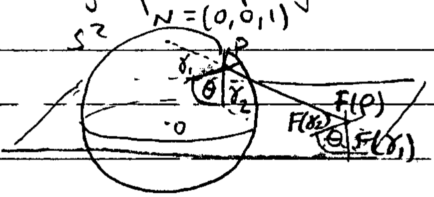
11/18/19

HW 7 due Wednesday

Office Hours today & tomorrow 4-5PM LGRT 1235H

Last Time

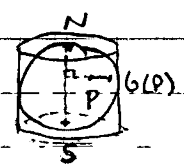
- Stereographic projection



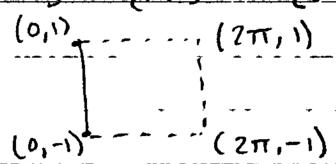
$F: S^2 \setminus \{N\} \to \mathbb{R}^2$ bijection

$F(x,y,z) = \frac{1}{1-z}(x,y)$ Inverse $F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$

- Gall-Peters projection



$G: S^2 \setminus \{N\} \to [0,2\pi) \times (-1,1) \subset \mathbb{R}^2$



a. Project radially outward from NS axis to cylinder with same axis & radius 1

b. Cut cylinder along line $x=1$, $y=0$ & roll out in plane

Facts

1. Stereographic projection preserves angles (although sense is reversed)
2. Gall-Peters projection preserves areas

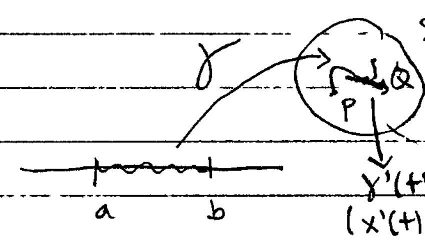
(No map of S^2 in the plane can preserve distances!)

Today

1. Finish proof that spherical lines give shortest paths (using triangle inequality)
2. Proof that S.P. preserves angles.

Theorem $P, Q \in S^2$. The shortest path from P to Q on S^2 is given by the shorter arc of the spherical line through P & Q.

Proof $\gamma : \underset{\mathbb{R}}{[a,b]} \to S^2 \subset \mathbb{R}^3$



$\gamma(t) = (x(t), y(t), z(t))$ $\qquad$ $\gamma'(t) = (x'(t), y'(t), z'(t))$

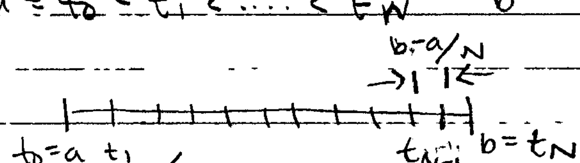
$\gamma(a) = P, \gamma(b) = Q$ $\qquad$ ("parametrized path")

Recall length(γ) $:= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$ (MATH 233)

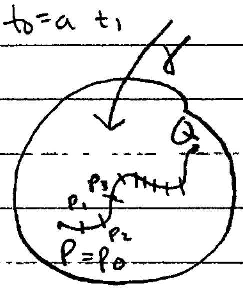
$$\left(= \int_a^b \|\gamma'(t)\|\, dt \right)$$

$a = t_0 < t_1 < \dots < t_N = b$. N large positive integer

$t_i - t_{i-1} = b-a/N$



$P_i = \gamma(t_i)$



(Assume γ is continuously differentiable) "smooth curve"

length$(\gamma) \approx d(P_0, P_1) + d(P_1, P_2) + \dots + d(P_{N-1}, P_N)$

approximate path γ by arcs of spherical lines

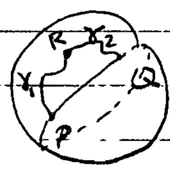
Spherical triangle inequality:
$d(P,Q) + d(Q,R) \geq d(P,R)$

$d(P_0, P_1) + \dots + d(P_{N-1}, P_N) \geq d(P_0, P_N)$ by Δ inequality applied several times.
$= d(P, Q)$

$\lim_{N \to \infty}$: approximation becomes exact & we get length$(\gamma) \geq d(P, Q)$.

When do we have equality?
We do have equality if γ is shorter arc of spherical line through P & Q (by definition of d_{S^2}).

Otherwise, pick a point R on $\gamma \subset S^2$ which is not on the shorter arc of the spherical line.

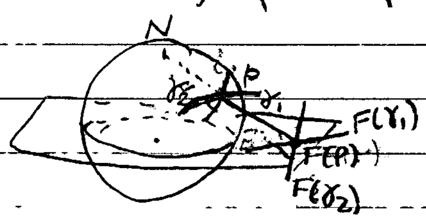


$$\begin{aligned} \text{length}(\gamma) &= \text{length}(\gamma_1) + \text{length}(\gamma_2) \\ &\geq d(P,R) + d(R,Q) \\ &> d(P,Q) \end{aligned}$$

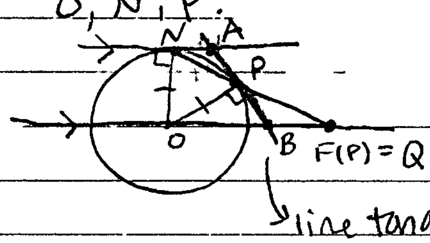
strict form of triangle inequality. ▨

Back to stereographic projection:

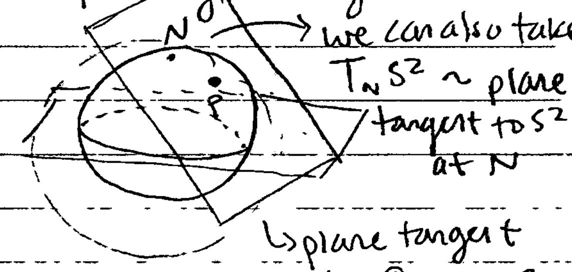
Theorem Given two smooth curves γ_1, γ_2 on S^2 intersecting at a point $P \neq N$, the angle between $F(\gamma_1)$ & $F(\gamma_2)$ at $F(P)$ is equal to the angle between γ_1 & γ_2 at P. (here $F: S^2 \setminus \{N\} \to \mathbb{R}^2$ is stereographic projection)



Warm up
Look at vertical slice by plane passing through O, N, P.



in 3D



Claim: $\triangle BPQ$ is isosceles.

<u>Proof</u> $\angle BPQ = \angle APN$ (vertical angles θ/θ)
$\angle ANP = \angle BQP$ (alternate angles)

Just need $\angle ANP = \angle APN$.
Tangent $\perp$ radius, $\triangle ONP$ isosceles.
$\angle ONP = \angle OPN$
$\angle ANP = \pi/2 - \angle ONP = \pi/2 - \angle OPN = \angle APN$. ▨

If Π is a plane in $\mathbb{R}^3$ passing through N, then $\Pi \cap S^2$ is a circle $C \subset S^2$ and its image
$F(N \setminus \{C\}) = L \subset \mathbb{R}^2$ is a line in $\mathbb{R}^2$,
$L = \Pi \cap (z=0)$