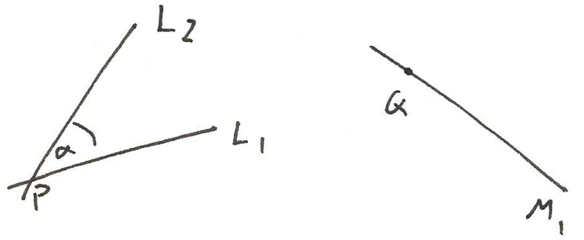
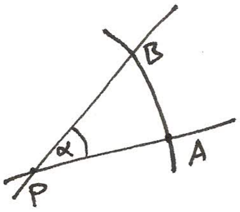
Wednesday 9/18/19.

MATH 461 HW1 SOLUTIONS.

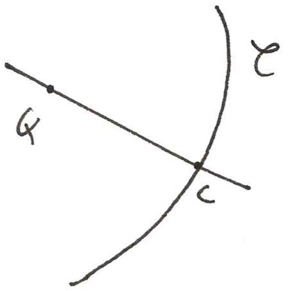
Q1.



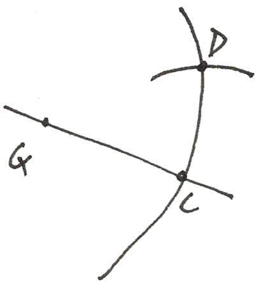
1. Draw a circle, center P, of some radius r, intersecting L_1 at a point A & L_2 at a point B as shown.



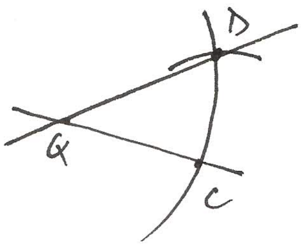
2. Draw a circle $\mathcal{C}$ with center Q of the same radius r, intersecting M_1 at a point C.



3. Draw a circle with center C and radius $|AB|$, intersecting $\mathcal{C}$ at a point D.



4. Join Q & D by a line.



<u>Claim</u>: $\angle CQD = \alpha$.

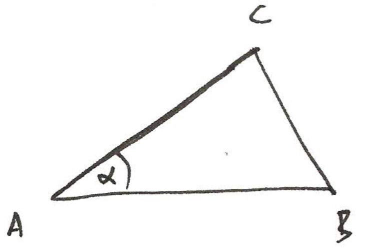
<u>Proof</u>: By construction $|QC| = |QD| = |PA| = |PB|$

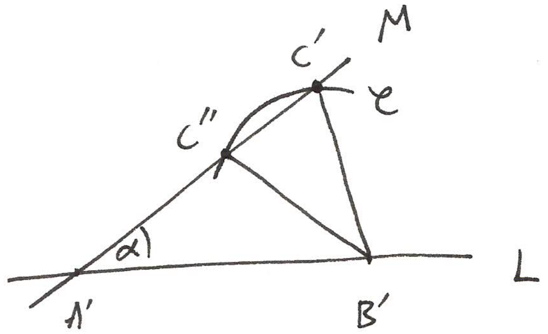
and $|CD| = |AB|$.

So $\triangle CQD \cong \triangle APB$ (SSS).

In particular $\angle CQD = \angle APB = \alpha$ $\square$

Q2. No. We follow the hint.





By construction, $|A'B'| = |AB|$

$|B'C'| = |B'C''| = |BC|$

and $\angle C'A'B' = \angle C''A'B' = \angle CAB$.

But $\triangle A'B'C' \not\cong \triangle A'B'C''$ (because $|A'C'| > |A'C''|$)

This gives a counterexample to the "ASS" criterion for congruence.

So ASS is <u>NOT</u> a valid congruence criterion.

[<u>Remark</u>: For the counterexample above, we require that $\angle CAB < \angle BCA$ and $\angle BCA \neq \pi/2$, so that $\mathcal{C}$ intersects the line M in two points C' & C'' on the same side of A']

Q3

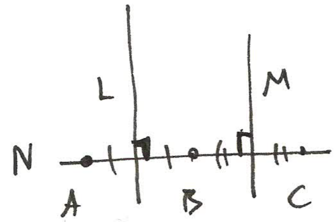
a) <u>Claim</u> : The perpendicular bisectors of AB & BC intersect

$\iff$ A, B, C do not lie on a line.

<u>Proof</u>: $\underline{\Rightarrow}$: We will prove the contrapositive statement (recall $X \Rightarrow Y$ is equivalent to NOT $Y \Rightarrow$ NOT X, the <u>contrapositive</u> statement (proved in MATH 300)).

That is, A, B, C do lie on a line $\Rightarrow$ the perpendicular bisectors of AB & BC do not intersect.

