Math 461 Lecture 30 11/14

homework 6 returned

homework 7 due today

last time: isometries of $\mathbb{R}^3$ fixing the origin ($\leftrightsquigarrow$ isometries of S^2)

T	Fix T
identity	$\mathbb{R}^3$
$Refl_\pi$	plane π
$Rot(\ell,\theta)$	line ℓ
Rotary reflection	$\{\bar{0}\}$

Rotary reflection = $Rot(\ell,\theta) \circ Refl_\pi$

θ: Trace$(A) = 2\cos\theta + 1$

$$A \sim \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ℓ: solve $T(\bar{x}) = \bar{x}$

θ: Trace$(A) = 2\cos\theta - 1$

$$A \sim \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \leftarrow \text{the only difference}$$

today:

classification of spherical isometries - proof

identity, reflection, rotation, rotary reflection via

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"3 reflections theorem"
(similar to $\mathbb{R}^2$)

Theorem: every isometry of S^2 is either identity, reflection, rotation, or a rotary reflection

3 reflections theorem: every isometry of S^2 is a composition of ≤ 3 reflections

GPS theorem: given $A, B, C \in S^2$ not lying on a spherical line then any point $P \in S^2$ is uniquely determined by its distances from A, B, C

proof: (essentially same as $\mathbb{R}^2$)
proof by contradiction:
assume have $P, Q \in S^2$, $P \neq Q$
$d(P, A) = d(Q, A)$
$d(P, B) = d(Q, B)$
$d(P, C) = d(Q, C)$
the perpendicular bisector of the spherical line segment joining P & Q is the ? of points $X \in S^2$ such that $d(P, X) = d(Q, X)$
(grant this for the moment)
now A, B, C lie on perpendicular bisector of the segment PQ

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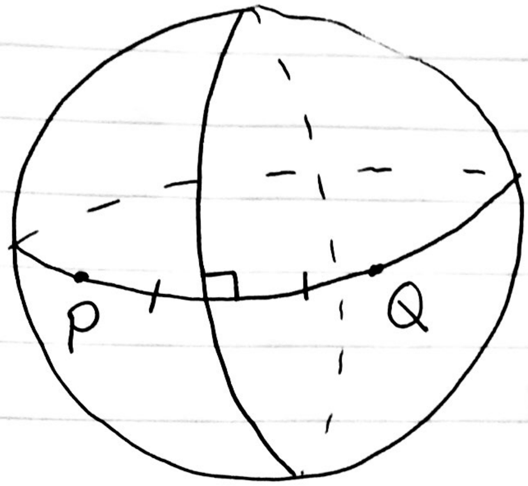
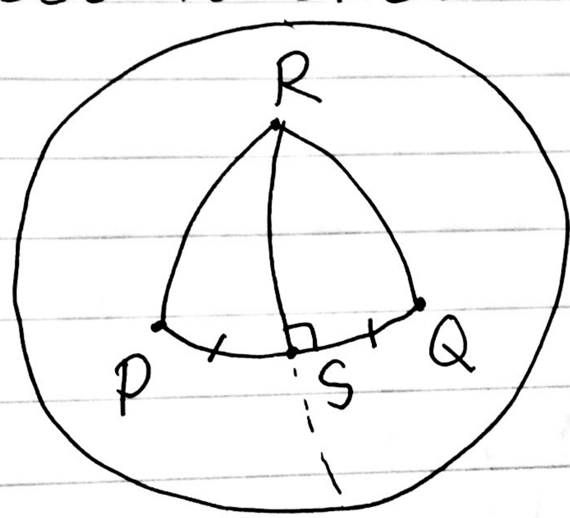
* A, B, C don't lie on spherical line $\square$



perpendicular bisector

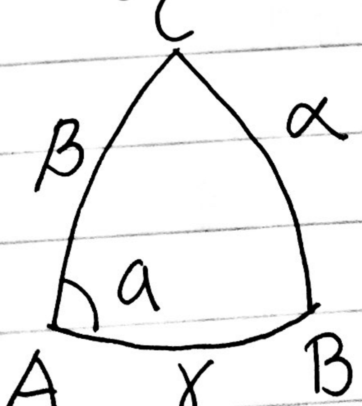
need to show the following:-



given $P, Q, R \in S^2$ draw perpendicular line through R to PQ, S the foot of the perpendicular

need to show that $d(P,R) = d(Q,R)$
$\Leftrightarrow d(P,S) = d(Q,S)$

why is this true?



