Math 461 Homework 4

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(1) Consider the circle with center the origin and radius 1 and the circle with center $(1, 2)$ and radius 2.

(a) Find the intersection points of the two circles.

(b) Eliminating x^2 and y^2 from the equations of the two circles gives a linear equation in x and y. What is the geometric meaning of this equation?

(2) Prove the *triangle inequality*: for a triangle ΔABC with side lengths

a, b, c, we have $a < b + c$, $b < a + c$, and $c < a + b$.

[Hint: It suffices to show $a < b + c$. One approach is to use the cosine rule together with the bounds $|\cos \alpha| \leq 1$ (which follow from the definition of cosine and Pythagoras' theorem).]

(3) Conversely, suppose a, b, c are positive real numbers such that $a < b+c$, $b < a + c$, and $c < a + b$. Show that there exists a triangle ΔABC with side lengths a, b, c.

[Hint: Draw a circle with center $(0, 0)$ and radius a and a circle with center $(c, 0)$ and radius b. Show that the two circles intersect in two points not lying on the x-axis if the inequalities above are satisfied by finding the common solutions of the equations of the two circles. Now deduce the existence of a triangle with side lengths a, b, c.]

(4) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. We define the *fixed locus* of T to be the set

$$\{(x, y) \in \mathbb{R}^2 \mid T(x, y) = (x, y)\}.$$

Determine the fixed locus of T in each of the following cases: (a) the identity, (b) a translation, (c) a rotation, (d) a reflection, (e) a glide reflection.

(5) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a glide reflection. Show that T^2 is a translation. (Here we write T^2 for the composi-

tion $T \circ T$.)

(6) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Suppose that $T^n = \mathrm{id}$ for some positive integer n. (Here T^n denotes the composition $T \circ T \circ \cdots \circ T$ where there are n copies of T.) What are the possibilities for T? Justify your answer carefully.

(7) Compute the formula
$$T(x, y) = (a(x, y), b(x, y))$$
for T in each of the following cases.

(a) T is rotation about the point $P = (1, 2)$ through angle $\pi/2$ counterclockwise.

(b) T is reflection in the line $y = 2$.

(c) T is a glide reflection given by reflection in the line $y = -x$ followed by translation by $(2, -2)$.

(8) In each of the following cases, describe the given isometry T as a translation, rotation, reflection, or glide reflection. For a translation, give the translation vector. For a rotation give the center, angle, and sense (counterclockwise or clockwise) of rotation. For a reflection give the line of reflection. For a glide reflection give the line of reflection and the translation vector.

(a) $T(x, y) = (-x, -y)$.

(b) $T(x, y) = \frac{1}{5}(4x+3y+2, 3x-4y-6)$.

(c) $T(x, y) = \frac{1}{5}(3x-4y+8, 4x+3y+4)$.

(d) $T(x, y) = (y + 4, x + 8)$.

[Hint: One possible approach is as follows: To determine the type of T, compute the fixed locus and use Q4. For a rotation, find the angle using the formula $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for the rotation matrix. For a glide reflection, find the translation vector using Q5, then use this to determine the line of reflection.]

(9) (Optional) You break a stick into 3 pieces. What is the probability that the pieces are the sides of a triangle?

(10) (Optional) Let L_1, L_2, L_3 be 3 distinct lines in the plane with equations $a_i x + b_i y = c_i$, $i = 1, 2, 3$. Show that the lines are concurrent

or parallel if and only if

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0.$$

[Hint: Consider the possible row echelon forms of the augmented matrix of the system of linear equations $a_i x + b_i y = c_i$, $i = 1, 2, 3$, as in MATH 235.]