Math 461 Lecture 10 9/26

Homework 1 returned

out of 45, not all problems graded

+5 points for completeness

Homework 2 due today

Solutions will be posted later today

Homework 3 available

due at start of class next wednesday 10/3

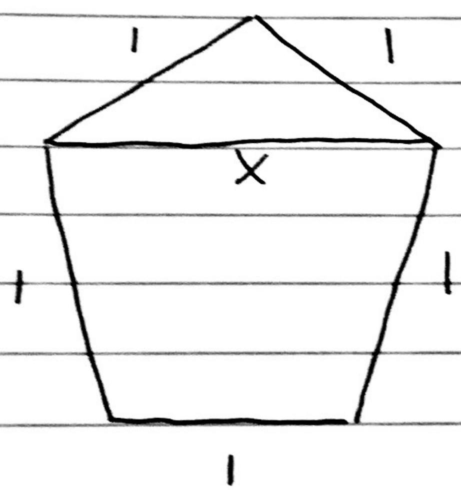
Last time:

A length is constructible <=> obtained from 1 by $+, -, \times, \div, \sqrt{\ }$

proved it <= this way

will prove => this way later using coordinates

The regular pentagon is constructible:



$$x = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

$x^2 = x+1$ "golden ratio"

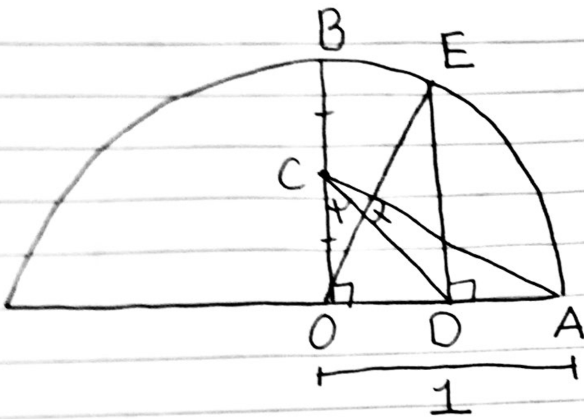
Explicit construction:

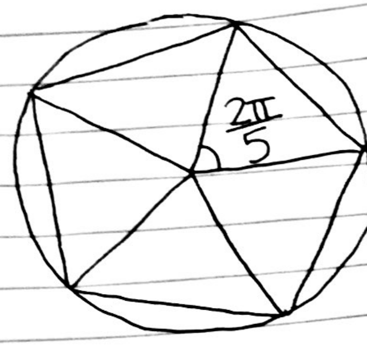
claim: $\angle AOE = \frac{2\pi}{5}$ equivalently $|OD| = \cos\left(\frac{2\pi}{5}\right)$

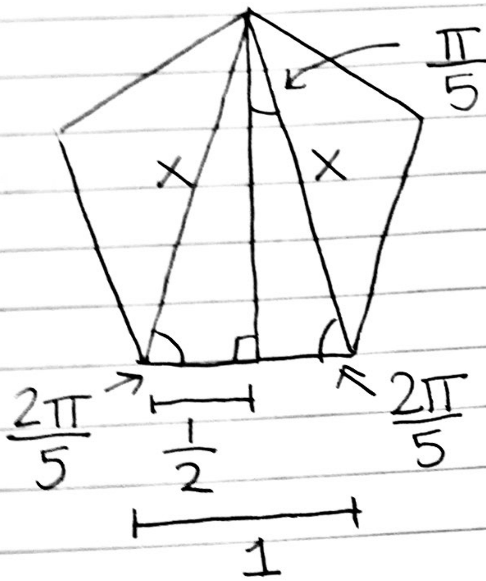
computed $|OD| = \frac{\sqrt{5}-1}{4}$

picture →

Math 461 Lecture 10 9/26







see $\cos\left(\frac{2\pi}{5}\right) = \frac{\frac{1}{2}}{x} =$

$$\frac{\frac{1}{2}}{\frac{1+\sqrt{5}}{2}} = \frac{1}{1+\sqrt{5}} =$$

$$\frac{1-\sqrt{5}}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{1-\sqrt{5}}{1-5} =$$

