Sunday 10/11/15 Math 421. Midterm 1 review solutions.

1. $z = x+iy$. $\bar{z} = x-iy$

$f: \mathbb{C} \to \mathbb{C}$ $f(z) = \bar{z}$ is reflection in the x-axis.

$z = x+iy$, $w = u+iv$

$\bar{z}\cdot\bar{w} = (x-iy)\cdot(u-iv) = (xu-yv) - i(xv+yu)$

$zw = (x+iy)\cdot(u+iv) = (xu-yv) + i(xv+yu)$

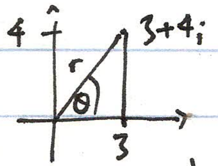
$\Rightarrow \overline{zw} = \bar{z}\cdot\bar{w}$.

2. $f: \mathbb{C} \to \mathbb{C}$, $f(z) = (3+4i)\cdot z$.

$3+4i = re^{i\theta} = r(\cos\theta + i\sin\theta)$

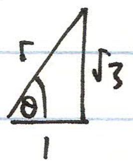
$r = \sqrt{3^2+4^2} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$



$\Rightarrow$ f is ccw rotation through angle ~~by~~ $\theta = \tan^{-1}(4/3)$ about the origin, followed by ~~and~~ scaling by factor $r=5$.

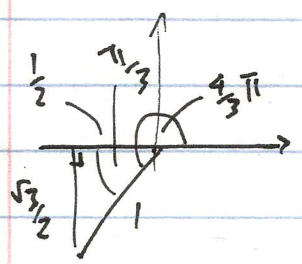
3. $(1+\sqrt{3}i) = re^{i\theta}$ $r = \sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$



~~$\theta = \tan^{-1}(1/\sqrt{3}) =$~~

$\theta = \tan^{-1}(\sqrt{3}/1) = \pi/3$.

$\Rightarrow (1+\sqrt{3}i)^{100} = 2^{100} e^{i\,100\cdot\pi/3} = 2^{100}\cdot e^{i 33\frac{1}{3}\pi}$



$= 2^{100}\cdot e^{i 1\frac{1}{3}\pi} = 2^{100}\cdot\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -2^{99}(1+\sqrt{3}i)$.

4. The range of f is $\mathbb{C}$ by the fundamental theorem of algebra: if $g(z)$ is a polynomial of degree $n>0$, then the equation $g(z)=0$ has a solution in $\mathbb{C}$.
Now apply this theorem to the polynomial $g(z)=f(z)-b$ for $b\in\mathbb{C}$ to see that the equation $f(z)=b$ has a solution (for any $b\in\mathbb{C}$), i.e., the range of f is $\mathbb{C}$.

The number of solutions of $f(z)=b$ equals the number of solutions of $g(z)=0$ (with $g(z)=f(z)-b$ as above).
Since $g(z)$ is a polynomial of degree n, there are at most n solutions, and there is at least 1 solution by F.T.A. (see above).
So the number of solutions is $1,2,3,\ldots,n-1$, or n.
In fact, if we count solutions "with multiplicities", then there are always exactly n solutions.

5. $f:\mathbb{C}\to\mathbb{C}$, $f(z)=e^z$.

$$f(z)=e^z=e^{x+iy}=e^x\cdot(\cos y+i\sin y)$$

So, writing $w=f(z)$, $w=se^{i\phi}$, we have

$$s=e^x,\quad \phi=y \quad \text{(up to multiples of } 2\pi). \quad \dagger$$

We see that the only restriction on s,ϕ is $s>0$, equivalently, $w\neq 0$.
So, the range of f equals $\mathbb{C}\setminus\{0\}$.

6. Using $\dagger$ above, we see that for $z_1,z_2\in\mathbb{C}$ we have

$$e^{z_1}=e^{z_2} \iff z_2=z_1+(2\pi i)\cdot k,\ \text{some integer } k.$$

7. a) $z^2+5z+7=0 \iff z = \dfrac{-5 \pm \sqrt{5^2-4.7}}{2} = \dfrac{-5\pm\sqrt{-3}}{2} = \dfrac{-5\pm\sqrt{3}i}{2}.$

b) $z^3+4z=0 \iff z\cdot(z^2+4)=0$

$\iff z=0$ OR $z^2+4=0$

$\iff z=0$ OR $z=\pm 2i$

c) $z^3-5z^2+4z+10=0$

By trial & error, find that $z=-1$ is a solution:-

$(-1)^3-5(-1)^2+4(-1)+10 = -1-5-4+10=0$ ✓

Factor: $(z+1)\cdot(z^2-6z+10)=0$

$\iff z=-1$ OR $z^2-6z+10=0$

$\iff z=-1$ OR $z = \dfrac{6\pm\sqrt{36-40}}{2} = \dfrac{6\pm\sqrt{-4}}{2} = \dfrac{6\pm 2i}{2} = 3\pm i.$

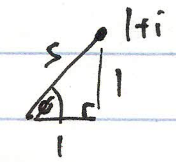
d) $z+{}^{5}/_{z}=2 \iff z^2+5=2z$

$\iff z^2-2z+5=0$

$\iff z = \dfrac{2\pm\sqrt{4-20}}{2} = \dfrac{2\pm\sqrt{-16}}{2} = \dfrac{2\pm 4i}{2} = 1\pm 2i$

e) $e^z = 1+i$

$1+i = se^{i\phi}$ $\quad s=\sqrt{1^2+1^2}=\sqrt{2}$, $\phi = \tan^{-1}(1/1) = \pi/4$



$e^z = e^{x+iy} = e^x\cdot e^{iy} \iff e^x = s,\ y=\phi+(2\pi)\cdot k,$ k integer

$= se^{i\phi}$

$\iff x=\log s,\ y=\phi+(2\pi)\cdot k,$ k integer.

