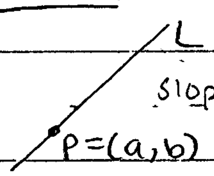
10/11/19

Midterm 1: Wed 10/16/19 7-9PM LGRC A301
Midterm review problems available (check e-mail/website)
HW4 solutions available " "
You are allowed one sheet of notes (letter-size, both sides) on exam.
No class Monday, but we have class Tuesday instead.

Last time • Equation of a line

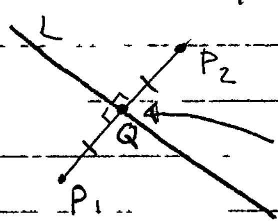


slope m $\rightsquigarrow (y-b)=m(x-a)$, or $y=mx+c$
If "$m=\infty$", $x=a$

both cases: $ax+by=c$, $(a,b)\neq(0,0)$

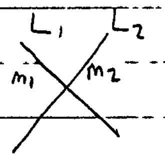
Today • Perpendicular bisector of a line segment
• Algebraic interpretation of ruler & compass constructions
• (isometries) $\rightsquigarrow$ constructible lengths: obtained by 1 and $+,-,\times,\div,\&\sqrt{\ }$

Q1: Given $P_1=(a_1,b_1)$, $P_2=(a_2,b_2)$
find the equation of the perpendicular bisector of P_1P_2.



$Q=\left(\frac{1}{2}(a_1+a_2), \frac{1}{2}(b_1+b_2)\right)$

Aside:



where m_1 & m_2 are slopes $\rightsquigarrow$ find eq. of perp. bisector L
$L_1 \perp L_2 \Leftrightarrow m_1 m_2 = -1$
$\Leftrightarrow m_2 = -1/m_1$

Q2: How can we characterize the perp. bisector?
$L=\{R \mid |RP_1| = |RP_2|\}$
$=\{(x,y) \mid \sqrt{(x-a_1)^2+(y-b_1)^2} = \sqrt{(x-a_2)^2+(y-b_2)^2}\}$
Simplify: $(x-a_1)^2+(y-b_1)^2=(x-a_2)^2+(y-b_2)^2$
$\cancel{x^2}-2a_1x+a_1^2+\cancel{y^2}-2b_1y+b_1^2 = \cancel{x^2}-2a_2x+a_2^2+\cancel{y^2}-2b_2y+b_2^2$

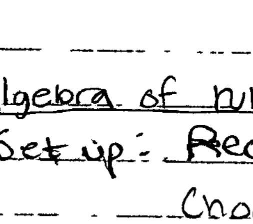
$Ax+By=C$ $\quad \underbrace{2(a_2-a_1)x+2(b_2-b_1)y=(a_2^2+b_2^2)-(a_1^2+b_1^2)}_{\text{equation of perp. bisector } L}$

Algebra of ruler & compass constructions

Set up: Recall we're given a line segment of length 1.

Choose coordinates so that this line segment has endpoints $(0,0)$, $(1,0)$.

Q: What points, $P=(a,b)$, in the plane can we construct using ruler & compass?



Allowed constructions:

1. Given two points $P_1=(a_1,b_1)$, $P_2=(a_2,b_2)$, draw line L through P_1 & P_2.

Equation of L: $\dfrac{(y-b_1)}{(x-a_1)} = m = \dfrac{(b_2-b_1)}{(a_2-a_1)}$

$(a_2-a_1)(y-b_1) = (b_2-b_1)(x-a_1)$ $\quad Ax+By=C$

coefficients A,B,C are obtained from a_1,b_1,a_2,b_2 (coords of P_1 & P_2) by $+,-,\times,\div$.

2. Given a point $P=(a,b)$ & a length r, draw circle $\mathcal{C}$ center P & radius r: $\sqrt{(x-a)^2+(y-b)^2} = r$

$(x-a)^2+(y-b)^2 = r^2$ eq. of $\mathcal{C}$

coefficients obtained from a,b,r by $+,-,\times,\div$.

3. Compute the intersection point of 2 lines.

" " line & a circle.

" " 2 circles.

