10/18/19

<u>Last Time</u>

$T: \mathbb{R}^2 \to \mathbb{R}^2$ an isometry, $L \subset \mathbb{R}^2$ a line $\Rightarrow T(L) \subset \mathbb{R}^2$ is a line.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ isometry, $T(\underline{0}) = \underline{0} \Rightarrow T$ is a linear transformation

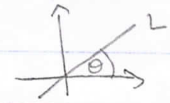
(WARNING: converse is not true!) $\left(T\begin{pmatrix}x\\y\end{pmatrix} = A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ax+by\\cx+dy\end{pmatrix}\right)$

<u>Ex</u>: T relation about (0,0) through angle θ ccw

$$\Rightarrow T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$

T reflection in a line L through (0,0) at angle θ to x-axis

$$\Rightarrow T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$



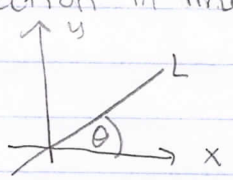
<u>Today</u>
- Another proof of formula for reflection
- Rotation about a point $P \neq (0,0)$
- Algebraic formulas for isometries

} "conjugation"

<u>Ex</u>: Rotation about the origin through angle $\theta = \pi/2$ ccw.

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\pi/2 & -\sin\pi/2\\ \sin\pi/2 & \cos\pi/2\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-y\\x\end{pmatrix}.$$

T = reflection in line L through (0,0) at angle θ to x-axis.



<u>Observed</u>: T = "first rotate about (0,0) through angle θ cw, then reflect in x-axis, finally rotate about (0,0) then angle θ ccw."

$\boxed{T = R \circ S \circ R^{-1}}$

where S = Reflection in x-axis

R = Rotation about (0,0) the angle θ ccw

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "R^{-1}"'] \arrow[rrr, bend left, "T"] & \mathbb{R}^2 \arrow[r, "S"'] & \mathbb{R}^2 \arrow[r, "R"'] & \mathbb{R} \end{tikzcd}$$

reminder, do compositions from right to left

Use this observation to compute formula for T. (again)

$R \leftrightsquigarrow \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ $\quad S: S\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \leftrightsquigarrow \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

we know formula: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

assuming $ad-bc \neq 0$ (otherwise A is NOT invertible)

$$R^{-1} \leftrightsquigarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \frac{1}{\cos\theta\cos\theta - \sin\theta\sin\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \rightsquigarrow \quad = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

<u>Alternative</u>: Clear that inverse of R = rotation by θ ccw is rotation by θ cw OR rotation by $(-\theta)$ ccw. So matrix for R^{-1} is $\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

$\cos(-\theta) = \cos(\theta)$

$\sin(-\theta) = -\sin(\theta)$

$$R \circ S \circ R^{-1} \leftrightsquigarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

<u>Aside</u>: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot5+2\cdot7 & 1\cdot6+2\cdot8 \\ 3\cdot5+4\cdot7 & 3\cdot6+4\cdot8 \end{pmatrix}$

also, $A \cdot B \cdot C = A \cdot (B \cdot C) = (A \cdot B) \cdot C$ "associative law"

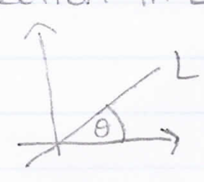
$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\cos\theta)^2 - (\sin\theta)^2 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & (\sin\theta)^2 - (\cos\theta)^2 \end{pmatrix}$$

Use "double angle formulas":

$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\rightsquigarrow \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ matrix for T = reflection in L



$\underline{Q}$: What's the formula for a rotation T about $P=(a,b) \neq (0,0)$ through angle θ ccw?

Observe $T =$ "first translate P to O,
then rotate through θ ccw about O,
then translate O to P."

$= U \circ R \circ U^{-1}$

$R =$ rotation about O through angle θ ccw

$U =$ translation sending O to P, i.e. $U(\underline{x}) = \underline{x} + \vec{OP}$, $\vec{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$

Now use known formula:

$$R(\underline{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underline{x} \qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightsquigarrow T(\underline{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \left(\underline{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right) + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\left(\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$

$\underline{Ex}$: $\theta = \pi/4$ ccw about $(1,2)$

$$T(\underline{x}) = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 + \frac{1}{2}\sqrt{2} \\ 2 - \frac{3}{2}\sqrt{2} \end{pmatrix}$$

$\underline{Q}$: Suppose given formula for rotation T.

$T(\underline{x}) = A \cdot \underline{x} + \underline{b}$ where A – 2×2 matrix, $\underline{b} \in \mathbb{R}^2$ vector

How to determine a geometric description of T? (center? angle?)

• The center P of rotation satisfies $T(P)=P$ ("doesn't move")
Solve $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix} \rightsquigarrow P$.

Ex: $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y+5\\x+7\end{pmatrix}$ $\quad T:\mathbb{R}^2\to\mathbb{R}^2$
This is a rotation (may assume this)
Find center & angle.

$$\begin{aligned}-y+5&=x\\x+7&=y\end{aligned} \rightsquigarrow \begin{aligned}x+y&=5\\x-y&=-7\end{aligned} \rightsquigarrow \boxed{x=-1,\ y=6}$$

$\rightsquigarrow$ center $P=(-1,6)$ ... angle?

Recall: $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}$

So, $\begin{pmatrix}-y+5\\x+7\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}+\begin{pmatrix}5\\7\end{pmatrix}$ ↳ constant term $=\begin{pmatrix}c\\d\end{pmatrix}$

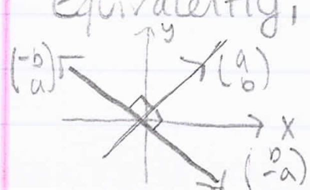
$$=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}5\\7\end{pmatrix}$$

Then, $\cos\theta=0$, $\sin\theta=1$ $\Rightarrow \boxed{\theta=\pi/2}$.

Theorem ① If $T:\mathbb{R}^2\to\mathbb{R}^2$ is an isometry, then $T(\underline{x})=A\cdot\underline{x}+\underline{b}$
Where A is a 2×2 matrix, and $b\in\mathbb{R}^2$
② Given $T(\underline{x})=A\underline{x}+\underline{b}$ & T is an isometry.
$\iff$ A is an orthogonal matrix.

↳ i.e. $A=(\underline{v}_1 \mid \underline{v}_2)$ ($\underline{v}_1,\underline{v}_2$ columns of A)
then $\|\underline{v}_1\|=\|\underline{v}_2\|=1$ & $\underline{v}_1\cdot\underline{v}_2=0$
($\underline{v}_1$ & $\underline{v}_2$ are orthogonal)

Equivalently, $A=\begin{pmatrix}a&-b\\b&a\end{pmatrix}$



OR $A=\begin{pmatrix}a&b\\b&-a\end{pmatrix}$ where $a^2+b^2=1$.