Math 412 Midterm 1 review questions

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Reading: Saracino, Chapters 16,17,18.

(1) Give the precise definition of a ring. Which of the following are rings? Justify your answer carefully. If it is not a ring, determine which of the ring axioms hold and which fail.

(a) $R = \mathbb{R}^2$ with the usual addition

$$\mathbf{a} \oplus \mathbf{b} := \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and multiplication given by

$$\mathbf{a} \otimes \mathbf{b} := \begin{pmatrix} a_1 b_1 \\ a_2 b_1 + a_1 b_2 \end{pmatrix}.$$

(b) $R = \mathbb{R}[x]$ the set of polynomials in the variable x with real coefficients, with the usual pointwise addition

$$(f \oplus g)(x) := f(x) + g(x)$$

and multiplication given by composition of functions

$$(f \otimes g)(x) := f(g(x)).$$

(c) $R = \mathbb{R}^3$ with the usual addition

$$\mathbf{a} \oplus \mathbf{b} := \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

and multiplication given by the cross product

$$\mathbf{a} \otimes \mathbf{b} := \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

(d) Let V be a real vector space and R the set of linear maps $T\colon V \to V$, with addition defined pointwise

$$(T_1 \oplus T_2)(\mathbf{x}) := (T_1 + T_2)(\mathbf{x}) := T_1(\mathbf{x}) + T_2(\mathbf{x})$$

and multiplication defined by composition

$$(T_1 \otimes T_2)(\mathbf{x}) := (T_1 \circ T_2)(\mathbf{x}) := T_1(T_2(\mathbf{x})).$$

(2) Let R be a ring and $S \subset R$ a subset of R. What does it mean to say that S is a subring of R? In each of the following cases, determine whether S is a subring.

(a) $R = \mathbb{R}^{2\times 2}$, $S = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$.

(b) $R = \mathbb{R}^{2\times 2}$, $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{R} \right\}$.

(c) $R = \mathbb{R}^{2\times 2}$, $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$.

(d) $R = \mathbb{C}$, $S = \{a + b\alpha \mid a, b \in \mathbb{Z}\}$ where $\alpha = e^{\pi i/3} = (1 + \sqrt{3}i)/2$.

(e) $R = \mathbb{C}$, $S = \{a + b\beta \mid a, b \in \mathbb{Z}\}$ where $\beta = (1 + \sqrt{7})/2$.

(f) $R = \mathbb{Q}$, $S = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, \quad b \text{ is not divisible by } 4\}$.

(g) $R = \mathbb{Q}$, $S = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, \quad b \text{ is not divisible by } 5\}$.

(h) $R = \mathbb{R}[x]$, $S = \{f \in \mathbb{R}[x] \mid f(1) = 1\}$.

(i) $R = \mathbb{R}[x]$, $S = \{f \in \mathbb{R}[x] \mid f(1) = f(-1)\}$.

(j) $R = \mathbb{R}[x]$, $S = \{f \in \mathbb{R}[x] \mid f'(0) = 0\}$.

(3) Let R be a ring. What does it mean to say that $a \in R$ is a zero divisor? Identify the zero-divisors in the following rings.

(a) $R = \mathbb{Z}/12\mathbb{Z}$.

(b) $R = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$

(c) $R = \mathbb{R}^{2\times 2}$.

(d) $R = \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$.

(e) $R = \mathbb{Z} \oplus \mathbb{Z}$.

(f) $R = \mathbb{R}[x]/(x^3)$.

(g) $R = \mathbb{R}[x]/(x^2 - 3x + 2)$.

(4) Let R be a ring. What does it mean to say that $a \in R$ is nilpotent? Identify the nilpotent elements in the following rings.

(a) $R = \mathbb{Z}/20\mathbb{Z}$.

(b) $R = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$.

(c) $S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$, a subring of $R = \mathbb{R}^{2\times 2}$.

(d) $\mathbb{R}[x]/(x^3)$.

(e) $\mathbb{R}[x]/(x^3 - x^2)$.

(5) Let R be a ring with 1. What does it mean to say that $a \in R$ is a unit? Identify the units in the following rings.

(a) $R = \mathbb{Z}/15\mathbb{Z}$.

(b) $R = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$.

(c) $R = \mathbb{Z}$.

(d) $R = \mathbb{R} \oplus \mathbb{R}$.

(e) $R = \mathbb{R}[x]$.

(f) $R = \mathbb{R}^{2\times 2}$.

(g) $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \;\middle|\; a, c, d \in \mathbb{R} \right\}$, a subring of $R = \mathbb{R}^{2\times 2}$.

(h) $R = \mathbb{Z}[i]$

(i) $S \subset \mathbb{C}$ the subring of Q2(d).

(6) Give the definition of an ideal of a ring R. Which of the following subsets $I \subset R$ are ideals?

(a) $R = \mathbb{Z}$, $I = \{a \in \mathbb{Z} \mid a \text{ is divisible by either 2 or 3}\}$.

(b) $R = \mathbb{Z}$, $I = \{6x + 15y \mid x, y \in \mathbb{Z}\}$.

(c) $R = \mathbb{R}[x]$, $I = \{f \in \mathbb{R}[x] \mid f(3) = 0\}$.

(d) $R = \mathbb{R}[x]$, $I = \{f \in \mathbb{R}[x] \mid f'(3) = 0\}$.

(e) $R = \mathbb{R}[x]$, $I = \{f \in \mathbb{R}[x] \mid f(3) = f'(3) = 0\}$.

(f) $R = \mathbb{Z}[x]$, $I = \{f \in \mathbb{Z}[x] \mid f(4) \equiv 0 \bmod 6\}$.

(7) List all the ideals in the following rings.

(a) $\mathbb{Z}$.

