Class 5: Building Optimal Portfolios I

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Today's Question

A vaccine CEO has their entire wealth in company stock.

How much risk should they hedge?

Positioning

Where We've Been	Where We Are	Where We're Going
Risk and return of individual assets	Combining assets; diversification	The "optimal" portfolio

“SPACs Aren’t Cheaper Than IPOs Yet”

(Vaccine Hedging section)

Key themes: Portfolio concentration risk and optimal hedging

Learning Objectives

By the end of today's class, you should be able to:

1. Define risk aversion and represent preferences using utility functions
2. Derive the Capital Allocation Line (CAL) for combining a risky and risk-free asset
3. Calculate the optimal portfolio weight based on risk aversion

Preview of our goal

Solve two problems:

1. How should we distribute our wealth between a risky portfolio of assets and the risk-free asset based on our risk aversion?
2. What portfolio of risky assets should we hold?

With many risky assets, there is an optimal portfolio of these assets that all investors prefer

- Lectures on active management will address departures from this optimal risky portfolio

Risk Aversion

- We now know from our historical data that risky asset returns tend to beat safe asset returns
  - “In Expectation”
- Since there’s a positive “premium” for risk, how much risk are you willing to bear?
  - This depends on risk tolerance

Consider the following gamble (where you pay me a dollar):

- With probability 0.5, I give you 2 dollars
- With probability 0.5, I give you 50 cents

What are the expected returns? Variance?

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- With probability 0.5, I give you 50 cents

What are the expected returns? Variance?

$$E\left(r_{\text{gamble}}\right) = 0.5 \times (2 - 1) + 0.5 \times (0.5 - 1) = 0.25$$

$$\sigma^2\left(r_{\text{gamble}}\right) = 0.5 \times (1 - 0.25)^2 + 0.5 \times (-0.5 - 0.25)^2 = 0.5625$$

$$\sigma\left(r_{\text{gamble}}\right) = \sqrt{0.5625} = 0.75$$

- Would you take it? What if you paid me 1.25?
- At what cost would you become unwilling to take this deal?
  - Known as the “certainty equivalence”

Ok, now consider another gamble (where you pay me a dollar):

- With probability 0.4, I give you 3 dollars
- With probability 0.2, I give you 1 dollar
- With probability 0.4, I give you 25 cents

What are the expected returns? Variance?

$$E\left(r_{\text{gamble}}\right) = ?$$

$$\sigma^2\left(r_{\text{gamble}}\right) = ?$$

$$\sigma\left(r_{\text{gamble}}\right) = ?$$

Preferences

Let's assume a simple model of preferences over returns and risk.

$$U(r) = E(r) - \frac{1}{2} \times A \times \sigma^2(r)$$

- $A > 0$ implies risk aversion
- $A = 0$ implies risk neutrality
- $A < 0$ implies risk seeking

- $U(r)$ provides an **ordering** over investments, given their returns and risks

Preferences

- This particular function can be derived under a number of different assumptions and is widely used by practitioners and academics

What does risk aversion mean?

$$U(r) = E(r) - \frac{1}{2} \times A \times \sigma^2(r)$$

Key insight: Recall from Class 3 that geometric mean $\approx$ arithmetic mean $-\frac{1}{2}\sigma^2$

When $A = 1$:

$$U(r) = E(r) - \frac{1}{2}\sigma^2(r) \approx \text{Geometric Mean}$$

You care about **long-run compound growth** — indifferent between any investments with the same geometric return

When $A < 1$: You weight arithmetic returns more heavily (more risk-tolerant)

What does risk aversion mean?

