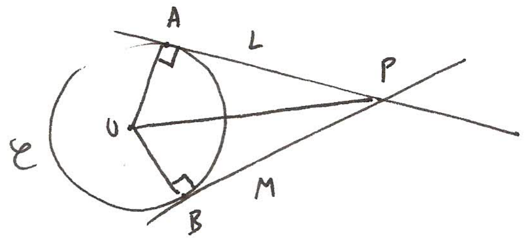
MATH 461 MIDTERM REVIEW PROBLEMS SOLUTIONS

Monday 10/14/19

1.

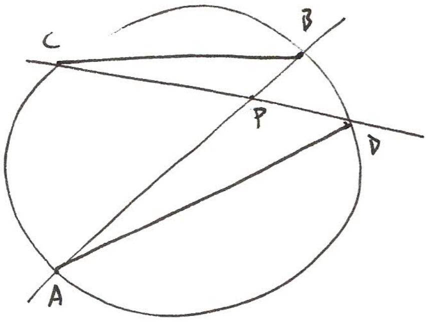


Let O be the center of the circle $\mathcal{C}$.

$\angle OAP = \angle OBP = \pi/2$ (angle between tangent & radius $= \pi/2$)

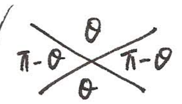
$$\therefore \underbrace{|PA| = \sqrt{|OP|^2 - |OA|^2}}_{\text{Pythagoras' thm}} \; \underbrace{=}_{\substack{|OA|=|OB|=r, \\ \text{radius of } \mathcal{C}}} \sqrt{|OP|^2 - |OB|^2} \; \underbrace{=}_{\text{Pythagoras' thm.}} |PB| \qquad \square$$

2.



$\angle ABC = \angle ADC$ (angles subtended by chord at circumference are equal)

$\angle BPC = \angle DPA$

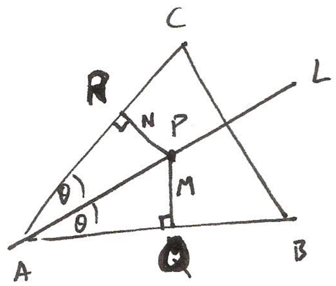


So $\triangle BPC \sim \triangle DPA$

(2 pairs of corresponding angles equal by above + angle sum of $\triangle = \pi$ $\Rightarrow$ corresponding angles equal).

Now $\dfrac{|BP|}{|DP|} = \dfrac{|CP|}{|AP|} \Rightarrow |AP| \cdot |BP| = |CP| \cdot |DP|$ $\square$.

3. a



$|PQ| = |PA| \sin\theta = |PR|$

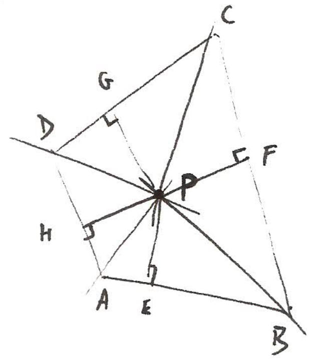
(Alternatively: $\triangle PAQ \cong \triangle PAR$: (ASA) :—

$|PA| = |PA|$, $\angle PAQ = \angle PAR$, $\angle PQA = \angle PRA$

& so $\angle APQ = \angle APR$ by angle sum of $\triangle$)

In particular, $|PQ| = |PR|$.

b.

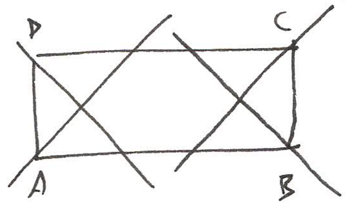


By part (a), $|PE| = |PF| = |PG| = |PH|$.

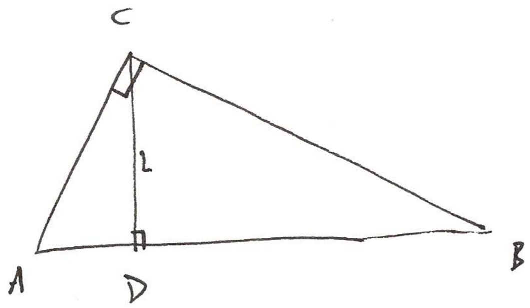
Draw a circle $\mathcal{C}$ w/ center P & radius $r = |PE|$.

