Math 461 Lecture 26 11/2

Homework 6 available and due next wednesday

Last time:

spherical lines give shortest paths

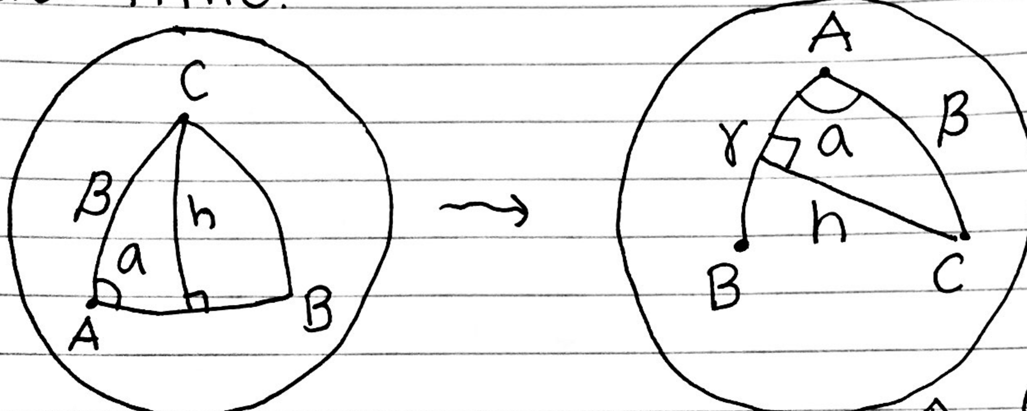
spherical sine rule

Today:

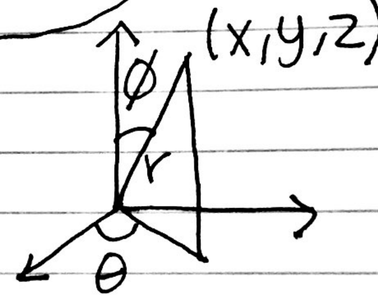
finish statement and proof of spherical sine rule

spherical isometries

Last time:



choose coordinates in xz plane



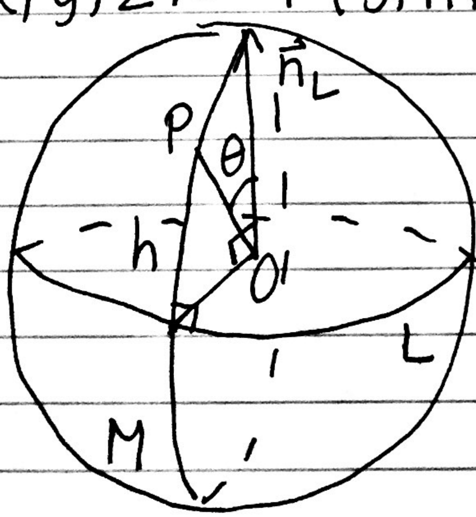
$A = (0,0,1)$

$B = (\sin\gamma, 0, \cos\gamma)$

$C = (\sin\beta\cos a, \sin\beta\sin a, \cos\beta)$

spherical polar coordinates

$(x,y,z) = r(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$



$h = \frac{\pi}{2} - \theta$ OR

$\vec{OP} \cdot \vec{n}_L = \cos\theta$

$= \cos\left(\frac{\pi}{2} - h\right)$

*

$= \sin(h)$

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L is a spherical line through A and B

$$\vec{n}_L = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \vec{OP}\cdot\vec{n}_L = \begin{pmatrix}\sin\beta\cos a\\ \sin\beta\sin a\\ \cos\beta\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \sin\beta\sin a$$

so by $*$ $\sin(h) = \sin\beta\sin a$

Similarly (reversing roles of A and B)

