Monday 12/7/15 MATH 611. HW8 Solutions

1. a. $M = \langle m \rangle_R$

$$R \xrightarrow{\varphi} M \qquad \text{hom.}$$
$$r \longmapsto r \cdot m$$

$I = \ker \varphi \subset R$ ideal

$$R/I \underset{\bar\varphi}{\xrightarrow{\sim}} M \qquad \text{by FIT.}$$

b. Characteristic polynomial

$$c_A(x) = \det(x \cdot I - A) = (x-1)(x-2)$$

$\Rightarrow$ Minimal polynomial

$$m_A(x) = c_A(x) = (x-1)(x-2)$$

$$\Rightarrow \quad M \cong \mathbb{R}[x] / (x-1)(x-2), \quad \text{cyclic}$$

(in general, if $M \cong F[x]/(d_1) \oplus \cdots \oplus F[x]/(d_r)$, w/ $d_1 | d_2 | \cdots | d_r$, monic

then $m_A(x) = d_r$, $c_A(x) = d_1 d_2 \cdots d_r$)

2. a

$$A = \begin{array}{c} -C2 \\ \begin{pmatrix} 3 & 2 \\ 2 & 0 \\ 8 & 4 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ -2R1 \\ -4R1 \end{array} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 4 & 4 \end{pmatrix} \sim \begin{array}{c} -C1 \\ \begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 0 & -4 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ \times -1 \\ \\ \end{array} \begin{pmatrix} 1 & 0 \\ 0 & -4 \\ 0 & -4 \end{pmatrix}$$

$$\sim \begin{array}{c} \\ \\ +R2 \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \quad \mathbb{Z}^3 / A \cdot \mathbb{Z}^2 \simeq \mathbb{Z}^3 / \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \cdot \mathbb{Z}^2 = \frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z} \oplus 4\mathbb{Z} \oplus 0}$$

$$= \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}.$$

b.

$$A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ +2R1 \\ -3R1 \end{array}\begin{pmatrix} 2 & 2 & 4 \\ -4 & -6 & 7 \\ 6 & 6 & 15 \end{pmatrix} \rightsquigarrow \begin{array}{ccc} & -C1 & -2C1 \end{array} \begin{pmatrix} 2 & 2 & 4 \\ 0 & -2 & 15 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ x-1 \\ \\ \end{array}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 15 \\ 0 & 0 & 3 \end{pmatrix} \rightsquigarrow \begin{array}{ccc} & & +8C2 \end{array}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -15 \\ 0 & 0 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ - \\ -3R1 \end{array}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{ccc} & -2C1 & \end{array}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & -6 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -6 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ x-1 \end{array}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Rightarrow \quad \mathbb{Z}^3/A\mathbb{Z}^3 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.$$

3.

$$A = \begin{array}{cc} & -2C1 \end{array}\begin{pmatrix} x+1 & 2x+3 \\ x^3+x+2 & 2x^3+x+6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x+1 & 1 \\ x^3+x+2 & -x+2 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ +(x-2)R1 \end{array}\begin{pmatrix} 1 & x+1 \\ -x+2 & x^3+x+2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & x+1 \\ 0 & x^3+x^2 \end{pmatrix}_{-(x+1)\cdot C1} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & x^3+x^2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbb{C}[x]^2/A\cdot\mathbb{C}[x]^2 \cong \mathbb{C}[x]/(x^3+x^2) \underset{CRT}{\cong} \mathbb{C}[x]/(x^2) \oplus \mathbb{C}[x]/(x+1)$$

4.

$$A = \begin{array}{cc} & -(1+i)\cdot C1 \end{array}\begin{pmatrix} 1+i & 3 \\ 2-i & 5i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1+i & 1 \\ 2-i & -1+8i \end{pmatrix} \rightsquigarrow \begin{array}{cc} & -(1+i)C1 \end{array}\begin{pmatrix} 1 & 1+i \\ -1+8i & 2-i \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ -(-1+8i)R1 \end{array}\begin{pmatrix} 1 & 0 \\ -1+8i & 11-8i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 11-8i \end{pmatrix}$$

$$\therefore \quad \mathbb{Z}[i]^2/A\cdot\mathbb{Z}[i]^2 \simeq \mathbb{Z}[i]/(11-8i)$$

To apply CRT, factor $11-8i$ into irreducibles in $\mathbb{Z}[i]$.

