Math 461 Lecture 15 10/9

Homework 4 due at start of class Wednesday (tomorrow)

Office Hours: 4:00 - 5:00 today LGRT 1235H

Last time:

Isometries of $\mathbb{R}^2$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $|T(P)T(Q)| = PQ$ for all $P, Q \in \mathbb{R}^2$

Translation Reflection

Rotation Identity

Glide reflection

Theorem: Every isometry is one of these. (Proof later)

Composition:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^2 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$

$T_2 \circ T_1(P) = T_2(T_1(P))$

Today:

Algebraic formulas for isometries

Examples of composition

Formula for rotation:

First consider rotation about origin through angle θ counterclockwise

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$T(x,y) = (\cos\theta \cdot x - \sin\theta \cdot y, \sin\theta \cdot x + \cos\theta \cdot y)$

Ex. $\theta = \frac{\pi}{2}$ $T(x,y) = (-y, x)$

How to prove the formula?

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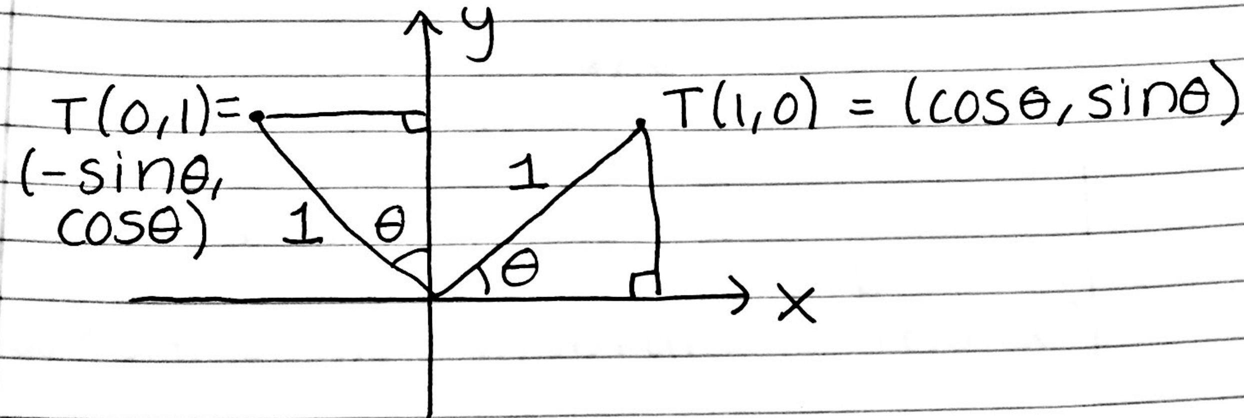
1. T is a linear transformation
   $T(\bar{v}+\bar{w}) = T(\bar{v}) + T(\bar{w})$ for all $\bar{v},\bar{w} \in \mathbb{R}^2$
   & $T(c\bar{v}) = cT(\bar{v})$ and $c \in \mathbb{R}$
2. $T(x,y) = T(x\cdot(1,0)+y\cdot(0,1)) =$ ①
   $T(x\cdot(1,0)) + T(y\cdot(0,1)) =$ ②
   $x\cdot T(1,0) + y\cdot T(0,1)$



$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
Rotation by A, followed by rotation by B about origin = rotation by A+B
Now compute using formula

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad * \quad T:\mathbb{R}^2\to\mathbb{R}^2$$

rotation about $P=(a,b)$ through angle θ counterclockwise
what is the algebraic formula?
Notice: T is a composition
$Rot(P,\theta)$ = rotation about P through θ counterclockwise
$Trans_{\bar{v}}$ = translation by vector $\bar{v}$
$T = Rot(P,\theta) = Trans_{\overrightarrow{OP}} \circ Rot(O,\theta) \circ Trans_{-\overrightarrow{OP}}$
(read from right to left)

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\left(\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}a\\b\end{pmatrix}\right) + \begin{pmatrix}a\\b\end{pmatrix}$$