When $A > 1$: You penalize variance even more than geometric returns suggest (very risk-averse)

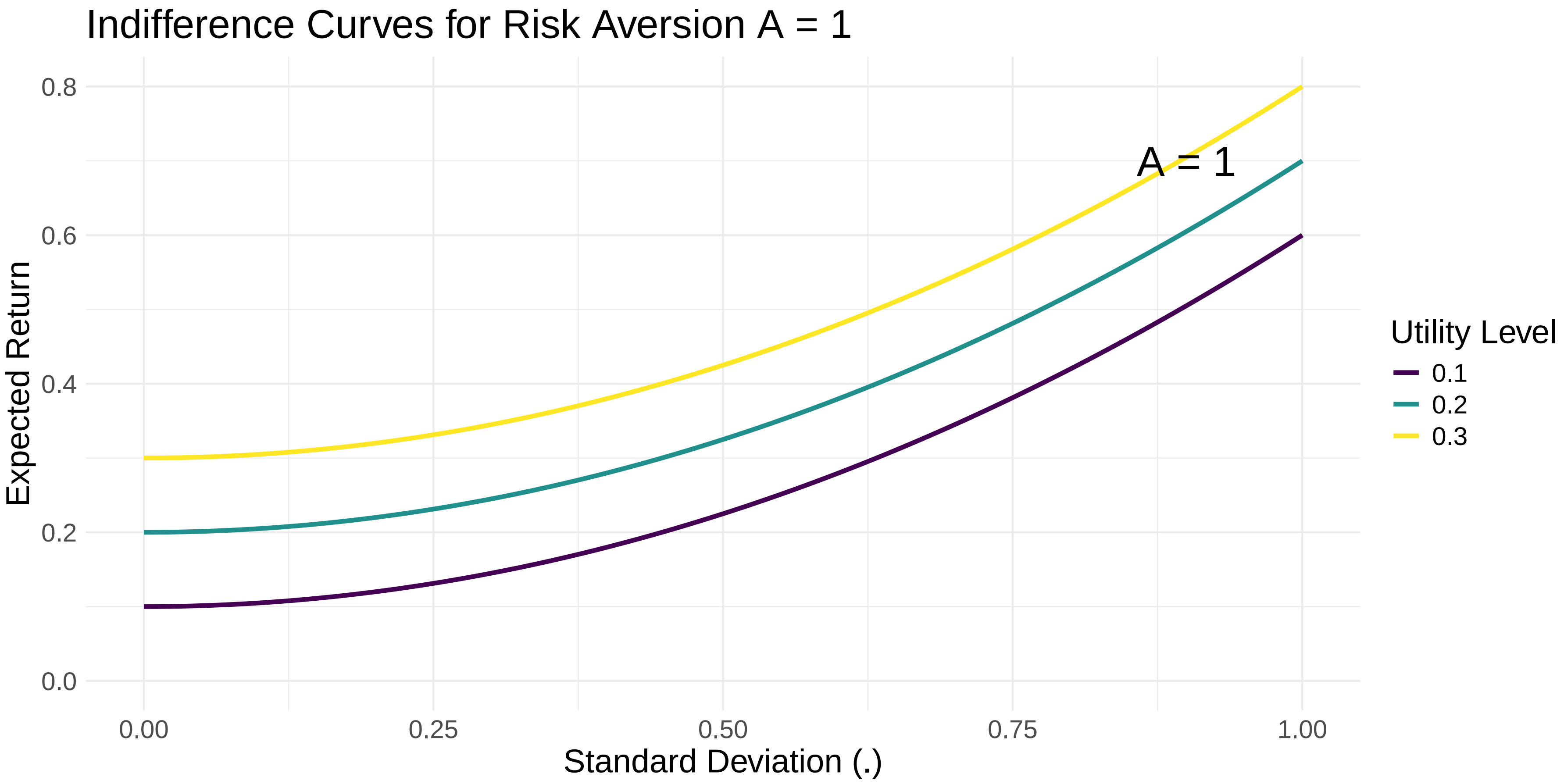
$A = 1$ is the "maximize long-run wealth" benchmark

