Math 461 Homework 5

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Reading: Stillwell, Sections 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6.

Justify your answers carefully. Complete proofs are expected (as in MATH 300).

(1) Consider the circle with center the origin and radius 1 and the circle with center $(1, 2)$ and radius 2.

  (a) Find the intersection points of the two circles.

  (b) Eliminating x^2 and y^2 from the equations of the two circles gives a linear equation in x and y. What is the geometric meaning of this equation?

(2) Find the center and radius of the circle passing through the points $A = (0, 0)$, $B = (1, 2)$, and $C = (-1, 3)$.

(3) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. We define the *fixed locus* of T to be the set

$$\{(x, y) \in \mathbb{R}^2 \mid T(x, y) = (x, y)\}.$$

Determine the fixed locus of T in each of the following cases: (a) the identity, (b) a translation, (c) a rotation, (d) a reflection, (e) a glide reflection.

(4) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a glide reflection. Show that T^2 is a translation. (Here T^2 denotes the composition $T \circ T$.)

(5) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Suppose that $T^n = \mathrm{id}$ for some positive integer n. (Here T^n denotes the composition $T \circ T \circ \cdots \circ T$ where there are n copies of T.) What are the possibilities for T? Justify your answer carefully.

(6) Compute the formula $T(x, y) = (a(x, y), b(x, y))$ for the isometry T in each of the following cases.

   (a) T is rotation about the point $P = (1, 2)$ through angle $\pi/2$ counterclockwise.

   (b) T is reflection in the line $y = 2$.

   (c) T is a glide reflection given by reflection in the line $y = -x$ followed by translation by $(2, -2)$.

(7) In each of the following cases, describe the given isometry T as a translation, rotation, reflection, or glide reflection. For a translation, give the translation vector. For a rotation give the center, angle, and sense (counterclockwise or clockwise) of rotation. For a reflection give the line of reflection. For a glide reflection give the line of reflection and the translation vector.

   (a) $T(x, y) = (-x, -y)$.

   (b) $T(x, y) = \frac{1}{5}(4x + 3y + 2, 3x - 4y - 6)$.

   (c) $T(x, y) = \frac{1}{5}(3x - 4y + 8, 4x + 3y + 4)$.

   (d) $T(x, y) = (y + 4, x + 8)$.

   [Hint: One possible approach is as follows: To determine the type of T, compute the fixed locus and use Q3. For a rotation, find the angle using the formula $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for the rotation matrix. For a glide reflection, find the translation vector using Q4, then use this to determine the line of reflection.]