Math 461 Lecture 34 11/30
homework 8 available due next friday 12/7
no office hours Monday and Tuesday
office hour: Thursday 4-5 pm
Professor Lai will teach class Monday
Last time:

$$F(x,y,z) = (u,v) = \frac{1}{1-z}(x,y)$$

$F: S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ stereographic projection

$$\sqrt{x'^2 + y'^2 + z'^2} = \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2} \Rightarrow$$

① can compute lengths of curves on S^2 in plane $\mathbb{R}^2$ via integral

$$\int \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}\,dt$$

② F preserves angles

$C \subset S^2$ spherical circle $C = \Pi \cap S^2$

$\Pi \subset \mathbb{R}^3$ plane (not necessarily thru origin)

$N \in C \Rightarrow F(C \setminus \{N\}) = \ell \subset \mathbb{R}^2$, $\ell = \Pi \cap$ xy-plane

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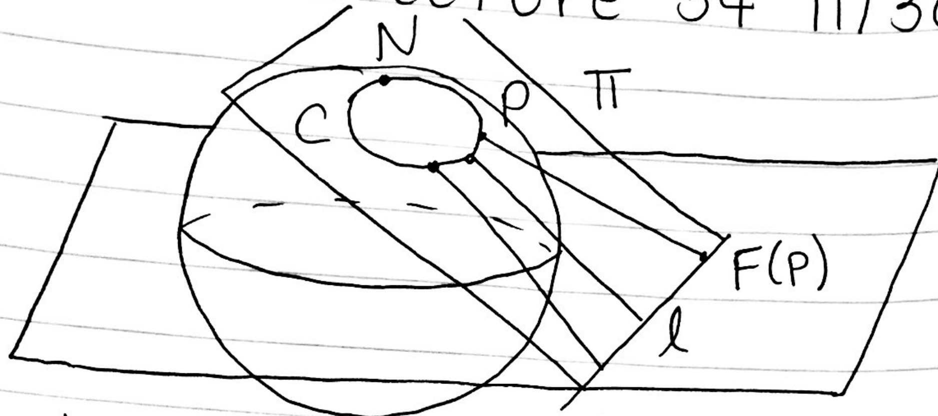
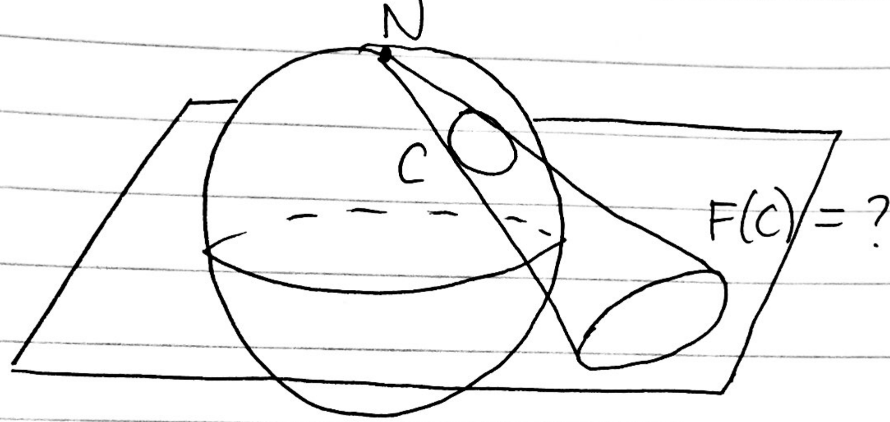
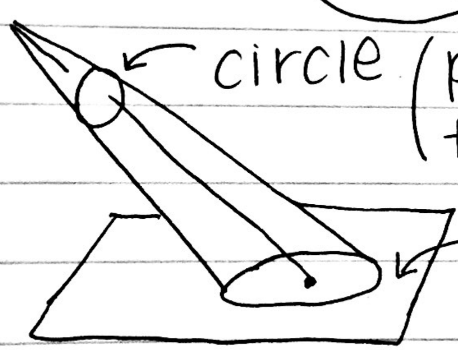


Today: ① F(C) for N ∉ C
② images of spherical lines under F (great circles)

