11/15/19

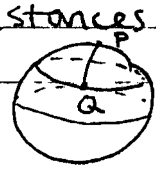
Stereographic Projection

Recall $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$
sphere centered at origin w/ radius 1 in $\mathbb{R}^3$

Stereographic Projection (Ptolemy, 100-170AD)
$\rightarrow$ a way to map portions of S^2 in the plane.

Note Any map of a portion of S^2 in plane cannot preserve distances.

Why?



"spherical circle"
↳ circle drawn on a sphere
$C(P,r) = \{Q \in S^2 \mid d_{S^2}(P,Q) = r\}$
↳ has circumference $2\pi \cdot \sin(r)$ (HW 7)

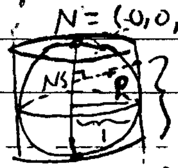
⇝ draw circle of same radius on the plane
↳ circumference is $2\pi r$

$* \ 2\pi\sin(r) < 2\pi r$!!!

Two projections

1. Stereographic projections: preserves angles between curves (but not distances or areas)

2. The Gall-Peters projections: preserves areas (but not angles or distances)

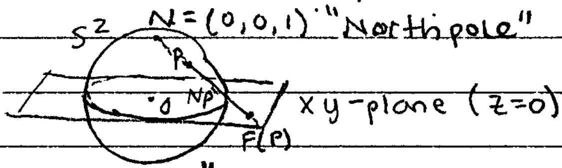


remove points N & S

$G: S^2 \setminus \{N,S\} \rightarrow [0,2\pi) \times (-1,1)$
bijection (no holes)

↳ map points on sphere to rectangle
project P, in a line perp to NS, to rectangle

Geometric Description of Stereographic Projection



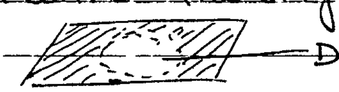
"S^2 with the single point N removed"

$F: S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$

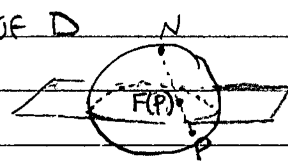
$F(P)$ = intersection of line NP with plane $z=0$ in $\mathbb{R}^3$.

Notes 1. As P approaches $N=(0,0,1)$, the distance of $F(P)$ from the origin tends to ~~∞~~ infinity

2. * The upper hemisphere maps to the compliment $\mathbb{R}^2 \setminus D$ of the disk $D = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 \leq 1\}$ with center the origin & radius 1.



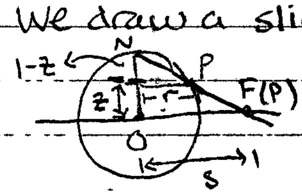
* The lower hemisphere maps to the interior of D



* The equator maps to the circle $C = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2=1\}$ (the boundary of D). These points "don't move" in $\mathbb{R}^3$ under stereographic projection.

Algebraic formula for stereographic projection

We draw a slice in the plane containing pts. O, N, P and F(P).



$P = (x,y,z)$
$r = \sqrt{x^2+y^2}$

Similar triangles $\Rightarrow \frac{s}{r} = \frac{1}{1-z}$ so $s = \frac{r}{1-z} = \frac{\sqrt{x^2+y^2}}{1-z}$

$s = \frac{1}{1-z} \cdot r$

Then $(u,v) = F(x,y,z) = \frac{1}{1-z}(x,y)$

(Reason: vector (u,v) has length $\frac{r}{1-z} = \frac{1}{1-z}\sqrt{x^2+y^2}$ and is in the same direction as (x,y).)

Quick check of formula:

As $P \to N$, $F(P) \to \infty$ in the plane

$$\underbrace{\|(u,v)\|}_{\text{length of }(u,v)} = \frac{r}{1-z} = \frac{\sqrt{x^2+y^2}}{1-z}$$

$$= \frac{\sqrt{1-z^2}}{1-z} \quad (\text{because } x^2+y^2+z^2=1)$$

$$= \sqrt{\frac{(1-z^2)}{(1-z)^2}}$$

$$= \sqrt{\frac{(1-z)(1+z)}{(1-z)^2}} = \sqrt{\frac{1+z}{1-z}} \to \infty \text{ as } z \to 1$$

(from below)

$F: S^2 \setminus \{N\} \to \mathbb{R}^2$ is a bijection, so it has an inverse $F^{-1}: \mathbb{R}^2 \to S^2 \setminus \{N\}$

To find a formula for F^{-1}, use $(u,v) = F(x,y,z) = \frac{1}{1-z}(x,y)$:

$$u^2+v^2 = \|(u,v)\|^2$$

$$= \frac{1+z}{1-z} \Rightarrow (1-z)(u^2+v^2) = 1+z$$

$$\Rightarrow (u^2+v^2-1) = (u^2+v^2+1)z$$

$$\text{so } z = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

Now $(u,v) = \frac{1}{1-z}(x,y)$, so $(x,y) = (1-z)(u,v)$

$$= \left(1 - \frac{(u^2+v^2-1)}{(u^2+v^2+1)}\right)(u,v)$$

$$= \frac{2}{u^2+v^2+1}(u,v)$$

Then $(x,y,z) = F^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$

i.e. $(x,y,z) = \boxed{F^{-1}(u,v) = \frac{1}{u^2+v^2+1}\left(2u, 2v, u^2+v^2-1\right)}$