Math 461 Lecture 36 12/5

Homework 8 due Friday

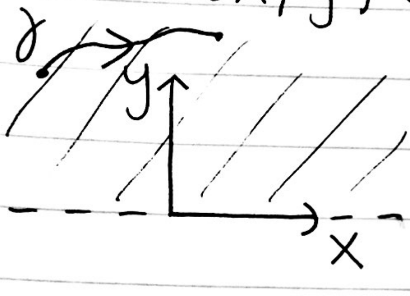
Office hours Thursday 4-5 pm

Last time:

Hyperbolic Geometry

Upper half plane model

$H = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$



γ curve in H, parametrization

$\bar{x}: [a,b] \to H,$

$t \longmapsto (x(t), y(t))$

$$\text{hyperbolic length}(\gamma) = \int_a^b \frac{\sqrt{x'^2 + y'^2}}{y}\, dt$$

by definition

$d_H(P,Q)$ = length of shortest path from P to Q

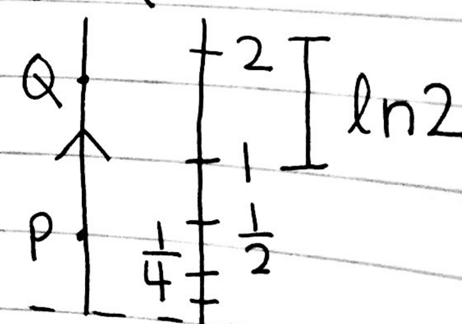
distances distorted, angles correct

Example:

$P = (c,a),\ Q = (\overset{c}{\cancel{a}}, b),\ 0 < a < b$

shortest path from P to Q = vertical line segment

$d_H(P,Q) = \ln\left(\frac{b}{a}\right)$



Today:

- shortest paths for any $P, Q \in H$
- hyperbolic isometries (examples)
- inversion

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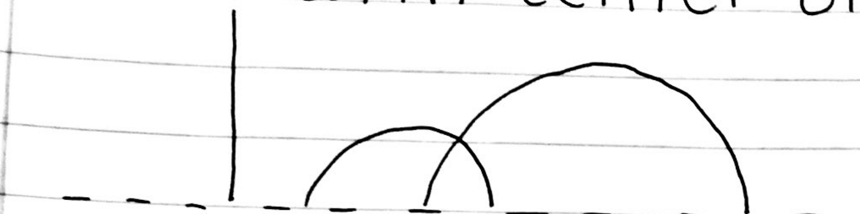
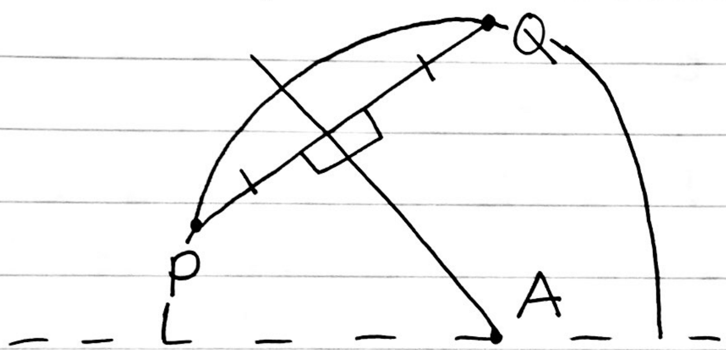
Question: what's the shortest path between P and Q $\in$ H?

Answer: Given by a segment of a hyperbolic line - either a vertical half line OR a semi-circle with center on the x-axis



examples of hyperbolic lines

First observe: There is an unique hyperbolic line through any two points P, Q $\in$ H



draw circle center A radius $|AP| = |AQ|$

We will prove hyperbolic lines give shortest paths by reducing to the case of vertical lines using hyperbolic isometries

$T: H \to H$ such that

$d_H(T(P), T(Q)) = d_H(P, Q)$

hyperbolic isometry

Questions: what are some examples of hyperbolic isometries

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$\text{length}(\gamma) = \int_a^b \frac{\sqrt{x'^2+y'^2}}{y}\,dt$

$d_H(P,Q)$ = length of shortest path

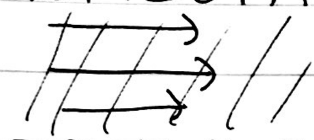
☆ $(x,y) \longmapsto T(x,y) = (f(x,y), g(x,y))$

T will be a hyperbolic isometry if $\frac{\sqrt{x'^2+y'^2}}{y}$ doesn't change under the substitution $x = f(x,y)$, $y = g(x,y)$

