11/4/19

HW6 available. Due Friday at start of class.
Office hours this week: Tu 4-5pm, Th 5-6, LGRT 1235H

Last Time • Every isometry of $\mathbb{R}^2$ is one of identity, translation, rotation, reflection, glide reflection.
Proof uses "3 reflections theorem".

Today • spherical geometry

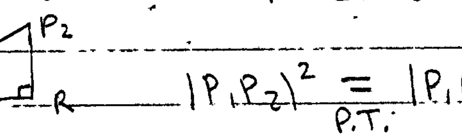
Spherical Geometry

Let S^2 denote the sphere in $\mathbb{R}^3$ with center the origin and radius 1.

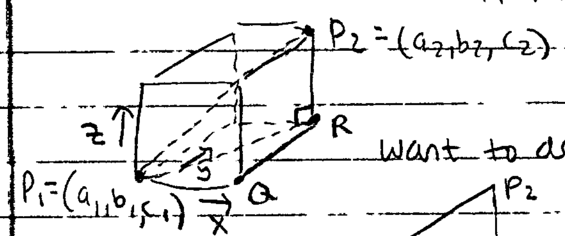
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2+z^2} = 1\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$$

(Recall for points $(a_1,b_1,c_1) = P_1$, $(a_2,b_2,c_2) = P_2$ in $\mathbb{R}^3$,
$|P_1P_2| = \sqrt{(a_1-a_2)^2+(b_1-b_2)^2+(c_1-c_2)^2}$)

↳ Why? In plane

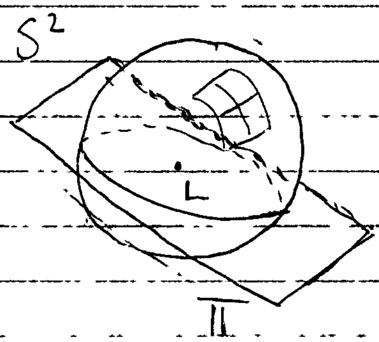


$\rightsquigarrow$ $|P_1P_2| = \sqrt{(a_1-a_2)^2+(b_1-b_2)^2}$
Pyth. Th.



want to determine $|P_1P_2|$ (diagonal

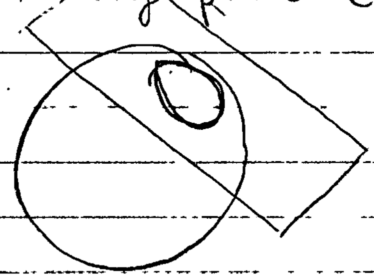
$$|P_1P_2|^2 \underset{P.T.}{=} |P_1R|^2 + |RP_2|^2$$
$$= |P_1R|^2 + (c_1-c_2)^2$$
$$\underset{P.T.}{=} |P_1Q|^2 + |QR|^2 + (c_1-c_2)^2$$
$$= (a_1-a_2)^2 + (b_1-b_2)^2 + (c_1-c_2)^2 \quad \blacksquare$$



A $\boxed{\text{spherical line}}$ or $\boxed{\text{great circle}}$ on S^2 is $L = \Pi \cap S^2 \rightarrow$ the intersection of S^2 with a plane $\Pi \subset \mathbb{R}^3$ with passes through the origin.

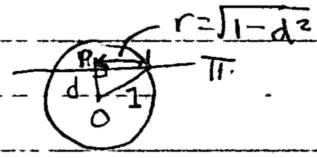
<u>Remark</u> L is a circle in the plane Π with center the origin and radius 1. (unit sphere with center the origin)

<u>Another Remark</u> Could also consider $\Pi \cup S^2$ where $\Pi \subset \mathbb{R}^3$ is any plane (not necessarily through origin).



This gives a circle in the plane Π,
center = ?? radius = ?? ≤ 1
$= P$ $= \sqrt{1-d^2}$ equal iff $(0,0,0) \in \Pi$

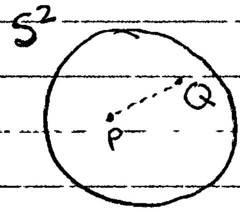
Cross section



<u>Fact</u> (to be proved later) The shortest path from P to Q along the surface of the sphere S^2, is given by a segment of a spherical line.

