ALGEBRA STRUCTURES FOR FINITE FREE RESOLUTIONS, AND SOME STRUCTURE THEOREMS FOR IDEALS OF CODIMENSION 3.

By David A. Buchsbaum and David Eisenbud.*

In this paper we begin to study the algebra structures that may be given to finite free resolutions of cyclic modules over a local noetherian ring. Using these algebra structures, we obtain structure theorems for two classes of ideals of codimension 3 analogous to Hilbert's structure theorem for perfect ideals of codimension 2, and we deduce various consequences for these classes of ideals.

In recent years a number of people have exploited the unique algebra structure that can be put on the minimal free resolution of the residue class field of a local ring R; and of course the algebra structure on the Koszul complex, which gives a minimal free resolution for an ideal that is generated by an R-sequence, has always been an important tool. But the idea of putting an algebra structure on a finite free resolution of a cyclic module, when that resolution is not the Koszul complex, seems to have received very little attention. Perhaps this is because the Koszul complex and the resolutions of residue class fields are all free graded commutative divided power algebras (so the algebra structure itself is rather trivial) whereas the algebra structures on other finite free resolutions are never free, and thus give rather subtle invariants.

When we say that a free resolution has an algebra structure, we mean that the resolution—which is, to begin with, a differential graded module—has the structure of a homotopy-associative, commutative differential graded algebra. For a few classes of resolutions, such as those of length $\leqslant 3$, the algebra structures can be chosen to be associative "on the nose"; but we do not know whether this is true in general.

Hilbert's structure theorem, referred to above, asserts (in a slightly special case) that if R is a regular local ring, and I is an ideal of codimension 2 in R requiring n generators, then I is perfect (that is, R/I is a Macaulay ring) if and only if I is generated by the $n \times n$ minors of an $n \times n+1$ matrix. Hilbert proved

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this only in a special case. The first modern treatment seems to be that of Burch [BUR]. A very fast proof may be found in [KAP, Exercise 8, p. 148]. Hilbert's theorem has recently found a number of applications, for instance, to the Lifting Problem of Grothendieck [B-E-0], in the theory of rational singularities, in deformation theory [SCH and P-S], and in the theory of linkage [A-N and P-S]. All these applications rest on the fact that Hilbert's theorem gives a sort of "generic form" for perfect ideals of codimension 2.

Our structure theorems are analogous to Hilbert's in that they give "generic forms"—this time involving both pfaffians and determinants—for two classes of perfect ideals of codimension 3.

We now describe the contents of this paper in more detail. Let R be a regular local ring (in the body of the paper we have often made the easy generalization to the case of ideals of finite homological dimension in arbitrary local rings.)

Section 1 is concerned with the definition of algebra structures on free resolutions, and their simplest properties, including one result on Gorenstein rings which is the key to our later structure theorems: If R/I is Gorenstein, then the free resolution (over R) of R/I is self-dual, the duality being induced by the algebra structure. (This is closely connected to duality results of Avramov [AVR] and Avramov-Golod [A-G].)

In section 2 we state our main result—the structure theorem for ideals I of codimension 3 in R such that R/I is a Gorenstein ring. We call these Gorenstein ideals. (Serre [SER] showed that every Gorenstein ideal of codimension 2 is a complete intersection.) We show that an ideal I of codimension 3 in R is Gorenstein if and only if it is the ideal of $(n-1)$th order pfaffians of some $n \times n$ alternating matrix of rank $n-1$. (An *alternating* matrix is a square skew-symmetric matrix whose diagonal entries are 0. The pfaffian of an alternating matrix is the square root of its determinant; the "$(n-1)^{\text{th}}$ order pfaffians of an $n \times n$ alternating matrix" are simply the pfaffians of the $(n-1)\times(n-1)$ submatrices obtained by deleting a row and the corresponding column of the matrix.) Thus for example, since the rank of an alternating matrix must be even, the minimal number, n, of generators for I, must be odd. In order to handle pfaffians succinctly, we require some multilinear algebra; this occupies the remainder of section 2. (Some fundamental tools of multilinear algebra are briefly reviewed in an appendix to this paper, whose results we refer to as $A1$, $A2, \ldots$.)

The proof of the structure theorem for Gorenstein ideals is undertaken in section 3; it leans heavily on the self duality and commutativity (not just up to homotopy) of the algebra structure on the minimal free resolution of R/I, where I is a Gorenstein ideal. As an application, we exhibit a formula that expresses the Hilbert function of an "arithmetically Gorenstein" subvariety of codimension 3 in projective space in terms of a minimal number of parameters.

In section 4 we apply our structure theorem to determine completely the algebra structure on the minimal free resolution of R/I, where I is Gorenstein of codimension 3, with a view to controlling the linkage process of Peskine-Szpiro [P-S] and Artin-Nagata [A-N], which we will now briefly describe.

Roughly speaking, two subvarieties of a smooth variety are said to be *linked* if their union is a complete intersection. If we work locally, and let R be the coordinate ring of the ambient variety, it turns out that the linkage of a variety corresponding to an ideal $I \subseteq R$ is determined by the comparison maps from various Koszul complex into the resolution of R/I.

Suppose now that the minimal free resolution $\mathbf{F}$ of R/I possesses an algebra structure which is associative "on the nose" instead of "up to homotopy". The minimal free resolution $\mathbf{K}$ of $R/(x_1,\dots,x_g)$ is the Koszul complex, which is free as a differential graded commutative associative algebra. Thus any expression for the x_i in terms of the generators for I yields a canonical comparison map from $\mathbf{F}$ to $\mathbf{K}$. This leads, for example, to certain interesting bounds on the ranks of the free modules in $\mathbf{F}$ (Proposition 1.5), and to a good grip on the linkage of I. If the length of $\mathbf{F}$ is $\leqslant 3$, then such an associative algebra structure always exists, so we can easily apply these ideas to the case where I is Gorenstein of codimension 3.

It is known [KUNZ] that the ideals which are linked to Gorenstein ideals of codimension n are the "almost complete intersections"—that is, perfect ideals of codimension g which can be generated by $g+1$ elements. Thus we are able to deduce, in section 5, a structure theorem for almost complete intersections of codimension 3. As an application, we partially solve a problem which may be stated geometrically as follows: When can a Macaulay subvariety of codimension 3 be obtained as a hypersurface section of one of codimension 2? With codimension 1 less, the answer is simple: a Macaulay subvariety of codimension 1 is a hypersurface, so a hypersurface section of it of codimension 2 is a complete intersection. But in our case the situation is more complicated. Based on our structure theorem, we are able to settle the matter for almost complete intersections.

In section 6 we use our structure theorems to exhibit the generic examples, and some specific examples, of Gorenstein ideals and almost complete intersections of codimension 3.

It seems natural to regard the structure theory outlined above as the first steps in a program that might eventually expose the structure of all perfect ideals of codimension 3. Define the *linkage class* of a perfect ideal I to be the set of all ideals which can be obtained from I by iterating the linkage procedure. The idea would be to describe a set of representatives for the linkage classes of ideals of codimension 3, and then to describe inductively how to build up the whole of the linkage classes by keeping hold of the algebra structures on resolutions. In codimension 2, this process works splendidly: Hilbert's theorem on perfect ideals of codimension 2 easily implies that every ideal is in the linkage class of a complete intersection, and the linkage process can be made quite explicit ([A-N], [P-S]). In codimension 3, Watanabe has shown [WAT] that every Gorenstein ideal is in the linkage class of a complete intersection; in fact, a Gorenstein ideal minimally generated by n elements is linked to an almost complete intersection which is in turn linked to a Gorenstein ideal with $n-2$ generators. (He uses this to show, independently of our proof, that n must be odd.) We do not know whether there are any perfect ideals of codimension 3 that are not in the linkage class of an almost complete intersection; but the square of the maximal ideal in a 3-dimensional regular local ring (which is always determinantal) seems to be a good candidate.

Note added in proof. L. Avramov has communicated to us the existence of a counterexample to our conjecture 1.2 in an appendix by V. Khimich to "On the Hopf Algebra of a local ring" (L. Avramov: Math. USSR Izv. **8** (1974), 259–284). However, the following weaker conjecture, which seems to remain open, would have substantially the same applications:

Conjecture 1.2′. Let I be an ideal in the local ring R, and suppose $x_1,\dots,x_g$ is an R-sequence in I. Then the minimal R-free resolution $\mathbf{P}$ of R/I possesses the structure of a differential graded module over the Koszul complex $\mathbf{K}$ on $x_1,\dots,x_g$ in such a way that the comparison map $\mathbf{K}\to\mathbf{P}$, extending the projection $R/(x_1,\dots,x_g)\to R/I$, may be taken to be a map of K-modules. (Here the underlying algebra of the Koszul complex is as usual the exterior algebra.)

We would like to extend our thanks to M. Artin, who first suggested to us that some sort of multiplicative structure might be the key to the self-duality of resolutions of Gorenstein ideals, and to S. MacLane, who clarified for us the material of section 1. We also profited from a number of discussions of the material of this paper with J. Herzog and E. G. Evans. Evans, in particular, helped us substantially with the details of the application to almost complete intersections.

Finally, we record some of the terminology that we will use throughout this paper. Let R be a local ring with maximal ideal J. The *grade* of a proper ideal I in R is the length of the maximal R-sequence contained in I (Bourbaki calls this the I-depth of R). We define grade $(R)=\infty$. In a Macaulay ring, grade I coincides with the codimension (=height=rank) of I (grade is the right replacement for codimension in most homological questions over non-Macaulay rings). The ideal I is *perfect* if grade $I=$p.d. (R/I), the projective dimension of R/I. If I is a perfect ideal of grade G, then the *type* of I is the dimension of the R/J-vectorspace $\mathrm{Ext}_R^g(R/J,R/I)$; this is in fact an invariant of the ring R/I [Her-Kunz]. An ideal of grade g is a *complete intersection* if it can be generated by g elements. Such an ideal is automatically perfect of type 1. More generally, we say that a perfect ideal is Gorenstein if it has type 1. A perfect ideal of grade g is an *almost complete intersection* if it can be generated by $g+1$ elements.

For any R module G, we write $G^*=\mathrm{Hom}(G,R)$. If $f:F\to G$ is a map of R-modules, we define $I_n(f)$ to be the image of the map $\overset{n}{\Lambda} F\otimes(\overset{n}{\Lambda} G)^*\to R$ induced by the map $\overset{n}{\Lambda} f\colon \overset{n}{\Lambda} F\to \overset{n}{\Lambda} G$; it is the "ideal of $n\times n$ minors" of f. The rank of f is by definition the largest integer r such that $\overset{r}{\Lambda} f\neq 0$.

1. Algebra Structures on Resolutions. Let R be a commutative ring, and let

$$\mathbf{P}\colon \qquad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} R$$

be a projective resolution of a cyclic R-module R/I, where $I=\mathrm{Im}\,(d_1)$. It is well-known that $\mathbf{P}\otimes_R\mathbf{P}$ has the structure of a complex, and that any comparison map $\mathbf{P}\otimes\mathbf{P}\to\mathbf{P}$ covering the natural map $R/I\otimes R/I\to R/I$ makes $\mathbf{P}$ into a homotopy-associative, homotopy-commutative differential graded algebra. (This algebra structure induces the associative, commutative algebra on the homology $\mathrm{Tor}_*(R/I,R/I)$.) But one can do slightly better: one can make $\mathbf{P}$ commutative "on the nose." It is precisely this improvement which will give us our results on Gorenstein ideals.

