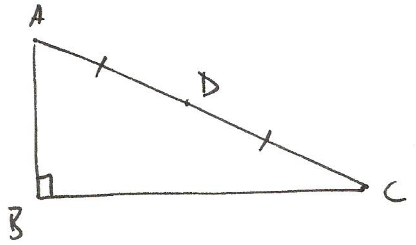
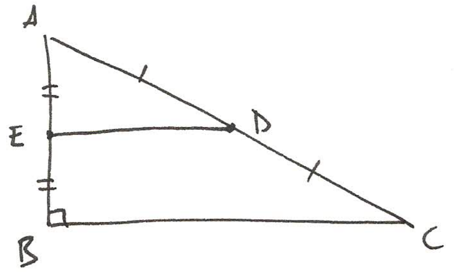
MATH 461 HW2 Solutions.

1.



Let E be the midpoint of AB



Then ED is parallel to BC by converse Thales' theorem

(because $\frac{|AE|}{|AB|} = \frac{|AD|}{|AC|} = \frac{1}{2}$).

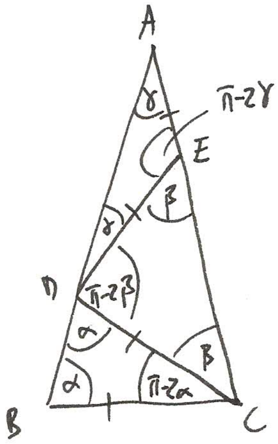
So $\angle AED = \angle ABC = \pi/2$.

Now $\Delta AED \cong \Delta BED$ by SAS:

$|AE| = |EB|$
$|ED| = |ED|$
$\angle AED = \angle BED = \pi/2$.

So $|BD| = |AD| = \frac{1}{2}|AC|$. $\square$.

2.



Angles are as shown by isosceles triangle theorem (equal sides $\Rightarrow$ equal angles) & angle sum of $\Delta = \pi$.

We deduce $\beta + \pi - 2\gamma = \pi$, $\beta = 2\gamma$)

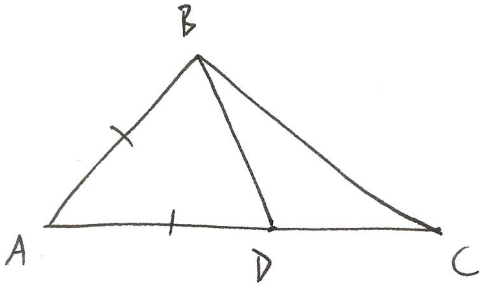
$\alpha + \pi - 2\beta + \gamma = \pi$, $\alpha + \gamma = 2\beta$) $\Rightarrow \alpha = 3\gamma$

$\alpha = (\pi - 2\alpha) + \beta$ (ΔABC is isosceles)

So $3\alpha = \pi + \beta$, $9\gamma = \pi + 2\gamma$, $\gamma = \pi/7 = \angle BAC$.

$\square$.

3.



Let D be the point on AC such that $|AB| = |AD|$

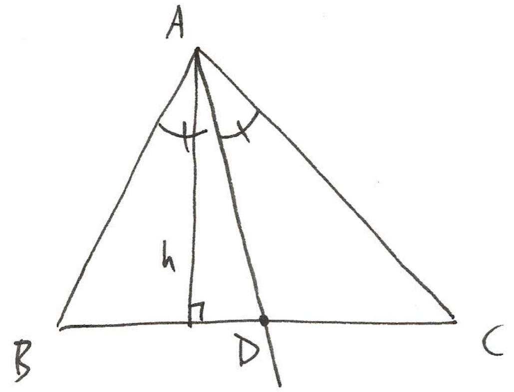
Then $\angle ADB = \pi - \angle BDC \overset{\text{angle sum of } \Delta BDC}{=} \pi - (\pi - \angle DBC - \angle DCB)$

$= \angle DBC + \angle DCB > \angle DCB = \angle ACB.$

And $\angle ABC > \angle ABD \underset{\Delta ABD \text{ is isoceles.}}{=} \angle ADB$

