Math 462 Final exam review questions

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April 25, 2013

(1) Find a Mobius transformation

$$f\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, \quad f(z) = \frac{az+b}{cz+d}$$

such that $f(2)=0$, $f(3)=\infty$ and $f(i)=1$.

(2) Let C be the circle with center $2i$ and radius 1. Find a Mobius transformation $f\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ such that $f(C)$ is the x-axis.

(3) Let $S^2 \subset \mathbb{R}^3$ be the sphere with center the origin and radius 1, and

$$F\colon S^2 \to \mathbb{C}\cup\{\infty\}$$

the map given by stereographic projection from the north pole $N=(0,0,1)$ to the xy-plane. Compute the image $F(C)$ of the circle $C \subset S^2$ given by $C = \Pi \cap S^2$ where $\Pi \subset \mathbb{R}^3$ is the plane given by the following equations.

(a) $2x+y+3z=3$.

(b) $2x+2y+3z=4$.

[Hint: Recall the formulas $F(x,y,z) = \frac{1}{1-z}(x,y)$ and $F^{-1}(u+iv) = \frac{1}{u^2+v^2+1}(2u,2v,u^2+v^2-1)$.]

(4) Write down the formula for the function $g\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ given by inversion in the circle with center the origin and radius 1. Describe the image of the following circles and lines under the map g.

(a) The circle with center the point $\frac{1}{2}$ and radius $\frac{1}{2}$.

(b) The line $y = 1$.

(c) The circle with center the point $\sqrt{2}$ and radius 1.

(d) The circle with center the point $2 + 2i$ and radius $\sqrt{2}$.

(5) Find the hyperbolic line in the upper half plane $\mathcal{H}$ through the following pairs of points.

(a) $2 + i$, $2 + 5i$.

(b) $1 + 2i$, $3 + 2i$.

(c) i, $2 + 3i$.

(6) Find the hyperbolic line in the upper half plane $\mathcal{H}$ passing through the point $3 + 4i$ and having tangent direction $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ at that point.

(7) Let L be a hyperbolic line in the upper half plane given by a semicircle with center the point 5 and radius 2. Find a hyperbolic isometry $f \colon \mathcal{H} \to \mathcal{H}$ such that $f(L)$ is the hyperbolic line given by the y-axis.

(8) Find a hyperbolic isometry f of $\mathcal{H}$ such that $f(1 + 2i) = 6 + 4i$

[Hint: Use an isometry of the form $f(z) = az + b$, $a, b \in \mathbb{R}$, $a > 0$ (a composition of a scaling and a translation).]

(9) Compute the hyperbolic length of the segment of the Euclidean line connecting the points i and $4 + 2i$.

[Hint: Recall that the hyperbolic length of a parametrized path

$$\gamma \colon [a, b] \to \mathcal{H}, \quad \gamma(t) = (x(t), y(t))$$

is given by the integral

$$\int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

Describe a parametrization of the line segment and compute the integral to determine its hyperbolic length.]

(10) Compute the angle between the hyperbolic lines L and M in $\mathcal{H}$ in each of the following cases.

(a) L is given by the y-axis and M is given by the circle with center 1 and radius $\sqrt{2}$.

(b) L is given by the circle with center the origin and radius 1 and M is given by the circle with center 2 and radius 2.

[Hint: Find the intersection point of the two hyperbolic lines by writing down the equation of each hyperbolic line and solving for x and y. Recall that for a (Euclidean) circle the tangent is perpendicular to the radius. Use this to write down tangent vectors to each hyperbolic line at the intersection point and compute the angle θ between the vectors using the dot product formula $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta)$.]

(11) Let $T \subset \mathcal{H}$ be a hyperbolic triangle.

(a) Show that the area of T is less than π.

(b) Now suppose two of the sides of T are given by the y-axis and the circle with center the origin and radius 1. Show that the area of T is less than $\pi/2$.

