Class 8: Arbitrage Pricing Theory and Factor Models

February 12, 2026

Paul Goldsmith-Pinkham
Yale University
https://paulgp.com

Yale SCHOOL OF MANAGEMENT

Today's Question

Your quant friend says “beta is dead.”

What does she mean?

Positioning

Where We've Been	Where We Are	Where We're Going
CAPM: one factor (market)	Multiple sources of systematic risk	Which factors actually matter? (Evidence)

"The Robots Can Handle the Factors"

Key themes: Systematic factor investing and the role of quantitative strategies

Learning Objectives

By the end of today's class, you should be able to:

1. Explain the logic of APT: multiple betas for multiple risk sources
2. Distinguish between macroeconomic factor models and characteristic-based models
3. Interpret a multi-factor regression output

Same Beta, Different Risks

Last class, we said market beta is the only risk that matters. Let's stress-test that claim.

Consider two stocks, both with a market beta of about 1.1:

ExxonMobil: When oil prices collapse (as in 2014–2016), Exxon drops sharply — even if the broad market is relatively stable. Exxon's fortunes are tied to energy prices.

JPMorgan: When credit spreads blow out (as in 2008), JPMorgan drops sharply — even holding the market index constant. JPMorgan's fortunes are tied to financial conditions.

Same Beta, Different Risks

These stocks have the same market beta. The CAPM says they should have the same expected return. But they are clearly exposed to **different kinds of bad times**.

The question: If there are multiple dimensions of systematic risk — energy shocks, credit crunches, recessions — can a single number really capture them all?

From One Beta to Many

Recall the CAPM's core logic (which we are **keeping**):

1. Only **systematic** risk is priced — diversifiable risk earns no premium
2. Expected returns compensate investors for bearing systematic risk
3. A stock's expected return depends on its **exposure** to that risk

The CAPM assumes all systematic risk is captured by one thing: the market portfolio. But what if the market return is too blunt a summary?

Think of it this way: saying "the temperature outside is 70°F" tells you something useful. But if you are planning a hike, you also want to know humidity, wind speed, and chance of rain. Temperature is not **wrong** — it is **incomplete**.

From One Beta to Many

Today's move: We keep the CAPM's logic — systematic risk exposures determine expected returns — but allow for **multiple** sources of systematic risk, each with its own beta and its own risk premium. This is Arbitrage Pricing Theory.

Arbitrage Pricing Theory

- Out of dissatisfaction with some of the assumptions in the CAPM, in the 1970s Steve Ross developed arbitrage pricing theory to show that the core insight — expected returns are determined by exposure to systematic risk — could be derived from weaker assumptions
- Combining his own assumptions about return distributions and no-arbitrage conditions, we will develop “factor models” to complement the CAPM
- We will begin with a single factor model very close to the CAPM and build from there

No Arbitrage Assumption

- What does no arbitrage imply?
  - No security/portfolio can assure a positive payoff, but have a negative (or zero) price
  - Any violation that arises cannot persist for long
  - i.e. no free lunch in financial markets
- Why do we believe no arbitrage is reasonable?
  - Otherwise, money-making machines exist that guarantee infinite riskless profits
- Unlike the CAPM assumptions, “absence of arbitrage” excludes preferences and requires only a handful of rational investors

Factor Model Assumption

- Suppose that returns on a security come from two sources:
  1. Common macro-economic factor
  2. Firm specific events
- Possible common macro-economic factors
  1. GDP growth
  2. Interest Rates
  3. Market performance

Factor Model Assumption

$$r_{i,t} = E(r_i) + \beta_i F_t + \varepsilon_{i,t}$$

$r_{i,t}$: Return on security of firm i at time t

β_i: Factor sensitivity or factor loading or factor beta

F_t: *Surprise* in macro-economic factor (zero expected value). E.g. if GDP growth was expected to be 3% and turns out to be 4%, then $F_t = 1\%$

$\varepsilon_{i,t}$: Firm specific events (zero expected value)

Critical assumption: firm specific events ($\varepsilon_{i,t}$) are independent across firms (no covariance). This means the chosen factor is the only common factor. If this fails — say, oil price shocks are a second common factor — the single-factor model is misspecified.

