Math 462 Homework 6

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(1) Let S^2 denote the sphere with center the origin and radius 1 in $\mathbb{R}^3$ and $N = (0,0,1) \in S^2$ the north pole. Let

$$F\colon S^2 \setminus \{N\} \to \mathbb{R}^2, \quad F(x,y,z) = \frac{1}{1-z}(x,y).$$

denote the stereographic projection of S^2 onto the xy-plane. Let $C \subset S^2$ be a circle given by $C = \Pi \cap S^2$ where $\Pi \subset \mathbb{R}^3$ is a plane. Compute the equation of the image of C under stereographic projection and describe the image geometrically in the following cases.

(a) $\Pi = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=1\}$.

(b) $\Pi = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=\frac{3}{2}\}$.

[Hint: If Π passes through the north pole N then the image of C is the line $L \subset \mathbb{R}^2$ given by $L = \Pi \cap \mathbb{R}^2$ (why?). If Π does not pass through N then the image of C is a circle in $\mathbb{R}^2$. Its equation can be computed from the equation of Π using the formula

$$F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$$

for the inverse of the stereographic projection F.]

(2) What is the image of (a) the northern hemisphere and (b) the southern hemisphere under stereographic projection $F\colon S^2 \setminus \{N\} \to \mathbb{R}^2$?

(3) Give a precise geometric description of the transformation $f\colon \mathbb{C} \to \mathbb{C}$ of the complex plane given by $f(z) = (1+i)z$.

(4) Let $C = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle with center the origin and radius 1 in the complex plane and $D = \{z \in \mathbb{C} \mid |z| = R\}$ the circle with center the origin and radius R for some $R > 0$. Recall that inversion in the circle C is by definition the transformation

$$g\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

$$g(z) = z/|z|^2 \text{ for } z \neq 0, \infty, \text{ and } g(0) = \infty,\ g(\infty) = 0.$$

We can similarly define inversion in the circle D to be the transformation

$$h\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

$$h(z) = R^2 z/|z|^2 \text{ for } z \neq 0, \infty, \text{ and } h(0) = \infty,\ h(\infty) = 0.$$

Prove the following statements carefully:

(a) $|h(z)| = R^2/|z|$ for all $z \in \mathbb{C}$, $z \neq 0$.

(b) The composition $h \circ h$ is the identity transformation, that is, $h(h(z)) = z$ for all $z \in \mathbb{C} \cup \{\infty\}$.

(c) $h(z) = z$ iff $z \in D$, and h exchanges the "inside" and "outside" of D.

(d) Consider the transformation

$$f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad f(z) = Rz.$$

[The transformation f is a dilation (or scaling) with center the origin and scaling factor R. In particular, $f(C) = D$.] Show that $h = f \circ g \circ f^{-1}$, that is, $h(z) = f(g(f^{-1}(z)))$ for all $z \in \mathbb{C} \cup \{\infty\}$.

(e) We showed in class that the inversion g preserves angles and sends circles and lines to circles and lines. [Here by convention each line includes the point ∞.] Using part (d) or otherwise, explain why the same is true for the inversion h.

(5) Let

$$g\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad g(z) = z/|z|^2$$

be inversion in the circle C with center the origin and radius 1. Compute the image of the following circles and lines under the inversion g (describe your results geometrically).

(a) D is the circle center $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.

(b) E is the line $y = 2$.

(c) F is the circle center $(3, 0)$ and radius 1.

(6) Let $g\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad g(z) = z/|z|^2$ be inversion in the circle C with center the origin and radius 1. Let

$$\bar{F}\colon S^2 \to \mathbb{C} \cup \{\infty\}$$

$$\bar{F}(x, y, z) = \frac{x + iy}{1 - z} \text{ for } (x, y, z) \neq (0, 0, 1),\ \bar{F}(0, 0, 1) = \infty.$$

be the stereographic projection from the sphere S^2 with center the origin and radius 1 to the xy-plane. Let $R\colon S^2 \to S^2$ be the isometry of the sphere given by reflection in the xy-plane.

(a) Describe R by an algebraic formula.

(b) Show that the reflection R corresponds to the inversion g under stereographic projection. That is, $\bar{F} \circ R \circ \bar{F}^{-1} = g$, or

$$\bar{F}(R(\bar{F}^{-1}(w)) = g(w) \text{ for all } w \in \mathbb{C} \cup \infty\}.$$

Equivalently, $\bar{F} \circ R = g \circ \bar{F}$, or

$$\bar{F}(R(x, y, z)) = g(\bar{F}(x, y, z)) \text{ for all } (x, y, z) \in S^2.$$

(c) Use part (b) to give another proof that g preserves angles and sends circles and lines to circles and lines.

(7) Let

$$A\colon S^2 \to S^2, \quad A(x, y, z) = (-x, -y, -z)$$

be the transformation of the sphere S^2 which sends a point to its antipode. What is the transformation $f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ corresponding to A under stereographic projection?