

Monday 11/11/13 Math 235. Linear algebra. Solutions to Common Midterm Review Questions.

1. We say v is a linear combination of v_1, v_2, \dots, v_m if

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

for some $c_1, c_2, \dots, c_m \in \mathbb{R}$.

$$2. v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

We try to solve $v = c_1 v_1 + c_2 v_2$ for $c_1, c_2 \in \mathbb{R}$.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1+3c_2 \\ c_1+c_2 \end{pmatrix}$$

$$\begin{matrix} c_1 = 1 \\ c_1+3c_2 = 2 \end{matrix}$$

$$\begin{matrix} c_1 + c_2 = 3 \end{matrix}$$

Either solve by hand: $\begin{matrix} c_1 = 1 \\ c_1+3c_2 = 2 \\ c_1 + c_2 = 3 \end{matrix} \Rightarrow c_1 = 1, c_2 = \frac{1}{3}(2-1) = \frac{1}{3}$

then $c_1+c_2 = 1 + \frac{1}{3} = \frac{4}{3} \neq 3$, i.e.,
 3 does not hold.

So there are no solutions

Or use Gaussian elimination:

$$\text{Augmented Matrix } \left(\begin{array}{cc|c} c_1 & c_2 & \text{RHS} \\ 1 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{array} \right) \xrightarrow{-R1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{\div 3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{-R2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{5}{3} \end{array} \right)$$

RREF

no solutions

So, v is NOT a linear combination of v_1 and v_2 .

3. The span of v_1, v_2, \dots, v_n is the set of all linear combinations of v_1, v_2, \dots, v_n :-

$$\text{Span}(v_1, v_2, \dots, v_n) = \left\{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n \mid c_1, \dots, c_n \in \mathbb{R} \right\} \subset \mathbb{R}^n.$$

4. a) $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

$\text{Span}(v_1, v_2) = \mathbb{R}^2$ (Note that v_1 & v_2 are NOT parallel, i.e., v_2 is NOT a multiple of v_1 .)

b) $v_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Note that $v_2 = \frac{3}{2} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{3}{2} v_1$

so $\text{Span}(v_1, v_2) = \text{Span}(v_1) = \text{line through the origin in the direction of } v_1$
 $= \text{line with equation } y=2x$

c) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\text{Span}(v_1, v_2) \subset \mathbb{R}^3$ is the plane Π through the origin

in \mathbb{R}^3 containing the vectors v_1 & v_2 .

(Note that v_1 & v_2 are NOT parallel so the span is a plane).

The equation of the plane Π is $ax+by+cz=0$
where $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a normal vector to the plane,

i.e., a nonzero vector such that $\underline{x} \cdot \underline{\lambda} = 0$ for all $\underline{x} \in \Pi$

To find a normal vector $\underline{\lambda}$, either use the cross product:

$$\underline{\lambda} = \underline{v}_1 \times \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

or solve the equations $\underline{v}_1 \cdot \underline{\lambda} = 0, \underline{v}_2 \cdot \underline{\lambda} = 0$ for $\underline{\lambda} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\therefore \begin{array}{ccc|cc} a & b & c & a+b=0 & b+c=0 \end{array}$

$$\text{GE: } -R2 \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|cc} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right) \text{ RREF.}$$

Coefficient Matrix

(RHS = 0 so omitted from row reduction algorithm)

$$a - c = 0 \rightsquigarrow a = c$$

$$b + c = 0 \rightsquigarrow b = -c$$

c is free

c free.

$$\underline{\lambda} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ -c \\ c \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad c \in \mathbb{R} \text{ arbitrary.}$$

So can take $\underline{\lambda} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

So equation of Π is $\underline{x} \cdot \underline{\lambda} = 0$.

i.e. $x - y + z = 0$.

d) $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

Clearly \underline{v}_1 & \underline{v}_2 are not parallel, so they span a plane $\Pi \subset \mathbb{R}^3$

If \underline{v}_3 lies in this plane (i.e., if \underline{v}_3 is a linear combination of \underline{v}_1 & \underline{v}_2) then $\text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{Span}(\underline{v}_1, \underline{v}_2) = \Pi$.

Otherwise $\text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \mathbb{R}^3$.

We try to solve $\underline{v}_3 = c_1 \underline{v}_1 + c_2 \underline{v}_2$ for $c_1, c_2 \in \mathbb{R}$.

GE:
 Augmented Matrix $\xrightarrow{-R_1} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \xrightarrow{-2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow{-2R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

- no solutions (last row corresponds to equation "0=1")

$\therefore \underline{v}_3 \notin \text{Span}(\underline{v}_1, \underline{v}_2)$, and $\text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \mathbb{R}^3$.

e) $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$

Similarly to (d), we find (either by inspection or Gaussian elimination)

$$\underline{v}_3 = 2 \cdot \underline{v}_1 + 6 \underline{v}_2$$

So $\text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{Span}(\underline{v}_1, \underline{v}_2) = \Pi \subset \mathbb{R}^3$,

the plane through the origin containing $\underline{v}_1, \underline{v}_2$.

Similarly to (c), the equation of the plane Π is $x+y+z=0$

($\underline{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a normal vector.)

A function

S. $| T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if

$$T(\underline{x}) = A \cdot \underline{x} \quad \text{for some } m \times n \text{ matrix } A$$

$(m = \# \text{ rows}, n = \# \text{ columns})$

Equivalently,

$$T(\underline{v} + \underline{w}) = T(\underline{v}) + T(\underline{w}) \quad \text{for all } \underline{v}, \underline{w} \in \mathbb{R}^n$$

$$\text{and } T(c\underline{v}) = cT(\underline{v}) \quad \text{for all } c \in \mathbb{R} \text{ and } \underline{v} \in \mathbb{R}^n.$$

A is called the standard matrix of T .

