Math 461 Lecture 14 10/5

Homework 3 solutions posted

Last time:

- Algebra of ruler and compass constructions
- A length is constructible <=> obtained from 1 by $+, -, \times, \div, \sqrt{\ }$

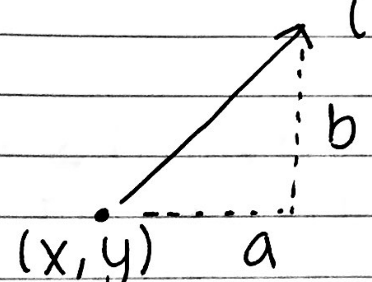
Today:

- Isometries

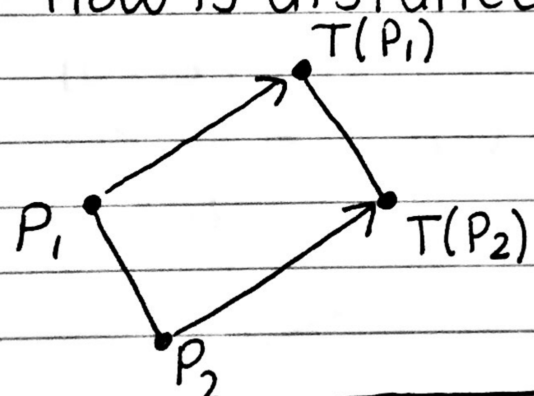
Isometries:

- An isometry of the plane is a function or transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which preserves distances, that is
  $|T(P_1)T(P_2)| = |P_1P_2|$ for all $P_1, P_2 \in \mathbb{R}^2$
- Examples:

1. Translation by (a, b):
   $T(x,y) = (x+a, y+b)$



- How is distance preserved?



$P_1 = (a_1, b_1)$

$P_2 = (a_2, b_2)$

$T(P_1) = (a_1 + a, b_1 + b)$

$T(P_2) = (a_2 + a, b_2 + b)$

$$|T(P_1)T(P_2)| = \sqrt{((a_2+a)-(a_1+a))^2 + ((b_2+b)-(b_1+b))^2}$$
$$= \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}$$
$$= |P_1P_2|$$

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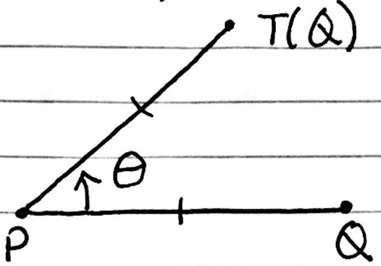
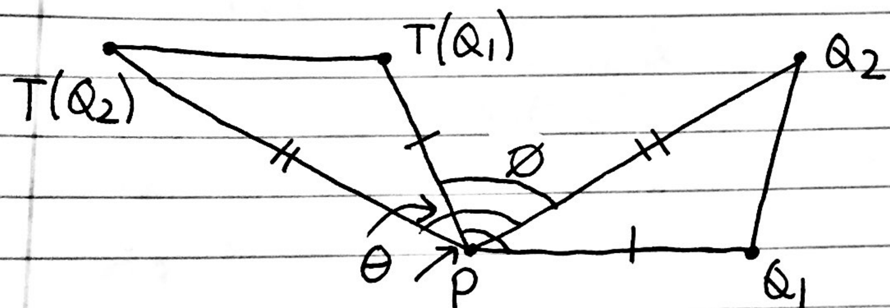
2. Rotation:

- Fix a point $P=(a,b)$ and an angle θ (counterclockwise)
- Define T = rotation about P through angle θ counterclockwise

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(Q)$ defined by diagram



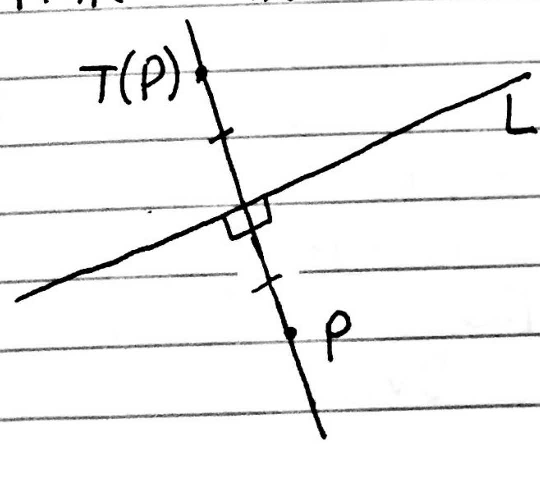
- Claim: This is an isometry.
  - Need to show for any $Q_1, Q_2 \in \mathbb{R}^2$, $|T(Q_1)T(Q_2)| = |Q_1Q_2|$



