Math 132.5. Power series (11.8)

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1 Section 11.8

1.1 Definition of power series

A *power series in x* is a series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

depending on a variable x. The c_i are real numbers called the *coefficients* of the series. A power series defines a function $f(x)$ with domain the set of real numbers x for which the series converges. Roughly speaking, a power series is like a polynomial but with infinitely many terms in the sum.

Example 1.1.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

is a geometric series with common ratio x, so is convergent for $|x| < 1$ and divergent for $|x| \geq 1$. The formula for the sum of a geometric series gives

$$\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)} \qquad \text{for } |x| < 1.$$

More generally we consider *power series in* $(x-a)$ (also called power series centered at $x = a$ or power series about $x = a$)

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a is a real number. (An expression for a function $f(x)$ as a power series in $(x-a)$ is often used to compute values of the function for x close to a.)

1.2 Radius of convergence

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series in $(x-a)$. Then one of the following is true:

(1) There is a positive real number R such that the series is absolutely convergent for $|x-a|<R$ and the series is divergent for $|x-a|>R$.

(2) The series is absolutely convergent for all real numbers x. ("$R=\infty$")

(3) The series is only convergent for $x=a$.("$R=0$")

The number R is called the *radius of convergence* of the power series. It can be found by applying the ratio test to the power series.

Example 1.2. For the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} a_n,$$

we compute

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\cdot\sqrt{\frac{n}{n+1}}\cdot|x-3| = \frac{1}{2}|x-3|.$$

So the series is absolutely convergent for $\frac{1}{2}|x-3|<1$ and divergent for $\frac{1}{2}|x-3|>1$ by the ratio test. Equivalently, the series is absolutely convergent for $|x-3|<2$ and divergent for $|x-3|>2$, so the radius of convergence $R=2$.

1.3 Interval of convergence

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series in $(x-a)$ with radius of convergence R (with $R\neq 0$ and $R\neq\infty$), the series may be either convergent or divergent when $|x-a|=R$. The two cases $x=a-R$ and $x=a+R$ must be checked separately using one of the tests for convergence (the ratio and root tests will *not* work here). The *interval of convergence* is the set of all values of x for which the series converges; it is equal to one of $(a-R,a+R)$, $[a-R,a+R)$, $(a-R,a+R]$, or $[a-R,a+R]$.

Example 1.3. The series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n \sqrt{n}}$$

from Example 1.2 with radius of convergence $R = 2$ is convergent for $x - 3 = -2$ by the alternating series test and divergent for $x - 3 = 2$ by the p-series convergence criterion ($p = 1/2 \leq 1$). So the interval of convergence of the power series is $[-2+3, 2+3) = [1, 5)$.

If $R = \infty$ the interval of convergence is $(-\infty, \infty) = \mathbb{R}$ and if $R = 0$ the interval of convergence is $\{a\}$.