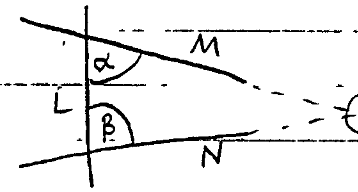
9/13/19

Euclid's Parallel Axiom

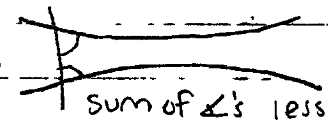
If a line (L) crosses two lines (M&N) and makes interior angles w/ sum less than π on one side, then the two lines meet on that side.



$\alpha + \beta < \pi$

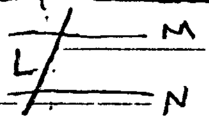
~1800 Gauss, Bolyai, Lobachevsky, etc constructed the "hyperbolic plane" which satisfies all of Euclid's axioms except parallel axiom.

$\Rightarrow$ parallel axiom is needed in Euclidean geometry

drawing parallel lines in hyperbolic plan $\Rightarrow$ 

sum of $\angle$'s less than π but lines DON'T intersect

Consequences of the parallel axiom

① 

IF M & N are parallel then $\alpha + \beta = \pi$.

We say lines M & N are parallel if they do not intersect.

Proof of ①: $A \Rightarrow B$, NOT A $\Leftarrow$ NOT B contrapositive

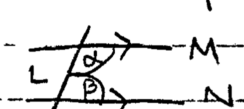
If $\alpha + \beta \neq \pi$ either $\alpha + \beta < \pi \underset{(PA)}{\Rightarrow}$ M&N meet on right side of L.

or $\alpha + \beta > \pi \Rightarrow (\pi - \alpha) + (\pi - \beta) < \pi \underset{(PA)}{\Rightarrow}$ M&N meet on left side of L.

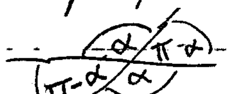
We've shown $\alpha + \beta \neq \pi \Rightarrow$ M&N are not parallel

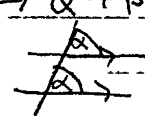
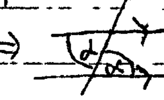
equivalently (contrapositive)

M&N parallel $\Rightarrow \alpha + \beta = \pi$ ∎

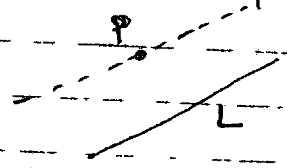


M&N parallel $\Rightarrow \alpha + \beta = \pi$



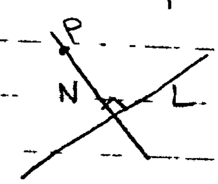
so equivalently  $\Rightarrow$  "alternate angles"

② Given a point P and a line L not passing through P, there is a unique line M parallel to L passing through P.



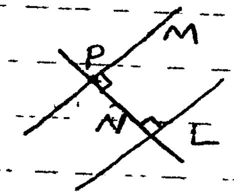
"Playfair's axiom"

Ruler & compass construction



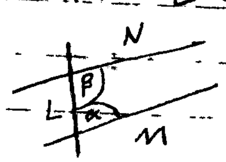
(a) construct perpendicular line N to L passing through P

(b) Construct line M through P, perpendicular to N



We claim L & M are parallel. Need converse of statement ①:

①'

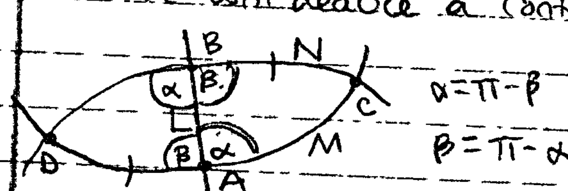


$\alpha + \beta = \pi \Rightarrow M$ is parallel to N

Proof of ①': (by contradiction)

Assume $\alpha+\beta=\pi$ & M is not parallel to N.

We will deduce a contradiction.



$\alpha = \pi - \beta$

$\beta = \pi - \alpha$

Recall: Euclid's axioms: Given 2 (distinct) points P & Q there is a unique line through P & Q

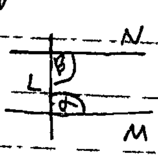
Let D be the point on the line AC on the opposite side of A to C with distance $|AD| = |BC|$. We claim D lies on the line BC.

Consider $\triangle ABD \cong \triangle BAD$ by SAS, $|AB| = |BA|$, $|BC| = |AD|$, $\angle ABC = \angle BAD = \beta$.

So $\angle ABD = \angle BAC = \alpha \Rightarrow$ lines BC & BD coincide.

Finally lines BC & AC intersect at 2 points C & D (contradiction)

Recap:



M & N are parallel $\Leftrightarrow \alpha + \beta = \pi$

$\rightsquigarrow$ construction of parallel line is valid