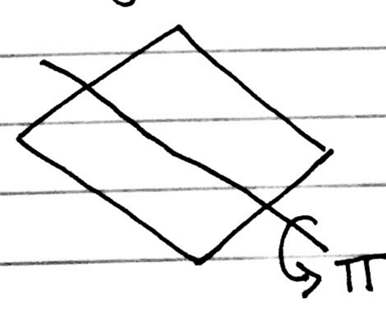
- $\angle T(P_1)QT(P_2) = \angle P_1QP_2 = \theta - \emptyset$
- by SAS $\Delta T(P_1)QT(P_2) \cong \Delta P_1QP_2 \Rightarrow$ $|T(P_1)T(P_2)| = |P_1P_2|$ $\square$

3. Reflection in a line $L \subset \mathbb{R}^2$ (axis of reflection)

- $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(P)$ defined by diagram



- In 3D can describe by rotation



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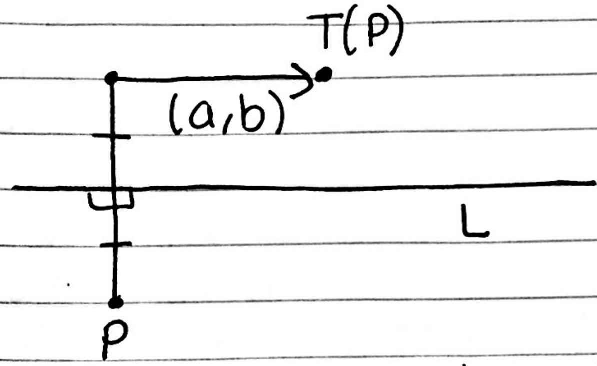
4. Glide reflection

- Line $L \subset \mathbb{R}^2$ and a vector (a,b) in the direction of L
- $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the composition of reflection in L followed by translation by (a,b) (parallel to L)



- In this case, order of composition is irrelevant
- Here note: If T_1 & T_2 are isometries, then so is the composition
  $T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T_2 \circ T_1(P) = T_2(T_1(P))$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^2 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

- Why does this work?
  $$|T_2(T_1(P))\,T_2(T_1(Q))| \underset{T_2 \text{ is an isometry}}{=} |T_1(P)\,T_1(Q)| \underset{T_1 \text{ is an isometry}}{=} |PQ|$$
- Also isometries form a group with group operation = composition of functions
- Identity: $id: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad id(x,y) = (x,y)$ ("do nothing")
- Associative: Composition of functions is associative

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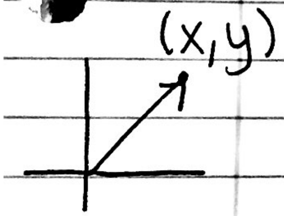
- Inverses: Every isometry has an inverse, which is also an isometry
- Theorem: Every isometry is either:
  - identity
  - rotation
  - translation
  - reflection
  - glide reflection
- surprising because it means composition of two known isometries is again on this list
- Describe isometries algebraically
  - Translation ✓
  - Rotations:
    - recall from Math 235: Rotation about origin through angle θ counterclockwise is given by



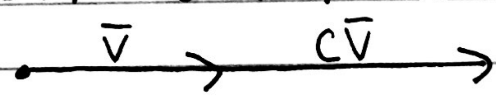
$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

i.e. $T(x,y) = (\cos\theta\cdot x - \sin\theta\cdot y, \sin\theta\cdot x - \cos\theta\cdot y)$

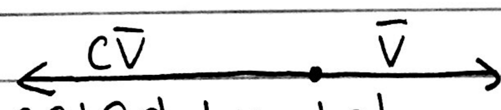
- How do we prove this formula?

