9/25/19

HW 1 returned

HW 2 due now

HW 3 available at people.math.umass.edu/~hacking/461F19

<u>Last Time</u>: • Constructible lengths – Multiplication & Division

• Subdivision of line segment into n equal parts

<u>Stated</u>: • Constructible lengths are those obtained from 1 by $+, -, \times, \div, \sqrt{\ }$

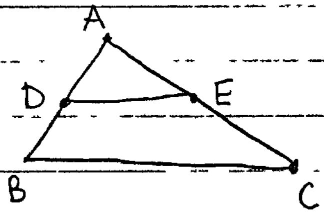
• It is impossible to subdivide an angle into n equal parts in general

(e.g. $n=3$ is impossible in general)

<u>Today</u>: • Converse of Thales' theorem

• Impossibility of trisection of angle (sketch of proof)

• Angles in a circle → didn't get to it



<u>Converse of Thales' theorem</u>

(*) $\frac{|AD|}{|AB|} = \frac{|AE|}{|AC|} \Rightarrow$ DE is parallel to BC

("$\Leftarrow$" is Thales' theorem)

<u>Proof</u> Draw parallel line to BC through D (intersects AC at point F)

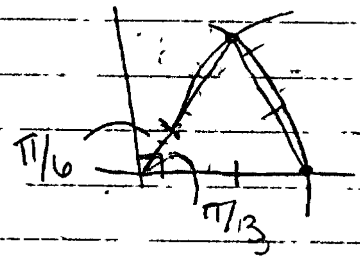
By Thales' theorem: $\frac{|AF|}{|AC|} = \frac{|AD|}{|AB|} = \frac{|AE|}{|AC|}$

by our assumption (*)

$\Rightarrow |AF| = |AE| \Rightarrow F = E$, so line DE is equal to DF, i.e. it's parallel to BC. ▨

Claim: It is impossible to trisect a given angle by ruler and compass in general.
↳ divide into 3 equal parts
↳ some angles can be trisected (but not all)

e.g. $\pi/2$

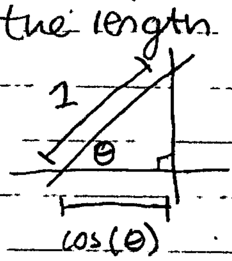


1. Draw equilateral Δ
2. Bisect angle $\pi/3$

We will explain why $\pi/3$ <u>cannot</u> be trisected.

Notice: An angle θ can be constructed using ruler & compass
$\iff$ the length $\cos\theta$ can be constructed.

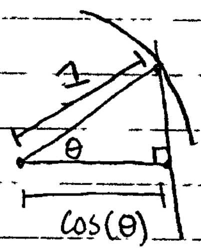
" $\Rightarrow$ "



SOHCAHTOA

$\cos\theta = \frac{a}{h}$

" $\Leftarrow$ "



→ drop perp. Find hyp. w/ length 1 w/ compass.

Notice $\pi/3$ can be constructed.
So $\pi/3$ can be trisected
$\iff$ $\pi/9$ can be trisected
(don't need to be given angle $\pi/3$)

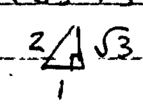
We will show $\pi/9$ cannot be constructed, equivalently length $\cos\pi/9$ can't be constructed.

Recall: $\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2 \Leftarrow \cos(A+B) = \cos A\cos B - \sin A \sin B$ ($A = B = \theta$)

$\sin 2\theta = 2\sin\theta\cos\theta$

To understand $\cos \pi/9$, use formula expressing $\cos 3\theta$ in terms of $\cos\theta$ (take $\theta = \pi/9$, then $\cos\theta = \cos \pi/9 = ??$, $\cos 3\theta = \cos \pi/3 = 1/2$)

$\rightsquigarrow$ equation satisfied by $\cos \pi/9$



$$\boxed{\cos(3\theta) = 4(\cos\theta)^3 - 3\cos\theta}$$

$\hookrightarrow \cos(3\theta) = \cos(2\theta+\theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta = \ldots = 4(\cos\theta)^3 - 3\cos\theta$

go to OH
to learn more

$x = \cos \pi/9 \Rightarrow \frac{1}{2} = \cos(\pi/3) = 4x^3 - 3x$, $\boxed{8x^3 - 6x - 1 = 0}$

Recall: $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\rightsquigarrow$ x is constructible

$ax^3 + bx^2 + cx + d = 0$ $\quad a, b, c, d$ constants

First divide by $a \rightsquigarrow$ $1 \cdot x^3 + ex^2 + fx + g = 0$ $\quad e, f, g$ constants

Second "completing the cube" $\rightsquigarrow$ $\boxed{y^3 + py + q = 0}$ $\quad p, q$ constants

$\hookrightarrow y = (x + 1/3 e)$

$$y = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$\rightarrow$ Tartaglia ~ 1500

$\hookrightarrow$ mathmetician

NOT constructible in general, because require $\sqrt[3]{\ }$ (not quite a proof)

Actual proof. Back to $x = \cos \pi/9$, satisfies $8x^3 - 6x - 1 = 0$.
Show the polynomial $8x^3 - 6x - 1$ is irreducible (can't be factored as a product of two polynomials of lower degree with rational coefficients — equivalently, because it's a cubic there are no rational solutions to $f(x) = 0$.)

$\longrightarrow$

__Fact:__

If you have some polynomial:

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$,

where $a_0, \dots, a_n$ are coefficients,

if $x = a/b$ is a rational solution of $f(x) = 0$,

then $a \mid a_0$ & $b \mid a_1$

ex: $3x^2 + 7x + 5 = 0$

a/b where $a \mid 5$, $b \mid 3$, either $\pm 5/3$, ± 5, $\pm 1/3$, ± 1

By this fact, check: $x = \pm 1, \pm 1/2, \pm 1/4, \pm 1/8$ are __not__ solutions.

Now use the theory of Field extensions (MATH 412) to conclude that the solutions of $f(x) = 0$ are not constructible.

$\sim$1800 (Galois) – sorta

Another example: $x^3 - 2$ is irreducible $\Rightarrow$ $\sqrt[3]{2}$ is not constructible.

("doubling the cube" is impossible)

__What happens if $3\theta = \pi/2$, $\theta = \pi/6$?__

$\cos 3\theta = 4(\cos\theta)^3 - 3\cos\theta$, $y = \cos \pi/6$

$0 = 4y^3 - 3y = y(4y^2 - 3) \rightsquigarrow y = 0, \pm\frac{\sqrt{3}}{2}$ constructible