Non-Robustness of Diffusion Estimates on Networks with Measurement Error

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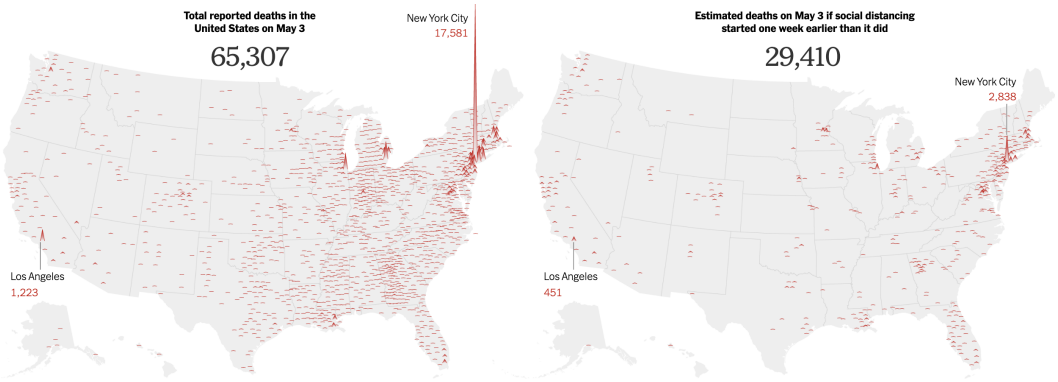
Models of Diffusion on Networks

- Researchers and policymakers studying the spread of ideas, technology, or disease often estimate models of diffusion using network data
  1. quantifying the extent of illness or technology take-up;
  2. summarizing diffusion dynamics (e.g., $\mathcal{R}_0$ of a disease);
  3. targeting interventions
     - where to seed new information to maximize spread
     - where to lockdown to prevent spread
  4. estimating counterfactuals
     e.g., estimates of peer effects.
- Goal today: show how tiny errors in network data can lead to large errors in diffusion models
- Natural example: SEIRD models during Covid-19

Counterfactual predictions of Covid-19 infections + deaths



By Lazaro Gamio · Source: "Differential Effects of Intervention Timing on COVID-19 Spread in the United States," by Sen Pei, Sasikiran Kandula and Jeffrey Shaman, Columbia University

Modelers find that tens of thousands of U.S. deaths could have been prevented.

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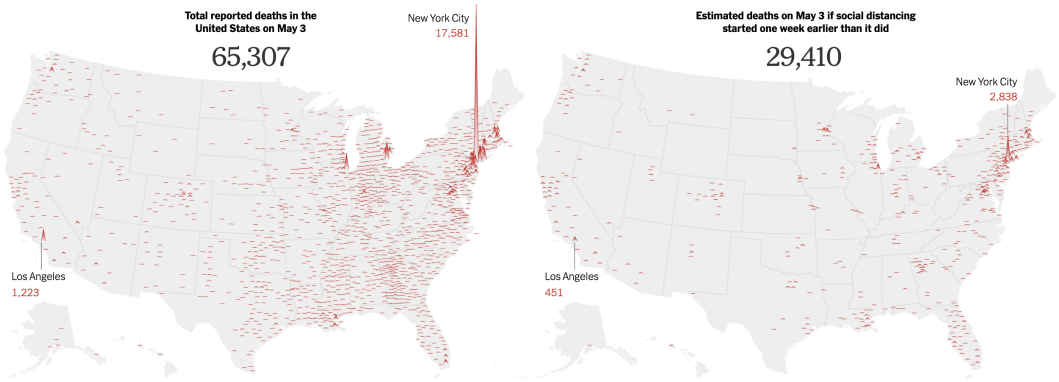

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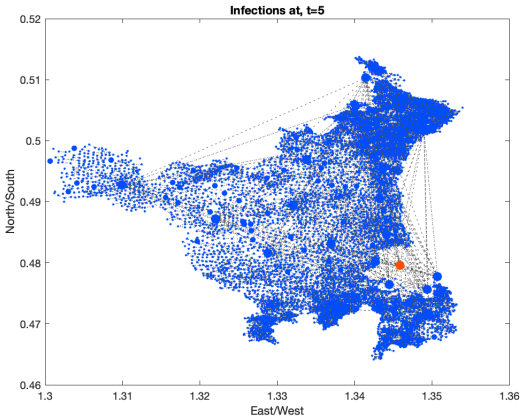
How did they model this?

