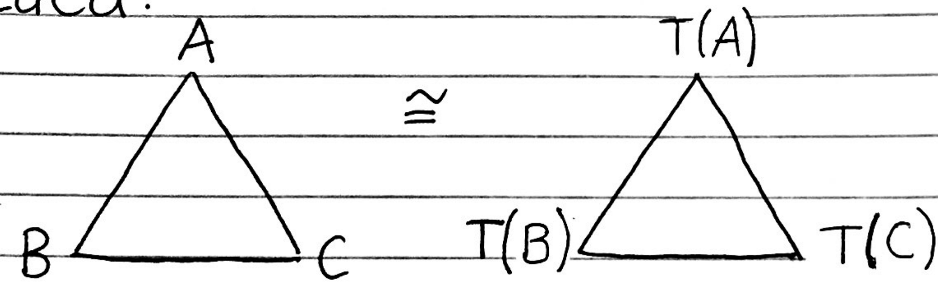
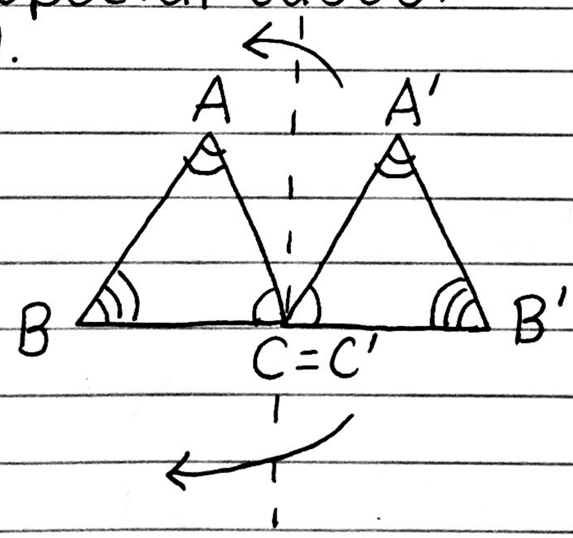
Math 461 Lecture 19 10/17
Homework is handed back

If $\triangle ABC \cong \triangle A'B'C'$ then you can construct an isometry $T: \mathbb{R}^2 \to \mathbb{R}^2$ that maps the first triangle to the second triangle $T(\triangle ABC) = \triangle A'B'C'$.
$T = Refl_M \circ Refl_N \circ Refl_L$

3-Reflection Theorem:
Every isometry $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of at most three reflections.
Idea:

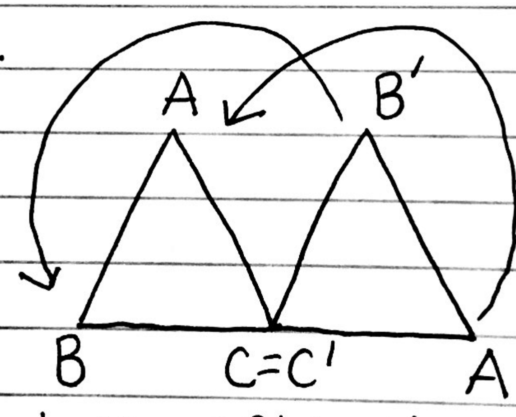


Special cases:
1.



one reflection

2.



two reflections
rotation = $Refl_N \circ Refl_L$
N not parallel to L

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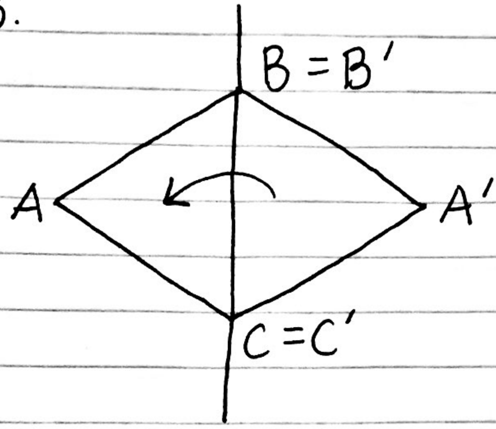
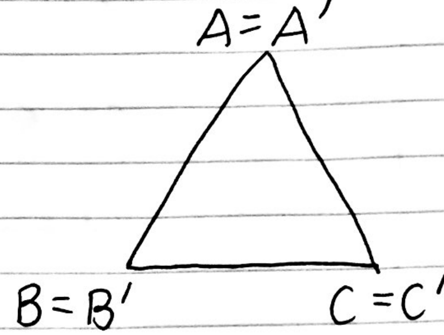
3.



one reflection

4.

