Math 300.2 Homework 3

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Reading: Gilbert and Vanstone, Chapter 4.

(1) Prove that $n! \geq 2^n$ for $n \geq 4$.

(2) We showed in class that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and we showed on the last homework that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. Deduce a formula for $\sum_{r=1}^{n} r^2$. Simplify your answer.

(3) Consider a $2^n \times 2^n$ checkerboard. [For example, a standard checkerboard is $8 \times 8 = 2^3 \times 2^3$.] Suppose one of the squares is covered by a single square tile. Show that the remaining squares can be covered by a collection of L-shaped tiles (each covering 3 squares). [Hint: Use induction. The board for $n+1$ can be divided into 4 copies of the board for n.]

(4) Define a sequence of integers $a_0, a_1, a_2, \ldots$ by $a_0 = 3$, $a_1 = 5$, and

$$a_{n+2} = 2a_{n+1} + 3a_n$$

for each $n \geq 0$. Show that

$$a_n = (-1)^n + 2 \cdot 3^n$$

for all $n \geq 0$.

(5) Let n be an integer such that $n \geq 18$. Show that there are nonnegative integers x and y such that $4x + 7y = n$. [Hint: We discussed a similar problem in class phrased in terms of postage stamps. Give a careful proof using (strong) induction.]

(6) Expand the following expressions using the binomial theorem. Simplify your answer as much as possible.

(a) $(1+x)^6$

(b) $(2x+3y)^3$

(c) $(x^2-1)^4$

(d) $(2xy-z^2)^5$

(7) A coin is tossed 7 times. What is the probability of getting exactly 3 heads?

(8) Prove the following identities

(a)

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n.$$

(b)

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\cdot\binom{n}{n}=0.$$