If we write $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ 5.
 for the i^{th} standard

basis vector of \mathbb{R}^n , $i=1, 2, \dots, n$, then the columns of A

are $T(e_1), T(e_2), \dots, T(e_n)$. †

6. a) NOT linear, because $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(Note: If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, we must have)
 $T(\underline{0}) = \underline{0}$.

b) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Linear, standard matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

c) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by rotation about the origin through

angle $\pi/4$ radians counterclockwise.

T is linear, standard matrix $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$\theta = \pi/4$$

OR, using \dagger ,

$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\xrightarrow{T} T(e_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\xrightarrow{T} T(e_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow A = (T(e_1) : T(e_2)) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

d) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by reflection in yz plane.

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow T$ linear, standard matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(OR compute A via $A = (T(e_1) \ T(e_2) \ T(e_3))$)

$$\begin{array}{ccc} e_1 \uparrow & & T(e_3) = e_3 \\ \downarrow \quad \rightarrow e_2 \rightarrow T \rightarrow T(e_1) = -e_1 & & \uparrow \quad \rightarrow T(e_2) = e_2 \\ e_1 & & \end{array} \quad \text{w/ } A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

e) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ orthogonal projection onto $y=x$.

$$\begin{array}{ccc} \text{Linear,} & \begin{matrix} e_2 \\ \uparrow \\ e_1 \end{matrix} & \longrightarrow \begin{matrix} T(e_1) = T(e_2) = \frac{1}{\sqrt{2}}(1,1) \end{matrix} \\ & & \end{array}$$

$$\therefore \text{standard matrix } A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

OR $T(x) = \left(\frac{x \cdot v}{v \cdot v} \right) v$ where $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is vector in direction of the line.

$$= \left(\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) v$$

$$= \left(\frac{x+y}{2} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

? $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation.

The kernel of T is $\ker(T) = \{x \in \mathbb{R}^n \mid T(x) = 0\} \subset \mathbb{R}^n$.

The image of T is $\text{image}(T) = \{ \underline{y} \in \mathbb{R}^M \mid \underline{y} = T(\underline{x}) \text{ for some } \underline{x} \in \mathbb{R}^n \}$

$$= \{ T(\underline{x}) \mid \underline{x} \in \mathbb{R}^n \} \subset \mathbb{R}^M.$$

If $T(\underline{x}) = A \cdot \underline{x}$, we can express $\ker(T)$ as the span of a set of vectors by solving the equation $A\underline{x} = \underline{0}$ using Gaussian elimination.

Also, the image of T is the span of the columns of A .

8. $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, linear transformation, $\forall \underline{v} \in \mathbb{R}^n$.

Note that, for $\underline{w} \in \mathbb{R}^n$, $T(\underline{w}) = T(\underline{v}) \iff \underline{w} - \underline{v} \in \ker(T)$

$$\left(T(\underline{w}) = T(\underline{v}) \iff T(\underline{w}) - T(\underline{v}) = \underline{0} \stackrel{T \text{ linear}}{\iff} T(\underline{w} - \underline{v}) = \underline{0} \right)$$

$$\iff \underline{w} - \underline{v} \in \ker(T)$$

So $\{ \underline{w} \in \mathbb{R}^n \mid T(\underline{w}) = T(\underline{v}) \} \subset \mathbb{R}^n$

$$\{ \underline{w} \in \mathbb{R}^n \mid \underline{w} = \underline{v} + \underline{x}, \underline{x} \in \ker(T) \}.$$

9. Describe kernel & image as span of a set of vectors.

(a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad T(\underline{x}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \underline{x}$

$\ker(T)$: Solve $T(\underline{x}) = \underline{0}$ by Gaussian elimination (or inspection)

coefficient matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow[-2R_1]{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ RREF

$$x + 2y = 0 \quad \underset{y \text{ free}}{\Rightarrow} \quad x = -2y, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\ker(T) = \text{Span} \left(\begin{pmatrix} -2 \\ 1 \end{pmatrix} \right).$$

$$\text{image}(T) = \text{Span}(\text{columns of } A)$$

$$= \text{Span} \left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \right) = \text{Span} \left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right)$$

because $\left(\frac{2}{4}\right)$ is a multiple of $\left(\frac{1}{2}\right)$.

b) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by orthogonal projection onto plane $\Pi \subset \mathbb{R}^3$
 with equation $x+2y+3z=0$.

$$\ker(\tilde{\pi})^\perp = \text{Span} \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) \quad , \quad \text{where } \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ is normal vector to } \pi.$$

$$\text{image}(T) = \Pi = \text{Span} \left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right)$$

given by solving $x+2y+3z=0$ via G.E.

$$\therefore x = -2y - 3z, \quad y, z \text{ free}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \quad y, z \in \mathbb{R}$$

arbitrary.

$$c) T: \mathbb{R}^3 \rightarrow \mathbb{R}^4, \quad T(\underline{x}) = A\underline{x}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix}$$

$\ker(T)$: Solve $A\underline{x} = \underline{0}$ using GE :-

$$\begin{array}{c}
 -R_1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ 1 & 2 & 3 & \\ 1 & 3 & 5 & \\ \hline 1 & 4 & 7 & \end{array} \right) \xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ 0 & 1 & 2 & \\ 0 & 2 & 4 & \\ \hline 1 & 4 & 7 & \end{array} \right) \xrightarrow{-2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ 0 & 1 & 2 & \\ 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & \end{array} \right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & \\ 0 & 1 & 2 & \\ 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & \end{array} \right) \text{ RREF}
 \end{array}$$

$$x - z = 0$$

$$y + 2z = 0$$

\approx free

$$x = 2$$

$$y = -2$$

\approx free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\ker(T) = \text{span} \left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right)$$

$$\text{Image}(T) = \text{span} \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} \right)$$

$$\downarrow = \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

10. $v_1, v_2, \dots, v_n \in \mathbb{R}^n$.

v_1, v_2, \dots, v_n are linearly independent if the only solution of the equation

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \underline{0}, \quad c_1, c_2, \dots, c_n \in \mathbb{R}$$

is

$$c_1 = c_2 = \dots = c_n = 0.$$

Equivalently, $v_1 \neq \underline{0}$, and for each $i = 2, 3, \dots, n$, v_i is NOT a linear combination of v_1, v_2, \dots, v_{i-1}

If $v_1, v_2, \dots, v_m \in \mathbb{R}^n$ are linearly independent,
we must have $m \leq n$.

