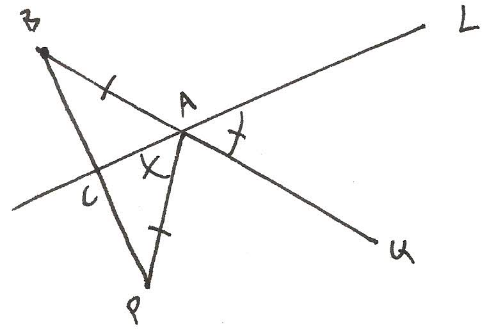
MATH 461 HW 6 SOLUTIONS

Monday 11/11/19

1.

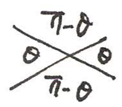


Let B be the perceived position of P for an observer at Q looking in the mirror. Let A be the point on the mirror L that light travelling from P to Q via the mirror hits the mirror.

Then $|AP| = |AB|$ and AP & AQ make equal angles with L.

We claim that $B = \text{Refl}_L(P)$, equivalently, letting C be the intersection point of BP with L, $|BC| = |CP|$ and $\angle ACB = \pi/2$.

Note that $\angle BAC =$ angle between L & AQ $= \angle PAC$

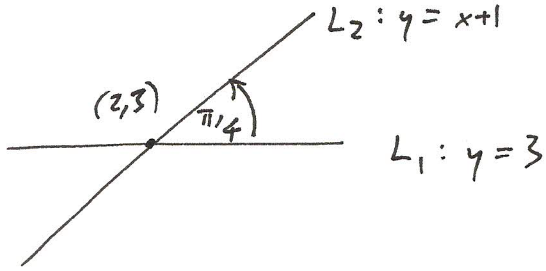


So $\triangle BAC \cong \triangle PAC$ (SAS)

$\Rightarrow |BC| = |CP|$, $\angle BCA = \angle PCA$

Also $\angle BCA + \angle PCA = \pi \Rightarrow \angle BCA = \pi/2$. $\square$

2 a.



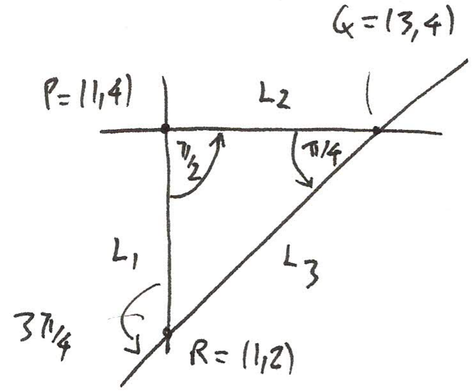
$L_1 \cap L_2 = \{(2,3)\}$

Angle from L_1 to L_2 $= \pi/4$ ccw

(because L_1 is horizontal & L_2 has slope 1)

$\Rightarrow$ $\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$ = rotation about $P=(2,3)$ through angle $2\cdot\pi/4 = \pi/2$ ccw. $\square$.

b.



$\mathrm{Rot}(Q, \pi/2) \circ \mathrm{Rot}(P, \pi)$

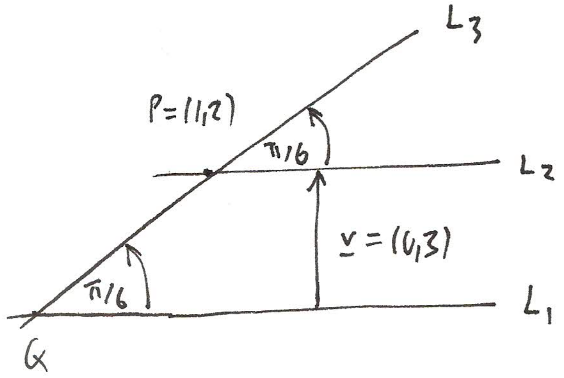
$= \left(\mathrm{Refl}_{L_3} \circ \cancel{\mathrm{Refl}_{L_2}}\right) \circ \left(\cancel{\mathrm{Refl}_{L_2}} \circ \mathrm{Refl}_{L_1}\right)$

$= \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_1}$

$= \mathrm{Rot}(R, 2\cdot 3\pi/4)$

$=$ Rotation about $R=(1,2)$ through $3\pi/2$ ccw. $\square$

c.

