Math 461 Lecture 37 12/7

Final review questions available later today

Last time:

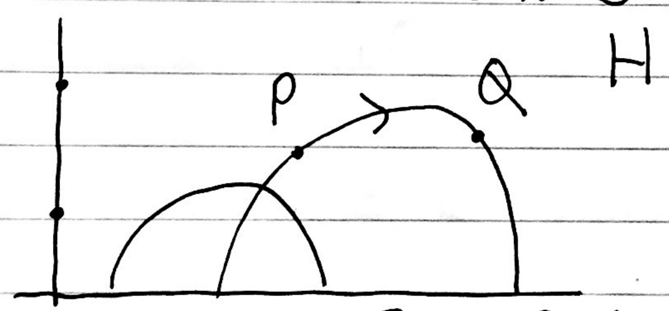
$H = \{(x,y) \mid y>0\} \subset \mathbb{R}^2$

upper half plane model of hyperbolic plane

(hyperbolic) length$(\gamma) = \int_a^b \frac{\sqrt{x'^2+y'^2}}{y}\,dt$

$(x,y) : [a,b] \to H$ parametrization of γ

A hyperbolic line $L \subset H$ is a vertical half line or semicircle with center on x-axis



Theorem: For $P, Q \in H$ there's a unique h-line through P and Q and this gives the shortest path from P to Q (for the hyperbolic length)

Hyperbolic isometries:

$T: H \to H$ preserving hyperbolic distances

Ex. horizontal translation

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$(x,y) \longmapsto (x+a, y)$

Scaling $(x,y) \longmapsto (cx, cy)$, $c>0$

reflection in a vertical line

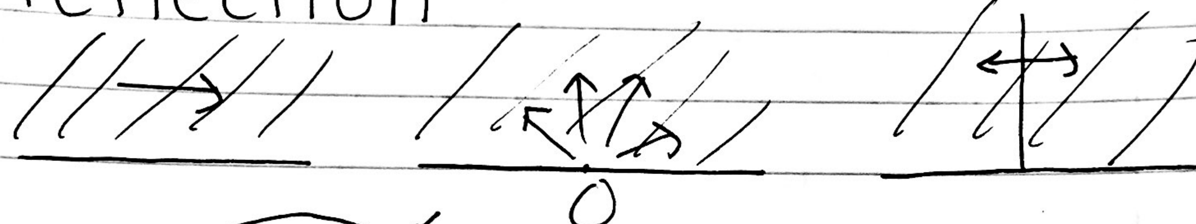
inversion in unit circle

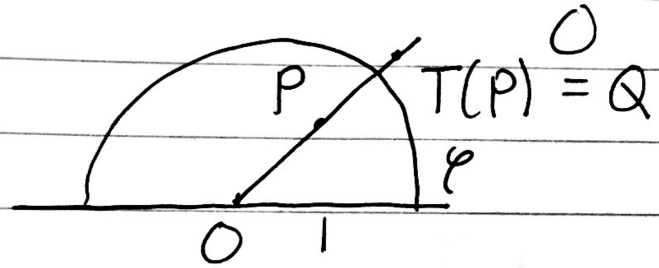
$\mathcal{C}: x^2+y^2=1$ $(x,y) \to \dfrac{(x,y)}{x^2+y^2}$

Today:

proof of theorem

inversion is a hyperbolic reflection





$|OP| \cdot |OQ| = 1$

Homework 8 Question 4:

$T(x,y) = \dfrac{(x,y)}{x^2+y^2}$ inversion in the unit circle

$T: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$

$\cup H \to \cup H$

$T = F \circ R \circ F^{-1}$

where $F: S^2 \setminus \{N\} \to \mathbb{R}^2$ is a stereographic projection and

