Math 461 Lecture 25 10/31

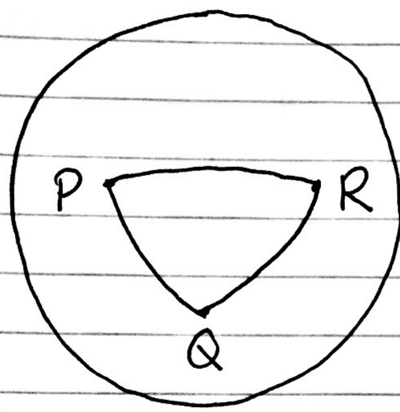
Homework 5 due today

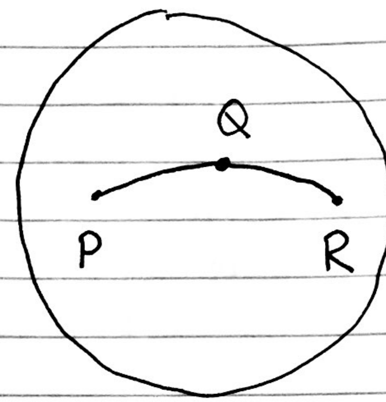
Last time:

Spherical triangle inequality

$d(P,Q) + d(Q,R) \geq d(P,R)$

equal if and only if Q lies on shorter arc of spherical line from P to R





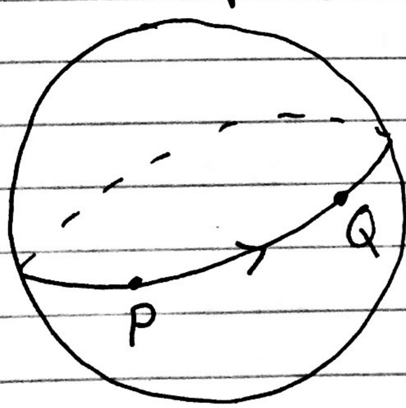
Today:

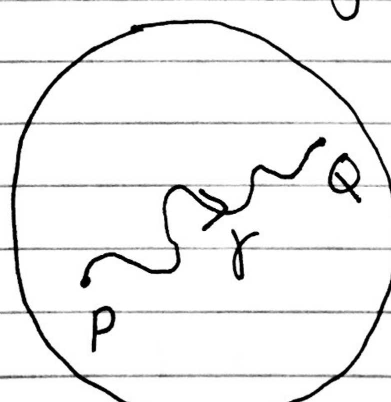
spherical lines give shortest paths

spherical sine rule

if time: spherical isometries

Theorem: $P, Q \in S^2$ the shortest path from P to Q is the shorter arc of the spherical line through P & Q





Proof: $\gamma : [a,b] \subset \mathbb{R} \to S^2 \subset \mathbb{R}^3$

$\gamma(t) = (x(t), y(t), z(t)) \quad x(t)^2 + y(t)^2 + z(t)^2 = 1$

$\gamma(a) = P, \ \gamma(b) = Q$

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continuously differentiable

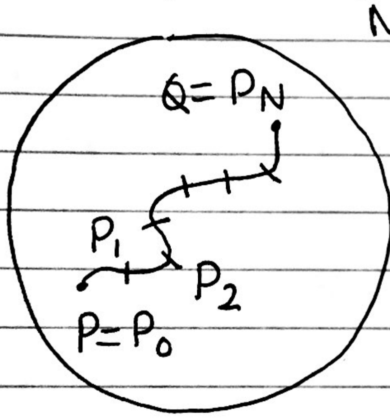
$x'(t), y'(t), z'(t)$ exist and are continuous

$$\text{length}(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$$
(233)
$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

subdivide $[a,b]$ into N equal parts

$a = t_0, \; t_1, \; t_2, \; \dots, \; t_{N-1}, \; b = t_N$ $\quad N \in \mathbb{N}$

$t_i - t_{i-1} = \frac{b-a}{N}$ $\quad P_i = \gamma(t_i)$



For N large, can approximate the path from P_i to P_{i+1} by the arc of the spherical line through those points

$$\text{length}(\gamma) \approx d(P_0, P_1) + d(P_1, P_2) + \dots + d(P_{N-1}, P_N) \geq d(P_0, P_N)$$
triangle inequality

$d(P_0, P_1) + d(P_1, P_2) \geq d(P_0, P_2)$

claim: the approximation becomes equality in limit

$\leadsto$ get $\text{length}(\gamma) \geq d(P, Q)$

remains to show:

if $\text{length}(\gamma) = d(P, Q)$

then γ is a parameterization of the shorter arc of the spherical line from P to Q

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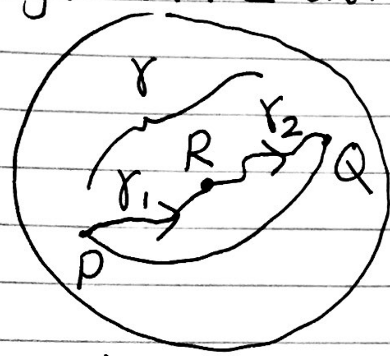
Contrapositive:

