# Math 462: Homework 7 solutions

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**1**. Consider the motion $T$ of $\mathbb{R}^3$ defined by $T(\mathbf{x}) = -\mathbf{x}$.

(a) Is $T$ a direct or opposite motion?

(b) We classified motions of $\mathbb{R}^3$ into 6 types: rotations, reflections, rotary reflections, translations, glides, and twists. What type of motion is $T$? Give a precise geometric description in these terms (include angle of rotation and/or plane of reflection etc.).

(c) Now let $P$ be a regular polyhedron in $\mathbb{R}^3$ with center at the origin (so $P$ is either a tetrahedron, cube, octahedron, dodecahedron, or icosahedron). In which cases is $T$ a symmetry of $P$?

(a) $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have $\det A = (-1)^3 = -1$, so $T$ is an opposite motion.

(b) $T$ is a rotary reflection. More precisely, let $L$ be any line through the origin, and $\Pi$ the plane through the origin which is normal to $L$. Then $T$ is given by rotation about the axis $L$ through an angle $\pi$ followed by reflection in the plane $\Pi$. (Note: in this case, the choice of the line $L$ is irrelevant. This is not true in general for rotary reflections, it only happens when the angle of rotation equals $\pi$.)

(c) $T$ is a symmetry for all cases except the tetrahedron.

**2**. Recall that the *dihedral group* $D_n$ is the group of symmetries of a regular $n$-sided polygon $P$. We showed in class that $D_n$ consists of $n$ rotations (including the identity transformation) and $n$ reflections. Moreover, if $a$ denotes a rotation about the center of $P$ through angle $2\pi/n$ anticlockwise and $b$ denotes reflection in a line of symmetry of $P$, then $D_n$ is generated by $a$ and $b$ (more precisely, the rotations are $1, a, \ldots, a^{n-1}$ and the reflections are $b, ab, \ldots, a^{n-1}b$). The symmetries $a$ and $b$ satisfy the relations $a^n = b^2 = 1$ and $ba = a^{-1}b$ (and all other relations can be derived from these). Express the following products in $D_5$ in the form $a^i b^j$ where $0 \leq i \leq 5$ and $0 \leq j \leq 1$ and describe them geometrically.

(a) $a^3bab$.

(b) $a^2ba^3b^2a$

(c) $a^2ba^{-1}b^{-1}a^3b^3$.

First observe that the relation $ba = a^{-1}b$ implies that $a^kb = a^{-k}b$ for any integer $k$.

(a) $a^3bab = a^3(ba)b = a^3(a^{-1}b)b = a^2b^2 = a^2$. Rotation about the origin through angle $4\pi/5$ anticlockwise.

(b) $a^2ba^3b^2a = a^2ba^3a = a^2(ba^4) = a^2a^{-4}b = a^{-2}b = a^3b$. Reflection in line through the origin making an angle of $3\pi/5$ with the line of reflection for $b$.

(c) $a^2ba^{-1}b^{-1}a^3b^3 = a^2(ba^4)ba^3b = a^2(a^{-4}b)ba^3b = a^{-2}b^2a^3b = a^{-2}a^3b = ab$. Reflection in line through the origin making an angle of $\pi/5$ with the line of reflection for $b$.

**3**. In this problem we see that the dihedral group $D_n$ can be realized as a subgroup of the group of *rotations* of $\mathbb{R}^3$. Let $P$ be a regular $n$-sided polygon. Position $P$ in the plane $(z = 0) \subset \mathbb{R}^3$ with center $O$ at the origin and one vertex $Q$ on the $x$-axis.

(a) What is the matrix $A$ of the rotation $S$ of $\mathbb{R}^3$ about the $z$-axis through angle $2\pi/n$ anticlockwise? What is the symmetry of $P$ induced by $S$?

(b) Describe a rotation $T$ of $\mathbb{R}^3$ which maps $P$ to itself and when regarded as a symmetry of $P$ is given by reflection in the line $OQ$. What is the matrix $B$ of $T$?

(c) Use parts (a) and (b) to describe the dihedral group $D_n$ (the group of symmetries of $P$) in terms of rotations of $\mathbb{R}^3$. What are the matrices of these rotations?

(a)

$$A = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation about the center of $P$ through angle $2\pi/n$ anticlockwise.

(b) Rotation about the line $OQ$ (the $x$-axis) through an angle of $\pi$.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(c) The dihedral group can be realized as the following rotations of $\mathbb{R}^3$: the rotations about the $z$-axis through angles $\pi k/n$ anticlockwise, for $k = 0, 1, \ldots, n$, and the rotations through angle $\pi$ about the lines in the $xy$-plane making angle $\pi k/n$ with the $x$-axis, for $k = 0, 1 \ldots, n-1$. The matrices are

$$\begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) & 0 \\ \sin(2\pi k/n) & \cos(2\pi k/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) & 0 \\ \sin(2\pi k/n) & -\cos(2\pi k/n) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

**4**. Recall that a *permutation* of $1, 2, \ldots, n$ is a function $f \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $f(i) \neq f(j)$ for $i \neq j$. The *symmetric group* $S_n$ is the group of all permutations with the group law being composition of functions. A *transposition* is a permutation $f$ that switches two elements $i$ and $j$ and leaves the remaining elements fixed, that is, $f(i) = j$, $f(j) = i$, and $f(k) = k$ for $k \neq i, j$. A permutation $f$ is a *cycle of length* $l$ if there are distinct numbers $i_1, \ldots, i_l$ such that $f(i_1) = i_2$, $f(i_2) = i_3$, ..., $f(i_{l-1}) = i_l$, $f(i_l) = i_1$, and $f(k) = k$ for $k \neq i_1, \ldots, i_l$ (for example, a transposition is a cycle of length 2). We say two cycles $f, g$ are *disjoint* if for each $k$ either $f(k) = k$ or $g(k) = k$. In class we explained that every permutation can be written as a product of transpositions. We also explained how to write a permutation as a composition of disjoint cycles.

