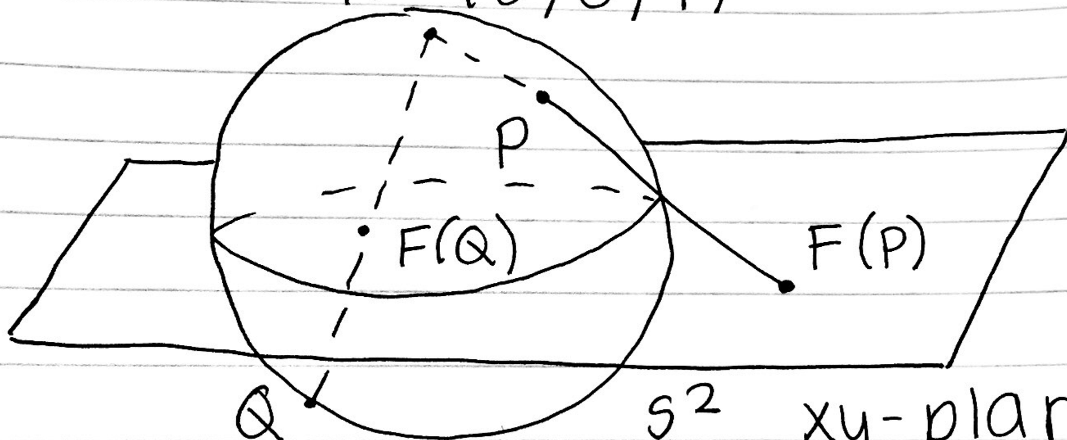
Math 461 Lecture 32 11/26

Last time: spherical isometries

- composition of 3 reflections = rotary reflection $\rightsquigarrow$ classification completed
- composition of 2 rotations is another rotation * (similar to $\mathbb{R}^2$ using spherical triangle)
- stereographic projection

$F: S^2 \setminus \{N\} \xrightarrow{\sim} \mathbb{R}^2$ bijection

$N = (0,0,1)$



Today:

algebraic formula for F, F^{-1}

comparison of distances on S^2 and $\mathbb{R}^2$

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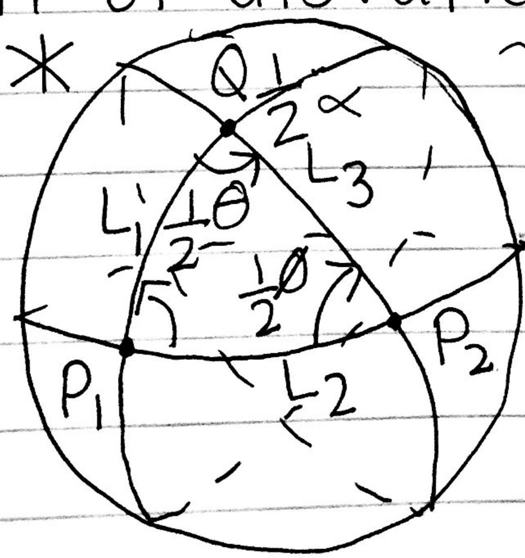

$\rightsquigarrow \mathrm{Rot}(P_2, \phi) \circ \mathrm{Rot}(P_1, \theta) = \mathrm{Rot}(Q, \alpha)$

