Thursday 10/22/15 MATH 421 HW5 solutions

1 a. Parametrization $z: [0, 2\pi] \longrightarrow \mathbb{C}$

$$z(t) = R\cos t + iR\sin t$$

$$\Rightarrow \text{length}(C) = \int_0^{2\pi} |z'(t)|\,dt$$

$$= \int_0^{2\pi} |-R\sin t + iR\cos t|\,dt$$

$$= \int_0^{2\pi} \sqrt{((-R\sin t)^2 + (R\cos t)^2)}\,dt$$

$$= \int_0^{2\pi} R\cdot\sqrt{(\sin t)^2 + (\cos t)^2}\,dt$$

$$= \int_0^{2\pi} R\,dt = 2\pi R \quad \checkmark$$

b. Parametrization $z: [1, 4] \longrightarrow \mathbb{C}$

$$z(t) = t + i t^{3/2}$$

$$\Rightarrow \text{length}(C) = \int_1^4 |z'(t)|\,dt = \int_1^4 \left|1 + i\tfrac{3}{2}t^{1/2}\right|dt$$

$$= \int_1^4 \sqrt{1 + \tfrac{9}{4}t}\,dt = \int_{13/4}^{10} \sqrt{u}\cdot \tfrac{4}{9}\,du$$

$u = 1 + \frac{9}{4}t$, $du = \frac{9}{4}\,dt$, $\frac{4}{9}\,du = dt$

$$= \tfrac{4}{9}\left[\tfrac{2}{3}u^{3/2}\right]_{13/4}^{10}$$

$$= \tfrac{8}{27}\left(10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right).$$

2. $|z-w| \geq \big||z| - |w|\big|$

$\Leftrightarrow \quad |z-w| \geq |z| - |w| \quad$ and $\quad |z-w| \geq |w| - |z|$

$\Leftrightarrow \quad |z-w| + |w| \geq |z| \quad$ and $\quad |z-w| + |z| \geq |w|$

$$|z-w| + |z| = |w-z| + |z|$$

Both inequalities are instances of the triangle inequality.

For example $\quad |z| = |(z-w)+w| \leq |z-w| + |w|.$ is the first inequality, & the second is similar.

3\. a $\quad f(z) = \dfrac{z^2+4}{z^4+3iz}$

C = circle center the origin & radius R in $\mathbb{C} = \mathbb{R}^2$
$= \{ z \in \mathbb{C} \mid |z| = R \}.$

Now for $z \in C$, $\quad |z^2+4| \underset{\text{T.E.}}{\leq} |z^2| + 4 = |z|^2 + 4 = R^2+4$

and $\quad |z^4+3iz| \underset{\text{Q2}}{\geq} \big| |z^4| - |3iz| \big| = \big| |z|^4 - |3i|\cdot|z| \big|$

$$= |R^4 - 3R|$$

So $\quad |f(z)| = \left| \dfrac{z^2+4}{z^4+3iz} \right| = \dfrac{|z^2+4|}{|z^4+3iz|} \leq \dfrac{R^2+4}{|R^4-3R|}$

Finally, $R^4 - 3R > 0$ for R sufficiently large,
so $|R^4-3R| = R^4-3R$ and $|f(z)| \leq \dfrac{R^2+4}{R^4-3R}.$

b. $\left| \int_C f(z)\,dz \right| \leq M \cdot \text{length}(C)$ where $|f(z)| \leq M$ for all $z \in C.$

Using $M = \dfrac{R^2+4}{R^4-3R}$ from (a), and length $(C) = 2\pi R$
(circumference of circle of radius R)

we find $\left| \int_C f(z)\,dz \right| \leq \left(\dfrac{R^2+4}{R^4-3R} \right) \cdot 2\pi R = 2\pi \cdot \left(\dfrac{R^3+4R}{R^4-3R} \right)$
for R suff. large.

c. $\lim\limits_{R \to \infty} 2\pi \cdot \left(\dfrac{R^3+4R}{R^4-3R} \right) = \lim\limits_{R \to \infty} 2\pi \cdot \left(\dfrac{\frac{1}{R} + \frac{4}{R^3}}{1 - \frac{3}{R^3}} \right) = 2\pi \cdot \left(\dfrac{0}{1} \right) = 0.$

divide numerator & denominator by R^4

So, since $\quad 0 \leq \left| \int_C f(z)\,dz \right| \leq 2\pi \cdot \left(\dfrac{R^3+4R}{R^4-3R} \right)$

we have $\quad \left| \int_C f(z)\,dz \right| \to 0 \quad$ as $\quad R \to \infty$

or equivalently $\quad \int_C f(z)\,dz \to 0 \quad$ as $\quad R \to \infty$.

(Note: By definition, $\quad z(t) \to \alpha \quad$ as $\quad t \to a$
$\iff \quad |z(t) - \alpha| \to 0 \quad$ as $\quad t \to a$.).

