Math 300.2 Homework 7

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Reading: Gilbert and Vanstone, Section 3.3 and Chapter 5.

(1) Let $S = \mathbb{Z}$ and let R be the equivalence relation on S defined by

$$aRb \iff a \equiv b \bmod 4.$$

How many different equivalence classes are there? Describe each class explicitly as in the example below:

$$[0] = \{a \in \mathbb{Z} \mid a = 4q \text{ for some } q \in \mathbb{Z}\} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}.$$

(2) Let R be a relation on a set S. What does it mean to say that R is an equivalence relation? In each of the following cases, determine whether R is an equivalence relation. (You should either give a careful proof that R is an equivalence relation or show by example that one of the required properties is not satisfied.)

(a) $S = \mathbb{R}$, $aRb \iff |a - b| < 4$.

(b) $S = \{1, 2, 3, 4, 5\}$ with relation R given by the table below. (In row a and column b we write Y if aRb and N otherwise.)

R	1	2	3	4	5
1	Y	N	N	Y	N
2	N	Y	Y	N	Y
3	N	Y	Y	N	Y
4	Y	N	N	Y	N
5	N	Y	Y	N	Y

(c) Let S be a set of triangles in the plane $\mathbb{R}^2$, and let R be the relation defined as follows: aRb if there exists $r \in \mathbb{R}$ such that $r > 0$ and the lengths of the sides of a are equal to the lengths of the sides of b multiplied by r (in some order). [What is the usual name for this relation?]

(d) $S = \mathbb{R}$, $aRb \iff a \leq b$.

(e) $S = \mathbb{R}$, $aRb \iff ab > 0$.

(3) Let $S = \mathbb{R}$. Define a relation R on S by

$$aRb \iff a - b \in \mathbb{Z}$$

(a) Show that R is an equivalence relation on S.

(b) Show that for every element $a \in \mathbb{R}$ there is a unique element $b \in \mathbb{R}$ such that aRb and $0 \leq b < 1$.

(4) Let $S = \mathbb{R}^2$ be the plane. Define a relation R on S by

$$(a, b)R(c, d) \iff a^2 + b^2 = c^2 + d^2$$

(a) Show that R is an equivalence relation.

(b) Describe the equivalence classes of R on S geometrically.

(5) Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$ be the plane with the origin removed. Define a relation R on S by

$$(a, b)R(c, d) \iff ad = bc$$

(a) Show carefully that R is an equivalence relation. [Hint: If $(a, b) \in S$, then either $a \neq 0$ or $b \neq 0$. Be careful not to divide by zero when proving transitivity!]

(b) Describe the equivalence classes of R on S geometrically.

(6) Let $S = \{(a, b) \mid a, b \in \mathbb{Z}, \quad b \neq 0\}$ and R be the equivalence relation on S defined by

$$(a, b)R(c, d) \iff ad = bc.$$

We explained in class that an equivalence class $[(a, b)]$ of R on S corresponds to a rational number $\frac{a}{b}$, with the different elements of the equivalence class corresponding to different ways of writing the same number as a fraction.

(a) As everybody knows, there is a unique way to write a fraction in its lowest terms. That is, given a rational number $x = \frac{a}{b}$, there exist a unique pair of integers (c, d) such that $\gcd(c, d) = 1$, $d > 0$, and $x = \frac{c}{d}$. In terms of the equivalence relation, this says that for every pair $(a, b) \in S$, there is a unique pair $(c, d) \in S$ such that $\gcd(c, d) = 1$, $d > 0$, and $(a, b)R(c, d)$. Prove this statement carefully. [Hint: First show that we can find (c, d) with the desired properties (cancel a factor $e = \gcd(a, b)$ and fix the sign of the denominator). Second show that if we have two such pairs (c, d) and (c', d') then $(c, d) = (c', d')$, using the basic result $((a \mid bc)\,\text{AND}(\gcd(a, b) = 1)) \Rightarrow a \mid c$.]

(b) If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers then we define the sum

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

To show that addition of rational numbers makes sense, we need to know it does not depend on the way in which we write the numbers as fractions. In terms of the equivalence relation, we need to know that if $(a, b)R(a', b')$ and $(c, d)R(c', d')$ then

$$(ad + bc, bd)R(a'd' + b'c', b'd').$$

Prove this statement carefully.