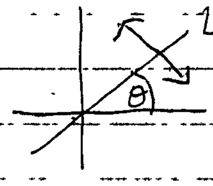
Math 461 Midterm

10/21/19

Office hours today & tomorrow 4-5PM LGRT 1235H
(No HW due this Wednesday)

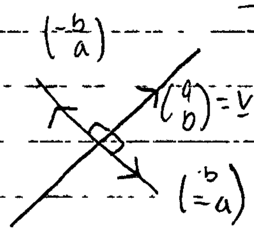
Last Time • Another proof of algebraic formula for reflection T in line L through origin at angle θ to x-axis



$T = R \circ S \circ R^{-1}$, R = rotation about origin through angle θ ccw, S = reflection in x-axis

$$\Rightarrow T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

• Similarly, formula for rotation center $P \neq (0,0)$
• Given algebraic formula for rotation, determine center & angle of rotation



Theorem ① $T: \mathbb{R}^2 \to \mathbb{R}^2$ isometry $\Rightarrow T\begin{pmatrix}x\\y\end{pmatrix} = A\begin{pmatrix}x\\y\end{pmatrix} + \underline{b}$
A, 2×2 matrix and $\underline{b} \in \mathbb{R}^2$ vector

② $T(\underline{x}) = A\underline{x} + \underline{b}$; T isometry $\Leftrightarrow$ A is $\boxed{\text{orthogonal}}$ matrix
i.e. $A = (\underline{v}_1 \vdots \underline{v}_2)$, $\|\underline{v}_1\| = \|\underline{v}_2\| = 1$ & $\underline{v}_1 \cdot \underline{v}_2 = 0$ (orthogonal)

equivalently, $A = \begin{pmatrix}a & -b\\ b & a\end{pmatrix}$ OR $\begin{pmatrix}a & b\\ b & -a\end{pmatrix}$, $a^2+b^2=1$.

Today • Proof of theorem
• Direct & Opposite isometries
• Algebraic formula ⇝ geometric description
• (compositions)

Proof of Theorem ① Recall, if $T(\underline{0}) = \underline{0}$ we showed T is a linear transformation, equivalently $T(\underline{x}) = A\underline{x}$ for some 2×2 matrix A.
If $T(\underline{0}) = P \neq \underline{0}$.
Compose T with U^{-1} where $U(\underline{x}) = \underline{x} + \overrightarrow{OP}$. (translation sending O to P)
Then $U^{-1} \circ T(\underline{0}) = \underline{0}$, $U^{-1} \circ T$ is an isometry. $\Rightarrow U^{-1} \circ T(\underline{x}) = A \cdot \underline{x}$
$\Rightarrow T(\underline{x}) = U(A\cdot\underline{x}) = A\cdot\underline{x} + \underbrace{\overrightarrow{OP}}_{\underline{b}}$ ▨

② We may assume that $\underline{b} = \underline{0}$.

$=$ can write $T = U \circ V$ (as before) where $V(\underline{x}) = A \cdot \underline{x}$,

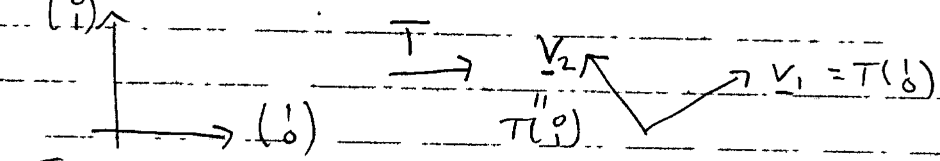
$U(\underline{x}) = \underline{x} + \underline{b}$ is a translation.

Then T is an isometry $\Leftrightarrow$ V is an isometry.

($\Rightarrow$) $T(\underline{x}) = A\underline{x} = (\underline{v}_1 \vdots \underline{v}_2)\binom{x}{y} = x \cdot \underline{v}_1 + y \cdot \underline{v}_2$.

We know T is an isometry, want to show

$\|\underline{v}_1\| = \|\underline{v}_2\| = 1$ & $\underline{v}_1 \perp \underline{v}_2$.



T is an isometry $\Rightarrow$ T preserves angles & distances $\Rightarrow$ $\|\underline{v}_1\| = \|\binom{1}{0}\| = 1$

$\|\underline{v}_2\| = \|\binom{0}{1}\| = 1$

and $\underline{v}_1 \perp \underline{v}_2$ because $\binom{1}{0} \perp \binom{0}{1}$.

($\Leftarrow$) Suppose A is an orthogonal 2×2 matrix.

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\underline{x}) = A\underline{x}$.

Want to show T is an isometry, i.e. T preserves distances.

$$\|T(\underline{x}) - T(\underline{y})\| \overset{?}{=} \|\underline{x} - \underline{y}\| \text{ for all } \underline{x}, \underline{y} \in \mathbb{R}^2.$$

$\| \| = \|A\underline{x} - A\underline{y}\|$

$\| \| = \|A \cdot (\underline{x} - \underline{y})\|$ (set $\underline{z} = \underline{x} - \underline{y}$)

i.e., need to show $\|A \cdot \underline{z}\| = \|\underline{z}\|$ for all $\underline{z} \in \mathbb{R}^2$

equivalently, $\|A \cdot \underline{z}\|^2 = \|\underline{z}\|^2$ for all $\underline{z} \in \mathbb{R}^2$.

