Math 461 Homework 7

Paul Hacking

November 7, 2018

(1)(a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane $\Pi \subset \mathbb{R}^3$ with equation $x+2y+z=0$. Find an algebraic formula for T. Express your formula in the form $T(\mathbf{x}) = A\mathbf{x}$ where A is a 3×3 matrix.

(b) Check your answer to part (a) by verifying that (i) $A\mathbf{n} = -\mathbf{n}$ where $\mathbf{n}$ is a normal vector to Π and (ii) the matrix A is orthogonal, that is, $A^T A = I$ where the superscript T is used to denote the transpose of the matrix (interchange rows and columns) and I is the identity matrix. (Property (ii) holds iff the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an isometry.)

(2)(a) Determine the matrix A_θ of a rotation about the z-axis through angle θ counterclockwise as viewed from $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (looking towards the origin).

(b) Determine the matrix B_φ of a rotation about the x-axis through angle φ counterclockwise as viewed from $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

[Hint: If $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation then the matrix A of T has columns the vectors $T\begin{pmatrix}1\\0\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix}$. Use this fact together with geometric reasoning to determine the matrices A_θ and B_φ.]

(3) Let $L \subset \mathbb{R}^3$ be the line in the direction $\begin{pmatrix}1\\1\\0\end{pmatrix}$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the line L through angle $\pi/2$ counterclockwise as viewed

from $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(a) Explain why T can be expressed as the composition $U \circ V \circ U^{-1}$ where U is the rotation about the z-axis through angle $\pi/4$ and V is the rotation about the x-axis through angle $\pi/2$.

(b) Now use the result of Q2 to compute the matrix A of T.

[Remark: One could also use the general method described in class on Wednesday 11/7 to compute A, but in this case the method described above is easier.]

(4) Let $\tilde{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry of

$\mathbb{R}^3$ fixing the origin and $T\colon S^2 \to S^2$ the induced isometry of S^2 (given by $T(P) = \tilde{T}(P)$ for $P \in S^2$). For each type of $\tilde{T}$ (the identity, reflection, rotation, or rotary reflection) describe the fixed locus of $\tilde{T}$ and the fixed locus of T.

(5) For each of the following isometries $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, classify the isometry as a reflection, rotation, or rotary reflection, and give a precise geometric description as follows: for a reflection give the plane of reflection, for a rotation give the axis and angle of rotation, for a rotary reflection give the plane of reflection and axis and angle or rotation. (The sense of rotation (counterclockwise or clockwise)

may be omitted.)

(a) $T(\mathbf{x}) = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}\mathbf{x}$.

(b) $T(\mathbf{x}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\mathbf{x}$.

(c) $T(\mathbf{x}) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}\mathbf{x}$.

(6) What type of isometry is the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbf{x}) = -\mathbf{x}$? Give a precise geometric description as in Q4.

[Hint: In this case the description of the isometry as in Q4 is not uniquely determined — you will need to make a choice.]

(7) A tetrahedron is a regular polytope with 4 faces. For example, there is a tetrahedron $T \subset \mathbb{R}^3$ with vertices the points

$$(1,1,1),(-1,-1,1),(1,-1,-1),(-1,1,-1).$$

It may help to visualize the tetrahedron by first drawing the cube with 8 vertices $(\pm 1, \pm 1, \pm 1)$. There are 24 symmetries of the tetrahedron (including the identity transformation) — because there is exactly one symmetry realizing each permutation of the 4 vertices, and $4! = 24$. Give a geometric description of each of the symmetries as a rotation, reflection, or rotary reflection.