Math 300.2 Homework 2

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Reading: Gilbert and Vanstone, Chapter 4.

(1) Translate each of the following mathematical statements into a english sentence. Are the statements true or false?

(a) $\forall x \in \mathbb{R} \quad x^2 + 2x + 3 > 0$.

(b) $\exists x \in \mathbb{Z} \quad x^3 + x = 10$.

(c) $\forall x \in \mathbb{Z} \quad \exists y \in \mathbb{Z} \quad y > x$.

(d) $\forall x \in \mathbb{Z} \quad \exists y \in \mathbb{Z} \quad \exists z \in \mathbb{Z} \quad x = 4y + 7z$.

(2) Translate each of the following sentences into a logical statement using quantifiers.

(a) For each real number x, if $x > 3$ then $e^x > 20$.

(b) There is a real number x such that $x^3 = 13$.

(c) There is a positive real number δ such that if x is a real number satisfying $|x - 1| < \delta$ then $|x^3 - 1| < 0.1$.

(d) For every positive integer n, $n^3 + 1$ is divisible by $n + 1$. [Hint: We say an integer a is divisible by an integer b if there is an integer c such that $a = bc$].

(3) Form the negation of each of the following statements.

(a) $\exists x \in \mathbb{Z} \quad x^3 + x + 1 = 0$.

(b) $\forall x \in \mathbb{R} \quad (1.01)^x < x^2 + 2$.

(c) $\forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \quad x^2 + y^2 = 4$.

(d) $\exists y \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad y \neq x^3 - x + 2$.

(4) Prove the following statements for every positive integer n using mathematical induction.

(a) Let r be a real number, $r \neq 1$. Then

$$\sum_{i=1}^{n} r^{i-1} = 1 + r + r^2 + \cdots + r^{n-1} = \frac{(1-r^n)}{(1-r)}.$$

(b) A sequence of numbers a_n is defined by $a_1 = 13$ and $a_{n+1} = 3a_n + 4$ for each $n \in \mathbb{N}$. Show that $a_n = 5 \cdot 3^n - 2$ for each n.

(c) $\sum_{i=1}^{n} i(i+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{1}{3} n(n+1)(n+2)$.

(5) Prove the following statement for each $n \in \mathbb{N}$ by induction. Suppose n lines are drawn in the plane so that no 2 lines are parallel and no 3 lines pass through the same point. The lines divide the plane into a number of regions. Show that it is possible to color each region either red or blue so that any two regions which are adjacent are different colors. (Here we say two regions are *adjacent* if they share a common edge given by a segment of one of the lines.) [Hint: In class we used induction to prove a formula for the number of regions. Some of the ideas we used there are useful for this problem too.]