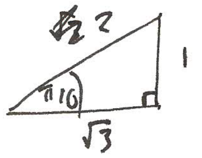


$\mathrm{Rot}(P, \pi/3) \circ \mathrm{Trans}_{(0,6)}$

$= \left(\mathrm{Refl}_{L_3} \circ \cancel{\mathrm{Refl}_{L_2}}\right) \circ \left(\cancel{\mathrm{Refl}_{L_2}} \circ \mathrm{Refl}_{L_1}\right)$

$= \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_1}$

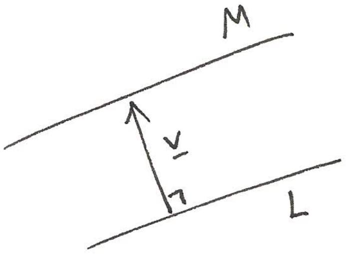
$= \mathrm{Rot}(Q, 2\cdot\pi/6)$ = Rotation about $Q = (1-3\sqrt{3}, -1)$ through $\pi/3$ ccw.



$\Rightarrow$ $Q =$

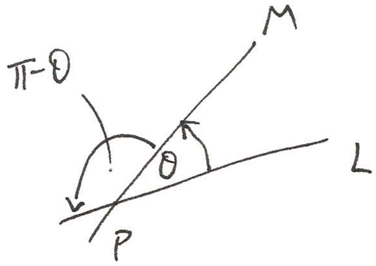
$(1-3\sqrt{3}, 2-3) = (1-3\sqrt{3}, -1)$

3. If L & M are parallel:



$$\mathrm{Refl}_M \circ \mathrm{Refl}_L = \mathrm{Trans}_{2\underline{v}}$$
$$\neq$$
$$\mathrm{Refl}_L \circ \mathrm{Refl}_M = \mathrm{Trans}_{2\cdot(-\underline{v})}$$

If L & M intersect at a point P:



$\mathrm{Refl}_M \circ \mathrm{Refl}_L = \mathrm{Rot}(P, 2\theta)$

= rotation about P through angle 2θ ccw

$\mathrm{Refl}_L \circ \mathrm{Refl}_M = \mathrm{Rot}(P, 2(\pi-\theta))$

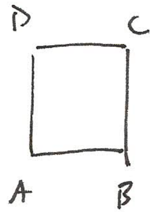
= rotation about P through angle $2(\pi-\theta)$ ccw

$\therefore$ $\mathrm{Refl}_M \circ \mathrm{Refl}_L = \mathrm{Refl}_L \circ \mathrm{Refl}_M \iff 2\theta = 2(\pi-\theta)$

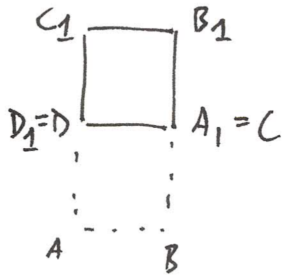
$\iff 4\theta = 2\pi$

$\iff \theta = \pi/2$ $\square$.

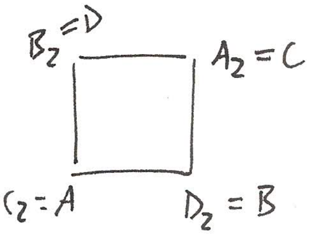
4.