PROPOSITION 1.1. *The resolution of a cyclic module may be given the structure of a commutative, homotopy-associative differential graded algebra.*

Proof. Let $\mathbf{P}$ be the resolution of R/I, as above. We define the symmetric square $S_2(\mathbf{P})$ to be

$$S_2(\mathbf{P})=(\mathbf{P}\otimes\mathbf{P})/M$$

where M is the graded submodule of $\mathbf{P}\otimes\mathbf{P}$ generated by

$$\left\{ f\otimes g-(-1)^{(\deg f)(\deg g)}g\otimes f \mid f,g \quad \text{homogeneous elements of} \quad \mathbf{P}\right\}.$$

Of course $\mathbf{P}\otimes\mathbf{P}$ is a complex with differential

$$d(f\otimes g)=df\otimes g+(-1)^{\deg f}f\otimes dg.$$

It is easy to check that $d(M)\subseteq M$, so $S_2(\mathbf{P})$ inherits the structure of a complex from $\mathbf{P}\otimes\mathbf{P}$. Moreover, each component of $S_2(\mathbf{P})$ is a projective R-module since for each k we have

$$S_2(\mathbf{P})_k \cong \left[\sum_{\substack{i+j=k\\ i<j}} P_i\otimes P_j\right] + T_k$$

where

$$T_k=\begin{cases} 0 \text{ if } k \text{ is odd.}\\ \overset{2}{\Lambda} P_{k/2} \text{ if } k \text{ is of the form } 4n+2.\\ S_2(P_{k/2}) \text{ if } k \text{ is of the form } 4n.\end{cases}$$

Thus $S_2(\mathbf{P})$ is a complex of projective R-modules, and is isomorphic to $\mathbf{P}$ in degrees 0 and 1. By the comparison theorem for projective complexes, there exists a map of complexes $S_2(\mathbf{P})\xrightarrow{\Phi}\mathbf{P}$ which extends the natural isomorphism $R/I\otimes R/I\to R/I$, and which is the identity on the subcomplex $R\otimes\mathbf{P}\subset S_2(\mathbf{P})$. If we write $f\cdot g=\Phi(\overline{f\otimes g})$, where $f,\ g\in\mathbf{P}$, and $\overline{f\otimes g}$ is the image, in $S_2(\mathbf{P})$, of $f\otimes g$, then the conditions that the map must satisfy are easily seen to be equivalent to the statement that $\cdot$ makes $\mathbf{P}$ into a (not necessarily associative) commutative differential graded algebra, with the differential d, and with structure map $R\to\mathbf{P}$ given by the isomorphism $R\cong\mathbf{P}_0$. (The commutativity follows from our use of $S_2(\mathbf{P})$ in place of $\mathbf{P}\otimes\mathbf{P}$). It follows from the uniqueness up to homotopy of comparison maps that this algebra structure is homotopy-associative.

Remark. For the general theory one really should use the "divided power square" $D_2(\mathbf{P})$ in place of the symmetric square $S_2(\mathbf{P})$. However, our primary concern in this paper is with resolutions of length at most 3, and the first non-trivial divided powers would occur in degree 4; so we have chosen to avoid this complication.

The algebra structure defined above depends on a great many choices, and it seems natural to hope that these could be made so that the algebra structure is actually associative. It is suggestive that this can be done if: 1) I is generated by an R-sequence (in which case $\mathbf{P}$ is the exterior algebra on P_1); 2) If R is a local ring and I is its maximal ideal and $\mathbf{P}$ is the minimal free resolution [GUL]; 3) If $P_i = 0$ for $i \geqslant 4$ (Prop. 1.3, below); 4) In a few scattered special cases, for example if I is the ideal of $(n-1)\times(n-1)$ minors of an $n \times n$ matrix, and I has grade 4 (the resolution $\mathbf{P}$ is given in [G-N]; the algebra structure was computed by J. Herzog (unpublished)). Thus we are led to

Conjecture 1.2. If R is a local ring, then the minimal free resolution of a cyclic R-module possesses the structure of a commutative, associative, differential graded algebra.

As mentioned above, the piece of this which is most relevant to the work of this paper is the following proposition, which shows that if $P_i = 0$ for $i \geqslant 4$, then associativity follows from homotopy-associativity.

PROPOSITION 1.3. *With the notation as above, if $P_n = 0$ for $n \geqslant 4$, then $\mathbf{P}$ is a commutative associative differential graded algebra.*

Proof. Give $\mathbf{P}$ the algebra structure specified in Prop. 1.1. There is only one associativity to be checked: if $a, b, c \in P_1$, we want $(a \cdot b)\cdot c = a \cdot (b \cdot c)$. Since d_3 is a monomorphism, it suffices to check this after applying d_3; and both sides then become

$$d_1(a)\cdot b \cdot c - a \cdot d_1(b) \cdot c + a \cdot b \cdot d_1(c),$$

since d is a derivation and $d_1(a)$, $d_1(b)$, $d_1(c) \in R$ are scalars.

One consequence of conjecture 1.2 which might be easier to verify, is given in the next proposition. It was conjectured independently by J. Herzog.

PROPOSITION 1.4. *Let R be a domain, let $I \subset R$ be an ideal, and suppose that $\mathbf{P}$ as above is a free resolution of R/I. Suppose that*

$$x_1, \ldots, x_g \in I$$

form an R-sequence and let

$$\mathbf{K}: 0 \to \overset{g}{\Lambda} R^g \to \overset{g-1}{\Lambda} R^g \to \cdots \to R^g \to R$$

be the Koszul complex resolving $R/(x_1, \ldots, x_g)$. Then if $\mathbf{P}$ has the structure of an associative commutative differential graded algebra, there is a monomorphism $\mathbf{K} \to \mathbf{P}$. In particular,

$$\operatorname{rank} P_k \geqslant \binom{g}{k}$$

$$\operatorname{rank} d_k \geqslant \binom{g-1}{k-1}$$

Remark. P. Hackman has conjectured [HAC] that, if p.d. $R/I=n<\infty$, then rank $d_k \geqslant k$ for $k<n$; he has proven this, for $k=1,2$ and $n-1$. Since $\binom{n}{k} \geqslant k$ for $k<n$, our conjecture implies his for perfect ideals I, at least if R is a domain. (It ought to be possible to prove the Proposition above without this hypothesis.)

Proof. Extending $x_1 \dots x_g$ to a maximal R-sequence if necessary, we may assume that grade $(I)=g$. **K**, with the structure of the exterior algebra, is a free, commutative associative differential graded algebra. Thus, under the hypothesis that **P** possesses an associative commutative algebra structure, the natural map $R/(x_1,\dots,x_g)\to R/I$ induces an algebra homomorphism $\mathbf{K}\xrightarrow{s}\mathbf{P}$. We claim that this must be a monomorphism. First we show that $s_g: R\cong \overset{g}{\Lambda} R^g \to P_g$ is a monomorphism. Since R is a domain and P_g is free, it suffices to show that $s_g \neq 0$. But the short exact sequence

$$0\to I/(x_1,\dots,x_g)\to R/(x_1,\dots,x_g)\to R/I\to 0$$

induces a long exact sequence

$$\cdots \to \operatorname{Ext}_R^{g-1}(I/(x_1,\dots,x_g),R)\to \operatorname{Ext}_R^g(R/I,R)\xrightarrow{s_g^*}\operatorname{Ext}_R^g(R/(x_1,\dots,x_g),R)\to\cdots$$

where s^* is the map induced by s on cohomology. Since $(x_1,\dots,x_g)$ annihilates $I/(x_1,\dots,x_g)$, we have

$$\operatorname{Ext}_R^{g-1}(I/(x_1,\dots,x_g),R)=0,$$

while

$$\operatorname{Ext}_R^g(R/I,R)\cong \operatorname{Hom}(R/I,R/(x_1,\dots,x_g))\neq 0,$$

and consequently $s_g^*\neq 0$; so $s_g\neq 0$.

Since s is a map of algebras, its kernel must be an ideal in ΛR^g. But any nonzero ideal of ΛR^g meets $\overset{g}{\Lambda} R^g$ nontrivially; thus $\ker s=0$ as claimed.

The second half of the proposition follows at once from the first half.

We now specialize and suppose that R is a local ring, that I is a Gorenstein ideal of R of grade g, and that

$$\mathbf{F}: 0\to F_g \xrightarrow{d_g} F_{g-1}\dots F_1 \xrightarrow{d_1} R$$

is a minimal free resolution of R/I, equipped with a commutative multiplication $\cdot$: $\mathbf{F}\otimes\mathbf{F}\to\mathbf{F}$. Since I is Gorenstein, we may make the identification $F_g=R$, so that, for each $k \leqslant g$, the map $\cdot : F_k \otimes F_{g-k} \to F_g = R$ induces a map

$$s_k : F_k \to F^*_{g-k}$$

THEOREM 1.5. *For each $k \leqslant g$, s_k is an isomorphism.*

Proof. We claim that the following diagram commutes up to sign

$$\begin{tikzcd}
\mathbf{F}: 0 \arrow[r] & F_g \arrow[r, "d_g"] \arrow[d, equal] & F_{g-1} \arrow[r] \arrow[d, "s_{g-1}"] & \dots \arrow[r] & F_2 \arrow[r, "d_2"] \arrow[d, "s_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "s_1"] & R \arrow[d, equal, "s_0"] \\
\mathbf{F}^*: 0 \arrow[r] & \mathrm{R} \arrow[r, "d_1^*"] & F_i^* \arrow[r] & \dots \arrow[r] & F^*_{g-2} \arrow[r] & F^*_{g-1} \arrow[r, "d_n^*"] & F^*_g .
\end{tikzcd}$$

For, if $a \in F_k$, $b \in F_{g-k+1}$, then $a \cdot b \in F_{g+1} = 0$, so

$$0 = d_{g+1}(a \cdot b) = d_k(a) \cdot b + (-1)^k a \cdot d_{g-k+1}(b),$$

or $d_k(a) \cdot b = \pm a \cdot d_{g-k+1}(b)$.
If, for $\alpha \in \mathbf{F}^*$ and $b \in \mathbf{F}$ we write $\langle \alpha, b \rangle \in R$ for the value of the linear functional α at b, we thus have

$$\begin{aligned}
\langle s_{k-1} d_k(a), \quad b \rangle &= d_k(a) \cdot b \\
&= \pm a \cdot d_{g-k+1}(b) \\
&= \langle s_k(a), \quad d_{g-k+1}(b) \rangle \\
&= \langle d^*_{g-k+1} s_k(a), \quad b \rangle.
\end{aligned}$$

Thus $s_{k-1} d_k = \pm d^*_{g-k+1} s_k$, as required.

Now, since I is Gorenstein, both $\mathbf{F}$ and $\mathbf{F}^*$ are minimal free resolutions of R/I and it follows that any map of complexes $\mathbf{F} \to \mathbf{F}^*$ which extends an isomorphism in degree 0 must be an isomorphism. Thus each s_k is an isomorphism.

Remark. Let K be the residue field of R. The multiplication we have defined on $\mathbf{F}$ induces the usual algebra structure on $\mathrm{Tor}^R_*(R/I, K)$. Thus Theorem 1 also yields the result of $[A-G]$ which states that if R is regular, and if R/I is Gorenstein, then $\mathrm{Tor}^R_*(R/I, K)$ is a Poincare algebra, that is, the pairings

$$\mathrm{Tor}^R_k(R/I, K) \otimes \mathrm{Tor}^R_{g-k}(R/I, K) \to \mathrm{Tor}^R_g(R/I, K)$$

are perfect (where $g = \text{grade } I = pd_R R/I$). The result of [AVR] can also be deduced from Theorem 1 in this way. (If R is regular, our theorem can be deduced from that of [A–G], via Nakayama's lemma.)

2. The statement of the Structure Theorem, and Tools from Multilinear Algebra. We first review some terminology. If R is a commutative ring, and F a finitely generated free R-module, then a map $f: F^* \to F$ is said to be alternating if, with respect to some (and therefore every) basis and dual basis of F and F^*, the matrix of f is skew symmetric, and all its diagonal entries are 0. More invariantly: Let $\varphi \in F \otimes F$ be the element corresponding to f under the isomorphism

$$\mathrm{Hom}(F^*, F) \cong (F^*)^* \otimes F \cong F \otimes F,$$

and identify $\overset{2}{\Lambda} F$ with the submodule of $F \otimes F$ that is the kernel of the natural map

$$F \otimes F \to S_2(F).$$

Then f is alternating if and only if $\varphi \in \overset{2}{\Lambda} F$.

If F has even rank, and $f: F^* \to F$ is alternating, then it turns out that $\det(f)$ is the square of a polynomial function of the entries of a matrix for f, called the pfaffian of f. We will write $\det(f) = (\mathrm{Pf}(f))^2$. (We shall shortly prove this, and the other facts about pfaffians that we need.)

More generally, if F has any rank, and G is a free summand of F of even rank n, with projection $\pi: F \to G$, then the composite

$$g: G^* \xrightarrow{\pi^*} F^* \xrightarrow{f} F \xrightarrow{\pi} G$$

is an alternating map, represented in $\overset{2}{\Lambda} G$ by $\overset{2}{\Lambda} \pi(\varphi)$, and we say that $\mathrm{Pf}(g)$ is a pfaffian of order n of f (it is well-defined up to the square of a unit of R). In analogy with $I_n(f)$ defined at the end of the introduction, we define $\mathrm{Pf}_n(f)$ to be the ideal generated by all the n^{th}-order pfaffians of f. (A finite set of generators for $\mathrm{Pf}_n(f)$ may be obtained by fixing a basis of F, and considering only the summmands G generated by subsets of that basis.)

