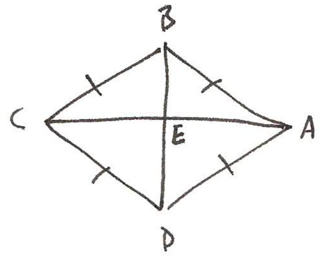
MATH 461 Midterm Review Solutions.

10/22/18

1a.



$\Delta ABC \cong \Delta ADC$ (SSS)

$\Rightarrow \angle BCE = \angle DCE$

$\Rightarrow \Delta BCE \cong \Delta DCE$ (SAS)

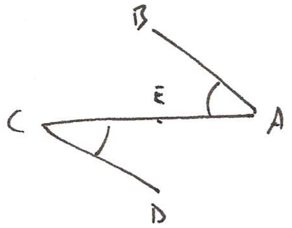
$\Rightarrow \angle BEC = \angle DEC$

Now $\angle BEC + \angle DEC = \pi \Rightarrow \angle BEC = \pi/2$ $\square$.

b. $\angle BCE = \angle BAE$ (ΔCBA is isosceles, $|CB| = |BA|$)

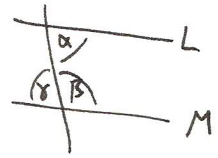
$\angle BCE = \angle DCE$ (proved in a. above)

So $\angle BAE = \angle DCE$



Now it follows that AB & CD are parallel.

(recall :



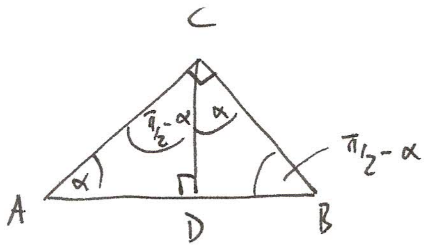
L, M parallel $\Leftrightarrow \alpha + \beta = \pi$

$\Leftrightarrow \alpha = \gamma$

because $\gamma = \pi - \beta$.)

Similarly, BC and AD are parallel.

2.



Using angle sum of triangle $= \pi$,

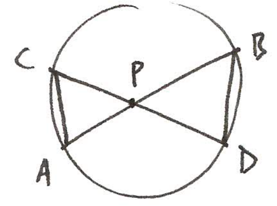
see $\angle CAD = \angle BCD$, $\angle ACD = \angle CBD$.

So $\Delta ADC \sim \Delta CDB$ (equal angles)

$\therefore \dfrac{|AD|}{|CD|} = \dfrac{|CD|}{|DB|}$ (ratios of corresponding sides for similar triangles are equal.)

$\Rightarrow |AD| \cdot |DB| = |CD|^2$ $\square$

3.



$\angle ACD = \angle ABD$ (angles subtended by chord AD)

$\angle CAB = \angle BDC$ (.. .. .. .. BC)

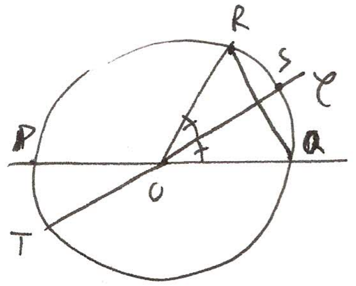
$\angle CPA = \angle BPD$

$\Rightarrow \Delta PAC \sim \Delta PDB$

$$\Rightarrow \frac{|PA|}{|PD|} = \frac{|PC|}{|PB|}$$

$\Rightarrow |PA|\cdot|PB| = |PC|\cdot|PD|$. □.

4.



Draw a diameter POQ of $\mathcal{C}$.

Mark a point R on $\mathcal{C}$ s.t. $|RQ| = |OQ|$

(the intersection of $\mathcal{C}$ w/ a circle center Q, radius $|OQ|$)

Then ΔOQR is equilateral, so $\angle QOR = \pi/3$.

Bisect angle $\angle QOR$; let S be the intersection point of the bisector with $\mathcal{C}$ between Q & R, and T the other intersection point.

So $\angle QOS = \frac{1}{2}(\pi/3) = \pi/6$.