So, assume A, B, C lie on a line N.

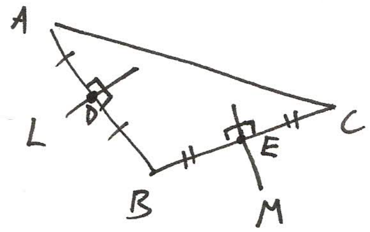


Sum of interior angles on one side of N

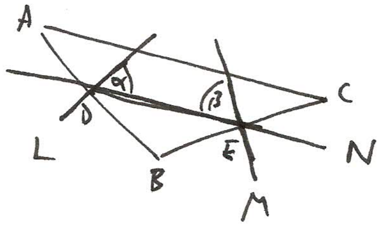
$:- \ \alpha + \beta = \pi/2 + \pi/2 = \pi \iff L$ & M are parallel, that is, L & M do not intersect, as required.

(proved in class using the parallel axiom)

$\underline{\Leftarrow}$:



Draw line DE connecting the midpoints of AB & BC.



$\alpha = \pi/2 - \angle EDB < \pi/2$

$\beta = \pi/2 - \angle DEB < \pi/2$

$\Rightarrow \alpha + \beta < \pi$

$\Rightarrow L$ & M intersect $\square$.

(parallel axiom).

b) Let O be the intersection point of the perpendicular bisectors L & M of AB & BC.

Then $|OA| = |OB|$ and $|OB| = |OC|$

(the perpendicular bisector of a line segment AB is the set of points P such that $|AP| = |BP|$).

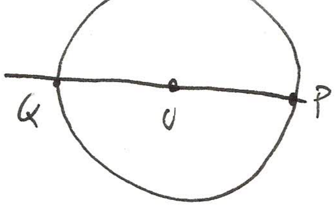
So $|OA| = |OB| = |OC| =: r$.

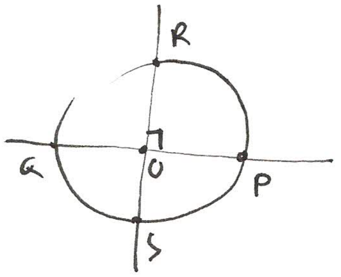
Now draw a circle $\mathcal{C}$ center O and radius r.

Then $\mathcal{C}$ passes through $A, B, \& C$.

(Also, the circle $\mathcal{C}$ is uniquely determined by this property:- first, the center of the circle must be equidistant from $A, B, \& C$, so must lie on both the perpendicular bisectors of AB & BC, and so is equal to their intersection point O. Now the radius must equal $|OA| = r$.)

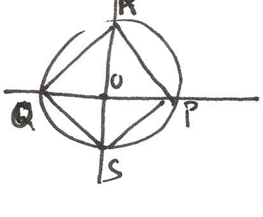
4. a) $n = 4$





1. Draw line OP, intersecting circle C at P & Q
2. Draw the perpendicular bisector of PQ, intersecting C at R & S (& passing through O, since $|OP| = |OQ|$).
3. Draw lines PR, RQ, QS, SP.

<u>Claim:</u> $PRQS$ is a square (regular 4-gon)



Proof: $\triangle POR \cong \triangle ROQ \cong \triangle QOS \cong \triangle SOP \quad (\dagger)$ (SAS)

$:-$ $|OP| = |OR| = |OQ| = |OS| = r$, the radius of C,

and $\angle POR = \angle ROQ = \angle QOS = \angle SOP = \pi/2$ by construction.

So $|PR| = |RQ| = |QS| = |SP|$, that is, $PRQS$ has equal side lengths.

Finally $\triangle POR$ is isosceles ($|OP| = |OR| = r$)

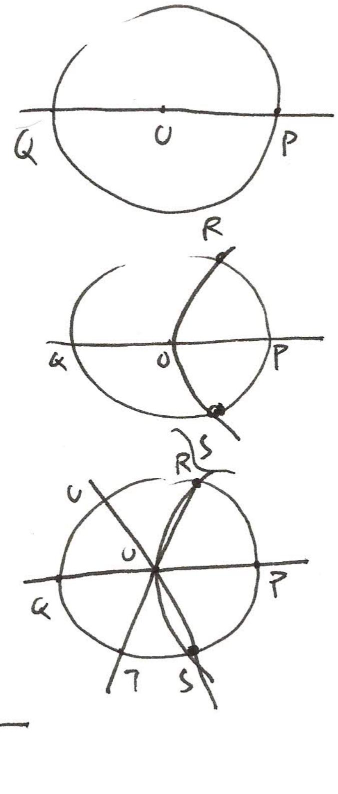
and $\angle POR = \pi/2 \implies \angle OPR = \angle ORP = \frac{1}{2}(\pi - \pi/2) = \pi/4$

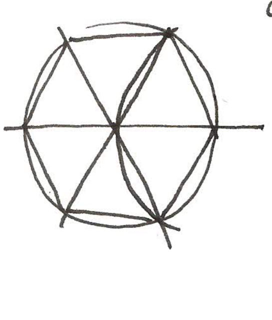
by the isosceles triangle theorem and "angle sum of a triangle $= \pi$".

