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Rational surfaces with an anti-canonical cycle

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Introduction

A curve D on a compact analytic surface Y is called an *anti-canonical cycle* if Y is smooth near D, $-D$ is a canonical divisor, and if either D is an irreducible rational curve with an ordinary node or D is reducible and the distinct irreducible components $D_0, \ldots, D_{s-1}$ of D are smooth rational curves forming a polygon.

This paper deals with rational surfaces Y (with at worst rational double points) endowed with an anti-canonical cycle D such that D contains no exceptional curve and the intersection matrix $(D_i . D_j)$ is negative. This amounts to $D.D \leq 0$ if D is irreducible and $D_i . D_i \leq -2$ for all i if D is reducible.

The Picard group of these surfaces is studied in Chapter I. One of our main findings is that for $s \leq 5$ the Picard group of such a surface contains a naturally defined infinite root system which enables us to give a precise description of the classes which represent exceptional curves. The restriction to $s \leq 5$ is there for technical reasons; it is not a very natural one.

The next chapter is devoted to the construction of a fine moduli space p_d: $(\mathcal{Y}_d, \mathcal{D}_d) \to M_d$ for rational surfaces Y endowed with an anti-canonical divisor $D = D_0 + \cdots + D_{s-1}$ with a given cycle of self-intersection numbers $d = (-D_0 . D_0, \ldots, -D_{s-1} . D_{s-1})$ of length $s \leq 5$. In order to get a reasonable space we must discard some pairs (Y, D), however. The base M_d is entirely given in terms of the (abstract) infinite root system defined in Chapter I. In fact, it is nothing but the orbit space M introduced and investigated in [19]. In that paper we proved that the global holomorphic functions separate the points of M_d and we gave a precise description of the holomorphic hull $\hat{M}_d$ of M_d. This holomorphic hull is a Stein manifold and the added material $\hat{M}_d - M_d$ is naturally stratified.

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In the last chapter we extend p_d to a proper flat family of surfaces with an anti-canonical cycle $\hat{p}_d$: $(\hat{\mathcal{Y}}_d, \hat{\mathcal{D}}_d) \to \hat{M}_d$. It appears that the fibres over $\hat{M}_d - M_d$ are surfaces without meromorphic functions which are birationally equivalent to surfaces constructed by M. Inoue. The surfaces over the unique stratum of lowest dimension (which is a point or isomorphic to Poincaré's upper half plane) have a unique singular point which is either a cusp or a simply-elliptic singularity. The map $\hat{p}_d$ exhibits there a semi-universal deformation of that singular point. Thus we obtain valuable information concerning these singularities (for instance on their monodromy group and on the topology of the complement of their discriminant). In fact, familiarity with some of these singularities led us to conjecture the results of Chapter I.

Perhaps a better overall picture of the contents is obtained by reading the introductions to the separate chapters.

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I. A ROOT SYSTEM IN THE PICARD GROUP

In order to put the results of this chapter into perspective we first review some properties (which we will not need however) of the rational surfaces which are probably best understood, namely the Del Pezzo surfaces. Recall that a Del Pezzo surface is by definition a rational surface (smooth, compact and connected) whose anti-canonical system is ample (e.g. a smooth cubic surface in $\mathbf{P}^3$). If E is an exceptional curve of the first kind on a surface Y, then, as is well-known, $E.E = -1$ and $E.K = -1$ (K denoting a canonical divisor). If Y is a Del Pezzo surface then we have a converse: any divisor E on Y satisfying these two equalities is linearly equivalent to a (unique) exceptional curve of the first kind. Moreover, the vectors of norm -2 in $\mathrm{Pic}(Y)$ orthogonal to $[K]$ form a (finite) root system R whose Weyl group $W(R)$ acts transitively on the classes of exceptional curves of the first kind. If $Y \not\cong \mathbf{P} \times \mathbf{P}$, then we can find a set $\mathcal{E}$ of disjoint exceptional curves of the first kind on Y such that separate contraction of each of them gives a surface isomorphic to the projective plane. The Weyl group $W(R)$ acts transitively on the sets $\mathcal{E}$ (or rather their images in $\mathrm{Pic}(Y)$) thus obtained. In other words, $W(R)$ permutes transitively the realizations of Y as a blown-up projective plane.

In this chapter we investigate a class of rational surfaces for which analogous properties hold. A typical subclass consists of the rational surfaces Y endowed with a negative canonical divisor K such that $-K$ is a rational curve with a node. Such a pair (Y, K) is obtained by blowing up points on the regular part of a cubic with a node in $\mathbf{P}^2$ ($-K$ being the strict transform of the cubic). If $K.K > 0$ we get in general a Del Pezzo surface; we therefore assume $K.K \leq 0$. Then we show that there is an infinite root system R *naturally* defined in the orthogonal complement of $[K]$ in $\mathrm{Pic}(Y)$. (In general, R doesn't contain all vectors of norm -2 in this complement.) We prove that if no root is the class of a nodal curve, then the Weyl group $W(R)$ acts transitively on the set of classes of exceptional curves of the first kind. We also show that then $W(R)$ permutes transitively the realizations of $(Y, -K)$ as a blown up projective plane with cubic. If some roots happen to represent nodal curves, similar, but more complicated, statements hold. Finally we prove a Torelli theorem for the pairs $(Y, -K)$, essentially in terms of the mixed Hodge structure on $H^2(Y - (-K))$. The formulation of this theorem is very similar to the corresponding theorem for kählerian $K3$ surfaces, although much easier to prove.

0. Rational surfaces

In this section, we review some well-known properties of rational surfaces.

(0.1) If Y is a smooth surface, then a curve C on Y is called *exceptional* if there exists a smooth surface Y' and a proper morphism $\pi\colon Y \to Y'$, which maps

C to a point c of Y' and maps $Y - C$ isomorphically onto $Y - \{c\}$. Briefly, C can be analytically contracted to a point such that the result is still smooth. Following Castelnuovo, an irreducible curve C on Y is exceptional if and only if C is a smooth rational curve with self-intersection -1. In that case, C is said to be *exceptional of the first kind*. Contraction of an exceptional curve of the first kind is precisely the reverse procedure of blowing up a point. Any exceptional curve has an exceptional curve of the first kind as one of its irreducible components.

A smooth compact connected surface free of exceptional curves is said to be *relatively minimal*. The rational relatively minimal surfaces are up to isomorphism: the projective plane $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ and the Hirzebruch surfaces Σ_i (a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ having a section C with $C.C = -i$), $i = 2,3,4,\ldots$. The anti-canonical system of $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ is without fixed components, but that of $\Sigma_i (i \geq 3)$, has the section C as a fixed component. The Chern number c_1^2 of the projective plane is 9, while for $\mathbf{P}^1 \times \mathbf{P}^1$ and the Hirzebruch surfaces it is 8.

(0.2) The Riemann-Roch theorem for a rational surface Y says that for any divisor C on Y,

$$h^0(C) - h^1(C) + h^2(C) = 1 + \tfrac{1}{2}C.(C - K)$$

where for short $h^i(C) = \dim H^i(\mathcal{O}(C))$ and K is a canonical divisor. Suppose K is a negative divisor and C is a positive divisor. By Serre duality, $h^2(C) = h^0(K - C) = 0$ and so

$$h^0(C) \geq 1 + \tfrac{1}{2}C.(C - K) \geq 1 + \tfrac{1}{2}C.C.$$

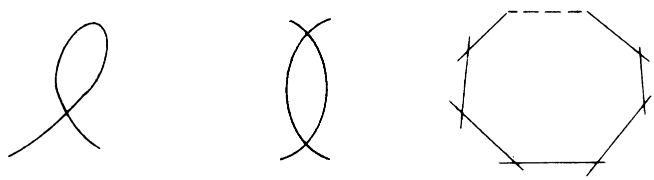
It follows, that if $C.C \geq 0$, the linear system $|C|$ has positive dimension. If C is irreducible, then its arithmetic genus $p_a(C)$ is given by

$$p_a(C) = 1 + \tfrac{1}{2}C.(C + K).$$

So if C is not a component of $-K$, then $C.C \geq -2$ and $C.C = -2$ implies $C.K = 0$, $p_a(C) = 0$. The last equality means that C is a smooth rational curve. Any smooth rational curve with self-intersection -2 is called a *nodal* curve. By what we just showed, a nodal curve is either an irreducible component of $-K$ or disjoint from K. If $C.C = -1$ and C is not a component of K, then the genus formula tells us that $C.K = -1$ and $p_a(C) = 0$. So C is then exceptional of the first kind and meets $-K$ simply in a unique irreducible component.

1. A partial classification

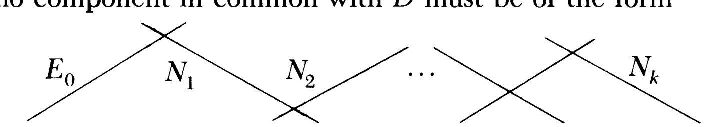
Let Y be a smooth rational surface. An *anti-canonical cycle* on Y is a reduced anti-canonical divisor D on Y such that either D is an irreducible rational curve with a node or D is reducible and the irreducible components of D are smooth rational curves intersecting each other transversally and the intersection diagram is a polygon. So we have the following picture:



one irreducible component | two irreducible components | $\geq$ three irreducible components

We say that D is *oriented* if we have chosen a generator of the infinite cyclic group $H^1(D,\mathbf{Z})$. If D has s irreducible components, we call s the *length* of D and then we index these components by $\mathbf{Z}/s$: $D_0,\dots,D_{s-1}$ such that for $s \geq 3$, $(D_0,\dots,D_{s-1})$ represents an (and in the oriented case, *the*) orientation.

If E is an exceptional curve on Y, then $E.D = 1$. So if E has no component in common with D, then E meets D in precisely one component D_i. So contraction of E gives a rational surface $\bar{Y}$ with an anti-canonical cycle $\bar{D} = \bar{D}_0 + \cdots + \bar{D}_{s-1}$, where $\bar{D}_i$ denotes the image of D_i. If E is irreducible, then an irreducible curve N on Y, not contained in D with $N.E \neq 0$, maps to an exceptional curve on $\bar{Y}$ if and only if N is a nodal curve with $N.E = 1$. It then follows with induction on the number of components that an exceptional curve E on Y having no component in common with D must be of the form



where E_0 is exceptional of the first kind and N_i is nodal ($i = 1,\dots,k$). Note that $E_0 + N_1 + \cdots + N_l$ is then exceptional for $l = 0,1,\dots,k$.

(1.1) THEOREM. *Let Y be a smooth rational surface endowed with an anti-canonical cycle $D = D_0 + \dots + D_{s-1}$ of length $s \leq 5$. Suppose that if $s \geq 2$, then $D_i.D_i \leq 4 - s$ for all $i \in \mathbf{Z}/s$. Then there exists a set of disjoint exceptional curves on Y, each having no component in common with D, such that contraction of these curves gives a smooth rational surface $\bar{Y}$ endowed with an anti-canonical cycle $\bar{D} = \bar{D}_0 + \dots + \bar{D}_{s-1}$ ($\bar{D}_i$ being the image of D_i) such that for*

$s = 1$, $\bar{Y} \cong \mathbf{P}^2$ and $\bar{D}$ is a cubic curve with a node,
$s = 2$, $\bar{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$ and $\bar{D}_0$ and $\bar{D}_1$ are of bidegree $(1,1)$,
$s = 3$, $\bar{Y} \cong \mathbf{P}^2$ and $\bar{D}$ is a triangle,
$s = 4$, $\bar{Y} \approx \mathbf{P}^1 \times \mathbf{P}^1$ and $\bar{D}$ is a square ($\cong \mathbf{P}^1 \times \{0,\infty\} \cup \{0,\infty\} \times \mathbf{P}^1$),
$s = 5$, $\bar{Y}$ is a Del Pezzo surface of degree five and each $\bar{D}_i$ is an exceptional curve of the first kind.

(Recall that a Del Pezzo surface of degree five is obtained by blowing up four points of a projective plane, no three of which are collinear. Such a surface contains exactly ten exceptional curves of the first kind: the inverse images of the four points and the strict transforms of the six lines passing through two of these points.)

Proof. Consider the collection of surfaces obtained from Y by contraction of disjoint exceptional curves having no component in common with D and having the property that the image of each component D_i of D has self-intersection $\leq 4 - s$ if $s \geq 2$. Among this collection we choose a surface $\bar{Y}$ (with anti-canonical cycle $\bar{D} = \bar{D}_0 + \ldots + \bar{D}_{s-1}$) of minimal Picard number. Then we claim that some component of $\bar{D}$, say $\bar{D}_0$, has self-intersection ≥ 8 if $s = 1$ and $= 4 - s$ if $s \geq 2$. Otherwise our minimality condition would imply that every exceptional curve of the first kind on $\bar{Y}$ is a component of D. Successive blowing down of these curves then gives a relatively minimal rational surface for which it is easily checked that the image of $\bar{D}$ is an anti-canonical divisor of self-intersection ≤ 7. But such a surface does not exist (0.1).

If $s = 1$, then $\bar{D}.\bar{D}$ is 8 or 9 and $\bar{Y}$ is relatively minimal. It cannot be a Hirzebruch surface of type Σ_i $(i \geq 3)$ for the anti-canonical system of such a surface has a fixed component of negative self-intersection. If $\bar{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$, then $\bar{D}.\bar{D} = -8$ and so the contraction mapping π: $Y \to \bar{Y}$ factors over the blow-up $Y' \to \bar{Y}$ of a point $p \in \bar{D}$. But Y' contains two *disjoint* exceptional curves (namely the strict transforms of the fibres through p of the two rulings of $\bar{Y}$) and contracting these yields a surface of lower Picard number. Hence $\bar{Y} \not\cong \mathbf{P}^1 \times \mathbf{P}^1$. Similarly, $\bar{Y} \not\cong \Sigma_2$. The remaining possibility is then $\bar{Y} \cong \mathbf{P}^2$. Clearly, $\bar{D}$ then has to be a cubic with a node.

If $s = 2$, let $\mathfrak{L}$ denote the linear system of curves on $\bar{Y}$ which are linearly equivalent to $\bar{D}_0$ and which pass through $\bar{D}_0 \cap \bar{D}_1$. The Riemann-Roch inequality (applied to the strict transform of $\bar{D}_0$ on the surface obtained from $\bar{Y}$ by blowing up the two points of $\bar{D}_0 \cap \bar{D}_1$) shows that $\dim \mathfrak{L} \geq 1$. So $\bar{D}_1$ is an irreducible component of some positive divisor $F \in \mathfrak{L}$. We claim that $F = \bar{D}_1$. If F were reducible, then it has an irreducible component C distinct from $\bar{D}_1$ which meets $\bar{D}_1$. Since $C.\bar{D}_0 = 0$, C is not movable and so $C.C < 0$ by (0.2). As C meets D, C has to be an exceptional curve by (0.2). Our minimality condition then forces $\bar{D}_1$ to have self-intersection 2. Contraction of C gives a rational surface Y' with anti-canonical cycle $D' = D_0' + D_1'$ such that $D_0'.D_0' = 2$ and $D_1'.D_1' = 3$. Hence $D'.D' = 9$, implying by (0.1) that $Y' \cong \mathbf{P}^2$. But this cannot be since $\mathbf{P}^2$ does not contain curves with self-intersection 2. Thus F is irreducible and so $F = m\bar{D}_1$. From $2 = \bar{D}_0.\bar{D}_0 = \bar{D}_0.F = 2m$ it then follows that $F = \bar{D}_1$.

Now the preceding argument also shows that $\bar{D}_0$ and $\bar{D}_1$ do not meet exceptional curves of the first kind. So $\bar{Y}$ is relatively minimal. For the same reasons as in the case $s = 1$, $\bar{Y}$ is not a Hirzebruch surface of type Σ_i ($i \geq 2$). As $\bar{D}.\bar{D} = -8$, we then must have $\bar{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$. Because the anti-canonical system of $\mathbf{P}^1 \times \mathbf{P}^1$ is of bidegree $(2,2)$, each $\bar{D}_i$ will be of bidegree $(1,1)$.

If $s = 3$, then $|\bar{D}_0|$ is a two-dimensional linear system which defines a birational regular map $\bar{Y} \to \mathbf{P}^2$. As it maps each component $\bar{D}_i$ onto a line, our minimality condition implies that this map is an isomorphism.

If $s = 4$, the Riemann-Roch inequality implies that $\dim|\bar{D}_0| \geq 1$. As $\bar{D}_0.\bar{D}_2 = 0$, $\bar{D}_2$ is contained in a fibre F of $|\bar{D}_0|$. If F were reducible, then it would contain an irreducible curve C distinct from $\bar{D}_2$ which meets $\bar{D}_2$. Since $C.\bar{D}_0 = 0$, C is not movable and so $C.C < 0$ by (0.2). Then C must be exceptional by (0.2) for $C.\bar{D} \neq 0$. Thus by successive contraction of exceptional curves of the first kind we can attain that the image of F becomes irreducible. Since $F.F = 0$ this would imply $\bar{D}_2.\bar{D}_2 < 0$ which obviously contradicts our minimality assumption. So $F = m\bar{D}_2$ for some $m \in \mathbf{N}$. In fact, $F = \bar{D}_2$, for $m = F.\bar{D}_1 = \bar{D}_0.\bar{D}_1 = 1$.

Now let E be an exceptional curve of the first kind on $\bar{Y}$. If E is not contained in $\bar{D}$, then E meets neither $\bar{D}_0$ nor $\bar{D}_2$ for otherwise $1 = E.\bar{D} \leq E.(\bar{D}_0 + \bar{D}_2) = 2E.\bar{D}_0 \geq 2$. If E meets $\bar{D}_1$, then our minimality hypothesis implies that $\bar{D}_1.\bar{D}_1 = 0$ and the argument used for $\bar{D}_2$ then shows that $\bar{D}_3 \sim \bar{D}_1$. But then we get a contradiction too: $1 = E.\bar{D} \geq E.(\bar{D}_1 + \bar{D}_3) \geq 2$. It follows that $E = \bar{D}_1$ or $E = \bar{D}_3$, say $E = \bar{D}_1$. In that case, $\bar{D}_3.\bar{D}_3 \neq 0$ (otherwise $\bar{D}_1 \sim \bar{D}_3$ by the preceding argument and so $-1 = \bar{D}_1.\bar{D}_1 = 0$). If $\bar{D}_3.\bar{D}_3 \leq -2$, then contraction of E gives a relatively minimal rational surface with $c_1^2 < 8$ and we know such a surface does not exist. So $\bar{D}_3.\bar{D}_3 = -1$; in other words, $\bar{D}_3$ is also exceptional. Contraction of $\bar{D}_1$ and $\bar{D}_3$ gives a surface Y' with anti-canonical cycle $D' = D_0' + D_2'$, $D_0'.D_0' = D_2'.D_2' = 2$. By the result for $s = 2$ treated earlier, $Y' \cong \mathbf{P}^1 \times \mathbf{P}^1$ and D_0', D_2' are of bidegree $(1,1)$. But then $\bar{Y}$, being obtained from Y' by blowing up the two points of $D_0' \cap D_2'$, contains exceptional curves of the first kind other than $\bar{D}_1$ and $\bar{D}_3$ (for instance the strict transform of a fibre through a point of $D_0' \cap D_2'$ of one of the rulings), which obviously contradicts our earlier findings. We therefore conclude that $\bar{Y}$ contains no exceptional curves of the first kind at all; in other words $\bar{Y}$ is relatively minimal. Then we must have that $\bar{D}.\bar{D} = 8 + \bar{D}_1.\bar{D}_1 + \bar{D}_3.\bar{D}_3$ is either 8 or 9. Since $D_i.D_i \leq 0$, it follows that $\bar{D}_i.\bar{D}_i = 0$ ($i = 1,3$). The map $|D_0| \times |D_1|$: $\bar{Y} \to \mathbf{P}^1 \times \mathbf{P}^1$ is then an isomorphism mapping $\bar{D}$ onto a square.

If $s = 5$, we contract $\bar{D}_0$: $\bar{Y} \to Y'$. Then $\bar{D}$ maps to an anti-canonical cycle $D' = D_1' + \cdots + D_4'$ on Y' with $D_i'.D_i' \leq 0$ for $i = 1,4$ and $D_i'.D_i' \leq -1$ for $i = 2,3$. By the case $s = 4$ treated above, we know that any D_i' with $D_i'.D_i' \leq -1$ meets an exceptional curve of the first kind not contained in D'. The minimality hypothesis then implies that $D_i'.D_i' = 0$ for $i = 1,4$, $D_i'.D_i' = -1$ for $i = 2,3$ and there exist disjoint exceptional curves of the first kind E_2, E_3, not contained in $\bar{D}$ which meet $\bar{D}$ in $\bar{D}_2$, resp. $\bar{D}_3$. Contraction of the disjoint exceptional curves $\bar{D}_1$, E_2, E_3 and $\bar{D}_4$ then gives a surface Y'' with anti-canonical cycle $D'' = D_0'' + D_2'' + D_3''$ and $D_i''.D_i'' = 1$ $(i = 0,2,3)$. We then know that Y'' is a projective plane and D'' a triangle on it. The image points of the four exceptional curves are the two vertices $D_0'' \cap D_2''$, $D_0'' \cap D_3''$ and a non-vertex point on the sides D_2'' and D_3''. In particular, no three of these points are collinear.

This finishes the proof of the theorem (1.2).

(1.2) It is perhaps worthwhile to note that the pairs $(\bar{Y}, \bar{D})$ occurring in Theorem (1.1) are all rigid (i.e. have no moduli), except when $s = 2$: the two rulings on $\bar{Y}$ define an automorphism of $\bar{D}_0$ which leaves the two points of $D_0 \cap D_1$ fixed. Such an automorphism is in terms of an affine coordinate t on D_0 (such that $D_0 \cap D_1 = \{t = 0, \infty\}$) given as multiplication by a scalar. The appearance of this modulus will not bother us, however.

We will be mainly interested in the case when the intersection matrix $(D_i.D_j)$ is negative. This property is preserved if we contract a component of D which is exceptional. We therefore say that an anti-canonical cycle is *negative* (resp. *negative definite*) if the matrix $(D_i.D_j)$ is such and no D_i is exceptional. We have the following simple criterion, the proof of which is left to the reader.

(1.3) Let $D = D_0 + \cdots + D_{s-1}$ be a reducible (anti-canonical) cycle. Then

(i) D is negative if and only if $D_i.D_i \leq -2$ for all i;

(ii) D is negative non-definite if and only if $D_i.D_i = -2$ for all i. In that case D generates the radical of the inner product space spanned by $D_0, \ldots, D_{s-1}$.

We shall also need

(1.4) LEMMA. *Let $D = D_0 + \cdots + D_{s-1}$ be a reducible negative definite (anti-canonical) cycle.*

(i) *If $F = f_0D_0 + \cdots + f_{s-1}D_{s-1}$ is an* **R**-*linear combination of the components of D with $F.D_i \geq 0$ for all $i \in \mathbf{Z}/s$, then $f_i < 0$ for all $i \in \mathbf{Z}/s$.*

(ii) *$D.D = -1$ if and only if there exists an $i \in \mathbf{Z}/s$ such that $D_i.D_i = -3$ and $D_j.D_j = -2$ for $j \neq i$.*

(iii) *Let D_i^* denote the* **Q**-*linear combination of the D_j's satisfying $D_i^*.D_j = \delta_{ij}$. If $D.D \leq -2$ and $D_i.D_i \leq -3$, then $D_i^*.D_i^* > -1$.*

Proof. To prove (i) we assume without loss of generality that $f_0 = \max(f_i)$. If $f_0 \geq 0$, then $F.D_0 \geq 0$ implies $f_1 + f_{-1} \geq (-D_0.D_0)f_0 \geq 2f_0$. Since f_0 is maximal, this can only be if $f_1 = f_{-1} = f_0$ and $D_0.D_0 = -2$. Continuing in this way we find that $f_0 = \cdots = f_{s-1}$ and $D_0.D_0 = \cdots = D_{s-1}.D_{s-1} = -2$, contradicting the negative definiteness.

Assertion (ii) is immediate from the equality $D.D = \Sigma_i(D_i.D_i + 2)$ and the assumption that $D_i.D_i \leq -2$ for all i.