Then, since the tangent is perpendicular to the radius, AB, BC, CD & DA are tangent to $\mathcal{C}$. □.

c. No, for example a rectangle which is not a square does not have this property.



4.



$\triangle ABC \sim \triangle CBD$

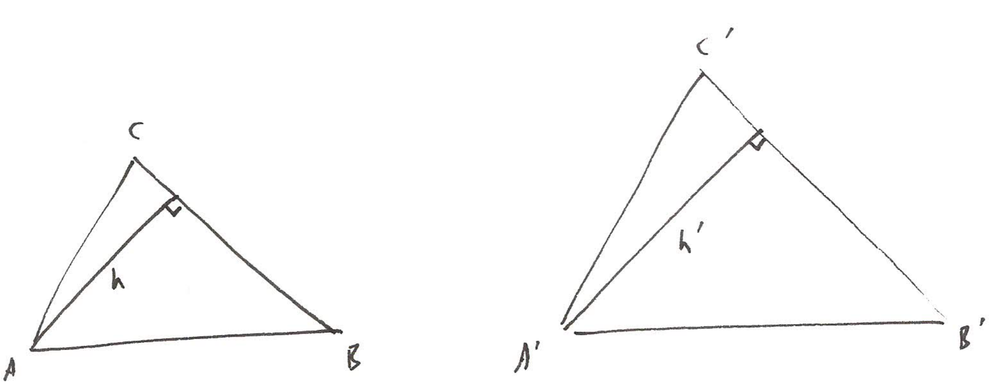
($\angle ABC = \angle CBD$

$\angle ACB = \angle CDB = \pi/2$

$\triangle$ angle sum of $\triangle = \pi$)

So $\dfrac{|AB|}{|CB|} = \dfrac{|BC|}{|BD|}$, $|AB| \cdot |BD| = |BC|^2$. □.

5. a.



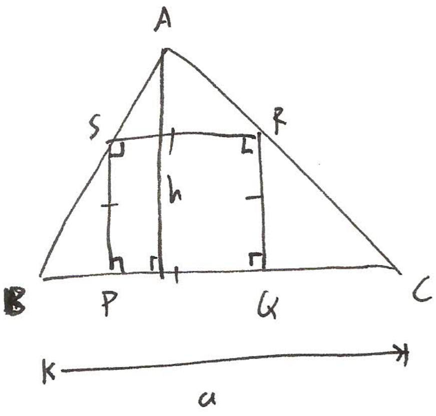
$\triangle ABC \sim \triangle A'B'C' \Rightarrow \angle ABC = \angle A'B'C' = \theta$, same angle θ.

Then $h = |AB| \sin\theta$, $h' = |A'B'| \sin\theta$, so $h'/h = \dfrac{|A'B'|}{|AB|} = \lambda$.

b.

$$\text{Area}(\Delta A'B'C') = \tfrac{1}{2}\text{ base} \times \text{height}$$
$$= \tfrac{1}{2}|B'C'| \cdot h'$$
$$= \tfrac{1}{2}\lambda \cdot |BC| \cdot \lambda h$$
$$= \lambda^2 \cdot \left(\tfrac{1}{2}|BC| \cdot h\right) = \lambda^2 \cdot \text{Area}(\Delta ABC).$$

6.



Write $x = |PQ|$, the side length of the square.

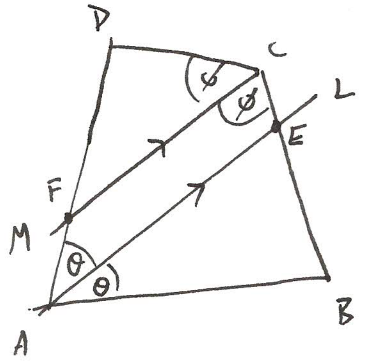
$\Delta ABC \sim \Delta ASR$ ($\angle BAC = \angle SAR$
$\angle ABC = \angle ASR$, $\angle ACB = \angle ARS$ } corresponding angles for the parallel lines BC & SR.)

So $\dfrac{|BC|}{|SR|} \overset{\text{GSa}}{=} \dfrac{h}{h-x}$ — perp. ht of ΔASR for base SR.

i.e. $\dfrac{a}{x} = \dfrac{h}{h-x}$

$$ah - ax = hx$$
$$ah = (a+h)x, \quad \boxed{x = \frac{ah}{a+h}}. \quad \square.$$

7.