1. Show that T is a linear transformation
   i.e. $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u},\bar{v} \in \mathbb{R}^2$
   and $T(c\cdot\bar{v}) = cT(\bar{v})$ $c \in \mathbb{R}$

- Recall:



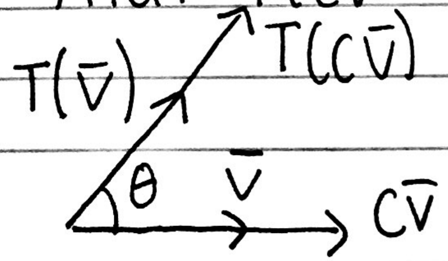
$c > 0$



$c < 0$

length scaled by $|c|$

- For a rotation about origin, now clear that $T(c\bar{v}) = cT(\bar{v})$:- $cT(\bar{v}) = T(c\bar{v})$



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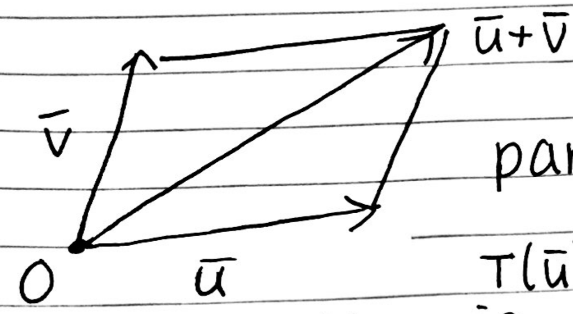
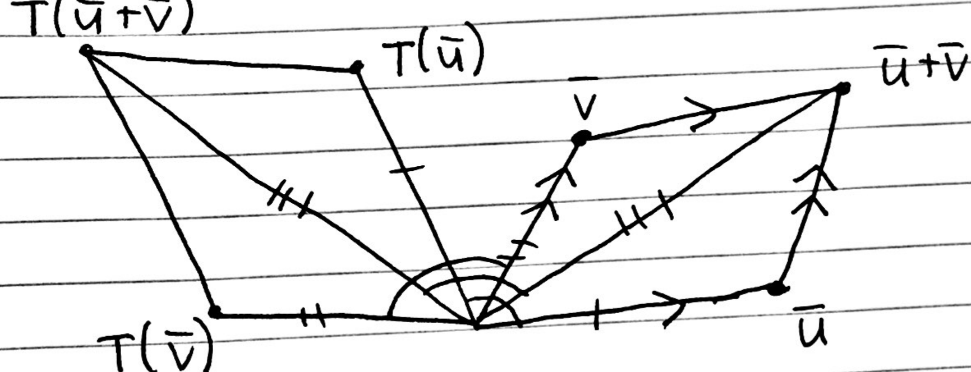
- Geometric additions of vectors ("parallelogram law")



parallelogram

$T(\bar{u}) + T(\bar{v}) = T(\bar{u}+\bar{v})$?

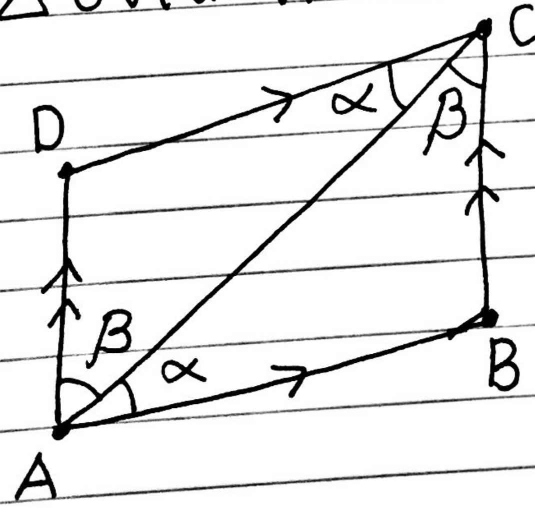
- Equivalently, is $0, T(\bar{u}), T(\bar{v}), T(\bar{u}+\bar{v})$ a parallelogram?



- By a previous argument

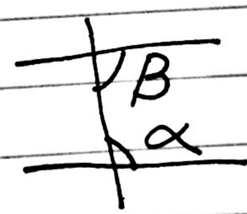
$\triangle O u (u+v) \cong \triangle O T(u) T(u+v)$

$\triangle O v (u+v) \cong \triangle O T(v) T(u+v)$



by ASA:

$\triangle ACB \cong \triangle CAD$



$\alpha + \beta = \pi$

$\Longleftrightarrow$ parallel