Example: Rich Covid-19 SIERD Model in India

Haryana



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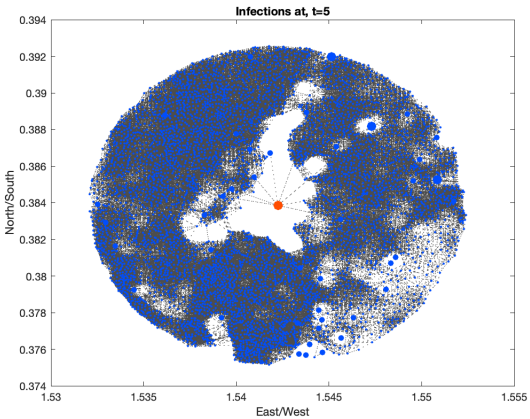

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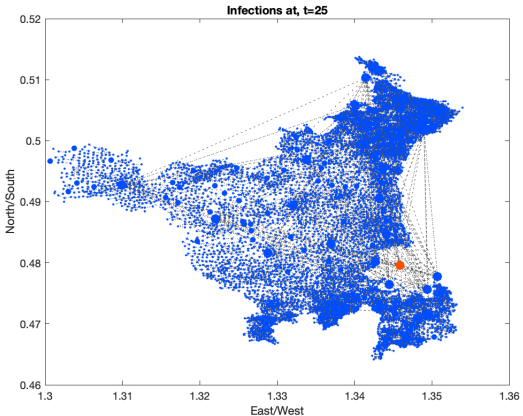

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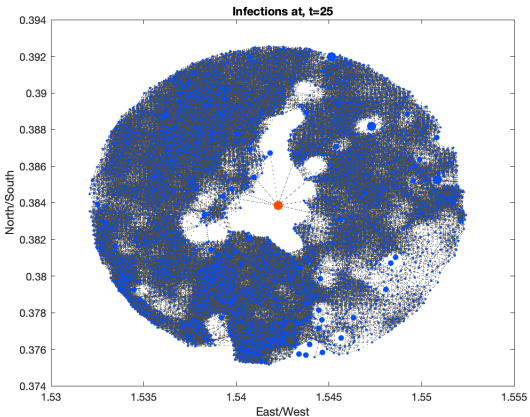

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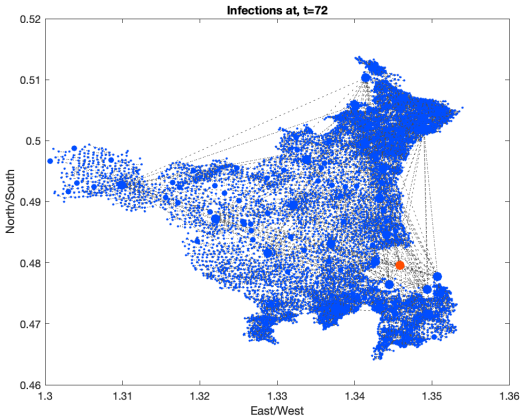

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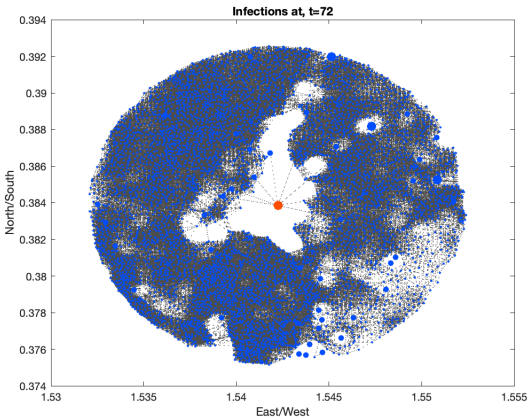

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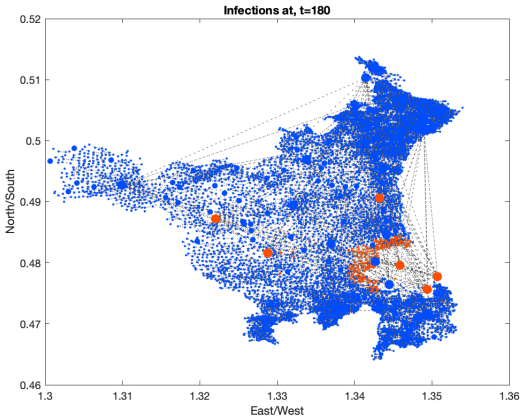

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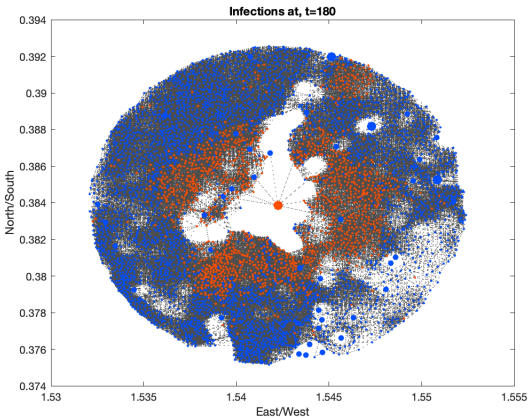

