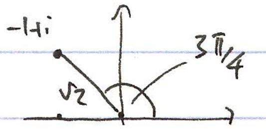
1. a. $e^z + 1 - i = 0$



$$\Leftrightarrow \quad e^z = -1+i = \sqrt{2}e^{i3\pi/4}$$

$$\Leftrightarrow \quad z = \log(-1+i)$$

$$= \log\sqrt{2} + i(3\pi/4 + 2\pi k), \quad k \text{ an integer.}$$

b. $z^3 + 8i = 0$

$$\Leftrightarrow \quad z^3 = -8i = 8e^{i3\pi/2}$$

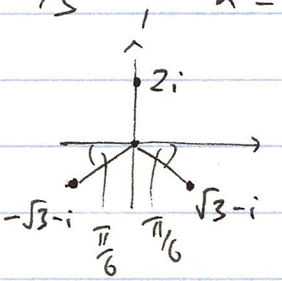
$z = re^{i\theta}$. $\quad z^3 = r^3 e^{i3\theta} = 8e^{i3\pi/2}$

$$\Rightarrow \quad r = \sqrt[3]{8} = 2, \quad 3\theta = 3\pi/2 + 2\pi k, \quad k \text{ an integer}$$

$$\theta = \pi/2 + 2\pi k/3, \quad k = 0,1,2$$

$$z = 2e^{i\pi/2},\ 2e^{i7\pi/6},\ 2e^{i11\pi/6}$$

$$= 2i,\ -\sqrt{3}-i,\ \sqrt{3}-i$$



c. $z + \frac{1}{z} = i$

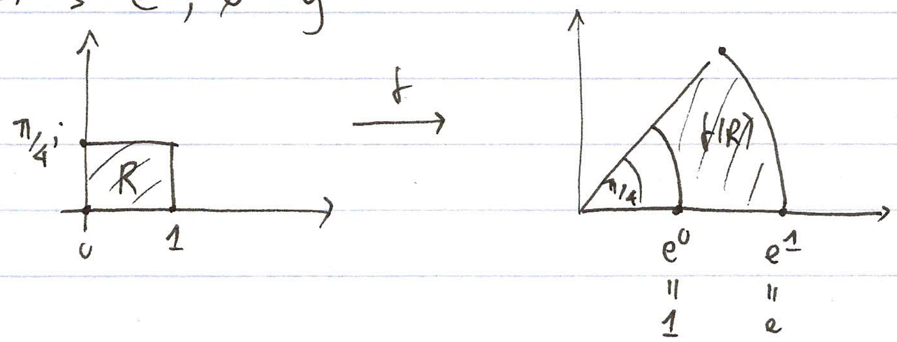
$$z^2 - iz + 1 = 0$$

$$z \underset{QF}{=} \frac{i \pm \sqrt{-1-4}}{2} = \frac{(1 \pm \sqrt{5})i}{2}$$

2. a. $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$, $R = \{z = x+iy \mid 0 < x < 1 \text{ \& } 0 < y < \pi/4\}$

$$se^{i\phi} = w = e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$\Rightarrow \quad s = e^x,\ \phi = y$$



b $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \text{Log}\, z$, $R = \{z = x+iy \mid |z| > 1 \ \& \ y > 0\}$

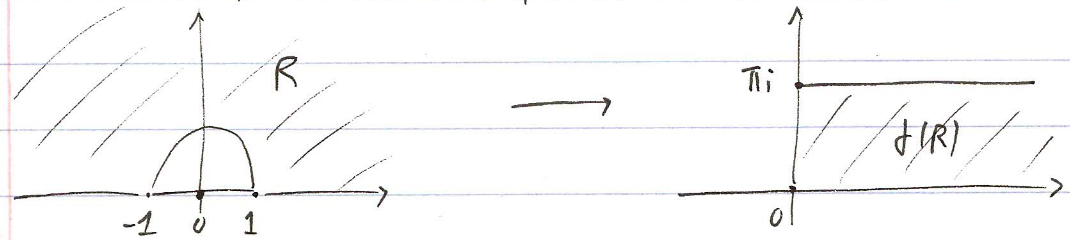
$z = re^{i\theta}$, $-\pi < \theta \leq \pi$

$\Rightarrow \text{Log}\, z = \log r + i\theta$ (principal value of complex logarithm

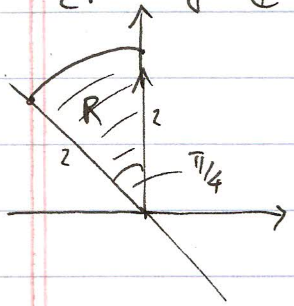
$= w = u + iv$, $u = \log r$, $v = \theta$.

$R = \{z = re^{i\theta} \mid r > 1 \ \& \ 0 < \theta < \pi\}$

$\Rightarrow f(R) = \{w = u+iv \mid u > 0 \ \& \ 0 < v < \pi\}$



c. $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = z^2$, $R = \{z = x+iy \mid |z| < 2, \ x < 0, \ \& \ x+y > 0\}$

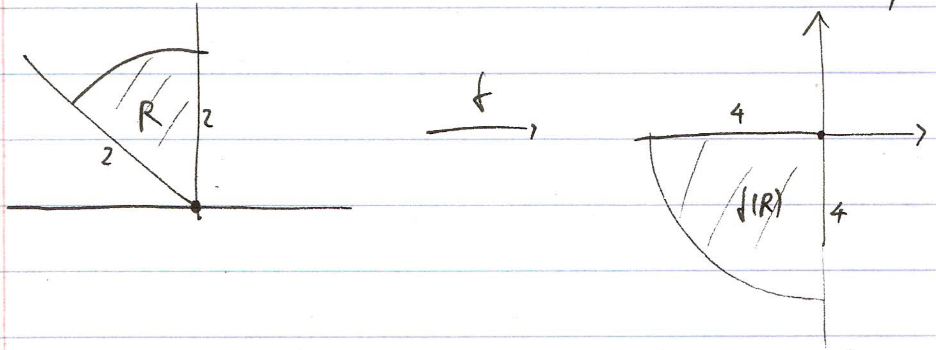


$= \{z = re^{i\theta} \mid r < 2, \ \pi/2 < \theta < 3\pi/4\}$

$z = re^{i\theta} \Rightarrow w = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$w = se^{i\phi}$, $s = r^2$, $\phi = 2\theta$.