11. a) Linearly independent, because v_2 is NOT a multiple of v_1 .

b) NOT linearly independent because $3 > 2$ (3 vectors in \mathbb{R}^2).

c) Linearly independent:-

Try to solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = \underline{0}$:-

$$\begin{pmatrix} c_1 \\ 2c_2 \\ 3c_1 + 5c_2 + 13c_3 \\ 7c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$$

d) Solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = \underline{0}$ using G.E.

$$\text{coefft. } -2R1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 5 & 0 \\ 3 & 5 & 7 & 0 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

NOT linearly independent because there is a free variable (c_3)
so there are solutions besides $c_1 = c_2 = c_3 = 0$.

To find solutions, continue GE to RREF

$$-R_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$c_1 + c_3 = 0$$

$$c_2 + 3c_3 = 0$$

c_3 free

$$c_1 = 2c_3$$

$$c_2 = -3c_3$$

c_3 free

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2c_3 \\ -3c_3 \\ c_3 \end{pmatrix} = c_3 \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

$c_3 \in \mathbb{R}$ is arbitrary.

$$\text{So } (2)v_1 + (-3)v_2 + v_3 = 0, \text{ or } v_3 = -2v_1 + 3v_2.$$

12. $W \subset \mathbb{R}^n$ is a subspace of \mathbb{R}^n if

$$(1) \underline{0} \in W$$

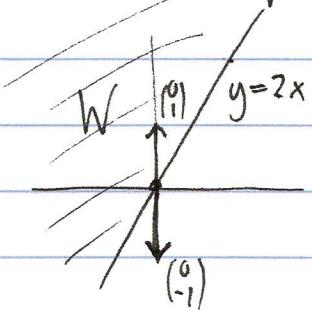
$$(2) \underline{v}, \underline{w} \in W \Rightarrow \underline{v} + \underline{w} \in W$$

$$(3) c \in \mathbb{R}, \underline{w} \in W \Rightarrow c\underline{w} \in W.$$

13. a) NOT a subspace. $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{0} \notin W$.

b) Is a subspace

c) NOT a subspace. e.g. $\begin{pmatrix} 1 \\ 3 \end{pmatrix} \in W$ but $(-1) \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \notin W$.



$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W \text{ but } (-1) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \notin W$$

d) $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, $\ker(T) \subset \mathbb{R}^n$ is a subspace and $\text{image}(T) \subset \mathbb{R}^m$ is a subspace.

e) $v_1, \dots, v_m \in \mathbb{R}^n \Rightarrow \text{Span}(v_1, v_2, \dots, v_m) \subset \mathbb{R}^n$ is a subspace.

$$f) W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{array} \right\} \subset \mathbb{R}^n \text{ is a subspace.}$$

14. $W \subset \mathbb{R}^n$ a subspace.

$v_1, \dots, v_m \in \mathbb{R}^n$ is a basis of W if

$$(1) \text{ Span } (v_1, v_2, \dots, v_m) = W$$

(2) v_1, v_2, \dots, v_m are linearly independent

The dimension of W is the number of elements in a basis of W .

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T(\underline{x}) = A \cdot \underline{x}$, is a linear transformation, we can determine a basis of $\ker(T)$ and $\text{image}(T)$ as follows:-

- Compute RREF(A) using Gaussian elimination.
- Use it to write all solutions of $A\underline{x} = \underline{0}$ as a linear combination of a set of vectors, with coefficients the free variables. This set of vectors is a basis for $\ker(T)$.
- A basis for $\text{image}(T)$ is given by the columns of A corresponding to columns of RREF(A) containing a pivot (or "leading 1").

15. Determine a basis of the subspace W .

a) $W \subset \mathbb{R}^3$ is the plane with equation $x+2y+4z=0$.

Since W is a plane, it is spanned by any two vectors

$v_1, v_2 \in W$ which are not parallel, & then v_1, v_2 is a basis of W (spans & is linearly independent).

e.g. we can take $v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$. $\dim W = 2$

(Alternatively, solve $x+2y+4z=0$ using G.E.)

b) $W = \ker(T)$, $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$, $T(\underline{x}) = A\underline{x}$, $A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

Notice $A = \text{RREF}(A)$.

$$\begin{aligned}
 A\underline{x} = \underline{0} &\iff x_1 + 3x_3 + 5x_5 = 0 \\
 &\quad x_2 + 2x_3 + 7x_5 = 0 \\
 &\quad x_4 + 2x_5 = 0 \\
 &\quad x_3 \text{ and } x_5 \text{ are free} \\
 &\iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_3 - 5x_5 \\ -2x_3 - 7x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} -5 \\ -7 \\ 0 \\ -2 \\ 1 \end{pmatrix}
 \end{aligned}$$

$$\therefore \ker(T) \text{ has basis } \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -7 \\ 0 \\ -2 \\ 1 \end{pmatrix}. \quad \dim(\ker(T)) = 2. \\
 (= \# \text{ vectors in basis})$$

$$c) T: \mathbb{R}^4 \rightarrow \mathbb{R}^3, T(\underline{x}) = A\underline{x}, \quad A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 1 & 4 \end{pmatrix}, \quad W = \text{image}(T).$$

