Sunday 9/20/15 MATH 421 HW1 Solutions

1. a $(2+i)(5+3i) = (2\cdot5+2\cdot3i+5\cdot i+i\cdot3i)$

$= (2\cdot5-3)+(2\cdot3+5)i$ using $i^2=-1$

$= 7+11i$

b $(3-4i)(1+2i) = (3\cdot1+4\cdot2)+(3\cdot2-4\cdot1)i = 11+2i$

c $(7+5i)(3+2i) = (21-10)+(14+15)i = 11+29i$

d $(a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2.$

2. a $\dfrac{3+4i}{2+i} = \dfrac{(3+4i)(2-i)}{(2+i)(2-i)} = \dfrac{(6+4)+(8-3)i}{2^2+1^2} = \dfrac{10+5i}{5} = 2+i$

b $\dfrac{7+i}{2+5i} = \dfrac{(7+i)(2-5i)}{2^2+5^2} = \dfrac{(14+5)+(2-35)i}{29} = \dfrac{19-33i}{29} = \dfrac{19}{29}-\dfrac{33}{29}i$

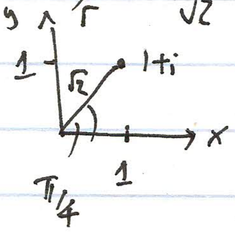
c $\dfrac{1+i}{1-i} = \dfrac{(1+i)\cdot(1+i)}{1^2+1^2} = \dfrac{(1-1)+2\cdot i}{2} = i$

3. a. $z=1+i = r(\cos\theta+i\sin\theta)$

$r=\sqrt{x^2+y^2} = \sqrt{1^2+1^2}=\sqrt{2}$

$\Rightarrow \cos\theta = x/r = 1/\sqrt{2}$, $\sin\theta = y/r = 1/\sqrt{2}$ $\Rightarrow \theta=\pi/4$

$1+i = \sqrt{2}\,(\cos(\pi/4)+i\sin\pi/4)$

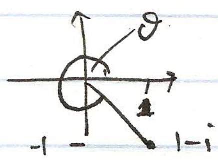


b. $z=\sqrt{3}+i$

$r=\sqrt{\sqrt{3}^2+1^2}=2$

$\cos\theta = x/r = \sqrt{3}/2$, $\sin\theta = y/r = 1/2$ $\Rightarrow$ $\theta=\pi/3.$

$\sqrt{3}+i = 2(\cos(\pi/3)+i\sin\pi/3)$

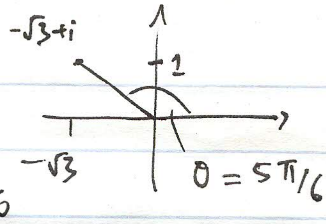


c. $z=1-i$

$r=\sqrt{1^2+(-1)^2}=\sqrt{2}$. $\cos\theta=1/\sqrt{2}$, $\sin\theta=-1/\sqrt{2}$ $\Rightarrow$ $\theta=-\pi/4$ OR $7\pi/4$

$1-i = \sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$

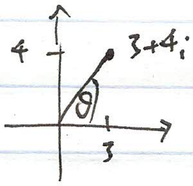
d. $z = -\sqrt{3} + i$



$r = 2$

$\cos\theta = -\sqrt{3}/2\ ,\ \sin\theta = 1/2 \implies \theta = 5\pi/6$

e. $z = 3+4i$



$r = \sqrt{3^2+4^2} = 5$

$x > 0 \implies -\pi/2 < \theta < \pi/2$

$\implies \theta = \tan^{-1}(y/x) = \tan^{-1}(4/3)$

$3+4i = 5\left(\cos(\tan^{-1}(4/3)) + i\sin(\tan^{-1}(4/3))\right)$

4 a. $z^2+3z+4=0$

$$\implies z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3\pm\sqrt{9-16}}{2} = \frac{-3\pm\sqrt{-7}}{2}$$