$\therefore f(R) = \{w = se^{i\phi} \mid s < 4, \ \pi < \phi < 3\pi/2\}$



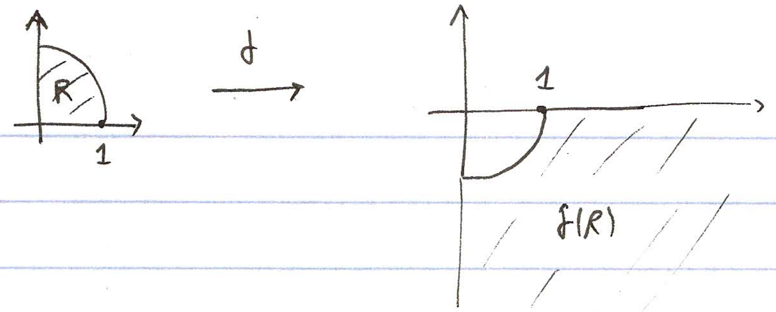
d. $f: \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$, $f(z) = 1/z$, $R = \{z = x+iy \mid |z| < 1, \ x > 0 \ \& \ y > 0\}$

$= \{z = re^{i\theta} \mid r < 1, \ 0 < \theta < \pi/2\}$

$z = re^{i\theta} \Rightarrow w = 1/z = (1/r) \cdot e^{-i\theta}$

$w = se^{i\phi}$, $s = 1/r$, $\phi = -\theta$.

$\therefore f(R) = \{w = se^{i\phi} \mid s > 1, \ -\pi/2 < \phi < 0\}$.



3. $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$

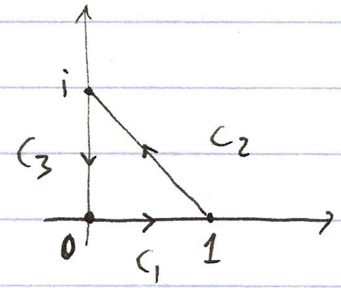
$$f(x+iy) = x - iy.$$

a. $f(x+iy) = u + iv$, $u = x$, $v = -y$

CR eqs $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Not satisfied : $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$ $\Rightarrow$ f not complex diff'ble.

b.



$C = C_1 + C_2 + C_3$

C_1 : $z(t) = t$, $z : [0,1] \to \mathbb{C}$

$$\int_{C_1} f(z)\,dz = \int_0^1 f(z(t))z'(t)\,dt = \int_0^1 t\,dt = \left[t^2/2\right]_0^1 = 1/2$$

C_2 : $z(t) = 1 + t \cdot (i-1)$, $z : [0,1] \to \mathbb{C}$

$$\int_{C_2} f(z)\,dz = \int_0^1 ((1-t) - ti) \cdot (-1+i)\,dt$$

$$= \int_0^1 ((2t-1) + i)\,dt = \left[t^2 - t + i \cdot t\right]_0^1 = i$$

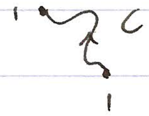
C_3 : $z(t) = i + t(0-i) = (1-t)i$

$$\int_{C_3} f(z)\,dz = \int_0^1 (-(1-t)i) \cdot -i\,dt = \int_0^1 t-1\,dt$$

$$= \left[t^2/2 - t\right]_0^1 = -1/2$$

$$\therefore \int_C f(z)\,dz = \int_{C_1} + \int_{C_2} + \int_{C_3} = i.$$

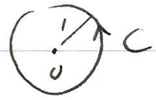
4 a. $\displaystyle\int_C z^3+3z+5 \, dz$



$$= \left[\frac{z^4}{4} + \frac{3z^2}{2} + 5z \right]_1^i = \left(\frac{1}{4} - \frac{3}{2} + 5i\right) - \left(\frac{1}{4} + \frac{3}{2} + 5\right)$$

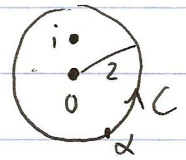
$$= -8 + 5i$$

b. $\displaystyle\int_C \cos(z^2) \, dz$



$\cos(z^2)$ is complex differentiable (by chain rule) on all of $\mathbb{C}$.
C is closed curve
$\Rightarrow \displaystyle\int_C \cos(z^2) \, dz = 0$ by Cauchy's theorem.

c.



$$\int_C \frac{1}{(z-i)^3} \, dz = \left[\frac{-1}{2(z-i)^2} \right]_\alpha^\alpha$$

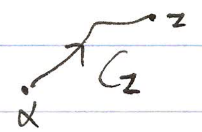
$$= 0.$$

i.e. integral is zero because integrand has an antiderivative, & C is a closed curve.

5. $f : \mathbb{C} \to \mathbb{C}, \quad f(z) = e^{(z^2)} \sin(z^3)$

f is complex differentiable (by chain rule & product rule). on all of $\mathbb{C}$.
$U = \mathbb{C}$ has "no holes" (simply connected) $\Rightarrow$ f has an antiderivative, defined by

$$F(z) = \int_{C_z} f(w) \, dw$$



($\alpha \in \mathbb{C}$ fixed.)

