1.a.i. $$F^{-1}(F(x,y,z)) = F^{-1}\left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \frac{1}{\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2+1}\left(\frac{2x}{1-z}, \frac{2y}{1-z}, \left(\frac{x}{1-z}\right)^2+\left(\frac{y}{1-z}\right)^2-1\right)$$

$$= \frac{(1-z)^2}{x^2+y^2+(1-z)^2}\cdot\left(\frac{2x}{1-z}, \frac{2y}{1-z}, \frac{x^2+y^2-(1-z)^2}{(1-z)^2}\right)$$

$$= \frac{(1-z)^2}{x^2+y^2+z^2-2z+1}\cdot\left(\frac{2x}{1-z}, \frac{2y}{1-z}, \frac{x^2+y^2-z^2+2z-1}{(1-z)^2}\right)$$

$$= \frac{(1-z)^2}{2-2z}\cdot\left(\frac{2x}{1-z}, \frac{2y}{1-z}, \frac{2z-2z^2}{(1-z)^2}\right)$$

$x^2+y^2+z^2=1$

OR $x^2+y^2=1-z^2$

$$= \frac{(1-z)}{2}\cdot\left(\frac{2x}{1-z}, \frac{2y}{1-z}, \frac{2z(1-z)}{(1-z)^2}\right)$$

$$= (x,y,z). \square$$

ii. $$F(F^{-1}(u,v)) = F\left(\frac{1}{u^2+v^2+1}(2u,2v,u^2+v^2-1)\right)$$

$$= \frac{\frac{1}{u^2+v^2+1}(2u,2v)}{1-\frac{u^2+v^2-1}{u^2+v^2+1}} = \frac{(2u,2v)}{(u^2+v^2+1)-(u^2+v^2-1)} = \frac{(2u,2v)}{2} = (u,v).$$

$\square$

b. $$(x,y,z) = F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$$

$$\Rightarrow x^2+y^2+z^2 = \frac{1}{(u^2+v^2+1)^2}\left((2u)^2+(2v)^2+(u^2+v^2-1)^2\right)$$

$$= \frac{1}{(u^2+v^2+1)^2}\left(4(u^2+v^2)+(u^2+v^2-1)^2\right) = \frac{1}{(u^2+v^2+1)^2}(u^2+v^2+1)^2 = 1.$$

$\square$.

$4ab+(a-b)^2 = (a+b)^2$

where $a=u^2+v^2$, $b=1$.

c. $$\frac{1}{(u^2+v^2+1)^2}\cdot\left((2u)^2+(2v)^2+(u^2+v^2-1)^2\right) = 1.$$

i.e. $$(2u)^2+(2v)^2+(u^2+v^2-1)^2 = (u^2+v^2+1)^2$$

Now choose $u,v\in\mathbb{N}$ to get a solution of $a^2+b^2+c^2=d^2$ in positive integers a,b,c,d.

e.g. $u=v=1$ $\qquad 2^2+2^2+1^2=3^2.$ $\square$.

2. a.

$$\begin{aligned} F(C_1) &= \{(u,v) \mid F^{-1}(u,v)\in C_1\} \\ &= \left\{(u,v) \;\middle|\; \frac{1}{u^2+v^2+1}(2u,2v,u^2+v^2-1)\in \Pi_1\right\} \\ &= \left\{(u,v) \;\middle|\; 1\cdot\frac{2u}{(u^2+v^2+1)} + 2\cdot\frac{2v}{(u^2+v^2+1)} + 3\cdot\frac{(u^2+v^2-1)}{(u^2+v^2+1)} = 3\right\} \end{aligned}$$

simplify

$$2u+4v+3u^2+3v^2-3 = 3u^2+3v^2+3$$

$$2u+4v=6$$

$$v=-\tfrac{1}{2}u+\tfrac{3}{2}.$$ line in (u,v) plane.

(Alternative solution: Observe $\underset{(0,0,1)}{\underset{\shortparallel}{N}}\in C_1$: $1.0+2.0+3.1=3$ ✓

So $F(C_1) = \underset{(x+2y+3z=3)}{\Pi_1}\cap\underbrace{(xy\text{-plane})}_{(z=0)} = (x+2y=3)\subset\mathbb{R}^2$

i.e. line $y=-\frac{1}{2}x+\frac{3}{2}$ in $\mathbb{R}^2$.)

$\square$.

b. As in part a, eq. of $F(C_2)$ given by

$$\frac{3\cdot 2u}{(u^2+v^2+1)} + \frac{4\cdot 2v}{(u^2+v^2+1)} + \frac{5\cdot(u^2+v^2-1)}{(u^2+v^2+1)} = 6.$$

$$6u + 8v + 5u^2 + 5v^2 - 5 = 6u^2 + 6v^2 + 6$$

$$0 = u^2+v^2 - 6u - 8v + 11$$

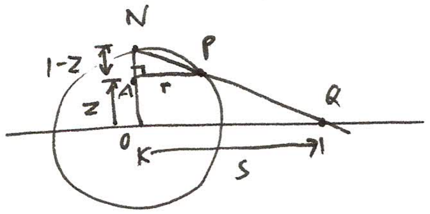
$$0 = (u-3)^2 + (v-4)^2 + 11 - 9 - 16$$

$$(u-3)^2 + (v-4)^2 = 14$$

circle, center $(3,4)$, radius $\sqrt{14}$.