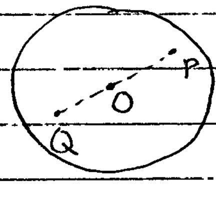
<u>Q</u>: Given two points P & Q, (1) is there a spherical line L passing through P & Q, and (2) if so, is it uniquely determined?

(1) YES
(2) NO



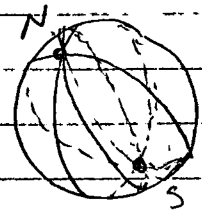
(1) Need a plane Π that contains P, Q, and the origin.
(2) If P, Q, & O do not lie on a line, then Π is uniquely determined.

Note, O, P, Q lie on a line $\Leftrightarrow$ P & Q are $\boxed{\text{antipodal points}}$
(i.e. $P = (a,b,c)$, $Q = (-a,-b,-c)$ or $\vec{OQ} = -\vec{OP}$)



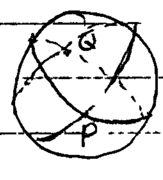
Recap: There's a spherical line through any two points $P, Q \in S^2$. It's unique unless P & Q are antipodal.

EX

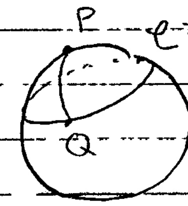


line of longitude on earth are spherical lines connecting the north & south poles (antipodal points)
(lines of latitude are not great circles)
↳ but Equator is a great circle)

Note also, If L_1 & L_2 are spherical lines then $L_1 \cap L_2 = \{P, Q\}$, pair of antipodal points.

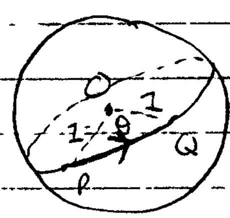


Aside:



$\ell = \Pi \cap S^2$ "spherical circle"
$\Pi \not\ni O$.

Spherical distance → Compute



Define the spherical distance $d_{S^2}(P,Q)$ to be the length of the shortest path from P to Q.
(which is the shorter segment of the great circle connecting P & Q)

Formula for $d_{S^2}(P,Q)$?? (given, $P = (a,b,c)$ $Q = (d,e,f)$)

Length of arc of $\underbrace{\text{great circle}}$ from P to Q.
radius $r = 1$
circumference $2\pi r = 2\pi$ →

So $d_{S^2}(P,Q) = 2\pi r \cdot \theta/2\pi$

$= \theta \cdot r$

$= \theta$

$\boxed{d_{S^2}(P,Q) = \theta}$ $\quad 0 \leq \theta \leq \pi$

<u>Ex</u> $P = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$ $\quad Q = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$

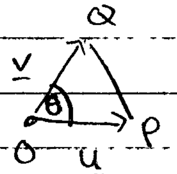
$d_{S^2}(P,Q) = ?$

Recall dot product of vectors $\underline{u}, \underline{v}$ in $\mathbb{R}^3$:

$u \cdot v = (a,b,c)\cdot(d,e,f) = ad + be + cf$ (length of vector $\underline{u}$)

(MATH 233) $= \|\underline{u}\| \cdot \|\underline{v}\| \cdot \cos\theta$ where $\|\underline{u}\| = \|(a,b,c)\|$

$= \sqrt{a^2+b^2+c^2}$



Cosine rule: $|PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP||OQ|\cos\theta$

$\|\underline{v} - \underline{u}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\| \cdot \|\underline{v}\| \cos\theta$

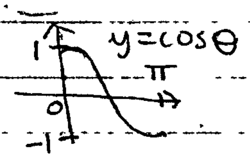
$\|$

Recall: $\|x\|^2 = \underline{x} \cdot \underline{x}$

$(\underline{v} - \underline{u})(\underline{v} - \underline{u}) = \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{v} - 2\underline{u} \cdot \underline{v}$

$\Rightarrow \underline{u} \cdot \underline{v} = \|\underline{u}\| \cdot \|\underline{v}\| \cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \cdot \|\underline{v}\|}\right)$



$0 \leq \theta \leq \pi$

Our exercise: $\theta = \cos^{-1}(\vec{OP} \cdot \vec{OQ}) = \cos^{-1}\left(\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}\right)$

$= \cos^{-1}\left(\frac{1}{9}(2+2+4)\right) = \cos^{-1}(8/9)$. ▨

note: $\|\vec{OP}\| = \|\vec{OQ}\| = 1$ because $P, Q \in S^2$!