6. $$f(z) = \frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}, \quad A,B \in \mathbb{C}$$

$$\begin{aligned} 1 &= A(z-3) + B(z-2) \\ &= (A+B)z + (-3A-2B) \end{aligned}$$

$$\Rightarrow A+B=0, \quad -3A-2B=1$$

$$B=-A \qquad -A=1$$

$$A=-1, \ B=1.$$

$$\begin{aligned} f(z) &= \frac{1}{z-3} - \frac{1}{z-2} \\ &= \frac{-1}{3\cdot(1-z/3)} + \frac{1}{2\cdot(1-z/2)} \\ &= \frac{-1}{3}\cdot\sum_{n=0}^{\infty}\left(z/3\right)^n + \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(z/2\right)^n \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)\cdot z^n \end{aligned}$$

$$\left(\text{using } \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\dots \text{ for } |z|<1\right)$$

Radius of convergence $R=2$ (largest radius of open disc centered at $z=0$ contained in domain of f.)

7. a. $$\frac{\sin(z)}{z^4} = \frac{z - z^3/3! + z^5/5! - \dots}{z^4}$$

$$= z^{-3} - z^{-1}/6 + z/120 - \dots$$

Pole of order 3.

$$\operatorname{Res}_{z=0}\left(\frac{\sin(z)}{z^4}\right) = -1/6.$$

b. $$\frac{e^z - 1 - z}{z^2} = \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) - 1 - z}{z^2}$$

$$= \frac{\frac{z^2}{2!} + \frac{z^3}{3!} + \dots}{z^2} = \frac{1}{2} + \frac{z}{6} + \frac{z^2}{24} + \dots$$

Removable singularity.

$$\operatorname{Res}_{z=0}\left(\frac{e^z - 1 - z}{z^2}\right) = 0.$$

c. $$z\cos(1/z) = z \cdot \left(1 - \frac{(1/z)^2}{2!} + \frac{(1/z)^4}{4!} - \dots\right)$$

$$= z \cdot \left(1 - \frac{z^{-2}}{2} + \frac{z^{-4}}{24} - \dots\right)$$

$$= z - \frac{z^{-1}}{2} + \frac{z^{-3}}{24} - \dots$$

Essential singularity

$$\operatorname{Res}_{z=0}\left(z\cos(1/z)\right) = -1/2.$$

using $\frac{1}{1-w} = 1 + w + w^2 + \dots$

d. $$\frac{1}{z^4(z+1)} = \frac{\frac{1}{1-(-z)}}{z^4} = \frac{1 - z + z^2 - z^3 + \dots}{z^4}$$

$$= z^{-4} - z^{-3} + z^{-2} - z^{-1} + 1 - \dots$$

Pole of order 4.

$$\operatorname{Res}_{z=0}\left(\frac{1}{z^4(z+1)}\right) = -1.$$

8 a. If $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ is the power series expansion of a complex differentiable function $f(z)$ about $z = \alpha$, then $a_n = \frac{f^{(n)}(\alpha)}{n!}$

So in our case, $\frac{\tan^{(5)}(0)}{5!} = \frac{2}{15}$

$\Rightarrow \tan^{(5)}(0) = \frac{2 \cdot 120}{15} = 16.$

b. $f(z) = \frac{(z+1)\tan z}{z^5} = \frac{(1+z)\cdot\left(z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots\right)}{z^5}$

$= \frac{\left(z + z^2 + \frac{1}{3}z^3 + \frac{1}{3}z^4 + \frac{2}{15}z^5 + \dots\right)}{z^5}$

$= z^{-4} + z^{-3} + \frac{1}{3}z^{-2} + \frac{1}{3}z^{-1} + \frac{2}{15} + \dots$

c. Pole of order 4.

$\operatorname{Res}_{z=0} (f(z)) = \frac{1}{3}$

d. $f(z) = \frac{(z+1)\tan z}{z^5} = \frac{(z+1)\sin z}{z^5 \cos z} = \frac{a(z)}{b(z)}$

Singularities at $z=0$, $\cos z = 0 \iff z = \pi/2 + k\pi$, k an integer.

$|z| < 2 \Rightarrow z = 0, \pm\pi/2.$

At $z = \alpha = \pm\pi/2$, $a(\alpha) \neq 0$, $b(\alpha) = 0$, $b'(\alpha) \neq 0$

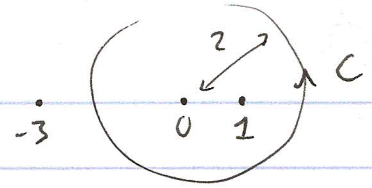
$\Rightarrow$ simple pole (pole of order 1), $\operatorname{Res}_{z=\alpha} f(z) = \frac{a(\alpha)}{b'(\alpha)}$

$b(z) = z^5 \cos z$, $b'(z) = 5z^4 \cos z + z^5 \cdot (-\sin z)$

$\alpha = \pi/2.$ $\operatorname{Res}_{z=\pi/2} f(z) = \frac{(\pi/2 + 1)\cdot 1}{(\pi/2)^5 \cdot -1} = -\frac{(1+\pi/2)}{(\pi/2)^5}$

$\alpha = -\pi/2$ $\operatorname{Res}_{z=-\pi/2} f(z) = \frac{(-\pi/2 + 1)\cdot -1}{(-\pi/2)^5 \cdot 1} = \frac{1-\pi/2}{(\pi/2)^5}.$

