Monday 11/23/15 MATH 421 Midterm 2. Solutions

1.a. $f(x+iy) = x - iy = u + iv,$

i.e. $u = x, \quad v = -y$

$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1. \quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow f$ is not complex differentiable.

(Recall the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

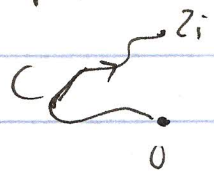
are satisfied by a complex differentiable function $f = u + iv$)

b. $z : [0,1] \longrightarrow \mathbb{C}, \quad z(t) = 1 + t\cdot(i-1) = (1-t) + i\cdot t$

c.
$$\int_C f(z)\,dz = \int_0^1 f(z(t))\,z'(t)\,dt$$
$$= \int_0^1 \left((1-t) - i\cdot t\right)\cdot(-1+i)\,dt$$
$$= \int_0^1 \left(-(1-t) + t\right) + i\cdot\left((1-t) + t\right)\,dt$$
$$= \int_0^1 (2t-1) + i \,dt$$
$$= \left[t^2 - t\right]_0^1 + i\cdot\left[t\right]_0^1 = \boxed{i.}$$

2.a. $f(z) = z^3 + iz + 3$

$F(z) = z^4/4 + iz^2/2 + 3z$ is an antiderivative of f.



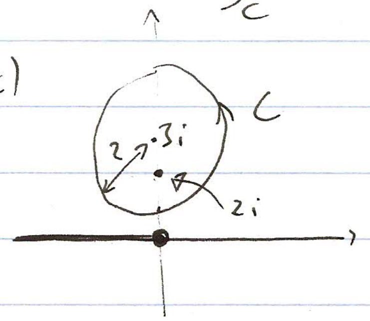
$$\therefore \int_C f(z)\,dz = \left[z^4/4 + iz^2/2 + 3z\right]_0^{2i}$$
$$= 16/4 - 4i/2 + 6i = \boxed{4+4i}$$

b. $f(z) = e^{3z}\cos(z^2)$ is complex differentiable on all of $\mathbb{C}$ by the chain rule & product rule.

C is a closed curve

So $\int_C f(z)\,dz = 0$ by Cauchy's theorem.

c)



$\mathrm{Log}(z)$ is complex differentiable on $U = \mathbb{C} \setminus (-\infty, 0]$,

C & the region bounded by C are contained in U, and $2i$ is inside C.

So, by the generalized Cauchy integral formula

$$\frac{1!}{2\pi i}\int_C \frac{\mathrm{Log}(z)}{(z-2i)^2}\,dz = \mathrm{Log}'(2i) = \frac{1}{2i} \qquad \left(\mathrm{Log}'(z) = \tfrac{1}{z} \text{ on } U\right)$$

$$\Rightarrow \int_C \frac{\mathrm{Log}(z)}{(z-2i)^2}\,dz = \frac{2\pi i}{2i} = \boxed{\pi.}$$

3 a. $F(z) = -1/z$ is an antiderivative of $f(z) = 1/z^2$

b. $\int_C \frac{1}{z-i}\,dz = 2\pi i$ for C a simple closed curve with i inside C, oriented ccw.

$\Rightarrow f(z) = \frac{1}{z-i}$ does not have an antiderivative on $U = \mathbb{C} \setminus \{i\}$

(because if F were an antiderivative then $\int_C f(z)\,dz = F(\beta) - F(\alpha)$ for a curve C in U with endpoints α & β, & in particular $\int_C f(z)\,dz = 0$ if C is a closed curve (so that $\alpha = \beta$)).

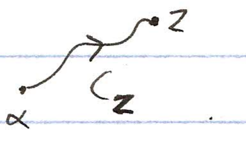
c. $f(z) = \sin(z^2)$ is complex differentiable by the chain rule.

The domain $U = \mathbb{C}$ of f is simply connected ("no holes").

So we can define an antiderivative by

$$F(z) = \int_{C_z} f(w)\,dw = \int_{C_z} \sin(w^2)\,dw$$

where C_z is a curve with endpoints α and z, oriented from α to z, and $\alpha \in \mathbb{C}$ is fixed.



4a. $f(z) = \dfrac{e^{iz}}{z^2+4}$

$z^2+4=0 \iff z = \pm 2i$

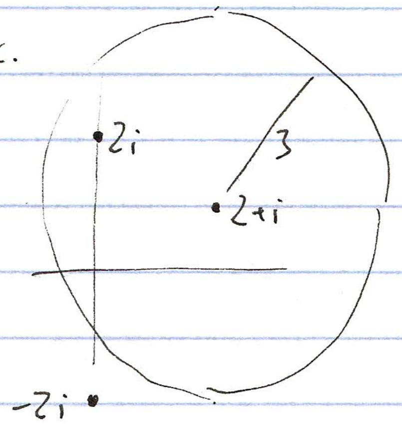
$\therefore$ the domain of f is $U = \mathbb{C} \setminus \{\pm 2i\}$

b.

