Math 412 Homework 3

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Reading: Saracino, Chapter 18.
Show your work and justify your answers carefully.

(1) Let R be a commutative ring with 1. Let a, b be elements of R and assume that a is not a zero divisor. Show that $(ba) = (a)$ iff b is a unit.

(2) Let R be the ring from HW1Q1(c):

$$R = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, \quad b \text{ is not divisible by } 3 \right\} \subset \mathbb{Q},$$

a subring of $\mathbb{Q}$.

   (a) Find all the units in R.

   (b) Find all the ideals I in R, identify the prime ideals, and describe the quotient rings R/I.

(3) Let R be the ring from HW1Q2(d):

$$R = \{a + bx \mid a, b \in \mathbb{R}\}$$

with addition

$$(a + bx) + (c + dx) = (a + c) + (b + d)x$$

and multiplication

$$(a + bx)(c + dx) = ac + (ad + bc)x.$$

(a) Find all the units in R.

(b) Find all the ideals I in R, identify the prime ideals, and describe the quotient rings R/I.

[Remark: Actually the ring R is isomorphic to the quotient ring $\mathbb{R}[x]/(x^2)$ (why?), but it is possible to do this question without using this.]

(4) Let $R = \mathbb{R}[x]/(x^2 - 3x + 2)$. Find all the ideals of R.

[Hint: Use the bijective correspondence between ideals K of a quotient ring S/I and ideals J of S containing I given by $J \mapsto K = J/I$.]

(5) Which of the following maps are ring homomorphisms?

(a) $\alpha\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$, $\alpha(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(b) $\beta\colon \mathbb{C} \to \mathbb{R}$, $\beta(a + bi) = a$.

(c) $\gamma\colon \mathbb{C} \to \mathbb{H}$, $\gamma(a + bi) = a + bj$.

(d) $\delta\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$, $\delta(A) = \det(A)$. [Here $\det A$ denotes the determinant of the 2×2 matrix A, given by $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.]

(e) $\epsilon\colon \mathbb{Z}/2\mathbb{Z}[x] \to \mathbb{Z}/2\mathbb{Z}[x]$, $\epsilon(f) = f^2$.

(6) Recall from class that every ideal I in the ring $R = \mathbb{Z}$ of integers is principal, equal to $(n) = n\mathbb{Z}$ for some $n \in \mathbb{Z}$, $n \geq 0$. Now suppose $a, b \in \mathbb{N}$, and consider the ideal

$$I = (a, b) = \{xa + yb \mid x, y \in \mathbb{Z}\} \subset \mathbb{Z}.$$

We know $I = (c)$ for some $c \in \mathbb{N}$. Identify (with proof) the number c.

(7) Let R be a commutative ring with 1. Recall that we say an ideal $I \subset R$ is *maximal* if $I \neq R$ and there does not exist an ideal $J \subset R$ such that $I \subsetneq J \subsetneq R$. [Then I is maximal iff R/I is a field.] Suppose $I \subset R$ is an ideal such that every element of $R \setminus I$ is a unit. Show that I is maximal, $U(R) = R \setminus I$, and I is the only maximal ideal of R. [In this case we say R is a *local ring*.]

(8) Let R_1 and R_2 be commutative rings with 1. Show that an ideal I of the direct sum $R = R_1 \oplus R_2$ is given by

$$I = I_1 \oplus I_2 = \{(i_1, i_2) \mid i_1 \in I_1 \text{ and } i_2 \in I_2\} \subset R_1 \oplus R_2$$

for some ideals $I_1 \subset R_1$ and $I_2 \subset R_2$. Which ideals I are prime?

[Hint: What is the quotient R/I?]

(9) (Optional) In this question, we will see that it is in general difficult to find ideals I of a noncommutative ring R other than $\{0\}$ and R.

(a) Let $R = \mathbb{R}^{2\times 2}$ be the ring of 2×2 real matrices and $I \subset R$ an ideal. Show that $I = \{0\}$ or $I = R$.

(b) Let R be the *Weyl algebra* generated by variables x and y over the real numbers such that x and y do not commute but satisfy the equation

$$yx = xy + 1. \quad (*)$$

Explicitly,

$$R = \left\{ f \;\middle|\; f = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{R} \right\}$$

with the obvious addition, and multiplication given by using the relation (*) repeatedly. Let $I \subset R$ be an ideal. Show that $I = \{0\}$ or $I = R$.

[Hint: Prove by induction that

$$yx^n - x^n y = nx^{n-1}$$

and

$$xy^n - y^n x = -ny^{n-1}$$

for all $n \in \mathbb{N}$. Deduce that

$$yf - fy = \frac{\partial f}{\partial x}$$

and

$$xf - fx = -\frac{\partial f}{\partial y}$$

for all $f \in R$ (regarding f as a function of x, y). Now use these identities to show that if $I \neq \{0\}$ then $I = R$.]