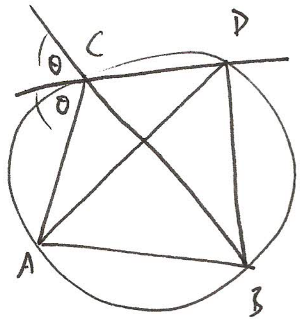


$\angle DFC = \angle DAE = \theta$, $\angle BEA = \angle BCF = \phi$ } corresponding angles for the parallel lines L & M.

$\Rightarrow \angle ABC = \pi - \theta - \phi = \angle ADC$

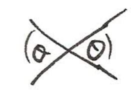
by angle sum of ΔABE & ΔFDC. $\square$.

8.



Required to prove $|AD| = |BD|$, equivalently, $\angle DAB = \angle DBA$ (isosceles triangle thm)

$\angle DAB = \angle DCB = \theta$ (angles subtended by a chord at circumference are equal)

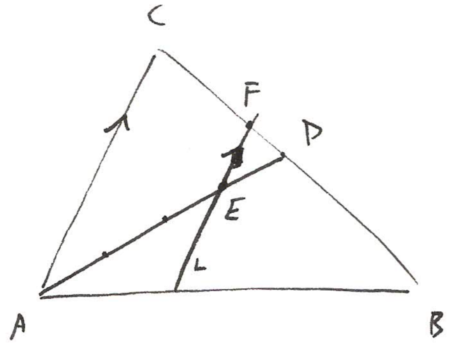


$\angle ADB = \angle ACB = \pi - 2\theta$ (.. .. ..)

$\therefore \angle DBA = \pi - (\theta + (\pi - 2\theta)) = \theta$ (angle sum of $\Delta = \pi$).

So $\angle DAB = \angle DBA$, & $|AD| = |BD|$. □.

9.



Thales' thm : $\dfrac{|DF|}{|FC|} = \dfrac{|DE|}{|EA|} = \dfrac{1}{3}$.

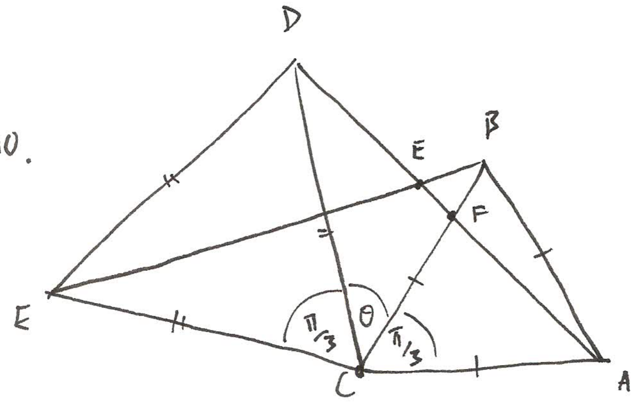
Also $\dfrac{|BD|}{|DC|} = 1$ by assumption.

So $\dfrac{|BF|}{|FC|} = \dfrac{|BD| + |DF|}{|FC|}$

$= \dfrac{|DC| + |DF|}{|FC|} = \dfrac{|FC| + 2|DF|}{|FC|}$

$= 1 + 2/3 = 5/3$. □.

10.



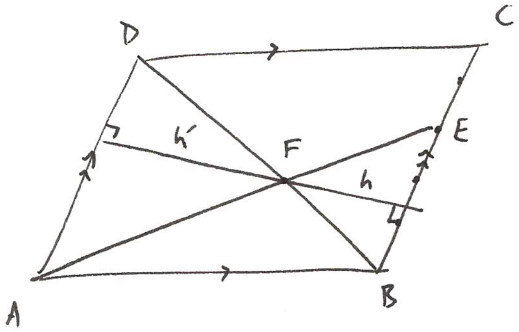
$\Delta ACD \cong \Delta BCE$ (SAS)

$\Rightarrow \angle CAD = \angle CBE$

Also $\angle CFA = \angle BFE$

$\Rightarrow \angle BEF = \angle FCA = \pi/3$ □. (angle sum of ΔCFA & ΔBFE)

11.



$$\text{Area}(CDFE) = \text{Area}(ABCD) - \text{Area}(\Delta ABD) - \text{Area}(\Delta BEF)$$
$$= 1 - \tfrac{1}{2} - \text{Area}(\Delta BEF)$$

$\Delta BEF \sim \Delta DAF$ (alternate angles)