~~compute RREF(A)~~ Perform first stage of GE algorithm (to row echelon form)

$$\begin{array}{l}
 -R1 \left(\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 1 & 4 \end{array} \right) \rightsquigarrow \left(\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{array} \right) \rightsquigarrow \left(\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right)
 \end{array}$$

pivots in cols 1 4 3.

$$\therefore \text{image}(T) \text{ has basis } \left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) \quad (\text{cols 1 4 3 of } A)$$

$$\dim(\text{image}(T)) = 2.$$

$$d) W = \mathbb{R}^n \text{ has basis } \underline{e}_1, \dots, \underline{e}_n, \quad \text{dimension } \dim W = n.$$

$$16. \quad \underline{v}_1, \underline{v}_2, \dots, \underline{v}_m \in \mathbb{R}^n.$$

(a) linearly independent $\Rightarrow M \leq n$

(b) span $\mathbb{R}^n \Rightarrow M \geq n$

(c) basis of $\mathbb{R}^n \Rightarrow M = n$.

$$17. \quad T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$T(\underline{x}) = A \cdot \underline{x}$$

$$\begin{aligned} \text{rank}(T) &= \# \text{ pivots in RREF}(A) \\ &= \dim(\text{image}(T)). \end{aligned}$$

Rank-nullity formula:-

$$n = \dim(\text{domain}(T)) = \dim(\ker(T)) + \dim(\text{image}(T))$$

"nullity" "rank"

$$T: \mathbb{R}^8 \rightarrow \mathbb{R}^3$$

$$18. (a) \quad \text{image}(T) \subset \mathbb{R}^3 \Rightarrow 0 \leq \underset{\dim(\text{image}(T))}{\text{rank}(T)} \leq 3$$

$$\ker(T) \subset \mathbb{R}^8 \Rightarrow 0 \leq \dim(\ker(T)) \leq 8.$$

$$\text{and (rank-nullity formula)} \quad \dim(\ker(T)) + \text{rank}(T) = 8.$$

-:	Possibilities:	$\dim(\ker(T))$	$\text{rank}(T)$
		8	0
		7	1
		6	2
		5	3

$$(b) \quad T: \mathbb{R}^4 \rightarrow \mathbb{R}^7$$

Similarly, possibilities:	$\dim(\ker(T))$	$\text{rank}(T)$
	4	0
	3	1
	2	2
	1	3
	0	4

$$(c) \quad T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

If $n \geq m$, $0 \leq \text{rank}(T) \leq m$ AND $\dim \ker(T) = n - \text{rank}(T)$

If $n \leq m$, ~~$0 \leq \text{rank}(T) \leq n$~~ AND ~~$\text{rank}(T) =$~~
 $0 \leq \text{rank}(T) \leq n$ AND $\dim \ker(T) = n - \text{rank}(T)$.

19. $W \subset \mathbb{R}^n$ subspace, $\dim W = m \Rightarrow m \leq n$.

$B = (\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m)$ basis of W .

$$\underline{v} \in W. [\underline{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \in \mathbb{R}^m$$

$$\text{Means } \underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_m \underline{v}_m.$$

20. $W \subset \mathbb{R}^3$ is the plane with equation $x+y+z=0$.

$$B = (\underline{v}_1, \underline{v}_2), \quad \underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(a) Since W is a plane, it is spanned by any two vectors $\underline{v}_1, \underline{v}_2 \in W$ which are not parallel, and then $\underline{v}_1, \underline{v}_2$ is a basis of W if span & linearly independent.

We check (1) $\underline{v}_1, \underline{v}_2 \in W$

(2) $\underline{v}_1, \underline{v}_2$ not parallel, i.e. $\underline{v}_2 \neq c \cdot \underline{v}_1$,

$$(1): 1 + (-1) + 0 = 0 \Rightarrow \underline{v}_1 \in W$$

$$0 + (1) + (-1) = 0 \Rightarrow \underline{v}_2 \in W.$$

(2): This is clear.

$$(b) \underline{w} \in W, [\underline{w}]_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Leftrightarrow \underline{w} = 2\underline{v}_1 + 5\underline{v}_2$$

$$= 2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

$$(c) \underline{v} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}. \text{ Show } \underline{v} \in W \text{ & compute } [\underline{v}]_B.$$

$$-\text{Solve } \underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \text{ for } c_1, c_2 \in \mathbb{R}$$

$$\text{By inspection } c_1 = 3, c_2 = 7 \quad (\text{or, use GE})$$

$$\therefore \underline{v} \in W, \text{ and } [\underline{v}]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

21. $B = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(a) Explain why B is a basis of \mathbb{R}^2 .

Because \mathbb{R}^2 has dimension 2, it is spanned by any two vectors which are not parallel, & the these vectors form a basis of \mathbb{R}^2 (span & linearly independent). Clearly v_1 & v_2 are NOT parallel, so B is a basis of \mathbb{R}^2 .

(b) $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $L(\underline{x}) = [\underline{x}]_B$, an invertible linear transformation.

$$L^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 v_1 + c_2 v_2 = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

by the definition of $[\underline{x}]_B$.

i.e., standard matrix of L^{-1} is the matrix S with columns v_1 & v_2 , i.e. $S = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Now standard matrix of L is $S^{-1} = \frac{1}{1-2-2} \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \frac{-1}{3} \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

using the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

22. $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, B basis of \mathbb{R}^n , $B = (v_1, v_2, \dots, v_n)$.

