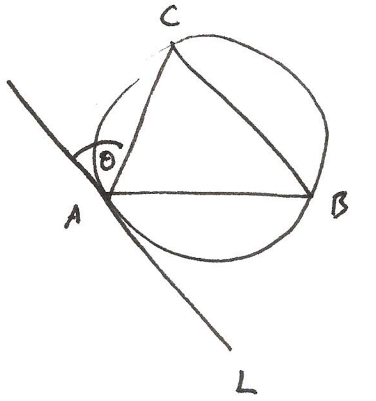
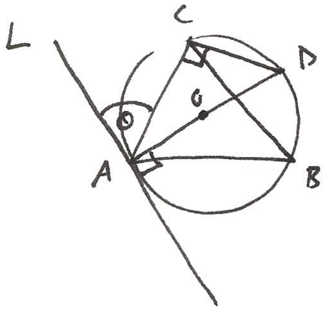
Wednesday 10/9/19

MATH 461 HW4 SOLUTIONS.

1.



Draw the diameter of the circle passing through A, meeting the circle $\mathcal{C}$ again at D.



L is perpendicular to AD (tangent line is perpendicular to radius)

So $\angle CAD = \pi/2 - \theta$.

$\angle ACD = \pi/2$ (angle in a semicircle)

$\therefore \angle ADC = \pi - (\pi/2 - \theta) - \pi/2 = \theta$ (angle sum of $\Delta ACD = \pi$)

$\therefore \angle ABC = \angle ADC = \theta$ (angles subtended by a chord at the circumference are equal).

2. a Recall that the perpendicular bisector L of a line segment AB is equal to the locus of points P such that $|AP| = |BP|$:- $\quad (*)$

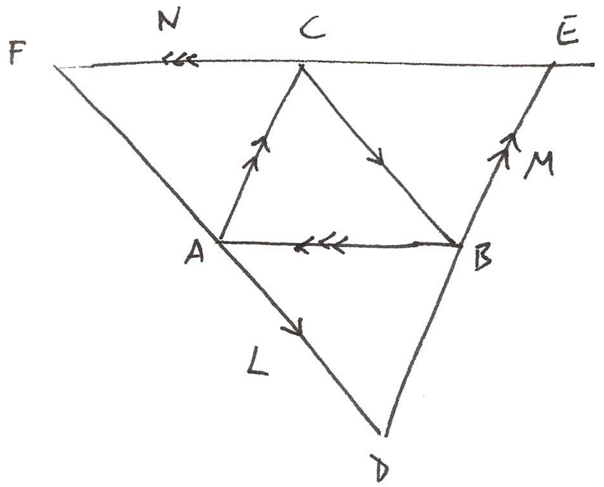
$$L = \{ P \mid |AP| = |BP| \}$$

Now, let P be the intersection point of the perpendicular bisectors of DE & EF.† Then $|PD| = |PE|$ and $|PE| = |PF|$ by $(*)$.

So $|PD| = |PF|$, and P lies on the perpendicular bisector of DF by $(*)$ again.

That is, the perpendicular bisectors of DE, EF & FD all meet at P. $\square$.

($\dagger$ these do intersect by HW1 Q3a) b.



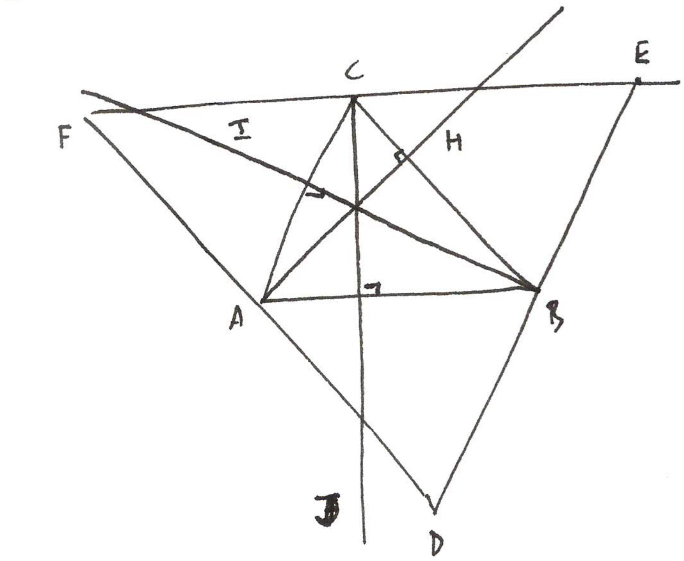
ADBC is a parallelogram $\overset{\text{WS1}}{\Longrightarrow}$ $|AD| = |BC|$