- Using the formula for variance,

$$\sigma_i^2 = \beta_i^2 \sigma_F^2 + \sigma^2(\varepsilon_i)$$

- Recall that in large portfolios, idiosyncratic risk is diversified away
  - No covariance by assumption (once you account for F)
  - For an equal-weighted portfolio of n stocks: $\sigma^2(\varepsilon_p) = \frac{1}{n}\overline{\sigma^2(\varepsilon)} \to 0$
  - Only variation in F remains
- Hence, for a portfolio (that is well-diversified)

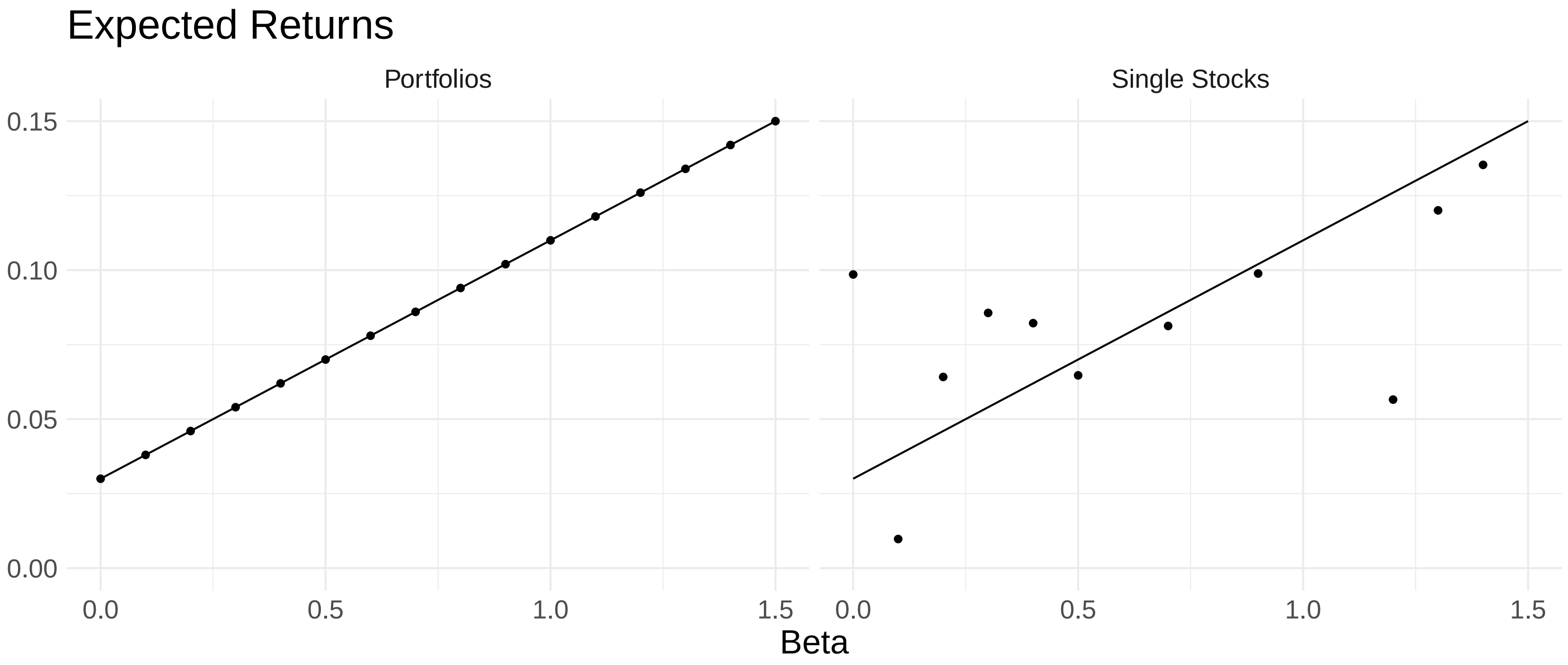
$$r_{p,t} \approx E(r_p) + \beta_p F_t$$

