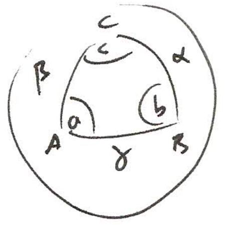
1. a.



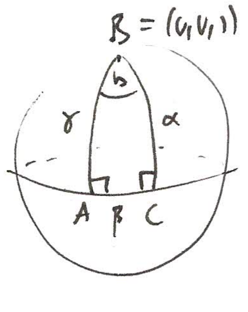
Spherical cosine rule:

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

$$a = \pi/2 \Rightarrow \cos a = 0 \Rightarrow \boxed{\cos\alpha = \cos\beta\cos\gamma}$$

spherical Pythagoras' theorem.

b.



$$\alpha = \gamma = \pi/2 \Rightarrow \cos\alpha = \cos\gamma = 0.$$

S.P.T. $\cos\alpha = \cos\beta\cos\gamma$

$$0 = \cos\beta \cdot 0 \quad \checkmark.$$

2. a. Recall from HW5&6 that a spherical circle is the intersection of a plane Π (not necessarily containing the origin) with the sphere S^2.

Given $A, B, C \in S^2$ there is a unique plane $\Pi \subset \mathbb{R}^3$ containing A, B, & C. (Otherwise, A, B, C are collinear, but a line intersects the sphere in at most 2 points. ※.)

Then $\mathcal{C} = \Pi \cap S^2$ is the unique spherical circle thru A, B, C.

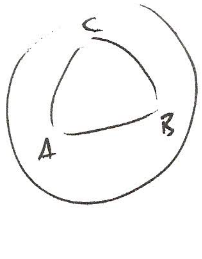
b. The unique plane Π passing thru $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$ has equation $x+y+z = 1$.

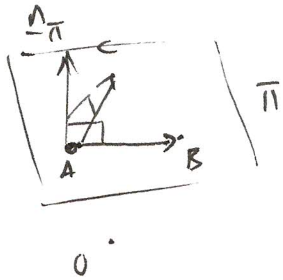
Then $\mathcal{C} = \Pi \cap S^2$ is the spherical circle passing thru A, B, C.

It has center the point P w/ $\overrightarrow{OP} = \underline{n} =$ unit normal to Π (i.e., length 1) $= \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$ (compare HW5&6.)

It has spherical radius $d(P,A) = \cos^{-1}(\overrightarrow{OP} \cdot \overrightarrow{OA}) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

c.





π $\quad \vec{OP} = \hat{n}_\pi = \dfrac{\vec{AB} \times \vec{AC}}{\|\vec{AB} \times \vec{AC}\|}$

length 1 normal vector to plane π.

3. a. $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \qquad 2 \le p \le q \le r.$

Notice $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

otherwise $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ #

It follows that, if $(p,q,r) \neq (3,3,3)$, then $p < 3$, so $\boxed{p=2.}$

$\frac{1}{2} + \frac{1}{q} + \frac{1}{r} = 1, \quad \frac{1}{q} + \frac{1}{r} = 1/2$

Notice $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

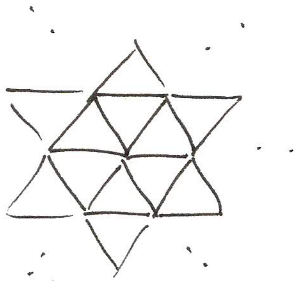
then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{r} > 1$.

It follows that, if $(p,q,r) \neq (2,4,4)$, $q < 4$, so $q = 2$ # or $\boxed{q=3.}$

Now $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} = 1 \Rightarrow r = 6.$

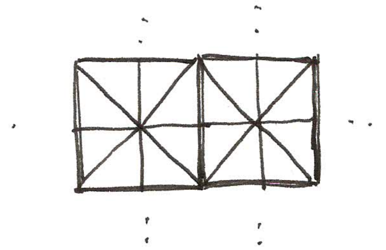
So, solutions are $(3,3,3)$, $(2,4,4)$, $(2,3,6)$

b. (3,3,3):



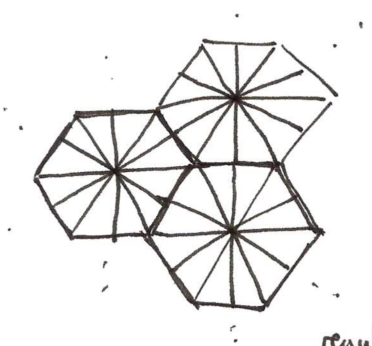
tiling by equilateral triangles.

(2,4,4)



(subdivide tiling by squares)

(2,3,6)



(subdivide tiling by regular hexagons)

4a. If $p=q=2$ then $r\geq 2$ is arbitrary: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1+\frac{1}{r} > 1$.

Notice $\frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$

So, must have $p=2$ (otherwise $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq 1$)

$$\frac{1}{2}+\frac{1}{q}+\frac{1}{r} > 1$$

$$\frac{1}{q}+\frac{1}{r} > \frac{1}{2}$$

We may assume $q>2$ ($q=2$ treated above)

Notice $\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

So, must have $q=3$.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{r} > 1, \qquad \frac{1}{r} > \frac{1}{6}.$$

So $r<6$, $r\geq q=3$, $r=3,4,5$.

