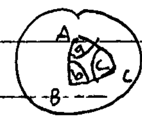
11/13/19

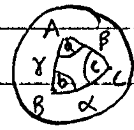
HW 7 available, due next Wednesday 11/20 at start of class

Last Time · Proof of Thm $a+b+c = \pi + \text{Area}(\triangle ABC)$



· Statement of spherical cosine rule

↳ $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$



↳ if $\triangle ABC$ is small (approximately planar) then SCR reduces to Euclidean cosine rule

Today · Proof of SCR (Review of spherical polar coords : Math 233)

⇒ Triangle Inequality

⇒ spherical lines give shortest paths

I will be away Friday, class will be taught by Prof Jennifer Li

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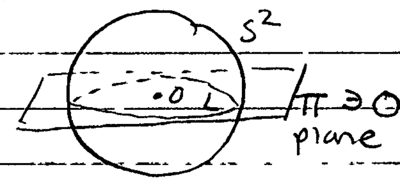
Review of spherical lines:



$L = \Pi \cap S^2$ great circle or spherical line

Π has an equation $ax+by+cz=d$

(because it is a plane in $\mathbb{R}^3$) $a,b,c,d \in \mathbb{R}$

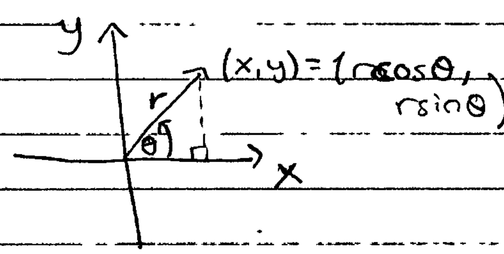
$0 \in \Pi \Rightarrow d=0 \quad \Pi : ax+by+cz=0$

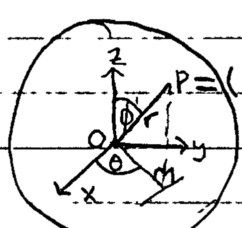
$L = \{(x,y,z) \in S^2 \mid ax+by+cz=0\}$

↳ $\boxed{\text{equation of } L}$

Spherical polar coords (Math 233)

Polar coords in plane (Math 132)

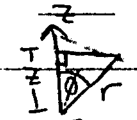




$r \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi$

$(x,y,z) = (?,?,?)$ in terms of r, θ, ϕ

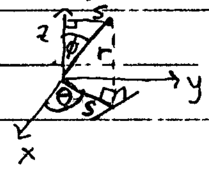
$z = ?$



$z = r\cos\phi \quad \left(\cos\phi = \frac{a}{h} = \frac{z}{r}\right)$

$x = ?$

$y = ?$



$x = s\cos\theta = r\sin\phi\cos\theta$

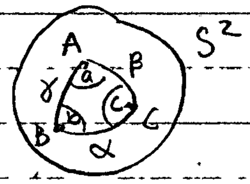
$y = s\sin\theta = r\sin\phi\sin\theta$

polar coords in (x,y) plane

$s = r\sin\phi$

$$\boxed{(x,y,z) = r\cdot(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)}$$

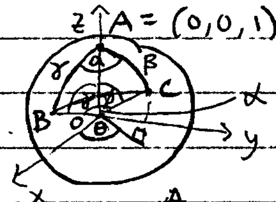
Proof of Spherical Cosine Rule : $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$



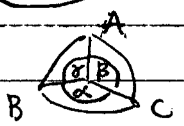
Want : quantities α, β, γ, a have clear relation to spherical coordinates (r, θ, ϕ) of vertices A, B, C of ΔABC.

① We may assume $A = (0,0,1)$ (i.e. A is on the positive z-axis)

Note : $r = \sqrt{x^2+y^2+z^2}$, $p \in S^2 \iff r = 1$



② We may assume B lies in the xz-plane, (i.e. $\theta = 0$) with $x > 0$.



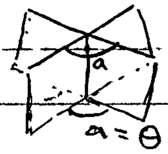
Spherical coords of B? $(r,\theta,\phi) = (1, 0, \gamma)$

$\rightsquigarrow (x,y,z) = (\sin\gamma, 0, \cos\gamma)$

using our formula.

Spherical coords of C :

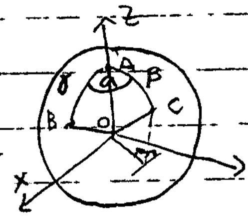
$(r,\theta,\phi) = (1, a, \beta)$



$\rightsquigarrow (x,y,z) = (\sin\beta\cos a, \sin\beta\sin a, \cos\beta)$

In (x,y,z) coords:

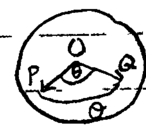
$$\left.\begin{aligned} A &= (0,0,1) \\ B &= (\sin\gamma, 0, \cos\gamma) \\ C &= (\sin\beta\cos a, \sin\beta\sin a, \cos\beta) \end{aligned}\right\} \text{expressed in terms of } \beta, \gamma, a \qquad (*)$$



Want to show: SCR: $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

How can we compute $d_{S^2}(B,C) = \alpha$ given $(*)$?

Recall



$\vec{OP}\cdot\vec{OQ} = \|\vec{OP}\|\cdot\|\vec{OQ}\|\cdot\cos\theta = \cos\theta$

$\theta = \cos^{-1}(\vec{OP}\cdot\vec{OQ})$

Our case: $\vec{OB}\cdot\vec{OC} = \cos\alpha$

$\qquad\qquad\ \ \|$

$(\sin\gamma, 0, \cos\gamma)\cdot(\sin\beta\cos a, \sin\beta\sin a, \cos\beta) = \sin\gamma\sin\beta\cos a + 0 + \cos\gamma\cdot\cos\beta$

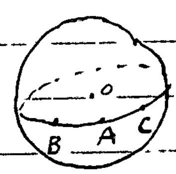
$\rightsquigarrow$ i.e. $\boxed{\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a} \Rightarrow$ SCR ∎

Corollary Spherical triangle inequality:

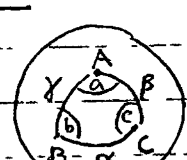
($d = d_{S^2}$ = spherical distance)

$A, B, C \in S^2$, then $d(B,A) + d(A,C) \geq d(B,C)$

with equality if and only if: B, A, C lie on a spherical line with A lying on the shorter segment connecting B & C. $(\dagger)$



Proof Consider $\triangle ABC$



Want to show $\gamma + \beta \geq \alpha$

(with equality iff $(\dagger)$).

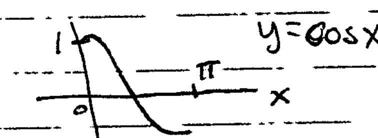
SCR: $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

Also, $\cos(\beta+\gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma$ (addition formula)

Notice: $\cos a \geq -1$, equal iff $a = \pi$

β, γ distances on $S^2 \Rightarrow 0 \leq \beta, \gamma \leq \pi$

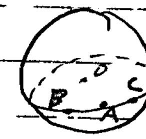
$\Rightarrow \sin\beta, \sin\gamma \geq 0$



This gives $\cos\alpha \geq \cos(\beta+\gamma)$ $(\dagger\dagger)$

$\cos x$ is decreasing on $[0,\pi]$. So if $\alpha, \beta+\gamma \leq \pi$, by $(\dagger\dagger) \Rightarrow \alpha \leq \beta+\gamma$.

Otherwise, $\alpha \leq \pi < \beta+\gamma$. Both cases, $\alpha \leq \beta+\gamma$.

Equality? need $\cos a = -1$, i.e. $a = \pi$ $\longrightarrow$ 

. $\blacksquare$