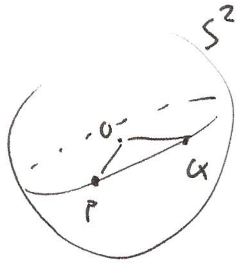
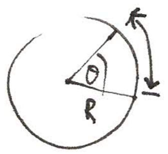
1. 

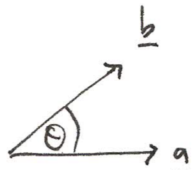
S^2 = sphere, center the origin & radius 1 in $\mathbb{R}^3$

$P, Q \in S^2$.

The shortest path on S^2 from P to Q is the shorter arc of the spherical line (or great circle) passing through P & Q, given by the intersection of the sphere S^2 with the plane Π passing through $O, P,$ & Q.

The length of the shortest path $d(P,Q) = \angle POQ$.

( $\text{length} = (2\pi R)\cdot \frac{\theta}{2\pi} = R\cdot\theta$; $R=1 \Rightarrow \text{length} = \theta$)

Dot product  $\underline{a}\cdot\underline{b} = \|\underline{a}\|\cdot\|\underline{b}\|\cos\theta$

So $d(P,Q) = \angle POQ = \cos^{-1}\left(\dfrac{\vec{OP}\cdot\vec{OQ}}{\|\vec{OP}\|\cdot\|\vec{OQ}\|}\right) = \cos^{-1}(\vec{OP}\cdot\vec{OQ})$

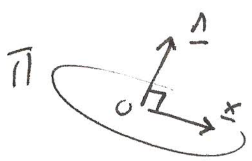
$\|\vec{OP}\| = \|\vec{OQ}\|$

$= \cos^{-1}\left(\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\frac{1}{9}\begin{pmatrix}4\\4\\7\end{pmatrix}\right) = \cos^{-1}\left(\frac{1}{27}(4+8+14)\right) = \cos^{-1}\left(\frac{26}{27}\right) = 0.273\ldots$

2. The spherical line through P & Q is $L = \Pi \cap S^2$

where Π is the plane through $O, P,$ & Q, with equation $ax+by+cz=0$ (or $\underline{x}\cdot\underline{n}=0$)

where $\underline{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is a normal vector for the plane Π.



So $L = \{(x,y,z)\in S^2 \mid ax+by+cz=0\}$.

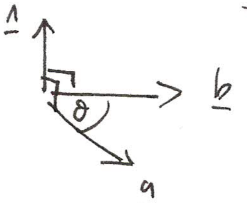
To find $\underline{n}$, we can take $\overrightarrow{OP} \times \overrightarrow{OQ}$ (cross product).

Recall
$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

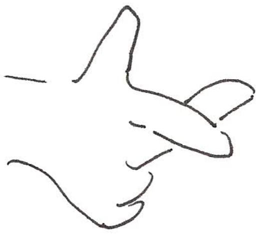
$$\left(= \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \right).$$

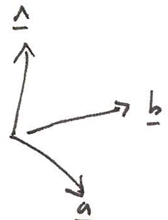
& $\underline{a} \times \underline{b} = \|\underline{a}\| \cdot \|\underline{b}\| \cdot \sin\theta \; \underline{n}$

where



θ is the angle between $\underline{a}$ & $\underline{b}$, & $\underline{n}$ is a unit vector perpendicular to $\underline{a}$ & $\underline{b}$ such that $\underline{a}, \underline{b}, \underline{n}$ is a right handed set of vectors





Our case
$$\overrightarrow{OP} \times \overrightarrow{OQ} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$
$$= \frac{1}{21}\begin{pmatrix} 2\cdot 6 - 1\cdot 3 \\ 1\cdot 2 - 2\cdot 6 \\ 2\cdot 3 - 2\cdot 2 \end{pmatrix} = \frac{1}{21}\begin{pmatrix} 9 \\ -10 \\ 2 \end{pmatrix}$$

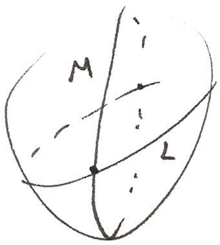
Take $\underline{n} = \begin{pmatrix} 9 \\ -10 \\ 2 \end{pmatrix}$ (the scalar $\frac{1}{21}$ can be omitted)

Equation of L : $9x - 10y + 2z = 0.$

3. a. $L \subset S^2$: $x+y+2z=0$.