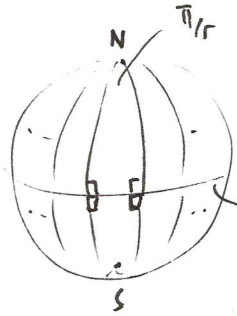
$\therefore$ Solutions: $(2,2,r)$, $r\geq 2$

$(2,3,3)$

$(2,3,4)$

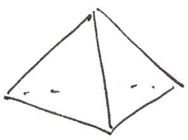
$(2,3,5)$.

b. $(2,2,r)$:

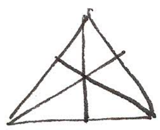


$\#$ triangles $= 2r+2r = 4r$.

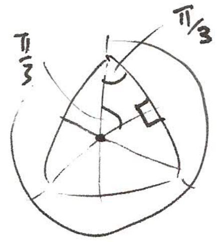
$(2,3,3)$



tetrahedron. Subdivide faces of tetrahedron as shown:



Now project onto sphere

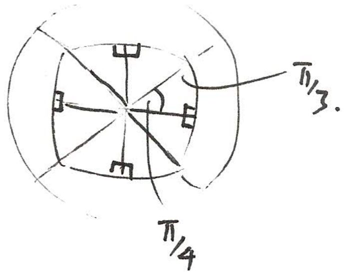


(angles at each vertex are equal & sum to 2π.)

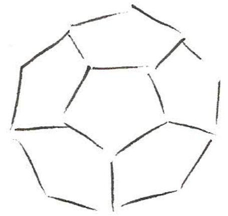
(2, 3, 4)

cube. Subdivide faces:

Project onto sphere



(2, 3, 5)

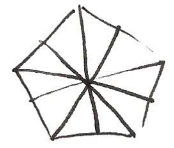


dodecahedron.

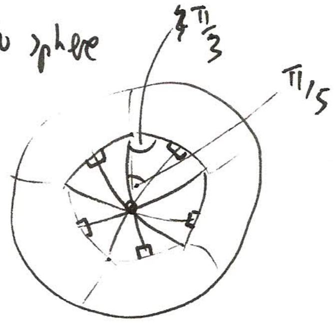
(12 faces, each a regular pentagon)

search images on google.

Subdivide faces.



Project onto sphere



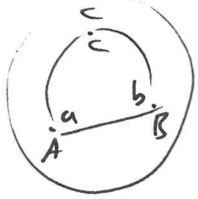
https://commons.wikimedia.org/wiki/File:Icosahedral_reflection_domains.png

c. $$\#\text{triangles} = \frac{\text{Area}(S^2)}{\text{Area}(\text{triangle})} = \frac{4\pi}{\pi/p + \pi/q + \pi/r - \pi} = \frac{4}{1/p + \frac{1}{q} + \frac{1}{r} - 1}$$

$$\left(\begin{array}{l} \text{recall: } a+b+c = \pi + \text{Area}(\Delta ABC) \\ \Rightarrow \text{Area}(\Delta ABC) = a+b+c-\pi \end{array}\right) \qquad = \frac{4pqr}{qr + pr + pq - pqr}.$$

__Ex.__ $p=2, q=3, r=5$. $$\#\text{triangles} = \frac{4\cdot 2\cdot 3\cdot 5}{3\cdot 5 + 2\cdot 5 + 2\cdot 3 - 2\cdot 3\cdot 5} = \frac{120}{15+10+6-30} = \boxed{120}.$$

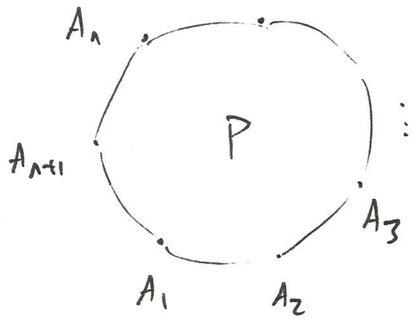
5a. Proof by induction on $n \geq 3$.



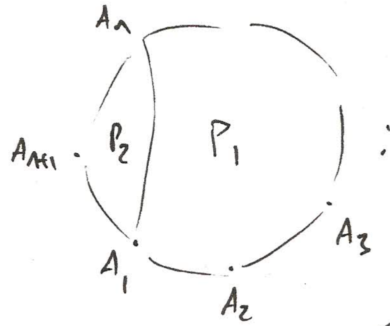
$\underline{n=3}$. $a+b+c = \pi + \text{Area}(\triangle ABC)$ proved in class

$= (n-2)\cdot\pi + \text{Area}(\triangle ABC)$ ✓

$\underline{n \Rightarrow n+1}$: Suppose true for n. We show it's true for $n+1$.



Subdivide P by joining A_n to A_1 by shortest path.



