Math 611 Homework 1

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Reading: Dummit and Foote, Sections 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, and 1.7.

Justify your answers carefully (complete proofs are expected).

(1) Let G be a group of order p, a prime. Show that G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(2) Let G be a group of order 4. Show that G is abelian

(3) An *isometry* or *rigid motion* of $\mathbb{R}^2$ is a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which preserves distances, that is, for all $p, q \in \mathbb{R}^2$, $d(T(p), T(q)) = d(p, q)$, where $d(p, q)$ denotes the distance from p to q. The isometries of $\mathbb{R}^2$ form a group with the group law given by composition of transformations.

Let a be the isometry of $\mathbb{R}^2$ given by rotation about a point p through angle θ counterclockwise and let b be isometry of $\mathbb{R}^2$ given by reflection about a line l through p. Show that $bab = a^{-1}$.

(4) Let G be a finite group of isometries of $\mathbb{R}^2$.

(a) Show that there exists a point $p \in \mathbb{R}^2$ such that for all $g \in G$ $g(p) = p$. Choosing coordinates we may assume p is the origin; then each $g \in G$ is a linear transformation given by an orthogonal matrix.

(b) Show that the subgroup of G consisting of rotations is a cyclic group.

(c) Show that G is either a group of rotations isomorphic to $\mathbb{Z}/n\mathbb{Z}$, or the dihedral group D_n of symmetries of a regular n-gon, for some $n \in \mathbb{N}$. (Here we include the cases $D_1 \simeq \mathbb{Z}/2\mathbb{Z}$ and $D_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of the dihedral group, which can (for example) be defined as the symmetries of an isosceles triangle and a rectangle respectively.)

(5) Show that the dihedral group D_4 of symmetries of the square is *not* isomorphic to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

(6) Prove the *Chinese remainder theorem*: If $\gcd(m, n) = 1$ then $\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

(7) Recall the structure theorem for finitely generated abelian groups G:

$$G \simeq \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \cdots \mathbb{Z}/p_r^{\alpha_r}\mathbb{Z} \times \mathbb{Z}^s$$

for some $r \in \mathbb{Z}_{\geq 0}$, primes $p_1, \ldots, p_r$, exponents $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$, and $s \in \mathbb{Z}_{\geq 0}$. Moreover, the factors in the product decomposition are uniquely determined up to reordering.

Now assume that G is finite. Prove the uniqueness statement for the factors in the product decomposition.

(8) Let D_n be the dihedral group of symmetries of the regular n-gon. Prove that D_{2n} is isomorphic to $D_n \times \mathbb{Z}/2\mathbb{Z}$ if and only if n is odd.

(9) (a) Suppose $\sigma \in S_n$ is a permutation which has cycle type $(l_1, \ldots, l_r)$ (that is, is a product of disjoint cycles of lengths $l_1, \ldots, l_r$). What is the sign $\operatorname{sgn}(\sigma)$ of σ?

(b) List the elements of S_3, A_3, S_4, and A_4 in cycle notation (that is, write each element as a product of disjoint cycles).

(c) Given a group G, we say $a, b \in G$ are *conjugate* if there exists a $g \in G$ such that $gag^{-1} = b$. Show that two elements of S_n are conjugate iff they have the same cycle type.

(d) Let $\sigma \in S_n$. Suppose we draw smooth paths

$$\gamma_i = \{(x_i(t), y_i(t)) \mid 0 \leq t \leq 1\}$$

in the xy-plane from $(i, 1)$ to $(\sigma(i), 0)$ for each $i = 1, \ldots, n$, such that

i. $y_i'(t) < 0$ for all i, t,

ii. at most two paths intersect at any point,

iii. at a point where two paths γ_i and γ_j intersect, the tangent vectors to the paths at that point are linearly independent (we say the paths intersect *transversely*), and

iv. the points of intersection of the paths have distinct y coordinates.

Let m be the total number of intersection points of the paths. Show that σ is a product of m transpositions. In particular, $\operatorname{sgn}(\sigma) = (-1)^m$.