$$|11-8i|^2 = 11^2+8^2 = 185 = 5\cdot 37 \quad \text{(prime factorization)}$$

In $\mathbb{Z}[i]$ $\quad 5 = (1+2i)(1-2i) \qquad (5 \,\&\, 37 \equiv 1 \bmod 4)$

$$37 = (1+6i)(1-6i)$$

$$\frac{11-8i}{1+2i} = \frac{(11-8i)\cdot(1-2i)}{5} = \frac{-5-30i}{5} = -1-6i$$

$\therefore$ $1+2i \mid 11-8i$, & $11-8i = -(1+2i)(1+6i)$ prime factorization in $\mathbb{Z}[i]$.

$$\therefore \quad \mathbb{Z}[i]/(11-8i) \underset{CRT.}{\simeq} \mathbb{Z}[i]/(1+2i) \oplus \mathbb{Z}[i]/(1+6i)$$

5. a. $M/N = \mathbb{Z}^3/A\cdot\mathbb{Z}^3$,

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 5 & 7 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{array}{c} \\ -4R1 \\ -2R1 \end{array}\begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 7 \\ 2 & 1 & 5 \end{pmatrix} \sim \begin{array}{c} \\ \times -1 \\ \\ \end{array}\begin{array}{c} -2R1 \ \ -R1 \\ \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ \\ +R2 \end{array}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix}$$

$$\sim \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \\ +C2 \end{array} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad M/N \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}. \qquad N \simeq \mathbb{Z}^2$$

b. No, because if $M = L \oplus N$ then $M/N \simeq L$.

But $\mathrm{Tors}(M/N) \neq \{0\}$ and $L \subset M$ free $\Rightarrow \mathrm{Tors}(L) = 0$.

$\#$.

6. a Given R-module M s.t. $x\cdot m = 0 \ \forall x \in I, m \in M$, define R/I-module structure via $\bar{r}\cdot m := r\cdot m$.

(well defined, because $\bar{r} = \bar{r}' \Leftrightarrow r - r' \in I \Rightarrow r\cdot m = r'\cdot m$.)

Conversely, given R/I-module M, define R-module structure via $r\cdot m := \bar{r}\cdot m$.

Then $x \cdot m = 0 \quad \forall\ x \in I,\ m \in M$.

b) M fg module over $F[x]/(x^2)$

$\iff$ M fg module over $F[x]$ s.t. $x^2 \cdot m = 0\ \forall\ m \in M$

$\iff$ $M \simeq \left(F[x]/(x)\right)^{\oplus a} \oplus \left(F[x]/(x^2)\right)^{\oplus b}$, some $a, b \geq 0$.

by structure theorem for fg modules over a PID.

7. This follows from the uniqueness statement in the structure theorem for fg modules over a PID:–

writing

$$L \simeq R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_t^{\alpha_t}) \oplus R^s$$

$$M \simeq R/(q_1^{\beta_1}) \oplus \cdots \oplus R/(q_v^{\beta_v}) \oplus R^u$$

$p_1, \ldots, p_t$ irred

$\alpha_1, \ldots, \alpha_t \in \mathbb{N}$

$s \geq 0$.

$$N \simeq R/(r_1^{\gamma_1}) \oplus \cdots \oplus R/(r_x^{\gamma_x}) \oplus R^w$$

$L \oplus N \simeq M \oplus N \implies s+w = u+w$

& $p_1^{\alpha_1}, \ldots, p_t^{\alpha_t}, r_1^{\gamma_1}, \ldots, r_x^{\gamma_x}$

$= q_1^{\beta_1}, \ldots, q_v^{\beta_v}, r_1^{\gamma_1}, \ldots, r_x^{\gamma_x}$

up to multiplication by units & reordering

$\implies s = u$ & $p_1^{\alpha_1}, \ldots, p_t^{\alpha_t} = q_1^{\beta_1}, \ldots, q_v^{\beta_v}$ up to units & reordering

$\implies L \simeq M$. $\square$.

8. Each conjugacy class contains a unique matrix in rational canonical form (uniqueness statement for RCF)

A has order 4 $\iff$ minimal polynomial $m_A \mid x^4 - 1$, $m_A \nmid x^2 - 1$.

$x^4 - 1 = (x-1)(x+1)(x^2+1)$ irred factorization in $\mathbb{Q}[x]$.

So $m_A = (x^2+1),\ (x-1)(x^2+1),\ (x+1)(x^2+1)$, or $(x-1)(x+1)(x^2+1) = (x^4-1)$

Recall $F[x]$-module $M = F^n \circlearrowleft A$, $x \cdot v = Av$

$M \simeq F[x]/(d_1) \oplus \cdots \oplus F[x]/(d_r)$, $d_1 \mid d_2 \mid \cdots \mid d_r$, monic.

$$\Rightarrow P^{-1}AP = \begin{pmatrix} \boxed{C_{d_1}} & & 0 \\ & \ddots & \\ 0 & & \boxed{C_{d_r}} \end{pmatrix} \qquad C_f = \begin{pmatrix} 0 & & & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & -a_{m-1} \end{pmatrix}$$

RCF

for $f = x^m + a_{m-1}x^{m-1} + \cdots + a_0$

"companion matrix"

$m \times m$.

