Math 412 Homework 2

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Reading: Saracino, Chapter 17.
Show your work and justify your answers carefully.

(1) Let R be a ring with 1. Show that the set $U(R)$ of units forms a group with operation given by multiplication. Identify the group $(U(R), \cdot)$ with a standard group in the following cases:

(a) $R = \mathbb{Z}$.

(b) $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$.

(c) $R = \mathbb{Z}/5\mathbb{Z}$.

(d) $R = \mathbb{Z}/8\mathbb{Z}$.

(e) (Optional, harder). $R = \mathbb{Z}/p\mathbb{Z}$, where p is a prime.

[Remark: Note that if R is commutative then $U(R)$ is an abelian group. But for example if R is the non-commutative ring of Q7(e) then $U(R)$ is a non-abelian group of order 8.]

(2) In each of the following cases, determine whether the subset S of the given ring R is a subring.

(a)

$$S = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \subset R = \mathbb{R}^{2\times 2}.$$

(b)

$$S = \left\{ a + b\frac{(1+\sqrt{5})}{2} \;\middle|\; a, b \in \mathbb{Z} \right\} \subset R = \mathbb{R}.$$

(c)

$$S = \{f \in \mathbb{R}[x] \mid f'(0) = f'''(0) = 0\} =$$
$$\{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_1 = a_3 = 0\} \subset R = \mathbb{R}[x].$$

(3) In each of the following cases, determine whether the subset I of the ring R is an ideal.

(a) $R = \mathbb{R}[x]$, $I = \{f \in R \mid f(5) = 0\}$.

(b) $R = \mathbb{R}[x]$, $I = \{f \in R \mid f'(2) = 0\}$.

(c) $R = \mathbb{R}^{2\times 2}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$, and $I = \{X \cdot A \mid X \in R\}$.

(d) $R = \mathbb{Z}[x]$, the ring of polynomials in the variable x with integer coefficients, and $I = \{f \in \mathbb{R} \mid f(2) \equiv 0 \bmod 3\}$.

(4) Let R be a commutative ring with 1. Show that an element $a \in R$ is a unit iff the principal ideal $(a) = \{xa \mid x \in R\}$ equals R.

(5) Let R be a commutative ring with 1, $R \neq \{0\}$. Show that R is a field iff the only ideals of R are $\{0\}$ and R.

(6) Let R be a ring and $I, J \subset R$ ideals.

(a) Show that $I \cap J$ and $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$ are ideals of R.

(b) Is $I \cup J$ always an ideal? Give a proof or counterexample.

(7) Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the ring of quaternions, with multiplication given by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i \quad ki = -ik = j,$$

so that

$$(a + bi + cj + dk) \cdot (a' + b'i + c'j + d'k) =$$
$$(aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i + (ac' + ca' + db' - bd')j + (ad' + da' + bc' - cb')k.$$

Equivalently, we can write a quaternion q as a formal sum

$$q = a + (bi + cj + dk) = t + \mathbf{v}$$

where $t = a \in \mathbb{R}$ and $\mathbf{v} = bi + cj + dk = \begin{pmatrix} b \\ c \\ d \end{pmatrix} \in \mathbb{R}^3$. Then

$$q_1 \cdot q_2 = (t_1 + \mathbf{v}_1) \cdot (t_2 + \mathbf{v}_2) = (t_1 t_2 - \mathbf{v}_1 \cdot \mathbf{v}_2) + t_1 \mathbf{v}_2 + t_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2$$

where $\mathbf{v}_1 \cdot \mathbf{v}_2$ is the dot product (or scalar product) and $\mathbf{v}_1 \times \mathbf{v}_2$ is the cross product (or vector product).

(a) For a quaternion $q = a + bi + cj + dk = t + \mathbf{v}$ define the conjugate $\bar{q}$ of q by

$$\bar{q} = a - bi - cj - dk = t - \mathbf{v}.$$

Show that

$$q\bar{q} = \bar{q}q = \|q\|^2$$

where $\|q\| := \sqrt{a^2 + b^2 + c^2 + d^2}$ is the length of q.

(b) Use (a) to give a formula for the multiplicative inverse of a nonzero quaternion q. (In particular, this shows that the quaternions are a *skew field*.)

(c) Show that

$$\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1.$$

(d) Show that $\|q_1 q_2\| = \|q_1\| \|q_2\|$. [Hint: Use (a) and (c).]

(e) Let

$$R = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}\} \subset \mathbb{H}.$$

Then R is a subring of $\mathbb{H}$. Find all the units in R. [Hint: Use (b) and (d), and generalize the proof for the Gaussian integers given class.]

(8) (Optional) In this question we give two proofs that multiplication of quaternions is associative. (This is needed to prove that $\mathbb{H}$ is a ring, and was omitted in class.)

(a) Let $\mathbb{C}^{2\times 2}$ be the ring of 2×2 complex matrices. Show that the map

$$\theta \colon \mathbb{H} \to \mathbb{C}^{2\times 2}, \quad \theta(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

is injective and satisfies $\theta(q_1q_2) = \theta(q_1)\theta(q_2)$ for $q_1, q_2 \in \mathbb{H}$. Explain why it follows that multiplication of quaternions is associative.

(b) Give a direct proof for associativity of multiplication of quaternions using the formula

$$q_1 \cdot q_2 = (t_1 + \mathbf{v}_1) \cdot (t_2 + \mathbf{v}_2) = (t_1 t_2 - \mathbf{v}_1 \cdot \mathbf{v}_2) + t_1 \mathbf{v}_2 + t_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2$$

together with the identities

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

and

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$$

where $(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ denotes the 3×3 matrix with columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$.