Math 412 Homework 6

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Reading: Saracino, Chapter 22.
Show your work and justify your answers carefully.

(1) Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$.

   (a) $f(x) = x^3 + 3x^2 - 2x + 3$.
   (b) $g(x) = x^5 + 35x^3 - 21x + 63$.

(2) Let K be a field and $f \in K[x]$ be a nonconstant polynomial. Show that if f is not irreducible then it has an irreducible factor g such that $\deg(g) \leq \deg(f)/2$.

(3) (a) List all the irreducible polynomials of degree ≤ 4 in $(\mathbb{Z}/2\mathbb{Z})[x]$.
   (b) Using your list from part (a) or otherwise, prove that the following polynomials are irreducible in $\mathbb{Q}[x]$.
      i. $f(x) = 5x^4 + 7x^3 + 2x^2 + 3$.
      ii. $g(x) = 3x^5 + 2x^4 + 6x + 10$.
      iii. $h(x) = 9x^4 + 3x^3 + x^2 + 5x - 1$.

   [Hint: Part (a) can be done fairly quickly using the "sieve of Eratosthenes" method discussed in class (the result of Q2 should be used here). Recall also that $(x - a)$ divides $f(x)$ iff $f(a) = 0$ (so it is easy to determine divisibility by linear factors).]

(4) Let p be a prime. Show that $f(x) = x^n - p$ is irreducible in $\mathbb{Q}[x]$ for all $n \in \mathbb{N}$.

(5) Compute the gcd of the following pairs of polynomials.

(a) $x^3 + x^2 + 2x + 2$, $x^2 + 4x + 3$ in $\mathbb{Q}[x]$.

(b) $x^4 + 2x^3 + x + 2$, $x^3 + 2x^2 + x + 2$ in $(\mathbb{Z}/3\mathbb{Z})[x]$

[Hint: For $f, g \in K[x]$ polynomials with coefficients in a field K, the *greatest common divisor* $d = \gcd(f, g)$ is the monic polynomial d such that (1) d divides both f and g, and (2) if e divides both f and g then e divides d. We require that d is monic (that is, has leading coefficient 1) so that it is uniquely determined. The gcd can be computed using the Euclidean algorithm.]

(6) Compute the irreducible polynomials over $\mathbb{Q}$ of the following numbers $\alpha \in \mathbb{R}$.

(a) $\alpha = 1 + \sqrt{2}$.

(b) $\alpha = \sqrt[3]{2}$.

(c) $\alpha = \sqrt{3} + \sqrt{5}$.

[Hint: If $K \subset L$ are fields and $\alpha \in L$, we say α is *algebraic* over K if there exists a nonzero polynomial $g(x) \in K[x]$ such that $g(\alpha) = 0$. In this case, the *irreducible polynomial of α over K* is the monic polynomial $f(x) \in K[x]$ of smallest degree such that $f(\alpha) = 0$. (Then f is irreducible, and any other polynomial $g(x) \in K[x]$ with $g(\alpha) = 0$ is a multiple of f.) To determine the irreducible polynomial f of α, first find some polynomial $g(x) \in K[x]$ such that $g(\alpha) = 0$. Then f is an irreducible factor of g. In particular, if g is irreducible then $f = g/c$ where $c \in K$ is the leading coefficient of g.]

(7) Let K be a field and $a, b \in K$, $a \neq 0$. For $f(x) \in K[x]$ a polynomial, show that $f(x)$ is irreducible in $K[x]$ iff $f(ay+b)$ is irreducible in $K[y]$.

(8) (Optional) Prove the following analogue of Gauss' Lemma: Let $f \in \mathbb{C}[x, y] = \mathbb{C}[x][y]$ be a polynomial in the variables x and y with complex coefficients. Let $\mathbb{C}(x)$ denote the fraction field of $\mathbb{C}[x]$, that is, $\mathbb{C}(x)$ is the field consisting of rational functions in the variable x with coefficients in $\mathbb{C}$. Suppose $f = gh$ for some $g, h \in \mathbb{C}(x)[y]$. Then $f = \tilde{g}\tilde{h}$ for some $\tilde{g}, \tilde{h} \in \mathbb{C}[x][y]$ such that $\tilde{g} = ag$ and $\tilde{h} = a^{-1}h$ for some $0 \neq a \in \mathbb{C}(x)$.

(9) (Optional) Prove the following analogue of Eisenstein's criterion: Let $f \in \mathbb{C}[x, y] = \mathbb{C}[x][y]$ be a polynomial in the variables x and y with complex coefficients, and write

$$f = a_n(x)y^n + \cdots + a_1(x)y + a_0(x)$$

where $a_n(x), \ldots, a_0(x) \in \mathbb{C}[x]$. Suppose that $a_n(x)$ is not divisible by x, each of $a_{n-1}(x), \ldots, a_1(x), a_0(x)$ is divisible by x, and $a_0(x)$ is not divisible by x^2. Then f is irreducible in $\mathbb{C}(x)[y]$.