Jenia Tevelev

MMP Intro   2:30   LGRT 1334

* $X \hookrightarrow \mathbb{A}^n$ affine alg. var. $k[X] = k[x_1 \ldots x_n]/I(X)$
domain
"coordinate algebra"
"algebra of reg. functions"

* $k(X) = $ f.f. $k[X]$ field of rational functions.

* X and Y are called birational if

$k(X) \simeq k(Y) \iff \exists$ rat. maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$

such that $f \circ g = g \circ f = Id$
(where defined)

$\Updownarrow$

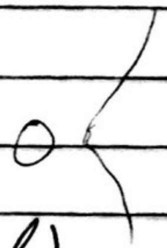
$\exists$ open $U \hookrightarrow X$, $V \hookrightarrow Y$ such that $U \simeq V$.

* Looks like a weak condition (for example in diff. geometry every 2 manifolds X and Y of the same dimension have isom. open sets $\simeq \mathbb{R}^k$) but ~~[illegible]~~ Zariski open sets are "large" (complements to closed subsets of smaller dim)

* X is called rational if X is birational to $\mathbb{A}^r$
$\Updownarrow$
$k(X) \simeq k(x_1 \ldots x_r)$

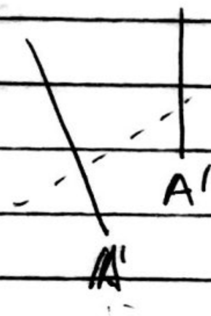
* Let $X=(f=0)\subset \mathbb{A}^{n+1}$ be a smooth cubic hypersurface of dimension n.

* $n=1 \Rightarrow$ elliptic curve $y^2=x^3+ax+b$



not rational (will review the proof from class)

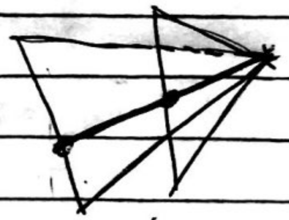
* $n=2 \Rightarrow$ cubic surface. We have seen in class that they contain 27 lines, some of which are skew.



third intersection point

$\mathbb{A}^1$ $\quad \mathbb{A}^2=\mathbb{A}^1\times\mathbb{A}^1 \dashrightarrow S$

$\mathbb{A}^1$



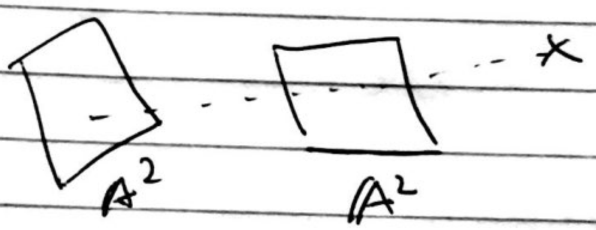
$\mathbb{A}^1\times\mathbb{A}^1 \dashleftarrow S$

Rational!

* $n=3$ Not rational (Clemens-Griffiths)

* $n=4$ (and >4) One of the biggest open questions: is a general cubic n-fold irrational for $n\geq 4$?

* To complicate matters, some cubic 4-folds are rational, for example suppose X contains 2 skew 2-planes



$\mathbb{A}^2 \times \mathbb{A}^2 \dashrightarrow X$ birational.

* Unfortunately, a general cubic 4-fold doesn't even contain one 2-plane.

$X = (f=0)$ $\quad f|_{\mathbb{A}^2} \in k[x,y]$, degree 3, 10 coefficients

$\Rightarrow \mathbb{A}^2 \subset X$ is a codimension 10 condition on coefficients of f.

parameter space for $\mathbb{A}^2 \hookrightarrow \mathbb{A}^5$

has dimension $\underbrace{(5+5+5)}_{3 \text{ points in } \mathbb{A}^5} - \underbrace{(2+2+2)}_{3 \text{ points in } \mathbb{A}^2} = 9$

* So a cubic 4-fold contains a plane is a codimension $10-9=1$ condition

(hypersurface = "divisor") in the space of all cubic 4-folds.

★ How did we prove that an elliptic curve is not rational?

★ Step 1 take closures $\bar{E} \hookrightarrow \mathbb{P}^2$, $\mathbb{P}^1$
$E = (f=0)$; $f \in k[x,y,z]$
homogeneous polynomial of degree 3.

★ Step 2 Check that E is nonsingular.

