Math 621 Homework 1

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Reading: Stein and Shakarchi, Chapter 1 and Chapter 2, Sections 1 and 2.

(1) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^n & \text{if } x \geq 0. \\ 0 & \text{otherwise.} \end{cases}$$

Here n is a positive integer. Show that f is differentiable $n-1$ times at $0 \in \mathbb{R}$, but not n times.

(2) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0. \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is infinitely differentiable and $f^{(n)}(0) = 0$ for all $n \geq 0$. Deduce that f does not have a power series expansion at $0 \in \mathbb{R}$.

(3) Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function on a open set $\Omega \subset \mathbb{C}$. Write $f(x+iy) = u(x,y)+iv(x,y)$. Show that u and v are harmonic functions, that is, they satisfy Laplace's equation

$$\nabla^2 F := \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

[You may assume that the second partial derivatives of u and v exist and are continuous. This follows from the fact that holomorphic functions are infinitely differentiable, which we will prove later in class.]

(4) (a) Show that in polar coordinates the Cauchy–Riemann equations are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

(b) Now define the logarithm $\log\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by

$$\log z = \log r + i\theta$$

where $z = re^{i\theta}$ with $-\pi < \theta < \pi$. Show that log is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

(5) (a) Use the definition of e^z as a power series to prove that $e^{z+w} = e^z \cdot e^w$.

(b) Use part (a) and the identity $e^z = \cos z + i \sin z$ to deduce the usual addition laws for sine and cosine (but now with the independent variables being complex).

(c) Show that, for $x, y \in \mathbb{R}$, $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$. Deduce that $|\cos(x + iy)|^2 = (\cos x)^2 + (\sinh y)^2$. In particular $|\cos(x + iy)| \to \infty$ as $y \to \pm\infty$. Prove analogous formulas for $\sin(x + iy)$.

(6) Let $\log z$ be the logarithm defined above on the open set $\Omega := \mathbb{C} \setminus (-\infty, 0]$. Derive the power series expansion for $\log z$ about $z = 1$. What is its radius of convergence? [Hint: The derivative of $\log z$ equals $\frac{1}{z}$ on Ω. The proof is the same as the real case: use $\log z$ is the inverse to e^z, $(e^z)' = e^z$, and the chain rule.]

(7) Find the radius of convergence of the following power series.

(a) $\sum_{n=0}^{\infty} n^\alpha z^n$, where $\alpha \in \mathbb{R}$ is fixed.

(b) $\sum_{n=0}^{\infty} n! z^n$.

(c) $\sum_{n=0}^{\infty} q^{n^2} z^n$, where $q \in \mathbb{C}$ is fixed, $|q| < 1$.

(8) If the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ equals R, what is the radius of convergence of the following series?

(a) $\sum_{n=0}^{\infty} a_n z^{2n}$.

(b) $\sum_{n=0}^{\infty} a_n^2 z^n$.

(c) $\sum_{n=0}^{\infty} n^5 a_n z^n$.

(9) (a) Let γ be the circle $\{z \in \mathbb{C} \mid |z| = R\}$ center the origin, radius R, with the counterclockwise orientation. Compute the integral

$$\int_\gamma z^n dz$$

from first principles, for each integer $n \in \mathbb{Z}$.

(b) Let γ be the circle center $a \in \mathbb{C}$, radius R, with the counterclockwise orientation. Compute the integral

$$\int_\gamma \frac{1}{z} dz$$

from first principles in the cases $R > |a|$ and $R < |a|$. [Hint: Expand

$$(a + Re^{it})^{-1} = a^{-1}(1 + (R/a)e^{it})^{-1} = (Re^{it})^{-1}((a/R)e^{-it} + 1)^{-1}$$

as a convergent power series in e^{it} or e^{-it}.]

(c) Let γ be the circle center the origin radius R. Using part (b) or otherwise, show that for $a, b \in \mathbb{C}$ with $|a| < R < |b|$ we have

$$\int_\gamma \frac{1}{(z-a)(z-b)} = \frac{2\pi i}{a-b}.$$