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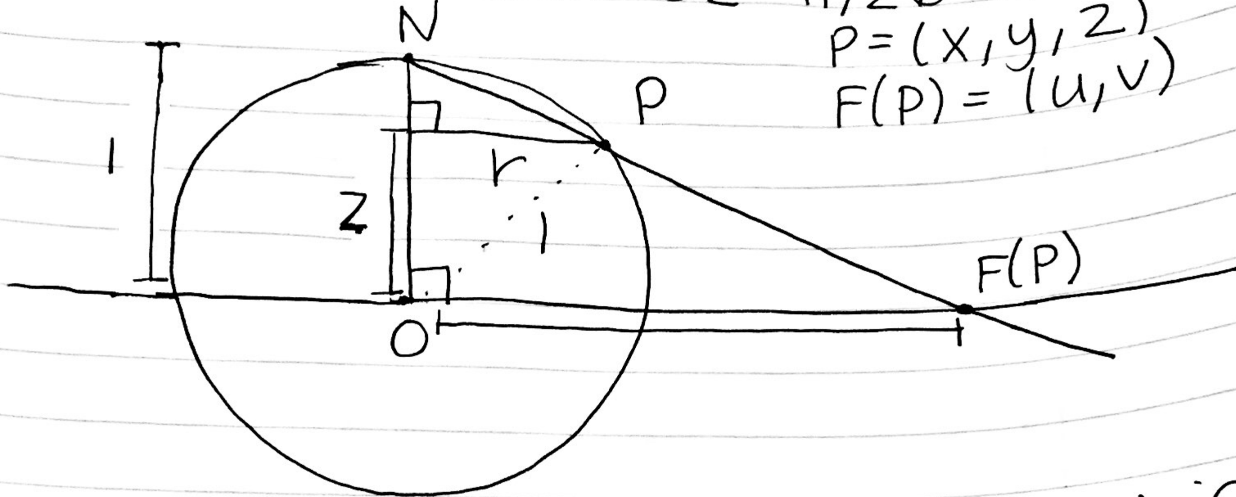
$P = (x, y, z)$

$F(P) = (u, v)$



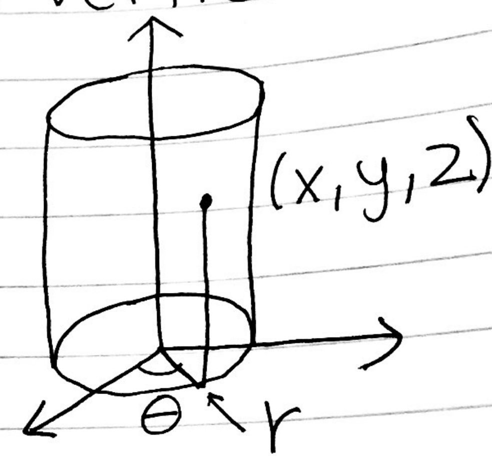
2D slice of picture for stereographic projection given by the vertical plane through O, N, P

