# Math 462: Homework 3 solutions

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**1**. Let $A, B, C$ be three points on a circle with center $O$ in the plane.

(a) Suppose that $B$ and $O$ are on the same side of the line $AC$. Show that the angle $\angle ABC$ equals half the angle $\angle AOC$. [Hint: Draw a diagram and use the angle sum of a triangle equals $\pi$ radians.]

(b) What about if $B$ and $O$ are on opposite sides of $AC$?

(a) Draw the triangle $ABC$ and the radii $OA$, $OB$, $OC$. The triangles $BOA$ and $BOC$ are isosceles (two sides of equal length). So writing $\alpha = \angle ABO = \angle BAO$ and $\beta = \angle BCO = \angle CBO$ we have $\angle BOA = \pi - 2\alpha$ and $\angle BOC = \pi - 2\beta$. Now

$$\angle BOA + \angle BOC + \angle AOC = 2\pi,$$

(this is where we use that $B$ and $O$ are on the same side of the line $AC$) so

$$\angle AOC = 2\pi - (\pi - 2\alpha) - (\pi - 2\beta) = 2(\alpha + \beta) = 2(\angle ABO + \angle CBO) = 2\angle ABC$$

as required.

(b) In this case we have

$$\angle BOA + \angle BOC = \angle AOC$$

so

$$\angle AOC = (\pi - 2\alpha) + (\pi - 2\beta) = 2\pi - 2(\angle ABO + \angle CBO) = 2\pi - 2\angle ABC.$$

**2**. Show that the sum of the angles of a polygon with $n$ sides equals $(n-2)\pi$ radians. [Hint: Subdivide the polygon into triangles.]

(There are many ways to do this, here is one possible solution.) Label the vertices of the polygon $P_1, P_2, \ldots, P_n$ in anticlockwise order. Now draw the lines $P_1P_3, P_1P_4, \ldots, P_1P_{n-1}$. This divides the polygon into $(n-2)$ triangles $P_1P_2P_3, P_1P_3P_4, \ldots, P_1P_{n-1}P_n$. We observe that the sum of the angles of the polygon equals the sum of the angles of all the triangles. We know that the sum of the angles of a triangle equals $\pi$. So the sum of the angles of the polygon equals $(n-2)\pi$.

**3**. Recall that in class Prof Urzua showed that any 3 points $A, B, C$ lie on a circle (the *circumcircle* of the triangle $ABC$). What condition must the angles of the triangle $ABC$ satisfy so that the center of the circumcircle lies inside the triangle? [Hint: Use Q1.]

From $Q1$ we see that if $ABC$ is a triangle with vertices on a circle with center $O$, then $O$ lies on the same side of $AC$ as $B$ if $\angle ABC < \frac{\pi}{2}$, on the opposite side if $\angle ABC > \frac{\pi}{2}$, and $O$ lies on the line $AC$ if $\angle ABC = \frac{\pi}{2}$. So the condition for the center of the circumcircle of a triangle to be contained in the triangle is that all the angles of the triangle are less than $\frac{\pi}{2}$ radians.

**4**.

(a) Show that if 4 points $A, B, C, D$ lie on a circle (in that order) then the sum of the angles $\angle ABC$ and $\angle CDA$ equals $\pi$ radians:

$$\angle ABC + \angle CDA = \pi.$$

[Hint: Use Q1.]

(b) Conversely, suppose we are given 4 points $A, B, C, D$ such that $\angle ABC + \angle CDA = \pi$. Does it follow that $A, B, C, D$ lie on a circle? Why?

(a) Let $O$ denote the center of the circle. The points $B$ and $D$ lie on opposite sides of the line $AC$. Suppose that $B$ lies on the same side of $AC$ as the center $O$ (the other case can be treated in the same way). Then by Q1 parts (a) and (b) we have

$$\angle AOC = 2\angle ABC = 2\pi - 2\angle ADC.$$

So

$$\angle ABC + \angle ADC = \frac{1}{2}\angle AOC + \frac{1}{2}(2\pi - \angle AOC) = \pi.$$