If $p \in X$ point on alg. variety $\mathcal{O}_p = \left\{ \begin{array}{l} f \in k(X) \\ f \text{ regular at } p \end{array} \right\}$
local ring of the point

★ For curves, $p \in C$ nonsingular $\Leftrightarrow$ $\mathcal{O}_p$ is a DVR,

i.e. there exists a uniformizer (= local parameter)

$t \in \mathcal{O}_p$ such that every non-zero $f \in \mathcal{O}_p$

can be written uniquely as $f = t^k u$ where $u(p) \neq 0$.

★ Step 3 Every rational map $\underset{\text{sm. curve}}{C} \dashrightarrow \mathbb{P}^n$

is regular, i.e. defined at every point.

In particular, smooth birational projective curves are in fact isomorphic

★ To see this, $\varphi: C \dashrightarrow \mathbb{P}^r_{z_0:\dots:z_r}$ is given

by $\varphi = (\varphi_0 : \dots : \varphi_r)$ $\quad \varphi_i \in k(C)$

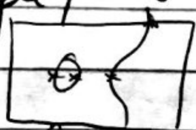
at $p \in C$ write $\varphi_i = t^{k_i} u_i$ and let $k_j := \min\{k_i\}$

$$\varphi = (t^{k_0 - k_j} u_0 : \dots : \underset{\neq 0}{u_j} : \dots : t^{k_r - k_j} u_j)$$

$\Rightarrow$ defined at p and sends p to the chart $z_j \neq 0$.

★ <u>Step 4</u> Checking that $E \not\cong \mathbb{P}^1$.

For example, E admits a hyperelliptic involution



$(x,y) \leftrightarrow (x,-y)$

with 4 fixed points but every involution of $\mathbb{P}^1$ is a Möbius transformation $x \mapsto \frac{ax+b}{cx+d}$ (and in fact every automorphism) with at most 2 fixedpoints.

★ <u>Goal of MMP</u>: For every projective variety X, construct a minimal model X^{min} (later people realized that in fact one needs what's called canonical model X^{can})

So that $X \underset{bir}{\dashleftarrow\dashrightarrow} Y \Leftrightarrow X^{can} \underset{iso}{\simeq} Y^{can}$

★ So MMP should really be called CMP ☺

★ In general, this can't be done (otherwise rationality would be an easy question) — we will need to impose some conditions on X.

★ Two big ingredients in MMP

Study of rational curves $\mathbb{P}^1 \underset{\text{bir}}{\leftrightarrow} C \hookrightarrow X$

Study of the canonical divisor K_X

★ Why rational curves? Reason 1: If $X \underset{\text{bir}}{\leftrightarrow} \mathbb{P}^n$ then X, just like $\mathbb{P}^1$, is rationally connected, i.e. given $p, q \in X$ general points, $\exists\, p, q \in \underset{\underset{X}{\cap}}{C} \underset{\text{bir}}{\leftrightarrow} \mathbb{P}^1$.

★ Unfortunately, this doesn't prove that a general cubic n-fold is irrational: its general 3-plane section is a smooth cubic surface $\Rightarrow X$ is rationally connected!

★ Reason 2 Why can't we just take smooth projective models as minimal models? like we did for curves?

Fields! ★ Theorem (Hironaka) $\forall\, X \hookrightarrow \mathbb{P}^r$ projective variety, $\exists\ Y \to X$ birational morphism such that Y is a smooth projective variety $\Rightarrow$ every variety has a smooth projective model

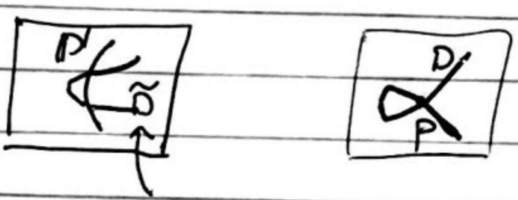
★ The reason is existence of blowups:
in dim ≥ 2 varieties can be birational but not isomorphic

★ $p \in S$ smooth point on a surface

$$\exists \quad \begin{array}{ccc} Bl_p S & \longrightarrow & S \\ \cup & & \cup \\ E = \mathbb{P}^1 & \longrightarrow & p \end{array} \quad \text{iso outside } p$$

"exceptional divisor"

birational but not isomorphic!



proper transform.

Remark: note how blow-up of S resolves singularity of D. In fact that's how Hironaka Th. is proved.