Preferences

$$U(r) = E(r) - \frac{1}{2} \times A \times \sigma^2(r)$$

For $A = 1$:

Preferences



Preferences

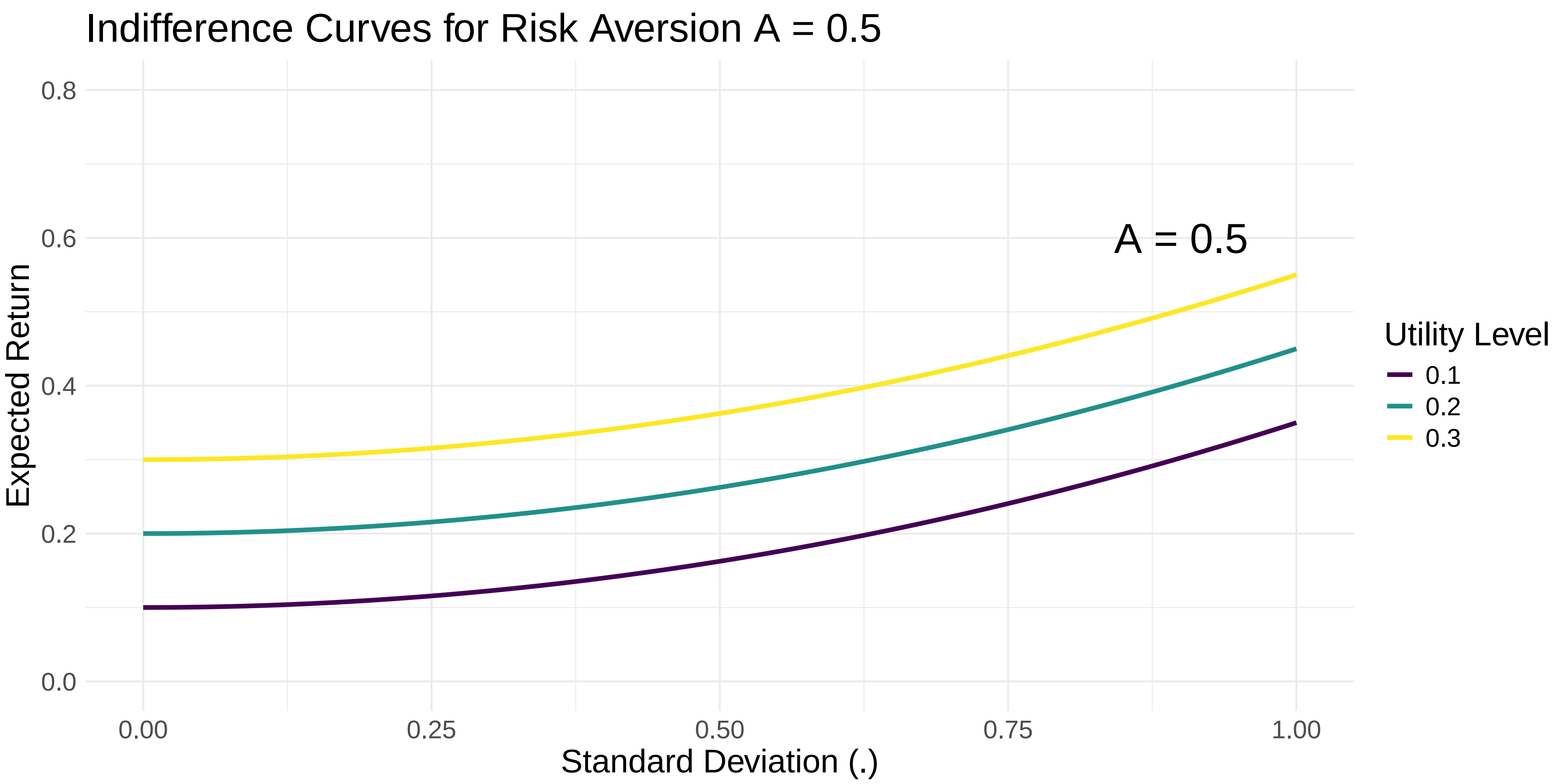
Each curve represents a fixed utility level. Higher curves (to the northwest) represent higher utility.