$\sin(h) = \sin\alpha\sin b$

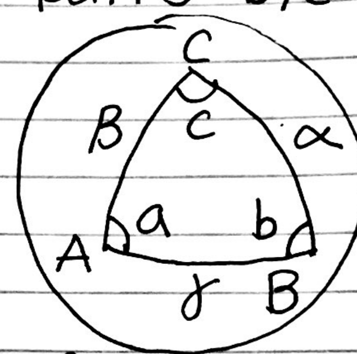
$\rightsquigarrow \sin\beta\sin a = \sin\alpha\sin b$

$$\Rightarrow \frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b}$$

now repeating forother pairs B, C and C, A and get:

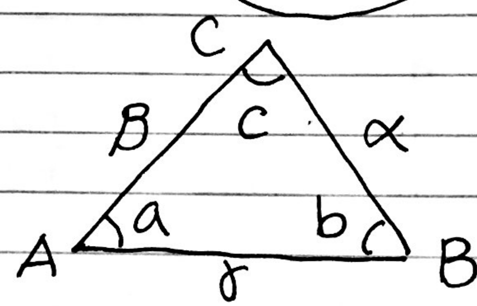
Spherical sine rule

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$$

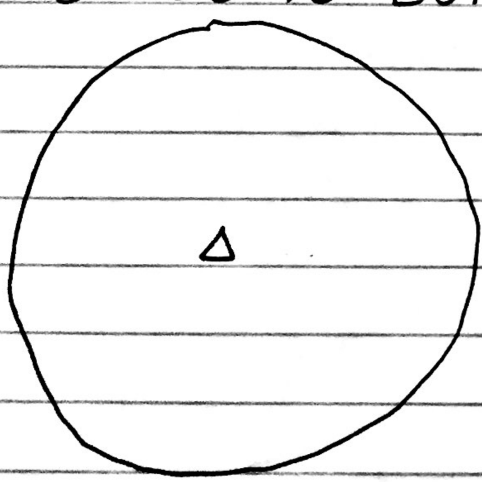


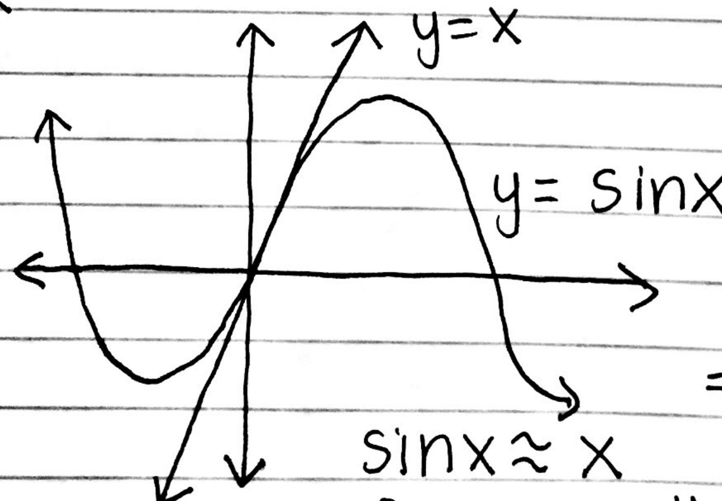
compare with Euclidean case

$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}$$



If $\triangle ABC$ on the sphere is small, its appoximately a plane triangle so SSR should reduce to ESR





$\Rightarrow$ ✓

$\sin x \approx x$ for x small

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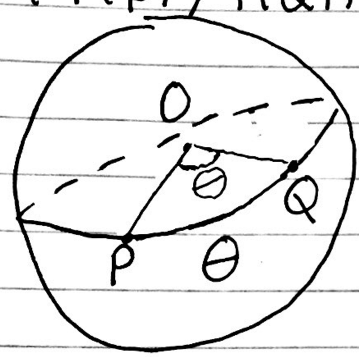
sperical isometries:

$S^2 = \{(x, y, z) \mid x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$

sphere of radius 1 and center $(0,0,0)$

an isometry of S^2 is a transformation $T: S^2 \to S^2$ such that T preserves distances

