Math 797W Algebraic geometry. Homework 2

Paul Hacking

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Throughout we work over an algebraically closed field k.

(1) Let $X \subset \mathbb{A}^{n+1}_{X_0,\ldots,X_n}$ be an algebraic set. Consider the action of $k^\times = k \setminus \{0\}$ on $\mathbb{A}^{n+1} = k^{n+1}$ by scalar multiplication:

$$k^\times \ni \lambda \colon (X_0, \ldots, X_n) \mapsto (\lambda X_0, \ldots, \lambda X_n).$$

Show that the following conditions are equivalent.

(a) The set X is preserved by the action of $k^\times$, and $X \neq \{0\}$.

(b) The ideal $I(X) \subset k[X_0, \ldots, X_n]$ is homogeneous, and $I(X) \neq (X_0, \ldots, X_n)$.

(c) The set X is a union of 1-dimensional subspaces of k^{n+1}. (More precisely, X is the union of the 1-dimensional subspaces of $\mathbb{A}^{n+1}$ corresponding to the points of the projective algebraic set $Y \subset \mathbb{P}^n$ defined by the homogeneous ideal $I(X)$. We say X is the *affine cone* over Y.)

(2) Recall that projective n-space $\mathbb{P}^n$ over k is the set $\mathbb{P}^n = (k^{n+1} \setminus \{0\})/k^\times$, together with some additional structure (the Zariski topology, and the k-algebra of regular functions on each open subset). Given a vector space V over k of dimension $n+1$, we can similarly define the associated projective space $\mathbb{P}(V) = (V \setminus \{0\})/k^\times$. Picking a basis of V defines an isomorphism $\mathbb{P}(V) \simeq \mathbb{P}^n$.

A *linear subspace* of $\mathbb{P}^n$ of dimension r is by definition the subset $\mathbb{P}(W) \subset \mathbb{P}^n$ associated to a linear subspace $W \subset k^{n+1}$ of dimension $r+1$. Linear subspaces in $\mathbb{P}^n$ of dimension 1 and $n-1$ are called lines and hyperplanes respectively.

Prove the following:

(a) A linear subspace $X \subset \mathbb{P}^n$ of dimension r is an irreducible algebraic set with ideal $I(X)$ generated by $n-r$ linear forms (homogeneous polynomials of degree 1).

(b) Let $X \subset \mathbb{P}^n$ be a linear subspace of dimension r. Let $U_i = (X_i \neq 0) \subset \mathbb{P}^n$ be the locus where the ith homogeneous coordinate is nonzero. Then the intersection $X \cap U_i$ is an affine linear subspace of $U_i = \mathbb{A}^n = k^n$ of dimension r (or is empty). That is, $X \cap U_i = V + c$ where $V \subset k^n$ is a linear subspace of dimension r and $c \in k^n$ is a vector (here V and c depend on the choice of i).

(c) Let $L_1, L_2 \subset \mathbb{P}^2$ be two lines, $L_1 \neq L_2$. Show that $L_1 \cap L_2$ is a point. (In particular, two parallel lines in $\mathbb{A}^2$ meet at infinity in the projective plane $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. The intersection point in $\mathbb{P}^1$ is given by the common slope.)

(d) More generally, show that if $X_1, \ldots, X_k$ are linear subspaces of $\mathbb{P}^n$ of dimensions $r_1, \ldots, r_k$, and $c := (n-r_1) + \cdots + (n-r_k) \leq n$, then the intersection $X_1 \cap \cdots \cap X_k$ is non-empty and is a linear subspace of dimension $r \geq n-c$.

(3) Recall that we say a topological space X is *irreducible* if there does not exist a decomposition $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are proper closed subsets.

(a) Show that X is irreducible iff every non-empty open subset $U \subset X$ is dense (that is, the closure of U equals X).

(b) Show that if X is irreducible and $f \colon X \to Y$ is a continuous map then $f(X)$ is irreducible.

(4) Compute the closure $\overline{X} \subset \mathbb{P}^n$ of the following affine varieties $X \subset \mathbb{A}^n$ in projective space $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$. Identify the set $\overline{X} \cap \mathbb{P}^{n-1} = \overline{X} \setminus X$.

(a) $X = (x_1^3 + x_1 x_2^2 + x_1^2 + x_2 + 1 = 0) \subset \mathbb{A}^2_{x_1,x_2}$.