Preferences

$$U(r) = E(r) - \frac{1}{2} \times A \times \sigma^2(r)$$

For $A = 0.5$:

Preferences



Preferences

Notice the curves are flatter—less risk averse investors tolerate more risk for the same utility increase.

Allocating between 1 risky and 1 riskless asset

Back to our first example:

- The risky asset's return is $r_p = 0.25$, $\sigma = 0.75$
- and there is a riskless asset with return $r_f = 0.03$, $\sigma = 0$

Utility ($U(r)$) across assets:

Risk Aversion	**A=0.04**	**A=0.5**	**A=0.78**	**A=1**
Risk Free	0.03	0.03	0.03	0.03
Risky	0.24	0.11	0.03	-0.03

- For risk aversion of 0.5, do you prefer r_f or investment r_p?
- Suppose the constant of risk aversion is 0.78?

Now what about if you can choose to invest in both assets, but with different weights?

What weight (w) should you allocate to the risky asset?

A real example: Consider a vaccine CEO

Scenario: You're the founder/CEO of a small biotech working on a coronavirus vaccine.

The market's view: After preliminary trials, the market interprets a **20% chance** of success

- Company stock jumps from \$200M to \$4B (20% × \$20B if successful)
- Your net worth went up 1,900% overnight!

But you might feel differently — You know your team, the science, the data...

What should you do?

1. **Keep all your stock** — Bet everything on success
2. **Sell half** — Lock in some wealth, keep some upside
3. **Sell everything** — \$4 billion is plenty; take the money and run

What would you choose? What would the model recommend?

Vaccine CEO: If you believe the market (20% chance)

Suppose you agree with the market: 20% chance of success

Binary outcome: 20% → \$20B, 80% → \$0. Current value: \$4B

Calculate the risky asset parameters (from your current \$4B position):

$$E(r_p) = 0.2 \times \left(\frac{20}{4} - 1\right) + 0.8 \times \left(\frac{0}{4} - 1\right) = 0.2 \times 4 + 0.8 \times (-1) = 0$$

$$\sigma^2(r_p) = 0.2 \times (4 - 0)^2 + 0.8 \times (-1 - 0)^2 = 3.2 + 0.8 = 4$$

$$\sigma(r_p) = 2 \quad \text{(200\% volatility!)}$$

Risk-free asset: $r_f = 0.03$

Optimal weight: $w^* = \frac{E(r_p - r_f)}{A\sigma^2(r_p)} = \frac{-0.03}{A \times 4} = -\frac{0.0075}{A}$

The model says: SELL! (w < 0 for any positive A)

Vaccine CEO: What if you're more optimistic?

Scenario: You believe there's a 50% chance of success (market thinks 20%)

From current \$4B position, calculate returns if **you're right**:

$$E(r_p) = 0.5 \times \left(\frac{20}{4} - 1\right) + 0.5 \times \left(\frac{0}{4} - 1\right) = 0.5 \times 4 + 0.5 \times (-1) = 1.5$$

$$\sigma^2(r_p) = 0.5 \times (4 - 1.5)^2 + 0.5 \times (-1 - 1.5)^2 = 6.25$$

$$\sigma(r_p) = 2.5$$

Optimal weight: $w^* = \frac{1.5-0.03}{A \times 6.25} = \frac{1.47}{6.25A} = \frac{0.235}{A}$

Risk Aversion (A)	Optimal w	Action
0.5	0.47	Sell 53%
1	0.235	Sell 76.5%
2	0.12	Sell 88%

Even believing 50% chance (2.5$\times$ the market!), model says sell most of it!