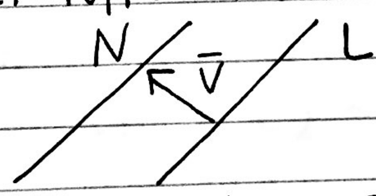


zero reflections

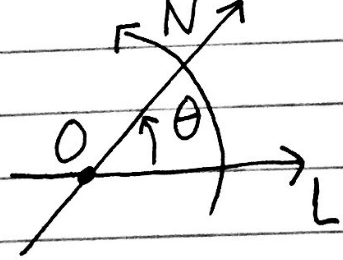
Classify isometries on $\mathbb{R}^2$:

# of reflections	isometry
0	identity
1	reflection
2	① translation - if lines are parallel ② rotation - if lines are not parallel
3	① reflection ② glide translation (translation and reflection)

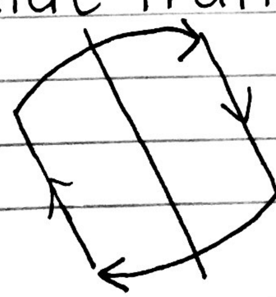
① If $N \parallel L$ then $Refl_N \circ Refl_L = Trans_{2\bar{v}}$



② If $N \not\parallel L$ then $Refl_N \circ Refl_L = Rot(O, 2\theta)$



② glide translation



translate and reflection

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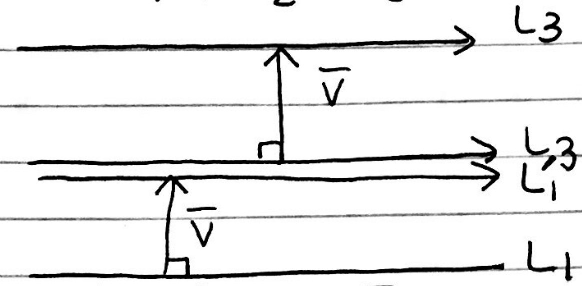
Proof: $T = \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$

Case a: $L_1 \parallel L_2 \parallel L_3$

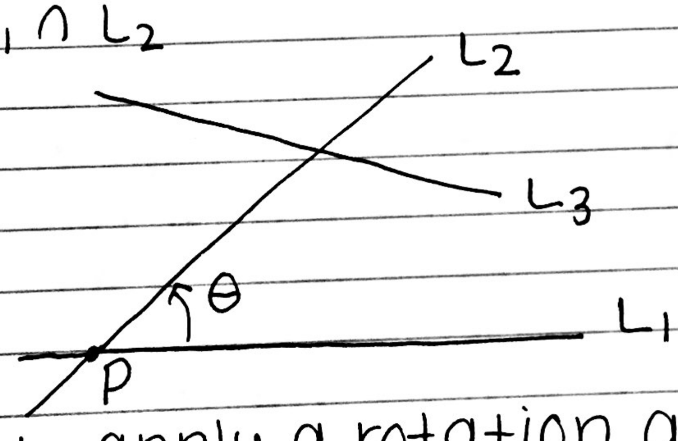


$\mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2} = \mathrm{Trans}_{2\bar{v}} = \mathrm{Refl}_{L_1'} \circ \mathrm{Refl}_{L_1'}$

$T = (\mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2}) \circ \mathrm{Refl}_{L_1}$
$= \mathrm{Trans}_{2\bar{v}} \circ \mathrm{Refl}_{L_1}$
$= (\mathrm{Refl}_{L_1'} \circ \mathrm{Refl}_{L_1'}) \circ \mathrm{Refl}_{L_1}$
$= \mathrm{Refl}_{L_1'} \circ \text{identity}$
$= \mathrm{Refl}_{L_1'}$