To prove (iii), write $D_i^* = \Sigma_{j=0}^{s-1} - x_j D_j$. Then all $x_j > 0$ by (i). $D_i^*.D_j = \delta_{ij}$ means that

$$- x_{j-1} - x_j D_j.D_j - x_{j+1} = \delta_{ij} \qquad (j \in \mathbf{Z}/s).$$

Summation over j gives $\Sigma_j(-D_j.D_j - 2)x_j = 1$. If x_i were ≥ 1 then the last equality shows that we must have $D_i.D_i = -3$ and $D_j.D_j = -2$ if $j \neq i$. But then $D.D = -1$, contradicting our assumption. It follows that $D_i^*.D_i^* = -x_i > -1$.

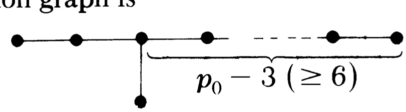
2. Root bases in the Picard group

From now on Y denotes a rational surface endowed with a negative anti-canonical cycle $D = D_0 + \cdots + D_{s-1}$ of length $s \leq 5$.

(2.1) Let π: $Y \to \bar{Y}$ be a contraction as described in Theorem (1.2). We denote the distinct exceptional curves of π which meet D_i by $E_i^1, \ldots, E_i^{p_i}$ (with $p_0 = 9 - D.D$ if $s = 1$ and $p_i = 4 - s - D_i.D_i$ if $s > 1$). We suppose that they are indexed in such a way that $E_i^j \subset E_i^{j'}$ implies $j \geq j'$. Any indexed set of exceptional curves (E_i^j) thus obtained will be called an *exceptional configuration*, unless $s = 2$: then we (often tacitly) assume that we are also given an ordering of the two rulings of $\bar{Y}$ ($\cong \mathbf{P}^1 \times \mathbf{P}^1$). As E_i^j is the unique positive divisor within its linear equivalence (or equivalently, homology) class e_i^j, we use the same terminology for the corresponding collection (e_i^j).

For reasons which will become clear in a moment, we call the orthogonal complement of the classes $[D_i]$ in Pic(Y) the *root lattice*; we denote it by Q. Our first aim is to describe an integral basis B of Q by means of the exceptional configuration. This basis will have the property that any of its elements is representable by a difference $E - E'$ of exceptional curves having no component in common with D, meeting D in the same component and satisfying $E \cap E' = \varnothing$ or $E' \subset E$. In particular, $\alpha.\alpha = -2$ for all $\alpha \in B$. Moreover, we will have either $\alpha.\beta = 0$ or $\alpha.\beta = 1$ for any *distinct* pair $\alpha, \beta \in B$. The intersection matrix $(\alpha.\beta)$ is conveniently read off from the intersection graph on B: this is a graph having B as its set of vertices and distinct $\alpha, \beta \in B$ are connected if and only if $\alpha.\beta = 1$.

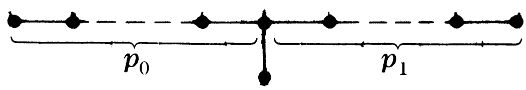
If $s = 1$, let $h \in$ Pic(Y) "define" the projection π: $Y \to \bar{Y} \cong \mathbf{P}^2$ (so h is represented by the total transform of any line in $\bar{Y}$). We take $B = \{e_0^1 - e_0^2, e_0^2 - e_0^3, \ldots, e_0^{p_0-1} - e_0^{p_0}, (h - e_0^1 - e_0^2) - e_0^3\}$. Note that $h - e_0^1 - e_0^2$ can be represented by the exceptional curve obtained by subtracting $E_0^1 + E_0^2$ from the total transform of the line $\bar{Y}$ which passes through $\pi(e_0^1)$ and $\pi(e_0^2)$ (at least when $\pi(e_0^1) \neq \pi(e_0^2)$; otherwise take the tangent line of $\bar{D}$ at $\pi(e_0^1) = \pi(e_0^2)$). The corresponding intersection graph is



If $s = 2$, let $h_0, h_1 \in \operatorname{Pic}(Y)$ define the two rulings determined by the projection $\pi\colon Y \to \bar{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$. We put

$$B = \left\{e_0^1 - e_0^2, \dots, e_0^{p_0-1} - e_0^{p_0}, e_1^1 - e_1^2, \dots, e_1^{p_1-1} - e_1^{p_1},\right.$$
$$\left.\left(h_0 - e_0^1\right) - e_1^1, \left(h_0 - e_0^1\right) - \left(h_1 - e_0^1\right)\right\}.$$

Note that $h_k - e_i^j$ can be represented by the exceptional curve obtained by subtracting E_i^j from the fibre of the ruling defined by h_k which contains E_i^j. The intersection graph is

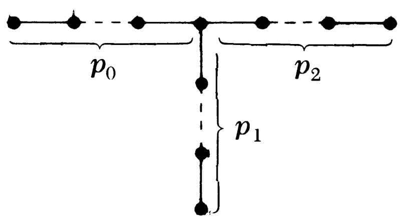


$(p_i \geq 4)$.

If $s = 3$, let $h \in \operatorname{Pic}(Y)$ define the projection $\pi\colon Y \to \bar{Y} \cong \mathbf{P}^2$. We take

$$B = \left\{e_i^{j-1} - e_i^j\colon i \in \mathbf{Z}/3, j = 2, \dots, p_i\right\} \cup \left\{\left(h - e_0^1 - e_1^1\right) - e_2^1\right\}.$$

As in the case $s = 1$, $h - e_0^1 - e_1^1$ is represented by an exceptional curve meeting D in D_2. The intersection graph will be

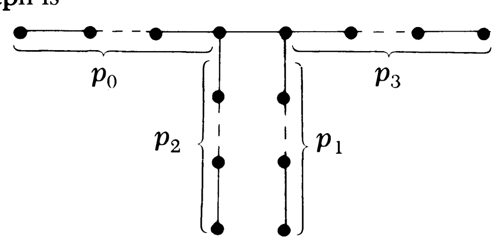


$(p_i \geq 3)$.

If $s = 4$, we denote the two rulings of Y defined by $\pi\colon Y \to \bar{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$ by $h_0, h_1 \in \operatorname{Pic}(Y)$ such that $h_i.[D_j] = 1$ if $i + j$ is even and 0 if $i + j$ is odd. Put

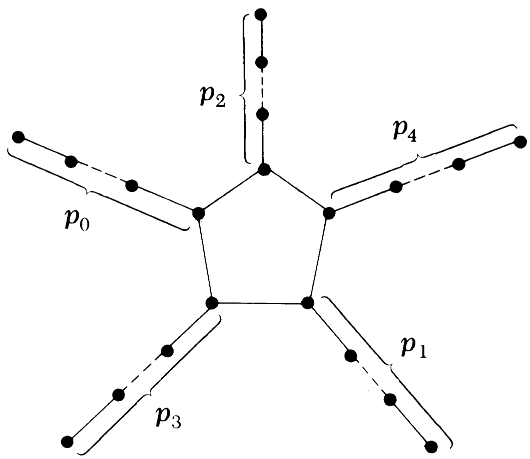
$$B = \left\{e_i^{j-1} - e_i^j\colon i \in \mathbf{Z}/4, j = 2, \dots, p_i\right\} \cup \left\{\left(h_0 - e_0^1\right) - e_2^1, \left(h_1 - e_1^1\right) - e_3^1\right\}.$$

The intersection graph is



$(p_i \geq 2)$.

If $s = 5$, let $\bar{E}_i$ $(i \in \mathbf{Z}/5)$ denote the exceptional curve of the first kind on $\bar{Y}$ which is linearly equivalent to $\bar{D}_{i-2} - \bar{D}_i + \bar{D}_{i-2}$. So $\bar{E}_i.\bar{D}_j = \delta_{ij}$ and $\bar{E}_i.\bar{E}_j = -1$ if $i = j$, 0 if $i - j = \pm 1$ and 1 if $i - j = \pm 2$. Let E_i^0 denote the total transform of $\bar{E}_i$ on Y. Clearly, E_i^0 is an exceptional curve which meets D in D_i. Denoting its class by e_i^0, we let B be the set $\{e_i^{j-1} - e_i^j\colon i \in \mathbf{Z}/5, j = 1, \dots, p_i\}$. The intersection graph then becomes



(2.2) Most of these intersection graphs have been encountered before (although in a different context): those for $s = 1,2,3$ occur in Arnol'd [1], those for $s = 1,2,3,4$ in Karras [13]. (While preparing this manuscript, I received a preprint by Nakamura [21] in which the intersection graph for $s = 5$ occurs. However, like Karras, he finds the diagram "by experiment" without offering an intrinsic characterization (as we do in Section 4).) The precise connection with their work will be explained in Part III.

Given π: $Y \to \bar{Y}$ as above, we define the corresponding *quasi-polarization* $p \in \mathrm{Pic}(Y)$ as the pull-back of the anti-canonical class of $\bar{Y}$. So for $s = 1,3$ we have $p = 3h$, for $s = 2,4$, $p = 2h_0 + 2h_1$ and for $s = 5$, p is the pull-back of $e_0^0 + e_1^0 + \cdots + e_4^0$. Clearly, an exceptional configuration determines a quasi-polarization. Conversely, the linear system of a quasi-polarization maps Y onto the corresponding $\bar{Y}$ and thus determines its exceptional configuration up to indexing.

3. Generalized root systems

(3.1) Let B be the set defined in the previous section. For any $\alpha \in B$ we define a linear form $\alpha^\vee$ on $\mathrm{Pic}(Y)$ by $\alpha^\vee(x) = -\alpha \cdot x$. Then the vector space $\mathrm{Pic}(Y)_{\mathbf{R}}$ together with the injection $^\vee$: $B \to \mathrm{Pic}(Y)^*_{\mathbf{R}}$ is a root basis in the sense of [19], which has Q as its root lattice. In the present situation this simply comes down to the facts that $\alpha.\alpha = -2$ for all $\alpha \in B$, $\alpha.\beta \in \mathbf{Z}_+$ for any distinct pair $\alpha, \beta \in B$, $\alpha.\beta = 0$ if and only if $\beta.\alpha = 0$, B and $B^\vee$ are linearly independent sets and B generates Q over $\mathbf{Z}$.

In general, a root basis determines a rich geometric structure in the vector space in which it is defined. We briefly describe its main features in terms of the present context. Any $\alpha \in B$ defines an orthogonal reflection s_α in $\mathrm{Pic}(Y)$ by $s_\alpha(x) = x + (x.\alpha)\alpha$, called a *fundamental reflection*. The fundamental reflections generate a discrete subgroup W of the orthogonal group of $\mathrm{Pic}(Y)_{\mathbf{R}}$ (or equivalently, of $Q_{\mathbf{R}}$), called the *Weyl group* of B. The *fundamental chamber* $C \subset \mathrm{Pic}(Y)_{\mathbf{R}}$ is defined by the inequalities $x.\alpha > 0$ for all $\alpha \in B$.[(*)] Each subset $X \subset B$ defines a *fundamental facet*

$$F_X := \{x \in \mathrm{Pic}(Y)_{\mathbf{R}} : x.\alpha = 0 \text{ if } \alpha \in X \text{ and } x.\alpha > 0 \text{ if } \alpha \in B - X\}.$$

Note that $F_\phi = C$ and that F_B is the $\mathbf{R}$-linear span of $[D_0], \ldots, [D_{s-1}]$. Clearly, the closure $\bar{C}$ of C is the disjoint union of the fundamental facets. The following property, proved in [4, Ch. V § 4], is particularly important.

(3.2) If $w \in W$ and $X \subset B$ are such that $w(F_X) \cap \bar{C} \neq \varnothing$, then w belongs to the subgroup W_X of W generated by the s_α, $\alpha \in X$. In particular, w leaves F_X pointwise fixed.

Among other things, this implies that $(W, \{s_\alpha : \alpha \in B\})$ is a *Coxeter system*.

(3.3) The W-orbit of $\bar{C}$ is called the *Tits cone* and denoted I. It follows from (3.2) that the W-translates of the fundamental facets define a partition of I. A member of this partition is called a *facet*, but a W-translate of C is usually called a *chamber*. It is also shown in [4, *loc. cit.*] that I is a convex cone. We proved in [19, §1] that a facet F of I belongs to its interior if and only if its stabilizer W_F is finite and that W acts properly discontinuously on $\mathring{I}$.

An element of the W-orbit of B is called a *root*. Since the elements of B are W-equivalent, the set of roots constitutes a single W-orbit; we denote it by R. If $\alpha \in R$, then the reflection orthogonal to α, $s_\alpha : x \mapsto x + (x.\alpha)\alpha$ clearly belongs to W. It follows from (3.2) that the linear form $x \mapsto x.\alpha$ $(\alpha \in R)$ has constant (non-zero) sign on C; if the sign is positive, resp. negative, we say that α is a positive, resp. negative, root. The corresponding partition of R is denoted by $R = R_+ \cup R_-$. For $x \in \mathrm{Pic}(Y)$ we let Φ_x denote the set $\alpha \in R_+$ with $\alpha.x < 0$. We proved in [19, §1]:

(3.3) An element $x \in \mathrm{Pic}(Y)_{\mathbf{R}}$ belongs to I if and only if Φ_x is a finite set. If so, then we can index the elements of Φ_x: $\alpha_1, \ldots, \alpha_k$ (with $k = \mathrm{card}\,\Phi_x$) such that the chambers $\{C_i = s_{\alpha_i} \cdots s_{\alpha_1}(C) : i = 0, \ldots, k\}$ form a *minimal gallery* from C to x; more precisely C_{i-1} and $C_i = s_{\alpha_i}(C_{i-1})$ have a facet in common which is open in the reflection hyperplane of s_{α_i}: $x.\alpha_i = 0$, and if we put $w = s_{\alpha_k} \cdots s_{\alpha_1}$, then $x \in w(\bar{C})$ and $x \in w^{-1}(x) + \mathbf{R}_+ . \Phi_x$.

(3.4) The Dynkin diagram of B (as defined in [19]) is just the intersection graph. We see from it that W is an irreducible infinite Coxeter group and that W

(*) Actually, this definition doesn't agree with the one in [19] for in that paper the fundamental chamber is just $-C$.

is affine precisely when $D.D = 0$ (we get the affine systems $\hat{E}_8$, $\hat{E}_7$, $\hat{E}_6$, $\hat{D}_5$ and $\hat{A}_4$).

Note that in all cases the quasi-polarization p is in $\bar{C} \cap \mathring{I}$. In fact, if $s = 1$ let $X \subset B$ denote the maximal string in the intersection diagram and if $s > 1$, let X denote the set of non-ramification vertices of B; then $p \in F_X$. The set X enables us to recover the exceptional configuration solely from the basis B. First we observe that X consists of s disjoint strings of length $p_0 - 1, \ldots, p_{s-1} - 1$ (so X is isomorphic to the Dynkin diagram of $A_{p_0-1} \times \cdots \times A_{p_{s-1}-1}$). Now $e_i^{p_i}$ is characterized by the property that $e_i^{p_i}[D_j] = \delta_{ij}$ and $e_i^{p_i}.\beta = \delta_{\alpha,\beta}$ ($\beta \in B$) for some (unique) $\alpha_i \in X$. It appears that α_i is an endpoint of a connected component X_i of X which is a string of $p_i - 1$ elements. If we index the elements in the obvious way: $X_i = \{\alpha_i^1, \ldots, \alpha_i^{p_i-1} = \alpha_i\}$, then $e_i^j = s_{\alpha_i^j} \cdots s_{\alpha_i^{p_i-1}}(e_i^{p_i})$. In case $s = 2$, we also reconstruct the ordering of the two rulings h_0 and h_1: if $Y \subset B$ denotes the maximal string in B (of length $p_p + p_1 - 1$), then $B - Y$ consists of a single root α and $h_0.\alpha = -1$, $h_1.\alpha = 1$.

A root $\alpha \in R$ is called *nodal* if α is representable by a nodal curve. We denote the set of nodal roots by B^n. Clearly, for any distinct pair $\alpha, \beta \in B^n$, we have $\alpha.\beta \in \mathbf{Z}_+$. We may not conclude, however, that B^n is a root basis, for B^n need not be a linearly independent set. Nevertheless we can define the *nodal* (= fundamental) *chamber* C^n, the Weyl group W^n and the set $R^n = W^n.B^n$.

(3.5) Lemma. *We have $C \subset C^n$ (equivalently, $B^n \subset R_+$).*

Proof. We observed that $p \in \bar{C}$. If N is a nodal curve not contained in any E_i^j ($j \geq 1$), then clearly $[N].p > 0$ and so $[N] \in R_+$. If N is a nodal curve contained in some E_i^j ($j \geq 1$), then $N = E_i^{j_1} - E_i^{j_2}$ for certain $j_1 > j_2 \geq 1$ and hence $[N]$ is a positive linear combination of elements of B: $[N] = (e_i^{j_1} - e_i^{j_1+1}) + \ldots + (e_i^{j_2-1} - e_i^{j_2})$. So in this case $[N] \in R_+$, too.

(3.6) In particular, $C^n \neq \varnothing$. This "implies" that most of the arguments and geometric constructions leading to (3.2) and (3.3) remain valid (with some minor modifications: not every subset of B^n need determine a facet of C^n); see Vinberg [27] for details. In any case, we have a partition of R^n into positive and negative roots $R^n = R^n_+ \cup R^n_-$ (with $R^n_\pm = R^n \cap R_\pm$ by (3.5)) and the analogue of (3.3) holds.

Recall that the positive cone C^+ of Y is the component of $\{x \in \mathrm{Pic}(Y)_{\mathbf{R}}: x.x > 0\}$ which contains ample divisor classes. Since $p.p > 0$ and p has positive inner product with any ample class, we must have $p \in C^+$.

(3.7) Lemma. *The Tits cone contains C^+.*

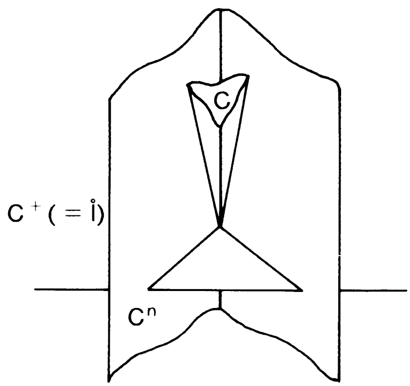


Figure I

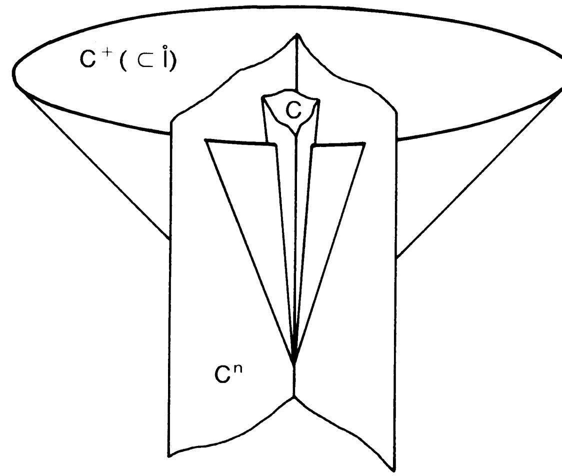


Figure 2

Relative position of the positive cone (C^+), the fundamental chamber (C) and the nodal chamber (C^n) in $\mathrm{Pic}(Y)_R$ (or rather their images under projection parallel to the subspace F_B) when (D_i, D_j) is degenerate, resp. nondegenerate.

Proof. Since $p \in \overline{C} \cap C^+$ and C is open, we have $C \cap C^+ \neq \varnothing$. Now let $x \in C^+$ be arbitrary. In order to prove that $x \in I$, it suffices to show by (3.3) that Φ_x is a finite set. The subgroup G (of index two) of the orthogonal group of $\mathrm{Pic}(Y)_{\mathbf{R}}$ which preserves C^+ acts with compact isotropy groups on C^+. Since W is discrete in G, W acts properly discontinuously on C^+. This implies that x and $C \cap C^+$ are separated from each other by only a finite number of reflection hyperplanes of W. Hence Φ_x is finite.

We will see in Chapter II that $\mathring{I} \cup -\mathring{I}$ is the maximal open subset of $\mathrm{Pic}(Y)_{\mathbf{R}}$ on which W acts properly discontinuously.

(3.8) A subset X of B is called *extremal* if for any connected component Y of X, W_Y is infinite and $\mathrm{card}(X) = \mathrm{card}(B) - 1$. The corresponding facet F_X, as well as any W-translate of F_X, is called an *extremal facet*. By simple inspection we find that any extremal subset of B is of the form $X_i = B_i - \{e_i^{p_i - 1} - e_i^{p_i}\}$. We then clearly have $e^{p_i} \in F_{X_i}$. This terminology is explained by the following.

(3.9) Proposition. *If $D.D = 0$, then I is the union of the half space defined by $[D].x > 0$ and the facet F_B. If $D.D < 0$, then I is the convex hull of F_B and the extremal facets.*

Proof. If $D.D = 0$, then as we observed earlier, W is an affine Weyl group. It then follows from [19, Lemma 5.2] that $\mathring{I}$ is the half space defined by $x.[D] > 0$. Since F_B is the only fundamental facet with an infinite stabilizer, we have $I - \mathring{I} = W.F_B = F_B$.

If $D.D \le -1$, let $\mathcal{H}$ denote the convex hull of F_B and the extremal facets of I. We prove that for any $\beta \in B$, $\mathcal{H} \cap F_{B-\{\beta\}} \ne \varnothing$. This will be sufficient since $\mathcal{H} \supset F_B$ then implies $\mathcal{H} \supset F_{B-\{\beta\}}$ for any $\beta \in B$ and so $\mathcal{H} \supset \bar{C}$ (since $\bar{C}$ is the convex hull of the $F_{B-\{\beta\}}$). As $\mathcal{H}$ is W-invariant it then follows that $\mathcal{H} \supset I$.

Write $B - \{\beta\} = X \cup Y$ where X, resp. Y, denotes the union of connected components of $B - \{\beta\}$ with infinite, resp. finite, Weyl group. There exists an extremal subset $B - \{\alpha\}$ of B containing X. If $X \ne \varnothing$, this is clear; if $X = \varnothing$, then it follows from the assumption $D.D \le -1$ that an extremal subset exists (look at the intersection diagrams!). If $\alpha = \beta$, then $F_{B-\{\alpha\}} \subset \mathcal{H}$ and we are done. We therefore assume that $\alpha \in Y$. Choose $u \in F_{B-\{\alpha\}}$ and put $v = \Sigma_{w \in W_Y} w(u)$. Then clearly $v \in \mathcal{H}$. If $\gamma \in Y$, then $v.\gamma = \Sigma_{w \in W_Y} w(u).\gamma = u.\Sigma_{w \in W_Y} w^{-1}(\gamma) = 0$. If $\gamma \in X$, then $v.\gamma = \Sigma_{w \in W_Y} w(u).\gamma = u.\Sigma_{w \in W_Y}\gamma = 0$. Finally, $v.\beta = \Sigma_{w \in W_Y} w(u).\beta = \Sigma_{w \in W_Y}(w(u) - u).\beta$. As $w(u) - u$ is a positive linear combination of elements of Y, we have $(w(u) - u).\beta \ge 0$. We claim that this inequality is strict for at least one $w \in W_Y$: if $\alpha = \alpha_0, \dots, \alpha_k$ is the shortest edge path in Y from α to a root which is connected with β, then $s_{\alpha_k} \cdots s_{\alpha_1} \in W_Y$ is as desired. It follows that $v.\beta > 0$ and hence $v \in F_{B-\{\beta\}} \cap \mathcal{H}$.

(3.10) Corollary. *If $x \in Q_{\mathbf{R}}$ is such that $x.w(e_i^{p_i}) \ge 0$ for all i, then $x.x \ge 0$.*

Proof. If $D.D = 0$, then W is of affine type. Then $W.\{e_i^{p_i}\}$ meets any coset $e_i^{p_i} + q + \mathbf{Z}.[D]$, $q \in Q$ (see [19], (5.1)). As $[D]$ spans the radical of Q, we then have $x.(e_i^{p_i} + q) \ge 0$ for all q. This implies $x.q \ge 0$ for all $q \in Q$ and hence $x \in \mathbf{R}.[D]$. In particular $x.x \ge 0$.