$\cos\alpha = \cos\beta\cos\gamma$

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

if $a = \frac{\pi}{2}$

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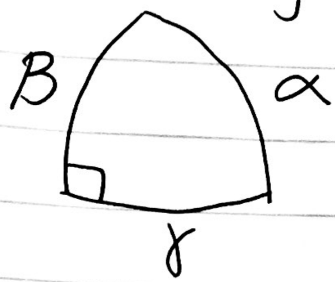
Why is this true?

<= ✓ by SPT => also by SPT

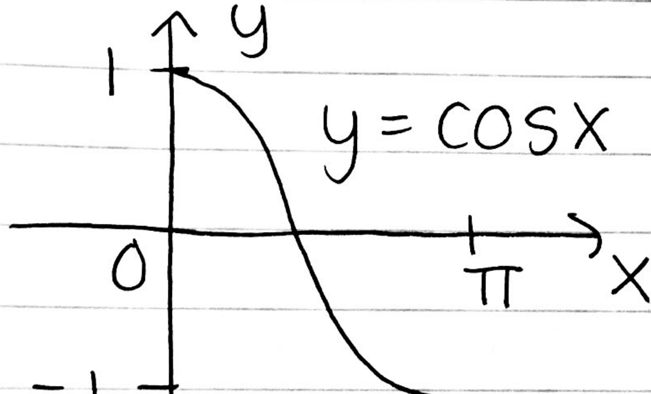


SPT:

$\cos\alpha = \cos\beta\cos\gamma$ =>

only two sides of spherical ? triangle determine the third

note:



$0 \le \alpha, \beta, \gamma \le \pi$

(distances on S^2)

$\cos x$ is invertible function

$[0, \pi] \to [-1, 1]$

corollary of GPS theorem:

a spherical isometry $T: S^2 \to S^2$ is determined by $T(A)$, $T(B)$, $T(C)$ where $A, B, C \in S^2$ are three points which do not lie on a spherical line

proof: for any point $P \in S^2$ we know $d(T(P), T(A)) = d(P, A)$ etc.

i.e know distances of $T(P)$ from $T(A)$, $T(B)$, $T(C)$

this determines $T(P)$ by GPS □

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proof of three reflections theorem:

let $T: S^2 \to S^2$ be an isometry

pick $A, B, C \in S^2$ three points not lying on a spherical line

will compose T with reflections until we obtain isometry T' such that $T'(A) = A$, $T'(B) = B$, $T'(C) = C$

then $T' =$ identity by corollary

$\rightsquigarrow$ T is a product of reflections

$R_n \circ \cdots \circ R_1 \circ T = T' =$ identity

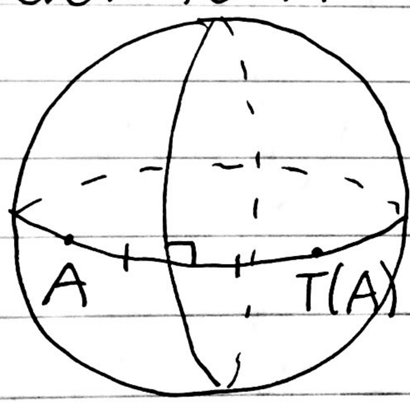
equivalently $T = R_1^{-1} \circ \cdots R_{n-1}^{-1} \circ R_n^{-1} \circ id$

