KU Leuven Summer School
Segment 6
Bayesian Calibration

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Need a simple sandbox to play in

A stripped-down misclassification problem.

Interested in the prevalance, r, of a disease in a population.

The diagnostic test for the condition is known to have perfect *specificity*, i.e., no false positives.

But there is ambiguity about the *sensitivity*, i.e., could be some false negatives.

The diagnostic test is applied to a random sample of n individuals from the target population.

Generative model (collapsed version)

- $r \sim \text{Unif}(0, 1)$
- $Sn \sim \text{Beta}(a, b)$
- $(Y^*|r, Sn) \sim \text{Bin}(n, r \times Sn)$

And let's make things quite focussed:

Inputs:

- Dataset (Y^*, n)
- Expert opinion (hyperparameters) (a, b)

Output:

- (general) posterior distribution of (r, Sn)
- (specific) 80% equal-tailed credible interval for r

Computational implementation

Would be easy to implement in JAGS.

But nice to have something **faster,** and less beholden to **diagnostics**, to support **simulation studies**.

Turns out this is a **nearly conjugate** situation

- possible to do *iid* Monte Carlo draws from a decent approximation to the posterior distribution
- possible to correct for the approximation error via **importance weights** that *do not depend on the data*

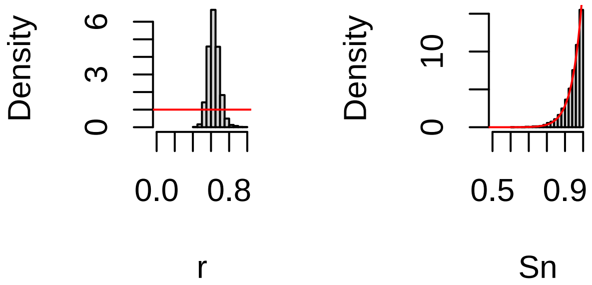
See the appendix for bespoke code, if interested.

Example

Say the data are $(Y^*, n) = (60{,}100)$.

Say the investigator believes that the diagnostic test could be *slightly* insensitive, chooses hyperparameters $(a, b) = (19, 1)$

Marginal posterior distributions of r and Sn, compared to prior



And then focus on the credible interval

From bespoke code, 80% credible interval for r:

```
cred.int(60,100, hyp=list(a=19,b=1))
```

```
## [1] 0.558 0.713
```

For comparison, the corresponding interval if the investigator assumes $Sn = 1$

```
qbeta(c(0.1,0.9), shape1=1+60, shape2=1+(100-60))
```

```
## [1] 0.536 0.660
```

Thought (well simulation) experiment #1A

Frequentist coverage of the 80% credible interval?

At a particular spot in the parameter space.

```
n <- 100; r.tr <- 0.7; sn.tr <- 0.85; m <- 1600

ystr <- cover <- rep(NA, m)
intrvl <- matrix(NA, m, 2)

for (i in 1:m) {
   ystr[i] <- rbinom(1, size=n, prob=r.tr*sn.tr)
   intrvl[i,] <- cred.int(ystr[i],n, hyp=list(a=19, b=1))
   cover[i] <- (intrvl[i,1]<r.tr) & (r.tr<intrvl[i,2])
}
```

Experiment #1A, continued

```
head(cbind(r.tr, sn.tr, ystr,intrvl,cover),8)
```

```
##      r.tr sn.tr ystr             cover
## [1,]  0.7  0.85   52 0.477 0.628     0
## [2,]  0.7  0.85   50 0.455 0.608     0
## [3,]  0.7  0.85   51 0.466 0.616     0
## [4,]  0.7  0.85   64 0.598 0.753     1
## [5,]  0.7  0.85   58 0.534 0.691     0
## [6,]  0.7  0.85   57 0.526 0.679     0
## [7,]  0.7  0.85   57 0.527 0.682     0
## [8,]  0.7  0.85   67 0.630 0.785     1
```

```
### frequentist coverage
mean(cover)
```

```
## [1] 0.571
```

```
### average width
mean(intrvl[,2]-intrvl[,1])
```

```
## [1] 0.154
```

Thought experiment #1B

Frequentist coverage of the 80% credible interval?

