Saturday 11/7/15 611 Midterm Solutions

1. $|G| = 52$.

$x \in G$, $|C(x)| = 4$.

Orbit-stabilizer $\Rightarrow$ $|C(x)| \cdot |Z(x)| = |G|$, $|Z(x)| = 52/4 = 13$

$x \in Z(x)$ $\Rightarrow$ order of $x = 13$ or 1

But $x \neq e$ ($C(e) = \{e\}$) so order of $x = 13$.

2. $|G| = 90$.

$G \curvearrowright X$, $|X| = 5$, non trivial.

$\rightsquigarrow \varphi : G \longrightarrow S_5$ hom.

$\ker \varphi \trianglelefteq G$, $\ker \varphi \neq G$ (action non trivial)

We claim $\ker \varphi \neq \{e\}$ $\Rightarrow$ G not simple.

Otherwise $G \xrightarrow{\sim} \varphi(G) \leq S_5$ $\underset{\text{Lagrange}}{\Rightarrow}$ $|G| \mid |S_5| = 5! = 120$ ※.

3. $|G| = 75 = 3 \cdot 5^2$

$s :=$ # Sylow 3-subgroups

$t :=$ # Sylow 5-subgroups

$s \equiv 1 \bmod 3$, $s \mid 5^2$ $\Rightarrow$ $s = 1$ or 25

$t \equiv 1 \bmod 5$, $t \mid 3$ $\Rightarrow$ $t = 1$

If $s = t = 1$ $G \cong H \times K$, H Sylow 3-subgroup, $H \cong \mathbb{Z}/3\mathbb{Z}$

K Sylow 5-subgroup, $K \cong (\mathbb{Z}/5\mathbb{Z})^2$ or $\mathbb{Z}/25\mathbb{Z}$

$\Rightarrow$ G abelian ※

because $H \trianglelefteq G$, $K \trianglelefteq G$, $H \cap K = \{e\}$ (by Lagrange), $|H| \cdot |K| = |G|$ $\Big\}$ $\Rightarrow$ $HK = G$.

classification of groups of order p^2

$\therefore$ $s = 25$.

$\therefore$ # elements of order $3 = s \cdot (3-1) = 50$.

elements of order 3 are non identity elements of Sylow 3-subgroups ($\cong \mathbb{Z}/3\mathbb{Z}$)

4.a. $|G| = 44 = 2^2 \cdot 11$

$\exists\ x \in G$ of order 4.

$s := \#$ Sylow 2-subgroups $\quad s \equiv 1 \bmod 2,\ s \mid 11 \Rightarrow s = 1$ or 11

$t := \#$ Sylow 11-subgroups $\quad t \equiv 1 \bmod 11,\ t \mid 4 \Rightarrow t = 1$

$\therefore$ $H :=$ Sylow 11-subgroup is normal, let K be a Sylow 2-subgroup.

$\exists$ element of order 4, & all Sylow p-subgroups are conjugate $\Rightarrow K \cong \mathbb{Z}/4\mathbb{Z}$.

$$\therefore \quad G \cong H \rtimes_\varphi K \cong \mathbb{Z}/11\mathbb{Z} \rtimes_\psi \mathbb{Z}/4\mathbb{Z},$$

$$\psi: \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/11\mathbb{Z}) \cong (\mathbb{Z}/11\mathbb{Z})^\times \cong \mathbb{Z}/10\mathbb{Z}.$$

G non abelian $\Rightarrow$ ψ non trivial

$\gcd(4,10) = 2 \Rightarrow \psi(1)$ is unique element of order 2,

i.e $\psi(1) = (x \mapsto -x) \in \mathrm{Aut}(\mathbb{Z}/11\mathbb{Z})$

Writing $K = \langle b \rangle$, $H = \langle a \rangle$, have

$$G \cong \langle a, b \mid a^{11} = b^4 = e,\ \underbrace{bab^{-1} = a^{-1}}_{\text{equiv., } ba = a^{-1}b} \rangle.$$

b. Elements of $G = H \rtimes_\varphi K$ can be written uniquely as $a^i b^j$ $\quad 0 \le i < 11$, $0 \le j < 4$

$Z(G) = ?$

Since G is generated by a & b, $z \in Z(G) \Leftrightarrow z$ commutes w/ a & b.

$b \cdot a^i b^j = a^{-i} b^{j+1}$

$a^i b^j \cdot b = a^i b^{j+1}$

$\therefore$ $a^i b^j$ commutes w/ b $\Leftrightarrow$ $i \equiv -i \bmod 11 \Leftrightarrow 2i \equiv 0 \bmod 11$

$\Leftrightarrow i \equiv 0 \bmod 11,\ i = 0.$

b^j commutes w/ a $\Leftrightarrow$ j is even $\Leftrightarrow$ $j = 0, 2$.

$\therefore$ $Z(G) = \{e, b^2\}$.

c. $G/Z(G) = \langle a, b \mid a^{11} = b^2 = e,\ bab^{-1} = a^{-1} \rangle \cong D_{11}$, dihedral group.

5. $G = \langle a, b \mid a^7 = b^3 = e,\ bab^{-1} = a^M \rangle$

a. $M=2$. $G \simeq \mathbb{Z}/7\mathbb{Z} \rtimes_\psi \mathbb{Z}/3\mathbb{Z}$, $a \leftarrow\!\shortmid (1,0)$, $b \leftarrow\!\shortmid (0,1)$

$$\psi : \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z}) \simeq (\mathbb{Z}/7\mathbb{Z})^\times = \mathbb{Z}/6\mathbb{Z}$$
$$1 \longmapsto 2 \in (\mathbb{Z}/7\mathbb{Z})^\times$$

Check: $2^3 \equiv 1 \mod 7$. ✓ In particular, $|G| = 21$

b. $M=3$ This is not a semi direct product as above, because $3^3 \equiv -1 \mod 7$, $\not\equiv 1 \mod 7$.

So $a = b^3ab^{-3} = a^{(3^3)} = a^{-1}$, $a^2 = e$.

Now $a^7 = e \Rightarrow a = e$.

Thus G is generated by b, $|G| \leq 3$.

By universal property of free group, have $G \longrightarrow \mathbb{Z}/3\mathbb{Z}$, $\Rightarrow G \cong \mathbb{Z}/3\mathbb{Z}$,
$a \longmapsto e$
$b \longmapsto 1$ $|G| = 3$.

6. $G = (\mathbb{Z}/q\mathbb{Z})^2 \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$.

$$\varphi : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Aut}\left((\mathbb{Z}/q\mathbb{Z})^2\right) \equiv GL_2(\mathbb{Z}/q\mathbb{Z})$$
$$1 \longmapsto A \in GL_2(\mathbb{Z}/q\mathbb{Z}), \quad A^p = I,\ A \neq I.$$

G not abelian $\Uparrow$

Such A exists because
$$|GL_2(\mathbb{Z}/q\mathbb{Z})| = (q^2-1)(q^2-q)$$
and $p \mid q^2-1$ by assumption.
(Cayley's theorem, follows from Sylow Thm 1.)