Math 412 Homework 1

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Reading: Saracino, Chapter 16.

(1) Which of the following are rings? Justify your answers carefully.

[Note: In examples (a),(b), and (c), R is a subset of a known ring S with addition and multiplication given by the addition and multiplication in S. In this situation, we don't need to check associativity of addition and multiplication and the distributive law for R because we know they are true for S, and $R \subset S$. If R is a ring, we say it is a *subring* of S.]

(a)
$$R = \mathbb{Z}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R},$$
with addition and multiplication given by addition and multiplication of real numbers. [Remark: For this question, it is useful to know that if $a + b\sqrt{2} = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Z}$ then $a = c$ and $b = d$ (why?)].

(b)
$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\} \subset \mathbb{R}^{2\times 2},$$
with addition and multiplication given by addition and multiplication of 2×2 matrices.

(c)
$$R = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \text{ is not divisible by } 3 \right\} \subset \mathbb{Q}$$
with addition and multiplication given by addition and multiplication of rational numbers.

(d) $R = \mathbb{R}$ with addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes b = a + b$. [To explain the notation, we are using $\oplus$ and $\otimes$ to denote the addition and multiplication operations on R, and $+$ denotes ordinary addition of real numbers. Also $\max(a, b)$ denotes the maximum of a and b, i.e, $\max(a, b) = a$ if $a \geq b$ and $\max(a, b) = b$ if $a \leq b$.]

(2) Let R be a ring. What does it mean to say that an element $a \in R$ is a zero divisor? Describe the set of all zero divisors in the following rings. (Justify your answers carefully.)

(a) $\mathbb{Z}/15\mathbb{Z}$ (i.e., integers modulo 15).

(b) $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$.

(c) $\mathbb{Z} \oplus \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}\}$, with addition

$$(a, b) + (c, d) = (a + c, b + d)$$

and multiplication

$$(a, b) \cdot (c, d) = (ac, bd).$$

(d) $R = \{a + bx \mid a, b \in \mathbb{Z}/2\mathbb{Z}\}$ with addition $(a + bx) + (c + dx) = (a+c)+(b+d)x$ and multiplication $(a+bx)(c+dx) = (ac+bd)+(ad+bc+bd)x$. [Hint: The set R has size 4, and R is commutative, so it is easy to write out the full multiplication table.]

(e) $\mathbb{R}^{2\times 2}$. [Hint: If a matrix A is a zero divisor, what can you say about the rank of A?]

(3) Let R be a ring. What does it mean to say that an element $a \in R$ is nilpotent? Describe the set of all nilpotent elements in the following rings. (Justify your answers carefully.)

(a) $\mathbb{Z}/18\mathbb{Z}$.

(b) $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$. [Hint: Consider the prime factorization of n.]

(c) $\mathbb{C}^{2\times 2}$. [Hint: Here it is useful to know the "Jordan normal form" of a matrix. For a 2×2 complex matrix A, this is the following: there is an invertible matrix P such that the matrix $P^{-1}AP$ equals either $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$ or $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$ for some $\lambda_1 = \lambda_2 \in \mathbb{C}$. (In both cases λ_1, λ_2 are the eigenvalues of A.)]

(4) Let R be a ring and S a set. Consider the set R^S of all functions $f\colon S \to R$. Define addition and multiplication "pointwise" on S, i.e.,

$$(f+g)(s) = f(s) + g(s)$$

and

$$(f \cdot g)(s) = f(s) \cdot g(s)$$

where on the right hand sides we use addition and multiplication in the ring R. Convince yourself that R^S is a ring (I won't ask you to write it out because it is rather tedious).

Now let $R = \mathbb{Z}/2\mathbb{Z}$. Let

$$\mathcal{P}(S) := \{A \mid A \subset S\}$$

denote the power set of S (the set of all subsets of S). Then there is a bijection

$$F\colon (\mathbb{Z}/2\mathbb{Z})^S \to \mathcal{P}(S), \quad f \mapsto \{s \in S \mid f(s) = 1\}.$$

with inverse

$$F^{-1}\colon \mathcal{P}(S) \to (\mathbb{Z}/2\mathbb{Z})^S, A \mapsto \chi_A$$

where

$$\chi_A\colon S \to \mathbb{Z}/2\mathbb{Z}, \quad \chi_A(s) = \begin{cases} 1 \text{ if } s \in A \\ 0 \text{ if } s \notin A \end{cases}$$

Describe the addition and multiplication of the ring $(\mathbb{Z}/2\mathbb{Z})^S$ in terms of set-theoretic operations using the bijection F. That is, for $A, B \subset S$, identify the sets $F(F^{-1}(A) + F^{-1}(B))$ and $F(F^{-1}(A) \cdot F^{-1}(B))$. These operations make the power set into a ring.

(5) Let R be a commutative ring. Show that if $a \in R$ and $b \in R$ are nilpotent, then so is $a + b$. Show by example that this can fail for noncommutative rings.