1. The index of H in G, $[G:H]$, is the number of (left) cosets of H in G.

(Note: the number of left cosets equals the number of right cosets, because $gH \mapsto (gH)^{-1} = H^{-1}g^{-1} = Hg^{-1}$ defines a bijection from the set of left cosets to the set of right cosets. Of course if $|G|$ is finite then $[G:H] = |G|/|H|$.)

In general, the left cosets define a partition of G, the right cosets define another partition of G, and H is normal iff these two partitions coincide.

If $[G:H]=2$, each partition consists of 2 sets,
$G = H \amalg G \setminus H$
disjoint union
(Note: $H = e\cdot H = H \cdot e$ is always both a left & right coset)
Here $G\setminus H := \{g \in G \mid g \notin H\}$ (set theory notation)

So the two partitions are equal, & H is normal.

If $[G:H]=3$ H need not be normal.
For example take $G = S_3$, $H = \langle (12) \rangle = \{e, (12)\}$,

then $gHg^{-1} = \langle (g(1)\,g(2)) \rangle \neq H$ for $g \notin H$.

So H is not normal.

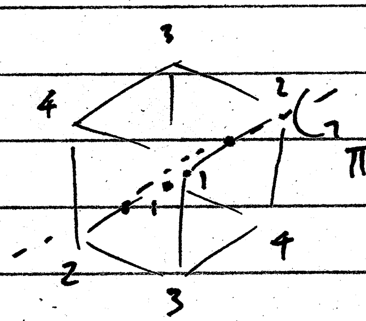
2. a) The conjugacy classes in S_n are the subsets of S_n consisting of all elements of a given cycle type. (Here the cycle type of an element $\sigma \in S_n$ is the (unordered) list of lengths of the cycles in the expression of σ as a product of disjoint cycles.)

So in S_4 there are the following conjugacy classes C:

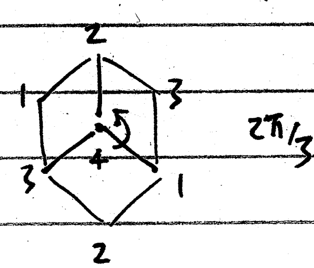
Representative element	$\lvert C \rvert$
e	1
(12)	$\binom{4}{2} = 6$
(123)	$4\cdot 3\cdot 2/3 = 8$
(1234)	$4\cdot 3\cdot 2\cdot 1/4 = 6$
$(12)(34)$	$\binom{4}{2}/2 = 3$

b) e = identity

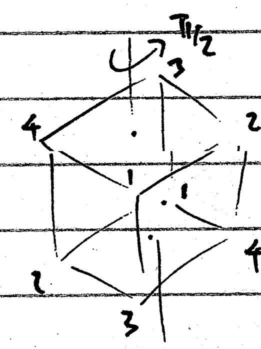
(12) = rotation by π about axis thru midpoint of edge 12.



(123) = rotation by $\pm 2\pi/3$ about axis thru vertex 4



(1234) = rotation by $\pm \pi/2$ about axis thru center of face 1234.



$(12)(34)$ = rotation by π about axis thru center of face 1324.

3. a) Conjugacy classes in A_4:

C	$\lvert C \rvert$
e	1
$(12)(34)$, $(13)(24)$, $(14)(23)$	3
(123), (142), (134), (243)	4
(132), (124), (143), (234)	4

b) $H \leq A_4$, $|H| = 6 \Rightarrow$ index 2 $\overset{Q1}{\Rightarrow}$ normal

A normal subgroup is a disjoint union of conjugacy classes, including $\{e\}$'s. So 6 is a sum of some of the terms $|C|$ listed above, including 1. This is a contradiction.

4. a) $a = (123) \in G = S_5$.

$$Z(a) = \{ g \in G \mid gag^{-1} = a \}$$
$$= \{ g \in S_5 \mid (g(1)\, g(2)\, g(3)) = (123) \}$$
$$= \langle (123) \rangle \times \langle (45) \rangle$$
$$|Z(a)| = 3 \cdot 2 = 6 \Rightarrow |C(a)| = |G| / |Z(a)| = 5!/6 = 20.$$

b) $a = (123)(456) \in G = S_7$.

$$Z(a) = \langle (123), (456), (14)(25)(36) \rangle$$
$$|Z(a)| = 2 \cdot 3^2 = 18. \Rightarrow |C(a)| = 7!/18 = 7 \cdot 5 \cdot 4 \cdot 2 = 280.$$

c) $a = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \in GL_2(\mathbb{Z}/5\mathbb{Z})$

$Z(a) = ?$ $\quad \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}$

(Note: $gag^{-1} = a$
$\Leftrightarrow$ $ga = ag$)

$\Leftrightarrow \begin{pmatrix} 2x & x+2y \\ 2z & z+2t \end{pmatrix} = \begin{pmatrix} 2x+z & 2y+t \\ 2z & 2t \end{pmatrix}$

$\Leftrightarrow \quad z = 0, \; x = t$

$\therefore \quad Z(a) = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \;\middle|\; 0 \neq x \in \mathbb{Z}/5\mathbb{Z}, \; y \in \mathbb{Z}/5\mathbb{Z} \right\}$

invertible!

