Math 461 Homework 6

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(1) Let ΔABC be a spherical triangle such that $\angle BAC = \pi/2$.

(a) Using the spherical cosine rule or otherwise, derive the spherical version of Pythagoras' theorem relating the side lengths α, β, γ of ΔABC.

(b) Check your formula in the case that $B = (0,0,1)$ is the north pole and A and C lie on the equator (the intersection of S^2 with the xy-plane).

(2) Let ΔABC be a spherical triangle.

(a) Show that there is a unique spherical circle passing through the vertices A,B,C. This is called the *circumscribed circle* of the spherical triangle.

(b) Find the spherical center and spherical radius of the circumscribed circle of the spherical triangle ΔABC with vertices $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$.

(c) Find a formula for the spherical center of the circumscribed circle of a spherical triangle ΔABC in terms of the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ using the cross product. (Assume that the vertices A, B, C of the triangle ΔABC occur in that order as we traverse the boundary of the triangle in the counter-clockwise direction.)

[See HW5Q6 for the definition of a spherical circle and some useful properties.]

(3)(a) Find all solutions of the equation $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ where $p, q, r \in \mathbb{N}$, $2 \le p \le q \le r$.

(b) Show that, for each solution (p, q, r) from part (a), there is a tiling of the plane $\mathbb{R}^2$ by congruent triangles with angles π/p, π/q, π/r, such that all the angles meeting at a vertex of the tiling are the same.

[Hint: (a) There are three solutions. (b) First tile the plane with regular polygons, then subdivide the polygons into triangles with the desired angles.]

(4)(a) Find all solutions of the inequality $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ where $p, q, r \in \mathbb{N}$, $2 \leq p \leq q \leq r$.

(b) Show that, for each solution (p, q, r) from part (a), there is a tiling of the sphere by spherical triangles with angles π/p, π/q, π/r, such that all the angles meeting at a vertex of the tiling are the same.

(c) Find a uniform formula for the number of triangles in the tiling in terms of p, q, r.

[Hint: (a) There is one infinite sequence of solutions and three additional solutions. (b) For the infinite sequence, use triangles as in Q1b (and their reflections in the xy-plane). For the three additional solutions, the following construction can be used: start with one of the following regular polytopes: the tetrahedron, the cube, or the dodecahedron, with vertices on the sphere. Now project the faces of the polytope onto the sphere from the origin O to obtain a tiling of the sphere by spherical polygons. Finally, subdivide the faces of the polytope into triangles and, projecting again, obtain a subdivision of the spherical polygons into spherical triangles. (c) What is the area of a spherical triangle?]

(5) A *spherical polygon* with vertices $A_1, \ldots, A_n \in S^2$ is a region $P \subset S^2$

bounded by the shortest paths from A_1 to A_2, A_2 to A_3,..., A_{n-1} to A_n, and A_n to A_1. We say that a spherical polygon is *convex* if for all points $A, B \in P$ the shortest path from A to B is contained in P.

(a) Let P be a convex spherical polygon with n vertices. Show that the sum of the interior angles of P equals $(n-2)\pi$ plus the area of P.

(b) Now suppose given a tiling of the sphere by convex spherical polygons. Use part (a) to show that the numbers V, E, F of vertices, edges, and faces of the tiling satisfy *Euler's formula*

$$V - E + F = 2.$$

[Hint: (a) Use the same argument as in the plane (see HW1Q5). (b) The area 4π of the sphere is equal to the sum of the areas of the spherical polygons in the tiling. Now express this sum in terms of V, E, F using the formula from part (a).]

(6) Let ΔABC be a spherical triangle, with angles a,b,c and side lengths α,β,γ. We define the *polar* spherical triangle $\Delta A'B'C'$ as follows: Let Π_{BC} be the plane passing through the points O, B and C. (So $S^2 \cap \Pi_{BC}$ is the spherical line passing through B and C.) The position vector $\overrightarrow{OA'}$ of A' is the normal vector to the plane Π_{BC} which has length 1 and lies on the same side of Π_{BC} as A. We define

B' and C' analogously.

(a) Show that the side lengths of the polar spherical triangle $\Delta A'B'C'$ are given by $\alpha' = \pi - a$, $\beta' = \pi - b$, $\gamma' = \pi - c$.

(b) Show that the polar spherical triangle of $\Delta A'B'C'$ is ΔABC. (That is, if we apply the polar construction twice we recover the original spherical triangle.)

(c) Deduce that the angles of the polar spherical triangle are given by $a' = \pi - \alpha$, $b' = \pi - \beta$, $c' = \pi - \gamma$.

(7) Given a spherical triangle, the spherical cosine rule can be used to determine the angles of the triangle given the side lengths. (The same is true

of the usual cosine rule for plane triangles.)

Given a spherical triangle, apply the spherical cosine rule to the polar triangle (see Q6 above) to obtain a formula (called the *second spherical cosine rule*), which can be used to determine the side lengths of the spherical triangle in terms of its angles. (Of course this is not possible for plane triangles!)