We are now ready to state the main result of this paper, which describes the structure of Gorenstein ideals of grade 3 in a local ring.

THEOREM 2.1. *Let R be a noetherian local ring with maximal ideal J.*

1) *Let $n \geq 3$ be an odd integer, and let F be a free R-module of rank n. Let $f: F^* \to F$ be an alternating map whose image is contained in JF. Suppose*

$\mathrm{Pf}_{n-1}(f)$ *has grade* 3. *Then* $\mathrm{Pf}_{n-1}(f)$ *is a Gorenstein ideal, minimally generated by n elements.*

2) *Every Gorenstein ideal of grade* 3 *arises as in* 1).

COROLLARY 2.2. *The minimal number of generators of a Gorenstein ideal of grade* 3 *is odd.*

Proof. This follows from part 2) of the theorem, since only odd n are considered in part 1).

Remarks. 1) In the proof of part 1), we will construct a complex whose existence shows that if F is a free module of odd rank n, and $f: F^* \to F$ is an alternating map of rank $n-1$, then

$$\operatorname{grade}(\mathrm{Pf}_{n-1}(f)) \leqslant 3.$$

2) A further question that can be raised in connection with Theorem 2.1 is the uniqueness question: what can be said about two maps f and f', both satisfying the conditions of part 1) of the theorem, and having

$$\mathrm{Pf}_{n-1}(f) = \mathrm{Pf}_{n-1}(f')?$$

In case R is complete and 2 is a unit in R, the most desirable conclusion holds: there exists an automorphism $a: F \to F$ such that

$$f' = afa^*.$$

If R is not complete, we have not been able to show the existence of such an a; but neither have we been able to find a counter-example.

In order to handle pfaffians succinctly we need some results from multilinear algebra which we will now develop. The proof of Theorem 2.1 itself will be postponed to the next section. Basic to what we do are the facts that ΛF and ΛF^* are modules over one another, and that they are divided power algebras. These things are reviewed in the appendix at the end of this paper, whose results we quote as $A1$, $A2, \ldots$. (A different treatment of pfaffians, which is unfortunately not very easily adaptable to our purposes, may be found in [HEY].)

First we must connect the divided powers of φ with the pfaffians and minors of the map f. To do this, we use

LEMMA 2.3. 1) *Suppose F is a free module of even rank $2m$, and let $f: F^* \to F$ be an alternating map represented by $\varphi \in \overset{2}{\Lambda} F$. Then the element*

$\overset{2m}{\Lambda} f \in \operatorname{Hom}\left(\overset{2m}{\Lambda} F^*, \overset{2m}{\Lambda} F\right) \approx \overset{2m}{\Lambda} F \otimes \overset{2m}{\Lambda} F$ *is* $(-1)^m \varphi^{(m)} \varphi^{(m)} \otimes \varphi^{(m)}$, *where* $\varphi^{(m)}$ *is the* m*th divided power of* φ. *Thus, if we choose an identification of* $\overset{2m}{\Lambda} F$ *with* R, *and let* $\mathrm{Pf}_{2m}(f) = \varphi^{(m)} \in R$, *we have*

$$\det(f) = (\mathrm{Pf}_{2m}(f))^2.$$

2) *More generally, if* F *is any free module of finite rank, and* $f: F^* \to F$ *is an alternating map represented by* $\varphi \in \overset{2}{\Lambda} F$, *then for every element* $a \in \overset{t}{\Lambda} F^*$ *we have*

$$(*) \quad \overset{t}{\Lambda} f(a) = \sum_{t \geq 0} (-1)^i \varphi^{(i)}(a)(\varphi^{(t-i)}).$$

Proof. Part 1) follows from part 2). For observe first of all that $\varphi^{(i)}$ or $\varphi^{(2m-i)}$ must be 0 unless $i = m$. Also $\varphi^{(m)}(a)$ is by definition $(-1)^{2m(2m-1)/2}\langle \varphi^{(m)}, a\rangle = (-1)^m \langle \varphi^{(m)}, a\rangle$ where we write $\langle\ ,\ \rangle$ for the action of $(\Lambda F^*)^*$ on ΛF^* and identify $\varphi^{(m)} \in \overset{2m}{\Lambda} F$ with its image under the canonical map $\overset{2m}{\Lambda} F \to \overset{2m}{\Lambda} F^{**} \to \left(\overset{2m}{\Lambda} F^*\right)^*$, as explained in the appendix. Thus (*) becomes

$$\overset{2m}{\Lambda} f(a) = \langle \varphi^{(m)}, a\rangle \varphi^{(m)}$$

and, letting a be a generator of $\overset{2m}{\Lambda} F$ we get 1).

To prove part 2), we use induction on t. For $t = 0$, the only non-vanishing term of (*) occurs where $i = 0$, so the formula is obvious since $\varphi^{(0)} = 1 \in \overset{0}{\Lambda} F = R$.

For $t = 1$, the formula becomes

$$f(a) = a(\varphi) - \varphi(a)(1).$$

Since $a \in F^*$ and $\varphi \in \overset{2}{\Lambda} F$, $\varphi(a) = 0$. Also, if $\{e_i\}$ is a basis for F, and if $\varphi = \sum_{i<j} f_{ij} e_i \hat{} e_j$, then

$$a(\varphi) = \sum_{i<j} f_{ij}(\langle a, e_i\rangle e_j - \langle a, e_j\rangle e_i) = f(a)$$

as required.

If now $t > 1$, it suffices, by the linearity of (*) in a, to prove (*) for elements of the form

$$a = b \wedge c, \quad b \in F^*, \quad c \in \overset{t-1}{\Lambda} F^*.$$

Since $\overset{t}{\Lambda} f(b \wedge c) = f(b) \overset{t-1}{\Lambda} f(c)$, what must be shown is that

$$\sum_{i \geqslant 0} (-1)^i \varphi^{(i)}(b \wedge c)(\varphi^{(t-i)}) = b(\varphi) \wedge \sum_{i \geqslant 0} (-1)^i \varphi^{(i)}(c)(\varphi^{(t-i-1)}).$$

Applying A.3 (see appendix) and remembering that $\varphi^{(i)}$ has even degree, we have

$$(**) \quad \sum_{i \geqslant 0} (-1)^i \varphi^{(i)}(b \wedge c)(\varphi^{(t-i)})$$

$$= \sum_{i \geqslant 0} (-1)^i \left[b \wedge \varphi^{(i)}(c) \right](\varphi^{(t-i)}) - \sum_{i > 1} (-1)^i \left[b\left(\varphi^{(i)}(c)\right) \right](\varphi^{(t-i)})$$

where the last summations is taken over $i > 1$ since $b(\varphi^{(0)}) = 0$. Changing the index of summation, we have

$$\sum_{i > 1} (-1)^i \left[b(\varphi^{(i)})(c) \right](\varphi^{(t-i)}) = \sum_{i \geqslant 0} (-1)^{i+1} \left[b(\varphi^{(i+1)})(c) \right](\varphi^{(t-i-1)}).$$

By A.4, $b(\varphi^{(i+1)}) = b(\varphi) \wedge \varphi^{(i)}$ so that (**) becomes:

$$\sum_{i \geqslant 0} (-1)^i \varphi^{(i)}(b \wedge c)(\varphi^{(t-i)})$$

$$= \sum_{i \geqslant 0} (-1)^i \left\{ \left[b \wedge \varphi^{(i)}(c) \right](\varphi^{(t-i)}) + \left(b(\varphi) \wedge \varphi^{(i)}(c) \right)(\varphi^{(t-i-1)}) \right\}.$$

Since $b \wedge \varphi^{(i)}(c) = (-1)^{t-2i-1} \varphi^{(i)}(c) \wedge b = (-1)^{t-1} \varphi^{(i)}(c) \wedge b$, and since $(\varphi^{(i)}(c) \wedge b)(\varphi^{(t-i)}) = \varphi^{(i)=\varphi(i)}(c)(b(\varphi^{(t-i)})) = \varphi^{(i)}(c)(b(\varphi) \wedge \varphi^{(t-i-1)})$, the above expression is:

$$\sum_{i \geqslant 0} (-1)^i \varphi^{(i)}(b \wedge c)(\varphi^{(t-i)})$$

$$= \sum_{i \geqslant 0} (-1)^i \left\{ (-1)^{t-1} \varphi^{(i)}(c)\left(b(\varphi) \wedge \varphi^{(t-i-1)}\right) + \left(b(\varphi) \wedge \varphi^{(i)}(c)(\varphi^{(t-i-1)}\right) \right\}.$$

However, since $b(\varphi)$ is of degree 1, we may apply A.3 to $\varphi^{(i)}(c)(b(\varphi)\wedge \varphi^{(t-i-1)})$ to get

$$(-1)^t\varphi^{(i)}(c)\big(b(\varphi)\wedge\varphi^{(t-i-1)}\big)$$

$$= -b(\varphi)\wedge\big(\varphi^{(i)}(c)(\varphi^{(t-i-1)}\big)+\big(b(\varphi)\wedge\varphi^{(i)}(c)(\varphi^{(t-i-1)}\big).$$

Thus, we finally get:

$$\sum(-1)^i\varphi^{(i)}(b\wedge c)(\varphi^{(t-i)})=\sum_{i\geqslant 0}(-1)^i b(\varphi)\wedge\big[\varphi^{(i)}(c)(\varphi^{(t-i-1)})\big]$$

$$=b(\varphi)\wedge\sum_{i\geqslant 0}(-1)^i\varphi^{(i)}(c)(\varphi^{(t-i-1)})$$

which is the result we wanted, and lemma 2.3 is proven.

In view of part 1 of the lemma, part 2 becomes more comprehensible. First, because $\varphi^{(i)}$ has degree $2i$ and $\varphi^{(i)}(a)$ has degree $t-2i$, we need only sum over $0<i<\frac{t}{2}$. Now choose a basis $e_1,\dots,e_n$ of F and a dual basis $\epsilon_1,\dots,\epsilon_n$ for F^*. Then the coefficient of $e_{i_1}\wedge\dots\wedge e_{i_t}$ in $\overset{t}{\Lambda} f(\epsilon_{j_1}\wedge\dots\wedge\epsilon_{j_t})$ is the minor of f involving columns $i_1,\dots,i_t$ and rows $j_1,\dots,j_t$. Also, it follows from part 1) that the coordinates of $\varphi^{(i)}(\epsilon_{j_1}\wedge\dots\wedge\epsilon_{j_t})$ are the $2i$-order pfaffians of the submatrix of f obtained from the rows and columns numbered $j_1,\dots,j_t$. Thus (*) is a formula expressing the $t\times t$ minors of f in terms of the pfaffians of f of order $<t$ (if t is even) or $<t-1$ (if t is odd).

In order to exploit Lemma 2.3 fully, we need to know something about the way in which pfaffians of order $2m$ of f can be expressed in terms of the pfaffians of order $2m-2$. This information is contained in the following formula:

LEMMA 2.4. *Suppose that $\varphi\in\overset{2}{\Lambda}F$, and $a\in\overset{2m+1}{\Lambda}F^*$. Then*

$$(***)\quad a\big(\varphi^{(m+1)}\big)=\varphi^{(m)}(a)(\varphi).$$

Proof. The case $m=0$ being trivial, we use induction on m. Since the lemma represents an identity, we need only prove it when F is a free module over a polynomial ring over the integers. Thus, it suffices to prove that

$$(m+1)a\big(\varphi^{(m+1)}\big)=(m+1)\varphi^{(m)}(a)(\varphi).$$

But $(m+1)\varphi^{(m+1)}=\varphi\wedge\varphi^{(m)}$, so $(m+1)a(\varphi^{(m+1)})=a(\varphi\wedge\varphi^{(m)})$. If $\Delta(a)=\Sigma a_i\otimes a_i'$, the measuring identity A.2 gives:

$$a(\varphi\wedge\varphi^{(m)})=\sum a_i(\varphi)\wedge a_i'(\varphi^{(m)}).$$

Now if $\deg a_i>2$, $a_i(\varphi)=0$, while if $\deg a_i=0$ then $\deg a_i'=2m+1$ so that $a_i'(\varphi^{(m)})=0$. Thus

$$a(\varphi\wedge\varphi^{(m)})=\sum_{1\leqslant \deg a_i\leqslant 2} a_i(\varphi)\wedge a_i'(\varphi^{(m)})$$

If $\deg a_i=1$, then $\deg a_i'=2m$, so $a_i'=2m$, so $a_i'(\varphi^{(m)})=\varphi^{(m)}(a_i')\in R$ and

$$\begin{aligned}\sum_{\deg a_i=1} a_i(\varphi)\wedge a_i'(\varphi^{(m)})&=\sum_{\deg a_i=1}\varphi^{(m)}(a_i')a_i(\varphi)\\&=\big(\varphi^{(m)}(a)\big)(\varphi).\end{aligned}$$

On the other hand if $\deg a_i=2$, then $\deg a_i'=2m-1$, so by induction $a_i'(\varphi^{(m)})=\varphi^{(m-1)}(a_i')(\varphi)$. In this case $a_i(\varphi)=\varphi(a_i)\in R$, and we have

$$\begin{aligned}&\sum_{\deg} a_i=2a_i(\varphi)\wedge a_i'(\varphi^{(m)})\\&=\sum_{\deg} a_i=2\varphi(a_i)\big(\varphi^{(m-1)}(a_1')(\varphi)\big)\\&=\sum_{\deg} a_i=2\varphi^{(m-1)}(\varphi(a_i)a_i')(\varphi)\\&=\varphi^{(m-1)}(\varphi(a))(\varphi)\\&=(\varphi^{(m-1)}\wedge\varphi)(a)(\varphi)\\&=m\varphi^{(m)}(a)(\varphi)\end{aligned}$$

Putting these two pieces together we get

$$(m+1)a(\varphi^{(m+1)})=(m+1)\varphi^{(m)}(a)(\varphi)$$

as required.

