Math 412 Homework 8

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Show your work and justify your answers carefully. All fields are assumed to have characteristic 0.

(1) Let L be the splitting field of $f(x) = (x^2 - 2x - 2)(x^2 - 2x - 4)$ over $\mathbb{Q}$.

(a) Describe L explicitly and write down a basis for L as a vector space over $\mathbb{Q}$.

(b) Determine all the automorphisms of L over $\mathbb{Q}$.

(c) Identify the Galois group $G(L/\mathbb{Q})$ with a standard group.

(2) Let K be a field and $K \subset L$ a field extension of degree 2. So $L = K(\sqrt{d})$ for some $d \in K$. Determine all the elements $\alpha \in L$ such that $\alpha^2 \in K$.

(3) Determine the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$.

[You may assume without proof that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q}] = 8$. (This can be proved by using the result of Q2 repeatedly.)]

(4) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 3. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ be the roots of f and suppose that $\alpha_1 \in \mathbb{R}$ and $\alpha_2, \alpha_3 \notin \mathbb{R}$. Determine all the automorphisms of $\mathbb{Q}(\alpha_1)$ over $\mathbb{Q}$. Is the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha_1)$ a Galois extension?

(5) Let $\zeta = \exp(2\pi i/5)$.

(a) Show that $\mathbb{Q}(\zeta)$ is the splitting field for the polynomial $x^5 - 1$ over $\mathbb{Q}$.

(b) Determine the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.

(6) (Optional) Let $\mathbb{C}(t)$ denote the field of rational functions with complex coefficients in the variable t (equivalently, $\mathbb{C}(t)$ is the fraction field of the polynomial ring $\mathbb{C}[t]$). Let $\varphi\colon \mathbb{C}(t) \to \mathbb{C}(t)$ be the automorphism of $\mathbb{C}(t)$ determined by $\varphi(t) = \zeta t$ where $\zeta = e^{2\pi i/3}$.

  (a) Show that φ has order 3, that is, $\varphi^3(f) := \varphi(\varphi(\varphi(f))) = f$ for all $f \in \mathbb{C}(t)$. Write $G = \langle \varphi \rangle = \{1, \varphi, \varphi^2\} \simeq \mathbb{Z}/3\mathbb{Z}$.

  (b) Compute the fixed field

$$\mathbb{C}(t)^G := \{f \in \mathbb{C}(t) \mid g(f) = f \text{ for all } g \in G\} = \{f \in \mathbb{C}(t) \mid \varphi(f) = f\}.$$

  (c) Compute the degree $[\mathbb{C}(t) : \mathbb{C}(t)^G]$ and so verify the fixed field theorem in this case.

  (d) Generalize to the case $\zeta = e^{2\pi i/n}$, $n \in \mathbb{N}$.

[Hint: (b) If $f(t) \in \mathbb{C}(t)$ then we can write $f(t) = p(t)/q(t)$ where $p(t), q(t) \in \mathbb{C}[t]$ and $\gcd(p(t), q(t)) = 1$. Then $g(f(t)) = f(\zeta t) = p(\zeta t)/q(\zeta t)$ and $\gcd(p(\zeta t), q(\zeta t)) = 1$ (why?). Now if $g(f(t)) = f(t)$ we must have $p(\zeta t) = \lambda p(t)$ and $q(\zeta t) = \lambda q(t)$ for some $\lambda \in \mathbb{C}$. Deduce that $\mathbb{C}(t)^G = \mathbb{C}(t^3)$. (c) Writing $K = \mathbb{C}(t)^G = \mathbb{C}(t^3)$ and $L = \mathbb{C}(t)$, explain why $L = K(t)$, that is, L is generated by the element t over K. Now determine the irreducible polynomial $f(x) \in K[x]$ of t over the field K and deduce the degree $[L : K]$.]

(7) Let $f(x) = x^4 + bx^2 + c \in \mathbb{Q}[x]$. Assume that $f(x)$ does not have any repeated roots in $\mathbb{C}$.

  (a) Show that the roots of $f(x)$ are $\pm\alpha_1, \pm\alpha_2$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$. Write $\alpha_3 = -\alpha_1$ and $\alpha_4 = -\alpha_2$.

  (b) Let $L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{Q}(\alpha_1, \alpha_2)$ be the splitting field for $f(x)$ over $\mathbb{Q}$. Recall that the Galois group $G = G(L/\mathbb{Q})$ of the extension $\mathbb{Q} \subset L$ is identified with a subgroup H of the symmetric group S_4 by sending an automorphism $g \in G$ to the permutation $\theta(g)$ of the roots $\alpha_1, \ldots, \alpha_4$ induced by g. Show that H is contained in the dihedral group $D_4 \subset S_4$ corresponding to the symmetries of the square with vertices labelled $1, 2, 3, 4$ in counterclockwise order.

[Hint: (b) Note that there are 3 ways of grouping the elements $1, 2, 3, 4$ into pairs: $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$. Explain why the dihedral group D_4 of symmetries of the square preserves the second of these pairings. Now use the orbit-stabilizer theorem to show that the subgroup of S_4 which preserves this pairing is equal to the dihedral group D_4. Finally, explain why the image $H \subset S_4$ of the Galois group $G(L/\mathbb{Q})$ also preserves the second pairing.]