Math 461 Lecture 27 11/5

Homework 6 due Wednesday

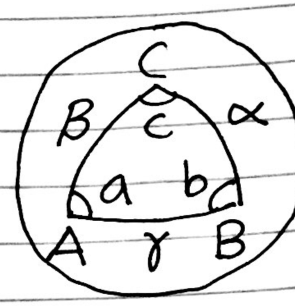
Solutions to Midterm are posted

Office hours: LGRT 1235H

today: 2:30-3:30 PM

tomorrow: 4:00-5:00 PM

Last time:



Spherical sine rule

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$$

Spherical isometries:

$T: S^2 \to S^2$

$d(T(P), T(Q)) = d(P, Q)$ for all $P, Q \in S^2$

if $\tilde{T}: \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ such that $\tilde{T}(\bar{0}) = \bar{0}$ it induces an isometry $T: S^2 \to S^2$ via

$T(P) = \tilde{T}(P)$ for $P \in S^2$

Examples:

$\tilde{T}$ = reflection in plane through $\bar{0}$

$\tilde{T}$ = rotation about line ℓ through $\bar{0}$

$\tilde{T}$ = rotary reflection

= reflection in plane π through $\bar{0}$ followed by rotation about line ℓ through $\bar{0}$ perpendicular through π

Theorem: every isometry of S^2 is either:

identity rotation

reflection rotary reflection

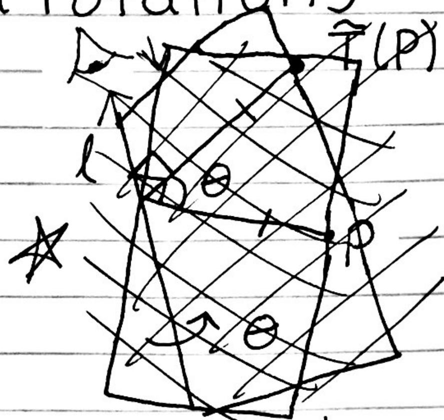
(proof later)

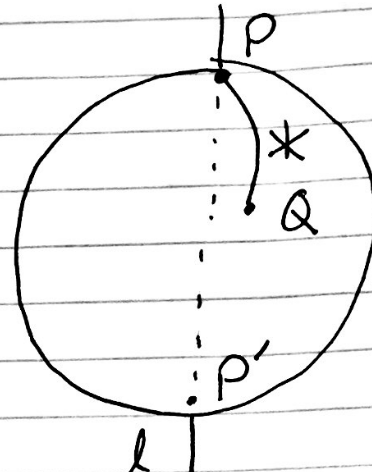
Today:

intrinsic description of rotation (from point of view of S^2)

