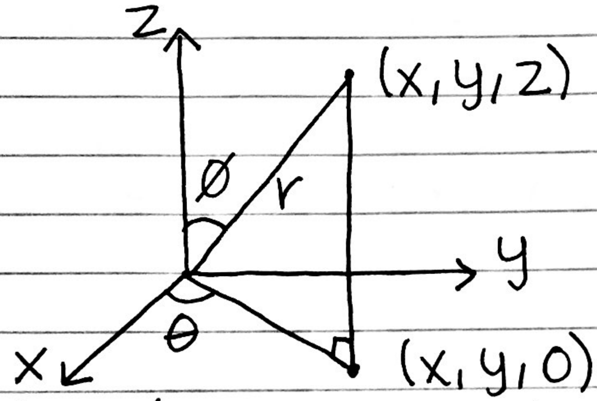
Math 461 Lecture 24 10/29
Midterms returned
average 70.2
3Q: 85% median: 74% 1Q: 54%
Homework 5 due wednesday

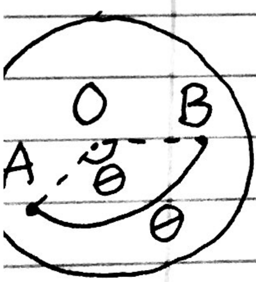
Last time:
spherical polar coordinates: (233?)

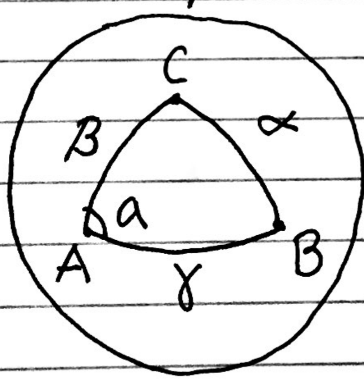


$S^2: r=1$

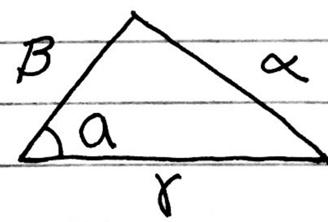
$(x,y,z) = r(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

spherical cosine rule:
$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$





compare to the Euclidean case
$\alpha^2 = \beta^2 + \gamma^2 - 2\beta\gamma\cos a$



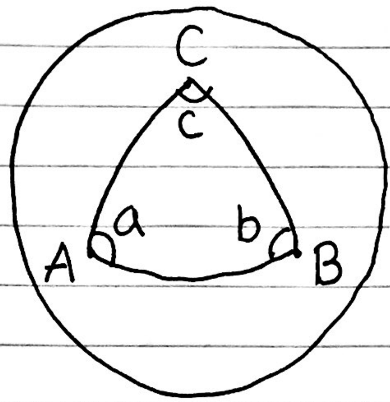
Today:
spherical triangle inequality
=> spherical lines give shortest paths
spherical sine rule

Office Hours:
Today: 2:30-3:30
Tomorrow: 4:00-5:00

Math 461 Lecture 24 10/29

Question: what's the maximum area of a spherical triangle?

we know $a+b+c = \pi + \text{Area}(\triangle ABC)$



equivalently, Question: what's the maximum value of $a+b+c$?

$S \subset \mathbb{R}$

S is a subset of $\mathbb{R}$

can say supremum instead of maximum.

sup S = least upper bound = smallest number $M \in \mathbb{R}$ such that $s \leq m$ for all $s \in S$

for us we can guess:

sup{areas of spherical triangles} $= 2\pi$ - can get arbitrarily close to 2π, but not equal