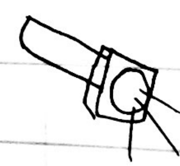


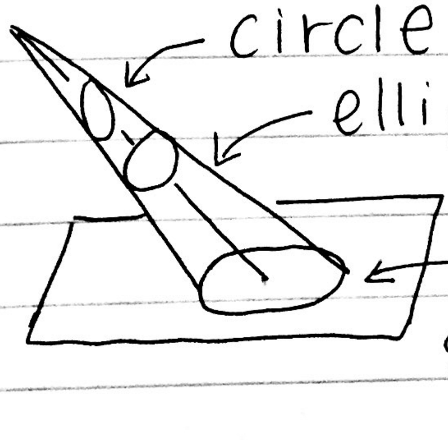


circle (perpendicular to axis of cone)

ellipse/oval

FLASH LIGHT





circle

ellipse perpendicular to axis

F(C) is a circle

stereographic projection

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Claim: for $C \subset S^2$ a spherical circle, $N \notin C$, then $F(C) \subset \mathbb{R}^2$ is a circle

Proof: use algebraic formula for stereographic projection (or rather its inverse)

$$F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$$

$$F(C) = \{(u,v) \mid F^{-1}(u,v) \in C\}$$

$C = \Pi \cap S^2 \quad \Pi: ax+by+cz=d$ plane in $\mathbb{R}^3$

$$F(C) = \{(u,v) \mid F^{-1}(u,v) \in C\}$$
$$= \{(u,v) \mid F^{-1}(u,v) \in \Pi\}$$

i.e. $a \cdot \dfrac{2u}{u^2+v^2+1} + b \cdot \dfrac{2v}{u^2+v^2+1} + c \cdot \dfrac{u^2+v^2-1}{u^2+v^2+1} = d$

$$2au + 2bv + c(u^2+v^2-1) = d(u^2+v^2+1)$$

$$(c-d)(u^2+v^2) + 2au + 2bv - (c+d) = 0$$

divide by $c-d$, where $c \neq d$ ☆

$$u^2+v^2 + \frac{2a}{c-d}u + \frac{2b}{c-d}v - \frac{(c+d)}{(c-d)} = 0$$

☆ $N \notin C \iff c \neq d$

complete the square:

$$\left(u+\frac{a}{c-d}\right)^2 + \left(v+\frac{b}{c-d}\right)^2 = \frac{a^2}{(c-d)^2} + \frac{b^2}{(c-d)^2} + \frac{c+d}{c-d}$$

$$\left(u+\frac{a}{c-d}\right)^2 + \left(v+\frac{b}{c-d}\right)^2 = \frac{a^2+b^2+c^2-d^2}{(c-d)^2}$$

circle with center $\left(\dfrac{-a}{c-d}, \dfrac{-b}{c-d}\right)$ and

~~radius~~

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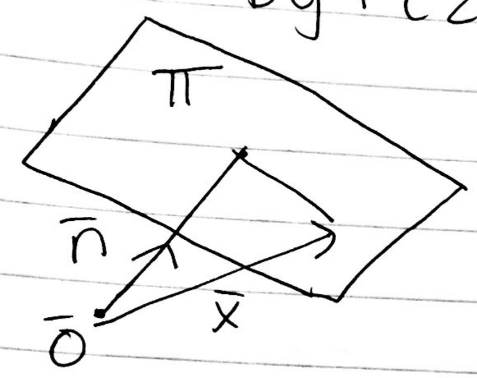
radius $= \sqrt{\dfrac{a^2+b^2+c^2-d^2}{(c-d)^2}} = \dfrac{\sqrt{a^2+b^2+c^2-d^2}}{c-d}$

$(u-\emptyset)^2 + (v-\beta)^2 = r^2$

question: why is $a^2+b^2+c^2-d^2 > 0$?

