Math 611 Homework 1

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Reading: Dummit and Foote, 1.1–4.3 (presumably much of this will be review).

Justify your answers carefully.

(1) Establish the relation $bab = a^{-1}$ between the generators a (rotation by $2\pi/n$) and b (a reflection) of the dihedral group D_n by identifying a and b with 2×2 matrices.

(2) Prove that S_n is generated by the transpositions $(12), (23), \ldots, (n-1, n)$ of adjacent elements.

(3) Let $\mathbb{Z}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables $x_1, \ldots, x_n$ with integer coefficients. Define

$$\Delta = \prod_{i<j} (x_i - x_j) \in \mathbb{Z}[x_1, \ldots, x_n].$$

Consider the action of S_n on $\mathbb{Z}[x_1, \ldots, x_n]$ given by permuting the variables. Show that a transposition τ satisfies $\tau(\Delta) = -\Delta$. Deduce that the notion of even and odd permutations is well defined.

(4) Let F be a field. Describe an injective group homomorphism $\varphi \colon S_n \to \mathrm{GL}_n(F)$. Show that $\det(\varphi(\sigma)) = \mathrm{sgn}(\sigma)$ for all $\sigma \in S_n$.

(5) Let G be the group of all symmetries of a regular tetrahedron, and H the subgroup of rotational symmetries.

(a) Describe an isomorphism $G \to S_4$.

(b) Describe an isomorphism $H \to A_4$.

[Hint for (b): The regular tetrahedron can be realized as the convex hull of the standard basis vectors in $\mathbb{R}^4$. This can be used to relate the determinant of the matrix of a symmetry and the sign of the associated permutation in part (a). (It is also possible to proceed directly.)]

(6) For each of the following statements, give a proof or a counterexample. (We write $H \leq G$ for "H is a subgroup of G" and $N \lhd G$ for "N is a normal subgroup of G".)

   (a) If $H \lhd G$ and $K \lhd H$, then $K \lhd G$.

   (b) If $H \lhd G$ and $K \leq G$ then $H \cap K \lhd K$.

(7) List the conjugacy classes in (a) S_4, (b) A_4. [Hint: The order of a conjugacy class divides the order of the group. Check your answer to (b) using this property.]

(8) Describe the conjugacy classes in the dihedral group D_n. [Hint: There are two cases depending on the parity of n.]

(9) If G is a group of order 21 and $x \in G$ is an element such that the conjugacy class $C(x)$ of x has size 3, what is the order of x?

(10) Let F be a field. Determine the center of $\mathrm{GL}_n(F)$.

(11) Let $a \in \mathbb{R}$. Determine whether the matrices $\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$ are conjugate in (a) $\mathrm{GL}_2(\mathbb{R})$, (b) $\mathrm{SL}_2(\mathbb{R})$.

(12) Show that there does not exist a subgroup of A_4 of order 6. [Hint: A subgroup of index 2 is necessarily normal. It follows that it contains all the elements of odd order (why?).] This example shows that the full converse of Lagrange's theorem is false.

(13) Let G be a group and $Z(G)$ its center. Show that if $G/Z(G)$ is cyclic then G is abelian.

(14) Classify all finite groups G having exactly (a) 2, (b) 3 conjugacy classes. [Hint: Use the class equation and divisibility properties.]