$|Z(a)| = (5-1)\cdot 5 = 20.$

$|G| = |GL_2(\mathbb{Z}/5\mathbb{Z})| = (5^2-1)(5^2-5) = 24\cdot 20 = 480$

$\Rightarrow \quad |C(a)| = |G|/|Z(a)| = 24.$

d) $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}/3\mathbb{Z})$

$Z(a)$? $\quad \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} y & -x \\ t & -z \end{pmatrix} = \begin{pmatrix} -z & -t \\ x & y \end{pmatrix}$

$\Leftrightarrow \quad x = t, \; y = -z.$

$Z(a) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \;\middle|\; \det = x^2 + y^2 = 1 \; (x, y \in \mathbb{Z}/3\mathbb{Z}) \right\}$

Possible values (x,y): $\pm(1,0), \pm(0,1)$

$\therefore \quad |Z(a)| = 4, \quad |G| = |SL_2(\mathbb{Z}/3\mathbb{Z})| = \dfrac{(3^2-1)(3^2-3)}{(3-1)} = \dfrac{8\cdot 6}{2} = 24,$

$|C(a)| = |G|/|Z(a)| = 6.$

5. $|G| = 21 \qquad |C(x)| = 3.$

$\Rightarrow |Z(x)| = |G|/|C(x)| = 7$

$x \in Z(x) \Rightarrow \text{ord}(x) = 1$ or $7.$

But $x \neq e$ $(|C(x)| \neq 1)$ so $\text{ord}(x) = 7.$

6. Since G is generated by a and b, to understand conjugacy in G it suffices to consider conjugation by a & b

$$a \cdot a^i b^j \cdot a^{-1} = a^{i+1-(-1)^j} \cdot b^j = \begin{cases} a^i b^j & j \text{ even} \\ a^{i+2} b^j & j \text{ odd} \end{cases}$$

$ba = a^{-1}b, \Rightarrow ba^{-1} = ab$

$$b \cdot a^i b^j \cdot b^{-1} = a^{-i} \cdot b \cdot b^j \cdot b^{-1} = a^{-i} b^j$$

$\Rightarrow$ conjugacy classes

$\{e\}$

$\{a, a^2\}$

$\{b, ab, a^2b\}$

$\{b^2\}$

$\{ab^2, a^2b^2\}$

$\{b^3, ab^3, a^2b^3\}$

7. a) $PGL_2(\mathbb{Z}/3\mathbb{Z})$ acts faithfully on $\mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}} = \mathbb{Z}/3\mathbb{Z} \cup \{\infty\}$

[For instance, one can check that any Mobius transformation $z \overset{f}{\longmapsto} \dfrac{az+b}{cz+d}$ fixing $0, 1$ & ∞ must be the identity:

$f(\infty) = \infty \Rightarrow c = 0, \quad f(z) = \dfrac{az+b}{d} = a'z + b'$

$f(0) = 0 \Rightarrow b' = 0, \quad f(z) = a'z$

$f(1) = 1 \Rightarrow a' = 1, \quad f(z) = z.$

OR one can argue more generally that $PGL_{n+1}(F)$ acts faithfully on projective space $\mathbb{P}^n_F = \left(F^{n+1} \setminus \{0\}\right)/\sim$:-

where $\sim$ is the equivalence relation given by $\underline{x} \sim \lambda \cdot \underline{x}$ for $0 \neq \lambda \in F$.

If a matrix $A \in GL_{n+1}(F)$ acts trivially on $\mathbb{P}^n_F$, then $A \cdot \underline{x} = \lambda \cdot \underline{x}$ † for every $\underline{x} \in F^{n+1}$, for some $0 \neq \lambda \in F$ (depending on $\underline{x}$ a priori).

Equivalently, every $0 \neq \underline{x} \in F^{n+1}$ is an eigenvector of A.

In particular, A is diagonal, $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$.

