Math 461 Lecture 17 10/12

Today:

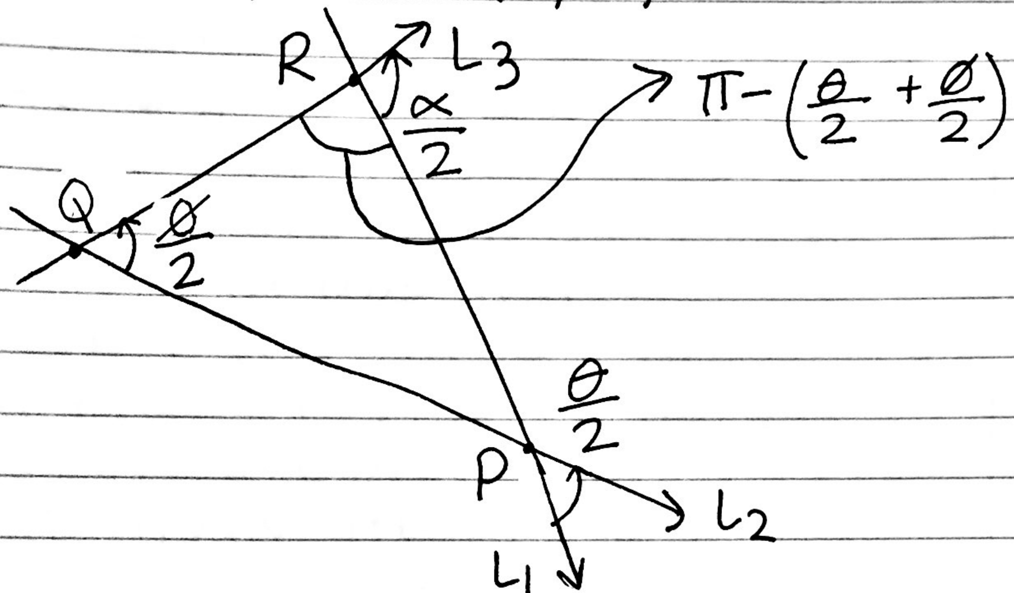
More compositions

GPS Theorem

Compositions of rotations:

$Rot(P, \emptyset) \circ Rot(P, \theta) = Rot(P, \emptyset + \theta)$

$Rot(Q, \emptyset) \circ Rot(P, \theta) =$



Now $Rot(Q, \emptyset) \circ Rot(P, \theta) =$

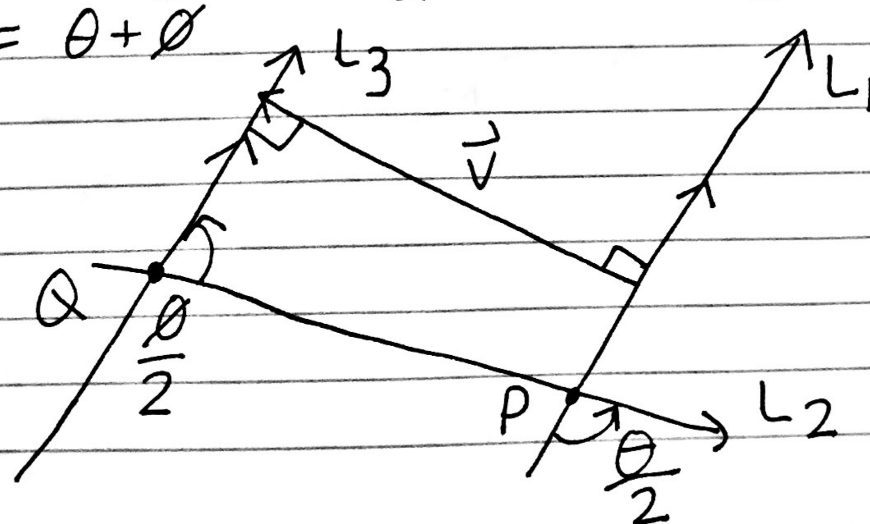
$(Refl_{L_3} \circ Refl_{L_2}) \circ (Refl_{L_2} \circ Refl_{L_1}) =$

$Refl_{L_3} \circ \text{identity} \circ Refl_{L_1} =$

$Refl_{L_3} \circ Refl_{L_1} = Rot(R, \alpha)$

$\alpha = ?$ $\frac{\alpha}{2} = \pi - \left[\pi - \left(\frac{\theta}{2} + \frac{\emptyset}{2}\right)\right] = \frac{\theta}{2} + \frac{\emptyset}{2}$

