Math 611 Homework 5

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Reading: Dummit and Foote, Chapter 8.
Justify your answers carefully.

(1) Let R be an integral domain and $a, b \in R$. Recall that we say a and b are *associates* if a divides b and b divides a. Show that the following are equivalent.

   (a) a and b are associates.

   (b) The principal ideals generated by a and b are equal: $(a) = (b)$.

   (c) $a = ub$ where $u \in R$ is a unit.

(2) Let $d \in \mathbb{N}$ be a positive integer, and consider the subring

$$R = \mathbb{Z}[\sqrt{-d}] := \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

Define the norm $N \colon R \to \mathbb{Z}_{\geq 0}$ as follows: for $\alpha = a + b\sqrt{-d} \in R$,

$$N(\alpha) = \alpha\bar{\alpha} = |\alpha|^2 = a^2 + db^2.$$

   (a) Show that the norm is multiplicative: $N(\alpha\beta) = N(\alpha)N(\beta)$.

   (b) Show that $\alpha \in R$ is a unit iff $N(\alpha) = 1$.

   (c) Determine the units in R for each d.

(3) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a unique factorization domain (UFD).

(4) Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a UFD.

(5) Exhibit an ideal $I \subset \mathbb{Z}[\sqrt{-5}]$ which is not principal.

(6) Let $d \in \mathbb{N}$, $d \equiv 3 \bmod 4$. Let

$$R = \mathbb{Z}[(1+\sqrt{-d})/2] := \{a + b((1+\sqrt{-d})/2) \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

(a) Show that $R \subset \mathbb{C}$ is a subring.

(b) Determine the units of R.

(c) Show that if $d = 3$ then R is a UFD.

(7) Let R be an integral domain.

(a) Let $a, b \in R$. Show that a divides b iff $(b) \subset (a)$.

(b) Let $a \in R$ be an irreducible element. Show that if R is a principal ideal domain (PID) then $R/(a)$ is a field.

(c) Using part (b) or otherwise, prove that $\mathbb{Z}[x]$ and $\mathbb{C}[x, y]$ are not PIDs.

(8) Prove that there are infinitely many prime elements in $\mathbb{F}_p[x]$.

(9) Let $R = \mathbb{C}[t]$ and $S \subset R$ the subset of polynomials

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

such that $a_1 = 0$ (equivalently, $f'(t) = 0$).

(a) Prove that $S \subset R$ is a subring.

(b) Show that S is not a UFD.