Math 461 Lecture 27 11/5

algebraic description of reflections and rotations





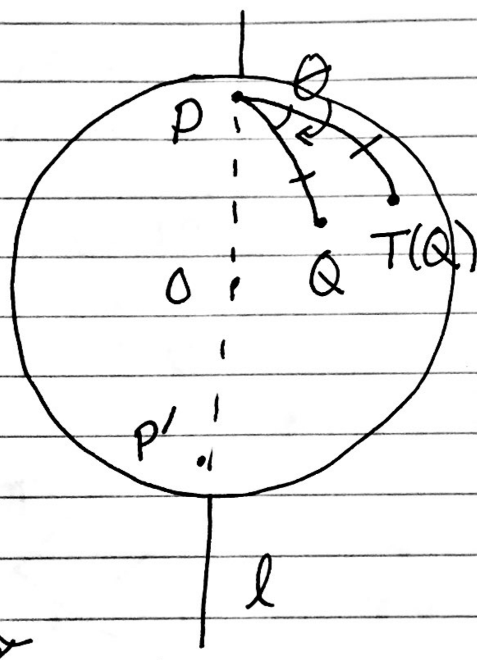
* shortest path on S^2 from P to Q

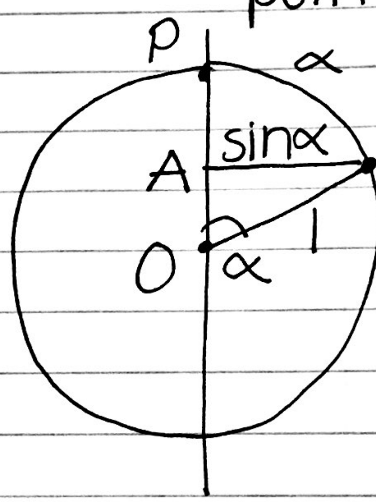
$T(Q) = ?$

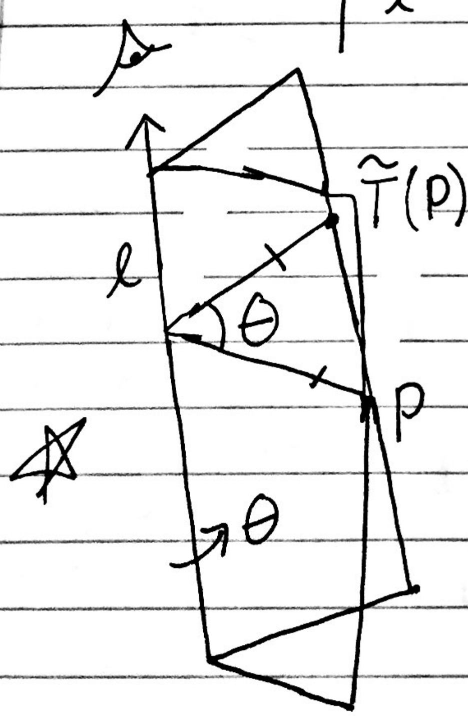
$S^2 \subset \mathbb{R}^3$

$\ell \cap S^2 = \{P, P'\}$

pair of antipodal points







lengths:

$\sin\alpha = \sin\beta \iff \alpha = \beta$

provided $0 \le \alpha \le \frac{\pi}{2}$

also note:

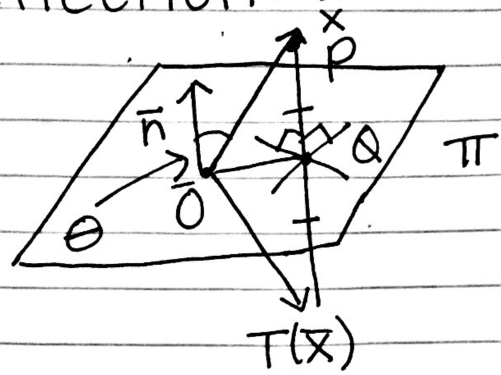
spherical rotation has two centers P & P' (antipodal points)

rotation by θ ccw about P = rotation by θ cw about P'

Math 461 lecture 27 11/5

algebraic description of isometries

reflection:



$T: \mathbb{R}^3 \to \mathbb{R}^3$ isometry given by reflection in plane π through $\bar{0}$

$T(\bar{x}) = ?$

$\pi: \bar{x} \cdot \bar{n} = \bar{0}$ where $\bar{n}$ is a normal vector to the plane π

$\overrightarrow{QP} = (?) \cdot \bar{n}$

$\|\overrightarrow{QP}\| = \|x\| \cdot \cos\theta$

$\overrightarrow{QP} = \dfrac{\|x\| \cos\theta}{\|n\|} \cdot \bar{n} = \left(\dfrac{\bar{x} \cdot \bar{n}}{\bar{n} \cdot \bar{n}}\right)\bar{n}$

$\bar{x} \cdot \bar{n} = \|x\| \cdot \|n\| \cos\theta$

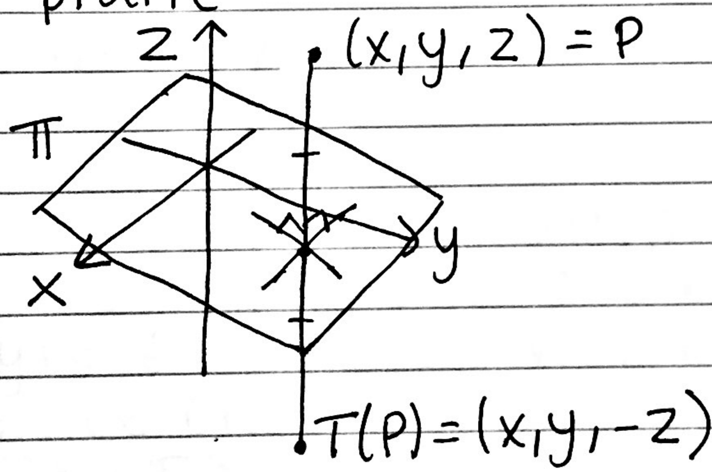
$\bar{n} \cdot \bar{n} = \|n\|^2$

$T(\bar{x}) = \bar{x} - 2\overrightarrow{QP}$

$T(\bar{x}) = \bar{x} - 2\left(\dfrac{\bar{x} \cdot \bar{n}}{\bar{n} \cdot \bar{n}}\right)\bar{n}$

easy example:

formula for reflection in the xy-plane



example #2:

reflection in the plane π (through origin) with equation $x+y+z=0$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{x} - 2\left(\dfrac{\bar{x} \cdot \bar{n}}{\bar{n} \cdot \bar{n}}\right)\bar{n}$

$n = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Math 461 lecture 27 11/5

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \bar{x} - 2\left(\frac{\bar{x}\cdot\bar{n}}{\bar{n}\cdot\bar{n}}\right)\bar{n}$$

$$= \begin{pmatrix}x\\y\\z\end{pmatrix} - 2\,\frac{(x+y+z)}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}3x-2x-2y-2z\\3y-2x-2y-2z\\3z-2x-2y-2z\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}x-2y-2z\\-2x+y-2z\\-2x-2y+z\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