$\alpha = \theta + \emptyset$



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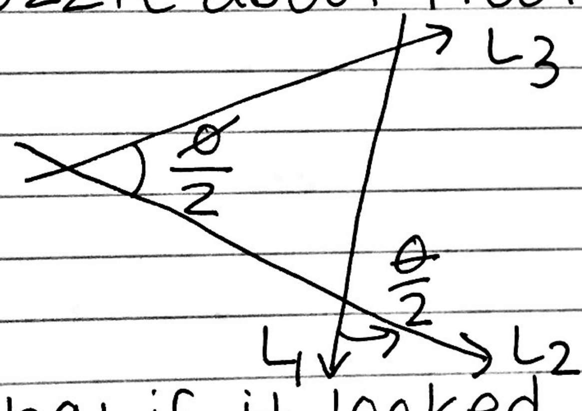
$\mathrm{Rot}(Q,\phi) \circ \mathrm{Rot}(P,\theta) = \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_1}$
$= \mathrm{Trans}_{2\vec{v}}$

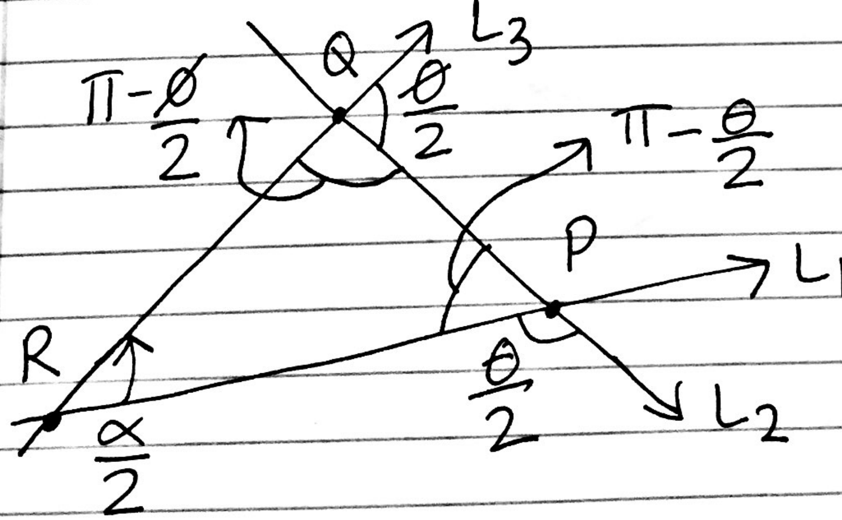
When does this happen?
L_1 and L_3 are parallel
$\iff \frac{\theta}{2} + \frac{\phi}{2} = \pi$
$\iff \theta + \phi = 2\pi$
$\iff \phi = -\theta \mod 2\pi$
(i.e. rotate by θ counter clockwise about P, then rotate by θ clockwise about Q)

Real world application:
moving a piece of furniture that's too heavy to pick up so you rotate it about one of the corners but you're essentially translating it.

Puzzle about proof:



What if it looked like this?



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$Rot(Q, \phi) \circ Rot(P, \theta) = Rot(R, \alpha)$

$$\frac{\alpha}{2} = \pi - \left(\pi - \frac{\phi}{2}\right) - \left(\pi - \frac{\theta}{2}\right)$$

$$= -\pi + \frac{\theta}{2} + \frac{\phi}{2}$$

$$\alpha = \theta + \phi - 2\pi \equiv \theta + \phi \mod 2\pi$$

GPS Theorem: (do not all lie on the same line)

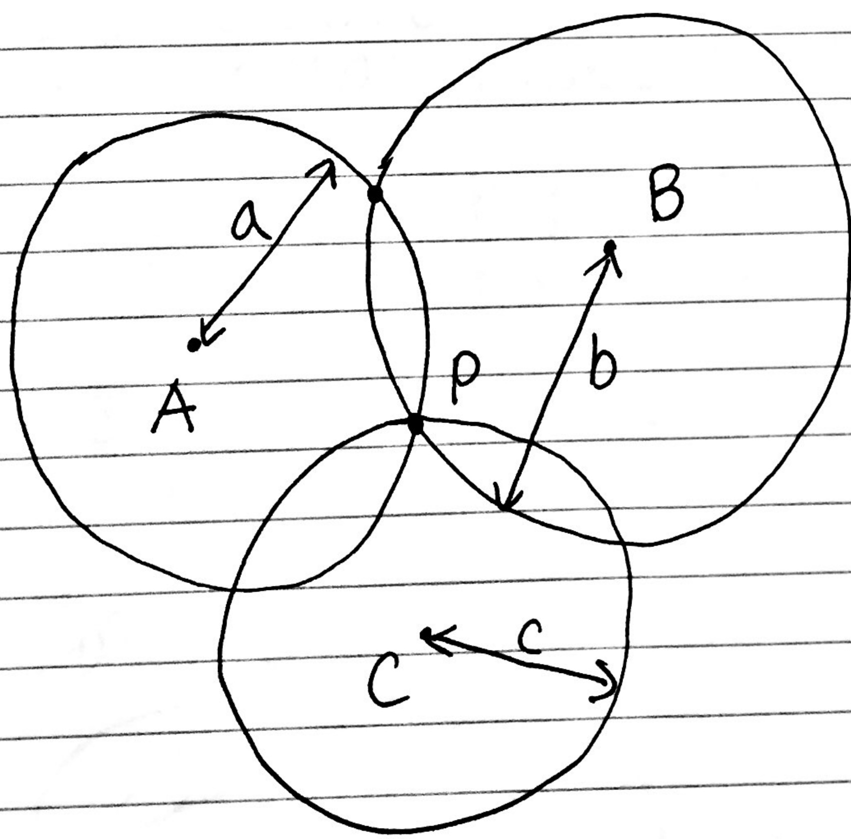
$A, B, C \in \mathbb{R}^2$ not collinear

