Q8. Let $|G| = p^e \cdot m$, $p \nmid m$.

$|N| = p^f \cdot m'$, $p \nmid m'$

$|G/N| = p^g \cdot m''$, $p \nmid m''$ where $e = f+g$ & $m = m'm''$.

Write $q : G \to G/N$ for the quotient hom.

We will construct an injective map

$$\{\text{Sylow } p\text{-subgroups of } G/N\} \xrightarrow[f]{} \{\text{Sylow } p\text{-subgroups of } G\}$$

as follows: given $K \leq G/N$, $|K| = p^g$

consider $q^{-1}K \leq G$, $|q^{-1}K| = p^g \cdot |N|$
$= p^e \cdot m'$

Let $L \leq q^{-1}K$ be a Sylow p-subgroup.

Then $|L| = p^e$ & so $L \leq G$ is a Sylow p-subgroup.

Define $f(K) = L$.

It remains to show that f is injective. This follows from the

<u>Claim</u>: $q(L) = K$.

<u>Proof</u>: $L \leq q^{-1}K \Rightarrow q(L) \leq q(q^{-1}K) = K$.

Also, $\ker(L \xrightarrow{q|_L} G/N) = \ker(q) \cap L = N \cap L$.

Since $|N| = p^f \cdot m'$ & $|L| = p^e$

we have $|N \cap L| \leq \gcd(|N|, |L|) = p^f$

So $|q(L)| = |L| / |N \cap L| \geq p^e / p^f = p^g = |K|$.

$\therefore q(L) = K$. $\square$