Math 300.3 Homework 5

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March 2, 2017

Reading: Sundstrom, Sections 3.4, 3.5 and 4.1.

Justify your answers carefully.

(1) Prove that $\sqrt{3}$ is irrational.

[Recall that for all $a.b \in \mathbb{Z}$, if $3 \mid ab$ then $3 \mid a$ or $3 \mid b$. In particular, for all $a \in \mathbb{Z}$, if $3 \mid a^2$ then $3 \mid a$. (This is the special case $a = b$ of the previous result.)]

(2) Suppose we are given an 8×8 chessboard and a collection of dominoes (2×1 tiles). Each domino can be used to cover two adjacent squares of the chessboard. Suppose we remove two opposite corner squares from the chessboard. Is it possible to cover the remaining squares using dominoes? (Either describe a tiling by dominoes or give a proof by contradiction that no tiling exists.)

[Hint: What colour are the squares we remove?]

(3) Prove the following statement: For all real numbers x, $x^2 + 1 \geqslant 2|x|$.

(4) Prove the following statements

(a) For all $x \in \mathbb{Z}$, $3 \mid x^3 + 1 \iff x \equiv 2 \bmod 3$.

(b) For all $x \in \mathbb{Z}$, $5 \mid (x^2 + x + 3) \iff (x \equiv 1 \bmod 5$ or $x \equiv 3 \bmod 5)$.

(5) Prove the following statements by induction.

(a) For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$.

(b) For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

(6) Prove the following statement by induction: For all $x \in \mathbb{R}$ such that $x \geqslant -1$ and all $n \in \mathbb{N}$,

$$(1+x)^n \geqslant 1 + nx.$$

(7) Recall that for all $a, b, c, d \in \mathbb{Z}$ and $m \in \mathbb{N}$, if $a \equiv b \bmod m$ and $c \equiv d \bmod m$ then $ac \equiv bd \bmod m$.

(a) Prove the following statement by induction: For all $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$, and $n \in \mathbb{N}$, if $a \equiv b \bmod m$ then $a^n \equiv b^n \bmod m$.

(b) Using part (a) or otherwise compute (i) the remainder when 2017^{100} is divided by 2016 and (ii) the remainder when 2016^{51} is divided by 2017.