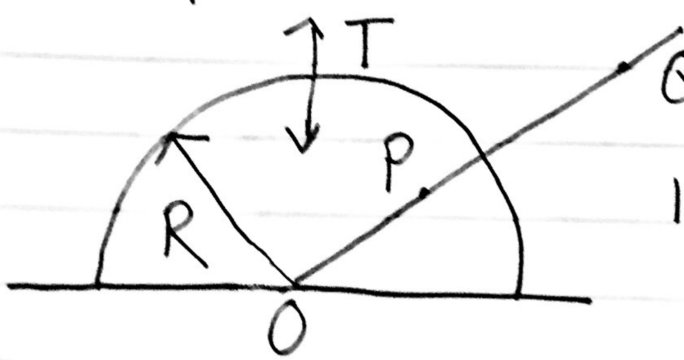
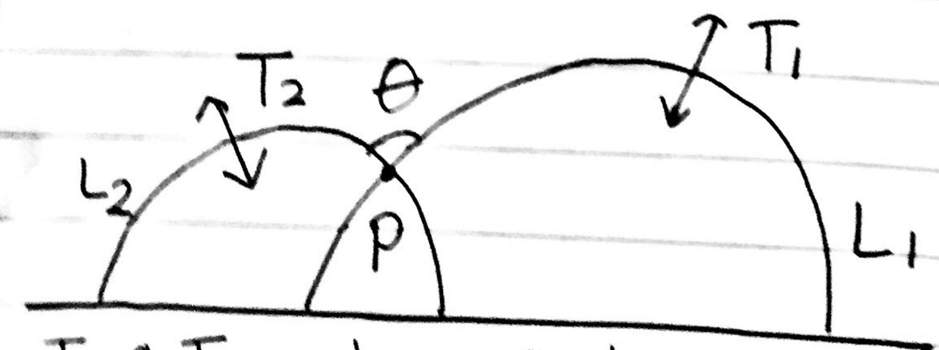
Math 461 Lecture 39 12/12
Final Friday 8-10 AM LGRT 141
Review session 3-4 PM
tomorrow LGRT 141
Last time:
inversion is hyperbolic reflection
3 reflections theorem:
any hyperbolic isometry is
composition of ≤ 3 reflections



inversion in circle
$|OP| \cdot |OQ| = R^2$

classification:
0 identity
1 reflection
2 a. hyperbolic rotation
b. ...
c. ...



$T_2 \circ T_1 =$ hyperbolic rotation about
P through 2θ

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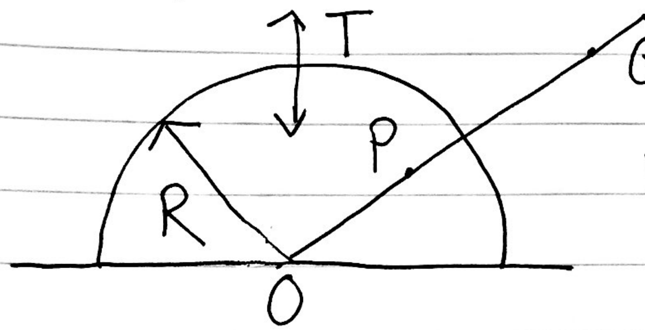

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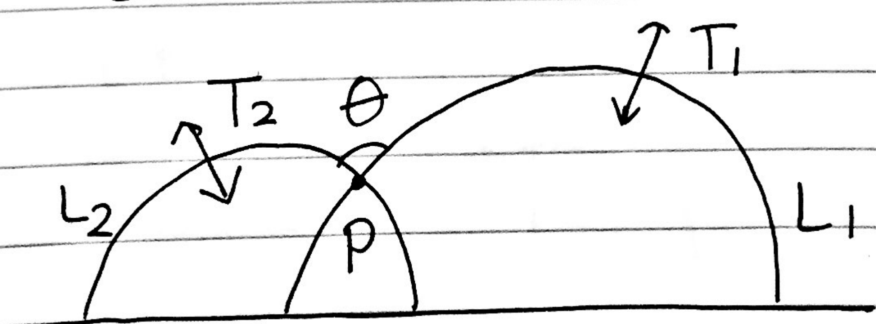

