Math 612 Homework 5

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Reading: Artin, Algebra, Chapter 10: Group representations. (Note: The treatment of representations in Dummit and Foote is in terms of modules over the group ring. We have described a more concrete approach in class, following Artin. The corresponding material in Dummit and Foote is 18.1,18.2,18.3, and 19.1.)

Another useful reference for representation theory is the book by James and Liebeck, Representations and characters of groups, CUP (2001).

Unless otherwise stated all groups G are finite and all representations ρ are finite dimensional complex representations. We say that a representation $\rho\colon G \to \mathrm{GL}(V)$ of a group G is *faithful* if ρ is injective. These problems will not be graded, but representation theory will be covered on the final exam.

Justify your answers carefully.

(1) Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible representation of a group G of dimension greater than 1. Let $v \in V$ be a vector. Show that

$$\sum_{g \in G} g \cdot v = 0.$$

(2) Show that every finite group G has a faithful representation on a finite dimensional complex vector space V.

(3) Let χ be a character of a group G. Show that $\bar{\chi}$ is also a character of G, where $\bar{\chi}(g) := \overline{\chi(g)}$ and the bar denotes complex conjugation. Show that if χ is the character of an irreducible representation then so is $\bar{\chi}$.

(4) (a) Compute the character table of D_4, the dihedral group of order 8.

(b) Let ρ be the permutation representation given by the action of D_4 on the vertices of the square. Determine the decomposition of ρ into irreducible representations.

[Hint: (a) Show that the abelianization of D_4 is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This determines the one dimensional representations.]

(5) Compute the character table of the group G of rotational symmetries of the cube.

[Hint: The group G is isomorphic to S_4 (why?). Both points of view are useful for describing the representations of G.]

(6) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G of dimension d and χ its character. Let $g \in G$. Show that $|\chi(g)| \leq d$ with equality iff $\rho(g) = \zeta \cdot \mathrm{id}_V$ for some root of unity ζ.

(7) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of dimension d and χ its character. Show that

$$\ker(\rho) = \{g \in G \mid \chi(g) = d\}.$$

(8) Show that G is simple iff for every nontrivial irreducible character χ and element $g \in G \setminus \{e\}$ we have $\chi(g) \neq \chi(e)$.

(9) Recall that the quaternion group Q is defined by

$$Q = \{\pm 1, \pm i, \pm j, \pm k\},$$

with group law given by multiplication of quaternions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Compute the character table of Q and compare it with the character table of D_4 from Q4.

(10) Let $\mathbb{F}_q$ be the finite field of order q. Let G be the group of affine linear maps

$$g\colon \mathbb{F}_q \to \mathbb{F}_q, \quad x \mapsto ax + b$$

where $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$, with the group law being composition of maps.

(a) List the conjugacy classes of G.

(b) Determine the abelianization of G.

(c) Compute the dimensions of the irreducible complex representations of G.

(d) By its definition G acts on the set $\mathbb{F}_q$. Let ρ be the associated permutation representation. Then we can write $\rho = \rho_1 \oplus \rho'$ where ρ is the trivial representation. Compute the character of ρ' and show that ρ' is irreducible.

(11) Let G be the subgroup of $\mathrm{GL}_3(\mathbb{F}_5)$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Describe the conjugacy classes of G. What is the center of G?

(b) Determine the abelianization of G.

(c) Compute the number of irreducible complex representations of G and their dimensions.