So $e^z=1+i \iff x=\log(\sqrt{2}),\ y=\pi/4+(2\pi)\cdot k,$ k integer

$\iff z=\log(\sqrt{2})+i\cdot(\pi/4+(2\pi)\cdot k),$ k integer

Alternatively,

$$e^z = 1+i \iff z = \log(1+i) = \log(\sqrt{2}\cdot e^{i\pi/4}) \quad \left(\begin{array}{c}\text{multivalued} \\ \text{complex logarithm}\end{array}\right)$$

$$= \log|\sqrt{2}| + i\left(\tfrac{\pi}{4} + (2\pi)\cdot k\right), \quad k \text{ integer.}$$

f) $\sin z = i$

Recall $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

So $\sin z = i \iff \dfrac{e^{iz} - e^{-iz}}{2i} = i \iff e^{iz} - e^{-iz} = -2$

$$\iff e^{2iz} - 1 = -2e^{iz} \iff e^{2iz} + 2e^{iz} - 1 = 0$$

This is a quadratic equation for e^{iz}. $(e^{iz})^2 + 2e^{iz} - 1 = 0$.

Solving, $e^{iz} = \dfrac{-2 \pm \sqrt{2^2+4}}{2} = -1 \pm \sqrt{2}$

$$\iff iz = \log(-1 \pm \sqrt{2}) \qquad \cancel{s = \sqrt{(-1)^2 + (\sqrt{2})^2}}$$

$$= \begin{cases} \log(\sqrt{2}-1) \\ \log(-1-\sqrt{2}) = \log\left((\sqrt{2}+1)\cdot e^{i\pi}\right) \end{cases} \quad \left(\begin{array}{c}\text{multivalued complex} \\ \text{logarithm}\end{array}\right)$$

(real logarithm)

$$= \begin{cases} \log(\sqrt{2}-1) + 2\pi i\cdot k \\ \log(\sqrt{2}+1) + i(\pi + 2\pi\cdot k) \end{cases} \quad k \text{ integer}$$

$$\iff z = \begin{cases} 2\pi k - i\log(\sqrt{2}-1) \\ (2k+1)\pi - i\log(\sqrt{2}+1) \end{cases} \quad k \text{ integer}$$

g) $z^2 + (2+2i)z + i = 0$.

$$\iff z = \frac{-(2+2i) \pm \sqrt{(2+2i)^2 - 4i}}{2} = \frac{-(2+2i) \pm \sqrt{4i}}{2} = -(1+i) \pm \sqrt{i}$$

Now compute $\sqrt{i}$.

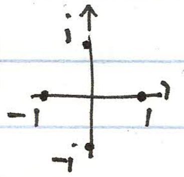
Write $i = re^{i\theta} = 1 \cdot e^{i\pi/2}$

$\Rightarrow \sqrt{i} = \pm\sqrt{1} \cdot e^{i\frac{1}{2}(\pi/2)} = \pm e^{i\pi/4} = \pm \frac{1}{\sqrt{2}}(1+i)$

So, $z = -(1+i) \pm \sqrt{i} = -(1+i) \pm \frac{1}{\sqrt{2}}(1+i)$

$= \left(-1 \pm \frac{1}{\sqrt{2}}\right) \cdot (1+i)$

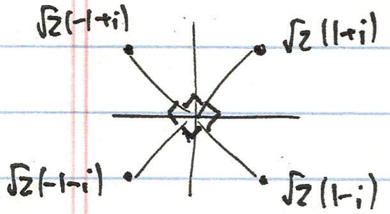
h) $z^4 - 1 = 0. \iff z^4 = 1$



$\iff z = e^{i\,2\pi k/4}, \quad k = 0,1,2,3$

$\iff z = 1, i, -1, -i$

i) $z^4 + 16 = 0 \iff z^4 = -16 = 16 \cdot e^{i\pi}$



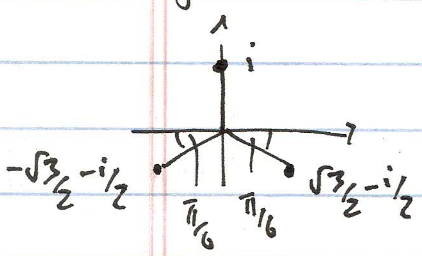
$\iff z = \sqrt[4]{16} \cdot e^{i(\pi + 2\pi k)/4}, \quad k = 0,1,2,3$

$\iff z = 2 \cdot e^{i(\pi+2\pi k)/4}, \quad k = 0,1,2,3$

$\iff z = 2 \cdot \left(\frac{1+i}{\sqrt{2}}\right), 2 \cdot \left(\frac{-1+i}{\sqrt{2}}\right), 2 \cdot \left(\frac{-1-i}{\sqrt{2}}\right), 2 \cdot \left(\frac{1-i}{\sqrt{2}}\right)$

$\iff z = \sqrt{2} \cdot (1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i).$

j) $z^3 + i = 0 \iff z^3 = -i = 1 \cdot e^{i(-\pi/2)}$



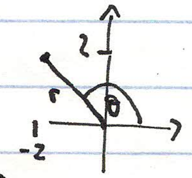
$\iff z = \sqrt[3]{1} \cdot e^{i(-\pi/2 + 2\pi k)/3}, \quad k = 0,1,2$