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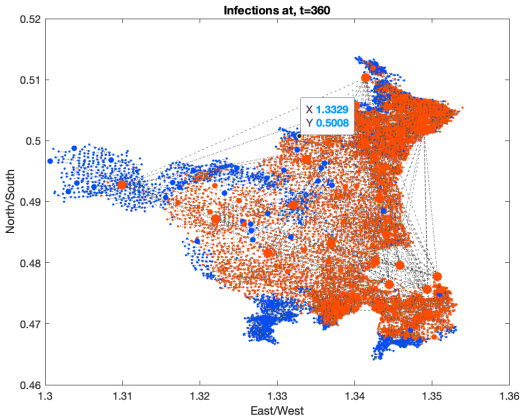

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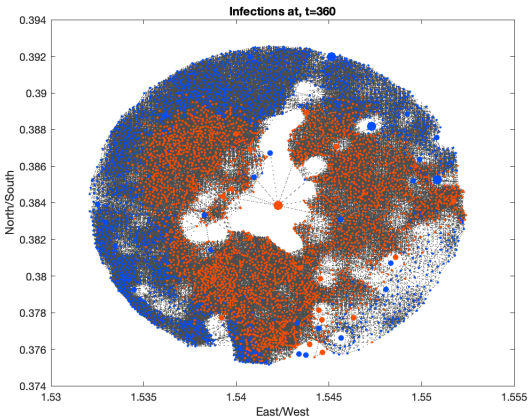
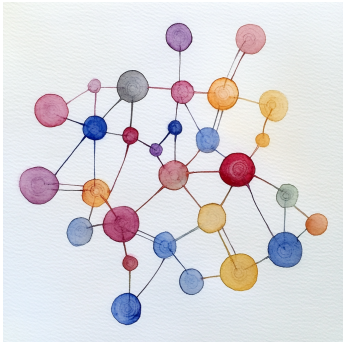


Two ingredients for diffusion on a network

Network connections

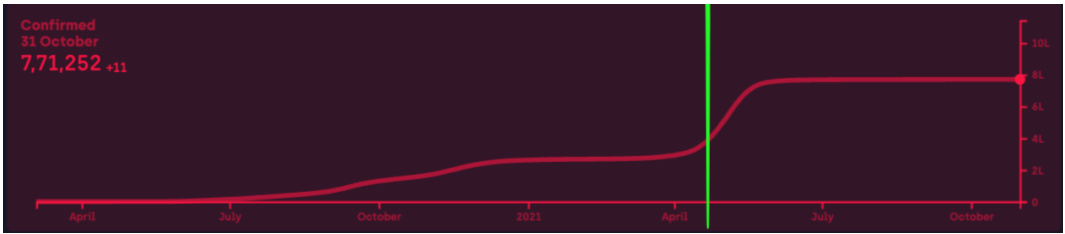


Diffusion process

- Quite plausible that the diffusion inputs or network connections are mismeasured
  - Who is patient zero?
  - Network connections measured with surveys or cell phone data

Forecasting a diffusion process in the medium run

- Goldilocks problem:
  - Day 1 is direct and uninteresting
  - In the long-run, diffuses everywhere
- scope for intervention is in the “sleeve” in the prior to the explosion



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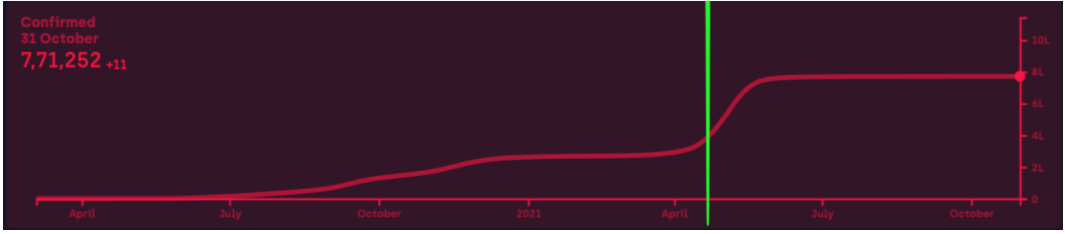

- Network structure and shape of diffusion intimately related to time

- Joe Schmoe: slow spread
  - time to respond
- Taylor Swift: info diffuses to new heights
  - too late to respond

Main Results

1. known network G, very locally perturbed seed i_0:
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3. in these regimes, aggregated quantities often ok:
   - e.g., transmission prob., $\mathcal{R}_0$
4. many (practical) data augmentation approaches ineffectual:
   - trivial size of measurement error makes it hard to estimate w/ extra data collection
   - and realistic testing protocols will be behind the curve $\implies$ reseeding elsewhere

1. Environment

General Setup

Society = seq. of undirected, unweighted graph G_n

$G_n := L_n \cup E_n$, where L_n is base with min. degree d_L, and the links in E_n are Ber($\beta_{ij,n}$).

- L_n perfectly observed by statistician
- E_n unobserved
  - e.g., sampling, compartmental smoothing, network shifting over time, ...
- $d_E/d_L \to 0$ (we assume much stronger vanishingness of E_n)
  - E_n *exceedingly* sparse..

