1. $\mathcal{C}_1: \overset{①}{x^2+y^2=1}$

$\mathcal{C}_2: \overset{②}{(x-1)^2+(y-2)^2=4}$, i.e., $x^2-2x+1+y^2-4y+4=4$.

a. ① − ② : $2x-1+4y-4=-3$

$2x+4y=2$

$x+2y=1.$

Substitute $x=1-2y$ in ① :

$(1-2y)^2+y^2=1$

$4y^2-4y+1+y^2=1$

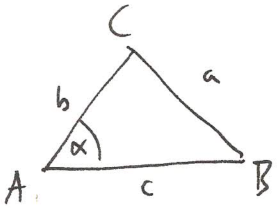
$5y^2-4y=0$

$y(5y-4)=0$

$y=0$ OR $y=4/5 \overset{x=1-2y}{\Rightarrow} (x,y)=(1,0)$ OR $(-3/5, 4/5)$.

b. This is the equation of the line passing through the two intersection points of the circles.

2. Recall the cosine rule: $a^2=b^2+c^2-2bc\cos\alpha$



$0<\alpha<\pi \Rightarrow -1<\cos\alpha<1 \Rightarrow a^2<b^2+c^2+2bc$

$=(b+c)^2$

$\Rightarrow a<b+c$. □.

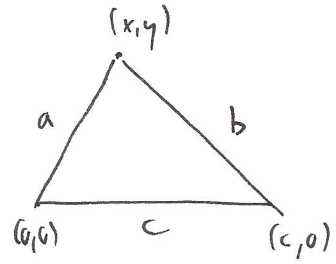
3. We follow the hint.

$\mathcal{C}_1$: ① $x^2+y^2=a^2$

$\mathcal{C}_2$: ② $(x-c)^2+y^2=b^2$, i.e., $x^2-2cx+c^2+y^2=b^2$

① − ② $\quad 2cx-c^2=a^2-b^2$

$$\Rightarrow \quad x=\frac{a^2-b^2+c^2}{2c}$$



$$\overset{①}{\Rightarrow} \quad y=\pm\sqrt{a^2-x^2}.$$

We require $y\in\mathbb{R}$, $y\neq 0$ to obtain a triangle,

equivalently $x^2<a^2$, or $|x|<a$, i.e. $-a<x<a$.

or $$-a<\frac{a^2-b^2+c^2}{2c}<a.$$

$$\frac{a^2-b^2+c^2}{2c}<a \iff a^2-b^2+c^2<2ac$$
$$\iff a^2-2ac+c^2<b^2$$
$$\iff (a-c)^2<b^2$$
$$\iff -b<a-c<b$$
$$\iff c<a+b \text{ and } a<b+c.$$

$$\frac{a^2-b^2+c^2}{2c}>-a \iff a^2-b^2+c^2>-2ac$$
$$\iff a^2+2ac+c^2>b^2$$
$$\iff (a+c)^2>b^2$$
$$\iff a+c>b$$

So we obtain a triangle w/ side lengths $a,b,c \iff a<b+c$, $b<c+a$, & $c<a+b$. □

4.

T	$\mathrm{Fix}(T)$
identity	$\mathbb{R}^2$
translation by $(a,b) \neq (0,0)$	$\emptyset$ (empty set)
rotation about point P thru angle θ, $0<\theta<2\pi$	$\{P\}$
reflection ~~abou~~ in a line L	L
glide reflection (reflection in a line L followed by a translation by $(a,b)\neq(0,0)$ parallel to L)	$\emptyset$

5. A glide reflection is a composition $T = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$ of reflection in a line L followed by translation by a vector $\underline{v}$ parallel to L. As noted in class $\mathrm{Trans}_{\underline{v}}$ & Refl_L commute, i.e.,

$$T = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L = \mathrm{Refl}_L \circ \mathrm{Trans}_{\underline{v}}$$

Now

$$\begin{aligned} T^2 = T \circ T &= (\mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L) \circ (\mathrm{Refl}_L \circ \mathrm{Trans}_{\underline{v}}) \\ &= \mathrm{Trans}_{\underline{v}} \circ (\mathrm{Refl}_L \circ \mathrm{Refl}_L) \circ \mathrm{Trans}_{\underline{v}} \\ &= \mathrm{Trans}_{\underline{v}} \circ id \circ \mathrm{Trans}_{\underline{v}} \\ &= \mathrm{Trans}_{\underline{v}} \circ \mathrm{Trans}_{\underline{v}} \\ &= \mathrm{Trans}_{2\underline{v}}, \quad \text{translation by } 2\underline{v}. \end{aligned}$$

6. If T = identity then $T^n = id$ for all $n \in \mathbb{N}$.

If T is a translation by a vector $\underline{v} \neq \underline{0}$, then T^n is translation by $n\underline{v}$, so $T^n \neq id$ for all $n \in \mathbb{N}$.

If T is a rotation about a point P thru angle θ ccw, then T^n is a rotation about P thru angle $n\theta$ ccw, so $T^n = id$ iff

$n\theta = 2\pi \cdot k$ for some integer $k \in \mathbb{Z}$, i.e. $\theta = \frac{2\pi k}{n}$, some $k \in \mathbb{Z}$.

If T is a reflection then $T^2 = id$, so $T^n = id$ if n is even (and $T^n = T \neq id$ if n is odd).

If T is a glide reflection then, using the notation of Q.5,

$$T^2 = \mathrm{Trans}_{2\underline{v}}, \quad \text{so} \quad T^n = T^{2k} = \mathrm{Trans}_{2k\underline{v}} = \mathrm{Trans}_{n\underline{v}} \quad \text{if } n = 2k \text{ is even}$$

$$\& \quad T^n = T^{2k+1} = \mathrm{Trans}_{2k\underline{v}} \circ T = \mathrm{Trans}_{(2k+1)\underline{v}} \circ \mathrm{Refl}_L \quad \text{if } n = 2k+1 \text{ is odd.}$$

So $T^n \neq id$ for all $n \in \mathbb{N}$.

