Math 611 Final Exam, Friday 12/11/15 – Friday 12/18/15.

Instructions: This is a take home exam. Please note that collaboration with other students is not allowed. Please turn in your exam either to my office LGRT 1235H or my mailbox in LGRT 1623D by 5PM on Friday 12/18/15.

Justify all your answers carefully. If you use a result proved in the textbook, class notes, or homework solutions, state the result precisely.

All rings are assumed commutative with 1.

(1) Let G be a finite group. Suppose that there is a conjugacy class $C \subset G$ of size $|C| = 2$. Prove that G is not a simple group.

(2) Let G be a non-abelian group of order 117 which does *not* contain an element of order 9.

  (a) Describe G explicitly (i) as a semidirect product and (ii) in terms of generators and relations. (You should in particular show that G is uniquely determined up to isomorphism by the above properties.)

  (b) Determine the center $Z(G)$ of G.

  (c) Determine the number of elements of G of order (i) 13 and (ii) 3.

(3) Let R be a ring and $I \subset R$ an ideal. Describe a bijective correspondence between ideals of the quotient ring R/I and ideals of R containing I. (This may be stated without proof.)

  Let n be a positive integer. Consider the quotient ring $S = \mathbb{R}[x]/(x^n)$.

  (a) Determine a basis of S as an $\mathbb{R}$-vector space.

  (b) Find all the ideals of S and identify the prime ideals.

  (c) Determine the units of S.

(4) Consider the ring homomorphism

$$\varphi\colon \mathbb{C}[x, y] \to \mathbb{C}[t], \quad \varphi(f(x, y)) = f(t^2, t^2 + t).$$

  (a) Determine an element $g \in \mathbb{C}[x, y]$ such that $\ker \varphi = (g)$.

  (b) Identify the quotient ring $\mathbb{C}[x, y]/(g)$ with a standard ring.

(5) Let

$$R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C},$$

a subring of $\mathbb{C}$. [Note: The ring R was studied in HW6 Q6.]

Let $p \in \mathbb{N}$ be a prime number. Show that the following conditions are equivalent.

(a) p is irreducible in R.

(b) The quotient ring $R/(p)$ is a field.

(c) The equation $x^2 + 2 = 0$ has no solutions in $\mathbb{Z}/p\mathbb{Z}$.

(6) Let M denote the abelian group $\mathbb{Z}^3$, and let $N \subset M$ be the subgroup generated by the elements

$$m_1 = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \ m_2 = \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix}, \ m_3 = \begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix}.$$

(a) Determine the isomorphism type of the abelian groups N and M/N. (Identify each group with a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/p^\alpha\mathbb{Z}$ for p a prime and $\alpha \in \mathbb{N}$.)

(b) Does there exist a subgroup $L \subset M$ such that $M = L \oplus N$? Justify your answer.

(7) Let R be a ring. We say an R-module M is *simple* if $M \neq \{0\}$ and the only submodules of M are $\{0\}$ and M.

(a) Show that M is simple iff M is isomorphic to R/I where $I \subset R$ is a maximal ideal.

(b) Using part (a) or otherwise, determine a matrix $A \in \mathrm{GL}_4(\mathbb{Q})$ with the following property: If $W \subset \mathbb{Q}^4$ is a $\mathbb{Q}$-vector subspace such that $A \cdot W \subset W$, then $W = \{0\}$ or $W = \mathbb{Q}^4$.