$T_2 \circ T_1$ = hyperbolic rotation about P through 2θ

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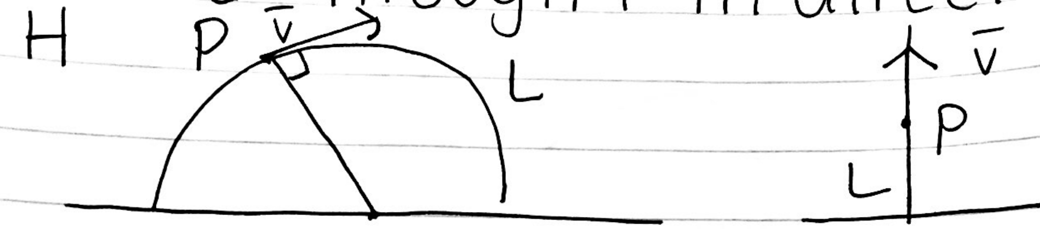
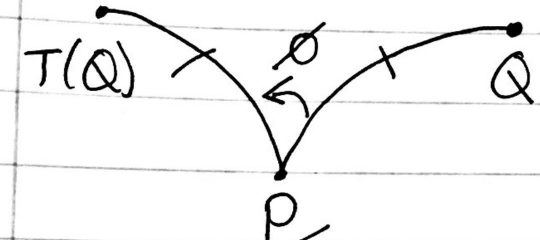
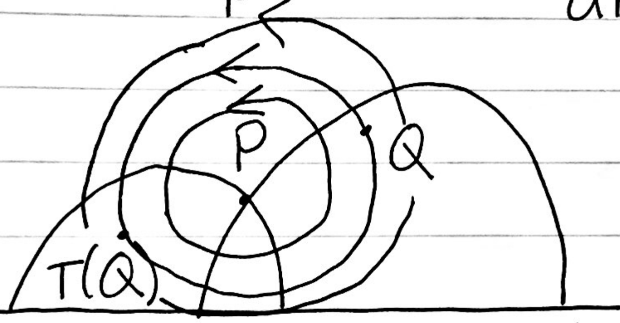
Today: finish classification
algebraic description of hyperbolic isometries

Lemma: Given $P \in H$, $\bar{v}$ tangent direction at P, there's a unique line L through P in direction $\bar{v}$





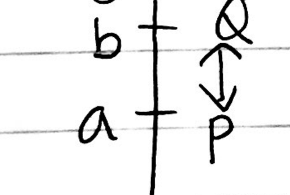
hyperbolic rotation about P through angle ϕ



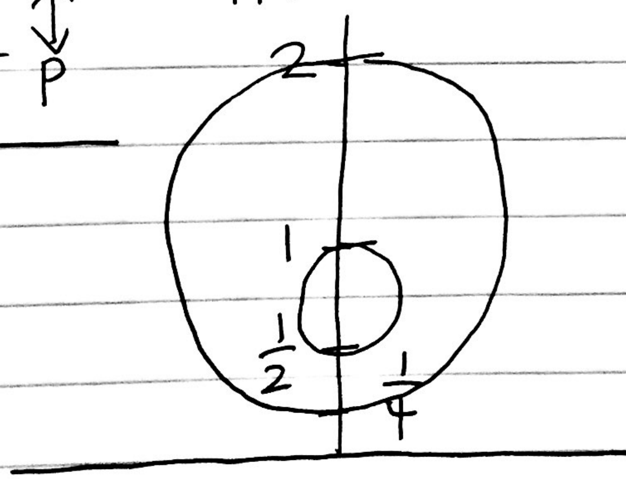
hyperbolic circles, hyperbolic center P

fact: Euclidean circles in $H \subset \mathbb{R}^2$

(Euclidean radius and center $\neq$ hyperbolic radius and center)



$d_H(P,Q) = \ln\left(\frac{b}{a}\right)$



two hyperbolic circles, center $(0,\frac{1}{2})$ hyperbolic radius $\ln 2$, $\ln 4$

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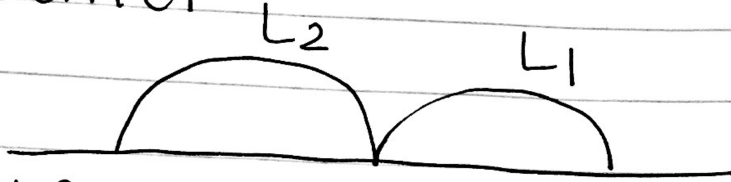
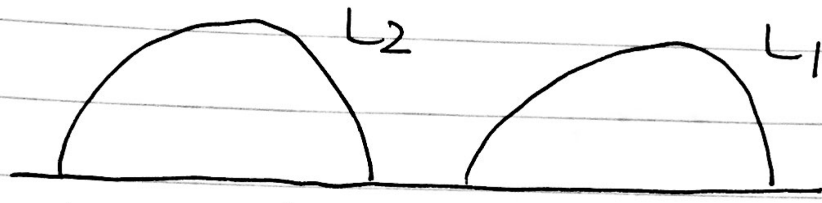
2 reflections: 2 more cases