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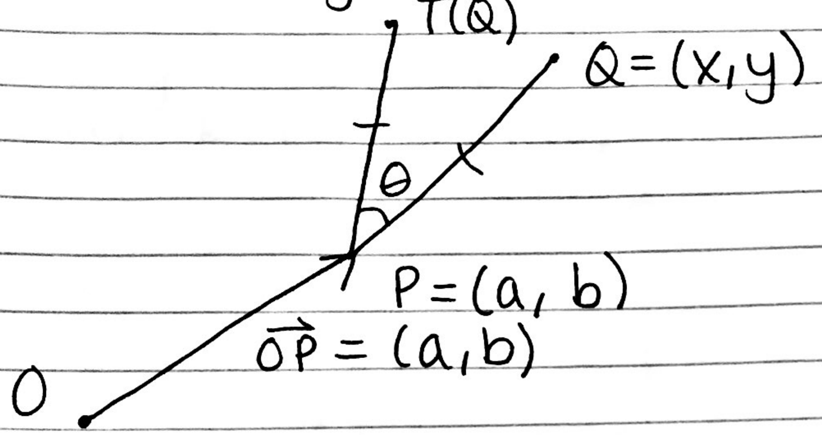
$$= \begin{pmatrix} x\cdot\cos\theta - y\cdot\sin\theta + c \\ x\cdot\sin\theta + y\cdot\cos\theta + d \end{pmatrix} \quad \text{some } c,d \in \mathbb{R}$$

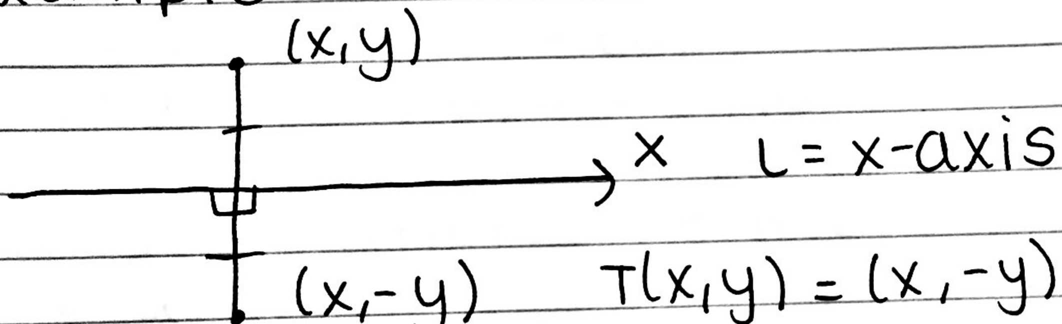


Reflection:

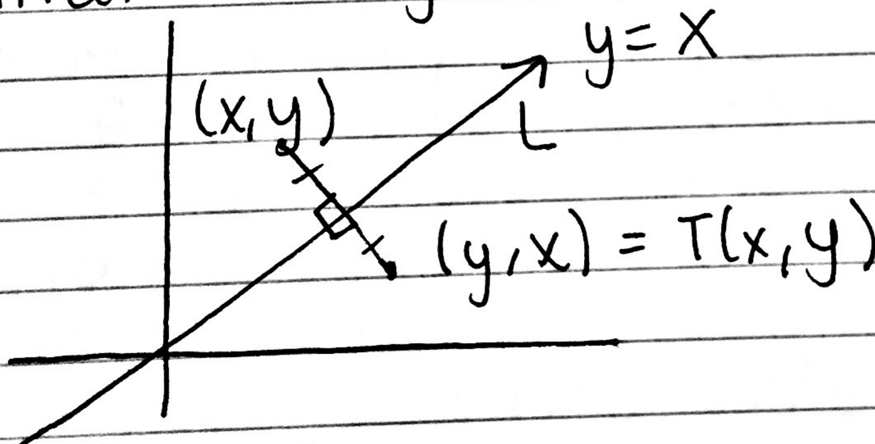
$T: \mathbb{R}^2 \to \mathbb{R}^2$ reflection in L

what is the algebraic formula?

Example:



Similiar if L = y-axis



General case:

First assume that line L passes through the origin O.

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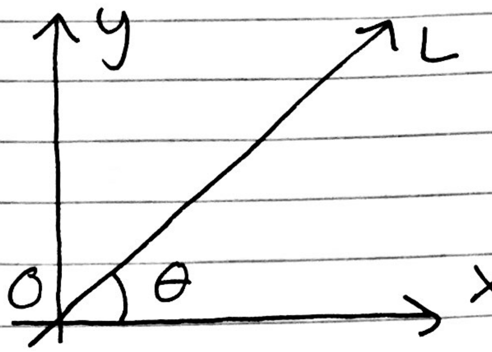
One way: First rotate by $-\theta$ about origin so that L maps to x-axis.
Then reflection over x-axis.
Finally rotate by $+\theta$ about origin.

