Practice questions for Midterm 1

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1. Find all solutions of the following systems of linear equations. Explain your results geometrically.

(a)

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + y + 2z &= 1 \\ x - y + 3z &= 2 \end{aligned}$$

(b)

$$\begin{aligned} x + y + z + t &= 2 \\ x + 2y + 3z + 2t &= 4 \\ 2x + 4y + 4z - t &= 5 \end{aligned}$$

2. Find all solutions of the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$$

where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Explain your result geometrically.

3. Find all solutions of the equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^3$? Explain your answer.

4. Let S, T, U, V be the linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

(a) Describe the maps geometrically. (It may help to draw the image of the unit square.)

(b) Compute the matrices of the compositions $S \circ U$, $T \circ T$, and $T^{-1} \circ U \circ T$. Interpret your results geometrically.

5. Find the matrix of the following linear maps.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ rotation about the origin through an angle of $\pi/3$ radians anticlockwise.

(b) $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ reflection in the line through the origin in direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) $V\colon \mathbb{R}^3 \to \mathbb{R}^3$ projection onto the plane $x + 2y + 3z = 0$.

(d) $W\colon \mathbb{R}^3 \to \mathbb{R}^3$ rotation about the y-axis through an angle of $\pi/3$ radians anticlockwise.

6.

(a) The linear map $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \frac{1}{13}\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$ is a projection onto a line. Find the line.

(b) The linear map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $B = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$ is a reflection in a plane. Find the plane.

(c) The linear map $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$ is a rotation. Find the axis of rotation.

7. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -1 & -1 & 2 \end{pmatrix}.$$

(a) Compute A^{-1}.

(b) Using your result from (a), solve the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 3x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

8. The unit cube in $\mathbb{R}^3$ has vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$. Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map such that $T(1,0,0) = (2,1)$, $T(0,1,0) = (1,2)$, and $T(0,0,1) = (1,1)$. Write down the matrix of T. Draw the image of the unit cube in $\mathbb{R}^2$ under the map T (draw the image of each edge of the cube).