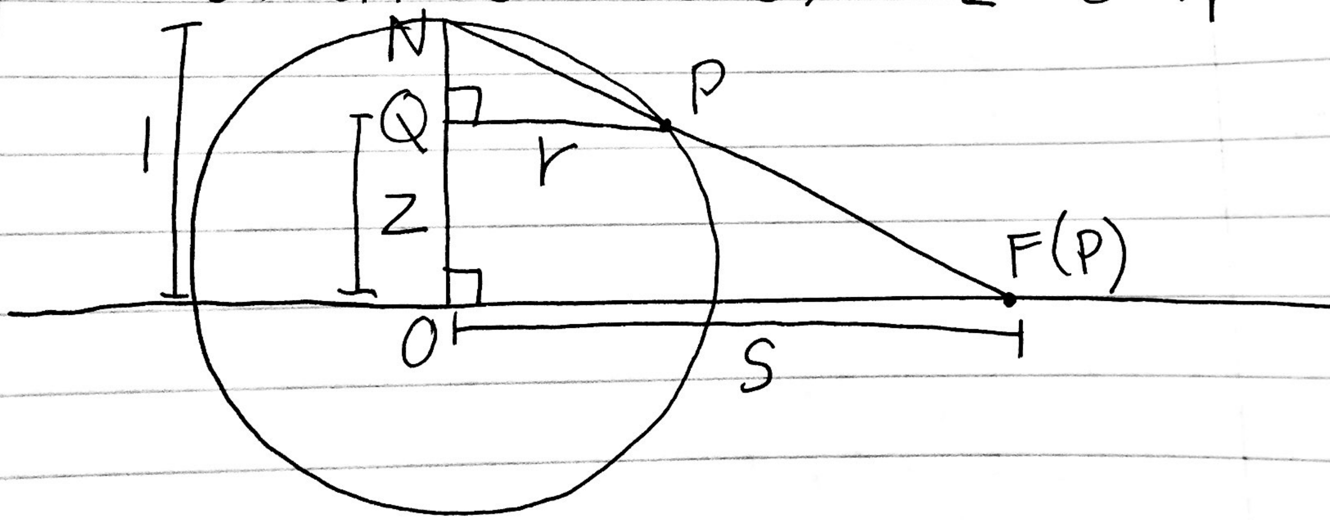
$P \in S^2 \quad x^2 + y^2 + z^2 = 1$

$r = \sqrt{1 - z^2} = \sqrt{x^2 + y^2}$

$(x, y, z) = (r\cos\theta, r\sin\theta, z)$

"cylindrical polar coordinates" (233)





$\Delta NQP \sim \Delta NOF(P) \quad \frac{|NQ|}{|NO|} = \frac{|QP|}{|OF(P)|} \quad \frac{1-z}{1} = \frac{r}{s}$

$\sqrt{u^2 + v^2} = \boxed{s = \frac{r}{1-z}}$

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$\rightsquigarrow F(x,y,z) = (u,v) = s \cdot \frac{(x,y)}{r}$

unit vector in direction (x,y)

$$\boxed{F(x,y,z) = \frac{1}{1-z} \cdot (x,y)}$$

$F^{-1}(u,v) = (x,y,z) = ?$

first compute z.

recall $s = \sqrt{u^2+v^2} = \frac{r}{1-z} = \frac{\sqrt{x^2+y^2}}{1-z} = \frac{\sqrt{1-z^2}}{1-z}$

square both sides:

$$s^2 = u^2+v^2 = \frac{1-z^2}{(1-z)^2} = \frac{(1+z)\cancel{(1-z)}}{(1-z)\cancel{(1-z)}}$$

$s^2 = \frac{1+z}{1-z}$ solve for z $s^2(1-z) = 1+z$

$s^2 - 1 = z(1+s^2)$

$$\boxed{z = \frac{s^2-1}{s^2+1} = \frac{u^2+v^2-1}{u^2+v^2+1}}$$

finally $F^{-1}(u,v) = (x,y,z) =$

using $(u,v) = \frac{1}{1-z}(x,y)$

$$(x,y) = (1-z)\cdot(u,v)$$

$$= \left(1 - \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)\right)\cdot(u,v)$$

$$= \left(\frac{2}{u^2+v^2+1}\right)\cdot(u,v)$$

$$\boxed{F^{-1}(u,v) = (x,y,z) = \frac{1}{u^2+v^2+1}\left(2u, 2v, u^2+v^2-1\right)}$$

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compare distances on S^2 and $\mathbb{R}^2$

$\bar{x} = (x, y, z) : [a,b] \to S^2 \subset \mathbb{R}^3$

curve on S^2

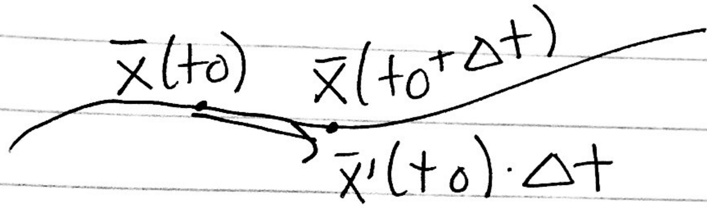
$t \mapsto (x(t), y(t), z(t))$

x, y, z continuously differentiable functions

$x(t)^2 + y(t)^2 + z(t)^2 = 1$ for all $t \in [a,b]$

$$\text{length}(\gamma) = \int_a^b \|\bar{x}'(t)\|\, dt$$

(233)
$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$



compare with length of image $F(\gamma)$ of curve γ in $\mathbb{R}^2$ under stereographic projection