$M \subset S^2$: $x+2y+3z=0$.

$L \cap M$



a). Two spherical lines intersect in two antipodal points, given by the intersection of the line $\ell = \Pi_L \cap \Pi_M$ with S^2, where Π_L & Π_M are the planes through the origin in $\mathbb{R}^3$ such that $L = \Pi_L \cap S^2$ & $M = \Pi_M \cap S^2$.

The line ℓ is spanned by the vector $\underline{v} = \underline{n}_L \times \underline{n}_M$, where $\underline{n}_L, \underline{n}_M$ are the normal vectors to Π_L & Π_M.

Our case : $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 - 2\cdot 2 \\ 2\cdot 1 - 1\cdot 3 \\ 1\cdot 2 - 1\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

(Alternatively, solve the equations for Π_L & Π_M using the row reduction algorithm to find $\underline{v}$)

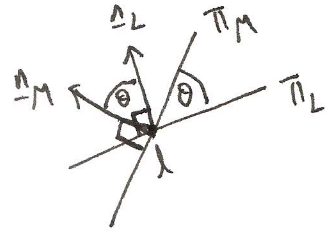
So $\ell = \left\{ c \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$

$$c \cdot \underline{v} \in S^2 \iff \|c\cdot \underline{v}\| = 1 \iff |c|\cdot\|\underline{v}\| = 1$$
$$\iff |c| = 1/\|\underline{v}\|$$
$$\iff c = \pm 1/\|\underline{v}\|.$$

So $\ell \cap S^2 = \frac{\pm 1}{\sqrt{3}} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

L & M intersect in the points $\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ & $\frac{-1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

b. The angle between the spherical lines L & M equals the dihedral angle between the planes Π_L & Π_M (the angle seen when looking along $\ell = \Pi_L \cap \Pi_M$)

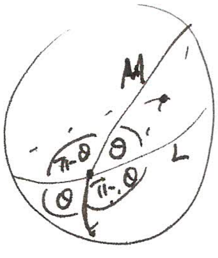


which equals the angle between the normal vectors $\underline{n}_L$ & $\underline{n}_M$.

$$\theta = \cos^{-1}\left(\frac{\underline{n}_L \cdot \underline{n}_M}{\|\underline{n}_L\| \cdot \|\underline{n}_M\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{6}\cdot\sqrt{14}}\right)$$

$$= \cos^{-1}\left(\frac{1+2+3}{\sqrt{84}}\right) = \cos^{-1}\left(\frac{6}{\sqrt{84}}\right) = 0.857\ldots \text{ radians}$$

c.

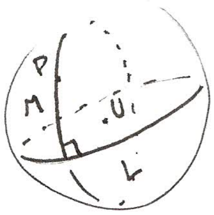


The spherical lines L & M divide S^2 into 4 lunes with angles $\theta, \pi-\theta, \theta, \pi-\theta$.

Area of a lune with angle α $= \underbrace{4\pi}_{\text{area of } S^2\ (4\pi \cdot R^2 = 4\pi \cdot 1 = 4\pi)} \cdot \dfrac{\alpha}{2\pi} = 2\alpha$

So, areas of regions are $2\theta, 2\pi-2\theta, 2\theta, 2\pi-2\theta$, where $\theta = \cos^{-1}\left(6/\sqrt{84}\right) = 0.857\ldots$

4.



$L \perp M \iff \underline{n}_M \perp \underline{n}_L$

$P \in M \iff \underline{n}_M \perp \overrightarrow{OP}$

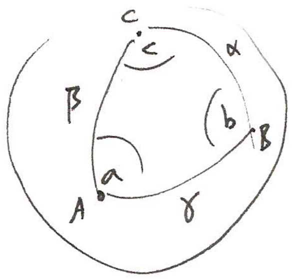
So, we can take $\underline{n}_M = \overrightarrow{OP} \times \underline{n}_L = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\3\end{pmatrix}$

$$= \frac{1}{3}\begin{pmatrix}1\cdot3 - 2\cdot2\\ 2\cdot1 - 2\cdot3\\ 2\cdot2 - 1\cdot1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1\\-4\\3\end{pmatrix}$$

or $\underline{n}_M = \begin{pmatrix}-1\\-4\\3\end{pmatrix}$

$M : \quad -x - 4y + 3z = 0.$

5. Recall the spherical cosine rule:



$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$$

Rearranging,

$$\cos a = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

Now suppose $\alpha = \beta$.

