Math 300.2 Homework 3

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Reading: Sundstrom, Sections 3.1 and 3.2.

Justify your answers carefully.

(1) Prove the following statements

(a) If n is odd then $n^2 + 4n + 7$ is even.

(b) If n^3 is even then n is even.

[Hint for (b): What is the contrapositive?]

(2) Recall that for $a, b \in \mathbb{Z}$, we say that a *divides* b and write $a \mid b$ if $b = qa$ for some integer q. Prove the following statements.

(a) For all $a, b, c \in \mathbb{Z}$, if $a^2 \mid b$ and $b^3 \mid c$ then $a^6 \mid c$.

(b) For all $a, b, c, d \in \mathbb{Z}$, if $a \mid b$, $a \mid c$, and $a \mid d$ then $a^2 \mid bc - d^2$.

(c) For all $a, b, c, x, y \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid bx + cy$.

(3) Prove or give a counterexample for each of the following statements.

(a) For all $x \in \mathbb{R}$, $x^2 \geq x$.

(b) For all $n \in \mathbb{N}$, $2n^2 + 5$ is prime.

(c) For all $n \in \mathbb{N}$, there exist $x, y \in \mathbb{Z}$ such that $n = x^2 + 2y^2$.

(4) Prove the following statement: For all $a, b \in \mathbb{Z}$ and non-zero $c \in \mathbb{Z}$, $ac \mid bc$ if and only if $a \mid b$.

[Reminder: To prove a biconditional statement "P if and only if Q" (also written $P \iff Q$) we must show $P \Rightarrow Q$ *and* $Q \Rightarrow P$.]

(5) Prove the following statement: For all $n \in \mathbb{Z}$, if n is odd then there exist integers a and b such that $n = a^2 - b^2$.

[Hint: What is the "difference of two squares" identity? How can we write any integer as the product of two integers?]

[Bonus problem (optional): Which integers can be expressed as a difference of two squares?]

(6) Prove the following statement: For all $n \in \mathbb{N}$, the integers

$$n! + 2, n! + 3, \ldots, n! + n$$

are *not* prime.

[Hint: Recall that $n! = n(n-1)(n-2)\cdots 2 \cdot 1$. The symbol $n!$ is pronounced "n factorial".]

[Remark: This shows that there are arbitrarily long gaps in the sequence of prime numbers.]

(7) Prove the following statements.

(a) For all $a, b \in \mathbb{R}$, $a^2 + b^2 \geq 2a + 4b - 5$.

(b) For all $a, b, c, d \in \mathbb{R}$, $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$.

(c) For all $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 \geq ab + bc + ca$.

[Hint: If $x \in \mathbb{R}$ what can we say about x^2 ? What is the expansion of the product $(x - y)^2$?]

(8) Prove the following statement: For all $n \in \mathbb{Z}$, $n^3 - n$ is a multiple of 3.