The B -matrix B of T is defined by $[T(\underline{x})]_B = B \cdot [\underline{x}]_B$.

Its columns are give by $B = ([T(v_1)]_B [T(v_2)]_B \dots [T(v_n)]_B)$

If A is the standard matrix of T then using the diagram.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ S \uparrow & \uparrow S & \\ \mathbb{R}^n & \xrightarrow{B} & \mathbb{R}^n \end{array} \quad \begin{array}{ccc} \underline{x} & \longrightarrow & T(\underline{x}) \\ \uparrow & & \uparrow \\ [\underline{x}]_B & \longrightarrow & [T(\underline{x})]_B \end{array} \quad \text{where } S = (v_1, v_2, \dots, v_n)$$

We find $B = S^{-1}AS$

$$\text{OR } A = SBS^{-1}$$

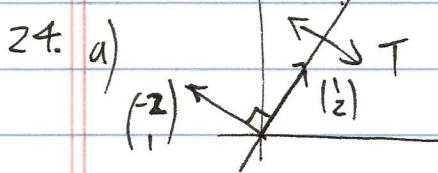
$$23. \quad T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad T(\underline{x}) = A \cdot \underline{x}, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$B = (\underline{v}_1, \underline{v}_2), \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{basis of } \mathbb{R}^2.$$

B -Matrix of T ?

$$B = S^{-1}AS = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\text{see Q22.} \quad = \frac{1}{1 \cdot 1 - 2 \cdot 3} \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ 11 & 13 \end{pmatrix} = \frac{-1}{5} \begin{pmatrix} -28 & -34 \\ 1 & 3 \end{pmatrix}$$



$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{reflection in line } L \text{ with equation } y=2x$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \left. \right\} \text{equation } y=2x$$

$$\underline{v}_1 \text{ in direction of line } L \Rightarrow T(\underline{v}_1) = \underline{v}_1$$

$$\underline{v}_2 \text{ orthogonal to line } L \quad (\text{because } \underline{v}_1 \cdot \underline{v}_2 = 0) \Rightarrow T(\underline{v}_2) = -\underline{v}_2.$$

$$\therefore B\text{-matrix} \quad B = ([T(\underline{v}_1)]_B \ [T(\underline{v}_2)]_B) = ([\underline{v}_1]_B \ [\underline{v}_2]_B)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$b) \quad T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad \text{orthogonal projection onto plane } \Pi \text{ with equation } 2x+y+z=0.$$

$$\underline{v}_1, \underline{v}_2 \text{ basis of } \Pi, \text{ e.g. } \underline{v}_1 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow T(\underline{v}_1) = \underline{v}_1, \quad T(\underline{v}_2) = \underline{v}_2.$$

$$\underline{v}_3 \text{ normal to } \Pi, \text{ e.g. } \underline{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad (\text{from equation of } \Pi)$$

$$\Rightarrow T(\underline{v}_3) = \underline{0}$$

Now $\mathcal{B} = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a basis of \mathbb{R}^3

$$\text{4 } \mathcal{B}\text{-matrix } \mathcal{B} = ([T(\underline{v}_1)]_{\mathcal{B}} [T(\underline{v}_2)]_{\mathcal{B}} [T(\underline{v}_3)]_{\mathcal{B}}) = ([\underline{v}_1]_{\mathcal{B}} [\underline{v}_2]_{\mathcal{B}} [\underline{v}_3]_{\mathcal{B}})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

c) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by rotation about axis the line through the origin w/ equations $x=y=z$, through angle π radians.

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ in direction of axis } \Rightarrow T(\underline{v}_1) = \underline{v}_1$$

$\underline{v}_2, \underline{v}_3$ basis of plane Π orthogonal to axis, e.g. $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow T(\underline{v}_2) = -\underline{v}_2, T(\underline{v}_3) = -\underline{v}_3 \quad (\text{because angle of rotation} = \pi \text{ radians} = 180^\circ)$$

Then $\mathcal{B} = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is basis of \mathbb{R}^3 ,

$$\mathcal{B}\text{-matrix } \mathcal{B} = ([T(\underline{v}_1)]_{\mathcal{B}} [T(\underline{v}_2)]_{\mathcal{B}} [T(\underline{v}_3)]_{\mathcal{B}}) = ([\underline{v}_1]_{\mathcal{B}} [-\underline{v}_2]_{\mathcal{B}} [-\underline{v}_3]_{\mathcal{B}})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

25 A linear space V is a set together with operations of addition: $\underline{v}, \underline{w} \in V \rightsquigarrow \underline{v} + \underline{w} \in V$

and scalar multiplication: $c \in \mathbb{R}, \underline{v} \in V \rightsquigarrow c\underline{v} \in V$

satisfying "same properties as \mathbb{R}^n ".

(see 4.1 of text for precise definition)

26. (a) P_4 = linear space of polynomials of degree ≤ 4

$$= \left\{ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \mid c_0, c_1, \dots, c_4 \in \mathbb{R} \right\}$$

Basis $\mathcal{B} = \{1, x, x^2, x^3, x^4\}$

dimension = # elements in basis = 5.

(b) $\mathbb{R}^{2 \times 3}$ = linear space of 2×3 matrices = $\left\{ \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{pmatrix} \mid c_{11}, \dots, c_6 \in \mathbb{R} \right\}$

$$\text{Basis } \mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Dimension = 6.

(c) P_1 = linear space of polynomials of degree ≤ 1

$$\text{Basis } \mathcal{B} = \langle 1, x, x^2, \dots, x^n \rangle$$

Dimension = $n+1$

(d) $\mathbb{R}^{M \times N}$

Basis \mathcal{B} given by matrices w/ one entry equal to 1,
& remaining entries equal to 0.

