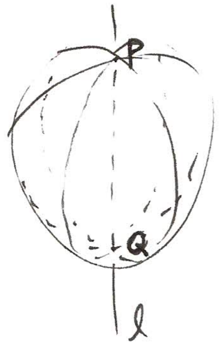
1. Because $\vec{OP}$ & $\vec{OQ}$ have length 1, if they are multiples of each other, then $\vec{OQ} = \pm \vec{OP}$, i.e., either $Q = P$ (which we assume is not the case), or P & Q are antipodal.

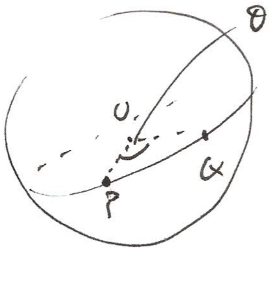
So, unless P & Q are antipodal, $\vec{OP}$ & $\vec{OQ}$ are not multiples of each other, so span a plane Π. Then $L = \Pi \cap S^2$ is the unique spherical line through P & Q.

If P & Q are antipodal, let l be the line spanned by $\vec{OP}$ (or $\vec{OQ} = -\vec{OP}$). Then <u>any plane</u> Π containing l gives a spherical line $L = \Pi \cap S^2$ through P & Q.



antipodal case

2.



$d(P,Q) = R \cdot \theta = 1 \cdot \theta = \theta$

where $\theta = \angle POQ$, $0 \leq \theta \leq \pi$, $\theta = \pi \iff P, O, Q$ are collinear $\iff P, Q$ antipodal.

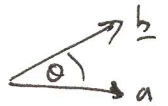
$\square$.

3. a) $d(P,Q) = \theta = \cos^{-1}(\vec{OP} \cdot \vec{OQ}) = \cos^{-1}\left(\frac{1}{3}(1,2,2) \cdot \frac{1}{3}(2,2,1)\right)$

(using $\|\vec{OP}\| = \|\vec{OQ}\| = 1$ & dot product formula $\underline{a} \cdot \underline{b} = \|\underline{a}\| \cdot \|\underline{b}\| \cdot \cos\theta$)

$= \cos^{-1}\left(\frac{1}{9}(1 \cdot 2 + 2 \cdot 2 + 2 \cdot 1)\right) = \cos^{-1}(8/9)$

$= 0.476\ldots$ radians



b : $L = \Pi_L \cap S^2$

$\Pi_L = \{\underline{x} \in \mathbb{R}^3 \mid \underline{x}\cdot\underline{n}_L = 0\}$, a plane thru 0

$\underline{n}_L$ normal vector to Π_L

We require $P, Q \in \Pi$, equivalently, $\overrightarrow{OP}$ & $\overrightarrow{OQ}$ perpendicular to $\underline{n}_L$,

so we can take $\underline{n}_L = \overrightarrow{OP} \times \overrightarrow{OQ} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} \times \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}$

$$= \frac{1}{9}\begin{pmatrix}-2\\3\\-2\end{pmatrix}$$

or (scaling) $\underline{n}_L = \begin{pmatrix}-2\\3\\-2\end{pmatrix}$

Equation $\underline{x}\cdot\underline{n}_L = 0$ is $-2x + 3y - 2z = 0$. $\square$.

4. a. Solve $x+y+z=0$
& $x+2y+3z=0$.

As in 235 : Augmented matrix $\begin{pmatrix}1 & 1 & 1 & \vdots & 0\\ 1 & 2 & 3 & \vdots & 0\end{pmatrix}$ $-R1$

Row reduce $\begin{array}{c} -R2 \\ \\ \end{array} \sim \begin{pmatrix}1 & 1 & 1 & \vdots & 0\\ 0 & 1 & 2 & \vdots & 0\end{pmatrix}$ echelon form

$\sim \begin{pmatrix}1 & 0 & -1 & \vdots & 0\\ 0 & 1 & 2 & \vdots & 0\end{pmatrix}$ reduced echelon form

$x - z = 0 \quad \rightsquigarrow \quad x = z$

$y + 2z = 0. \quad\quad y = -2z$

z free variable.

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\-2z\\z\end{pmatrix} = z\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $z \in \mathbb{R}$ arbitrary.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2 \quad : \quad x^2+y^2+z^2=1$

$\Rightarrow$ require $z = \dfrac{\pm 1}{\sqrt{1^2+(-2)^2+1^2}} = \dfrac{\pm 1}{\sqrt{6}}$.

$$L \cap M = \left\{ \frac{\pm 1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\} \quad \square.$$

b. Angle between L & M = angle between normal vectors

$L: \; x+y+z=0 \quad \Rightarrow \quad \underline{n}_L = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$M: \; x+2y+3z=0 \quad \Rightarrow \quad \underline{n}_M = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