9. $$\int_C \frac{1}{z^2+2z-3}\,dz$$



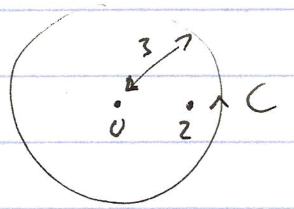
$$\frac{1}{z^2+2z-3} = \frac{1}{(z-1)(z+3)}\,,$$ simple poles at $z=1,-3$.
1 inside C, -3 outside C

$$\therefore \int_C \frac{1}{z^2+2z-3}\,dz \overset{R.T.}{=} 2\pi i \operatorname{Res}_{z=1} \frac{1}{z^2+2z-3}$$

$$= 2\pi i \operatorname{Res}_{z=1} \frac{1/z+3}{z-1} = 2\pi i \cdot \frac{1}{+1+3}$$

$$= \frac{\pi i}{2}.$$

10. $$\int_C \frac{e^z}{z^3-2z^2}\,dz$$



$$\frac{e^z}{z^3-2z^2} = \frac{e^z}{z^2(z-2)}$$ simple pole at $z=2$
pole of order 2 at $z=0$.

$$\operatorname{Res}_{z=2} \frac{e^z}{z^3-2z^2} = \operatorname{Res}_{z=2} \frac{e^z/z^2}{(z-2)} = \frac{e^2}{2^2} = \frac{e^2}{4}$$

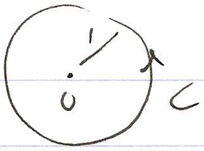
$$\operatorname{Res}_{z=0} \frac{e^z}{z^3-2z^2} = \operatorname{Res}_{z=0} \frac{e^z/z-2}{z^2} = \left(e^z/z-2\right)'(0)$$

$$= \left.\frac{e^z\cdot(z-2) - e^z}{(z-2)^2}\right|_{z=0} = \frac{-3}{4}$$

$$\int_C \frac{e^z}{z^3-2z^2}\,dz \overset{RT}{=} 2\pi i \left(\operatorname{Res}_{z=0} \frac{e^z}{z^3-2z^2} + \operatorname{Res}_{z=2} \frac{e^z}{z^3-2z^2} \right)$$

$$= 2\pi i \left(\tfrac{1}{4}(e^2-3)\right) = \frac{\pi i}{2}(e^2-3)$$

11. $$\int_C \frac{\cos z}{e^z - 1}\, dz$$



$\dfrac{\cos z}{e^z-1}$ has singularities at $z = (2\pi i)\cdot k$, k integer
(solutions of $e^z - 1 = 0$)

Only singularity inside C is $z = 0$.

$$\therefore \int_C \frac{\cos z}{e^z-1}\, dz = 2\pi i \cdot \operatorname{Res}_{z=0}\left(\frac{\cos z}{e^z-1}\right)$$

$$\frac{\cos z}{e^z - 1} = \frac{a(z)}{b(z)} \qquad \begin{aligned} a(0) &= 1 \neq 0 \\ b(0) &= 0 \\ b'(0) &= e^z \big|_{z=0} = 1 \neq 0. \end{aligned}$$

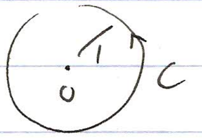
$$\Rightarrow \operatorname{Res}_{z=0} \frac{\cos(z)}{e^z - 1} = \frac{a(0)}{b'(0)} = 1 \qquad (\text{\& } z=0 \text{ is simple pole})$$

$$\therefore \int_C \frac{\cos z}{e^z-1}\, dz = 2\pi i$$

12. a. $$e^{1/z} = 1 + (1/z) + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \ldots$$

$$= 1 + z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{6} + \ldots$$

b. $$\int_C \left(z^{-2} + z + z^3\right) \cdot e^{1/z}\, dz$$



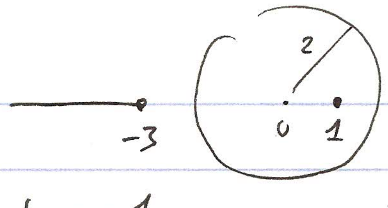
$$\overset{RT.}{=} 2\pi i \cdot \operatorname{Res}_{z=0} \left(z^{-2}+z+z^3\right)\cdot e^{1/z}$$

(the only singularity of the integrand is at $z=0$)

$$f(z) = \left(z^{-2} + z + z^3\right)\cdot e^{1/z} = \left(z^{-2}+z+z^3\right) \cdot \left(1 + z^{-1} + \tfrac{1}{2} z^{-2} + \tfrac{1}{6} z^{-3} + \tfrac{1}{24} z^{-4} + \ldots\right)$$
$$= \sum_{n=-\infty}^{\infty} a_n z^n$$

$$\therefore \operatorname{Res}_{z=0} f(z) = a_{-1} = \cancel{1\cdot\tfrac{1}{6}}\ 1 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{24} = 13/24, \quad \int_C f(z)\,dz = 2\pi i \cdot 13/24$$

13 $\displaystyle\int_C \frac{\mathrm{Log}(z+3)}{(z-1)^n}\,dz$



Only singularity of integrand inside C is at $z=1$.

$$\therefore \int_C \frac{\mathrm{Log}(z+3)}{(z-1)^n}\,dz \underset{RT}{=} 2\pi i \cdot \mathrm{Res}_{z=1} \frac{\mathrm{Log}(z+3)}{(z-1)^n}$$

$$= 2\pi i \cdot \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{(n-1)}\left(\mathrm{Log}(z+3)\right)\Bigg|_{z=1}$$

$$\frac{d}{dz}\left(\mathrm{Log}(z+3)\right) = \frac{1}{z+3}$$

$$\frac{d}{dz}\left(\frac{1}{z+3}\right) = \frac{-1}{(z+3)^2}$$

$$\frac{d}{dz}\left(\frac{-1}{(z+3)^2}\right) = \frac{-1\cdot -2}{(z+3)^3}, \text{ etc.}$$

$$= 2\pi i \cdot \frac{1}{(n-1)!} \frac{-1\cdot -2 \cdots -(n-2)}{(z+3)^{n-1}}\Bigg|_{z=1}$$