$F: S^2 \setminus \{N\} \to \mathbb{R}^2$ (assuming: $\gamma \not\ni N$)

$(x, y, z) \mapsto (u, v) = \frac{1}{1-z}(x, y)$

$$\begin{tikzcd} {[a,b]} \arrow[r, "{(x,y,z)}"] \arrow[rr, bend right, "{(u,v)}"'] & S^2 \setminus \{N\} \arrow[r] & \mathbb{R}^2 \end{tikzcd}$$

$$\text{length}(F(\gamma)) = \int_a^b \|(u', v')\|\, dt$$

$$= \int_a^b \sqrt{u'^2 + v'^2}\, dt$$

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in fact, find:

$$\boxed{\sqrt{x'^2+y'^2+z'^2} = \frac{2}{u^2+v^2+1}\cdot\sqrt{u'^2+v'^2}}$$

so $\text{length}(\gamma) = \int_a^b \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}\,dt$

vs. $\text{length}(F(\gamma)) = \int_a^b \sqrt{u'^2+v'^2}\,dt$

example:

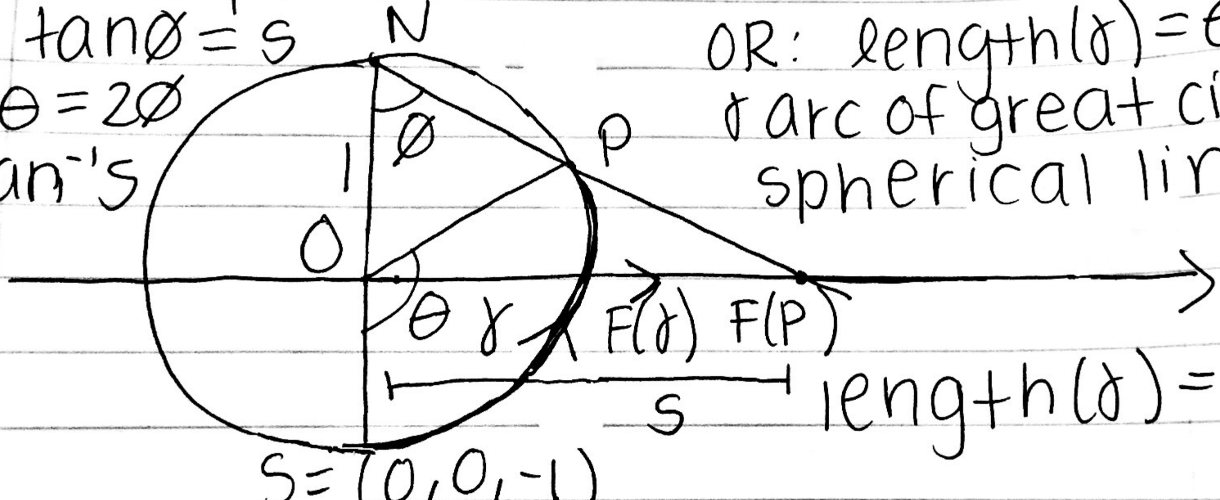
$\tan\phi = s$

$\theta = 2\phi$

$= 2\tan^{-1}s$

OR: $\text{length}(\gamma)=\theta$

γ arc of great circle/spherical line



$\text{length}(\gamma) = ?$

F(γ) parametization

$(u,v): [0,s] \to \mathbb{R}^2 \quad (u,v) = (t,0)$

$\text{length}(\gamma) = \int_a^b \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}\,dt$

$= \int_0^s \frac{2}{1+t^2}(1)\,dt = 2\tan^{-1}(t)\Big|_0^s$

$= 2\tan^{-1}(s)$