Math 462 Homework 7

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(1) Give a precise geometric description of the following transformation of the complex plane:

$$T\colon \mathbb{C} \to \mathbb{C}, \quad T(z) = (1+i)z.$$

(2) Recall the description of rotations of $\mathbb{R}^3$ using quaternions of length 1: Let $q = a + bi + cj + dk$ be a quaternion with length $|q| = 1$. Then we can write q uniquely as

$$q = \cos(\theta) + \sin(\theta)\mathbf{v},$$

where $0 \leq \theta \leq \pi$ and $\mathbf{v} \in \mathbb{R}^3 \subset \mathbb{H}$ has length 1. Consider the transformation

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T(\mathbf{x}) = q\mathbf{x}\bar{q}.$$

Then T is given by rotation about the axis L passing through the origin in direction $\mathbf{v}$ through angle 2θ counterclockwise as viewed from $\mathbf{v}$.

(a) Write $q = (\sqrt{2} + i + j)/2$ in the form $q = \cos(\theta) + \sin(\theta)\mathbf{v}$.

(b) Let T_1 be rotation about the line L through the origin in direction $(1, 1, 0)$ through angle π. Let T_2 be rotation about the x-axis through angle $\pi/2$ as viewed from the positive x-axis. Use quaternions to obtain a geometric description of the composite $T_2 \circ T_1$.

(c) Let T_1 be rotation about the y-axis through angle $\pi/2$ counterclockwise as viewed from the positive y-axis. Let T_2 be rotation about the z-axis through angle $\pi/2$ counterclockwise as viewed from the positive z-axis. Use quaternions to obtain a geometric description of the composite $T_2 \circ T_1$.

(3) Recall the interpretation of the quaternion multiplication in terms of the dot and cross product:

$$q_1q_2 = (t_1 + \mathbf{v}_1)(t_2 + \mathbf{v}_2) = (t_1t_2 - \mathbf{v}_1 \cdot \mathbf{v}_2) + (t_1\mathbf{v}_2 + t_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2).$$

(a) Let $\mathbf{v} \in \mathbb{R}^3$ be a vector with length 1. Show that the quaternions of the form $a + b\mathbf{v}$, $a, b \in \mathbb{R}$ multiply in the same way as complex numbers. That is, regarding $\mathbf{v} \in \mathbb{R}^3 \subset \mathbb{H}$ as a quaternion with zero real part, we have $\mathbf{v}^2 = -1$.

(b) Explain why (a) is consistent with the description of rotations of $\mathbb{R}^3$ about the axis L through the origin in direction $\mathbf{v}$ in terms of the quaternions q of the form $q = a + b\mathbf{v}$ with $|q| = 1$.

(4) (a) Suppose given $q = \cos(\theta) + \sin(\theta)\mathbf{v}$, with $0 \le \theta \le \pi$ and $\|\mathbf{v}\| = 1$. Write $-q = \cos(\phi) + \sin(\phi)\mathbf{w}$ with $0 \le \phi \le \pi$ and $\|\mathbf{w}\| = 1$. How are ϕ and $\mathbf{w}$ related to θ and $\mathbf{v}$?

(b) Use your answer to part (b) and the geometric description of the rotation associated to q recalled in Q2 above to check that q and $-q$ correspond to the same rotation.

(c) Conversely, show that if unit quaternions q_1 and q_2 define the same rotation then $q_2 = \pm q_1$.

(5) Let $\bar{F} \colon S^2 \to \mathbb{C} \cup \{\infty\}$ be the stereographic projection of the sphere S^2 onto the (extended) complex plane. In class we showed that a rotation $T \colon S^2 \to S^2$ induced by a quaternion $q = a + bi + cj + dk$ of length 1 corresponds to the linear fractional transformation (LFT)

$$f \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad f(w) = \frac{(a + di)w + (-c + bi)}{(c + bi)w + (a - di)}$$

(a) Determine the images of the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$ under stereographic projection $\bar{F}$.

(b) Let $T \colon S^2 \to S^2$ be rotation about the x-axis through angle π. Compute the corresponding LFT f.

(c) Check your answer to part (b) using your answer to part (a).

(6) Let $C \subset \mathbb{R}^3$ be the cube with vertices $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$. By a *rotational symmetry* of the cube C we mean a rotation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$

such that $T(C) = C$. There are 24 rotational symmetries of the cube including the identity transformation (this can be checked using the orbit–stabilizer theorem from 411). List the rotational symmetries T of the cube and identify the corresponding pairs $\pm q$ of quaternions.

[Hint: The possible axes of rotation are of three types.]