10/2/19

HW 2 returned
HW 3 due now

Q	2	3abc	4	5ab	6	
	5	15	5	10	5	+5 = 45

HW 4 available, due next Wed. 10/9/19 at start of class
people.math.umass.edu/~hacking/461F19

Last Time : - Construction of regular pentagon using ruler & compass
Today : - Statement of Gauss' theorem on constructibility of regular polygons
- Sine rule
- Cosine rule
- Center of mass
- Coordinates?

Gauss Theorem

The regular n-gon is constructible by ruler & compass if and only if $n = 2^k \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_r$ where the p_i are distinct Fermat primes.

A Fermat prime is a prime number of the form $2^{(2^m)}+1$.

Remark : If $p = 2^\ell + 1$ is prime, must have $\ell = 2^m$ is a power of 2. (HW4 optional problem)

Fermat (~1600) conjectured that conversely, $2^{(2^m)}+1$ is always prime (this is FALSE)

Ex :

m	$2^{(2^m)}+1$
0	3
1	5
2	17
3	257
4	65537

all prime

In fact, there are only known primes of this form.

5 - Fermat prime, explains why regular pentagon can be constructed.

power of 2

F. prime ↑ F. prime

not allowed in Gauss thm

$9 = 3^{\textcircled{2}}$

$3 \cdot 5 = 15$

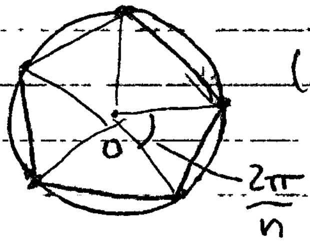
n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
constructible?	✓	✓	✓	✓	X	✓	X	✓	X	✓	X	X	✓	✓	✓

How is this theorem proved?

1. Observe regular n-gon is constructible

$\iff$ can construct angle $2\pi/n$

$\iff$ can construct length $\cos(2\pi/n)$.



(n=5 here)

n	$\cos(2\pi/n)$
3	$-1/2$
4	0
5	$(\sqrt{5}-1)/4$ (last time)
6	$1/2$
7	? (not constructible)

2. Galas theory (MATH 412)

$\cos(2\pi/2)$ is constructible

$\iff \cos(2\pi/n)$ can be obtained from 1 by $+, -, \times, \div$ & $\sqrt{\ }$

$\overset{G.T.}{\iff} \phi(n)$ is a power of 2 $\overset{\text{(optional HW Q)}}{\iff} n = 2^k \cdot p_1 \cdot p_2 \cdot \dots \cdot p_r$

Here, $\phi(n)$ = # of positive integers $a \leq n$ such that $\gcd(a, n) = 1$.

"Euler's totient function"

Ex: $n=15$

$\phi(n) = ?$ ↱ remove all numbers w/ common factors w/ 15.

$1, 2, \cancel{3}, 4, \cancel{5}, \cancel{6}, 7, 8, \cancel{9}, 1\cancel{0}, 11, \cancel{12}, 13, 14, \cancel{15}$

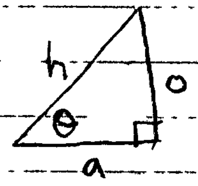
$\phi(n) = 8 = 2^3$

↳ # of remaining numbers

$$\cos\left(\frac{2\pi}{17}\right) = \frac{1}{16}\left(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}\right)$$

To read if interested: - Artin Algebra (~412 level)
- Hardy & Wright; An introduction to the theory of numbers pg. 57 (~ easier level)

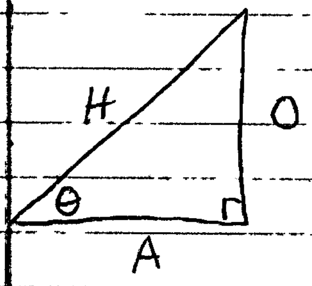
Trigonometry



$\sin\theta = \frac{o}{h}$ (SOHCAHTOA)

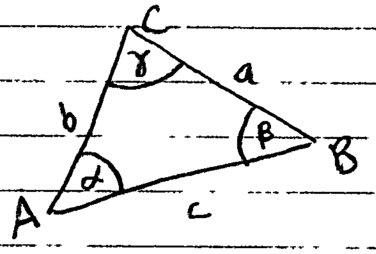
$\cos\theta = \frac{a}{h}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{o}{a}$



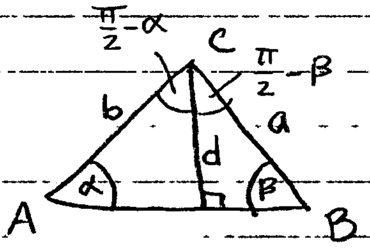
same definitions for both triangles?
↳ YES, well-defined b/c of similar triangles.

Sine Rule



$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Proof   Enough to show $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$ (others follow by same argument)

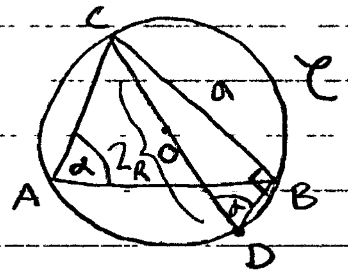


$\sin\alpha = \frac{d}{b}$ $\sin\beta = \frac{d}{a}$

$\frac{\sin\alpha}{a} = \frac{d}{ab} = \frac{\sin\beta}{b} = \frac{d}{ab}$ ▨

Sine Rule$^+$ (Director's Cut)

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$ (where R is the radius of the circumscribed circle)

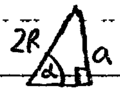


$|OA| = |OB| = |OC| = R$

$\angle CDB = \alpha$ by  theorem proved in class.

Proof of Sine Rule$^+$

Enough to show $\frac{a}{\sin\alpha} = 2R$

Recall $\sin\theta = \frac{o}{h}$ ⤳ $\sin\alpha = \frac{a}{2R}$, does  exist?

Yes, $\triangle CDB$. ▨