$= R_1 \circ R_2 \circ \cdots \circ R_n$

details: choose first reflection R_1:

if $T(A) = A$ ✓ ok

otherwise choose R so sends $T(A)$ back to A



i.e reflect in perpendicular bisector of the line segment $A - T(A)$

$U = R_1 \circ T$ $A \mapsto A$

keep going: if $U(B) = B$

otherwise compose $V = R_2 \circ U$

R_2 reflection in perpendicular of $B - U(B)$

key point: $U(A) = A \Rightarrow B$ and $U(B)$ are equidistant from A:-

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$d(A,B) = d(U(A),U(B)) = d(A,U(B))$

A is a perpendicular bisector of B & $U(B)$, so fixed by reflection R_2

and so $V = R_2 \circ U$ still fixes A

finally $V: A \mapsto A \quad B \mapsto B$

if $V(C) = C$ ok otherwise $W = R_3 \circ V$

R_3 reflection in perpendicular bisector of $C - V(C)$

now $T' = W$ fixes A, B, C, $T' =$ identity $\square$

proof of classification:

$\leq$ three reflections

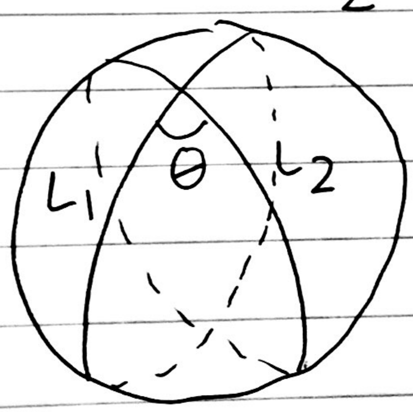
cases: 0 reflections: identity

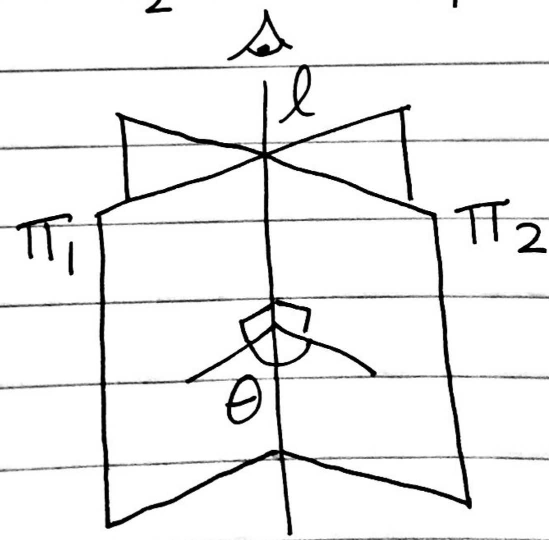
1 reflection: reflection

2 reflections:

$Refl_{L_2} \circ Refl_{L_1} : S^2 \to S^2$

$Refl_{\Pi_2} \circ Refl_{\Pi_1} : \mathbb{R}^3 \to \mathbb{R}^3$





$\ell = \Pi_1 \cap \Pi_2$

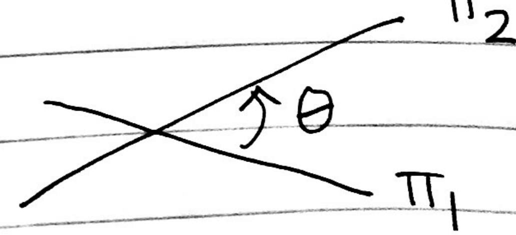
$L_1 = \Pi_1 \cap S^2$

$L_2 = \Pi_2 \cap S^2$

$Refl_{\Pi_2} \circ Refl_{\Pi_1}$

$= Rot(\ell, 2\theta)$

same as in $\mathbb{R}^2$



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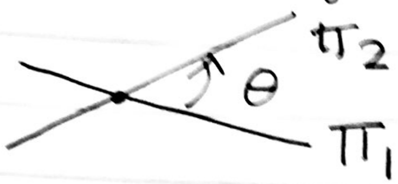
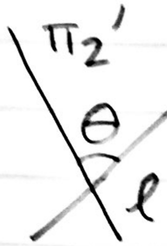
3 reflections: rotary reflection

