Math 621 Homework 4

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Reading: Stein and Shakarchi, Chapter 8.

(1) Let f be a holomorphic function on $D(0,R) = \{z \in \mathbb{C} \mid |z| < R\}$. Suppose that $|f(z)| < M$ for all $z \in D(0,R)$ and $f(0) = 0$.

   (a) Prove that $|f(z)| \leq \frac{M}{R}|z|$ for all $z \in D(0,R)$, with equality holding for some $z_0 \neq 0$ iff $f(z) = e^{i\theta}\frac{M}{R}z$ for some $\theta \in \mathbb{R}$.

   (b) Suppose f has a zero of order k at 0. Prove that $|f(z)| \leq \frac{M}{R^k}|z|^k$ for all $z \in D(0,R)$, with equality holding for some $z_0 \neq 0$ iff $f(z) = e^{i\theta}\frac{M}{R^k}z^k$ for some $\theta \in \mathbb{R}$.

(2) Let $f(z)$ be a holomorphic function on the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose that $\mathrm{Im}(f(z)) > 0$ for all $z \in D$, and that $f(0) = i$. Show that for all $z \in D$ we have

$$|f(z)| \leq \frac{1+|z|}{1-|z|}.$$

(3) Let $f \colon D \to D$ be a holomorphic map from the unit disc to itself. Prove that for all $a \in D$ we have

$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}.$$

[Hint: Let g be an automorphism of D such that $g(0) = a$ and let h be an automorphism of D such that $h(f(a)) = 0$. Let $F = h \circ f \circ g$. Compute $F'(0)$ and apply the Schwarz Lemma. This result is called the Schwarz-Pick Lemma and is used to prove that holomorphic maps from the disc to itself are distance decreasing for the Poincaré hyperbolic metric on the disc, see Problem 3 on p. 256 of the text.]

(4) Recall that a *fractional linear transformation* is an automorphism of the extended complex plane $\mathbb{C} \cup \{\infty\}$, given by

$$f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. In each of the following cases, either find a fractional linear transformation with the desired properties or explain why no such transformation exists.

(a) Maps the circle $\{z \in \mathbb{C} \mid |z| = 1\}$ to itself and maps 0 to 1/2.

(b) Maps the circles $\{z \in \mathbb{C} \mid |z| = 1\}$ and $\{z \in \mathbb{C} \mid |z| = 2\}$ to parallel lines.

(c) Maps the region between the circles $\{z \in \mathbb{C} \mid |z+1| = 1\}$ and $\{z \in \mathbb{C} \mid |z| = 2\}$ to the region between the horizontal lines $\{z \in \mathbb{C} \mid \operatorname{Im} z = 0\} \cup \{\infty\}$ and $\{z \in \mathbb{C} \mid \operatorname{Im} z = 1\} \cup \{\infty\}$.

(5) (From the qualifying exam, Fall 2011). Let U be the portion of the open unit disc given in polar coordinates by

$$U \ := \ \{re^{i\theta} \mid 0 < r < 1 \text{ and } 0 < \theta < \pi/3\}.$$

The boundary of U consists of the line segment L_0 from 0 to 1, the line segment $L_{\pi/3}$ from 0 to $e^{\pi i/3}$, and a curve Γ on the unit circle. Prove that there exists a unique fractional linear transformation f satisfying $f(1) = i$, $f(e^{\pi i/3}) = 0$, f maps Γ into the imaginary line $\mathbb{R}i$, and f maps $L_{\pi/3}$ into the real axis. Give an explicit, simple formula for $f(z)$. Justify your answer. [Hint: Find $f^{-1}(\infty)$ first.]

(6) (a) Let $z_1, \ldots, z_4$ be 4 distinct points in the extended complex plane $\mathbb{C} \cup \{\infty\}$. Give a careful definition of the cross ratio

$$\operatorname{CR}(z_1, z_2, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4} \bigg/ \frac{z_2 - z_3}{z_2 - z_4}$$

in the case where one of the z_i equals ∞.

(b) Show that if $\operatorname{CR}(z_1, z_2, z_3, z_4) = \xi$ then the $4! = 24$ permutations of $z_1, \ldots, z_4$ yield the following 6 values of the cross ratio: $\xi, 1/\xi, 1-\xi, 1/(1-\xi), \xi/(\xi-1), (\xi-1)/\xi$. [Hint: We may assume $z_1, z_2, z_3, z_4 = \xi, 1, 0, \infty$ (why?)]

(7) From the text book: Chapter 8, Section 5, Exercises 4,5,12.