Then $$\cos b \underset{SCR}{=} \frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma} \underset{SCR}{=} \cos a$$

$0 \leq a, b \leq \pi \implies a = b.$ $\square$.

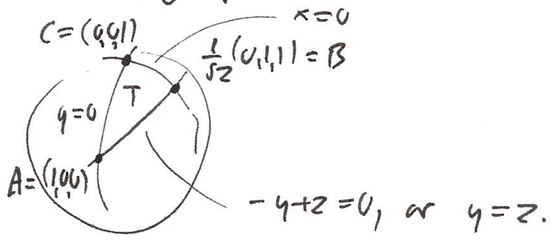
6. a)

$$\Delta_{AB} = \vec{OA} \times \vec{OB} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}, \text{ or } \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\Delta_{BC} = \vec{OB} \times \vec{OC} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\end{pmatrix}, \text{ or } \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Delta_{CA} = \vec{OC} \times \vec{OA} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

w/ equations
$$-y + z = 0$$
$$x = 0$$
$$y = 0.$$



b).

$$b = \cos^{-1}\left(\frac{\Delta_{AB}\cdot\Delta_{BC}}{\|\Delta_{AB}\|\cdot\|\Delta_{BC}\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}0\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\left\|\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}1\\0\\0\end{pmatrix}\right\|}\right) = \cos^{-1}(0) = \pi/2.$$

$$\cancel{b = \cos^{-1}\left(\frac{\Delta_{AB}\cdot\Delta_{BC}}{\|\Delta_{AB}\|\cdot\|\Delta_{BC}\|}\right) =}$$

$$c = \cos^{-1}\left(\frac{\Delta_{BC}\cdot\Delta_{CA}}{\|\Delta_{BC}\|\cdot\|\Delta_{CA}\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\left\|\begin{pmatrix}1\\0\\0\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\|}\right) = \cos^{-1}(0) = \pi/2.$$

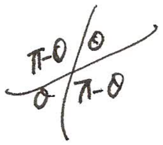
$$a = \cos^{-1}\left(\frac{n_{CA}\cdot n_{AB}}{\|n_{CA}\|\cdot\|n_{AB}\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}}{\left\|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$= 3\pi/4.$$

Actually, given that $a \leq \pi/2$ so $\pi - a = 3\pi/4$, $\boxed{a = \pi/4}$

(when we compute the angle between two spherical lines, always have ambiguity

θ vs. $\pi - \theta$:



.)

d) $\text{Area}(T) = a+b+c-\pi = \pi/2 + \pi/2 + \pi/4 - \pi = \pi/4.$

7. a)

$$\text{Area}(T) = \pi/3 + \pi/3 + 2\pi/5 - \pi = \pi\left(\tfrac{2}{3} + \tfrac{2}{5} - 1\right) = \pi\cdot\frac{1}{15}$$

$$\text{Area}(S^2) = 4\pi \quad \Rightarrow \quad \#\text{ triangles} = \frac{4\pi}{\pi\cdot 1/15} = 60.$$

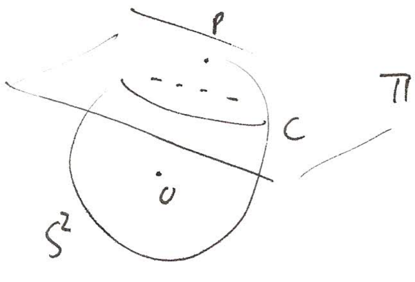
b).

$$\text{Area}(T) = \frac{\text{Area}(S^2)}{20} = \frac{4\pi}{20} = \frac{\pi}{5} \quad (*)$$

$$\text{Area}(T) = a+b+c-\pi = 3a - \pi \overset{(*)}{=} \pi/5$$

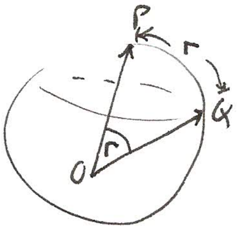
$$\Rightarrow \quad 3a = \pi + \pi/5 = 6\pi/5, \quad \boxed{a = 2\pi/5}.$$

