Math 461 Lecture 22 10/24

Midterm 7-9 PM LGRT 121 TODAY

You are allowed 1 sheet of notes (letter size, both sides)

Review problems and solutions on webpage

syllabus: up to classification of isometries of $\mathbb{R}^2$

Last time:

spherical geometry

compute: equation of spherical line through two points
spherical distance
angles between spherical lines

Theorem: $a+b+c = \pi + \text{Area}(\Delta ABC)$

Lemma: $\text{Area}(\text{lune}) = 2a$

Today: proof of theorem
review for midterm

Proof of Theorem:

area of sphere $= 4\pi =$
$2(2a+2b+2c) - 2\text{Area}(\Delta ABC) - 2\text{Area}(\Delta A'B'C')$

$\text{Area}(\Delta ABC) = \text{Area}(\Delta A'B'C')$

$4a+4b+4c = 4\pi + 4\text{Area}(\Delta ABC)$

divide by 4 $\square$

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Isometries:

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$|T(P)T(Q)| = |PQ|$ for all $P, Q \in \mathbb{R}^2$

Classification:

identity nothing

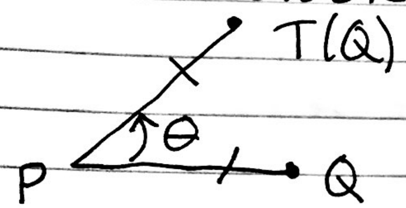
translation $T(x,y) = (x+a, y+b)$

rotation

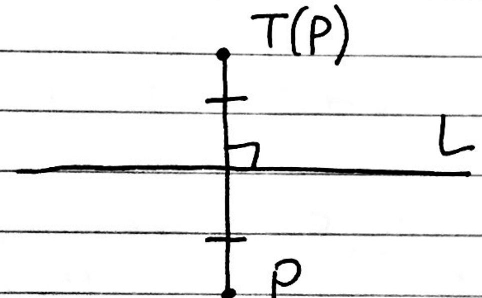
reflection

glide reflection

Rotation about P through angle θ counterclockwise



Reflection in a line L



Glide: reflection in a line followed by translation by a vector $\bar{v} = (a,b)$ in direction of line

Homework 4: fixed lines of isometry T

$\text{Fix}(T) = \{P \in \mathbb{R}^2 \mid T(P) = P\}$

isometry	fixed locus
identity	$\mathbb{R}^2$
translation	$\emptyset$
rotation	P, center of rotation
reflection	L, the line of reflection
glide	$\emptyset$

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Question 12b on practice problems

$T(x,y) = (1-y, 3-x)$

want to find the fixed locus

solve $T(x,y) = (x,y)$

$(1-y, 3-x) = (x,y)$

$1-y = x \rightsquigarrow x+y = 1$

$3-x = y \rightsquigarrow x+y = 3$

there are no solutions

$Fix(T) = \emptyset$

either a translation or glide reflection

not a translation $\Rightarrow$ glide reflection

Homework 4:

If T is a glide with line of reflection L and translation vector $\bar{v}$ then $T^2 = T \circ T$ is translation by $2\bar{v}$

$T = Trans_{\bar{v}} \circ Refl_L = Refl_L \circ Trans_{\bar{v}}$

$T^2 = Trans_{\bar{v}} \circ Refl_L \circ Refl_L \circ Trans_{\bar{v}}$

$= Trans_{2\bar{v}}$

$T(x,y) = (1-y, 3-x)$

$T(T(x,y)) = (1-(3-x), 3-(1-y))$

$= (1-3+x, 3-1+y)$

$= (x-2, y+2)$

translation by $(-2,2)$

so $\bar{v} = \frac{1}{2}(-2,2) = (-1,1)$

$T = Trans_{\bar{v}} \circ Refl_L$

so $Refl_L = Trans_{\bar{v}}^{-1} \circ T = Trans_{-\bar{v}} \circ T$

$Refl(x,y) + \bar{v} = T(x,y)$

$Refl_L(x,y) = T(x,y) - \bar{v}$

$= (1-y, 3-x) - (-1,1)$

$= (1-y+1, 3-x-1)$

$= (2-y, 2-x)$

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What is L? compute fixed locus

$(x,y) = \mathrm{Refl}_L(x,y)$

$(x,y) = (2-y,\ 2-x)$

$\left.\begin{array}{l} x = 2-y \\ y = 2-x \end{array}\right\}$ $x+y=2$ $\quad L:\ y=-x+2$