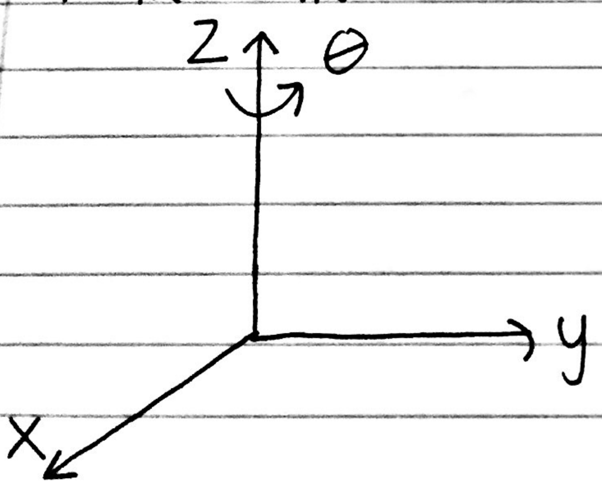
$$\bar{x}\cdot\bar{n} = \begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = x+y+2$$

$$\bar{n}\cdot\bar{n} = \begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 1^2+1^2+1^2 = 1+1+1 = 3$$

rotations: easy example
rotate about the z-axis through angle θ ccw (as viewed from positive end of the z-axis)

$T: \mathbb{R}^3 \to \mathbb{R}^3$

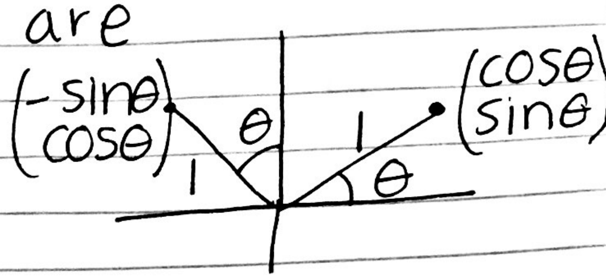
$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}?\\?\\z\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

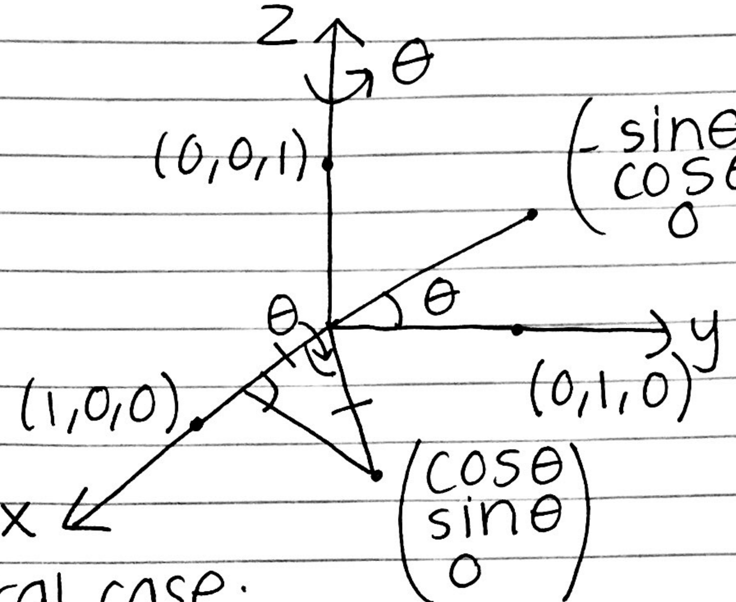


Math 461 Lecture 27 11/5

one way: in general if $T(\bar{x}) = A\bar{x}$ then columns of A are

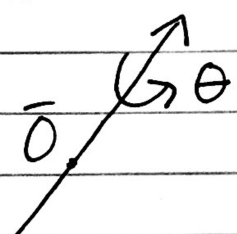
$T\begin{pmatrix}1\\0\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix}, T\begin{pmatrix}0\\0\\1\end{pmatrix}$





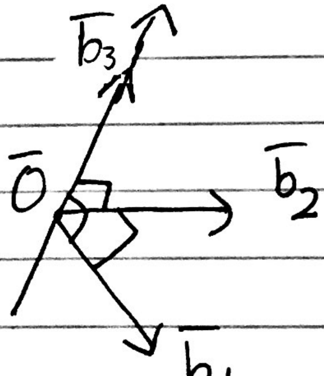
general case:

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(\bar{x}) = ?$



depends on 3 parameters:
2 for direction of line (given by vector of length 1)
1 for angle

basic idea in linear algebra: reduce to the easy case by "change of basis"



write down vectors $\bar{b}_1, \bar{b}_2, \bar{b}_3$ such that
- pairwise perpendicular
- length 1
- b_3 in the reflection of ℓ

then $B = \{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ is an orthonormal basis of $\mathbb{R}^3$

Math 461 Lecture 27 11/5

B-matrix of T is:
(matrix of T with respect to coordinates determined by the basis B)

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then $T(\bar{x}) = A\bar{x}$ where $A = PMP^T$

$P = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$

"change of basis matrix"

$P^T = P^{-1}$ because P is orthogonal