$R: S^2 \to S^2$ is a reflection in the xy-plane

$R(x,y,z) = (x,y,-z)$

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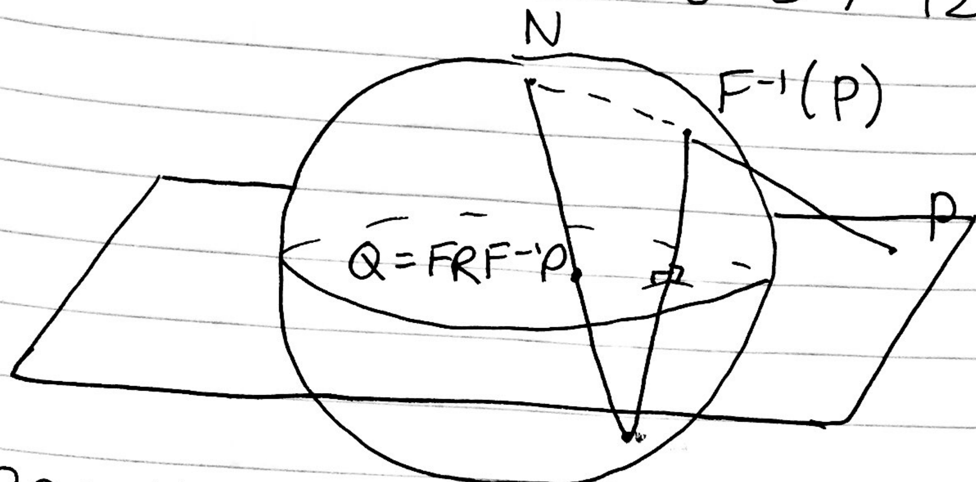

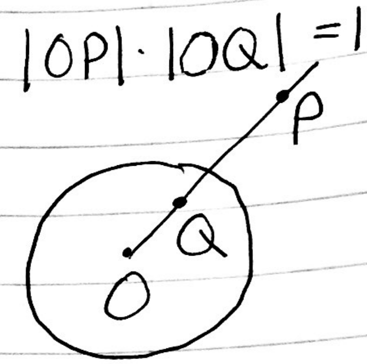

Recall: stereographic projection gives a bijection

$$\begin{tikzcd} \{\text{spherical circles}\} \arrow[r, shift left, "F"] \arrow[loop below, "R"] & \{\text{circles and lines in } \mathbb{R}^2\} \arrow[l, shift left, "F^{-1}"] \arrow[loop below, "T"] \end{tikzcd}$$

similarly, T preserves angles because F and R do

also under F $C\subset S^2$

C maps to a line $\iff N\in C$

$\Rightarrow$ for $A\subset\mathbb{R}^2$ circle or line,

$A \xrightarrow{T}$ line $\iff O\in A$

$$\begin{tikzcd} O \arrow[r, mapsto, "F^{-1}"'] \arrow[rrr, bend left, "T"] & S \arrow[r, mapsto, "R"'] & N \arrow[r, mapsto, "F"'] & \text{"}\infty\text{"} \end{tikzcd}$$

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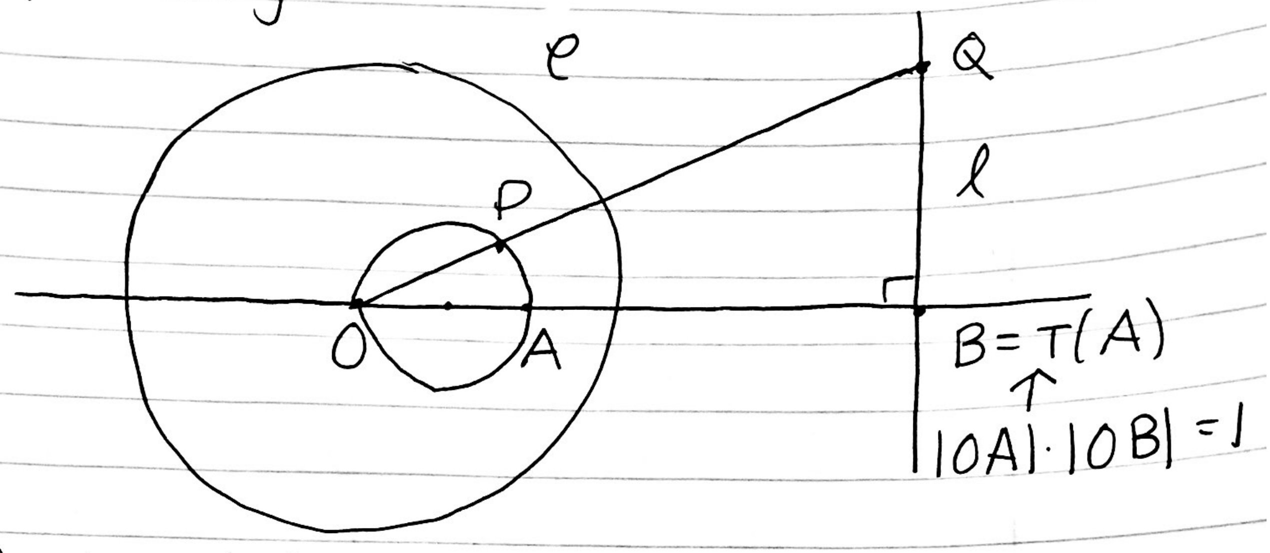
Explicitly:



c circle with center on x-axis

claim: $\ell = T(c)$

equivalently: $|OP| \cdot |OQ| = 1$

(for any point P on c, $P \neq O$)

