Math 797W Homework 3

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Justify your answers carefully.

(1) Let $f \in k[x,y]$ be an irreducible polynomial and $X = V(f) \subset \mathbb{A}^2_{x,y}$ an affine plane curve. Let $p \in X$ be a smooth point. Show that $x - x(p)$ is a local parameter at p if $\frac{\partial f}{\partial y}(p) \neq 0$ and $y - y(p)$ is a local parameter at p if $\frac{\partial f}{\partial x}(p) \neq 0$.

[Recall that we say t is a *local parameter* at a smooth point p of a curve X if t is a generator of the maximal ideal of the local ring $\mathcal{O}_{X,p}$ of X at p.]

(2) Let $X = V(x^2 + y^2 - 1) \subset \mathbb{A}^2_{x,y}$ and $f = (x-1)/y \in k(X)$. For $p \in X$, let $\nu_p \colon k(X)^\times \to \mathbb{Z}$ denote the associated valuation, given by "order of vanishing at p". Compute $\nu_p(f)$ for all $p \in X$.

(3) Let $X = V(f) \subset \mathbb{A}^2_{x,y}$ as in Q1 and consider the morphism $g \colon X \to \mathbb{A}^1$ given by $(x,y) \mapsto x$. Let $p \in X$ be a smooth point. After changing coordinates on $\mathbb{A}^2$ by a translation we may assume that $p = (0,0)$. Show that the ramification index e of g at p is then equal to the smallest number m such that the monomial cy^m is a term in the polynomial f for some $0 \neq c \in k$.

[Recall that the *ramification index* of a morphism $f \colon X \to Y$ of smooth curves at a point $p \in X$ is $\nu_p(f^*t)$ where t is a local parameter at $f(p) \in Y$.]

(4) Let X be an affine variety and $A = k[X]$ its coordinate ring. Let M be a finitely generated A-module. For $p \in X$, let $k(p)$ denote the A-module $A/I(p)$ (the quotient of A by the maximal ideal $I(p) \subset A$

corresponding to $p \in X$). Thus $k(p) = k$ as a ring, but its A-module structure depends on p, and is given by $f \cdot \lambda = f(p)\lambda$ for $f \in A$ and $\lambda \in k = k(p)$. Suppose that for all $p \in X$, the k-vector space $M \otimes_A k(p) = M/I(p)M$ has dimension r. Prove the following: There exist $f_1, \ldots, f_s \in A$ such that, writing $U_i = X \setminus V(f_i)$ (an affine open subset of X with coordinate ring $k[U_i] = A_{f_i}$), we have $X = U_1 \cup \cdots \cup U_s$ and $M_{f_i} \simeq A_{f_i}^{\oplus r}$ for each i.

[Remark: The module M corresponds to an algebraic vector bundle of rank r over the variety X.]

[Hint: Study the proof that the A-module $M = \Omega_{A/k}$ is locally free (in the above sense) of rank $r = \dim X$ when X is smooth. The method used there can be used to prove this general result.]

(5) Assume $\mathrm{char}(k) \neq 2$. Let $X = V(y^2 - x(x^2+1)) \subset \mathbb{A}^2_{x,y}$, a smooth affine curve . Let $A = k[X]$ be its coordinate ring, $p = (0,0) \in X$, and $M = I(p) = (x, y) \subset A$ the maximal ideal of A corresponding to the point $p \in X$.

 (a) Prove that for all $q \in X$, $M \otimes_A k(q)$ is a k-vector space of dimension 1, so that the conclusion of Q4 applies with $r = 1$.

 (b) Show that M is *not* isomorphic to A as an A-module.

[Remark: The module M corresponds to an algebraic line bundle over X which is not isomorphic to the trivial line bundle $X \times \mathbb{A}^1 \to X$.]

(6) Assume $\mathrm{char}(k) \neq 2$. Let $X = V(y^2 - x^2(x+1)) \subset \mathbb{A}^2_{x,y}$, an affine curve with a unique singular point $p = (0,0) \in X$.

 (a) Show that the assignment $t \mapsto (t^2-1, t(t^2-1))$ defines a morphism $f\colon \mathbb{A}^1_t \to X$ which restricts to an isomorphism
$$\mathbb{A}^1_t \setminus \{\pm 1\} \xrightarrow{\sim} X \setminus \{p\}.$$

 (b) Prove that the image of $k[X]$ in $k[t]$ under f^* is the subring of polynomials g such that $g(1) = g(-1)$.

 (c) Show that, if we identify $k[X]$ with its image in $k[t]$ under f^*, then the integral closure of $k[X]$ in its fraction field $k(X)$ is identified with $k[t]$.

(7) (a) Let $f\colon X \to Y$ be a morphism of affine varieties corresponding to a k-algebra homomorphism $f^*\colon B := k[Y] \to A := k[X]$. Observe that f induces a homomorphism of B-modules

$$\Omega_{B/k} \to \Omega_{A/k}$$

(also denoted f^*) given by

$$f^*(\sum_{i=1}^{s} g_i dh_i) = \sum_{i=1}^{s} f^*(g_i)d(f^*(h_i)).$$

(Here we regard $\Omega_{A/k}$ as a B-module via the ring homomorphism $f^*\colon B \to A$ ("restriction of scalars")). This is the algebraic analogue of the pullback $f^*\omega$ of a differential form ω on a smooth manifold Y under a smooth map $f\colon X \to Y$ in differential geometry. [You don't need to write a solution for this part but do convince yourself it makes sense using the definition of Ω.]