8. $C = \Pi \cap S^2$, Π: $3x + 4y + 5z = 6.$



Spherical center of C, P, determined by

$$\vec{OP} = \frac{n}{\|n\|} = \frac{(3,4,5)}{\sqrt{3^2+4^2+5^2}} = \frac{1}{\sqrt{50}}(3,4,5).$$



Spherical radius $r = \cos^{-1}(\overrightarrow{OP} \cdot \overrightarrow{OQ})$

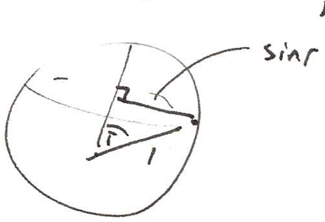
for $Q \in C$.
$\quad\;\, \|$
(x,y,z)

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = \frac{1}{\sqrt{50}}(3x+4y+5z) \overset{\text{eq. of } \Pi}{=} \frac{6}{\sqrt{50}}$$

$$r = \cos^{-1}\left(\frac{6}{\sqrt{50}}\right) = 0.558..$$

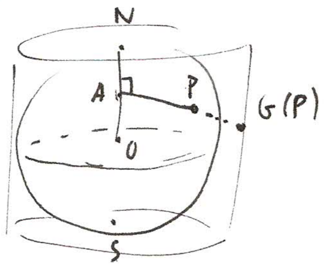
~~Sph~~ Circumference $= 2\pi \sin r = 2\pi \cdot \sqrt{(1-(\cos r)^2)}$

$$= 2\pi \cdot \sqrt{\left(1 - \frac{36}{50}\right)} = 2\pi\sqrt{\frac{14}{50}}.$$



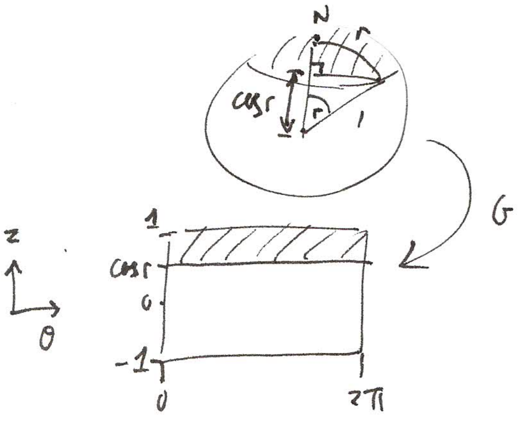
spherical circle of
radius r is
Euclidean circle of
radius $\sin r$ in plane Π.

9. Choose coordinates so that the center of the spherical disc is the north pole $N = (0,0,1)$



$G: S^2 \setminus \{N,S\} \longrightarrow [0,2\pi) \times (-1,1)$

Project radially outward from NS axis to the cylinder, then "roll out" the cylinder. (after cutting along line $x=1, y=0$.)



G preserves areas (HW8 Q7).

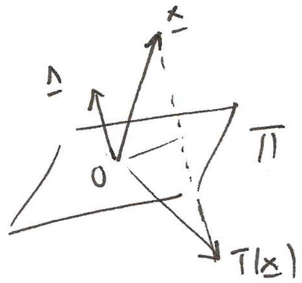
G(spherical disc, center N, spherical radius r)

$= $ rectangle $[0,2\pi) \times (\cos r, 1)$,

area $2\pi \cdot (1-\cos r)$.

So, spherical disc has area $2\pi \cdot (1-\cos r)$ $\square$.

10.



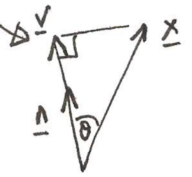
$\Pi \subset \mathbb{R}^3$ plane, $0 \in \Pi$.

$\underline{n}$ normal vector to Π.

