Math 611 Homework 7

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Reading: Dummit and Foote, 12.2–12.3 (rational canonical form and Jordan normal form), 11.1–11.3 (review of vector spaces including dual space). Unfortunately the material on bilinear forms is not covered in Dummit and Foote. I recommend Artin, Algebra, Chapter 7 (1st ed.) / Chapter 8 (2nd ed.).

All vector spaces are assumed finite dimensional.

(1) Classify matrices $A \in \mathrm{GL}_4(\mathbb{Q})$ of orders 4 and 5 up to conjugacy $A \rightsquigarrow PAP^{-1}$.

(2) Let $J(n, \lambda)$ denote the $n \times n$ matrix

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

We call $J(n, \lambda)$ the Jordan block of size n with eigenvalue λ. (Note: In DF they use the transpose of the above matrix. This corresponds to a change of basis given by reversing the order of the basis.)

Show that λ is the unique eigenvalue of $J(n, \lambda)$ and

$$\dim \ker(J(n, \lambda) - \lambda I)^k = \min(k, n).$$

(3) Let V be a vector space and $T \colon V \to V$ be a linear transformation from V to itself. What is the *Jordan normal form* of T? Suppose that

the eigenvalues of T are λ_i, $i = 1, \dots, r$, and the sizes of the Jordan blocks with eigenvalue λ_i are $m_{i1} \leq m_{i2} \leq \dots \leq m_{is_i}$.

(a) What is the characteristic polynomial of T? What is the minimal polynomial of T?

(b) Let $d_{ik} = \dim \ker(T - \lambda_i \operatorname{id}_V)^k$. Using Q2 or otherwise, describe an algorithm to determine the block sizes m_{ij} in terms of the dimensions d_{ik}.

(c) Determine the Jordan normal form of the matrix

$$A = \begin{pmatrix} 5 & 1 & 1 \\ -10 & -2 & -5 \\ 6 & 3 & 6 \end{pmatrix}$$

(4) (This question will not be graded. Its purpose is to review our notation for bilinear forms and to understand various concepts explicitly in terms of a choice of coordinates.)

Let V be a finite dimensional vector space over a field F and $b \colon V \times V \to F$ a bilinear form. Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V and

$$L_{\mathcal{B}} \colon V \xrightarrow{\sim} F^n, \quad v = c_1 v_1 + \dots + c_n v_n \mapsto [v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

the corresponding isomorphism from V to F^n. Then

$$b(v, w) = [v]_{\mathcal{B}}^T A [w]_{\mathcal{B}} = \sum a_{ij} c_i d_j$$

where $A = (a_{ij}) = (b(v_i, v_j))$, $[v]_{\mathcal{B}} = (c_i)$, $[w]_{\mathcal{B}} = (d_i)$. We call A the matrix of b with respect to the basis $\mathcal{B}$ of V.

We say b is *symmetric* if $b(v, w) = b(w, v)$ for all $v, w \in V$. We say b is *skew-symmetric* if $b(v, v) = 0$ for all $v \in V$. (Note: This condition implies $b(v, w) = -b(w, v)$ for all $v, w \in V$, and the two conditions are equivalent unless $\operatorname{char}(F) = 2$.)

If b is symmetric or skew-symmetric, the *null-space* $N \subset V$ of b is defined by

$$N = \{v \in V \mid b(v, w) = 0 \text{ for all } w \in V\}.$$

We say b is *non-degenerate* if $N = \{0\}$.

There is a bijective correspondence between bilinear forms $b\colon V \times V \to F$ and linear transformations $\varphi\colon V \to V^*$, where V^* denotes the dual of the vector space V (i.e., the vector space of linear transformations $\theta\colon V \to F$), given by

$$\varphi(v)(w) = b(v, w).$$

Establish the following facts.

(a) b is symmetric iff $A^T = A$.

(b) b is skew symmetric iff $A^T = -A$ and the diagonal entries of A are zero. (Note: The second condition follows from the first unless $\operatorname{char}(F) = 2$.)

(c) For b symmetric or skew-symmetric, the isomorphism

$$L_{\mathcal{B}}\colon V \xrightarrow{\sim} F^n$$

induces an isomorphism $N \xrightarrow{\sim} \ker(A)$. In particular, b is non-degenerate iff A is invertible.

(d) The basis $\mathcal{B} = (v_1, \ldots, v_n)$ of V determines a dual basis $\mathcal{B}^* = (v_1^*, \ldots, v_n^*)$ of V^* given by $v_i^*(v_j) = 1$ if $i = j$ and 0 if $i \neq j$. If $\varphi\colon V \to V^*$ is the linear transformation determined by b, then the matrix of φ with respect to the basis $\mathcal{B}$ of V and the basis $\mathcal{B}^*$ of V^* is the matrix A^T.

(5) Let $b\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the symmetric bilinear form with matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that the matrix of b with respect to $\mathcal{B}$ is diagonal and determine the rank and signature of b.