& $m_A = d_r$.

So, cases (recall $n=4$ here):

$d_1 = d_2 = (x^2+1)$

$d_1 = (x-1),\ d_2 = (x-1)(x^2+1)$

$d_1 = (x+1),\ d_2 = (x+1)(x^2+1)$

$d_1 = (x^4-1)$

A has order 5. $m_A \mid x^5 - 1$, $m_A \nmid x - 1$.

$$x^5 - 1 = (x-1)\underbrace{(x^4+x^3+x^2+x+1)}_{\text{irred.}}$$

$\therefore\ m_A = x^4+x^3+x^2+x+1$.

$$\Rightarrow d_1 = x^4+x^3+x^2+x+1 \text{ (single block)}, \quad P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$\square$.

9. $A^8 = 9I \Rightarrow m_A \mid x^8 - 9$

$$x^8 - 9 = (x^4+3)(x^4-3)$$

irred factorization (by Eisenstein) in $\mathbb{Q}[x]$

$\therefore$ invariant factors $d_1, \ldots, d_r$ of A are divisors of m_A & so of degree 4 or 8.

$\Rightarrow 4 \mid n \qquad (A \in GL_n(\mathbb{Q}), \ \sum_{i=1}^{r} \deg d_i = n)$

Example for $n=4$: $A = C_{x^4-3} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

10. Determine possible RCF's :-

$d_1 = d_2 = (x-\alpha) \qquad P^{-1}AP = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad \det A \neq 0 \Leftrightarrow \alpha \neq 0$

OR $d_1 = x^2 + bx + c \qquad P^{-1}AP = C_{d_1} = \begin{pmatrix} 0 & -c \\ 1 & -b \end{pmatrix}. \qquad \det A \neq 0 \Leftrightarrow c \neq 0$

$\therefore$ # conjugacy classes $= (p-1) + p\cdot(p-1) = (p+1)\cdot(p-1) = p^2-1.$

11. $\det(xI - J) = (x-\lambda)^m \Rightarrow \lambda$ is unique eigenvalue.

$$(J-\lambda I)\cdot e_i = \begin{cases} e_{i+1} & i<m \\ 0 & i=m. \end{cases}$$

$\Rightarrow \ker((J-\lambda I)^k) = \langle e_{m-k+1}, \ldots, e_m \rangle_F$ for $k \leq m$, dimension k

& $(J-\lambda I)^m = 0.$ $\square$.

12. a $\quad c_A = \prod_i (x-\lambda_i)^{m_{i1}+\cdots+m_{is_i}}$

$m_A = \prod_i (x-\lambda_i)^{m_{is_i}}$

$$d_{i,k} - d_{i,k-1} = \# \ \lambda_i\text{-blocks of size} \geqslant k.$$

$$\Rightarrow (d_{i,k} - d_{i,k-1}) - (d_{i,k+1} - d_{i,k}) = \# \ \lambda_i\text{-blocks of size } k$$

c. $c_A = \det \begin{pmatrix} x-4 & -1 & -1 \\ 10 & x+3 & 5 \\ -6 & -3 & x-5 \end{pmatrix}$

$\det(xI - A)$

$$= (x-4) \begin{vmatrix} x+3 & 5 \\ -3 & x-5 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 10 & 5 \\ -6 & x-5 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 10 & x+3 \\ -6 & -3 \end{vmatrix}$$

$$= (x-4)(x^2 - 2x - 15 + 15) + (10x - 50 + 30) - (6x + 18 - 30)$$

$$= (x-4)(x^2 - 2x) + 4x - 8$$

$$= (x-2)(x(x-4) + 4) = (x-2)^3.$$

$\therefore$ $x = 2$ is unique eigenvalue.

$$A - 2I = \begin{pmatrix} 2 & 1 & 1 \\ -10 & -5 & -5 \\ 6 & 3 & 3 \end{pmatrix} \neq 0.$$

$$(A-2I)^2 = \begin{pmatrix} 2 & 1 & 1 \\ -10 & -5 & -5 \\ 6 & 3 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -10 & -5 & -5 \\ 6 & 3 & 3 \end{pmatrix} = 0.$$

$$\therefore \quad P^{-1}AP = \begin{pmatrix} \boxed{\begin{matrix} 2 & 0 \\ 1 & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{2} \end{pmatrix} \quad \text{JNF.}$$

B.

$$0 \to R \xrightarrow[\alpha]{\binom{g}{-f}} R^2 \xrightarrow[\beta]{} I \to 0 \qquad \text{exact:-}$$

$$\binom{a}{b} \mapsto af + bg$$

β surj ✓ $\quad I = (f, g)$.