$\iff z = e^{i(-\pi/6)}, e^{i\pi/2}, e^{i7\pi/6}$

$= \sqrt{3}/2 - i/2, \quad i, \quad -\sqrt{3}/2 - i/2.$

8. a $f : \mathbb{C} \to \mathbb{C} \qquad f(z) = (-2+2i) \cdot z.$

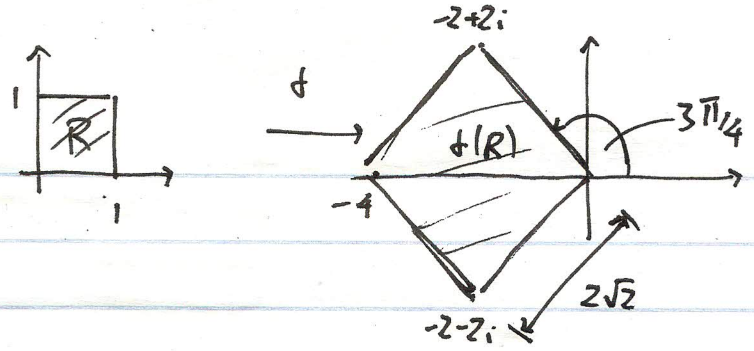
$-2+2i = re^{i\theta}$



$r = \sqrt{(2)^2 + 2^2} = 2\sqrt{2}$

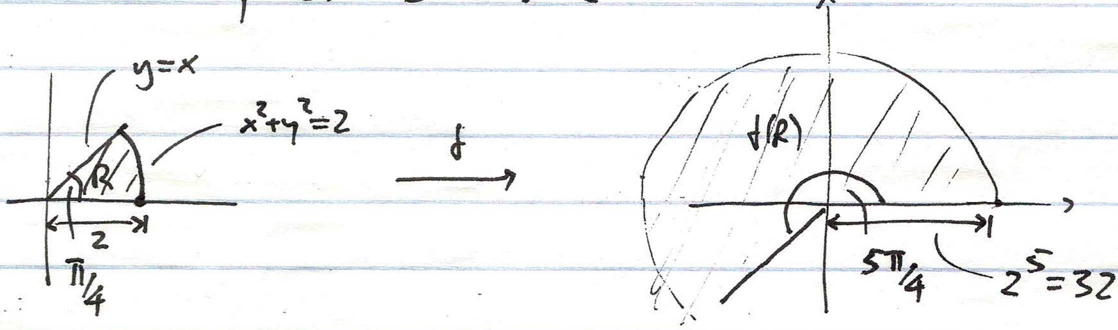
$\theta = 3\pi/4.$

$\Rightarrow$ f is rotation through angle $\theta = 3\pi/4$ ccw about the origin, followed by scaling by factor $r = 2\sqrt{2}$.



b $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z^5$.

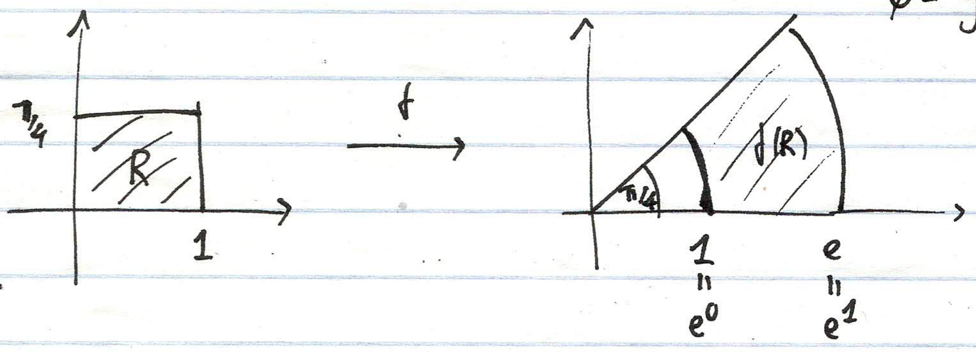
Write $z = re^{i\theta}$, then $z^5 = r^5 e^{i5\theta}$



c. $f: \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$.

$e^z = e^{x+iy} = e^x \cdot e^{iy} = se^{i\phi}$ $\quad s = e^x$

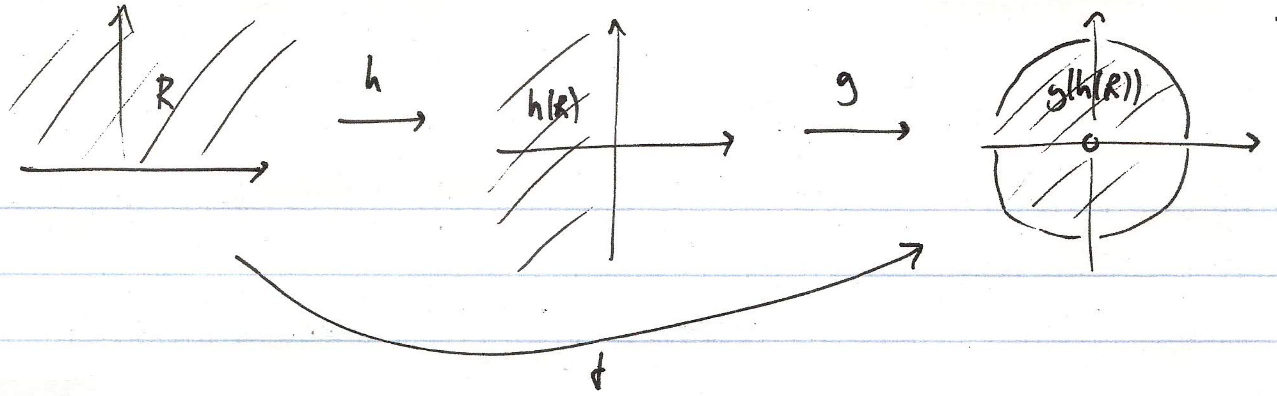
$\phi = y$



d. $f: \mathbb{C} \to \mathbb{C}$, $f(z) = e^{iz}$

$f(z) = g(h(z))$, $h(z) = i \cdot z$ rotation thru angle $\pi/2$ ccw about the origin.

