697B Example Sheet 1

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(1) Let $X = (y^2 = x) \subset \mathbb{C}^2$. Consider the inclusion

$$\mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}, \quad (x, y) \mapsto (X : Y : Z) = (x : y : 1).$$

Let $L_\infty = (Z = 0) \subset \mathbb{P}^2$, the line at infinity.

(a) What is the closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$? (What is its homogeneous equation?)

(b) What is $\overline{X} \cap L_\infty$?

(c) Show that $\overline{X}$ is smooth and identify it with a standard Riemann surface.

(2) Let $f(z) = p(z)/q(z)$ be a rational function of a complex variable z. Here $p(z)$ and $q(z)$ are polynomials with no common factors.

(a) Show that $f(z)$ defines a holomorphic map

$$F \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}.$$

[If $F \colon X \to Y$ is a continuous map between Riemann surfaces X and Y with charts $\phi_i \colon U_i \to \mathbb{C}$, $\psi_j \colon V_j \to \mathbb{C}$, we say F is *holomorphic* if $\psi_j \circ F \circ \phi_i^{-1}$ is holomorphic on $\phi_i(U_i \cap F^{-1}(V_j))$ for each i and j.]

(b) What is $F(\infty)$?

(c) What is the size of $F^{-1}(\alpha)$ for $\alpha \in \mathbb{C} \cup \{\infty\}$ a general point?

(3) Find the singular points of the following curves. Draw the locus of real points $X \cap \mathbb{R}^2 \subset \mathbb{R}^2$. Which of the singular points are nodes (ordinary double points)?

(a) $X = (y^2 = x^2(x + 1)) \subset \mathbb{C}^2$.

(b) $X = (y^2 = x^3) \subset \mathbb{C}^2$.

(c) $X = ((x^2 + y^2)^2 + 3x^2y - y^3 = 0) \subset \mathbb{C}^2$. [Hint: To draw the real locus use polar coordinates and the identity $\sin 3\theta = 3(\cos\theta)^2 \sin\theta - (\sin\theta)^3$.]

(4) Let $X = (y^2 = x(x-1)(x-\lambda)) \subset \mathbb{C}^2$ where $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

(a) Compute the closure $\overline{X} \subset \mathbb{P}^2$ and show that $\overline{X}$ is smooth.

(b) Show that the map $X \to \mathbb{C}$ given by $(x, y) \mapsto x$ extends to a map $F\colon \overline{X} \to \mathbb{P}^1_{\mathbb{C}}$.

(c) By considering the map F determine the topological type of $\overline{X}$.