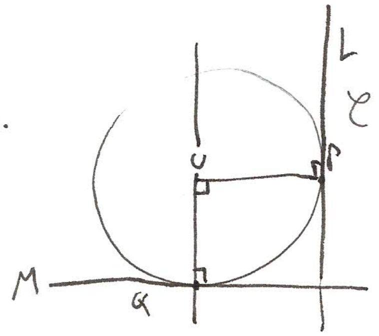
Claim: $PTQS$ is a rectangle.

Proof: PQ & TS are diameters of $\mathcal{C}$.

So $\angle PTQ = \angle PSQ = \angle TPS = \angle TQS = \pi/2$ (angle in a semicircle). □.

So $PTQS$ is a rectangle w/ vertices on $\mathcal{C}$, s.t. the diagonals meet at angle $\pi/6$. □.

5.



Draw a radius OP of $\mathcal{C}$.

Draw the perpendicular line L to OP thru P. (construction of perpendicular lines by ruler & compass was explained in class)

This is the tangent line to $\mathcal{C}$ at P

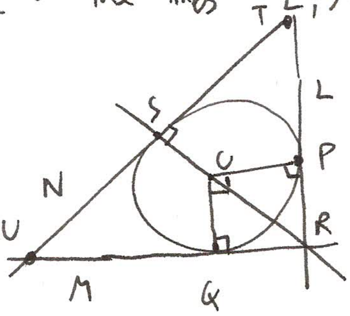
Draw the perpendicular line to OP at O, intersecting $\mathcal{C}$ at Q

Draw the perpendicular line M to OQ thru Q; this is the tangent to $\mathcal{C}$ at Q.

Let R be the intersection point of L & M; draw the line thru O & R and let S be the other intersection point w/ $\mathcal{C}$.

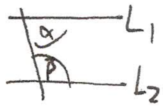
Finally draw the perpendicular N to OS thru S, this is the tangent to $\mathcal{C}$ at S.

<u>Claim</u>: The lines L, M, N form an isosceles right angled triangle.



<u>Proof</u>: OQRP is a square:

OQ $\parallel$ PR & OP $\parallel$ QR

(

$\alpha + \beta = \pi \Leftrightarrow L_1, L_2$ parallel)

$\Rightarrow |OQ| = |PR|, \quad |OP| = |QR|$

(opposite sides of parallelogram have equal lengths)

& $|OP| = |OQ|$ (radii of $\mathcal{C}$)

$\Rightarrow$ 4 equal sides.

Also, angle sum of quadrilateral $= 2\pi \Rightarrow$ 4 equal angles.

$\Rightarrow \angle PRQ = \pi/2 \Leftarrow$

Now $\triangle OQR \cong \triangle OPR$ (SSS)

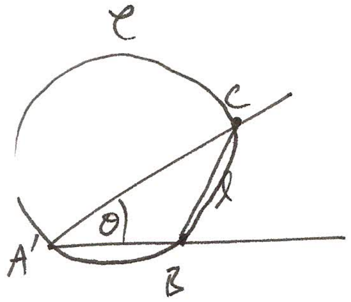
$\Rightarrow \angle PRO = \angle QRO = \pi/4$ (using $\angle PRO + \angle QRO = \pi/2$)

$\Rightarrow \angle RTU = \angle RUT = \pi/4$ (using angle sum of $\triangle RTS, \triangle RUS = \pi$)

$\Rightarrow \triangle RTU$ isosceles.

(& $\angle TRU = \angle PRQ = \pi/2$, see above). $\square$.

6.