(6) Let $V = \mathbb{R}^{n \times n}$ be the $\mathbb{R}$-vector space of $n \times n$ real matrices. Let

$$b\colon V \times V \to \mathbb{R}$$

be the symmetric bilinear form given by

$$b(A, B) = \operatorname{trace}(AB).$$

(a) Show that b is indeed symmetric.

(b) For $n = 2$, find a basis $\mathcal{B}$ of V such that the matrix of b with respect to $\mathcal{B}$ is diagonal, with diagonal entries equal to $1, -1$, or 0. Deduce the rank and signature of b.

(c) Repeat for n arbitrary.

(7) Let $b \colon V \times V \to F$ be a symmetric or skew-symmetric bilinear form. Recall that for $W \subset V$ a subspace we define the orthogonal $W^\perp$ of W by

$$W^\perp = \{v \in V \mid b(v, w) = 0 \text{ for all } w \in W\}.$$

(a) Let $b|_W$ denote the restriction of b to W, that is, the bilinear form on W given by $b|_W(x, y) = b(x, y)$ for $x, y \in W$. In class we showed that if $b|_W$ is non-degenerate then $V = W \oplus W^\perp$. Give a direct proof of this fact in the case that W is 1-dimensional.

(b) Let b be a symmetric bilinear form which is positive definite (i.e. $b(x, x) > 0$ for all $0 \neq x \in V$). Show that for any subspace $W \subset V$ we have $V = W \oplus W^\perp$.

(c) Let b be the bilinear form on $V = F^3$ with matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $W \subset V$ the 1-dimensional subspace spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Determine $W^\perp$. Draw a picture showing W, $W^\perp$, and the quadratic cone $(q(x) = 0) \subset \mathbb{R}^3$, where $q(x) = b(x, x) = x_1^2 + x_2^2 - x_3^2$ is the quadratic form associated to b.

(d) Suppose b is non-degenerate. Prove that

$$\dim W^\perp = \dim V - \dim W.$$

[Hint: Let $\varphi \colon V \to V^*$ be the linear transformation determined by b, and $\psi \colon V^* \to W^*$ the linear transformation given by restriction to W (i.e. $\psi(\theta) = \theta|_W$, where $\theta|_W \colon W \to F$, $\theta|_W(x) = \theta(x)$ for $x \in W$). Show that φ is an isomorphism, ψ is surjective, and $W^\perp = \ker(\psi \circ \varphi)$.]

(8) Let $b \colon V \times V \to F$ be a non-degenerate symmetric or skew-symmetric bilinear form. We say a subspace $W \subset V$ is *isotropic* if $b|_W = 0$, i.e., $b(x,y) = 0$ for all $x, y \in W$.

(a) Suppose b is skew-symmetric. First explain why the dimension of V is even, say $\dim V = 2m$. Second prove that the maximal dimension of an isotropic subspace $W \subset V$ equals m. (Remark: A vector space V with nondegenerate skew-symmetric bilinear form b is called a *symplectic* vector space, and a maximal isotropic subspace W of V is called a *Lagrangian* subspace.)
[Hint: $W \subset V$ is isotropic iff $W \subset W^{\perp}$.]

(b) Suppose b is symmetric and $F = \mathbb{R}$. Show that the maximal dimension of an isotropic subspace is $\min(s,t)$ where (s,t) is the signature of b.
[Hint: If $U \subset V$ is a subspace such that $b|_U$ is positive definite (i.e. $b(x,x) > 0$ for all $0 \neq x \in U$) or negative definite (i.e. $b(x,x) < 0$ for all $0 \neq x \in U$) and W is isotropic then $U \cap W = \{0\}$.]

(9) Let A be a square matrix with entries in $\mathbb{Q}$ such that A is skew-symmetric, i.e., $A^T = -A$. Prove that the determinant of A is a square in $\mathbb{Q}$.

[Hint: Use the structure theorem for skew-symmetric bilinear forms.]

(10) Let $b \colon V \times V \to F$ be a non-degenerate skew-symmetric bilinear form. Let $W \subset V$ be a subspace of codimension 1 (i.e. $\dim W = \dim V - 1$). Show that the restriction $b|_W$ of b to W is degenerate, and its null space has dimension 1.

(11) Let V be a complex vector space and $h \colon V \times V \to \mathbb{C}$ a positive definite hermitian form. We may regard V as a real vector space (using the inclusion $\mathbb{R} \subset \mathbb{C}$). Let $g \colon V \times V \to \mathbb{R}$ and $\omega \colon V \times V \to \mathbb{R}$ be the real and imaginary parts of h.

(a) Show that g is a positive definite symmetric bilinear form and ω is a skew-symmetric bilinear form on the real vector space V.

(b) Let $J \colon V \to V$ be the linear transformation of the real vector space V corresponding to scalar multiplication by $i = \sqrt{-1}$ on the complex vector space V, i.e., $J(v) = iv$. (So $J^2 = -\operatorname{id}_V$.) Find a formula relating g, ω and J.

(c) Show that ω is non-degenerate.