$$= \frac{-3\pm\sqrt{7}\,i}{2}.$$

b. $z^2-4z+13=0$

$$\implies z = \frac{4\pm\sqrt{16-52}}{2} = \frac{4\pm\sqrt{-36}}{2} = \frac{2\pm 6i}{2} = 1\pm 3i$$

c. $z^3+6z^2+10z=0$

$\iff z\cdot(z^2+6z+10)=0$

$\iff z=0$ OR $z^2+6z+10=0.$

$\Downarrow$

$$z = \frac{-6\pm\sqrt{36-40}}{2} = \frac{-6\pm 2i}{2} = -3\pm i$$

$z = 0, -3\pm i$

d. $z^3-4z^2+6z-4=0.$

Observe $z=2$ is a solution. $8-16+12-4=0$ ✓

Factor $z^3 - 4z^2 + 6z - 4 = (z-2)(z^2 - 2z + 2)$

$$z^2 - 2z + 2 = 0 \Rightarrow z = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$z = 2,\ 1 \pm i.$

5. $$z^2 + bz + c = (z-\alpha_1)(z-\alpha_2) = z^2 - (\alpha_1+\alpha_2)z + \alpha_1\alpha_2$$

$\Rightarrow b = -(\alpha_1+\alpha_2),\quad c = \alpha_1\alpha_2.$

If $\alpha_1 = A + Bi,\ \alpha_2 = A - Bi$ then

$$b = -(\alpha_1+\alpha_2) = -2A$$

$$c = \alpha_1\alpha_2 = (A+Bi)(A-Bi) = A^2 + B^2.$$

6. a. $z = (x+iy)$

$$z^n = (x+iy)^n \overset{\text{B.T.}}{=} \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(iy)^k$$

$$= \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} x^{n-2l}\, i^{2l}\, y^{2l} + \sum_{l=0}^{\lfloor n-1/2 \rfloor} \binom{n}{2l+1} x^{n-2l-1}\, i^{2l+1}\, y^{2l+1}$$

$$= \sum_{l=0}^{\lfloor n/2 \rfloor} (-1)^l \binom{n}{2l} \cdot x^{n-2l} \cdot y^{2l} + i \cdot \sum_{l=0}^{\lfloor n-1/2 \rfloor} (-1)^l \binom{n}{2l+1} x^{n-2l-1} \cdot y^{2l+1}$$

$$= A(x,y) + i \cdot B(x,y)$$

Here we split the sum into two parts corresponding to $k=2l$ even & $k=2l+1$ odd. (These give the real & imaginary parts of the expansion) The symbol $\lfloor x \rfloor$ for $x \in \mathbb{R}$ denotes the largest integer $\leq x$.

For example, $\lfloor n/2 \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n-1/2 & \text{if } n \text{ is odd} \end{cases}$

Particular example: $n = 4$

$$z^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$
$$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$$

b $\quad z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^n = s(\cos\phi + i\sin\phi)$ where $s = r^n$ and $\phi = n\theta$.

7. $\quad (1+i) = \sqrt{2}\,(\cos \pi/4 + i\sin \pi/4)$

$$\Rightarrow (1+i)^{2015} = \sqrt{2}^{2015}\left(\cos\frac{2015\pi}{4} + i\sin\frac{2015\pi}{4}\right)$$
$$= \sqrt{2}\cdot\sqrt{2}^{2014}\left(\cos\left(251\cdot 2\pi + \frac{7\pi}{4}\right) + i\sin\left(251\cdot 2\pi + \frac{7\pi}{4}\right)\right)$$
$$= \sqrt{2}\cdot 2^{1007}\,(\cos(7\pi/4) + i\sin(7\pi/4))$$
$$= \sqrt{2}\cdot 2^{1007}\left(\frac{1}{\sqrt{2}} - i/\sqrt{2}\right)$$
$$= 2^{1007}\cdot(1-i)$$

