Math 797W Homework 1

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September 15, 2014

Reading: David Mumford, The red book of varieties and schemes, Chapter I, Sections 1–5.

Justify your answers carefully.

(1) Let X be the union of the coordinate axes in $\mathbb{A}^3$

(a) Compute the ideal $I(X)$.

(b) Prove that $I(X)$ cannot be generated by 2 elements.

(2) Let X be an affine variety and $f \in k(X)$ a rational function on X. Define

$$\operatorname{domain}(f) = \{p \in X \mid f \in \mathcal{O}_{X,p}\}.$$

(a) Prove $\operatorname{domain}(f) \subset X$ is an open subset.

(b) Let $p \in X$. Suppose $f = g/h$, $g, h \in k[X]$, and $g(p) \neq 0$, $h(p) = 0$. Show that $p \notin \operatorname{domain}(f)$.

(c) Compute $\operatorname{domain}(f)$ in the following cases:

i. $X = V(x_1x_3 - x_2^2) \subset \mathbb{A}^3$, $f = x_1/x_2$.

ii. $X = V(x_2^2 - x_1^3) \subset \mathbb{A}^2$, $f = x_2/x_1$.

(3) Let $U = \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2$.

(a) Compute $\mathcal{O}_{\mathbb{A}^2}(U)$.

(b) Prove that U is not isomorphic to an affine variety.

[Hint: (b) Argue by contradiction using (a) and the equivalence of categories for affine varieties and their coordinate rings.]

(4) Show directly using the standard affine charts that $\mathcal{O}_X(X) = k$ for $X = \mathbb{P}^1$.

(5) We say a topological space X is *irreducible* if there does not exist a decomposition $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are proper closed subsets. Prove the following statements.

(a) X is irreducible iff for all nonempty open sets $\emptyset \neq U_1, U_2 \subset X$ we have $U_1 \cap U_2 \neq \emptyset$.

(b) If X is irreducible and $f \colon X \to Y$ continuous then $f(X)$ is irreducible.

(c) If Y is irreducible, $Y \subset X$, and $\overline{Y}$ is the closure of Y in X, then $\overline{Y}$ is irreducible.

(d) If X is irreducible and $\emptyset \neq U \subset X$ is a non-empty open set, then U is dense (i.e. $\overline{U} = X$) and U is irreducible.

(6) (a) Let $J \subset S = k[X_0, \ldots, X_n]$ be a homogeneous ideal. Show that if J is not prime then there exist *homogeneous* elements $a, b \in S$ such that $ab \in J$ and $a, b \notin J$.

(b) Let $X \subset \mathbb{P}^n$ be an algebraic set. Show that X is irreducible iff $I(X) \subset S$ is prime.

(7) Let $X = V(x_1^3 + x_1x_2^2 + x_1^2 + x_2 + 1) \subset \mathbb{A}^2$. Let $\overline{X}$ denote the closure of X in

$$\mathbb{P}^2 = (X_0 \neq 0) \cup (X_0 = 0) = \mathbb{A}^2 \cup \mathbb{P}^1.$$

(a) Write down the homogeneous equation of $\overline{X}$ and identify the set $\overline{X} \setminus X = \overline{X} \cap \mathbb{P}^1$.

(b) Find another affine chart $Y \subset \mathbb{A}^2$ for X such that $\overline{X} = X \cup Y$, write down the equation of $Y \subset \mathbb{A}^2$, and describe the transition map between the two charts explicitly.

(8) Let $F \in k[X_0, X_1, X_2]$ be an irreducible homogeneous polynomial of degree d. Let $X = V(F) \subset \mathbb{P}^2$ be the associated projective variety, a projective plane curve of degree d. Let $L \subset \mathbb{P}^2$ be a line (i.e. $L = V(a_0X_0 + a_1X_1 + a_2X_2) \subset \mathbb{P}^2$ is the zero locus of a linear form). Show that $X \cap L$ consists of exactly d points counting multiplicities (unless $d = 1$ and $X = L$).

(9) Let $X = V(f) \subset \mathbb{A}^2$. Suppose

$$f = a_1x_1 + a_2x_2 + \cdots$$

where $\cdots$ denotes higher order terms in x_1, x_2, and $(a_1, a_2) \neq (0, 0)$. (Geometrically, we have $(0, 0) \in X$, and X is smooth at $(0, 0)$ with tangent line $V(a_1x_1 + a_2x_2) \subset \mathbb{A}^2$.) Consider the morphism

$$q \colon \mathbb{A}^2 \setminus \{(0, 0)\} \to \mathbb{P}^1, \quad (x_1, x_2) \mapsto (x_1 : x_2).$$

(a) Show that the restriction of q to $X \setminus \{(0, 0)\}$ extends to a morphism $g \colon X \to \mathbb{P}^1$.

(b) What is the geometric interpretation of the point $g(0, 0) \in \mathbb{P}^1$?

(10) Consider the map

$$f \colon \mathbb{P}^1 \to \mathbb{P}^3, \quad (Y_0 : Y_1) \mapsto (Y_0^3 : Y_0^2Y_1 : Y_0Y_1^2 : Y_1^3).$$

(a) Check that f is a morphism.

(b) Let $X = f(\mathbb{P}^1) \subset \mathbb{P}^3$. Show that $X = V(J)$ where

$$J := (X_0X_2 - X_1^2, X_1X_3 - X_2^2, X_0X_3 - X_1X_2).$$

(c) Prove that f is an isomorphism onto its image.

(d) Show that J is the kernel of the ring homomorphism

$$k[X_0, \ldots, X_3] \to k[Y_0, Y_1], \quad X_0, X_1, X_2, X_3 \mapsto Y_0^3, Y_0^2Y_1, Y_0Y_1^2, Y_1^3.$$

Deduce that J is prime and hence $J = I(X)$.

(11) Let $n \in \mathbb{Z}$. Let $X = X(n) = U_1 \cup U_2$ where $U_1 = \mathbb{A}^2_{x_1,y_1}$, $U_2 = \mathbb{A}^2_{x_2,y_2}$ and the glueing is given by

$$U_1 \supset (x_1 \neq 0) \xrightarrow{\sim} (x_2 \neq 0) \subset U_2, \quad (x_1, y_1) \mapsto (x_1^{-1}, x_1^n y_1).$$

(a) Show that $C \subset X$ defined by $C \cap U_i = V(y_i)$ for $i = 1, 2$ is a closed subvariety isomorphic to $\mathbb{P}^1$.

(b) Show that the morphisms

$$p_i \colon U_i \to \mathbb{A}^1, \quad (x_i, y_i) \mapsto x_i$$

patch to give a morphism $p \colon X \to \mathbb{P}^1$. Moreover there is a section $s \colon \mathbb{P}^1 \to X$ of p with image C.

(c) Now assume $n > 0$. Compute $\mathcal{O}_X(X)$ as a subring of $k[x_1, y_1]$. Find a set of $n+1$ generators for $\mathcal{O}_X(X)$ as a k-algebra.

(d) Let $f \colon X \to \mathbb{A}^{n+1}$ be the morphism defined by the generators for $\mathcal{O}_X(X)$ found in (c). Show that $f(X)$ is closed, $f(C)$ is a point, and the restriction of f to $X \setminus C$ is an isomorphism onto its image.

[Hint: (c),(d). If you have trouble first try $n = 1$ and $n = 2$.]