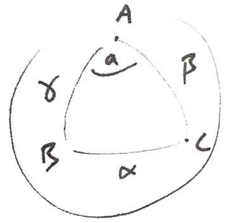
Friday 12/6/19.

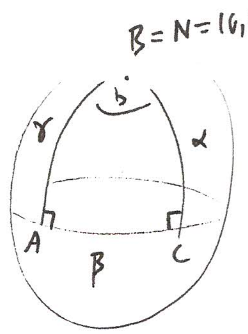
MATH 461 HW 8 SOLUTIONS.

1. a. Spherical Cosine Rule: $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$



$a = \pi/2 \Rightarrow \cos a = 0 \Rightarrow \boxed{\cos\alpha = \cos\beta\cos\gamma}$

b.



$\alpha = \gamma = \pi/2 \Rightarrow \cos\alpha = \cos\gamma = 0.$

So $\cos\alpha = \cos\beta \cdot \cos\gamma$

becomes $0 = \cos\beta \cdot 0$ ✓.

2. a A spherical circle $C \subset S^2$ is given by $C = \Pi \cap S^2$, where $\Pi \subset \mathbb{R}^3$ is a plane. (see HW7Q6).

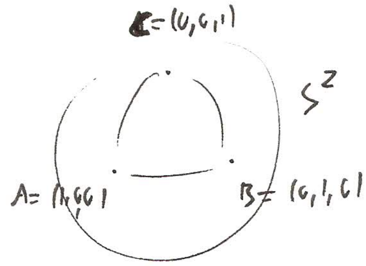
3 points $A, B, C \in \mathbb{R}^3$ determine a unique plane $\Pi \subset \mathbb{R}^3$ such that $A, B, C \in \Pi$ unless A, B, C lie on a line.

But if $A, B, C \in S^2$ then A, B, C cannot lie on a line (because a line intersects the sphere S^2 at ≤ 2 points)

So there's a unique plane $\Pi \subset \mathbb{R}^3$ containing A, B, C

& so a unique spherical circle $C = \Pi \cap S^2$ passing through A, B, C. □.

b.



The circumscribed circle $\mathcal{C}$ is given by
$\mathcal{C} = \Pi \cap S^2$
where $\Pi \subset \mathbb{R}^3$ is the plane containing A, B, C.

$\Pi : \quad ax + by + cz = d.$

Substituting $(x,y,z) = (1,0,0), (0,1,0)$ & $(0,0,1)$

gives $\quad a = d, \; b = d, \; c = d$

i.e. $\Pi: \quad d \cdot x + d \cdot y + d \cdot z = d$,

or $\Pi: \quad x + y + z = 1 \quad (\dagger) \quad$ (dividing by d)

The spherical center of $\mathcal{C}$ is the point P given by $\vec{OP} = \pm \underline{n} / \|\underline{n}\|$, where

$\underline{n}$ is the normal vector of Π. From the equation $(\dagger)$ of Π, we see

$\underline{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $\vec{OP} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, i.e. $P = \frac{1}{\sqrt{3}}(1,1,1)$.

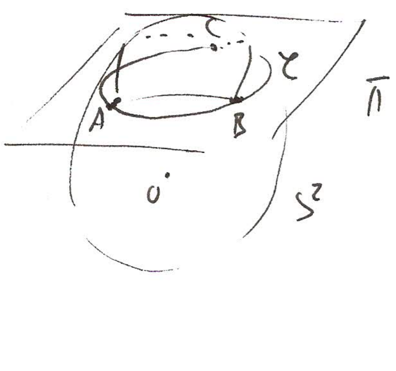
The spherical radius r of $\mathcal{C}$ is given by

$r = d_{S^2}(P, Q) \quad$ for $Q \in \mathcal{C}$,

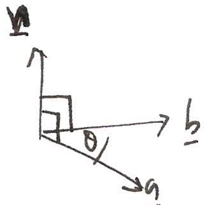
e.g. $r = d_{S^2}(P, A) = \cos^{-1}\left(\vec{OP} \cdot \vec{OA}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$

$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.955$ radians.

c.



