MATH 611 HW4 Solutions

1. $|G| = m \cdot n$, $\gcd(m,n) = 1$.

a. $H \trianglelefteq G$, $|H| = m$

b. $K \leq G$, $|K| = n$.

c. $H \cap K = \{e\}$ (because $|H \cap K| \mid \gcd(m,n) = 1$ by Lagrange Thm)

It follows that the map of sets $H \times K \longrightarrow HK \subset G$
$(h,k) \longmapsto hk$

is injective

(Proof: $h_1 k_1 = h_2 k_2 \Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K = \{e\}$
$\Rightarrow h_1 = h_2,\ k_1 = k_2$ $\blacksquare$.)

So d. $HK = G$ (because $|HK| = |H| \cdot |K| = |G|$).

Now a, b, c, d $\Rightarrow$ $G \cong H \rtimes_\varphi K$

where $\varphi : K \longrightarrow \mathrm{Aut}(H)$
is defined by $k \longmapsto (h \longmapsto khk^{-1})$

2. a. $H := SO(2) \trianglelefteq O(2)$, subgroup of orthogonal 2×2 matrices of determinant 1 (matrices of rotations about the origin)

$K := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle = \left\{ I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ $\leq O(2)$ subgroup generated by a single reflection.

Then $H \cap K = \{I\}$, $HK = O(2)$ (using $\det A = \pm 1$ for $A \in O(2)$).

$\Rightarrow O(2) \cong H \rtimes_\varphi K$, $\varphi : K \longrightarrow \mathrm{Aut}\, H$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right)$

i.e., $\varphi : K \to \mathrm{Aut} H$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto (h \mapsto h^{-1})$.

b. $H := A_n \trianglelefteq S_n$.

$K := \langle (12) \rangle = \{e, (12)\}$ subgroup generated by a single transposition.

$H \cap K = \{e\}$; $HK = S_n$

$\Rightarrow S_n \cong A_n \rtimes_\varphi K$, $\varphi : K \to \mathrm{Aut}(A_n)$

$(12) \mapsto (\sigma \mapsto (12)\sigma(12)^{-1})$

c $H := SL_n(F) \trianglelefteq GL_n(F)$

$K := \left\{ \begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} \;\middle|\; \lambda \in F^* \right\} \leq GL_n(F)$, $K \simeq F^*$

$H \cap K = \{e\}$, $HK = GL_n(F)$ $\left(\text{using } \det \begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} = \lambda \right)$

$\Rightarrow GL_n(F) \cong SL_n(F) \rtimes_\varphi K$

$\varphi : K \to \mathrm{Aut}(SL_n(F))$, $\begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} \mapsto \left(A \mapsto \begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} A \begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}^{-1} \right)$

3. $G = GL_n(F)$

$H = SL_n(F) \trianglelefteq G$

$K = Z(GL_n(F)) = \{ \lambda \cdot I \mid \lambda \in F^* \} \trianglelefteq G$

$\cong$

F^*.

$H \cap K = \{ \lambda I \mid \lambda^n = 1 \}$

$H \cdot K = \{ A \mid \det A = \lambda^n, \text{ some } \lambda \in F \} \leq G.$

a. $F = \mathbb{R}$, n odd.

Then $\lambda^n = 1 \implies \lambda = 1$, so $H \cap K = \{I\}$

Also $\mathbb{R}^* \to \mathbb{R}^*$, $\lambda \mapsto \lambda^n$ is surjective, so $HK = G$

Now $H \triangleleft G$, $K \triangleleft G \implies G \cong H \times K$.

b. $F = \mathbb{Z}/p\mathbb{Z}$, $\gcd(n, p-1) = 1.^\dagger$

Recall F^* is cyclic, $F^* \cong \mathbb{Z}/(p-1)\mathbb{Z}$.

