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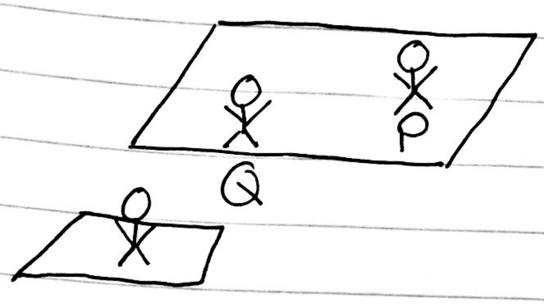
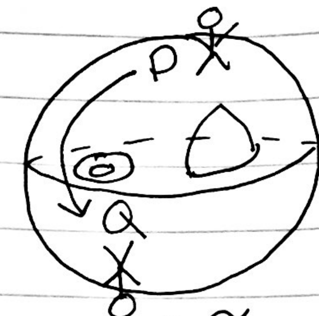
Hyperbolic Geometry

technically $\forall P, Q \in \mathbb{R}^2$
$\exists t : \mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}^2 \quad P \mapsto Q$

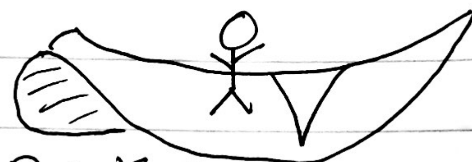




S^2

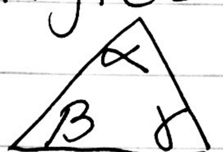
$P, Q \quad t : S^2 \xrightarrow{\sim} S^2 \quad P \mapsto Q$
homogeneous/symmetric

hyperbolic plane H^2
locally looks like a saddle or
(a pringle)



triangles:

$\mathbb{R}^2$:

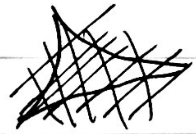


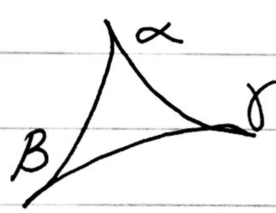
$\alpha + \beta + \gamma = \pi$

S^2:

$\alpha + \beta + \gamma > \pi$

H^2:





$\alpha + \beta + \gamma < \pi$

question: how to construct H^2?
the str. of a space is determined
by "geodesics"
is determined by how you
measure the length of a curve

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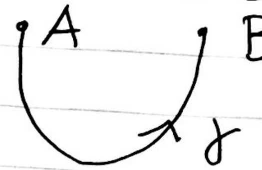
$\mathbb{R}^2$



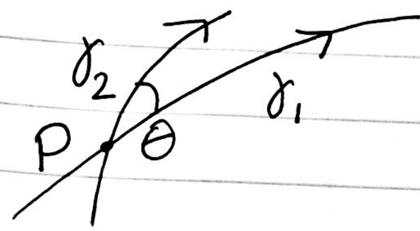
$\gamma: \bar{x}(t) = (x(t), y(t))$

eg. $y = x^2$

$\bar{x}(t) = (t, t^2) \quad a \le t \le b$

$$\text{length}(\gamma) = \int_a^b |\bar{x}'(t)|\, dt$$

$$= \int_a^b \sqrt{x'^2 + y'^2}\, dt$$

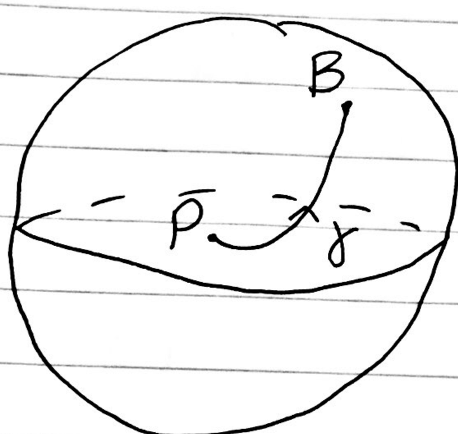


geodesic

angle

$$\cos\theta = \left.\frac{\gamma_1' \cdot \gamma_2'}{|\gamma_1'| \cdot |\gamma_2'|}\right|_p$$

S^2



I: $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$

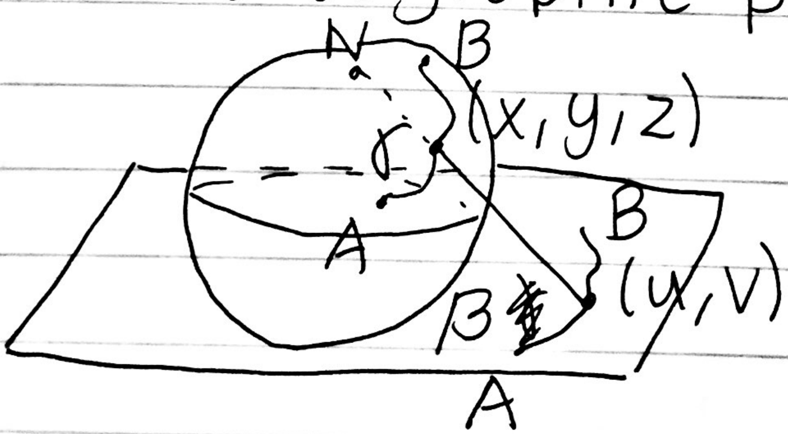
$\gamma: \bar{x}(t) = (x(t), y(t), z(t))$