Lemma 2.4 may perhaps be elucidated by an illustration. Suppose $\operatorname{rank} F=2m+2$. Let $e_1,\ldots,e_{2m+2}$ and $\epsilon_1,\ldots,\epsilon_{2m+2}$ be dual bases for F and F^*, and identify f with its matrix with respect to these bases, so that we have:

$$\varphi = \sum_{i<j} f_{ij} e_i \wedge e_j, \quad \varphi^{(m+1)} = \mathrm{Pf}(f) e_1 \wedge \cdots \wedge e_{2m+2}.$$

Let $a = \epsilon_1 \wedge \cdots \wedge \epsilon_{2m+1}$. With this notation, the left-hand side of (***) becomes $\pm \mathrm{Pf}(f) e_{2m+2}$, while the right-hand side of (***) is $\sum_{i \leqslant 2m+1} \pm p_i f_{ij} e_j$, where we have written p_i for the pfaffian of the matrix obtained from f by omitting the i^{th} and $(2m+2)^{\text{th}}$ rows and columns. Thus Lemma 2.4 says, in this case, that

$$\mathrm{Pf}(f) = \sum_{i \leqslant 2m+1} \pm p_i f_{i,2m+2},$$

while various other linear combinations of pfaffians and elements f_{ij} are 0.

As suggested by this illustration, Lemma 2.4 has the following useful corollary:

COROLLARY 2.5. *Let $f: F^* \to F$ be an alternating map. For each integer $m \geqslant 1$ we have*

$$\mathrm{Pf}_{2m+2}(f) \subseteq \mathrm{Pf}_{2m}(f)$$

Proof. The generators of $\mathrm{Pf}_{2m+2}(f)$ can be made to appear as coefficients on the left-hand side of (***) by letting a run over a basis for $\overset{2m+1}{\Lambda} F$, while the right-hand side has linear combinations of pfaffians of order $2m$ as coefficients.

If $f: F^* \to F$ is an alternating map, we can use Lemma 2.3 and Corollary 2.5 to relate the ideals $\mathrm{Pf}_{2m}(f)$ to the ideals $I_n(f)$, which are easier to interpret homologically (see Theorem 3.1). We will write $\mathrm{Rad}\, I$ for the radical of an ideal I.

COROLLARY 2.6. *Let $f: F^* \to F$ be an alternating map, and let $m \geqslant 1$ be an integer. Then*

1) $I_{2m}(f) \subseteq \mathrm{Pf}_{2m}(f) \subseteq \mathrm{Rad}(I_{2m}(f))$
2) $I_{2m-1}(f) \subseteq \mathrm{Pf}_{2m}(f)$
3) If rank $(f) = 2m-1$ is odd, then $I_{2m-1}(f)$ is nilpotent. If $\mathrm{rank}\, F = 2m-1$, then $\det f = 0$.

Proof. Consider the identity (*). For any t, $I_t(f)$ is generated by the coordinates of $\overset{t}{\Lambda} f(a)$, for various a, while the coordinates of $\varphi^{(i)}$ and $\varphi^{(t-i)}$ involve pfaffians of order $2i$ and $2(t-i)$ respectively. Thus if $t = 2m$ or $2m-1$, each term on the right-hand side of (*) is contained in $\mathrm{Pf}(f)$ for some $n \geqslant m$. By

Corollary 2.4, $\mathrm{Pf}_{2n}(f) \subseteq \mathrm{Pf}_{2m}(f)$, so the first inequality of Part 1), and all of Part 2) are established.

To prove the second inequality of Part 1), we note that by Lemma 2.3 1), the square of each pfaffian of f of order $2m$ is a minor of order $2m$. If $\operatorname{rank} F = r$, then $\mathrm{Pf}_{2m}(f)$ is generated by $\binom{r}{2m}$ pfaffians of order $2m$, so

$$(\mathrm{Pf}_{2m}(f))^{\binom{r}{2m}+1} \subseteq I_{2m}(f),$$

which finishes the proof of part 1).

The first statement of part 3) follows easily from Parts 1) and 2), while the second statement can be deduced from (*) directly: If $\operatorname{rank} F = 2m-1$, then $\varphi^{(i)} = 0$ for $i \geqslant m$; but one of i and $2m-1-i$ must be $\geqslant m$.

3. The proof of the Main Theorem, and a Formula for Hilbert Functions

Proof of part 1 of Theorem 2.1. We will prove that $\mathrm{Pf}_{n-1}(f)$ is a Gorenstein ideal under the hypotheses of the theorem, by describing a minimal free resolution for $R/\mathrm{Pf}_{n-1}(f)$. Let $\varphi \in \overset{2}{\Lambda} F$ be the element corresponding to $f: F^* \to F$, so that, by Lemma 2.3, we have

$$f(a) = a(\varphi) \quad \text{for} \quad a \in F^*.$$

Let $e \in \overset{n}{\Lambda} F^*$ be a generator (e is often called an *orientation* of F), and let $g: F \to R$ be the map defined by

$$g(b) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(b) \quad \text{for} \quad b \in F.$$

We claim that

$$\operatorname{Im} g = \mathrm{Pf}_{n-1}(f).$$

For, by A3,3), $\varphi^{\left(\frac{n-1}{2}\right)}(e)(b) = b(e)(\varphi^{\left(\frac{n-1}{2}\right)})$, and as b ranges over a basis for F, $b(e)$ ranges over a basis of $\overset{n-1}{\Lambda} F$. Thus $\operatorname{Im}(g)$ is the ideal generated by the coordinates of $\varphi^{\left(\frac{n-1}{2}\right)}$—that is, $\operatorname{Im}(g) = \mathrm{Pf}_{n-1}(f)$.

We will now show that, under the hypothesis of the theorem, the sequence

$$\mathbf{E}: 0 \to R \xrightarrow{g^*} F^* \xrightarrow{f} F \xrightarrow{g} R$$

is a minimal free resolution of $R/\mathrm{Pf}_{n-1}(f)$.

First $\mathbf{E}$ is a complex. In view of the fact that $f^* = -f$, it suffices to show $gf = 0$. But, for $a \in F^*$,

$$gf(a) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(a(\varphi)) = \left(\varphi^{\left(\frac{n-1}{2}\right)}(e) \wedge a\right)(\varphi)$$

since ΛF^* is a ΛF-module. By the commutative law, this is

$$= \left(a \wedge \varphi^{\left(\frac{n-1}{2}\right)}(e)\right)(\varphi)$$
$$= a\left(\varphi^{\left(\frac{n-1}{2}\right)}(e)(\varphi)\right).$$

By lemma 2.3,

$$\varphi^{\left(\frac{n-1}{2}\right)}(e)(\varphi) = e\left(\varphi^{\left(\frac{n-1}{2}\right)}\right) = 0,$$

since $\varphi^{\left(\frac{n-1}{2}\right)} \in \overset{n+1}{\Lambda} F = 0$.

To show that $\mathbf{E}$ is exact, we employ the criterion of [B-E-1], which we now recapitulate. If $f: F \to G$ is a map of rank r we define $I(f) = I_r(f)$, the ideal of "$r \times r$ minors" of f.

THEOREM 3.1. *Let R be a noetherian ring, and let*

$$\mathbf{F}: O \to F_n \xrightarrow{f_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{f_1} F_0$$

be a complex of free R-modules. $\mathbf{F}$ is exact if and only if for each k,

1) $\operatorname{rank} f_k + \operatorname{rank} f_{k+1} = \operatorname{rank} F_k$
2) $\operatorname{grade} I(f_k) \geqslant k$ or $I(f_k) = R$.

Since we have already shown that $\mathbf{E}$ is a complex, we see from theorem 3.1 that it suffices to prove that

1) $\operatorname{rank} f + \operatorname{rank} g = n$
2) $\operatorname{grade} I(g)$ is $\geqslant 3$, and grade $I(f) \geqslant 2$ (note that $I(g) = I(g^*)$).

By Corollary 2.6, 1), $I_1(g) = \operatorname{Pf}_{n-1}(f)$ has the same radical as $I_{n-1}(f)$. By Corollary 2.6, 3), $\operatorname{rank} f = n-1$, and $I_{n-1}(f)$ has grade 3 so rank $g = 1$, and $I(g) = I_1(g)$ has grade 3 as well. This proves 1) and 2), and thus shows that $\mathbf{E}$ is exact.

Finally, $\mathbf{E}$ is a *minimal* free resolution because $f(F^*) \subset JF$, so $\operatorname{Pf}_{n-1}(f) = g(F) \subseteq JR$ and $g^*(R) \subseteq JF^*$. This concludes the proof of part 1 of Theorem 2.1.

Proof of part 2) of Theorem 2.1. Suppose that I is a Gorenstein ideal of grade 3, which is minimally generated by n elements, and let

$$\mathbf{F}: 0 \to F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R$$

be a minimal free resolution of R/I, so that $F_1 \cong R^n$. In section 1 we saw that $\mathbf{F}$ has the structure of a commutative (associative) differential graded algebra. By Theorem 1.5, the multiplication on this algebra gives a perfect pairing $F_2 \otimes F_1 \to F_3$. If we identify F_3 with R, this pairing yields an identification of F_2 with F_1^*. We will first show that, with respect to this identification, f_2 is alternating. Let $\{e_i\}$ and $\{\epsilon_i\}$ be dual bases for F_1 and $F_2 = F_1^*$ respectively, and write f_{ij} for the i,j^{th} element of a matrix of f_2 with respect to these bases. Thus, writing $\langle\ ,\ \rangle$ for the pairing between F_2 and F_1,

$$f_{ij} = \langle \epsilon_i, f_2 \epsilon_j \rangle = \epsilon_i \cdot f_2(\epsilon_j) \in R = F_3$$

where $\cdot$ is the product for the algebra structure on F. We must show that $f_{ij} = -f_{ji}$ and $f_{ii} = 0$. Applying f_3 to the above equation, and using the formula for the differentiation of a product, we obtain

$$\begin{aligned} f_3(f_{ij}) &= f_2(\epsilon_i) \cdot f_2(\epsilon_j) + \epsilon_i \cdot f_1 f_2(\epsilon_j) \\ &= f_2(\epsilon_i) \cdot f_2(\epsilon_j), \end{aligned}$$

since $f_1 f_2 = 0$. Since $\cdot$ satisfies the graded commutative law and $f_2(\epsilon_i)$ and $f_2(\epsilon_j)$ have degree 1, we have

$$f_3(f_{ij}) = -f_3(f_{ij}), \qquad f_3(f_{ii}) = 0.$$

Since f_3 is a monomorphism, this shows that f_2 is alternating.

We next show that f_2 satisfies the other conditions of part 1) of Theorem 2.1. By Theorem 3.1, we see that rank $f_2 = n-1$, and $I_{n-1}(f_2)$ has grade $\geqslant 2$. By Corollary 2.6, part 3), it follows that $n-1$ is even, so n is odd. Since $\mathbf{F}$ is a minimal resolution, $f_2(F_2) \subseteq JF_1$. To see that grade $\mathrm{Pf}_{n-1}(f_2) \geqslant 3$, we first note that by Corollary 2.6, 1), $\mathrm{Rad}(\mathrm{Pf}_{n-1}(f_2)) = \mathrm{Rad}(I_{n-1}(f_2))$, so it suffices to show that grade $I_{n-1}(f_2) \geqslant 3$. Since $I_{n-1}(f_2) = I(f_2)$, and since $I(f_1) = I_1(f_1) = I$ has grade 3 by hypothesis, the following Proposition will do this for us. (The proof is an easy localization argument, which may be found, for example, in [B-E-3, Theorem 2.1a].)

