MATH 461 HW1 solutions.

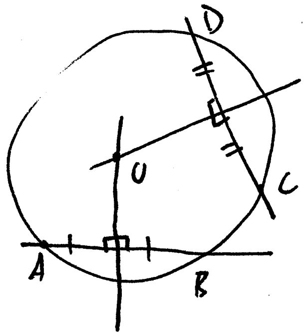
1. Draw two lines L & M intersecting the circle at points A, B & C, D.

Let O be the center of the circle.

Since $|OA| = |OB| = r$, the radius of the circle, the point O lies on the perpendicular bisector of AB.

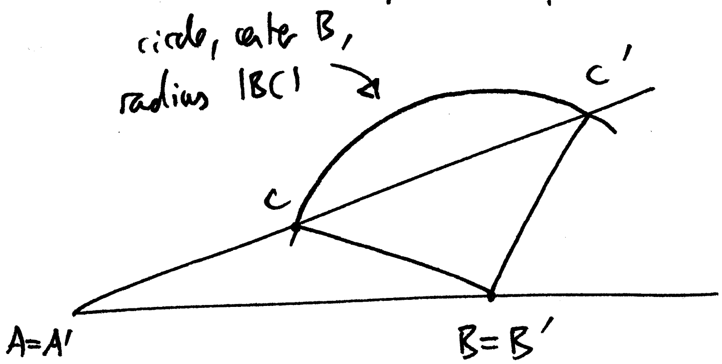
Similarly, O lies on the perpendicular bisector of CD.

So we can construct O as the intersection of the two perpendicular bisectors.



2. The triangles are not necessarily congruent.

To describe a counterexample, we proceed as in the hint:-

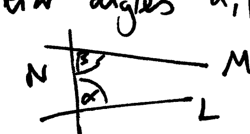


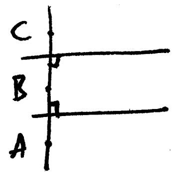
We have $|AB| = |A'B'|$, $|BC| = |B'C'|$ & $\angle CAB = \angle C'A'B'$

but $\Delta ABC \not\cong \Delta A'B'C'$ (because for example $|AC| \neq |A'C'|$.

3. a. If the points A, B, C lie on a line, then the perpendicular bisectors of AB & BC are parallel (do not intersect) using the criterion proved in class:

If two lines L & M cross a 3rd line N making interior angles α, β on one side of N, then L & M are parallel $\Leftrightarrow \alpha + \beta = \pi$



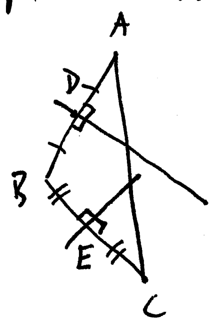
In our case  $\alpha+\beta = \pi/2 + \pi/2 = \pi$ $\Rightarrow$ perpendicular bisectors of AB & BC are parallel.

Passing to the contrapositive statement, we have shown:

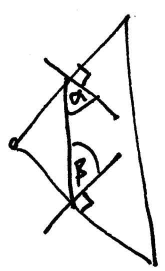
If the perpendicular bisectors of AB & BC do intersect, then A, B, C do not lie on a line.

Now we prove the converse.

So we assume A, B, C do not lie on a line, and consider the triangle ABC & the perpendicular bisectors of AB & BC



Draw the line DE & mark angles α & β at D & E as shown



We see $\alpha+\beta < \pi/2 + \pi/2 = \pi$

so the perpendicular bisectors meet (by the parallel axiom).

b) If A, B, C do not lie on a line, by part (a) the perpendicular bisectors of AB & BC meet at a point P.

Since P lies on the perpendicular bisector of AB, we have $|AP| = |BP|$

" " " " " " " " BC, " " $|BP| = |CP|$.

So $|AP| = |BP| = |CP|$, & the points A, B, C lie on the circle with center P and radius $|AP|$.

If C is a circle passing through A, B, C then the center O of the circle satisfies $|OA| = |OB| = |OC| = r$, the radius of the circle.

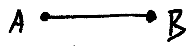
Since $|OA| = |OB|$, O lies on the perpendicular bisector of AB.

$|OB| = |OC|$, 〃 〃 〃 〃 〃 〃 〃 BC.

So $O = P =$ the intersection point of the two perpendicular bisectors.

And $r = |OA| = |AP|$. So the circle is uniquely determined by A, B, C.

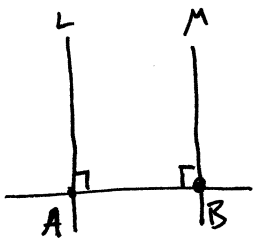
4. a.



given line segment.