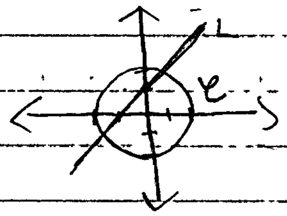
(a) $\left.\begin{array}{l} L_1: A_1x+B_1y=C_1 \\ L_2: A_2x+B_2y=C_2 \end{array}\right\}$ MATH 235 or highschool $\rightsquigarrow (x,y)=(a,b)$

obtained from coefficients A_1,B_1,C_1,A_2,B_2,C_2 by $+,-,\times,\div$

(b) line & a circle

$\mathscr{C}: x^2+y^2=4 \rightsquigarrow$ circle center $(0,0)$, radius 2

$L: y=x+1 \rightsquigarrow$ line through $(0,1)$, slope 1



Substitute for y in first equation, using second equation.

$$x^2+(x+1)^2=4$$
$$2x^2+2x+1=4$$
$$2x^2+2x-3=0 \rightsquigarrow x=\frac{-2\pm\sqrt{4+24}}{4}=\frac{-2\pm\sqrt{28}}{4}=x$$

quadratic formula

$$y=x+1$$
$$y=\frac{-2\pm\sqrt{28}}{4}+1$$

See that coordinates of intersection points are obtained from coefficients of equations by $+,-,\times,\div,\sqrt{\ }$ (same method always works)

$\hookrightarrow$ $\emptyset$ no intersection points

(c) 2 circles

$\mathscr{C}_1: (x-a_1)^2+(y-b_1)^2=r_1^2$

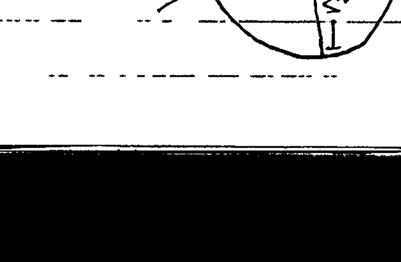
$\mathscr{C}_2: (x-a_2)^2+(y-b_2)^2=r_2^2$

same method as before:

$$r_1^2=x^2-2a_1x+a_1^2+y^2-2b_1y+b_1^2$$
$$0=y^2-2b_1y+(x^2-2a_1x+a_1^2+b_1^2-r_1^2)$$
$$\rightsquigarrow y=\frac{(2b_1)\pm\sqrt{(2b_1)^2-4(x^2-2a_1x+a_1^2+b_1^2-r_1^2)}}{2}$$

quad. formula

} too long and messy



observe geometrically there's a line L such that $\mathscr{C}_1\cap L=\mathscr{C}_2\cap L=\mathscr{C}_1\cap\mathscr{C}_2$
$=\{P_1,P_2\}$ ($L=P_1P_2$!)

What is equation of L?

$\sqrt{r_1^2-d^2}+\sqrt{r_2^2-d^2}=|O_1O_2| \rightarrow$ might work

but there is an easier way ... $\rightarrow$

Subtract second equation from first equation:

$$\text{①}\ (x-a_1)^2+(y-b_1)^2=r_1^2 = x^2-2a_1x+a_1^2+y^2-2b_1y+b_1^2$$
$$\text{②}\ (x-a_2)^2+(y-b_2)^2=r_2^2 = x^2-2a_2x+a_2^2+y^2-2b_2y+b_2^2$$

$$\text{①}-\text{②} = 2(a_2-a_1)x+2(b_2-b_1)y = r_1^2-r_2^2-\left((a_1^2+b_1^2)-(a_2^2+b_2^2)\right)$$

$Ax+By=C$ — this is eq. of L.

See 3c reduces to 3b: circle $\cap$ line.

Again coordinates of intersection points are obtained from coefficients by $+, -, \times, \div, \sqrt{\ }$.

Conclusion: If a point (a,b) can be constructed, then a,b are obtained from 1 by $+, -, \times, \div, \sqrt{\ }$.

(Converse is also true: we showed how to effect $+, -, \times, \div, \sqrt{\ }$ using ruler & compass)

Finally, if ℓ is a length which can be constructed by ruler & compass, then $\ell = |PQ|$ where P, Q are constructible points.

$P = (a,b)$, $Q = (c,d)$

$\ell = \sqrt{(a-c)^2+(b-d)^2}$ a,b,c,d obtained from 1 by $+, -, \times, \div, \sqrt{\ }$

$\rightsquigarrow$ same for ℓ

This proves: length is constructible $\Rightarrow$ obtained from 1 by $+, -, \times, \div, \sqrt{\ }$

$\Leftarrow$ (saw earlier)

Ex: $5+\sqrt{\dfrac{7+\sqrt{29}}{3\sqrt{17}}}$ etc.