PROPOSITION 3.2. *Let $F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$ be an exact sequence of free R-modules. Then* $\mathrm{Rad}(I(f_2)) \geqslant \mathrm{Rad}(I(f_1))$.

This shows that f_2 satisfies all the conditions on the map f in part 1). Proceeding as in the proof of part 1), we can construct an exact sequence

$$\mathbf{E}: R \xrightarrow{g^*} F_1^* \xrightarrow{f_2} F_1 \xrightarrow{g} R.$$

Now under our identifications $F_3 = R$, $F_2 = F_1^*$, we have $f_3 = f_1^*$. Since both f_1^* and g^* are kernels of f_2, there is a unit $u \in R$ so that the diagram

$$\begin{tikzcd} R \arrow[d, "u"'] \arrow[r, "g^*"] & F_1^* \\ R \arrow[ur, "f_1^*"'] & \end{tikzcd}$$

is commutative, whence $ug = f_1$, and

$$\mathrm{Pf}_{n-1}(f_2) = \mathrm{Im}(g) = \mathrm{Im}(ug) = \mathrm{Im}(f_1) = I.$$

This concludes the proof.

Now suppose I is a homogeneous Gorenstein ideal of grade 3 in the ring $R = k[X_0 \dots X_m]$, where k is a field. Since, as is well-known, graded polynomial rings behave just like local rings for the purposes of free resolutions, we may use the explicit description of a free resolution for R/I contained in Theorem 2.1 to describe the Hilbert function of R/I.

Writing $R(m)$ for the free R-module on one generator of degree m, we may use the homogeneity of I to write a free resolution of R/I, as in the proof of Theorem 2.1, in the form

$$\mathbf{E}_{\mathrm{hom}}: 0 \to R(s) \xrightarrow{g^*} \sum_{i=1}^{n} R(p_i) \xrightarrow{f} \sum_{i=1}^{n} R(q_i) \xrightarrow{g} R(0).$$

Here the maps f and g are homogeneous of degree 0, and if we write g_i and f_{ij} for the forms which are the entries of the matrices of g and f, then

$$q_i = \deg g_i, \quad p_j = q_i + \deg f_{ij}, \quad s = p_i + q_i.$$

(Of course we may also have $f_{ij} = 0$ for certain ij, for instance when $i = j$.) If we define $r_i = p_i - q_i$, and use the fact that g_i is a certain Pfaffian of f, we get

$$\deg g_i = \frac{1}{2}(s - r_i), \quad \deg f_{ij} = \frac{1}{2}(r_i + r_j), \qquad s = \sum_1^n r_i.$$

It follows that the numbers r_i are either all even or all odd. Recalling that the

Hilbert function $H_I(t)$ is defined to be the dimension of the space of forms of degree t in R/I, and using Hilbert's original method of computing Hilbert functions [Hil], we get

PROPOSITION 3.3. *With the above notation, we have*

$$H_I(t)=\binom{m+t}{m}-\sum_{i=1}^{n}\binom{m+t-\frac{1}{2}(s-r_i)}{m}$$
$$+\sum_{i=1}^{n}\binom{m+t-\frac{1}{2}(s+r_i)}{m}-\binom{m+t-s}{m},$$

where $\binom{h}{k}$ *is the binomial coefficient, which we take to be* 0 *if* $h<k$.

The construction of Section 6 can be used to show that any set of integers $r_1 \leqslant \ldots \leqslant r_n$, either all even or all odd, with n odd, may actually be realized as coming from a Gorenstein ideal, as long as

$$m=\sum_1^n r_i, \quad m-r_n, \quad \text{and} \quad r_i-r_{i-1}, \qquad \text{for} \quad i\geqslant 3,$$

are strictly positive.

4. The Multiplication on the Minimal Resolution of a Gorenstein Ideal of Grade 3. We are now in a good position to continue our study, begun in section 1, of the algebra structure on the minimal free resolution of R/I, where I is a Gorenstein ideal of grade 3. If I requires n generators, then by Theorem 2.1 there is an alternating map $f:F^*\to F$ such that $I=\mathrm{Pf}_{n-1}(f)$ where $n=\operatorname{rank} F$. Recall that in section 3 the resolution of R/I was written as

$$\mathbf{E}:0\to R\xrightarrow{g^*}F^*\xrightarrow{f}F\xrightarrow{g}R$$

with $g(a)=\varphi^{\left(\frac{n-1}{2}\right)}(e)(a)$

$$f(b)=b(\varphi),$$

where $\varphi\in\overset{2}{\Lambda}F$ is the element corresponding to f, and e is a generator of $\overset{n}{\Lambda}F^*$.

We want to bring out the fact that the pairing $F\otimes F^*\to R$, which induces the duality of $\mathbf{E}$, is really part of the algebra structure on $\mathbf{E}$, and to emphasize the analogy of $\mathbf{E}$ with the Koszul complex (which $\mathbf{E}$ must be if $n=3$). To this end we will use the isomorphisms

$$\overset{k}{\Lambda}F\to\overset{n-k}{\Lambda}F^*:a\mapsto a(e),$$

to rewrite $\mathbf{E}$ as

$$\mathbf{E}': 0 \to \overset{n}{\Lambda} F \xrightarrow{g'} \overset{n-1}{\Lambda} F \xrightarrow{f'} F \xrightarrow{g} R$$

where f' is the composite

$$\overset{n-1}{\Lambda} F \overset{\approx}{\to} F^* \xrightarrow{-f} F$$

and g' is the composite

$$\overset{n}{\Lambda} F \overset{\approx}{\to} R \xrightarrow{g^*} F^* \overset{\approx}{\to} \overset{n-1}{\Lambda} F.$$

Note that these identifications ensure that the pairing on $\mathbf{E}'$, which gives the duality expressed by Theorem 1.5, is the multiplication $F \otimes \overset{n-1}{\Lambda} F \to \overset{n}{\Lambda} F$, $R \otimes \overset{n}{\Lambda} F \to \overset{n}{\Lambda} F$ in the exterior algebra. One advantage of these identifications is that the formulas for g, f', and g' have a particularly suggestive form:

$$g(a) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(a) \qquad \text{for} \quad a \in F,$$

$$f'(a) = \varphi(e)(a) \qquad \text{for} \quad a \in \overset{n-1}{\Lambda} F,$$

$$g'(a) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(a) \qquad \text{for} \quad a \in \overset{n}{\Lambda} F.$$

To check that these formulas are right, note first that the formula for $g(a)$ is the one given initially. Next, since we have defined f' as a compositie, we have

$$f'(a) = -a(e)(\varphi) = \varphi(e)(a) \qquad \text{(by A.3,3)}.$$

Finally, since the pairing $\mathbf{E}_1' \otimes \mathbf{E}_2' \to \mathbf{E}_3'$ is now given by exterior multiplication, it suffices to show that $b \wedge g'(a) = g(b) \wedge a$, i.e.

$$b \wedge \left[\varphi^{\left(\frac{n-1}{2}\right)}(e)(a)\right] = \left[\varphi^{\left(\frac{n-1}{2}\right)}(e)(b)\right] \wedge a$$

for all $b \in F$, $a \in \overset{n}{\Lambda} F$.

Now $\varphi^{\left(\frac{n-1}{2}\right)}(e)$ is an element of degree 1 in ΛF^*, and therefore acts as a deriviation on ΛF. Since $b \wedge a \in \overset{n+1}{\Lambda} F = 0$, we have

$$0 = \varphi^{\left(\frac{n-1}{2}\right)}(e)(b \wedge a) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(b) \wedge a - b \wedge \varphi^{\left(\frac{n-1}{2}\right)}(e)(a)$$

which is what we wanted to show.

We have constructed $\mathbf{E}'$ in such a way that, for some algebra structure, the product $F \otimes \overset{n-1}{\Lambda} F \to \overset{n}{\Lambda} F$ of elements of degree 1 with elements of degree 2 of $\mathbf{E}'$ is given by the usual exterior product. We will now make such an algebra structure explicit:

THEOREM 4.1. *With the notation as above, the following products make $\mathbf{E}'$ into an associative, commutative, graded differential algebra*:

$$F \otimes F \xrightarrow{m} \overset{2}{\Lambda} F \xrightarrow{\lambda} \overset{n-1}{\Lambda} F$$

$$F \otimes \overset{n-1}{\Lambda} F \xrightarrow{m} \overset{n}{\Lambda} F,$$

where m is the exterior multiplication, and $\lambda : \Lambda F \to \Lambda F$ is the map $\lambda(a) = a \wedge \varphi^{\left(\frac{n-3}{2}\right)}$. Moreover, with this algebra structure, the product of three elements of degree 1 is given by the composite map:

$$F \otimes F \otimes F \xrightarrow{m} \overset{3}{\Lambda} F \xrightarrow{\lambda} \overset{n}{\Lambda} F.$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
F \otimes \overset{n-1}{\Lambda} F \arrow[r, "1 \wedge f'"] \arrow[d, "m"'] \arrow[dr, "g \otimes 1"] & \overset{2}{\Lambda} F \arrow[d, "\lambda"] \arrow[dr, "k"] & \\
\overset{n}{\Lambda} F \arrow[r, "g'"] & \overset{n-1}{\Lambda} F \arrow[r, "f'"] & F
\end{tikzcd}$$

where $k : \Lambda F \to \Lambda F$ is the map defined by $k(a) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(a)$. Then $k : \overset{2}{\Lambda} F \to F$ is the differential of the Koszul complex associated to $g : F \to R$. In order to prove the first assertion of the theorem, we must show that $f'\lambda = k$, and that $g'm = g \otimes 1 - \lambda(1 \wedge f')$.

The first of these formulas says that, for $a \in \overset{2}{\Lambda} F$,

$$\varphi(e)\left[a \wedge \varphi\left(\frac{n-3}{2}\right)\right] = \varphi\left(\frac{n-1}{2}\right)(e)(a).$$

But

$$\begin{aligned}\varphi(e)\Big[a\wedge\varphi^{\left(\frac{n-3}{2}\right)}\Big] &= -\Big(\Big[a\wedge\varphi^{\left(\frac{n-3}{2}\right)}\Big](e)\Big)(\varphi) && \text{(by A.3,3)}\\ &= -\Big(\Big[\varphi^{\left(\frac{n-3}{2}\right)}\wedge a\Big](e)\Big)(\varphi)\\ &= -\Big(\varphi^{\left(\frac{n-3}{2}\right)}(a(e))(\varphi)\Big)\\ &= -a(e)\Big(\varphi^{\left(\frac{n-1}{2}\right)}\Big) && \text{(by lemma 2.4)}\\ &= \varphi^{\left(\frac{n-1}{2}\right)}(e)(a) && \text{(by A.3,3).}\end{aligned}$$

The second formula, $g'm = g\otimes 1 - \lambda(1\wedge f')$, says that for $a\in F$, $b\in \overset{n-1}{\Lambda} F$, we should have

$$\varphi^{\left(\frac{n-1}{2}\right)}(e)(a\wedge b) = \Big(\varphi^{\left(\frac{n-1}{2}\right)}(e)(a)\Big)b - a\wedge\varphi(e)(b)\wedge\varphi^{\left(\frac{n-3}{2}\right)}.$$

However, since $\varphi^{\left(\frac{n-1}{2}\right)}(e)$ has degree 1 it acts as a derivation so that

$$\varphi^{\left(\frac{n-1}{2}\right)}(e)(a\wedge b) = \Big(\varphi^{\left(\frac{n-1}{2}\right)}(e)(a)\Big)b - a\wedge\varphi^{\left(\frac{n-1}{2}\right)}(e)(b).$$

But

$$\begin{aligned}a\wedge\varphi^{\left(\frac{n-1}{2}\right)}(e)(b) &= -a\wedge b(e)\Big(\varphi^{\left(\frac{n-1}{2}\right)}\Big) && \text{(by A.3,3)}\\ &= -a\wedge b(e)(\varphi)\wedge\varphi^{\left(\frac{n-3}{2}\right)} && \text{(by A.4)}\\ &= a\wedge\varphi(e)(b)\wedge\varphi^{\left(\frac{n-3}{2}\right)} && \text{(by A.3,3)}\end{aligned}$$

as required.

