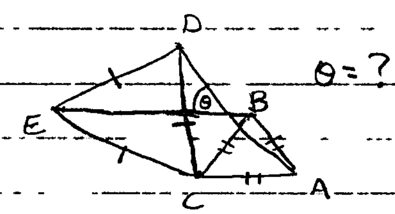
10/15/19

Midterm tomorrow 7-9PM LGRC A301
Midterm review problems & solutions available
people.math.umass.edu/~hacking/461F19
Office hours today: 4-5PM LGRT 1235H

<u>Last Time</u> • Algebraic interpretation of ruler & compass constructions
• Constructible lengths are those obtained from 1 by $+, -, \times, \div, \sqrt{\ }$

<u>Today</u> • Isometries of $\mathbb{R}^2$ (rigid motions / symmetries)

Review problem (10)



$\theta = ?$

<u>Exam</u> Everything up to but not including coordinates.

An $\boxed{\text{isometry}}$ of $\mathbb{R}^2$ is a function (or transformation) $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that for all $P_1, P_2 \in \mathbb{R}^2$

$$|T(P_1)T(P_2)| = |P_1P_2|$$

i.e. T preserves distances.

<u>Examples</u>

1. $\boxed{\text{Translation}}$

Fix $\underline{v} = (a,b)$ vector

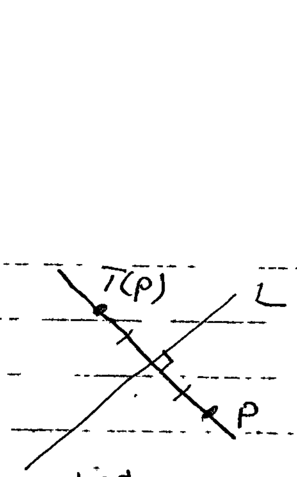
$$T(x,y) = (x+a, y+b)$$

Check T is an isometry: $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$.

$T(P_1) = (x_1+a, y_1+b)$ $T(P_2) = (x_2+a, y_2+b)$

$$|T(P_1)T(P_2)| = \sqrt{((x_1+a)-(x_2+a))^2 - ((y_1+b)-(y_2+b))^2}$$
$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = |P_1P_2| \checkmark$$

2. **Reflection**

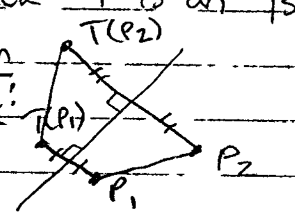


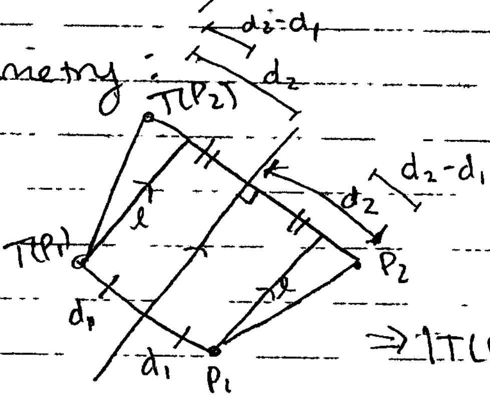
L line in $\mathbb{R}^2$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Check T is an isometry:

Euclidean Geometry:

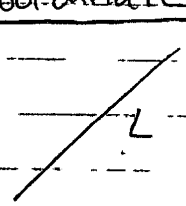


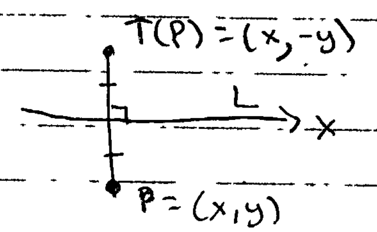


$\Rightarrow |T(P_1P_2)| = |P_1P_2| = \sqrt{\ell^2 + (d_1 - d_2)^2}$

(Also check case where P_1 & P_2 are on opposite sides of L)

Coordinate proof:





i.e. reflections in x-axis are given by $T(x, y) = (x, -y)$

Choose coordinates such that L is the x-axis.