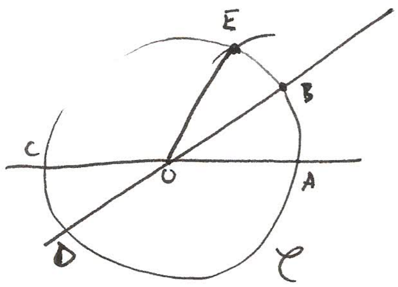
$$\Rightarrow \frac{h}{h'} = \frac{|BE|}{|AD|} = \frac{|BE|}{|BC|} = \frac{3}{5} \qquad \Rightarrow \frac{h}{h+h'} = \frac{3}{3+5} = \frac{3}{8}$$

opposite sides of parallelogram have equal lengths.

$$\Rightarrow \text{Area}(\Delta BEF) = \tfrac{1}{2} \cdot |BE| \cdot h = \tfrac{1}{2} \cdot \tfrac{3}{5} \cdot |BC| \cdot \tfrac{3}{8} \cdot (h+h')$$
$$= \frac{9}{80} \cdot \underbrace{|BC| \cdot (h+h')}_{\text{Area}(ABCD)} = \tfrac{9}{80}.$$

So $\text{Area}(CDFE) = 1 - \tfrac{1}{2} - \tfrac{9}{80} = \tfrac{31}{80}$. $\square$.

12.



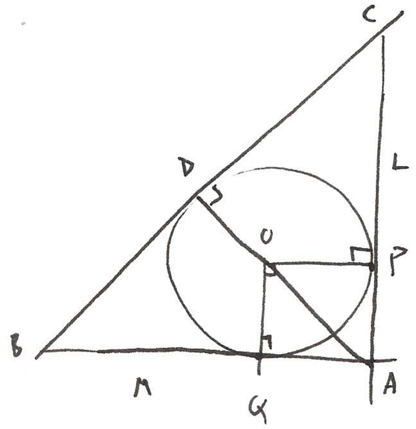
1. Draw line OA, intersecting $\mathcal{C}$ again at C
2. Draw circle center A, radius OA, intersecting $\mathcal{C}$ at E.
3. Bisect the angle AOE, let B, D be intersection points of the bisector & $\mathcal{C}$.

Then $ABCD$ is a rectangle & its diagonals AC & BD meet at angle $\pi/6$:-

$\angle ABC = \angle BCD = \angle CDA = \angle DAB = \pi/2$ (angle in a semicircle)

$\angle AOB = \tfrac{1}{2} \angle AOE = \tfrac{1}{2} \cdot \pi/3 = \pi/6$ (ΔOAE is equilateral $\Rightarrow$ angles $= \pi/3$).

13.



1. Draw radius OP.
2. Construct perpendicular line to OP through P – this is the tangent L to $\mathcal{C}$ at P.
3. Construct perpendicular line to OP through O, intersecting $\mathcal{C}$ at G
4. Construct perpendicular line to OG through G – the tangent line M to $\mathcal{C}$ through G.
5. Let A be the intersection point of L & M. Draw the line OA intersecting the circle $\mathcal{C}$ again at D.
6. Draw the perpendicular to OD through D (the tangent to $\mathcal{C}$ at D), meeting L and M at A & B.

Claim: ΔABC is isoceles & $\angle BAC = \pi/2$.

Proof: $\angle BAC = \pi/2$ by angle sum of quadrilateral $APOG = 2\pi$.

$|AG| = |OP| = |OG| = |AP|$ $\Rightarrow$ $\angle OAG = \angle OAP = \pi/4$

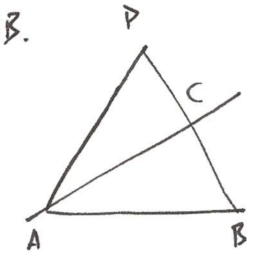
opposite sides of parallelogram have equal lengths. radius of $\mathcal{C}$.

(using isosceles triangle theorem & angle sum of triangle)

$\Rightarrow$ $\angle ABD = \angle ACD = \pi/4$ (using angle sum of triangle)

$\Rightarrow$ ΔABC is isosceles. □

14 a. Construct an equilateral triangle ΔABD with base AB.



Bisect angle BAD, let C be the intersection point w/ BD.