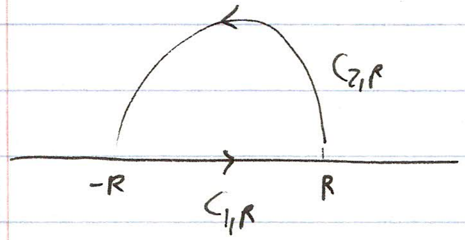
$$= 2\pi i \cdot \frac{(-1)^{n-2}\cdot (n-2)!}{(n-1)!\, 4^{n-1}}$$

$$= \frac{2\pi i \cdot (-1)^n}{(n-1)\cdot 4^{n-1}} \qquad \text{for } n \geq 2$$

If $n=1$, $\displaystyle\mathrm{Res}_{z=1} \frac{\mathrm{Log}(z+3)}{(z-1)} = \mathrm{Log}(1+3) = \log 4$,

$$\int_C \frac{\mathrm{Log}(z+3)}{z-1}\,dz = 2\pi i \log 4.$$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$



$C_R = C_{1,R} + C_{2,R}$.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \int_{C_{1,R}} \frac{1}{(z^2+1)^2}\,dz.$$

Also $\displaystyle\lim_{R\to\infty} \int_{C_{2,R}} \frac{1}{(z^2+1)^2}\,dz = 0$:-

$$\left| \int_{C_{2,R}} \frac{1}{(z^2+1)^2}\, dz \right| \leq \text{length}(C_{2,R}) \cdot \frac{1}{(R^2-1)^2} = \pi R \cdot \frac{1}{(R^2-1)^2} \to 0$$

as $R \to \infty$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = \lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+1)^2}\, dz$$

$$\overset{RT}{=} 2\pi i \cdot \operatorname{Res}_{z=i} \left(\frac{1}{(z^2+1)^2} \right)$$

Only singularity of $1/(z^2+1)^2$ inside C_R is $z=i$

$$= 2\pi i \cdot \operatorname{Res}_{z=i} \left(\frac{1/(z+i)^2}{(z-i)^2} \right)$$

$$= 2\pi i \left(\frac{1}{(z+i)^2} \right)'(i)$$

$$= 2\pi i \left(\frac{-2}{(z+i)^3} \right) \Big|_{z=i}$$

$$= 2\pi i \cdot \frac{-2}{(2i)^3} = \frac{2\pi i \cdot -2}{-8i} = \frac{\pi}{2}$$

15. Use same contour $C_R = C_{1,R} + C_{2,R}$ as in Q14.

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx = \lim_{R\to\infty} \int_{C_{1,R}} \frac{z^2}{z^4+1}$$

$$\lim_{R\to\infty} \int_{C_{2,R}} \frac{z^2}{z^4+1}\, dz = 0 \quad :- \quad \left| \int_{C_{2,R}} \frac{z^2}{z^4+1}\, dz \right| \leq \pi R \cdot \frac{R^2}{R^4-1} \longrightarrow 0$$

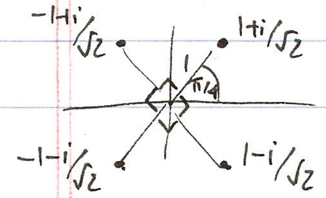
as $R \to \infty$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx = \lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+1}\, dz$$

$$\overset{RT}{=} 2\pi i \left(\operatorname{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{z^2}{z^4+1} + \operatorname{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{z^2}{z^4+1} \right)$$

$:- \ z^4+1 = 0$ (i.e. $z^4 = -1 = e^{i\pi}$) $\iff z = re^{i\theta}$, $r^4 = 1$, $r=1$

$4\theta = \pi + (2\pi)\cdot k$, $\theta = \pi/4 + \pi/2 \cdot k$, $k = 0,1,2,3$



$\iff z = \frac{\pm 1 \pm i}{\sqrt{2}}$. Only $\frac{\pm 1 + i}{\sqrt{2}}$ inside C_R.

$$\operatorname{Res}_{z=\frac{1+i}{\sqrt2}} \frac{z^2}{z^4+1} = \operatorname{Res}_{z=\frac{1+i}{\sqrt2}} \frac{z^2}{\left(z-\left(\frac{-1+i}{\sqrt2}\right)\right)\left(z-\left(\frac{-1-i}{\sqrt2}\right)\right)\left(z-\left(\frac{1-i}{\sqrt2}\right)\right)\cancel{\left(z-\left(\frac{1+i}{\sqrt2}\right)\right)}}$$

$$= \frac{\left(\frac{1+i}{\sqrt2}\right)^2}{\frac{2}{\sqrt2}\cdot\frac{2}{\sqrt2}(1+i)\cdot\frac{2i}{\sqrt2}} = \frac{1+i}{4\sqrt2\, i} = \frac{1-i}{4\sqrt2}$$

Similarly, $\operatorname{Res}_{z=-1+i/\sqrt2} \frac{z^2}{z^4+1} = \operatorname{Res}_{z=-1+i/\sqrt2} \frac{z^2}{\left(z-\left(\frac{1+i}{\sqrt2}\right)\right)\left(z-\left(\frac{-1-i}{\sqrt2}\right)\right)\left(z-\left(\frac{1-i}{\sqrt2}\right)\right)\cancel{\left(z-\left(\frac{-1+i}{\sqrt2}\right)\right)}}$

$$= \frac{\left(\frac{-1+i}{\sqrt2}\right)^2}{-2/\sqrt2\cdot 2i/\sqrt2\cdot 2/\sqrt2\cdot(-1+i)} = \frac{(-1+i)}{-4\sqrt2\, i} = \frac{-1-i}{4\sqrt2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\pi i\left(\frac{1-i}{4\sqrt2} + \frac{-1-i}{4\sqrt2}\right)$$

$$= 2\pi i\cdot\frac{-2i}{4\sqrt2} = \frac{\pi}{\sqrt2}$$

16. If ϕ has continuous 2nd partial derivatives and $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ (Laplace's equation)

we say ϕ is <u>harmonic</u>.

