KU Leuven Summer School
Segment 1
Jumping into Bayes

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Example story

Going to be getting some patient-level data:

- take standard ($X = 0$) or new ($X = 1$) drug;
- symptoms improve ($Y = 1$) or not ($Y = 0$).

Define $p_x = Pr(Y = 1|X = x)$, $x = 0, 1$.

In advance of receiving data, experts feels that $p_x < 0.05$ is unlikely, as is $p_x > 0.33$, for both $x = 0$ and $x = 1$.

This expert knowledge is encoded in a **prior distribution**:

$p_0, p_1 \overset{\text{iid}}{\sim} \text{beta}(a, b)$.

```r
### declare our hyperparameter specification
hyp <- list(a=3, b=15)

### and confirm this specification is fit-for-purpose
qbeta(c(.05,.95),hyp$a, hyp$b)
```

```
## [1] 0.0499 0.3262
```

Arrival of the data

```
dim(dat)
```

```
## [1] 37  2
```

```
head(dat)
```

```
##   x y
## 1 1 0
## 2 0 0
## 3 0 0
## 4 0 0
## 5 0 0
## 6 0 0
```

```
table(dat)
```

```
##    y
## x    0  1
##   0 22  1
##   1  9  5
```

Statistical model for data given parameters

Sufficient statistics $S_j = \sum_{i:x_i=j} y_i$, $j = 0, 1$.

$S_j \sim Bin(n_j, p_j)$, independently for $j = 0, 1$.

```
n <- as.vector(table(dat$x))
s.dat <- c(sum(dat$y[dat$x==0]), sum(dat$y[dat$x==1]))
```

```
n
```

```
## [1] 23 14
```

```
s.dat
```

```
## [1] 1 5
```

Scalar parameter of most interest:

$\psi = \text{logit}(p_1) - \text{logit}(p_0)$

Generative model

The amalgamation of the prior distribution and the statistical model can be referred to as the *generative model*.

- A completely specified *joint* dist. for data and parameters.
- May feel weird, since from the investigator's perspective parameters are **fixed but unknown**, not **random**. Need to remember: prior distribution is being used to describe **investigator knowledge** about these fixed but unknown quantities.
- Perhaps helpful heuristic: 'Mother Nature' uses the generative model to generate both the state of the world and the ensuing data. But she keeps the former private, only shares the data with the investigator.

Posterior distribution

Describes investigator's knowledge about the state of the world after Nature has shared the data.

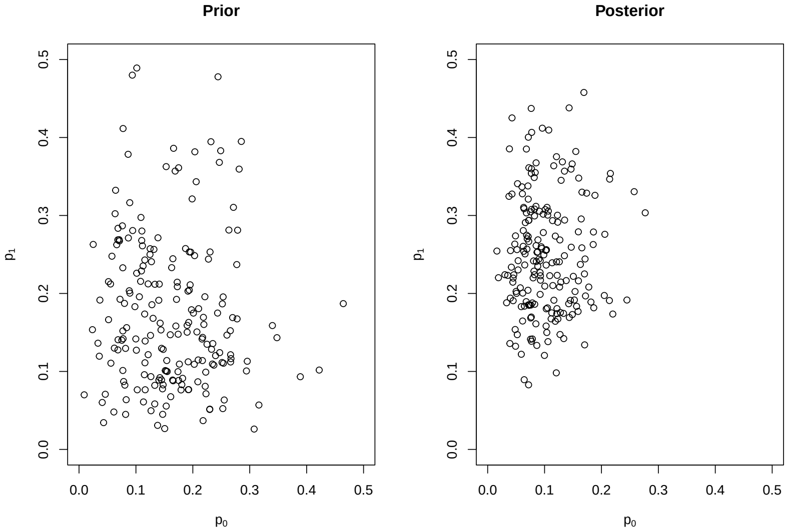
$$
\begin{aligned}
f(p|s) &= \frac{f(s|p)f(p)}{f(s)} \\
&\propto f(s|p)f(p)
\end{aligned}
$$

