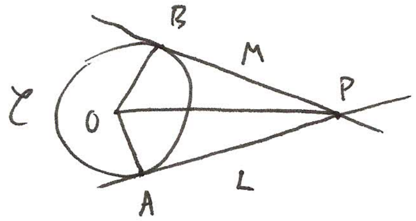
Q1.



Let O be the center of the circle $\mathcal{C}$.

$\angle OAP = \angle OBP = \pi/2$

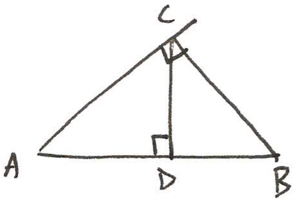
(radius is perpendicular to tangent)

$|OA| = |OB| = r$, radius of $\mathcal{C}$.

$|OA|^2 + |AP|^2 = |OP|^2 = |OB|^2 + |BP|^2$ (Pythagoras' thm)

$\therefore |AP| = \sqrt{|OP|^2 - |OA|^2} = \sqrt{|OP|^2 - |OB|^2} = |BP|$. $\square$.

Q2.



$\angle BAC = \pi - \pi/2 - \angle ACD$ (angle sum of triangle)

$= \pi/2 - \angle ACD$

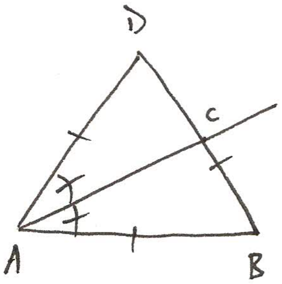
$= \angle BCD$.

$\Rightarrow \triangle ABC \sim \triangle CBD$ (equal angles)

$\Rightarrow \dfrac{|AB|}{|CB|} = \dfrac{|CB|}{|DB|}$

$\Rightarrow |AB| \cdot |BD| = |BC|^2$ $\square$.

Q3. a.



1. Construct an equilateral triangle $\triangle ABD$ with base AB (draw circles with centers at A & B & radius $|AB|$, let D be one of the intersection points.)
2. Bisect the angle $\angle BAD$; let C be the intersection point of the bisector with BD.

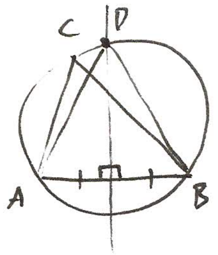
Then $\angle ABD = \angle BAD = \pi/3$ (angles of an equilateral triangle are equal by the isosceles triangle theorem, so each equal to $\pi/3$ by angle sum of a triangle $= \pi$).

& $\angle BAC = \frac{1}{2} \angle BAD = \pi/6$ by construction (angle bisected),

$\angle ABC = \angle ABD = \pi/3$, so $\angle ACB = \pi - \pi/3 - \pi/6 = \pi/2$ (by angle sum of $\triangle ABC$).

$\square$.

3b.



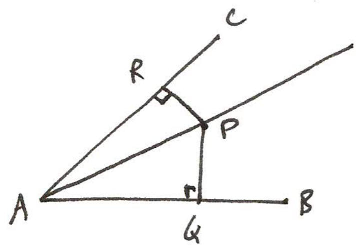
1. Draw the circumscribed circle $\mathscr{C}$ of ΔABC
(Draw the perpendicular bisectors of two of the sides, these intersect at a point O, then $\mathscr{C}$ is the circle w/ centre O & radius $r = |OA| = |OB| = |OC|$).
2. Let D be the intersection point of $\mathscr{C}$ & the perpendicular bisector of AB on the same side of AB as C.

Then $\angle ACB = \angle ADB$ (angles subtended by a chord at the circumference)

$|AD| = |BD|$ (D lies on the perpendicular bisector of AB)

$\therefore \angle ABD = \angle BAD$ (isosceles triangle theorem) $\square$.

4a.



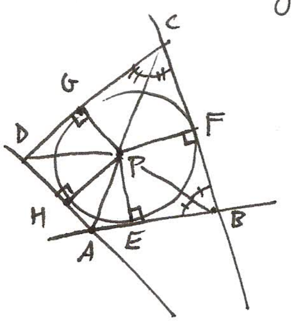
$\angle PAR = \angle PAQ$ by construction
(AP bisects the angle $\angle BAC$)

So $\angle APR = \pi - \frac{\pi}{2} - \angle PAR = \pi - \frac{\pi}{2} - \angle PAQ = \angle APQ$
by angle sum of a triangle.