Expected returns from a blended portfolio

Expected returns on a portfolio combining the risky (p) and risk-free asset are:

$$\begin{aligned} E(r_{\text{blended}}) &= E\left(wr_p + (1-w)r_f\right) \\ &= wE\left(r_p\right) + (1-w)E\left(r_f\right) \\ &= r_f + wE\left(r_p - r_f\right) \end{aligned}$$

Note, expected returns on the complete portfolio are equal to two parts:

1. the risk free return (r_f)

2. the compensation for exposure to the risk in the risky asset ($E(r_p - r_f)$)
    - Sometimes referred to as the "premium"

Return variance from a blended portfolio

The variance of returns from the blended portfolio is:

$$
\begin{aligned}
\sigma^2(r_{\text{blended}}) &= \text{Var}\big(wr_p + (1-w)r_f\big) \\
&= w^2\sigma^2(r_p) + (1-w)^2\sigma^2(r_f) + 2w(1-w)\sigma(r_p, r_f) \\
&= w^2\sigma^2(r_p) \\
\sigma(r_{\text{blended}}) &= w\sigma(r_p)
\end{aligned}
$$

When we combine these two equations, we get the **“capital allocation line”**

Return variance from a blended portfolio

$$E(r_{\text{blended}}) = r_f + \sigma(r_{\text{blended}}) \frac{E(r_p - r_f)}{\sigma_p}$$

When we combined these two equations, we get the **“capital allocation line”**

$$E(r_{\text{blended}}) = r_f + \sigma(r_{\text{blended}})\frac{E(r_p - r_f)}{\sigma_p}$$

In our simple example,

$$E(r_{\text{blended}}) = r_f + \sigma(r_{\text{blended}})\frac{E(r_p - r_f)}{\sigma_p} = 0.03 + \sigma(r_{\text{blended}})\frac{0.22}{0.75}$$

