Wednesday 10/21/15.

MATH G11 HW4. Solutions (continued)

11 (b) $|G| = 28 = 2^2 \cdot 7$

$s = \#$ Sylow 2-subgroups $\equiv 1 \bmod 2$, $s \mid 7 \Rightarrow s = 1$ OR 7.

$t = \#$ Sylow 7-subgroups $\equiv 1 \bmod 7$, $t \mid 4 \Rightarrow t = 1$

$\therefore \quad G \cong H \rtimes_\varphi K \qquad H \simeq \mathbb{Z}/7\mathbb{Z}$

$K \simeq \mathbb{Z}/4\mathbb{Z}$ OR $(\mathbb{Z}/2\mathbb{Z})^2$.

$\varphi : K \to \operatorname{Aut} H$.

a. $K \simeq \mathbb{Z}/4\mathbb{Z}$.

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & \operatorname{Aut}(H) \\ \wr\wr & & \wr\wr \\ \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \operatorname{Aut}(\mathbb{Z}/7\mathbb{Z}) = (\mathbb{Z}/7\mathbb{Z})^\times = \mathbb{Z}/6\mathbb{Z} \end{array}$$

Only non-trivial hom is given by

$$i \longmapsto 3 \cdot i$$

$3 \in \mathbb{Z}/6\mathbb{Z}$ corresponds to $-1 \in (\mathbb{Z}/7\mathbb{Z})^\times$ (element of order 2)

& so to $(x \mapsto -x) \in \operatorname{Aut}(\mathbb{Z}/7\mathbb{Z})$

$\therefore$

$$G = \langle a, b \mid a^7 = b^4 = e,\ bab^{-1} = a^{-1} \rangle.$$

b. $K \simeq (\mathbb{Z}/2\mathbb{Z})^2$

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & \operatorname{Aut}(H) \\ \wr\wr & & \wr\wr \\ (\mathbb{Z}/2\mathbb{Z})^2 & \longrightarrow & \operatorname{Aut}(\mathbb{Z}/7\mathbb{Z}) \simeq (\mathbb{Z}/7\mathbb{Z})^\times = \mathbb{Z}/6\mathbb{Z} \end{array}$$

~~Cases~~ $\qquad (i,j) \longmapsto 3i$

Only non trivial hom up to change of basis in $(\mathbb{Z}/2\mathbb{Z})^2$.

$$\therefore \quad G = \langle a, b, c \mid a^7 = b^2 = c^2 = e,\ bc = cb,\ bab^{-1} = a^{-1},\ cac^{-1} = a \rangle$$

$$= \langle a, b \mid a^7 = b^2 = e,\ bab^{-1} = a^{-1} \rangle \times \langle c \mid c^2 = e \rangle$$

$$\cong D_7 \times \mathbb{Z}/2\mathbb{Z} \qquad (\simeq D_{14})$$

HW1 Q7b

Also have abelian cases $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/7\mathbb{Z}$

(corresponding to trivial hom φ).