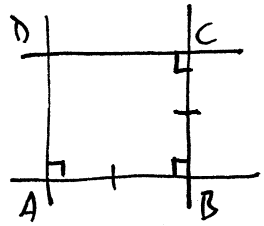
1. Construct a perpendicular line L to the line AB passing thru A
2. 〃 〃 〃 〃 M 〃 〃 〃 〃 〃 〃 〃 B.

we showed in class how to do this using ruler & compass



3. Mark off a point C on M at distance $|AB|$ from B. (using the compass)

4. Construct a perpendicular line to the line $M = BC$ through C; let D be the intersection point of this line with L.



<u>Claim</u>: the quadrilateral ABCD is a square.

<u>Proof</u>: AB is parallel to DC because $\angle ABC + \angle BCD = \pi/2 + \pi/2 = \pi$.

BC is parallel to AD because $\angle DAB + \angle ABC = \pi/2 + \pi/2 = \pi$.

So $ABCD$ is a parallelogram.

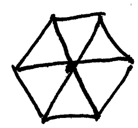
It follows that $\angle ADC = \pi/2$ (because $\angle ADC + \angle DCB = \pi$ and $\angle DCB = \pi/2$)

And $|AB| = |CD|$, $|BC| = |AD|$ (opposite sides of a parallelogram have equal lengths)

So, since $|AB| = |BC|$ by construction, $|AB| = |BC| = |CD| = |DA|$.

So $ABCD$ has equal side lengths & equal angles. $\square$.

4b. [The idea is to observe that the regular hexagon can be divided into 6 equilateral triangles:



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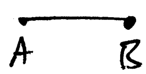
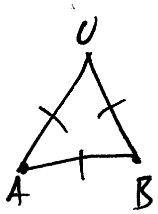
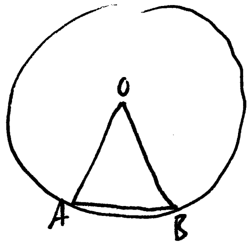


given line segment.

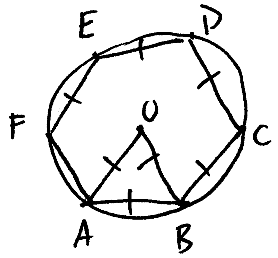
1. Construct an equilateral triangle with base AB and apex O.
(we showed in class how to do this with ruler and compass)



2. Draw a circle with center O and radius $|OA| = |OB| = |AB|$



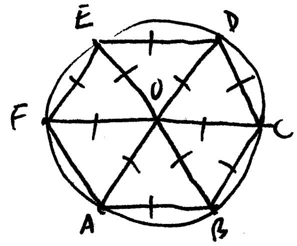
3. Mark off points C, D, E, F on the circle such that
$|AB| = |BC| = |CD| = |DE| = |EF|$ (using the compass).



(16-sided polygon)

Claim. ABCDEF is a regular hexagon

Proof



Each of the triangles ΔOAB, ΔOBC, ΔOCD, ΔODE, ΔOEF is equilateral (all sides have equal lengths).

So, all their angles are $\pi/3$.

(because the angle sum of a triangle equals π, and all the angles are equal using isosceles triangle thm (equal side lengths $\Rightarrow$ equal angles))

Now it follows that $\angle FOA = 2\pi - 5 \cdot \pi/3 = \pi/3$.

So ΔOFA is also equilateral (SAS congruence criterion).

We deduce that all the side lengths of the hexagon are equal, and all the angles are equal to $\pi/3 + \pi/3 = 2\pi/3$. $\square$.

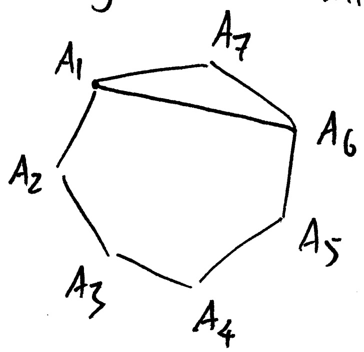
5. Proof by induction

Base case: $n=3$ (note: we are proving the assertion for $n \geq 3$)

We showed in class that a triangle (i.e. a 3 sided polygon) has angle sum π. And $(n-2)\pi = (3-2)\pi = \pi$ for $n=3$ ✓

Induction step. We assume the statement is true for $n=k$, and prove it is true for $n=k+1$. (Here k is a positive integer, $k \geq 3$).

Given a convex $(k+1)$ sided polygon, let $A_1, A_2, \ldots, A_{k+1}$ be the vertices of the polygon. The line segment $A_k A_1$ (which is contained in the polygon by convexity) divides the polygon into a k sided polygon $A_1 A_2 \cdots A_k$ and a triangle $A_k A_{k+1} A_1$:-

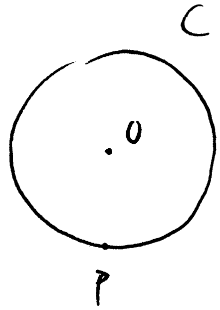


($k=6$ pictured)

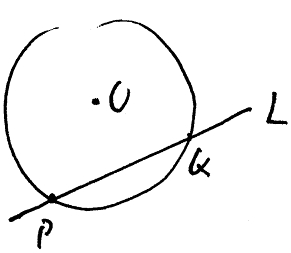
Now we see that the angle sum of the polygon $A_1 \cdots A_{k+1}$ equals the angle sum of the polygon $A_1 \cdots A_k$ plus the angle sum of the triangle $A_k A_{k+1} A_1$, which, by the induction hypothesis, equals

$$(k-2)\pi + \pi = (k-1)\pi = ((k+1)-2)\pi. \quad \square.$$

6.



