Math 462: Homework 2 Solutions

Paul Hacking

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1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Check that A is orthogonal.

(b) Determine whether A is a rotation or a reflection / rotary reflection. [Hint: What is the determinant of A?]

(c) Find the eigenvalues and eigenvectors of A (including the complex ones if there are any).

(d) Describe the motion $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$ geometrically. (If T is a rotation, give the axis and angle of rotation. If T is a reflection in a plane, find the plane.)

(e) Find an orthonormal basis of $\mathbb{R}^3$ such that the matrix B of T with respect to this basis has the form described in the Theorem on p.12 of the textbook, and write down the new matrix B.

(a) Just compute $A^T A = I$.

(b) The determinant of A is -1. It follows that A is either a reflection or rotary reflection. To decide which, we need to compute the eigenvalues of A, see (c) below.

(c) We compute the eigenvectors of a matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$ (this is a polynomial equation of degree n in λ, where A is a $n \times n$ matrix). In our case we find that $\lambda = -1, 1, 1$. (At this point it follows that A is a reflection. The eigenvalues for a rotary reflection are $-1, \cos\theta \pm i\sin\theta$, where $\pm\theta$ is the angle of rotation.) To find the eigenvectors $\mathbf{x}$ with given eigenvalue λ, we solve the equation $(A - \lambda I)\mathbf{x} = 0$ (by Gaussian elimination or otherwise). In our case we find that the eigenvectors with eigenvalue 1 are

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for $c_1, c_2 \in \mathbb{R}$ and the eigenvectors with eigenvalue -1 are

$$c \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

for $c \in \mathbb{R}$.

(d) The transformation T is reflection in the plane through the origin with normal vector $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (the eigenvector with eigenvalue -1).

(e) We need to find an orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$ such that the matrix B of T with respect to $\mathcal{B}$ is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(Recall that an *orthonormal basis* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ is a set of n vectors such that $\mathbf{v}_i \cdot \mathbf{v}_j = 1$ if $i = j$ and 0 otherwise. That is, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ each have length 1 and are pairwise orthogonal.) The vector $\mathbf{v}_1$ is a normal vector to the plane of reflection (equivalently, an eigenvector with eigenvalue -1), scaled so that it has length 1. The vectors $\mathbf{v}_2, \mathbf{v}_3$ are an orthonormal basis for the plane of reflection. We have the basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

for the plane of reflection (equivalently, the space of eigenvectors with eigenvalue 1). We can use the Gram Schmidt process to replace this with an orthonormal basis $\mathbf{v}_2, \mathbf{v}_3$. Combining we find

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

2. Repeat Q1 for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) $\det A = 1$ so A is a rotation.

(c) The eigenvalues of A are the solutions λ of the characteristic equation $\det(A-\lambda I) = 0$. We compute $\det(A-\lambda I) = 1-\lambda^3$, so the eigenvalues are $\lambda = 1, \cos(2\pi/3) \pm i\sin(2\pi/3)$. The eigenvector with eigenvalue $\lambda = 1$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the eigenvectors with eigenvalues $\lambda = \cos(2\pi/3) \pm i\sin(2\pi/3) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ are

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \mp \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{pmatrix}.$$

(d) T is a rotation about the line through the origin in the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (the eigenvector with eigenvalue 1) through angle $\theta = \pm 2\pi/3$ (the angle is determined up to sign by the complex eigenvalues of A).

(e) We want to find an orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$ such that the matrix B of T with respect to this basis has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where θ is the angle of rotation. This is obtained by taking $\mathbf{v}_1$ to be an eigenvector with eigenvalue 1, scaled so that it has length 1, and

$\mathbf{v}_2, \mathbf{v}_3$ the real and imaginary parts of one of the complex eigenvectors, again scaled so that they have length 1. (See the proof of the theorem in your class notes or in the textbook.) We find

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{pmatrix} \right\}.$$

The matrix B is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(the angle $\theta = -2\pi/3$).

3. The matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

defines a rotation of $\mathbb{R}^3$. Find the axis and angle of rotation (give the angle in radians to 2 decimal places.) [Hint: What are the eigenvectors and eigenvalues of A?]

We compute the characteristic polynomial

$$\det(A - \lambda I) = \left(\frac{1}{\sqrt{2}}\right)^3 (1-\lambda)(2\sqrt{2} + (2+2\sqrt{2})\lambda + 2\sqrt{2}\lambda^2)$$

Note that we know $\lambda = 1$ is an eigenvalue of A because A is a rotation, so that $\det(A - \lambda I)$ has a factor $(1-\lambda)$. Now we can solve for the remaining eigenvalues using the quadratic formula applied to the second factor of the characteristic polynomial. We have

$$\lambda^2 + \left(1 + \frac{1}{\sqrt{2}}\right)\lambda + 1 = 0,$$

so

$$\lambda = \frac{-(1+\frac{1}{\sqrt{2}}) \pm \sqrt{(1+\frac{1}{\sqrt{2}})^2 - 4}}{2} = \cos\theta \pm i \sin\theta,$$

where $\pm\theta$ is the angle of rotation. We deduce

$$\theta = \cos^{-1}\left(-\left(1 + \frac{1}{\sqrt{2}}\right)/2\right) = 2.59 \text{ radians.}$$

The axis of rotation is the line through the origin in the direction of the eigenvector $\mathbf{v}$ of A with eigenvalue $\lambda = 1$. We solve $(A - I)\mathbf{x} = \underset{\sim}{0}$ to find

$$\mathbf{v} = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 1+\sqrt{2} \end{pmatrix}.$$

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a motion of $\mathbb{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. Describe T geometrically (as a translation, rotation, twist, reflection, glide, or rotary reflection) in the following cases.