- Thus, in theory, by forming portfolios and eliminating idiosyncratic risk, we get

$$r_{p,t} \approx E(r_p) + \beta_p F_t$$

- Recall that $\beta_p = \sum_i w_i \beta_i$

Single Factor Models



- Consider the implications of $r_{p,t} \approx E(r_p) + \beta_p F_t$
- Assume $r_f = 0.04$ and two portfolios:
  1. Portfolio A has $\beta_A = 1$ and $E(r_A) = 0.10$
  2. Portfolio B has $\beta_B = 0.5$ and $E(r_B) = 0.06$
- This provides an arbitrage opportunity! How?

Portfolio	Expected Return	Beta
A	0.10	1
B	0.06	0.5
Riskless	0.04	0

- Combine the riskless asset and A to get a 0.5 beta portfolio (to match B):
  - Solve $0.5 = w_A \times 1 + (1 - w_A) \times 0$
  - $w_A \to E(r_p) = (0.5 \times 0.10) + (0.5 \times 0.04) = 0.07$

- Two portfolios, identical betas, but different payoffs implies arbitrage

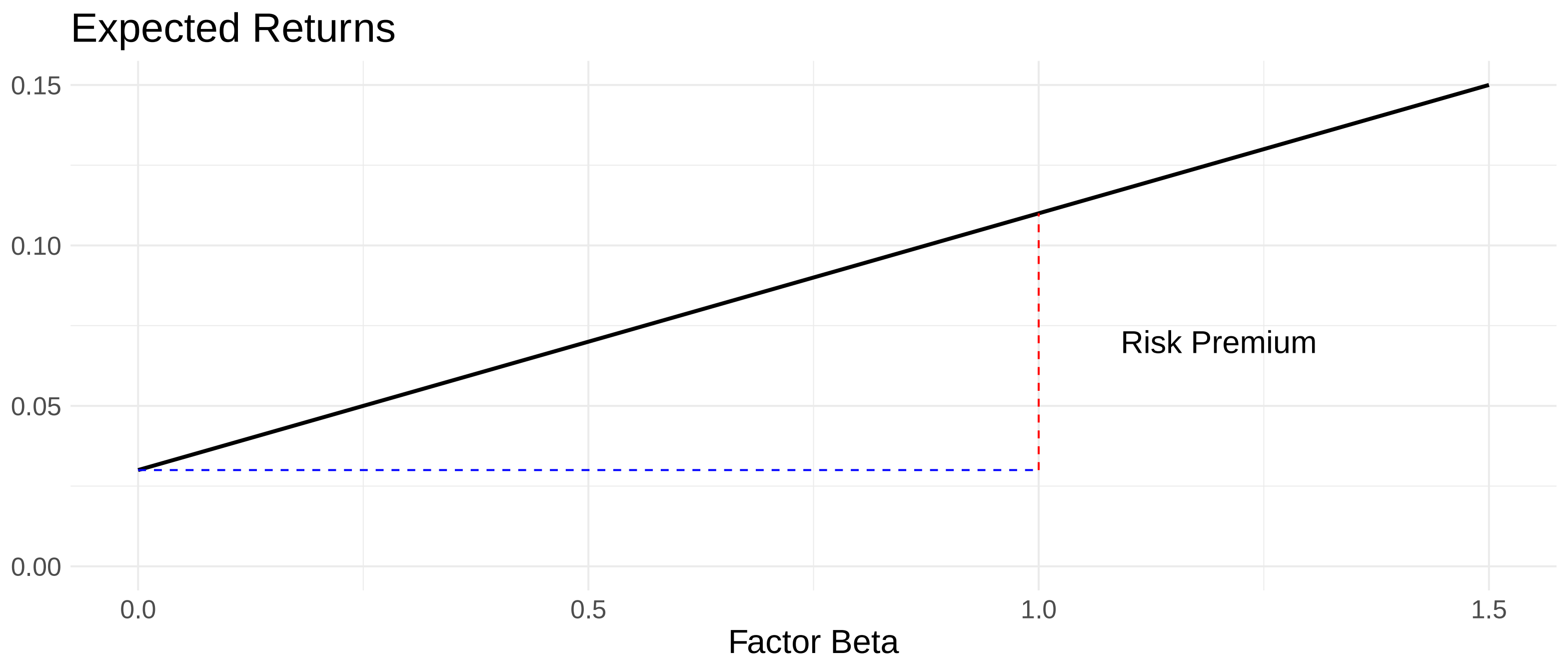
Portfolio	Expected Return	Beta
A	0.10	1
B	0.06	0.5
Riskless	0.04	0
C	0.07	0.5