Diffusion process: standard SIR on G_n with i.i.d. passing probability p_n.

Percolation $P_n(G_n)$: directed graph, each link in G_n activated i.i.d. w/ prob. p_n.

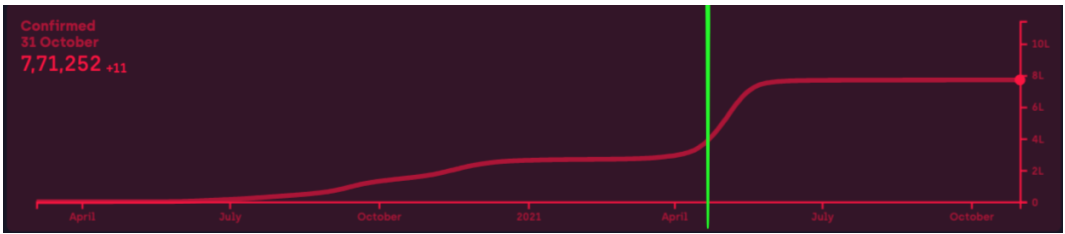
The Diffusion Process

The ever-activated set: $\mathcal{E}_t := \mathbb{E}\,|\{j \mid j \text{ ever activated by the diffusion on } L_n\}|$.
The shell: expected increment of new activations, $\mathcal{S}_t = \mathcal{E}_t - \mathcal{E}_{t-1}$.

Assumption 1. (Growth rates)
For some constant $q > 1$ and $t \in \mathbb{N}$,

1. $\mathcal{E}_t = \Theta(t^{q+1})$ and $\mathcal{S}_t = \Theta(t^q)$.
2. $p_n \in \left((\log n)^{-q/(2q+2)}, 1\right]$

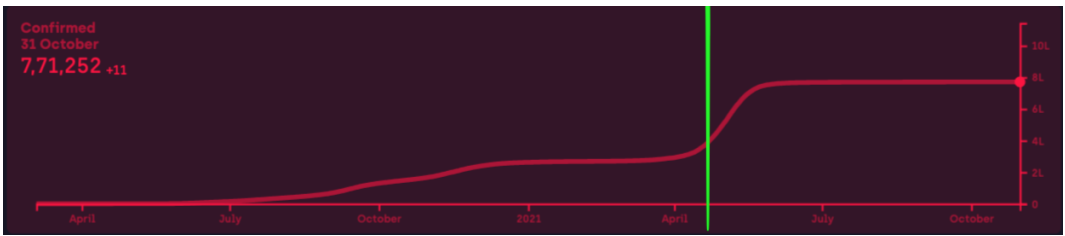
1. $\mathcal{E}_t$: polynomial growth period
2. p_n: some diffusion happens



Policy Relevant Time Period

Assumption 2. (Time Period of Focus)
Let a be any positive constant satisfying $2a > 1/(q+1)$.
T_n has for each n, $T_n \in [\underline{T}_n, \overline{T}_n]$ where the following holds:
(1) $\overline{T}_n = n^{\frac{1}{q+1}}$ and
(2) $\underline{T}_n = (\log n)^a$.

1. $\overline{T}_n$: the diffusion has not reached the edge of the graph
   - more expansive ($q \uparrow$), the earlier medium run ends
2. $\underline{T}_n$: the party has to get started...
   - more expansive ($q \uparrow$), the earlier the party gets started



Missing Mechanism

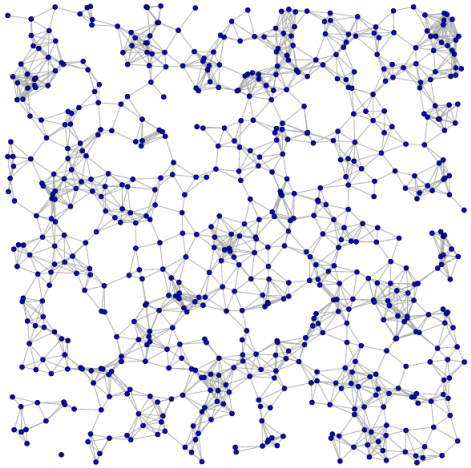
A given node i can link to fraction $\delta_n \in (\underline{\delta}_n, 1]$. of nodes through E_n.