3.

a.



$$r = \sqrt{x^2+y^2}$$

$$s = |OQ| = ?$$

$$\Delta NAP \sim \Delta NOQ$$

$$\Rightarrow \frac{s}{r} = \frac{1}{1-z}, \quad s = \frac{r}{1-z}.$$

$$|OQ| = \frac{r}{1-z} = \frac{\sqrt{x^2+y^2}}{1-z} \underset{x^2+y^2+z^2=1}{=} \frac{\sqrt{1-z^2}}{1-z} = \frac{\sqrt{(1-z)(1+z)}}{(1-z)} = \sqrt{\frac{(1+z)}{(1-z)}}. \quad \square$$

b.

$$\lim_{z\to 1^-} \sqrt{\frac{(1+z)}{(1-z)}} = \infty \quad \text{because} \quad \lim_{z\to 1^-} 1-z = 0^+$$

$$\& \lim_{z\to 1} 1+z = 2 > 0.$$

4. a $\quad R(x,y,z) = (x,y,-z).$

b. $\quad T(u,v) = F\circ R\circ F^{-1}(u,v)$

$$= F\circ R\left(\frac{1}{u^2+v^2+1}\,(2u,2v,u^2+v^2-1)\right)$$

$$= F\left(\frac{1}{u^2+v^2+1}\,(2u,2v,-(u^2+v^2-1))\right)$$

$$= \frac{\frac{1}{u^2+v^2+1}\,(2u,2v)}{1+\frac{u^2+v^2-1}{u^2+v^2+1}} = \frac{(2u,2v)}{u^2+v^2+1+(u^2+v^2-1)} = \frac{(2u,2v)}{2u^2+2v^2} = \frac{(u,v)}{u^2+v^2}. \quad \square$$

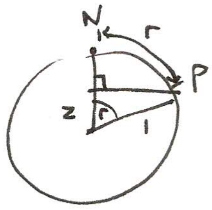
c. $\quad u^2+v^2=1 \Rightarrow T(u,v)=(u,v).$

$$\|T(u,v)\|^2 = \left\|\frac{(u,v)}{u^2+v^2}\right\|^2 = \frac{u^2+v^2}{(u^2+v^2)^2} = \frac{1}{(u^2+v^2)}.$$

So, $\|(u,v)\|<1 \Rightarrow \|T(u,v)\|>1$

$\|(u,v)\|>1 \Rightarrow \|T(u,v)\|<1. \quad \square$

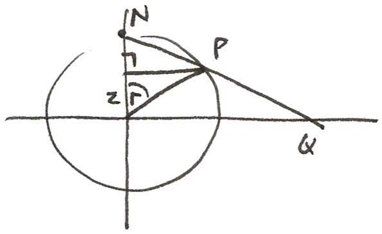
5a.



$z=\cos r$

$\Rightarrow C = S^2\cap\Pi, \quad \Pi: z=\cos r$

b.



$$|OQ| \overset{3a}{=} \sqrt{\frac{1+z}{1-z}} = \sqrt{\frac{1+\cos r}{1-\cos r}}$$

$\Rightarrow$ $F(C)$ is a circle, center O, radius $\sqrt{\dfrac{1+\cos r}{1-\cos r}}$.

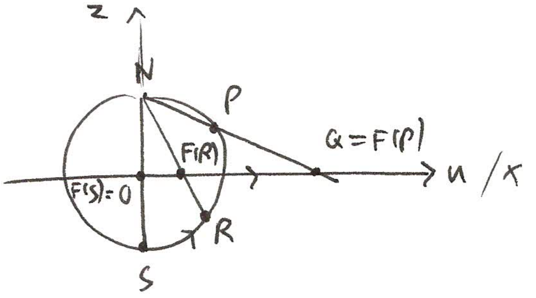
c. $\quad \text{length}(C) = \int_a^b \frac{2}{u^2+v^2+1}\cdot\sqrt{u'^2+v'^2}\,dt \qquad (u,v):[a,b]\to\mathbb{R}^2$ parametrization of $F(C)$

$$= \frac{2}{\frac{1+\cos r}{1-\cos r}+1}\cdot\int_a^b\sqrt{u'^2+v'^2}\,dt = \frac{2\cdot(1-\cos r)}{1+\cos r+(1-\cos r)}\cdot\overbrace{2\pi\cdot\sqrt{\frac{1+\cos r}{1-\cos r}}}^{\text{circumference of } F(C)}$$

$$= (1-\cos r)\cdot 2\pi \cdot \sqrt{\frac{1+\cos r}{1-\cos r}}$$

$$= 2\pi\sqrt{(1+\cos r)(1-\cos r)} = 2\pi\sqrt{1-(\cos r)^2} = 2\pi \sin r \quad \square.$$

6. a.