Suppose first L is not tangent to C, so L intersects C at another point Q in addition to P:-



The triangle ΔOPQ is isosceles ($|OP| = |OQ| = r =$ radius of C).

So $\angle OPQ = \angle OQP$

Now the angle sum of a triangle equals π.

So $\pi = \angle OPQ + \angle OQP + \angle POQ > \angle OPQ + \angle OQP = 2\angle OPQ$

So $\angle OPQ < \pi/2$, in particular OP is <u>not</u> perpendicular to L.

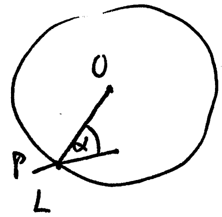
Passing to the contrapositive, we have shown

If OP is perpendicular to L, then L is tangent to C.

We now prove the converse. Again we will use the contrapositive form:

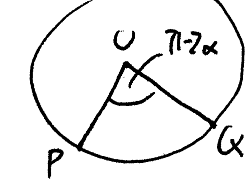
<u>Converse</u>: If L is tangent to C, then OP is perpendicular to L.

<u>Contrapositive of converse</u>: If OP is not perpendicular to L, then L is not tangent to C.



So, suppose OP makes angle $\alpha < \pi/2$ with L.

Let Q be the point on C, (on the same side of the line OP as the angle α) such that $\angle POQ = \pi - 2\alpha$.



The triangle ΔOPQ is isosceles, so $\angle OPQ = \angle OQP$.

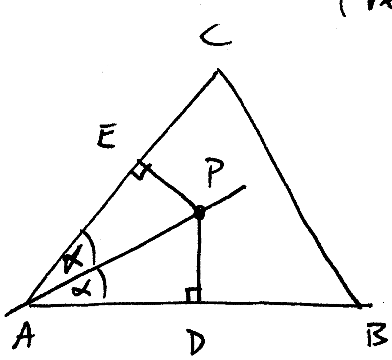
Now since $\angle OPQ + \angle OQP + \angle POQ = \pi$,

we find $\angle OPQ = \alpha$.

So $L = PQ$, and L is not tangent to C

(because it intersects C at $Q \neq P$).

7. a



Let D & E be the feet of the perpendicular lines from P to AB & AC as shown.

We claim $\Delta APD \cong \Delta APE$.

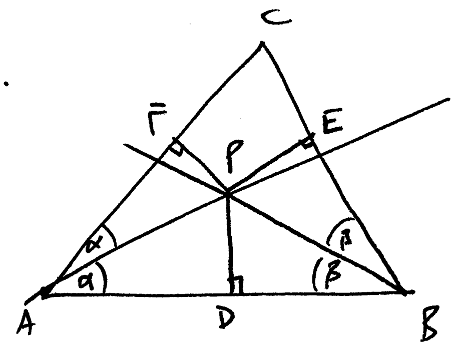
This follows by the ASA congruence criterion (using the angle sum of a triangle $= \pi$):-

$|AP| = |AP|$

$\angle EAP = \angle DAP = \alpha$, $\angle APE = \angle APD = (\pi - \pi/2 - \alpha) = \pi/2 - \alpha$.

So $|PD| = |PE|$. $\square$

b.



Let P be the intersection point of the angle bisectors of $\angle CAB$ & $\angle ABC$.

Let D, E, F be the feet of the perpendiculars from P to AB, BC, CA as shown.

Then $|PD| = |PF|$ & $|PD| = |PE|$ by part (a)

So $|PE| = |PD| = |PF|$.

Now consider the line segment PC.

We claim that $\Delta PCF \cong \Delta PCE$.

We have $|PC| = |PC|$

and $|PF| = |PE|$ (see above)

also $\angle PFC = \angle PEC = \pi/2$

so $\underset{\sqrt{|PC|^2 - |PF|^2}}{\underset{\shortparallel}{|CF|}} = \underset{\sqrt{|PC|^2 - |PE|^2}}{\underset{\shortparallel}{|CE|}}$ by Pythagoras' theorem

So $\Delta PCF \cong \Delta PCE$ (SSS).

So $\angle PCF = \angle PCE$

So PC is the angle bisector of $\angle ACB$.

This shows that the 3 angle bisectors are concurrent, meeting at P.

c. With notation as in part b, draw the circle C with center P and radius $|PD| = |PE| = |PF|$.

Then C passes through D, E, F by construction and is tangent to the lines AB, BC, CA at those points because the radii PD, PE, PF are perpendicular to the lines AB, BC, CA.

(here we are using Q6).