(a) $\phi = 4xy^3 - 4x^3y$

$$\frac{\partial\phi}{\partial x} = 4y^3 - 12x^2y \qquad \frac{\partial\phi}{\partial y} = 12xy^2 - 4x^3$$

$$\frac{\partial^2\phi}{\partial x^2} = -24xy \qquad \frac{\partial^2\phi}{\partial y^2} = 24xy$$

$\therefore \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$, ϕ is harmonic.

If $f=\phi+i\psi$ is cx diffble,

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \qquad \text{(CR eqs)}$$

So $\dfrac{\partial \psi}{\partial y} = 4y^3 - 12x^2y$

$\Rightarrow \quad \psi = y^4 - 6x^2y^2 + a(x)$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -12xy^2 + 4x^3 \quad \Rightarrow \quad a'(x) = 4x^3$$

$$\Rightarrow \quad a(x) = x^4 + c.$$

$\quad\parallel$

$-12xy^2 + a'(x)$

$\therefore \quad \psi = y^4 - 6x^2y^2 + x^4$ is a harmonic conjugate.

b. $\phi = e^{-x}\sin y$

$$\frac{\partial \phi}{\partial x} = -e^{-x}\sin y \qquad \frac{\partial \phi}{\partial y} = e^{-x}\cos y$$

$$\frac{\partial^2 \phi}{\partial x^2} = +e^{-x}\sin y \qquad \frac{\partial^2 \phi}{\partial y^2} = -e^{-x}\sin y$$

$\therefore \quad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$, ϕ harmonic.

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = -e^{-x}\sin y \quad \Rightarrow \quad \psi = e^{-x}\cos y + a(x)$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -e^{-x}\cos y \quad \Rightarrow \quad a'(x) = 0 \quad \Rightarrow \quad a(x) = c.$$

$\therefore \quad \psi = e^{-x}\cos y$ is a harmonic conjugate

17. No, because u is <u>not</u> harmonic : $\dfrac{\partial^2 u}{\partial x^2} = 0$, $\dfrac{\partial^2 u}{\partial y^2} = xe^y$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0.$$

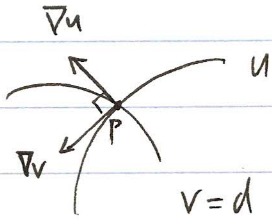
18. $\nabla u \cdot \nabla v = \left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) \cdot \left(\dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right)$

$$\overset{CR}{=} \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

$\Rightarrow$ ∇u & ∇v are orthogonal.

The level curve $u=c$ of u is orthogonal to ∇u, & similarly the level curve $v=d$ of v is orthogonal to ∇v.

Now, assuming $\nabla u \neq 0$ & $\nabla v \neq 0$ at the point p of intersection of the two curves (equivalently, p is not a critical point of f, i.e. $f'(p) \neq 0$) we see that ∇u & ∇v orthogonal $\Rightarrow$ $u=c$ & $v=d$ orthogonal.

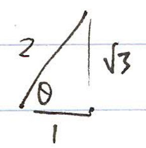


(& also ∇u is tangent to $v=d$ & ∇v is tangent to $u=c$ } at the point p)

19. $f: \mathbb{C} \to \mathbb{C}$ $\quad f(z) = (1+\sqrt{3}i)\cdot z$

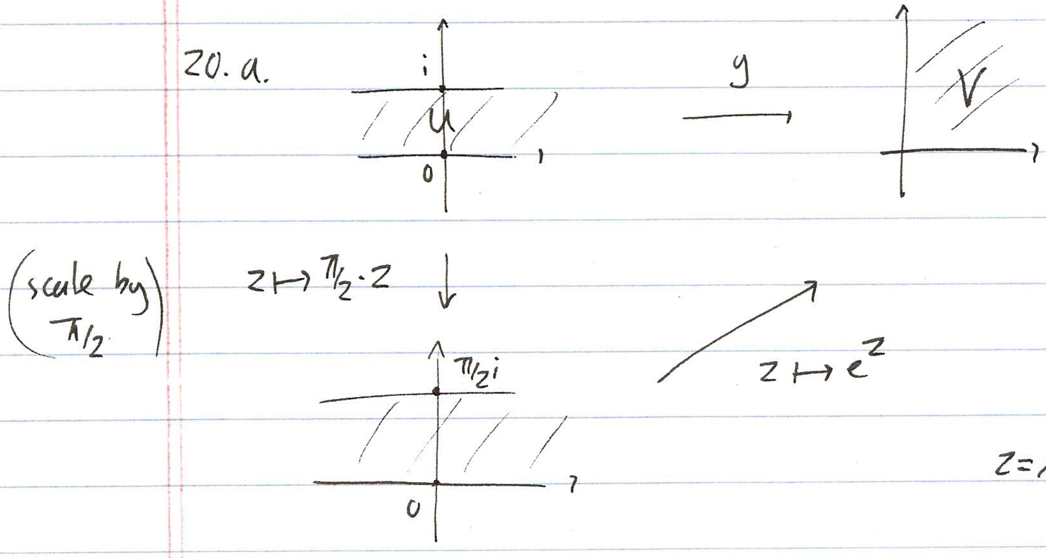
$1+\sqrt{3}i = re^{i\theta}$, $\quad r = \sqrt{1^2+\sqrt{3}^2} = 2.$

$\theta = \tan^{-1}(\sqrt{3}/1) = \pi/3.$



$\therefore$ f is scaling by 2 followed by rotation by $\pi/3$ ccw about the origin.