α inj ✓ $\quad R$ ID.

$\ker\beta = \operatorname{im}\alpha$: $\quad \ker\beta \supset \operatorname{im}\alpha$ ✓ $\quad \beta\alpha = 0.$ $\quad \ker\beta \subset \operatorname{im}\alpha$:-

$\begin{pmatrix} a \\ b \end{pmatrix} \in \ker\beta \iff af+bg=0$

$\Rightarrow f \mid b,\ g \mid a$ (gcd$(f,g)=1$, R UFD)

$b = cf$, $a = d\cdot g$,

then $(d+c)fg = 0$, $d=-c$.

$\begin{pmatrix} a \\ b \end{pmatrix} = -c\cdot\begin{pmatrix} g \\ -f \end{pmatrix} \in \operatorname{im}\alpha$. $\square$.

14.

$M = (2, 1+\delta) \subset R = \mathbb{Z}[\delta]$, $\delta = \sqrt{-5}$.

$$R^2 \overset{\varphi}{\twoheadrightarrow} M$$
$$e_1 \longmapsto 2$$
$$e_2 \longmapsto 1+\delta$$

$\ker\varphi = ?$ Examples: $2\cdot 3 = (1+\delta)(1-\delta) \Rightarrow \begin{pmatrix} 3 \\ -1+\delta \end{pmatrix} \in \ker\varphi$

Also, clearly $\begin{pmatrix} 1+\delta \\ -2 \end{pmatrix} \in \ker\varphi$.

<u>Claim</u>: $\ker\varphi = \left\langle \begin{pmatrix} 1+\delta \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1+\delta \end{pmatrix} \right\rangle_R$

i.e. $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in R^2$ $2\alpha + (1+\delta)\beta = 0$

$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} 1+\delta \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -1+\delta \end{pmatrix}$, some $\lambda, \mu \in R$.

<u>Proof</u>: Observe $R/(-2,-1+\delta) = \mathbb{Z}[\delta]/(-2,-1+\delta) = \mathbb{Z}[x]/(x^2+5, 2, x-1)$

$= \mathbb{Z}/2\mathbb{Z}[x]/(x^2+1, x+1) = \mathbb{Z}/2\mathbb{Z}[x]/(x+1) \cong \mathbb{Z}/2\mathbb{Z}$.

So, subtracting multiples of $\begin{pmatrix} 1+\delta \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1+\delta \end{pmatrix}$, WMA $\beta = 0$ or 1.

But $\beta = 1 \Rightarrow 2 \mid (1+\delta)$ #.

$\therefore \quad \beta = 0, \ \& \ \alpha = 0 \qquad \square.$

So, presentation
$$R^2 \xrightarrow{\begin{pmatrix} 1+\delta & 3 \\ -2 & -1+\delta \end{pmatrix}} R^2 \xrightarrow{\varphi} M \longrightarrow 0$$

15.

(a) γ surj. ✓ $\quad M = (x,y,z)$.

$\ker\gamma = \operatorname{im}\beta$: $\quad \underline{\ker\gamma \supset \operatorname{im}\beta}$: $\gamma\circ\beta = 0$ ✓

$\underline{\ker\gamma \subset \operatorname{im}\beta}$: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in R^3, \quad ax+by+cz = 0. \ \dagger$

$$\overset{?}{\Rightarrow} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + \mu \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} + \nu \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}$$

some $\lambda, \mu, \nu \in R = F[x,y,z]$.

Subtracting multiples of $\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \& \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}$, WMA $a \in F[x] \subset R$

Then $\dagger \Rightarrow a = 0 \Rightarrow \begin{pmatrix} b \\ c \end{pmatrix} = \nu \begin{pmatrix} z \\ -y \end{pmatrix}$, cf. Q13.

$\square$.

(b) Similarly to Q13, find $\ker\beta = \left\langle \begin{pmatrix} z \\ -y \\ x \end{pmatrix} \right\rangle_R$.

$\rightsquigarrow$ exact sequence
$$0 \longrightarrow R \xrightarrow[\begin{pmatrix} z \\ -y \\ x \end{pmatrix}]{\alpha} R^3 \xrightarrow[\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}]{\beta} R^3 \xrightarrow{\gamma} M \longrightarrow 0$$