Answers to practice questions for Midterm 1

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1.

(a) The RREF (reduced row echelon form) of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} \end{pmatrix}$$

So the system of linear equations has exactly one solution given by $x = \frac{-3}{2}$, $y = 1$, and $z = \frac{3}{2}$. Geometrically, each of the equations defines a plane in $\mathbb{R}^3$, and the intersection of the three planes is one point.

(b) The RREF of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & \frac{5}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -4 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

So the system of linear equations has a line of solutions given by $x = \frac{3}{2} - \frac{5}{2}t, y = -1 + 4t, z = \frac{3}{2} - \frac{5}{2}t$ (the variable t is free). Geometrically, each equation defines a linear space of dimension 3 in $\mathbb{R}^4$, and the intersection of these is a line in $\mathbb{R}^4$.

2. The vector equation is equivalent to a system of 3 linear equations in 3 variables. The RREF of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So there are no solutions (the last row corresponds to the equation $0 = 1$). Geometrically, the 3 vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ lie in a plane P in $\mathbb{R}^3$, and the vector

$\mathbf{b}$ does not lie in P. So any combination $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$ also lies in P and in particular cannot be equal to $\mathbf{b}$.

3. The RREF of the augmented matrix is

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the solutions are given by $x_1 = 1 - 2x_2 - x_4$ and $x_3 = 3 - 2x_4$ (the variables x_2 and x_4 are free).

The equation $A\mathbf{x} = \mathbf{c}$ does *not* have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^3$ — changing $\mathbf{b}$ to $\mathbf{c}$ will only change the last column of the RREF above, and for some choice of $\mathbf{c}$ we will get last row of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & d \end{pmatrix}$$

for some $d \neq 0$. Then for this choice of $\mathbf{c}$ the equation $A\mathbf{x} = \mathbf{c}$ has no solutions.

4.

(a) S is reflection in the y-axis, T is rotation about the origin through an angle of $\pi/4$ radians clockwise, U is reflection in the line $y = -x$, V is a vertical shear (see p. 63–64 of the textbook).

(b) The matrix of $S \circ U$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix of $T \circ T$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix of $T^{-1} \circ U \circ T$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Geometrically, $S \circ U$ and $T \circ T$ are both rotation about the origin through an angle of $\pi/2$ radians clockwise, $T^{-1} \circ U \circ T$ is reflection in the x-axis.

5.

(a) The matrix of a rotation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ about the origin through an angle of θ radians anticlockwise is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In our case $\theta = \pi/3$ so the matrix is

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(b) The reflection $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ in the line through the origin with direction $\mathbf{v}$ is given by the formula

$$U(\mathbf{x}) = 2\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} - \mathbf{x}.$$

In our case $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so, writing $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ we have

$$U(\mathbf{x}) = 2\left(\frac{x+2y}{5}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2x+4y \\ 4x+8y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3x+4y \\ 4x+3y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\mathbf{x}.$$

So the matrix of U is $\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$.

Alternatively, we can first write down a vector perpendicular to the line, for example $\mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, and use the formula

$$U(\mathbf{x}) = \mathbf{x} - 2\left(\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}$$

for $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ the reflection in the line through the origin perpendicular to the vector $\mathbf{n}$.

(c) The equation $x + 2y + 3z = 0$ of the plane can be written as $\mathbf{x}\cdot\mathbf{n} = 0$, where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. So we see that the vector $\mathbf{n}$ given by the coefficients of the equation is a normal vector to the plane (a vector perpendicular to every vector lying in the plane). The projection $V\colon \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane through the origin with normal vector $\mathbf{n}$ is given by the formula

$$V(\mathbf{x}) = \mathbf{x} - \left(\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}.$$

In our case this gives matrix

$$\frac{1}{14}\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}.$$

(d) Recall that the matrix of a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has columns $T\mathbf{e}_1, \dots, T\mathbf{e}_n$, where $\mathbf{e}_i$ denotes the vector with ith entry 1 and all other entries 0. Now using this recipe for W (draw a picture!) gives matrix

$$\begin{pmatrix} \cos(\pi/3) & 0 & \sin(\pi/3) \\ 0 & 1 & 0 \\ -\sin(\pi/3) & 0 & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

6.

(a) The line is the line through the origin in some direction. (Note: if we project onto a line that does not go through the origin then we don't get a linear map, because for a linear map T we always have $T(\underline{0}) = \underline{0}$.) To find the direction, just compute the image $T(\mathbf{v})$ of any vector under the map T — that will be a vector in the direction of the line. For example if we take $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $T(\mathbf{v}) = \frac{1}{13}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \frac{2}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

So the line has direction $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(b) To find the plane we observe that for any vector $\mathbf{v}$ in $\mathbb{R}^3$ the difference $T(\mathbf{v}) - \mathbf{v}$ will be a normal vector to the plane. Taking $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ we find

$$T(\mathbf{v}) - \mathbf{v} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

So the plane of reflection is the plane through the origin with normal vector $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, in other words, the plane with equation

$$-x + y - z = 0.$$

(c) A vector $\mathbf{x}$ in the direction of the axis of rotation satisfies $U(\mathbf{x}) = \mathbf{x}$. This is a system of linear equations for $\mathbf{x}$. Solving for $\mathbf{x}$ we find that $\mathbf{x}$ is a multiple of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. So the axis of rotation is the line through the origin in the direction $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

7.

(a) To find the inverse of A we apply the row reduction algorithm to the matrix $(A\ I)$ formed by A and the identity matrix. This gives a matrix $(I\ B)$, and then $B = A^{-1}$ is the inverse of A. In this way we compute

$$\begin{pmatrix} -11 & 5 & -2 \\ 7 & -3 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

(b) The system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. This has solution $\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -11 \\ 7 \\ -2 \end{pmatrix}$.

8. The matrix of T is the matrix with columns given $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, that is, the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Now we can compute the images of the vertices

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

of the unit cube under the map T: they are

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix}.$$

The image of an edge of the cube is the line segment joining the images of the two vertices on the edge.