(10) (a) Show that the dihedral group D_3 of symmetries of an equilateral triangle is isomorphic to S_3. What is the image of the subgroup of rotations under this isomorphism?

(b) Show that the group G of symmetries of a regular tetrahedron is isomorphic to S_4. What is the image of the subgroup of rotations under this isomorphism? For each of the cycle types of S_4, give a geometric description of the corresponding symmetry. Show that the 3-cycles in A_4 form two conjugacy classes in A_4, and describe how to distinguish them geometrically.

(11) Given two groups H and K, we say a group G is a *semi-direct product* of H and K and write $G = H \rtimes K$ if

(a) H is a normal subgroup of G,

(b) K is a subgroup of G,

(c) $HK := \{hk \mid h \in H, k \in K\} = G$, and

(d) $H \cap K = \{e\}$.

In this situation, the isomorphism type of G is determined by the homomorphism $\varphi\colon K \to \operatorname{Aut}(H)$ from K to the group of automorphisms of H, given by $k \mapsto (h \mapsto khk^{-1})$; we write $G = H \rtimes_\varphi K$ if we want to make the dependence on φ explicit.

Now let $G = \mathbb{Z}/3\mathbb{Z} \rtimes_\varphi \mathbb{Z}/4\mathbb{Z}$, where $\varphi\colon \mathbb{Z}/4\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/3\mathbb{Z})$ is the homomorphism given by $1 \mapsto (x \mapsto -x)$.

Show that no two of the groups D_6, A_4, and G are isomorphic.

(12) Let F be a field and $n \in \mathbb{N}$. Let $\sim$ be the equivalence relation on $F^{n+1} \setminus \{0\}$ defined by $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if there exists $0 \neq \lambda \in F$ such that $(x_0, \dots, x_n) = \lambda \cdot (y_0, \dots, y_n)$. Define $\mathbb{P}^n_F$, the *projective n-space over F*, to be the set of equivalence classes $\mathbb{P}^n_F = (F^{n+1} \setminus \{0\})/\sim$.

(a) Show that there is a bijection of sets $\mathbb{P}^1_F \to F \cup \{\infty\}$ given by

$$[(x, y)] \mapsto \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ \infty & \text{if } y = 0. \end{cases}$$

(b) (Optional) Show that there is a bijection of sets $\mathbb{P}^n_F \to F^n \cup \mathbb{P}^{n-1}_F$.

(c) The action of $\mathrm{GL}_{n+1}(F)$ on F^{n+1} induces an action of $\mathrm{PGL}_{n+1}(F)$ on $\mathbb{P}^n_F$. Deduce that, for $|F| = q < \infty$ there is a natural homomorphism $\theta \colon \mathrm{PGL}_2(F) \to S_{q+1}$.

(d) Show that θ is injective.

(e) Deduce that θ is an isomorphism for $q = 2, 3$. What is the image of θ for $q = 4$?

Hints:

2 There exist elements $a, b \in G$ such that $G = \{e, a, b, ab\}$ (why?). Now use cases to show that $ba = ab$. Deduce that G is abelian.

3 By choosing coordinates we may assume p is the origin and l is the x-axis. Now compute using matrices. Alternatively, one can try to argue geometrically.

4 (a) Pick a point $q \in \mathbb{R}^2$, consider its orbit $\mathcal{O} = \{g \cdot q \mid g \in G\}$ under the action of G, and the center of mass of the orbit.

5 What are the orders of the elements of these groups?

6 First write down a natural homomorphism $\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and show it is injective. What is the pigeonhole principle?

7 Consider the orders of elements.

8 Recall that the *center* $Z(G)$ of a group G is the subgroup of elements $z \in G$ such that $zg = gz$ for all $g \in G$. What is the center of the dihedral group D_n?

10 (a) The action of the group of symmetries on the vertices of the triangle defines a homomorphism $D_3 \to S_3$. (b) The regular tetrahedron can be realized as the polyhedron with vertices $(1, 1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ in $\mathbb{R}^3$ (a subset of the vertices $(\pm 1, \pm 1, \pm 1)$ of a cube). What are the types of isometry of $\mathbb{R}^3$? (What is a rotary reflection?)