20. a.



So $g(z) = e^{\frac{\pi}{2}z}$

Justification:

$V = \{w = u+iv \mid u>0 \,\&\, v>0\}$

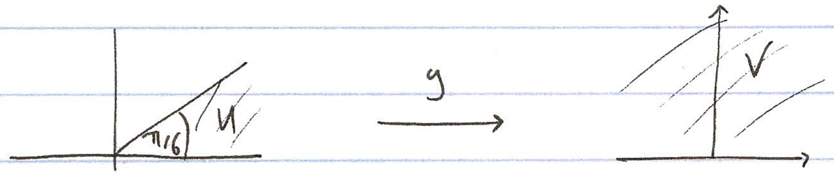
$= \{w = se^{i\phi} \mid s>0 \,\&\, 0<\phi<\pi/2\}$

$z = x+iy \Rightarrow e^z = e^x \cdot e^{iy} = se^{i\phi} = w,$

$s = e^x, \ \phi = y$

So $0<y<\pi/2 \Rightarrow 0<\phi<\pi/2$.

b.



$g(z)=z^6$.

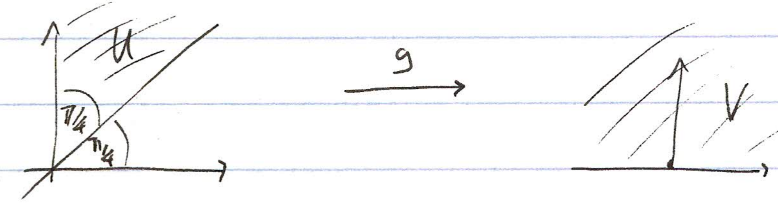
Justification: $z=re^{i\theta} \Rightarrow w=z^6=r^6e^{i6\theta}$

i.e. $w=se^{i\phi}$, $s=r^6$, $\phi=6\theta$

So $0<\theta<\pi/6 \Rightarrow 0<\phi<\pi$

And $V=\{w=se^{i\phi} \mid s>0 \text{ \& } 0<\phi<\pi\}$.

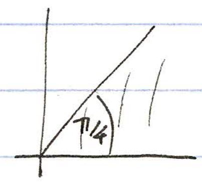
c.



$U=\{z=re^{i\theta} \mid r>0 \text{ \& } \pi/4<\theta<\pi/2\}$

$z\mapsto e^{-i\frac{\pi}{4}}\cdot z$

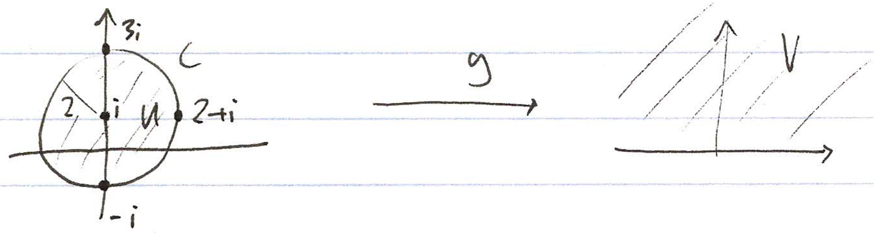
rotate ccw by $-\pi/4$, i.e. cw by $\pi/4$



$z\mapsto z^4$ (compare b.)

$\therefore\ g(z)=\left(e^{-i\pi/4}\cdot z\right)^4 = e^{-i\pi}\cdot z^4 = -z^4$.

d.



$-i\mapsto 0$

$2+i\mapsto 1$

$3i\mapsto\infty$

$$g(z)=\frac{z-(-i)}{z-3i}\bigg/\frac{(2+i)-(-i)}{(2+i)-3i}$$

$$=\frac{z+i}{z-3i}\cdot\frac{2-2i}{2+2i}=\frac{z+i}{z-3i}\cdot\frac{1-i}{1+i}=\frac{z+i}{z-3i}\,\frac{(1-i)^2}{2}$$

$$\overset{\displaystyle -i\cdot\left(\frac{z+i}{z-3i}\right)}{\shortparallel}$$

This LFT (linear fractional transformation) is constructed so that the circle C (the boundary of U) is mapped to the real axis (the boundary of V).

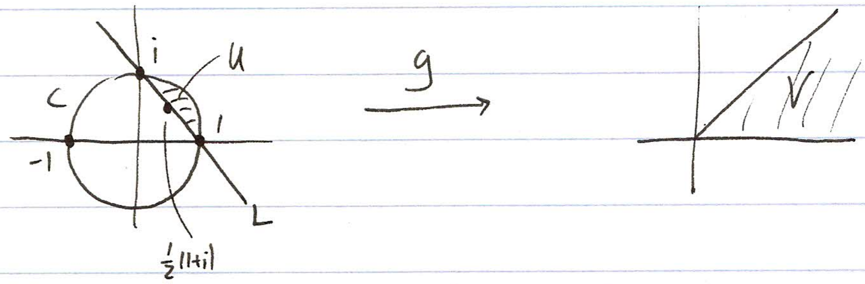
It follows that either U maps to V or U maps to the lower half plane W (the region below the real axis).