The Holy Grail of Bayesian Statistics

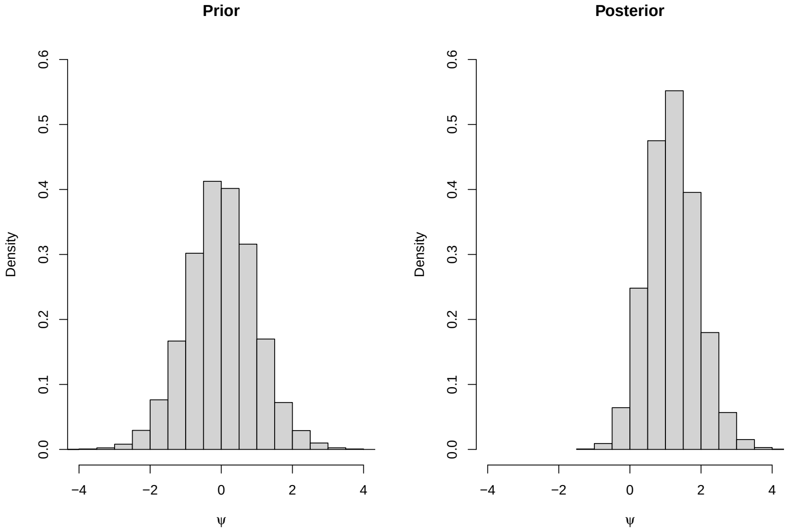
Determining the posterior distribution - in a literal, brute-force manner

```r
### represent the posterior via m MC realizations
m <- 20000; opt.brute <- matrix(NA, m, 2)
colnames(opt.brute) <- c("p0","p1")

have <- 0
while (have < m) {

  ### simulate from generative model
  p.0.sim <- rbeta(1, hyp$a, hyp$b)
  p.1.sim <- rbeta(1, hyp$a, hyp$b)
  s.sim <- c(rbinom(1, size=n[1], prob=p.0.sim),
             rbinom(1, size=n[2], prob=p.1.sim))

  ### only keep if the simulated data matches observed
  if (all(s.sim==s.dat)) {
    have <- have + 1
    opt.brute[have,] <- c(p.0.sim, p.1.sim)
  }
}
```

Prior and posterior distributions of (p_0, p_1)



Estimate the target



Do some inference

A point estimate, $E(\psi|\text{Data})$:

```
mean(opt.brute[,"psi"])
```

```
## [1] 1.2
```

A "standard error" $SD(\psi|\text{Data})$:

```
sqrt(var(opt.brute[,"psi"]))
```

```
## [1] 0.705
```

A 95% interval estimate:

```
quantile(opt.brute[,"psi"], c(0.025, 0.975))
```

```
##   2.5%  97.5%
## -0.111  2.644
```

Nuance: Statistical error versus numerical / Monte Carlo error

$SD(\psi|\text{Data})$ is a 'standard-error-like' quantification of how well the sample quantity $E(\psi|\text{Data})$ estimates the population parameter ψ.

That said, carefully distinguish the roles of:

```
sqrt(var(opt.brute[,"psi"]))
```

```
## [1] 0.705
```

```
sqrt(var(opt.brute[,"psi"])/m)
```

```
## [1] 0.00498
```

Answer by package - JAGS

```
genmod.JAGS <- "model{

  ### prior distribution
  p0 ~ dbeta(a,b)
  p1 ~ dbeta(a,b)

  ### statistical model
  s0 ~ dbinom(p0, n0)
  s1 ~ dbinom(p1, n1)

  ### and for conveience, store the target param also
  psi <- logit(p1)-logit(p0)
}"
```

Answer by JAGS, continued

```r
require(rjags)
```

```r
### generative model, data go in
mod <- jags.model(textConnection(genmod.JAGS),
                  data=list(s0=s.dat[1], n0=n[1],
                            s1=s.dat[2], n1=n[2],
                            a=hyp$a, b=hyp$b),
                  n.chains=4)

### bit of burn-in
update(mod,2000)

###  MC output comes out
opt.JAGS <- coda.samples(mod,
                 variable.names=c("psi"),
                 n.iter=10000)
```

Can we find our friendly neighbourhood estimates in the package output?

```
summary(opt.JAGS)
```

```
##
## Iterations = 3001:13000
## Thinning interval = 1
## Number of chains = 4
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##           Mean             SD       Naive SE Time-series SE
##        1.19930        0.69515        0.00348        0.00450
##
## 2. Quantiles for each variable:
##
##   2.5%    25%    50%    75%  97.5%
## -0.105  0.724  1.179  1.649  2.625
```

FYI, lots of output options

```
require(MCMCvis)
MCMCsummary(opt.JAGS)
```

```
##      mean    sd   2.5%  50% 97.5% Rhat n.eff
## psi  1.2 0.695 -0.105 1.18  2.63    1 23849
```

ASIDE: off-the-shelf **non-Bayesian** analysis of these data?

```
### ML estimate of psi
logit(s.dat[2]/n[2]) - logit(s.dat[1]/n[1])
```

```
## [1] 2.5
```

```
### standard error
sqrt(sum(1/c(s.dat[1], n[1]-s.dat[1], s.dat[2], n[2]-s.dat[2])))
```

```
## [1] 1.16
```

```
### or if you prefer, get these from...
glm(y~x, family=binomial)
```

Can speculate on **two** reasons why our present Bayesian answer is not so close to this

Black-box package actually overkill for problem we just did

Our problem has **conjugate** structure, ergo a clean math description of the posterior dist. for $(p_0, p_1 | S_0, S_1)$:

$(p_j | S = s) \sim \text{beta}(a + s_j, b + (n_j - s_j))$, independently for $j = 0, 1$.