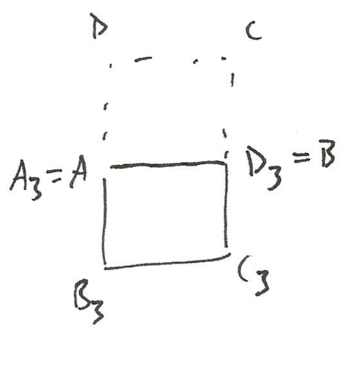
$\rightsquigarrow$ $\mathrm{Rot}(D, \pi/2)$



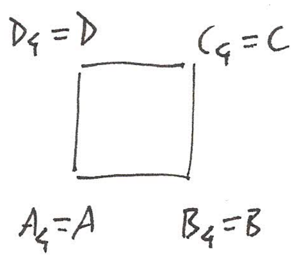
$\rightsquigarrow$ $\mathrm{Rot}(C, \pi/2)$



$\rightsquigarrow$ $\mathrm{Rot}(B, \pi/2)$



$\rightsquigarrow$ $\mathrm{Rot}(A, \pi/2)$



we see that the composition $T = \mathrm{Rot}(A, \pi/2) \circ \mathrm{Rot}(B, \pi/2) \circ \mathrm{Rot}(C, \pi/2) \circ \mathrm{Rot}(D, \pi/2)$

sends A to A, B to B, C to C, & D to D.

Recall that, given a triangle $\triangle ABC$ and a congruent triangle $\triangle A'B'C'$,

there is a unique isometry T such that $T(A) = A'$, $T(B) = B'$, & $T(C) = C'$.

So, applying this result in the case $A = A'$, $B = B'$ & $C = C'$, we see that

$T =$ identity. $\square$. (Alternatively, express the rotations as composites of two reflections, similar to Q2b.)

5\. A similar calculation to Q4 shows that the composition equals the identity.

6\. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (x+6, 4-y)$

$\mathrm{Fix}(T)$: $x = x+6$, $y = 4-y$ $\Rightarrow$ $\mathrm{Fix}(T) = \emptyset$

$\Rightarrow$ T is a glide reflection.

(T is not a translation by inspection)

$$T^2(x,y) = T(T(x,y)) = T(x+6, 4-y) = ((x+6)+6, 4-(4-y))$$
$$= (x+12, y) = (x,y) + (12,0)$$

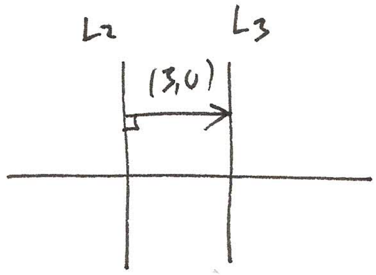
So $T = \mathrm{Trans}_{(6,0)} \circ \mathrm{Refl}_L$, where L is a line in the

direction of $\underline{v} = (6,0) = \frac{1}{2}(12,0)$ (compare HW5 Q4).

$\Rightarrow$ $\mathrm{Refl}_L = \mathrm{Trans}_{-(6,0)} \circ \cancel{\mathrm{Refl}_L}\ T$ $\Rightarrow$ $\mathrm{Refl}_L(x,y) = (x+6, 4-y) - (6,0)$
$= (x, 4-y)$

$\mathrm{Fix}(\mathrm{Refl}_L)$: $x = x$, $y = 4-y$, i.e. $\underline{y=2}$ is eq. of L.

Now



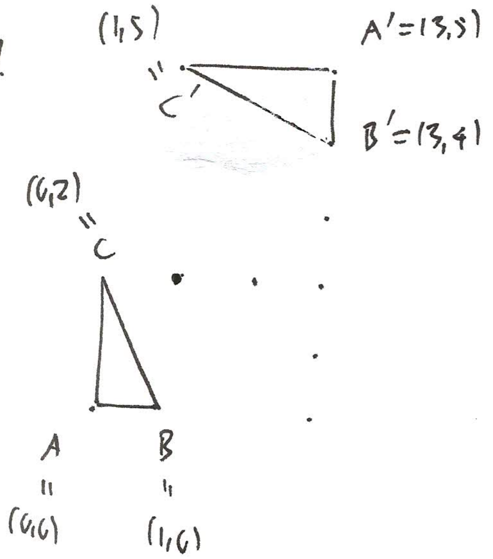
$L_1 = L : y = 2$

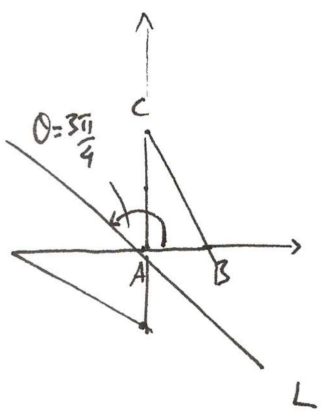
$$T = \underbrace{Refl_{L_3} \circ Refl_{L_2}}_{= \; Trans_{(6,0)}} \circ \underbrace{Refl_{L_1}}_{= \; Refl_L}$$