Claim: $\mathrm{Refl}_L = \mathrm{Rot}(0,\theta) \circ \mathrm{Refl}_{x\text{-axis}} \circ \mathrm{Rot}(0,-\theta)$



Matrices:

$R(0,\theta)$:
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$\mathrm{Refl}_{x\text{-axis}}$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathrm{Rot}(0,-\theta)$: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

"even" $\cos(-\theta) = \cos\theta$

"odd" $\sin(-\theta) = -\sin\theta$

Recall: composition of linear transformations $\leftrightsquigarrow$ multiplication of matrices

$\mathrm{Refl}_L = \mathrm{Rot}(0,\theta) \circ \mathrm{Refl}_{x\text{-axis}} \cdot \mathrm{Rot}(0,-\theta)$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

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Matrix for rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

vs.

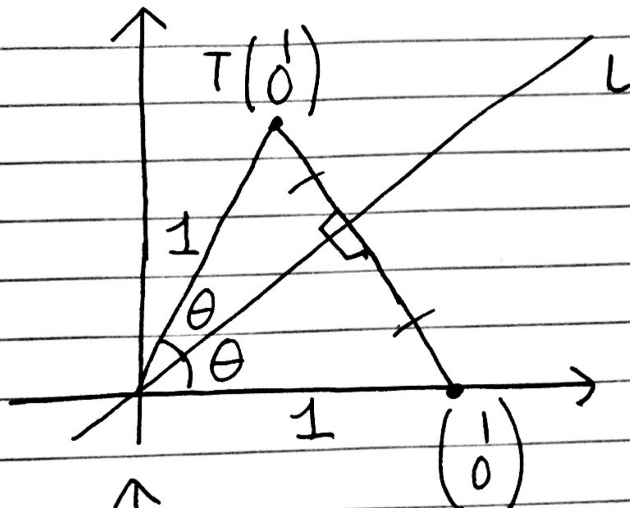
Matrix for reflection

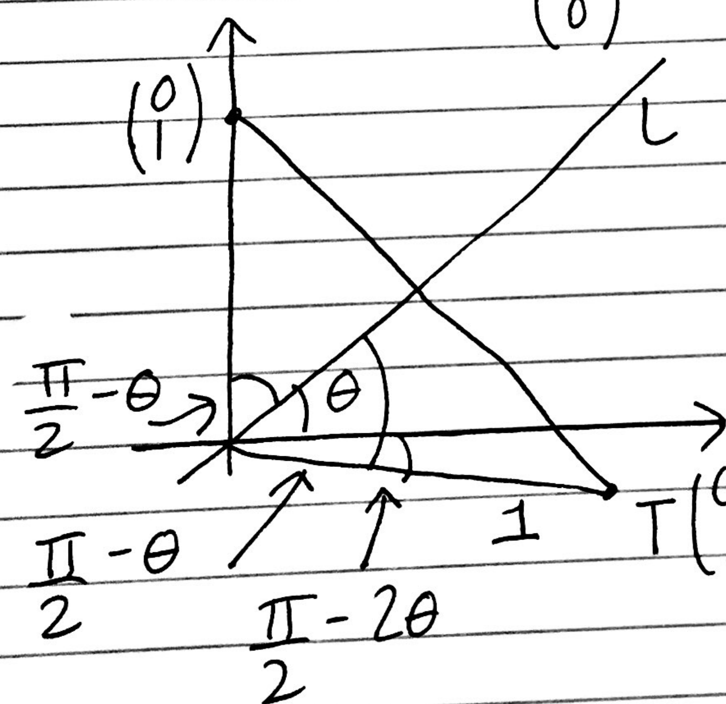
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

what is the geometric proof?

If its given in a matrix then it is a linear transformation.

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix} \quad T = \mathrm{Refl}_L$$





$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{2} - 2\theta\right) \\ -\sin\left(\frac{\pi}{2} - 2\theta\right) \end{pmatrix} = \begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix}$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$