Math 462: Homework 5

Paul Hacking

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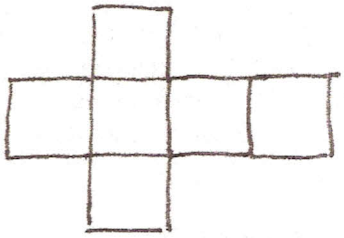
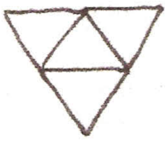
(1) One way to describe a polyhedron is by cutting along some of the edges and folding it flat in the plane. The diagram obtained in this way is called a net. For example here is a net for the cube:

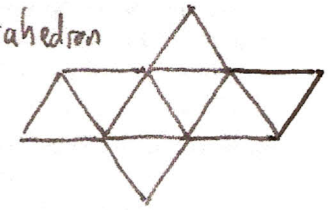


Draw nets for the tetrahedron, octahedron, and dodecahedron.

tetrahedron

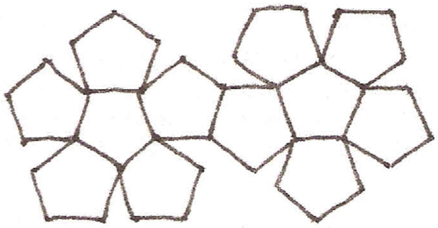


octahedron

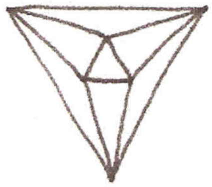


[Note: There is more than one correct answer here.]

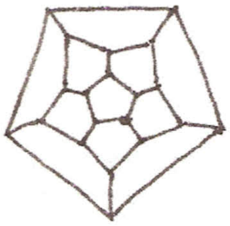
dodecahedron.



(2) Another way to describe a polyhedron is as follows: imagine that the faces of the polyhedron are made of glass, look through one of the faces and draw the image you see of the remaining faces of the polyhedron. This image is called a Schlegel diagram. For example here is a Schlegel diagram for the octahedron.



Note that the face you are looking through is not drawn, but the boundary of that face corresponds to the boundary of the diagram. Now draw a Schlegel diagram for the dodecahedron.



(3) (a) A *pyramid* is a polyhedron obtained from a polygon with some number n of sides by joining every vertex of the polygon to a point lying above the plane of the polygon. The polygon is called the base of the pyramid and the additional vertex is called the apex. For example the Egyptian pyramids have base a square (so $n = 4$) and the tetrahedron is a pyramid with base an equilateral

triangle (so $n = 3$). Compute the number of vertices, edges, and faces of a pyramid with base a polygon with n sides.

(b) A *prism* is a polyhedron obtained from a polygon (called the base of the prism) as follows: translate the polygon some distance in the direction normal to the plane of the polygon, and join the vertices of the original polygon to the corresponding vertices of the translated polygon. For example, a Toblerone box is a prism with base an equilateral triangle. Compute the number of vertices, edges, and faces of a prism with base a polygon with n sides.

(c) An *antiprism* is a polyhedron obtained from a regular n-sided polygon (called the base of the antiprism) as follows: translate the polygon some distance in the direction normal to the plane of the polygon, and rotate the polygon through an angle of π/n either clockwise or anticlockwise about its center. (Note that the choice of the direction of rotation does not change the image of the polygon (because rotation by $2\pi/n$ is a symmetry of the polygon), but it does affect the way the vertices are matched up.) Now join the original polygon to the new polygon by connecting each vertex to its image under the two transformations. For example, an octahedron is an antiprism with base an equilateral triangle, and the middle portion of an icosahedron is an antiprism with base a regular pentagon (check for yourself by drawing a picture). Compute the number of vertices, edges, and faces of an antiprism with base a regular polygon with n sides.

(d) Compute the number of vertices, edges, and faces of the regular polyhedrons (the cube, the octahedron, the tetrahedron, the icosahedron, and the dodecahedron).

(e) Do you notice any patterns? Can you explain them?

We write V, E, F for the numbers of vertices, edges, and faces.