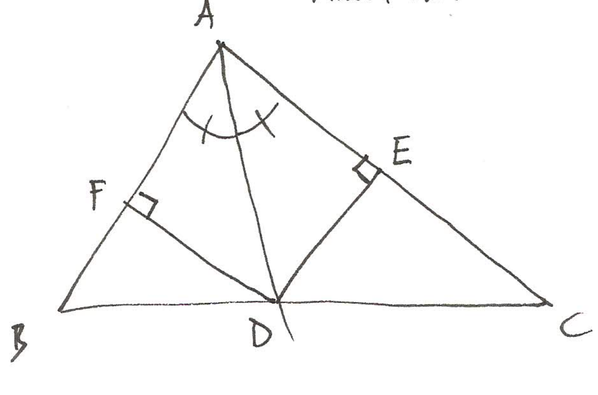
So, combining, $\angle ABC > \angle ACB$. □.

4.

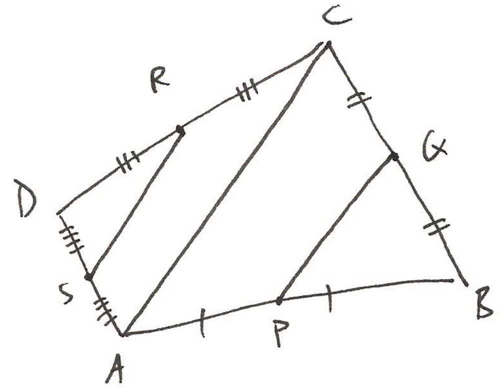


a. $$\frac{\text{Area}(\Delta ACD)}{\text{Area}(\Delta ABD)} = \frac{\frac{1}{2} \cdot |CD| \cdot h}{\frac{1}{2} \cdot |BD| \cdot h} = \frac{|CD|}{|BD|} \quad \square$$

b. $$\frac{|CD|}{|DB|} \overset{(a)}{=} \frac{\text{Area}(\Delta ACD)}{\text{Area}(\Delta ABD)} = \frac{\frac{1}{2}|AC| \cdot |DE|}{\frac{1}{2}|AB| \cdot |DF|} \underset{|DE| = |DF| \text{ by HW1Q7a.}}{=} \frac{|AC|}{|AB|} \quad \square.$$



5.

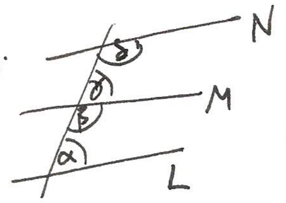


PQ is parallel to AC by converse Thales' theorem

& RS " " " " " " " "

So PQ is parallel to RS. (Note: If L is parallel to M & M is parallel to N then L is parallel to N.)

[e.g.



$\alpha+\beta=\pi$ & $\gamma+\delta=\pi$

$\Rightarrow \alpha+\delta = (\alpha+\beta)+(\gamma+\delta) - (\beta+\gamma)$

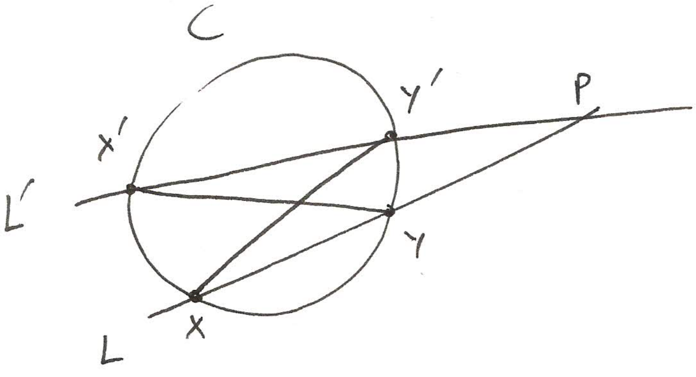
$= \pi+\pi-\pi = \pi.$]

Similarly QR is parallel to BD & PS is parallel to BD,

so QR is parallel to PS.