Assumption 3. (Missing Mechanism) Let $2a > 1/(q+1)$ and $\nu := a - 1/(2q+2)$. The lower bound on the share of nodes that can be linked to is given by

$$\underline{\delta}_n = (\log n)^{-q\nu}.$$

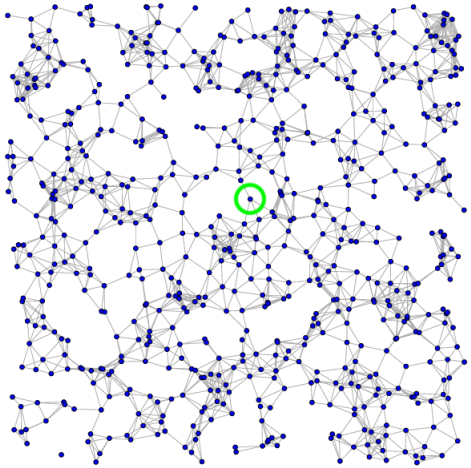
1. Take somewhat flexible stand as to which js constitute this δ_n-share.
2. May be that the entire δ_n share of nodes are in a highly localized neighborhood (in L_n) of i.
3. Results not about long-range "shortcuts."

Missing Mechanism



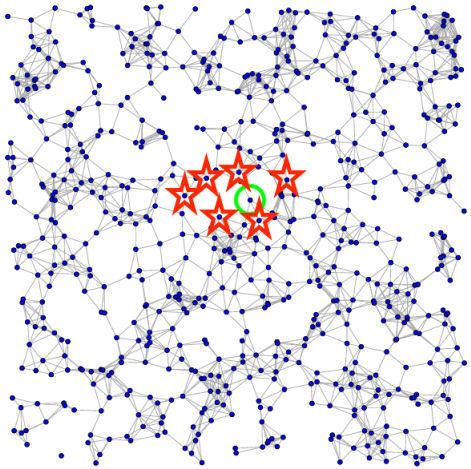
stylized L_n

Missing Mechanism



seed i_0

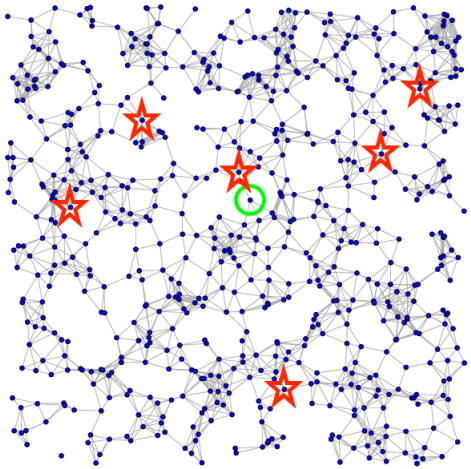
Missing Mechanism



local support:

- i_0 can only make links in E_n with ★
- local in L_n

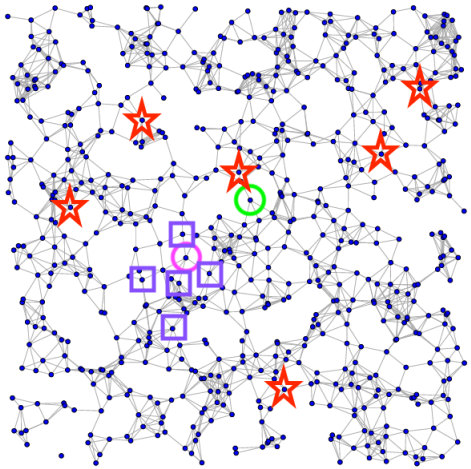
Missing Mechanism



global support:

- i_0 can only make links in E_n with ★
- but reside anywhere

Missing Mechanism



two seeds i_0 and j_0:

- i_0 can only make links with ★
- j_0 can only make links with □

Sense of Scale

California: pop 38.9 million

- $q = 2$: upper bound 11 months
- $q = 3$: 3 months
- lower bound close to 1 in either case
- $\underline{\delta}_n < 0.001$ and even lower ok
- structures w/ rare links, local in L_n,

Haryana: pop 25.4 million

- $q = 2$: upper bound 10 months
- $q = 3$: 2.4 months
- lower bound close to 1 in either case
- $\underline{\delta}_n < 0.001$ and even lower ok
- structures w/ rare links, local in L_n,

Missing Probability

Assumption 4. (Missing Probability) For every n, i, j, $E_{ij} \sim \mathrm{Ber}(\beta_n)$ for up to some share δ_n of the n nodes and is zero otherwise. Further,

1. $\delta_n \in (\underline{\delta}_n, 1]$
2. $\beta_n \in \left(\frac{1}{p_n T^q \delta_n n}, \frac{1}{n}\right)$.

- live in interesting case: $\beta_n = o(n^{-1})$ means E_n not connected / no giant comp.