If $D.D \le -1$, then any extremal facet of I is of the form $\mathbf{R}_+.w(e_i^{p_i}) + \mathbf{R}.[D_0] + \dots + \mathbf{R}.[D_{s-1}]$. By assumption the linear form $y \mapsto y.x$ on $\operatorname{Pic}(Y)_{\mathbf{R}}$ is ≥ 0 on any such set. It then follows from (3.9) that this form is ≥ 0 on I. As I contains C^+, this implies that $x \in \bar{C}^+$, in particular $x.x \ge 0$.

4. The Weyl group as a Cremona group

In this section we shall give a precise description of $\operatorname{Pic}^+(Y)$, the semi-group of effective divisor classes. This includes an algorithm for determining whether or not an element of $\operatorname{Pic}(Y)$ represents an exceptional curve of the first kind (at least when we are given the set of nodal roots). Our starting point is the following (key) lemma.

(4.1) Lemma. *If $\alpha \in B - R_+^n$, then $s_\alpha(e_i^j)$ is an exceptional configuration having $s_\alpha(p)$ as its quasi-polarization.*

Proof. If $\alpha = e_i^{j-1} - e_i^j$ ($j \geq 2$), then $\alpha \notin R_+^n$ just means that E_i^j is not contained in E_i^{j-1}. Hence E_i^{j-1} and E_i^j are disjoint. Since s_α interchanges E_i^{j-1} and E_i^j and leaves all the other $E_{i'}^{j'}$ fixed, everything is obvious in this case.

Now suppose for instance that $s = 1$ and $\alpha = h - e^1 - e^2 - e^3$ (we dropped the index 0). The assumption $\alpha \notin R_+^n$ implies that E^i does not contain E^j for $1 \leq i < j \leq 3$. Hence the image points $\pi(E^1)$, $\pi(E^2)$ and $\pi(E^3)$ in $\bar{Y}$ are distinct. Moreover, each E^i is a maximal exception curve in the sense that $E^i = \pi^{-1}\pi(E^i)$. It also follows that $\pi(E^1), \pi(E^2), \pi(E^3)$ are not collinear (otherwise $\alpha \in B^n$!). Now, $s_\alpha(e^1) = h - e^2 - e^3$ is represented by the full transform of the line in $\bar{Y}$ which passes through $\pi(E^2)$ and $\pi(E^3)$ minus $E^2 + E^3$. If we denote this representative $'E^1$ (and $'E^2$, resp. $'E^3$, the corresponding representatives of $s_\alpha(e^2)$, resp. $s_\alpha(e^3)$), then it is clear that $'E^1$, $'E^2$ and $'E^3$ are disjoint and that any E^j ($j > 3$) which meets $'E^k$ ($k \leq 3$) is actually contained in $'E^k$. Hence $s_\alpha(e^j)$ is an exceptional configuration.

The proof for the remaining cases is similar.

This lemma has some interesting corollaries.

(4.2) Corollary. *If $w \in W$ is such that $w(C) \subset C^n$, then $w(e_i^j)$ is an exceptional configuration.*

Proof. Since C and $w(C)$ are both subsets of C^n, no reflection hyperplane separating C from $w(C)$ is orthogonal to some $\alpha \in R^n$. So if $C_0 = C, \ldots, C_{i+1} = s_{\alpha_i}(C_i), \ldots, C_k = w(C)$ denotes the gallery of (3.3) then $\alpha_i \notin R^n$ for all i. With induction on i it now easily follows from Lemma (4.1) that $s_{\alpha_i} \cdots s_{\alpha_1}(e_i^j)$ is an exceptional configuration. As $w = s_{\alpha_k} \cdots s_{\alpha_1}$, the corollary follows.

(4.3) Corollary. *Any $e \in W.\{e_i^{p_i}\}$ is representable by a connected positive divisor of the form $E = E_0 + n_1N_1 + \cdots + n_kN_k$ ($n_i \in \mathbf{Z}_+$), where E_0 is an exceptional curve of the first kind with $[E_0] \in W.\{e_i^{p_i}\}$ and $N_1, \ldots, N_k$ are nodal curves whose classes belong to R.*

If, moreover, $e \in \bar{C}^n$, then $E = E_0$.

Proof. First observe that $e_i^{p_i} \in \bar{C}$. If $e = w(e_i^{p_i})$ for some $w \in W$, then by [19, Lemma (1.13)], we may write $w = w''w'$ with $w'' \in W^n$ and $w'(C) \subset C^n$. By (4.2), $w'(e_k^j)$ is an exceptional configuration and so $w'(e_i^{p_i})$ is represented by an exceptional curve of the first kind E_0. If $e \in \bar{C}^n$, then we have $w'' = 1$ and so $e = w'(e_i^{p_i}) = [E_0]$. Otherwise, it follows from (3.3) that e is of the form $[E_0] + \beta_1 + \cdots + \beta_k$ with $\beta_i \in R_+^n$ and $([E_0] + \beta_1 + \cdots + \beta_{i-1}.\beta_i).\beta_i > 0$ for all i.

(4.4) Proposition. *If N is a nodal curve on Y, not contained in D, then $[N] \in B^n$.*

Proof. Since $N.D_i = 0$ for all i, we must have $[N] \in Q$. If $[N] \notin B^n$, then it follows from (4.3) that $[N].w(e_i^{p_i}) \geq 0$ for all $i \in \mathbf{Z}/s$ and $w \in W$. By (3.10) this implies that $N.N \geq 0$ which contradicts the fact that $N.N = -2$.

We now prove a converse to (4.2).

(4.5) PROPOSITION. *Let $('E_i^j)$ be an exceptional configuration on Y. Then $'E_i^j = w(E_i^j)$ for some $w \in W$.*

We divide the proof in four steps.

Step 1. If $D.D = 0$ or $D.D = -1$, then any $\alpha \in Q$ with $\alpha.\alpha = -2$ is in R.

Proof. If $D.D = 0$, then W is an affine Weyl group (of type $\hat{E}_8, \hat{E}_7, \hat{E}_6, \hat{D}_5, \hat{A}_4$ resp.) and R projects onto a finite root system R_0 in $Q/\mathbf{Z}.[D]$ (of type E_8, E_7, E_6, D_5, A_4 resp.). These finite root systems are known to contain all elements of $Q/\mathbf{Z}.[D]$ of square length -2. Let $\alpha_0 \in B$ be such that $B - \{\alpha_0\}$ projects onto a basis for R_0 and let $\tilde{\alpha}$ denote the longest root for $B - \{\alpha_0\}$. Then $\alpha_0 - [D] = -\tilde{\alpha}$, as is easily checked in each case. Since W acts transitively on R, it then follows that for any $\alpha \in R$ and any $k \in \mathbf{Z}$, $\alpha + k[D]$ belongs to R. Hence the result.

Next assume that $D.D = -1$. If D is reducible, then $D_i.D_i = -3$ for some i and $D_j.D_j = -2$ for all $j \neq i$, by (1.4)-ii. In all cases we have that for any proper subset $X \subset B$ either W_X is finite or $B - X$ consists of a single root and W_X is an affine Weyl group. By Vinberg's criterion [27, §2.4], W is then a subgroup of the orthogonal group of Q of finite index and contains all reflections orthogonal to elements of square length -2. In fact, he proves that any such reflection is W-conjugate to some s_α, $\alpha \in B$. This gives the desired result.

Step 2. Proof of the proposition in case $D.D = 0$ or $D.D = -1$.

Proof. Let $'B \subset Q$ denote the root basis associated to $('E_i^j)$ and $'C$ the corresponding chamber. Since $'C \cap C^+ \neq \varnothing$, we have $'C \cap I \neq \varnothing$ by (3.5) and so there is a $w \in W$ with $w(C) \subset {'C}$. For reasons of symmetry the opposite inclusion also holds and so $'C = w(C)$ and hence $'B = w(B)$. It then follows from the discussion in (3.4) that $w(e_i^j) = {'e_i^j}$.

Step 3. Suppose $D.D \leq -2$ and let E be an irreducible exceptional curve on Y which meets D in D_i. If $s \geq 2$, assume also that $D_i.D_i \leq -3$. Then $[E] \in W.\{e_i^{p_i}\}$.

Proof. If not, then $[E].w(e_j^{p_j}) \geq 0$ for all $j \in \mathbf{Z}/s$, $w \in W$ by (4.3). Let $D_i^* = \sum_{j=0}^{s-1} - x_j D_j$ be such that $D_i^*.D_j = \delta_{ij}$. Then we have $x_j > 0$ for all j and $x_i = -D_i^*.D_i^* < 1$ by (1.4). Since $E.D_j = \delta_{ij}$, we have $[E - D_i^*] \in Q_{\mathbf{R}}$. Clearly,

$(E - D_i^*).w(e_j^{p_i}) = E.w(e_j^{p_i}) + x_j \geq 0$. It then follows from (3.10) that $(E - D_i^*)^2 \geq 0$. In other words, $0 \leq E.E - 2E.D_i^* + D_i^*.D_i^* = -1 + x_i$, which contradicts the fact that $x_i < 1$. So $[E] \in W.\{e_i^{p_i}\}$.

Step 4. Proof of (4.5) in case $D.D \leq -2$.

We proceed with induction on $-D.D$. Let $i \in \mathbf{Z}/s$ be such that $D_i.D_i \leq -3$. Since $'E_i^{p_i}$ is irreducible, there exists by step 3 a $w \in W$ such that $w(E_i^{p_i}) = 'E_i^{p_i}$. We may choose $w \in W$ such that $w(C) \subset C^n$ (for the W-stabilizer of $['E_i^{p_i}]$ acts transitively on the set of chambers containing $['E_i^{p_i}]$ in their closure). Then $(w(E_k^j))$ is an exceptional configuration by (4.2). Now blow down $E_i^{p_i}$: $Y \to Y'$. Then $(w(E_k^j$: $(k, j) \neq (i, p_i))$ and $('E_k^j$: $(k, j) \neq (i, p_i))$ map to exceptional configurations on Y', so that we can apply our inductive hypothesis. It says that there exists a w_0 (in the subgroup of W generated by the s_α, $\alpha \in W(B - \{e_i^{p_i - 1} - e_i^{p_i}\})$), such that $w_0 w(E_k^j) = 'E_k^j$.

Note that proposition (4.5) implies that the set of roots R, the Weyl group W, the Tits cone I with its partition into facets and the nodal chamber C^n are intrinsic to the pair (Y, D). If we combine (4.2) through (4.5) we find:

(4.6) THEOREM. *Let Y be rational surface endowed with a negative anti-canonical cycle $D = D_0 + \ldots + D_{s-1}$ of length $s \leq 5$. Then there is a natural bijective correspondence between the set of exceptional configurations and the set of chambers of I contained in the nodal chamber. The set of classes of exceptional curves of the first kind meeting D in a fixed irreducible component D_i is the intersection of a W-orbit (contained in I) with the nodal chamber.*

An orthogonal transformation of $\mathrm{Pic}(Y)$ which leaves the semi-group of effective classes $\mathrm{Pic}^+(Y)$ invariant is called a *Cremona isometry*. We denote the group of Cremona isometries which leave the classes $[D_i]$ $(i \in \mathbf{Z}/s)$ fixed by $\mathrm{Cr}(Y, D)$.

(4.7) PROPOSITION. $\mathrm{Cr}(Y, D) = \{w \in W\colon w(B^n) = B^n\}$.

Proof. As a semi-group, $\mathrm{Pic}^+(Y)$ is generated by the integral elements in C^+, the classes of exceptional curves of the first kind, the nodal classes and the classes of the components of D. If $w \in W$ is such that $w(B^n) = B^n$, then it follows from (4.2)–(4.5) that w permutes these generators.

Conversely, if $\sigma \in \mathrm{Cr}(Y, D)$, then it follows from (4.6) that σ permutes the chambers (of I). So there exists a $w \in W$ such that $\sigma(C) = w(C)$. This implies that $w^{-1}\sigma$ induces an automorphism of the intersection graph on B. Any subset X_i of B as defined in (3.4) corresponds in a one-one manner to an irreducible component D_i of D. As $w^{-1}\sigma$ leaves the classes $[D_i]$ fixed, it follows that it leaves each X_i invariant. It is easily checked that this implies that $w^{-1}\sigma$ acts trivially on B. Hence $w = \sigma$.

5. A theorem of Torelli type

(5.1) The root system R^n generated by the nodal roots can be recovered in much the same way as for $K3$ surfaces. For this purpose we assume that D has been oriented. Let ω be a meromorphic 2-form on Y with $\mathrm{div}(\omega) = -D$. Clearly the double residue of ω at a node of D is nonzero. Now consider the exact homology sequence of the pair $(Y, Y - D)$ (with integral coefficients):

$$\begin{array}{l}\cdots \to H_3(Y) \to H_3(Y, Y-D) \xrightarrow{\partial_*} H_2(Y-D) \xrightarrow{i_*} H_2(Y) \xrightarrow{j_*} H_2(Y, Y-D) \to \cdots . \\ \quad\;\; \| \\ \quad\;\; 0\end{array}$$

We identify $H_k(Y, Y-D)$ with $H^{4-k}(D)$ by Poincaré duality. The corresponding map $\partial_*\colon H^1(D) \to H_2(Y-D)$ is then dual to the residue homomorphism; in particular, if we think of ω as a linear form $H_2(Y-D) \to \mathbf{C}$, then ω is nontrivial on $\partial_*(\varepsilon)$, where $\varepsilon \in H^1(D)$ denotes the orientation. We scale ω by the condition that $\omega(\partial_*(\varepsilon)) = 1$. This just means that $\mathrm{Res}_{D_{i-1}\cap D_i}\mathrm{Res}_{D_i}\omega = (1/2\pi i)^2$ for some (or equivalently, all) i. Since $H_2(Y, Y-D) \cong H^2(D)$ is the free $\mathbf{Z}$-module spanned by the irreducible components of D, it follows that the kernel of j_* coincides with the root lattice. Hence in the short exact sequence

$$0 \to H^1(D) \to H_2(Y-D) \to Q \to 0,$$

the linear form ω determines a character $\chi\colon Q \to \mathbf{C}^*$ uniquely determined by the condition that $\exp 2\pi i \omega = \chi \circ i_*$.

(5.2) Proposition. *Let α be a root. Then $\alpha \in R^n$ if and only if $\chi(\alpha) = 1$.*

Proof. If N is a nodal curve on Y not contained in D, then $N \subset Y - D$ and the restriction of ω to N is identically zero. Hence $\chi([N]) = 1$.

Next suppose that $\alpha \in R$ is such that $\chi(\alpha) = 1$. Then $\alpha = w(\beta)$ for some $\beta \in B$, $w \in W$. Write $w = w''w'$ with $w'(C) \subset C^n$ and $w'' \in W^n$. According to (4.2), $w'(\beta)$ is representable by $E^1 - E^2$, where E^1, E^2 are exceptional curves which have no component in common with D, satisfy $E^1 \cap E^2 = \varnothing$ or $E^2 \subset E^1$, and meet D in the same component D_i. Since $w''(E^i) \equiv E^i \pmod{\mathbf{Z}.B^n}$, it follows that $1 = \chi(w''([E^1 - E^2])) = \chi([E^1 - E^2])$.

In order to describe the geometric meaning of $\chi([E^1 - E^2])$, let γ be an injective path in $D_i \cap D_{\mathrm{reg}}$ from $D \cap E^1$ to $D \cap E^2$. Choose a closed tubular neighbourhood τ of D_i in Y such that $\tau^i := \tau \cap E^i$ is a fibre ($i = 1,2$) and let c denote the set $(E^1 - \tau^1) \cup (\partial\tau|\gamma) \cup (E^2 - \tau^2)$. Then c is a topological manifold which can be oriented in such a way that its fundamental class $[c]$ is homologous to $[E^1 - E^2]$ in Y. Now

$$\int_c \omega = \int_{\partial\tau|\gamma} \omega = \int_\gamma 2\pi i \,\mathrm{Res}_{D_i}\omega,$$

by the Cauchy residue formula. Choose an affine coordinate t on D_i such that $\mathrm{Res}_{D_i}\omega = (1/2\pi i)^2 dt/t$. Then the last integral equals $(1/2\pi i)\log(\gamma(1)/\gamma(0))$ $(\mathrm{mod}\,\mathbf{Z})$ and hence $\chi[E^1 - E^2]) = \gamma(1)/\gamma(0)$. More intrinsically, $\chi(\omega)$ is the cross ratio of $D_{i-1} \cap D_i$, $D_{i+1} \cap D_i$, $E^2 \cap D_i$, $E^1 \cap D_i$ (at least when $s \geq 2$). In particular, $\chi([E^1 - E^2]) = 1$ implies that $E^1 \cap D_i = E^2 \cap D_i$. This implies $E^2 \subset E^1$ and hence $[E^1 - E^2] \in R^n_+$. It follows that $\alpha = w''([E^1 - E^2]) \in w''(R^n) = R^n$.

We now prove a Torelli theorem for the surfaces considered in the previous section.

(5.3) THEOREM. *Let Y and $'Y$ be smooth rational surfaces endowed with negative oriented anti-canonical cycles $D = D_0 + \cdots + D_{s-1}$ and $'D = 'D_0 + \cdots + 'D_{s-1}$ of length $s \leq 5$ respectively. Denote the positive cone by C^+, resp. $'C^+$, the set of roots by R, resp. $'R$, the set of nodal roots by B^n, resp. $'B^n$ and the character of the root lattice induced by the 2-form by χ, resp. $'\chi$. If ϕ:* $\mathrm{Pic}(Y) \to \mathrm{Pic}('Y)$ *is an isometry such that*

(i) $\phi([D_i]) = ['D_i]$,

(ii) $\phi(R) = 'R$, $\phi(B^n) = 'B^n$,

(iii) $\phi(C^+) = 'C^+$,

(iv) $\phi^*('\chi) = \chi$,

then ϕ is induced by a unique isomorphism Φ: $Y \to 'Y$ which maps D_i onto $'D_i$ $(i \in \mathbf{Z}/s)$ and preserves the orientation.

Proof. Choose an exceptional configuration (E_i^j) on Y and let C denote the corresponding chamber of I. Since $C \cap C^+ \neq \varnothing$, we also have $\phi(C) \cap 'C^+ \neq \varnothing$ and as $'I$ contains $'C^+$, it follows that there exists a chamber $'C$ of $'I$ with $\phi(C) \cap 'C \neq \varnothing$. This implies that $\phi(C) = 'C$ (for $'R = \phi(R)$). Since $\phi([D_i]) = ['D_i]$, it follows from the discussion (3.4) that $\phi(e_i^j)$ is an exceptional configuration. We let $'E_i^j$ denote the corresponding set of exceptional curves. If (as usual) $\overline{Y}$, resp. $'\overline{Y}$, denotes the surface obtained by contraction of the E_i^j, resp. $'E_i^j$, let $u_i^j \in \overline{D}_i$, resp. $'u_i^j \in '\overline{D}_i$, be the image point of E_i^j, resp. $'E_i^j$, in $\overline{Y}$, resp. $'\overline{Y}$. We put $u_i(0) = D_{i-1} \cap D_i$ and $u_i(\infty) = D_{i+1} \cap D_i$ (this is ambiguous if $s = 1$ or 2 but then the orientation saves us: in case $s = 2$ it determines an ordering of the two points of $D_0 \cap D_1$ and if $s = 1$ we let $u_i(0), u_i(\infty)$ be the two pre-image points (ordered by the orientation) of the node of $\overline{D}$ under normalization). Then the argument used in the proof of (5.2) shows that the cross-ratio $(u_i(0), u_i(\infty), u_i^{j-1}, u_i^j)$ and the corresponding expression for $'\overline{Y}$ are equal. If $s = 5$, then this is even true for $j = 1$ and from that it is easily seen that there is a unique isomorphism $\overline{\Phi}$ of $\overline{Y}$ onto $'\overline{Y}$ carrying $\overline{D}_i$ onto $'\overline{D}_i$ and mapping u_i^j to $'u_i^j$ $(i \in \mathbf{Z}/s,\ j = 1, \ldots, p_i)$. If $s < 5$, then it at least follows that there is a unique isomorphism $\overline{\Phi}_0$ of $\overline{D}$ onto $'\overline{D}$ which maps $\overline{D}_i$ to $'\overline{D}_i$, u_i^j to $'u_i^j$ and preserves the orientation. To prove that $\overline{\Phi}_0$ uniquely extends to an isomorphism $\overline{\Phi}$ of $\overline{Y}$ onto $'\overline{Y}$, take for instance the case $s = 1$. Then $\chi(h - e^1 - e^2 - e^3)$ is simply the cross-ratio $(u(0), u(\infty), u^3, u)$, where u denotes the third point of intersection of the line u^1u^2 ($=$ tangent to $\overline{D}$ at u^1 if $u^1 = u^2$) with the cubic $\overline{D}$. Note that the linear system $|u^1 + u^2 + u|$ on $\overline{D}$ "defines" the embedding $\overline{D} \subset \overline{Y} \cong \mathbf{P}^2$. As $\overline{\Phi}_0$ maps this system to the corresponding system on $'\overline{D}$, it follows that $\overline{\Phi}_0$ extends to an isomorphism $\overline{\Phi}$: $\overline{Y} \to {}'\overline{Y}$. This extension is unique, of course.

Clearly, $\overline{\Phi}$ lifts to an isomorphism Φ: $Y \to {}'Y$ with the required properties. Any such isomorphism must map the exceptional configuration (E_i^j) onto $('E_i^j)$ and hence induce an isomorphism $\overline{Y} \to {}'\overline{Y}$ with the same properties as $\overline{\Phi}$. Thus the uniqueness of $\overline{\Phi}$ implies the uniqueness of Φ.

(5.4) Corollary. *Let Y be a smooth rational surface endowed with a negative anti-canonical cycle D of length $s \leq 5$. Any automorphism of Y which preserves the classes of the irreducible components of D and acts trivially on $H^1(D)$ is the identity. In particular, if $D.D < 0$, the group of automorphisms of Y is a subgroup of $\mathbf{Z}/s \times \mathbf{Z}/2$.*

Proof. The first part is immediate from (5.3). The last part then follows from the fact that any automorphism of Y must preserve D (for it is the only effective anti-canonical divisor).

An argument used in the proof of (5.2) shows that, roughly speaking, any character of the root lattice comes from a rational surface with anti-canonical cycle:

(5.5) Proposition. *Let l denote the rank of the root lattice of a rational surface with negative anti-canonical cycle of type $(d_0, \ldots, d_{s-1})$, $s \leq 5$. Then for any $(t_1, \ldots, t_l) \in (\mathbf{C}^*)^l$ there exists a rational surface Y with anti-canonical cycle of type $(d_0, \ldots, d_{s-1})$ and root basis $\alpha_1, \ldots, \alpha_l$ such that $\chi(\alpha_\lambda) = t_\lambda$ $(\lambda = 1, \ldots, l)$.*

Proof. For the sake of definiteness we assume that $s = 1$. Let $\overline{D} \subset \mathbf{P}^2$ be a cubic curve with a node. Let u_0 and u_∞ denote the two points above the node in the normalization of $\overline{D}$ and choose $u_1, \ldots, u_l$ on $\overline{D}$ such that the cross ratios $(u_0, u_\infty u_2, u_1), \ldots, (u_0, u_\infty, u_l, u_{l-1}), (u_0, u_\infty, u_1, u_2 \cap \overline{D}, u_3)$ are $t_1, \ldots, t_l$ respectively. The surface Y obtained from $\mathbf{P}^2$ by successively blowing up $u_1, \ldots, u_l$ is then as required.

The cases $s = 2, \ldots, 5$ are handled in a similar way.

We close this chapter with a brief comment on the nature of the restriction $s \leq 5$. In the first place our results extend without much additional effort to the case $s = 6$, although there is an essential difference with the situation for $s \leq 5$:

the natural equivalent of a root basis turns out to be a generating set of the root lattice whose cardinal is one more than the rank of the root lattice. So the fundamental chamber is a polyhedral rather than a simplicial cone. (This is reflected by the fact that the holomorphic hull of a moduli space which will be defined in Chapter III fails to be smooth for $s = 6$, as is the case when $s \leq 5$.) But for $s = 8$ another difficulty appears: the rational surfaces Y with an anti-canonical cycle D of type $(2, \ldots, 2)$ of length 8 come in two distinct types: those for which $Y - D$ is simply-connected and those for which $Y - D$ has fundamental group $\mathbf{Z}/2$; compare Pinkham [23, (9.2)].

