Math 462 Homework 6

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(1) Use the Gall–Peters projection to give another proof that the area A of a spherical circle of radius r on a sphere of radius R is given by $A = 2\pi R^2(1 - \cos(r/R))$.

[Hint: Recall that the Gall–Peters projection preserves areas. Now choose the axis of the projection carefully so that it is easy to determine the image of the spherical circle and compute its area. This gives a quicker proof than the one in HW1Q6.]

(2) Let S^2 be the sphere of radius $R = 1$ and center the origin in $\mathbb{R}^3$. Let $F \colon S^2 \backslash \{N\} \to \mathbb{R}^2$ be the stereographic projection of the sphere S^2 from the north pole $N = (0, 0, 1)$ onto the xy-plane. In class we showed that the image of a spherical circle C on S^2 under stereographic projection is either a circle or a line in the plane. Describe the image precisely in the following cases.

(a) $C_1 = \Pi_1 \cap S^2$ where Π_1 is the plane given by equation

$$x + 2y + 3z = 3.$$

(b) $C_2 = \Pi_2 \cap S^2$ where Π_2 is the plane given by equation

$$3x + 4y + 5z = 6.$$

[Hint: Recall that the image of C is a line if C contains the north pole N and a circle otherwise. In the first case the line is just the intersection of the plane Π containing C with the xy-plane. In the second case we can find the equation of the image circle using the algebraic formula for the inverse of F found in class: $F^{-1}(u, v) = (2u, 2v, u^2+v^2-1)/(u^2+v^2+1)$.]

(3) Let $F\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection as in Q2. Let $P = (x, y, z) \in S^2$ and $Q = F(P)$.

(a) Compute the distance $d(O, Q)$ from the origin O to Q as a function of z.

(b) Deduce from your formula in part (a) that $d(O, Q) \to \infty$ as $Q \to N$ (equivalently, as $z \to 1$).

(4) Let $F\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection. The equator is the intersection of S^2 with the xy-plane.

(a) What happens to the equator under stereographic projection?

(b) Fix two antipodal points P and P' on the equator, for example $P = (1, 0, 0)$ and $P' = (-1, 0, 0)$, and consider all the great circles on S^2 passing through P and P'. Describe the image of these great circles under stereographic projection.

[Hint: Recall stereographic projection sends circles on S^2 to circles and lines in the xy-plane. Also stereographic projection preserves angles. Use these two properties and your answer to part (a) to do part (b).]

(5) Recall that a *linear fractional transformation* is a function

$$f\colon \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}, \quad f(t) = \frac{at + b}{ct + d}$$

for some $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$. Here we define $f(\infty) = a/c$ and $f(-d/c) = \infty$. The condition $ad - bc \neq 0$ is the condition that f is not constant. To an invertible 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can associate the linear fractional transformation $f(t) = \frac{at+b}{ct+d}$. Notice that the linear fractional transformation associated to a nonzero scalar multiple $e \cdot A$ of an invertible matrix A, where $0 \neq e \in \mathbb{R}$, is the same as the linear fractional transformation associated to A.

(a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be two invertible 2×2 matrices and let f and g be the associated linear fractional transformations. Show that the matrix product $B \cdot A$ corresponds to the composite linear fractional transformation $g \circ f$.

(b) Compute a formula for the inverse f^{-1} of the linear transformation $f(t) = \frac{at+b}{ct+d}$. Check your answer using part (a) and the formula for the inverse of a 2×2 matrix.

(6) Let S^1 be the circle with radius $R = 1$ and center the origin in $\mathbb{R}^2$. Let $F\colon S^1 \setminus \{N\} \to \mathbb{R}$ be the (1-dimensional) stereographic projection of the circle S^1 from the north pole $N = (0,1)$ onto the x-axis. Let $\overline{F}\colon S^1 \to \mathbb{R} \cup \{\infty\}$ be the extension of F given by defining $\overline{F}(N) = \infty$. In class we showed that if $T\colon S^1 \to S^1$ is the transformation given by rotation about the origin through angle θ counterclockwise, then the corresponding transformation $U = F \circ T \circ F^{-1}$ of $\mathbb{R} \cup \{\infty\}$ is the linear fractional transformation $f(t) = \frac{at+b}{ct+d}$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

is the matrix for a rotation through angle $\theta/2$ clockwise.

(a) Compute $\overline{F}(1,0)$, $\overline{F}(-1,0)$, $\overline{F}(0,1)$, and $\overline{F}(0,-1)$.

(b) Compute the linear fractional transformations corresponding to rotation by $\theta = \pi$ and $\theta = \pi/2$ and check your answers using part (a).