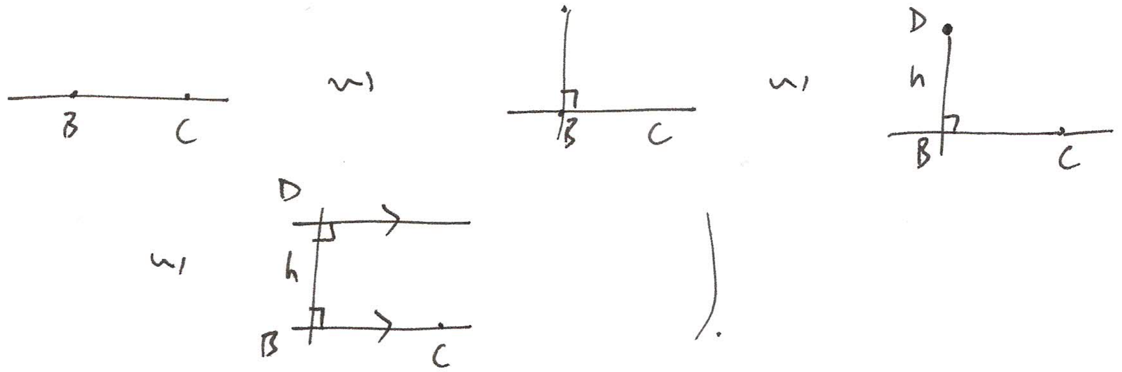


Given the angle θ, let A' be the vertex of the angle & let B be a point on one of the lines ~~so that~~ close enough to A' so that the circle with center B & radius l intersects the other line at a point C.

Draw the circumscribed circle $\mathcal{C}$ of the triangle $\Delta A'BC$.

Then for any point P on the arc of $\mathcal{C}$ from B to C which contains A' has $\angle BPC = \theta$ (angles subtended by chord BC are equal)

Now draw a line L parallel to BC at distance h from BC on the same side as A' (construction of parallels was explained in class:–



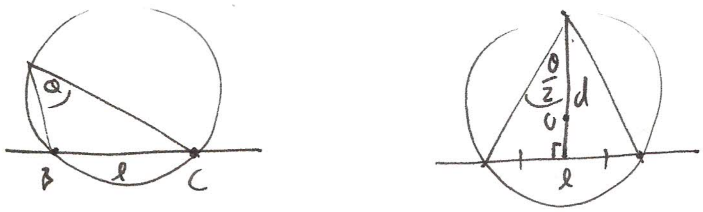
(distance h is given)

)

Finally, let A be an intersection point of L & $\mathcal{C}$.*

The ΔABC has $|BC| = l$, height h from A to BC, and $\angle CAB = \theta$. □.

* here require $\tan \theta/2 \leq l/2h$ for $L \cap \mathcal{C} \neq \emptyset$:–



$h \leq d$, where $\dfrac{l/2}{d} = \tan \theta/2$

i.e. $h \leq \dfrac{l/2}{\tan\theta/2}$, $\tan\dfrac{\theta}{2} \leq \dfrac{l}{2h}$.

7\.

C_1 $x^2+y^2=3^2$ ①

C_2 $(x-1)^2+(y-2)^2=2^2$ ②

Subtract $2x-1+4y-4=5$

$2x+4y=10$

$\div 2$ $x+2y=5$

Substitute $x=5-2y$ in ①.

$(5-2y)^2+y^2=9$

$5y^2-20y+25=9$

$5y^2-20y+16=0$

$$y \underset{QF}{=} \frac{20\pm\sqrt{20^2-4.5.16}}{10} = 2\pm\frac{\sqrt{400-320}}{10}$$

$$= 2\pm\frac{\sqrt{80}}{10} = 2\pm\frac{2}{5}\sqrt{5} = \begin{cases}2.894\\1.105\end{cases}$$

$$x=5-2y = 1\mp\tfrac{4}{5}\sqrt{5} = \begin{cases}-0.788\\2.788\end{cases}$$

Intersection points: $(1-\frac{4}{5}\sqrt{5},\ 2+\frac{2}{5}\sqrt{5})$ & $(1+\frac{4}{5}\sqrt{5},\ 2-\frac{2}{5}\sqrt{5})$.

8\. $A=(0,0)$, $B=(1,2)$, $C=(-1,3)$

Perpendicular bisector of AB: $x^2+y^2=(x-1)^2+(y-2)^2 \leadsto 2x-1+4y-4=0$

(locus of points equidistant from A & B)

AC: $x^2+y^2=(x+1)^2+(y-3)^2 \leadsto -2x\mp1+6y-9=0$

① $2x+4y=5$

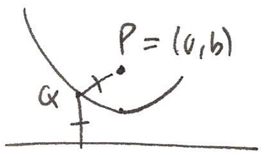
② $-2x+6y=10$

①+② $\leadsto 10y=15$, $y=3/2$, $x=-5+9/2=-1/2$.