★ More generally, $Y \hookrightarrow X$
smooth smooth
$r = \text{codim} \geq 2$

$$\begin{array}{ccc} Bl_Y X & \longrightarrow & X \\ \cup & & \cup \\ E & \xrightarrow[\mathbb{P}^{r-1}\text{-bundle}]{} & Y \end{array}$$

(projectivized normal bundle of $N_{Y/X}$)

★ Note: E covered by rational curves.

★ Philosophy: MMP should ~~[illegible]~~ "contract" rational curves to get rid of blow-ups.

★ We have to rethink how to embed $X \hookrightarrow \mathbb{P}^n$

★ Weil divisor $D = \sum_{i=1}^{s} a_i D_i$ $a_i \in \mathbb{Z}$
$D_i \subset X$ irred. hypersurfaces.

[Eg. $X = C$ curve $\Rightarrow D = \sum a_i P_i$
$P_i \in C$]

★ ~~Linear system~~
If $a_i \geq 0$ $\forall i$ then we say that D is effective,
write $D \geq 0$

★ Linear system $\Gamma(X, D) = \{ f \in k(X) : (f) + D \geq 0 \}$.

Here (f) is the divisor of zeros-poles of f (with multiplicities)

Fields! ★ Theorem (Serre) $\Gamma(X, D)$ is finite-dimensional as long as X is projective.

★ ~~The~~ Choosing a basis $\varphi_0, \dots, \varphi_n \in \Gamma(X, D)$
Gives a rational map $X \overset{\varphi_D}{\dashrightarrow} \mathbb{P}^n$
$x \longmapsto (\varphi_0(x) : \dots : \varphi_n(x))$

★ D is called very ample if φ_D is an embedding

★ D is called ample if rD is very ample for some $r > 0$.

★ Note that $R(X, D) = \bigoplus_{r \geq 0} \Gamma(X, rD)$ is an algebra

called the coordinate algebra of D

★ D is called linearly equivalent to D', $D \sim D'$, if $D - D' = (f)$ for some $f \in k(X)$.

★ Note that in this case $\Gamma(X,D) \underset{\cdot \frac{1}{f}}{\overset{\cdot f}{\rightleftarrows}} \Gamma(X,D')$

$$(g) + D \geq 0 \iff (g) + D' + (f) \geq 0 \iff D' + (gf) \geq 0.$$

★ Therefore $\varphi_D = \varphi_{D'}$ (homogeneous coordinates!)
If choose corresponding bases or related by projective transformation of $\mathbb{P}^r$ if choose different bases.

★ Next we are going to construct a canonical divisor.

★ Let X be a smooth projective n-fold.

Recall that it means that $\exists\, n$ algebraically independent $f_1, \dots, f_n \in k(X)$ but every $n+1$ are algebraically dependent.

★ A rational n-form is an expression

$$\omega = f_0\, df_1 \wedge \dots \wedge df_n \qquad \begin{matrix} f_0, \dots, f_n \in k(X) \\ f_1 \dots f_n \text{ alg. indep.} \end{matrix}$$

computed using usual rules of exterior algebra and derivations (Leibniz)

* Take 2 of those: $f_0\, df_1 \wedge \dots \wedge df_n = \omega$
  and $g_0\, dg_1 \wedge \dots \wedge dg_n = \omega'$

<u>Claim</u>: $\omega' = h\,\omega$ for some $h \in k(X)$.

Indeed, $\forall k$ $f_1, f_2, \dots, f_n, g_k$ are alg. dependent

$F(f_1, f_2, \dots, f_n, g_k) = 0$, differentiate

$$\sum \frac{\partial F}{\partial f_i} df_i + \frac{\partial F}{\partial g_k} dg_k = 0.$$

$\forall k$ express dg_k in terms of df_i's, plug on ω'

$\Rightarrow \omega' = h\omega$.

* A canonical divisor K_X is a divisor of a rational n-form ω defined as follows:

Choose a system of local parameters $t_1 \dots t_n$ at $p \in X \Rightarrow$ in some Zariski open $p \in U \subset X$

write $\omega = f\, dt_1 \wedge \dots \wedge dt_n$

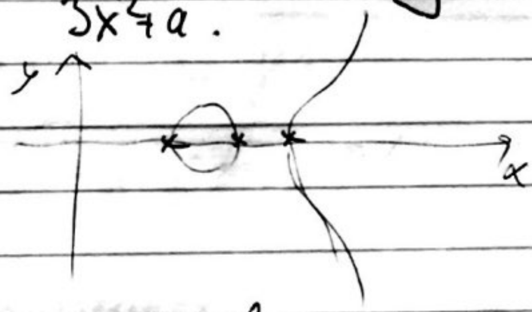
Then $(\omega) \cap U = (f) \cap U$.