(12) Find a formula for the hyperbolic reflection $f \colon \mathcal{H} \to \mathcal{H}$ in the hyperbolic line L given by the circle with center 3 and radius 2.

[Hint: First find a hyperbolic isometry $h \colon \mathcal{H} \to \mathcal{H}$ such that $h(L)$ is the hyperbolic line given by the circle C with center the origin and radius 1. (Here we can take $h(z) = az + b$ for some $a, b \in \mathbb{R}$, $a > 0$.) Now $f = h^{-1} \circ g \circ h$ where $g \colon \mathcal{H} \to \mathcal{H}$ is the hyperbolic reflection in $h(L)$ (why?), that is, g is given by inversion in the circle C. Finally use the known formula $g(z) = 1/\bar{z} = z/|z|^2$ to obtain a formula for f.]

(13) Find a formula for the hyperbolic rotation $f \colon \mathcal{H} \to \mathcal{H}$ about the point $2 + i$ through angle $\pi/2$ counter clockwise.

[Hint: First find a hyperbolic isometry $h \colon \mathcal{H} \to \mathcal{H}$ such that $h(2+i) = i$. Then $f = h^{-1} \circ g \circ h$ where $g \colon \mathcal{H} \to \mathcal{H}$ is the hyperbolic rotation about the point i through the same angle $\theta = \pi/2$ (why?). Now use the known formula $g(z) = \frac{\cos(\theta/2)z + \sin(\theta/2)}{-\sin(\theta/2)z + \cos(\theta/2)}$ to obtain a formula for f.]

(14) Let $C \subset \mathcal{H}$ be the hyperbolic circle with center $1+i$ and radius 2. (Then $C \subset \mathbb{R}^2$ is also a Euclidean circle (why?).) Compute the Euclidean center and Euclidean radius of C.

(15) Let $C = \{w \in \mathbb{C} \mid |w| = \frac{1}{2}\}$ be the Euclidean circle with center the origin and radius $\frac{1}{2}$ in the disc $D = \{w \in \mathbb{C} \mid |w| < 1\}$ (the Poincaré disc model of the hyperbolic plane). Compute the hyperbolic length of the shorter arc of C between the points $\frac{1}{2}$ and $\frac{1}{2}i$.

[Hint: Recall that the hyperbolic length of a parametrized path

$$\gamma\colon [a,b] \to D, \quad \gamma(t) = u(t) + iv(t)$$

in the disc D is given by the integral

$$\int_a^b \frac{2\sqrt{u'(t)^2 + v'(t)^2}}{1 - u(t)^2 - v(t)^2}\,dt.$$

]

(16) Let $F\colon \mathcal{H} \to D$ be the bijection from the upper half plane $\mathcal{H}$ to the disc $D = \{w \in \mathbb{C} \mid |w| = 1\}$ given by

$$F(z) = \frac{z-i}{z+i}.$$

(a) Let $g\colon \mathcal{H} \to \mathcal{H}$ be the hyperbolic isometry given by $g(z) = 2z$. Compute a formula for the corresponding isometry $h\colon D \to D$ given by $h = F \circ g \circ F^{-1}$.

(b) What are the images of the Euclidean circles with center 0 and the Euclidean lines passing through 0 under the bijection F?

(c) Use your answer to part (b) to describe the isometry h.

[Hint: (b) Recall that F preserves angles and sends circles and lines to circles and lines because F is a Mobius transformation.]

(17) In each of the following cases, determine the fixed points of the hyperbolic isometry $f\colon \mathcal{H} \to \mathcal{H}$ in $\mathcal{H}$ and $\partial\mathcal{H}$. (Here $\partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$ denotes the boundary of $\mathcal{H}$ in $\mathbb{C} \cup \{\infty\}$.) Deduce the type of the isometry f in the classification of hyperbolic isometries.

(a) $f(z) = \frac{1+z}{1-z}$.

(b) $f(z) = \frac{z}{2z-1}$.

(c) $f(z) = \frac{5z-18}{2z-7}$.