$H = \{(x,y) \mid y>0\}$ x-axis $\notin H$

2b. parallel

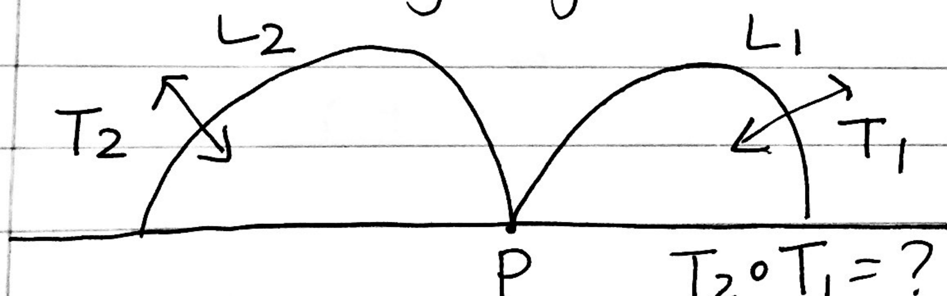


2c. ultra parallel



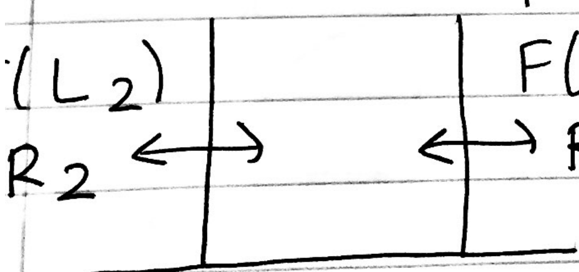
try to understand by reducing to an easier case using a "change of coordinates"

auxilary hyperbolic isometry

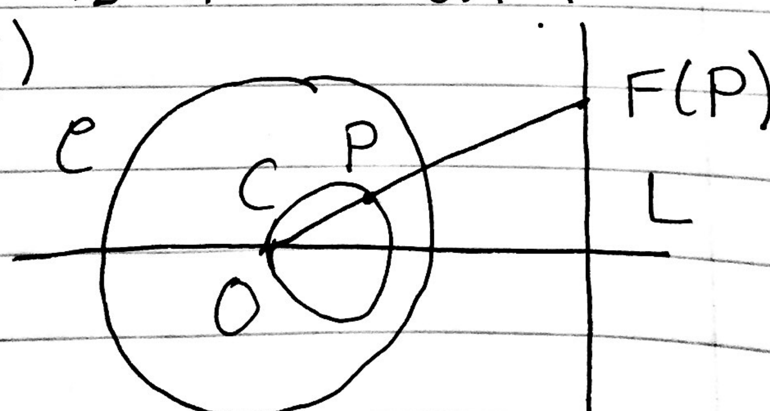


$T_2 \circ T_1 = ?$

invert F in a semicircle e with center at P $\rightsquigarrow$



$R_2 \circ R_1 = ?$



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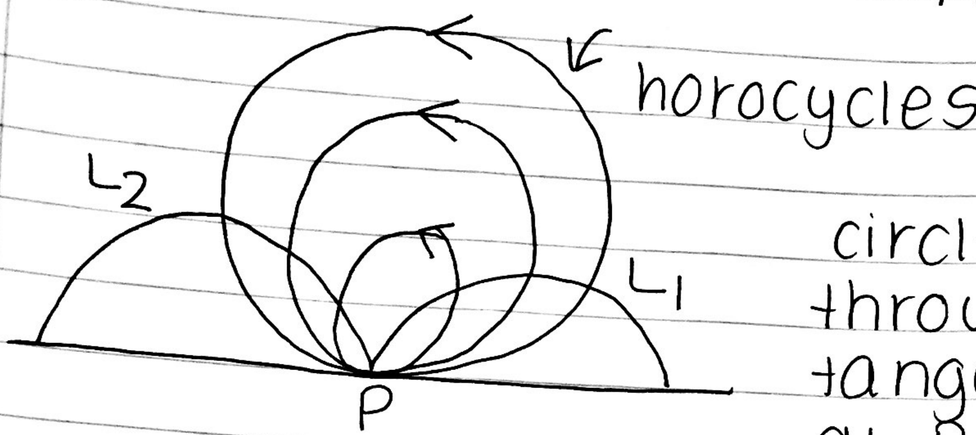
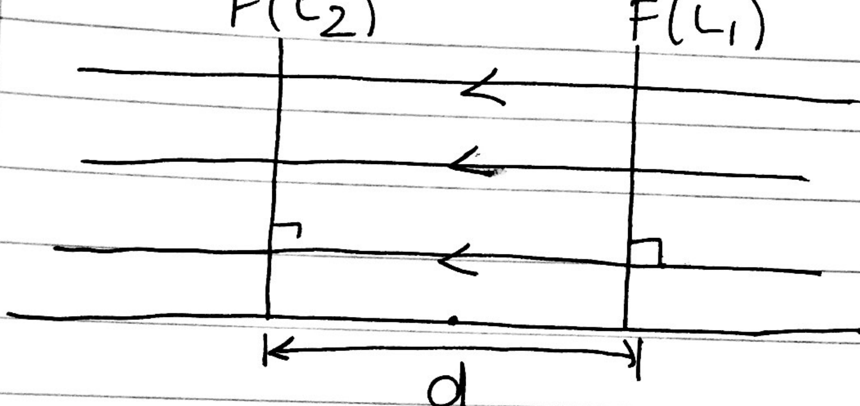