Dimension = $M \cdot N$

$$27) \text{ a) } V = P_2, \quad W = \{ f(x) \mid f(1) = 0 \} \subset P_2$$

$$W \text{ is a subspace : } W = \{ c_0 + c_1 x + c_2 x^2 \mid c_0 + c_1 + c_2 = 0 \}$$

$$\text{Basis } \mathcal{B} = (-1+x, -1+x^2) \quad \left(\text{corresponding to basis } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$\dim W = 2$ of plane $c_0 + c_1 + c_2 = 0$ in \mathbb{R}^3

$$\text{b) } V = \mathbb{R}^{2 \times 2}, \quad W = \{ X \mid AX = XA \} \subset \mathbb{R}^{2 \times 2}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\text{Write } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a+c & b+d \\ a-c & b-d \end{pmatrix} = \begin{pmatrix} a+b & a-b \\ c+d & c-d \end{pmatrix}$$

$$\Leftrightarrow a+c = a+b \Rightarrow b=c$$

$$b+d = a-b \quad \Leftrightarrow -b+c=0 \quad \text{Solve using GE.}$$

$$a-c = c+d$$

$$-a+2b+d=0$$

$$b-d = c-d$$

$$a-2c-d=0$$

$$b-c=0$$

coefft.
Matrix

$$\xrightarrow{\text{R1} \rightarrow \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 2 & 0 & +1 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}} \xrightarrow{\text{R1} \rightarrow \begin{pmatrix} 1 & 0 & -2 & -1 \\ -1 & 2 & 0 & +1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}} \xrightarrow{\text{R2} \rightarrow \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}} \xrightarrow{\text{R2} \rightarrow \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}} \xrightarrow{\text{R2} \rightarrow \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}}$$

$$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a b c d

$$a - 2c + d = 0 \quad a = 2c - d$$

$$b - c = 0 \quad b = c$$

c, d free

c, d free

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2c-d & c \\ c & d \end{pmatrix} = c \cdot \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$c, d \in \mathbb{R}$ arbitrary

W is subspace, basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$c) V = P_4, W = \{ f(x) \mid f(-x) = -f(x) \} \subset P_4.$$

$$\begin{aligned} W &= \left\{ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \mid \begin{array}{l} c_0 - c_1 x + c_2 x^2 - c_3 x^3 + c_4 x^4 \\ = -c_0 - c_1 x - c_2 x^2 - c_3 x^3 - c_4 x^4 \end{array} \right\} \\ &= \left\{ c_0 + c_1 x + \dots + c_4 x^4 \mid \begin{array}{l} 2c_0 + 2c_2 x^2 + 2c_4 x^4 = 0 \\ (\text{for all } x \in \mathbb{R}) \end{array} \right\} \\ &= \left\{ c_0 + c_1 x + \dots + c_4 x^4 \mid c_0 = c_2 = c_4 = 0 \right\} \\ &= \left\{ c_1 x + c_3 x^3 \mid c_1, c_3 \in \mathbb{R} \right\}. \end{aligned}$$

Subspace, basis $\mathcal{B} = (x, x^3)$. ("odd functions" $\subset P_4$)

$$28. (a) T: P_3 \rightarrow P_3 \quad T(f(x)) = f'(x).$$

$$\begin{aligned} T \text{ is linear : } T(f+g) &= (f+g)' = f' + g' && \text{for } f, g \in P_3 \\ T(cf) &= (cf)' = c \cdot f' && \text{for } c \in \mathbb{R}, f \in P_3. \end{aligned}$$

$$\ker(T) = \{ f \in P_3 \mid f' = 0 \} = \{c \mid c \in \mathbb{R}\} \quad \text{constant functions, basis } \mathcal{B} = (1)$$

$$\begin{aligned}
 \text{image}(T) &= \left\{ T(f) \mid f \in P_3 \right\} = \left\{ f' \mid f = c_0 + c_1x + c_2x^2 + c_3x^3 \right\} \\
 &= \left\{ c_1 + 2c_2x + 3c_3x^2 \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \\
 &\text{basis } (1, x, x^2).
 \end{aligned}$$

$$(b) \quad T: P_2 \rightarrow \mathbb{R}^2 \quad T(f) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$$

T is linear.

$$T(c_0 + c_1x + c_2x^2) = \begin{pmatrix} c_0 + c_1 + c_2 \\ c_0 + 2c_1 + 4c_2 \end{pmatrix}$$

$$\ker(T) ? \quad \text{Solve } c_0 + c_1 + c_2 = 0 \quad \text{using GE:—}$$

$$c_0 + 2c_1 + 4c_2 = 0$$

$$-R1 \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \rightsquigarrow -R2 \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} +2c_2 \\ -3c_2 \\ c_2 \end{pmatrix} = c_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad c_2 \text{ free}$$

$$\therefore \ker(T) = \text{Span}(2-3x+x^2).$$

$$\text{image}(T) = \mathbb{R}^2$$

$$\begin{aligned}
 (\text{e.g. equations.}) \quad & \begin{pmatrix} c_0 + c_1 + c_2 \\ c_0 + 2c_1 + 4c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{have a solution for all } b_1, b_2 \text{ because} \\
 & \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}}_{\text{RREF } A \text{ has pivot in every row (see above)}}
 \end{aligned}$$

$$(c) \quad T: P_2 \rightarrow P_2, \quad T(f(x)) = xf'(x) - f(x)$$

T is linear.

$$\begin{aligned}
 T(c_0 + c_1x + c_2x^2) &= x(c_1 + 2c_2x) - (c_0 + c_1x + c_2x^2) \\
 &= -c_0 + (0 \cdot x + c_2)x^2
 \end{aligned}$$

$$\begin{aligned}
 \ker(T) &= \left\{ c_0 + c_1x + c_2x^2 \mid -c_0 + c_2x^2 = 0 \text{ (for all } x \in \mathbb{R}) \right\} \\
 &= \left\{ c_1x \mid c_1 \in \mathbb{R} \right\}, \quad \text{basis } \{x\}.
 \end{aligned}$$

$$\text{image}(T) = \{ -c_0 + c_2 x^2 \mid c_0, c_2 \in \mathbb{R}\}, \text{ basis } (1, x^2).$$

29. $T: V \rightarrow W$ linear transformation, $\dim V = \dim W < \infty$

The T is invertible $\Leftrightarrow \ker(T) = \{0\} \Leftrightarrow \text{image}(T) = W$.