Finally, to prove the last part of the theorem, we note that if $\lambda' : \overset{3}{\Lambda} F \to \overset{n}{\Lambda} F$ represents the multiplication of three elements of $\mathbf{E}_1'$ into $\mathbf{E}_3'$, then λ' makes the following diagram commute

$$(*) \qquad \begin{array}{ccccccc} \overset{3}{\Lambda} F & \xrightarrow{k} & \overset{2}{\Lambda} F & \xrightarrow{k} & F & \xrightarrow{g} & R \\ \downarrow{\lambda'} & & \downarrow{\lambda} & & \| & & \| \\ \overset{n}{\Lambda} F & \xrightarrow{g'} & \overset{n-1}{\Lambda} F & \xrightarrow{f'} & F & \xrightarrow{g} & R. \end{array}$$

On the other hand, since g' is a monomorphism, the map λ' making (*)

commutative is unique. Consequently it suffices to show that the choice $\lambda' = \lambda$ makes (*) commute, that is, that

$$\varphi^{\left(\frac{n-1}{2}\right)}(e)\left(a\wedge\varphi^{\left(\frac{n-3}{2}\right)}\right)=\left(\varphi^{\left(\frac{n-1}{2}\right)}(e)(a)\right)\wedge\varphi^{\left(\frac{n-3}{2}\right)} \qquad \text{all} \quad a\in \overset{3}{\Lambda} F.$$

We have

$$\varphi^{\left(\frac{n-1}{2}\right)}(e)\left(a\wedge\varphi^{\left(\frac{n-3}{2}\right)}\right)$$
$$=\left[\varphi^{\left(\frac{n-1}{2}\right)}(e)(a)\right]\wedge\varphi^{\left(\frac{n-3}{2}\right)}-a\wedge\left(\varphi^{\left(\frac{n-1}{2}\right)}(e)\left(\varphi^{\left(\frac{n-3}{2}\right)}\right)\right)$$

since $\varphi^{\left(\frac{n-1}{2}\right)}(e)$, being of degree 1, acts as a derivation. Thus it suffices to prove that $\varphi^{\left(\frac{n-1}{2}\right)}(e)(\varphi^{\left(\frac{n-3}{2}\right)})=0$. By A.4, we have $\varphi^{\left(\frac{n-1}{2}\right)}(e)(\varphi^{\left(\frac{n-3}{2}\right)}) = \varphi^{\left(\frac{n-1}{2}\right)}(e)(\varphi)\wedge\varphi^{\left(\frac{n-5}{2}\right)}$ and this latter term is, by lemma 2.4, equal to $e(\varphi^{\left(\frac{n+1}{2}\right)})\wedge\varphi^{\left(\frac{n-5}{2}\right)}$. Since F has rank n, and since $\varphi^{\left(\frac{n+1}{2}\right)}\in \overset{n+1}{\Lambda} F$, this last term is zero and the proof is complete.

As for the question of uniqueness of the algebra structure on the resolution $\mathbf{E}'$, the following is true, though because of its incomplete nature we will not prove it here: Suppose either that R is a graded ring, $R=\sum_{i\geq 0} R_i$, and I is a Gorenstein ideal of grade 3 contained in $R_+=\sum_{i>0} R_i$, or that R is a complete local ring, 2 is a unit in R, and I is a Gorenstein ideal of grade 3 in R. Then the structure of a commutative differential graded algebra on the resolution $\mathbf{E}'$ is unique up to isomorphism (in the graded case we require the maps giving the multiplication on $\mathbf{E}'$ to be homogeneous).

Remarks. 1) We do not know whether or not the hypothesis on R can be dropped.

2) In the local case it is possible to construct a (non-Gorenstein) ideal I so that the algebra structure on the minimal free resolution of R/I is not unique. In fact, I may be taken to be an almost complete intersection of grade 3. However, we know of no example of a homogeneous ideal I in a graded ring such that the homogeneous algebra structure on the minimal free resolution of R/I is not unique.

5. Liaison and Almost Complete Intersections of Grade 3. We begin by reviewing the definition and basic properties of Liaison [A-N; P-S].

Definition. Let I and J be ideals of a Macaulay ring R of pure height ($=$grade) g. We will say that I is linked to J if there is an R-sequence $x_1,\dots,x_g\in I\cap J$ such that $J=(x_1,\dots,x_g)\colon I$.

Though the linkage relation is not in general symmetric, it is for perfect ideals in a Gorenstein ring:

PROPOSITION 5.1. ([P-S], *Proposition* 1.3). *Let I be a perfect ideal of grade g in a Gorenstein local ring R, and let $x_1,\ldots,x_g \in I$ be an R-sequence of length g. Then*

$$(x_1,\ldots,x_g):I=J$$

is a perfect ideal of grade g, and we have

$$(x_1,\ldots,x_g):J=I.$$

We will need a more precise version of this result, namely:

PROPOSITION 5.1a. ([P-S], *Proposition* 2.6). *Let I be a perfect ideal of grade g in the Gorenstein local ring R, and let $\mathbf{P}$ be a minimal free resolution of R/I. Let $x_1,\ldots,x_g \in I$ be an R-sequence, let $\mathbf{K}$ be the Koszul complex which is the minimal free resolution of $R/(x_1,\ldots,x_g)$, and let ψ: $\mathbf{K}\to\mathbf{P}$ be a comparison map induced by the inclusion $(x_1,\ldots,x_g)\subset I$:*

$$\begin{tikzcd}
\mathbf{P}: & 0 \arrow[r] & P_g \arrow[r] & P_{g-1} \arrow[r] & \cdots \arrow[r, "\varphi_2"] & P_1 \arrow[r, "\varphi_1"] & R \\
\mathbf{K}: \arrow[u, "\psi"] & 0 \arrow[r] & \overset{g}{\Lambda} R^g \arrow[r] \arrow[u, "\psi_g"'] & \overset{g-1}{\Lambda} R^g \arrow[r] \arrow[u, "\psi_{g-1}"'] & \cdots \arrow[r, "f_2"] & R^g \arrow[r, "f_1"] \arrow[u, "\psi_1"'] & R. \arrow[u, equal, "\psi_0=1"']
\end{tikzcd}$$

Then the dual of the mapping cylinder of ψ, modulo the subcomplex $\psi_0:R\to R$, is a resolution of length g of R/J, where $J=(x_1,\ldots,x_g):I$.

Note that if $\mathbf{P}$ has an algebra structure, then a map $\psi:\mathbf{K}\to\mathbf{P}$ as in the above proposition may be defined by simply setting $\psi_k(y_1\wedge\cdots\wedge y_k)=\psi_1(y_1)\psi_2(y_2)\cdots\psi_k(y_k)$.

We can thus apply our knowledge of the multiplicative structure for resolutions of Gorenstein ideals of grade 3 to the study of almost complete intersections, using:

PROPOSITION 5.2. *Let I and J be perfect ideals of the same grade g in a Gorenstein local ring R, and suppose I is linked to J by an R-sequence $x_1,\ldots,x_g$. Then*

1) *If J is Gorenstein, then I is an almost complete intersection*;

2) *If I is an almost complete intersection and $x_1,\ldots,x_g$ form part of a minimal set of generators for I, then J is Gorenstein.*

Proof. 1) This follows from the fact that a free resolution of R/I is obtained from the dual of the mapping cylinder of a map between the Koszul complex resolving $R/(x_1,\ldots,x_g)$ and the minimal free resolution of R/J, as in Proposition 5.1a).

2) With the notation of Proposition 5.1a), we have $P_1 = R^{g+1}$ and our hypothesis on the elements x_i is equivalent to the statement that ψ_1 may be chosen to be a split monomorphism.

If M is the mapping cylinder of ψ, it has the form:

$$M: \qquad \ldots \xrightarrow{u_3} P_2 \oplus R^g \xrightarrow{u_2} P_1 \oplus R \xrightarrow{u_1} R$$

where $u_1(p_1, r) = \varphi_1(p_1) + r$, and $u_2(p_2, x) = (\varphi_2(p_2) - \psi_1(x), f_1(x))$.

Define M_0 to be the complex:

$$M_0: \qquad \ldots \to 0 \to 0 \to R^g \xrightarrow{v_2} R^g \oplus R \xrightarrow{v_1} R \to 0$$

where $v_1(x,r) = f_1(x) + r$ and $v_2(y) = (y, -f_1(y))$. Then M_0 is clearly exact and the map $w: M_0 \to M$ defined by:

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & R^g \arrow[r, "v_2"] \arrow[d, "w_2"] & R^g \oplus R \arrow[r, "v_1"] \arrow[d, "w_1"] & R \arrow[d, equal] \\
\ldots \arrow[rr] & & P_2 \oplus R^g \arrow[r, "u_2"] & P_1 \oplus R \arrow[r, "u_1"] & R
\end{tikzcd}$$

where $w_1(x,r) = (\psi_1(x), r)$ and $w_2(y) = (0, y)$, injects M_0 as a direct summand of M. M/M_0 has the form:

$$M/M_0: \qquad \ldots \to P_2 \to R \to 0.$$

It is clear that $(M/M_0)^*$ is a free resolution of R/J of length g and its term in dimension g is R. This shows that J is a Gorenstein ideal.

We now come to the main result of this section, which illustrates the way in which the knowledge of the algebra structure on a free resolution can be applied to the linkage process.

Let R be a regular local ring with maximal ideal $\mathfrak{M}$, and let J be a Gorenstein ideal of R of grade 3, generated by n elements. By Theorems 2.1 and 4.1, J is generated by the $(n-1)$ order Pfaffians of an alternating map $\overset{n-1}{\Lambda} R^n \approx R^{n^*} \to R^n$, and the product of 2 or 3 elements of degree 1 in the minimal free resolution of R/J is given by exterior multiplication by $\varphi^{\left(\frac{n-3}{2}\right)}$, which we denote by λ. We thus have the commutative diagram:

$$\begin{tikzcd}
& \overset{3}{\Lambda} R^n \arrow[r, "k"] \arrow[d, "\lambda"] & \overset{2}{\Lambda} R^n \arrow[d, "\lambda"] \arrow[dr, "k"] & & \\
\mathbf{E}': & \overset{n}{\Lambda} R^n \arrow[r, "g'"'] & \overset{n-1}{\Lambda} R^n \arrow[r, "f'"'] & R^n \arrow[r, "g"'] & R
\end{tikzcd}$$

where the bottom row is the minimal free resolution of R/J, and all the notation is as in section 4. If x_1, x_2, x_3 is an R-sequence contained in J, and $\alpha: R^3 \to R^n$ is any lifting of the inclusion $(x_1, x_2, x_3) \subset J$, i.e. the diagram:

$$\begin{tikzcd}
R^n \arrow[r, "g"] & R \\
R^3 \arrow[u, "\alpha"] \arrow[r, "\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}"'] & R \arrow[u, equal]
\end{tikzcd}$$

is commutative, then we can express the map $\psi: \mathbf{K} \to \mathbf{E}'$ required by Proposition 5.1a) completely in terms of α and λ:

$$\psi_0 = 1, \quad \psi_1 = \alpha, \quad \psi_2 = \lambda \overset{2}{\Lambda} \alpha, \quad \psi_3 = \lambda \overset{3}{\Lambda} \alpha.$$

Thus we obtain the first part of the following theorem:

THEOREM 5.3. *With the above notation and hypotheses*:

1) *The almost complete intersection $I = (x_1, x_2, x_3)$: J is generated by x_1, x_2, x_3 and the image of the composite*:

$$R \cong \overset{n}{\Lambda} R^{n*} \xrightarrow{\lambda^*} \overset{3}{\Lambda} R^{n*} \xrightarrow{\overset{3}{\Lambda} \alpha^*} \overset{3}{\Lambda} R^{3*} \cong R.$$

2) *Every almost complete intersection I of grade 3 in R can be obtained in this way for various J and x_1, x_2, x_3. If I has type t, then x_1, x_2, x_3 and J can be chosen so that*

$$n = \begin{cases} t & \text{if } t \text{ is odd} \\ t+1 & \text{if } t \text{ is even.} \end{cases}$$

(Recall that the type of a perfect ideal I of grade g in R is the dimension over $R/\mathfrak{M}$ of $\mathrm{Ext}_R^g(R/I,R/\mathfrak{M})$; *equivalently, if*

$$0\to F_g\to F_{g-1}\to\dots\to F_1\to R\to R/I\to 0$$

is a minimal free resolution of R/I, then the type of I is the rank of F_g.)