- The new portfolio, C, offers an additional 1% return
  - Buy \$100 of C that pays $7 + 0.5 \times (100 \times F)$
  - Sell \$100 of B that pays $6 + 0.5 \times (100 \times F)$

- This is arbitrage: (1) zero net investment, (2) zero factor risk (the F terms cancel), (3) positive payoff ($1)
- Regrettably, arbitrage pricing theory assumes no free money

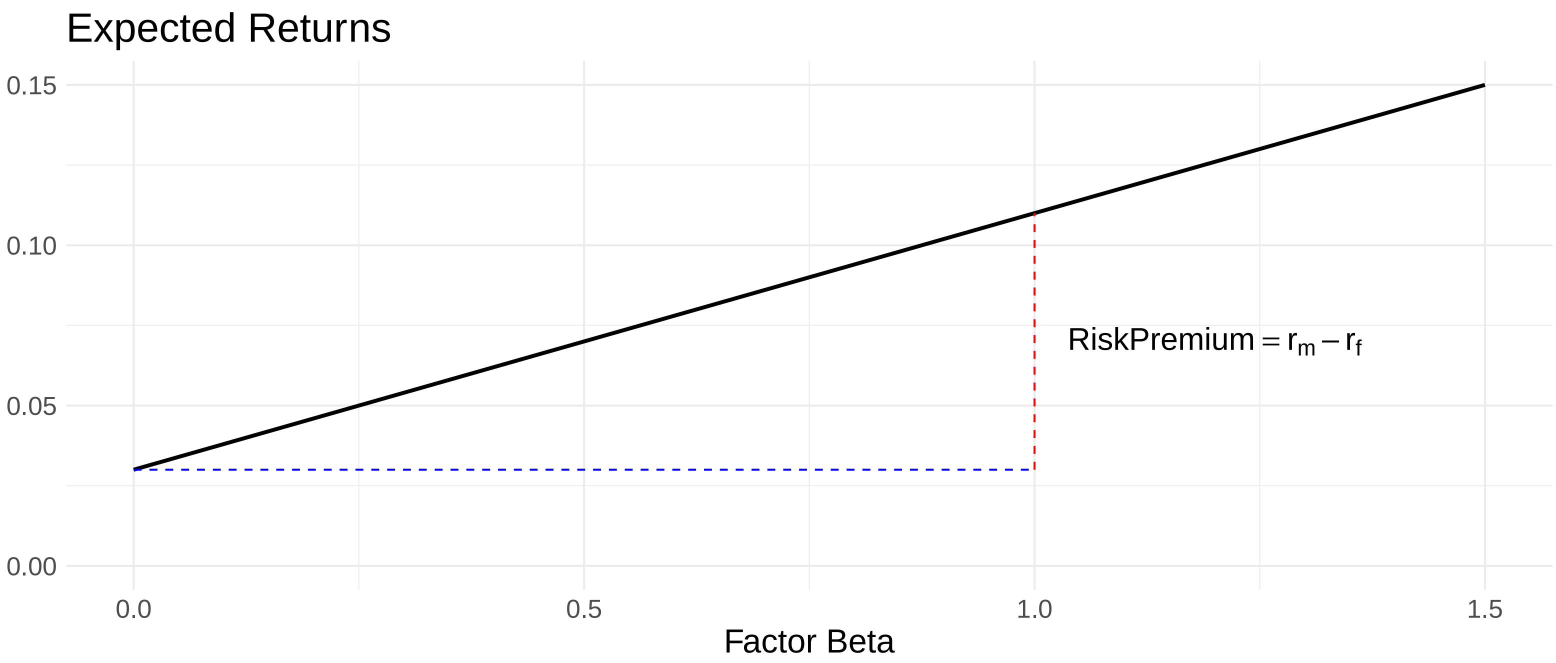
- No arbitrage condition implies portfolios with equal factor betas must have equal expected returns
- This implies a new security market line $E(r_i) = r_f + \beta_i \mathrm{RP}_F$, where the slope RP_F is the risk premium on factor F