$\Rightarrow x=-5+3y$

So, circle has center $P=(-\tfrac{1}{2},\tfrac{3}{2})$, radius $|AP| = \sqrt{(-\tfrac{1}{2})^2+(\tfrac{3}{2})^2} = \sqrt{\tfrac{1}{4}+\tfrac{9}{4}} = \tfrac{1}{2}\sqrt{10}.$ $\square$

9.



$Q = (x,y)$

$|PQ|$ = distance from Q to L.

$$\sqrt{x^2+(y-b)^2} = |y|$$

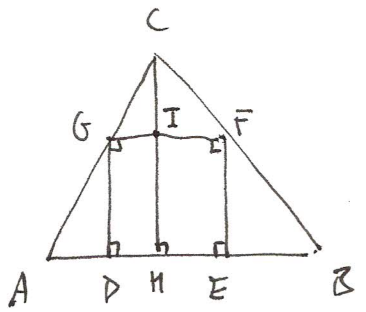
$$x^2+(y-b)^2 = y^2$$

$$x^2-2by+b^2 = 0.$$

$$y = \frac{1}{2b}(x^2+b^2)\quad\text{, parabola.}$$

(Note $b\neq 0$: $P \notin L$).

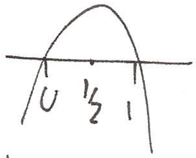
10.



$\triangle ADG \sim \triangle AHC \sim \triangle CGI$

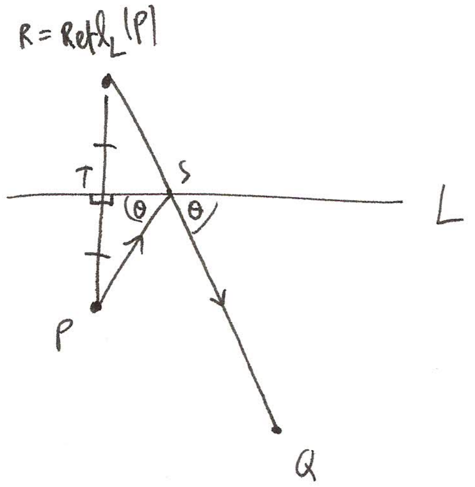
$$\begin{aligned}
\text{Area}(GDHI) &= \text{Area}(\triangle AHC) - \text{Area}(\triangle ADG) - \text{Area}(\triangle CGI)\\
&= \text{Area}(\triangle AHC)\cdot\left(1-\left(\frac{|DG|}{|CH|}\right)^2-\left(\frac{|CI|}{|CH|}\right)^2\right)\\
&= \text{Area}(\triangle AHC)\cdot\left(1-\left(\frac{|IH|}{|CH|}\right)^2-\left(\frac{|CI|}{|CH|}\right)^2\right)\\
&= \text{Area}(\triangle AHC)\cdot\left(1-x^2-(1-x)^2\right) \qquad 0\leq x\leq 1.\\
&= \text{Area}(\triangle AHC)\cdot\left(2x\cdot(1-x)\right)
\end{aligned}$$

Similarly for Area(FEHI). max value when $x=\tfrac{1}{2}$



$\leadsto$ Area(GDEF) $= \text{Area}(\triangle ABC)\cdot(2x(1-x)) \leq \tfrac{1}{2}\cdot\text{Area}(\triangle ABC)$,
equal when $x = |IH|/|CH| = \tfrac{1}{2}$. $\square$

11.



$\Delta STP \cong \Delta STR$ (SAS)

$\Rightarrow |PS| = |RS|$.

& $\angle TSR = \angle TSP = \theta$

$\Rightarrow$ R, S, Q are collinear.

12.

a. Translation by $\underline{v} = (5,7)$

b. Compute $\mathrm{Fix}(T)$:

$$T(x,y) = (x,y).$$

$$(1-y,\ 3-x) = (x,y)$$

$$\begin{matrix} 1-y = x \\ 3-x = y \end{matrix} \Rightarrow \begin{matrix} 1 = x+y \\ 3 = x+y \end{matrix} \Rightarrow \text{※},\ \text{i.e., } \mathrm{Fix}(T) = \emptyset.$$

So T is either a translation (X) or <u>glide reflection</u>.