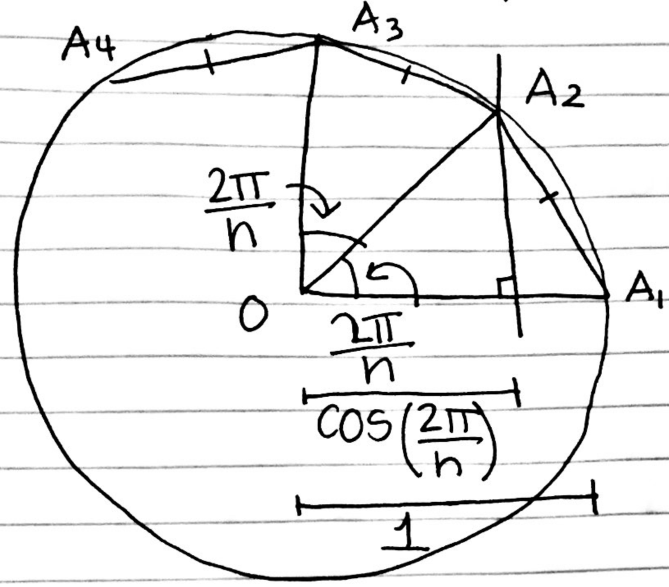
$$\frac{1-\sqrt{5}}{-4} = \frac{\sqrt{5}-1}{4} \checkmark$$

Today:
finish proof of construction
construction of regular polygons
  Gauss' Theorem
center of mass
orthocenter
if time: coordinates

In general, to construct a regular n-sided polygon it's equivalent to construct the length $\cos\left(\frac{2\pi}{n}\right)$

picture →

Math 461 Lecture 10 9/26



regular n-gon $A_1 A_2 \cdots A_n$

$n \quad \cos\left(\frac{2\pi}{n}\right)$

$3 \quad \cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$4 \quad \cos\left(\frac{2\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$5 \quad \cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$

$6 \quad \cos\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$7 \quad \cos\left(\frac{2\pi}{7}\right) = ??$ in fact, not constructible

The complex solutions of the equation $z^n = 1$ are:

$$z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

$$= e^{i\left(\frac{2\pi k}{n}\right)} \quad k = 0, 1, \ldots, n-1$$

The points on the unit circle in the xy plane $(z = x + iy)$ including 1,

Math 461 Lecture 10 9/26

equally spaced vertices of regular n-gon

i.e. $z = \zeta^k \quad k = 0, 1, \ldots, n-1$

where $\zeta = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

$z^n - 1$ can be factored

$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$

root $z = 1 = \zeta^0$ roots $\zeta, \zeta^2, \ldots, \zeta^{n-1}$

$\zeta + \zeta^{-1} = \zeta + \bar{\zeta} = 2 \cdot \mathrm{Re}(\zeta) = 2\cos\left(\frac{2\pi}{n}\right)$

$\cos\left(\frac{2\pi}{7}\right) = ?$

$\zeta = e^{\frac{2\pi i}{7}}$ (case $n=7$)

we know $\zeta^6 + \zeta^5 + \zeta^4 + \ldots + \zeta + 1 = 0$ (*)

want equation for $\zeta + \zeta^{-1}$ (and $\zeta^7 = 1$)

divide (*) by ζ^3 then multiply by ζ^{-3}

$\zeta^3 + \zeta^2 + \zeta + 1 + \zeta^{-1} + \zeta^{-2} + \zeta^{-3} = 0$ (+)

