Math 300.2 Homework 9

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Reading: Gilbert and Vanstone, Chapter 8.

Note: This homework will not be graded, but the material will be covered on the final exam.

(1) Write the following complex numbers in polar form

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

   (a) $z = 5i$.
   (b) $z = 3 + 3i$.
   (c) $z = 1 - \sqrt{3}i$.

(2) Recall that for a complex number $z = x + iy$ we write $|z| = \sqrt{x^2 + y^2}$.

   (a) Show directly that for $z, w \in \mathbb{C}$ we have $|zw| = |z| \cdot |w|$. [Write $z = x + iy$ and $w = u + iv$ and compute both sides.]
   (b) Show that $|z| = 0 \iff z = 0$.
   (c) Use parts (a) and (b) to show that

$$zw = 0 \iff ((z = 0)\,\mathrm{OR}(w = 0)).$$

   [You can assume the corresponding property for real numbers.]
   (d) Show that if $z \neq 0$ then writing $z' = \bar{z}/|z|^2$ we have $zz' = 1$. Use this to give another proof for part (c).

(3) Find all the solutions $z \in \mathbb{C}$ of the following equations.

   (a) $z^2 - 4z + 13 = 0$.

(b) $z^3 = 27i$.

(c) $z^2 + (2 - 6i)z + (-8 + 2i) = 0$.

(4) Find all solutions $z \in \mathbb{C}$ of the equation $z^4 + z^2 + 1 = 0$. [Hint: Write $w = z^2$ and solve a quadratic equation to find w. Now take square roots to find z.]

(5) Recall De Moivre's theorem: $\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$, for $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$.

(a) Use De Moivre's theorem and the binomial theorem to write $\cos(4\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

(b) Use the identity $(\cos\theta)^2 + (\sin\theta)^2 = 1$ to write $\cos(4\theta)$ in terms of $\cos\theta$ only.

(6) Let $\theta = 2\pi/5$.

(a) Explain why $\cos(2\theta) = \cos(3\theta)$.

(b) Use De Moivre's theorem and part (a) to write down a cubic polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with real coefficients a_0, a_1, a_2, a_3 such that $p(\cos\theta) = 0$.

(c) Now find all the solutions of $p(x) = 0$, and deduce an exact formula for $\cos(\theta)$. [Hint: One can check that $p(1) = 0$, so we have $p(x) = (x - 1)q(x)$ for some quadratic polynomial $q(x)$. Remark: Note that $p(1) = 0$ because $1 = \cos(0)$ and the equation in part (a) is also satisfied for $\theta = 0$.]