612 Example Sheet 5

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(1) Let $A \subset \mathbb{Q}$ be a subring. Show that $A = S^{-1}\mathbb{Z}$, where $S \subset \mathbb{Z}$ is the multiplicative subset generated by a set of primes $T = \{p_i \mid i \in I\} \subset \mathbb{Z}$. (Note that T is not necessarily finite.)

(2) Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal. Let $S = A \setminus \mathfrak{p}$ and write $A_\mathfrak{p} = S^{-1}A$ and $\mathfrak{p}_\mathfrak{p} = S^{-1}\mathfrak{p}$. (Then $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}_\mathfrak{p}$, called the *localization of A at $\mathfrak{p}$*.) Show that the map

$$\varphi\colon A \to S^{-1}A, \quad a \mapsto a/1$$

induces an inclusion $A/\mathfrak{p} \subset A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$, and that $A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is the field of fractions of the integral domain $A/\mathfrak{p}$. In particular if $\mathfrak{p} = \mathfrak{m}$ is maximal then $A/\mathfrak{m} = A_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. [Hint: For A a ring and $S \subset A$ a multiplicative set, the operation S^{-1} from A-modules to $S^{-1}A$-modules is exact, so in particular $S^{-1}M/S^{-1}N = S^{-1}(M/N)$ for A-modules M, N.]

(3) Let k be a field. Let $k[[x]] := \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in k\}$ denote the formal power series ring in the variable x over k. Show that the inclusion $k[x] \subset k[[x]]$ extends to an inclusion $k[x]_{(x)} \subset k[[x]]$, where $k[x]_{(x)}$ denotes the localization of $k[x]$ at the prime ideal $(x) \subset k[x]$.

(4) Let k be a field, $A = k[x, y]$, and $\mathfrak{m} = (x, y) \subset k[x, y]$, a maximal ideal of A. Consider the localization $A_\mathfrak{m}$ of A at $\mathfrak{m}$. Show that the map $\theta\colon A_\mathfrak{m}^{\oplus 3} \to A_\mathfrak{m}^{\oplus 2}$ of $A_\mathfrak{m}$-modules defined by the matrix

$$\begin{pmatrix} xy+1 & x^3 & x^2+2 \\ x^2 & y & y^3+1 \end{pmatrix}$$

is surjective.

(5) Let A be a UFD, K its field of fractions, and L/K a finite extension of K. Show carefully that an element $\alpha \in L$ is integral over A iff the minimal polynomial of α over K has coefficients in A. [Hint: Use the Gauss Lemma, DF p. 303.]

(6) Let $n \in \mathbb{Z}$ be square-free, that is, n is *not* divisible by the square of any prime. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$ is $\mathbb{Z}[(1+\sqrt{n})/2]$ if $n \equiv 1 \bmod 4$ and $\mathbb{Z}[\sqrt{n}]$ otherwise.

(7) Let k be a field and $A = k[x,y]/(y^2 - x^2(x+1))$.

(a) Show that A is an integral domain.

(b) Compute the integral closure of A in its field of fractions.

(8) Consider the integral extension of rings $\mathbb{Z} \subset B := \mathbb{Z}[i]$, where $i^2 = -1$. We recall that the ring B of Gaussian integers is a PID, and the units in B are $\pm 1, \pm i$. Thus the prime ideals $\mathfrak{q} \subset B$ are of the form $\mathfrak{q} = (\alpha)$ where $\alpha = a + bi$ is irreducible, and is determined up to multiplication by a unit.

(a) Let $\mathfrak{q} = (\alpha) \subset B$ be a prime ideal and let $\mathfrak{p} = (p) := \mathfrak{q} \cap \mathbb{Z}$ be the corresponding prime ideal in $\mathbb{Z}$. Show that either $\alpha = a \in \mathbb{Z}$ and $p = \pm a$ or $\alpha = a + bi \notin \mathbb{Z}$ and $p = (a+bi)(a-bi) = a^2 + b^2$.

(b) Now suppose given a prime ideal $\mathfrak{p} = (p) \subset \mathbb{Z}$ and consider prime ideals $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap \mathbb{Z} = \mathfrak{p}$. Show the following:

i. If $p \equiv 3 \mod 4$ there is a unique such prime $\mathfrak{q} = (p)$.

ii. If $p \equiv 1 \mod 4$ there are exactly two such primes $\mathfrak{q} = (a + bi), (a - bi)$, and $p = a^2 + b^2$.

iii. If $p = 2$ there is a unique such prime $\mathfrak{q} = (1+i) = (1-i)$.

[Hint: Consider the ring extension

$$\mathbb{Z}/\mathfrak{p} = \mathbb{F}_p \subset B/\mathfrak{p}B = \mathbb{F}_p[x]/(x^2+1).$$

The prime ideals $\mathfrak{q}$ lying over $\mathfrak{p}$ correspond to the maximal ideals of $B/\mathfrak{p}B$.]

(9) Let k be a field and $A = k[x,y]/(x^2y^3, x^5)$. Find the nilradical of the ring A.

(10) Let k be an algebraically closed field. Let $J \subset k[x_1, \ldots, x_n]$ be an ideal and define $A = k[x_1, \ldots, x_n]/J$. Let X denote the zero locus $Z(J) \subset \mathbb{A}^n$ of J.

(a) Let N be the nilradical of A. Show that $N = \sqrt{J}/J$. Deduce that the reduced ring $A_{\text{red}} := A/N$ is the coordinate ring $k[X] := k[x_1, \ldots, x_n]/I(X)$ of X.

(b) Show that A is Artinian iff X is a finite set.

(11) Let A be an Artinian ring. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of A. Show that the natural maps $A \to A_{\mathfrak{m}_i}$ give an isomorphism

$$A \xrightarrow{\sim} A_{\mathfrak{m}_1} \times \cdots A_{\mathfrak{m}_r}.$$