$a \le t \le b$

$$\text{length}(\gamma) = \int_a^b |\bar{x}'(t)|\, dt$$

$$= \int_a^b \sqrt{x'^2 + y'^2 + z'^2}\, dt$$

II: stereographic projection



$F: S^2 \setminus \{N\} \xrightarrow{\sim} \mathbb{R}^2$

$u = \dfrac{x}{1-z} \quad v = \dfrac{y}{1-z}$

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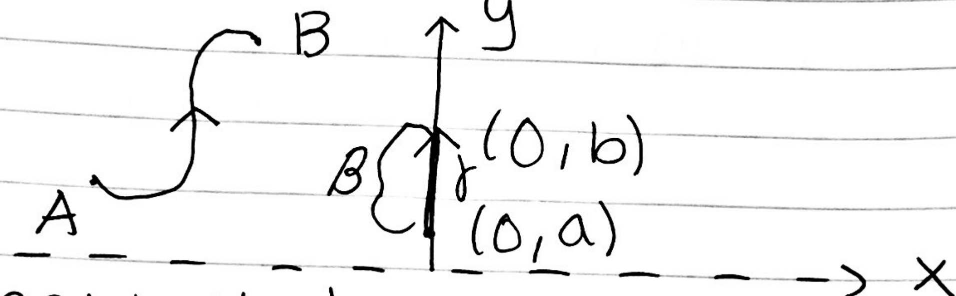
$$\text{length}(\gamma) = \int_a^b |\bar{x}'(t)|\,dt$$

$$= \int_a^b \sqrt{x'^2 + y'^2 + z'^2}\,dt$$

$$= \int_a^b \left(\frac{2}{u^2+v^2+1}\right)\left(\underbrace{\sqrt{u'^2+v'^2}}_{|\beta'|}\right) dt$$

H^2: Poincare half plane model

$H^2 = \{y > 0\}$



not including the x-axis

γ: $\bar{x}(t) = (x(t), y(t))$ $\quad a \le t \le b$

$$\text{length}(\gamma) = \int_a^b \frac{|\bar{x}'(t)|}{y(t)}\,dt$$

$$= \int_a^b \frac{\sqrt{x'^2 + y'^2}}{y}\,dt$$

example: $\gamma: (0,t),\ t > 0 \quad a \le t \le b$

in Euclidean geometry:

$\text{length}_{\mathbb{R}^2}(\gamma) = b - a$

in H^2: $\text{length}_{H^2}(\gamma) = \int_a^b \frac{\sqrt{x'^2+y'^2}}{y}\,dt$

$$= \int_a^b \frac{\sqrt{0^2+1^2}}{t}\,dt = \int_a^b \frac{1}{t}\,dt = \ln(b) - \ln(a)$$

$$= \ln\left(\frac{b}{a}\right)$$

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in fact, γ is the geodesic between $(0,a)$ and $(0,b)$

$\beta: (x(t), t) \quad a \le t \le b \quad \underline{x(a) = 0} = x(b)$

$$\text{length}_{H^2}(\beta) = \int_a^b \frac{\sqrt{x'^2 + y'^2}}{y}\, dt$$

if $x^2 \ge 0$ then $\displaystyle\int_a^b \frac{\sqrt{x'^2 + y'^2}}{y} dt \ge \int_a^b \frac{\sqrt{y'^2}}{y} dt$

$$= \int_a^b \frac{1}{t} dt = \ln\left(\frac{b}{a}\right)$$

γ is the shortest path (geodesic) between the points $(0,a)$ and $(0,b)$

example: if $a = b$, a is the same point then $\text{length}_{H^2}(\gamma) = \ln\left(\frac{a}{a}\right) = \ln(1) = 0$ as b

if they are the same point then the distance between them is zero

if fix $(0,b)$ move $(0,a) \rightsquigarrow (0,0)$

$\text{length}_{H^2}(\gamma) = \ln\left(\frac{b}{a}\right) \rightsquigarrow \infty$

as $a \to 0$ then $\frac{b}{a} \to \infty$ so $\ln\left(\frac{b}{a}\right) \to \infty$

the points $\{(0, 2^n), n \in \mathbb{Z}\}$

$(0,a) = (0, 2^n)$

$(0,b) = (0, 2^{n+1})$

$\text{length}_{H_2}(\gamma) = \ln\left(\frac{2^{n+1}}{2^n}\right) = \ln(2)$ a constant

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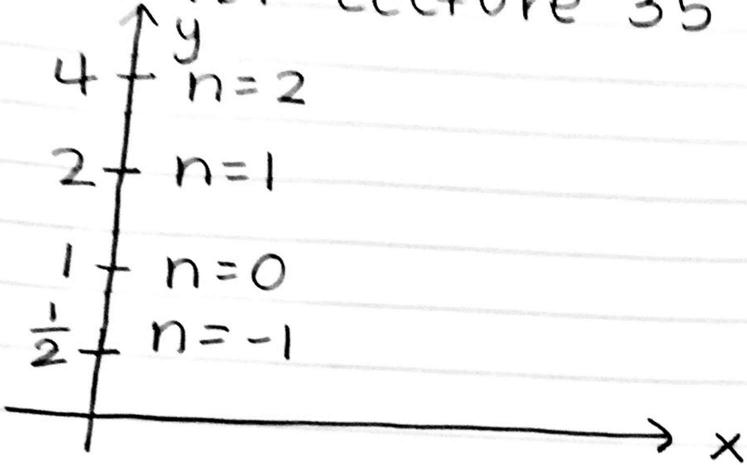

points $(0, 3^n)$ equally spaced by $\ln(3)$
points $(0, e^n)$ equally spaced by $\ln($