# Math 797W Algebraic geometry. Homework 4

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(1) Show that a product of proper varieties is proper.

(2) Consider the map

$$F\colon \mathbb{P}^n_{(X_i)} \times \mathbb{P}^m_{(Y_j)} \to \mathbb{P}^{(n+1)(m+1)-1}_{(Z_{ij})}$$

given by

$$((X_i)_{0\le i\le n}, (Y_j)_{0\le j\le m}) \mapsto (X_iY_j)_{0\le i\le n, 0\le j\le m}.$$

(a) Check that $F$ is a well defined map of sets.

(b) Show that $F$ is a closed embedding of algebraic varieties (that is, an isomorphism onto a closed subvariety of the target). $F$ is called the *Segre embedding* of $\mathbb{P}^n \times \mathbb{P}^m$.

(3) Let $X = F(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ be the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ (defined in Q2).

(a) Show that $I(X) = (Z_{00}Z_{11} - Z_{01}Z_{10})$.

(b) Show that the projections $\mathrm{pr}_1\colon \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1$ and $\mathrm{pr}_2\colon \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1$ are given by

$$p_1\colon X \to \mathbb{P}^1, \quad (Z_{00} : Z_{01} : Z_{10} : Z_{11}) \mapsto (Z_{00} : Z_{10}) = (Z_{01} : Z_{11})$$

and

$$p_2\colon X \to \mathbb{P}^1, \quad (Z_{00} : Z_{01} : Z_{10} : Z_{11}) \mapsto (Z_{00} : Z_{01}) = (Z_{10} : Z_{11})$$

in the Segre embedding.

(c) Show that the image of a fiber of $\mathrm{pr}_1$ or $\mathrm{pr}_2$ under $F$ is a line $L \subset X \subset \mathbb{P}^3$, and every line $L$ on $X$ arises in this way.

(4) (Optional) Let $X = F(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^{(n+1)(m+1)-1}$ be the image of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$. Show that the homogeneous ideal $I(X)$ is generated by the $2 \times 2$ minors of the matrix $(Z_{ij})_{0 \le i \le n, 0 \le j \le n}$.

[Hint: Let $J$ be the ideal generated by the minors. First check $X = Z(J)$. Second, show $J$ is prime by identifying it as the kernel of the ring homomorphism

$$k[\{Z_{ij}\}] \to k[\{X_i\}] \otimes k[\{Y_j\}], \quad Z_{ij} \mapsto X_i Y_j.$$

So $I(X) = I(Z(J)) = \sqrt{J} = J$.]

(5) Consider the closed subvariety

$$X = Z(x_i X_j - x_j X_i, 1 \le i < j \le n) \subset \mathbb{A}^n_{x_1,\dots,x_n} \times \mathbb{P}^{n-1}_{(X_1:\cdots:X_n)}.$$

Let $\pi\colon X \to \mathbb{A}^n$ and $p\colon X \to \mathbb{P}^{n-1}$ be the restrictions of the projections of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ onto the two factors. ($\pi$ is called the *blowup* of $0 \in \mathbb{A}^n$.)

(a) Show that $X$ is covered by $n$ open affines

$$U_i = (X_i \neq 0) \simeq \mathbb{A}^n_{y_{i1},\dots,y_{i,i-1},x_i,y_{i,i+1},\dots,y_{in}}$$

and the morphism $\pi$ is given in the charts $U_i$ by

$$(y_{i1},\dots,y_{i,i-1},x_i,y_{i,i+1},\dots,y_{in}) \mapsto (x_i y_{i1},\dots,x_i y_{i,i-1},x_i,x_i y_{i,i+1},\dots,x_i y_{in}).$$

(b) Show that if we identify $\mathbb{P}^{n-1} = (\mathbb{A}^n \setminus \{0\})/k^\times$ with the set of lines through the origin in $\mathbb{A}^n$ (i.e., one dimensional subspaces $L \subset \mathbb{A}^n = k^n$), then $X$ is identified with the *incidence variety*

$$X = \{(P, L) \mid P \in L\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

(c) Let $E = \pi^{-1}(0)$. Show that $E = \{0\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}$.

(d) Show that $\pi$ restricts to an isomorphism $X \setminus E \xrightarrow{\sim} \mathbb{A}^n \setminus \{0\}$.