II. CONSTRUCTION OF A FINE MODULI SPACE

The goal of this chapter is to construct a fine moduli space, for the rational surfaces studied in the previous chapter, which is finite over the corresponding coarse moduli space. The main obstacle to this is the fact that the surfaces in question are not in any way polarized, that is, are not endowed with a prescribed embedding, up to projective equivalence, in some $\mathbf{P}^n$. To see what problems may arise, let Y be a smooth surface with a nodal curve $N \subset Y$ and let π: $Y \to Y'$ contract this curve. The semi-universal deformations p: $(\mathcal{Y}, Y) \to (T, *)$ of Y, resp. p': $(\mathcal{Y}', Y') \to (T', *)$ of Y', are related by a "simultaneous resolution"

$$\begin{array}{ccc} (\mathcal{Y}, Y) & \xrightarrow{\tilde{\Phi}} & (\mathcal{Y}', Y') \\ p \downarrow & & \downarrow p' \\ (T, *) & \xrightarrow{\Phi} & (T', *) \end{array}$$

in which Φ has degree two. If σ_N denotes the involution in $(T, *)$ which permutes the fibres of Φ, then Y_t and $Y_{\sigma_N(t)}$ are isomorphic. If there exists a coarse moduli space M for the surfaces of type Y, then the canonical map $(T, *) \to M$ must be constant on the orbits of the group generated by the σ_N. If this group is infinite (which easily happens), then this map is not finite and hence maps a curve through $*$ to a single point. But this is impossible since p is semi-universal. Therefore, the best we can do is to restrict our attention to surfaces Y which have a semi-universal deformation represented by a proper map p: $\mathcal{Y} \to T$, none of whose fibres contains a nodal curve. On the other hand, it is then reasonable to allow these surfaces to have rational double points. This is precisely what we do and we are then able to show that for this special class of rational surfaces with anti-canonical cycle, a fine moduli space with the required properties can be constructed.

The base of this moduli space admits a description completely in terms of the (abstract) root system introduced in Chapter I. To prove this we need a characterization of the Tits cone, which is the subject of Section 1.

1. A characterization of the Tits cone

(1.1) Let Y be a smooth rational surface endowed with a negative anti-canonical cycle $D = D_0 + \cdots + D_{s-1}$ of length $s \leq 5$. An element $\delta \in H_2(Y - D)$ is called a *vanishing cycle* if it maps under the natural projection $H_2(Y - D) \to H_2(Y)$ to a root. Any such δ determines a reflection s_δ in $H_2(Y - D)$ orthogonal to δ: $s_\delta(x) = x + (x.\delta)\delta$. Let Δ denote the set of vanishing cycles and let $\tilde{W}_0$ denote the subgroup of Aut $H_2(Y - D)$ generated by the reflections s_δ, $\delta \in \Delta$. We have an obvious projection $\Delta \to R$ and an obvious epimorphism $\tilde{W}_0 \to W$. We further let Q_0 denote the quotient of Q by its radical. There is an evident inclusion of Q_0 in the dual Q^* of Q and in $H_2(Y - D)^*$.

(1.2) LEMMA. *The group $\tilde{W}_0$ acts trivially on both the image of $H^1(D)$ in $H_2(Y - D)$ and the quotient $H_2(Y - D)^*/Q_0$. The epimorphism $\tilde{W}_0 \to W$ fits in a short (non-canonically split) exact sequence*

$$0 \to Q_0 \to \tilde{W}_0 \to W \to 1$$

in which W acts on Q_0 in the canonical way. The kernel of $\tilde{W}_0 \to W$ is the group of automorphisms of $H_2(Y - D)$ which leave the image of $H^1(D)$ invariant and act trivially on the quotients $H_2(Y - D)/H^1(D) \cong Q$ and $H_2(Y - D)^*/Q_0$.

Proof. If ε is a generator of the image of $H^1(D)$ in $H_2(Y - D)$ and $\delta \in H_2(Y - D)$ a vanishing cycle, then $\varepsilon.\delta = 0$ and so $s_\delta(\varepsilon) = \varepsilon$. This implies that $\tilde{W}_0$ acts trivially on the image of $H^1(D)$. If ξ: $H_2(Y - D) \to \mathbf{Z}$ is a linear form, then $s_\delta^*(\xi) = \xi + \xi(\delta)(\delta.) \equiv \xi(\operatorname{mod} Q_0)$ and so $\tilde{W}_0$ acts also trivially on $H_2(Y - D)^*/Q_0$.

Let U denote the group of automorphisms of $H_2(Y - D)$ which leave the image of $H^1(D)$ invariant and act trivially on Q and $H_2(Y - D)^*/Q_0$. We just proved that $\operatorname{Ker}(\tilde{W}_0 \to W) \subset U$. If $g \in U$, then $g(\varepsilon) = \varepsilon$ for $H_2(Y - D)^*/Q_0$ maps onto $H^1(D)^*$. Since g acts also trivially on Q, it must be of the form $g(x) = x + \phi(x)\varepsilon$ for some linear form ϕ on $H_2(Y - D)$ with $\phi(\varepsilon) = 0$. The fact that g acts trivially on $H_2(Y - D)^*/Q_0$ then implies that $\phi \in Q_0$. We thus obtain an inclusion of U in Q_0 which is easily checked to be a group homomorphism. Next we show that $\operatorname{Ker}(\tilde{W}_0 \to W)$ contains Q_0. If we put $\delta' = \delta + \varepsilon$, then

$$\begin{aligned} s_\delta s_{\delta'}(x) &= x + (x.\delta')\delta' + (x.\delta)\delta + (x.\delta')(\delta'.\delta)\delta \\ &= x + (x.\delta)(\delta + \varepsilon) + (x.\delta)\delta - 2(x.\delta)\delta \\ &= x + (x.\delta)\varepsilon, \end{aligned}$$

which shows that under the inclusion $\mathrm{Ker}(\tilde{W}_0 \to W) \subset Q_0$, $s_\delta s_{\delta'}$ corresponds to the image of δ in Q_0. As the image of Δ in Q_0 generates Q_0, it follows that $\mathrm{Ker}(\tilde{W}_0 \to W)$ maps onto Q_0.

It remains to show that $\tilde{W}_0 \to W$ admits a section. Choose a subset $\tilde{B}$ of Δ which maps bijectively to a (root) basis B of R. Then the subgroup of $\tilde{W}_0$ generated by the s_δ, $\delta \in \tilde{B}$, maps isomorphically to W.

(1.3) We now orient D by the choice of a generator ε of the image of $H^1(D)$ in $H_2(Y - D)$ and let $A \subset H_2(Y - D)^*$ denote the "affine lattice" defined by $\xi(\varepsilon) = 1$. By this we mean that A is a principal homogeneous space for a free finitely generated $\mathbf{Z}$-module (in this case $(H_2(Y - D)/\mathbf{Z}.\varepsilon)^* \cong Q^*$). Note that $\tilde{W}_0$, acting in the contragradient way on $H_2(Y - D)^*$ leaves A invariant by (1.2) and thus acts as an affine transformation group on A. It is easily checked that $\mathrm{Ker}(\tilde{W}_0 \to W) \cong Q_0 \subset Q^*$ is its translation subgroup.

If $A_{\mathbf{C}}$ denotes the complexification of A, then for any $\omega \in A_{\mathbf{C}}$, its imaginary part vanishes on ε and thus defines an $\mathbf{R}$-linear form $\mathrm{Im}(\omega)$ on Q. This defines an $\mathbf{R}$-linear map $\mathrm{Im}\colon A_{\mathbf{C}} \to \mathrm{Hom}(Q, \mathbf{R})$.

If we identify $\mathrm{Pic}(Y)$ with its dual by means of the intersection pairing, then any facet of I is the pre-image of a subset of $\mathrm{Hom}(Q, \mathbf{R})$ under the restriction map $\mathrm{Hom}(\mathrm{Pic}(Y), \mathbf{R}) \to \mathrm{Hom}(Q, \mathbf{R})$. We thus find a convex cone J in $\mathrm{Hom}(Q, \mathbf{R})$ which is partitioned into facets. As opposed to I, J is a proper convex cone; i.e. J does not contain a linear subspace of dim > 0. If $D.D = 0$, then by (I.3.9), $J = \{\eta \in \mathrm{Hom}(Q, \mathbf{R})\colon \eta([D]) > 0\} \cup \{0\}$.

The following proposition is the main result of this section.

(1.4) Proposition. *For any vanishing cycle $\delta \in \Delta$, let $H^\delta_{\mathbf{C}}$ denote the affine hyperplane in $A_{\mathbf{C}}$ defined by $\omega(\delta) = 0$. Then $\{H^\delta_{\mathbf{C}}\}_{\delta \in \Delta}$ is locally finite at $\omega \in A_{\mathbf{C}}$ if and only if $\pm\mathrm{Im}(\omega) \in \mathring{J}$.*

We derive this from

(1.5) Theorem. *Let $\mathfrak{M} \subset \mathrm{Pic}(Y)_{\mathbf{R}}$ denote the union of hyperplanes which are orthogonal to some root. Then $\mathrm{Pic}(Y)_{\mathbf{R}} - \overline{\mathfrak{M}}$ is the union of chambers in I and $-I$.*

Proof that (1.5) implies (1.4). We identify $A_{\mathbf{C}}$ with $A_{\mathbf{R}} \times \mathrm{Hom}(Q, \mathbf{R})$ by taking the real, resp. imaginary, part. Let $H^\alpha_{\mathbf{R}}$ $(\alpha \in R)$ denote the hyperplane in $\mathrm{Hom}(Q, \mathbf{R})$ defined by $\eta(\alpha) = 0$. Then (1.5) implies that $\{H^\alpha_{\mathbf{R}}\}_{\alpha \in R}$ is locally finite in $\mathring{J} \cup -\mathring{J}$ and so $\{H^\delta_{\mathbf{C}}\}_{\delta \in \Delta}$ is locally finite in $A_{\mathbf{R}} \times (\mathring{J} \cup -\mathring{J})$.

Next let $U \times V \subset A_{\mathbf{R}} \times \mathrm{Hom}(Q, \mathbf{R})$ be a non-void open subset with V not contained in $\mathring{J} \cup -\mathring{J}$. We prove that $U \times V$ meets some $H^\Delta_{\mathbf{C}}$. Let $\mathfrak{M}_V$ denote the union of hyperplanes $H^\alpha_{\mathbf{R}}$ which meet V. Assume first $D.D < 0$. Then according to (1.5), $\overline{\mathfrak{M}_V}$ has non-void interior and so $\overline{\mathfrak{M}_V}$ contains an open cone in

$\mathrm{Hom}(Q,\mathbf{R})$. Since $\overline{Q} = Q_0$ is a lattice of maximal rank in $\mathrm{Hom}(Q,\mathbf{R})$, it follows that $\mathfrak{M}_V + Q_0 = \mathrm{Hom}(Q,\mathbf{R})$. This implies that there is a $\delta \in \Delta$ such that $H^{\delta}_{\mathbf{C}} \cap (U \times V) \neq \varnothing$. If $D.D = 0$, then V contains an η with $\eta([D]) = 0$. Since the hyperplane defined by $[D]$ is the only accumulation hyperplane of the $\{H^{\alpha}_{\mathbf{R}}\}_{\alpha\in R'}$, there exists an $\alpha \in R$ such that $\mathfrak{M}_V$ contains $H^{\alpha+k[D]}_{\mathbf{R}}$ for $k = 0, 1, 2, \ldots$. As the corresponding union of affine hyperplanes $\{\xi \in \mathrm{Hom}(Q,\mathbf{R})\colon \xi(\alpha + k[D]) = l$ for some $k \in 0, 1, 2, \ldots,$ and $l \in \mathbf{Z}\}$ is dense in $\mathrm{Hom}(Q,\mathbf{R})$, it follows that there exists a $\delta \in \Delta$ projecting on some $\alpha + k[D]$ such that $H^{\delta}_{\mathbf{C}} \cap (U \times V) \neq \varnothing$.

Proof of (1.5). If $D.D = 0$, then $\mathring{I}$ is the half space in $\mathrm{Pic}(Y)_{\mathbf{R}}$ defined by $[D].x > 0$. As the collection of reflection hyperplanes is infinite, but locally finite in $\mathring{I}$, the boundary of $\mathring{I}$, defined by $[D].x = 0$, must be a limit of reflection hyperplanes and so (1.5) holds in this case.

If $D.D = -1$, then by inspection of the diagrams we see that for any proper subset X of B, W_X is finite or of affine type. Moreover, the last case really occurs. Then according to [4, exerc. 13, 18 of Ch. V §4], W is of finite index in the orthogonal group of Q and $\mathring{I} \cap Q_{\mathbf{R}} = C^+ \cap Q_{\mathbf{R}}$. Choose $X \subset B$ such that W_X is an affine Weyl group and let n denote the unique indivisible $\mathbf{N}$-linear combination of X which is orthogonal to X. The line $\mathbf{R}.n$ is isotropic of course and since W is of finite index in $\mathrm{Aut}(Q)$, its W-orbit is dense in the set of isotropic vectors. Now if $x \in \mathrm{Pic}(Y)_{\mathbf{R}}$ is not in $\mathring{I} \cup (-\mathring{I}) = (C^+ \cup C^-) + F_B$, then there exists an isotropic line l in $Q_{\mathbf{R}}$ which is orthogonal to x. As l is an accumulation line of $\{\mathbf{R}.w(n)\}_{w\in W}$ and as the hyperplane orthogonal to $w(n)$ is a limit of reflection hyperplanes, it follows that $x \in \overline{\mathfrak{M}}$.

We do the remaining cases ($D.D \leq -2$) with induction on the rank of $\mathrm{Pic}(Y)$. If $D.D \leq -2$, then a look at the diagrams in I.2 tells us that B contains a subset X_0 of affine type such that $\mathrm{card}(B - X_0) \geq 2$. Let n denote the unique indivisible $\mathbf{N}$-linear combination of X_0 which is orthogonal to X_0. If U is any open connected nonvoid subset of $\mathrm{Pic}(Y)_{\mathbf{R}} - \overline{\mathfrak{M}}$, then U avoids the hyperplane orthogonal to n, since the latter is a limit of reflection hyperplanes. After replacing U by $-U$ if necessary, we may assume that $x.n > 0$ for all $x \in U$. Our aim is to prove that U is then contained in a chamber of I.

Let $\mathfrak{F}_\infty$ denote the collection of facets of $I - F_B$ which have an infinite irreducible stabilizer. Denote by $\sim$ the equivalence relation on $\mathfrak{F}_\infty$ generated by the incidence relation. We need three simple lemmas. If $F \in \mathfrak{F}_\infty$, let $\mathbf{R}.F$ denote the $\mathbf{R}$-linear span of F.

LEMMA 1. *For any $F \in \mathfrak{F}_\infty$, either $U \subset \mathring{I} + \mathbf{R}.F$ or $U \subset -\mathring{I} + \mathbf{R}.F$.*

Proof. If X is a connected subset of B such that W_X is infinite, then we see from the diagrams of I.2 that X is the root basis corresponding to a rational surface of lower Picard number. By [19, Lemma 1.13], the Tits cone of the root basis X in $\mathrm{Pic}(Y)_{\mathbf{R}}$ equals $I + \mathbf{R}.F_X$. So by our inductive hypothesis either $U \subset \mathring{I} + \mathbf{R}.F_X$ or $U \subset (-\mathring{I}) + \mathbf{R}.F_X$.

LEMMA 2. *If $F \in \mathfrak{F}_\infty$ and $F \sim F_{X_0}$, then $U \subset \mathring{I} + \mathbf{R}.F$.*

Proof. It suffices to show that if $F, F' \in \mathfrak{F}_\infty$ are incident and $U \subset \mathring{I} + \mathbf{R}.F$, then $U \subset \mathring{I} + \mathbf{R}.F'$. By Lemma 1, either $U \subset \mathring{I} + \mathbf{R}.F'$ or $U \subset (-\mathring{I}) + \mathbf{R}.F'$. As $(-\mathring{I}) + \mathbf{R}.F'$ is disjoint from $\mathring{I} + \mathbf{R}.F$ (for F and F' are incident), we must have $U \subset \mathring{I} + \mathbf{R}.F'$.

LEMMA 3. *We have $w(U) \subset \mathbf{R}.F_{X_0} + \mathring{I}$ for all $w \in W$.*

Proof. By Lemma 2 this is the case if $w = 1$. We proceed with induction on the length of w. So it suffices to prove the lemma for $w = s_\alpha$, $\alpha \in B$. If s_α leaves $\mathbf{R}.F_{X_0}$ invariant, there is nothing to prove. If not, then $X_0 \cup \{\alpha\}$ is a connected proper subset of B and so $F_{X_0 \cup \{\alpha\}} \in \mathfrak{F}_\infty$. Since $F_{X_0} > F_{X_0 \cup \{\alpha\}} < s_\alpha(F_{X_0})$, it follows from Lemma 2 that $U \subset \mathbf{R}.s_\alpha(F_{X_0}) + \mathring{I}$.

We now finish the proof of (1.5). As the reflection hyperplanes are locally finite in $\mathring{I}$, the linear form $x \mapsto x.n$ is nonzero on $\mathring{I}$. Since n is zero on F_{X_0} and positive on C, it follows that n is positive on $\mathbf{R}.F_{X_0} + \mathring{I}$. Choose $z \in U$ such that $z.\alpha \in \mathbf{Q}$ for all $\alpha \in B$. By Lemma 3, we then have $w(z).n > 0$ for all $w \in W$. On the other hand, these inner products lie in a finitely generated subgroup of $\mathbf{Q}$. So there is a $w_0 \in W$ such that $w_0(z).n$ is minimal. Let Y_0 denote the set of $\alpha \in B$ orthogonal to X_0. Then W_{Y_0} is finite (check!). As $ww_0(z).n = w_0(z).n$ for any $w \in W_{Y_0}$, we may assume that w_0 is such that $w_0(z).\alpha \geq 0$ for all $\alpha \in Y_0$. We also have $ww_0(z).n = w_0(z).n$ for all $w \in W_{X_0}$. As z is contained in the Tits cone of X and W_{X_0} commutes with W_{Y_0} we may moreover assume that $w_0(z).\alpha \geq 0$ for all $\alpha \in X_0$. We then claim that $w_0(z) \in \bar{C}$. For, if $\alpha \in B - X_0 - Y_0$, then $0 \leq s_\alpha w_0(z).n - w_0(z).n = (w_0(z).\alpha)(\alpha.n)$ and as $\alpha.n > 0$ it follows that $w_0(z).\alpha \geq 0$.

So we have shown that $w_0(U) \cap \bar{C} \neq \varnothing$. As U is connected and disjoint from $\mathfrak{M}$, we then must have $U \subset w_0^{-1}(C)$.

(1.6) *Remark.* Theorem (1.5) holds more generally for any root basis (V, B) with the properties that (i) any $X \subset B$ with W_X infinite contains a subset of affine type and (ii) W leaves invariant a nontrivial quadratic form on the root lattice. The above proof then requires only minor modifications. It seems likely that (1.5) holds for any root basis which is free of hyperbolic rank two summands.

2. Universal deformations

(2.1) From now on we shall allow Y to have rational double points outside its negative anti-canonical cycle $D = D_0 + \cdots + D_{s-1}$. Minimal resolution of these singular points then gives a smooth rational surface with negative anti-canonical cycle as considered before. We are interested in (germs of) *deformations* of Y which preserve D. By definition such a deformation consists of a flat map-germ p: $(\mathcal{Y}, Y_*) \to (S, *)$ between germs of analytic spaces, a relative anti-canonical cycle $\mathcal{D} = \mathcal{D}_0 + \cdots + \mathcal{D}_{s-1}$ on $\mathcal{Y}$ and an isomorphism ι of Y onto Y_* such that $D_j = \iota^{-1}(\mathcal{D}_j)$ $(j \in \mathbf{Z}/s)$. A *morphism* from a deformation (p', ι') to a deformation (p, ι) of (Y, D) will be a pair $(\tilde{\Phi}, \Phi)$ of analytic map-germs which make the diagram below cartesian:

$$\begin{array}{ccc} (\mathcal{Y}', Y'_*) & \overset{\tilde{\Phi}}{\to} & (\mathcal{Y}, Y_*) \\ p' \downarrow & & \downarrow p \\ (S', *) & \overset{\Phi}{\to} & (S, *) \end{array}$$

and satisfy $\tilde{\Phi} \circ \iota' = \iota$. Thus the deformations for (Y, D) form a category. A final object in this category is called a *universal deformation of* (Y, D). The following lemma will ensure that these exist.

Let θ_Y denote the sheaf of analytic vector fields on Y and let $\theta_Y(\log D)$ be the subsheaf of θ_Y of vector fields which preserve D. The dual of $\theta_Y(\log D)$ is denoted $\Omega_Y(\log D)$.

(2.2) LEMMA. *Let Y be a rational surface with negative anti-canonical cycle $D = D_0 + \cdots + D_{s-1}$ having at worst rational double points on $Y - D$ as singularities. Then*

(i) $H^j(\theta_Y(\log D)) = 0$ *for* $j \neq 1$,

(ii) *The cup product*

$$\text{cup: } H^0\big(\Omega_Y^2(\log D)\big) \otimes H^1(\theta_Y(\log D)) \to H^1(\Omega(\log D))$$

is an isomorphism

(iii) *There is a canonical identification of $H^1(\Omega(\log D))$ with the image of the restriction map $H^2(Y; \mathbf{C}) \to H^2(Y - D; \mathbf{C})$ and via this identification,*

$$H^2(Y - D; \mathbf{C}) = H^1(\Omega(\log D)) \oplus H^0\big(\Omega^2(\log D)\big).$$

Proof. Fix a generator ω of $H^0(\Omega_Y^2(\log D))$. Taking the inner product with ω defines an isomorphism of $\theta_Y(\log D)$ onto $\Omega_Y(\log D)$. Now consider the exact sequence

$$0 \to \Omega_Y \to \Omega_Y(\log D) \overset{\text{Res}}{\to} \oplus_i \mathcal{O}_{D_i} \to 0.$$

Its associated cohomology sequence begins with

$$0 \to H^0(\Omega_Y) \to H^0(\Omega_Y(\log D)) \to \oplus_i \mathbf{C}_{D_i} \to H^1(\Omega_Y) \to \cdots.$$

Since Y is rational, $H^0(\Omega_Y) = 0$. The map $H^0(\Omega_Y(\log D)) \to \oplus_i \mathbf{C}_{D_i}$ factors over the differential $H^1(Y - D; \mathbf{C}) \to H^2(Y, Y - D; \mathbf{C}) \cong H_2(D; \mathbf{C}) \cong \oplus_i \mathbf{C}_{D_i}$. As $H_2(D) \to H^2(Y)$ is injective (for any component D_i there exists an exceptional curve E such that $D_j.E = \delta_{ij}$), it follows that the map $H^1(Y - D) \to H_2(D)$ is trivial and hence $H^0(\Omega_Y(\log D)) = 0$. So $H^0(\theta_Y(\log D)) = 0$ and Serre duality gives the fact that $H^2(\theta_Y(\log D)) \cong H^0(\theta_Y(\log D))^* = 0$, too. This proves (i). Since $\Omega_Y^2(\log D) \cong \mathcal{O}_Y$, (ii) is immediate. The first part of (iii) follows from the commutative diagram

$$\begin{array}{ccccccc} 0 \to \oplus_i \mathbf{C}_{D_i} & \to & H^1(\Omega_Y) & \to & H^1(\Omega_Y(\log D)) & \to & 0 \\ \cong\downarrow & & \cong\downarrow & & & & \\ H_2(D; \mathbf{C}) & \to & H^2(Y; \mathbf{C}) & \to & H^2(Y - D; \mathbf{C}) & \to & H_1(D; \mathbf{C}). \end{array}$$

We found in (I.5.1) that the linear form on $H_2(Y - D)$ defined by ω is nonzero on the image of $H^1(D)$. This implies that the last map in the bottom sequence of the above diagram is nonzero on $[\omega] \in H^2(Y - D; \mathbf{C})$. This proves the last part of (iii).

