612 Example Sheet 1

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30 January 2011

Notation: For K/F a field extension and $\alpha \in K$ an element that is algebraic over F, the *degree of* α *over* F is the degree of the minimal polynomial of α over F, or, equivalently, the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ of the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$.

(1) Let K be a subfield of $\mathbb{C}$ which is not contained in $\mathbb{R}$. Prove that K is a dense subset of $\mathbb{C}$.

(2) Let R be an integral domain containing a field F such that R is a finite dimensional when viewed as a vector space over F. Show that R is a field.

(3) Let F be a field with 27 elements. What is the characteristic of F? Justify your answer.

(4) Let α be the real cube root of 2. Compute the minimal polynomial for $1 + \alpha^2$ over $\mathbb{Q}$.

(5) Determine the minimal polynomial of $\alpha = \sqrt{3} + \sqrt{5}$ over each of the following fields:(a) $\mathbb{Q}$, (b) $\mathbb{Q}(\sqrt{5})$, (c) $\mathbb{Q}(\sqrt{10})$, (d) $\mathbb{Q}(\sqrt{15})$.

(6) Let $\alpha \in \mathbb{C}$ be a root of the irreducible polynomial $x^3 - 3x + 4$. Express the inverse of $\alpha^2 + \alpha + 1$ in $\mathbb{Q}(\alpha)$ in the form $a + b\alpha + c\alpha^2$, $a, b, c \in \mathbb{Q}$.

(7) Let K/F be a field extension and $\alpha \in K$ an element of degree 5 over F. Show that $F(\alpha^2) = F(\alpha)$.

(8) Let a be a positive rational number such that a is not a square in $\mathbb{Q}$. Prove that $a^{1/4}$ has degree 4 over $\mathbb{Q}$.

(9) Determine whether i is in the following fields (a) $\mathbb{Q}(\sqrt{-2})$, (b) $\mathbb{Q}((-2)^{1/4})$, (c) $\mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha + 1 = 0$.

(10) Let α, β be complex numbers of degree 3 over $\mathbb{Q}$. Determine the possibilities for the degree $[F(\alpha, \beta) : F]$. (You should show by example that each case occurs.)

(11) Let $\alpha, \beta \in \mathbb{C}$ be roots of irreducible polynomials $f(x), g(x) \in \mathbb{Q}[x]$. Show that f is irreducible in $\mathbb{Q}(\beta)$ iff g is irreducible in $\mathbb{Q}(\alpha)$.

Hints: In some of the problems you need to show that a polynomial $f(x) \in F[x]$ is irreducible, where F is some field. Here are some ways to do this:

(1) If f has degree 2 or 3 then f is irreducible over F iff f does not have a root in F.

(2) Suppose $f \in \mathbb{Z}[x]$ and the highest common factor of all the coefficients of f equals 1. Then f is irreducible in $\mathbb{Q}[x]$ iff f is irreducible in $\mathbb{Z}[x]$ (Gauss' lemma, DF p. 303).

(3) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$. Let p be a prime such that p does not divide a_n. Let $\bar{f} \in \mathbb{F}_p[x]$ denote the reduction of f modulo p. If $\bar{f}$ is irreducible over $\mathbb{F}_p$ then f is irreducible over $\mathbb{Q}$.

(4) This is an application of (3) above. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$. Suppose p is a prime such that p does not divide a_n, p divides $a_{n-1}, \ldots, a_1$ and p^2 does not divide a_0. Then f is irreducible (Eisenstein's Criterion, DF p. 309).