Sunday. 9/20/15

MATH 611 HW1 Solutions

1. $a^3 b = b a^3 \Rightarrow a^{3n} b = b a^{3n}$ † for all $n \in \mathbb{Z}$.

$a^7 = e$, and $\gcd(3,7) = 1$, so we can solve $3n \equiv 1 \bmod 7$

finding $n \equiv 5 \bmod 7$.

So, putting $n=5$ in † gives $ab = ba$.

2. $|G| = 22 = 2 \cdot 11$ (prime factorization).

We are given $e \neq a \in G$ and $b \notin \langle a \rangle$.

So $\{e\} \underset{\neq}{<} \langle a \rangle \underset{\neq}{<} \langle a, b \rangle \leq G$

& $1 < |\langle a \rangle| < |\langle a, b \rangle|$ (*)

Lagrange's theorem states that the order $|H|$ of a subgroup H of a finite group G divides the order $|G|$ of G.

Now, since $|G|$ has only 2 factors in its prime factorization, (*) $\Rightarrow |\langle a,b \rangle| = |G|$, i.e., $\langle a,b \rangle = G$, as required.

3. $\varphi : G \to G'$ homomorphism, $|G| = 18 = 2 \cdot 3^2$, $|G'| = 15 = 3 \cdot 5$

1/ $|G| = |\varphi(G)| \cdot |\ker \varphi|$ and 2/ $|\varphi(G)| \mid |G'|$.

φ nontrivial, i.e., $|\varphi(G)| \neq 1$.

By 1/ & 2/, $|\varphi(G)| \mid \gcd(|G|, |G'|) = 3$.

So $|\varphi(G)| = 3$, & $|\ker \varphi| = |G| / |\varphi(G)| = 6$.

4. $ab = a(ba)a^{-1}$. So ab & ba are conjugate.

In general, conjugate elements have the same order

because the map $\begin{aligned} G &\to G \\ x &\mapsto gxg^{-1} \end{aligned}$

given by conjugation by $g \in G$ is an isomorphism

from G to itself (an <u>automorphism</u> of G).

5. Suppose G has no proper subgroups.

If G is not the trivial group $\{e\}$, choose $e \neq a \in G$.

Then $\langle a \rangle \leq G$, $\langle a \rangle \neq \{e\} \Rightarrow \langle a \rangle = G$

So G is cyclic, $G \simeq \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$
or $G \simeq \mathbb{Z}$

$\mathbb{Z}/n\mathbb{Z}$ has no proper subgroups iff n is prime
(subgroups of $\mathbb{Z}/n\mathbb{Z}$ are given by $\langle d \rangle$ where $d \mid n$.)

$\mathbb{Z}$ does have proper subgroups (given by $\langle d \rangle = d \cdot \mathbb{Z}$, $d \geq 2$)

So G has no proper subgroups $\Leftrightarrow$ $G \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p
OR $G = \{e\}$, the trivial group.

6 a) Let E_{kl} denote the elementary $n \times n$ matrix with entry 1 in position (k,l) and zeroes elsewhere.

WARNING: E_{kl} is not invertible!

But we still have

$$A \in Z(GL_n(\mathbb{R})) \iff A \text{ commutes w/ } E_{kl} \text{ for each } k,l.$$

<u>Proof</u> $\Leftarrow$: $A = \sum_{k,l} a_{kl} \cdot E_{kl}$ (where a_{kl} denotes the k,l entry of A)

and clearly $A \cdot B = B \cdot A \Rightarrow A \cdot (\lambda B) = (\lambda B) \cdot A$, $\lambda \in \mathbb{R}$

$A \cdot B = B \cdot A$ & $A \cdot C = C \cdot A \Rightarrow A \cdot (B+C) = (B+C) \cdot A$.

$\Rightarrow$: $I + E_{kl}$ is invertible.

Now A commutes w/ I & $I + E_{kl}$

$\Rightarrow A$ commutes w/ E_{kl}. $\square$

Now compute: $E_{kl} \cdot A$ = Matrix w/ k^{th} row equal to l^{th} row of A & all other rows zero.

$A \cdot E_{kl}$ = Matrix w/ l^{th} col. equal to k^{th} col. of A & all other columns zero.

Thus $E_{kl} \cdot A = A \cdot E_{kl} \iff \begin{cases} a_{lj} = 0 & \text{for } j \neq l \\ a_{ik} = 0 & \text{for } i \neq k \\ \& \ a_{ll} = a_{kk} \end{cases}$

If this holds for all k,l, we find $A = \lambda \cdot I$, $\lambda \in \mathbb{R}$.

Thus $Z(GL_n(\mathbb{R})) = \{ \lambda \cdot I \mid 0 \neq \lambda \in \mathbb{R} \}$.

b) If ~~n~~ $n \leq 2$, S_n is abelian, $Z(S_n) = S_n$.

Suppose $n \geq 3$. We show $Z(S_n) = \{e\}$

Given $e \neq \sigma \in S_n$ we must find $\tau \in S_n$ such that $\tau \sigma \tau^{-1} \neq \sigma$.