Now applying A to $\underline{x} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, † $\Rightarrow \lambda_1 = \dots = \lambda_n =: \lambda$

So $A = \lambda \cdot I$ & $[A] = e \in PGL_{n+1}(F) := GL_{n+1}(F)/Z$,

$Z = \{\lambda I \mid 0 \neq \lambda \in F\}$]

Now $PGL_2(\mathbb{Z}/3\mathbb{Z}) \circlearrowright \{0, 1, 2, \infty\} = \mathbb{Z}/3\mathbb{Z} \cup \{\infty\}$

(faithful action)

$\rightsquigarrow \varphi : PGL_2(\mathbb{Z}/3\mathbb{Z}) \longrightarrow S_4$ injective homomorphism

$$|PGL_2(\mathbb{Z}/3\mathbb{Z})| = \frac{(3^2-1)(3^2-3)}{(3-1)} = \frac{8\cdot 6}{2} = 24 = 4! = |S_4|$$

$\Rightarrow$ φ isomorphism.

b) $PSL_2(\mathbb{Z}/3\mathbb{Z}) \triangleleft PGL_2(\mathbb{Z}/3\mathbb{Z})$

$$PSL_2(\mathbb{Z}/3\mathbb{Z}) = SL_2(\mathbb{Z}/3\mathbb{Z}) \Big/ Z \cap SL_2(\mathbb{Z}/3\mathbb{Z})$$

$Z = \{\lambda \cdot I \mid 0 \neq \lambda \in \mathbb{Z}/3\mathbb{Z}\}$

$Z \cap SL_2(\mathbb{Z}/3\mathbb{Z}) = \{\lambda \cdot I \mid 0 \neq \lambda \in \mathbb{Z}/3\mathbb{Z},\ \lambda^2 = 1\}$

$= Z$, $(1^2 = 1,\ 2^2 = 1 \mod 3 \ \checkmark)$

So $|Z \cap SL_2(\mathbb{Z}/3\mathbb{Z})| = 2$, $|PSL_2(\mathbb{Z}/3\mathbb{Z})| = \dfrac{|SL_2(\mathbb{Z}/3\mathbb{Z})|}{|Z \cap SL_2(\mathbb{Z}/3\mathbb{Z})|} = \dfrac{24}{2}$

$= 12.$

So $\varphi(PSL_2(\mathbb{Z}/3\mathbb{Z})) < S_4$ is a subgroup of index 2.

$$PSL_2(\mathbb{Z}/3\mathbb{Z}) \xrightarrow[\cong]{\varphi}$$

In particular it is a normal subgroup, so a disjoint union of conjugacy classes (including $\{e\}$)

The class equation of S_4 is $24 = 1 + 6 + 8 + 6 + 3$ (see p. 2)

$= 1 + 3 + 6 + 6 + 8$

There's only one way to write 12 as a sum of terms from RHS (including 1).

So there's only one subgroup of S_4 of index 2, namely A_4,

and $PSL_2(\mathbb{Z}/3\mathbb{Z}) \xrightarrow[\sim]{\varphi} A_4$.

8. $\{e\} < G$ is a conjugacy class

So if G has 1 conjugacy class, then G is the trivial group $\{e\}$.

Suppose G has 2 conjugacy classes, and consider the class equation

$|G| = 1 + a$

$a \mid |G| \Rightarrow a \mid 1 \Rightarrow a = 1 \Rightarrow |G| = 2 \Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$.

Suppose G has 3 conjugacy classes. Class equation:-

$|G| = 1 + a + b$, $1 \leq a \leq b$.

$b \mid |G| \Rightarrow b \mid 1+a \Rightarrow b \leq 1+a$, $b = a$ OR $a+1$

If $b = a$, $b \mid 1+a \Rightarrow b = 1 \Rightarrow a = b = 1 \Rightarrow |G| = 3$
$\Rightarrow G \cong \mathbb{Z}/3\mathbb{Z}$.

If $b = a+1$, $|G| = 2(1+a)$.
$a \mid |G| \Rightarrow a \mid 2 \Rightarrow a = 1$ OR 2
$\Rightarrow |G| = 4$ OR 6, w/ class equation
$1+1+2$ OR $1+2+3$.

But $|G| = 4 \overset{\dagger}{\Rightarrow} G$ abelian $\Rightarrow$ class eq. $4 = 1+1+1+1$ #.

So $|G| = 6$, class eq $6 = 1+2+3$ $\Rightarrow$ G non-abelian $\overset{\dagger\dagger}{\Rightarrow}$ $G \cong S_3$.

$\dagger$: Suppose $|G| = 4$, $x, y \in G$, $xy \neq yx$.
Then e, x, y, xy & yx are distinct elements of G #.