2. Sensitive Dependence to Seed Set

known G, locally perturbed $i_0 \to j_0$

Notation for keeping track of activations

- $B^G_{i_0}(T_n)$: set of nodes j that could have been activated by T_n is the ball in G_n (sometimes call these “catchment regions”)

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Notation for keeping track of activations

- $B^P_{i_0}(T_n) \subset B^G_{i_0}(T_n)$: set of nodes j that are activated by T_n is the ball in P_n
- local neighborhood $U_{a_n}(i_0) \subset B^G_{i_0}(T_n)$: a ball of radius $a_n/T_n \to 0$
- set of j

$$J_{i_0} := \left\{ j : \exists k \notin B^G_{i_0}(T) \text{ with } U_{a_n}(i_0) \cap U_{b_n}(k) \right\}$$

  (a) local to i_0;
  (b) local to some other k;
  (c) k can't be reached by i_0 in T periods

- Compare $B^P_{i_0}(T_n)$ to $B^P_{j_0}(T_n)$ and show they can differ considerably

$$\Delta(i_0, j_0) := |B^P_{i_0}(T_n) \cap B^P_{j_0}(T_n)| / |B^P_{i_0}(T_n) \cup B^P_{j_0}(T_n)|, \text{ Jaccard Index}$$

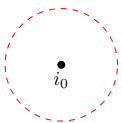
Sensitive Dependence

Theorem 1. Let Assumptions 1-3 hold and i_0 be a seed. The local neighborhood vanishes– $|U_{a_n}(i_0)|/n \to 0$ and wpa1 (over (P_n, E_n))

1. a non-vanishing share belongs to J_{i_0}: $|J_{i_0}|/|U_{a_n}(i_0)| > c$
2. if we counterfactually seed $j_0 \in J_{i_0}$, a agreement is bounded from above

$$\Delta(i_0, j_0) < c' < 1.$$

3. many disjoint catchment areas form



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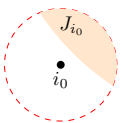

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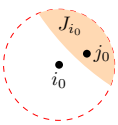

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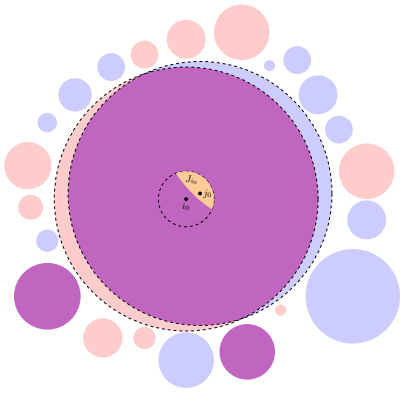

3. Forecasting Difficulties

imperfect knowledge of G, known i_0

Forecasting Setup

- Assume i_0 and L_n are known perfectly.
- Econometrician mistakenly uses the observed L_n instead of G_n:

$$\hat{Y}_T(L_n) := \mathbb{E}_{P_n(L_n)}\left[\sum_{j=1}^{n} y_{jT} \;\middle|\; L_n, i_0\right].$$

- Benchmark is targeting G_n (integrating over E_n error):

$$\tilde{Y}_T(G_n) := \mathbb{E}_{E_n, P_n(G_n)}\left[\sum_{j=1}^{n} y_{jT} \;\middle|\; L_n, i_0\right].$$

Forecasting Error

Theorem 2. (Extent of undercounting)
Under Assumptions 1-4, as $n \to \infty$, $\frac{\hat{Y}_T(L_n)}{\hat{Y}_T(G_n)} \to 0$.

- Despite perfect knowledge of L_n, i_0, T and q, the error dominates as $n \to \infty$.
- Why? Small errors caused by E_n recursively compound on themselves.
- As time grows, the volume around the seed grows.
- Then, the likelihood of hitting a link in E_n increases.

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- Why? Small errors caused by E_n recursively compound on themselves.
- As time grows, the volume around the seed grows.
- Then, the likelihood of hitting a link in E_n increases.
- This creates new activated regions elsewhere on the graph.
- These new regions created by jumps repeat the same process.
- The error caused by these new region swamp the forecast process.

3b. Forecasting in the Exponential Case

imperfect knowledge of G, known i_0

New assumptions for the exponential regime

Assumption 1→5

For some constant $q > 1$ and $t \in \mathbb{N}$,

1. $\mathcal{E}_t = \Theta(q^t)$ and $\mathcal{S}_t = \Theta(q^t)$.
2. $p_n \delta_n > \frac{1}{\log(n)}$

1. $\mathcal{E}_t$: Exponential growth period
2. p_n: much lower passing rate allowed

New assumptions for the exponential regime

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2. $p_n \delta_n > \frac{1}{\log(n)}$

Assumption 2 → 6

$T_n \in [\underline{T}_n, \overline{T}_n]$ where:

1. $\overline{T}_n = \log(n)$ and
2. $\underline{T}_n = (\log \log n)$.

1. $\mathcal{E}_t$: Exponential growth period
2. p_n: much lower passing rate allowed

1. $\underline{T}_n$: $\log \log n$ is much earlier
2. $\overline{T}_n$: $n^{\frac{1}{q+1}}$ changed to $\log(n)$

New assumptions for the exponential regime

Assumption 1→5

For some constant $q > 1$ and $t \in \mathbb{N}$,

1. $\mathcal{E}_t = \Theta(q^t)$ and $\mathcal{S}_t = \Theta(q^t)$.
2. $p_n \delta_n > \frac{1}{\log(n)}$

1. $\mathcal{E}_t$: Exponential growth period
2. p_n: much lower passing rate allowed

Assumption 2 → 6

$T_n \in [\underline{T}_n, \overline{T}_n]$ where:

1. $\overline{T}_n = \log(n)$ and
2. $\underline{T}_n = (\log \log n)$.

