Math 132.5. Parametric curves (10.1); Calculus with Parametric curves: Tangents and Arc Length (10.2); Polar coordinates (10.3).

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1 Section 10.1

1.1 Parametric curves

Given equations $x = f(t)$ and $y = g(t)$ expressing x and y in terms of a parameter t, as t varies the point (x, y) traces out a curve in the plane called a *parametric curve*.

Example 1.1. Let $x = \cos t$ and $y = \sin t$ for $0 \leq t \leq 2\pi$. Then (x, y) traces out the circle C with center the origin and radius 1, traversed once in the counterclockwise direction.

To give a rough sketch of a parametric curve we can form a table of x and y values corresponding to a selection of values of t and connect the points (x, y). We usually indicate the direction the curve is traced out as t increases by an arrow.

1.2 Eliminating the parameter

Sometimes it is possible to eliminate the parameter to obtain an equation $h(x, y) = 0$ defining the curve. (However, even when this is possible it may be easier to use the parametric description.)

Example 1.2. Continuing Example 1.1, note that

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

The equation $x^2 + y^2 = 1$ defines the circle C.

Example 1.3. Consider the parametric curve defined by $x = t^2 + 1$ and $y = 2t + 3$. We can use the second equation to solve for t in terms of y, then substitute into the first equation to obtain the equation of the curve:

$$t = \frac{y-3}{2}$$

so

$$x = \left(\frac{y-3}{2}\right)^2 + 1,$$

simplifying

$$4x = y^2 - 6y + 13.$$

2 Section 10.2

2.1 Tangents

For a parametric curve given by $x = f(t)$ and $y = g(t)$, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

by the chain rule. If $P = (a, b) = (f(c), g(c))$ is the point on the curve corresponding to $t = c$, then the slope m of the tangent line to the curve at P is given by $\frac{dy}{dx}$ evaluated at $t = c$. Then the equation of the tangent line is given by the point–slope formula

$$(y - b) = m(x - a).$$

Example 2.1. We compute the tangent line to the parametric curve $x = t^2$, $y = t^3$ at the point $(4, 8)$ corresponding to $t = 2$. The slope of the line is given by

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \left.\left(\frac{dy}{dt} \Big/ \frac{dx}{dt}\right)\right|_{t=2} = \left.\frac{3t^2}{2t}\right|_{t=2} = \left.\frac{3t}{2}\right|_{t=2} = 3.$$

So the equation of the tangent line is

$$(y - 8) = 3(x - 4)$$

or

$$y = 3x - 4.$$

The tangent line to a parametric curve at a point is horizontal when $\frac{dy}{dt} = 0$ and vertical when $\frac{dx}{dt} = 0$. (If the parameter t is time then $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the horizontal and vertical components of the velocity of the point (x, y).)

We can also compute $\frac{d^2y}{dx^2}$ using the chain rule:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}.$$

2.2 Arc length

The parametric curve given by $x = f(t)$ and $y = g(t)$ for $a \leq t \leq b$ has length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 2.2. The parametric curve given by $x = t - t^3/3$ and $y = t^2$ for $0 \leq t \leq 1$ has length

$$L = \int_0^1 \sqrt{(1 - t^2)^2 + (2t)^2}\, dt = \int_0^1 \sqrt{1 - 2t^2 + t^4 + 4t^2}\, dt$$

$$= \int_0^1 \sqrt{1 + 2t^2 + t^4}\, dt = \int_0^1 1 + t^2\, dt = \left[t + t^3/3\right]_0^1 = \frac{4}{3}.$$

3 Section 10.3

3.1 Polar coordinates

A point P in the plane may be represented by either Cartesian coordinates (x, y) or Polar coordinates (r, θ). Here r is the distance from the origin O to P and θ is the angle between the x-axis and the line OP measured in the counterclockwise direction. We can convert between the two types of coordinates using the formulas

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Note that (r, θ) and $(r, \theta + 2\pi k)$ for any integer k represent the same point. Also, θ is not defined for the origin. Finally, we sometimes allow r to be negative, with the understanding that $(-r, \theta)$ and $(r, \theta + \pi)$ represent the same point.

3.2 Polar curves

A *polar curve* is defined by an equation $r = f(\theta)$ in polar coordinates.

We can give a rough sketch of a polar curve by making a table of values of r and θ for selected values of θ and connecting the points.

Example 3.1. $r = 1 + \cos\theta$ defines a cardioid (heart-shaped curve).

3.3 Tangent lines

Note that a polar curve is a special case of a parametric curve (with parameter θ): since $x = r\cos\theta$, $y = r\sin\theta$ and $r = f(\theta)$, we have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. So we can apply the method of Section 10.2 to compute the tangent line to a polar curve at a point. Since $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\frac{dy}{dx} = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) \bigg/ \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)$$

where we have used the product rule.