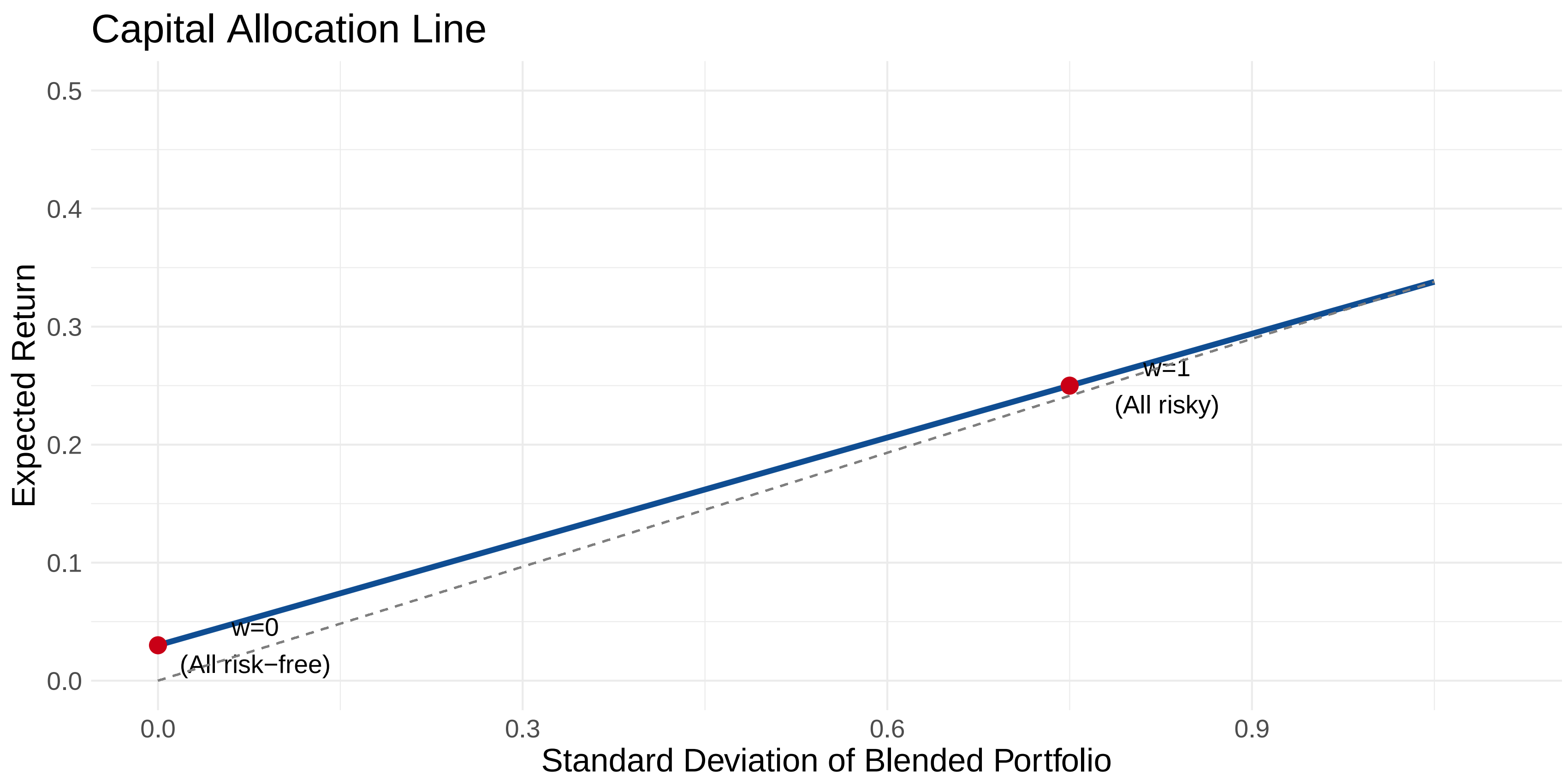
Capital Allocation Line

- The capital allocation line shows all risk-return combinations available based on choice of w.
- The slope of the capital allocation line (the Sharpe Ratio) prices the risk-return tradeoff
- In our example, the Sharpe Ratio is 0.29 (0.22/0.75)

Capital Allocation Line

$$E(r_{\text{blended}}) = r_f + \sigma(r_{\text{blended}})\frac{E(r_p - r_f)}{\sigma_p}$$

Capital Allocation Line



- Point at origin: $w = 0$ (all risk-free asset)

