612 Example Sheet 4

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(1) Let k be an infinite field. Let $f \in k[x_1, \ldots, x_n]$ be a nonzero polynomial. Show that there exist $a_1, \ldots, a_n \in k$ such that $f(a_1, \ldots, a_n) \neq 0$. [Hint: Use $k[x_1, \ldots, x_n] = k[x_1, \ldots, x_{n-1}][x_n]$ and induction on n].

(2) Let $R = k[x, y]$. Let $S \subset R$ be the k-algebra generated by the monomials $x^i y^j$ for $j > \sqrt{2}i$. Show that S is *not* Noetherian. [Hint: If $\alpha \in \mathbb{R}$ is irrational then there are infinitely many rational numbers j/i such that $\alpha < j/i < \alpha + 1/i^2$.]

(3) Let k be a field, $S = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k, and $\mathbb{A}^n_k = k^n$ the affine n-space over k with coordinates $x_1, \ldots, x_n$. Recall from class that $Z(J) \subset \mathbb{A}^n$ denotes the zero locus of an ideal $J \subset S$ and $I(X) \subset S$ denotes the ideal of polynomials vanishing on a subset $X \subset \mathbb{A}^n$.

  (a) Let $X_1, X_2 \subset \mathbb{A}^n$. Verify that $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

  (b) Let $\{J_i\}$ be a collection of ideals. Verify that $\bigcap Z(J_i) = Z(\sum J_i)$.

  (c) Let $J_1, J_2 \subset S$ be ideals. Show that $Z(J_1 \cap J_2) = Z(J_1 \cdot J_2) = Z(J_1) \cup Z(J_2)$.

  (d) Assume k is algebraically closed. Let $\{X_i\}$ be a collection of algebraic subsets of $\mathbb{A}^n$. Show that $I(\bigcap X_i) = \sqrt{\sum I(X_i)}$.

  (e) Give an example of algebraic sets $X_1, X_2 \subset \mathbb{A}^2$ over an algebraically closed field k such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$.

(4) (The Zariski topology)

  (a) Show that the sets $Z(J)$ for $J \subset S = k[x_1, \ldots, x_n]$ an ideal are the closed sets of a topology on $\mathbb{A}^n_k$, the *Zariski topology.*

  (b) Describe the Zariski topology on $\mathbb{A}^1$.

(c) Let k be an infinite field. Show that, for the Zariski topology, a non empty open set $U \subset \mathbb{A}^n$ is dense (that is, the closure of U equals $\mathbb{A}^n$). In particular, the Zariski topology is *not* Hausdorff (because any two nonempty open sets intersect).

(d) Let $k = \mathbb{C}$. Show that a Zariski open set of $\mathbb{A}^n = \mathbb{C}^n$ is an open set for the Euclidean topology. Show that a nonempty Zariski open set $U \subset \mathbb{A}^n$ is dense for the Euclidean topology.

(5) Let k be an algebraically closed field. Recall that an algebraic set $X \subset \mathbb{A}^n$ is *irreducible* if there does *not* exist a decomposition $X = X_1 \cup X_2$, where $X_1, X_2 \subsetneq X$ are proper algebraic subsets of X. We showed in class that any algebraic subset admits a decomposition

$$X = X_1 \cup \cdots \cup X_N$$

where each X_i is an irreducible algebraic set, and $X_i \not\subseteq X_j$ for $i \neq j$. Show that this decomposition is unique. (Then $X_1, \ldots, X_N$ are called the *irreducible components* of X.)

(6) Let k be an algebraically closed field. Let J be the ideal $(xy, yz, zx) \subset k[x, y, z]$. Describe the irreducible components of the associated algebraic set $X = Z(J) \subset \mathbb{A}^3$. Use your result to express $\sqrt{J}$ as an intersection of prime ideals.

(7) Let k be an algebraically closed field. Let $\mathfrak{p} \subset k[x, y]$ be a prime ideal. Show that either $\mathfrak{p} = 0$, $\mathfrak{p} = (f)$ for some irreducible $f \in k[x, y]$, or $\mathfrak{p} = (x - a, y - b)$ for some $a, b \in k$. [Hint: If $\mathfrak{p}$ is not principal then there exist $f, g \in \mathfrak{p}$ with no common factor in $k[x, y]$. Use Gauss' Lemma to show that f, g have no common factors in $k(x)[y]$. Conclude that there exists a nonzero $h \in k[x] \cap (f, g)$. Finally deduce that $Z(f, g)$ is a finite set of points, so $\mathfrak{p} \supset (f, g)$ is the maximal ideal corresponding to one of these points.]

(8) (Transcendence degree of a field extension) Let $k \subset K$ be a finitely generated field extension. We say that the *transcendence degree* of K/k equals n if there exist elements $t_1, \ldots, t_n \in K$ such that $t_1, \ldots, t_n$ are algebraically independent over k (that is, they do not satisfy a polynomial equation over k, so the subfield $k(t_1, \ldots, t_n) \subset K$ is isomorphic to the field of rational functions in n variables over k) and $K/k(t_1, \ldots, t_n)$ is finite. Show that the transcendence degree is well defined. [Hint: Imitate the Steinitz exchange lemma used to prove that the dimension of a vector space is well defined.]