Math 611 Homework 6

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Reading: Dummit and Foote, 10.1-10.3 and 12.1.

All rings are commutative with 1. We say M is a *free R-module of rank n* if $M \simeq R^n$. (We only consider free modules of finite rank.) Justify your answers carefully.

(1) Let R be a ring and M an R-module. We say M is *cyclic* if it is generated as an R-module by a single element $m \in M$. Prove that if M is cyclic then M is isomorphic to R/I for some ideal $I \subset R$.

(2) Let R be a ring, M an R-module, and $M_1, M_2 \subset M$ be submodules. Then $M_1 \cap M_2$ and

$$M_1 + M_2 := \{m_1 + m_2 \mid m_1 \in M_1 \text{ and } m_2 \in M_2\}$$

are submodules of M. (You don't need to prove this but convince yourself that it's true.) Define a homomorphism

$$\theta \colon M_1 \cap M_2 \to M_1 \oplus M_2$$

such that

$$M_1 \oplus M_2/\theta(M_1 \cap M_2) \simeq M_1 + M_2.$$

Justify your answer carefully.

(3) Let $R = F[x]$ be the ring of polynomials in the variable x with coefficients in a field F. As we described in class, there is a bijective correspondence between R-modules M and pairs (V, T) consisting of a vector space V over F and a linear transformation $T \colon V \to V$ given by $V = M$ (with the induced structure of a vector space over F) and

$T(v) = x \cdot v$. Now let M be the R-module $R/(x^n)$. Describe a basis $\mathcal{B}$ of $V = M$ (as a vector space over F) and compute the matrix of T with respect to this basis.

(4) Let R be an integral domain and $I \subset R$ an ideal. Prove that I is a free R-module iff I is principal.

(5) Let R be a ring and $M = R^n$ a free R-module. For each of the following statements, give a proof or a counterexample.

(a) If $S = \{m_1, \ldots, m_k\}$ generates M as an R-module then some subset of S is a basis of M.

(b) If $S = \{m_1, \ldots, m_k\} \subset M$ is independent (that is, $\sum r_i m_i = 0 \Rightarrow r_i = 0$ for all i) then S can be extended to a basis of M.

(c) If $N \subset M$ is a submodule of M then N is a free R-module.

(6) Let R be a PID. Let $\varphi \colon R^n \to R^m$ be the homomorphism of free R-modules given by a matrix $A \in R^{m \times n}$.

(a) Define the *rank* of A to be the rank of A regarded as a matrix with entries in the fraction field $F = \operatorname{ff} R$ of R. Show that φ is injective iff the rank of A equals n.

(b) Show that φ is surjective iff the gcd of the determinants of the $m \times m$ minors of A equals 1.

[Hint: (a) Consider the associated linear map $F^n \to F^m$ and use the basic results of linear algebra over F. (b) Use change of basis of the domain and codomain replace the matrix A by a matrix $B = PAQ$ of a simple form, as in the proof of the structure theorem for finitely generated modules over a PID. (You should explain why the condition in (b) is preserved under change of basis).]

(7) For each of the following abelian groups, determine an isomorphic direct sum of cyclic groups.

(a) $\mathbb{Z}^3/\langle (3,2,8)^T, (2,0,4)^T \rangle$.

(b) $\mathbb{Z}^3/\langle (7,5,1)^T, (3,0,3)^T, (13,11,2)^T \rangle$.

(8) Prove that $(\mathbb{Q}, +)$ is not a finitely generated abelian group.

(9) Let $R = \mathbb{C}[x]$ and let M be the R-module

$$M = R^2/\langle (t-2, t^2-3t+2)^T, (t^2-3t+1, t^3-3t^2+3t-1)^T \rangle.$$

Determine a direct sum of cyclic R-modules which is isomorphic to M. Use the Chinese remainder theorem to further decompose M if possible.

(10) Let $R = \mathbb{Z}[i]$ and let M be the R-module

$$M = R^2/\langle (1+i, 2-i)^T, (3, 5i)^T \rangle.$$

Determine a direct sum of cyclic R-modules which is isomorphic to M. Use the Chinese remainder theorem to further decompose M if possible.

(11) Let $\delta = \sqrt{-5}$, $R = \mathbb{Z}[\delta]$ and $M = (2, 1+\delta) \subset R$. Determine a presentation for the R-module M, that is, a matrix $A \in R^{m \times n}$ for some m, n such that $M \simeq R^m / A \cdot R^n$. [Warning: R is not a UFD].

(12) (a) Let R be a ring and $I \subset R$ an ideal. Describe a bijective correspondence between R/I-modules and R-modules M such that $x \cdot m = 0$ for all $x \in I$ and $m \in M$. (You do not need to write a careful justification.)

(b) Let F be a field. Describe the classification of finitely generated modules over the ring $F[x]/(x^2)$.

(13) Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers.

(a) Describe a bijective correspondence between R-modules and pairs (A, φ) consisting of an abelian group A and a homomorphism $\varphi\colon A \to A$ such that $\varphi \circ \varphi(x) = -x$ for all $x \in A$.

(b) For which primes p can the abelian group $\mathbb{Z}/p\mathbb{Z}$ be given the structure of a $\mathbb{Z}[i]$ module? What about $(\mathbb{Z}/p\mathbb{Z})^2$?