697B Midterm exam, 28 October 2010

Show all your work and justify your answers carefully.

(1) Let X be the algebraic curve given by $X = (y^2 + 2xy + x^4 = 0) \subset \mathbb{C}^2$ and $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ its closure in $\mathbb{P}^2_{\mathbb{C}}$.

   (a) Find all the points in $\overline{X} \setminus X$.

   (b) Find all the singular points of $\overline{X}$.

   (c) Which of the singular points are nodes?

(2) Let X be the algebraic curve defined by $X = (y^3 + y + x^2 = 0) \subset \mathbb{C}^2$ and $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ its closure in $\mathbb{P}^2_{\mathbb{C}}$. Then $\overline{X}$ is a compact Riemann surface. Let f be the meromorphic function on $\overline{X}$ given by $f = x/y$.

   (a) Find the zeroes and poles of f on $\overline{X}$ and their multiplicities.

   (b) The meromorphic function $f\colon \overline{X} \dashrightarrow \mathbb{C}$ extends to a holomorphic map $F\colon \overline{X} \to \mathbb{C} \cup \{\infty\}$. What is $F^{-1}(0)$ and $F^{-1}(\infty)$? What is the degree of F? [Hint: This follows easily from part (a).]

(3) Let X be the algebraic curve defined by $X = (y^5 = x^4 + 1) \subset \mathbb{C}^2_{x,y}$ and $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ its closure in $\mathbb{P}^2_{\mathbb{C}}$. Then $\overline{X}$ is a compact Riemann surface. Let ω be the meromorphic differential on $\overline{X}$ defined by $\omega = dx/y^4$.

   (a) Find the zeroes and poles of ω on $\overline{X}$ and their multiplicities.

   (b) Use the Poincaré-Hopf theorem to determine the genus of $\overline{X}$.

(4) Let $\overline{X}$ be the projective algebraic curve of degree d defined by

$$\overline{X} = (X^d + Y^d + Z^d = 0) \subset \mathbb{P}^2_{\mathbb{C}}$$

and $F\colon \overline{X} \to \mathbb{P}^1_{\mathbb{C}}$ the holomorphic map given by $(X : Y : Z) \mapsto (Y : Z)$.

   (a) Show that $\overline{X}$ is a smooth curve.

   (b) Find the degree of F, the ramification points of F, and the ramification indices.

   (c) Use the Riemann–Hurwitz formula to determine the genus of $\overline{X}$.