Proof. Part 1) has just been established.

To prove the first statement of part 2), we recall that, by a standard "general position" (or "prime avoidance") argument, I may be generated by 4 elements x_1, x_2, x_3, x_4 any 3 of which form an R-sequence ([B-E-3], lemma 8.2). The first statement of part 2) now follows from part 1) and the fact that, by Proposition 5.2, $J=(x_1,x_2,x_3):x_4$ is a Gorenstein ideal. To finish the proof, we must show that x_1, x_2, x_3, x_4 can be chosen so that J has t or $t+1$ generators, according to whether t is odd or even. If $x_1,\dots,x_4$ is a set of generators for I, any 3 of which form an R-sequence, then we will show that this can be achieved merely by permuting the the x_i; to do this we follow an idea of Watanabe ([WAT], Theorem).

Let

$$\mathbf{F}:0\to R^t\xrightarrow{f_3}R^m\xrightarrow{f_2}R^4\xrightarrow{f_1}R\to R/I\to 0$$

be the minimal free resolution of R/I. By theorem 3.1, $m=t+3$.

Let

$$\mathbf{L}:0\to\overset{4}{\Lambda}R^4\to\overset{3}{\Lambda}R^4\to\overset{2}{\Lambda}R^4\xrightarrow{f_2'}R^4\xrightarrow{f_1}R$$

be the Koszul complex of $(x_1,x_2,x_3,x_4)=I$. Since grade $I=3$, we have $H_j(\mathrm{L})=0$ for $j>4\text{-}3=1$, and $H_1(\mathrm{L})\approx\mathrm{Ext}_R^3(R/I,R)$, which is minimally generated by t elements. However,

$$H_1(\mathbf{L})=\mathrm{Ker}f_1/\mathrm{Im}f_2'=\mathrm{Im}f_2/\mathrm{Im}f_2',$$

and the minimal number of generators of $\mathrm{Im}f_2$ is $m=t+3$. Thus three of the "Koszul relations" can be taken to be among a minimal set of generators of $\mathrm{Ker}f_1$ and, after a change of basis in R^m and a permutation of the x_i, the first three rows of the matrix of f_2 may be taken to be either

$$\begin{bmatrix} x_2 & -x_1 & 0 & 0\\ x_3 & 0 & -x_1 & 0\\ 0 & x_3 & -x_2 & 0\end{bmatrix}\quad\text{or}\quad\begin{bmatrix} x_2 & -x_1 & 0 & 0\\ x_3 & 0 & -x_1 & 0\\ 0 & x_4 & 0 & -x_2\end{bmatrix}.$$

(Here the rows of the matrix of f_2 are supposed to represent the images of the basis elements of R^m; we are waiting our matrices on the right of their arguments.) In either case, the last column of the matrix of f_2 (whose entries generate the Gorenstein ideal $J=(x_1,x_2,x_3):x_4$) has at most $t+3-2=t+1$ elements. However, by Theorem 2.1, the minimal number of generators of a Gorenstein ideal must be odd, so we see that the minimal number of generators of J does not exceed t if t is odd, or $t+1$ if t is even. On the other hand, the construction of a free resolution of R/I as a mapping cylinder involving the minimal free resolution of R/J shows that J cannot have fewer than t generators if the type of I is t. This concludes the proof of Theorem 5.3.

One way to get a perfect ideal of grade 3 is to take a perfect ideal I_0 of grade 2, and adjoin an element x which is a non-zero divisor modulo I_0, to get a perfect ideal $I=(I_0,x)$. Since Hilbert's theorem completely describes perfect ideals of grade 2 and their free resolutions, it is easy to compute a free resolution for the grade 3 perfect ideal $I=(I_0,x)$. In particular, it is easy to see that, if n is the minimal number of generators of I, then the type of R/I is $n-2$. It is natural to ask whether every perfect ideal of grade 3, generated by n elements and having type $n-2$, is of the form $I=(I_0,x)$ as above. That the answer is "no" may be seen from the following example:

$R=k[[W,X,Y,Z]]$

$I=(X^2-WY, YZ-W^3, Z^2-WXY, XY^2-W^2Z, Y^3-WXZ)$, a 5-generator perfect ideal of grade 3, with type 3. (I is in fact a one-dimensional prime ideal obtained as the kernel of the map

$R\to k[[t]]$ defined by:

$W\mapsto t^7$

$X\mapsto t^8$

$Y\mapsto t^9$

$Z\mapsto t^{12}$.)

The minimal free resolution of R/I is:

$$0\to R^3\xrightarrow{f_3}R^7\xrightarrow{f_2}R^5\xrightarrow{f_1}R$$

where the maps f_i have the following matrices:

$$f_3=\begin{pmatrix} Z & Y & 0\\ -Y & 0 & W\\ -WX & Z & 0\\ X^2-WY & 0 & 0\\ 0 & X & Y\\ 0 & -W & -X\\ -W^2 & 0 & Z\end{pmatrix}$$

$$f_2=\begin{bmatrix} YZ-W^3 & Z^2-WXY & Y^2 & 0 & 0 & -W^2Y & -WZ \\ -X^2+WY & 0 & 0 & Z & XY & Y^2 & 0 \\ 0 & -X^2+WY & 0 & -Y & -W^2 & -WX & 0 \\ 0 & 0 & -X & -W & -Z & 0 & Y \\ 0 & 0 & W & 0 & 0 & -Z & -X \end{bmatrix}$$

$$f_1=(X^2-WY \quad YZ-W^3 \quad Z^2-WXY \quad XY^2-W^2Z \quad Y^3-WXZ).$$

(Here we are writing our maps on the left of their arguments.)

To see that I is not of the form (I_0,x), one first notes that if it were, then $I_2(f_3)$ would contain four elements from a minimal generating set for I, and one then checks, by a degree argument, that this is not the case. However, for almost complete intersections of grade 3, the desirable result does hold:

Corollary 5.4. *Let I be an almost complete intersection of grade 3. Then I has the form $I=(I_0,x)$ where I_0 is a perfect ideal of grade 2 and x is a non-zero divisor modulo I_0, if and only if type $(R/I)=2$.*

Proof. If $I=(I_0,x)$ where I_0 is perfect of grade 2 and x is a non-zero divisor modulo I_0, then it is easy to see that I_0 is minimally generated by 3 elements and thus has a resolution of the form:

$$\mathbf{P}_0: 0\to R^2\to R^3\to R\to R/I_0\to 0.$$

Since x is a non-zero divisor modulo I_0, the minimal resolution of R/I will be $\mathbf{P}_0\otimes(0\to R\xrightarrow{x} R)$, and $(\mathbf{P}_0\otimes(0\to R\xrightarrow{x} R))_3=R^2\otimes R=R^2$, so type $(R/I)=2$.

Conversely, suppose I is an almost complete intersection with type $(R/I)=2$. By Theorem 5.3, we may write $I=(x_1,x_2,x_3):J$ where J is a Gorenstein ideal generated by 3 elements, that is, J is generated by an R-sequence: y_1,y_2,y_3. Let

$$\begin{tikzcd}[column sep=small]
\mathbf{K}_1: & 0 \arrow[r] & \overset{3}{\Lambda} R^3 \arrow[r] & \overset{2}{\Lambda} R^3 \arrow[r] & R^3 \arrow[r] & R \arrow[r] & R/J \arrow[r] & 0 \\
\mathbf{K}_2: \arrow[u, "\Lambda\alpha"] & 0 \arrow[r] & \overset{3}{\Lambda} R^3 \arrow[r] \arrow[u, "\overset{3}{\Lambda}\alpha"'] & \overset{2}{\Lambda} R^3 \arrow[r] \arrow[u, "\overset{2}{\Lambda}\alpha"'] & R^3 \arrow[r] \arrow[u, "\alpha"'] & R \arrow[r] \arrow[u, equal] & R/(x_1,x_2,x_3) \arrow[r] & 0
\end{tikzcd}$$

be the resolutions of $R/(x_1,x_2,x_3)$ and R/J, and let $\Lambda\alpha$ be a comparison map between them, so that $I=(x_1,x_2,x_3,\det\alpha)$. A free resolution of R/I is now given by the dual of the mapping cylinder of $\Lambda\alpha$. Since type $(R/I)=2$ by hypothesis, the image of α must contain one of the basis elements of R^3. Thus, after a change of basis we may assume:

$$\alpha=\begin{bmatrix}1 & 0 & 0\\ 0 & \alpha_{22} & \alpha_{32}\\ 0 & \alpha_{23} & \alpha_{33}\end{bmatrix}; \qquad \begin{aligned}x_1&=y_1\\ x_2&=\alpha_{22}\,y_2+\alpha_{23}\,y_3\\ x_3&=\alpha_{32}\,y_2+\alpha_{33}\,y_3.\end{aligned}$$

An easy computation now shows that the minimal free resolution of R/I has the form:

$$0\to R^2\xrightarrow{\begin{bmatrix}\alpha_{32} & -\alpha_{33}\\ -\alpha_{22} & \alpha_{23}\\ y_3 & y_2\\ 0 & -x_1\\ -x_1 & 0\end{bmatrix}} R^5\xrightarrow{\begin{bmatrix}x_2 & x_3 & \det\alpha & 0 & 0\\ -x_1 & 0 & 0 & \alpha_{33} & -\alpha_{32}\\ 0 & -x_1 & 0 & -\alpha_{23} & \alpha_{22}\\ 0 & 0 & -x_1 & -y_2 & -y_3\end{bmatrix}} R^4\xrightarrow{(x_1,x_2,x_3,\det\alpha)} R$$

Since $(x_2,x_3,\det\alpha):x_1$ is generated by the elements in the first row of middle matrix, we see that $x_1=y_1$ is a non-zero divisor modulo $(x_2,x_3,\det\alpha)$. Thus $(x_2,x_3,\det\alpha)$ is a perfect ideal of grade 2; in fact it is the ideal of 2×2 minors of the matrix

$$\begin{pmatrix}\alpha_{33} & -\alpha_{23} & -y_2\\ -\alpha_{32} & \alpha_{22} & -y_3\end{pmatrix}.$$

This proves the corollary.

Remark. In [B-E-O], the connection between the lifting problem of Grothendieck and the explicit study of finite free resolutions was discussed. It is clear from the structure theorems 2.1 and 5.3, that the lifting problem is solvable for cyclic modules R/I if I is either Gorenstein of grade 3 or an almost complete intersection of grade 3.

6. Some Examples of Gorenstein Ideals and Almost Complete Intersections. a) *Generic Gorenstein ideals of height 3.* Let k be a commutative ring, and let $G_n(k)$ be a generic alternating $(2n+1)\times(2n+1)$ matrix over $k[\{x_{ij}\}_{1\leqslant i\leqslant j\leqslant 2n+1}]=R_n(k)$:

$$G_n(k)=\begin{bmatrix}0 & x_{12} & \cdots & x_{1\,2n+1}\\ -x_{12} & 0 & \ddots & \vdots\\ \vdots & & \ddots & x_{2n\,2n+1}\cdots\\ -x_{1\,2n+1} & \cdots & -x_{2n\,2n+1} & \ddots\ 0\end{bmatrix}$$

PROPOSITION 6.1. *Let k be a field. For every $n \geqslant 1$, $\mathrm{Pf}_{2n}(G_n(k))$ is a Gorenstein ideal of height* 3 *in $R_n(k)$, so that $R_n(k)/\mathrm{Pf}_{2n}(G_n(k))$ is a Gorenstein ring. Moreover, $R_n(k)/\mathrm{Pf}_{2n}(G_n(k))$ is a normal domain, non-singular in codimension* 6.

Proof. By 3.5 and that fact that $G_n(k)$ is "generic", we have $\operatorname{rank} G_n(k) = 2n$. Thus part 1 of Theorem 2.1 shows that $\mathrm{Pf}_{2n}(G_n(k))$ is a Gorenstein ideal if it has grade 3. Since $R_n(k)$ is a Macaulay ring, grade $(\mathrm{Pf}_{2n}(G_n(k)))$ = height $(\mathrm{Pf}_{2n}(G_n(k)))$. However, it is known [ROO, p. 196] that

$$\operatorname{height}\big(\mathrm{Pf}_{2n-2l}(G_n(k))\big) = (l+1)(2l+3).$$

In particular, height $(\mathrm{Pf}_{2n}(G_n(k))) = 3$. Thus we have the first assertion of Proposition 5.1.