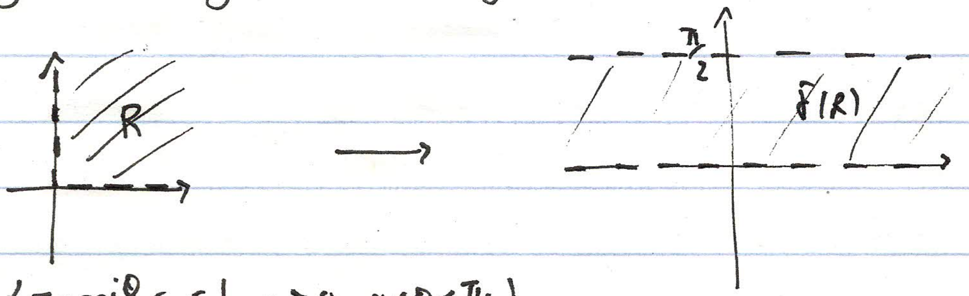
$g(w) = e^w$.



$$f(R) = g(h(R)) = \{ w \in \mathbb{C} \mid |w| \leq 1, \ w \neq 0 \}.$$

e. $f(z) = \operatorname{Log}(z)$

$$\operatorname{Log}(z) = \operatorname{Log}(re^{i\theta}) = \log r + i\theta \qquad \text{for} \quad -\pi < \theta \leq \pi.$$



$R = \{ z = re^{i\theta} \in \mathbb{C} \mid r > 0, \ 0 < \theta < \pi/2 \}$

$\Rightarrow \ f(R) = \{ w = u + iv \in \mathbb{C} \mid 0 < v < \pi/2 \}$

4. a) $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$

TRUE. Proved using definition $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and manipulation of power series.

b) $e^{-z} = 1/e^z$ for all $z \in \mathbb{C}$.

TRUE:

$$(a) \Rightarrow \quad e^z \cdot e^{-z} = e^{z+(-z)} = e^0 = 1$$

$$\Rightarrow \quad e^{-z} = 1/e^z.$$

c) $e^z = e^w \Rightarrow z = w.$

FALSE. $e^z = e^w \iff z = w + (2\pi i) \cdot k$, k an integer (see Q6.)

d $\sin(-z) = -\sin(z)$ & $\cos(-z) = \cos(z)$ for all $z \in \mathbb{C}$.
(i.e. $\sin(z)$ is <u>odd</u> and $\cos(z)$ is <u>even</u>)

TRUE.
Proved using definition as power series: $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$$

e $|\sin z| \leq 1$ for all $z \in \mathbb{C}$

FALSE. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y} \cdot e^{ix},$$

$\Rightarrow |e^{iz}| = e^{-y}$
Similarly $|e^{-iz}| = e^{y}$.
Now $e^y \to 0$ as $y \to -\infty$ & $e^y \to \infty$ as $y \to \infty$.

$$\Rightarrow |\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right| = \frac{1}{2} |e^{iz} - e^{-iz}|$$

$$\approx \begin{cases} \frac{1}{2} e^{-y} & \text{for } y \text{ large \& negative} \\ \frac{1}{2} e^{y} & \text{for } y \text{ large \& positive} \end{cases}$$

i.e. $|\sin z| \simeq \frac{1}{2} e^{|y|}$ for $|y|$ large.

In particular, $|\sin z| \to \infty$ as $y \to \pm\infty$.

f $\cos(z + 2\pi) = \cos z$ for all $z \in \mathbb{C}$.

TRUE: $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$$\Rightarrow \cos(z + 2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{iz+2\pi i} + e^{-iz-2\pi i}}{2} \overset{\text{G6}}{=} \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

(Similarly, we can show $\sin(z+2\pi) = \sin z$ for all $z \in \mathbb{C}$)

g $\quad \sin(z+\pi) = -\sin(z)$ for all $z \in \mathbb{C}$.

TRUE. $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\qquad e^{i\pi} = e^{-i\pi} = -1.$

$$\sin(z+\pi) = \frac{e^{i(z+\pi)} - e^{-i(z+\pi)}}{2i} = \frac{e^{iz}\cdot e^{i\pi} - e^{-iz}\cdot e^{-i\pi}}{2i} = \frac{-e^{iz} + e^{-iz}}{2i}$$

$$= -\sin z$$

h $\quad \sin(z) \in \mathbb{R} \Rightarrow z \in \mathbb{R}$.

FALSE.

$$\sin(z) = \sin(x+iy) \overset{\text{addition formula}}{=} \sin x \cos(iy) + \cos(x)\sin(iy)$$

$$= \sin x \cosh(y) + i \cos x \sinh(y)$$

where $\cosh(y) = \frac{1}{2}(e^{y}+e^{-y})$, $\sinh(y) = \frac{1}{2}(e^{y}-e^{-y})$

"hyperbolic cosine" "hyperbolic sine".

So $\sin(z) \in \mathbb{R} \iff \cos x \sinh y = 0 \iff \cos x = 0$ OR $\sinh y = 0$

$\iff x = \pi/2 + k\cdot\pi$, k integer

OR $y = 0$

$\iff z = (\pi/2 + k\pi) + iy$, k integer, $y \in \mathbb{R}$

OR $z = x \in \mathbb{R}$.

i $\quad (\cos(z))^2 + (\sin(z))^2 = 1$ for all $z \in \mathbb{C}$.

TRUE. $\quad \cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$\Rightarrow (\cos z)^2 + (\sin z)^2 = \frac{e^{2iz} + 2 + e^{-2iz}}{4} + \frac{e^{2iz} - 2 + e^{-2iz}}{-4}$$

$$= \frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz} - (e^{2iz} - 2 + e^{-2iz})\right) = 4/4 = \underline{1}.$$

j. $e^{\mathrm{Log} z} = z$ for all $z \in \mathbb{C}$

TRUE

Write $z = re^{i\theta}$, $-\pi < \theta \leq \pi$.