4. a)

$$\int_C x\,dy \underset{\text{G.T.}}{=} \int_R (1-0)\,dxdy$$
$$= \int_R dxdy = \text{Area}(R) = \pi \cdot (1)^2 = \pi.$$

b)

$$\int_C 2xy\,dx + (x^2-y^2)\,dy$$
$$\underset{\text{G.T.}}{=} \int_R \left(\frac{\partial}{\partial x}(x^2-y^2) - \frac{\partial}{\partial y}(2xy) \right) dxdy$$
$$= \int_R (2x - 2x)\,dxdy = \int_R 0\,dxdy = 0.$$

c) $$\int_C e^x \sin y\,dx + e^x \cos y\,dy = \int_R \left(\frac{\partial}{\partial x}(e^x \cos y) - \frac{\partial}{\partial y}(e^x \sin y) \right) dxdy$$
$$= \int_R (e^x \cos y - e^x \cos y)\,dxdy = \int_R 0 \cdot dxdy = 0.$$

5. $f(z) = e^{(z^2)} \cos(2z) \sin(3z)$ is complex differentiable on $\mathbb{C}$, $\quad f: \mathbb{C} \to \mathbb{C}$.
(because $z^2, e^z, \cos z, \sin z$ are complex differentiable and we have the product rule & chain rule).

Now by Cauchy's theorem $\int_C f(z)\,dz = 0.$

6. $f(z) = \dfrac{3z}{z^2+4z+13}$

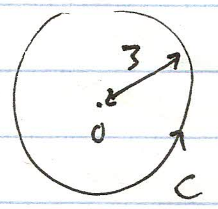
a) The domain U of f is $\mathbb{C} \setminus \{\text{zeros of denominator}\}$.
Compute : $z^2+4z+13 = 0$

$$\Rightarrow \quad z = \frac{-4 \pm \sqrt{4^2 - 4\cdot 13}}{2}$$

$$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$\therefore \quad U = \mathbb{C} \setminus \{-2+3i, -2-3i\}.$

b)



The points $-2 \pm 3i$ are outside the circle C.
(because $|-2 \pm 3i| = \sqrt{2^2+3^2} = \sqrt{13} > 3.$)

So $\int_C f(z)\,dz = 0$ by Cauchy's theorem

(because f is complex differentiable on U, and the region bounded by C is contained in U.)

7. a. $f(z) = \dfrac{1}{z^2 - iz} = \dfrac{1}{z(z-i)} = \dfrac{A}{z} + \dfrac{B}{z-i}$

$A, B \in \mathbb{C}$ (to be determined)

Clearing denominators:

$$1 = A(z-i) + Bz = A\cdot(-i) + (A+B)\cdot z$$

$$\Leftrightarrow \begin{cases} 1 = A\cdot(-i) \\ 0 = A+B \end{cases} \Leftrightarrow A = i,\ B = -i$$

So $f(z) = \dfrac{i}{z} - \dfrac{i}{z-i}$

b. $$\int_C f(z)\,dz = \int_C \frac{i}{z} - \frac{i}{z-i}\,dz$$

$$= i\cdot\left(\int_C \frac{1}{z}\,dz\right) - i\cdot\left(\int_C \frac{1}{z-i}\,dz\right)$$

$$= i\cdot\begin{cases} 2\pi i & \text{if } 0 \text{ inside } C \\ 0 & \text{if } 0 \text{ outside } C \end{cases} - i\cdot\begin{cases} 2\pi i & \text{if } i \text{ inside } C \\ 0 & \text{if } i \text{ outside } C \end{cases}$$

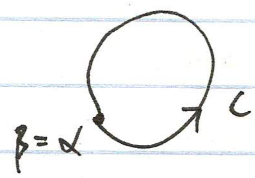
$$= \begin{cases} -2\pi & \text{if } 0 \text{ inside } C \\ 0 & \text{if } 0 \text{ outside } C \end{cases} + \begin{cases} 2\pi & \text{if } i \text{ inside } C \\ 0 & \text{if } i \text{ outside } C \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 \,\&\, i \text{ both inside or both outside } C \\ -2\pi & \text{if } 0 \text{ inside}, \ i \text{ outside} \\ 2\pi & \text{if } 0 \text{ outside}, \ i \text{ inside.} \end{cases}$$

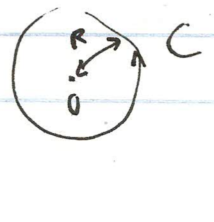
8. $f: \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $\quad f(z) = 1/z$.

Suppose f has an antiderivative $F: \mathbb{C}\setminus\{0\} \to \mathbb{C}$.

Then for any closed curve $C \subset \mathbb{C}\setminus\{0\}$,
we have $\int_C f(z)\,dz = F(\alpha) - F(\alpha) = 0$.



BUT we know that $\int_C f(z)\,dz = 2\pi i$



C circle center origin, radius R.

This is a contradiction.

So f does NOT have an antiderivative $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.