(a) $V = n + 1$, $E = 2n$, $F = n + 1$.

(b) $V = 2n$, $E = 3n$, $F = n + 2$.

(c) $V = 2n$, $E = 4n$, $F = 2n + 2$.

(d) Tetrahedron: $V = 4$, $E = 6$, $F = 4$. Cube: $V = 8$, $E = 12$, $F = 6$. Octahedron: $V = 6$, $E = 12$, $F = 8$. Icosahedron: $V = 12$, $E = 30$, $F = 20$. Dodecahedron: $V = 20$, $E = 30$, $F = 12$.

(e) The numbers V, E, F for the cube are equal to the numbers F, E, V for the octahedron. This is because the cube and octahedron are dual polyhedra. The same is true for the icosahedron and dodecahedron. A pyramid as in part (a) is self dual so $V = F$ (this includes the case of the tetrahedron for $n = 3$).

In all the examples above we have $V - E + F = 2$. We will show in class that this is true for any polyhedron

(4) Recall that if P is a regular polyhedron then the *dual polyhedron* P^* has vertices the centers of the faces of P. Two vertices of P^* are joined by an edge if the corresponding faces of P meet at an edge. Identify the dual of a cube, an octahedron, and a tetrahedron. In each case, find the ratio of the length of the edges of the original polyhedron P to the length of the edges of its dual P^*.

We just compute in coordinates: For the cube we can take the vertices to be the 8 points $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$. Then the centers of the faces are the points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$. These are the vertices of dual polyhedron which is an octahedron. The side lengths of the cube equal 2, and the side lengths of the octahedron equal $\sqrt{2}$, so the ratio is $2/\sqrt{2} = \sqrt{2}$. Here we have used the distance formula in 3-dimensions

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

For the octahedron we can take the vertices to be the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ in $\mathbb{R}^3$. The centers of the faces are $\frac{1}{3}(\pm 1, \pm 1, \pm 1)$. (Note that the position vector of the center of a face is given by the average of the position vectors of the vertices of the face.) These are the vertices of the dual polyhedron which is a cube. The side lengths of the octahedron equal $\sqrt{2}$, and the side lengths of the cube equal $2/3$, so the ratio is $\sqrt{2}/(2/3) = 3\sqrt{2}/2$.

For the tetrahedron we can take the vertices to be the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ in $\mathbb{R}^4$. (Note that they lie in the 3-dimensional space given by the equation $x + y + z + t = 1$ in $\mathbb{R}^4$.) The centers of the faces are $\frac{1}{3}(1, 1, 1, 0), \frac{1}{3}(1, 1, 0, 1), \frac{1}{3}(1, 0, 1, 1), \frac{1}{3}(0, 1, 1, 1)$. These are the vertices of the dual polyhedron which is another (smaller) tetrahedron. The side lengths of the original tetrahedron are $\sqrt{2}$, the side lengths of the dual tetrahedron are $\sqrt{2}/3$, so the ratio is $\sqrt{2}/(\sqrt{2}/3) = 3$. Here we have used the distance formula in 4 dimensions

$$d((x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (t_1 - t_2)^2}.$$

(5) Describe the duals of a pyramid, a prism, and an antiprism (see Q3 above).

The pyramid is self dual. The dual of a prism is a *bipyramid* obtained by gluing two pyramids together along their base. The dual of an antiprism is a polyhedron which looks like a diamond: it has $2n$ faces each with 4 edges, with n meeting at the top vertex and n at the bottom.

Note: For a polyhedron P which is not regular, we cannot always construct the dual polyhedron P^* by taking the vertices of P^* to be the centers of the faces of P as in Q4. The problem is that we want a vertex v of P to correspond to a face f of P^*, so that the vertices of P^* corresponding to the faces of P meeting at v are the vertices of the face f, in particular, these vertices must lie in a plane. The original construction does work for the pyramid and the prism. For the antiprism, if we choose the height carefully, then we can take the vertices of P^* to be points on the faces of P, but these points will not be the centroids of the triangular faces of P (except in the case $n = 3$ when the antiprism is an octahedron and so the dual is a cube).