Math 462 Homework 2

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In the problems below S^2 denotes the sphere of some radius $R > 0$ in $\mathbb{R}^3$ with center the origin O. Justify your answers carefully.

(1) Recall the spherical cosine rule: For a spherical triangle on a sphere of radius $R = 1$ with vertices A, B, C, angles a, b, c, and opposite side lengths α, β, γ, we have

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$$

Now suppose that the triangle is small, i.e., the side lengths α, β, γ are much smaller that $R = 1$. Use the approximations $\cos x \approx 1 - x^2/2$ and $\sin x \approx x$ for $x = \alpha, \beta, \gamma \approx 0$ in the spherical cosine rule and simplify the resulting equation. Explain your result.

(2) Consider a spherical triangle on the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

of radius $R = 1$ with vertices A, B, C, angles a, b, c and opposite side lengths α, β, γ. In this question we will prove the *spherical sine rule*:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}.$$

(a) Position the triangle on the sphere so that the vertex A is at the north pole and the vertex B lies in the xz-plane. Compute the coordinates of the vertices A, B, C in terms of $a, b, c, \alpha, \beta, \gamma$ using spherical coordinates.

(b) Using part (a), show that $\vec{OA} \cdot (\vec{OB} \times \vec{OC}) = \sin a \sin\beta\sin\gamma$.

(c) We have the identity

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \det(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$$

where $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$ and $(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ denotes the 3×3 matrix with columns $\mathbf{a},\mathbf{b},\mathbf{c}$ (why?). Use this identity together with (b) (and the corresponding equations for cyclic permutations of A,B,C) to deduce the spherical sine rule.

(3) Let A be a 2×2 orthogonal matrix. Then the linear map

$$T\colon \mathbb{R}^2\to\mathbb{R}^2, \quad T(\mathbf{x}) = A\mathbf{x}$$

is either a rotation about the origin, or a reflection in a line through the origin.

(a) Show that $\det A = +1$ if T is a rotation and $\det A = -1$ if T is a reflection.

(b) Show that if T is a reflection then the eigenvalues of A are ± 1. Describe the eigenvectors geometrically.

(c) Show that if T is a rotation through angle θ counter-clockwise, then the eigenvalues of A are $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

(4) Describe the following isometries geometrically.

(a) $R\colon \mathbb{R}^2\to\mathbb{R}^2$, $R(x,y) = \frac{1}{\sqrt{2}}(x+y, x-y)$.

(b) $S\colon \mathbb{R}^2\to\mathbb{R}^2$, $S(x,y) = \frac{1}{5}(3x-4y, 4x+3y)$.

(c) $T\colon \mathbb{R}^3\to\mathbb{R}^3$, $T(x,y,z) = \frac{1}{3}(x-2y-2z, -2x+y-2z, -2x-2y+z)$. [Hint: T is given by reflection in a plane.]

(5) Consider the sphere $S^2 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 = R^2\}$ with center the origin and radius R. Consider the cylinder

$$C = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2 = R, \quad -R<z<R\}$$

with radius R and axis the interval $(-R,R)$ on the z-axis. (So the sphere S^2 lies inside the cylinder.) We can “roll out” the cylinder to obtain a rectangle. Mathematically, let R denote the rectangular region

$$R = \{(u,v)\in\mathbb{R}^2 \mid 0\le u<2\pi R, \quad -R<v<R\}\subset\mathbb{R}^2.$$

We define a bijection

$$F\colon R \to C, \quad F(u,v) = (R\cos(u/R), R\sin(u/R), v)$$

which identifies the rectangle R with the cylinder C. Now let $N = (0,0,R)$ and $S = (0,0,-R)$ be the north and south poles of the sphere S^2, and define a bijection

$$G\colon S^2 \setminus \{N,S\} \to C, \quad G(x,y,z) = \left(\frac{Rx}{\sqrt{x^2+y^2}}, \frac{Ry}{\sqrt{x^2+y^2}}, z\right)$$

given by projecting radially outward from the z-axis. Combining, we get a bijection

$$F^{-1} \circ G\colon S^2 \setminus \{N,S\} \to R$$

which can be used to draw a map of the surface of the sphere in the plane. [It is called the Gall-Peters projection.]

(a) Find an explicit formula for the bijection

$$H := G^{-1} \circ F\colon R \to S^2 \setminus \{N,S\}.$$

(This is the inverse of the Gall Peters projection $F^{-1} \circ G$ considered above.)

(b) Show that the function H preserves areas. [Hint: To do this, write $H(u,v) = \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$. Then (from Math 233) the area of the image $H(T)$ of a region $T \subset R$ under the function H is given by the integral

$$\int\int_T \left\| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right\| du\, dv.$$

So H preserves areas iff the function $\|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\|$ is constant, equal to 1 (why?).]