1. $\underline{T}_n$: $\log \log n$ is much earlier
2. $\overline{T}_n$: $n^{\frac{1}{q+1}}$ changed to $\log(n)$

Assumption 4 → 7

For every n, i, j, $E_{ij} \sim \mathrm{Ber}(\beta_n)$ for up to some share δ_n of the n nodes and is zero otherwise. Further,

1. $\delta_n \in (\underline{\delta}_n, 1]$
2. $\beta_n = \Omega(\frac{1}{p_n \delta_n n})$

1. If $p_n \delta_n < 1$, then E_n will have a giant component

Forecasting Error

Theorem 2b. (Extent of undercounting)
Under Assumptions 4, 5-7, as $n \to \infty$, $\frac{\hat{Y}_T(L_n)}{\hat{Y}_T(G_n)} \to 0$.

- The same logic holds here, but now we need more idiosyncratic links
- Due to dominant explosion from expontential growth, need errors to grow fast
- In fact, giant component in E_n is crucial

Forecasting Error

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Proposition. (Partial converse) Assume that L_n is made up of K_n independent regions, which each satisfy Assumption 5. Then, if $\beta_n = O(\frac{1}{p_n \delta_n n})$, we have that $\frac{\hat{Y}_T(L_n)}{\tilde{Y}_T(G_n)} \to 1$.

4. Estimation and Possible Solutions

aggregate quantities often ok; sampling/testing not

Estimating Parameters of the Process

Example: $\mathcal{R}_0$

- say $\hat{p}$ consistent for p_n and d_L (mean degree of L) known
- then $\hat{\mathcal{R}}_0 := \hat{p} d_L$ is consistent
- but still under Assumptions 1-4: sensitive dependence + forecasting problems

Bigger point: aggregative quantities (e.g., $\mathcal{R}_0$, p_n) may be easy to get

- still not enough for targeted policy
- maybe good retrospective descriptives

Maybe we can “take better measurements”?

Possible Solution by Estimating β_n

Let's try to estimate β_n naively. Let $\delta_n = 1$ (i.i.d. random error in E_n).

- Sample m_n nodes uniformly at random out of the n and perfectly observe $G_{ij,n}$.
- A sample of size m_n nodes will deliver $\binom{m_n}{2}$ possible links.
- New info in E_n can supplement the information of the known L_n.

Does this solve the problem for our our very small β_n?

Failure of Estimating β_n

Proposition 1. If:

1. $m_n = o(\sqrt{n})$,
   $\mathbb{P}\left(\text{No links amongst } \binom{m_n}{2} \text{ found}\right) \to 1$.
2. $m_n = O(1/\sqrt{\beta_n})$, there exists $\epsilon > 0$ and $c \in (0,1)$ such that $\mathbb{P}(|\hat{\beta}_n/\beta_n - 1| < \epsilon) < c$.

San Jose, pop. approx 1 million

- 1000 surveyed
  - detects essentially no links in E_n
- 41,000 surveyed
  - volatile estimates

Can we be more clever?

- perhaps... but the problem is also harder
- recall: assumed away the problem of *where* the E_n links could potentially go

Widespread Testing

Another potential solution is the use of widespread testing:

- goal: forecast where the activation (disease) has gotten as of period T
  - Consider the network split into disjoint regions (e.g. states or cities)
  - Where has it gotten?
- Instantaneously perform random samples throughout the n nodes
- Detect the activations with i.i.d. probability α_n
  - $\alpha_n \to 0$ with increasing n
  - thought experiment: combination of limited supply, testing consent, and test power

$\to$ Number of true regions that are activated at some time period will be grossly underestimated.

Failure of Widespread Testing

Theorem 3.

1. Detection prob. $\alpha_n \to 0$
2. Time $T < \alpha_n^{-1/(q+1)}$
3. $K_T^\star$ expected number of regions activated at T
4. $\hat{K}_T$ expected number w/ observed activated agent

As $n \to \infty$,

$$\frac{\hat{K}_T}{K_T^\star} \leq \underbrace{\alpha_n}_{\text{supply} \times \text{consent} \times \text{test power}} T^{q+1} < 1.$$

Ex.: Haryana, India – first 30 days

- Govt. very proactive
- Back of envelope calculation
  - conservative: maximum num. tests over first 3 months assumed to be done every day over the first month
  - actual policy: $\hat{K}_T / K_T^\star < 0.1$
- Counterfactuals
  - perfect power: $\hat{K}_T / K_T^\star < 0.15$
  - quintuple budget: $\hat{K}_T / K_T^\star < 0.75$