So $\lambda \in F$, $\lambda^n = 1 \implies \lambda = 1$ using $\dagger$, i.e. $H \cap K = \{I\}$

Also $F^* \to F^*$, $\lambda \mapsto \lambda^n$ is surjective, so $HK = G$

$\therefore G \cong H \times K$.

c. $d := \gcd(n, p-1)$, assume $\gcd(d, \frac{p-1}{d}) = 1.^{\dagger\dagger}$

$\lambda \in F$, $\lambda^n = 1 \implies \lambda^d = 1$

$$\begin{array}{ccccc} \text{So} \quad H \cap K & = & \{ \lambda I \mid \lambda^d = 1 \} & < & K \\ & & \shortparallel & & \shortparallel \\ & & \mathbb{Z}/d\mathbb{Z} & < & \mathbb{Z}/(p-1)\mathbb{Z} \end{array}$$

$\therefore$ Here, need to change definition of K to obtain direct product decomposition:-

$$\begin{array}{ccc} K \cong \mathbb{Z}/(p-1)\mathbb{Z} & \cong & \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/(\frac{p-1}{d})\mathbb{Z} \quad (\text{using } \dagger\dagger) \\ \cup & & \cup \\ H \cap K & \xrightarrow{\sim} & \mathbb{Z}/d\mathbb{Z} \times \{0\}. \end{array}$$

So define $K' \subset K$ as the inverse image of $\{0\} \times \mathbb{Z}/_{(p-1/d)\mathbb{Z}} \subset \mathbb{Z}/_{d\mathbb{Z}} \times \mathbb{Z}/_{(p-1)\mathbb{Z}}$ under the above isomorphism.

Then $H \cap K' = \{e\}$

But $HK' = HK = \{A \mid \det A = \lambda^n, \text{ some } \lambda \in F_p^*\} \lneq G$

because $F^* \to F^*$ $\lambda \mapsto \lambda^n$ is NOT surjective.

So, only obtain $H \times K' \cong HK' \lneq G$.

4. a. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

The group operation is determined by $i^2 = j^2 = k^2 = ijk = -1$.

Note that $\pm i, \pm j, \pm k$ have order 4

-1 has order 2, & 1 is the identity element.

So, any automorphism of Q_8 must fix 1 & -1,
and permute $\pm i, \pm j, \pm k$.
Also the pairs $\pm i, \pm j, \pm k$ are pairs of inverse elements so must also be preserved.

Let G be the group of rotational symmetries of the cube.
(Then $G \cong S_4$).
Consider the cube w/ vertices $(\pm 1, \pm 1, \pm 1)$.
Then the centers of the faces are $\pm i = \pm(1,0,0)$, $\pm j = \pm(0,1,0)$, $\pm k = \pm(0,0,1)$

Define a group homomorphism

$$\varphi: G \longrightarrow S_{Q_8} = \{\text{permutations of } Q_8\}$$

as follows: $\varphi(g)$ fixes ± 1, & permutes $\pm i, \pm j, \pm k$ according to the action of g on the centers of the faces of the cube.

We claim that in fact $\varphi(g): Q_8 \to Q_8$ is an automorphism of Q_8 for all $g \in G$.

We must check $\varphi(g)(q_1 q_2) = \varphi(g)(q_1) \cdot \varphi(g)(q_2)$ $\dagger$ $\forall q_1, q_2 \in Q_8$ (i.e. $\varphi(g)$ is a homomorphism).

If q_1 OR $q_2 = 1$, this is trivially true.

If q_1 OR $q_2 = -1$, this holds because g sends opposite faces to opposite faces.

If $q_1 = \pm q_2$, this follows from $i^2 = j^2 = k^2 = -1$.

In the remaining cases, $q_1 q_2 = q_1 \times q_2$ (cross product).

And the cross product is invariant under rotations.

(i.e. $\theta(q_1 \times q_2) = \theta(q_1) \times \theta(q_2)$ for θ a rotation)

This shows $\dagger$ in these cases

So φ defines a homomorphism $\varphi: S_4 \to \mathrm{Aut}(Q_8)$.

Finally, this homomorphism is clearly injective.

And $|\mathrm{Aut}(Q_8)| \leq 24$: an auto is given by a choice of permutation of i, j, k, together with a choice of 3 signs, with one condition imposed by $ijk = -1$.

