Math 462: Homework 3

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(1) Let A, B, C be three points on a circle with center O in the plane.

(a) Suppose that B and O are on the same side of the line AC. Show that the angle $\angle ABC$ equals half the angle $\angle AOC$. [Hint: Draw a diagram and use the angle sum of a triangle equals π radians.]

(b) What about if B and O are on opposite sides of AC?

(2) Show that the sum of the angles of a polygon with n sides equals $(n-2)\pi$ radians. [Hint: Subdivide the polygon into triangles.]

(3) Recall that in class Prof Urzua showed that any 3 points A, B, C lie on a circle (the *circumcircle* of the triangle ABC). What condition must the angles of the triangle ABC satisfy so that the center of the circumcircle lies inside the triangle? [Hint: Use Q1.]

(4) (a) Show that if 4 points A, B, C, D lie on a circle (in that order) then the sum of the angles $\angle ABC$ and $\angle CDA$ equals π radians:

$$\angle ABC + \angle CDA = \pi.$$

[Hint: Use Q1.]

(b) Conversely, suppose we are given 4 points A, B, C, D such that $\angle ABC + \angle CDA = \pi$. Does it follow that A, B, C, D lie on a circle? Why?

(5) (a) Show that the rotation $T = \mathrm{Rot}(\mathbf{c}, \theta)$ of $\mathbb{R}^2$ about a point $\mathbf{c}$ through angle θ anticlockwise is given by the formula

$$T(\mathbf{x}) = A(\mathbf{x} - \mathbf{c}) + \mathbf{c}$$

where

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the matrix defining the rotation about the origin through angle θ anticlockwise. [Hint: Draw a picture.]

(b) Let P, Q be points in $\mathbb{R}^2$. Show that the composition

$$\mathrm{Rot}(Q, -\theta) \circ \mathrm{Rot}(P, \theta)$$

of rotation about P through angle θ anticlockwise followed by rotation about Q through angle θ clockwise is a translation. What is the translation vector? Explain geometrically what happens if $\theta = \pi$. Show that if θ is small, then the translation vector has length approximately $\theta \cdot d(P, Q)$ and its direction is approximately perpendicular to PQ. [Hint: To make things easier, we can choose coordinates so that P is the origin and $Q = \mathbf{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$ is a point on the x-axis. Now use the formula from part (a).]