Then $\mathrm{Log} z = \log r + i\theta$ ("principal value" of complex logarithm)

$e^{\mathrm{Log} z} = e^{\log r} \cdot e^{i\theta} = \cancel{e} \; r \cdot e^{i\theta} = z.$

k. $\mathrm{Log}(e^z) = z$ for all $z \in \mathbb{C}$.

FALSE $z = x+iy$. $e^z = e^{x+iy} = e^x \cdot e^{iy} = se^{i\phi}$

$s = e^x$, $\phi = y + (2\pi)\cdot k$, k integer.

We can chose k so that $-\pi < \phi \leq \pi$.

Then $\mathrm{Log}(se^{i\phi}) = \log s + i\phi = x + i\phi$

So $\mathrm{Log}(e^z) = x + i\phi = x + iy + (2\pi i)k = z + (2\pi i)k$,

and $\mathrm{Log}(e^z) = z \iff k = 0 \iff -\pi < y \leq \pi$.

l. $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$ for all $0 \neq z, w \in \mathbb{C}$.

FALSE. Write $z = re^{i\theta}$, $w = se^{i\phi}$, where $-\pi < \theta, \phi \leq \pi$.

Then RHS $= \log r + i\theta + \log s + i\phi = \log(rs) + i(\theta + \phi)$

LHS $= \mathrm{Log}(rse^{i(\theta+\phi)}) = \log(rs) + i \cdot \left(\theta + \phi + \begin{Bmatrix} 2\pi \\ 0 \\ -2\pi \end{Bmatrix}\right)$

chosen to put angle in range $(-\pi, \pi]$.

For example, $z = w = e^{-i3\pi/4}$

$\Rightarrow$ $\mathrm{Log}(z) = \mathrm{Log}(w) = -i3\pi/4$

$\mathrm{Log}(zw) = \mathrm{Log}(e^{-i3\pi/2}) = \mathrm{Log}(e^{i\pi/2}) = i\pi/2 = i(-3\pi/2 + 2\pi)$

$= \mathrm{Log}(z) + \mathrm{Log}(w) + 2\pi i$

m. $\log(zw) = \log(z) + \log(w)$ for all $0 \neq z, w \in \mathbb{C}$.

TRUE. Here $\log(z)$ denotes multivalued complex logarithm, defined by $\log(re^{i\theta}) = \log r + i(\theta + 2\pi k)$, k integer.

Now $\log(zw) = \log(re^{i\theta} se^{i\phi}) = \log(rse^{i(\theta+\phi)})$
$= \log(rs) + i(\theta + \phi + 2\pi k)$, k integer.

$\log(z) + \log(w) = \log(re^{i\theta}) + \log(se^{i\phi})$
$= \log r + i(\theta + 2\pi l) + \log s + i(\phi + 2\pi m)$, l, m integers
$= \log(rs) + i(\theta + \phi + 2\pi(l+m))$
$= \log(rs) + i(\theta + \phi + 2\pi k)$ $k = l+m$ integer
$= \log(zw)$.

n. $\cos(z) = \cos(w) \Rightarrow z = \pm w + (2\pi)k$, k integer.

TRUE:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = f(g(z)),$$

$$g(z) = e^{iz}, \quad f(w) = \frac{1}{2}\left(w + \frac{1}{w}\right), \quad \begin{array}{l} g: \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\} \\ f: \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C} \end{array}$$

$$\begin{aligned} f(w) = \alpha \in \mathbb{C} \quad &\Leftrightarrow \quad \tfrac{1}{2}\left(w + \tfrac{1}{w}\right) = \alpha \\ &\Leftrightarrow \quad w^2 + 1 = 2\alpha w \\ &\Leftrightarrow \quad w^2 - 2\alpha w + 1 = 0 \quad \dagger \\ &\Leftrightarrow \quad w = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4}}{2} = \alpha \pm \sqrt{\alpha^2 - 1} \end{aligned}$$

Notice that the two solutions $w = \beta_1, \beta_2$ satisfy $\beta_1 \beta_2 = 1$, i.e. $\beta_2 = 1/\beta_1$.
In fact, the equation $\dagger$ can be factorized as
$(w - \beta_1) \cdot (w - \beta_2) = 0$, so $\beta_1 \beta_2 =$ constant term $= 1$.

So we see that $f(w_1) = f(w_2) \iff w_2 = w_1$ or $1/w_1$. (A)

Next, $g(z_1) = g(z_2) \iff e^{iz_1} = e^{iz_2}$ (B)

$\underset{Q6}{\iff} z_2 = z_1 + (2\pi)\cdot k$, k integer.

Combining, $\cos(z_1) = \cos(z_2) \iff f(g(z_1)) = f(g(z_2))$

$\underset{(A)}{\iff} g(z_1) = g(z_2)$ or $1/g(z_2) = g(-z_2)$

$\underset{(B)}{\iff} z_1 = z_2 + (2\pi)\cdot k$, k integer

OR $z_1 = -z_2 + (2\pi)\cdot k$

$\iff z_1 = \pm z_2 + (2\pi)\cdot k$, k integer.

10. $\sin(z + \pi/2) = \cos z$ for all $z \in \mathbb{C}$.