These are all the types of isometries of $\mathbb{R}^2$.

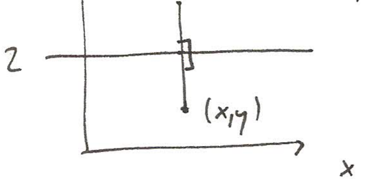
7.

a. $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \theta = \pi/2$$

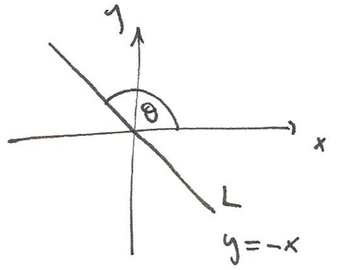
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y-2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= (-y+2,\ x-1) + (1,\ 2) = \begin{pmatrix} -y+3 \\ x+1 \end{pmatrix}$$

b. $T(x,y) = (x,\ 2+(2-y)) = (x,\ 4-y)$



c.

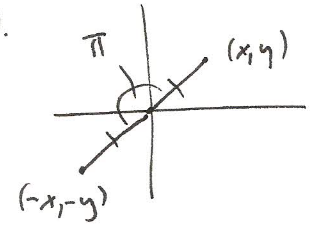


$$\mathrm{Refl}_L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\theta = 3\pi/4 \qquad = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}.$$

$$\therefore\ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2-y \\ -2-x \end{pmatrix}.$$

8. a)



Rotation, center O, thru angle π ccw.

b) $T(x,y) = \frac{1}{5}(4x+3y+2,\ 3x-4y-6)$

Find $\mathrm{Fix}(T)$ & use Q4.

Solve $T(x,y) = (x,y)$

$$\begin{aligned} 4x+3y+2 &= 5x \\ 3x-4y-6 &= 5y \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} {}^{1}/\ \ -x+3y &= -2 \\ {}^{2}/\ \ 3x-9y &= 6 \end{aligned}$$

$^{2}/ + 3\cdot{}^{1}/ \qquad 0 = 0.$

i.e. $\mathrm{Fix}(T) = \{(x,y) \in \mathbb{R}^2 \mid -x+3y = -2\}$

$\therefore$, by Q4, T is reflection in the line $-x+3y=-2$

OR $y = \frac{1}{3}x - \frac{2}{3}$

c) $T(x,y) = \frac{1}{5}(3x-4y+8,\ 4x+3y+4)$.

Solve $T(x,y) = (x,y)$

$$\begin{aligned} 3x-4y+8 &= 5x \\ 4x+3y+4 &= 5y \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} -2x-4y &= -8 \quad \div -2 \\ 4x-2y &= -4 \quad \div 2 \end{aligned}$$

$$\rightsquigarrow \quad \begin{aligned} {}^{1}/\ \ x+2y &= 4 \\ {}^{2}/\ \ 2x-y &= -2 \end{aligned}$$

$^{2}/ - 2\cdot{}^{1}/ \qquad -5y = -10,\ y = 2,\ x \overset{1/}{=} 4-2y = 0.$

$\therefore$ By Q4, T is a rotation about $P = (0,2)$. $\Big\}$ $\mathrm{Fix}(T) = \{(0,2)\}$.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$\Rightarrow \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \theta = \tan^{-1}(4/3) \text{ ccw.}$$

T is a rotation about $(0,2)$ thru $\theta = \tan^{-1}(4/3)$ ccw.

d. $T(x,y) = (y+4, x+8)$

$\mathrm{Fix}(T)$: Solve $T(x,y) = (x,y)$

$$\begin{aligned} y+4 &= x \\ x+8 &= y \end{aligned} \sim \begin{aligned} &1. \quad -x+y = -4 \\ &2. \quad x-y = -8 \end{aligned} \qquad 1.+2. : \quad 0 = -12 \text{ ※}.$$

So $\mathrm{Fix}(T) = \emptyset$.

Clearly, T is not a translation (because then $T(x,y) = (x+a, y+b)$, for some $a,b \in \mathbb{R}$)

So, by Q4, T is a glide reflection.

Now by Q5, $T^2 = \mathrm{Trans}_{2\underline{v}}$ where $T = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$

So compute: $T^2(x,y) = T(T(x,y)) = T(y+4, x+8) = ((x+8)+4, (y+4)+8)$
$= (x+12, y+12)$

$\therefore \quad 2\underline{v} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$

Now $\mathrm{Refl}_L = \cancel{T \circ \mathrm{Refl}}\ \mathrm{Trans}_{\underline{v}}^{-1} \circ T = \mathrm{Trans}_{\begin{pmatrix} -6 \\ -6 \end{pmatrix}} \circ T$

$\mathrm{Refl}_L(x,y) = T(x,y) - (6,6) = (y-2, x+2)$

$\mathrm{Fix}(\mathrm{Refl}_L)$: Solve $\cancel{T(x)}$ $\mathrm{Refl}_L(x,y) = (x,y)$

$$\left.\begin{aligned} y-2 &= x \\ x+2 &= y \end{aligned}\right\} \qquad \begin{aligned} &x-y = 2 \\ \text{OR} \quad &y = x-2 \end{aligned}$$

So L is the line $y = x-2$.

T is the glide reflection given by reflection in the line L w/ equation $y=x-2$ followed by translation by $\underline{v} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$ parallel to L. $\square$.