8. a. $\quad z^2 - 1 = (z+1)(z-1)$

$$z^3 - 1 = (z-1)(z^2+z+1) = (z-1)\left(z - \left(\frac{-1+\sqrt{-3}}{2}\right)\right)\left(z - \left(\frac{-1-\sqrt{-3}}{2}\right)\right)$$
$$= (z-1)(z - (-1+\sqrt{3}i/2))(z - (-1-\sqrt{3}i/2))$$
$$z^4 - 1 = (z^2)^2 - 1 = (z^2+1)(z^2-1)$$
$$= (z+i)(z-i)(z+1)(z-1)$$

b. $\quad z^6 - 1 = (z^3)^2 - 1 = (z^3+1)(z^3-1)$

$$z^3 + 1 = (z+1)(z^2 - z + 1) = (z+1)(z - (1+\sqrt{3}i/2))(z - (1-\sqrt{3}i/2))$$

$z^3 - 1$: see part a

$$\Rightarrow (z^6 - 1) = (z-1)(z - (-1+\sqrt{3}i/2))(z - (-1-\sqrt{3}i/2))(z+1)(z - (1+\sqrt{3}i/2))(z - (1-\sqrt{3}i/2))$$

$$z^8 - 1 = (z^4)^2 - 1 = (z^4+1)(z^4-1)$$
$$z^4 + 1 = (z^2)^2 + i^2 = (z^2+i)(z^2-i) \overset{\dagger}{=} \left(z - \frac{1-i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right)\left(z - \frac{1+i}{\sqrt{2}}\right)\left(z + \frac{1+i}{\sqrt{2}}\right)$$

$z^4 - 1$: see above.

$$\Rightarrow z^8 - 1 = (z-1)(z+1)(z-i)(z+i)\left(z - \frac{1+i}{\sqrt{2}}\right)\left(z + \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right)$$

Note: In † I computed the square roots of i & $-i$ using polar coordinates. (Unfortunately, this is not quite in the spirit of part a ...)

c. i. $(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0.$

$z \cdot z \cdots z = a_n z^n \Rightarrow a_n = 1$

$-\alpha_1 \cdot z \cdots z + z\cdot(-\alpha_2)\cdot z \cdots z + \cdots + z\cdot z \cdots z \cdot (-\alpha_n) = a_{n-1}z^{n-1}$

$\Rightarrow a_{n-1} = -(\alpha_1 + \alpha_2 + \cdots + \alpha_n).$

ii. $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$ are equally spaced points on the circle, center the origin, radius 1.

So, by symmetry*, the center of mass $\frac{1}{n}(1+\zeta+\cdots+\zeta^{n-1})$ is the origin

i.e. $\frac{1}{n}(1+\zeta+\cdots+\zeta^{n-1}) = 0$, so $1+\zeta+\cdots+\zeta^{n-1} = 0$,

and $a_{n-1} = -(1+\zeta+\cdots+\zeta^{n-1}) = 0$ in part i above.

* In more detail: the rotation with center the origin through angle $2\pi/n$ preserves the set of points $\{1, \zeta, \ldots, \zeta^{n-1}\}$. So it must fix the center of mass. But the only point fixed by the rotation is the origin, so the center of mass equals the origin.

9. $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$

a. $w = f(z) = u + iv$, $z = x + iy$

$f(z) = z^2 = (x^2-y^2) + 2xyi$. So $u = x^2-y^2$, $v = 2xy$.

b. $L_1 = \{(x,y) \mid x = 1, y \in \mathbb{R}\} \subset \mathbb{R}^2$

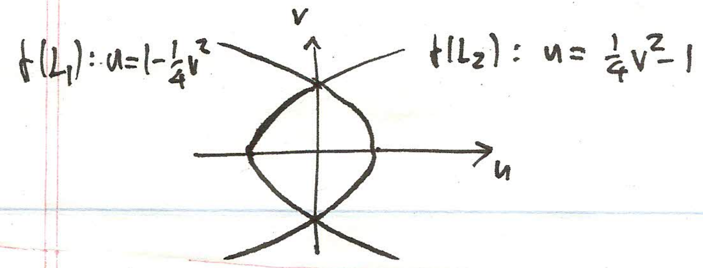
$\Rightarrow f(L_1) = \{(u,v) \mid u = 1-y^2, v = 2y, y \in \mathbb{R}\} \subset \mathbb{R}^2.$

