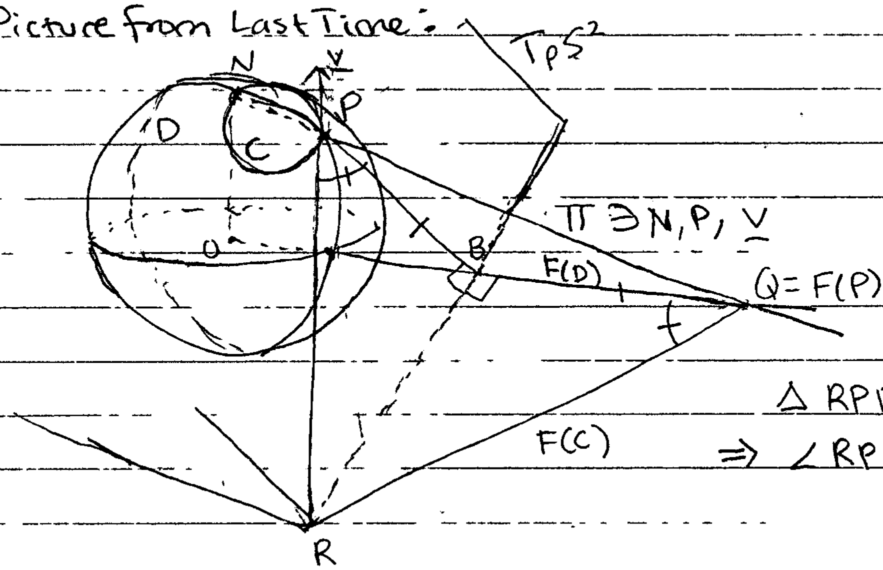
11/22/19

Picture from Last Time:



$\triangle RPB \cong \triangle RQB$ by SAS

$\Rightarrow \angle RPB = \angle RQB$

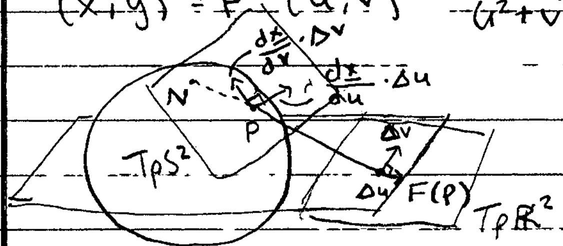
Last Time • Stereographic projection preserves angles (geometric proof)

Today 1. Algebraic/Calculus proof

2. S.P. sends spherical circles to circles & lines in $\mathbb{R}^2$

$F^{-1} : \mathbb{R}^2 \xrightarrow{\sim} S^2 \setminus \{N\} \subset \mathbb{R}^3$

$(x,y) = F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$



We'll compute $\frac{d\underline{x}}{du}$ & $\frac{d\underline{x}}{dv}$, and use this to show F^{-1} preserves angles.

$$\underline{x} = \underline{x}(u,v) = \frac{1}{u^2+v^2+1}\begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1 \end{pmatrix}$$

$$\frac{d\underline{x}}{du} = \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} 2\cdot(u^2+v^2+1) - 2u\cdot 2u \\ 0 - 2v \cdot 2u \\ 2u(u^2+v^2+1) - (u^2+v^2-1)\cdot 2u \end{pmatrix}$$

$\boxed{\text{quotient rule: } \frac{d}{dx}\left(\frac{a}{b}\right) = \frac{\frac{da}{dx}\cdot b - a\frac{db}{dx}}{b^2}}$

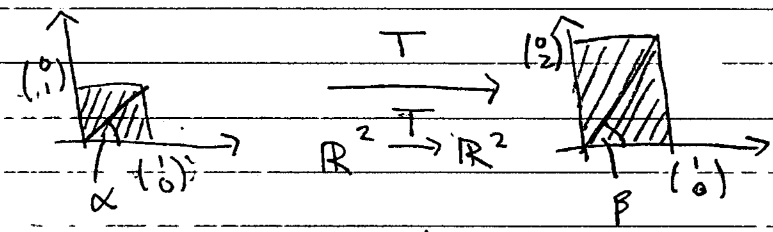
$$= \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} 2\cdot(-u^2+v^2+1) \\ -4uv \\ 4u \end{pmatrix} = \frac{2}{(u^2+v^2+1)^2}\begin{pmatrix} -u^2+v^2+1 \\ -2uv \\ 2u \end{pmatrix}$$

Similarly $\dfrac{dx}{dv} = \dfrac{2}{(u^2+v^2+1)^2}\begin{pmatrix} -2uv \\ u^2-v^2+1 \\ 2v \end{pmatrix}$.

First, check $\dfrac{dx}{du} \perp \dfrac{dx}{dv}$. Equivalently, $\dfrac{dx}{du}\cdot\dfrac{dx}{dv} = 0$.

$$\frac{dx}{du}\cdot\frac{dx}{dv} = \left(\frac{2}{(u^2+v^2+1)^2}\right)^2 \begin{pmatrix} (-u^2+v^2+1)\cdot(-2uv) \\ +(-2uv)\cdot(u^2-v^2+1) \\ +4uv \end{pmatrix} = 0 \checkmark$$

Ex:



$\alpha \neq \beta$!