Sum of interior angles of P

$=$ Sum of interior angles of P_1 & P_2

$= \big((n-2)\pi + \text{Area}(P_1)\big) + \big(\pi + \text{Area}(P_2)\big)$

using cases n & 3.

$= (n-1)\pi + \text{Area}(P_1) + \text{Area}(P_2)$

$= ((n+1)-2)\pi + \text{Area}(P)$. □.

b.

Area of $S^2 = 4\pi =$ Sum of areas of tiles

$$\overset{a.}{=} \sum_{i=1}^{F} (\text{sum of interior angles of } i^{\text{th}} \text{ tile}) - (\#\text{edges of } i^{\text{th}} \text{ tile} - 2)\cdot\pi$$

$$= (\text{sum of interior angles of all tiles}) - \left(\sum_{i=1}^{F} (\#\text{edges of } i^{\text{th}} \text{ tile})\right)\cdot\pi + \sum_{i=1}^{F} 2\pi$$

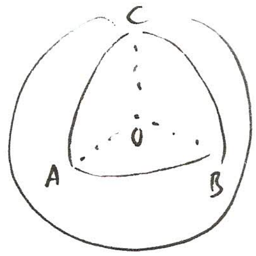
sum of angles at each vertex equals 2π.

each edge is contained in 2 tiles

$$= 2\pi\cdot V - (2E)\cdot\pi + F\cdot 2\pi$$

i.e. $4\pi = 2\pi\cdot(V-E+F)$ $\overset{\div 2\pi}{\rightsquigarrow}$ $\boxed{2 = V-E+F}$ Euler's formula.

6 a.



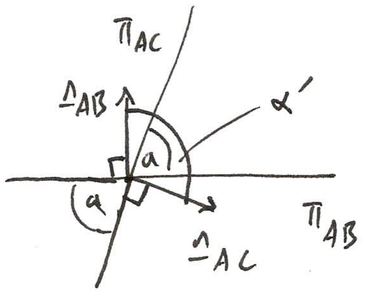
$\vec{OA'} = \Delta_{BC}$, normal vector to plane Π_{BC} containing O, B, C, of length 1 & on same side of Π_{BC} as A.

$\vec{OB'} = \Delta_{AC}$

$\vec{OC'} = \Delta_{AB}$.

a) $\alpha' = d(B', C') =$ angle between $\vec{OB'}$ & $\vec{OC'}$ = angle between Δ_{AC} & Δ_{AB}.

View looking along line $\Pi_{AC} \cap \Pi_{AB}$, i.e. from A towards O.



We see $a + \pi/2 + \alpha' + \pi/2 = 2\pi$.

$\Rightarrow \quad \alpha' = \pi - a$.

Similarly $\beta' = \pi - b$, $\gamma' = \pi - c$.

b). $\Delta ABC \overset{\text{polar}}{\rightsquigarrow} \Delta A'B'C' \overset{\text{polar}}{\rightsquigarrow} \Delta A''B''C''$.

Want to show $A'' = A$, $B'' = B$, $C'' = C$.

$\vec{OA''}$ is the normal vector to the plane spanned by $\vec{OB'} = \Delta_{AC}$ & $\vec{OC'} = \Delta_{AB}$.

(of unit length, & on same side of the plane as A').

But note that $\vec{OA}$ is perpendicular to $\vec{OB'}$ & $\vec{OC'}$

(because $\vec{OB'} = \Delta_{AC}$ is perpendicular to $\vec{OA}$ & $\vec{OC}$ by definition, & similarly for $\vec{OC'}$.)

So $\vec{OA} = \pm \vec{OA''}$ & checking our picture, we see $\vec{OA} = \vec{OA''}$, so $A = A''$.

Similarly $B = B''$, $C = C''$.

c). By b. ΔABC is the polar of $\Delta A'B'C'$. So by a. $\alpha = \pi - a'$, $\beta = \pi - b'$, $\gamma = \pi - c'$, i.e.,

$a' = \pi - \alpha, \quad b' = \pi - \beta, \quad c' = \pi - \gamma.$ $\square$

7. SCR: $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$

Apply to polar triangle

$$\cos\alpha' = \cos\beta'\cos\gamma' + \sin\beta'\sin\gamma'\cos a'$$

$\alpha' = \pi - a, \quad \beta' = \pi - b, \quad \gamma' = \pi - c, \quad a' = \pi - \alpha$ by Q6 a & c

So $$\cos(\pi - a) = \cos(\pi - b)\cos(\pi - c) + \sin(\pi - b)\sin(\pi - c)\cos(\pi - \alpha)$$

$$\cos(\pi - x) = -\cos x$$
$$\sin(\pi - x) = \sin x$$

$$\Rightarrow \quad -\cos a = (-\cos b)(-\cos c) + \sin b \sin c(-\cos\alpha)$$

$$\Rightarrow \boxed{\cos a + \cos b\cos c = \sin b\sin c\cos\alpha}.$$

$$\left(\text{i.e.} \quad \alpha = \cos^{-1}\left(\frac{\cos a + \cos b\cos c}{\sin b\sin c}\right). \right)$$