So yet another route to our inference

```
opt.MC <-cbind(
  "p0"=rbeta(m, hyp$a+s.dat[1], hyp$b+n[1]-s.dat[1]),
  "p1"=rbeta(m, hyp$a+s.dat[2], hyp$b+n[2]-s.dat[2]))

opt.MC <- cbind(opt.MC,
  "psi"=logit(opt.MC[,"p1"])-logit(opt.MC[,"p0"]))
```

```
## posterior mean and SD of target
c(mean(opt.MC[,"psi"]), sqrt(var(opt.MC[,"psi"])))
```

```
## [1] 1.196 0.697
```

```
### quality of numerical approximation
sqrt(var(opt.MC[,"psi"])/m)
```

```
## [1] 0.00493
```

In fact, for problem at hand, can skip MC approximation, compute exactly

```r
### posterior mean of psi
(digamma(hyp$a+s.dat[2])-digamma(hyp$b+n[2]-s.dat[2])) -
(digamma(hyp$a+s.dat[1]) - digamma(hyp$b+n[1]-s.dat[1]))
```

```
## [1] 1.2
```

```r
### posterior SD of psi
sqrt(trigamma(hyp$a+s.dat[2]) + trigamma(hyp$b+n[2]-s.dat[2]) +
     trigamma(hyp$a+s.dat[1]) + trigamma(hyp$b+n[1]-s.dat[1]))
```

```
## [1] 0.698
```

Back to the holy grail

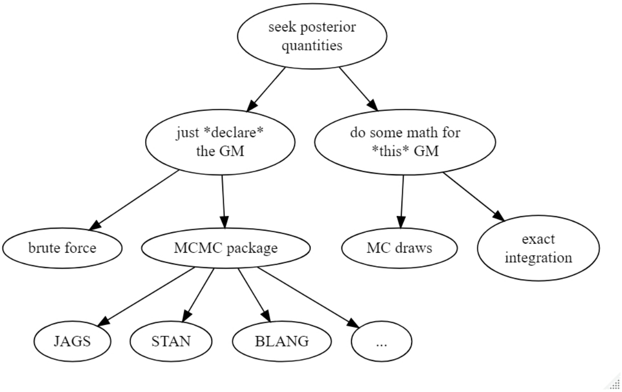
What *might* a platinum grail spit out?

What *might* a golden grail spit out?

What *might* a silver grail spit out?

What **does** a bronze grail spit out?

Taxonomy of the computing options we've seen today



Since we are typically stuck with a bronze grail (i.e., MCMC) . . .

- Users can/should exert control over how much computing to do.
- Users can/should monitor how successful this computing is.

APPENDIX: Could you stand to see one more computational route to the same end?

```
genmod.STAN <- "
data {
  int<lower=0> n[2];
  int<lower=0> s[2];
  real<lower=0> a[2];
  real<lower=0> b[2];
}
parameters {
  real<lower=0,upper=1> p[2];
}
transformed parameters {
  real psi;
  psi =logit(p[2])-logit(p[1]);
}
model {
  p ~ beta(a,b);
  s ~ binomial(n,p);
}
```

STAN, continued

```r
require("rstan"); rstan_options(auto_write = TRUE)
```

```r
opt.STAN <- stan(model_code=genmod.STAN,
  data=list(s=s.dat, n=n, a=rep(hyp$a,2), b=rep(hyp$b,2)),
  iter=12000)
```

STAN, continued

```r
summary(opt.STAN)$summary
```

```
##           mean  se_mean     sd     2.5%      25%      50%     75%
## p[1]    0.0965 0.000304 0.0447   0.0286   0.0631   0.0902   0.123
## p[2]    0.2496 0.000529 0.0746   0.1200   0.1961   0.2442   0.298
## psi     1.2074 0.004873 0.6817  -0.0863   0.7423   1.1863   1.654
## lp__  -32.1083 0.009706 1.0006 -34.7985 -32.4934 -31.8057 -31.400
##         97.5% n_eff Rhat
## p[1]    0.199 21618    1
## p[2]    0.405 19883    1
## psi     2.601 19572    1
## lp__  -31.127 10627    1
```

Due diligence: Two different black-boxes give same answer?

```
MCMCsummary(opt.STAN)
```

```
##          mean     sd     2.5%      50%   97.5% Rhat n.eff
## p[1]   0.0965 0.0447   0.0286   0.0902   0.199    1 21618
## p[2]   0.2496 0.0746   0.1200   0.2442   0.405    1 19883
## psi    1.2074 0.6817  -0.0863   1.1863   2.601    1 19572
## lp__ -32.1083 1.0006 -34.7985 -31.8057 -31.127    1 10627
```

```
MCMCsummary(opt.JAGS)
```

```
##      mean    sd   2.5%  50% 97.5% Rhat n.eff
## psi   1.2 0.695 -0.105 1.18  2.63    1 23849
```