Class 6: Building Optimal Portfolios II

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Today's Question

Two investors have very different risk tolerances.

Should they hold different risky portfolios?

Positioning

Where We've Been	Where We Are	Where We're Going
Diversification, efficient frontier	Capital allocation; the tangency portfolio	What determines expected returns? (CAPM)

“Playing the Game of Infinite Leverage”

Key themes: The risks and rewards of leverage in portfolio management

Learning Objectives

By the end of today's class, you should be able to:

1. Explain why portfolio risk isn't just the average of individual risks
2. Calculate portfolio expected return and standard deviation with multiple risky assets
3. Define the efficient frontier and explain **why** it's efficient

Now suppose instead of choosing a mix between a risky and a risk-free asset, we have two risky assets (but no risk free asset).

The expected return for the portfolio of risky assets A and B is

$$E(r_p) = w_a E(r_a) + (1 - w)E(r_b)$$

where w_a is the weight on stock A.

Two Risky Assets - Variance

The variance and standard deviation of the portfolio are:

$$\begin{aligned}\sigma^2(r_p) &= \mathrm{Var}(w_a r_a + (1 - w_a) r_b) \\ &= w_a^2 \sigma^2(r_a) + (1 - w)^2 \sigma^2(r_b) + 2 w_{a(1-w_a)} \underbrace{\sigma(r_a, r_b)}_{\text{covariance}} \\ &= w_a^2 \sigma^2(r_a) + (1 - w)^2 \sigma^2(r_b) + 2 w_{a(1-w_a)} \rho_{a,b} \sigma(r_a) \sigma(r_b)\end{aligned}$$

Now suppose assets A and B are characterized as follows:

Asset	$E(r)$	$\sigma(r)$
A	0.25	0.75
B	0.1	0.25

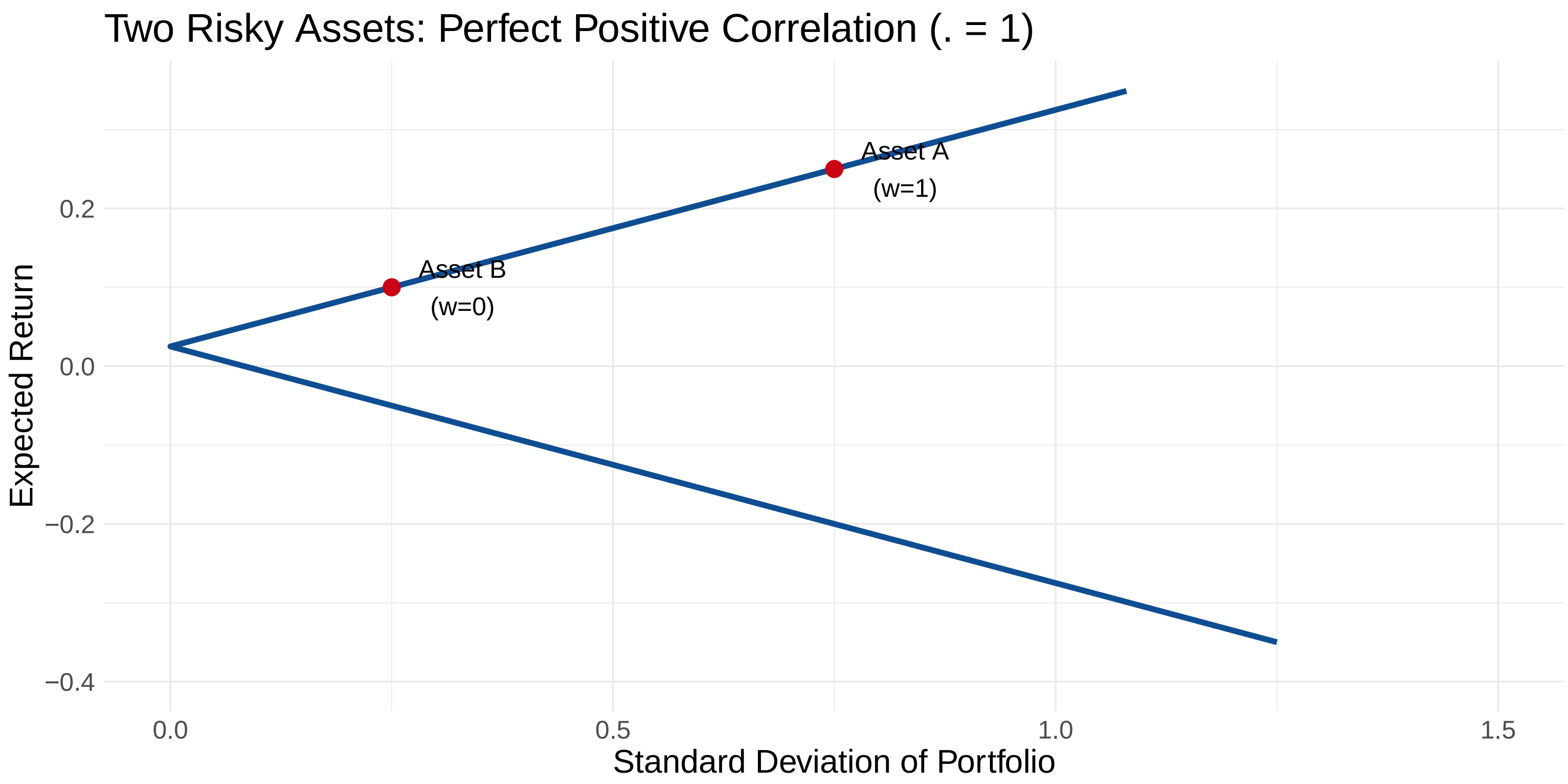
- Consider the frontier of returns/standard deviations available by allocating different amounts to the two assets (i.e. varying w)
- Notice the variance calculation now includes the correlation between A and B
  - https://learn-investments.rice-business.org/portfolios/two-assets

