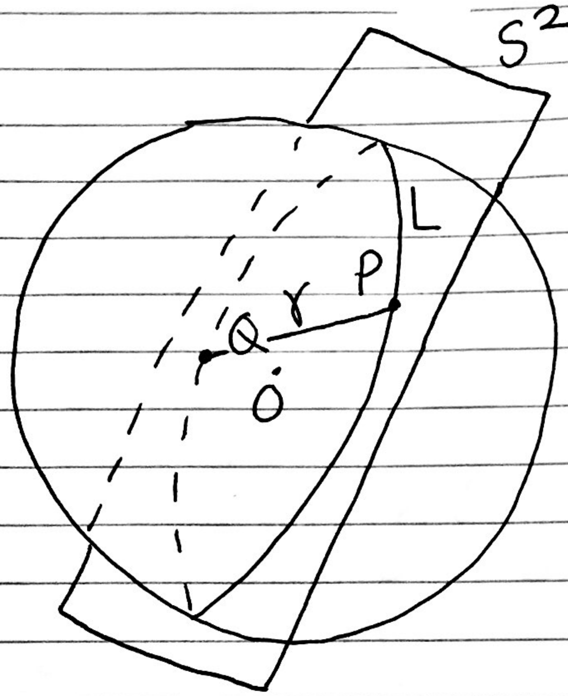
Math 461 Lecture 20 10/19

P and Q are not antipodal $\Rightarrow$
There's a unique great circle passing through P and Q

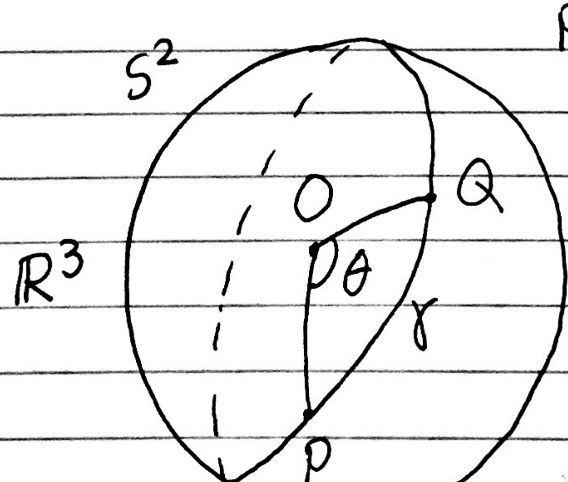
P and Q are antipodal $\Rightarrow$



There are infinite many great circles passing through P and Q

Distances on S^2:

$d_{S^2}(P,Q) := |\gamma|$ γ is the shortest distance between P and Q

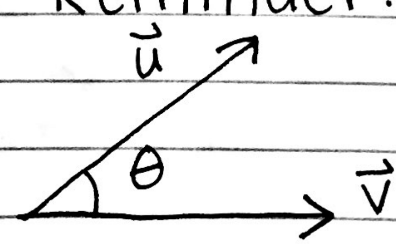


$$d(P,Q) = 2\pi \cdot \frac{\theta}{2\pi} = \theta$$

θ in radians

$\theta = \cos^{-1}(\vec{OP} \cdot \vec{OQ})$

Reminder:



$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

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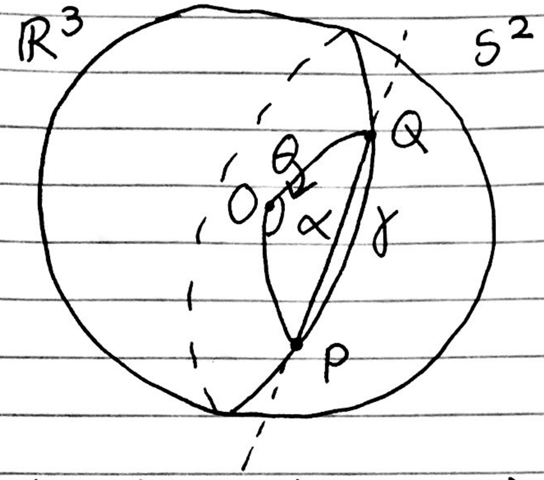
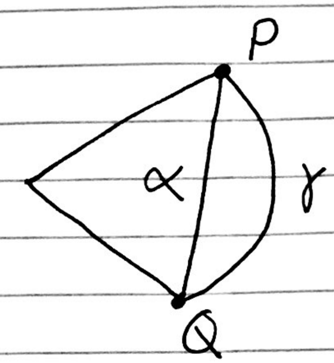
In our case: $|\vec{OP}| = |\vec{OQ}| = 1 \Rightarrow$

