Math 412 Homework 4

Paul Hacking

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Reading: Saracino, Chapter 19.
Show your work and justify your answers carefully.

(1) Identify the quotient $\mathbb{R}[x]/(x^3+x)$ with a standard ring.

(2) Let $R = \mathbb{Z}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$. Identify the fraction field of R with a subring of $\mathbb{R}$.

(3) Let R be a ring with 1 such that R is a finite set of order p, a prime. Identify R with a standard ring. [Hint: Consider the ring homomorphism $\varphi\colon \mathbb{Z} \to R$ determined by $\varphi(1) = 1$ and use Lagrange's theorem.]

(4) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ be a polynomial in the variable x with integer coefficients $a_n, \ldots, a_1, a_0$. Suppose $\alpha \in \mathbb{Q}$ is a root of f, i.e., $f(\alpha) = 0$. Write $\alpha = a/b$ with $a, b \in \mathbb{Z}$, $b > 0$, and $\gcd(a, b) = 1$. Show that a divides a_0 and b divides a_n. [In particular, if $a_n = 1$ then $\alpha \in \mathbb{Z}$.]

(5) Describe the quotient ring $R = (\mathbb{Z}/p\mathbb{Z})[x]/(x^2+1)$ in the following cases:

(a) $p = 5$.

(b) $p = 3$.

[Hint: In case (b) show that R is a field with 9 elements. In case (a) it is possible to identify R with a standard ring.]

(6) Find an ideal in $\mathbb{R}[x,y]$ which is not principal.

[Here

$$\mathbb{R}[x,y] = \left\{ f(x,y) \;\middle|\; f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}x^i y^j,\ n,m \in \mathbb{Z},\ n,m \geq 0,\ a_{ij} \in \mathbb{R} \right\}$$

denotes the ring of polynomials in two variables x and y with real coefficients.]

(7) Consider the ring homomorphism

$$\varphi\colon \mathbb{R}[x,y] \to \mathbb{R}[t], \quad \varphi(f(x,y)) = f(t^2,t^5).$$

(a) Describe the image $\varphi(\mathbb{R}[x,y]) \subset \mathbb{R}[t]$ of φ (a subring of $\mathbb{R}[t]$) explicitly.

(b) Compute the kernel of φ.

(c) Identify the subring of part (a) with a quotient of $\mathbb{R}[x,y]$.

(8) (Optional) Let R be a ring with 1 such that R is a finite set of order 4. Give a complete list of the possibilities for R up to isomorphism, and show how to distinguish them using the characteristic of R, zero divisors, and nilpotent elements.

[Hint: If R has characteristic 2 then we can write

$$R = \{a + bx \mid a,b \in \mathbb{Z}/2\mathbb{Z}\}$$

for some $x \in R$, and the multiplication on R is determined by a rule $x^2 = c + dx$ for some $c,d \in \mathbb{Z}/2\mathbb{Z}$ (why?). (In particular, R is commutative.)]

(9) (Optional) The ring of formal power series $\mathbb{R}[[x]]$ with real coefficients is the set

$$\mathbb{R}[[x]] := \left\{ f(x) \;\middle|\; f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i, \quad a_i \in \mathbb{R} \right\}$$

with the obvious addition, and multiplication given by

$$(a_0 + a_1x + a_2x^2 + \cdots) \cdot (b_0 + b_1x + b_2x^2 + \cdots) =$$
$$a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots ,$$

that is,

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} a_i b_j\right) x^k.$$

[So the definition is the same as the polynomial ring $\mathbb{R}[x]$ except that we allow infinitely many nonzero terms $a_i x^i$. The adjective "formal" means that we do *not* require that the power series converge for sufficiently small values of x.]

(a) Show that $\mathbb{R}[[x]]$ is an integral domain.

(b) Let $f \in \mathbb{R}[[x]]$ be a power series with nonzero constant term a_0. Show that f is a unit.

[Hint: Because nonzero constants are clearly units in $\mathbb{R}[[x]]$ we can assume for simplicity that $a_0 = 1$. So we have $f = 1 + x \cdot g$ for some $g \in \mathbb{R}[[x]]$. Show that the sum

$$1 - xg + x^2g - x^3g + \cdots = \sum_{i=0}^{\infty} (-1)^i x^i g^i$$

is a well defined element of $\mathbb{R}[[x]]$ and is the multiplicative inverse of f.]

(c) Show that the fraction field of $\mathbb{R}[[x]]$ can be identified with the ring of formal Laurent series

$$\mathbb{R}((x)) := \left\{ f(x) \;\middle|\; f(x) = \sum_{i=c}^{\infty} a_i x^i, \quad c \in \mathbb{Z}, \quad a_i \in \mathbb{R} \right\}$$

[Note that the integer c is allowed to be negative.]

(10) (Optional) Let R be a commutative ring with 1 and $a \in R$ an element. Consider the ring homomorphism

$$\varphi \colon R \to R[x]/(ax - 1), \quad \varphi(b) = b + (ax - 1).$$

[Equivalently, φ is the composite $q \circ i$ of the injective homomorphism $i\colon R \hookrightarrow R[x]$ (given by regarding an element of R as a constant polynomial) and the quotient map $q\colon R[x] \to R[x]/(ax-1)$.]

Show that

$$\ker(\varphi) = \{b \in R \mid a^n b = 0 \text{ for some } n \in \mathbb{N}\}.$$

[Remark: Informally, the ring $R[x]/(ax-1)$ is obtained from the ring R by introducing a multiplicative inverse x of a. For this to make sense elements $b \in R$ such that $a^n b = 0$ must be identified with 0 in the new ring.]