# 612 Example Sheet 6

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Unless otherwise stated all groups $G$ are finite and all representations $\rho$ are finite dimensional complex representations. We say that a representation $\rho\colon G \to \mathrm{GL}(V)$ of a group $G$ is *faithful* if $\rho$ is injective. These problems will not be graded, but representation theory will be covered on the final exam.

(1) Show that every finite group $G$ has a faithful representation on a finite dimensional complex vector space $V$.

(2) Determine all finite groups $G$ having a faithful representation on a real vector space of dimension 2. [Hint: We may assume that the action of $G$ preserves an inner product, so that, choosing an orthonormal basis of $V$, the image of $\rho\colon G \to \mathrm{GL}_2(\mathbb{R})$ lies in the orthogonal group $\mathrm{O}(2)$.]

(3) Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation of a group $G$ of dimension greater than 1. Let $v \in V$ be a vector. Show that $\sum_{g\in G} gv = 0$.

(4) Compute the dimensions of the irreducible representations of the group $G$ of rotational symmetries of the cube.

(5) Recall that the *quaternion group* $Q$ is defined by $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ with group law given by multiplication of quaternions, that is, $i^2 = j^2 = k^2 = ijk = -1$. Compute the character table of $Q$ and compare it with the character table of $D_4$.

(6) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of $G$ of dimension $d$ and $\chi$ its character. Let $g \in G$. Show that $|\chi(g)| \leq d$ with equality iff $\rho(g) = \zeta \cdot \mathrm{id}_V$ for some root of unity $\zeta$.

(7) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of dimension $d$ and $\chi$ its character. Show that

$$\ker(\rho) = \{g \in G \mid \chi(g) = d\}.$$

(8) Show that $G$ is simple iff for every nontrivial irreducible character $\chi$ and element $g \in G \setminus \{e\}$ we have $\chi(g) \neq \chi(e)$.

(9) Show that if $\chi$ and $\psi$ are characters of $G$ then so is the product $\chi \cdot \psi$, where $(\chi \cdot \psi)(g) := \chi(g) \cdot \psi(g)$. [Hint: Consider the action of $G$ on a tensor product $V \otimes W$.]

(10) The table below is part of the character table of a finite group (the conjugacy classes are all there but some of the irreducible characters are missing). The numbers in brackets indicate the number of elements in each conjugacy class, and $\omega = (-1 + \sqrt{3}i)/2$, $\gamma = (-1 + \sqrt{7}i)/2$.

| | (1) | (3) | (3) | (7) | (7) |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | $\omega$ | $\overline{\omega}$ |
| $\chi_2$ | 3 | $\gamma$ | $\overline{\gamma}$ | 0 | 0 |
| $\chi_3$ | 3 | $\overline{\gamma}$ | $\gamma$ | 0 | 0 |

(a) Compute the order of the group and the number and dimensions of the irreducible representations.

(b) Find the remaining characters.

(c) Describe the group by generators and relations.