$d(T(P), T(Q)) = d(P,Q)$ for all $P, Q \in S^2$



$d(P,Q) - \theta =$ length of shortest path on S^2 from P to Q

Remark:

given an isometry of S^2 it will extend to an isometry of $\mathbb{R}^3$

i.e. have transformation $\tilde{T}: \mathbb{R}^3 \to \mathbb{R}^3$ such that $d_{\mathbb{R}^3}(\tilde{T}(P), \tilde{T}(Q)) = d_{\mathbb{R}^3}(P,Q)$ and $\tilde{T}(P) = T(P)$ for $P \in S^2$

$d_{\mathbb{R}^3}(\bar{v} \cdot \bar{w}) = \|\bar{v} - \bar{w}\|$

$= \sqrt{(v_1-w_1)^2 + (v_2-w_2)^2 + (v_3-w_3)^2}$

example: rotation of $\mathbb{R}^3$

ℓ line in $\mathbb{R}^3$ (axis of rotation)

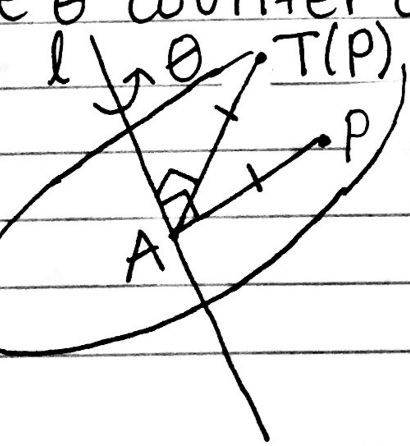
θ angle

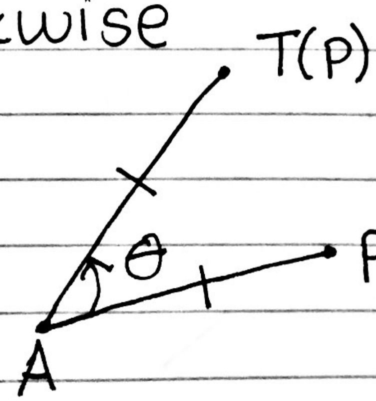
rotate about axis ℓ through angle θ

$T: \mathbb{R}^3 \to \mathbb{R}^3$

Reminder: $T: \mathbb{R}^2 \to \mathbb{R}^2$

rotation about a point A through angle θ counter clockwise





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if you look at it from the top it looks like it rotates counter clockwise

if you look at it from the bottom it looks like it rotates clockwise

to completely specify a rotation in three dimensions need:

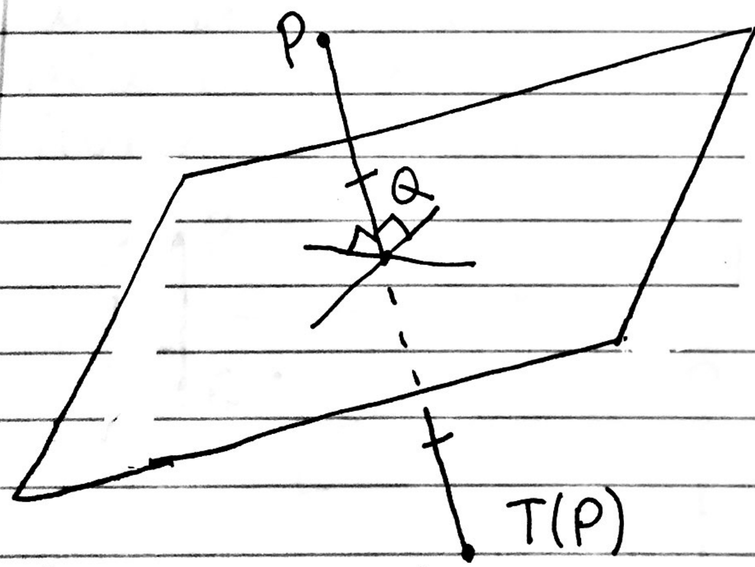
1. an axis ℓ (line)
2. a choice of direction for ℓ
3. an angle θ of rotation about ℓ (with convention that rotation is counter clockwise when viewed from positive end of ℓ)