$\rightsquigarrow$ $3! \cdot (2^3/2) = 24$.

So φ is surjective, & φ is an isomorphism.

b. Have hom. $\text{Aut}(Q_8) \twoheadrightarrow S_3$ τ given by sending an auto θ to the associated permutation of the pairs $\pm i, \pm j, \pm k$ of inverse elements of order 4.

The kernel is give by the even sign changes

$i \mapsto \pm i, \quad j \mapsto \pm j, \quad k \mapsto \pm k$

(i.e. have even number of "$-$" signs, to preserve $ijk=-1$.)

Thus the kernel is isomorphic to $\ker\left((\mathbb{Z}/2\mathbb{Z})^3 \xrightarrow{\Sigma} \mathbb{Z}/2\mathbb{Z}\right)$

$\simeq (\mathbb{Z}/2\mathbb{Z})^2$

To express $\text{Aut}(Q_8) \simeq S_4$ as a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^2$ and S_3 we first need to realize S_3 as a subgroup of $\text{Aut}(Q_8)$ (which maps isomorphically to S_3 under the hom τ). For this, it's better to work directly with S_4:-

$(\mathbb{Z}/2\mathbb{Z})^2 \simeq H = \{e, (12)(34), (13)(24), (14)(23)\} \lhd S_4$.

$K = S_3 = S_{\{1,2,3\}} \leq S_{\{1,2,3,4\}} = S_4$.

$H \cap K = \{e\}$ ✓, $|H| \cdot |K| = S_4 \Rightarrow H \rtimes_\varphi K \cong S_4$,

where $\varphi: K \to \text{Aut}(H)$, $k \mapsto khk^{-1}$

In fact, as we showed in class $\text{Aut}(H) \cong \text{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$

$\cong GL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$

and $\varphi: K \to \text{Aut}(H)$ is an isomorphism.

In fact, if we label the non identity elements of H as $a = (14)(23)$, $b = (13)(24)$, $c = (12)(34)$ and identify $\text{Aut}\, H = S_{\{a,b,c\}}$, then $\varphi: S_3 \xrightarrow{\sim} S_{\{a,b,c\}}$ is given by

$\sigma \mapsto f \sigma f^{-1}$ where $f: \{1,2,3\} \to \{a,b,c\}$

$1 \mapsto a,\ 2 \mapsto b,\ 3 \mapsto c.$

5. a.

$|D_{60}| = 2 \cdot 60 = 2^3 \cdot 3 \cdot 5$

H Sylow 2-subgroup, i.e $|H| = 2^3 = 8$.

We can construct an example $H \cong D_4$ by inscribing a square in the regular 60-gon†. Then H is subgroup preserving the square.

† w/ vertices a subset of the vertices of the 60-gon

According to Sylow Thm 2, all Sylow p-subgroups of $G = D_{60}$ are conjugate.

So all Sylow 2-subgroups are obtained in this way, for some choice of inscribed square.

$\therefore$ # Sylow 2-subgroups = # inscribed squares = 15

b. $G = S_6$. $|S_6| = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 2^4 \cdot 3^2 \cdot 5$

$H \leq G$, $|H| = 3^2$.

Ex: $H = \langle (123), (456) \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2$.

All other Sylow 3-subgroups are conjugate.

$$\therefore \# = \frac{\dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2} \cdot \dfrac{3 \cdot 2 \cdot 1}{3 \cdot 2}}{2} = 10$$

c. $G = GL_3(\mathbb{Z}/5\mathbb{Z})$, $p = 5$.

In general, a Sylow p-subgroup in $GL_n(\mathbb{Z}/p\mathbb{Z})$ is given by

$$H = U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \leq GL_n(\mathbb{Z}/p\mathbb{Z}) = G, \quad |U| = p^{\frac{1}{2}n(n-1)}$$

(upper triangular matrices w/ 1's on diagonal)

Moreover, the normalizer

$$N(U) = B = \left\{ \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} \right\} \leq GL_n(\mathbb{Z}/p\mathbb{Z})$$

(upper triangular invertible matrices).

