Math 462 Homework 1

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In the problems below S^2 denotes the sphere of some radius $R > 0$ in $\mathbb{R}^3$ with center the origin O. Justify your answers carefully.

(1) Let $P, Q \in S^2$ be the points $P = (1, 2, 1)$ and $Q = (2, 1, 1)$. Determine the radius R of the sphere and compute the spherical distance $d(P, Q)$.

(2) Let $P = (2, 3, 4)$ and $Q = (0, 2, 5)$. Compute the equation of the plane Π such that $\Pi \cap S^2$ is the great circle through P and Q.

(3) Let L and M be the great circles on S^2 given by $L = \Pi_L \cap S^2$ and $M = \Pi_M \cap S^2$ where Π_L and Π_M are the planes

$$\Pi_L = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 2y + z = 0\}$$

and

$$\Pi_M = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

Compute the points of intersection of L and M and the angle between L and M.

(4) (a) Let P be a polygon in the plane such that each of its angles is less than π (equivalently, P is *convex*). Show that the sum of the angles of P equals $(n - 2)\pi$, where $n \geq 3$ is the number of sides of P.

(b) Let P be a spherical polygon on the sphere S^2 of radius $R = 1$ such that each of its angles is less than π. (So, the sides of P are segments of great circles.) Show that the sum of the angles of P equals $(n - 2)\pi + A$, where $n \geq 3$ is the number of sides of P and A is the area of P.

(c) Show that the formula in part (b) also works in the case $n = 2$.

(5) A spherical circle with center a point P on S^2 and radius r is the locus of points Q on S^2 such that the spherical distance $d(P, Q)$ equals r. Note: The spherical distance between two points P and Q on S^2 is at most πR (why?). So it only makes sense to talk about spherical circles of radius r for $0 < r < \pi R$.

(a) Show that the circumference of a spherical circle of radius r equals $2\pi R \sin(r/R)$. [Hint: A spherical circle with center P is a Euclidean circle in $\mathbb{R}^3$ obtained by intersecting the sphere S^2 with a plane normal to the line OP. Notice that the Euclidean circumference is equal to the spherical circumference, but the Euclidean center and radius are different from the spherical center P and radius r.]

(b) What happens to the circumference of a spherical circle of radius r as r approaches πR? Explain your answer geometrically.

(c) Show that the circumference of a spherical circle of radius r is less than the circumference of a Euclidean circle of the same radius.

(d) If r is small, use the approximation $\sin(x) \approx x - x^3/6$ to give an approximate value for the circumference.

(6) Given a spherical circle with center P, the associated spherical disc is the region on S^2 which is enclosed by the spherical circle and contains P.

(a) Show that the area of a spherical disc of radius r equals $2\pi R^2(1 - \cos(r/R))$.

(b) What happens to the area of a spherical disc of radius r as r approaches πR? Explain your answer geometrically.

(c) Show that the area of a spherical disc of radius r is less than the area of a Euclidean disc in $\mathbb{R}^2$ of the same radius.

(d) If r is small, use the approximation $\cos(x) \approx 1 - x^2/2 + x^4/24$ to give an approximate value for the area.

(7) Let L be a great circle on S^2 and P a point on S^2 not lying on L.

(a) Show how to construct a great circle M through P and perpendicular to L.

(b) Is the great circle M uniquely determined by P and L?

(c) Carry out your construction explicitly in the case $P = (1, 1, 1)$ and $L = \Pi_L \cap S^2$ where

$$\Pi_L = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 4y + z = 0\}.$$