Math 612 Homework 4

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Reading: Dummit and Foote, 15.4 (rings of fractions and localization), 15.3 (integral dependence), 15.1–15.2 (basic algebraic geometry). The following topics were not covered and may be skipped: Grobner basis techniques (15.1), primary decomposition (15.2).

A useful reference for commutative algebra is the book by Atiyah and MacDonald. Rings of fractions and integral dependence are covered in Chapters 3 and 5.

Justify your answers carefully.

(1) Recall that given a ring A and a multiplicative set S, the *ring of fractions* $S^{-1}A$ is defined by

$$S^{-1}A = \left\{ \frac{a}{s} \mid a \in A, s \in S \right\} / \sim,$$

where $\sim$ is the equivalence relation

$$\frac{a}{s} \sim \frac{b}{t} \iff \exists u \in S \text{ such that } u(at - bs) = 0$$

and addition and multiplication of fractions are defined as usual.

(a) Verify that $\sim$ is an equivalence relation.

(b) Show by example that the relation R defined by $\frac{a}{s} R \frac{b}{t} \iff at = bs$ does not define an equivalence relation in general.

[Hint: (b) Consider a set S containing zero-divisors. See e.g. Q2 below]

(2) Let $n, m \in \mathbb{N}$. Let $A = \mathbb{Z}/n\mathbb{Z}$ and $S \subset A$ the multiplicative set generated by $\bar{m} \in A$. (Here $\bar{m}$ denotes the image of m under the quotient homomorphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by reduction modulo n.) Write $n = ab$, where a is a product of primes dividing m, and $\gcd(b, m) = 1$. Show that $S^{-1}A$ is identified with $\mathbb{Z}/b\mathbb{Z}$.

(3) Let $A \subset \mathbb{Q}$ be a subring. Show that $A = S^{-1}\mathbb{Z}$, where $S \subset \mathbb{Z}$ is the multiplicative subset generated by a set of primes $T \subset \mathbb{Z}$. (Note that T is not necessarily finite.)

(4) Let $R = \mathbb{Z}[\sqrt{-5}]$ and $\mathfrak{p} = (2, 1 + \sqrt{-5}) \subset R$.

(a) Show that $\mathfrak{p}$ is a prime ideal.

(b) Show that $\mathfrak{p}$ is *not* principal.

(c) Let $R_{\mathfrak{p}}$ be the localization of R at $\mathfrak{p}$. (That is, $R_{\mathfrak{p}} = S^{-1}R$ where $S = R \setminus \mathfrak{p}$.) Then R is a local ring with maximal ideal $\mathfrak{p}_{\mathfrak{p}} = S^{-1}\mathfrak{p}$. Show that the ideal $\mathfrak{p}_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is principal.

[Hint: (a) Compute the quotient $R/\mathfrak{p}$ in two stages: first compute the quotient $R/(1 + \sqrt{-5})$, then compute the quotient of this ring by the ideal generated by the image of 2. (b) e.g. consider the norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$ as in 611HW5Q2.]

(5) (a) State Nakayama's Lemma.

(b) Let k be a field, $A = k[x, y]$, and $\mathfrak{m} = (x, y) \subset k[x, y]$, a maximal ideal of A. Consider the localization $A_{\mathfrak{m}}$ of A at $\mathfrak{m}$. Show that the homomorphism

$$\theta \colon A_{\mathfrak{m}}^3 \to A_{\mathfrak{m}}^2$$

of $A_{\mathfrak{m}}$-modules defined by the matrix

$$\begin{pmatrix} xy + 1 & x^3 & x^2 + 2 \\ x^2 & y & y^3 + 1 \end{pmatrix}$$

is surjective.

(6) Let R be a UFD, K its field of fractions, and $L \supset K$ a finite extension of K. Show carefully that an element $\alpha \in L$ is integral over R iff the minimal polynomial of α over K has coefficients in R.

[Hint: Use the Gauss Lemma, DF p. 303.]

(7) Let $n \in \mathbb{Z}$ be square-free, that is, n is not divisible by the square of any prime. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$ is $\mathbb{Z}[(1+\sqrt{n})/2]$ if $n \equiv 1 \bmod 4$ and $\mathbb{Z}[\sqrt{n}]$ otherwise.

