Math 462: Homework 8

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(1) What is the largest possible order of an element of the symmetric group S_9? What is the largest possible order of an element of the alternating group A_6? [Hint: What are the possible cycle types?]

(2) A card dealer shuffles a pack of $2n$ cards as follows: He first divides the pack into 2 halves consisting of n cards each. He then interleaves the cards so that the new top card is the top card from the bottom half, the new second card is the top card from the top half, the new third card is the second card from the bottom half, the new fourth card is the second card from the top half, and so on.

(a) Number the positions of the cards in the deck $1, 2, \ldots, 2n$, starting at the top, and consider the permutation f of $\{1, 2, \ldots, 2n\}$ corresponding to the shuffle. Express f as a composite of disjoint cycles for $n = 4$ and $n = 5$. What is the order of f in each case? (In other words, how many shuffles are required before the cards return to their original positions?)

(b) Check the following: when one such shuffle is performed, the card originally in position j is moved to position $2j \bmod (2n + 1)$.

(c) Using part (b), show that the order of the shuffle permutation is the least number k such that $2^k \equiv 1 \bmod (2n + 1)$.

(d) Now find the order of the shuffle for an ordinary pack of cards (52 cards in the deck). [Hint: Fermat's little theorem asserts that $a^{p-1} \equiv 1 \bmod p$ for p a prime and a a number not divisible by p.]

(3) In class we showed that the group G of symmetries of the cube can be identified with $S_4 \times \{\pm 1\}$. The identification is given by associating to a symmetry $g \in G$ the induced permutation of the 4 interior diagonals of the cube together with its determinant $\det(g)$. (Here we choose

coordinates so the center of the cube is at the origin. Then $g(\mathbf{x}) = A\mathbf{x}$ for some 3×3 orthogonal matrix A, and $\det(g) := \det(A)$.)

(a) For each cycle type of permutation in S_4, describe the corresponding *direct* symmetries ($\det(g) = +1$) geometrically.

(b) For each cycle type of permutation in S_4, describe the corresponding *opposite* symmetries ($\det(g) = -1$) geometrically.

(4) Describe the group G of symmetries of a baseball, taking into account the stitching. [Hint: It may be easiest to first consider the rotational symmetries.]

(5) In class we showed that a finite subgroup of rotations in $\mathbb{R}^3$ is either a cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$, a dihedral group D_n, or the group of rotational symmetries of a regular polyhedron (tetrahedron, cube / octahedron, or dodecahedron / icosahedron). Give examples of polyhedra whose groups of rotational symmetries are C_n and D_n, for each integer $n \geq 3$.

(6) (This question is optional and is harder than the others). We are going to study the game pictured below (you have probably seen it before). The 8 tiles can be moved around by repeatedly sliding a tile into the empty space. Now consider the permutations of the 8 tiles that can be obtained by a sequence of moves of this type, such that at the end of the sequence the empty space is again at the bottom right. These permutations form a subgroup of the symmetric group S_8. What is this subgroup?