<u>Proof:</u> $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

So $$\sin(z + \pi/2) = \frac{e^{i(z+\pi/2)} - e^{-i(z+\pi/2)}}{2i}$$

$$= \frac{e^{iz}\cdot e^{i\pi/2} - e^{-iz}\cdot e^{-i\pi/2}}{2i}$$

$$= \frac{e^{iz}\cdot i - e^{-iz}\cdot(-i)}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

$\square$

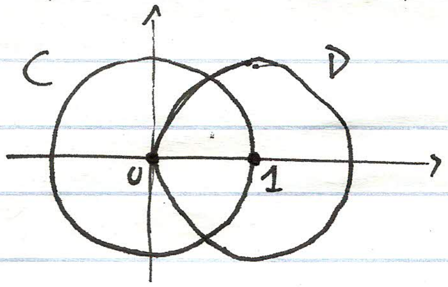
11. For $w, z \in \mathbb{C}$, $w \neq 0$, we define

$w^z = e^{z \log w}$, where log is the multivalued complex logarithm.

$$(-1)^i = e^{i\log(-1)} = e^{i(\log(1\cdot e^{i\pi}))} = e^{i(\log 1 + i(\pi + 2\pi k))}, \text{ k integer}$$

$$= e^{i(0 + i(\pi + 2\pi k))} = e^{-\pi - 2\pi k}, \text{ k integer}$$

12. $f: \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}\setminus\{0\}$, $\quad f(z) = 1/z$



a. $C = \{z \in \mathbb{C} \mid |z| = 1\}$.

$|f(z)| = \left|1/z\right| = 1/|z| \quad \Rightarrow \quad f(C) = C.$

b. Using the hint: $\quad g: (-\pi, \pi) \longrightarrow \mathbb{C}$

$$g(\theta) = 1 + e^{i\theta}$$

is a parametrization of $D\setminus\{0\}$

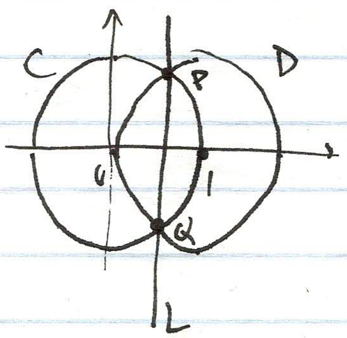
Now $f(g(\theta)) = \dfrac{1}{1+e^{i\theta}} = \dfrac{1}{(1+\cos\theta) + i\sin\theta} = \dfrac{(1+\cos\theta) - i\sin\theta}{(1+\cos\theta)^2 + (\sin\theta)^2}$

$$= \frac{(1+\cos\theta) - i\sin\theta}{1 + 2\cos\theta + (\cos\theta)^2 + (\sin\theta)^2} = \frac{(1+\cos\theta) - i\sin\theta}{2 + 2\cos\theta}$$

$$= \frac{1}{2} - i\,\frac{\sin\theta}{2(1+\cos\theta)}$$

As $\theta \searrow -\pi$, $\dfrac{-\sin\theta}{2(1+\cos\theta)} \longrightarrow \infty$, & as $\theta \nearrow \pi$, $\dfrac{-\sin\theta}{2(1+\cos\theta)} \longrightarrow -\infty$

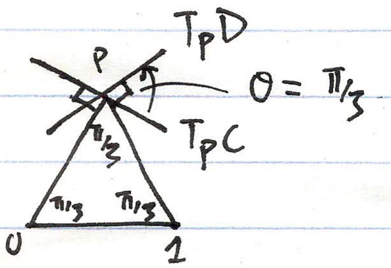
So, see $f(D\setminus\{0\}) = L$, vertical line given by equation $x = 1/2$.



c. The points $0, 1$ & $P \in C \cap D$ (see diagram above) form an equilateral triangle with side length 1 (because C & D have radius 1)

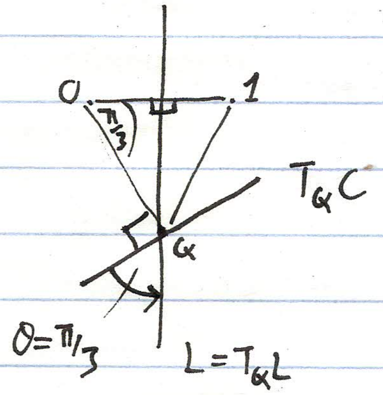
So we see that $P = e^{i\pi/3}$

At P, we have picture



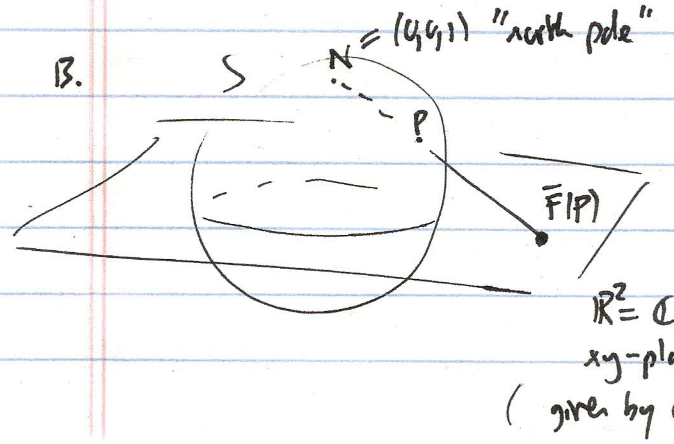
where T_pC, T_pD are the tangent lines to C & D at P,
and we've used the fact that for a circle the tangent is perpendicular to the radius.

At $Q = f(P) = 1/e^{i\pi/3} = e^{-i\pi/3}$, have picture.



So the angle between C and D at P is equal to the angle between $f(C) = C$ and $f(D) = L$ at Q, given by $\pi/3$ ccw measured from C to D (or $f(C)$ to $f(D)$).
(This is because f is complex differentiable at P, and $f'(P) \neq 0$).

B.



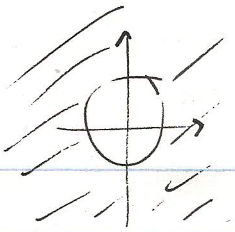
$F: S \longrightarrow \mathbb{C} \cup \{\infty\}$

$F(N) = \infty$

$F(P) =$ point in xy-plane $\mathbb{R}^2 = \mathbb{C}$ given by intersection with line thru N & P, for $P \neq N$.