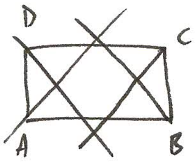
$\Rightarrow \Delta PAQ \cong \Delta PAR$ (ASA)

$\Rightarrow |PQ| = |PR|$ $\square$

b. Draw perpendicular lines to AB, BC, CD, DA through P, meeting AB, BC, CD, DA at E, F, G, H. By part (a) $|PE| = |PF| = |PG| = |PH| =: r$.
Draw a circle $\mathscr{C}$ w/ center P and radius r. Then $\mathscr{C}$ is tangent to AB, BC, CD, DA at E, F, G, H (tangent is perpendicular to radius)



c. Counterexample: If ABCD is a rectangle which is not a square, then the angle bisectors are not concurrent:



5a. $T(x,y) = (-y+3,\ x-3)$

Fix(T): $\begin{array}{l} x = -y+3 \\ y = x-3 \end{array} \rightsquigarrow \begin{array}{l} x+y=3 \\ x-y=3 \end{array} \rightsquigarrow (x,y) = (3,0)$

$\Rightarrow$ T is a rotation about $(3,0)$ thru angle θ ccw, where

$$\begin{pmatrix} -y+3 \\ x-3 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad \Rightarrow \quad \theta = \pi/2. \ \square.$$

b. $T(x,y) = \frac{1}{5}(-3x-4y+4,\ -4x+3y+2)$

Fix(T): $\begin{array}{l} x = \frac{1}{5}(-3x-4y+4) \\ y = \frac{1}{5}(-4x+3y+2) \end{array} \rightsquigarrow \begin{array}{l} -8x-4y=-4 \\ -4x-2y=-2 \end{array} \rightsquigarrow 2x+y=1.$

$\Rightarrow$ T is a reflection in the line L w/ equation $y = -2x+1$. $\square$.

c. $T(x,y) = (y+5,\ x+1)$

Fix(T): $\begin{array}{l} x = y+5 \\ y = x+1 \end{array} \rightsquigarrow \begin{array}{l} x-y=5 \\ x-y=-1 \end{array}$ $\#$. Fix$(T) = \emptyset$.

$\Rightarrow$ T is a translation or a glide. Clearly not a translation $(T(x,y) \neq (x+a, y+b))$ so a glide.

$T^2(x,y) = ((x+1)+5,\ (y+5)+1) = (x+6, y+6)$ translation by $(6,6)$.

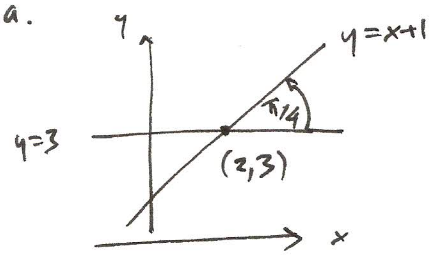
$\Rightarrow$ $T = \text{Trans}_{\underline{v}}\ \text{Refl}_L$ $\quad \underline{v} = \frac{1}{2}(6,6) = (3,3)$, L a line parallel to $\underline{v}$.

Now $\text{Refl}_L(x,y) = T(x,y) - \underline{v} = (y+2, x-2)$

Fixed locus: $\left.\begin{array}{l} x = y+2 \\ y = x-2 \end{array}\right\} \rightsquigarrow y = x-2$. this is the eq. of the line L.

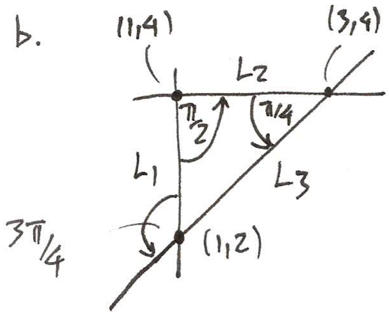
So T is a glide reflection given by reflection in the line $L: y = x-2$ followed by translation by $(3,3)$. $\square$.

Q6. a.



$\Rightarrow$ Rotation about $(2,3)$ thru angle $2 \cdot \pi/4 = \pi/2$ ccw. $\square$.

b.



$$T = (\text{Refl}_{L_3} \circ \text{Refl}_{L_2}) \circ (\text{Refl}_{L_2} \circ \text{Refl}_{L_1})$$
$$= \text{Refl}_{L_3} \circ \text{Refl}_{L_1}$$
$$= \text{rotation about } (1,2) \text{ thru } 2 \cdot \tfrac{3\pi}{4} = \tfrac{3\pi}{2} \text{ ccw.}$$

$\square$.