Compute the image of a point in U to find out which case occurs:–

$$g(i) = \frac{-i \cdot 2i}{-2i} = i \in V. \qquad \text{So } U \text{ maps to } V$$

(If instead we found $g(U) = W$, can compose with $z \mapsto -z$ (switches V & W)).

e



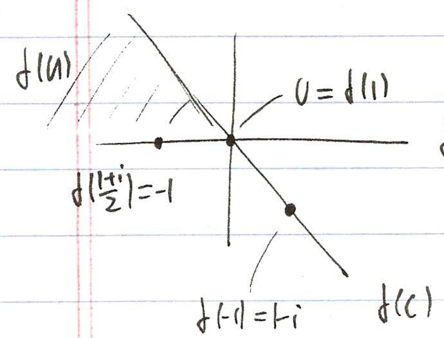
First, write down LFT sending the 2 intersection points of the circle C and line L bounding U to ∞ & 0 (then the image of these curves will be two lines through the origin).

$$f(z) = \frac{z-1}{z-i} \qquad \begin{array}{l} 1 \mapsto 0 \\ i \mapsto \infty \end{array}$$

Compute the images of the curves by computing the image of another point on each curve:

$1+i/2 \in L \qquad f(1+i/2) = \dfrac{(-1+i)/2}{(1-i)/2} = -1. \qquad \therefore f(L)$ is the real axis

$-1 \in C \qquad f(-1) = \dfrac{-2}{-1-i} = \dfrac{2}{1+i} = \dfrac{2\cdot(1-i)}{2} = 1-i$

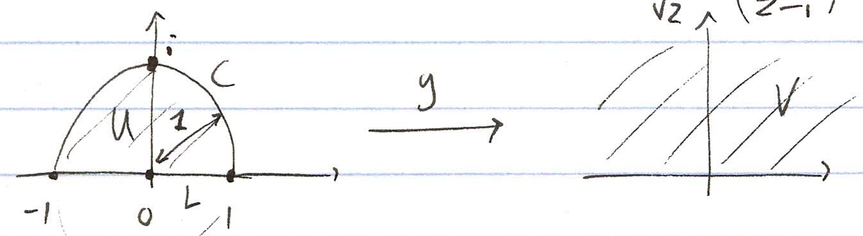


Now see $f(U)$ is the shaded region shown – it must have $f(\frac{1}{2}(1+i))$ on its boundary, & the angle at $f(1)$ must be $\pi/4$ (= the angle at 1 for original region)

Finally, rotate by $-3\pi/4$ to obtain

$$g(z) = e^{-i3\pi/4} \cdot f(z) = -\frac{(1+i)}{\sqrt{2}} \cdot \left(\frac{z-1}{z-i}\right)$$

f.



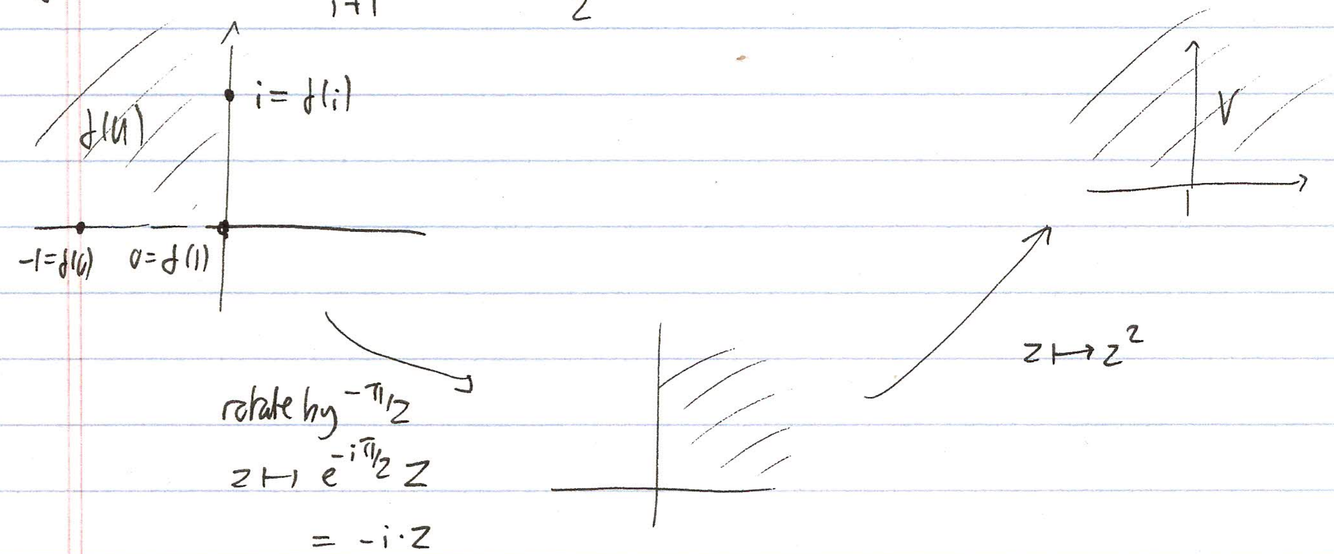
First send two intersection points of circle C & line L bounding U to 0 & ∞ by a LFT. f.

Then $f(C)$ & $f(L)$ will be two lines thru origin.

$$f(z) = \frac{z-1}{z-(-1)} = \frac{z-1}{z+1}$$

$f(0) = -1 \implies f(L) = $ real axis

$f(i) = \frac{i-1}{i+1} = \frac{(-1+i)(1-i)}{2} = i \implies f(C) = $ imaginary axis



So, $g(z) = \left(-i \cdot \left(\frac{z-1}{z+1}\right)\right)^2 = -\left(\frac{z-1}{z+1}\right)^2$

<u>WARNING</u>: For all parts of Q20, there is more than one solution!
(The function g is <u>not</u> uniquely determined.)