At a different spot in the parameter space.

```
n <- 100; r.tr <- 0.8; sn.tr <- 0.97; m <- 1600

ystr <- cover <- rep(NA, m)
intrvl <- matrix(NA, m, 2)

for (i in 1:m) {
   ystr[i] <- rbinom(1, size=n, prob=r.tr*sn.tr)
   intrvl[i,] <- cred.int(ystr[i],n, hyp=list(a=19, b=1))
   cover[i] <- (intrvl[i,1]<r.tr) & (r.tr<intrvl[i,2])
}
```

Experiment #1B, continued

```
head(cbind(r.tr, sn.tr, ystr,intrvl,cover),8)
```

```
##      r.tr sn.tr ystr             cover
## [1,]  0.8  0.97   83 0.801 0.935     0
## [2,]  0.8  0.97   80 0.767 0.909     1
## [3,]  0.8  0.97   82 0.789 0.928     1
## [4,]  0.8  0.97   74 0.703 0.854     1
## [5,]  0.8  0.97   70 0.660 0.815     1
## [6,]  0.8  0.97   81 0.778 0.918     1
## [7,]  0.8  0.97   75 0.716 0.864     1
## [8,]  0.8  0.97   80 0.767 0.912     1
```

```
### frequentist coverage
mean(cover)
```

```
## [1] 0.894
```

```
### average length
mean(intrvl[,2]-intrvl[,1])
```

```
## [1] 0.146
```

Thought experiment #2A

Repeated sampling of **(parameter,data) pairs**

```
n <- 100; m <- 6400

r.tr <- sn.tr <- ystr <- cover <- rep(NA, m)
intrvl <- matrix(NA, m, 2)

for (i in 1:m) {
   r.tr[i] <- runif(1)
   sn.tr[i] <- rbeta(1, shape1=19, shape2=1)

   ystr[i] <- rbinom(1, size=n, prob=r.tr[i]*sn.tr[i])
   intrvl[i,] <- cred.int(ystr[i],n, hyp=list(a=19, b=1))
   cover[i] <- (intrvl[i,1]<r.tr[i]) & (r.tr[i]<intrvl[i,2])
}
```

Thought experiment #2A, continued

```
head(cbind(r.tr, sn.tr, ystr,intrvl,cover),10)
```

```
##        r.tr sn.tr ystr             cover
##   [1,] 0.260 0.877   16 0.127 0.230     0
##   [2,] 0.508 0.947   37 0.325 0.466     0
##   [3,] 0.142 0.962   13 0.101 0.194     1
##   [4,] 0.806 0.964   80 0.768 0.911     1
##   [5,] 0.150 0.984   16 0.127 0.232     1
##   [6,] 0.639 0.964   63 0.590 0.744     1
##   [7,] 0.194 0.916   21 0.172 0.286     1
##   [8,] 0.854 0.994   83 0.800 0.934     1
##   [9,] 0.402 0.956   38 0.336 0.475     1
##  [10,] 0.689 0.978   70 0.662 0.813     1
```

```
mean(cover)
```

```
## [1] 0.793
```

Is this general?