$\cos\theta = \vec{OP} \cdot \vec{OQ}$

$\vec{u} = (u_1, u_2, u_3) \quad \vec{v} = (v_1, v_2, v_3)$

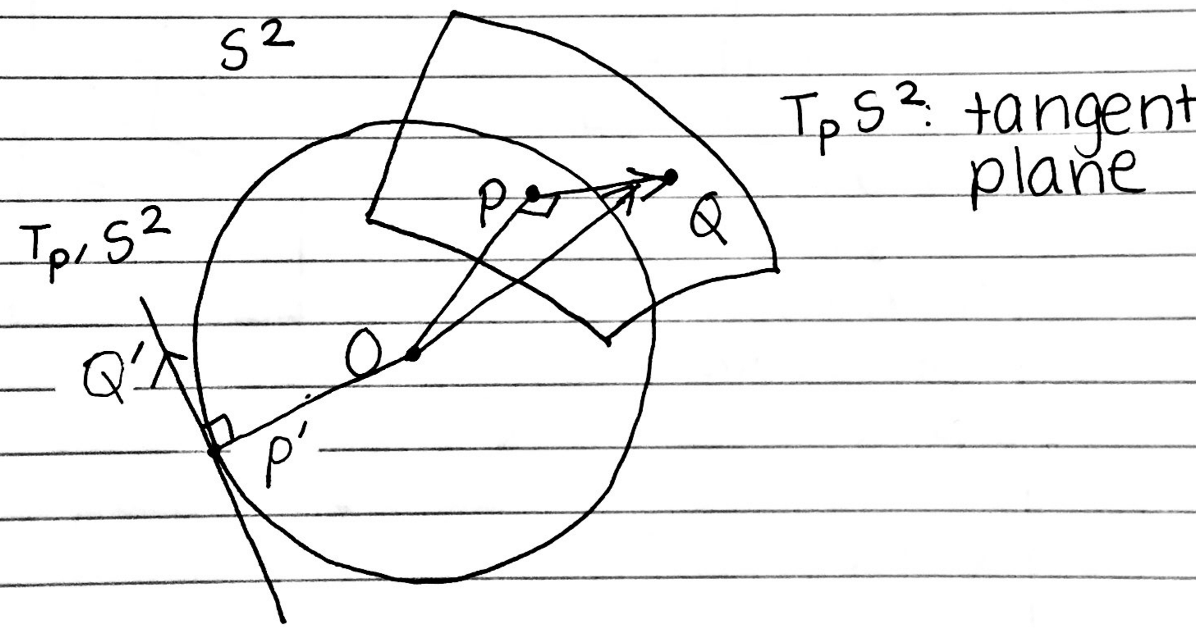
$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$





$\alpha = d_{\mathbb{R}^3}(P, Q) \leq d_{S^2}(P, Q)$

$\cos\theta = \vec{OP} \cdot \vec{OQ}$ holds if and only if $P = Q$



$T_P S^2 = \{Q \in \mathbb{R}^3 \mid \vec{OP} \cdot \vec{OQ} = 1\}$

Proof: $\vec{OQ} = \vec{OP} + \vec{PQ}$

$\vec{OP} \cdot \vec{OQ} = \vec{OP}(\vec{OP} + \vec{PQ}) = 1 + 0 = 1$

Conclusion: Every $Q \in T_P S^2$ satisfies $\vec{OP} \cdot \vec{OQ} = 1$

Conversely, want to show every $Q \in \mathbb{R}^3$

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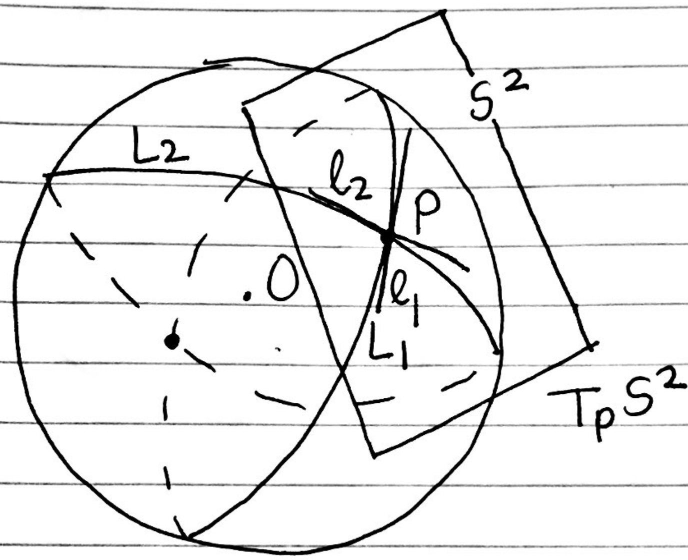
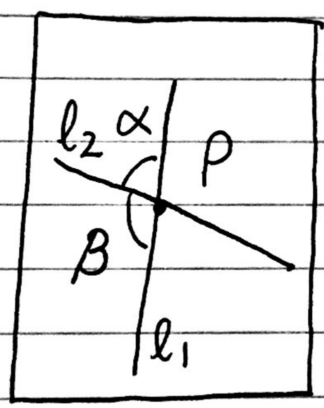
$\vec{OP} \cdot \vec{OQ} = 1$ lies on $T_p S^2$

