Math 462: Homework 4 solutions

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In the problems below S^2 denotes the sphere of radius 1 in $\mathbb{R}^3$ with center the origin O.

1. A *spherical circle* with center a point P on S^2 and radius r is the locus of points Q on S^2 such that the spherical distance $d(P,Q)$ equals r.

(a) Show that the circumference of a spherical circle of radius r equals $2\pi \sin r$. [Hint: A spherical circle with center P is a Euclidean circle in $\mathbb{R}^3$ obtained by intersecting the sphere S^2 with a plane normal to the line OP. Notice that the Euclidean circumference is equal to the spherical circumference, but the Euclidean center and radius of the circle are different from the spherical center P and radius r.]

(b) Recall that the spherical distance between two points P and Q on S^2 is at most π. So it only makes sense to talk about spherical circles of radius $r \leq \pi$. What happens to the circumference of a spherical circle of radius r as r approaches π? Explain your answer geometrically.

(c) Show that the circumference of a spherical circle of radius r is less than the circumference of a Euclidean circle of the same radius.

(a) As in the hint, a spherical circle with center P is a Euclidean circle in a plane Π normal to the line OP given by intersecting the sphere S^2 with the plane Π (note that Π does not necessarily pass through O). Let Q be a point on the spherical circle. Then the angle $\angle POQ$ is equal to the spherical distance $d(P,Q)$, which is the radius r of the spherical circle. Let O' be the intersection point of OP with the plane Π — this is the center of the Euclidean circle in Π. Then the Euclidean radius is

$$O'Q = OQ \sin(\angle POQ) = 1 \cdot \sin r = \sin r.$$

So the circumference of the circle is $2\pi(\sin r)$.

(b) As r approaches π the circumference of the circle approaches zero. To visualize geometrically, let P be the center of the spherical circle and Q the antipodal point (that is, P and Q lie on a line through the center O of the sphere). Then a spherical circle of radius r and center P coincides with a spherical circle of radius $\pi - r$ and center the antipodal point Q. So a spherical circle of radius close to π and center P coincides with a spherical circle of radius close to 0 and center Q.

(c) In part (a) we showed that the spherical circle of radius r coincides with a Euclidean circle of radius $\sin r$. Now $\sin r < r$ for $r > 0$: to see this geometrically, in the notation of part (a), consider the points O, O', P, Q. The length of the arc PQ (the spherical distance) equals r, which is greater than the length of the line segment PQ (the Euclidean distance), which in turn is greater than the length $\sin r$ of the line segment $O'Q$ (because $O'PQ$ is a right angled triangle with hypotenuse PQ). So the circumference $2\pi \sin r$ of the spherical circle of radius r is less than the circumference $2\pi r$ of a Euclidean circle of radius r.

2. A *spherical disc* is the region on S^2 enclosed by a spherical circle.

(a) Show that the area of a spherical disc of radius r equals $2\pi(1 - \cos r)$. [Hint: Use Q1(a) and integration.]

(b) What happens to the area of a spherical disc of radius r as r approaches π? Explain your answer geometrically.

(c) Show that the area of a spherical disc of radius r is less than the area of a Euclidean disc in $\mathbb{R}^2$ of the same radius.

(a) Let $A(r)$ denote the area and $C(r)$ the circumference of a spherical disc of radius r. Then for a small increase δr in the radius, $A(r + \delta r)$ is approximately equal to $A(r) + C(r)\delta r$, because a spherical disc of radius $r + \delta r$ obtained from a spherical disc of radius r by adding a thin circular strip at the boundary of width δr and inner circumference $C(r)$, which has area approximately $C(r)\delta r$. Rearranging gives

$$\frac{A(r + \delta r) - A(r)}{\delta r} \approx C(r)$$

and passing to the limit as $\delta r \to 0$ gives $A'(r) = C(r)$, where $A'(r)$ denotes the derivative of $A(r)$ with respect to r. Now we know from Q1(a) that $C(r) = 2\pi \sin r$, so we can integrate to find $A(r)$:

$$A(r) = \int_0^r A'(t)dt = \int_0^r C(t)dt = \int_0^r 2\pi \sin t\, dt$$
$$= 2\pi[-\cos t]_0^r = 2\pi(1 - \cos r).$$

(b) As r approaches π the area $A(r) = 2\pi(1 - \cos r)$ of the spherical disc approaches 4π. Geometrically, the spherical disc grows until it covers the whole sphere S^2, which has surface area 4π.

