10/16/19

Midterm today 7-9PM LGRC A301

HW 4 returned

Worksheet returned (not graded)

Last Time • Isometries of $\mathbb{R}^2$, $T:\mathbb{R}^2 \to \mathbb{R}^2$, $|T(P_1)T(P_2)| = |P_1P_2|$ for all $P_1, P_2 \in \mathbb{R}^2$

Translation, Reflection, Rotation, Glide reflection

• Theorem - every isometry is one of these 4 types

Today • More on isometries

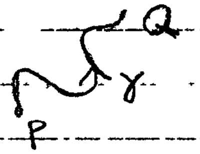
Isometries preserve lines:

If $L \subset \mathbb{R}^2$ is a line, the $T(L) \subset \mathbb{R}^2$ is also a line.

$\{T(P) \mid P \in L\}$

Recall (common sense notion)

(*) The straight line from P to Q is the shortest path from P to Q.

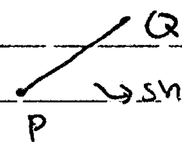


$\text{length}(\gamma) = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt$ (MATH 233)

$\gamma : [0,1] \to \mathbb{R}^2$ parametrization of curve, $\gamma(0) = P$

$\gamma(t) = (x(t), y(t))$ $\gamma(1) = Q$

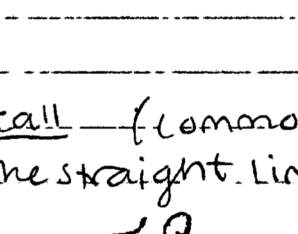
$x(t), y(t)$ differentiable



shortest path

This suggests that isometries should preserve straight lines.

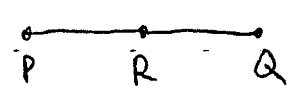
Instead we will use the triangle inequality. (HW4 Q6, relied on cosine rule)



$|PR| + |RQ| > |PQ|$

(special case of (*))

Restate triangle inequality: P, R, Q lie on a line in that order $\Leftrightarrow$ $|PR| + |RQ| = |PQ|$.



Proof that isometries preserve lines

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry & $L \subset \mathbb{R}^2$ is a line.

Pick two points $P, Q \in L$.

Then for $R \in L$ have $|PR| + |RQ| = |PQ|$ if R lies between P & Q.

$\Rightarrow |T(P)T(R)| + |T(R)T(Q)| = |T(P)T(Q)|$

isometry preserves distances

$\Rightarrow T(R)$ lies between $T(P)$ & $T(Q)$ on the line joining them.

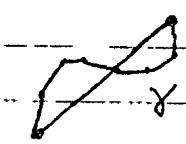
restated triangle inequality

Shown: line segment PQ maps to line segment $T(P)T(Q)$

$\rightsquigarrow$ line PQ maps to line $T(P)T(Q)$. ∎

Alternative proof: Use T preserves angles. ∎

Remark: Triangle inequality is key ingrediant in proof of "straight line is shortest path"



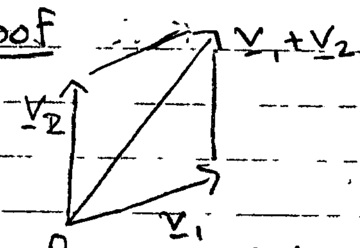
(approximate γ by union of line segments)

Lemma: T isometry & $T(0,0) = (0,0)$ then T is a linear transformation

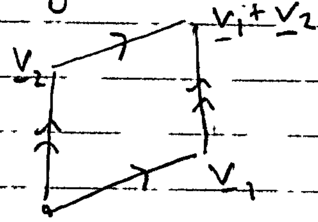
i.e. 1. $T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$ for all $\underline{v}_1, \underline{v}_2 \in \mathbb{R}^2$

2. $T(c\underline{v}) = c\,T(\underline{v})$ for all $\underline{v} \in \mathbb{R}^2, c \in \mathbb{R}$