Left-hand-side: $\|A \cdot \underline{z}\|^2 = \|z_1 \underline{v}_1 + z_2 \underline{v}_2\|^2$ where $\underline{z} = \binom{z_1}{z_2}$.

> Aside: properties of dot products:
> $\underline{v} \in \mathbb{R}^2$, $\underline{v} \cdot \underline{v} = \|\underline{v}\|^2$, $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$
> $\underline{u} \cdot (c\underline{v}) = c(\underline{u} \cdot \underline{v})$, etc.

$\|A \cdot \underline{z}\|^2 = \|z_1 \underline{v}_1 + z_2 \underline{v}_2\|^2 = (z_1 \cdot \underline{v}_1 + z_2 \underline{v}_2) \cdot (z_1 \underline{v}_1 + z_2 \underline{v}_2)$

$= z_1^2 \underline{v}_1 \underline{v}_1 + z_1 z_2 \underline{v}_1 \underline{v}_2 + z_2 z_1 \underline{v}_2 \underline{v}_1 + z_2^2 \underline{v}_2 \underline{v}_2$

$= z_1^2 \|\underline{v}_1\|^2 + \cancel{2 z_1 z_2 \underline{v}_1 \underline{v}_2} + z_2^2 \|\underline{v}_2\|^2 = z_1^2 + z_2^2 = \|\underline{z}\|^2$. ∎

$\underline{v}_1 \cdot \underline{v}_2 = 0$ by assumption

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<u>Remark</u> T isometry $\Leftrightarrow T(\underline{x}) = A\underline{x} + \underline{b}$ where

$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ OR $A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where $a^2+b^2=1$

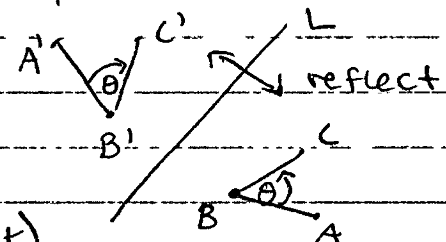
$\left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\right) \rightsquigarrow \det(A) = a^2+b^2 = 1$

OR $\det(A) = -a^2-b^2 = -1$.

In MATH 235, when $\det(A) = 1 \rightsquigarrow$ $\boxed{\text{"orientation preserving"}}$ $\rightsquigarrow$ $\boxed{\text{direct}}$, i.e. preserves sense (ccw/cw) of angles.

when $\det(A) = -1 \rightsquigarrow$ $\boxed{\text{"orientation reversing"}}$ $\rightsquigarrow$ $\boxed{\text{opposite}}$, i.e. does not preserve sense (ccw/cw) of angles.

ex: reflection



<u>Isometries</u> (not proved yet)

T	direct/opp.	Fix(T) (defined on the next page)
Translation	direct	$\emptyset$ (no points are fixed)
Rotation	direct	P = center of rotation
Reflection	opposite	L = line of reflection
Glide reflection (composition of reflection & translation parallel to line of reflection)	opposite	$\emptyset$
Identity	direct	$\mathbb{R}^2$

Compositions

direct ∘ direct = direct

direct ∘ opposite = opposite

opposite ∘ direct = opposite

opposite ∘ opposite = direct

<u>Q</u>: Rotate P through angle θ ccw (T_1), then reflect in line L (T_2). What is $T_2 \circ T_1$?

$T_1(\underline{x}) = A\underline{x} + \underline{b}$ $\qquad T = T_2 \circ T_1$

$T_2(\underline{x}) = C\underline{x} + \underline{d}$ $\qquad = T_2(T_1(\underline{x})) = T_2(A\underline{x} + \underline{b})$

$= C \cdot (A\underline{x} + \underline{b}) + \underline{d} = (CA) \cdot \underline{x} + (C \cdot \underline{b} + \underline{d})$

$= E\underline{x} + \underline{f}$

$\det(E) = \det(CA) = \det(C) \cdot \det(A) = (-1) \cdot (1) = -1$

Thus, must be opposite, either reflection or glide reflection.

$\underline{Q}$: Suppose given algebraic formula $T(\underline{x}) = A\underline{x} + \underline{b}$ for an isometry. How to describe T geometrically?

Can write : $T = U \circ V$, $V(\underline{x}) = A \cdot \underline{x}$ & $U(\underline{x}) = \underline{x} + \underline{b}$
i.e. first do V then do U (translate by $\underline{b}$)
(either rotation or reflection fixing origin)

Define : $\underline{Fix}(T) = \{ P \in \mathbb{R}^2 \mid T(P) = P \}$
Translation by (a,b) : $T(x,y) = (x+a, y+b)$. → very clearly a translation
Distinguish using table.