To prove that $S = R_n(k)/\mathrm{Pf}_{2n}(G_n(k))$ is a normal domain, we will use the Krull-Serre characterization [MAT, Th. 39] and verify the criteria S_2 and R_1. Since S is Gorenstein and thus, in particular, Macaulay, we need only show that R_1 holds for S; that is, that every localization of S at a prime of height 1 is regular. What we will actually see is that if P is any prime not containing $I = \mathrm{Pf}_{2n-2}(G_n(k))/\mathrm{Pf}_{2n}(G_n(k))$, then S_P is regular. Since height $(\mathrm{Pf}_{2n-2}(G_n(k)))$ $= 10$, the height of I is 7 so this will show that S is non-singular in codimension 6 as well.

Let A be any $(2n-2)\times(2n-2)$ pfaffian of $G_n(k)$ and let μ be the multiplicatively closed set generated by A. It is not difficult to see that $R_n(k)_\mu/\mathrm{Pf}_{2n}(G_n(k))$ is isomorphic to $R_n(k)_\mu/\mathrm{Pf}_2(G_1(k))$. For, assume that A is the $(2n-2)\times(2n-2)$ pfaffian of the submatrix of $G_n(k)$ obtained by omitting the first three columns and rows. Since A is invertible in $R_n(k)_\mu$, $G_n(k)$ may be transformed over $R_n(k)_\mu$ to

$$\left(\begin{array}{ccc|c} 0 & y_{12} & y_{13} & \\ -y_{12} & 0 & y_{23} & \mathbf{0} \\ -y_{13} & -y_{23} & 0 & \\ \hline & \mathbf{0} & & \mathbf{1} \end{array}\right)$$

where the y_{ij} are polynomials of the form

$$y_{ij} = x_{ij} + (\text{higher order terms}).$$

We therefore see that $R_n(k)_\mu/\mathrm{Pf}_{2n}(G_n(k))$ is a localization of a polynomial ring. Thus if P is a prime of S not containing all the pfaffians of order $2n-2$ of

$G_n(k)$, then one of these pfaffians is invertible in S_P so that S_P is a localization of a regular ring. This shows that S is a normal ring. Since $\mathrm{Pf}_{2n}(G_n(k))$ is a homogeneous ideal, Spec S is connected, so S is a normal domain, non-singular in codimension 6.

This investigation began from a desire to know about the Gorenstein ideals of height (= grade) 3 in regular local rings. In particular, we may consider rings of the form $k[[X, Y, Z]]$, where k is a field and X, Y, Z are indeterminates. By part 2) of Theorem 2.1, any grade 3 Gorenstein ideal in this ring must have as its minimal number of generators an odd number $\geqslant 3$. We now give an example to show that all these odd numbers do occur.

Let $n \geqslant 3$ be odd, and let $R = k[[X, Y, Z]]$. Let H_n be the $n \times n$ alternating matrix whose above-diagonal entries are defined by:

$$(H_n)_{ij} = \begin{cases} X \text{ if } i \text{ is odd and } j = i+1 \\ Y \text{ if } i \text{ is even and } j = i+1\,. \\ Z \text{ if } j = n-i+1 \end{cases}$$

For example,

$$H_7 = \begin{bmatrix} 0 & X & 0 & 0 & 0 & 0 & Z \\ -X & 0 & Y & 0 & 0 & Z & 0 \\ 0 & -Y & 0 & X & Z & 0 & 0 \\ 0 & 0 & -X & 0 & Y & 0 & 0 \\ 0 & 0 & -Z & -Y & 0 & X & 0 \\ 0 & -Z & 0 & 0 & -X & 0 & Y \\ -Z & 0 & 0 & 0 & 0 & -Y & 0 \end{bmatrix}$$

PROPOSITION 6.2. *$\mathrm{Pf}_{n-1}(H_n)$ is a Gorenstein ideal of height 3 in R with n generators.*

Proof. An easy computation shows that $\mathrm{Pf}_{n-1}(H)$ contains powers of $X, Y,$ and Z. Thus grade $\mathrm{Pf}_{n-1}(H_n) = 3$, so an application of part 1) of Theorem 2.1 finishes the proof.

b) *Almost complete intersections.* Let $t \geqslant 3$ be an odd number, and let $\bar{t} = (t-1)/2$. Let S_t be the ring $R_{\bar{t}}(k)[\{y_{ij}\} i = 1,2,3; j = 1,\ldots,t]$. We consider $G_{\bar{t}} = G_{\bar{t}}(k)$ as a matrix defined over S_t, and let p_i be the $(t-1)$ order pfaffian of $G_{\bar{t}}$ obtained by omitting the ith row and column. Thus p_i is the ith generator of the Gorenstein ideal $\mathrm{Pf}_{t-1}(G_{\bar{t}})$. If $i < j < k$ are integers, we also let p_{ijk} be the $(t-3)$ order pfaffian of $G_{\bar{t}}$ obtained by omitting the i^{th} and k^{th} rows and columns of $G_{\bar{t}}$. Let Y be the $3\times t$ matrix over S_t with entries y_{ij}, and let

$$d_{ijk}=\det\begin{bmatrix} y_{1i} & y_{1j} & y_{1k}\\ y_{2i} & y_{2j} & y_{2k}\\ y_{3i} & y_{3j} & y_{3k}\end{bmatrix}.$$

Finally, let I_t be the ideal of S_t generated by the three elements

$$Z_i=\sum_j y_{ij}p_j, \qquad i=1,2,3,$$

and the fourth element

$$Z_4=\sum_{i<j<k} \pm p_{ijk}d_{ijk}.$$

For an even integer $t\geqslant 2$, let

$$S_t=S_{t+1}/(y_{11}-1, \quad y_{12},\dots,y_{1t+1}, y_{21}, y_{31})$$

and let I_t be the image of I_{t+1} in S_t. The proof of the following proposition, which follows from Theorem 5.3, resembles the proof of Proposition 6.1, and we shall omit it:

Proposition 6.3. *For each integer $t\geqslant 2$, the ideal $I_t\subset S_t$ defined above is a (generic) almost complete intersection of height 3 with type $(S_t/I_t)=t$. It is a normal prime ideal.*

Appendix: Review of Multilinear Algebra. In this section we will recall those facts of multilinear algebra that are the basis of the computations done in this paper. An exposition in which all these facts are explained in detail may be found in [B-E-4]; for related material, see [BOU, Ch. III], [BUC] and [SWE]. Throughout this section, F and G will denote finitely generated modules over a commutative ring R.

a) *The exterior algebra as a divided power bialgebra.* We write ΛF for the exterior (or Grassman) algebra generated by F. In addition to its usual commutative algebra structure, ΛF has the structure of a graded co-commutative co-algebra. That is, there are maps

$$\epsilon:\Lambda F\to \overset{0}{\Lambda} F=R \qquad \text{and}$$
$$\Delta:\Lambda F\to\Lambda F\otimes\Lambda F,$$

satisfying the duals of the usual commutative and associative laws. Here ϵ is given by projection onto degree 0, while Δ is uniquely specified by the requirements that, for $a \in F$,

$$\Delta(a) = a \otimes 1 + 1 \otimes a \in \Lambda F \otimes \Lambda F,$$

and that Δ be an algebra homomorphism. The fact that Δ is an algebra homomorphism says that ΛF is a bialgebra. (It is even a Hopf algebra, but we will have no need of its antipode.)

Moreover, ΛF possesses a (unique) system of divided powers. This means that for every homogeneous element $a \in \Lambda F$ of even degree, there is a sequence of elements

$$a^{(0)}, a^{(1)}, a^{(2)}, \ldots$$

called the divided powers of a, satisfying:

1) $a^{(0)} = 1$; $a^{(1)} = a$; $\deg(a^{p)}) = p \deg(a)$.
2) $a^{(p)} a^{(q)} = \binom{p+q}{q} a^{(p+q)}$.
3) $(a+b)^{(p)} = \sum a^{(p-k)} b^{(k)}$ if $\deg(a) = \deg(b)$ is even.
4) $(ab)^{(p)} = \begin{cases} 0 \text{ if } \deg(a) \text{ and } \deg(b) \text{ are odd} \\ a^p b^{(p)} \text{ if } \deg(a) \text{ and } \deg(b) \text{ are even} \end{cases}$
5) $(a^{(p)})^{(q)} = [p,q] a^{(pq)}$ where $[p,q] = \dfrac{(pq)!}{q! p^q!}$

If $a = \sum_{i=1}^{k} a_i$, where each a_i is a product of elements of degree 1, then

$$a^{(p)} = \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant k} a_{i_1} \wedge \cdots \wedge a_{i_p}.$$

b) ΛF *and* $\Lambda(F^*)$ *are modules over each other.* Because $(\Lambda F)^* = \sum_k \left(\overset{k}{\Lambda} F \right)^*$, we may define an operation $(\Lambda F)^*$ on ΛF as follows: For $a \in \left(\overset{p}{\Lambda} F \right)^*$ and $b \in \overset{q}{\Lambda} F$, we let $\Delta_{p,q-p}(b) = \sum b_i \otimes b_i'$ be the component of $\Delta(b)$ in $\overset{p}{\Lambda} F \otimes \overset{q-p}{\Lambda} F$, and set

$$a(b) = \sum_i \langle a, b_i \rangle b_i' \in \overset{q-p}{\Lambda} F$$

where $\langle a, b_i \rangle \in R$ is the value of the linear functional a on b_i. There is a natural bialgebra homomorphism

$$\Lambda(F^*) \xrightarrow{\alpha} (\Lambda F)^*$$

so that, if $a \in \overset{p}{\Lambda}(F^*)$ and $b \in \overset{q}{\Lambda} F$, we may write $a(b)$ for $[\alpha(a)](b) \in \overset{q-p}{\Lambda} F$. Similarly, the natural map $F \to F^{**}$ induces a bialgebra map

$$\Lambda F \to \Lambda(F^{**}) \to (\Lambda F^*)^*,$$

so we may also write $b(a) \in \overset{p-q}{\Lambda} F^*$. We will generally write ΛF^* for $\Lambda(F^*)$. We have

PROPOSITION A.1. *The operations defined above make ΛF and ΛF^* into modules over each other.*

It follows immediately from the definition that if $a \in \Lambda F$ and $b \in \Lambda F^*$ have the same degree, then $a(b) = b(a) = \langle a, b\rangle \in R$, and that elements of degree 1 act as derivations.

The most important of the formulas which connect all these data is the *measuring identity*, which says that if $\Lambda F \otimes \Lambda F$ is regarded as a ΛF^*-module via the diagonal $\Lambda F^* \to \Lambda F^* \otimes \Lambda F^*$, then the multiplication map $\Lambda F \otimes \Lambda F \to \Lambda F$ is a map of modules. Explicitly,

PROPOSITION A.2. *Let $a \in \Lambda F^*$, and let b, $c \in \Lambda F$. Write $\Delta(a) = \sum a_i \otimes a_i'$. Then*

$$a(b \wedge c) = \sum (-1)^{\deg(b)\deg(a_i')} a_i(b) \wedge a_i'(c).$$

If $a, b \in \Lambda F$ and $c \in \Lambda F^*$, we write $a(c)(b)$ for $(a(c))(b)$. This sort of expression is rather important in our calculations. The following proposition makes such things manageable (it is something like a Laplace expansion). Fortunately we do not have to deal separately with expressions $a(c)(b)$ with $a, b \in \Lambda F^*$, $c \in \Lambda F$. Because of the way we defined the actions of ΛF and ΛF^* on each other, the rules for dealing with this case are the same.

PROPOSITION A.3. *Let $a, b \in \Lambda F$, $c \in \Lambda F^*$, and write $\Delta(a) = \sum a_i \otimes a_i'$. Then*

1) $a(c)(b) = \sum (-1)^{(1+\deg(c))\deg(a_i')} a_i \wedge c(a_i' \wedge b)$.
2) *In particular, if* $\deg(a) = 1$, *then*

$$a(c)(b) = a \wedge c(b) + (-1)^{1+\deg(c)} c(a \wedge b).$$

3) *If* $\overset{n+1}{\Lambda} F^* = 0$ *and* $c \in \overset{n}{\Lambda} F^*$, *then*

$$a(c)(b) = (-1)^{\nu} b(c)(a)$$

where

$$\nu = (1 + \deg(a) + \deg(b)) + \deg(a)\deg(b).$$

It should be remarked that the signs in the above "take care of themselves," and represent no difficulties in calculation.

Finally, the system of divided powers in ΛF is related to the action of ΛF^* on ΛF by the following:

PROPOSITION A.4. *Let $a \in \Lambda F$ have even degree, and let $b \in F^*$. Then*

$$b(a^{(p)}) = b(a) \wedge a^{(p-1)}.$$

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