then $\text{length}(\gamma) = \text{length}(T(\gamma))$ for all γ

1. $T(x,y) = (x+a, y)$ for some constant a

horizontal translation



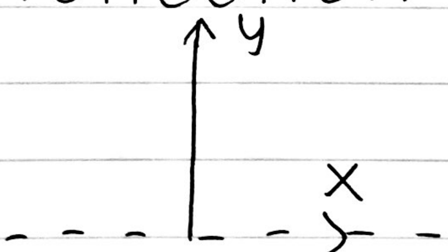
$T: H \to H$

$$\frac{\sqrt{((x+a)')^2+y'^2}}{y} = \frac{\sqrt{x'^2+y'^2}}{y} \checkmark$$

$a' = 0$ because a is a constant

2. $T: H \to H$ $(x,y) \to (-x,y)$

reflection in y-axis

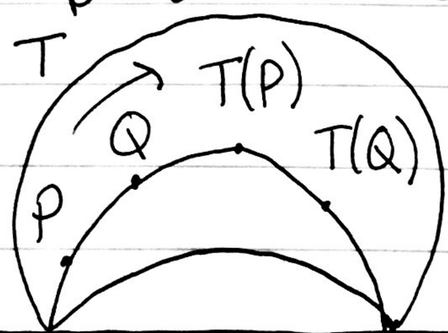


$$\frac{\sqrt{((-x)')^2+y'^2}}{y} = \frac{\sqrt{x'^2+y'^2}}{y} \checkmark$$

$(-x)' = -x'$

also can reflect in any vertical line

composition of types 1 and 2



hyperbolic translation will explain later

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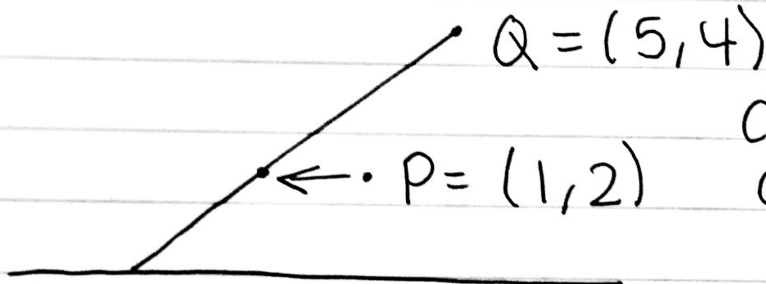
3. Scaling by constant $c>0$

$T(x,y) = (cx, cy)$

$$\frac{\sqrt{((cx)')^2+((cy)')^2}}{cy} = \frac{\sqrt{c^2x'^2+c^2y'^2}}{cy}$$

$$= \frac{\sqrt{x'^2+y'^2}}{y} \checkmark$$

At this point, can already show that the group of isometries of H acts transitively:
given two points P and Q, there is a hyperbolic isometry that sends P to Q



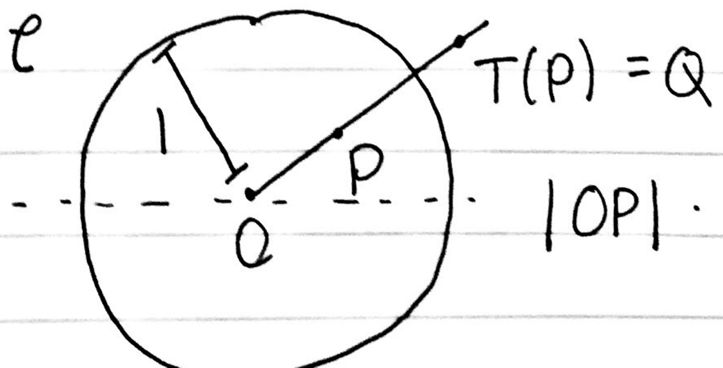
combination of 1 and 3

4. (Homework 8 Question 4)

Inversion $T: \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$

$$(x,y) \longmapsto \frac{1}{x^2+y^2}(x,y)$$

in the unit circle $\ell: x^2+y^2=1$
center O, radius 1



$|OP| \cdot |OQ| = 1$

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$$\|T(x,y)\| = \frac{\|(x,y)\|}{x^2+y^2} = \frac{\sqrt{x^2+y^2}}{x^2+y^2} =$$

$$\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{\|(x,y)\|}$$

Properties:

1. T fixes $\mathcal{C}$ pointwise
   ($T(P) = P$ for $P \in \mathcal{C}$)
2. P inside $\mathcal{C} \Rightarrow T(P)$ outside $\mathcal{C}$
   P outside $\mathcal{C} \Rightarrow T(P)$ inside $\mathcal{C}$
3. $T^2 =$ identity

infer: $T: H \to H$ is reflection in the hyperbolic line $L = \mathcal{C} \cap H$

check T is a hyperbolic isometry

$$\frac{\sqrt{x'^2+y'^2}}{y} \qquad T(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

$$\left(\frac{x}{x^2+y^2}\right)' = \frac{x'(x^2+y^2) - (2xx' + 2yy')x}{(x^2+y^2)^2}$$

$$\frac{d}{dt}\left(\frac{x}{x^2+y^2}\right) = \frac{(y^2-x^2)x' - (2xy)y'}{(x^2+y^2)^2}$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{\frac{du}{dt}v - \frac{dv}{dt}u}{v^2}$$

similar: $$\left(\frac{y}{x^2+y^2}\right)' = \frac{(x^2-y^2)y' - (2xy)x'}{(x^2+y^2)^2}$$

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$$\left(\left(\frac{x}{x^2+y^2}\right)'\right)^2 + \left(\left(\frac{y}{x^2+y^2}\right)'\right)^2 =$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\frac{1}{(x^2+y^2)^4}\left[\left((y^2-x^2)x' - 2xyy'\right)^2 + \left((x^2-y^2)y' - 2xyx'\right)^2\right] =$$

$$\frac{1}{(x^2+y^2)^4}\left[(y^2-x^2)^2(x'^2+y'^2) + 4x^2y^2(x'^2+y'^2)\right] =$$

$$\frac{1}{(x^2+y^2)^4}(x'^2+y'^2)\left((y^2-x^2)^2 + 4x^2y^2\right) =$$

$$(a-b)^2 + 4ab = (a+b)^2$$

$$\frac{1}{(x^2+y^2)^4}(x'^2+y'^2)(x^2+y^2)^2 = \frac{(x'^2+y'^2)}{(x^2+y^2)^2}$$

now
$$\frac{\sqrt{\left(\frac{x}{x^2+y^2}\right)'^2 + \left(\frac{y}{x^2+y^2}\right)'^2}}{\frac{y}{x^2+y^2}} = \frac{\sqrt{\frac{x'^2+y'^2}{(x^2+y^2)^2}}}{\frac{y}{x^2+y^2}} = \frac{x'^2+y'^2}{y}$$