$\dagger\dagger$: $|G| = 6$, class eq $G = 1+2+3$.
Let $a \in G$, $|C(a)| = 2$, $b \in G$, $|C(b)| = 3$.
$\Rightarrow |Z(a)| = 3$, $|Z(b)| = 2 \Rightarrow$ order$(a) = 3$, order$(b) = 2$.

Now $\langle a \rangle \leq G$ has index 2 $\Rightarrow$ normal

$\Rightarrow$ $bab^{-1} = a$ OR a^2

$bab^{-1} = a \Rightarrow G$ abelian #. So $bab^{-1} = a^2$, i.e. $ba = a^{-1}b$.

Now see $G \simeq D_3 \simeq S_3$.

9. a. This is false.

e.g. $G = S_4$, $H = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq G$.

(H is a subgroup by direct calculation: $(12)(34)\cdot(13)(24) = (14)(23) = (13)(24)\cdot(12)(34)$)

(H is normal because it's a union of conjugacy classes)

$|H| = 4 \Rightarrow H$ abelian.

So any subgroup K is normal, e.g. $K = \langle (12)(34) \rangle = \{e, (12)(34)\}$

But K is not normal in G (it's not a union of conj. classes).

b. It's enough to show $g(H \cap K)g^{-1} \subset H \cap K \quad \forall g \in K$

(see the Hints*)

Equivalently $^{1/}\ gHg^{-1} \subset H$ & $^{2/}\ gKg^{-1} \subset K$

$^{1/}$ holds because $H \trianglelefteq G$. $^{2/}$ holds because $g \in K$. □

*$H \trianglelefteq G \iff gHg^{-1} \subset H \ \forall g \in G$. Because $g^{-1}Hg \subset H \Rightarrow H \subset gHg^{-1}$, so $H = gHg^{-1}$.

[Another proof: $q: G \to G/H$ has kernel H. So the restriction $K \xrightarrow{q} G/H$ has kernel $H \cap K$. Thus $H \cap K \trianglelefteq K$, and $K/H\cap K \leq G/H$.]

10. a
$$
\begin{aligned}
gC_H(a)g^{-1} &= g\{hah^{-1} \mid h \in H\}g^{-1} \\
&= \{(gh)a(gh)^{-1} \mid h \in H\} \\
&\overset{\dagger}{=} \{(hg)a(hg)^{-1} \mid h \in H\} \\
&= \{h(gag^{-1})h^{-1} \mid h \in H\}. \\
&= C_H(gag^{-1})
\end{aligned}
$$

† $H \trianglelefteq G \Rightarrow gH = Hg$

Note: $gag^{-1} \in H$ because $H \trianglelefteq G$.

b. Consider the action of G on conjugacy classes of H by conjugation.

Clearly $C_G(a)$ is the union of the classes $g\,C_H(a)\,g^{-1} \overset{(a)}{=} C_H(gag^{-1})$ in the orbit of $C_H(a)$.

Now compute: $|C_G(a)| = |G| / |Z_G(a)|$

$$|C_H(a)| = |H| / |Z_H(a)|$$

$$\Rightarrow \#\text{ classes} = |C_G(a)| / |C_H(a)| = \frac{|G|}{|H|} \Big/ \frac{|Z_G(a)|}{|Z_H(a)|}$$

$$= |G/H| \,/\, |q(Z_G(a))|$$

$$Z_G(a)/Z_H(a) \overset{\bar{q}}{\xrightarrow{\sim}} q(Z_G(a)) \quad \text{F.I.T.}$$

c. $G = S_n$, $H = A_n$.

$$q: S_n \longrightarrow S_n/A_n \simeq \mathbb{Z}/2\mathbb{Z}$$

$$\therefore \#\text{ classes} \overset{(b)}{=} \begin{cases} 1 & \text{if } q(Z_G(a)) \neq \{e\}, \text{ i.e. } Z_G(a) \not\subset A_n \\ 2 & \phantom{\text{if }} q(Z_G(a)) = \{e\}, \text{ i.e. } Z_G(a) \subset A_n. \end{cases}$$

<u>Examples</u>:
- $(123) \in A_4$. $Z_{S_4}((123)) = \langle (123) \rangle \subset A_4$.
  $C_{S_4}((123)) = C_{A_4}((123)) \cup C_{A_4}((213))$ (see Q2 HW3)
- $(12)(34) \in A_4$. $Z_{S_4}((12)(34)) = \langle (12), (34), (13)(24) \rangle \not\subset A_4$.
  $C_{S_4}((12)(34)) = C_{A_4}((12)(34))$