$\pi: ax+by+cz = d \quad (a,b,c) = \bar{n}$



equation: $\bar{n} = (a,b,c)$

$\bar{x}\cdot\bar{n} = d$

closest point to origin is $\bar{x} = \left(\dfrac{d}{\bar{n}\cdot\bar{n}}\right)\cdot\bar{n}$

scaled to satisfy equation of π

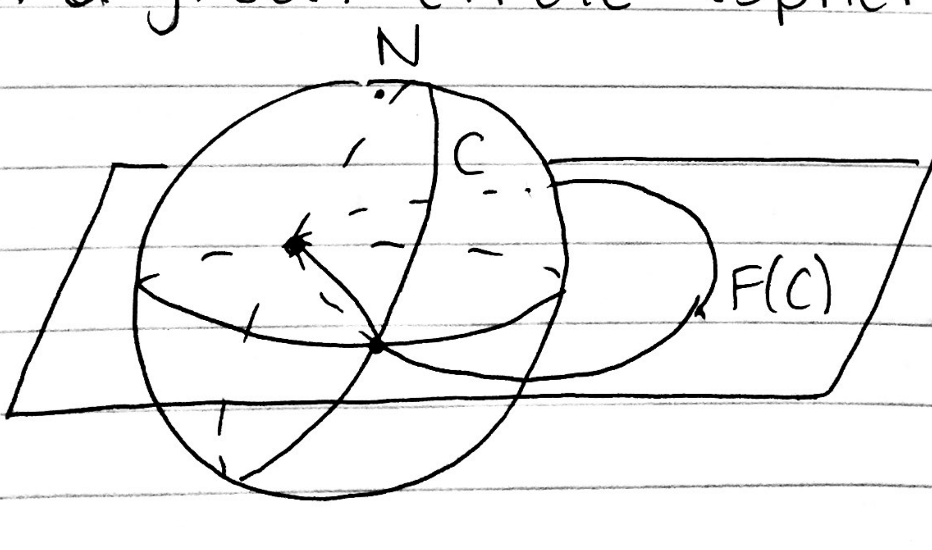
need $\|\bar{x}\| < 1 \quad \|\bar{x}\| = \dfrac{d}{\bar{n}\cdot\bar{n}}\|\bar{n}\| = \dfrac{d}{\sqrt{a^2+b^2+c^2}} < 1$

$\|\bar{n}\| = \sqrt{a^2+b^2+c^2}$

$\bar{n}\cdot\bar{n} = a^2+b^2+c^2$

~~then~~ Spherical lines:

question: what is special about image of a great circle (spherical line)



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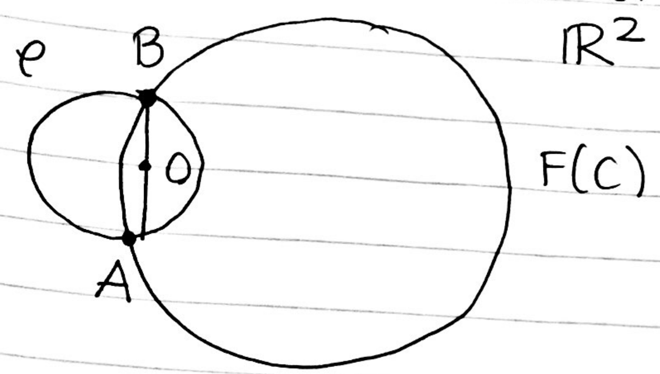
$e: u^2+v^2=1$ unit circle (equator of S^2)



$\Pi: ax+by+cz=0$
$d=0$ because
$\Pi \ni 0$

$$\left(u+\frac{a}{c-d}\right)^2+\left(v+\frac{b}{c-d}\right)^2=\frac{a^2+b^2+c^2-d^2}{(c-d)^2}$$

$d=0$: $$\left(u+\frac{a}{c}\right)^2+\left(v+\frac{b}{c}\right)^2=\frac{a^2+b^2+c^2}{c^2}$$

equation of F(C)

answer: intersection points A and B of F(C), C great circle/spherical line with unit circle e (equator) are antipodal points of e

i.e $\vec{OB}=-\vec{OA}$

$$\int \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}\,dt=\text{length}(\gamma)$$

remark: the two arcs from A to B on F(C) have the same length

"spherical length"

defined by $\displaystyle\int \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}\,dt$

corresponding to the two arcs from A to B on C having same length

(2 semicircles)

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we know through any two points on S^2, have a spherical line (great circle) unique if the points are not antipodal

in $\mathbb{R}^2$, given points P and Q $\in \mathbb{R}^2$ there should exist a circle C through P and Q intersecting unit circle $\mathcal{C}$ in two antipodal points