(b) $X = \{(t, t^2, t^4) \mid t \in k\} \subset \mathbb{A}^3_{x_1,x_2,x_3}$.

(5) Let $X = \mathbb{P}^2$ and $U = X \setminus \{(1:0:0)\}$. Show that $\mathcal{O}_X(U) = k$. [Hint: Use the open covering $U = U_1 \cup U_2$, where $U_i = (X_i \neq 0)$.]

(6) Let $X \subset \mathbb{P}^n$ be a projective variety, and $k[X] = k[X_0, \ldots, X_n]/I(X)$ its homogeneous coordinate ring. Recall that $k[X]$ is a graded ring (the

grading being induced by the usual grading on the polynomial ring $k[X_0, \ldots, X_n]$). Let $0 \neq F \in k[X]$ be a homogeneous element of degree d, and write $D(F) = X \setminus Z(F)$. Show that $\mathcal{O}_X(D(F)) = (k[X]_F)_0$, the degree 0 piece of the graded ring $k[X]_F$ (the localization of $k[X]$ at the element F). [Hint: Imitate the proof of the analogous assertion in the affine case.] Remark: In the case $F = X_i$ this is the coordinate ring of the affine variety $D(F) = X \cap U_i \subset U_i = \mathbb{A}^n$, where $U_i = (X_i \neq 0) \subset \mathbb{P}^n$.

(7) Let $f = x_2^2 - (x_1^3 + ax_1 + b)$, $a, b \in k$, and $X = Z(f) \subset \mathbb{A}^2_{x_1,x_2}$, an affine variety.

   (a) Compute the closure $\overline{X} \subset \mathbb{P}^2$ of X in $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$.

   (b) Describe the affine open subset V_2 of $\overline{X}$ given by $V_2 = \overline{X} \cap U_2 \subset U_2 = \mathbb{A}^2_{y_{20},y_{21}}$, where $U_2 = (X_2 \neq 0) \subset \mathbb{P}^2$.

   (c) Show that the morphism $\tilde{g}\colon \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x_1, x_2) \mapsto x_1$ extends to the morphism $\tilde{G}\colon \mathbb{P}^2 \setminus \{(0:0:1)\} \to \mathbb{P}^1$ given by $(X_0 : X_1 : X_2) \mapsto (X_0 : X_1)$, but does *not* extend to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. [The map $\tilde{G}$ is called the *projection* from the point $(0 : 0 : 1)$.]

   (d) Show that the morphism $g\colon X \to \mathbb{A}^1$ given by $(x_1, x_2) \mapsto x_1$ extends to a morphism $G\colon \overline{X} \to \mathbb{P}^1$. [Hint: Use the chart V_2 to define G in an open neighborhood of $\overline{X} \setminus X$ and check that it is a morphism.]

(8) Let $F\colon \mathbb{P}^1 \to \mathbb{P}^n$ be the map of sets given by

$$(Y_0 : Y_1) \mapsto (Y_0^n : Y_0^{n-1}Y_1 : \cdots : Y_1^n).$$

   (a) Show that F is a well-defined map and is a morphism of varieties. (That is, F is continuous for the Zariski topology and pullback of functions defines a map $F^*\colon \mathcal{O}_{\mathbb{P}^n}(U) \to \mathcal{O}_{\mathbb{P}^1}(F^{-1}U)$ for every open set $U \subset \mathbb{P}^n$.)

   (b) Show that $X := F(\mathbb{P}^1) \subset \mathbb{P}^n$ is the irreducible algebraic set with ideal $I(X)$ equal to the ideal J generated by the 2×2 minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}$$

The variety $X \subset \mathbb{P}^n$ is called the *rational normal curve of degree* n. [Hint: To show $X = Z(J)$ as a set, note that $Z(J) \subset \mathbb{P}^n$ is the locus where the above matrix has rank ≤ 1. To show $J = I(X)$, check that F induces an injection $k[X_0, \ldots, X_n]/J \to k[Y_0, Y_1]$, thus J is prime (cf. HW1 Q10).]

(c) Show that the map $G\colon X \to \mathbb{P}^1$ given by $(X_0 : \cdots : X_n) \mapsto (X_0 : X_1) = (X_{n-1} : X_n)$ is a well defined morphism of projective varieties which is inverse to F. Thus F is an isomorphism onto its image X.

(d) Let $H = (\sum a_i X_i = 0) \subset \mathbb{P}^n$ be a hyperplane. Show that the set $H \cap X$ consists of n points (counted with multiplicities).