(2.3) Corollary. *With Y and D as in* (2.2) *we have that* $\operatorname{Ext}^n(\Omega_Y(\log D), \mathcal{O}_Y) = 0$ *if $n \neq 1$ and there is a natural exact sequence*

$$0 \to H^1(\theta_Y(\log D)) \to \operatorname{Ext}^1\left(\operatorname{Ext}^1(\Omega_Y^1(\log D), \mathcal{O}_Y\right) \to \oplus_{y \in Y_{\text{sing}}} \operatorname{Ext}^1\left(\Omega_{Y,y}, \mathcal{O}_{Y,y}\right)$$

Proof. Consider the spectral sequence

$$E_2^{pq} = H^p\left(\mathcal{E}xt^q(\Omega_Y(\log D), \mathcal{O}_Y\right) \Rightarrow \operatorname{Ext}^{p+q}(\Omega_Y(\log D), \mathcal{O}_Y)$$

It collapses at the E_2-term since $\mathcal{E}xt^q(\Omega_Y(\log D), \mathcal{O}_Y)$ is the trivial sheaf for $q > 1$ (for rational double points are complete intersections), supported by a finite set if $q = 1$ and equal to $\theta_Y(\log D)$ if $q = 0$. The corollary is immediate from this and (2.2)-i.

(2.4) According to Douady, Grauert and Forster-Knorr (see e.g. [8], [25]) the vanishing of $\operatorname{Ext}^i(\Omega_Y(\log D), \mathcal{O}_Y)$, for $i = 0, 2$, guarantees the existence of a universal deformation π: $(\mathcal{Y}, Y_*) \to (S, *)$ such that $(S, *)$ is a smooth germ whose tangent space T_*S is naturally isomorphic to $\operatorname{Ext}^1(\Omega_Y(\log D), \mathcal{O}_Y)$. The subspace of T_*S defined by $H^1(\theta_Y(\log D))$ (via (2.3)) can be interpreted as the tangent space to the *equisingular stratum* in $(S, *)$, while the fact that T_*S maps *onto* $\oplus_{y \in Y_{\text{sing}}} \operatorname{Ext}^1(\Omega_{Y,y}, \mathcal{O}_{Y,y})$ means that π deforms the singular points of Y in a versal and independent manner. This last observation has an important consequence. If we compare π: $(\mathcal{Y}, Y_*) \to (S, *)$ with a universal deformation $\tilde{\pi}$: $(\tilde{\mathcal{Y}}, \tilde{Y}_*) \to (\tilde{S}, *)$ of a minimal resolution $\tilde{Y}$ of the singularities of Y, then we have a commutative diagram

$$\begin{array}{ccc} (\tilde{\mathcal{Y}}, \tilde{Y}_*) & \xrightarrow{\tilde{\phi}} & (\mathcal{Y}, Y_*) \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ (\tilde{S}, *) & \xrightarrow{\phi} & (S, *) \end{array}$$

which defines a simultaneous resolution in the sense of [5]: ϕ is a finite surjective mapping and $\tilde{\phi}$: $\tilde{Y}_{\tilde{s}} \to Y_{\phi(s)}$ is a minimal resolution of the singularities of $Y_{\phi(s)}$ for all $s \in (S, *)$.

(2.5) If Y is smooth, then we can be more precise about the identification between T_*S and $H^1(\theta_Y(\log D))$. For that purpose, let π: $(\mathcal{Y}, Y_*) \to (S, *)$ be any deformation of the pair (Y, D). Then associated to it is the *Kodaira-Spencer homomorphism*

$$\rho_\pi \colon T_*S \to H^1(\theta_Y(\log D))$$

which is simply the first differential in the cohomology sequence of

$$0 \to \theta_Y(\log D) \to \iota^*\theta_{\mathcal{Y}}(\log \mathcal{D}) \otimes_{\mathcal{O}_{S,*}} \mathbf{C} \to T_*S \to 0.$$

By Kuranishi's criterion, π is universal if and only if ρ_π is an isomorphism. The second part of (2.2) gives a nice geometric interpretation of ρ_π: Choose a proper representative of π, which we continue to denote by π: $\mathcal{Y} \to S$, such that D extends to a relative anti-canonical cycle $\mathcal{D} = \mathcal{D}_0 + \cdots + \mathcal{D}_{s-1}$ on $\mathcal{Y}/S$. We further assume that S is simply connected so that we have natural identifications $H^2(Y_t - D_t) \cong H^2(Y - D)$ for all $t \in S$. Then the *period mapping* P: $S \to \mathbf{P}H^2(Y - D; \mathbf{C})$ is defined by assigning to $t \in S$ the line $H^0(\Omega_{Y_t}(\log D_t))$ in $H^2(Y_t - D_t; \mathbf{C}) \cong H^2(Y - D; \mathbf{C})$. In view of (2.2)-iii, we may think of the differential $dP(*)$ of P at $* \in S$ as taking values in $\mathrm{Hom}(H^0(\Omega_Y^2(\log D)), H^1(\Omega_Y(\log D))$. By the naturality of this construction, $dP(s_0)$ factors over ρ_π. The induced mapping

$$H^1(\theta_Y(\log D)) \to \mathrm{Hom}\big(H^0\big(\Omega_Y^2(\log D)\big), H^1(\Omega_Y(\log D))\big)$$

is then just the adjoint of the cup product pairing of (2.2)-ii; see Deligne's account of Griffiths' work [7]. We conclude that the deformation π of (Y, D) is universal if and only if the associated period map-germ P: $(S, *) \to \mathbf{P}H^2(Y - D; \mathbf{C})$ is a local isomorphism.

An orientation of D defines an affine hyperplane $A_{\mathbf{C}}$ in $H^2(Y - D; \mathbf{C})$; see (1.3). As P maps into the affine part of $\mathbf{P}H^2(Y - D; \mathbf{C})$ parametrized by $A_{\mathbf{C}}$, we often think of P as mapping to $A_{\mathbf{C}}$.

(2.7) Let Y be a rational surface with negative anti-canonical cycle $D = D_0 + \cdots + D_{s-1}$ (possibly with rational double points outside D). We call the pair (Y, D) *good* (I have not been able to find a more descriptive name) if every deformation of the pair (Y, D) admits a representative such that no smooth fibre contains nodal curves outside its anti-canonical cycle. Of course, it is equivalent to this being the case for a universal deformation of (Y, D).

Before we characterize good pairs (Y, D) by means of our period map-germ, we note a simple, but useful consequence of (I.5.2): If Y is smooth, let Δ_Y denote the set of vanishing cycles orthogonal to $H^0(\Omega_Y^2(\log D)) = \mathbf{C}.\omega$. Then (I.5.2) implies that the classes in $H_2(Y - D)$ of the nodal curves in $Y - D$ form a "basis" for Δ_Y. Moreover, the projection $H_2(Y - D) \to Q$ maps Δ_Y isomorphically onto the nodal root system R^n.

(2.8) THEOREM. *Let $\tilde{Y}$ be a rational surface with anti-canonical cycle $\tilde{D} = \tilde{D}_0 + \cdots + \tilde{D}_{s-1}$ of period $s \leq 5$. Then $(\tilde{Y}, \tilde{D})$ is the minimal resolution of a good pair (Y, D) if and only if for a suitable orientation ε of $\tilde{D}$ we have $\mathrm{Im}(\omega) \in \mathring{J}$, where ω denotes the unique element of $H^0(\Omega_Y^2(\log D))$ with $\omega(\varepsilon) = 1$. In that case, Y is obtained by contracting all the nodal curves on $\tilde{Y} - \tilde{D}$.*

Proof. If the pair $(\tilde{Y}, \tilde{D})$ is obtained by minimally resolving the rational double points of a pair (Y, D) (where Y is non-singular at D), let $\tilde{\Delta} \subset H_2(\tilde{Y} - \tilde{D}, \mathbf{Z})$ denote the set of vanishing cycles which map to zero under the projection $\tilde{Y} - \tilde{D} \to Y - D$. Then $\tilde{\Delta}$ is direct sum of root systems of type A, D or E, each summand corresponding to a rational double point of corresponding type on Y. Now fix an orientation ε of $\tilde{D}$, choose a representative of a universal deformation $\pi: \tilde{\mathcal{Y}} \to \tilde{S} \ni *$ and $P: \tilde{S} \to A_{\mathbf{C}}$ be the period map defined above. For any $t \in \tilde{S}$, let $\tilde{B}_t$ denote the set of $\delta \in \tilde{\Delta}$ which can be represented by a nodal curve on $\tilde{Y}_t - \tilde{D}_t$ (as before, we identified $H_2(\tilde{Y}_t - \tilde{D}_t)$ with $H_2(\tilde{Y} - \tilde{D})$) and let Y_t denote the surface (with rational double points) we get by contracting these nodal curves. Then $\{Y_t\}_{t \in \tilde{S}}$ is the set of fibres of (some representative of) a universal deformation of (Y, D). Hence (Y, D) is a good pair if and only if (after shrinking $\tilde{S}$ if necessary) the classes of the nodal curves on $\tilde{Y}_t - \tilde{D}_t$ all belong to $\tilde{\Delta}$. By the observation made in (2.7) above, this is the case if and only if for any $t \in \tilde{S}$ and any vanishing cycle δ with $P(t)(\delta) = 0$, we have $\delta \in \tilde{\Delta}$. As P maps S to a neighbourhood of $P(*)$ in $A_{\mathbf{C}}$, this in turn is equivalent to $\tilde{\Delta} = \Delta_{\tilde{Y}}$ and $\mathcal{H}_{\mathbf{C}}$ being locally finite at $P(*)$. The theorem now follows from (2.4).

Comparing this result with (I.5.5) shows that there is a big set of bad pairs.

3. Good pairs

(3.1) Throughout this section we fix a sequence $d = (d_0, \ldots, d_{s-1})$ of integers ≥ 2 with $s \leq 5$. We shall consider rational surfaces Y with an anti-canonical cycle D whose irreducible components are indexed by $\mathbf{Z}/s$ (in an oriented manner): $D = D_0 + \cdots + D_{s-1}$ such that $-D_i.D_i = d_i$.

If (Y, D) is such a pair with Y smooth and $e_i \in \operatorname{Pic}(Y)$ is such that $e_i.[D_j] = \delta_{ij}$, then the image of the linear form $x \in Q \to x.e_i$ in Q^*/Q_0 depends only on $i \in \mathbf{Z}/s$. Denoting this image by ε_i, we have the following simple lemma:

(3.2) Lemma (i). *The classes* $\varepsilon_0, \ldots, \varepsilon_{s-1}$ *generate* Q^*/Q_0.

(ii). *Any automorphism of Q acting trivially on Q^*/Q_0 extends (uniquely) to an automorphism of* $\operatorname{Pic}(Y)$ *leaving each of the classes* $[D_i]$ *fixed.*

Proof. For any $\xi \in Q^*$ we can choose $x \in \operatorname{Pic}(Y)$ such that $\xi(y) = x.y$ for all $y \in Q$. Then $x \equiv \Sigma_i(x.[D_i])e_i \bmod Q$.

If ψ is an automorphism of Q, then we can always extend ψ (uniquely) to an automorphism $\tilde{\psi}$ of $\operatorname{Pic}(Y)_{\mathbf{Q}}$ which leaves each of the $[D_i]$ fixed. If moreover, ψ acts trivially on Q^*/Q_0, then for any $x \in \operatorname{Pic}(Y)$, $\tilde{\psi}(x) - x + q_0$ is orthogonal to Q for some $q_0 \in Q$. In other words, $\tilde{\psi}(x) - x + q_0 = \Sigma_i x_i[D_i]$ for certain $x_i \in \mathbf{Q}$. Taking the inner product with e_i gives $x_i \in \mathbf{Z}$ and so $\tilde{\psi}(x) \in \operatorname{Pic}(Y)$.

(The second part of (3.2) is a special case of a (well-known) general result about unimodular lattices.)

(3.3) We next fix any pair $(Y^0, D^0 = D^0_0 + \cdots + D^0_{s-1})$ as in Lemma (3.2) (the *reference pair*) and we put $\tilde{Q} := H_2(Y^0 - D^0; \mathbf{Z})$. We let $\varepsilon \in \tilde{Q}$ define an orientation (compatible with the indexing of the irreducible components of D^0 if $s \geq 3$) and we let $Q = \tilde{Q}/\mathbf{Z}\varepsilon$, $Q_0 = Q/(\text{radical of } Q)$, $R \subset Q$, $\varepsilon_0, \ldots, \varepsilon_{s-1} \in Q^*/Q_0$ and $J \subset Q^*_{\mathbf{R}}$ have the same meaning as before. As usual $\Delta \subset \tilde{Q}$ denotes the preimage of R and $\tilde{W}_0$, resp. W, the subgroup of $\operatorname{Aut}(\tilde{Q})$, resp. $\operatorname{Aut}(Q)$, generated by the s_δ, $\delta \in \Delta$, resp. s_α, $\alpha \in R$. These data depend (up to isomorphism) only on the sequence $d = (d_0, \ldots, d_{s-1})$.

If $(Y, D = D_0 + \cdots + D_{s-1})$ is a smooth rational surface with a negative anti-canonical cycle with $-D_i.D_i = d_i$, then a *marking* of the pair (Y, D) is a lattice isomorphism $\tilde{\phi}: \tilde{Q} \to H_2(Y - D; \mathbf{Z})$ such that the following hold:

(i) $\tilde{\phi}(\Delta)$ is the set of vanishing cycles in $H_2(Y - D; \mathbf{Z})$.

(ii) If $\phi: Q \to \operatorname{Pic}(Y)$ denotes the isomorphism induced by $\tilde{\phi}$, then ϕ^* maps the dual Tits cone of (Y, D) onto J.

(iii) If $\xi_i \in \operatorname{Pic}(Y)^*$ is such that $\xi_i([D_j]) = \delta_{ij}$, then $\phi^*(\xi_i) + Q_0 = \varepsilon_i$ ($i \in \mathbf{Z}/s$).

If we put $A := \{f \in \tilde{Q}^* : f(\varepsilon) = 1\}$, then any marked pair $(Y, D; \phi)$ as above determines a *period point* $P(Y, D; \phi)$ of $A_{\mathbf{C}}$, being the unique point of intersection of the complex line $\tilde{\phi}^*(H^0(\Omega^2_Y(\log D)))$ with $A_{\mathbf{C}}$.

Note that a marking determines an orientation of D: namely $\tilde{\phi}^{-1}(\varepsilon)$. If (Y, D) is a minimal resolution of a good pair (this is in fact the only case we are interested in), then we always assume that $\tilde{\phi}^{-1}(\varepsilon)$ is the canonical orientation. So then $\operatorname{Im} P(Y, D; \phi) \in \mathring{J}$ by (2.8). We let Ω_0 denote the set of $\omega \in A_{\mathbf{C}}$ with $\operatorname{Im}(\omega) \in \mathring{J}$. Clearly, Ω_0 is $\tilde{W}_0$-invariant and $\tilde{W}$ acts properly discontinuously on it. Since $\tilde{W}_0$ acts trivially on Q^*/Q_0 by (1.2), $\tilde{W}_0$ permutes the markings.

(3.4) LEMMA. *Let $(\tilde{Y}, \tilde{D})$, resp. $('\tilde{Y}, '\tilde{D})$, be minimal resolutions of good pairs (Y, D), resp. $('Y, 'D)$. Assume they are marked by $\tilde{\phi}$, resp. $'\tilde{\phi}$, such that these markings define the same period point. If G denotes the subgroup of $\tilde{W}$ generated by the reflections s_δ which leave this common period point fixed, then* (i) *via $\tilde{\phi}$, G is identified with the group generated by the reflections orthogonal to the nodal classes of $\tilde{Y} - \tilde{D}$ and* (ii) *there exists a unique $g \in G$ and a unique isomorphism Ψ of $\tilde{Y}$ onto $'\tilde{Y}$ which maps D_i onto $'D_i$ such that $\Psi_* \circ \phi = '\phi \circ g$.*

Proof. Let $\tilde{B} \subset H^2(Y - D; \mathbf{Z})$, resp. $'\tilde{B} \subset H^2('Y - 'D; \mathbf{Z})$, denote the set of classes of nodal curves. Then it follows from (I.5.2) that $\tilde{\phi}^{-1}(\tilde{B})$ and $'\tilde{\phi}^{-1}('\tilde{B})$ are bases for the finite root system $\{\delta \in \Delta \colon \delta . P(Y, D; \tilde{\phi}) = 0\}$. So there exists a $g \in G$ such that $g \circ \tilde{\phi}^{-1}(\tilde{B}) = '\tilde{\phi}^{-1}('\tilde{B})$. Put $\tilde{\psi} := '\tilde{\phi} \circ g \circ \tilde{\phi}^{-1}$. Then $\tilde{\psi}$ maps $H_2(\tilde{Y} - \tilde{D})$ onto $H_2('\tilde{Y} - '\tilde{D})$ and by construction $\tilde{\psi}^*$ maps $H^0(\Omega^2_{'\tilde{Y}}(\log '\tilde{D}))$ onto $H^0(\Omega^2_{\tilde{Y}}(\log \tilde{D}))$. It follows from (3.2) that $\tilde{\psi}$ determines a unique isomorphism $\psi \colon \operatorname{Pic}(Y) \to \operatorname{Pic}('Y)$ which maps $[D_i]$ onto $['D_i]$ such that the obvious commutation relation holds. As ψ preserves the classes of nodal curves, it follows from the Torelli theorem (I.5.3) that ψ is induced by a unique isomorphism $\Psi \colon \tilde{Y} \to '\tilde{Y}$. This proves both parts of the lemma.

(3.5) LEMMA. *If $d_i > 2$ for some i (this corresponds to having a negative definite anti-canonical cycle by* (I.1.3)), *then the $\tilde{W}_0$-stabilizer of any $\omega \in \Omega_0$ is generated by the reflections s_δ $(\delta \in \Delta)$ it contains.*

Proof. Write $\omega = \xi + i\eta$ with $\eta \in \mathring{J}$. Upon replacing ω by a different element in its $\tilde{W}_0$-orbit, we may assume that the (finite) W-stabilizer of η is W_X for some subset $X \subset B$. If σ is any section to $\tilde{W}_0 \to W$, then the $\tilde{W}_0$-stabilizer of ω coincides with the $\sigma(W_X).Q$ stabilizer of ξ. On the other hand it is clear that the latter is contained in the semi-direct product $\sigma(W_X).Q_X$, where Q_X denotes the root lattice $\mathbf{Z}.X$. As $\sigma(W_X).Q_X$ acts as an ordinary affine Weyl group on $A_{\mathbf{R}}$ the stabilizer of ξ is generated by the reflections it contains; see [4, Ch. VI §2].

(3.6) Unfortunately, the analogue of (3.5) in case $d_i = 2$ (all i) does not hold in general; compare [16, Remark (3.6)]. But this can be remedied as follows. Let $\tilde{W}$ denote the semi-direct product of W by Q (with the obvious W-action on Q). Then the kernel of the canonical epimorphism $\tilde{W} \to W$ equals $Z.[D]$ (this is in fact the centre of $\tilde{W}$). According to [19] (see also [16]) we can find an affine $\mathbf{C}$-bundle $\Omega \to \Omega_0$ over Ω_0 endowed with a properly discontinuous $\tilde{W}$-action on its total space such that

(a) the projection $\Omega \to \Omega_0$ is equivariant with respect to the epimorphism $\tilde{W} \to \tilde{W}_0$ and

(b) if $\omega \in \Omega$ lies over $\omega_0 \in \Omega_0$, then the $\tilde{W}$-stabilizer $\tilde{W}_\omega$ of ω maps isomorphically to the subgroup of $\tilde{W}_0$ generated by the reflections which fix ω_0; in particular $\tilde{W}_\omega$ is generated by reflections.

Following a theorem of Chevalley [4], the orbit space $M := \Omega/\tilde{W}$ is then in a natural way an analytic manifold. Note that the **C**-action on Ω induces a $\mathbf{C}^*$-action on M. The orbit space of this $\mathbf{C}^*$-action is naturally isomorphic to the orbit space $M_0 := \Omega_0/\tilde{W}_0$ (which is *not* smooth, in general). The fibres of the projection $\Omega_0 \to \mathcal{H}_+$ (= Poincaré upper half plane) defined by $\omega \to \omega([D])$ are invariant under the action of $\tilde{W}_0$, so we have induced projections $M_0 \to \mathcal{H}_+$ and $M \to \mathcal{H}_+$. We shall write Ω_d, resp. $\tilde{W}_d$, instead of Ω, resp. $\tilde{W}$.

Similarly, we write in the negative definite case, Ω_d, resp. $\tilde{W}_d$, for Ω_0, resp. $\tilde{W}_0$. By (3.5), the $\tilde{W}_d$-stabilizers of points of Ω_d are generated by reflections, so the theorem of Chevalley cited above implies that $M_d := \Omega_d/\tilde{W}_d$ is an analytic manifold. In both cases, we let $D_d \subset M_d$ denote the discriminant of the orbit map $\Omega_d \to M_d$. So D_d is the image of the union of reflection hyperplanes in Ω_d. Our aim is to construct a family of good pairs over M_d.

(3.7) If $\omega \in \Omega_d$ and $\omega_0 \in \Omega_0$ denotes its image in Ω_0 (so $\omega = \omega_0$ in the negative definite case), then by (I.5.5) there exists a marked pair $(\tilde{Y}, \tilde{D}; \phi)$ whose period point is ω_0. Let $\tilde{p}' : (\tilde{\mathcal{Y}}, \tilde{Y}'_*) \to (\tilde{S}, *)$ be a universal deformation of $(\tilde{Y}, \tilde{D})$. Then by (2.5) the induced period germ P: $(\tilde{S}', *) \to (\Omega_0, \omega_0)$ is an isomorphism. This enables us to identify $(\tilde{S}, *)$ with (Ω_0, ω_0). Let $\tilde{p}$: $(\tilde{\mathcal{Y}}, \tilde{Y}_\omega) \to (\Omega, \omega)$ denote the family induced over (Ω, ω). Since such a family is obtained from simultaneous resolution, we have a commutative diagram

$$\begin{array}{ccc} (\tilde{\mathcal{Y}}, \tilde{Y}_\omega) & \to & (\mathcal{Y}, Y) \\ \downarrow \tilde{p} & & \downarrow p \\ (\Omega, \omega) & \to & (\Omega, \omega)/G \end{array}$$

where the upper horizontal arrow contracts the nodal curves outside the anticanonical cycle of every fibre of p (in particular p is a family of good pairs) and G is the group appearing in Lemma (3.4). Since $G \cong \tilde{W}_\omega$, we may identify $(\Omega, \omega)/G$ with the germ (M, z) where z denotes the $\tilde{W}$-orbit of ω. In the negative definite case, p is a universal deformation of Y; in the non-definite case, the restriction of p to any transversal slice through the $\mathbf{C}^*$-orbit of z will be universal.

Thus we get for any $z \in M$ a family of good pairs over a neighbourhood of z such that the period mapping is essentially the projection to M. By (3.4) these families can be glued together to form a global family of good surfaces. We denote this family by $p_d\colon \mathcal{Y}_d \to M_d$. In the non-definite case, the $\mathbf{C}^*$-action on M_d lifts to $\mathcal{Y}_d$ by construction. We further observe that this family is endowed with an oriented relative anti-canonical cycle $\mathcal{D}_d$ whose irreducible components have been indexed by $\mathbf{Z}/s$: $\mathcal{D}_d = \mathcal{D}_{d,0} + \cdots + \mathcal{D}_{d,s-1}$.

However, the family $p_d\colon \mathcal{Y}_d \to M_d$ has more "constant structure". To describe it, let (Y, D) be a pair as usual and let T_D be a closed regular neighbourhood of D in Y. Then the exact cohomology sequence of the pair $(Y - \mathring{T}_D, \partial T_D)$,

$$\begin{array}{ccccccc} \to H_2(Y - \mathring{T}_D) & \to & H_2(Y - \mathring{T}_d, \partial T_D) & \to & H_1(\partial T_D) & \to & H_1(Y_D - \mathring{T}_D) \to \\ \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong \\ H_2(Y - D) & & H_2(Y, D) & & & & H_1(Y - D) = 0 \end{array}$$

shows that $H_1(\partial T_D)$ is naturally isomorphic to the cokernel of the canonical map $H_2(Y - D) \to H_2(Y, D)$, which, in case Y is smooth, is adjoint to the intersection form on $H_2(Y - D)$. We denote this cokernel by $H_1^\infty(Y - D)$. Now if $z \in M$, then $H_1^\infty(Y_z - D_z)$ is naturally isomorphic to $\tilde{Q}^*/Q_0$. Note that the last group contains the elements ε_i $(i \in \mathbf{Z}/s)$.