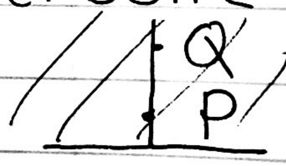
proof: $\triangle OAP \sim \triangle OQB$

$$\frac{|OP|}{|OB|} = \frac{|OA|}{|OQ|} \Rightarrow |OP||OQ| = |OA||OB| = 1 \checkmark$$

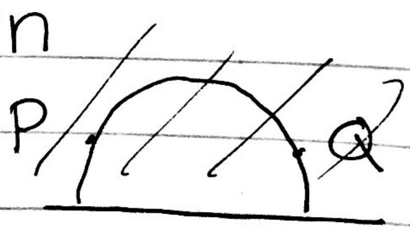
$\square$

now prove theorem that hyperbolic lines give shortest paths

know it for vertical lines

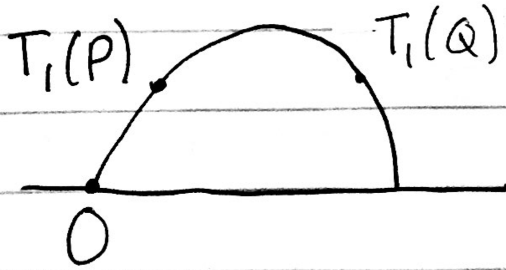


want to show it holds in semicircle case

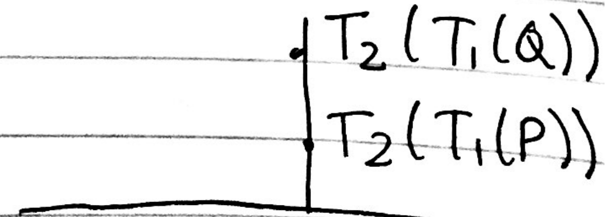


horizontal translate

$\xrightarrow{T_1}$



invert $\xrightarrow{T_2}$



$\square$

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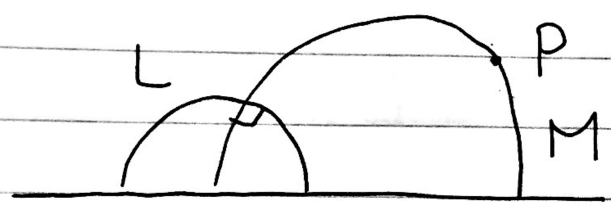
T_1 and T_2 are hyperbolic isometries
$T = T_2 \circ T_1$ preserves hyperbolic lengths

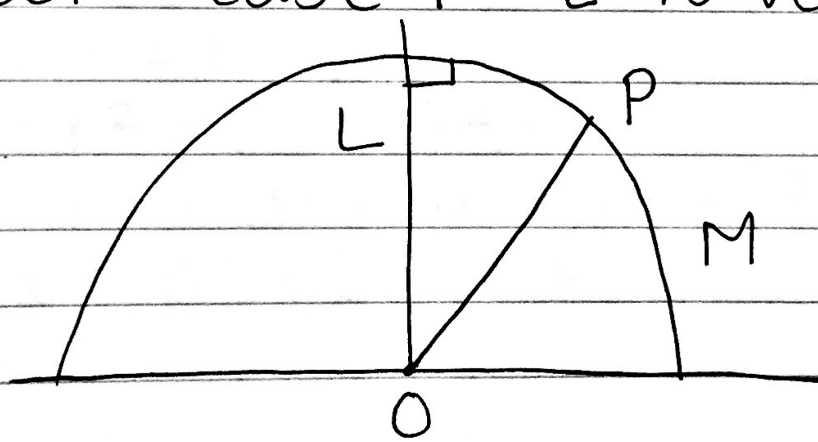
So $T(L)$ gives shortest path from $T(P)$ to $T(Q)$ $\Rightarrow$ L shortest path from P to Q $\square$

Hyperbolic reflections:

Claim: given hyperbolic line L and point $P \in H$, there is a unique hyperbolic line M through P and perpendicular to L



proof: case 1: L is vertical line



L perpendicular to M $\Rightarrow$ center of circle M lies on L (tangent $\perp$ radius)