2 Risky Assets (correlation $\rho = 1$)

Asset	$\boldsymbol{E(r)}$	$\boldsymbol{\sigma(r)}$
A	0.25	0.75
B	0.1	0.25

$$\rho_{a,b} = 1$$

2 Risky Assets (correlation ρ = 1)



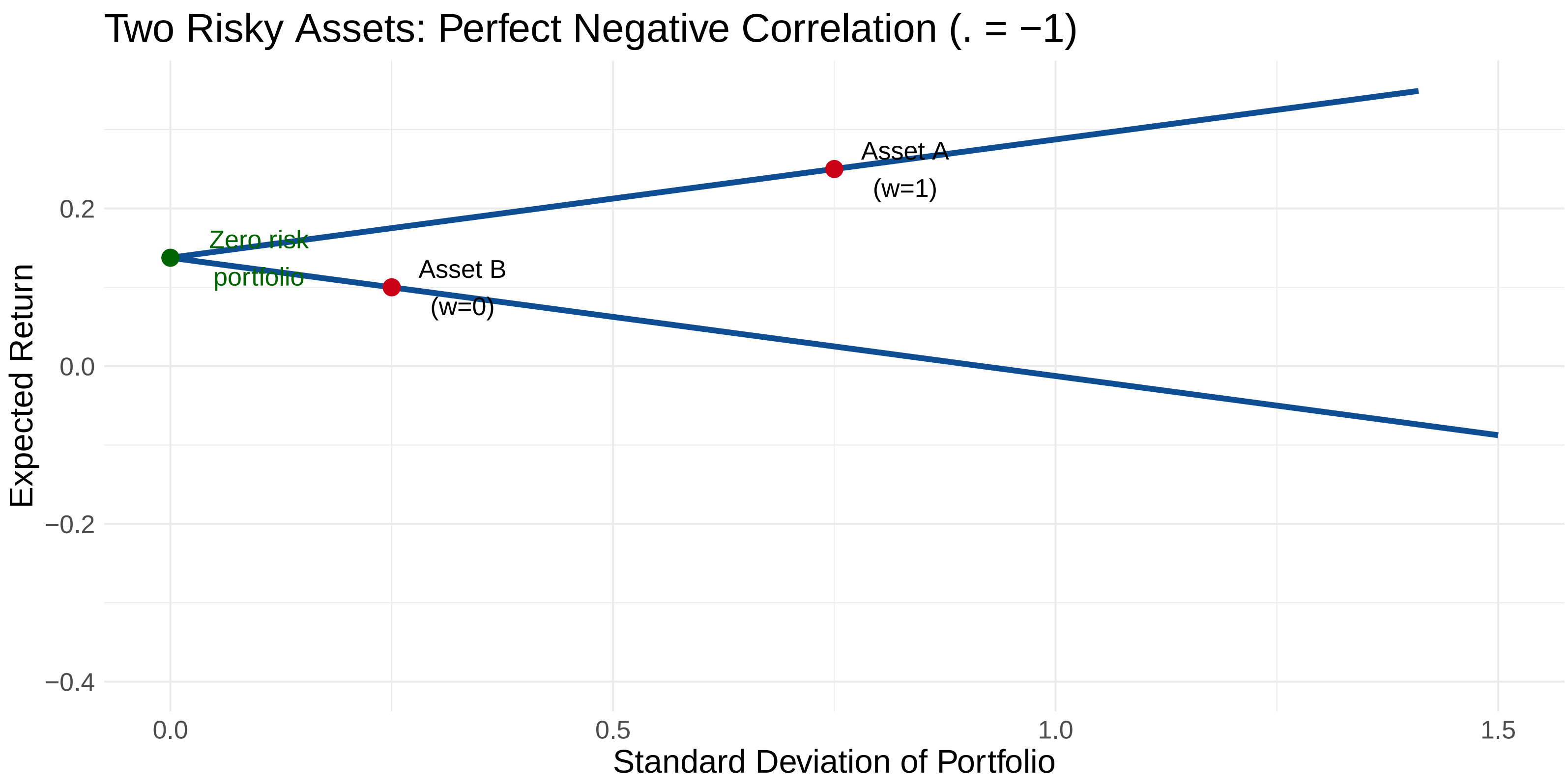
With perfect positive correlation, no diversification benefit—the frontier is linear.

