Math 611 Final, Friday 12/6/13 – Friday 12/13/13.

Instructions: This is a take-home exam. Solutions are due by 5PM on Friday 12/13/13. Show all your work and justify your answers carefully.

(1) (a) Let G be a group and $x \in G$ an element. Define the centralizer $Z(x)$ and the conjugacy class $C(x)$, and state a result relating $|Z(x)|$ and $|C(x)|$.

(b) Let $C \subset S_n$ be a conjugacy class.

i. Show that if $C \cap A_n \neq \emptyset$ then $C \subset A_n$.

ii. If $C \subset A_n$ show that either C is a conjugacy class of A_n or a disjoint union of two conjugacy classes of A_n. Show by examples that both cases occur.

(2) (a) Show that a group G of order 45 is abelian.

(b) Show that a group G of order 36 is not simple.

(3) Let p be a prime. Let G be a finite group of order p^α for some $\alpha \in \mathbb{N}$. Let $\theta \colon G \to \mathrm{GL}_n(\mathbb{F}_p)$ be a group homomorphism. Prove that there exists $A \in \mathrm{GL}_n(\mathbb{F}_p)$ such that for every $g \in G$ the matrix $A\theta(g)A^{-1}$ is upper triangular with 1's on the diagonal.

[Hint: Identify a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ and apply the Sylow theorems.]

(4) (a) Let R be a ring and $I \subset R$ an ideal. Write down a bijective correspondence between ideals of the quotient ring R/I and ideals of R containing I.

(b) Let n be a positive integer. Consider the quotient ring $S = \mathbb{R}[x]/(x^n)$.

i. Determine a basis of S as an $\mathbb{R}$-vector space.

ii. Find all the ideals of S and identify the prime ideals.

iii. Determine the units of S.

(5) Let $M = \mathbb{Z}^3$ and let $N \subset M$ be the subgroup generated by the elements

$$m_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, m_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, m_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

(a) Determine the isomorphism type of the abelian groups N and M/N. (Identify each group with a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/p^\alpha\mathbb{Z}$ for p prime.)

(b) Does there exist a submodule $L \subset M$ such that $M = L \oplus N$? Justify your answer.

(6) Suppose $A \in \mathrm{GL}_n(\mathbb{Q})$ satisfies $A^8 = 9I$. Show that n is divisible by 4 and give an explicit example of such a matrix A for $n = 4$.

(7) Let $b\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be the skew-symmetric bilinear form given by

$$b(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}$$

where

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & -2 & 0 \end{pmatrix}$$

(a) State the structure theorem for skew-symmetric bilinear forms.

(b) Find a basis $\mathcal{B}$ of $\mathbb{R}^4$ such the matrix of b with respect to this basis is of the standard form described in the structure theorem.

(8) Let R be a ring.

(a) What does it mean to say that an R-module M is finitely generated?

(b) Let L, M, N be R-modules and let $\alpha\colon L \to M$ and $\beta\colon M \to N$ be R-module homomorphisms. What does it mean to say that the sequence

$$0 \to L \stackrel{\alpha}{\to} M \stackrel{\beta}{\to} N \to 0$$

is exact?

(c) Let

$$0 \to L \stackrel{\alpha}{\to} M \stackrel{\beta}{\to} N \to 0$$

be an exact sequence of R-modules. Show that if L and N are finitely generated, then M is finitely generated.

(9) (a) State the structure theorem for finitely generated abelian groups (or, equivalently, $\mathbb{Z}$-modules).

(b) Show that if M is a finitely generated abelian group and $M \neq \{0\}$ then $M \otimes_{\mathbb{Z}} M \neq \{0\}$.

(c) Now consider the abelian group $M = \mathbb{Q}/\mathbb{Z}$. Show that $M \otimes_{\mathbb{Z}} M = \{0\}$. (In particular, it follows that M is *not* finitely generated.)