Now using the congruences $(\dagger)$ we see that each internal angle of quadrilateral $PRQS$ equals $\pi/4 + \pi/4 = \pi/2$.

So $PRQS$ is a square. $\square$.

b) $n = 6$.





1. Draw line OP, intersecting the circle C at P & Q.
2. Draw a circle center P, radius $|OP| =: r$,
3. ~~Draw a circle center Q, radius $|OQ| = r$~~

intersecting C at R & S.

4. Draw line OR, intersecting C at T, and line OS, intersecting C at U.
5. Draw lines PR, RU, UQ, QT, TS, SP.

Claim: $PRUQTS$ is a regular hexagon (6-gon)

Proof: By construction $|PR| = |PS| = r$

and $|OP| = |OR| = |OS| = r$ (radius of C).

So ΔOPR & ΔOSP are equilateral.

In particular, the interior angles of these triangles are equal to $\pi/3$ $(*)$

(because "angle sum of triangle $= \pi$" and the angles are equal by the isosceles triangle theorem)

Now, using the fact that π is the angle on a straight line,

we find that $\angle ROU = \pi - \angle POR - \angle SOP = \pi - \pi/3 - \pi/3 = \pi/3$,

and deduce $\angle POR = \angle ROU = \angle UOQ = \angle QOT = \angle TOS = \angle SOP = \pi/3$.

Also $|OP| = |OR| = |OU| = |OQ| = |OT| = |OS| = r$. (the radius of C)

so $\Delta POR \cong \Delta ROU \cong \Delta UOQ \cong \Delta QOT \cong \Delta TOS \cong \Delta SOP$ (SAS), $(\dagger)$

in particular $|PR| = |RU| = |UQ| = |QT| = |TS| = |SP|$, that is,

PRUQTS has equal side lengths.

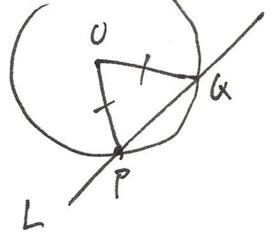
Finally, since ΔPOR has angles $\pi/3$ (see $(*)$ above), using the congruences $(\dagger)$

we find that PRUQTS has interior angles equal to $\pi/3 + \pi/3 = 2\pi/3$.

So hexagon PRUQTS is regular. $\square$.

5. a We follow the hint.

The contrapositive statement is: If L is not tangent to C then OP is not perpendicular to L.



Proof: If L is not tangent to C then L intersects C at another point Q (besides P).

The triangle OPQ is isosceles $(|OP| = |OQ| = \text{radius of } C)$

so by the isosceles triangle theorem $\angle OPQ = \angle OQP$

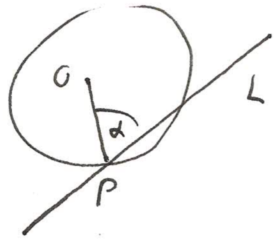
and by "angle sum of triangle $= \pi$", $\angle OPQ = \frac{1}{2}(\pi - \angle POQ) < \pi/2$.

So L is NOT perpendicular to OP. $\square$

b) We will again prove the contrapositive statement:-

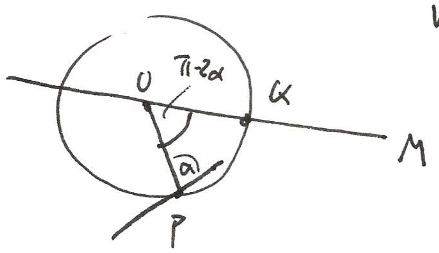
If OP is NOT perpendicular to L then L is NOT tangent to C

(that is, L intersects C at another point Q.)



Let α be the (acute) angle L makes with OP.

Draw the line M through O making angle $\pi - 2\alpha$ with OP, intersecting the circle at Q



The triangle OPQ is isosceles (because $|OP| = |OQ| =$ radius of C) so $\angle OPQ = \angle OQP$, and

$\angle OPQ = \frac{1}{2}(\pi - (\pi - 2\alpha)) = \alpha$

(by "angle sum of triangle $= \pi$").

So PQ coincides with the line L.

So L intersects C at Q. $\square$.