Math 462 Homework 8

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(1) Determine the unique Mobius transformation

$$f\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, \quad f(z) = \frac{az+b}{cz+d}$$

such that $f(1) = 0$, $f(i) = \infty$, and $f(-i) = 1$. (Here $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.)

(2) Determine the unique Mobius transformation f such that $f(1) = i$, $f(2) = 1 - i$ and $f(3) = 1 + i$.

[Hint: First find g and h such that $g(1) = h(i) = 0$, $g(2) = h(1-i) = \infty$, and $g(3) = h(1+i) = 1$. Then $f = h^{-1} \circ g$. Also recall that composition of Mobius transformations corresponds to multiplication of matrices (where$f(z) = \frac{az+b}{cz+d}$ corresponds to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determined up to a scalar factor $0 \neq \lambda \in \mathbb{C}$). So if g has matrix A and h has matrix B, then $f = h^{-1} \circ g$ has matrix $B^{-1}A$.]

(3) In class we showed that if $C \subset \mathbb{C}\cup\{\infty\}$ is a circle or line and

$$f\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}, \quad f(z) = \frac{az+b}{cz+d}$$

is a Mobius transformation, then $f(C)$ is a circle or line. Moreover $f(C)$ is a line precisely when f sends a point of C to ∞, i.e., the point $f^{-1}(\infty) = -d/c \in C$. (Here we use the convention that a line L in the extended complex plane $\mathbb{C}\cup\{\infty\}$ is a line in the usual complex plane $\mathbb{C}$ together with the point ∞.) For each of the following cases, determine whether the image $f(C)$ is a circle or a line. If the image is a line describe it precisely.

(a) C the circle with center the origin and radius 1, $f(z) = \frac{z-1}{z-i}$.

(b) C the line through the origin with slope 1, $f(z) = \frac{iz+2}{z-3}$.

(c) C the circle with center $1+i$ and radius 1, $f(z) = \frac{z+1}{z+(2-3i)}$.

(4) Find a Mobius transformation f which sends the circle C with center the point $i \in \mathbb{C}$ and radius 1 to the line $L = \mathbb{R} \cup \{\infty\}$ (the x-axis).

[Hint: Here is one possible approach. Choose 3 points z_1, z_2, z_3 on C and w_1, w_2, w_3 on L and determine the Mobius transformation f such that $f(z_j) = w_j$ for each $j = 1, 2, 3$. Then $f(C) = L$ (why?). Also, choosing the points z_j and w_j carefully will make the calculation easier.]

(5) Let C be the circle with center the origin O and radius 1, and D the circle with center the point $(r, 0)$ and radius r, where $0 < r < 1/2$. (So D lies inside C, has center on the x-axis, and passes through the origin O. Let $g \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be *inversion* in the circle C (identifying $\mathbb{R}^2 = \mathbb{C}$) given by $g(z) = z/|z|^2$. Show directly that the image of C under g is the line L perpendicular to the x-axis through the point $(1/2r, 0)$ as follows:

Let $P \in D$ be a point such that $P \neq O$. Draw the line through O and P, and let Q be the point where it meets the line L. Using similar triangles or otherwise, prove that $OP \cdot OQ = 1$. Deduce that $g(P) = Q$.

(6) Let $g \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be inversion in the unit circle C as in Q5. In class we showed that, under the stereographic projection $\bar{F} \colon S^2 \to \mathbb{C} \cup \{\infty\}$, the inversion g corresponds to the reflection $T \colon S^2 \to S^2$ in the xy-plane. Here we will give another more direct proof of this fact.

(a) Write down a formula for $T(x, y, z)$.

(b) Check that $g(\bar{F}(x, y, z)) = \bar{F}(T(x, y, z))$ for all points $(x, y, z) \in S^2$.

[Hint: Use the formula $F(x, y, z) = (x + iy)/(1 - z)$ and the equation $x^2 + y^2 + z^2 = 1$ of the sphere S^2.]

(7) Let $a, b \in \mathbb{C}$ be two distinct points in the complex plane. Consider the set S of all the circles C in $\mathbb{C}$ passing through both a and b.

(a) Explain how to construct the circles in the set S. Draw a picture.

(b) Now consider the Mobius transformation $f(z) = \frac{z-a}{z-b}$. What are the images $f(C)$ of the circles in S?

(c) We can also consider the "degenerate circle" given by the line L through a and b (together with the point ∞). What is $f(L)$?

(8) In class we defined the cross ratio $\mathrm{CR}(A, B, C, D)$ for 4 distinct points $A, B, C, D \in \mathbb{C}$ by the formula

$$\mathrm{CR}(A, B, C, D) = \frac{(C-A)(D-B)}{(C-B)(D-A)}$$

and we showed that the 4 points A, B, C, D lie on a circle or a line if and only if the cross ratio $CR(A, B, C, D) \in \mathbb{R}$. Here we will relate this result to a well known theorem in Euclidean geometry.

(a) Suppose A, B, C, D are 4 points on a circle, and A and B lie on the same side of the line segment CD. Then $\angle CAD = \angle CBD$. Use this fact to show directly that $\mathrm{CR}(A, B, C, D)$ is real and positive.

(b) What happens if A and B lie on opposite sides of the line segment CD?

[Hint: Addition of complex numbers is the same as for vectors in $\mathbb{R}^2$. So for example $C - A$ corresponds to the vector $\overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{AC}$. Also, given a complex number z, using polar coordinates we can write

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ is the length of z and $\theta = \arg(z)$ is the angle z makes with the x-axis. Then $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, i.e., under multiplication, the angles θ add (and similarly $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$).]