Math 612 Final, Friday 4/25/14 – Thursday 5/8/14.

Instructions: This is a take-home exam. Solutions are due by 5PM on Thursday 5/8/14. Show all your work and justify your answers carefully.

(1) Let $\alpha = \sqrt{2+\sqrt{2}}$.

   (a) Show that the field extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is Galois and determine its Galois group.

   (b) Using part (a) or otherwise, prove that there do *not* exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{Q}$, and $c_1, \ldots, c_n \in \mathbb{Q}$ such that
$$\alpha = c_1\sqrt{a_1} + c_2\sqrt{a_2} + \cdots + c_n\sqrt{a_n}.$$

(2) For $p \in \mathbb{N}$ a prime, let $K_p \subset \mathbb{C}$ be the splitting field of $x^3 - p$ over $\mathbb{Q}$.

   (a) Compute the Galois group $G = \mathrm{Gal}(K_p/\mathbb{Q})$. For each element $\varphi \in G$ compute $\varphi(\sqrt[3]{p})$ and $\varphi(\omega)$, where $\omega = e^{2\pi i/3}$.

   (b) Using part (a) or otherwise, show that if $p \neq q$ then $K_p \cap K_q = \mathbb{Q}(\omega)$.

(3) (a) What does it mean to say that a ring A is Noetherian?

   (b) Let A be a Noetherian ring and $\varphi\colon A \to A$ a ring homomorphism. Show that if φ is surjective then φ is an isomorphism.

(4) (a) Let B be an integral domain and G a finite group of automorphisms of B. Let $A = B^G$ be the fixed ring, that is,
$$A := \{b \in B \mid \varphi(b) = b \text{ for all } \varphi \in G\}.$$
   Recall that we say an integral domain R is *integrally closed* if it is integrally closed in its field of fractions. Show that if B is integrally closed then A is integrally closed.

   (b) Let $A = \mathbb{C}[u, v, w]/(uw - v^2)$.

      i. Show that A is isomorphic to the subring $\mathbb{C}[x^2, xy, y^2]$ of $\mathbb{C}[x, y]$ generated by x^2, xy, y^2 over $\mathbb{C}$.

      ii. Using part (a) or otherwise, prove that A is integrally closed.

(5) Let G be the group with presentation

$$G = \langle a, b \mid a^3 = b^4 = e, \quad bab^{-1} = a^2 \rangle.$$

(a) Compute the abelianization $G_{\mathrm{ab}} = G/[G,G]$ of G. (Here $[G,G]$ denotes the normal subgroup of G generated by all commutators $[g,h] = ghg^{-1}h^{-1}$.)

(b) Using your answer to part (a) or otherwise, describe all one-dimensional representations $\rho\colon G \to \mathbb{C}^\times$ of G explicitly.

(c) Determine the number of isomorphism types of irreducible representations of G and their dimensions.

(6) Let G be the group of rotational symmetries of the cube. Recall that $G \simeq S_4$.

(a) Determine the character table of G.

(b) Consider the action of G on the set S of faces of the cube. Let $\rho\colon G \to \mathrm{GL}(V)$ be the associated permutation representation, where V is the vector space with basis labelled by the elements of S.

i. Compute the character χ of ρ.

ii. Using part (a) or otherwise, express ρ as a direct sum of irreducible representations.