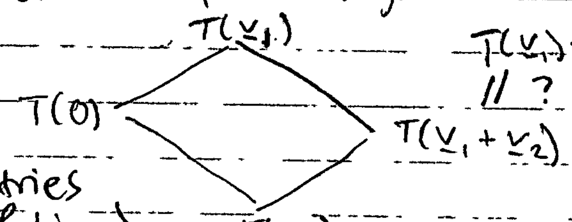
Proof



"parallelogram law" for addition of vectors



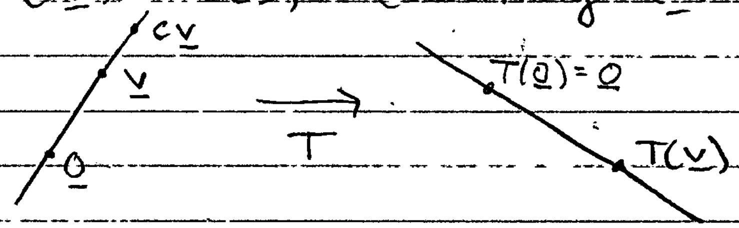
1. amounts to: T sends parallelogram to another parallelogram



$T(\underline{v}_1) + T(\underline{v}_2)$
$/\!/\,?$

Ok because isometries preserve angles (& lines).

2. $T(c\underline{v}) = cT(\underline{v})$ (assuming $T(\underline{0}) = \underline{0}$)



T sends line through $\underline{0}$ & $\underline{v}$ to line through $T(\underline{0})$ & $T(\underline{v})$

$\Rightarrow T(c\underline{v}) = \lambda T(\underline{v})$ some $\lambda \in \mathbb{R}$,

T preserves distances $\Rightarrow \lambda = c$. ∎

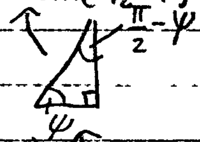
Ex Rotation about $(0,0)$ $\boxed{T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}}$

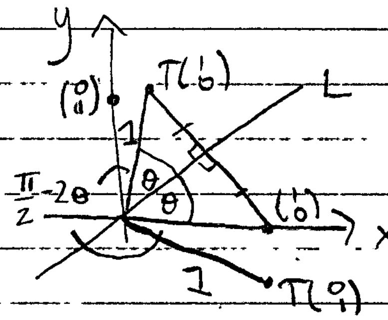
Reflection in a line L passing through the origin

$(\Rightarrow T(0,0) = (0,0))$

$\cos(\pi/2 - \psi) = \sin\psi$

$\sin(\pi/2 - \psi) = \cos\psi$





$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}? & ?\\ ? & ?\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}\cos 2\theta\\ \sin 2\theta\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix}$ $\quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\cos(\pi/2 - 2\theta)\\ -\sin(\pi/2 - 2\theta)\end{pmatrix} \overset{\uparrow}{=} \begin{pmatrix}\sin 2\theta\\ -\cos 2\theta\end{pmatrix}$

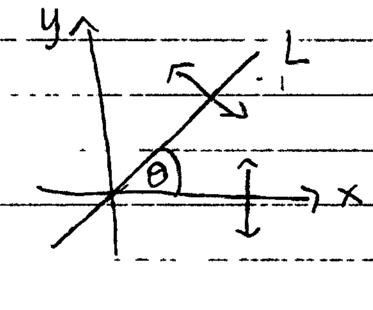
$\boxed{T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}}$

↳ reflection in line L through origin, making angle θ with x-axis.

Alternative proof

Formula for reflection in x-axis is easy: $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Q: How is reflection in L (making angle θ with x-axis ↳ & passing through 0) related to reflection in x-axis?



Trick

$\left.\begin{array}{l}\text{Rotate L to x-axis}\\ \text{Reflect in x-axis}\\ \text{Rotate x-axis back to L}\end{array}\right\} = \text{Reflection in L}$