(b) The answer is yes if we assume that $B$ and $D$ lie on opposite sides of the line $AC$ (otherwise the answer is no in general): Suppose that $\angle ABC \leq \frac{\pi}{2}$ (the other case $\angle CDA \leq \frac{\pi}{2}$ can be treated the same way). The circle passing through $A, B, C$ has center $O$ the point on the perpendicular bisector of $AB$ such that $\angle AOC = 2\angle ABC$ and $O$ lies on the same side of $AC$ as $B$. (Note that this determines $O$ and the radius of the circle uniquely). Similarly the circle passing through $A, D, C$ has center $O'$ the point on the perpendicular bisector of $AB$ such that $\angle AO'C = 2\pi - 2\angle ADC$ and $O'$ lies on the opposite side of $AC$ to $D$ (note $\angle ADC = \pi - \angle ABC \geq \frac{\pi}{2}$ by assumption). Now since $\angle ABC + \angle ADC = \pi$ and $B$ and $D$ lie on opposite sides of $AC$ we see that the points $O$ and $O'$ coincide, and so the two circles also coincide, that is, there is a circle through all 4 points $A, B, C, D$.

**5.**

(a) Show that the rotation $T = \mathrm{Rot}(\mathbf{c}, \theta)$ of $\mathbb{R}^2$ about a point $\mathbf{c}$ through angle $\theta$ anticlockwise is given by the formula

$$T(\mathbf{x}) = A(\mathbf{x} - \mathbf{c}) + \mathbf{c}$$

where

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the matrix defining the rotation about the origin through angle $\theta$ anticlockwise. [Hint: Draw a picture.]

(b) Let $P, Q$ be points in $\mathbb{R}^2$. Show that the composition

$$\mathrm{Rot}(Q, -\theta) \circ \mathrm{Rot}(P, \theta)$$

of rotation about $P$ through angle $\theta$ anticlockwise followed by rotation about $Q$ through angle $\theta$ clockwise is a translation. What is the translation vector? Explain geometrically what happens if $\theta = \pi$. Show that if $\theta$ is small, then the translation vector has length approximately $\theta \cdot d(P, Q)$ and its direction is approximately perpendicular to $PQ$. [Hint: To make things easier, we can choose coordinates so that $P$ is the origin and $Q = \mathbf{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$ is a point on the $x$-axis. Now use the formula from part (a).]

(a) We write $\mathbf{x} = \mathbf{c} + (\mathbf{x} - \mathbf{c})$. The effect of rotation about $\mathbf{c}$ on the point $\mathbf{x}$ is to rotate the vector $(\mathbf{x} - \mathbf{c})$ through angle $\theta$. This gives $T(\mathbf{x}) = \mathbf{c} + A(\mathbf{x} - \mathbf{c})$.

(b) Choose $P$ as the origin and write $\mathbf{c}$ for the vector $\overrightarrow{PQ}$. Then

$$\mathrm{Rot}(P, \theta)(\mathbf{x}) = A\mathbf{x}$$

and

$$\mathrm{Rot}(Q, -\theta)(\mathbf{x}) = A^{-1}(\mathbf{x} - \mathbf{c}) + \mathbf{c}$$

by part (a), so the composition

$$\mathrm{Rot}(Q, -\theta) \circ \mathrm{Rot}(P, \theta)(\mathbf{x}) = \mathrm{Rot}(Q, -\theta)(A\mathbf{x})$$

$$= A^{-1}(A\mathbf{x} - \mathbf{c}) + \mathbf{c} = \mathbf{x} + (\mathbf{c} - A^{-1}\mathbf{c}).$$

is translation by the vector $\mathbf{d} = \mathbf{c} - A^{-1}\mathbf{c}$. To compute $\mathbf{d}$ in coordinates, assume $\mathbf{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$ (by choosing coordinates appropriately). Then

$$\mathbf{d} = \begin{pmatrix} c \\ 0 \end{pmatrix} - \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = c \begin{pmatrix} 1 - \cos\theta \\ \sin\theta \end{pmatrix}.$$

If $\theta = \pi$, we find $\mathbf{d} = c \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{c}$. (To see why, draw a picture.) If $\theta$ is small, then

$$\cos\theta = 1 - \frac{\theta^2}{2} + \cdots \approx 1$$

and

$$\sin\theta = \theta - \frac{\theta^3}{6} + \cdots \approx \theta,$$

so

$$\mathbf{d} \approx c \begin{pmatrix} 0 \\ \theta \end{pmatrix}.$$

This is a vector perpendicular to $\mathbf{c} = \overrightarrow{PQ}$ of length $c\theta = d(P, Q) \cdot \theta$.