circles passing through P and tangent to x-axis at P

$\xrightarrow{F}$

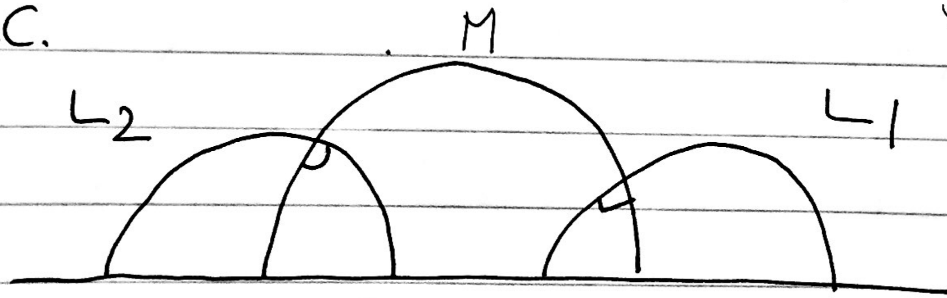
$\xleftarrow{F^{-1}}$

"horolation"



horizontal translation by 2d

2c.



Lemma: There is unique hyperbolic M orthogonal to both L_1 and L_2

Proof: Reduce to easier case: Invert in circle $\mathcal{C}$ with center one of the endpoints of L_1

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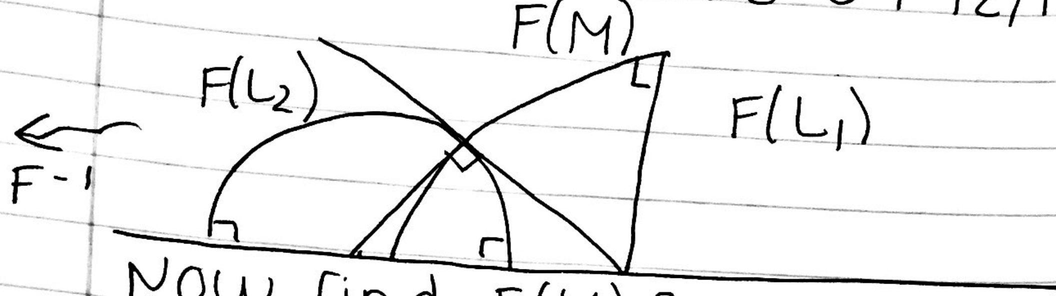
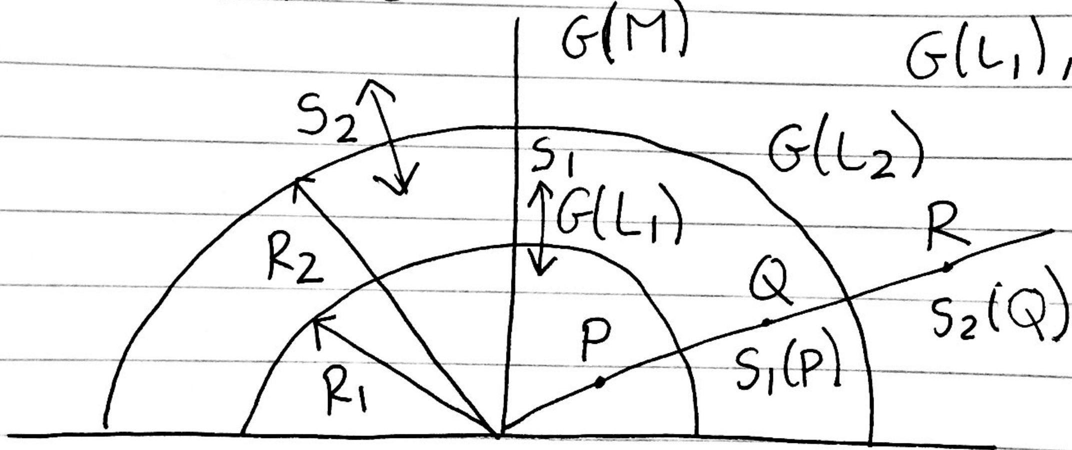