(8) Let R be an integral domain. Recall that we say R is *integrally closed* if it is integrally closed in its fraction field K. That is, if $\alpha \in K$ satisfies a monic polynomial equation with coefficients in R then $\alpha \in R$.

(a) Show that a UFD is integrally closed.

(b) Give an example of an integral domain R such that R is integrally closed but R is not a UFD.

(9) For each of the following integral domains, determine its integral closure (in its fraction field).

(a) $k[x,y]/(y^2 - x^5)$

(b) $k[x,y]/(y^2 - x^2(x+1))$.

[Hint: In both cases, the given ring R can be identified with a subring S of $k[t]$ by an appropriate substitution $x = x(t)$, $y = y(t)$. Compare 611HW4Q9. Show that the integral closure of R is identified with $k[t]$.]

(10) Let k be an algebraically closed field. Let $\mathfrak{p} \subset k[x,y]$ be a prime ideal. Show that either $\mathfrak{p} = 0$, $\mathfrak{p} = (f)$ for some irreducible $f \in k[x,y]$, or $\mathfrak{p} = (x-a, y-b)$ for some $a, b \in k$.

[Hint: If $\mathfrak{p}$ is not principal then there exist $f, g \in \mathfrak{p}$ with no common factor in $k[x,y]$. Use Gauss' Lemma to show that f, g have no common factors in $k(x)[y]$. Conclude that there exists a nonzero $h \in k[x] \cap (f,g)$.]

(11) Let $B = \mathbb{Z}[\sqrt{-2}]$.

(a) For each of the prime ideals $\mathfrak{p} = (0), (2), (3), (5) \subset \mathbb{Z}$, find all prime ideals $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap \mathbb{Z} = \mathfrak{p}$, and describe the ring extension $\mathbb{Z}/\mathfrak{p} \subset B/\mathfrak{q}$.

(b) (Optional). For $p \in \mathbb{N}$ prime, $p \neq 2$, the equation $x^2 + 2 = 0$ has a solution mod p iff $p \equiv 1$ or $3 \bmod 8$. Use this fact to extend part (a) to all prime ideals $\mathfrak{p} \subset \mathbb{Z}$.

[Hint: Recall (from 611) that B is a Euclidean domain (ED), and we have the general result ED $\Rightarrow$ PID $\Rightarrow$ UFD. So every ideal in B is principal, and the prime ideals are given by (0) and (α) for α an irreducible element.]

(12) Let k be a field and $J \subset k[x,y]$ an ideal. Compute the radical $\sqrt{J}$ of J in each of the following cases:

(a) (x^2+y^3, x^5).

(b) (x^2y^3, y^4).

(c) (x^3y^4, x^5y^2).

[Hint: In each case, identify powers f^n in the given ideal (for some simple elements $f \in k[x,y]$), replace J by the ideal $J+(f)$, and repeat. Then prove that the ideal K you have obtained is radical (that is, $K = \sqrt{K}$), so $\sqrt{J} = K$.]

(13) Let k be an algebraically closed field and write $S = k[x_1, \ldots, x_n]$ and $\mathbb{A}^n_k = k^n$. This question studies the correspondences $J \mapsto Z(J)$ and $X \mapsto I(X)$ between ideals $J \subset S$ and subsets $X \subset \mathbb{A}^n$.

(a) For subsets $X, Y \subset \mathbb{A}^n_k$, observe that $I(X \cup Y) = I(X) \cap I(Y)$. Suppose now that X and Y are algebraic subsets, i.e., $X = Z(J)$ and $Y = Z(K)$ for some ideals $J, K \subset S$. Show that $I(X \cap Y) = \sqrt{I(X) + I(Y)}$. Show by example that $I(X \cap Y) \neq I(X) + I(Y)$ in general.

(b) Let $J = (xy, yz, zx) \subset S = k[x,y,z]$. Compute $Z(J)$ and use your answer to write J as an intersection of prime ideals.

[Hint: (a) The Nullstellensatz shows that Z and I define a bijective correspondence between radical ideals and algebraic subsets. (b) Express $\sqrt{J}$ as an intersection of primes, and finally check $J = \sqrt{J}$.]