Math 300.2 Homework 6

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Reading: Sundstrom, Sections 4.2, 4.3, and 8.1.

Justify your answers carefully.

(1) Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively by $a_1 = 5$, $a_2 = 13$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geqslant 3$. Prove using strong induction that $a_n = 2^n + 3^n$ for all $n \in \mathbb{N}$.

(2) Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively by $a_1 = 1$ and $a_n = 1 + a_1 + a_2 + \ldots + a_{n-1}$ for $n \geqslant 2$. Guess a formula for a_n and prove your formula is correct using strong induction.

Hint: For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $x \neq 1$,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

(3) Let $f_1, f_2, f_3, \ldots$ be the Fibonacci numbers defined recursively by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ be the two solutions of the quadratic equation $x^2 = x + 1$. Prove by strong induction that $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$.

(4) Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geqslant 1$.

(a) Let $f_1, f_2, \ldots$ be the Fibonacci numbers. Prove by induction that $a_n = \frac{f_{n+1}}{f_n}$ for all $n \in \mathbb{N}$.

(b) What is the limit of a_n as $n \to \infty$?
[Hint: Use the result of Q3 and the fact that $|\frac{\beta}{\alpha}| < 1$.]

(5) Let $f_1, f_2, \ldots$ be the Fibonacci numbers. Prove the following statement by induction: $f_{n+1}^2 - f_n f_{n+2} = (-1)^n$ for all $n \in \mathbb{N}$.

(6) Find the greatest common divisor of each of the following pairs of integers. Use the Euclidean algorithm and show your work.

   (a) 126, 91.
   (b) 253,143.
   (c) 113, 51.

(7) For each of the following pairs of integers a and b, find integers x and y such that $ax + by = \gcd(a, b)$. Use the Euclidean algorithm and show your work.

   (a) 100, 31.
   (b) 169, 65.

(8) Prove the following statement: For all $a, b, c \in \mathbb{Z}$ such that $a, b \neq 0$, if $a \mid c$ and $b \mid c$ and $\gcd(a, b) = 1$ then $ab \mid c$.

   [Hint: Recall the following result (proved in class): For all $l, m, n \in \mathbb{Z}$ such that $l \neq 0$, if $l \mid mn$ and $\gcd(l, m) = 1$, then $l \mid n$. Now write down the definition of $a \mid c$ and use this result to construct a direct proof of the statement.]

(9) Prove the following statement: For all $n \in \mathbb{N}$, $\gcd(7n + 17, 2n + 5) = 1$.

   [Hint: Use the Euclidean algorithm.]

(10) Let $f_1, f_2, \ldots$ be the Fibonacci numbers. Prove by induction that $\gcd(f_{n+1}, f_n) = 1$ for all $n \in \mathbb{N}$.