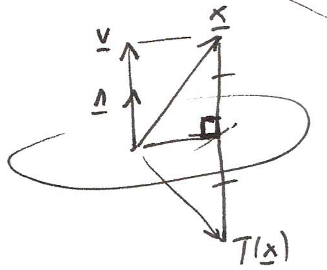
$T: \mathbb{R}^3 \to \mathbb{R}^3$ reflection in Π

then $T(\underline{x}) = \underline{x} - 2\left(\dfrac{\underline{x}\cdot\underline{n}}{\underline{n}\cdot\underline{n}}\right)\underline{n}$

$$\underline{v} = \frac{\|\underline{x}\|\cos\theta}{\|\underline{n}\|}\cdot\underline{n} = \frac{\underline{x}\cdot\underline{n}}{\|\underline{n}\|^2}\cdot\underline{n} = \frac{\underline{x}\cdot\underline{n}}{\underline{n}\cdot\underline{n}}\cdot\underline{n}$$



, now



$$T(\underline{x}) = \underline{x} - 2\underline{v} = \underline{x} - 2\left(\frac{\underline{x}\cdot\underline{n}}{\underline{n}\cdot\underline{n}}\right)\underline{n}$$

Our case. $\Pi: \quad x + 2y + 2z = 0.$

$$\underline{n} = (1,2,2)$$

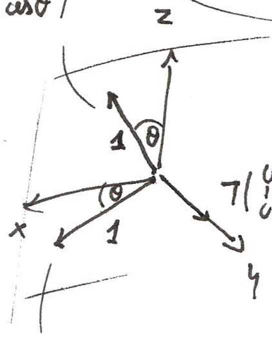
$$T(\underline{x}) = \underline{x} - 2\,\frac{(x,y,z)\cdot(1,2,2)}{1^2+2^2+2^2}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \begin{pmatrix}x\\y\\z\end{pmatrix} - \frac{2}{9}(x+2y+2z)\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}9x - (2x+4y+4z)\\ 9y - (4x+8y+8z)\\ 9z - (4x+8y+8z)\end{pmatrix} = \frac{1}{9}\begin{pmatrix}7x-4y-4z\\ -4x+y-8z\\ -4x-8y+z\end{pmatrix} = \frac{1}{9}\begin{pmatrix}7 & -4 & -4\\ -4 & 1 & -8\\ -4 & -8 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix}$$

11.



$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\0\\-\sin\theta\end{pmatrix}$$

$$T(\underline{x}) = A\cdot\underline{x} = \left(T\begin{pmatrix}1\\0\\0\end{pmatrix}\; T\begin{pmatrix}0\\1\\0\end{pmatrix}\; T\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\cdot\underline{x}$$

$$= \begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\cdot\underline{x}$$

12. a. $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$

Recall

T	$\text{Fix}(T)$
identity	$\mathbb{R}^3$
reflection	plane Π of reflection
rotation	axis l of rotation
rotary reflection	$\{\underline{0}\}$

Also, to find axis l of rotation for rotary reflection, solve $T(\underline{x}) = -\underline{x}$.
(then plane of reflection is the plane Π through $\underline{0}$ perpendicular to l).
Finally, to find angle θ of rotation for rotation $T(\underline{x}) = A\underline{x}$:

$$A \sim M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{trace } A = \text{trace } M = 2\cos\theta + 1$$

similar Matrices

For rotary reflection, $A \sim M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \text{trace } A = \text{trace } M = 2\cos\theta - 1.$

In our case $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $T(\underline{x}) = A\underline{x}$ $\quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -y \\ x \end{pmatrix}$

$\text{Fix}(T)$: $\begin{pmatrix} -z \\ -y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\begin{pmatrix} x+z \\ 2y \\ -x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

$\text{Fix}(T) = \{\underline{0}\}$, T is a <u>rotary reflection</u>.

Axis of rotation: Solve $T(\underline{x}) = -\underline{x}$ $\quad \begin{pmatrix} -z \\ -y \\ x \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$, $\begin{pmatrix} -x+z \\ 0 \\ -x-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow$ $x = z = 0$. $\quad$ Axis l = y-axis.

Plane of reflection Π = xz plane

Angle θ of rotation: $\quad 2\cos\theta - 1 = \text{trace } A = -1$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2. \quad \square.$$

b. $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$

Fix T. $\quad \frac{1}{3}\begin{pmatrix} x+2y-2z \\ 2x+y+2z \\ 2x-2y-z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} -2x+2y-2z \\ 2x-2y+2z \\ 2x-2y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\div 2$

augmented matrix

$$\begin{array}{c} \\ +R1 \\ +R1 \end{array}\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x = y$, $z = 0$. line l in direction $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

So T is a rotation, axis l.