(a) Show that order of $S_n$ (the number of permutations of $1, 2, \ldots, n$) equals $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

(b) Write each of the following permutations as a product of disjoint cycles.

  (i) $f\colon \{1, \ldots, 5\} \to \{1, \ldots, 5\}$, $f(1) = 3$, $f(2) = 4$, $f(3) = 5$, $f(4) = 1$, $f(5) = 2$.

  (ii) $g\colon \{1, \ldots, 7\} \to \{1, \ldots, 7\}$, $g(1) = 7$, $g(2) = 6$, $g(3) = 1$, $g(4) = 2$, $g(5) = 3$, $g(6) = 4$, $g(7) = 5$.

  (iii) $h\colon \{1, \ldots, 8\} \to \{1, \ldots, 8\}$, $h(1) = 6$, $h(2) = 2$, $h(3) = 5$, $h(4) = 7$, $h(5) = 8$, $h(6) = 1$, $h(7) = 3$, $h(8) = 4$.

(c) Using part (b) or otherwise, write each of the permutations $f, g, h$ as a composition of transpositions.

(d) The cycle type of a permutation $f$ is the (unordered) list of the lengths of the cycles in the description of $f$ as a composition of disjoint cycles. For example $f = (123)(45)(67)$ has cycle type $2, 2, 3$. List the possible cycle types for $S_4$ and $S_5$. How can we determine the order of a permutation from its cycle type? (The *order* of a permutation is the least number $r \geq 1$ such that applying the permutation $r$ times gives the identity permutation.)

(e) We say that a permutation is *even* if it can be written as a product of an even number of transpositions. The set of even permutations is a subgroup of $S_n$ called the *alternating group* $A_n$. What are the possible cycle types for elements of $A_4$ and $A_5$? What are their orders?

(a) Consider choosing the values $f(1), f(2), \ldots$ of a permutation $f$ in turn. There are $n$ choices for $f(1)$, then $n-1$ choices for $f(2)$ (remember that we require that $f(2) \neq f(1)$), then $n-2$ choices for $f(3)$, and so on. This shows that the number of permutations of $\{1, 2, \ldots, n\}$ equals $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$

(b) (i) (13524)

  (ii) (1753)(264)

  (iii) (16)(35847)

(c) (i) (13524) = (13)(35)(52)(24).

  (ii) (1753)(264)=(17)(75)(53)(26)(64).

(iii) (16)(35847)=(16)(35)(58)(84)(47).

(d) The cycle types for $S_4$ are the identity, (2), (3), (4) and $(2,2)$. For $S_5$ they are the identity, (2), (3), (4), (5), $(2,2)$, and $(2,3)$. (Note: we do not list the cycles of length 1 to make the notation more concise.) The order of a permutation with cycle type $(l_1,\ldots,l_r)$ equals $\operatorname{lcm}(l_1,\ldots,l_r)$, the least common multiple of the lengths $l_1,\ldots,l_r$ of the cycles (that is, the smallest positive integer $m$ such that each of $l_1,\ldots,l_r$ divides $m$).

(e) Recall that a cycle of length $l$ is a composition of $l$ transpositions: $(123\cdots l) = (12)(23)\cdots(l-1,l)$ where the composition is read from right to left (this is the usual notation for composition of functions, but beware that some texts use the opposite convention for permutations). So a permutation of cycle type $(l_1,\ldots,l_r)$ is a composition of $\sum_{i=1}^{r}(l_i-1)$ transpositions. Using this and part (d) we see that the cycle types for $A_4$ are the identity, (3), and $(2,2)$ (of orders 1, 3, and 2), and the cycle types for $A_5$ are the identity, (3), (5), and $(2,2)$ (of orders 1, 3, 5 and 2).

**5**. In HW6 we showed that the group of symmetries of the tetrahedron can be identified with the symmetric group $S_4$ of permutations of 4 objects by considering the permutations of the vertices of the tetrahedron induced by its symmetries. For each cycle type of $S_4$ describe the corresponding symmetries geometrically. What is the geometric significance of the subgroup $A_4 \subset S_4$?

Cycle type (2) corresponds to reflection in a plane. Cycle type (3) corresponds to a rotation about an axis joining a vertex to the center of the opposite face through an angle of $\pm 2\pi/3$. Cycle type (2,2) corresponds to a rotation about an axis joining the midpoints of two opposite sides through angle $\pi$. Cycle type (4) corresponds to a rotary reflection with axis of rotation the line joining the midpoints of two opposite sides and rotation angle $\pm\pi/2$. The subgroup $A_4 \subset S_4$ corresponds to the subgroup of rotational symmetries.