(c) We know $C(r) < 2\pi r$ for $r > 0$ by Q1(c). Integrating as in part (a) gives $A(r) < \pi r^2$.

3. Show that if $R \subset S^2$ is any region, then there is no map $T\colon R \to \mathbb{R}^2$ from R to the plane which preserves distances, that is $d(T(P), T(Q)) = d(P, Q)$ (here we are using the spherical distance on S^2 and the usual Euclidean distance on $\mathbb{R}^2$). [Hint: Use Q1(c).] Note: It follows that any map of a portion of the earth's surface distorts distances, that is, distances are not exactly to scale.

Let P be a point in the interior of the region R and C a spherical circle of small radius r and center P contained in R. Suppose that $T\colon R \to \mathbb{R}^2$ is a map that preserves distances. Then $T(C)$ is a Euclidean circle with center $T(P)$ and radius r. But then $T(C)$ has circumference $2\pi r$ which is greater than the circumference $2\pi \sin r$ of C. This is a contradiction, so such a map T does not exist. [Remark: Here we have used the following fact: If T preserves distances, then it also preserves lengths of curves. This can be proved by approximating a curve by a large number of spherical line segments (which are the shortest paths between points).]

4. Let L be a spherical line (great circle) on S^2 and P a point on S^2 not lying on L. Show how to construct a spherical line M through P and perpendicular to L. [Hint: Give a construction in terms of planes in $\mathbb{R}^3$]. Is the line M uniquely determined by P and L?

Let $L = \Pi_L \cap S^2$ where $\Pi_L \subset \mathbb{R}^3$ is a plane through the origin. Let n be the line through the origin normal to Π_L. Then a spherical line

M perpendicular to L and passing through P is given by $\Pi_M \cap S^2$ where Π_M is a plane through the origin containing the point P and the line n. It is uniquely determined unless P lies on the line n (in this case, choosing coordinates appropriately, L is the equator and P is the north pole, and the possible lines M are all the lines of longitude).

5. In class we proved that the sum of the angles of a Euclidean triangle in $\mathbb{R}^2$ equals π radians. [See p. 19–20 of the textbook.] What goes wrong when you try to prove that the sum of the angles of a spherical triangle equals π by the same method? More precisely, let ABC be a spherical triangle, and assume (by choosing coordinates appropriately) that the edge AC lies on the equator of the sphere. Now let T be the rotation about the axis NS joining the north and south poles through an angle θ chosen so that $T(A) = C$. Write A', B', C' for $T(A), T(B), T(C)$. Is the spherical triangle $B'A'B$ congruent to the spherical triangle ABC?

No, the argument breaks down because the spherical distance $d(B, B')$ is less than the spherical distance $d(A, C)$. To see this, note that B and B' lie on a line of latitude obtained by intersecting the sphere with a plane Π normal to the axis NS, not passing through the center of the sphere. Note that this is *not* a great circle, it has radius strictly smaller than the radius of the sphere (in fact it is a spherical circle as in Q1). Now the arc length BB' measured along the line of latitude is strictly smaller than the arc length AC measured along the equator (which equals the spherical distance $d(A, C)$), because they are arcs of circles corresponding to the same angle (the angle of the rotation T) and the line of latitude has smaller radius. Now $d(B, B')$ is the length of the shortest path on the sphere from B to B' (given by a part of a great circle), so it is strictly smaller than the arc length BB' along the line of latitude and so smaller than $d(A, C)$.