Recall the cross product



$\underline{a} \times \underline{b} = \|\underline{a}\| \cdot \|\underline{b}\| \cdot \sin\theta \cdot \underline{n}$

where $\underline{n}$ is a vector of length 1 perpendicular to $\underline{a}$ & $\underline{b}$, such that $\underline{a}, \underline{b}, \underline{n}$ is a right handed set of vectors (

)

So, a normal vector to Π is given by

$$\overrightarrow{AB} \times \overrightarrow{AC} = (\overrightarrow{OB} - \overrightarrow{OA}) \times (\overrightarrow{OC} - \overrightarrow{OA}),$$

& the center of the circumscribed circle is given by

$$\frac{\overrightarrow{AB} \times \overrightarrow{AC}}{\| \overrightarrow{AB} \times \overrightarrow{AC} \|} \quad \square.$$

3a. $$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

$$2 \leq p \leq q \leq r$$

$$\Rightarrow \frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{r} \Rightarrow \frac{3}{p} \geq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \text{ equal iff } p=q=r$$

$$\Rightarrow p \leq 3, \text{ equal iff } (p,q,r) = (3,3,3).$$

If $p=2$, $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $\frac{2}{q} \geq \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, equal iff $q=r$

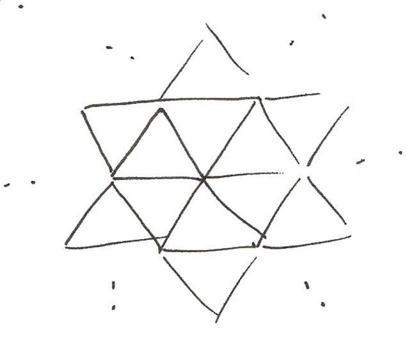
$$\Rightarrow q \leq 4, \text{ equal iff } (p,q,r) = (2,4,4).$$

Remaining cases: $q = 2 \Rightarrow \#$ $\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{r} = 1 \Rightarrow \frac{1}{r} = 0 \Rightarrow r = \infty\right)$

$q = 3 \Rightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{r} = 1$, $r = 6$.

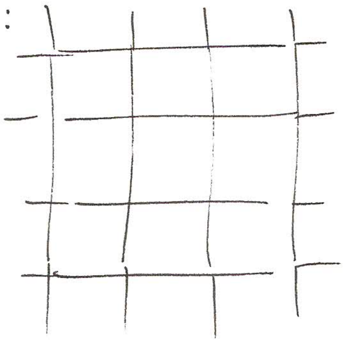
So the possibilities are $(p,q,r) = (3,3,3), (2,4,4), (2,3,6)$.

b. Tilings: $(3,3,3)$:



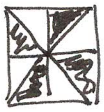
(equilateral triangles)

$(2,4,4)$:

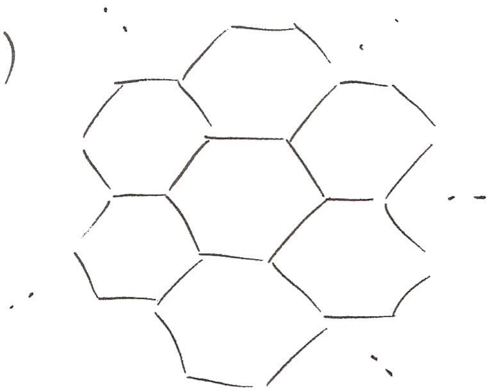


square tiling

$\rightsquigarrow$ subdivide each square as shown

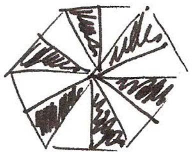


$(2,3,6)$



hexagonal tiling

subdivide each hexagon as shown :



$\square$.

4a. $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1 \qquad 2 \leq p \leq q \leq r$

Similar to 3a, $\frac{3}{p} > 1$, $p < 3$, $\underline{p=2}$.

$\frac{1}{q}+\frac{1}{r} > \frac{1}{2} \qquad \frac{2}{q} > \frac{1}{2}$, $q < 4$, $\underline{q=2 \text{ or } q=3}$.