Then any point $P \in \mathbb{R}^2$ is uniquely determined by $|PA|, |PB|, |PC|$.

Proof:

Need to show: if $P, Q \in \mathbb{R}^2$ are two points such that $|PA| = |QA|$, $|PB| = |QB|$, $|PC| = |QC|$ then in fact $P = Q$.

$|PA| = a$, $|PB| = b$, $|PC| = c$



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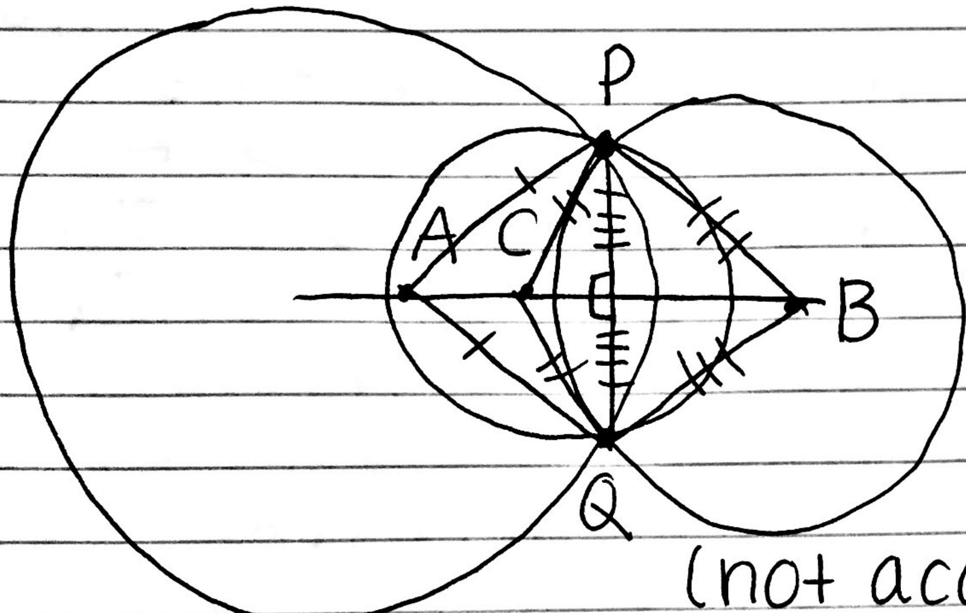

(not accurate)

A, B, C are supposed to be in the centers of the circles

Proof by contradiction:
Suppose we have points P and Q such that $P \neq Q$ and
$|PA| = |QA|$, $|PB| = |QB|$, $|PC| = |QC|$ *
Then claim A, B, C are collinear $\rightsquigarrow$ #
More precise, claim A, B, C lie on perpendicular bisector of the line segment PQ.
We proved earlier perpendicular bisector is set of points equidistant from P and Q.
So $A, B, C \in$ perpendicular bisector of PQ by * $\square$

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Corollary:

$A, B, C \in \mathbb{R}^2$ are not collinear

$T: \mathbb{R}^2 \to \mathbb{R}^2$ an isometry

Then T is uniquely determined by $T(A), T(B), T(C)$.

Proof: Given $P \in \mathbb{R}^2$. Need to show $T(P)$ is uniquely determined (given $T(A), T(B), T(C)$)

$|T(P)T(A)| = |PA|$ know

$|T(P)T(B)| = |PB|$ these

$|T(P)T(C)| = |PC|$

Theorem $\Rightarrow$ $T(P)$ uniquely determined provided $T(A), T(B), T(C)$ are not collinear