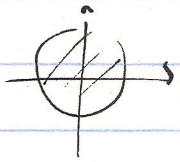
a $\bar{F}(H_1) = \{w \in \mathbb{C} \mid |w| > 1\} \cup \{\infty\}$

"outside" of circle, center origin, radius 1.



b. $\bar{F}(H_s) = \{w \in \mathbb{C} \mid |w| < 1\}$

"inside" of circle, center origin, radius 1.

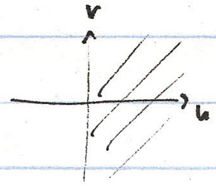


c) $\bar{F}(E) = E = \{w \in \mathbb{C} \mid |w| = 1\}$,

circle, center origin, radius 1.

d) $\bar{F}(H_e) = \{w = u+iv \in \mathbb{C} \mid u > 0\}$

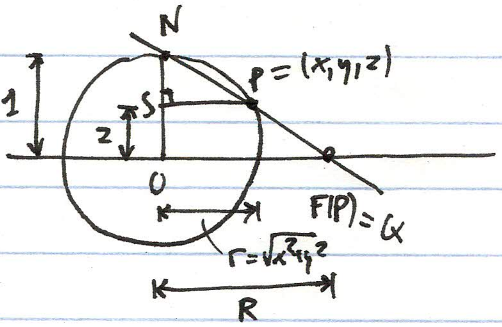
the right half plane



e) $\bar{F}(I) = \{w \in \mathbb{C} \mid |w| > R\} \cup \{\infty\}$

"outside" of circle, center origin, radius R,

where R is determined as follows :-



$\Delta NSP \sim \Delta NOQ$ (similar triangles)

$\Rightarrow \quad SP/SN = OQ/ON$

$(x,y,z) \in S \Rightarrow x^2+y^2+z^2 = 1$

$$\frac{\sqrt{x^2+y^2}}{1-z} = \frac{R}{1}$$

$$R = \frac{\sqrt{x^2+y^2}}{1-z} = \frac{\sqrt{1-z^2}}{1-z} = \sqrt{\frac{1+z}{1-z}}$$

In our example, $z = 3/4 \Rightarrow R = \sqrt{\dfrac{1+3/4}{1-3/4}} = \sqrt{7}$

14. $\underset{\mathbb{C}}{\overset{\cup}{U}}$ $f: U \to \mathbb{C}$ is complex differentiable at $a \in U$ if the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

In parts a, b, c we use the hint: Writing $f = u + iv$

f is complex differentiable $\Leftrightarrow$ f is real differentiable and $\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{Cauchy Rieman eqs}}$

f is real differentiable $\Leftarrow$ partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists and are continuous.

a $\quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x+4y) = 2 \quad \neq \quad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(3x+5y) = 5.$

$\therefore$ f is NOT complex differentiable.

b. $\quad u = 3x^2 - 2xy - 3y^2 \qquad \frac{\partial u}{\partial x} = 6x - 2y \qquad \frac{\partial v}{\partial y} = 6x - 2y$

$v = x^2 + 6xy - y^2$

$\frac{\partial u}{\partial y} = -2x - 6y \qquad \frac{\partial v}{\partial x} = 2x + 6y$

$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (CR eqs).

Also $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist & are continuous $\Rightarrow$ f real differentiable.

So f is complex differentiable.

c $\quad f(x+iy) = e^y \cos x - (e^y \sin x) i$

$u = e^y \cos x \qquad \frac{\partial u}{\partial x} = -e^y \sin x = \frac{\partial v}{\partial y}$

$v = -e^y \sin x$

$\frac{\partial u}{\partial y} = e^y \cos x \qquad \frac{\partial v}{\partial x} = -e^y \cos x.$

$\therefore$ CR eqns hold, partial derivs exist & continuous, $\Rightarrow$ f complex differentiable.

15. $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$, $f(z) = 1/z$.

$R = \{z = x+iy \mid x \geq 1\}$

$f(R) = ?$

Write $f(z) = w = u+iv$.

Equivalently, $z = f^{-1}(w) = f(w) = \dfrac{1}{u+iv} = \dfrac{u-iv}{u^2+v^2}$

$f = f^{-1}$!

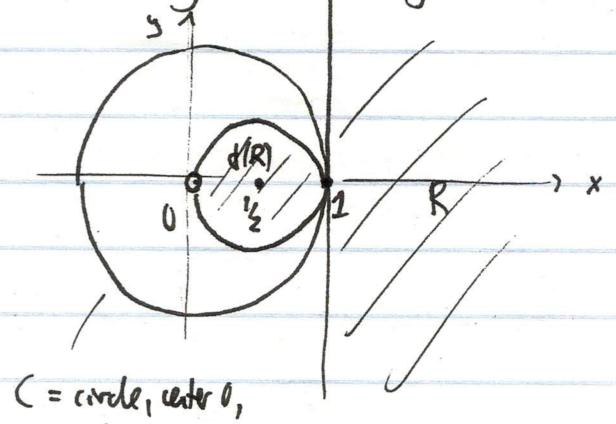
$$w \in f(R) \iff z = \frac{u-iv}{u^2+v^2} \in R \iff \frac{u}{u^2+v^2} \geq 1$$

$$\iff 0 \geq u^2+v^2-u$$

$$\iff \tfrac{1}{4} \geq (u-\tfrac{1}{2})^2 + v^2 \qquad \text{"complete the square"}$$

$$\iff |w - \tfrac{1}{2}| \leq \tfrac{1}{2}.$$

i.e. $f(R)$ is inside of circle center $\frac{1}{2}$, radius $\frac{1}{2}$ (including the boundary, except for the point $0 \in \mathbb{C}$).