Let $A(Y^*)$ be the credible interval

Under the **generative model:**

$$\begin{aligned} Pr\{\theta \in A(Y^*)\} &= E\{I_{A(Y^*)}(\theta)\} \\ &= \end{aligned}$$

Thought experiment #2B

Repeated sampling of (parameter,data) pairs *from a different distribution.*

```
n <- 100; m <- 1600

r.tr <- sn.tr <- ystr <- cover <- rep(NA, m)
intrvl <- matrix(NA, m, 2)

for (i in 1:m) {
   r.tr[i] <- runif(1)
   sn.tr[i] <- rbeta(1, shape1=15, shape2=5)

   ystr[i] <- rbinom(1, size=n, prob=r.tr[i]*sn.tr[i])
   intrvl[i,] <- cred.int(ystr[i],n, hyp=list(a=19, b=1))
   cover[i] <- (intrvl[i,1]<r.tr[i]) & (r.tr[i]<intrvl[i,2])
}
```

TE #2B, continued

```r
head(cbind(r.tr, sn.tr, ystr,intrvl,cover),10)
```

```
##         r.tr sn.tr ystr               cover
##  [1,] 0.8037 0.724   53 0.4864 0.637     0
##  [2,] 0.3861 0.734   25 0.2103 0.332     0
##  [3,] 0.5380 0.804   47 0.4267 0.574     1
##  [4,] 0.0894 0.848   13 0.0996 0.193     0
##  [5,] 0.6420 0.738   55 0.5061 0.661     1
##  [6,] 0.5261 0.824   39 0.3456 0.489     0
##  [7,] 0.8525 0.882   79 0.7568 0.902     1
##  [8,] 0.2517 0.675   27 0.2293 0.357     1
##  [9,] 0.6042 0.779   46 0.4162 0.563     0
## [10,] 0.5692 0.764   47 0.4238 0.573     1
```

```r
mean(cover)
```

```
## [1] 0.397
```

Frequentist coverage of the Bayesian interval revisited

Same as 1A and 1B, just with more data now.

Thought Experiment #1A*

```
n <- 200; r.tr <- 0.7; sn.tr <- 0.85; m <- 1600

ystr <- cover <- rep(NA, m)
intrvl <- matrix(NA, m, 2)

for (i in 1:m) {
   ystr[i] <- rbinom(1, size=n, prob=r.tr*sn.tr)
   intrvl[i,] <- cred.int(ystr[i],n, hyp=list(a=19, b=1))
   cover[i] <- (intrvl[i,1]<r.tr) & (r.tr<intrvl[i,2])
}
```

```
mean(cover)
```

```
## [1] 0.429
```

```
mean(intrvl[,2]-intrvl[,1])
```

```
## [1] 0.123
```

Thought Experiment #1B*

```
n <- 200; r.tr <- 0.8; sn.tr <- 0.97; m <- 1600

ystr <- cover <- rep(NA, m)
intrvl <- matrix(NA, m, 2)

for (i in 1:m) {
   ystr[i] <- rbinom(1, size=n, prob=r.tr*sn.tr)
   intrvl[i,] <- cred.int(ystr[i],n, hyp=list(a=19, b=1))
   cover[i] <- (intrvl[i,1]<r.tr) & (r.tr<intrvl[i,2])
}
```

```
mean(cover)
```

```
## [1] 0.922
```

```
mean(intrvl[,2]-intrvl[,1])
```

```
## [1] 0.125
```

In textbook problems, what do we expect?

textbook = full information = fully identified

- Frequentist coverage of x-percent credible interval is approximately x, at *every* point in the parameter space.

(Approximately meaning asymptotically.)

(Asymptotically, Bayesian and frequentist match up.)

- Width of interval scales as $\frac{1}{\sqrt{n}}$

In problems like the one here, what happens instead (1 of 3)

- Frequentist coverage of x-percent credible interval *varies widely* across the parameter space
- But no matter what, the average (in a certain sense) of the frequentist coverage is *exactly* x.

In problems like the one here, what happens instead (2 of 3)

- At some places in the parameter space, the frequentist coverage (of the x-percent credible interval) goes to as n goes to infinity.
- And at all the other places, it goes to
- And we can say something very specific about the set of parameter values for which the limiting frequentist coverage is , namely that

In problems like the one here, what happens instead (3 of 3)

- The width of the x-percent interval will scale like:
- This will not win you friends with your subject-area collaborators, but it is what it is.
- Check this matches up with $\#1A \rightarrow \#1A^*$, $\#1B \rightarrow \#1B^*$

Musings about Bayesian coverage (1 of 4)

Clearly having calibration in Bayesian coverage sense is not as strong as having calibration in a frequentist coverage sense

In math terms, say a given interval estimation procedure applied at a given sample size has frequentist coverage $fc(\theta)$, when the parameter value is θ.