(b) Now consider the morphism $f\colon \mathbb{A}^1_t \to X$ of Q6. Write $B = k[X]$. Compute $\Omega_{B/k}$ and describe the homomorphism

$$f^*\colon \Omega_{B/k} \to \Omega_{k[t]/k}$$

in this case.

(c) Determine a (nonzero) torsion element ω of the B-module $\Omega_{B/k}$ from part (b).

[Remark: This example shows that if X is an affine variety, and $A = k[X]$ is its coordinate ring, then the A-module $\Omega_{A/k}$ need not be torsion-free. However, as shown in class, if X is smooth then $\Omega_{A/k}$ is locally free in the sense of Q4 and so in particular torsion-free.]

[Hint: (c) If $\omega \in \Omega_{B/k}$ is torsion then ω lies in the kernel of the homomorphism f^* described in part (b) (why?).]

(8) Assume $\mathrm{char}(k) \neq 2$. Let $f\colon \overline{X} \to \mathbb{P}^1$ be the finite morphism from the smooth proper curve $\overline{X}$ to $\mathbb{P}^1$ constructed in HW2Q9. Thus $\overline{X}$ is a union $U \cup V$ of two open affine sets given by

$$U = V(y^2 - f(x)) \subset \mathbb{A}^2_{x,y}$$

and

$$V = V(t^2 - g(z)) \subset \mathbb{A}^2_{z,t}$$

with glueing given by

$$U \supset (x \neq 0) \xrightarrow{\sim} (z \neq 0) \subset V, \quad (x, y) \mapsto (x^{-1}, x^{-l}y).$$

Here $f(x) \in k[x]$ is a polynomial of degree $d \geq 1$ with distinct roots, $l = \lceil d/2 \rceil$, and $g(z) = z^{2l} f(1/z) \in k[z]$. Write $\mathbb{P}^1 = \mathbb{A}^1_x \cup \mathbb{A}^1_z$, with glueing given by

$$\mathbb{A}^1_x \setminus \{0\} \xrightarrow{\sim} \mathbb{A}^1_z \setminus \{0\}, \quad x \mapsto x^{-1}.$$

Then the morphism f is given in charts by

$$U \to \mathbb{A}^1_x, \quad (x, y) \mapsto x$$

and

$$V \to \mathbb{A}^1_z, \quad (z, t) \mapsto z.$$

Compute an explicit basis for the k-vector space $\Omega_{\overline{X}}(\overline{X})$ of global 1-forms on $\overline{X}$, and deduce that $\dim_k \Omega_{\overline{X}}(\overline{X}) = l - 1$.

[Hint: We did a similar calculation in class for a smooth projective plane curve $X \subset \mathbb{P}^2$.]

(9) Let $f \colon \overline{X} \to \mathbb{P}^1$ be as in Q8 above. Let $S \subset \mathbb{P}^1 = \mathbb{A}^1_x \cup \{\infty\}$ be the set of roots of $f(x)$ together with ∞ if d is odd. Then $|S| = 2l$, and (as shown in HW2Q9) $|f^{-1}(p)| = 1$ if $p \in S$ and $|f^{-1}(p)| = 2$ otherwise. Now suppose $k = \mathbb{C}$. Join the points of S in pairs by l smooth disjoint paths, with union Γ. Observe that $\sqrt{f(x)}$ can be consistently defined on $\mathbb{C}^1_x \setminus \Gamma$, and similarly $\sqrt{g(z)}$ can be defined on $\mathbb{C}^1_z \setminus \Gamma$. Now explain a construction of $\overline{X}$ as a topological space (for the analytic topology) given by glueing two copies of $\mathbb{P}^1_{\mathbb{C}}$ cut along Γ. Deduce that the genus of $\overline{X}$ equals $l - 1$.

(10) For varieties X and Y, a *rational map* $f \colon X \dashrightarrow Y$ is a morphism $f \colon U \to Y$ defined on a Zariski open subset $U \subset X$, up to the following equivalence relation: $f \colon U \to Y$ and $g \colon V \to Y$ are equivalent if $f|_{U \cap V} = g|_{U \cap V}$. A rational map f has a unique maximal domain of definition U, denoted domain(f).

Let $X = V(XY - ZT) \subset \mathbb{P}^3$, a smooth quadric surface, $Y = \mathbb{P}^2$, and $f\colon X \dashrightarrow Y$ the rational map defined by

$$(X : Y : Z : T) \mapsto (X : Y : Z)$$

(a) Determine $\mathrm{domain}(f)$.

(b) Show that f has a rational inverse g (that is a rational map $g\colon Y \dashrightarrow X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$), and determine $\mathrm{domain}(g)$.

(c) Find maximal open subsets $U \subset X$ and $V \subset Y$ such that f restricts to an isomorphism $U \xrightarrow{\sim} V$.