(a) A the matrix from Q1 and $\mathbf{b} = (3, 3, -1)^T$.

(b) A the matrix from Q2 and $\mathbf{b} = (0, -3, -3)^T$.

(c) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$.

(a) Recall that $U(\mathbf{x}) = A\mathbf{x}$ defines a reflection in the plane through the origin with normal vector

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

We write

$$\mathbf{b} = c\mathbf{v} + \mathbf{w}$$

where $c \in \mathbb{R}$ is a scalar and $\mathbf{w}$ is orthogonal to $\mathbf{v}$ (equivalently, the dot product $\mathbf{v} \cdot \mathbf{w} = 0$). To find c (and hence $\mathbf{w}$) we dot the above equation with $\mathbf{v}$ and solve for c. We obtain

$$c = \frac{\mathbf{b} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = 2$$

so

$$\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

Now it follows that T is a glide given by reflection in the plane with the same normal $\mathbf{v}$, through the point $\frac{1}{2}c\mathbf{v}$, followed by translation by $\mathbf{w}$ (parallel to the plane).

(b) Recall that $U(\mathbf{x}) = A\mathbf{x}$ is a rotation about the line through the origin in direction

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We write

$$\mathbf{b} = c\mathbf{v} + \mathbf{w}$$

where $c \in \mathbb{R}$ and $\mathbf{v} \cdot \mathbf{w} = 0$. We find $c = -2$ and

$$\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

It follows that T is a twist given by rotation about an axis in direction $\mathbf{v}$ (through some point to be determined) followed by translation by $c\mathbf{v}$ (parallel to the axis). To find the axis, we need to find the fixed line of the rotation

$$S(\mathbf{x}) = A\mathbf{x} + \mathbf{w}$$

(the first ingredient of the twist), that is, we need to solve the linear equation

$$A\mathbf{x} + \mathbf{w} = \mathbf{x}.$$

We find (by Gaussian elimination or otherwise)

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for $t \in \mathbb{R}$. This is a parametric equation for the axis of rotation: it is the line through the point $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ in the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c) The motion $U(\mathbf{x}) = A\mathbf{x}$ is a rotary reflection given by reflection in the plane $(x = 0)$ followed by rotation by $\pi/2$ about the x-axis. (Notice that A is already in the standard form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

so we can describe A geometrically straight away.) It follows that $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ will also be a rotary reflection with parallel plane and parallel axis about the same angle. We just need to determine the "center" $\mathbf{c}$ of the rotary reflection, that is, the intersection point of the reflection plane and the axis of rotation. The point $\mathbf{c}$ is the only point fixed by T, so can be found by solving the linear equation

$$A\mathbf{x} + \mathbf{b} = \mathbf{x}.$$

We find

$$\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

5. In this question, we find the group G of all rotational symmetries of the cube. (For concreteness, you might like to use the cube with center the origin and vertices the points $(\pm 1, \pm 1, \pm 1)$.)

(a) Find the order of G. (Hint: Use the orbit-stabilizer theorem.)

(b) Find an element of G of order 4, and an element of order 3.

(c) Describe all elements of G geometrically (give the axis and angle of rotation).

(d) Show that the group G is isomorphic to the symmetric group S_4. (Hint: G permutes the 4 diagonals of the cube joining opposite vertices.)

(a) Consider the center P of a face of the cube. The stabilizer of P is the group of rotations about the axis OP (where O is the center of the cube) through multiples of $\pi/2$ radians, of order 4. The orbit of P is the set of all centers of the faces of the cube, which has size 6. So by the orbit-stabilizer theorem, the order of G is $4 \cdot 6 = 24$.

(b) An element of order 4 is given by a rotation by $\pi/2$ about the axis joining the center of a face of the cube to the center of the cube (as used in part (a)). An element of order 3 is given by rotation by $2\pi/3$ about the axis joining two opposite vertices of the cube.

(c) There are 9 elements given by rotations about axes joining the center of the cube to the center of a face (3 possible axes each with 3 possible angles $\pm\pi/2, \pi$), 8 elements given by rotations about axes joining opposite vertices (4 possible axes each with 2 possible angles $\pm 2\pi/3$), 6 elements given by rotation by π about an axis joining the midpoints of 2 opposite edges of the cube (6 possible axes). Together with the identity transformation, this gives $9 + 8 + 6 + 1 = 24$.

(d) As suggested we consider the set of 4 diagonals of the cube (lines joining opposite vertices). These are permuted by the action of G and this defines a group homomorphism

$$\theta \colon G \to S_4$$

from G to the symmetric group on 4 letters. Note that G and S_4 have the same order 24 (S_4 has order $4! = 24$). So it suffices to show that θ is surjective. Recall also that S_n is generated by transpositions (the permutations which switch 2 numbers and leave the rest fixed). The transpositions are realized by the rotations with axes the lines joining the midpoints of two opposite edges of the cube. It follows that θ is surjective and so an isomorphism.