$

$h(x,y) = x - y = 0$

$g(x,y) = 4 - x^2 + y^2 \geq 0$

KKT

$\mathcal{L}(x, y, \lambda_1, \lambda_2) = \ln(x) + y + \lambda_1 x - \lambda_1 y - \lambda_2 x^2 - \lambda_2 y^2 + 4\lambda_2$

$\nabla_x \mathcal{L} = 0 \iff \frac{1}{x} + \lambda_1 - 2\lambda_2 = 0$ (1)

$\nabla_y \mathcal{L} = 0 \iff 1 - 2\lambda_2 y - \lambda_1 \iff y = \frac{1-\lambda_1}{2\lambda_2}$

- $\lambda_2 \geq 0$
- $4\lambda_2 - \lambda_2 x^2 - \lambda_2 y^2 = 0$
- $x = y$

- $\lambda_2 \neq 0$ alors $x = \pm\sqrt{2}$, $x = \sqrt{2}$ car $x > 0$

- Si $\lambda_2 = 0$ $\quad x = -\frac{1}{\lambda_1} \Rightarrow x = -1$ IMPOSSIBLE.

Donc $S = (\sqrt{2}, \sqrt{2})$