Math 461 Lecture 13 10/3

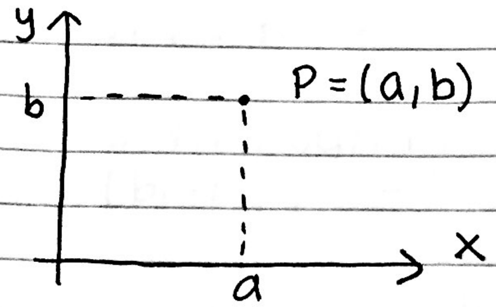
Homework 2 returned

Homework 3 due now

Homework 4 available and due next Wednesday 10/10 at start of class

Last time:

Coordinates



Distance

$P_1 = (a_1, b_1)$ $\qquad$ $P_2 = (a_2, b_2)$

$$|P_1P_2| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

Equation of circle

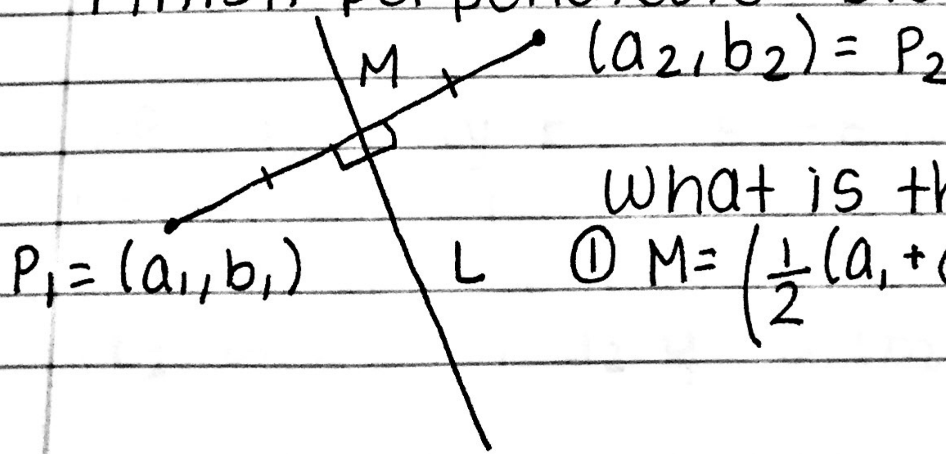
Equation of line, slope $\quad$ parallel $\Leftrightarrow$ same slope

Perpendicular bisector

Today:

- Algebra of ruler and compass constructions
- constructible numbers
- Isometries

Finish perpendicular bisector:



What is the equation of L?

① $M = \left(\frac{1}{2}(a_1 + a_2), \frac{1}{2}(b_1 + b_2)\right)$

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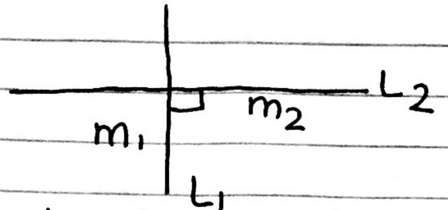

slopes satisfy $m_1 m_2 = -1$

$m_1 = \frac{b_2-b_1}{a_2-a_1}$ = slope of P_1P_2

slope of $L = m_2 = \frac{-1}{m_1} = \frac{-(a_2-a_1)}{(b_2-b_1)}$

passes through M

$$y - \left(\frac{1}{2}(b_1+b_2)\right) = -\frac{(a_2-a_1)}{b_2-b_1}\left(x - \frac{1}{2}(a_1+a_2)\right)$$

Recall: The perpendicular bisector of P_1P_2 is the set of points P in the plane such that $|PP_1| = |PP_2|$ $\quad P_1 = (a_1, b_1)$

$L = \{P \in \mathbb{R}^2 \mid |PP_1| = |PP_2|\}$ $\quad P_2 = (a_2, b_2)$

Let $P = (x, y)$ then

$$\sqrt{(x-a_1)^2 + (y-b_1)^2} = \sqrt{(x-a_2)^2 + (y-b_2)^2}$$

$$(x-a_1)^2 + (y-b_1)^2 = (x-a_2)^2 + (y-b_2)^2$$

$$\cancel{x^2} - 2a_1x + a_1^2 + \cancel{y^2} - 2b_1y + b_1^2 =$$

$$\cancel{x^2} - 2a_2x + a_2^2 + \cancel{y^2} - 2b_2y + b_1^2$$

$$2(a_2-a_1)x + 2(b_2-b_2)y = (a_2^2+b_2^2) - (a_1^2+b_1^2)$$

"$ax + by = c$" equation of line L

Algebra of ruler and compass constructions:

3 basic equations $P_1 = (a_1, b_1)$ $P_2 = (a_2, b_2)$

1 Draw a line between two points P_1, P_2

equation: $(y-b_1) = \frac{(b_2-b_1)}{(a_2-a_1)}(x-a_1)$

OR $(a_2-a_1)(y-b_1) = (b_2-b_1)(x-a_1)$

Notice: The coefficients of the equation of the line ("$ax+by=c$")

$$(b_2-b_1)x + (a_1-a_2)y = a_1(b_2-b_1) - b_1(a_2-a_1)$$

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are obtained from a_1, b_1, a_2, b_2 by $+, -, \times, \div$

② Draw a circle with center $P=(a,b)$, radius r

equation: $(x-a)^2 + (y-b)^2 = r^2$

③ Intersect lines and circles

3 cases: a. two lines

b. one line and one circle

c. two circles

Problem: Understand how coordinates of intersection points are related to coefficients of equations

(want to know: can be obtained from coefficients by $+, -, \times, \div, \sqrt{\ }$)

a. By linear algebra, here solutions are from $+, -, \times, \div, \sqrt{\ }$

b. one line and one circle

Ex. Find intersection point of a circle with center the origin and radius 2 and the line through point $(0,1)$ with slope 1

equation of circle: $x^2 + y^2 = 4$ ①

equation of line: $y = x+1$ ②

$x^2 + (x+1)^2 = 4$ (plug $y = x+1$ in ①)

$2x^2 + 2x + 1 = 4$

$2x^2 + 2x - 3 = 0$

solve using quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(2)(-3)}}{2(2)} = \frac{-2 \pm \sqrt{28}}{4} = \frac{-1 \pm \sqrt{7}}{2}$$

Finally $y = x + 1 = \frac{1 + \sqrt{7}}{2}$

Two points: $\left(\frac{-1+\sqrt{7}}{2}, \frac{1+\sqrt{7}}{2}\right), \left(\frac{-1-\sqrt{7}}{2}, \frac{1-\sqrt{7}}{2}\right)$

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In general, coordinates of the intersection points of a line and a circle are obtained from the coefficients of the equation of the line and the circle by $+, -, \times, \div, \sqrt{\ }$

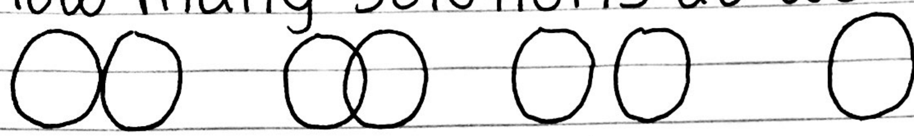
c. Intersection of two circles

$(x-a_1)^2 + (y-b_1)^2 = r_1^2$ ①

$(x-a_2)^2 + (y-b_2)^2 = r_2^2$ ②

Find points of intersection

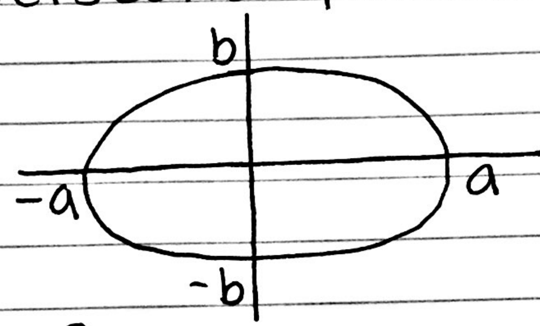
How many solutions do we except?

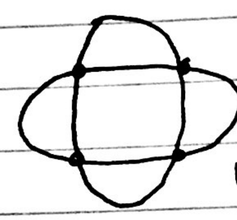


1 2 0 ∞

How many intersection points for two ellipses?

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$





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0, 1, 2, 3, 4

How do we solve?

① - ② is a linear equation

$x^2 - 2a_1x + a_1^2 + y^2 - 2b_1y + b_1^2 = r_1^2$

$x^2 - 2a_2x + a_2^2 + y^2 - 2b_2y + b_2^2 = r_2^2$

①, ② $\equiv$ ①, ① - ②

circle and a line → back to case b.

So again, coordinates of intersection points obtained from coefficients of equation by $+, -, \times, \div, \sqrt{\ }$

This proves Theorem: A length is constructible by ruler and compass $\Leftarrow\Rightarrow$ obtained from 1 by $+, -, \times, \div, \sqrt{\ }$. *

$\Leftarrow$ ✓ $\Rightarrow$ we just showed can only

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construct points (a,b) where (a,b) satisfy *

Distance $d = \sqrt{(a_1-a_2)^2 + (b_1-b_2)^2}$

also have * $\square$