Math 611 Homework 2

Paul Hacking

September 25, 2019

Reading: Dummit and Foote, Sections 2.1, 2.2, 3.1, 3.2, 3.5, 4.1, 4.2, 4.3.

Justify your answers carefully (complete proofs are expected).

(1) (Optional) Suppose given a group G and an action of G on a set X. Then we have a partition of X given by the orbits of the action, and the action of G on X is given by a transitive action of G on each orbit.

Let $x \in X$ be a point. Show that there is a bijection $f \colon G/G_x \xrightarrow{\sim} \mathcal{O}_x$ from the set G/G_x of left cosets of the stabilizer G_x of x to the orbit $\mathcal{O}_x$ of x that is compatible with the G-actions, i.e., $f(g \cdot a) = g \cdot f(a)$. (Here, the action of G on the set of left cosets G/H of a subgroup H of G is defined by $g \cdot aH = gaH$.)

In particular, any action of G can be understood in terms of the action of G on the set of left cosets of H for various subgroups H of G.

(2) Let G be a finite group of order p^n for some prime p and $n \in \mathbb{N}$. (We say G is a *p-group*.) Suppose G acts on a finite set X such that $|X|$ is not divisible by p. Show that there is a *fixed point* for the action, that is, a point $q \in X$ such that $g \cdot q = q$ for all $g \in G$.

(3) (Optional) Let G be a group. Recall we say two elements $a, b \in G$ are *conjugate* if there is a $g \in G$ such that $b = gag^{-1}$. Consider the two matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Are A and B conjugate in $\mathrm{GL}_2(\mathbb{R})$? Are A and B conjugate in $\mathrm{SL}_2(\mathbb{R})$? (Recall that $\mathrm{SL}_n(\mathbb{R})$ denotes the normal subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices with determinant 1.)

(4) Let G be a group and $a \in G$ an element. Determine the centralizer $Z(a)$ of a in G and the size of the conjugacy class $C(a)$ of a in G in the following cases.

(a) $(123) \in S_5$.

(b) $(123)(456) \in S_7$.

(c) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$.

(d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$.

(5) A group G of order 21 contains a conjugacy class $C(x)$ of size 3. What is the order of x in G?

(6) (Optional) Determine the conjugacy classes in the group

$$G = \langle a, b \mid a^3 = b^4 = e, \quad ba = a^{-1}b \rangle.$$

(That is, G is the group generated by two elements a and b subject to the given relations.) Here you may assume without proof that $|G| = 12$ so that the elements of G can be expressed uniquely as $a^i b^j$ for $0 \leq i < 3$ and $0 \leq j < 4$.

(7) Enumerate all the normal subgroups of the symmetric group S_4.

(8) (Optional)

(a) Determine the conjugacy classes in the alternating group A_4. Check your answer using the fact that the order of a conjugacy class divides the order of the group.

(b) Show that A_4 does not have a subgroup of order 6.

(9) Let G be a p-group. In class we used the class equation to show that the center of G is non-trivial ($Z(G) \neq \{e\}$). As a corollary we showed that any group of order p^2 is abelian. In this question we will study a non-abelian group G of order p^3 (the *Heisenberg group*).

Let G be the subgroup of $\mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices with all diagonal entries equal to 1.

(a) Determine the center $Z(G)$ of G.

(b) Construct an isomorphism from $G/Z(G)$ to a standard group.

(10) Classify finite groups G with at most 3 conjugacy classes.

(11) (Optional) Let G be a finite group and H a proper subgroup of G.

(a) Show that the union of the conjugate subgroups of H is not equal to G.

(b) Deduce that there is a conjugacy class which is disjoint from H.

(12) For each of the following statements, give a proof or a counterexample.

(a) If $H \triangleleft G$ and $K \triangleleft H$ then $K \triangleleft G$.

(b) If $H \triangleleft G$ and $K \leq G$ then $H \cap K \triangleleft K$.

(13) (Optional) Let G be a finite group and $H \triangleleft G$ a normal subgroup. Let $a \in H$ be an element. Let $C_H(a)$ denote the conjugacy class of a in H and $C_G(a)$ the conjugacy class of a in G. Let $Z_H(a)$ denote the centralizer of a in H and $Z_G(a)$ the centralizer of a in G. (Then $Z_H(a) = Z_G(a) \cap H$.) Let $q \colon G \to G/H$ be the quotient homomorphism.

(a) Show that $gC_H(a)g^{-1} = C_H(gag^{-1})$ for all $g \in G$.

(b) Show that $C_G(a)$ is a union of $[G/H : q(Z_G(a))]$ distinct conjugacy classes in H of equal size (the orbit of the conjugacy class $C_H(a)$ under conjugation by elements of G).

(c) Now suppose $G = S_n$, the symmetric group on n objects ($n \geq 2$), and $H = A_n$, the alternating group. Deduce that if $Z_{S_n}(a)$ is not contained in A_n then $C_{S_n}(a) = C_{A_n}(a)$, while if $Z_{S_n}(a)$ is contained in A_n then $C_{S_n}(a) = C_{A_n}(a) \cup C_{A_n}((12)a(12))$ is a union of two distinct conjugacy classes in A_n. Give examples showing that both cases occur.

Hints:

7 What is the class equation of S_4?

8b Recall that a subgroup of index 2 is necessarily normal.

10 Consider the class equation of G. The order of a conjugacy class divides the order of the group.

11a Establish an upper bound for the cardinality of the union of the conjugate subgroups.