FABC is a parallelogram $\overset{\text{WS1}}{\Longrightarrow}$ $|FA| = |BC|$

$\Big\}\Longrightarrow |AD| = |AF|$

Similarly $|BD| = |BE|$, & $|CE| = |CF|$ $\square$.

c.



Draw $\triangle DEF$ as in b.

Now observe that, since H is perpendicular to BC, and DF is parallel to BC, H is perpendicular to DF.

And, since $|AD| = |AF|$ by part b, H is the perpendicular bisector of DF.

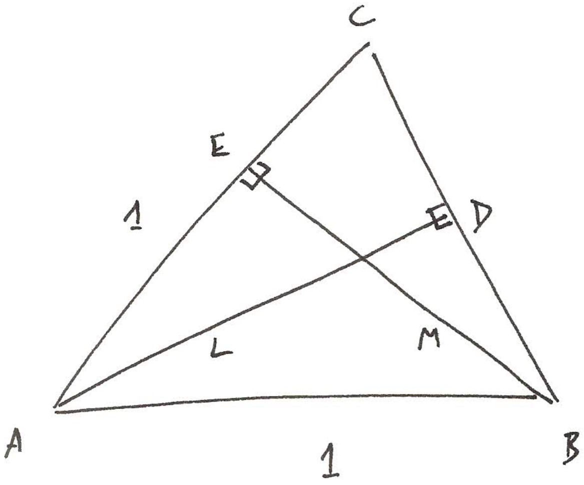
Similarly I is the perpendicular bisector of DE

& J is " " " " EF

So, by part a., H, I, & J all meet at a point.

$\square$.

3.



$\angle BAC = 2\theta.$

$\Delta ABD \cong \Delta ACD$ (SSS) :- $|AB| = |AC| = 1$

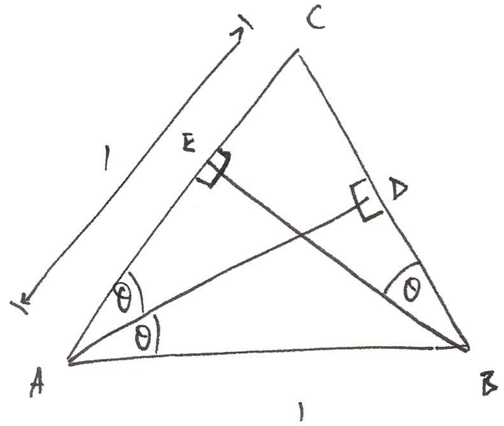
$|AD| = |AD|$

$|BD| \underset{P.T.}{=} \sqrt{|AB|^2 - |AD|^2} = \sqrt{|AC|^2 - |AD|^2} \underset{P.T.}{=} |CD|$

So $\angle BAD = \angle CAD = \frac{1}{2}(2\theta) = \theta.$

Now $\angle ACD = \pi - (\pi/2 + \theta) = \pi/2 - \theta$ (angle sum of $\Delta ACD = \pi$)

& $\angle CBE = \pi - (\pi/2 + (\pi/2 - \theta)) = \theta$ (angle sum of $\Delta CBE = \pi$)



Now, using "SOHCAHTOA"

$|BE|/1 = \sin 2\theta$ 1

$|AE|/1 = \cos 2\theta.$ 2

$|BD|/1 = |CD|/1 = \sin\theta$

So $|BC| = |BD| + |CD| = 2\sin\theta$ 3

$\frac{|BE|}{|BC|} = \cos\theta$ 4

So $\sin 2\theta \overset{1}{=} |BE| \overset{4}{=} |BC|\cos\theta \overset{3}{=} 2\sin\theta\cos\theta.$