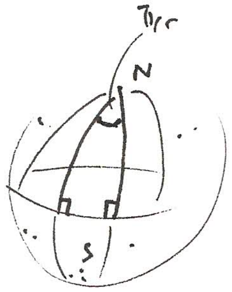
If $p=q=2$, $\frac{1}{2}+\frac{1}{2}+\frac{1}{r} > 1$ for any r, so have infinite series $(2,2,r)$ of possibilities.

If $p=2, q=3$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{r} > 1 \iff \frac{1}{r} > \frac{1}{6} \iff r < 6$, $r = 3,4,5$.

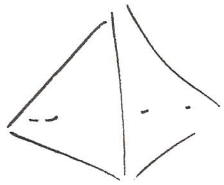
so $(p,q,r) = (2,3,3), (2,3,4), (2,3,5)$. $\square$.

b. We follow the hint.

$(2,2,r)$:

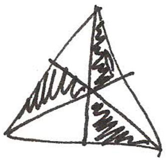


$(2,3,3)$

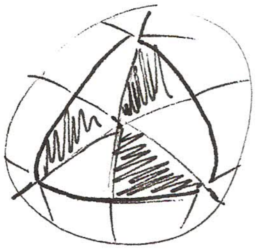


tetrahedron

subdivide faces as shown :



Project onto circumscribed sphere of tetrahedron :

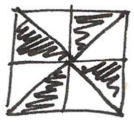


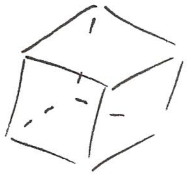
This gives a tiling of S^2 by congruent spherical triangles (the triangles are congruent by the symmetry of the tetrahedron)

We can compute the angles using "angle at a point $= 2\pi$" & all angles at a point are equal.

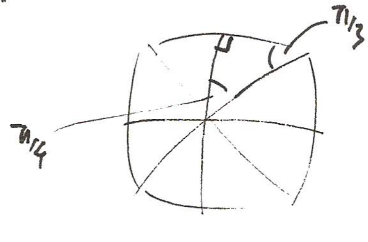
$\rightsquigarrow$ angles $2\pi/6$, $2\pi/6$, $2\pi/4$ $=$ $\pi/3, \pi/3, \pi/2$ $\square$.

$(2,3,4)$ Similar to $(2,3,3)$, subdivide faces of cube :



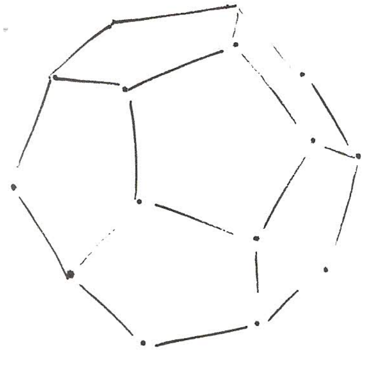


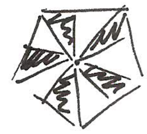
& project



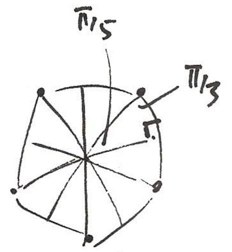
$\square$.

$(2,3,5)$. Subdivide faces of dodecahedron





, & project.



(compare handout from class) □.

4c. Area of $S^2 = 4\pi r^2 = 4\pi$ (radius $r=1$)

Area of triangle $= a+b+c-\pi = \frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{r}-\pi$

$= \pi\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right).$

$\therefore$ # triangles $= \dfrac{4\pi}{\pi\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right)} = \dfrac{4pqr}{qr+pr+pq-pqr}$