Slice of S^2 given by $(y=0)$ / xz plane.

$F: S^2 \setminus \{N\} \to \mathbb{R}^2$

sends $\underbrace{\text{arc } SP}_{\gamma}$ to line segment OQ.

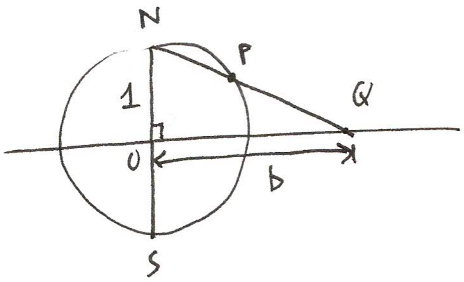
b.

$$\text{length}(\gamma) = \int_0^b \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}\,dt$$

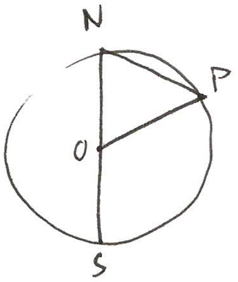
$$= \int_0^b \frac{2}{t^2+0^2+1}\sqrt{1^2+0^2}\,dt = \int_0^b \frac{2}{t^2+1}\,dt$$

$$= 2\cdot\left[\tan^{-1}t\right]_0^b = 2\tan^{-1}b. \quad \square.$$

c.



$\Rightarrow \tan(\angle ONP) = \tan(\angle ONQ) = b/1 = b. \quad \dagger$



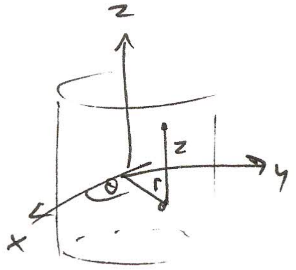
$\angle ONP = \angle SNP = \frac{1}{2}\angle SOP \quad *$

(angle subtended by chord SP at center O is twice the angle subtended at the circumference.)

$\text{length}(\gamma) = \angle SOP$ (γ is arc of spherical line subtending angle $\angle SOP$ at O)

$= 2\angle ONP \underset{\dagger}{=} 2\tan^{-1}b.$ This checks with (b).

7. a.



cylindrical polar coordinates:

$$(x,y,z) = (r\cos\theta, r\sin\theta, z)$$

$$G^{-1}: R = [0,2\pi) \times (-1,1) \xrightarrow[\sim]{\text{bijection}} S^2 \setminus \{N,S\}$$

$G^{-1}(u,v)$ has cylindrical polar coords $\theta = u$, $z = v$, & $r = \sqrt{x^2+y^2} = \sqrt{1-z^2}$

$x^2+y^2+z^2=1$ (eq. of S^2).

i.e. $G^{-1}(u,v) = (\sqrt{1-v^2}\cos u, \sqrt{1-v^2}\sin u, v)$.

b. $\underset{\sim}{x}(u,v) = (\sqrt{1-v^2}\cos u, \sqrt{1-v^2}\sin u, v)$

$$\frac{\partial \underset{\sim}{x}}{\partial u} = (\sqrt{1-v^2}\cdot(-\sin u), \sqrt{1-v^2}\cdot\cos u, 0)$$

$$\frac{\partial \underset{\sim}{x}}{\partial v} = \left(\frac{-v}{\sqrt{1-v^2}}\cdot\cos u, \frac{-v}{\sqrt{1-v^2}}\cdot\sin u, 1\right) \qquad \left(\frac{d}{dv}\left(\sqrt{1-v^2}\right) = \frac{1}{2}\cdot -2v\cdot(1-v^2)^{-1/2} \text{ by C.R.}\right)$$

$$\frac{\partial \underset{\sim}{x}}{\partial u} \times \frac{\partial \underset{\sim}{x}}{\partial v} = \begin{pmatrix} -\sqrt{1-v^2}\cdot\sin u \\ \sqrt{1-v^2}\cdot\cos u \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{-v}{\sqrt{1-v^2}}\cos u \\ \frac{-v}{\sqrt{1-v^2}}\sin u \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1-v^2}\cdot\cos u \\ \sqrt{1-v^2}\sin u \\ v(\sin u)^2 + v(\cos u)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-v^2}\cos u \\ \sqrt{1-v^2}\sin u \\ v \end{pmatrix}.$$

$$\therefore \left\| \frac{\partial \underset{\sim}{x}}{\partial u} \times \frac{\partial \underset{\sim}{x}}{\partial v} \right\| = \sqrt{(1-v^2)(\cos u)^2 + (1-v^2)(\sin u)^2 + v^2} = \sqrt{(1-v^2)+v^2} = \sqrt{1} = 1.$$

$$\text{Area}(G^{-1}(T)) = \int_T \left\| \frac{\partial \underset{\sim}{x}}{\partial u} \times \frac{\partial \underset{\sim}{x}}{\partial v} \right\| du\,dv = \int_T 1\cdot du\,dv = \text{Area}(T). \quad \square.$$