Math 462: Homework 7

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(1) Consider the motion T of $\mathbb{R}^3$ defined by $T(\mathbf{x}) = -\mathbf{x}$.

(a) Is T a direct or opposite motion?

(b) We classified motions of $\mathbb{R}^3$ into 6 types: rotations, reflections, rotary reflections, translations, glides, and twists. What type of motion is T? Give a precise geometric description in these terms (include angle of rotation and/or plane of reflection etc.).

(c) Now let P be a regular polyhedron in $\mathbb{R}^3$ with center at the origin (so P is either a tetrahedron, cube, octahedron, dodecahedron, or icosahedron). In which cases is T a symmetry of P?

(2) Recall that the *dihedral group* D_n is the group of symmetries of a regular n-sided polygon P. We showed in class that D_n consists of n rotations (including the identity transformation) and n reflections. Moreover, if a denotes a rotation about the center of P through angle $2\pi/n$ anticlockwise and b denotes reflection in a line of symmetry of P, then D_n is generated by a and b (more precisely, the rotations are $1, a, \ldots, a^{n-1}$ and the reflections are $b, ab, \ldots, a^{n-1}b$). The symmetries a and b satisfy the relations $a^n = b^2 = 1$ and $ba = a^{-1}b$ (and all other relations can be derived from these). Express the following products in D_5 in the form a^ib^j where $0 \leq i \leq 5$ and $0 \leq j \leq 1$ and describe them geometrically.

(a) a^3bab.

(b) $a^2ba^3b^2a$

(c) $a^2ba^{-1}b^{-1}a^3b^3$.

(3) In this problem we see that the dihedral group D_n can be realized as a subgroup of the group of *rotations* of $\mathbb{R}^3$. Let P be a regular n-sided polygon. Position P in the plane $(z = 0) \subset \mathbb{R}^3$ with center O at the origin and one vertex Q on the x-axis.

(a) What is the matrix A of the rotation S of $\mathbb{R}^3$ about the z-axis through angle $2\pi/n$ anticlockwise? What is the symmetry of P induced by S?

(b) Describe a rotation T of $\mathbb{R}^3$ which maps P to itself and when regarded as a symmetry of P is given by reflection in the line OQ. What is the matrix B of T?

(c) Use parts (a) and (b) to describe the dihedral group D_n (the group of symmetries of P) in terms of rotations of $\mathbb{R}^3$. What are the matrices of these rotations?

(4) Recall that a *permutation* of $1, 2, \ldots, n$ is a function $f\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $f(i) \neq f(j)$ for $i \neq j$. The *symmetric group* S_n is the group of all permutations with the group law being composition of functions. A *transposition* is a permutation f that switches two elements i and j and leaves the remaining elements fixed, that is, $f(i) = j$, $f(j) = i$, and $f(k) = k$ for $k \neq i, j$. A permutation f is a *cycle of length* l if there are distinct numbers $i_1, \ldots, i_l$ such that $f(i_1) = i_2$, $f(i_2) = i_3$, …, $f(i_{l-1}) = i_l$, $f(i_l) = i_1$, and $f(k) = k$ for $k \neq i_1, \ldots, i_l$ (for example, a transposition is a cycle of length 2). We say two cycles f, g are *disjoint* if for each k either $f(k) = k$ or $g(k) = k$. In class we explained that every permutation can be written as a product of transpositions. We also explained how to write a permutation as a composition of disjoint cycles.

(a) Show that order of S_n (the number of permutations of $1, 2, \ldots, n$) equals $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

(b) Write each of the following permutations as a product of disjoint cycles.

(i) $f\colon \{1, \ldots, 5\} \to \{1, \ldots, 5\}$, $f(1) = 3$, $f(2) = 4$, $f(3) = 5$, $f(4) = 1$, $f(5) = 2$.

(ii) $g\colon \{1, \ldots, 7\} \to \{1, \ldots, 7\}$, $g(1) = 7$, $g(2) = 6$, $g(3) = 1$, $g(4) = 2$, $g(5) = 3$, $g(6) = 4$, $g(7) = 5$.

(iii) $h\colon \{1, \ldots, 8\} \to \{1, \ldots, 8\}$, $h(1) = 6$, $h(2) = 2$, $h(3) = 5$, $h(4) = 7$, $h(5) = 8$, $h(6) = 1$, $h(7) = 3$, $h(8) = 4$.

(c) Using part (b) or otherwise, write each of the permutations f, g, h as a composition of transpositions.

(d) The cycle type of a permutation f is the (unordered) list of the lengths of the cycles in the description of f as a composition of disjoint cycles. For example $f = (123)(45)(67)$ has cycle type $2, 2, 3$. List the possible cycle types for S_4 and S_5. How can we determine the order of a permutation from its cycle type? (The *order* of a permutation is the least number $r \geq 1$ such that applying the permutation r times gives the identity permutation.)

(e) We say that a permutation is *even* if it can be written as a product of an even number of transpositions. The set of even permutations is a subgroup of S_n called the *alternating group* A_n. What are the possible cycle types for elements of A_4 and A_5? What are their orders?

(5) In HW6 we showed that the group of symmetries of the tetrahedron can be identified with the symmetric group S_4 of permutations of 4 objects by considering the permutations of the vertices of the tetrahedron induced by its symmetries. For each cycle type of S_4 describe the corresponding symmetries geometrically. What is the geometric significance of the subgroup $A_4 \subset S_4$?