2 Risky Assets (correlation ρ = -1)

Asset	$\boldsymbol{E(r)}$	$\boldsymbol{\sigma(r)}$
A	0.25	0.75
B	0.1	0.25

$$\rho_{a,b} = -1$$

2 Risky Assets (correlation ρ = -1)



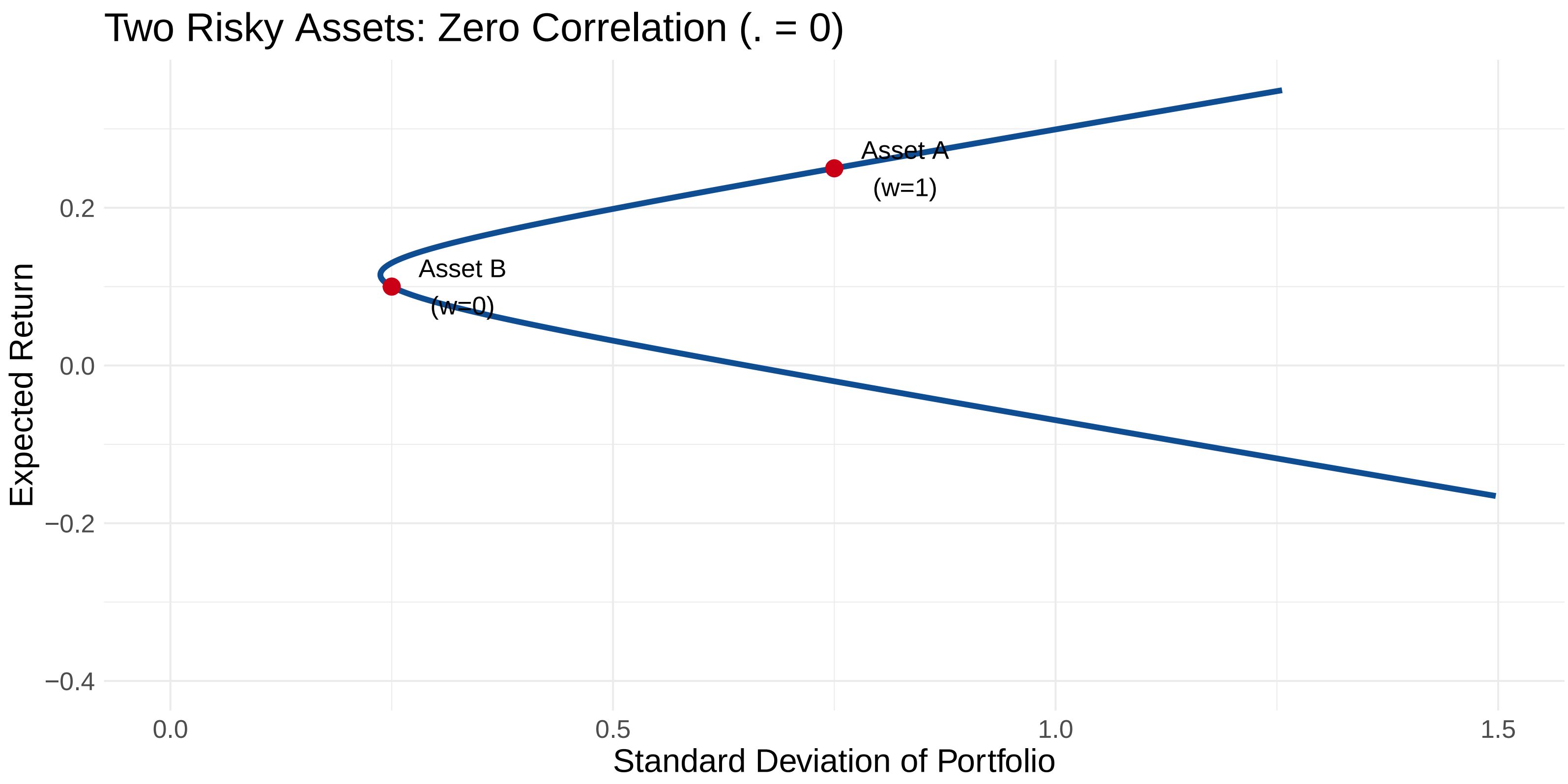
With perfect negative correlation, you can eliminate all risk at the apex of the V!