T is a glide reflection: $y=-x+2$, translation by $(-1,1)$

12c. $T(x,y) = (y+2,\ 8-x)$

Fix T:

$y+2 = x \rightsquigarrow -x+y=-2 \qquad y=3$

$8-x = y \rightsquigarrow x+y=8 \qquad x=5$

$P=(5,3) \Rightarrow$ rotation center $P=(5,3)$

Formula for rotation about (a,b) through angle θ ccw

$$T(x,y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right) + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\left(\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)}_{\begin{pmatrix} c \\ d \end{pmatrix} \text{ constants}}$$

$$T(x,y) = \begin{pmatrix} y+2 \\ 8-x \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \theta = \frac{3\pi}{2} \text{ ccw}$$

or equivalently $\frac{\pi}{2}$ clockwise

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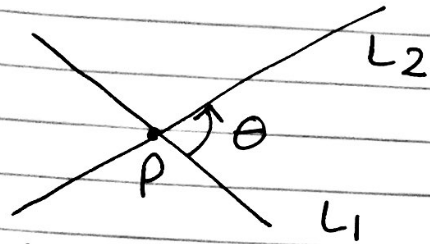
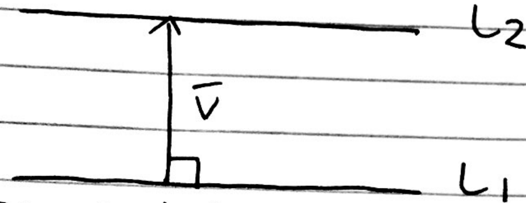
$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} = \mathrm{Rot}(P, 2\theta)$



$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} = \mathrm{Trans}_{2\bar{v}}$



Compositions

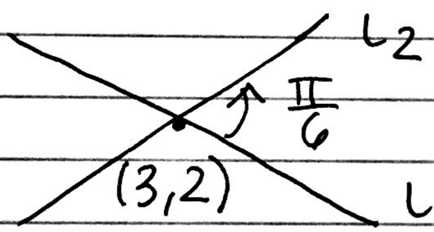
rotation about $(3,2)$ through angle $\pi/3$ followed by translation by $(4,0)$

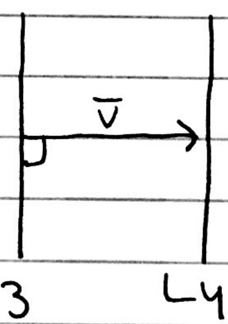
(Question 15d from practice problems)

T_1 = rotation about $(3,2)$ through angle $\pi/3$

T_2 = translation by $(4,0)$

$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} = T_1$

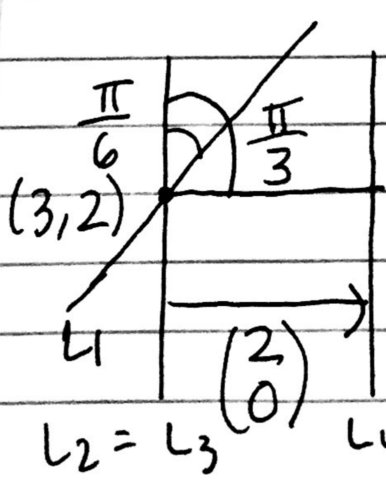


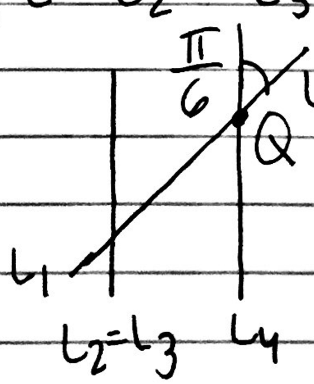


$\bar{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$T = T_2 \circ T_1 = \mathrm{Refl}_{L_4} \circ \cancel{\mathrm{Refl}_{L_3}} \circ \cancel{\mathrm{Refl}_{L_2}} \circ \mathrm{Refl}_{L_1}$

can arrange $L_2 = L_3$





$= \mathrm{Refl}_{L_4} \circ \mathrm{Refl}_{L_1}$

$= \mathrm{Rot}(Q, \pi/3)$

$Q = (3,2) + (2, 2\sqrt{3})$

$= (5, 2+2\sqrt{3})$

slope of $L_1 = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$