Math 797W Algebraic geometry. Homework 5

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(1) Let $X = (a_0X_0^4 + a_1X_0^3X_1 + \cdots + a_NX_3^4) \subset \mathbb{P}^3$ be a quartic surface. Show that there is a homogeneous polynomial R such that X contains a line iff $R(a_0, \ldots, a_N) = 0$. [Hint: Adapt the proof that every cubic surface contains a line given in class.]

(2) Let $X = Z(f) \subset \mathbb{A}^n_{x_1,\ldots,x_n}$ where $f \in k[x_1, \ldots, x_n]$ is an irreducible polynomial. Recall that the singular locus $\operatorname{Sing} X \subset X$ (the locus where X is *not* smooth) is given by

$$\operatorname{Sing} X = Z\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \mathbb{A}^n.$$

Now let $Y = Z(F) \subset \mathbb{P}^n_{(X_0 : \cdots : X_n)}$ where $F \in k[X_0, \ldots, X_n]$ is an irreducible homogeneous polynomial. Assume $\operatorname{char}(k) = 0$.

(a) Show that $\operatorname{Sing} Y = Z(\frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n}) \subset \mathbb{P}^n$. [Hint: Reduce to the affine case by working in affine charts. You will also need the identity $\sum_{i=0}^n X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F$ where d is the degree of F.]

(b) Let d be a positive integer such that $\operatorname{char}(k)$ does not divide d. Show that the Fermat hypersurface

$$Y = Z(X_0^d + X_1^d + \cdots + X_n^d) \subset \mathbb{P}^n$$

is smooth.

(c) Show that the Klein quartic curve

$$Y = Z(X_0^3X_1 + X_1^3X_2 + X_2^3X_0) \subset \mathbb{P}^2$$

is smooth.

(d) More generally, show that the hypersurface

$$Y = Z(X_0^d X_1 + X_1^d X_2 + \cdots + X_n^d X_0) \subset \mathbb{P}^n$$

is smooth for $d, n \in \mathbb{Z}$, $d, n \geq 2$.

(3) Let $X = Z(Q) \subset \mathbb{P}^n$ where $Q \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial of degree 2 (a quadratic form). X is called a *quadric hypersurface*. Assume $\operatorname{char} k \neq 2$.

(a) By quoting a result from Algebra I (Math 611), show that for an appropriate choice of homogeneous coordinates $Y_0, \ldots, Y_n$ on $\mathbb{P}^n$, we have

$$Q = Y_0^2 + \cdots + Y_m^2$$

for some $m \leq n$ (here $m+1$ is the *rank* of Q).

(b) Show the following: If $m = 0$ then X is a hyperplane (with "multiplicity 2"), if $m = 1$ then X is a union of two distinct hyperplanes, and if $m > 1$ then X is irreducible.

(c) Suppose $m > 1$ (so X is a variety). Show that $\operatorname{Sing} X$ equals the linear subspace $Z(Y_0, \ldots, Y_m) \subset \mathbb{P}^n$. In particular X is smooth iff $m = n$, i.e., Q has maximal rank.

(d) Deduce from HW4 that for $n = 2$, 3, and 5, a smooth quadric hypersurface $X \subset \mathbb{P}^n$ is isomorphic to $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $G(2,4)$ respectively.

(4) (a) Let $X = Z(f) \subset \mathbb{A}^2_{x,y}$ where $f(x,y) = x^4 + x^2y - y^3 \in k[x,y]$. The polynomial f is irreducible and X has a unique singular point $P = (0,0)$. Construct an explicit resolution of singularities $p \colon \tilde{X} \to X$ by blowing up the ambient space $\mathbb{A}^2_{x,y}$.

(b) Repeat for $f(x,y) = y^2 + x^5$. [Note: In this case, more than one blowup is required.]

(5) (a) Let $X = Z(f) \subset \mathbb{A}^3_{x,y}$ where $f(x,y) = xy - z^2$. Construct a resolution of singularities $p \colon \tilde{X} \to X$ and describe the exceptional locus $F = \pi^{-1}(P)$, where $P \in X$ is the singular point.

(b) (Optional) Repeat for $f = xy - z^n$, where $n \in \mathbb{Z}$, $n \geq 2$. [Hint: Use induction on n (several blowups are required). The case $n = 3$ was described in class. In general the exceptional locus $F = p^{-1}(P)$ is a chain of $n-1$ copies of $\mathbb{P}^1$.]

[Remark: The singularity $P \in X$ is the *Du Val singularity* of type A_{n-1}. For $\operatorname{char}(k)$ coprime to n it is isomorphic to the quotient $\mathbb{A}^2/(\mathbb{Z}/n\mathbb{Z})$ where the action is given by $(u,v) \mapsto (\zeta u, \zeta^{-1} v)$ for ζ a primitive nth root of unity; here $x = u^n$, $y = v^n$, and $z = uv$ generate the ring of invariants $k[u,v]^{\mathbb{Z}/n\mathbb{Z}}$.]

(6) Let $X = Z(X_0X_1 - X_2X_3) \subset \mathbb{P}^3$.

(a) Let $P = (1:0:0:0) \in X$. Let $f\colon X \setminus \{P\} \to \mathbb{P}^2$ be the restriction of the morphism

$$\mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2, \quad (X_0 : X_1 : X_2 : X_3) \mapsto (X_1 : X_2 : X_3)$$

given by projection from P.

(b) Let $p\colon \tilde{X} \to X$ denote the blowup of $P \in X$ and write $F = p^{-1}(P)$. Let $\phi\colon \tilde{X} \setminus F \to X \setminus \{P\}$ be the isomorphism given by the restriction of p. Show that there is a morphism $g\colon \tilde{X} \to \mathbb{P}^2$ such that $g|_{\tilde{X}\setminus F} = f \circ \phi$. [Hint: It may help to identify $\tilde{X}$ with $\overline{\pi^{-1}(X \setminus \{P\})}$ in $\operatorname{Bl}_P \mathbb{P}^3$.]

(c) Show that the morphism g is birational, and identify the maximal open set $U \subset \tilde{X}$ such that g maps U isomorphically onto its image.