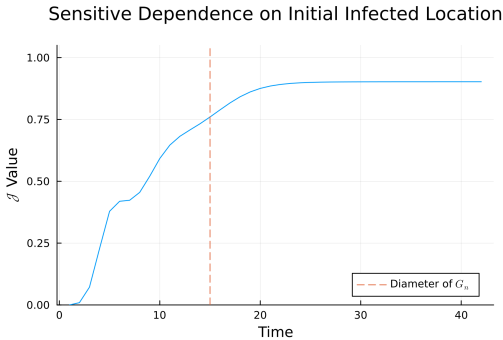
Govt. misses large share of regions with active agents over the first month

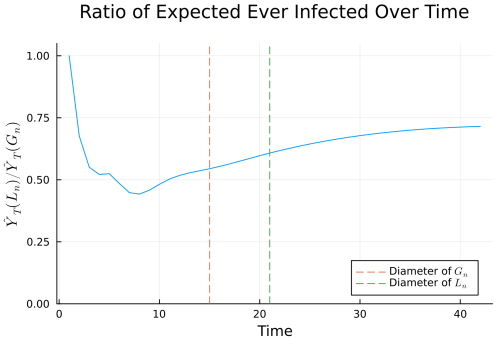
5. Empirical Applications

California Covid-19, ~~India marketing~~, China insurance

Mobility Flow Data for COVID-19 Pandemic

- Mobile phone data mapping origin-to-destination flows
  - anonymized data (Kang et al., '20)
- We use tract-to-tract flows from March 1st, 2020 in the Southwest US (CA, NV, AZ).
  - Network is extremely dense
- Construct L_n across census tracts
  - by linking tracts if the average flow between them (averaging over directions) is greater than six trips (the 93rd percentile of all flows).
- G_n^{92} links tracts if the average flow exceeds five trips (the 92nd percentile), meaning that E_n^{92} includes links of exactly 6 trips (18% from the G_n^{92} graph).
- $\mathcal{J}(t)$ is a Jaccard index tracking the set of ever-activated nodes infected by an epidemic that begins from i_0 and j_0





- U_{i_0}: 1.57% of pop.
- J_{i_0}: 82% of U_{i_0}
- $\mathcal{J}$ ($\approx \Delta(i_0, j_0)$): agreement quite low!

- Error: cutoff at 92 vs 93 percentile of cross- census tract flows
- Get as bad as only estimating 48% of actual diffusion

More provocative: Cai et al.: Informal Insurance in China

- Insurance products very important
- Seed info., generate a diffusion
- Outcome: take-up
- Core regressor: diffusion exposure

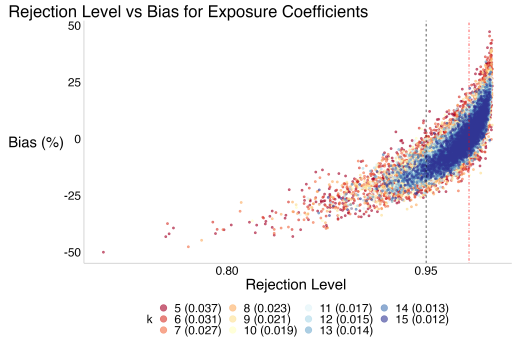
$$\text{diffusion exposure} = \left(\sum_{t=1}^{T} (p_n G)^t \right) s, \; s = \text{ seed indicator vector}$$

- take-up $= \beta$diffusion exposure + stuff $+ \epsilon$
- $H_0 : \beta = 0$?
- look at tiny amounts of measurement error 1-4%
- note: in data, top-code at 5, avg degree 4.5

take-up $\sim$ diffusion exposure $+$ stuff

	Insurance Uptake
Diffusion Exposure	0.029
	(0.012)
Household Controls	Yes
Village FE	Yes
Num Obs.	2676
Uptake Mean	0.459

$p = 0.02$ in the data



- even with 1% error, bias has std. dev. of 8pp
- 3.7% bias, biases over 20% common
- fail to rej. H_0: no peer effect 15% of time!

6. Discussion

Discussion

- Small error in i_0 or G can cause major problems for:
  - where the diffusion goes? how much diffusion there is?
  - devising (practical) ambitious, localized policy solutions
    - difficult for “welfare analysis” exercises (not this talk)
- Contrast of aggregate vs. non-aggregate estimands:
  - Some aggregated quantities like $\mathcal{R}_0$ or p are still estimable
  - Local prediction is very hard
  - Suggests limited policy relevance for targeted prediction
- Implications beyond our diffusion setting...
  - lots of behavior has “percolation-like” foundations to exposure maps
    - coalition proof risk-sharing, public goods, p-common knowledge, referrals...
  - similar errors in exposure maps are almost guaranteed
    - how bad can they get?
    - Perhaps a question of rate of diffusion vs. rate of error