compare $\mathbb{R}^2$

$L_1, L_2, L_3 \subset S^2$ spherical lines

$T = Refl_{L_3} \circ Refl_{L_2} \circ Refl_{L_1} = ?$



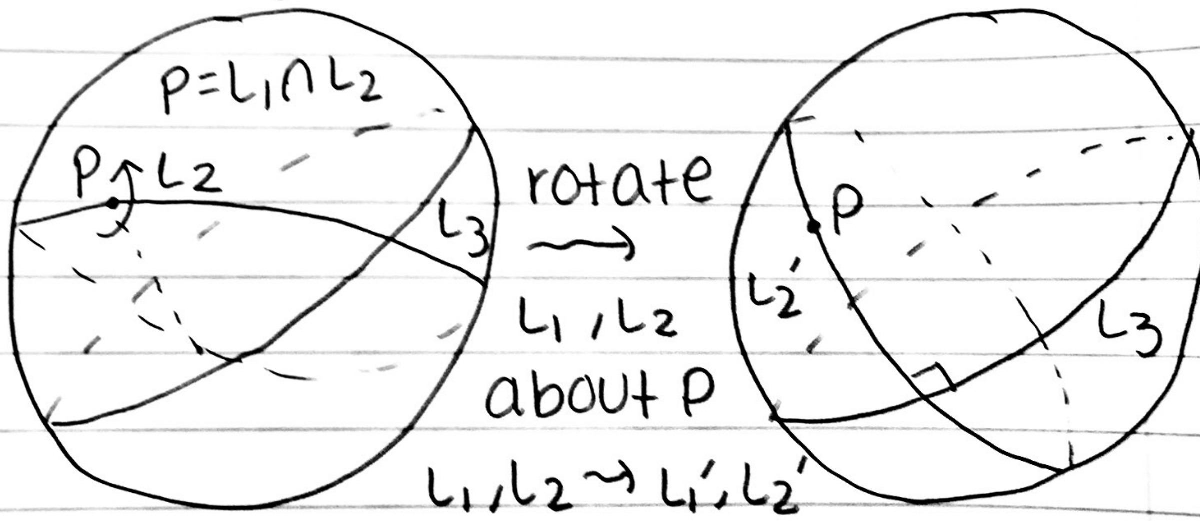


same as in $\mathbb{R}^2$

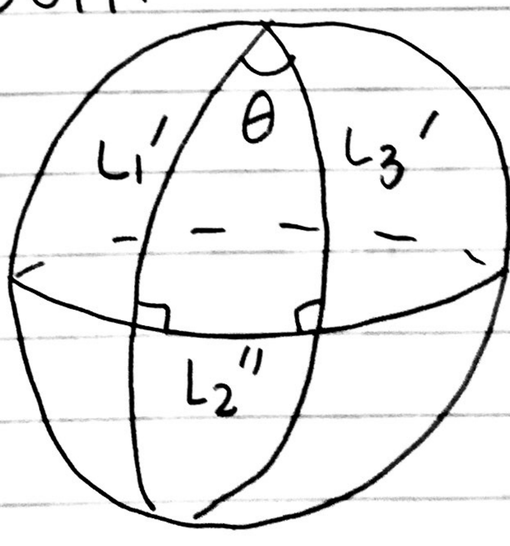
useful trick:

$Refl_{L_2} \circ Refl_{L_1} = Refl_{L_2'} \circ Refl_{L_1'}$

apply rotation about point $P \in L_1 \cap L_2$ to $L_1, L_2 \rightsquigarrow L_1', L_2'$ such that $L_2' \perp L_3$



result:



$T = Refl_{L_3'} \circ Refl_{L_2''} \circ Refl_{L_1'}$