T <u>not</u> of form $T(x,y) = (x+a, y+b)$.

$\therefore T^2$ is translation:

$$T^2(x,y) = T(1-y, 3-x) = (1-(3-x),\ 3-(1-y)) = (x-2,\ y+2)$$

Now T is the composition of a reflection followed by a translation by $\frac{1}{2}(-2,2) = (-1,1)$

The reflection R has formula $R(x,y) = T(x,y) - (-1,1) = (2-y,\ 2-x)$

$\mathrm{Fix}(R)$: $\left.\begin{matrix} x = 2-y \\ y = 2-x \end{matrix}\right\}\ x+y=2.$ So R is reflection in line $x+y=2$.

<u>Recap</u>: T is reflection in $x+y=2$ followed by translation by $(-1,1)$ (a glide reflection).

c. $T(x,y) = (y+2,\ 8-x)$

Fix (T) : $\left.\begin{array}{l} x = y+2 \\ y = 8-x \end{array}\right\}$ $\rightsquigarrow$ $\begin{array}{l} x-y=2 \\ x+y=8 \end{array}$ $\rightsquigarrow$ $\begin{array}{l} x=5 \\ y=3 \end{array}$

$\therefore$ T rotation, centre $(5,3)$.

Angle ? $T(x,y) = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\substack{\parallel \\ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$\Rightarrow \theta = -\pi/2.$

$\therefore$ T rotation about $(5,3)$ thru angle $-\pi/2$ ccw

i.e. angle $\pi/2$ cw. $\square$.

d. $T(x,y) = \frac{1}{13}(5x+12y+4,\ 12x-5y-6)$

Fix T ? . $\begin{array}{l} \frac{1}{13}(5x+12y+4) = x \\ \frac{1}{13}(12x-5y-6) = y \end{array}$ $\rightsquigarrow$ $\begin{array}{l} -8x+12y = -4 \\ 12x-18y = 6 \end{array}$ $\rightsquigarrow$ $\begin{array}{l} -2x+3y=-1 \\ 2x-3y=1 \end{array}$

equiv, $2x-3y=1$

i.e. $\text{Fix}\,T = \{(x,y) \mid 2x-3y=1\}$, line in $\mathbb{R}^2$

$\rightsquigarrow$ T is reflection in line $2x-3y=1$. $\square$.

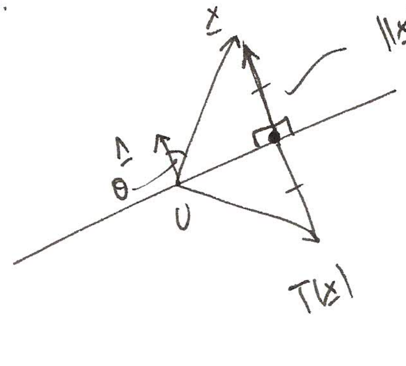
13. $T(x,y) = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right) + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right) + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4-\sqrt{2} \\ 2-3\sqrt{2} \end{pmatrix}$

14. a.



$$\underbrace{\|\underline{x}\| \cos\theta \cdot \frac{\underline{n}}{\|\underline{n}\|}} = \frac{\|\underline{x}\| \cdot \|\underline{n}\| \cos\theta}{\|\underline{n}\|^2} \cdot \underline{n} = \frac{(\underline{x} \cdot \underline{n})}{(\underline{n} \cdot \underline{n})} \cdot \underline{n}$$

$$\rightsquigarrow \quad T(\underline{x}) = \underline{x} - 2 \cdot \left(\frac{\underline{x} \cdot \underline{n}}{\underline{n} \cdot \underline{n}} \right) \underline{n}$$

b.