Arbitrage Pricing Theory



- Suppose the single factor in the factor model is the unexpected return on the market portfolio, $r_m - E(r_m)$
- We are back to the CAPM security market line, $E(r_i) = r_f + \beta_i E\left(r_m - r_f\right)$

Arbitrage Pricing Theory



- Remember that the arbitrage only holds at the portfolio level where there is no idiosyncratic risk, such that $r_{p,t} = E(r_{p,t}) + \beta_p F_t$
- Painting with a broad brush, if a factor model holds for all portfolios, then its implications are unlikely to be violated for more than a handful of individual assets

Multifactor Models

- Suppose that returns on a security come from multiple common factors
- The “idiosyncratic” returns $\varepsilon_{i,t}$ in a misspecified single factor model will not be independent across firms
  - E.g. if you use only the market factor but oil price shocks are also systematic, then Exxon and Chevron’s residuals will be correlated
- This implies risk cannot be diversified away and the APT will not hold
- Solution: add more factors!

Multifactor SML

$$E(r_i) = r_f + \beta_{i,F1}\mathrm{RP}_{F1} + \beta_{i,F2}\mathrm{RP}_{F2}$$

- $\beta_{i,F1}$: Factor sensitivity for F_1, asset i
- RP_{F1}: Risk premium for F_1
- $\beta_{i,F2}$: Factor sensitivity for F_2, asset i
- RP_{F2}: Risk premium for F_2

In general, APT implies for K different factors,

$$E(r_i) = r_f + \sum_{f=1}^{K} \beta_{i,f}\mathrm{RP}_f$$

Where Should We Look for Factors?

- Need important systematic risk factors — two broad approaches:
- **Macro factors (top-down):** Chen, Roll, and Ross used industrial production, expected inflation, unanticipated inflation, excess return on corporate bonds, and excess return on long-term government bonds
- **Characteristic-sorted portfolios (bottom-up):** Fama and French used returns from portfolios built on firm characteristics that proxy for systematic risk factors (book-to-market and size)

Fama-French Three-Factor Model

- SMB: Small Minus Big (firm size) portfolio
- HML: High Minus Low (book-to-market ratio) portfolio
- Are these portfolios correlated with actual (but currently unknown) systematic risk factors?

$$E(r_i - r_f) = \beta_{i,m} E(r_m - r_f) + \beta_{i,\text{SMB}} E(r_{\text{SMB}}) + \beta_{i,\text{HML}} E(r_{\text{HML}})$$

- Note: SMB and HML are long-short (zero-cost) portfolios, so their expected returns are already excess returns — no r_f subtraction needed
- It is the securities' *exposure* (beta) that matters, not its actual characteristic

- Debate: do these premia reflect compensation for risk, or mispricing?

Recap

- The APT was a response to strong assumptions embedded in the CAPM
  - Only needs a well-specified factor model and arbitrageurs
  - APT is flexible, but is largely silent on where to look for priced sources of risk
- We should think of the CAPM, augmented with APT style factors, as a starting point for otherwise elusive expected return inputs

Next Class: February 17

Topics: Evidence from CAPM and APT

- Tests of the CAPM
- Fama-Macbeth regressions
- Anomalies and factor models

Matt Levine Reading: TBD