30. a) NOT an isomorphism because $\dim P_3 = 3+1=4 \neq \dim \mathbb{R}^2 = 2$.

b) $T: P_2 \rightarrow P_2$, $T(f) = f - f'$.

$$\begin{aligned} T(c_0 + c_1 x + c_2 x^2) &= c_0 + c_1 x + c_2 x^2 - (c_1 + 2c_2 x) \\ &= (c_0 - c_1) + (c_1 - 2c_2)x + c_2 x^2. \end{aligned}$$

$$\begin{aligned} \ker(T): \text{ Solve } \quad c_0 - c_1 &= 0 \\ c_1 - 2c_2 &= 0 \\ c_2 &= 0 \end{aligned} \quad \begin{array}{l} \text{coefft} \\ \text{matrix} \end{array} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \ker(T) = \{0\}.$$

So T is an isomorphism (note domain & codomain have same dimension, see Q29.)

c) $T: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$

$$T(X) = AXB, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Note A & B are invertible 2×2 matrices,

$$\text{because } \det A = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$$

$$\text{and } \det B = 5 \cdot 8 - 6 \cdot 7 = -2 \neq 0.$$

So T is an isomorphism with inverse $T^{-1}(Y) = A^{-1}YB^{-1}$

(because $AXB = Y \Leftrightarrow X = A^{-1}YB^{-1}$).

31. V linear space, $B = \{f_1, f_2, \dots, f_n\}$ basis of V .

$$f \in V, \quad [f]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \text{ means } f = c_1 f_1 + c_2 f_2 + \dots + c_n f_n.$$

32. $V = P_2$, $B = (1, x-1, (x-1)^2)$ basis of V .

$$f = x^2 + 2x + 3 \in P_2.$$

$$[f]_B = ? \quad f = x^2 + 2x + 3 = c_1 \cdot 1 + c_2(x-1) + c_3 \cdot (x-1)^2$$

i.e. solve $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Find $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$, $[f]_B = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$.

33. $T: V \rightarrow V$ linear transformation, B basis of V , $B = (f_1, f_2, \dots, f_n)$

B -matrix B of T defined by $[T(f)]_B = B \cdot [f]_B$.

$$B = ([T(f_1)]_B \dots [T(f_n)]_B)$$

34. (a) P_3 has basis $B = (1, x, x^2, x^3)$

$$T: P_3 \rightarrow P_3 \quad T(f) = f'$$

$$(c_0 + c_1x + c_2x^2 + c_3x^3) \xrightarrow{T} T(c_0 + c_1x + c_2x^2 + c_3x^3) = c_1 + 2c_2x + 3c_3x^2$$

$$\begin{array}{ccc} \xrightarrow{\text{L}_B} & & \xrightarrow{\text{L}_B} \\ \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} & \xrightarrow{B} & \begin{pmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \end{array}$$

$$B\text{-matrix } B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) P_2 has basis $B = (1, x, x^2)$

$$T: P_2 \rightarrow P_2, T(f) = xf' - f.$$

$$(c_0 + c_1x + c_2x^2) \xrightarrow{T} T(c_0 + c_1x + c_2x^2) = -c_0 + c_2x^2 \quad (\text{see Q28(c)})$$

$$\xrightarrow{\text{L}_B} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} -c_0 \\ 0 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

35. $C^\infty(\mathbb{R}, \mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ has derivatives of all orders}\}$.

$$V = \text{Span}(\cos(x), \sin(x)) \subset C^\infty(\mathbb{R}, \mathbb{R})$$

(a) $B = (\cos(x), \sin(x))$ is a basis of V :-

span by definition of V

$\cos(x)$ & $\sin(x)$ are linearly independent :-

if $c_1 \cos(x) + c_2 \sin(x) = 0$, (for all $x \in \mathbb{R}$),

$$\begin{aligned} \text{setting } x=0 &\Rightarrow c_1 = 0 \\ x=\frac{\pi}{2} &\Rightarrow c_2 = 0. \end{aligned} \quad \left. \begin{array}{l} c_1 = c_2 = 0. \\ \end{array} \right\} \quad \left. \begin{array}{l} c_1 = c_2 = 0. \\ \end{array} \right\} \quad \left. \begin{array}{l} c_1 = c_2 = 0. \\ \end{array} \right\}$$

So $\cos(x)$ & $\sin(x)$ are linearly independent.

(b) $T: V \rightarrow V$ linear transformation.

$$T(f) = f' + f$$

$$B\text{-matrix: } B = ([T(\cos x)]_B \ [T(\sin x)]_B)$$

$$= ([-\sin x]_B \ [\cos x]_B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

B is invertible $\Rightarrow T$ is invertible.

$$(\text{e.g. } \det B = 0 \cdot 0 - 1 \cdot (-1) = 1 \neq 0)$$

36. V linear space. B basis, C another basis.

$S_{B \rightarrow C}$ "change of basis matrix" defined by $S_{B \rightarrow C} \cdot [x]_B = [x]_C$.

$$B = (f_1, \dots, f_n) \text{ & } C = (g_1, \dots, g_n)$$

$$\Rightarrow S_{B \rightarrow C} = ([f_1]_C \ [f_2]_C \ \dots \ [f_n]_C)$$

37. $V \subset \mathbb{R}^3$ plane with equation $x+2y+z=0$.

