Math 462 Homework 3

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(1) Let $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(a) Show that A is an orthogonal matrix.

(b) Compute the eigenvalues of A.

(c) Let $T(\mathbf{x}) = A\mathbf{x}$ be the isometry of $\mathbb{R}^3$ defined by A. Describe T geometrically as a reflection, rotation, or rotary reflection, specifying the plane and/or rotation angle and axis.

[Hint: For part (c), a reflection plane or rotation axis is determined by an eigenvector with eigenvalue $\lambda = \pm 1$. A rotation angle is determined by the complex eigenvalues.]

(2) Repeat Q1 for the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$.

(3) Let $r_L \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $r_M \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometries given by reflection in lines L and M in $\mathbb{R}^2$. Suppose L and M meet in a point P such that the angle from L to M is α (measured counterclockwise). Show that the composition $r_M \circ r_L$ is the rotation about P through angle 2α counterclockwise.

[Hint: One way to do this is to choose coordinates so that the point P is the origin and the line L is the x-axis. Now compute using matrices: writing $r_L(\mathbf{x}) = A\mathbf{x}$ and $r_M(\mathbf{x}) = B\mathbf{x}$, we have $r_M \circ r_L(\mathbf{x}) = BA\mathbf{x}$.]

(4) Let A be a 3×3 orthogonal matrix and

$$T \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T(\mathbf{x}) = A\mathbf{x}$$

the corresponding isometry. Show that the determinant $\det A = +1$ if T is the identity or a rotation and $\det A = -1$ if T is a reflection or rotary reflection.

[Hint: If $B = P^{-1}AP$ then $\det B = \det A$ (why?). Now refer to the structure theorem for orthogonal matrices.]

(5) Let L and M be two lines passing through the origin in $\mathbb{R}^3$. Let S be the isometry given by rotation about L through an angle θ and T the isometry given by rotation about M through an angle ϕ. Show that the composite isometry $T \circ S$ is either the identity or a rotation about some line N passing through the origin. When is $T \circ S$ the identity?

[Hint: Use Q4.]

(6) This question explains some of the linear algebra that is needed to prove the structure theorem for orthogonal matrices. For $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ complex vectors, we define the dot product

$$\mathbf{z} \cdot \mathbf{w} = \sum_{i=1}^{n} \bar{z}_i w_i.$$

[Here for $z = x + iy \in \mathbb{C}$ we write $\bar{z} = x - iy$ for the complex conjugate of z.]

(a) Show that $\mathbf{z} \cdot \mathbf{z} = \|\mathbf{z}\|^2$ for all $\mathbf{z} \in \mathbb{C}^n$, where $\|\mathbf{z}\| := \sqrt{\sum_{i=1}^{n} |z_i|^2}$ is the length of $\mathbf{z}$. [This is the reason we use the complex conjugate in the definition of the dot product for complex vectors.]

(b) Show that $(\lambda \mathbf{z}) \cdot \mathbf{w} = \bar{\lambda}(\mathbf{z} \cdot \mathbf{w})$ and $\mathbf{z} \cdot (\lambda \mathbf{w}) = \lambda(\mathbf{z} \cdot \mathbf{w})$ for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.

(c) Now let A be an $n \times n$ orthogonal matrix. Show that $(A\mathbf{z}) \cdot (A\mathbf{w}) = \mathbf{z} \cdot \mathbf{w}$ for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$.

(d) Let $\mathbf{v} \in \mathbb{C}^n$ be an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$. Show that $|\lambda| = 1$. [So $\lambda = e^{i\theta} = \cos\theta + i \sin\theta$ for some θ, and if $\lambda \in \mathbb{R}$ then $\lambda = \pm 1$.]

(e) Let $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ be eigenvectors of A with eigenvalues λ, μ such that $\lambda \neq \mu$. Show that $\mathbf{v} \cdot \mathbf{w} = 0$.

(f) Finally, let $\mathbf{v} \in \mathbb{C}^n$ be an eigenvector with eigenvalue $\lambda \in \mathbb{C}$, and write $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then the conjugate vector $\bar{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$ is an eigenvector with eigenvalue $\bar{\lambda}$ (why?). So, assuming $\lambda \notin \mathbb{R}$, we have $\bar{\mathbf{v}} \cdot \mathbf{v} = 0$ by part (e). Deduce that $\|\mathbf{a}\| = \|\mathbf{b}\|$ and $\mathbf{a} \cdot \mathbf{b} = 0$.

[Remark: The real eigenvectors and the real and imaginary parts $\mathbf{a}$ and $\mathbf{b}$ of the pairs of complex conjugate eigenvectors (scaled so that $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$) can be used to form an orthogonal basis of $\mathbb{R}^n$. If P is the associated change of basis matrix (with columns given by the vectors of the basis) then P is orthogonal and the matrix $B = P^{-1}AP = P^TAP$ has the form described in the structure theorem.]