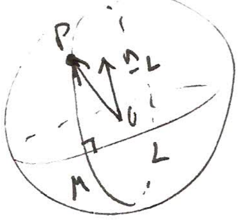
$$\Rightarrow \theta = \cos^{-1}\left(\frac{\underline{n}_L \cdot \underline{n}_M}{\|\underline{n}_L\| \cdot \|\underline{n}_M\|} \right)$$

$$= \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{3} \cdot \sqrt{1^2+2^2+3^2}} \right)$$

$$= \cos^{-1}\left(\frac{1+2+3}{\sqrt{42}} \right) = \cos^{-1}\left(\frac{6}{\sqrt{42}} \right)$$

$$= 0.388\ldots \text{ radians}$$

5. a&b

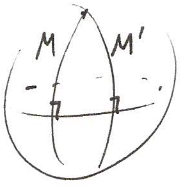


$$M = \Pi_M \cap S^2, \quad \Pi_M = \{ \underline{x} \in \mathbb{R}^3 \mid \underline{x} \cdot \underline{n}_M = 0 \}$$

$M \perp L \iff \underline{n}_M \perp \underline{n}_L \iff \underline{n}_M \cdot \underline{n}_L = 0.$

$P \in M \iff \underline{n}_M \cdot \overrightarrow{OP} = 0.$

So, can take $\underline{n}_M = \underline{n}_L \times \overrightarrow{OP}$, unless $\underline{n}_L$ & $\overrightarrow{OP}$ are parallel, in which case can take $M = \Pi_M \cap S^2$, Π_M any plane containing $\overrightarrow{OP}$:-



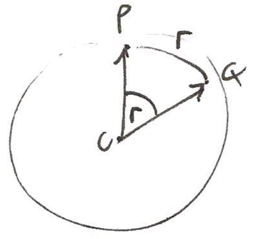
So M is unique provided $\underline{n}_L$ & $\overrightarrow{OP}$ not parallel; otherwise it is not uniquely determined.

c. As above, $\underline{n}_M = \underline{n}_L \times \vec{OP} = \begin{pmatrix}2\\4\\1\end{pmatrix} \times \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix}3\\-1\\-2\end{pmatrix}$$

or, scaling, $\underline{n}_M = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$, eq. of M is $3x - y - 2z = 0$. □.

6. a.



$\Rightarrow \vec{OP}\cdot\vec{OQ} = \cos r \quad (\|\vec{OP}\| = \|\vec{OQ}\| = 1)$

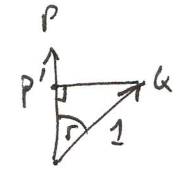
i.e. $C(P,r) = \Pi \cap S^2$

$$\Pi = \{\underline{x} \in \mathbb{R}^3 \mid \underline{x}\cdot\vec{OP} = \cos r\}$$

– a plane (not thru 0 in general) w/ normal vector $\vec{OP}$. □.

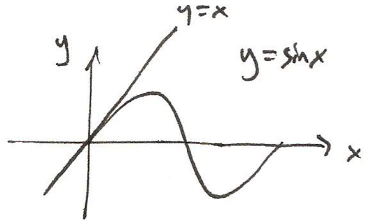
b. Let P' be the intersection of the plane Π w/ the line OP.

Then $|P'Q| = \sin r$, $C(P,r)$ is a Euclidean circle in Π w/ center P' & radius $\sin r$.



In particular, circumference of $C(P,r) = 2\pi\cdot(\sin r)$
$= 2\pi \sin r$

c.



$x > \sin x$ for $x > 0$:-

$f(x) := x - \sin x$

$f(0) = 0$.

$f'(x) = 1 - \cos x \geq 0 \quad \forall x \in \mathbb{R}$

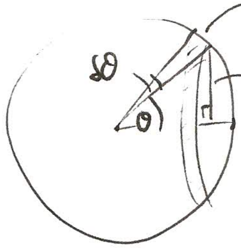
$(\,\&\ > 0$ unless $x = 2\pi k,\ k \in \mathbb{Z})$

$\Rightarrow f(x) > 0$ for $x > 0$. ✓.

So $2\pi \sin r < 2\pi r$ for $r > 0$. □.

d. $\lim\limits_{r\to\pi_-} 2\pi \sin r \underset{\substack{\sin\\ \text{cts}}}{=} 2\pi \sin \pi = 0.$ As r approaches π from below, the circle $C(P,r)$ shrinks to a point at the antipodal point to P.

7. a.



surface area of this strip $\approx (2\pi \sin\theta)\cdot \delta\theta.$

$$\rightsquigarrow \quad A = \int_0^r 2\pi \sin\theta \, d\theta$$

$$= 2\pi \left[-\cos\theta\right]_0^r$$

$$= 2\pi\cdot(1-\cos r)$$

b.

$$2\pi(1-\cos r) \overset{?}{<} \pi r^2$$

i.e. $1-\cos r \overset{?}{<} r^2$ for $r>0$.

$$f(x) = x^2 - (1-\cos x)$$

$$f(0) = 0.$$

$$f'(x) = 2x - \sin x = x + (x-\sin x) \overset{\text{by Q6c}}{\geqslant} x \geqslant 0 \quad \text{for } x \geqslant 0$$

$\Rightarrow$ $f(x) > 0$ for $x>0$ $\Rightarrow$ $\square$.

c.

$$\lim_{r\to\pi_-} 2\pi(1-\cos r) = 2\pi(1-\cos\pi) = 2\pi(1-(-1)) = 4\pi = \text{Area}(S^2)$$

As r approaches π from below, the spherical disc grows to cover the whole sphere.