C = circle, center 0, radius 1.

16. $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \cos(z)$. $f(z) = w = u+iv$.

$$\cos(z) = \cos(x+iy) = \cos(x)\cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y$$

addition formula

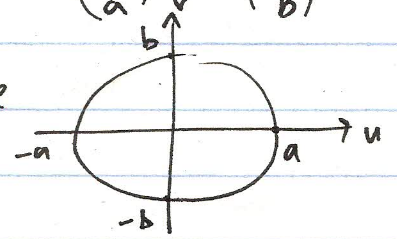
a. $y = c \neq 0$

$\Rightarrow \quad u = \cos x \cosh c \;, \quad v = -\sin x \sinh c$

$\Rightarrow \quad \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1 \qquad a = \cosh c > 0$

$\qquad b = |\sinh c| > 0 \qquad (c \neq 0)$

ellipse



The ellipse is traversed once as x varies from 0 to 2π
(as in case of circle)

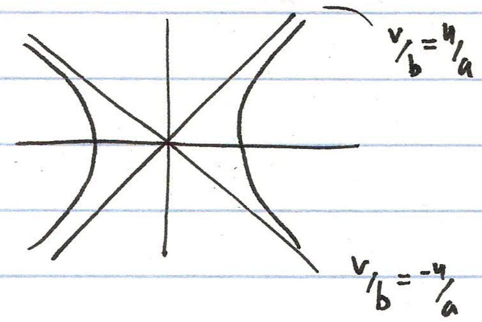
b. $y = 0 \Rightarrow u = \cos x \,,\; v = 0.$

So image is interval $[-1, 1] \subset \mathbb{R} \subset \mathbb{C}$.

c. $x = c \neq k \cdot \pi/2^{\dagger}$, k integer

$\Rightarrow \quad u = \cos c \cosh x \;, \quad v = -\sin c \sinh x$

$\Rightarrow \quad \left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 = 1 \qquad \left.\begin{array}{l} a = |\cos c| > 0 \\ b = |\sin c| > 0 \end{array}\right\}$ using $\dagger$



"hyperbola".

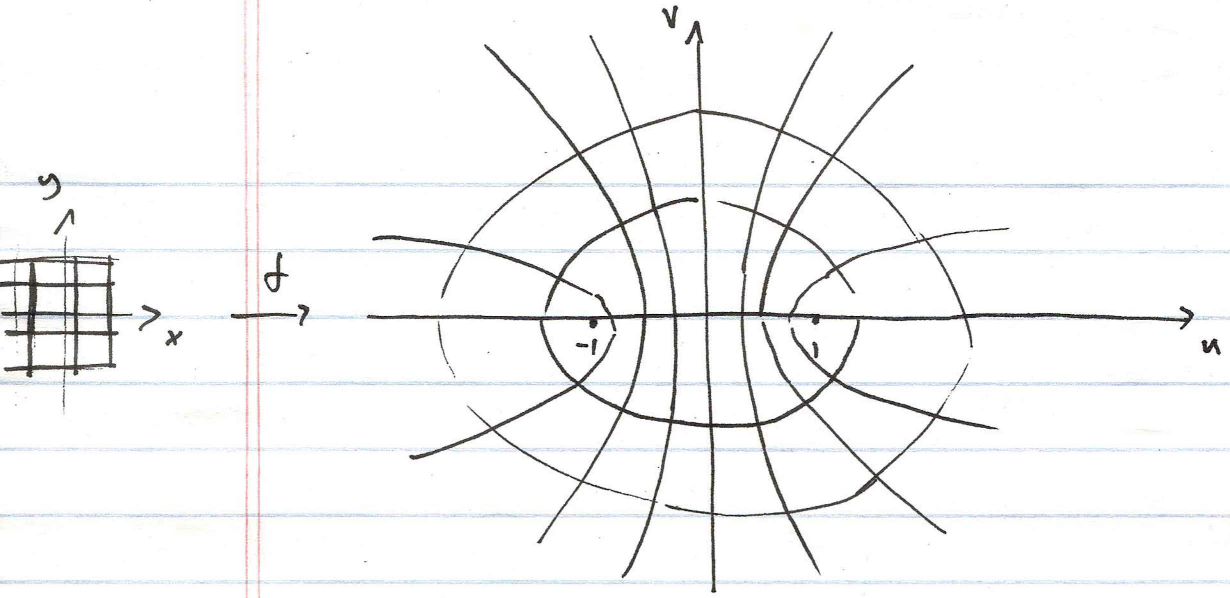
image is right hand piece for $\cos c > 0$
left hand piece for $\cos c < 0$.

d. Since $\cos(z)$ is periodic w/ period 2π (Q9f)
we need only consider $x = 0, \pi/2, \pi, 3\pi/2$.

$x = 0 \Rightarrow u = \cosh x \,,\; v = 0 \Rightarrow$ image $[1, \infty) \subset \mathbb{R} \subset \mathbb{C}$

$x = \pi \Rightarrow u = -\cosh x \,,\; v = 0 \Rightarrow$ image $(-\infty, -1] \subset \mathbb{R} \subset \mathbb{C}$

$\left.\begin{array}{l} x = \pi/2 \Rightarrow u = 0 \,,\; v = -\sinh x \\ x = 3\pi/2 \Rightarrow u = 0 \,,\; v = \sinh x \end{array}\right\} \Rightarrow$ image $i\mathbb{R} \subset \mathbb{C}$ (imaginary axis = v-axis)



The curves give by the images of the grid lines $x = c$ and $y = d$ meet at right angles (hyperbola & ellipse), except at points ± 1 corresponding to the zeroes of $f'(z) = -\sin z$ under the transformation $f(z) = \cos z$.