Cauchy's integral formula: $f(\alpha) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{z-\alpha}\,dz$

where f is complex differentiable on $U \subset \mathbb{C}$, C is a simple closed curve, oriented ccw, U contains C and the region bounded by C, and α is inside C.

c.



$|2i - (2+i)| = |-2+i| = \sqrt{5} < 3$

$|-2i - (2+i)| = |-2-3i| = \sqrt{13} > 3$

So $2i$ is inside C, $-2i$ is outside C.

Write $f(z) = \dfrac{e^{iz}}{z^2+4} = \dfrac{e^{iz}}{(z-2i)(z+2i)} = \dfrac{g(z)}{(z-2i)}$

where $g(z) = \dfrac{e^{iz}}{z+2i}$

Then $\frac{1}{2\pi i}\int_C \frac{g(z)}{(z-2i)}\,dz = g(2i) = \frac{e^{-2}}{4i}$

by CIF.

So $\int_C f(z)\,dz = \int_C \frac{g(z)}{z-2i}\,dz = 2\pi i \cdot \frac{e^{-2}}{4i}$

$= \frac{\pi e^{-2}}{2}$

5a. $f(z) = z + e^z$

$f'(z) = 1 + e^z = 0 \iff e^z = -1$

$\iff z = \pi i + (2\pi i)k$, k integer

b.

$f(x+iy) = x+iy + e^{x+iy}$

$= x+iy + e^x \cdot e^{iy}$

$= x+iy + e^x(\cos y + i\sin y)$

$= \underbrace{(x+e^x\cos y)}_{u} + i\underbrace{(y+e^x\sin y)}_{v}$

c. The critical points of u are the same as the critical points of f found in a: $z = \pi i + (2\pi i)k$, k an integer

i.e. $x=0$, $y = \pi + 2\pi k$.

A critical point of u is a saddle point if

$\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 < 0$ at the point.

In our case $\frac{\partial u}{\partial x} = 1 + e^x\cos y$ $\quad \frac{\partial u}{\partial y} = 0 + e^x(-\sin y) = -e^x\sin y$

$\frac{\partial^2 u}{\partial x^2} = e^x\cos y$, $\frac{\partial^2 u}{\partial x\partial y} = -e^x\sin y$, $\frac{\partial^2 u}{\partial y^2} = -e^x\cos y$

So $\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 = e^x\cos y\cdot(-e^x\cos y) - (-e^x\sin y)^2 \rightarrow$

Evaluating at $x=0$, $y=\pi+2\pi k$

gives $\quad (-1)\cdot(+1) - 0^2 = -1 < 0.$

So each critical point is a saddle point.

6. $$f(z) = \frac{z}{(z-1)(2z-1)}$$

a. $$f(z) = \frac{A}{z-1} + \frac{B}{2z-1}$$

$$\Rightarrow \quad z = A(2z-1) + B(z-1)$$

$$z = (2A+B)z + (-A-B)$$

$$\Rightarrow \quad 2A+B = 1, \quad -A-B = 0$$

$$\Rightarrow \quad B = -A, \quad 1 = 2A - A = A \quad \Rightarrow \quad A=1,\ B=-1.$$

$$f(z) = \frac{1}{z-1} - \frac{1}{2z-1}.$$

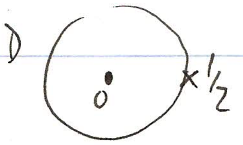
b. Recall $\displaystyle \frac{1}{1-z} = 1+z+z^2+\ldots = \sum_{n=0}^{\infty} z^n$, for $|z|<1$

So $$f(z) = \frac{1}{z-1} - \frac{1}{2z-1} = \frac{-1}{1-z} + \frac{1}{1-2z}$$

$$= -(1+z+z^2+\ldots) + (1+(2z)+(2z)^2+\ldots)$$

$$= \sum_{n=0}^{\infty} (2^n-1)z^n$$

c. The domain of $f(z)$ is $U = \mathbb{C} \setminus \{1, 1/2\}$.
The largest open disc $D = \{z \in \mathbb{C} \mid |z| < R\}$ with center $z=0$ contained in U has radius $R = 1/2$



. So the radius of convergence $R = 1/2$.

Alternatively, the radius of convergence R of the power series

$\sum_{n=0}^{\infty} a_n z^n$ is given by $R = 1/L$ if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

In our case, $a_n = 2^n - 1$

$$\text{So} \quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}-1}{2^n-1} = \lim_{n\to\infty} \frac{2-1/2^n}{1-1/2^n}$$

$$= \frac{2-0}{1-0} = 2. = L$$

and $R = 1/L = 1/2$.