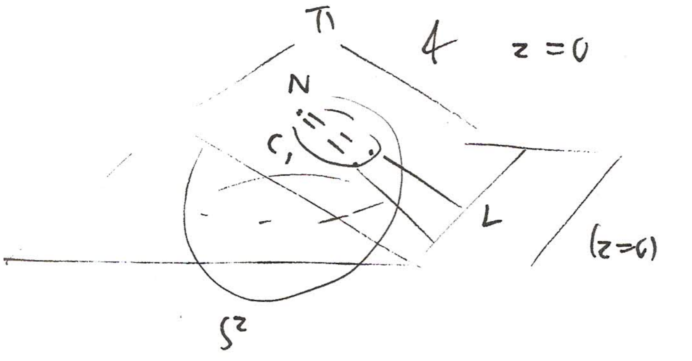
5. a. $C_1 = \Pi_1 \cap S^2$

$\Pi_1 : \ x+2y+3z=3.$

$(0,0,1) = N \in C_1 : \quad 0+2\cdot 0+3\cdot 1 = 3$ ✓.

So $F(C_1 \setminus \{N\}) = L = \Pi_1 \cap (z=0)$, line in xy-plane.

Eq. of L: $\quad x+2y+3z=3$ & $z=0$ $\Rightarrow (x+2y=3) \subset \mathbb{R}^2$, line in xy-plane.



b. $C_2 = \Pi_2 \cap S^2$

$\Pi_2 : \ 3x+4y+5z=6.$

$N \notin C_2$. So $F(C_2)$ is a circle in the xy-plane

Use formula $F^{-1}(u,v) = \dfrac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$

to compute eq. of $F(C_2)$:

$(u,v) \in F(C_2) \iff F^{-1}(u,v) \in C_2 = \Pi_2 \cap S^2$

$\iff F^{-1}(u,v) \in \Pi_2$

$\iff \dfrac{1}{u^2+v^2+1}\left(3\cdot 2u + 4\cdot 2v + 5\cdot(u^2+v^2-1)\right) = 6$

Rearrange: $6u + 8v + 5u^2 + 5v^2 - 5 = 6u^2 + 6v^2 + 6$

$$0 = u^2 + v^2 - 6u - 8v + 11$$

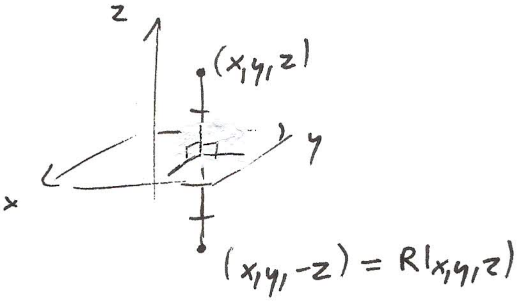
"complete the square" ×2 $\quad 0 = (u-3)^2 + (v-4)^2 + 11 - 9 - 16$

$$(u-3)^2 + (v-4)^2 = 14$$

circle, center $(3,4)$, radius $\sqrt{14}$. □.

6.

a. $R(x,y,z) = (x,y,-z)$



b.

$$T(u,v) = F(R(F^{-1}(u,v))) = F\left(R\left(\frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)\right)\right)$$

$$= F\left(\frac{1}{u^2+v^2+1}(2u, 2v, -(u^2+v^2-1))\right)$$

$$= \frac{1}{1+\frac{u^2+v^2-1}{u^2+v^2+1}} \cdot \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}\right)$$

$\left[F(x,y,z) = \frac{1}{1-z}(x,y) \right]$

$$= \frac{\cancel{u^2+v^2+1}}{2\cdot(u^2+v^2)} \cdot \left(\frac{\cancel{2}u}{\cancel{u^2+v^2+1}}, \frac{\cancel{2}v}{\cancel{u^2+v^2+1}}\right)$$

$$= \frac{1}{u^2+v^2} \cdot (u,v).$$

c) $u^2+v^2=1 \Rightarrow T(u,v) = (u,v)$

$u^2+v^2<1 \Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{v}{u^2+v^2}\right)^2 = \frac{u^2+v^2}{(u^2+v^2)^2} = \frac{1}{u^2+v^2} > 1$

& similarly $u^2+v^2>1 \Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{v}{u^2+v^2}\right)^2 < 1$.

So T fixes $\mathcal{C}$ pointwise & exchanges the inside & outside of $\mathcal{C}$. □.

(Alternatively, argue geometrically using $T = F \circ R \circ F^{-1}$.)