$\vec{OQ} = \vec{OP} + \vec{PQ}$

$1 = \vec{OP}(\vec{OP} + \vec{PQ}) = 1 + \vec{OP} \cdot \vec{PQ} \Rightarrow 0 = \vec{OP} \cdot \vec{PQ}$

i.e $\vec{OP} \perp \vec{PQ}$ $\square$





$T_p S^2$

Angles between L_1 and $L_2 = \alpha, \beta$

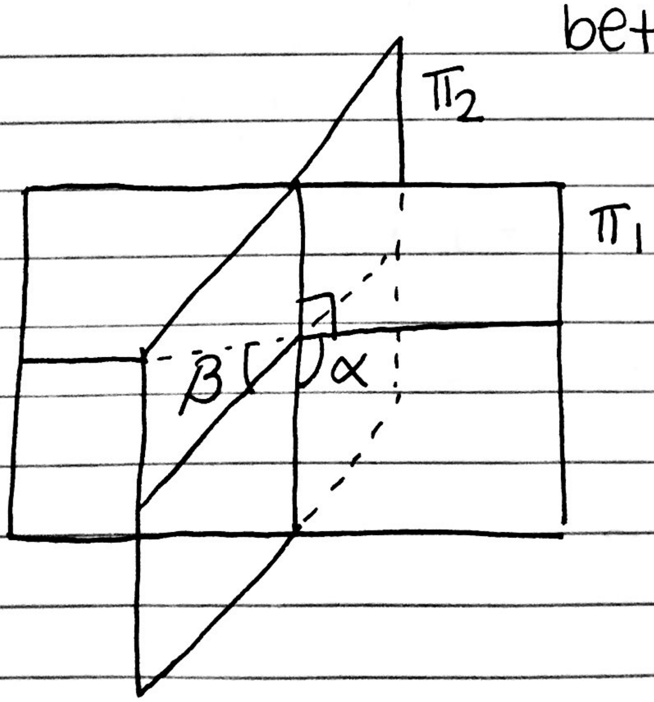
$L_1 = \Pi_1 \cap S^2$

$L_2 = \Pi_2 \cap S^2 \Rightarrow$

angles between L_1 and L_2

= Dihedral angles between Π_1 and Π_2

= angles between the normal vectors of Π_1 and Π_2



write Π_1 as

$ax + by + cz = 0$

$\Rightarrow$ normal vector $= (a, b, c)$

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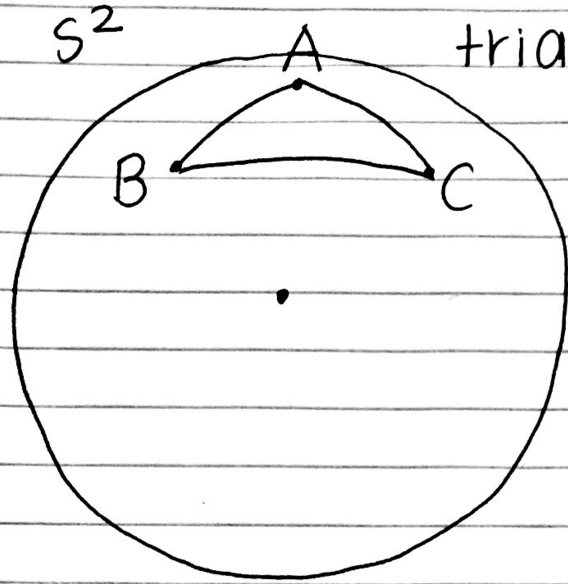
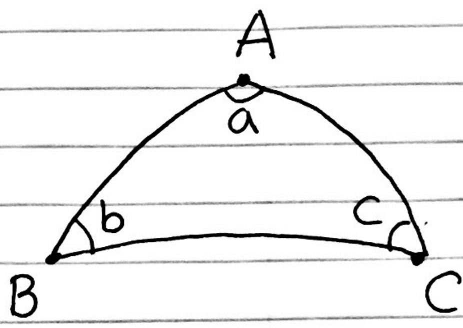
Spherical Triangles:

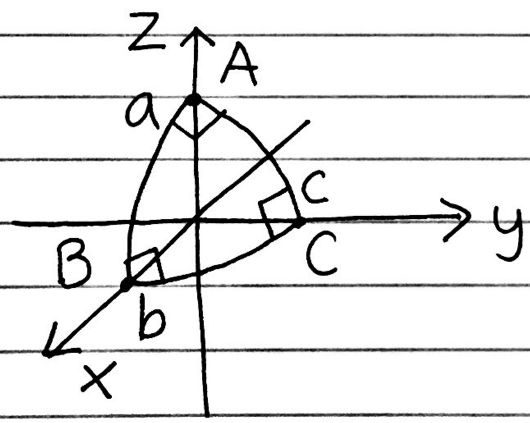


triangles on S^2



Fact: The shortest path between two points $P, Q \in S^2$ is the shortest arc γ of the great circle passing through P, Q

Question: $a+b+c > \pi$



$\Rightarrow a+b+c = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{3\pi}{2} > \pi$

Theorem: $a+b+c = \pi + \text{area}(\triangle ABC)$