And

$$\cos 2\theta \overset{2}{=} |AE| = |AC| - |EC| = 1 - |EC| = 1 - |BC| \sin\theta$$

$$\frac{|EC|}{|BC|} = \sin\theta$$

$$\underset{3}{=} 1 - 2\sin\theta \cdot \sin\theta$$

$$= 1 - 2(\sin\theta)^2. \quad \square.$$

4. a. $$\cos 3\theta = \cos(2\theta + \theta)$$

$$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$

$$= \cancel{\cos 2\theta}((\cos\theta)^2 - (\sin\theta)^2)\cos\theta - 2\sin\theta\cos\theta \cdot \sin\theta$$

$$= (2(\cos\theta)^2 - 1)\cdot\cos\theta - 2\cos\theta\cdot(1 - (\cos\theta)^2)$$

$(\sin\theta)^2 = 1 - (\cos\theta)^2$

$$= 4(\cos\theta)^3 - 3\cos\theta. \quad \square.$$

Alternative: $$\cos 3\theta = \frac{e^{i3\theta} + e^{i(-3\theta)}}{2}$$

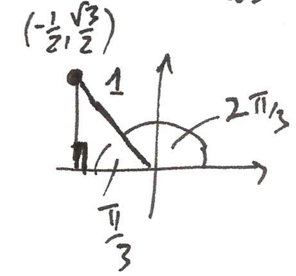
$$4(\cos\theta)^3 - 3\cos\theta = 4\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 - 3\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$

$$= 4\cdot\left(\frac{1}{2}\right)^3\cdot\left((e^{i\theta})^3 + 3(e^{i\theta})^2\cdot e^{-i\theta} + 3e^{i\theta}\cdot(e^{-i\theta})^2 + (e^{-i\theta})^3\right) - 3\cdot\frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2}\left(e^{i3\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-i3\theta}\right) - \frac{3}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2}(e^{i3\theta} + e^{-i3\theta}) = \cos 3\theta. \quad \square$$

b. $$\cos(2\pi/3) = -1/2$$



So $$-1/2 = \cos(3\cdot 2\pi/9) \overset{a.}{=} 4\cdot\left(\cos\frac{2\pi}{9}\right)^3 - 3\cdot(\cos 2\pi/9)$$

So, for $X = \cos 2\pi/9$, $\quad -\frac{1}{2} = 4X^3 - 3X$.

$$\text{or} \quad 0 = \underbrace{8X^3 - 6X + 1}_{p(X)}.$$

5. a. If we can construct angle $2\pi/n$, we can construct a regular n-gon:-

Let $\mathcal{C}$ be a circle center O and P a point on $\mathcal{C}$.

Construct points $P = P_1, P_2, \ldots, P_n$ on $\mathcal{C}$ such that $\angle P_i O P_{i+1} = 2\pi/n$ $\quad i = 1, \ldots, n-1$.

(& the $\angle P_n O P_1 = 2\pi/n$ because the total angle at a point equals 2π)

Then $P_1 P_2 \cdots P_n$ is a regular n-gon.

Now, $2\pi/5$ can be constructed (as proved in class).

So $2\pi/10 = \frac{1}{2} \cdot 2\pi/5$ can be constructed (because we can bisect any angle using ruler & compass).

So the regular 10-gon can be constructed.

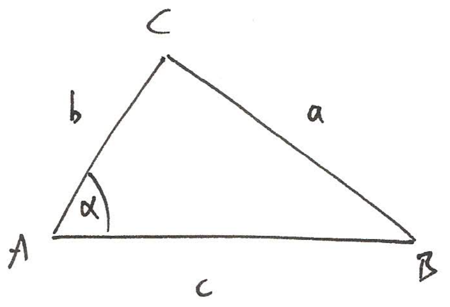
b. $$\frac{2\pi}{15} = 2\pi \cdot \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{\pi}{3} - \frac{2\pi}{10}$$

Angle $\pi/3$ can be constructed (angle of equilateral triangle)

Angle $2\pi/10$ can be constructed (by a.)

So angle $2\pi/15$ can be constructed (can copy angles as in HW1Q1, so can add & subtract angles.), & the regular 15-gon can be constructed.

6.



$$a^2 = b^2+c^2-2bc\cos\alpha \qquad \text{(Cosine rule)}$$

Note $\cos\alpha > -1$ $(*)$: $0<\alpha<\pi$, $0<\cos\alpha<1$ for $0<\alpha<\pi/2$ by P.T.

