Math 611 Homework 2

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Reading: Dummit and Foote, 4.4, 4.5, 5.5.
Justify your answers carefully.

(1) (a) Is $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ isomorphic to S_4?

(b) Is $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ isomorphic to A_4?

[Hint: Consider the action on the set $\mathbb{P}^1(\mathbb{Z}/3\mathbb{Z})$ of one dimensional subspaces of $(\mathbb{Z}/3\mathbb{Z})^2$.]

(2) Let G be a non abelian group of order p^3 for p a prime. Determine the class equation of G.

(3) Classify groups of order 8.

(4) Let G be the subgroup of $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices with 1's on the diagonal. Describe explicitly a series

$$\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_N = G$$

of subgroups of G such that $G_{i+1}/G_i \simeq \mathbb{Z}/p\mathbb{Z}$ for each $i = 0, 1, \ldots, N-1$.

(5) Let G be a p-group and H a proper subgroup of G. Show that H is a proper subgroup of its normalizer $N(H)$. [Hint: Argue by contradiction and use $Z(G) \neq \{e\}$ and induction.]

(6) Let G be the group of isometries of the plane $\mathbb{R}^2$. Let H be the subgroup consisting of all rotations with center $0 \in \mathbb{R}^2$. Determine the normalizer $N(H)$.

(7) (a) Compute the normalizer of the subgroup H of S_4 generated by the 4-cycle (1234).

(b) Check your answer by verifying the formula

$$|G| = |N(H)| \cdot (\text{number of conjugate subgroups}).$$

(c) Identify $N(H)$ with a standard group.

(d) Interpret your answer geometrically in terms of the group O of rotational symmetries of the cube. [Note: Labelling the diagonals of the cube induces an isomorphism $O \simeq S_4$.]

(8) Let G be a finite group and H a proper subgroup of G.

(a) Show that the union of the conjugate subgroups of H is not equal to G.

(b) Deduce that there is a conjugacy class which is disjoint from H.

(9) Let G be a finite group. Let p be the smallest prime dividing $|G|$. Suppose H is a normal subgroup of G of order p. Show that H is contained in the center of G.

(10) Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Show that G is abelian.