The family $p_d\colon \mathcal{Y}_d \to M_d$ is universal in some sense:

(3.8) THEOREM. *Let $p\colon (\mathcal{Y}, Y) \to (S, *)$ be a family of good pairs of type $d = (d_0, \ldots, d_{s-1})$. Suppose we are given an isomorphism $f\colon H_1^\infty(Y - D) \to \tilde{Q}^*/Q_0$ which maps the residue class of D_i to ε_i $(i \in \mathbf{Z}/s)$. Then there exists a pair of morphisms $(\tilde{\Phi}, \Phi)$ which makes the diagram*

$$\begin{array}{ccc} (\mathcal{Y}, Y) & \xrightarrow{\tilde{\Phi}} & \mathcal{Y}_d \\ p \downarrow & & \downarrow p_d \\ (S, *) & \xrightarrow{\Phi} & M_d \end{array}$$

cartesian such that $\tilde{\Phi}$ preserves the relative anti-canonical cycles and $\tilde{\Phi}_*\colon Y \to Y_{\Phi(*)}$ induces f. Moreover, this pair is unique for these properties in the negative case. In the non-definite case, it is unique up to the $\mathbf{C}^*$-action (that is, up to a morphism $(S, *) \to \mathbf{C}^*$).

Proof. Let $t \in S$. If $\tilde{Y} \to Y$ is a minimal resolution of the rational double points on Y, then $H_1^\infty(\tilde{Y} - \tilde{D}) \cong H_1^\infty(Y - D)$. Lift $f\colon H_1^\infty(\tilde{Y} - \tilde{D}) \to \tilde{Q}^*/Q_0$ to a marking $\phi\colon H_2(\tilde{Y} - \tilde{D}) \to \tilde{Q}$. If $z \in M$ denotes the $\tilde{W}$-orbit of an $\omega \in \Omega$ which represents the period point (in the non-definite case: a lift thereof) of $(\tilde{Y}, \tilde{D}; \phi)$, then by construction, we have an isomorphism $\tilde{\Phi}_*\colon Y \to Y_z$ inducing ϕ. Since $(\mathcal{Y}_d, Y_z) \to (M_d, z)$ contains a universal deformation, $\tilde{\Phi}_*$ extends to a pair $(\tilde{\Phi}, \Phi)$ with the required properties. The (partial) uniqueness follows from (1.2) and (3.2).

(3.9) COROLLARY. *If $p\colon (\mathcal{Y}, \mathcal{D}) \to S$ is a family of rational surfaces, with anti-canonical cycle, which blow down to good pairs of type $d = (d_0, \ldots, d_{s-1})$ with at least one $d_i > 2$, $s \leq 5$ and $f\colon H_1^\infty(\mathcal{Y} - \mathcal{D}) \to \tilde{Q}^*/Q_0$ is a homomorphism whose restriction to any $H_1^\infty(Y_t - D_t)$, $t \in T$, is an isomorphism, then there exists a unique pair of morphisms $(\tilde{\Phi}, \Phi)$ such that the square*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\tilde{\Phi}} & \mathcal{Y}_d \\ p\downarrow & & \downarrow p_d \\ S & \xrightarrow{\Phi} & M_d \end{array}$$

is a simultaneous resolution, and $\tilde{\Phi}$ respects the relative anti-canonical cycles and induces f.

(3.10) It is clear from our construction that $D_d \subset M_d$ parametrizes the singular fibres. We can describe precisely which configurations of rational double points may occur. For if $z \in M_d$ is represented by $\omega \in \Omega$, then the rational double points on Y_z are in a natural bijective correspondence with the irreducible summands of the $\tilde{W}$-stabilizer $\tilde{W}_\omega$ and this correspondence preserves the type. We may choose $\omega = \xi + i\eta$ such that the (finite) W-stabilizer of η equals W_X for some subset $X \subset B$. As was also pointed out in the proof of (3.5), the $\tilde{W}$-stabilizer of ω is then the stabilizer of ξ in the *affine Weyl group* $W_X.Q_X$. Such stabilizers are classified as follows (see [4]): Let $\hat{X}$ denote the graph obtained from X by completing each of its connected components (in the sense of [4, Ch. VI §4.3)]. Then the stabilizers of $W_X.Q_X$ (as acting on the affine space $A_{\mathbf{R}}$) are in bijective correspondence with the subsets Y of $\hat{X}$ which meet each connected component of $\hat{X}$ in a proper subset.

III. THE HOLOMORPHIC HULL OF THE MODULI SPACE

When Kodaira carried out his classification of analytic surfaces the only known surfaces without non-constant meromorphic functions were certain complex tori and certain K3 surfaces. However, in recent years M. Inoue has constructed an abundance of new examples [10], [11], [12], which all belong to Kodaira's class VII_0. The deformation theory of the surfaces constructed by

Inoue has also received attention [23], one reason being that some yield further new examples of surfaces of class VII_0. In this chapter we investigate the deformation theory of some of Inoue's surfaces, but only *after* we have contracted a number of curves of them. The resulting surfaces have a unique singular point (either simply-elliptic or a cusp) and come naturally with a negative anti-canonical cycle on it. We prove that a semi-universal deformation of such a surface (respecting the anti-canonical cycle) induces a semi-universal deformation of the singular point. This makes these surfaces very well suited for the study of the deformation theory of simply-elliptic singularities and cusps. (For that purpose related surfaces have already been exploited by J. Wahl [28].) We show that if the generic fibre of the semi-universal deformation is smooth, then it is a rational surface canonically endowed with a negative anti-canonical cycle, which is good in the sense of Chapter II. We thus obtain families of rational surfaces degenerating to a surface without meromorphic functions (we have been told that this contradicts a conjecture asserting a semi-continuity property of algebraic dimension).

We use these singular Inoue surfaces to extend the fine moduli space $p_d\colon \mathcal{Y}_d \to M_d$ of good pairs, constructed in the previous chapter, over the holomorphic hull $\hat{M}_d$ of M_d. This holomorphic hull is a Stein manifold which has been explicitly described in a more general context in [19]. We thus obtain valuable information pertaining to cusp singularities of multiplicity smaller than six (e.g. monodromy group, discriminant etc.).

1. Parabolic Inoue surfaces

(1.1) We begin by briefly recalling how these surfaces are obtained (for more details, consult Inoue [11] and Oda [23, §14]). Fix a positive integer s and a complex number τ with $\mathrm{Im}(\tau) > 0$. Then the transformation σ_τ^s of the torus $\mathbf{C}^2/\mathbf{Z}^2$ obtained by composing the linear transformation $(z_1, z_2) \mapsto (z_1, sz_1 + z_2)$ with the affine translation over $(s\tau, 0)$ generates an infinite cyclic group which acts freely and properly discontinuously on $\mathbf{C}^2/\mathbf{Z}^2$. The orbit manifold (denoted Y') admits a natural analytic compactification $Y' \subset Y$. This adds to Y' a point c (which will be singular on Y) and a cycle $D = D_0 + \cdots + D_{s-1}$ of length s of rational curves canonically oriented and indexed by $\mathbf{Z}/s$ (near which Y will be smooth). A neighbourhood basis of c in Y is for instance the family of subsets

$$U_V := \{c\} \cup \{x + iy \in \mathbf{C}^2/\mathbf{Z}^2 \colon y \in V\}/(\sigma_\tau^s),$$

where V runs over the non-void convex $\sigma_{\mathrm{Im}(\tau)}^s$-invariant subsets of $\mathbf{R}^2$. The analytic structure of Y near c is determined by calling a continuous complex function U_V analytic if its restriction to $U_V - \{c\}$ is. This makes c a simply-elliptic singularity of degree s, which means that a minimal resolution $\pi: \tilde{Y} \to Y$ of this singularity has a smooth elliptic curve $\tilde{C}$ with self-intersection $-s$ as exceptional divisor. We will make frequent use of the fact that such a singularity is Gorenstein. The curve $\tilde{C}$ is isomorphic to $\mathbf{C}/(\mathbf{Z} + s\tau\mathbf{Z})$. The topology and analytic structure near D is best described with the help of toroidal embeddings for which we refer to Oda, *loc. cit.* All irreducible components of D have self-intersection number -2 (unless $s = 1$: then $D.D = 0$). The surface Y has no non-constant meromorphic functions and any curve on Y is a union of irreducible components of D.

The translations $(z_1, z_2) \mapsto (z_1, z_2 + \lambda)$ in $\mathbf{C}^2/\mathbf{Z}^2$ $(\lambda \in \mathbf{C}/\mathbf{Z})$ commute with σ_τ^s and thus determine $\mathbf{C}^* \cong \mathbf{C}/\mathbf{Z}$-actions on Y and on $\tilde{Y}$. This action leaves each D_i invariant, but not pointwise. Let ξ, resp. $\tilde{\xi}$, denote the vector field on Y, resp. $\tilde{Y}$, corresponding to $d/d\lambda$. Then ξ, resp. $\tilde{\xi}$, is a section of $\theta_Y(\log D)$, resp. $\theta_{\tilde{Y}}(\log \tilde{D})$, where $\tilde{D}$ stands for $\pi^{-1}(D)$. Similarly, the 2-form $dz_1 dz_2$ on $\mathbf{C}^2/\mathbf{Z}^2$ is invariant under σ_τ^s and thus determines a holomorphic 2-form on Y'. This 2-form extends to a meromorphic section ω of the dualizing sheaf $\mathcal{K}_Y$ of Y, whose divisor equals $-D$. The form ω lifts to a meromorphic section $\tilde{\omega}$ of $\Omega^2_{\tilde{Y}}$ with $\operatorname{div}(\tilde{\omega}) = -\tilde{C} - \tilde{D}$. In particular, $\mathcal{K}_Y \cong \mathcal{O}_Y(-D)$ and $\Omega^2_{\tilde{Y}} \cong (-\tilde{C} - \tilde{D})$.

We shall need further the following homological data concerning $\tilde{Y}$ (all of which can be found in [11] or [23]; note however that in these references only the case $s = 1$ is considered. It is a simple matter to check that their arguments remain valid for $s > 1$. Alternatively, we could note that the surfaces we consider are just s-fold cyclic coverings of the surfaces with $s = 1$): $H^2(\tilde{Y}; \mathbf{Q})$ is generated by the classes $[\tilde{D}_0], \dots, [\tilde{D}_{s-1}]$ and $[\tilde{C}]$, subject to the single relation $[\tilde{D}] = \Sigma_i [\tilde{D}_i] = 0$. So $c_1^2(\tilde{Y}) = (-\tilde{C} - \tilde{D})^2 = -s$. We also use the fact that $H^2(\mathcal{O}_{\tilde{Y}}) \cong H^1(\tilde{Y}; \mathbf{C})$ is of dim 1 and that $H^0(\Omega_{\tilde{Y}}) = 0$.

(1.2) PROPOSITION. *We have* $H^i(\theta_Y(\log D)) = 0$ *for* $i > 0$ *and* $H^0(\theta_Y(\log D))$ *is generated by* ξ.

This has important consequences for the deformation theory of the pair (Y, D):

(1.3) COROLLARY. *The natural map* $\operatorname{Ext}^i(\Omega_Y(\log D), \mathcal{O}_Y) \to \operatorname{Ext}^i(\Omega_{Y,c}, \mathcal{O}_{Y,c})$ *is an isomorphism for* $i > 0$, *while* $\operatorname{Ext}^0(\Omega_Y(\log D), \mathcal{O}_Y) = H^0(\theta_Y(\log D))$ *is generated by* ξ. *If* $p: (\mathcal{Y}, Y_*) \to (S, *)$, $\iota: Y \overset{\cong}{\to} Y_*$ *is a deformation of* Y, *semi-universal for the condition that the cycle* D *be preserved, then the germ of* p *at* $\iota(c)$ *defines a semi-universal deformation of the simply-elliptic singularity* (Y, c).

Proof that (1.2) implies (1.3). The first clause is immediate from the fact that in the Leray spectral sequence

$$E_2^{pq} = H^p\big(\mathcal{E}xt^q(\Omega_Y(\log D), \mathcal{O}_Y\big) \Rightarrow \operatorname{Ext}^{p+q}\big(\Omega_Y(\log D), \mathcal{O}_Y\big),$$

we have $E_2^{pq} = 0$ for $p > 0$ (by (1.2) and by noting that $E_2^{0q} = \operatorname{Ext}^q(\Omega_{Y,c}, \mathcal{O}_{Y,c})$ for $q > 0$. The second part is a formal consequence of this; see for instance [26].

The proof of (1.2) requires some preparation:

(1.4) LEMMA. *$H^0(\theta_{\tilde{Y}}(\log \tilde{D}))$ is generated by $\tilde{\xi}$.*

Proof. Since $\tilde{C}$ is contractible, we have $H^0(\theta_{\tilde{Y}}(\log \tilde{D})) = H^0(\theta_{\tilde{Y}}(\log(\tilde{C} + \tilde{D})))$. Taking inner product with $\tilde{\omega}$ makes the latter isomorphic to $H^0(\Omega_{\tilde{Y}}(\log(\tilde{C} + \tilde{D})))$. Now consider the exact sequence

$$0 \to \Omega_{\tilde{Y}} \to \Omega_{\tilde{Y}}\big(\log(\tilde{C} + \tilde{D})\big) \xrightarrow{\text{Res}} \mathcal{O}_{\tilde{C}} \oplus \oplus_i \mathcal{O}_{\tilde{D}_i} \to 0$$

(where Res denotes the residue map). Since $H^0(\Omega_{\tilde{Y}}) = 0$,

$$\text{Res}^0 : H^0\Big(\Omega^1_{\tilde{Y}}\big(\log(\tilde{C} + \tilde{D})\big)\Big) \to \mathbf{C}_{\tilde{C}} \oplus \oplus_i \mathbf{C}_{\tilde{D}_i}$$

is injective. But this map has a topological interpretation: it factors over the Leray coboundary δ, which appears in the exact sequence

$$\cdots \to H^1(\tilde{Y} - \tilde{C} - \tilde{D}) \xrightarrow{\delta} H_2(\tilde{C} + \tilde{D}) \xrightarrow{k} H^2(\tilde{Y}) \to \cdots$$

(with complex coefficients). Following (1.1), k is surjective and has one-dimensional kernel generated by $[\tilde{D}]$. This implies that $h^0(\Omega_{\tilde{Y}}(\log(\tilde{C} + \tilde{D})) \leq 1$. So $\tilde{\xi}$ generates $H^0(\theta_{\tilde{Y}}(\log(\tilde{C} + \tilde{D})))$.

(1.5) COROLLARY. *We have $h^1(\theta_{\tilde{Y}}(\log \tilde{D})) = 1 + s$.*

Proof. We have an exact sequence

$$0 \to \theta_{\tilde{Y}}\big(\log \tilde{D}\big) \to \theta_{\tilde{Y}} \to \oplus_i \nu_{D_i} \to 0,$$

where ν_{D_i} denotes the sheaf of sections of the normal bundle of D_i. By Riemann-Roch [2], we have

$$\chi\big(\theta_{\tilde{Y}}(\log \tilde{D})\big) = \chi(\theta_{\tilde{Y}}) - \sum_i \chi(\nu_{\tilde{D}_i})$$

$$= \big(\tfrac{3}{2}c_1(\tilde{Y})^2 - \tfrac{1}{2}c_2(\tilde{Y})\big) - \sum_i (D_i.D_i + 1)$$

$$= \big(\tfrac{3}{2}.(-s) - \tfrac{1}{2}.s\big) - (-s) = -s.$$

By Serre duality, the dual of $H^2(\theta_{\tilde{Y}}(\log\tilde{D}))$ is isomorphic to $H^0(\Omega_{\tilde{Y}}(\log\tilde{D})\cdot(-\tilde{C}-\tilde{D}))$, which is contained in $H^0(\Omega_{\tilde{Y}})$ and is therefore trivial. So

$$h^1\left(\theta_{\tilde{Y}}(\log\tilde{D})\right) = -\chi\left(\theta_{\tilde{Y}}(\log\tilde{D})\right) + h^0\left(\theta_{\tilde{Y}}(\log\tilde{D})\right) = s+1.$$

Proof of (1.2). There is a Leray spectral sequence

$$E_2^{pq} = H^p\left(R^q\pi_*\theta_{\tilde{Y}}(\log\tilde{D})\right) \Rightarrow H^{p+q}\left(\theta_{\tilde{Y}}(\log\tilde{D})\right).$$

Since $R^q\pi_*(\theta_{\tilde{Y}}(\log\tilde{D})) = 0$ for $q > 1$ and $\pi_*\theta_{\tilde{Y}}(\log\tilde{D}) = \theta_Y(\log D)$ by [6], it collapses to a long exact sequence

(*)

$$0 \to H^1(\theta(\log D)) \to H^1\left(\theta_{\tilde{Y}}(\log\tilde{D})\right) \to H^0\left(R^1\pi_*\theta_{\tilde{Y}}\right) \to H^2(\theta_Y(\log D)) \to \cdots.$$

By Serre duality, the dual of $H^2(\theta_Y(\log D))$ is isomorphic to $H^0(\Omega_Y(\log D)(-D))$. In view of the inclusions $H^0(\Omega_Y(\log D)(-D)) \subset H^0(\Omega_Y) \subset H^0(\Omega_{\tilde{Y}}) = 0$, it follows that $H^2(\theta_Y(\log D)) = 0$. We further note that $H^0(R^1\pi_*\theta_{\tilde{Y}}) = \varprojlim H^1(\theta_Y \otimes_{\mathcal{O}_Y}\mathcal{O}_Y/\mathcal{J}_C^k)$, where $\mathcal{J}_C$ is the ideal sheaf in $\mathcal{O}_Y$ defining C. Clearly the epimorphism $\theta_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_Y/\mathcal{J}_C^k \to \theta_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_C$ induces an epimorphism on H^1 (for the sheaves are supported by C). From the exact cohomology sequence of

$$0 \to \theta_C \to \theta_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_C \to \nu_C \to 0$$

(ν_C denoting the sheaf of sections of the normal bundle of C) and the fact that $C.C = -s$ (<0) we find a short exact sequence

$$0 \to H^1(\theta_C) \to H^1(\theta_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_C) \to H^1(\nu_C) \to 0.$$

As $h^1(\theta_C) = h^1(\mathcal{O}_C) = 1$ and $h^1(\nu_C) = -\chi(\nu_C) = s$, by Riemann-Roch it follows that $h^1(\theta_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_C) = 1+s$. Hence $h^0(R^1\pi_*\theta_{\tilde{Y}}) \geq 1+s$. It then follows from (1.5) and the exactness of (*) that $H^1(\theta_Y(\log D)) = 0$.

Finally, the fact that $\pi_*\theta_{\tilde{Y}} = \theta_Y$ implies that π_* maps $H^0(\theta_{\tilde{Y}}(\log\tilde{D}))$ isomorphically onto $H^0(\theta_Y(\log D))$ and so by (1.4), the latter is generated by ξ.

(1.6) It is a general fact [24] that a $\mathbf{C}^*$-action on a compact variety naturally extends (that is, up to non-canonical isomorphism) to its semi-universal deformation. So the semi-universal deformation in Corollary (1.3) admits a representative (which we still denote by p: $(\mathcal{Y}, \mathcal{D}) \to S$) to which the $\mathbf{C}^*$-action on $Y \cong Y_*$ extends. The $\mathbf{C}^*$-action on S is actually inherited from the $\mathbf{C}^*$-action on the singularity (Y, c). Following Pinkham [22] the orbit structure in S is as follows: the fixed point set of the $\mathbf{C}^*$-action is a smooth curve $S_0 \subset S$ passing through $*$ which parametrizes simply-elliptic singularities (in the present global case: parabolic Inoue surfaces) with varying $\tau \in \mathcal{H}_+$ and there is a $\mathbf{C}^*$-equivariant retraction $\rho\colon S \to S_0$ such that for all $t \in S$, $e^{2\pi i\lambda}.t \to \rho(t)$ as $\mathrm{Im}(\lambda) \to \infty$. The fibres over $S - S_0$ have at worst rational double points.

It is clear from (1.3) that p will have smooth fibres if and only if the simply-elliptic singularity (Y, c) is smoothable. According to Pinkham [22] this is the case if and only if its degree s is ≤ 8 (for $s > 8$ we have $S = S_0$). Our interest in the parabolic Inoue surfaces is explained by the following.

(1.7) PROPOSITION. *If $t \in S - S_0$, then the pair (Y_t, D_t) is a rational surface with an anti-canonical cycle of type* $(2,\ldots,2)$.

First, we prove

(1.8) LEMMA. *The parabolic Inoue surface Y is regular.*

Proof. The Leray spectral sequence $H^p(R^q\pi_*\mathcal{O}_{\tilde{Y}}) \to H^{p+q}(\mathcal{O}_{\tilde{Y}})$ collapses to a long exact sequence

$$0 \to H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_{\tilde{Y}}) \to H^0\big(R^1\pi_*\mathcal{O}_{\tilde{Y}}\big) \to H^2(\mathcal{O}_Y) \to \cdots.$$

By Serre duality, $H^2(\mathcal{O}_Y) = H^0(\mathcal{O}_Y(-D))^* = 0$. The 4^{th} term equals $(R^1\pi_*\mathcal{O}_{\tilde{Y}})_c$ and is known to be one-dimensional [15]. As $H^1(\mathcal{O}_{\tilde{Y}}) \cong \mathbf{C}$, it follows that $H^1(\mathcal{O}_Y) = 0$.

Proof of (1.7). If $t \in S - S_0$, then Y_t has at worst rational double points. Let $\tilde{Y}_t \to Y_t$ resolve these minimally. Then the pre-image $\tilde{D}_t$ of D_t is anti-canonical and so $\tilde{Y}_t$ has Kodaira dimension $-\infty$. On the other hand, $H^1(\mathcal{O}_{\tilde{Y}_t}) = H^1(\mathcal{O}_{Y_t}) = 0$, because of $H^1(\mathcal{O}_Y) = 0$ and the semi-continuity of $t \mapsto h^1(\mathcal{O}_{Y_t})$. Following the classification, $\tilde{Y}_t$ must then be rational.

(1.8) *Remark.* It is not hard to show that there are no rational surfaces with a negative nondefinite anti-canonical cycle of length > 9. Those for which the length equals 9 are parametrized by a j-invariant. Compare this with conjecture (2.11) below).

2. Hyperbolic Inoue surfaces

(2.1) As with the parabolic Inoue surfaces, we recall their construction (the basic references are Inoue [12] and Oda [23, §15]). Let $\sigma \in \mathrm{SL}_2\mathbf{Z}$ have trace > 2. Then σ has two distinct *positive* eigenvalues so the line in $\mathbf{R}^2$ spanned by an eigenvector bounds an open σ-invariant half plane H in $\mathbf{R}^2$. Let $\mathfrak{D}$ denote the $z = x + iy \in \mathbf{C}^2/\mathbf{Z}^2$ with $y \in H$. Clearly, σ determines an automorphism of $\mathfrak{D}$ and the finite cyclic group generated by σ acts freely and properly discontinuously on $\mathfrak{D}$. The orbit manifold denoted by X' admits a natural analytic compactification $X' \subset X$ (of Baily-Borel type) which is obtained by adding two points c, d to X'. To describe X, let Q_c and Q_d denote the two disjoint open σ-invariant sectors contained in H (so they are bounded by σ-invariant half lines). Then a neighbourhood basis of c in X is given by the family of subsets

$$U_V = \{c\} \cup \{x + iy \in \mathbf{C}^2/\mathbf{Z}^2 \colon y \in V\}/\sigma$$

where V runs over the non-void convex σ-invariant subsets of Q_c. The analytic structure of X at c is determined by calling a continuous complex-valued function on U_V analytic if its restriction to $U_V - \{c\}$ is. The topology and analytic structure at d are defined in the same way. As a topological space, X is homeomorphic to the suspension of a torus bundle over a circle (with its characteristic map $\mathbf{R}^2/\mathbf{Z}^2 \to \mathbf{R}^2/\mathbf{Z}^2$ induced by σ), with c and d corresponding to the two suspension points. So the germs (X, c) and (X, d) are homeomorphic. Nevertheless, they can be very different analytically.

