Math 462 Homework 2

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In the problems below we consider the sphere S^2 of radius $R = 1$ in $\mathbb{R}^3$ with center the origin O. Justify your answers carefully.

(1) Let ABC be a spherical triangle with angles a, b, c and side lengths α, β, γ. (Here α denotes the length of the side opposite A, etc.) Suppose angle a is a right angle, i.e., $a = \pi/2$.

  (a) Prove the spherical Pythagorean formula: $\cos(\alpha) = \cos(\beta)\cos(\gamma)$.

  (b) Check the formula in part (a) for triangles ABC for which C is the north pole and A and B lie on the equator.

  (c) Suppose now the spherical triangle ABC is very small. Use the approximation $\cos(x) \approx 1 - x^2/2$ for x small to show that the formula in part (a) is approximately equivalent to the usual Pythagorean formula in $\mathbb{R}^2$ in this case.

(2) Let ABC be a spherical triangle.

  (a) Prove that $\text{Area}(ABC) < 2\pi$.

  (b) Show that the area can be arbitrarily close to 2π (so that the bound in part (a) is sharp).

  [Hint for part (b): Draw the lune given by two sides of the triangle. How does the area vary as you move the position of the third side?]

(3) Let ABC be a spherical triangle.

  (a) Show that there is a unique spherical circle passing through the vertices A, B, C. This is called the *circumcircle* of the spherical triangle.

(b) Find the spherical center and spherical radius of the circumcircle of the triangle ABC with vertices $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$.

(c) Find a formula for the spherical center of the circumcircle in the general case using the cross product.

[Hints: A spherical circle is obtained as the intersection of the sphere S^2 with a plane Π (not necessarily passing through the origin O). If P is the spherical center of the spherical circle then $\overrightarrow{OP}$ is a normal vector to the plane Π.]

(4) Let ABC be a spherical triangle, with angles a, b, c and side lengths α, β, γ. We can define the *polar spherical triangle* $A'B'C'$ as follows: Let Π_{BC} be the plane passing through the points O, B and C. (So $S^2 \cap \Pi_{BC}$ is the spherical line passing through B and C.) The position vector $\overrightarrow{OA'}$ of A' is the normal vector to the plane Π_{BC} which has length 1 and lies on the same side of Π_{BC} as A. We define B' and C' analogously.

(a) Show that the side lengths of the polar spherical triangle $A'B'C'$ are given by $\alpha' = \pi - a$, $\beta' = \pi - b$, $\gamma' = \pi - c$.

(b) Show that the polar spherical triangle of $A'B'C'$ is ABC. (That is, if we apply the polar construction twice we recover the original spherical triangle.)

(c) Deduce that the angles of the polar spherical triangle are given by $a' = \pi - \alpha$, $b' = \pi - \beta$, $c' = \pi - \gamma$.