$B = (v_1, v_2)$ basis of V , $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$,

$C = (w_1, w_2)$ another basis of V , $w_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$$(a) S_{B \rightarrow C} = ([v_1]_C \ [v_2]_C)$$

By inspection (or F.E.) $v_1 = w_1 - w_2$, $v_2 = w_1 - 2w_2$

$$[v_1]_e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad [v_2]_e = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$S_{B \rightarrow e} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

$$(b) S_{e \rightarrow B} = (S_{B \rightarrow e})^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} = \frac{1}{1 \cdot (-2) - 1 \cdot (-1)} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

38. $\text{Area}(T(S)) = |\det A| \cdot \text{Area}(S)$

where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is matrix of T.

$$\therefore \text{Area}(T(S)) = |1 \cdot 4 - 2 \cdot 3| \cdot (1)$$

$$= |-2| \cdot (1)$$

Similarly, $= 2$.

$$\begin{aligned} \text{Area}(T(D)) &= +2 \cdot \cancel{\text{Area}(D)} = \\ &= |\det A| \cdot \text{Area}(D) \\ &= 2 \cdot (\pi(1)^2) \\ &= 2\pi. \end{aligned}$$

39. a) $\det \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} = 2 \cdot 1 - 3 \cdot 5 = -13, \neq 0 \Rightarrow A \text{ invertible}$

b) $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

$$\begin{array}{ccc|cc} + & + & + & - & - \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 \end{array}$$

Sarrus' rule $\Rightarrow \det A = 0 + 1 \cdot (-1) \cdot 1 + (-1) \cdot 1 \cdot (-1) - 0 - 0 - 0$
 $= 0.$

$\Rightarrow A \text{ NOT invertible}$

$$c) A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{pmatrix}$$

$$\begin{array}{ccc|cc} + & + & + & - & - \\ \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{2} & \cancel{1} & \cancel{2} \\ \cancel{2} & \cancel{3} & \cancel{5} & \cancel{2} & \cancel{3} \end{array} \quad \text{Sarrus' rule}$$

$$\begin{aligned} \det A &= 1 \cdot 1 \cdot 5 + 1 \cdot 2 \cdot 2 + 1 \cdot 1 \cdot 3 - 1 \cdot 2 \cdot 3 - 1 \cdot 1 \cdot 5 - 1 \cdot 1 \cdot 2 \\ &= 5 + 4 + 3 - 6 - 5 - 2 \\ &= -1 \neq 0 \Rightarrow A \text{ invertible.} \end{aligned}$$

$$d) A \text{ upper triangular} \Rightarrow \det A = \text{product of diagonal entries} \\ = 2 \cdot 3 \cdot 5 \cdot 2 = 60, \neq 0 \\ \Rightarrow A \text{ invertible.}$$

$$e) A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 4 & 7 & 9 \\ 3 & 3 & 3 & 8 \end{pmatrix}$$

Compute using row operations

$$\begin{array}{l} -R1 \\ -2R1 \\ -3R1 \end{array} \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 4 & 7 & 9 \\ 3 & 3 & 3 & 8 \end{array} \right) \xrightarrow{\text{row swap}} \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 0 & 5 \end{array} \right) \xrightarrow{\text{row swap}} \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{array} \right) = B \text{ upper triangular}$$

$$\Rightarrow \det A = (-1) \cdot \underbrace{(1 \cdot 2 \cdot 2 \cdot 5)}_{\substack{\text{row swap} \\ \text{product of diag. entries of } B}} = -20.$$

$$40. T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & x \\ 1 & 2 & y \\ 1 & 3 & z \end{pmatrix}$$

$$(a) T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} - y \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + z \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{array}{l} \text{Laplace expansion} \\ \text{along 3rd col.} \end{array} = x - 2y + z, \quad \text{linear.} \\ = \underbrace{(1 - 2 1)}_{\text{standard matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

standard matrix

(b) The kernel of T is the plane $x-2y+z=0$ in \mathbb{R}^3 .

This is the plane \mathbb{P} spanned by the first two cols of the matrix $A = \begin{pmatrix} 1 & 2 & x \\ 1 & 3 & y \\ 1 & 3 & z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{P} \iff \text{matrix } A \text{ is not invertible} \iff \det A = 0.$$

41. (a) $\det(AB) = (\det A) \cdot (\det B) = 5 \cdot 7 = 35$

(b) $\det(A^{-1}) = 1/\det A = 1/5$ (A is invertible because $\det A \neq 0$)

(c) $\det(SAS^{-1}) = \cancel{\det(S)} \cdot \det A \cdot \cancel{\det(S)^{-1}}$
 $= (\det S) \cdot (\det A) \cdot (\det S)^{-1}$
 $= \det A = 5.$

42 $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad T(x) = A \cdot x$

$B = (v_1, v_2, v_3)$ basis of \mathbb{R}^3 .

$$T(v_1) = 3v_1, \quad T(v_2) = 2v_2, \quad T(v_3) = 2v_3 + v_2.$$

w) B -matrix of T $B = \left([T(v_1)]_B \ [T(v_2)]_B \ [T(v_3)]_B \right) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$$\Rightarrow \det B = 3 \cdot 2 \cdot 2 = 12 \quad (B \text{ upper triangular})$$

$$\Rightarrow \det A = \det B = 12 \quad \left(\text{because } A = SBS^{-1}, \text{ see Q22 \& Q41} \right).$$