Let $\pi \colon \tilde{Y} \to X$ minimally resolve the singularities c and d. Then the exceptional curves $\pi^{-1}(c)$, resp. $\pi^{-1}(d)$, are negative definite cycles of rational curves $\tilde{C} = \tilde{C}_0 + \cdots + \tilde{C}_{r-1}$, resp. $\tilde{D} = \tilde{D}_0 + \cdots + \tilde{D}_{s-1}$. The above construction endows both cycles with an orientation (for if l_+ denotes the unique σ-invariant half line in H, then $il_+ \subset \mathfrak{D}$ maps to a circle in X', which, when lifted to $\tilde{Y}$ is homologous to a generator of $H_1(\tilde{C})$, resp. $H_1(\tilde{D})$). Any normal surface singularity whose minimal resolution has a (necessarily negative definite) cycle of rational curves as exceptional divisor is called a *cusp*. By the above discussion, both (X, c) and (X, d) are cusps. Any cusp is obtained in this way (that is, up to isomorphism), for if $(d_0, \ldots, d_{s-1})$ is a sequence of integers ≥ 2 with at least one $d_i > 2$, then

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & d_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & d_{s-1} \end{pmatrix}$$

with the "correct" choice of H leading to a $\tilde{Y}$ with $\tilde{D}$ of type $(d_0, \ldots, d_{s-1})$. The isomorphism type of a cusp is completely determined by its cycle of self-intersection numbers. There is an interesting duality between the cusps (X, c) and (X, d) which has been independently observed by Nakamura [20] and is implicit in a paper of Hirzebruch and Zagier [9]. For instance, $r + C.C = -s - D.D$; emb.dim$(X, c) = \max\{s, 3\}$ and emb.dim$(X, d) = \max\{r, 3\}$; mult$(X, c) = \max(s, 2)$ and mult$(X, d) = \max\{r, 2\}$.

The 2-form $dz_1 dz_2$ on $\mathfrak{D}$ is clearly σ-invariant and thus determines a 2-form on X'. This 2-form extends to a generating section of the dualizing sheaf $\mathcal{K}_X$ of X. The lift of ω to $\tilde{Y}$ is a meromorphic 2-form $\tilde{\omega}$ with $\operatorname{div}(\tilde{\omega}) = -\tilde{C} - \tilde{D}$. So $c_1(\tilde{Y})^2 = \tilde{C}.\tilde{C} + \tilde{D}.\tilde{D} = -r - s$. Any curve on $\tilde{Y}$ is contained in $\tilde{C} \cup \tilde{D}$ (so X

contains no curves at all) and the classes of $\tilde{C}_0, \ldots, \tilde{C}_{r-1}, \tilde{D}_0, \ldots, \tilde{D}_{s-1}$ form a basis of $H_2(\tilde{Y}, \mathbf{Q})$. We also need that $H^0(\Omega_{\tilde{Y}}) = 0$ and that $H^1(\mathcal{O}_{\tilde{Y}}) \cong H^1(\tilde{Y}, \mathbf{C})$ is of dimension one. The surface $\tilde{Y}$ has no non-constant meromorphic functions.

(2.2) PROPOSITION. *The cohomology groups $H^i(\theta_X)$ vanish for all i.*

(2.3) COROLLARY. *The canonical maps*

$$\operatorname{Ext}^i(\Omega_X, \mathcal{O}_X) \to \operatorname{Ext}^i(\Omega^1_{X,c}, \mathcal{O}_{X,c}) \oplus \operatorname{Ext}^i(\Omega^1_{X,d}, \mathcal{O}_{X,d})$$

are isomorphisms for all i. The analytic space X admits a universal deformation $\iota: X \xrightarrow{\cong} X_*, q: (\mathcal{X}, X_*) \to (T, *)$ and the germ of q at $\{c, d\}$ defines a semi-universal deformation of $(X, \{c, d\})$.

This corollary is obtained from (2.2) in the same way as (1.3) is derived from (1.2) and the set-up of the proof of (2.2) is similar to the one of (1.2). We begin with a

(2.4) LEMMA. $H^0(\theta_{\tilde{Y}}) = 0$.

Proof. Since $\tilde{C}$ is contractible, we have $H^0(\theta_{\tilde{Y}}(\log \tilde{D})) = H^0(\theta_{\tilde{Y}}(\log(\tilde{C} + \tilde{D})))$. Taking inner product with $\tilde{\omega}$ makes the latter isomorphic to $H^0(\Omega_{\tilde{Y}}(\log(\tilde{C} + \tilde{D})))$. Now consider the exact sequence

$$0 \to \Omega_{\tilde{Y}} \to \Omega_{\tilde{Y}}\big(\log(\tilde{C} + \tilde{D})\big) \xrightarrow{\text{Res}} \mathcal{O}_{\tilde{C}} \oplus \oplus_i \mathcal{O}_{\tilde{D}_i} \to 0$$

Since $H^0(\Omega_{\tilde{Y}}) = 0$, this defines an injective map

$$\text{Res}^0 : H^0\Big(\Omega_{\tilde{Y}}\big(\log(\tilde{C} + \tilde{D})\big)\Big) \to \oplus_i \mathbf{C}_{\tilde{C}_i} \oplus \oplus_i \mathbf{C}_{\tilde{D}_i}.$$

This map factors over the Leray coboundary δ in the exact sequence

$$\cdots \to H^1(\tilde{Y} - \tilde{C} - \tilde{D}; \mathbf{C}) \xrightarrow{\delta} H_2(\tilde{C} + \tilde{D}; \mathbf{C}) \xrightarrow{k} H^2(\tilde{Y}; \mathbf{C}) \to \cdots.$$

As k is an isomorphism (by (2.1)), δ (and hence Res^0) is the zero map. It follows that $H^0(\Omega_{\tilde{Y}}(\log(\tilde{C} + \tilde{D}))) = 0$.

(2.5) COROLLARY. *We have $h^1(\theta_{\tilde{Y}}) = 2(r + s)$.*

Proof. By Serre duality, the dual of $H^2(\theta_{\tilde{Y}})$ is isomorphic to $H^0(\Omega_{\tilde{Y}}(-\tilde{C} - \tilde{D}))$, which is contained in $H^0(\theta_{\tilde{Y}})$ and hence trivial. By Riemann-Roch

$$\chi(\theta_{\tilde{Y}}) = \tfrac{3}{2}c_1(\tilde{Y})^2 - \tfrac{1}{2}c_2(\tilde{Y}) = \tfrac{3}{2}(-r - s) - \tfrac{1}{2}(r + s) = -2(r + s)$$

and so $h^1(\theta_{\tilde{Y}}) = 2(r + s)$.

Proof of (2.2). As the dualizing sheaf of X is trivial, Serre duality provides an isomorphism between $H^2(\theta_X)$ and $H^0(\Omega_X)$. Since $H^0(\Omega_X) \subset H^0(\Omega_{\tilde{Y}}) = 0$, it follows that $H^2(\theta_X) = 0$. So it remains to show that $H^1(\theta_X) = 0$. For this, we consider the Leray spectral sequence $E_2^{pq} = H^p(R^q\tilde{\pi}_*\theta_{\tilde{Y}}) \to H^{p+q}(\theta_{\tilde{Y}})$ which for the same reasons as in (1.2) collapses to

$$(*) \qquad 0 \to H^1(\theta_X) \to H^1(\theta_{\tilde{Y}}) \to H^0\big(R^1\pi_*\theta_{\tilde{Y}}\big) \to \underset{\substack{\| \\ 0}}{H^2(\theta_X)} \to \cdots.$$

We have

$$H^0\big(R^1\tilde{\pi}_*\theta_{\tilde{Y}}\big) = \varprojlim H^1\big(\theta_{\tilde{Y}} \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}/\mathcal{I}_{\tilde{C}}^k\big) \oplus \varprojlim H^1\big(\theta_{\tilde{Y}} \otimes_{\mathcal{O}_{\tilde{D}}} \mathcal{O}_{\tilde{D}}/\mathcal{I}_{\tilde{D}}^k\big).$$

The first summand maps onto $\oplus_i H^1(\nu_{C_i})$ (compare the proof of (1.2)) whose dimension is $\Sigma_i(-C_i.C_i - 1) = r - C.C$. Similarly, the second summand has $\dim \geq s - D.D$ and so

$$h^0\big(R^1\pi_*\theta_{\tilde{Y}}\big) \geq r + s - C.C - D.D = 2(r+s) = h^1(\Omega_{\tilde{Y}})$$

(by (2.5)).

It then follows from the exactness of $(*)$ above that $H^1(\theta_X) = 0$.

The resolution $\tilde{\pi}$: $\tilde{Y} \to X$ factors over π': $Y \to X$ which only resolves the singular point $d \in X$. The exceptional divisor of π is an (oriented, negative definite) anti-canonical cycle $D = D_0 + \cdots + D_{s-1}$. We call the surface Y a *hyperbolic Inoue surface*. We continue to denote the pre-image of c (the unique singular point of Y) by c.

(2.6) LEMMA. *The surfaces X and Y are regular.*

Proof. Let $\pi: \tilde{Y} \to Y$ denote the obvious map. We have an exact sequence

$$0 \to H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_{\tilde{Y}}) \to H^0\big(R^1\pi_*\mathcal{O}_{\tilde{Y}}\big) \to H^2(\mathcal{O}_{\tilde{Y}}) \to \cdots.$$

By Serre duality, $H^2(\mathcal{O}_{\tilde{Y}}) \cong H^0(\mathcal{O}_{\tilde{Y}}(-C-D)) = 0$. Now $H^0(R^1\pi_*\mathcal{O}_{\tilde{Y}}) = (R^1\pi_*\mathcal{O}_{\tilde{Y}})_c$ is of dimension 1 (by [15]) and has therefore the same dimension as $H^1(\mathcal{O}_{\tilde{y}})$. This implies that $H^1(\mathcal{O}_Y) = 0$. From the exactness of $0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_Y) \to \cdots$, it follows that $H^1(\mathcal{O}_X) = 0$, also.

(2.7) Let $q: \mathcal{X} \to T$ be a proper representative of the universal deformation described in Corollary (2.3). The fibres X_t of q which preserve the singularity (X, d) are parametrized by a subvariety S of T which passes through the base point $* \in T$. (As in (II.2.4), the Zariski tangent space of $(T, *)$ is naturally isomorphic to $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$; under the isomorphism of (2.3) the Zariski tangent space of $(S, *)$ corresponds to the first summand in $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$.) The bundle q admits an analytic section over S which assigns to $t \in S$ the singular point of X_t corresponding to (X, d). Minimal resolution along this section yields a proper family $p: \mathcal{Y} \to S$ endowed with a relative anti-canonical cycle $\mathcal{D} = \mathcal{D}_0 + \cdots + \mathcal{D}_{s-1}$. We clearly have a natural isomorphism ι: $(Y, D) \to (Y_*, D_*)$. It follows from (2.3) that the pair $(\iota, p$: $(\mathcal{Y}, Y) \to S)$ is a universal deformation of (Y, D) (where D is preserved) and that the germ of p at $\iota(c)$ is a semi-universal deformation of the singularity (Y, c). Note that p, resp. q, has smooth fibres near Y_*, resp. X_*, if and only if (Y, c), resp. (X, c) *and* (X, d) are smoothable.

(2.8) PROPOSITION. *Let S_f, resp. T_f, denote the set of $t \in S$, resp. T, such that Y_t, resp. X_t, has only rational double points as singularities. If T is small enough, then:*

(i) *For any $t \in T_f$, X_t is after minimal resolution of its singularities a minimal K3 surface and the germ $q: (\mathcal{X}, X_t) \to (T, t)$ is a universal deformation of X_t. Moreover T is smooth of dimension 20 at t.*

(ii) *For any $t \in S_f$, Y_t is a rational surface endowed with a negative definite anti-canonical cycle D_t and $p: (\mathcal{Y}, Y_t) \to (S, t)$ is a universal deformation of the pair (Y_t, D_t).*

Proof. If $t \in T_f$, then X_t is regular by (2.6) (and the semi-continuity of $h^1(\mathcal{O}_{\tilde{X}_t})$). Moreover, X_t has trivial canonical sheaf. As these properties are shared by $\tilde{X}_t$, $\tilde{X}_t$ is a minimal $K3$ surface by definition. The germ $q: (\mathcal{X}, X_t) \to (T_f, t)$ will be universal since by (2.3) X possesses no global vector fields [26]. As is well-known, the base of such a deformation is smooth of dim 20; see for instance [14].

If $t \in S_f$, then Y_t and its minimal resolution $\tilde{Y}_t$ are regular by (2.6). The pre-image of $D_t \subset Y_t$ is a negative definite anti-canonical cycle on $\tilde{Y}_t$ and so $\tilde{Y}_t$ is a rational surface by Kodaira's classification. The universality property of $p: (\mathcal{Y}, Y_t) \to (S_f, t)$ follows as before.

(2.9) Proposition (2.8) has a number of surprising implications for the deformation theory of the singularities (X, c) and (X, d). Let us first assume that both are smoothable. Then it follows from (2.8)-i that the sum of their Milnor numbers is 22 (the proof uses the fact that the second Betti number of a K3 surface is 22 and is based on a Mayer-Vietoris argument; we omit the details). Also, the sum of the dimensions of a smoothing component of (X, c) and a smoothing component of (X, d) is 20 (being the dimension of the moduli space of $K3$ surfaces). Most of this was also observed by Nakamura [20].

If we merely suppose that (X, c) is smoothable, then (2.8) guarantees the existence of a smooth rational surface Y' with anti-canonical cycle $D' =$

$D'_0 + \cdots + D'_{s-1}$ with $D'_i.D'_i = D_i.D_i$. For this it is important to note that the cycle of numbers $(D_i.D_i : i \in \mathbf{Z}/s)$ can be calculated from the cycle $(C_j.C_j : j \in \mathbf{Z}/r)$ by a very simple and amusing algorithm, due to Hirzebruch-Zagier [9, 2.3]: Define a cycle of natural numbers $(a_0, \ldots, a_{2k-1})$ by

$$(-C_0.C_0, \ldots, -C_{r-1}.C_{r-1}) \equiv \left(a_0 + 2, \underbrace{2, \ldots, 2}_{a_1 - 1}, a_2 + 2, \underbrace{2, \ldots, 2}_{a_3 - 1}, \ldots, \underbrace{2, \ldots, 2}_{a_{2k-1} - 1}\right)$$

(as oriented cycles). Then

$$(-D_0.D_0, \ldots, -D_{s-1}.D_{s-1}) \equiv \left(\underbrace{2, \ldots, 2}_{a_0 - 1}, a_1 + 2, \underbrace{2, \ldots, 2}_{a_2 - 1}, \ldots, a_{2k-1} + 2\right).$$

The base germ of a universal deformation of the pair (Y', D') is by (II.2.4) smooth, of $\dim b_2(Y') - s$, where $b_2(Y')$ denotes the second Betti number of Y'. If we use the Noether formula and the algorithm of Hirzebruch-Zagier, it is easily shown that $b_2(Y') - s = 10 - D.D - s = 10 + C.C + r$. Thus we find the following:

(2.10) (Wahl [28]). If (X, c) is smoothable, then all the smoothing components of (X, c) have the same dimension $10 + C.C + r$.

In particular, $10 + C.C + r > 0$. We conjecture a converse to (2.10):

(2.11) *Conjecture.* Assume $10 + C.C + r > 0$. Then the singularity (X, c) is smoothable if and only if there exists a smooth rational surface with anti-canonical cycle of type $(d_0, \ldots, d_{s-1})$ (this cycle of numbers being the dual of the cycle $(-C_0.C_0, \ldots, -C_{r-1}.C_{r-1})$ in the sense of Hirzebruch-Zagier).

The interest of the conjecture lies in the fact that the existence of a rational surface with anti-canonical cycle of given type $(d_0, \ldots, d_{s-1})$ is a decidable problem, for such a surface is obtained by successive blowing up of points on a cubic curve with a node in the projective plane. Wahl [28] showed that there exist smoothable cusps with arbitrarily large $-C.C$.

The remainder of this section is devoted to a proof of the following

(2.12) PROPOSITION. *With the notation and hypothesis of* (2.8), *the minimal resolution of any fibre* (Y_t, D_t) *with* $t \in S_f$ *is the minimal resolution of a good pair in the sense of* (II.2).

In Section 3 we shall prove the stronger assertion that the pair (Y_t, D_t) is good. For the proof of Proposition (2.12) we need a lemma which is probably well-known.

(2.13) LEMMA. *Let N be a compact manifold with boundary ∂N and η a closed k-form on N such that the cohomology class $[\eta|\partial N]$ is zero. Then, given a neighbourhood U of ∂N in N there exists a closed k-form η' which is zero near ∂N and such that $\eta - \eta'$ is zero on $N - U$. In particular, if* $\dim N = 2k$, *then there exists a constant c_0 which depends in a continuous way on $\eta|U$ such that $|\int_N \eta \wedge \eta - [\eta].[\eta]| < c_0$, where $[\eta]$ denotes the cohomology class of η in $H^k(N, \mathbf{R})$.*

Proof. Choose a collar neighbourhood of ∂N in U, that is, a diffeomorphism h of $\partial N \times [0,1)$ onto a neighbourhood U' of ∂N in U such that $h(x,0) = x$. We identify U' with $\partial N \times [0,1)$ by means of h. Then we can write $\eta|_{U'} = \tilde{\alpha} + dt \wedge \beta$ where $\tilde{\alpha}$, resp. β, is a k, resp. $(k-1)$, -form on U' which does not involve dt. As $\eta|\partial N \times \{t\}$ is cohomologous to zero for all $t \in [0,1)$ we can write according to (a parametrized version of) de Rham's theorem $\tilde{\alpha} = d'\alpha$, where d' denotes exterior derivation with respect to the ∂N parameters. Let ρ: $[0,1) \to [0,1]$ be a C^∞-function which is zero on $[0, \frac{1}{3}]$ and one on $[\frac{2}{3}, 1)$. Then the k-form η' on N defined by

$$\eta' = \begin{cases} \eta \quad \text{on} \quad N - U' \\ \rho\eta - \dfrac{d\rho}{dt} dt \wedge \alpha \quad \text{on} \quad U \end{cases}$$

is as required; clearly $\operatorname{supp}(\eta - \eta') \subset U$, $\operatorname{supp}(\eta') \subset N - \partial N$ and we have

$$\begin{aligned} d(\eta'|U') &= \rho\, d\eta + \frac{d\rho}{dt} dt \wedge \eta - \frac{d\rho}{dt} dt \wedge d\alpha \\ &= \frac{d\rho}{dt}(dt \wedge \eta - dt \wedge \tilde{\alpha}) \\ &= \frac{d\rho}{dt} dt \wedge dt \wedge \beta = 0. \end{aligned}$$

As $N - U' \subset N$ is a homotopy equivalence, we then have $[\eta'] = [\eta]$. So the last part of the lemma follows from the fact that $\int_N \eta' \wedge \eta' = [\eta'].[\eta']$.

Proof of (2.12). On each fibre (Y_t, D_t) we have a meromorphic 2-form characterized by the properties that $\operatorname{div}(\omega_t) = -D_t$ and ω_t takes the value 1 on the canonical generator of $H^1(D_t) \cong H^1(D)$.

Let $\mathcal{C} \subset \mathcal{Y}$ denote the set $x \in \mathcal{Y}$ where p has a singularity which is not a rational double point. Note that ω_t, resp. $\operatorname{Im} \omega_t$, defines a cohomology class $[\omega_t] \in H^2(Y_t - \mathcal{C}_t - D_t; \mathbf{C})$, resp. $\operatorname{Im}[\omega_t] \in H^2(Y_t - \mathcal{C}_t; \mathbf{R})$. If $t \in T_f$ (i.e. if $\mathcal{C}_t = \varnothing$) then $\operatorname{Im}[\omega_t].\operatorname{Im}[\omega_t] > 0$ implies according to (II.2.8) and (I.3.7) that the minimal resolution of (Y_t, D_t) is the minimal resolution of a good pair. So it suffices to show that $\mathrm{Im}[\omega_t].\mathrm{Im}[\omega_t] > 0$ for all $t \in T$ (T sufficiently small). Write $\omega_t = \xi_t + i\eta_t$ where ξ_t and η_t are real 2-forms on $Y_t - \mathcal{C}_t - D_t$ (or rather on its minimal resolution). Then $\omega_t \wedge \omega_t = 0$ implies that $\xi_t \wedge \xi_t = \eta_t \wedge \eta_t$ and so $\eta_t \wedge \eta_t = \frac{1}{2}\omega_t \wedge \bar{\omega}_t$ is a volume form. Let V resp. U be a regular neighbourhood of $\mathcal{C}$ resp. $\mathcal{D}$ in $\mathcal{Y}$ such that $\bar{U} \cap \bar{V} \neq \varnothing$ and define a continuous function f on T by

$$f(t) = \int_{Y_t - U_t - V_t} \omega_t \wedge \bar{\omega}_t.$$

The 2-form ω_* on $Y_* - \mathcal{C}_* - D_* \cong Y - \{c\} - D$ lifts under the resolution $\tilde{Y} \to Y$ to a meromorphic 2-form with a simple pole along $\tilde{C}$. So by taking V_* sufficiently small, we can make $f(*)$ arbitrarily large. Following Lemma (2.13) there exists a constant c_0 independent of t such that

$$\left| \int_{Y_t - U_t} \eta_t \wedge \eta_t - [\eta_t].[\eta_t] \right| < c_0$$

for all $t \in T_f$. Since $\int_{Y_t - U_t} \eta_t \wedge \eta_t \geq \frac{1}{2} f(t) > c_0$ if V_* and T are sufficiently small, it follows that $[\eta_t].[\eta_t] > 0$.

3. Extension of the moduli space

(3.1) For any sequence $d = (d_0, \ldots, d_{s-1})$ of integers ≥ 2 with $s \leq 5$, we constructed a family of good pairs $p_d : (\mathcal{Y}_d, \mathcal{D}_d) \to M_d$ of rational surfaces with anti-canonical cycle of type d, for which the base M_d was given as an orbit manifold $\Omega_d / \tilde{W}$. In [19] we investigated such quotients $\Omega_d / \tilde{W}$ in detail. Among other things we found that the global holomorphic functions separate the points of M_d (even infinitesimally) and that the holomorphic hull $\hat{M}_d$ of M_d is smooth. Moreover, we obtained a precise description of $\hat{M}_d$. For instance, $\hat{M}_d - M_d$ is partitioned into *strata* $M_d(X)$, where X runs over the so-called *special subsets* of the intersection graph associated to d. In the present situation, X is special if and only if X is connected and the Weyl group W_X is infinite. The stratum $M_d(X)$ is a Zariski-constructible submanifold of $\hat{M}_d$ of codimension card(X). For each pair X, Y of special subsets of B we either have $M_d(Y) \subset \bar{M}_d(X)$ or $M_d(Y) \cap \bar{M}_d(X) = \varnothing$, this according to whether or not $X \subset Y$. Our aim is to extend the family $\mathcal{Y}_d \to M_d$ over $\hat{M}_d$ such that $\hat{M}_d - M_d$ parametrizes blown-up Inoue surfaces. As a first step we show that any reasonable extension is in fact unique.

Let $\hat{U}$ be an open subset of $\hat{M}_d$ and denote $U := \hat{U} \cap M_d$. An *extension* of the restriction $(\mathcal{Y}_d, \mathcal{D}_d)$ to U is by definition a proper flat family of surfaces $p : (\mathcal{Y}, \mathcal{D}) \to \hat{U}$ endowed with a relative anti-canonical cycle $\mathcal{D}$ together with an isomorphism of families $\Phi : (\mathcal{Y}, \mathcal{D})|_U \to (\mathcal{Y}_d, \mathcal{D}_d)|_U$.

