Math 621 Midterm, Wednesday 2/29/12, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 6 questions for a total of 50 points. Calculators, notes, and textbooks are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1. (10 points) Let $f(z) = \tan(z) := \sin(z)/\cos(z)$

(a) (5 points) Compute the zeroes and poles of $f(z)$ and their multiplicities. (Justify your answer carefully.)

(b) (2 points) What is the radius of convergence of the power series expansion of $f(z)$ about $z = 0$?

(c) (3 points) Let γ be the circle with center the origin and radius 3, oriented positively (i.e., traversed counterclockwise). Compute $\int_\gamma f(z)dz$.

Q2. (10 points)

(a) (5 points) Let $\Omega \subset \mathbb{C}$ be an open set containing the closure of the disc $D = \{z \in \mathbb{C} \mid |z| < R\}$. Let $f \colon \Omega \to \mathbb{C}$ a holomorphic function. Suppose $|f(z)| \leq B$ for $|z| = R$. Show that

$$|f^{(n)}(0)| \leq \frac{(n!) \cdot B}{R^n}.$$

(b) (5 points) Now suppose $f \colon \mathbb{C} \to \mathbb{C}$ is a holomorphic function on $\mathbb{C}$ and suppose there exist $C, r \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|f(z)| \leq C \cdot |z|^N$ for $|z| \geq r$. Show that f is a polynomial of degree $\leq N$.

Q3. (10 points) Compute the integral

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 5x^2 + 4} dx.$$

Q4. (5 points) Compute the Laurent series expansion of $f(z) = \frac{1}{z^2(z-1)}$ centered at $z = 0$ in the regions $0 < |z| < 1$ and $|z| > 1$.

Q5. (10 points) Compute the number of zeroes of $f(z) = 3z^{100} - e^z$ in the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Are the zeroes simple (multiplicity 1)? Justify your answer.

Q6. (5 points) Compute the integral $\int_\gamma \frac{1}{z} dz$ where γ is a path from $2 - 2i$ to $3i$ which is contained in $\mathbb{C} \setminus [0, \infty)$. (Justify your answer.)