2 Risky Assets (correlation $\rho = 0$)

Asset	$\boldsymbol{E(r)}$	$\boldsymbol{\sigma(r)}$
A	0.25	0.75
B	0.1	0.25

$$\rho_{a,b} = 0$$

2 Risky Assets (correlation ρ = 0)



With zero correlation, we get the typical curved efficient frontier with diversification benefits.

The Efficient Frontier (2 Assets)

Looking at the frontier we just traced out, notice two key features:

1. **Minimum Variance Portfolio (MVP)**: The leftmost point on the frontier
   - This portfolio has the lowest possible risk given these two assets
   - Found by minimizing $\sigma^2(r_p)$ with respect to w_a
2. **Upper vs. Lower Portions**: The frontier has two branches
   - **Upper portion**: Higher return for each level of risk → **efficient**
   - **Lower portion**: Lower return for same risk → **inefficient** (dominated)

No rational investor would choose a portfolio on the lower portion!

The Minimum Variance Portfolio

The weight that minimizes portfolio variance (regardless of expected return):

$$w_{a,\mathrm{MVP}} = \frac{\sigma_b^2 - \rho_{a,b}\sigma_a\sigma_b}{\sigma_a^2 + \sigma_b^2 - 2\rho_{a,b}\sigma_a\sigma_b}$$

Key observations:

- This formula depends **only** on variances and correlation—not expected returns
- The MVP is the “safest” portfolio you can construct from these assets
- When $\rho = -1$, you can achieve **zero** variance (perfect hedge)

But is the MVP the **best** portfolio? Not necessarily—it ignores returns!

A portfolio is **efficient** if:

- No other portfolio offers higher expected return for the same risk, OR
- No other portfolio offers lower risk for the same expected return

The **efficient frontier** is the set of all efficient portfolios—the upper boundary of the feasible region.

Without a risk-free asset, different investors choose different points on this frontier based on their risk aversion A.

Given the available risk return trade-offs, which portfolio provides the highest utility level for a given investor?

- Depends on their risk aversion (A)!