M semicircle center $L \cap (x\text{-axis}) =: O$ radius $|OP|$

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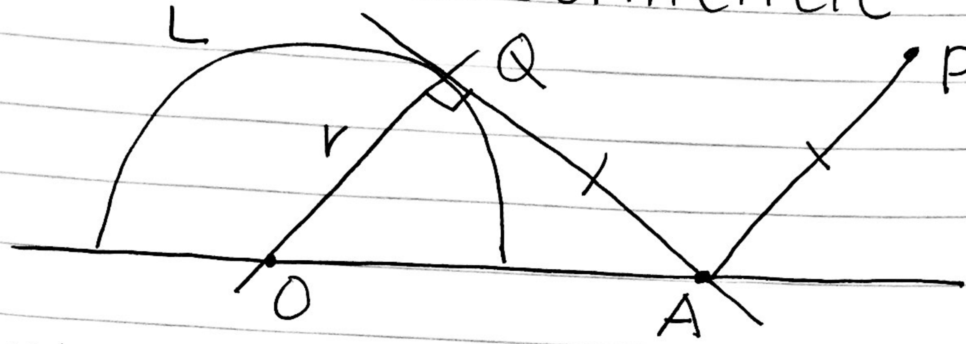
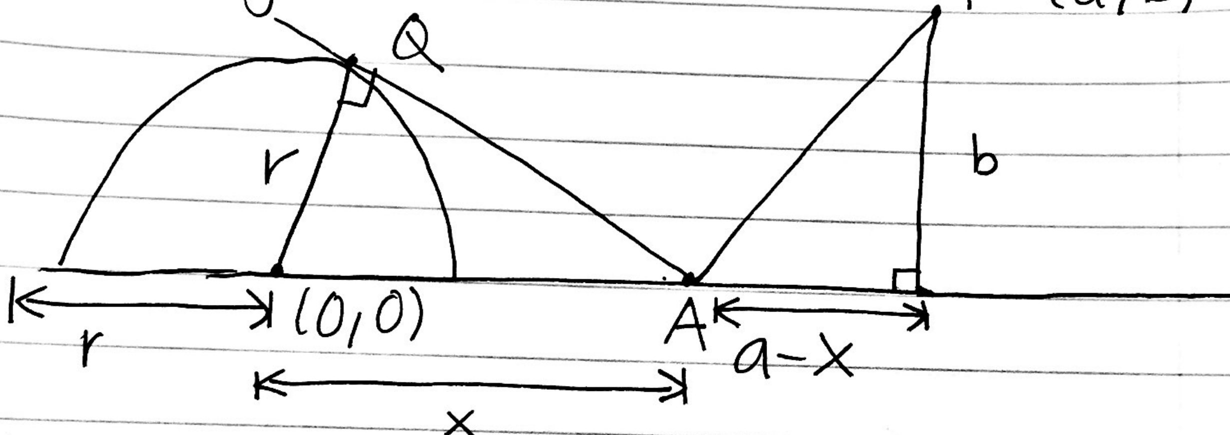
case 2: L is a semicircle



center A on x-axis has property $|AP| = |AQ|$ where AQ is tangent to L through A



$|AQ|^2 = |AP|^2$

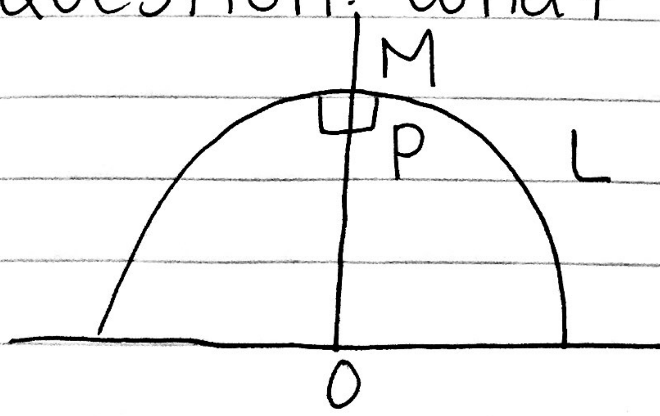
$x^2 - r^2 = (a-x)^2 + b^2$

$\cancel{x^2} - r^2 = a^2 - 2ax + \cancel{x^2} + b^2$

$2ax = a^2 + b^2 + r^2$

$$\boxed{x = \frac{a^2+b^2+r^2}{2a}}$$

Question: what if $a=0$?



case: $a=0$