612 Final Exam

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This is a take home exam, due Monday May 9 at 5PM in my mailbox. You are allowed to consult your notes and textbooks. Please do not *discuss the exam with other students. There are* 5 *questions worth* 10 *points each. Show all your work and justify your answers carefully.*

(1) (a) Let A_1 and A_2 be commutative rings. The direct product $A_1 \times A_2$ is the ring given by the set $A_1 \times A_2$ of pairs (a_1, a_2) with addition

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and multiplication

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, a_2 b_2).$$

Show that the prime ideals of $A_1 \times A_2$ are of the form $\mathfrak{p}_1 \times A_2$ and $A_1 \times \mathfrak{p}_2$ for prime ideals $\mathfrak{p}_1 \subset A_1$ and $\mathfrak{p}_2 \subset A_2$.

(b) Let $p_1, \ldots, p_r$ be prime numbers (not necessarily distinct) and $m_1, \ldots, m_r$ positive integers. Consider the ring

$$A = \mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{m_r}\mathbb{Z}.$$

Describe the primes $\mathfrak{p}$ of A and the corresponding localizations $A_{\mathfrak{p}}$.

(2) Let A be a ring and $J \subset A$ an ideal.

(a) Define the radical $\sqrt{J}$ of J.

(b) Suppose A is Noetherian. Show carefully that $(\sqrt{J})^N \subseteq J$ for some positive integer N.

(c) Give an example of a ring A and ideal J such that $(\sqrt{J})^N$ is not contained in J for any N.

(3) Let $A = \mathbb{C}[x, y]/(y^5 - x^{19})$.

  (a) Show that A is an integral domain.

  (b) Compute the integral closure of A in its field of fractions. Justify your answer carefully.

(4) Let $G = D_4$ be the dihedral group of order 8 (the symmetries of the square).

  (a) Compute the character table of G.

  (b) Consider the action of G on the set of vertices of the square. Let ρ be the associated permutation representation. Compute the decomposition of ρ into irreducible representations.

(5) Let $\mathbb{F}_q$ be the finite field of order q. Let G be the finite group of affine linear maps

$$g\colon \mathbb{F}_q \to \mathbb{F}_q, \quad x \mapsto ax + b$$

where $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$, with the group law being composition of maps.

  (a) List the conjugacy classes of G.

  (b) Find the abelianization of G.

  (c) Compute the dimensions of the irreducible complex representations of G.

  (d) By its definition G acts on the set $\mathbb{F}_q$. Let ρ be the associated permutation representation. Then we can write $\rho = \rho_1 \oplus \rho'$ where ρ_1 is the trivial representation. Compute the character of ρ' and show that ρ' is irreducible.