(3.2) PROPOSITION. *Two such extensions over $\hat{U}$ are naturally isomorphic.*

Proof. Let $p^i:(\mathcal{Y}^i, \mathcal{D}^i) \to \hat{U}, \Phi^i:(\mathcal{Y}^i, \mathcal{D}^i)|_U \to (\mathcal{Y}_d, \mathcal{D}_d)|_U$ be two such extensions. We put $\Psi := (\Phi^2)^{-1} \circ \Phi^1$ and let $\Gamma \subset \mathcal{Y}^1 \times_{\hat{U}} \mathcal{Y}^2$ denote the graph of Ψ. If $V \subset \mathcal{Y}^1 \times_{\hat{U}} \mathcal{Y}^2$ denotes the pre-image of $\hat{U} - U$, then V is analytic in $\mathcal{Y}^1 \times_{\hat{U}} \mathcal{Y}^2$ and Γ is analytic in $(\mathcal{Y}^1 \times_{\hat{U}} \mathcal{Y}^2) - V$. As $\operatorname{codim} \Gamma = 4$ and $\operatorname{codim} V = \min\{\operatorname{card}(X): X \text{ special}\} \geq 5$, it follows from the Remmert-Stein theorem that the closure $\bar{\Gamma}$ of Γ in $\mathcal{Y}^1 \times_{\hat{U}} \mathcal{Y}^2$ is analytic in the last space.

Without loss of generality we may assume $\hat{U}$ so small that $-\mathcal{D}^i$ is the divisor of a meromorphic volume form ω^i on $\mathcal{Y}^i$ and that there exists a holomorphic nowhere vanishing volume form ζ on $\hat{U}$ such that the family of 2-forms on the fibres of p^i defined by the "quotient" ω^i/ζ corresponds under Φ^i to the family of 2-forms on the fibres of p^i defined in Chapter II. Then obviously, $\Psi^*(\omega^2) = \omega^1$. Let $\tilde{\omega}^i$ denote the pull-back on $\bar{\Gamma}$ of ω. As $\tilde{\omega}^1$ and $\tilde{\omega}^2$ coincide on Γ, we have $\tilde{\omega}^1 = \tilde{\omega}^2$. Clearly the zero set of $\tilde{\omega}^i$ is the exceptional set of the projection $\bar{\Gamma} \to \mathcal{Y}^i$. Hence the exceptional sets of $\bar{\Gamma} \to \mathcal{Y}^1$ and $\bar{\Gamma} \to \mathcal{Y}^2$ are equal, which can only hold if they are vacuous. In other words, $\bar{\Gamma}$ is the graph of an isomorphism $\mathcal{Y}^1 \to \mathcal{Y}^2$, which clearly extends Ψ.

We first consider the parabolic case: $d_0 = \cdots = d_{s-1} = 2$ (and as before, $s \leq 5$). Then the corresponding intersection graph B is of affine type, so $\hat{M}_d - M_d = M(B)$ is isomorphic to Poincaré's upper half plane.

(3.3) THEOREM. *In this case, the family p_d: $(\mathcal{Y}_d, \mathcal{D}_d) \to M_d$ constructed in Chapter II extends to a proper flat family $\hat{p}_d$: $(\hat{\mathcal{Y}}_d, \hat{\mathcal{D}}_d) \to \hat{M}_d$ such that $M_d(B)$ parametrizes parabolic Inoue surfaces with an anti-canonical cycle of type d.*

Proof. This follows from Corollary (1.5) and the discussion (1.6), in combination with [22].

(3.4) For what follows we need more precise knowledge about the way $\hat{M}_d$ is constructed. A facet F of the Tits cone J is called *special* if it is a W-translate of a facet F_X for some special $X \subset B$. Let $\Omega_d(F)$ denote the image of Ω_d in the affine space $A_{\mathbf{C}}/Q_{X,\mathbf{C}}$ (note that $Q_{X,\mathbf{C}} = \mathbf{C}.X$ acts on $A_{\mathbf{C}}$ as a translation group) and denote by $\hat{\Omega}_d$ the disjoint union of the $\Omega(F)$, F special. The action of $\tilde{W}$ on Ω_d extends naturally to one on $\hat{\Omega}_d$. A topology on $\hat{\Omega}_d$ is defined by postulating that for any $\omega \in \Omega(F)$ the family of subsets U of $\hat{\Omega}_d$ satisfying (i) $\omega \in \Omega$, (ii) U is $\tilde{W}_\omega$-invariant and (iii) for any special facet F' of J with $F' > F$ the intersection $U \cap \Omega(F')$ is a nonvoid convex subset of $\Omega(F')$, is a neighbourhood basis of ω. Then the orbit space $\hat{M}_d := \hat{\Omega}_d/\tilde{W}$ is locally compact Hausdorff, the analytic structure on $M_d \subset \hat{M}_d$ extends uniquely to a normal analytic structure on $\hat{M}_d$, making it a Stein manifold. The added material $M_{d,\infty} = \hat{M}_d - M_d$ is a subvariety of $\hat{M}_d$ of codim > 1. The stratum $M_d(X)$ is just the image of $\Omega_d(F_X)$ in $\hat{M}_d$. The closure $\hat{D}_d$ of D_d in $\hat{M}_d$ is an analytic hypersurface which contains $M_{d,\infty}$.

The stratification $\{M_d(X)\}$ of $\hat{M}_d$ is refined by the stratification by orbit type. This is the partition of $\hat{M}_d$ induced by the partition of $\hat{\Omega}_d$ defined by the assignment $\omega \in \hat{\Omega} \mapsto \tilde{W}$-conjugacy class of $\tilde{W}_\omega$. The stabilizer of any $\omega \in \Omega(F_X)$ has been determined in [19, 2.17]: it is of the form $(W_X.Q_X) \times W'_\omega$, where W'_ω is the stabilizer of ω in $W_{X^*}.(Q/Q_X)$ acting on $\Omega(F_X)$. Here X^* stands for the set of vertices in B which are not connected with X. The group W'_ω is finite and its conjugacy class in $W_{X^*}.(Q/Q_X)$ is determined as in [19], *loc. cit.* We will use the fact that any $\tilde{W}_\omega$ leaves a nontrivial sublattice of Q pointwise fixed, unless $\{\omega\} = M(B)$.

(3.5) Let (Y, D) be a hyperbolic Inoue surface with anti-canonical cycle D of type $(d_0, \ldots, d_{s-1})$, $s \leq 5$, and let $p: (\mathcal{Y}, \mathcal{D}) \to S$ be a proper representative of a universal deformation of (Y, D) where we take for S a contractible Stein space. Let $\Delta \in S$ denote the set of $t \in S$ for which Y_t is singular. This is a subvariety of S which is everywhere of codimension one. We let S_∞ denote the set of $t \in S$ for which Y_t has a singular point which is *not* a rational double point and as before we put $S_f = S - S_\infty$. It is known that S_∞ is a subvariety of S of codim > 2. We also write Δ_f for $\Delta - \Delta_\infty$. For a natural class of choices for S (which we do not wish to describe here) the homotopy type of $S - \Delta$ is independent of S (that is, for two such S there is a natural homotopy equivalence between the corresponding complements). We always assume that S belongs to this class. Let $\tilde{\Gamma}$ resp. Γ denote the monodromy group of the local system $t \in S - \Delta \mapsto H_2(Y_t - D_t)$, resp. $H_2(Y_t)$. It is a general fact that any sublattice of $H_2(Y_t - D_t)$, resp. $H_2(Y_t)$, which is left pointwise fixed by $\tilde{\Gamma}$, resp. Γ, is contained in its radical, resp. the image of $H_2(D_t)$ in $H_2(Y_t)$.

According to (III.2.12), we may shrink S so that for all $t \in S_f$, (Y_t, D_t) blows down to a good pair. The local system over S_f defined by $t \mapsto H_1^\infty(Y_t - D_t)$ is evidently trivial and so (II.3.9) implies that we have a simultaneous resolution

$$\begin{array}{ccc} \mathcal{Y}_f & \xrightarrow{\tilde{\Phi}_f} & \mathcal{Y}_d \\ p_f \downarrow & & \downarrow p_d \\ S_f & \xrightarrow[\Phi_f]{} & \hat{M}_d. \end{array}$$

Since S and $\hat{M}_d$ are Stein spaces and codim $S_\infty > 1$, the map Φ_f extends to $\Phi: S \to \hat{M}_d$.

(3.6) THEOREM. *We have* $\{\Phi(*)\} = M(B)$ *and after possible shrinking of* S, Φ *will map* S *isomorphically onto an open neighbourhood* $\hat{U}$ *of* $M(B)$ *and* $\tilde{\Phi}_f$ *will map* $\mathcal{Y}_f$ *isomorphically onto* $p_d^{-1}(\hat{U} \cap M_d)$ (*in other words*, p *defines an extension of* $(\mathcal{Y}_d, \mathcal{D}_d)|_{\hat{U}\cap M_d}$ *over* $\hat{U}$.

(3.7) COROLLARY. *The pair* $(\tilde{\Phi}_f, \Phi_f)$ *determines an isomorphism of* Γ *onto* W *and of* $\tilde{\Gamma}$ *onto* $\tilde{W}$.

Proof. Clearly, $(\tilde{\Phi}_f, \Phi_f)$ identifies Γ with the monodromy group of $\mathcal{Y}_d|_{(M_d - D_d)\cap \hat{U}}$. But the latter is the monodromy group of $\mathcal{Y}_d|_{M_d - D_d}$ which is just W. Similarly, $\tilde{\Gamma}$ is identified with $\tilde{W}$.

For $s \leq 3$, this description of $\tilde{\Gamma}$ as an extension of a Coxeter group by a lattice had been earlier obtained by Gabrielov [1]. In case $s = 4$, Corollary (3.7) answers a question asked by Karras [13] affirmatively.

We prove (3.6) in three steps.

Step 1. After possible shrinking of S, $(\Phi_f, \tilde{\Phi}_f)$ will be cartesian.

Proof. Let $N \subset \mathcal{Y}_f$ denote the exceptional locus of the map

$$\mathcal{Y}_f \to S_f \times_{M_d} \mathcal{Y}_d$$

induced by $(\Phi_f, \tilde{\Phi}_f)$. So $N_t := N \cap Y_t$ is the (unique) curve which after separate contraction of its connected components yields a good pair. Let $\bar{N}$ be the closure of N in $\mathcal{Y}$. Since $\dim \mathcal{Y}_\infty = 2 + \dim S_\infty < \dim S = \dim N$, the Remmert-Stein theorem implies that $\bar{N}$ is analytic in $\mathcal{Y}$.

We claim that $\bar{N}$ does not meet Y_*. Otherwise, we can find an analytic curve E in $p(\bar{N})$ such that $E \cap S_\infty = \{*\}$. Let N_E be an irreducible component of $p^{-1}(E) \cap N$ not contained in Y_∞. Then N_E will meet Y_* in a curve. This curve will not be contained in D_* as can be seen by "simultaneous contraction" of the anti-canonical cycles D_t, $t \in E$. But according to Inoue [12] all curves on Y_* are contained in D_*. As this is a contradiction we have that $\bar{N} \cap Y_* = \varnothing$ indeed.

So by shrinking S, we will have $N = \varnothing$. This implies that $(\Phi_f, \tilde{\Phi}_f)$ is cartesian.

Step 2. Φ is a local isomorphism at $*$.

Proof. As universality is an open property [A], we may shrink S so that p defines a universal deformation of its fibres. By step 1, we may also suppose that $(\Phi_f, \tilde{\Phi}_f)$ is cartesian. As p_d too is a universal deformation of its fibres, Φ_f must be a local isomorphism. Then Φ is a local isomorphism because codim $S_\infty > 1$.

Step 3. $\{\Phi(*)\} = M(B)$.

Proof. It follows from steps 1 and 2 that the monodromy group Γ will correspond under $(\Phi_f, \tilde{\Phi}_f)$ to a subgroup of $\tilde{W}$ whose conjugacy class is described by the orbit type represented by $\Phi(*)$ in $\hat{\Omega}_d/\tilde{W}$. If $\omega \in \hat{\Omega}_d$ maps to $\Phi(*)$, then $\tilde{W}_\omega$ fixes a nontrivial sublattice of Q unless $\{\omega\} = \Omega(F_B) = \Omega(\{0\})$. As Γ does not fix a nontrivial sublattice in Image$(H_2(Y_t - D_t) \to H_2(Y_t))$, it follows that $\{\omega\} = \Omega(\{0\})$ and so $\{\Phi(*)\} = M(B)$.

Clearly steps 1, 2 and 3 imply the theorem.

(3.8) Next we are going to extend $p_d: (\mathcal{Y}_d, \mathcal{D}_d) \to M_d$ over a neighbourhood of $M(X)$ in $\hat{M}_d$, where X is any special subset of B. Many of the arguments will be similar to those previously used, which is why they are presented in a somewhat less detailed manner.

Let (Y_t, D_t) be a smooth fibre of p_d, $B \subset \mathrm{Pic}(Y_t)$ a root basis and X a special subset of B. The root basis B determines an exceptional configuration (E_i^j). Let (Y_t'', D_t'') be obtained from (Y_t, D_t) by contracting all exceptional curves E_i^j whose class is orthogonal to X. Then D_t'' may contain exceptional curves. By subsequent contraction of these we end up with a pair (Y_t', D_t') with D_t' negative. The special subset X projects to a root basis in $\mathrm{Pic}(Y_t')$. Let U_t' be a regular neighbourhood of D_t' in Y_t' and let U_t denote its pre-image in Y_t. Then the natural map $H_2(U_t - D_t; \mathbf{Q}) \to H_2(Y_t - D_t; \mathbf{Q})$ is injective and its image is just the subspace of $H_2(Y_t - D_t; \mathbf{Q})$ orthogonal to X. So $\Omega_d(X)$ may be identified with an open subset of the hyperplane of $H^2(U_t - D_t, \mathbf{C})$ defined by $\varepsilon = 1$.

In a sense we are going to reverse this procedure. Let (Y', D') be a parabolic or hyperbolic Inoue surface such that D' has the same cycle d' of absolute self-intersection numbers as D_t'. After successive blowing up of singular points on (the strict transform of) D' we get a pair (Y'', D'') such that $D'' = D_0'' + \cdots + D_{s-1}''$ has the same cycle d'' of absolute self-intersection numbers as D_t''. Let Z denote a universal covering of the product of the spaces $(D_i'' \cap D_{\mathrm{reg}}'')^{d_i - d_i''}$ taken over the indices $i \in \mathbf{Z}/s$ for which $d_i > d_i''$. Since $D_i'' \cap D_{\mathrm{reg}}''$ is isomorphic to $\mathbf{C}^*$, Z will be an affine space of dimension $\Sigma_i(d_i - d_i'')$. Any $z \in Z$ determines a pair $(\tilde{Y}_z, \tilde{D}_z)$, obtained from (Y'', D'') by blowing up successively the points of D'' determined by z. Since Y_{reg}' is a regular neighbourhood of D' in Y', it is homeomorphic to U_t' and hence the pair $(\tilde{Y}_{z,\mathrm{reg}}, \tilde{D}_z)$ is homeomorphic to (U_t, D_t). This homeomorphism induces an isomorphism $h_z: H_2(\tilde{Y}_{z,\mathrm{reg}} - \tilde{D}_z) \to H_2(U_t - D_t)$ which, since Z is simply-connected, we may assume to depend in a continuous manner on z. Let ω_z denote the unique holomorphic 2-form on $\tilde{Y}_{z,\mathrm{reg}} - \tilde{D}_z$ which takes the value 1 on the orientation of $\tilde{D}_z$. Then its class $[\omega_z]$ is via h_z identified with an element of $H^2(U_t - D_t; \mathbf{C})$. In the same way as in (I.5.3) it is shown that the map $Z \to H^2(U_t - \tilde{D}_t; \mathbf{C})$ thus defined is an isomorphism. So if we restrict out attention to pairs $(\tilde{Y}_z, \tilde{D}_z)$ with $h_z^*([\omega_z]) \in \Omega_d(F_X)$, we obtain a family

$$(\tilde{\mathcal{Y}}_X, \tilde{\mathcal{D}}_X) \to \Omega_d(F_X).$$

The stratum $M_d(X)$ is the orbit space of $\Omega_d(F_X)$ for the action of $W_X.(Q/Q_X)$ on it. We claim that we have a diagram which simultaneously resolves rational double points:

$$\begin{array}{ccc} (\tilde{\mathcal{Y}}_X, \tilde{\mathcal{D}}_X) & \overset{\tilde{\Phi}_X}{\to} & (\mathcal{Y}_X, \mathcal{D}_X) \\ \downarrow & & \downarrow \\ \Omega_d(F_X) & \overset{\Phi_X}{\to} & M_d(X), \end{array}$$

for the stabilizer of any $\omega \in \Omega_d(F_X)$ is generated by reflections and hence a finite Weyl group. Each irreducible summand of this Weyl group is of type A and corresponds to a maximal string of nodal curves on $\tilde{Y}_{\omega,\mathrm{reg}} - \tilde{D}_\omega$. Contraction of such a maximal string creates a rational double point of type A and this is what $\tilde{\Phi}_X$ does.

Now let p': $(\mathcal{Y}', \mathcal{D}') \to S'$ be a proper representative of a semi-universal deformation of (Y', D') such that S' is contractible and choose a trivialization $\mathcal{D}' \to D'$ of $p' : \mathcal{D}' \to S'$. Then the above construction can be repeated for the whole family p' (for it takes place in an arbitrary small neighbourhood of D) and so yields a proper flat family

$$p \colon (\mathcal{Y}, \mathcal{D}) \to S' \times M_d(X)$$

in which $S'_f \times M_d(X)$ parametrizes rational surfaces with an anti-canonical cycle of type d.

(3.9) Proposition. *There exists a neighbourhood V of $\{*'\} \times M_d(X)$ in $S' \times M_d(X)$, an isomorphism Φ_V of V onto a neighbourhood of $M_d(X)$ in $\hat{M}_d$ such that $\Phi_V(*', z) = z$ and a cartesian diagram*

$$\begin{array}{ccc} (\mathcal{Y}_{V_f}, \mathcal{D}_{V_f}) & \overset{\tilde{\Phi}_{V_f}}{\to} & \mathcal{Y}_d \\ p_{V_f} \downarrow & & \downarrow p_d \\ V_f & \overset{\Phi_{V_f}}{\to} & M_d \end{array}$$

where $\Phi_{V_f} = \Phi | V_f$.

Proof. As with (3.6), we prove this in three steps.

Step 1. There exists a neighbourhood V of $\{*'\} \times M_d(X)$ in $S' \times M_d(X)$ and a cartesian diagram as in the statement of (3.8).

Proof. If V is sufficiently small each fibre (Y_v, D_v) over V_f blows down to a good pair. This follows from (2.13) in the same way as (2.12) does. So by (II.3.10) there is a diagram

$$\begin{array}{ccc} \mathcal{Y}_{V_f} & \xrightarrow{\tilde{\Phi}_{V_f}} & \mathcal{Y}_d \\ p_{V_f}\downarrow & & \downarrow p_d \\ V_f & \xrightarrow[\Phi_{V_f}]{} & \hat{M}_d \end{array}$$

in which $\tilde{\Phi}_{V_f}$ blows down the fibres of p_{V_f} to good pairs. We claim that by shrinking V_f this diagram becomes cartesian. If not, the argument used in step 1 of (3.6) shows the existence of an irreducible curve N_0 in Y_z with $N_0 \not\subset D_z$ for some $z \in M_d(X)$ such that N_0 displaces to a nodal curve in nearby fibres. Let

$$Y_z \leftarrow \tilde{Y}_z \rightarrow Y'$$

resolve the rational double points of Y_z minimally, resp. blow down to Y'. Since any curve on Y is contained in its anti-canonical cycle, the strict transform $\tilde{N}_0$ of N_0 in $\tilde{Y}_z$ must be an exceptional curve of $\tilde{Y}_z \rightarrow Y'$. So $\tilde{N}_0$ is either a nodal curve or exceptional of the first kind. As $\tilde{Y}_z \rightarrow Y_z$ contracts the nodal curves, only the last case is possible. But then N_0 displaces to nearby fibres as an exceptional curve which cannot be (for such a curve meets the anti-canonical cycle).

Step 2. If V is sufficiently small, Φ_{V_f} extends to a local isomorphism $\Phi_V: V \rightarrow \hat{M}_d$.

The proof is analogous to the proof of step 2 of (3.6).

Step 3. We have $\Phi(*', z) = z$ for all $z \in M_d(X)$.

Proof. If z is such that Y_z has no rational double point then the local monodromy group of the family p_V near Y_z is according to (3.6) isomorphic to W_X. So $\Phi(*', z)$ has a neighbourhood $\hat{U}$ in $\hat{M}_d$ such that the monodromy group of $\mathcal{Y}_d|\hat{U} - (\hat{D}_d \cap \hat{U})$ is W_X. This can only be if $\Phi(*', z) \in M_d(X)$. Since this holds for generic z, it follows that $\Phi(*', z) \in M_d(X)$ for all $z \in M_d(X)$.

Let Φ'_{V_f}: $V_f \rightarrow \Omega_d$ be a multivalued lift of Φ_{V_f}. By construction the composite of Φ'_{V_f} with the natural projection $\Omega_d \rightarrow \Omega_d(X) \rightarrow M_d(X)$ is univalued and is in fact nothing but the projection of V in $M_d(X)$. Hence $\Phi(*', z) = z$.

These three steps clearly prove (3.9).

If we combine (3.2), (3.6) and (3.9) we obtain the main result of this chapter (which however is less precise than the union of these results).

(3.10) THEOREM. *The fine moduli space* $p_d\colon (\mathcal{Y}_d, \mathcal{D}_d) \to M_d$ *of good pairs of type* $d = (d_0, \ldots, d_{s-1})$, $s \leq 5$, *constructed in Chapter* II *extends to a proper flat family* $\hat{p}_d\colon (\hat{\mathcal{Y}}_d, \hat{\mathcal{D}}_d) \to \hat{M}_d$, *where* $\hat{M}_d$ *is the holomorphic hull of* M_d. *The space* $\hat{M}_d$ *is a Stein manifold and* $\hat{M}_d - M_d$ *is a subvariety of* $\hat{M}_d$, *naturally equipped with a stratification into constructible analytic submanifolds* $M_d(X)$, *where* X *runs over the non-void special subsets of the intersection diagram associated to* d. *We have* codim $M_d(X) =$ card(X) *and* $M_d(X)$ *parametrizes blown-up Inoue surfaces* (*parabolic if* X *is of affine type, hyperbolic otherwise*) *which have no nodal curves outside their anti-canonical cycle.*

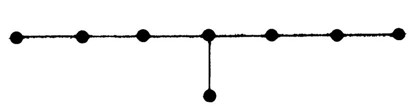
(3.11) The stratification of $\hat{M}_d$ by orbit type is of interest if one wishes to know which singularities may occur on a single fibre of $\hat{p}_d$. According to (2.17) in [19] we have the following recipe: choose a subset Z of the intersection graph B corresponding to d. Let Y denote the union of connected components of Z with the finite Weyl group, $\hat{Y}$ the graph obtained by completing each connected component of Y, and put $X := Z - Y$ (so X is special). Choose further a subset Y' of $\hat{Y}$ which meets each connected component of $\hat{Y}$ in a proper subset. The Y' is the diagram of a finite Weyl group and the pair (Z, Y') determines a connected stratum in $M_d(X)$ which parametrizes surfaces with a singularity of type X (that is, no singularity if $X = \varnothing$, a simply-elliptic or cusp singularity otherwise) and whose remaining singular locus is a set of rational double points of type Y'. These strata define a partition of $\hat{M}_d$ (see [19], *loc. cit* for details) by orbit type (but different pairs (Z, Y') may define the same stratum!).

This does not agree with a claim I made in [17, (7.8)]. Indeed, as P. Giblin pointed out to me, this claim is false because of the following:

(3.12) *Example.* The simply-elliptic singularity of type $\hat{E}_7$ deforms to a fibre with six ordinary double points by means of the family

$$Y_t\colon \{xy(x + y + t)(x - y + 2t)\}.$$

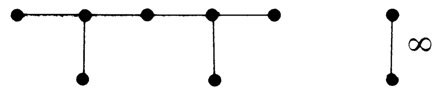
There are no six disconnected vertices in the $\hat{E}_7$-graph



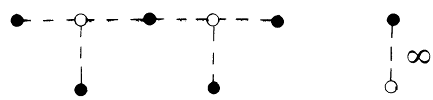
but if we take for $Z = Y$ the subdiagram

$Y =$ 

and complete it componentwise:

$\hat{Y} =$ 

then there is indeed a $Y' \subset \hat{Y}$ consisting of six disconnected vertices (see also Mérindol [22]):

$Y' =$ 

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