Check T is an isometry:

$P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $T(P_1) = (x_1, -y_1)$, $T(P_2) = (x_2, -y_2)$

$$|T(P_1)T(P_2)| = \sqrt{(x_1 - x_2)^2 + (-y_1 - (-y_2))^2}$$
$$= \sqrt{(x_1 - x_2)^2 + (-(y_1 - y_2))^2}$$
$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |P_1P_2| \checkmark$$

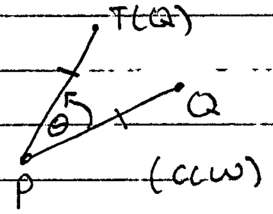
3. **Rotation**

Fix point P and an angle θ and a "sense" (counter clockwise or clockwise)

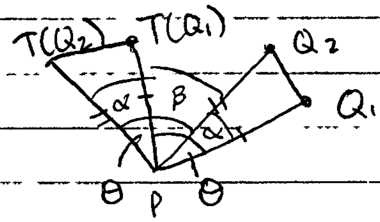
CCW (counter clockwise) — usual assumption

CW (clockwise)

$\rightarrow$



Check T is an isometry:



Want to show $|T(Q_1)T(Q_2)| = |Q_1Q_2|$

$\alpha = \theta - \beta$, $\triangle PQ_1Q_2 \cong \triangle PT(Q_1)T(Q_2)$ by SAS

$\Rightarrow |T(Q_1)T(Q_2)| = |Q_1Q_2|$

Formula for a rotation?

Choose coordinates such that ... $P = (0,0)$.

Then, T is a linear transformation (MATH 235)

$T(0,0) = (0,0)$ & T isometry $\Uparrow$

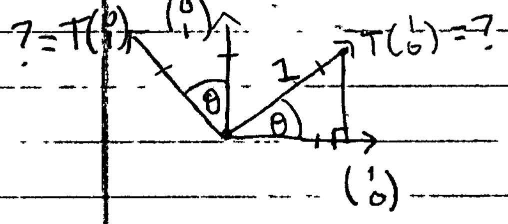
i.e. $T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$ for all $\underline{v}_1, \underline{v}_2 \in \mathbb{R}^2$

$T(c\underline{v}) = cT(\underline{v})$ for all $\underline{v} \in \mathbb{R}^2$, $c \in \mathbb{R}$

Assuming this, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

where $\begin{pmatrix} a \\ c \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix} = T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} = T\left(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= T\left(x\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + T\left(y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \end{aligned}$$



$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot \cos\theta \\ 1 \cdot \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

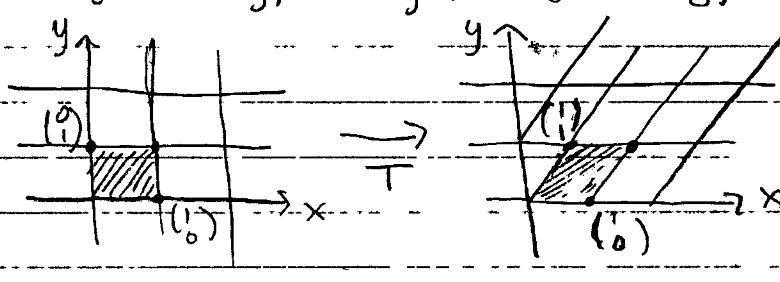
$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$\rightsquigarrow T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\cos\theta)x - (\sin\theta)y \\ (\sin\theta)x + (\cos\theta)y \end{pmatrix}$

Rotation about origin through angle θ ccw.

Q: Is a shear an isometry?

e.g. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

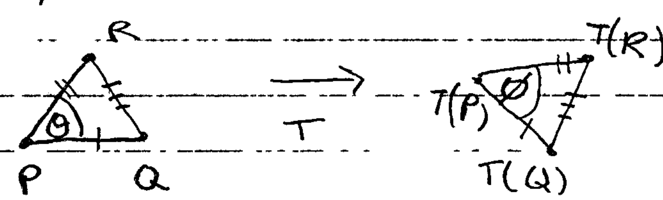


No.

In fact, isometries preserve angles.

(Recall, isometries preserve distances by definition)

Why?



Claim: T isometry $\Rightarrow \phi = \theta$

Proof by SSS, $\Delta T(P)T(Q)T(R) \cong \Delta PQR \Rightarrow \phi = \theta$ ■

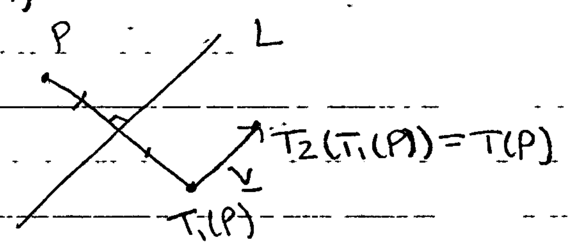
4. $\boxed{\text{Glide reflection}}$

L line, $\underline{v}$ vector in the direction of L.

$T = T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T(P) = T_2(T_1(P))$

T_1 = reflection in L

T_2 = translation by $\underline{v}$



Remark: If T_1 & T_2 are isometries, so is the composition $T_2 \circ T_1$.