$\Rightarrow f(L_1)$ has equation $u = 1 - (v/2)^2$ in the uv-plane (eliminating y)

c. Similarly $L_2 = \{(x,y) \mid y = 1, x \in \mathbb{R}\} \subset \mathbb{R}^2$

$\Rightarrow f(L_2) = \{(u,v) \mid u = x^2-1, v = 2x, x \in \mathbb{R}\} \subset \mathbb{R}^2$

$\Rightarrow f(L_2)$ has equation $u = (v/2)^2 - 1$



d. $u = 1 - \frac{1}{4}v^2$ & $u = \frac{1}{4}v^2 - 1$

$\cancel{\Rightarrow u = 0}$, $\frac{1}{4}v^2 - 1 = 1 - \frac{1}{4}v^2$

$v^2 = 4$, $v = \pm 2$, $u = 0$.

So $f(L_1)$ & $f(L_2)$ intersect at $(0, \pm 2) = \pm 2i$

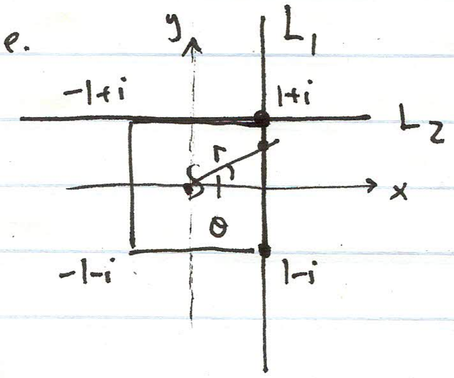
Slope of tangent lines:

$$M_1 = \frac{1}{(1 - \frac{1}{4}v^2)'(\pm 2)} = \frac{1}{-\frac{1}{2}v\,(\pm 2)} = \mp 1$$

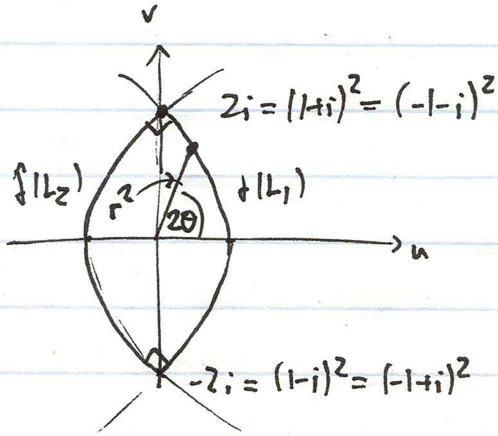
$$M_2 = \frac{1}{(\frac{1}{4}v^2 - 1)'(\pm 2)} = \frac{1}{\frac{1}{2}v\,(\pm 2)} = \pm 1.$$

$M_1 \cdot M_2 = -1 \Rightarrow$ perpendicular.

e.



$\xrightarrow{f}$



$$f(r(\cos\theta + i\sin\theta)) = r^2(\cos 2\theta + i\sin 2\theta) = s(\cos\phi + i\sin\phi)$$

$$s = r^2, \ \phi = 2\theta$$

First observe that because $f(z) = f(-z)$, the image of the boundary of the square equals the image of the union of the two sides contained in L_1 & L_2 (since the other two sides are obtained from these by $z \mapsto -z$). Thus the image of the boundary of the square is the union of the arcs of $f(L_1)$ & $f(L_2)$ between the two intersection points (which correspond under f to the ends of the ~~edges~~ sides of the square).

Now because the radial coordinate s in the w-plane is given by $s=r^2$, an increasing function of r, we see that the square S itself maps to the region bounded by $f(L_1)$ & $f(L_2)$.