Now find F(M)?
hyperbolic line meeting $F(L_1)$ and $F(L_2)$ orthogonally $\square$

Given Lemma, we change coordinates as follows :-
Apply inversion G with center one of the endpoints of M, so G(M) is vertical line
Q

$G(L_1), G(L_2) = ?$

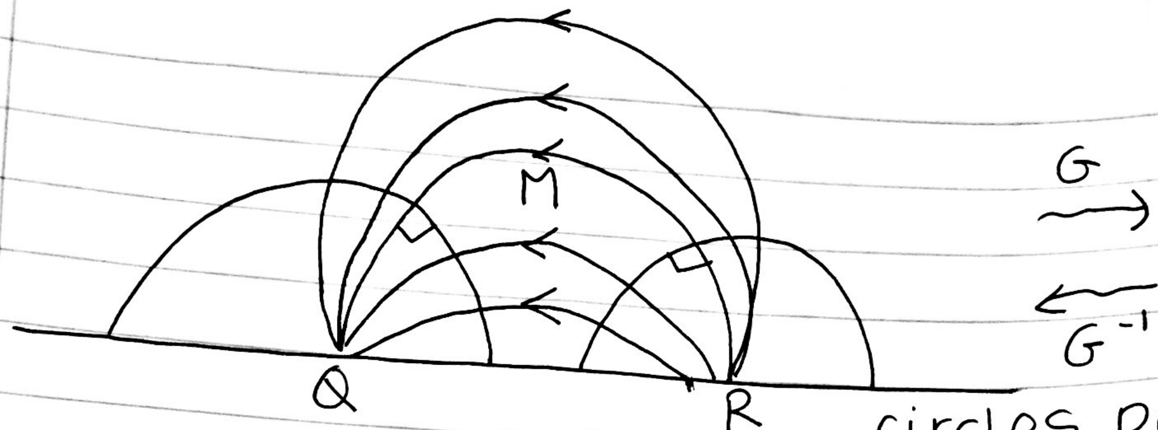


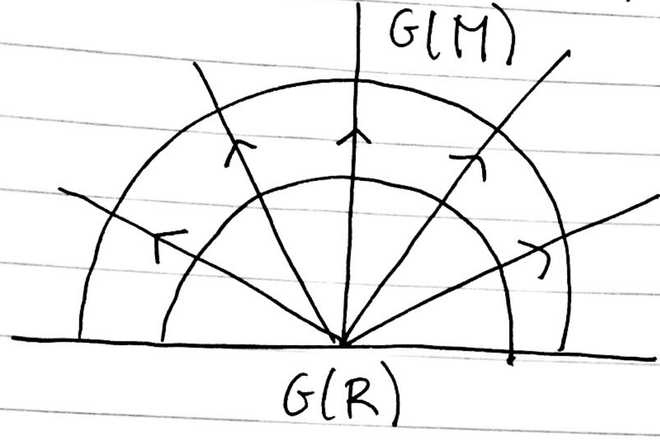
$|OP| \cdot |OQ| = R_1^2 \quad |OQ| \cdot |OR| = R_2^2$

i.e. $|OR| = \dfrac{R_2^2}{|OQ|} = \dfrac{R_2^2}{R_1^2} |OP|$

i.e scaling, center O, factor $c = \dfrac{R_2^2}{R_1^2}$

$(x,y) \longmapsto (cx, cy) \qquad c > 1$





circles passing through endpoint Q and R of M
"$G(Q) = \infty$"

"hyperbolic translation"

3 reflections:
Similar to $\mathbb{R}^2$ (and S^2), any composition of 3 reflections is a hyperbolic translation (with hyperbolic line M preserved by the translation) followed by hyperbolic reflection in M
hyperbolic glide reflection
algebraic description of hyperbolic isometries?
$H = \{(x,y) \mid y > 0\} \subset \mathbb{R}^2 = \mathbb{C}$
$z = x + iy$
Theorem: orientation preserving hyperbolic isometries ar

$$T(z) = \frac{az+b}{cz+d} \quad \text{where } a, b, c, d \in \mathbb{R}, \quad ad - bc > 0$$

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orientation preserving
preserves sense (ccw/cw) of angles, equivalent product of 0 or 2 reflections
given this formula: its easy to find Fix(T) and classify