and sense (ccw/cw) when viewed from positive end of ℓ

in general, given an isometry $\tilde{T}: \mathbb{R}^3 \to \mathbb{R}^3$ it will induce an isometry of S^2 precisely when $\tilde{T}(\vec{0}) = \vec{0}$

in the case of a rotation means ℓ passes through $\vec{0}$.

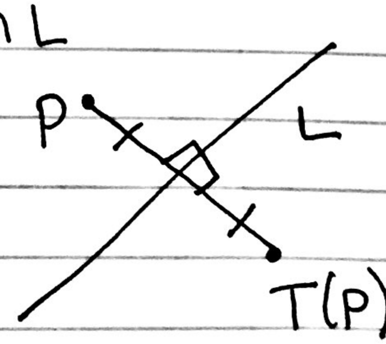
$T: \mathbb{R}^3 \to \mathbb{R}^3$ reflection in π



π plane

$\subset \mathbb{R}^3$

reminder: 2D reflection in L



will give an isometry of S^2 when $\vec{0} \in \pi$

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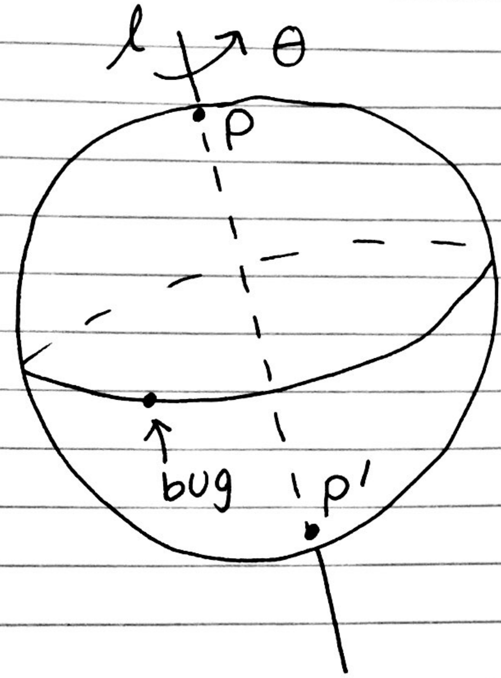
isometries in $\mathbb{R}^2$:

identity ✓ reflection ✓ glide

translation rotation ✓



translation
rotation in $\mathbb{R}^2$
are like a
rotation in S^2

glide:
rotary reflection
rotate about
an axis ℓ and
then reflect in
plane $\pi \perp$ to ℓ
ℓ and $\pi \in \vec{O}$

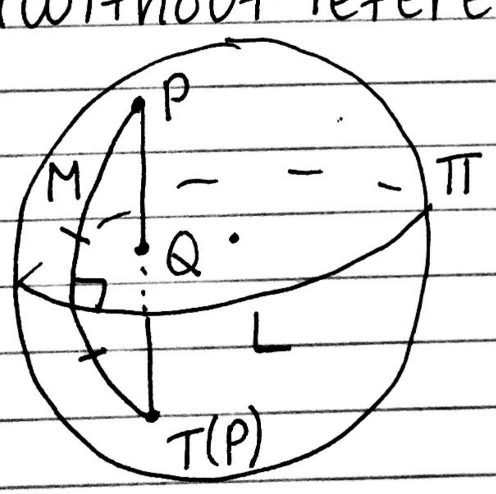
Theorem:
every isometry of S^2 is one of the following:

identity rotation

reflection rotary reflection

will prove later

how can these be defined on S^2?
(without reference to $\mathbb{R}^3$)



reflection in plane
π through origin in
$\pi \cap S^2 = L$