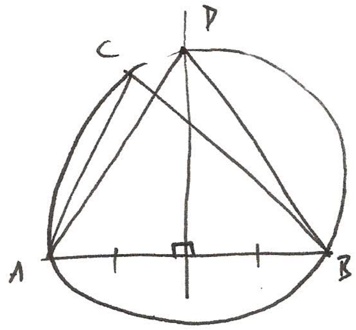
Then $\angle BAC = \frac{1}{2} \angle BAD = \frac{1}{2} \cdot \frac{\pi}{3} = \pi/6$

& $\angle ABC = \angle ABD = \pi/3$

$\Rightarrow$ $\angle ACB = \pi - \pi/3 - \pi/6 = \pi/2$ □.

(NB. An equilateral triangle has all angles $\pi/3$ by isosceles triangle thm.)

b.



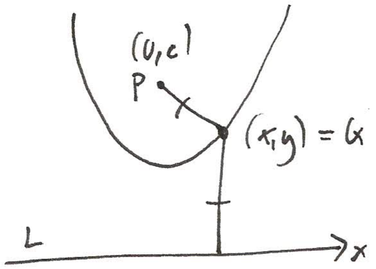
1. Construct the circumscribed circle of ΔABC (the unique circle $\mathcal{C}$ passing through $A, B, \& C$)
2. Construct the perpendicular bisector of the line segment AB, & let D be its intersection point w/ $\mathcal{C}$ on the same side of AB as C.

Then $\angle ADB = \angle ACB$ (angles subtended by a chord at the circumference are equal)

$|AD| = |BD|$ (D lies on perpendicular bisector of AB)

So $\angle DAB = \angle DBA$ (isosceles triangle thm). $\square$.

15. We follow the hint.



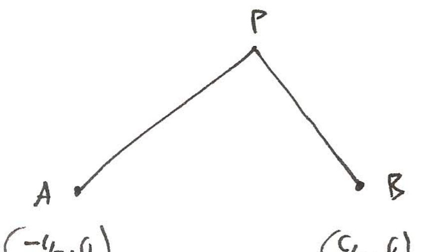
$$\sqrt{x^2 + (y-c)^2} = |y| \qquad \text{Square both sides.}$$

$$x^2 + (y-c)^2 = y^2$$

$$x^2 + \cancel{y^2} - 2cy + c^2 = \cancel{y^2}$$

$$y = \frac{1}{2c}(x^2+c^2) = \frac{1}{2c} \cdot x^2 + \frac{1}{2}c \qquad \square.$$

16. We follow the hint.



$$|PA| + |PB| = d.$$

$$\sqrt{(x+\tfrac{c}{2})^2 + y^2} + \sqrt{(x-\tfrac{c}{2})^2 + y^2} = d. \quad \text{Square both sides}$$

$$(x+\tfrac{c}{2})^2 + y^2 + (x-\tfrac{c}{2})^2 + y^2 + 2\sqrt{((x+\tfrac{c}{2})^2+y^2)\cdot((x-\tfrac{c}{2})^2+y^2)} = d^2$$

Rearrange

$$2x^2 + 2y^2 + \frac{c^2}{2} - d^2 = -2\sqrt{(x^2+cx+\tfrac{c^2}{4}+y^2)(x^2-cx+\tfrac{c^2}{4}+y^2)}$$

Square both sides

$$\cancel{4}\,(x^2+y^2+\tfrac{c^2}{4} - \tfrac{d^2}{2})^2 = \cancel{4}\,(x^2+y^2+\tfrac{c^2}{4}+cx)(x^2+y^2+\tfrac{c^2}{4}-cx)$$

$$\cancel{(x^2+y^2+\tfrac{c^2}{4})^2} - 2\tfrac{d^2}{2}(x^2+y^2+\tfrac{c^2}{4}) + \tfrac{d^4}{4} = \cancel{(x^2+y^2+\tfrac{c^2}{4})^2} - c^2x^2$$

(Here we used "difference of two squares" $(A+B)(A-B) = A^2 - B^2$.)

$$d^2 \cdot (x^2+y^2) - c^2x^2 = \frac{d^4}{4} - \frac{c^2d^2}{4}$$

$$(d^2-c^2)x^2 + d^2y^2 = \frac{1}{4}\, d^2(d^2-c^2)$$

$$\frac{x^2}{(d^2/4)} + \frac{y^2}{((d^2-c^2)/4)} = 1.$$

i.e. $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

where $a = d/2$, $b = \frac{1}{2}\sqrt{d^2-c^2}$. □