Frequentist x-percent confidence interval: $fc(\theta) = x$, for every θ.

Bayesian x-percent credible interval using prior $\pi(\theta)$?

Only full/general guarantee is that $\int fc(\theta)\Pi(\theta)d\theta = x$

Musings (2 of 4)

But in many low-info problems, there **do not exist** interval estimation procedures satisfying $fc(\theta) = x$. (Either for fixed n, or in the large n limit)

In such problems, the Bayesian calibration is the only game in town?

Sidebar: Considerable technical literature (mostly in econometrics) on trying to construct procedures with $fc(\theta) \geq x$, for every θ.

Musings (3 of 4)

What's the narrative to practitioners?

I choose prior π as my pre-data "projection" about the state of the world (i.e., the underlying parameter values).

Post-data, I will be reporting an x-percent credible interval for a (scalar) target parameter.

Then with respect to my joint projection (of parameters and data), there is an x-percent chance I will cover the truth.

Musings (4 of 4)

Or phrased a little differently ...

I direct a lab that will, over time, study the relationship between **different** exposure, disease pairs

I aspire to specify my prior distribution to correctly reflect the pair-to-pair variation in these associations.

If I meet my aspiration, then, in the long-run, x percent of the x-percent credible intervals the lab reports will contain the truth.

And now that you are primed to think about operating characteristics under repeated sampling of (parameter, data) pairs ...

There is a sense of best possible estimation, as well as a sense of correct coverage

Let $\pi_{Nature}(\theta)$ be the distribution giving rise to the repeated sampling (along with the distribution of data D given θ)

Amongst **any and all estimators** the minimum mean-squared error (across the repeated sampling) is achieved by the posterior mean of ψ when the investigator's choice of prior distribution matches that of nature.

Thoughts?

Appendix

```
show(full.pst)
```

```
## function(ystr, n, hyp, m=10000) {
##
##   ### draws from approx posterior in (rstr, sn) parameterization
##   rstr <- rbeta(m, 1+ystr, 1+(n-ystr))
##   sn <- rbeta.trnc(m, rstr, hyp$a-1, hyp$b)
##
##   ### importance weights to correct for approximation
##   impwht <- 1-pbeta(rstr, hyp$a-1, hyp$b)
##   impwht <- m*impwht/sum(impwht)
##
##   ### back to (r, sn) parameterization
##   r <- rstr/sn
##
##   # resample according to the weights,
##   # to get MC representation of actual posterior
##   tmp <- sample(1:m, replace=T, prob=impwht)
##   list(r=r[tmp], sn=sn[tmp])
## }
## <bytecode: 0x000001880a5cde90>
```

Appendix, continued

```
show(cred.int)
```

```
## function(ystr, n, hyp, m=10000, cr.lev=0.8) {
##
##   rstr <- rbeta(m, 1+ystr, 1+(n-ystr))
##
##   sn <- rbeta.trnc(m, rstr, hyp$a-1, hyp$b)
##
##   r <- rstr/sn
##
##   impwht <- 1-pbeta(rstr, hyp$a-1, hyp$b)
##   impwht <- m*impwht/sum(impwht)
##
##   c(weighted.quantile(r, impwht, (1-cr.lev)/2),
##     weighted.quantile(r, impwht, (1+cr.lev)/2))
## }
```

Appendix, continued

```r
show(rbeta.trnc)
```

```
## function(m, lwr, a, b) {
##   qbeta(  runif(m, pbeta(lwr,a, b), 1), a,b)
## }
```