$y = 3x$

$-3x + y = 0 \qquad \begin{pmatrix} -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0.$

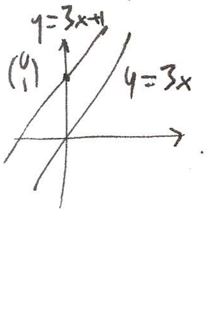
$$\underline{n} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \rightsquigarrow \quad T(\underline{x}) = \underline{x} - 2 \cdot \left(\frac{\underline{x} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\begin{pmatrix} -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}} \right) \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} - 2 \, \frac{(-3x+y)}{10} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 5x & + & -9x + 3y \\ 5y & + & 3x - y \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -4x + 3y \\ 3x + 4y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

c. $L: y = 3x + 1.$

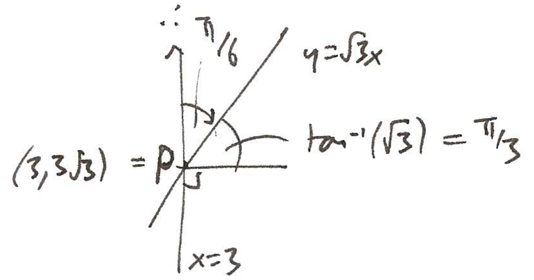


$$\mathrm{Refl}_L(\underline{x}) = T\left(\underline{x} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

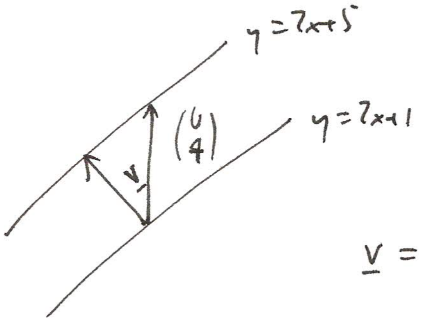
$$= \frac{1}{5} \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{5} \begin{pmatrix} -3 \\ 1 \end{pmatrix}. \quad \square$$

15a.



$\therefore$ Rotation about $P = (3, 3\sqrt{3})$ thru angle $2 \cdot \pi/6 = \pi/3$ cw.

b.



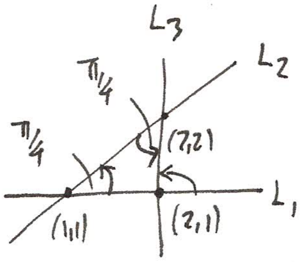
$y = 2x \rightsquigarrow -2x+y = \begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = 0.$

$\underline{n} = \begin{pmatrix}-2\\1\end{pmatrix}$

$$\underline{v} = \frac{\begin{pmatrix}0\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}}{\begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}} \cdot \begin{pmatrix}-2\\1\end{pmatrix} = \frac{4}{5}\cdot\begin{pmatrix}-2\\1\end{pmatrix}$$

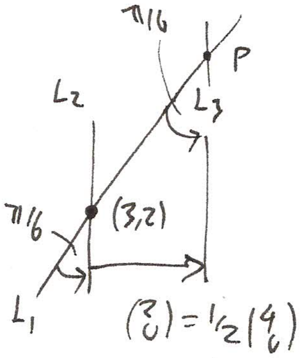
$\therefore$ Translation by $2\underline{v} = 8/5 \begin{pmatrix}-2\\1\end{pmatrix}$

c.



$$T = (\mathrm{Refl}_{L_3} \circ \cancel{\mathrm{Refl}_{L_2}}) \circ (\cancel{\mathrm{Refl}_{L_2}} \circ \mathrm{Refl}_{L_1})$$
$$= \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_1}$$
$= $ Rotation by π about $(2,1)$. $\square$.

d.



$$T = (\mathrm{Refl}_{L_3} \circ \cancel{\mathrm{Refl}_{L_2}}) \circ (\cancel{\mathrm{Refl}_{L_2}} \circ \mathrm{Refl}_{L_1})$$
$$= \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_1}$$
$= $ Rotation by $\pi/3$ ccw about $P = (3,2) + 2\cdot(1, \tan\frac{\pi}{3})$
$= (5, 2+2\sqrt{3})$

16. $\mathrm{Refl}_M \circ \mathrm{Refl}_L = $
a. rotation about $P = L \cap M$ thru angle 2θ where θ measured from L to M (if L & M intersect)
b. translation by $2\underline{v}$, where $\underline{v}$ perpendicular vector to L & M from L to M. (if L & M parallel)

Now see $\mathrm{Refl}_L \circ \mathrm{Refl}_M = \mathrm{Refl}_M \circ \mathrm{Refl}_L \iff$ a. $2\theta \equiv -2\theta \mod 2\pi$