(b) $\mathbb{Z}/15\mathbb{Z}$.

(c) $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$.

(d) $\mathbb{R}[x]$.

(e) $\mathbb{R}[x]/(x^2 + 5x + 6)$

(f) $\mathbb{R}[x]/(x^3 + 4x)$

(8) Let R be a commutative ring with 1. Give the definition of a prime ideal of R. List all the prime ideals in the following rings.

(a) $\mathbb{Z}$.

(b) $\mathbb{C}[x]$.

(c) $\mathbb{Z}/12\mathbb{Z}$.

(d) $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$.

(e) $\mathbb{C}[x]/(x^2 + 2x + 2)$.

(f) $\mathbb{R}[x]$.

(g) $\mathbb{R}[x]/(x^n)$, $n \in \mathbb{N}$.

(9) Give the definition of a ring homomorphism $\varphi\colon R \to S$. Which of the following are ring homomorphisms?

(a) $\varphi\colon \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^{2\times 2}$, $\varphi(a_1, a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$.

(b) $\varphi\colon \mathbb{C} \to \mathbb{R}[x]$, $\varphi(a+bi) = a+bx$.

(c) $\varphi\colon S \to \mathbb{R}$, $\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a$, where $S \subset \mathbb{R}^{2\times 2}$ is the subring of Q4(c).

(d) $\varphi\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, \varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$,

(e) $\varphi\colon (\mathbb{Z}/3\mathbb{Z})[x] \to (\mathbb{Z}/3\mathbb{Z})[x]$, $\varphi(f) = f^3$.

(10) Let $\varphi\colon R \to S$ be a homomorphism of rings. Define the kernel $\ker(\varphi)$ of φ. For each of the following ring homomorphisms describe the kernel as explicitly as possible.

(a) $\varphi\colon \mathbb{R}[x] \to \mathbb{R}$, $\varphi(f(x)) = f(2)$.

(b) $\varphi\colon \mathbb{R}[x] \to \mathbb{C}$, $\varphi(f(x)) = f(2+3i)$.

(c) $\varphi\colon R \to R/I$, $\varphi(a) = a + I$.

(d) $\varphi\colon \mathbb{Z} \to \mathbb{Z}[i]/(1+3i)$, φ the homomorphism determined by $\varphi(1) = 1$.

(e) $\varphi\colon \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$, φ the homomorphism determined by $\varphi(1) = 1$.

(f) $\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}/7\mathbb{Z}$, $\varphi(f(x)) = f(3) \bmod 7$.

(11) State the first isomorphism theorem for ring homomorphisms. Use this theorem to identify the following quotient rings with a standard ring.

(a) $\mathbb{R}[x]/(x-5)$.

(b) $\mathbb{R}[x]/(x^2+2x+5)$.

(c) $\mathbb{R}[x]/(x^2-3x+2)$.

(d) S/I where $S \subset \mathbb{R}^{2\times 2}$ is the subring of Q5(g) and $I \subset S$ is the ideal given by

$$I = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}.$$

(12) Let R be a ring. Show that if $a \in R$ is not a zero divisor then $ab = ac \Rightarrow b = c$.

(13) Let R be a ring with 1 and $a \in R$ a unit. Show that a is not a zero divisor.

(14) Let R be a ring with 1. Show that if $I \subset R$ is an ideal and I contains a unit then $I = R$.

(15) Let R be a ring with 1. Show that if $a \in R$ is nilpotent then $1 + a$ is a unit.

[Hint: Recall that $(1 + x)^{-1}$ may be expanded as a power series in x which converges for $|x| < 1$, $x \in \mathbb{R}$. Use this to guess a formula for $(1 + a)^{-1}$, and prove your guess is correct.]

(16) Let R_1 and R_2 be rings.

(a) Define the direct sum $R_1 \oplus R_2$.

(b) Assume $R_1 \neq \{0\}$ and $R_2 \neq \{0\}$. Show that $R_1 \oplus R_2$ is not an integral domain.

(17) Show that the following definition of a field is equivalent to the one given in class and the text: A field K is a set with two operations $(a, b) \mapsto a + b$ (addition) and $(a, b) \mapsto a \cdot b$ (multiplication) such that (1) $(K, +)$ is an abelian group, (2) $(K \setminus \{0\}, \cdot)$ is an abelian group (where 0 denotes the additive identity), and (3) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in K$.

(18) Let R and S be rings (commutative, with 1) and $\varphi \colon R \to S$ a ring homomorphism such that $\varphi(1) = 1$. Let $I \subset S$ be an ideal. Define

$$\varphi^{-1}(I) = \{a \in R \mid \varphi(a) \in I\}.$$

(a) Show that $\varphi^{-1}(I) \subset R$ is an ideal.

(b) Show that if I is a prime ideal of S then $\varphi^{-1}(I)$ is a prime ideal of R.

(19) Let R be a commutative ring with 1.

(a) Give the definition of a maximal ideal of R.

(b) Show that a maximal ideal is a prime ideal.

(c) Give an example of a prime ideal that is not a maximal ideal.

(20) Let K be a field, R a ring, and $\varphi\colon K \to R$ a ring homomorphism. Show that either φ is injective or $\varphi(a) = 0$ for all $a \in R$.

(21) Let R be a commutative ring with 1.

   (a) Give the definition of the principal ideal (a) generated by an element $a \in R$.

   (b) Give an example of an ideal I of a ring R (commutative, with 1) such that I is not a principal ideal.

(22) Give an example of a ring R and an element $a \in R$ such that there exists $b \in R$ with $ba = 1$ but there does not exist $c \in R$ with $ac = 1$.

[Hint: Consider the ring of Q1(d) for the vector space V of infinite real sequences $(a_1, a_2, \ldots)$.]