(e) Show that $p\colon X \to \mathbb{P}^{n-1}$ is a locally trivial fibration with fiber $\mathbb{A}^1$ (this is the *tautological line bundle* over $\mathbb{P}^{n-1}$).

(6) Construct a morphism $f\colon X \to \mathbb{A}^3$ such that $f$ is surjective, each fiber $f^{-1}(P)$ is irreducible, and

$$\dim f^{-1}(P) = \begin{cases} 2 & \text{if } P = (0,0,0) \\ 1 & \text{if } x(P) = y(P) = 0 \text{ and } z(P) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

[Hint: One way is to use blowups (see Q5). Note that we can define the blowup of the line $Z(x,y) \subset \mathbb{A}^3_{x,y,z}$ as the morphism

$$\pi \times \mathrm{id}\colon Y \times \mathbb{A}^1_z \to \mathbb{A}^2_{x,y} \times \mathbb{A}^1_z = \mathbb{A}^3_{x,y,z}$$

where the morphism $\pi\colon Y \to \mathbb{A}^2$ is the blowup of $(0,0) \in \mathbb{A}^2$.]

(7) Let $X \subset \mathbb{P}^n$ be a closed subset. Let $Y$ be a variety and $f\colon X \to Y$ a morphism. (That is, write $X = X_1 \cup \cdots \cup X_m$ for the decomposition of $X$ into its irreducible components. Then $f\colon X \to Y$ is a map of sets such that $f$ restricts to a morphism $X_i \to Y$ of varieties for each $i = 1, \ldots, m$.) Assume $\overline{f(X)}$ is irreducible and every non-empty fiber of $f$ is irreducible of dimension $r$ for some fixed $r$. Show that $X$ is irreducible.

[Hint: Use properness of the components of $X$ and the theorems on the dimension of fibers of morphisms.]

(8) Let $V$ be a $k$-vector space of dimension $n$ and let $G(r,V)$ denote the Grassmannian of subspaces of $V$ of dimension $r$. (So, picking a basis of $V$ defines an isomorphism $G(r,V) \simeq G(r,n)$ where $G(r,n)$ is the Grassmannian of subspaces of $k^n$ of dimension $r$ studied in class.) If $U$ is a subspace of $V$ of dimension $m \geq r$ then $G(r,U) \subset G(r,V)$ is a closed subvariety. Similarly if $U$ is a subspace of $V$ of dimension $l \leq r$ then $G(r-l, V/U) \subset G(r,V)$ is a closed subvariety (the inclusion being given by $W \mapsto q^{-1}W$ where $q\colon V \to V/U$ is the quotient map). Now suppose $r = 2, n = 4$. Recall that the image of the Plücker embedding of $G(2,4)$ is the quadric hypersurface

$$X = Z(P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23}) \subset \mathbb{P}^5.$$

Use the above constructions to define two families of planes $\Pi \subset X$ (that is, $\Pi \subset \mathbb{P}^5$ is a projective linear subspace of dimension 2 contained in $X$), and describe one member of each family explicitly.

[Remark: Compare your results with Q3(c).]

(9) Let $X = F(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ studied in Q3, a smooth quadric hypersurface in $\mathbb{P}^3$. Let $Y \subset G(2,4)$ be the subset of $G(2,4)$ corresponding to the lines $L$ lying on $X$. (Note that a line $L \subset \mathbb{P}^3$ is by definition the projectivization of a 2-dimensional subspace $V \subset k^4$, so $L$ corresponds to a point of $G(2,4)$.)

We say a curve $C \subset \mathbb{P}^2$ is a *conic* if $I(C)$ is generated by a homogeneous polynomial of degree 2. We say a curve $C \subset \mathbb{P}^N$ is a conic if there is a plane $\Pi \subset \mathbb{P}^N$ such that $C \subset \Pi \simeq \mathbb{P}^2$ is a conic (equivalently, the ideal of $C \subset \mathbb{P}^N$ is generated by $N-2$ homogeneous polynomials of degree 1 and one homogeneous polynomial of degree 2).

(a) Show that a conic is isomorphic to $\mathbb{P}^1$. [Hint: WLOG $C \subset \mathbb{P}^2$. Consider the projection from a point $P \in C$. Compare also HW2Q8, case $n = 2$.]

(b) Show that under the Plücker embedding of $G(2,4)$, the locus $Y \subset G(2,4)$ defined above is identified with a disjoint union of two conics in $\mathbb{P}^5$.