Still need to show $\left\|\dfrac{dx}{du}\right\| = \left\|\dfrac{dx}{dv}\right\|$

Then, if we take basis of T_pS^2 given by the unit vectors in the directions of $\dfrac{dx}{du}$, $\dfrac{dx}{dv}$, then the linear map $T_Q\mathbb{R}^2 \longrightarrow T_pS^2$ for this basis is given by

$\mathbb{R}^2 \to \mathbb{R}^2$;

$\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix} \longmapsto \lambda\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}$, so preserves angles.

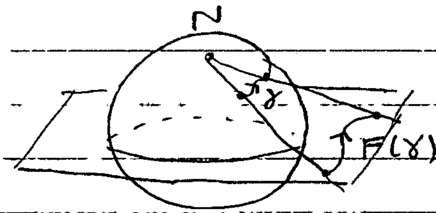
$$\left\|\frac{dx}{du}\right\| = \frac{2}{(u^2+v^2+1)^2}\cdot\left\|\begin{pmatrix} -u^2+v^2+1 \\ -2uv \\ -2u \end{pmatrix}\right\|$$

Recall: $(A-B)^2+4AB = (A+B)^2$

$$= \frac{2}{(u^2+v^2+1)^2}\sqrt{(\underbrace{-u^2}_{B}+\underbrace{v^2+1}_{A})^2 + \underbrace{4u^2v^2+4u^2}_{4AB = 4(v^2+1)\cdot u^2}}$$

$$= \frac{2}{(u^2+v^2+1)^2}\sqrt{(u^2+v^2+1)^2} = \frac{2}{u^2+v^2+1}$$

Similarly, $\left\|\dfrac{dx}{dv}\right\| = \dfrac{2}{u^2+v^2+1}$. $\blacksquare$



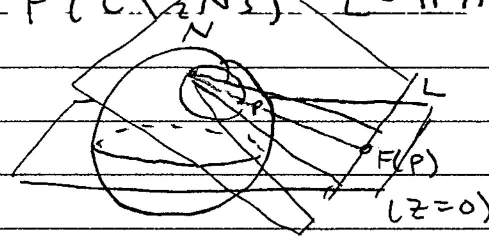
Q: What is the notion of distance / length of paths on $\mathbb{R}^2$ corresponding to the spherical distance under stereographic projection?

Corollary If $\gamma : [a,b] \longrightarrow S^2 \subset \mathbb{R}^3$
$\gamma(t) = (x(t), y(t), z(t))$
is a parametrized curve on the sphere, then we can compute the length of γ in terms of its image $F(\gamma)$ under S.P. by length$(\gamma) \underset{\text{def.}}{=} \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$

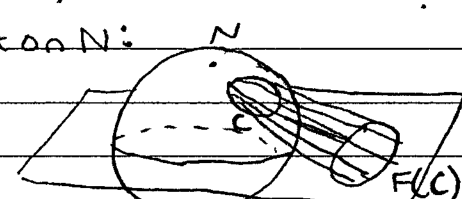
$= \int_a^b \frac{2}{u^2+v^2+1} \cdot \sqrt{u'(t)^2 + v'(t)^2}\, dt$.

Stereographic projection sends spherical circles to circles & lines in $\mathbb{R}^2$?

Already seen:
If $C = \Pi \cap S^2$ is a spherical circle, & $N \in C$ then $F(C \setminus \{N\}) = L = \Pi \cap (z=0)$ is a line in the xy-plane.



Not on N:



Claim: $F(C)$ is a circle.

Algebraic Proof: Use formula,
$F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$

$\longrightarrow$

Let $C = \Pi \cap S^2$, $\Pi = (ax+by+cz = d) \subset \mathbb{R}^3$ plane.

$\hookrightarrow a,b,c,d \in \mathbb{R}$ constants

$$F(C) = \{F(P) \mid P \in C\}$$
$$= \{Q \in \mathbb{R}^2 \mid F^{-1}(Q) \in C\}$$
$$= \{Q = (u,v) \in \mathbb{R}^2 \mid \underbrace{F^{-1}(Q) = \tfrac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1) \in \Pi}\}$$

$$\frac{1}{u^2+v^2+1}\left(a\cdot 2u + b\cdot 2v + c(u^2+v^2-1)\right) = d$$

$\hookrightarrow$ rearrange this to get equation of a circle

Rearrange: $a\cdot 2u + b\cdot 2v + c(u^2+v^2-1) = d(u^2+v^2+1)$

$$(c-d)(u^2+v^2) + a\cdot 2u + b\cdot 2v = c+d$$

$$(u^2+v^2) + \frac{a}{c-d}2u + \frac{b}{c-d}\cdot 2v = \frac{c+d}{c-d}$$

"Complete the square"

$$\left(u+\frac{a}{c-d}\right)^2 + \left(v+\frac{b}{c-d}\right)^2 = \frac{c+d}{c-d} + \left(\frac{a}{c-d}\right)^2 + \left(\frac{b}{c-d}\right)^2$$
$$= \frac{a^2+b^2+c^2-d^2}{(c-d)^2}$$

Circle, center $\left(\frac{-a}{c-d}, \frac{-b}{c-d}\right)$, radius $\frac{\sqrt{a^2+b^2+c^2-d}}{c-d}$. ▨