All Sylow p-subgroups are conjugate.

$\therefore$ By orbit-stabilizer,

$$\# \text{ Sylow } p\text{-subgroups} = |G| / |G_H| \quad \text{where } G \curvearrowright X = \{\text{Sylow } p\text{-subgroups}\} \text{ by conjugation}$$

$$= |G| / |N(H)|$$

$$= |G| / |B|.$$

$$= \frac{(p^n - 1)(p^n - p^2) \cdots (p^n - p^{n-1})}{(p-1)^n \cdot p^{\frac{1}{2}n(n-1)}}$$

$$= \frac{(p^n - 1)}{(p-1)} \cdot \frac{(p^{n-1} - 1)}{(p-1)} \cdots \frac{(p^2 - 1)}{(p-1)}$$

In our case

$$\# = \frac{(5^3 - 1)}{(5-1)} \cdot \frac{(5^2 - 1)}{(5-1)} = (1 + 5 + 5^2) \cdot (1+5) = 31 \cdot 6$$

$$= \boxed{186.}$$

6. $|G| = 50 = 2 \cdot 5^2$.

An element of order 5 is contained in a Sylow 5-subgroup (by Sylow Thm 2b.)

$s = \#$ Sylow 5-subgroups. $\quad s \equiv 1 \bmod 5$ & $s \mid 2 \implies s = 1$.

$|H| = 5^2 \implies H \simeq \mathbb{Z}/25\mathbb{Z}$ OR $(\mathbb{Z}/5\mathbb{Z})^2$

So $\#$ elements of order 5 $= 4$ OR 24.

Examples are give by the abelian cases

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \quad , \quad \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/5\mathbb{Z})^2 .$$

7. $|G| = 45 = 3^2 . 5$

$s = \#$ Sylow 3-subgroups. $s \equiv 1 \bmod 3$, $s \mid 5 \Rightarrow s = 1$

$t = \#$ Sylow 5-subgroups. $t \equiv 1 \bmod 5$, $t \mid 9 \Rightarrow t = 1$.

$\therefore \exists\ H, K$ s.t. $|H| = 9$, $|K| = 5$, $H \triangleleft G$ & $K \triangleleft G$.

Then $^{1.}$ $H \cap K = \{e\}$ (since $\gcd(9,5) = 1$),

& $^{2.}$ $HK = G$ ($|HK| \overset{!}{=} |H| \cdot |K| = |G|$)

$\Rightarrow$ $G \simeq H \times K$ (both H & K are normal!)

So, finally, since $H \simeq \mathbb{Z}/9\mathbb{Z}$ OR $(\mathbb{Z}/3\mathbb{Z})^2$ & $K \simeq \mathbb{Z}/5\mathbb{Z}$

have $G \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ OR $(\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/5\mathbb{Z}$.

8. $|G| = 57 = 3 \cdot 19$, G non abelian.

This is a case we discussed in class: if $|G| = p \cdot q$,
where $p < q$ are primes then either G is abelian, $G \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$,
OR $q \equiv 1 \bmod p$ and $G \cong \mathbb{Z}/q\mathbb{Z} \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$,

where $\varphi : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$.

$1 \longmapsto (q-1)/p$.

an element of order p.

In our case $p = 3$, $q = 19$.

First, must find a generator of $(\mathbb{Z}/19\mathbb{Z})^\times \simeq \mathbb{Z}/18\mathbb{Z}$.

We find $2 \in (\mathbb{Z}/19\mathbb{Z})^\times$ is a generator

So $2^{(19-1)/3} = 2^6 = 7 \bmod 19$ is an element of order 3. in $(\mathbb{Z}/19\mathbb{Z})^\times$.