* Note: K_X is defined uniquely up to linear equivalence.

★ Example: $y^2 = x^3 + ax + b$

$$2y\,dy = (3x^2+a)\,dx$$

$$\omega = \frac{2\,dy}{3x^2+a} = \frac{dx}{y}$$



we didn't have time to do that example

★ x is a local parameter outside $y=0$

$\Rightarrow$ ω has no zeros or poles there.

★ y is a local parameter near ramification points.

$\omega = \frac{2\,dy}{3x^2+a}$ Note: $y=0 \Rightarrow 3x^2+a \neq 0$ (x^3+ax+b has no multiple roots)

$\Rightarrow$ $(\omega) = 0$!

* Of course if $X \simeq Y$ then $R(X, K_X) \simeq R(Y, K_Y)$ but even better,

* <u>Theorem (easy)</u> $R(X, K_X)$ is a <u>birational invariant</u>, i.e. $X \underset{bir}{\longleftrightarrow} Y \Rightarrow R(X, K_X) \cong R(Y, K_Y)$.

* In particular, <u>geometric genus</u>

$$p_g := \dim \Gamma(X, K_X)$$

and <u>plurigenera</u> $P_k := \dim \Gamma(X, kK_X)$ are birational invariants.

* <u>Example</u>: E elliptic curve

$\Rightarrow K_E \sim \emptyset \Rightarrow kK_E \sim \emptyset \Rightarrow$

$$\Gamma(E, kK_E) = k[E] = k$$

(the only regular functions on a projective variety are constants).

$\Rightarrow P_k = 1 \quad \forall k.$

* <u>Example</u> $\mathbb{P}^1 = \mathbb{A}^1_x \cup \mathbb{A}^1_y$ $\quad \omega = dx = d\frac{1}{y} = \frac{-dy}{y^2}$

$\Rightarrow (\omega) = -2(\infty) \Rightarrow P_k = 0 \quad \forall k > 0.$

$\Rightarrow E \not\cong \mathbb{P}^1$ again

Definition The rate of growth of P_m is called Kodaira dimension κ:

$$P_m \sim \alpha\, m^{\kappa} \quad m >> 0, \text{ sufficiently divisible.}$$

Example: $\kappa(\mathbb{P}^1) = -\infty$
$\kappa(E) = 0$.

Fields! Theorem (BCHM very hard).

$R(X, K_X)$ is finitely generated.

Fields! (Mori in dimension 3)

If R is a finitely generated graded k algebra, one can construct a projective variety called $\operatorname{Proj} R$. If D is ample and $R = R(X, D)$ then $\operatorname{Proj} R \simeq X$

$X^{can} := \operatorname{Proj} R(X, K_X)$.

Clearly $X \underset{bir.}{\dashrightarrow} Y \Rightarrow X^{can} \simeq Y^{can}$.

Def X is called a variety of general type if $X \underset{bir}{\dashleftarrow\dashrightarrow} X^{can}$. $\Leftrightarrow \kappa = \dim X$ (max possible)

For varieties of general type,

$$X \underset{\text{bir.}}{\leftarrow\!-\!\to} Y \iff X^{can} \underset{\text{iso}}{\simeq} Y^{can}$$

Even better, we can check this isomorphism by studying canonical rings.

For curves:
- $g=0$ $\quad \mathbb{P}^1 \quad -K_C$ ample $\quad$ Fano
- $g=1$ $\quad E \quad K_C \sim 0 \quad$ CY
- $g \geq 2$ $\quad K_C \quad K_C$ ample $\Rightarrow$ general type

In particular, curves ($g \geq 2$) are their own canonical model

For hypersurfaces $X \subset \mathbb{P}^n$ degree d.

- $d \leq n \quad -K_X$ ample $\quad$ Fano
- $d = n+1 \quad K_X \sim 0 \quad$ CY
- $d > n+1 \quad K_X$ ample $\Rightarrow$ general type.

In particular, $X \simeq X^{can} \iff d > n+1$.

~~Remarks / Strategy~~

In general one is looking for an algorithm $X \dashrightarrow \dots \dashrightarrow X^{can}$

X^{can} can be singular ("canonical singularities").

Theorem (Kollár-Miyaoka-Mori)
Fano varieties are rationally connected.