More risk averse investors will choose a relatively safer portfolio

- This changes once you have a riskless asset to invest in

The Optimal Risky Portfolio without riskless asset

Formally, we can derive it using calculus, as

$$\max_{w_a} U(r_p) = E(r_p) - \frac{1}{2}A\sigma^2(r_p)$$

$$\text{s.t.} \quad E(r_p) = w_a E(r_a) + (1-w)E(r_b)$$

$$\text{and} \quad \sigma^2(r_p) = w_a^2\sigma^2(r_a) + (1-w)^2\sigma^2(r_b) + 2w_{a(1-w_a)}\rho_{a,b}\sigma(r_a)\sigma(r_b)$$

Solving directly (by plugging in and taking derivative w.r.t. w_a):

$$w_a^* = \frac{E(r_a) - E(r_b)}{A(\sigma_a^2 + \sigma_b^2 - 2\rho_{a,b}\sigma_a\sigma_b)} + \frac{\sigma_b^2 - \rho_{a,b}\sigma_a\sigma_b}{\sigma_a^2 + \sigma_b^2 - 2\rho_{a,b}\sigma_a\sigma_b}$$

$$= \frac{E(r_a) - E(r_b) + A(\sigma_b^2 - \rho_{a,b}\sigma_a\sigma_b)}{A(\sigma_a^2 + \sigma_b^2 - 2\rho_{a,b}\sigma_a\sigma_b)}$$

We can do **better** – just need a riskless asset

Two risky and one riskless asset

Add back our riskless asset

- $r_f = 0.03$
- $\sigma(r_f) = 0$

What is our Capital Allocation Line when we combine either A or B with our riskless asset?

To illustrate more clearly the point, let $\sigma(r_A) = 0.5$:

Asset	$\boldsymbol{E(r)}$	$\boldsymbol{\sigma(r)}$
A	0.25	0.50
B	0.1	0.25

Review:

The Capital Allocation Line (CAL) shows risk-return combinations from mixing:

- A risk-free asset (at r_f on the y-axis)
- A risky portfolio (some point on or inside the frontier)

The slope of the CAL is the **Sharpe Ratio** of the risky portfolio:

$$\text{Slope of CAL} = \frac{E(r_p) - r_f}{\sigma_p}$$

Key insight: We can draw a CAL from r_f to **any** portfolio on the frontier.

Which CAL is best? The one with the **steepest slope** (highest Sharpe ratio)!

Finding the Tangency Portfolio

The **tangency portfolio** is where the CAL just touches the efficient frontier—the point of tangency.

Why is this special?

- **Steepest CAL**: No other portfolio on the frontier gives a higher Sharpe ratio
- **Dominates all others**: Any point on this CAL beats points on flatter CALs
- **Universal optimum**: This is the best risky portfolio for **all** investors

Graphically: Draw a line from r_f that is tangent to the frontier. The point where it touches is the Mean-Variance Efficient (MVE) portfolio.

Why Tangency Dominates

Consider two investors choosing different risky portfolios:

Investor 1: Picks portfolio P (not the tangency)

- CAL has Sharpe ratio = 0.40
- To get 15% expected return, must accept 30% standard deviation

Investor 2: Picks the tangency portfolio T

- CAL has Sharpe ratio = 0.55
- To get 15% expected return, only needs 22% standard deviation

Same return, less risk! The tangency portfolio dominates.

This is why we call it the Mean-Variance Efficient (MVE) portfolio.

Finding the MVE Portfolio

The solution for $w_{a,\text{MVE}}$ is:

$$w^*_{a,\text{MVE}} = \frac{E(r_a - r_f)\sigma_b^2 - E(r_b - r_f)\sigma_a\sigma_b\rho_{a,b}}{E(r_a - r_f)\sigma_b^2 + E(r_b - r_f)\sigma_a^2 - \left[E(r_a - r_f) + E(r_b - r_f)\right]\sigma_a\sigma_b\rho_{a,b}}$$

Key thing to notice – this optimal portfolio does not include anything related to investor risk aversion (A)

Takeaway: All investors do best by choosing the same risky portfolio and **then** deciding how much to allocate to the riskless asset based on individual preferences

Now we have a simple two step recipe for an optimal portfolio, based on our taste for risk!

1. Specify the expected returns, standard deviations and covariance between our risky assets. Also define the riskless rate of return.
    - Using these parameters, solve for the unique MVE portfolio.
    - This is the tangency portfolio!
    - Everyone wants this risky portfolio, regardless of taste for risk.
2. Choose weights between the MVE portfolio and the riskless asset based on your taste for risk A.