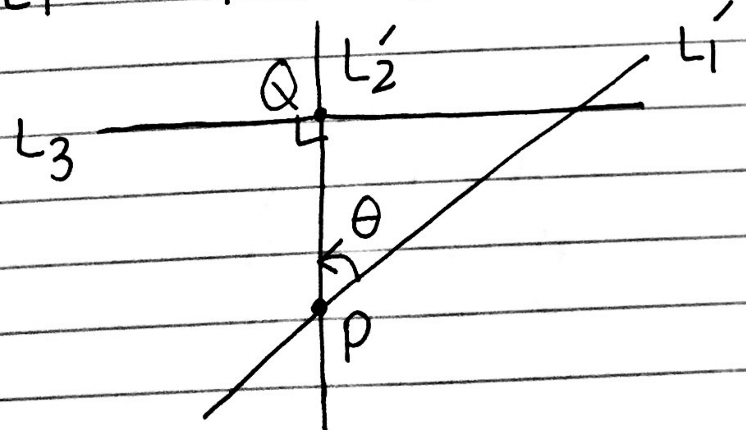
Case b: without loss of generality (WLOG) assume L_1 and L_2 are not parallel

$P = L_1 \cap L_2$



Step 1: apply a rotation about P

$L_1 \rightsquigarrow L_1'$, $L_2 \rightsquigarrow L_2'$



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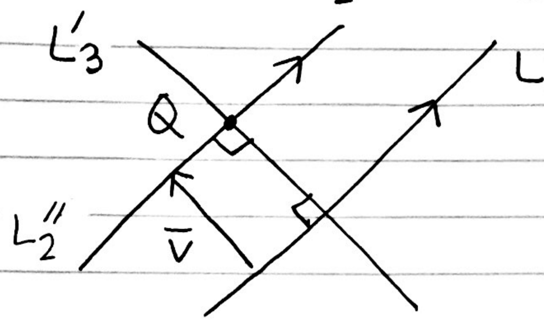
$T = Refl_{L_3} \circ (Refl_{L_2} \circ Refl_{L_1})$
$\quad = Refl_{L_3} \circ Rot(P, 2\theta)$
$\quad = Refl_{L_3} \circ (Refl_{L_2'} \circ Refl_{L_1'})$

Step 2: apply another rotation about Q $\quad L_2' \rightsquigarrow L_2''$, $L_3 \rightsquigarrow L_3'$



$L_2'' \parallel L_1'$

$T = (Refl_{L_3} \circ Refl_{L_2'}) \circ Refl_{L_1'}$
$\quad = Rot(Q, \pi) \circ Refl_{L_1'}$
$\quad = (Refl_{L_3'} \circ Refl_{L_2''}) \circ Refl_{L_1'}$
$\quad = Refl_{L_3'} \circ Trans_{2\bar{v}} \qquad \bar{v} \parallel L_3'$

glide translation

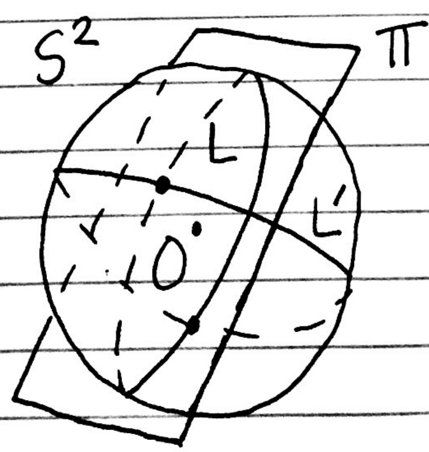
Spherical Geometry

S^2: the sphere is $\mathbb{R}^3$ with center $O = (0,0,0)$ and radius 1

$S^2 = \{P \in \mathbb{R}^3 \mid |OP| = 1\}$
$\quad = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

"lines" on S^2 are called a spherical line or a great circle on S^2 is $L = \Pi \cap S^2$

Π: plane through the origin O



no parallel "lines" on S^2

more precisely,

$L = \Pi \cap S^2$,

$L' = \Pi' \cap S^2$

$L \cap L' = (\Pi \cap \Pi') \cap S^2$
$\qquad\qquad = \ell \cap S^2$

pass through origin

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$= \{P, P'\}$ P and P' are antipodal