Let $y = \zeta + \zeta^{-1} = 2\cos\left(\frac{2\pi}{7}\right)$

$(\zeta^2 + \zeta^{-2}) + 2$

Then $y^2 = (\zeta + \zeta^{-1})^2 = \zeta^2 + 2\zeta\zeta^{-1} + \zeta^{-2} = \zeta^2 + 2 + \zeta^{-2}$

$y^3 = (\zeta + \zeta^{-1})^3 = \zeta^3 + 3\zeta^2\zeta^{-1} + 3\zeta\zeta^{-2} + \zeta^{-3}$

$= (\zeta^3 + \zeta^{-3}) + 3(\zeta + \zeta^{-1})$

$(\zeta^3 + \zeta^{-3}) + (\zeta^2 + \zeta^{-2}) + (\zeta + \zeta^{-1}) + 1 = 0$ (+)

$y^3 + y^2 - 2y - 1 = 0$

Gauss' Theorem:

regular n-sided polygon is constructible

$\Leftrightarrow \cos\left(\frac{2\pi}{n}\right)$ is a constructible length

$\Leftrightarrow n = 2^k \cdot p_1 \cdot p_2 \cdots p_r$ where $p_1, p_2, \ldots, p_r$ are distinct Fermat primes

Math 461 Lecture 10 9/26

What's a Fermat Prime?

It's a prime p such that $p-1$ is a power of 2

i.e. $p = 2^{\ell}+1$ for some $\ell \geq 1$

Observation: (Fermat)

if $2^{\ell}+1$ is prime then $\ell = 2^m$ for some $m \geq 0$

Why? $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots + x + 1)$

n is odd $\quad x^n + 1 = (x+1)(x^{n-1} - x^{n-2} + x^{n-3} - \dots - x + 1)$ (**)

If ℓ is not a power of 2, then $\ell = q \cdot n$ where n is odd and $n > 1$ and q is a positive integer

(take any odd prime factor of ℓ)

now $2^{\ell}+1 = 2^{q \cdot n}+1 = (2^q)^n + 1 =$ $\quad x = 2^q$ in (**)

$(2^q+1)((2^q)^{n-1} - \dots + 1)$ $\quad 2^{\ell}+1$ is not a prime

Fermat conjectured:

converse is true

i.e. $2^{(2^m)}+1$ is prime for all $m \geq 0$

$\cancel{m}$	$\cancel{2^{2^m}+1}$
$\cancel{0}$	$\cancel{2^{2^0}+1}$
$\cancel{1}$	
$\cancel{2}$	
$\cancel{3}$	

m	$2^{(2^m)}+1$
0	$2^{(2^0)}+1 = 2^1+1 = 2+1 = 3$
1	$2^{(2^1)}+1 = 2^2+1 = 4+1 = 5$
2	$2^{(2^2)}+1 = 2^4+1 = 16+1 = 17$
3	$2^{(2^3)}+1 = 2^8+1 = 256+1 = 257$
4	$2^{(2^4)}+1 = 2^{16}+1 = 65537$

very false - these are the only known primes in sequence

Math 461 Lecture 10 9/26

Recap:

A Fermat prime is a number of the form $2^{(2^m)}+1$ which is prime

Known examples: 3, 5, 17, 257, 65537

Gauss: regular n-gon constructible $\Leftrightarrow$ $n = 2^k \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_r$ $k \geq 0$ $p_1, \ldots, p_r$ distinct Fermat primes

n	3	4	5	6	7	8	9	10
constructible	✓	✓	✓	✓	X	✓	X	✓

n	11	12	13	14	15	16	17	18
constructible	X	✓	X	X	✓	✓	✓	X

Ex. $\cos\left(\frac{2\pi}{17}\right)$ can be written in terms of $(+, -, \times, \div, \sqrt{\ })$

$$\cos\left(\frac{2\pi}{17}\right) = \frac{1}{16}\left(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}}\right)$$

How to get this expression

start with similar analysis to n=7 and show can find solution of equation by solving sequence of quadratic equations

Hardy & Wright

introduction to theory of numbers page 5 and section 5.8