Capital Allocation Line

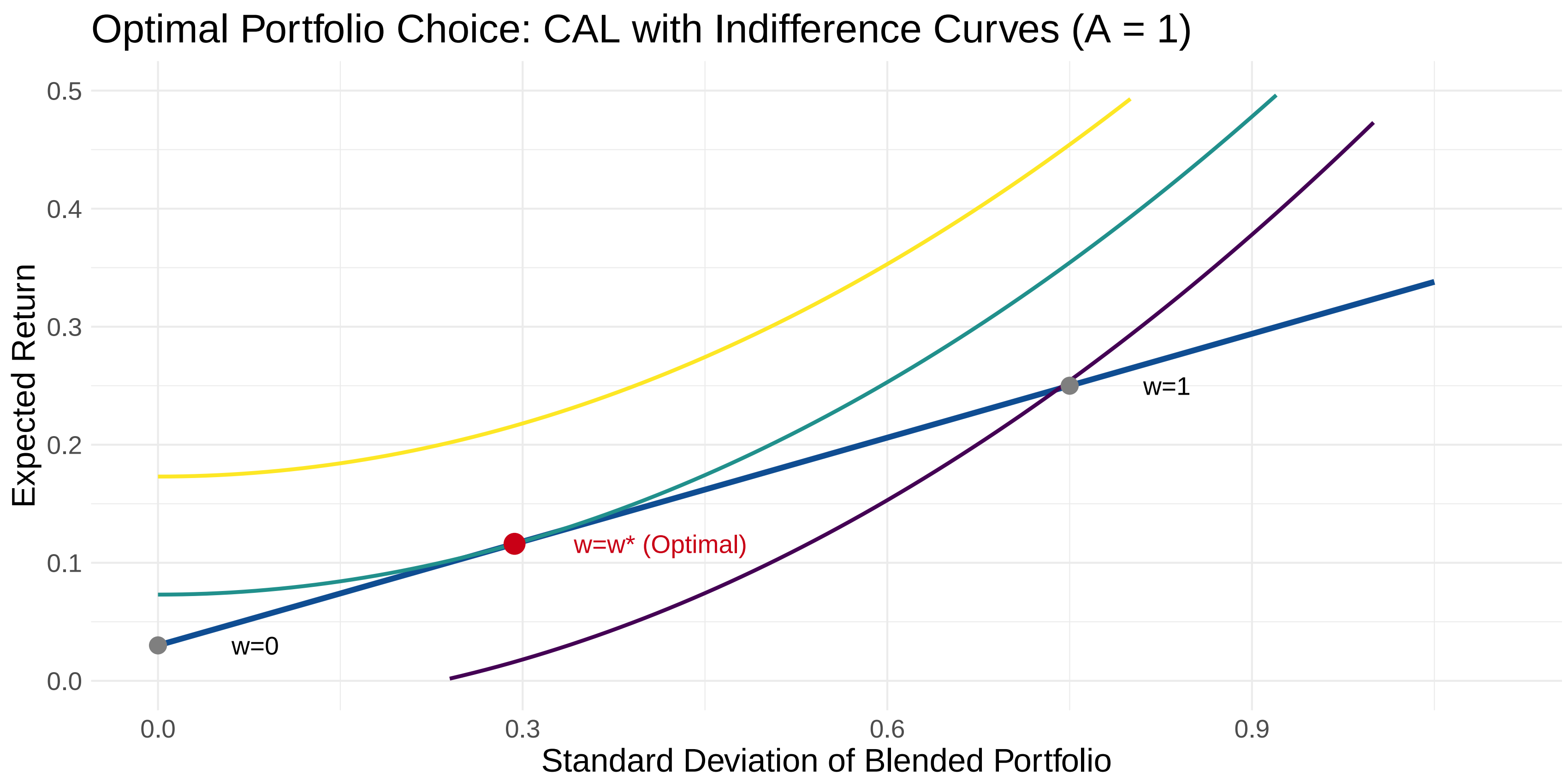
- Point at $\sigma = 0.75$: $w = 1$ (all risky asset)
- Points beyond: $w > 1$ (leveraged position)

Choosing w

Which risk return combination do we want?

- Find point where utility cannot move any further to the northwest, but still touches the CAL

Choosing w



The optimal portfolio is where the indifference curve just touches the CAL.

Choosing w

We can solve this problem more formally as

$$\max_w U(r_{\text{blend}}) = r_f + wE(r_p - r_f) - \frac{1}{2}Aw^2\sigma^2(r_p)$$

This is solved by finding the value of w that sets the derivative equal to zero:

$$w^* = \frac{E(r_p - r_f)}{A\sigma^2(r_p)}$$

Back to our example:

A	w^*	$E(r_{\text{blended}})$	σ_{blended}
0.25	1.56	0.37	1.17
0.5	0.78	0.20	0.51
0.78	0.49	0.14	0.37
1	0.39	0.12	0.29

- What is the meaning of 1.56 in this table? Can you ever get a negative w?
- How are the standard deviations and expected returns for these optimal portfolios computed?

Next Class: February 5

Topics: Building Optimal Portfolios (continued)

- Two risky assets
- Efficient frontier
- Tangency portfolio
- Many risky assets

Matt Levine Reading: “Playing the Game of Infinite Leverage”