$\cos(\pi/2) = 0$

$\cos\alpha = -\cos(\pi-\alpha)$ for $\pi/2<\alpha<\pi$

$\Rightarrow -1<\cos\alpha<0$ for $\pi/2<\alpha<\pi$.

So $$a^2 = b^2+c^2-2bc\cos\alpha \overset{\dagger}{\underset{(*)}{<}} b^2+c^2-2bc\cdot(-1) = b^2+c^2+2bc = (b+c)^2$$

$\Rightarrow a < (b+c)$ (taking square roots & noting $a,\ b+c>0$)

□.

[NB. $a<b$ & $c<0 \Rightarrow ca>cb$. This was used to obtain the inequality † above:- $-2bc<0$, $\cos\alpha>-1$

$\Rightarrow -2bc\cos\alpha < -2bc\cdot(-1)$.]

7. a. $x^3+3x-14=0$.

If $x = a/b$ is a rational solution of the equation, with $a,b\in\mathbb{Z}$, $b>0$, & $\gcd(a,b)=1$,

then $b \mid 1$ & $a \mid -14$ by Q8, so $x = \pm1, \pm2, \pm7$, or ±14.

Checking we find $x=2$ is a solution: $2^3+3\cdot2-14 = 8+6-14=0$.

b. Tartaglia's formula gives:

$$x = \sqrt[3]{7+\sqrt{50}} - \sqrt[3]{-7+\sqrt{50}} = \sqrt[3]{7+5\sqrt{2}} - \sqrt[3]{-7+5\sqrt{2}}$$

$$= (1+\sqrt{2}) - (-1+\sqrt{2}) = 2.$$

Remark: To get the other two (complex) solutions, one needs to take complex cube roots in Tartaglia's formula.

8. $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$

Suppose $x = a/b$, $a,b \in \mathbb{Z}$, $b>0$, $\gcd(a,b)=1$, is a solution of $p(x)=0$.

i.e. $a_n\left(\frac{a}{b}\right)^n + \dots + a_1\left(\frac{a}{b}\right) + a_0 = 0.$

Clearing denominators $a_n \cdot a^n + a_{n-1} a^{n-1} b + \dots + a_1 \cdot a \cdot b^{n-1} + a_0 \cdot b^n = 0.$

$$\Rightarrow b \mid a_n \cdot a^n \;\&\; a \mid a_0 \cdot b^n$$

$$\underset{\gcd(a,b)=1}{\Rightarrow} b \mid a_n \;\&\; a \mid a_0 \quad \square.$$

9. a) $\gcd(a,p)=1$ unless $p \mid a$ for p prime.

So $\phi(p) = |\{1,2,\dots,p-1\}| = p-1.$

b) Similarly $\gcd(a,p^\alpha)=1 \iff p \nmid a$

So $\phi(p^\alpha) = |\{1,2,\dots,p-1,p+1,\dots,2p-1,2p+1,\dots\dots,p^\alpha-1\}|$

$$= p^\alpha - \underbrace{p^{\alpha-1}}_{\#\text{ multiples of } p \,\le\, p^\alpha} = p^{\alpha-1}\cdot(p-1)$$

c) $\phi(n) = \phi(p_1^{\alpha_1}\cdots p_r^{\alpha_r}) = p_1^{\alpha_1-1}(p_1-1)\cdots p_r^{\alpha_r-1}(p_r-1)$

So $\phi(n)$ is a power of 2

$\Leftrightarrow$ for each prime p_i dividing n,

either $p_i = 2$ & α_i is arbitrary

or p_i is odd, $\alpha_i = 1$, & $p_i - 1$ is a power of 2. $\square$.

d. We follow the hint:

If m is not a power of 2, write $m = ab$, b odd, $b > 1$.

$$x^b + 1 = (x+1)(x^{b-1} - x^{b-2} + - \cdots + x^2 - x + 1)$$

set $x = 2^a$:

$$p = 2^m + 1 = 2^{ab} + 1 = (2^a)^b + 1 = \underset{1}{\underset{\vee}{(2^a+1)}}\underset{1}{\underset{\vee}{((2^a)^{b-1} - (2^a)^{b-2} + \cdots + 1)}}$$

$\Rightarrow$ p is NOT prime $\#$ $\square$.