Angle θ of rotation $\quad 2\cos\theta + 1 = \text{trace } A = \frac{1}{3}(1+1+(-1)) = 1/3$

$$\cos\theta = -1/3, \quad \theta = \cos^{-1}(-1/3) = 1.91.. \text{ radians.}$$

c. $A = \frac{1}{11}\begin{pmatrix} 9 & -6 & -2 \\ -6 & -7 & -6 \\ -2 & -6 & 9 \end{pmatrix}$

Fix T: $\quad \frac{1}{11}\begin{pmatrix} 9x-6y-2z \\ -6x-7y-6z \\ -2x-6y+9z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} -2x-6y-2z \\ -6x-18y-6z \\ -2x-6y-2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \sim \quad x+3y+z=0.$$

(all eqs are multiples of this eq.)

So T is a reflection in plane $\Pi: x+3y+z=0$. $\square$.

13.

$T = T_2 \circ T_1$

Matrices $A = A_2 \cdot A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

The composite of two rotations is another rotation (or the identity).

Axis: Solve $T(\underline{x}) = \underline{x}$ $\quad \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x=y=z,$
line in direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Angle: Solve $2\cos\theta + 1 = \text{trace } A = 0$

$\theta = \cos(-1/2) = 2\pi/3$.

14.

The composite of two reflections in planes Π_1 & Π_2 is a rotation about the axis $l_? = \Pi_1 \cap \Pi_2$ thru angle 2θ where θ is the dihedral angle between the planes.

$l = \text{Span}(\underline{n}_1 \times \underline{n}_2) = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \text{Span}\left(\begin{pmatrix} 1.3-1.2 \\ 1.1-1.3 \\ 1.2-1.1 \end{pmatrix}\right)$

$= \text{Span}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, i.e. line thru 0 in direction $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\theta = \cos^{-1}\left(\dfrac{\underline{n}_1 \cdot \underline{n}_2}{\|\underline{n}_1\| \cdot \|\underline{n}_2\|}\right) = \cos^{-1}\left(\dfrac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{3} \cdot \sqrt{14}}\right) = \cos^{-1}\left(\dfrac{6}{\sqrt{42}}\right) = 0.388..$ radians.

So, $T_2 \circ T_1$ is a rotation with axis l the line thru 0 in direction $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ thru angle $2\theta = 2\cos^{-1}\left(\frac{6}{\sqrt{42}}\right) = 0.775..$ radians.

15. a. Notice that $\underset{(0,0,1)}{\underset{\parallel}{N}} \in \Pi$: $3\cdot 0 + 5\cdot 0 + 7\cdot 1 = 7$ ✓

So $F(C \setminus \{N\text{'s}\}) = \Pi \cap (z=0) = (3x+5y=7) \subset \mathbb{R}^2$

or $y = -\frac{3}{5}x + \frac{7}{5}$

$\boxed{v = -\frac{3}{5}u + \frac{7}{5}}$ (writing $u=x$, $v=y$).

b.

C: $x + 3y + z = 2$. Substitute $(x,y,z) = \dfrac{(2u,\ 2v,\ u^2+v^2-1)}{u^2+v^2+1}$ to find equation of $F(C)$. :—

$$\frac{1}{u^2+v^2+1}(2u + 6v + u^2+v^2-1) = 2$$

$$2u + 6v + u^2+v^2-1 = 2u^2+2v^2+2$$

$$0 = u^2+v^2-2u-6v+3$$

$$0 = (u-1)^2 + (v-3)^2 + 3 - 1^2 - 3^2$$ "complete the square".