Finally, recall that the identification $\mathrm{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^\times$
is given by $(x \mapsto c \cdot x) \leftarrow\!\shortmid c$,

so
$$\varphi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/19\mathbb{Z})$$
$$1 \mapsto (x \mapsto 7 \cdot x)$$

Writing $\mathbb{Z}/3\mathbb{Z} \xrightarrow{\sim} \langle b \rangle$, $i \mapsto b^i$. $\mathbb{Z}/19\mathbb{Z} \xrightarrow{\sim} \langle a \rangle$, $i \mapsto a^i$

we have
$$\mathbb{Z}/19\mathbb{Z} \rtimes_\varphi \mathbb{Z}/3\mathbb{Z} \simeq \langle a, b \mid a^{19} = e, b^3 = e, bab^{-1} = a^7 \rangle$$

9. $|G| = p^a q^b$.

$s = \#$ Sylow q-subgroups.
$s \equiv 1 \bmod q$, $s \mid p^a$

We are given that $p, p^2, \ldots, p^a \not\equiv 1 \bmod q$.
So $s = 1$, & a Sylow q-subgroup H is normal.
Now let K be a Sylow p-subgroup.
Then, as in Q1, $G \simeq H \rtimes_\varphi K$.

10. $|G| = pqr$, p,q,r distinct primes. WLOG $p<q<r$

Let s,t,u be # Sylow p,q,r subgroups respectively.
Suppose $s,t,u \neq 1$.

$s \equiv 1 \mod p$, $s \mid qr \Rightarrow s = q, r$ OR qr
$t \equiv 1 \mod q$, $t \mid pr \Rightarrow t = r$ OR pr
$u \equiv 1 \mod r$, $u \mid pq \Rightarrow u = pq$

$\therefore \ |G| = pqr \geq \underbrace{pq\cdot(r-1)}_{\text{elts of order } r} + \underbrace{r\cdot(q-1)}_{\text{elts of order } q} + \underbrace{q\cdot(p-1)}_{\text{elts of order } p} + \underbrace{1}_{e}$
$= pqr + (q-1)(r-1)$ ※.

$\therefore \ s=1$ or $t=1$ or $u=1$. $\square$.

11. a. $|G| = 18 = 2\cdot 3^2$

$s = $ # Sylow 3-subgroups. $s \equiv 1 \mod 3$ & $s \mid 2 \Rightarrow s = 1$
So, let H be Sylow 3-subgroup, then $H \trianglelefteq G$, & $H \cong \mathbb{Z}/9\mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z})^2$
Let K be Sylow 2-subgroup, then $K \leq G$, $K \cong \mathbb{Z}/2\mathbb{Z}$.
As in Q1, $G \cong H \rtimes_\varphi K$, $\varphi: K \to \operatorname{Aut} H$.

If φ is trivial ($\varphi(k) = id_H \ \forall k \in K$) G is abelian:
$G \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$ OR $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$

Now assume φ non-trivial.
Case a. $H \cong \mathbb{Z}/9\mathbb{Z}$.

$$\varphi: K \longrightarrow \operatorname{Aut} H$$

$K \cong \mathbb{Z}/2\mathbb{Z}$, $\operatorname{Aut} H \cong \operatorname{Aut}(\mathbb{Z}/9\mathbb{Z}) \cong (\mathbb{Z}/9\mathbb{Z})^\times \cong (\mathbb{Z}/6\mathbb{Z})$, $2^i \longleftarrow i$

$(\mathbb{Z}/9\mathbb{Z})^\times = \{a \in \mathbb{Z}/9\mathbb{Z} \mid \gcd(a,9) = 1\} = \{1,2,4,5,7,8\}$

The only non trivial hom $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ is given by $i \mapsto 3i$.