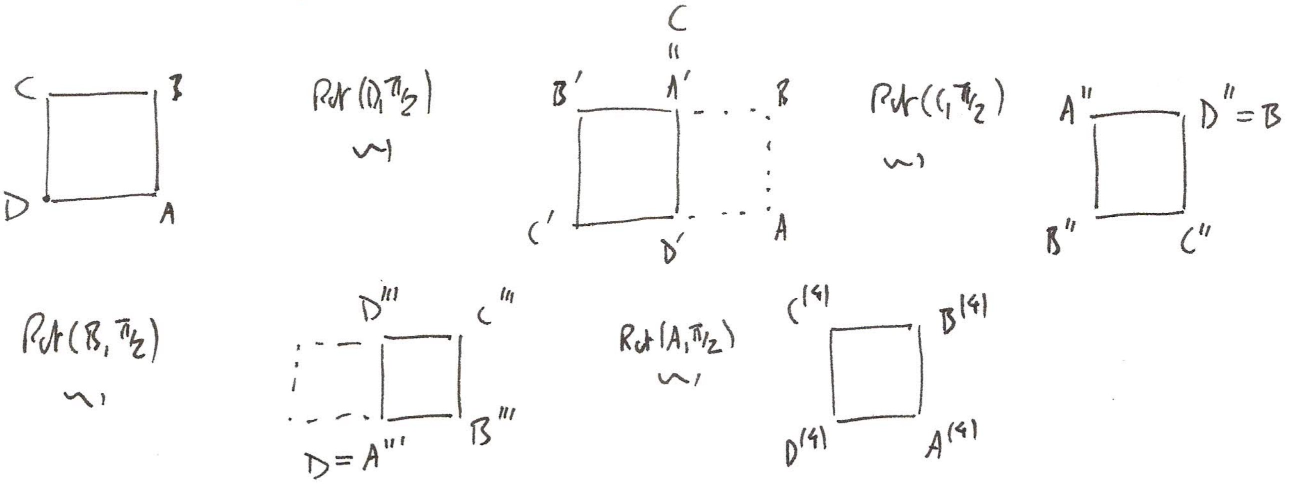
i.e. $4\theta \equiv 0 \mod 2\pi$

i.e. $\theta = \pi/2$ ~~$(0 < \theta \leq \pi/2)$~~ $(0 < \theta < \pi)$

b. $2\underline{v} = -2\underline{v}$

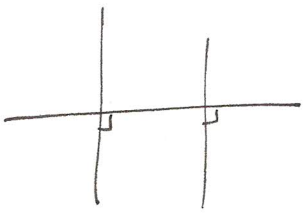
$\underline{v} = \underline{0}$ ※ $\square$.

17.



See $T(A) = A$, $T(B) = B$, ..., $T(D) = D$ $\Rightarrow$ T = identity. $\square$.

18. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+6, 4-y)$



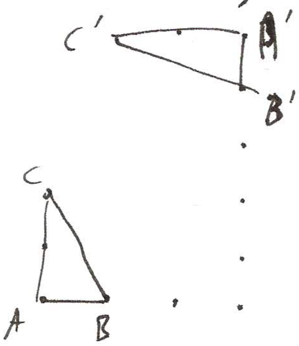
$y=2 : L_1$

$L_2 : x=0$ $\quad$ $x=3 : L_3$

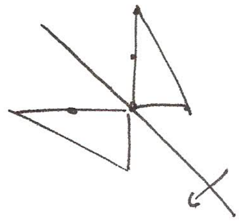
T is reflection in $y=2$ followed by translation by $\begin{pmatrix}6\\0\end{pmatrix}$ (glide reflection

$T = \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$. $\square$.

19.

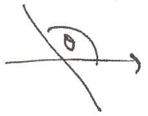


1. Reflect in $y=-x$. 2. translate by $(3,5)$.



(Remark: In fact, this is a glide reflection : Reflect in $y = -x+4$ followed by translation by $\begin{pmatrix}-1\\1\end{pmatrix}$.)

Algebraic formula



$\theta = 3\pi/4$

$$T(x,y) = \begin{pmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}3\\5\end{pmatrix}.$$