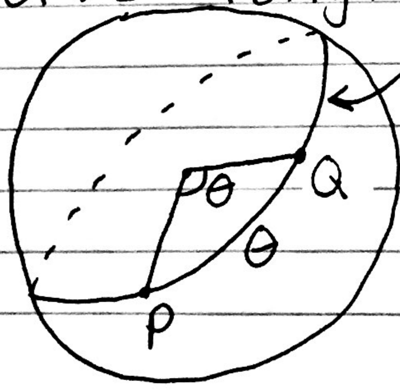
a spherical triangle is:

take points A, B, C on S^2, assume they do not lie on a single spherical line

draw shorter arc of spherical line from A to B, B to C, C to A

and get spherical triangle $\triangle ABC$

observe: length of each side is $\leq \pi$



spherical line is circle of radius 1

Remark: similar spherical triangles don't exist

Math 461 Lecture 24 10/29

more precisely, similar triangles will be congruent

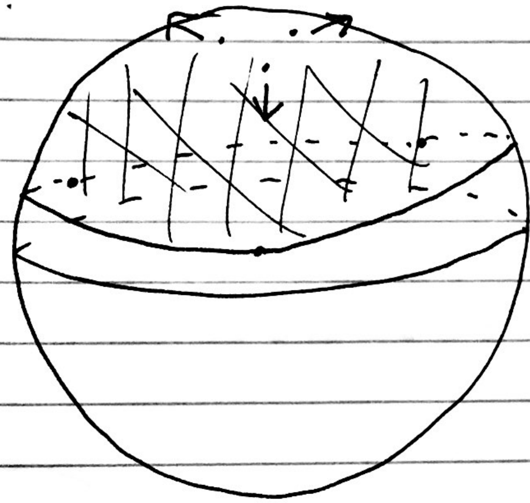
congruent: equal sides and equal angles

equivalently, there is an isometry of S^2

$T: S^2 \to S^2$

$d(T(P), T(Q)) = d(P, Q)$ for all $P, Q \in S^2$

which sends ΔABC to $\Delta A'B'C'$



(*) $a+b+c = \pi + \text{Area}(\Delta ABC)$

know $a, b, c < \pi$

(*) $\Rightarrow$ $\text{Area}(\Delta ABC) < 3\pi - \pi = 2\pi$

and construction shows that have sequence of triangles $\Delta A_+ B_+ C_+$

such that $\text{Area}(\Delta A_+ B_+ C_+) \to$

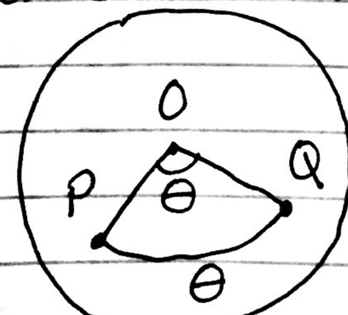
$\text{Area}(\text{northern hemisphere}) = \frac{1}{2}(4\pi) = 2\pi$

Spherical Triangle Inequality:

$d(P,Q) + d(Q,R) \geq d(P,Q)$ for $P, Q, R \in S^2$

$d(P,Q)$ = spherical distance from P to $Q = \theta$ $\quad 0 \leq \theta \leq \pi$

last time we changed notation in order to use SCR

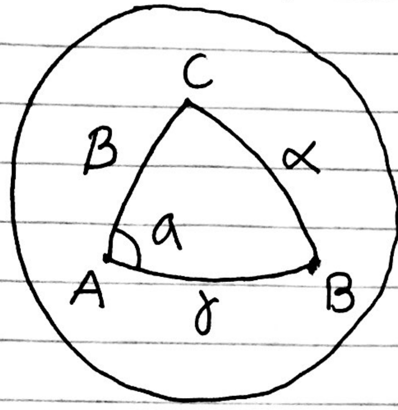


Math 461 Lecture 24 10/29

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

want to prove:

$d(B,C) \leq d(B,A) + d(A,C)$ i.e. $\alpha \leq \gamma + \beta$



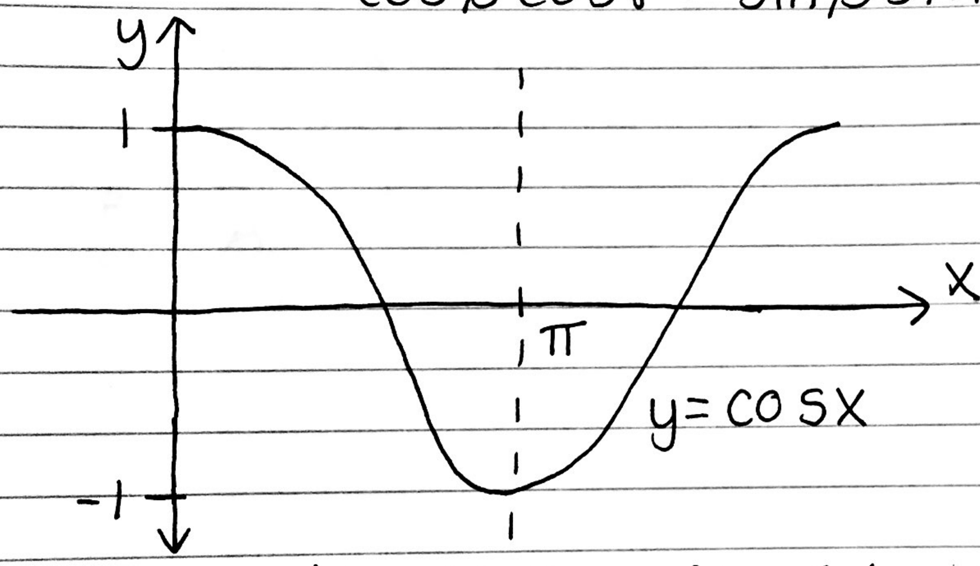
$\alpha \leq \gamma + \beta$ is the claim

want to differentiate the claim and what you want to prove

proof: recall:

$\cos(\beta+\gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma$

$= \cos\beta\cos\gamma + \sin\beta\sin\gamma\cdot(-1)$



1. $\cos x$ is decreasing for $0 \leq x \leq \pi$
2. $-1 \leq \cos x \leq 1$ for all x

property 1 shows that

$\alpha \leq \gamma + \beta \Leftarrow \cos\alpha \geq \cos(\gamma+\beta)$

why: suppose $\cos\alpha \geq \cos(\gamma+\beta)$

if $\alpha, \gamma+\beta \leq \pi$, get $\alpha \leq \gamma + \beta$

because $\cos x$ decreasing for $0 \leq x \leq \pi$

always have $\alpha, \gamma, \beta \leq \pi$ but might have $\gamma + \beta > \pi$

in that case $\alpha \leq \pi < \gamma + \beta$ ✓

Math 461 lecture 24 10/29

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

$\cos(\beta+\gamma) = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cdot(-1)$

want $\cos\alpha \geq \cos(\beta+\gamma)$

use $\cos a \geq -1$

(and $\sin\beta, \sin\gamma \geq 0$ $\quad 0 \leq \beta, \gamma \leq \pi$

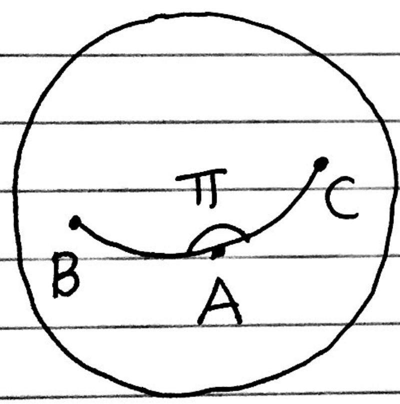
$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a \geq$

$\cos\beta\cos\gamma + \sin\beta\sin\gamma(-1) = \cos(\beta+\gamma)$ $\square$

triangle inequality +

$d(B,C) \leq d(B,A) + d(A,C)$

with equality if and only if A lies on the shorter arc of the spherical line from B to C



from proof:
equal if and only if $\cos a = -1$ i.e. $a = \pi$