where $L_1 = L : y = 2$

& $L_2 = (x = 0)$
$L_3 = (x = 3)$ e.g.

7.





$L = (y = -x)$

$T = Trans_{(3,5)} \circ Refl_L$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} 3-y \\ 5-x \end{pmatrix}.$$

$Fix(T)$: $\begin{matrix} x = 3-y \\ y = 5-x \end{matrix} \Rightarrow \begin{matrix} x+y=3 \\ x+y=5 \end{matrix}$

$\Rightarrow Fix(T) = \emptyset \Rightarrow T$ glide reflection.
(T not a translation)

$T^2(x,y) = T(3-y, 5-x) = T(3-(5-x), 5-(3-y))$
$= (x-2, y+2) = (x,y) + (-2,2)$

$\Rightarrow T = Trans_{(-1,1)} \circ Refl_M$, some line M in the direction of $(-1,1)$

$\Rightarrow Refl_M = Trans_{(1,-1)} \circ T$ $\quad Refl_M(x,y) = (3-y, 5-x) + (1,-1) = (4-y, 4-x)$

$\Rightarrow$ $M = \mathrm{Fix}(\mathrm{Refl}_M)$: $x = 4-y,\ y = 4-x$, i.e. $M: y = 4-x$.

So T is reflection in $M: y = 4-x$ followed by translation by $(-1,+1)$ parallel to M. $\square$

8. $T: \mathbb{R}^2 \to \mathbb{R}^2$ isometry

$\Rightarrow T(\underline{x}) = A\underline{x} + \underline{b}$ where A is an orthogonal matrix

T is direct $\Leftrightarrow$ $\det A = +1$ $\Leftrightarrow$ $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, some $0 \leq \theta < 2\pi$.

$\Leftrightarrow$ T is a translation (if $A = \underset{\text{identity matrix}}{I}$, equivalently $\theta = 0$)

or T is a rotation ccw through angle θ about some point P, $0 < \theta < 2\pi$.

(note that rotation ccw thru angle θ about the origin is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

& rotation ccw thru angle θ about (a,b) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right) + \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{same matrix!}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$

for some vector $\begin{pmatrix} c \\ d \end{pmatrix}$).

Now $\mathrm{Rot}(G, \phi) \circ \mathrm{Rot}(P, \theta) = ?$

Write $\mathrm{Rot}(P,\theta)(\underline{x}) = A\underline{x} + \underline{b}$ $\quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\mathrm{Rot}(G,\phi)(\underline{x}) = C\underline{x} + \underline{d}$ $\quad C = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

Then $\mathrm{Rot}(G,\phi) \circ \mathrm{Rot}(P,\theta)(\underline{x}) = C(A\underline{x} + \underline{b}) + \underline{d} = (CA)\underline{x} + (C\underline{b} + \underline{d})$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \cdot \underline{x} + (C\underline{b} + \underline{d})$$

$= \begin{cases} \mathrm{Rot}(R, \theta+\phi)(\underline{x}) & \text{for some point } R \text{ if } \theta+\phi \text{ is not a multiple of } 2\pi \\ \text{A Translation or the identity} & \text{if } \theta+\phi \text{ is a multiple of } 2\pi. \end{cases}$ $\square$

because $\mathrm{Rot}(0,\phi) \circ \mathrm{Rot}(0,\theta) = \mathrm{Rot}(0, \theta+\phi)$ (alternatively, use addition formulae for sine & cosine)