$\gamma \neq$ shorter arc of spherical line through P to Q $\Rightarrow$ length$(\gamma) > d(P,Q)$

in this case, have a point $R \in \gamma$, $R \notin$ arc of spherical line

$\gamma = \gamma_1 \cup \gamma_2$ length(γ) = length(γ_1) + length(γ_2)

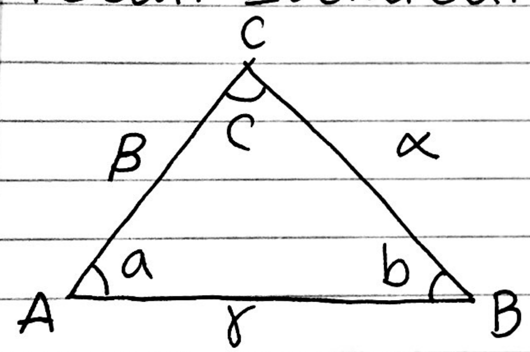
length$(\gamma) \geq d(P,R) + d(R,Q) > d(P,Q)$ $\square$

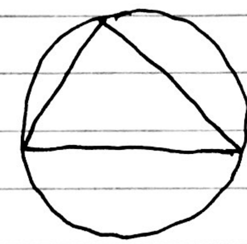


because PQR is a spherical triangle

Spherical sine rule:

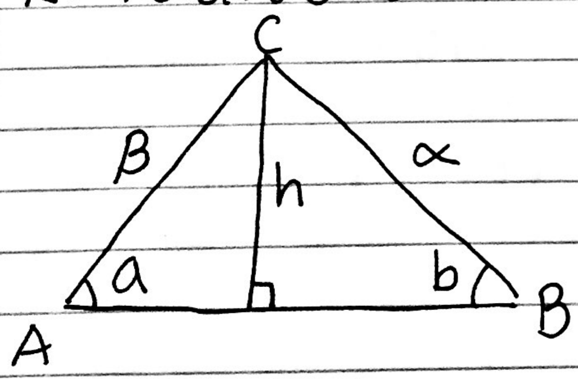
recall Euclidean sine rule:





sine rule: $\dfrac{\alpha}{\sin a} = \dfrac{\beta}{\sin b} = \dfrac{\gamma}{\sin c} = 2R$

R = radius of circumscribed circle



$\sin a = \dfrac{h}{\beta}$

$\sin b = \dfrac{h}{\alpha}$

$\beta \sin a = h = \alpha \sin b$

$\Rightarrow \dfrac{\beta}{\sin b} = \dfrac{\alpha}{\sin a}$

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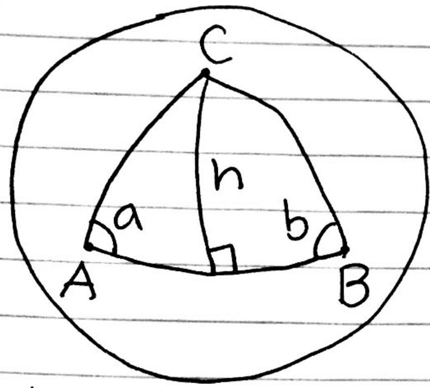
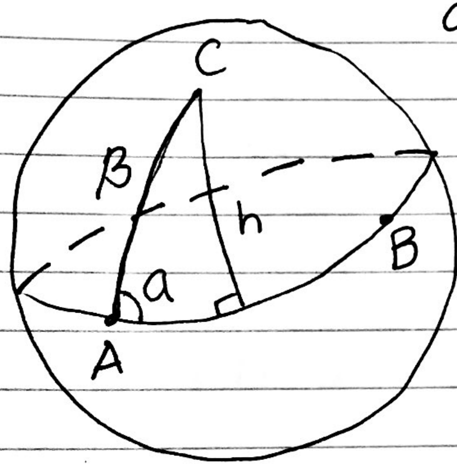
S^2

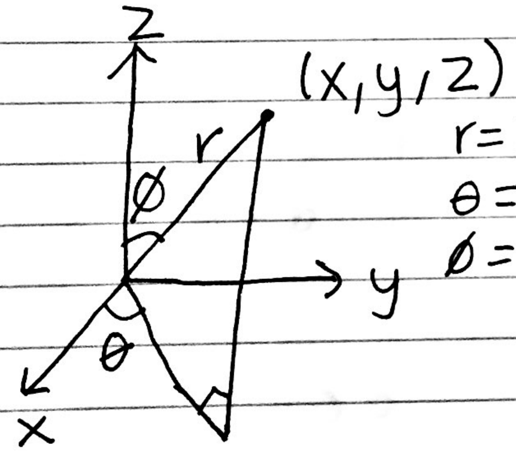


consider spherical line passing through C and perpendicular to the spherical line AB

need to work out: how to express h in terms of $\frac{1}{a}$ and β, $\frac{2}{b}$ and α