So PQRS is a parallelogram (opposite sides are parallel).

6.



"similar"

Claim: $\Delta PXY' \sim \Delta PX'Y$

Proof: $\angle XPY' = \angle X'PY$ ✓

$\angle PXY' = \angle PX'Y$

(angles subtended by the chord YY' of the circle at a point on the circumference).

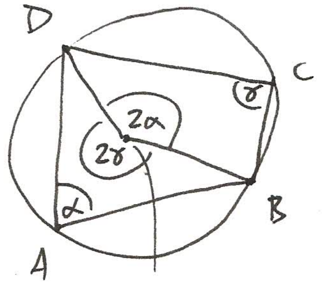
Now $\angle XY'P = \angle X'YP$

(by angle sum of triangle $=\pi$).

So $\Delta PXY'$ & $\Delta PX'Y$ have equal angles that is, they are similar triangles.

So $\dfrac{|PX|}{|PX'|} = \dfrac{|PY'|}{|PY|}$, & $|PX|\cdot|PY| = |PX'|\cdot|PY'|$. $\square$.

7.



O = center of the circle

The angle subtended by a chord at the center of the circle equals twice the angle subtended at the circumference.

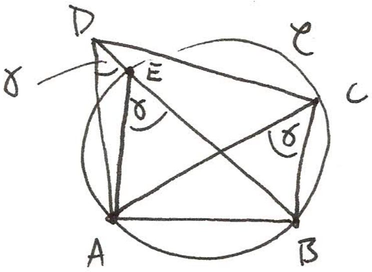
So, as in the diagram

$$2\cdot\angle DAB + 2\cdot\angle BCD = 2\pi \quad \text{(sum of angles at center O)}$$

$$\div 2: \quad \angle DAB + \angle BCD = \pi.$$

Similarly $\angle ABC + \angle CDA = \pi$. $\square$.

8.



Let $\mathscr{C}$ be the unique circle passing through A, B, C. (HW1 Q3b)

We need to show that $\mathscr{C}$ passes through D.

Let E be the intersection point of the line BD with the circle $\mathscr{C}$ (besides B).

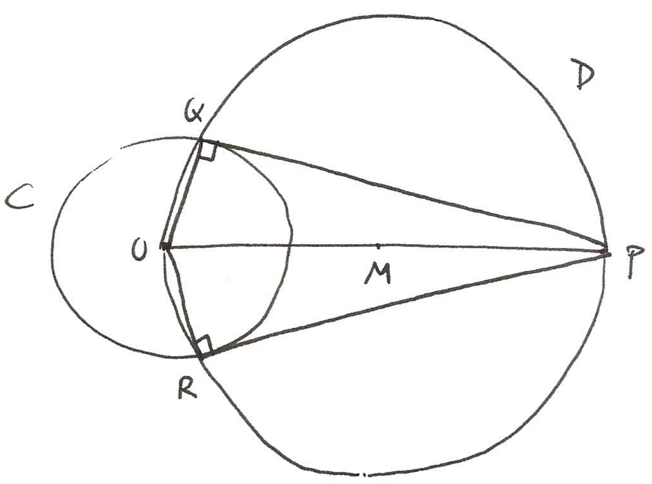
Then $\angle AEB = \angle ACB$ (angles subtended by a chord at the circumference)

If $D \neq E$, then ΔADE has angles γ, $\pi-\gamma$ and $\angle DAE$,

so angle sum $\gamma + \pi - \gamma + \angle DAE = \pi + \angle DAE > \pi$ ※.

So $D = E$. $\square$.

9.



Let O be the center of C.

Draw the line OP.

Let M be the midpoint of OP.

Draw the circle D with center M & radius $|OM| = |MP|$.

Let Q, R be the intersection points of C & D.

Then $\angle OQP = \angle ORP = \pi/2$ (angle in a semicircle)

So PQ & PR are tangent to C by HW1 Q6.