Note: this statement holds for as many risky assets as we would like!

Playing the game of infinite leverage

What if you want **more** than 100% in the risky portfolio? Borrow at r_f and invest more!

It did! Here's **Version 3**, which is quoted from ControlTheNarrative's description on Reddit:

> With Robinhood Gold, you can use what's called Margin to trade with increased Buying Power. So I did an instant deposit of 2000 dollars (the minimum required to access Gold Trading), and then I bought 100 shares of AMD for 3,800 or so, with margin. Then I sold an AMD Call Contract with a 2 dollar price strike to get almost all my money back (it's important to find a stock that has Call Options with such low strike prices like Ford, GE, etc). Then I use that money to buy TWO hundred shares of AMD because remember, margin doubles my buying power, then I sell 2 Call Contracts with the same 2 dollar strike price to get almost all my money back, which is then doubled again thanks to Robinhood's Margin. I repeat this until I am sufficiently leveraged for my Personal Risk Tolerance. Right now, I am at 25x leverage because I had 2000 dollars in Instant Deposits.

Ah. Super. If you are reading this I hope I do not need to tell you that this is not investing advice. First of all it really shouldn't be allowed; you shouldn't be able to get infinite leverage this way, and I assume that Robinhood will close this loophole quickly. Second, even if you *can* do this, it's not … it's not like it's free *money* or anything. You get more leverage than you ought to, but why would you *want* infinite leverage? Leverage isn't an absolute good; it increases your potential return but also your risk. You probably *shouldn't* make hugely levered single-stock bets with your savings! I don't know, maybe it's fun for you. Certainly it's not *investing*.

- Low risk aversion → want $w > 1$ (leverage)
- High risk aversion → want $w < 1$ (mostly safe assets)
- **Everyone wants the same risky portfolio**, just different amounts!
- Why can't you do this forever, in reality? (What is *ControlTheNarrative* avoiding?)

From Two Assets to Many Assets

Everything we've learned generalizes to N risky assets:

Portfolio expected return:

$$E(r_p) = \sum_{i=1}^{N} w_i E(r_i)$$

Portfolio variance:

$$\sigma^2(r_p) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{i,j} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_{i,j}$$

The key insight: With N assets, we need $\frac{N(N-1)}{2}$ covariances!

- 5 assets → 10 covariances
- 100 assets → 4,950 covariances
- 500 assets → 124,750 covariances

The Efficient Frontier with Many Assets

With more assets, the efficient frontier:

1. **Expands outward** (or stays the same)—never shrinks
   - More assets = more diversification opportunities
   - Worst case: put zero weight on unhelpful assets
2. **Becomes smoother**
   - With 2 assets: a curve or line segment
   - With many assets: a continuous, smooth boundary
3. **Still has one tangency portfolio**
   - The MVE portfolio now has weights across all N assets
   - Still maximizes the Sharpe ratio
   - Still optimal for all investors (combined with r_f)