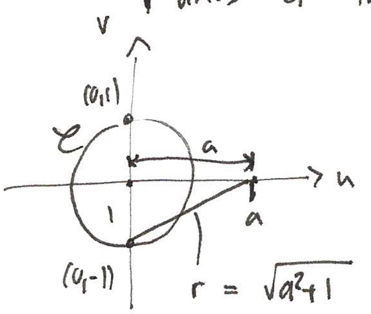
$$(u-1)^2 + (v-3)^2 = 7$$

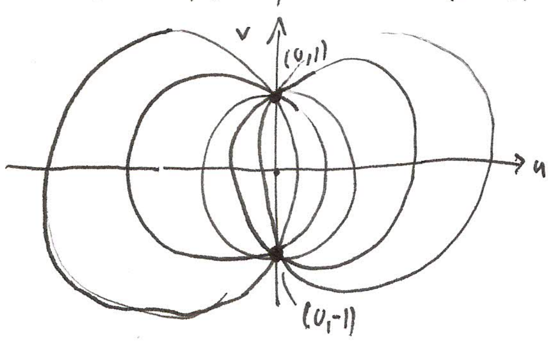
circle, center $(1,3)$, radius $\sqrt{7}$.

16. a. A spherical circle maps to a circle or a line in $\mathbb{R}^2$ under stereographic projection. A spherical line L (or great circle) maps to a circle or line in $\mathbb{R}^2$ meeting the unit circle $\mathcal{C}: u^2+v^2=1$ in two antipodal points given by $L \cap \mathcal{C}$.

So, the spherical lines passing through $(0,1,0)$ & $(0,-1,0)$ map to the circles & lines in $\mathbb{R}^2$ passing through $(0,1)$ & $(0,-1)$.

i.e the v-axis & the circles $(u-a)^2 + v^2 = (a^2+1)$ with center $(a,0)$ & radius $\sqrt{a^2+1}$, for $a \in \mathbb{R}$.





b. The spherical circles (with center $(0,\pm 1,0)$ are given by the planes $\Pi : y = c$, some $c \in \mathbb{R}$, $-1 < c < 1$.

$\rightsquigarrow$ $F(C)$ has equation $\dfrac{2v}{u^2+v^2+1} = c$

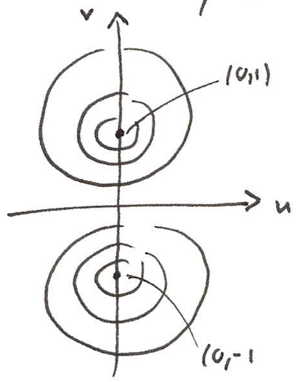
$$cu^2 + cv^2 - 2v + c = 0.$$

If $c=0$: $v = 0$, u-axis.

Otherwise: $u^2 + v^2 - 2/_c\, v + 1 = 0$

$$u^2 + (v - 1/_c)^2 = 1/_{c^2} - 1$$

circle, center $(0, 1/c)$, radius $\sqrt{1/_{c^2} - 1}$.



c. Note the spherical circles from parts a & b are pairwise orthogonal (like lines of longitude & latitude) so their images in $\mathbb{R}^2$ are also pairwise orthogonal since F preserves angles.

To check this:

(a) : $(u-a)^2 + v^2 = a^2 + 1.$

(b) : $u^2 + (v-b)^2 = b^2 - 1$ (writing $b = \frac{1}{c}$)

Slope of tangent line at a point: use implicit differentiation

(a) $2(u-a) + 2v \cdot \dfrac{dv}{du} = 0$, $\dfrac{dv}{du} = \dfrac{-(u-a)}{v}$

(b) $2u + 2(v-b) \cdot \dfrac{dv}{du} = 0$, $\dfrac{dv}{du} = \dfrac{-u}{v-b}.$

Required to prove at an intersection point (u,v), $m_1 \cdot m_2 \overset{?}{=} -1$ (product of slopes $= -1 \Rightarrow$ orthogonal)

i.e. $\dfrac{-(u-a)}{v} \cdot \dfrac{-u}{(v-b)} \overset{?}{=} -1$, or $u^2 - au \overset{?}{=} -(v^2 - bv)$

$$u^2 + v^2 - au - bv \overset{?}{=} 0.$$

we have the equations of the two circles:

(a) $$u^2 - 2au + a^2 + v^2 = a^2 + 1$$

(b) $$u^2 + v^2 - 2bv + b^2 = b^2 - 1$$

(a)+(b) $$2(u^2+v^2 - au - bv) + a^2 + b^2 = a^2+b^2$$

$$\Rightarrow \quad u^2+v^2 - au - bv = 0 \,. \checkmark \; \square.$$