So, writing $H = \langle a \rangle$, $K = \langle b \rangle$ we find

$$\begin{aligned} G = H \rtimes_\varphi K &= \langle a, b \mid a^9 = b^2 = e,\ bab^{-1} = a^{(2^3)} \rangle \\ &= \langle a, b \mid a^9 = b^2 = e,\ bab^{-1} = a^{-1} \rangle \\ &\cong D_9. \end{aligned}$$

Case b. $H \simeq (\mathbb{Z}/3\mathbb{Z})^2$.

$$\begin{array}{ccc} \varphi: K & \longrightarrow & \mathrm{Aut}\, H \\ \parallel & & \parallel \\ \mathbb{Z}/2\mathbb{Z} & & \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2) \simeq GL_2(\mathbb{Z}/3\mathbb{Z}) \end{array}$$

A hom $\mathbb{Z}/2\mathbb{Z} \to GL_2(\mathbb{Z}/3\mathbb{Z})$ is given by $A \in GL_2(\mathbb{Z}/3\mathbb{Z})$, w/ $A^2 = I$.

Such a matrix is diagonalizable over $\mathbb{Z}/3\mathbb{Z}$

(because its minimal polynomial has distinct linear factors:
$A^2 - I = (A-I)(A+I) = 0$.)

So after a change of basis in $(\mathbb{Z}/3\mathbb{Z})^2$ (i.e., changing the identification $H \simeq (\mathbb{Z}/3\mathbb{Z})^2$) we may assume A is diagonal.

Then $A^2 = I$, $A \neq I \Rightarrow A = \overset{\text{i.}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \overset{\text{ii.}}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$

(or $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, but again can switch diagonal entries by change of basis)

Now translating into generators & relations:

Write $H = \langle a, b \rangle \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $K = \langle c \rangle \simeq \mathbb{Z}/2\mathbb{Z}$

$a^i b^j \leftarrow (i, j)$

i. $G = H \rtimes_\varphi K = \langle a, b, c \mid a^3 = b^3 = c^2 = e,\ ab = ba,\ cac^{-1} = a,\ cbc^{-1} = b^{-1} \rangle$

$\simeq \langle a \rangle \times \langle b, c \mid b^3 = c^2 = e,\ cbc^{-1} = b^{-1} \rangle \cong \mathbb{Z}/3\mathbb{Z} \times D_3$

ii. $G = H \rtimes_\varphi K = \langle a, b, c \mid a^3 = b^3 = c^2 = e,\ ab = ba,\ cac^{-1} = a^{-1},\ cbc^{-1} = b^{-1} \rangle$

Finally we show the groups are not isomorphic.
Using G abelian / non abelian & Sylow 3-subgroup $H \cong \mathbb{Z}/9\mathbb{Z}$ / $(\mathbb{Z}/3\mathbb{Z})^2$
can distinguish all cases except (b)(i) & (b)(ii).
But observe that $Z(G) \cong \mathbb{Z}/2\mathbb{Z}$ in case (b)(i)
whereas $Z(G) = \{e\}$ in case (b)(ii). $\square$.

12. $\varphi: G \hookrightarrow S_G$ via left multiplication.

a. If g has order m and $|G| = n$
then the cycle type of $\varphi(g)$ is a product of n/m disjoint cycles of length m.
This is a product of $n/m \cdot (m-1)$ transpositions.
So $\varphi(g)$ is odd $\iff$ m is even & n/m is odd.

b. Now suppose $|G| = 2m$, where m is odd.
$\exists\, g \in G$ s.t. g has order 2 (because 2 is prime & $2 \mid |G|$)
Then by (a), $\varphi(g)$ is an odd permutation.
So the homomorphism

$$\begin{tikzcd} G \arrow[r, "\varphi"'] \arrow[rr, bend left, "\psi"] & S_G \arrow[r, "\mathrm{sgn}"'] & \{\pm 1\} \end{tikzcd}$$

is surjective
Now $\ker(\psi) \trianglelefteq G$ is a normal subgroup of index 2.

13. $G = GL_n(\mathbb{Z}/p\mathbb{Z})$
$H \leq G$, $|H| = p^\alpha$, some $\alpha \in \mathbb{N}$.
A Sylow p-subgroup of G is given by $U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \leq G$.

upper triangular matrices w/ 1's on diagonal

By Sylow Thm 2b, $\exists\ g \in G$ s.t. $gHg^{-1} \leq U$.

i.e. ghg^{-1} is upper triangular $\forall\ h \in H$ $\square$.