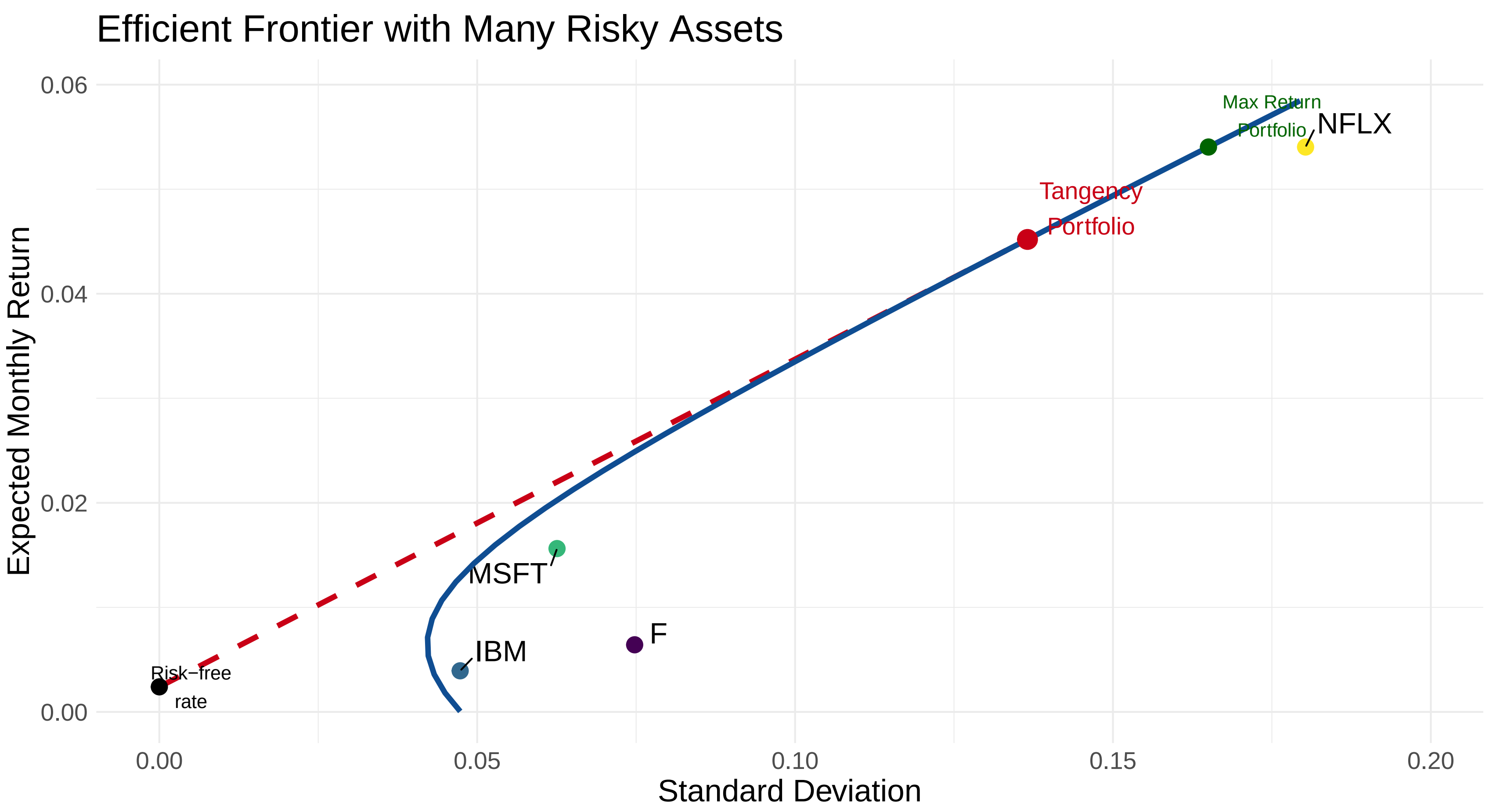
Many (N) risky assets

https://learn-investments.rice-business.org/portfolios/three-assets

https://learn-investments.rice-business.org/portfolios/frontier

https://learn-investments.rice-business.org/portfolios/tangency

Many Risky Assets



Blue curve: Efficient frontier of risky assets
Red dashed line: Capital Allocation Line

Many Risky Assets

The tangency portfolio is where the CAL has the steepest slope (highest Sharpe ratio).

All investors should hold some combination of:

- The risk-free asset
- The tangency portfolio

Diversification benefits

So why does diversification shift the frontier?

- Adding assets can never make us worse off (if they did, we would put zero weight in them)
- Can only help reduce risk via imperfect or negative correlation
- In spite of being risk averse, we'd like to get exposure to as many different types of risks as possible

Back to Our Opening Question

Two investors have very different risk tolerances.

Should they hold different risky portfolios?

Conclusions

- Mean-variance analysis works off the assumption that all investors care about is the mean and variance of returns
- This delivers some powerful takeaways
  - Portfolio management can be broken down into two steps:
    1. Find the unique mean-variance efficient portfolio
    2. Combine this with the riskless asset to suit preferences for risk
- What's wrong with this?
  - Not much! There is a lot of agreement with these points.
  - However, **measuring** the inputs is really hard! Garbage in, garbage out
  - Need to have models of expected returns!

Next Class: February 10

Topics: The Capital Asset Pricing Model (CAPM)

- From portfolio theory to equilibrium
- The market portfolio
- Beta and expected returns

Matt Levine Reading: “Buying the Good Stocks Can Be Bad”