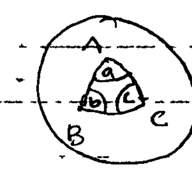
11/8/19

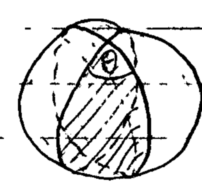
HW6 due now

<u>Last Time</u>: · Angles between spherical lines
· Spherical triangles
· <u>Theorem</u> $a+b+c = \pi + \text{Area}(\triangle ABC)$



<u>Today</u>: · Proof of Theorem
· spherical cosine rule $\Rightarrow$ spherical triangle inequality
$\Rightarrow$ great circles give shortest paths

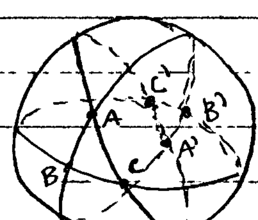
<u>Proof</u> (warm-up)

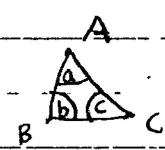


"lune"
- made with great circles

Area(lune) = ?
$= \underbrace{\text{Area}(S^2) \cdot \theta/2\pi}$ $= 4\pi\,\theta/2\pi = \underline{\underline{2\theta}}$

Area(sphere) $= 4\pi r^2$
Our case $r=1$





A, A'
B, B'
C, C' } antipodal pairs of points

Use fact: Area(lune w/ angle θ) $= 2\theta$

See 6 lunes in picture. (2 each with angle a, b, c)

Area(lunes) $= 2(2a) + 2(2b) + 2(2c) = 4a + 4b + 4c$
$= \text{Area}(S^2) + 2 \cdot \text{Area}(\triangle ABC) + 2\,\text{Area}(\triangle A'B'C')$
↳ <u>overcounted</u> $\triangle ABC$ & $\triangle A'B'C'$

$\text{Area}(\triangle ABC) = \text{Area}(\triangle A'B'C')$?

$\ulcorner$ A, A' antipodal means $\overrightarrow{OA'} = -\overrightarrow{OA}$

So, $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T(x,y,z) = (-x,-y,-z)$ $\Rightarrow$ preserves spherical distances

Then T is an isometry (preserves distances $\Rightarrow$ preserves angles)

and $T(A) = A'$, $T(B) = B'$, $T(C) = C'$ $\rightsquigarrow$ $\text{Area}(\triangle ABC) = \text{Area}(\triangle A'B'C')$. $\lrcorner$

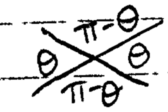
Aside: could use "tunnel" argument, haven't proved SSS for spherical triangles yet.

Thus, Area(lunes) $= 4\pi + 4\,\text{Area}(\Delta ABC)$
$\div 4 \Rightarrow a + b + c = \pi + \text{Area}(\Delta ABC)$

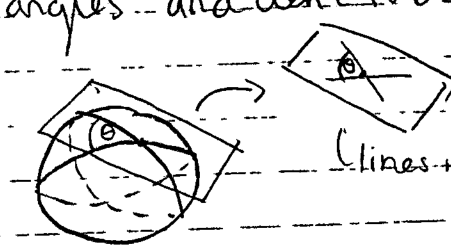
Note, need Area(ΔABC) if have 2 angles and want to determine third angle.

Recall

In $\mathbb{R}^2$,



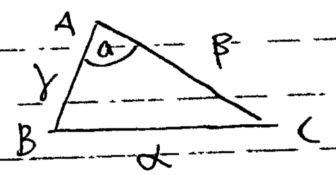
On S^2?



(lines in a plane

Spherical Cosine Rule

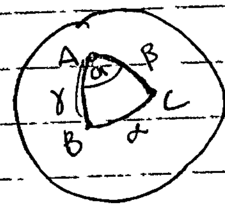
Recall cosine rule in $\mathbb{R}^2$:



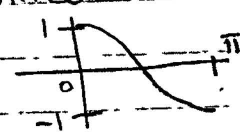
$$\alpha^2 = \beta^2 + \gamma^2 - 2\beta\gamma\cos a$$

SCR:

$$\boxed{\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a}$$

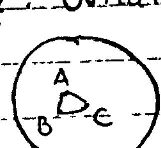


If $\cos\alpha$ is known, since $0 < \alpha < \pi$, α is determined



cannot be π because ~~if two are antipodal~~ I can find a great circle that all three points lie on.

Q: What happens if ΔABC is small?



SCR becomes...?

ECR (Euclidean Cosine Rule) $\longrightarrow$ if zoom into sphere, get's closer & closer to the plane

If x is small: $\sin x \approx x \Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\cos x \approx 1 - \frac{x^2}{2} \Rightarrow \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} \ldots$

$\longrightarrow$

If ΔABC is small:

$$1 - \alpha^2/2 \hat{=} (1 - \beta^2/2)(1 - \gamma^2/2) + \beta\gamma\cos a$$

$$= 1 - \beta/2 - \gamma^2/2 + \underbrace{\beta^2\gamma^2/4}_{} + \beta\gamma\cos a$$ → β & γ very small

get rid of ~~1's &~~ multiply by -2

$$\alpha^2 \hat{=} \beta^2 + \gamma^2 - 2\beta\gamma\cos a$$