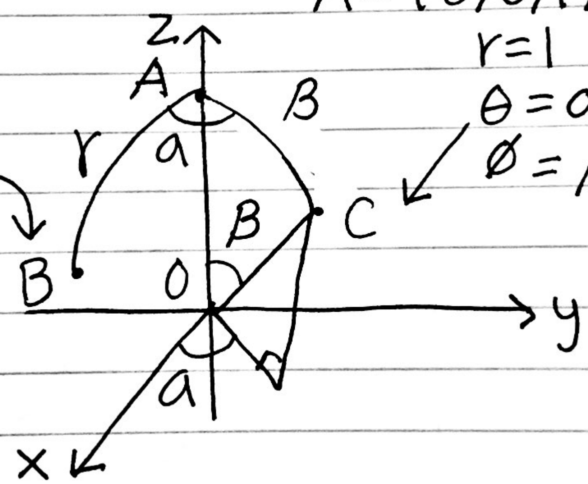


we will compute h in spherical coordinates, chosen carefully to make calculations easier



$r=1$
$\theta=0$
$\phi=\gamma$ B

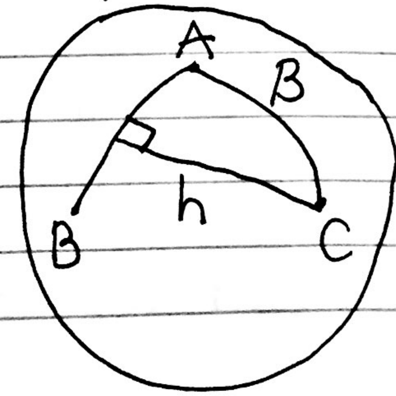
$A=(0,0,1)$



$r=1$
$\theta=a$
$\phi=\beta$

$(x,y,z)=r(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$
$C=(\sin\beta\cos a, \sin\beta\sin a, \cos\beta)$
$B=(\sin\gamma, 0, \cos\gamma)$

$h=?$



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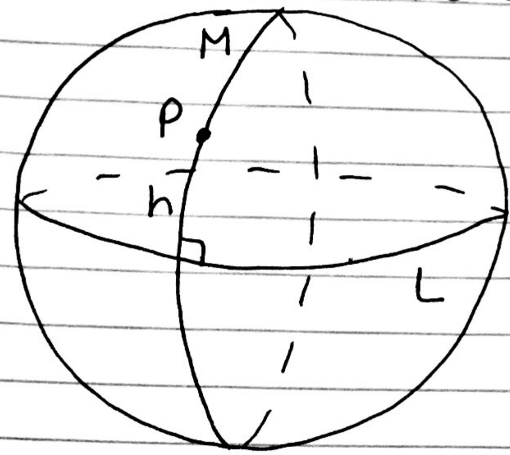


Question 1: how do we construct spherical line M through P perpendicular to L

Question 2: compute h

$L = \Pi_L \cap S^2 \quad \Pi_L = \{\bar{x} \in \mathbb{R}^3 \mid \bar{x} \cdot \bar{n}_L = 0\}$

$M = \Pi_M \cap S^2 \quad \Pi_M = \{\bar{x} \in \mathbb{R}^3 \mid \bar{x} \cdot \bar{n}_M = 0\}$

need: (Question 1)

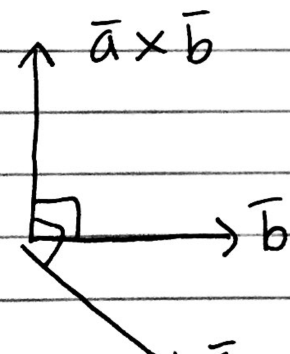
a. $\bar{n}_L \cdot \bar{n}_M = 0$

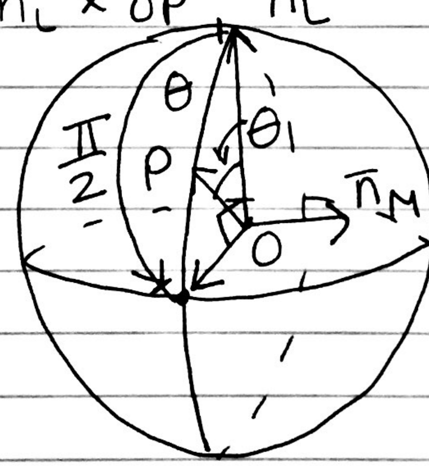
i.e. normal vectors perpendicular $\Rightarrow$ planes perpendicular $\Rightarrow$ L and M are perpendicular

b. $\overline{OP} \cdot \bar{n}_M = 0 \quad (P \in M)$

i.e lacking for a vector $\bar{n}_M$ such that $\bar{n}_M \cdot \bar{n}_L = \bar{n}_M \cdot \overline{OP} = 0$

can take $\bar{n}_M = \bar{n}_L \times \overline{OP}$





Question 2:

we may assume $\|\bar{n}_L\| = 1$ (scale)

then $\overline{OP} \cdot \bar{n}_L = \cos\theta$

$(\bar{a} \cdot \bar{b} = \|\bar{a}\| \|\bar{b}\| \cos\theta)$

$\Rightarrow h = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}(\overline{OP} \cdot \bar{n}_L)$