If the expression of σ as a product of disjoint cycles contains a cycle of length ≥ 3, say $(a_1 a_2 \cdots a_\ell)$,

Then $(a_1 a_2)\sigma(a_1 a_2)^{-1} \neq \sigma$.

$$\begin{array}{ccc} (a_1 a_2)\sigma(a_1 a_2)^{-1} & \neq & \sigma \\ \| & & \| \\ (a_2 a_1 a_3 \cdots a_\ell)\cdots & , & (a_1 a_2 \cdots a_\ell)\cdots \end{array}$$

Otherwise, the expression of σ as a product of disjoint cycles contains a transposition $(a_1 a_2)$. $(\sigma \neq e)$.

Let $a_3 \in \{1,2,\ldots,n\} \setminus \{a_1, a_2\}$ $\quad (n \geq 3)$

Then $(a_2 a_3)\sigma(a_2 a_3)^{-1} \neq \sigma$

$$\begin{array}{ccc} (a_2 a_3)\sigma(a_2 a_3)^{-1} & \neq & \sigma \\ \| & & \| \\ (a_1 a_3)\cdots & & (a_1 a_2)\cdots \end{array}$$

$\square$

c) Recall $D_n = \langle a, b \mid a^n = b^2 = e,\ ba = a^{-1}b \rangle$
$= \{e, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$

Since D_n is generated by a & b, $g \in D_n$ is in the center $Z(D_n)$ iff it commutes w/ a & b.

$(ba = a^{-1}b \Rightarrow ba^{-1} = ab)$

Now compute: 1/ $a \cdot (a^i b) a^{-1} = a^{i+1} b a^{-1} = a^{i+1} \cdot ab = a^{i+2} b$

$n \geq 3$ by assumption (D_n is the symmetries of regular n-gon)

So $a^i b \neq a^{i+2} b$ (because $n \nmid 2$),

$a^i b$ does NOT commute w/ a.

2/ $b \cdot a^i \cdot b^{-1} = a^{-i} \cdot b \cdot b^{-1} = a^{-i}$.

So a^i commutes w/ b iff $a^i = a^{-i}$, or $a^{2i} = e$.

Assuming $0 < i < n$, this gives $n = 2m$, $i = m$.

(Also, a^i commutes w/ a $\forall i$ of course.)

So $Z(D_n) = \begin{cases} \{e\} & n \text{ odd} \\ \{e, a^m\} & n = 2m \text{ even.} \end{cases}$

<u>Remark</u>: In the second case $a^m = -I \in GL_2(\mathbb{R})$ regarding $D_n \leq GL_2(\mathbb{R})$.

7\. a) Regarding $D_n \leq GL_2(\mathbb{R})$

(choosing coordinates on $\mathbb{R}^2$ such that the origin is the center of mass of the regular n-gon and the axis of reflection of b is the x-axis)

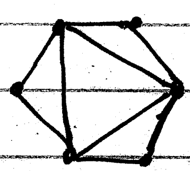
we have $a = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta = 2\pi/n$

$b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Now compute $ba = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$

$a^{-1}b = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$

b) The regular n-gon is inscribed in the regular $2n$-gon.



$n=3$.

Choose a vertex of the regular n-gon and let b denote reflection in the axis of symmetry passing through that vertex.

Then $\underset{\langle a', b\rangle}{\overset{\|}{D_n}} \leq \underset{\langle a, b\rangle}{\overset{\|}{D_{2n}}}$, where $a' = a^2$.

(a is rotation about the center of mass thru angle $2\pi/2n$ counterclockwise)

The element $a^n \in D_{2n}$ is central (i.e commutes w/ all elements of D_{2n}), see 6b c.

It follows that the map $\varphi: D_n \times \mathbb{Z}/2\mathbb{Z} \longrightarrow D_{2n}$

$(g, i) \longmapsto g \cdot a^{ni}$

is a homomorphism of groups.

The kernel of φ is trivial, $\ker\varphi = \{e\}$, because $D_n \cap \langle a^n \rangle \overset{\dagger}{=} \{e\}$ (using n odd) :–

$$\varphi(g,i) = e \iff g \cdot a^{ni} = e \iff g = a^{-ni}$$
$$\overset{\dagger}{\iff} g = e,\ i = 0.$$

Since $|D_n \times \mathbb{Z}/2\mathbb{Z}| = |D_{2n}| = 4n$,

φ is an isomorphism.

c) For example, D_{2n} has an element a of order $2n$, but $D_n \times \mathbb{Z}/2\mathbb{Z}$ has no such element

(Note: the order of $(a,b) \in G \times H$ equals the lcm of the order of $a \in G$ & the order of $b \in H$).

8. Q_8 has 6 elements of order 4 $(\pm i, \pm j, \pm k)$

D_4 has 2 elements of order 4 $(a^{\pm 1})$

$6 \neq 2 \Rightarrow Q_8 \not\cong D_4$.

9. $G = \langle a, b \mid a^3 = b^4 = e,\ ba = a^{-1}b \rangle$

a) By the universal property stated in the question, we have a group homomorphism

$$\varphi : G \longrightarrow \mathbb{Z}/4\mathbb{Z}$$

given by $\varphi(a) = 0$, $\varphi(b) = 1$

(We just need to check, using additive notation in $\mathbb{Z}/4\mathbb{Z}$, that

$$3\varphi(a) = 4\varphi(b) = 0 \quad \& \quad \varphi(b) + \varphi(a) = -\varphi(a) + \varphi(b)$$

i.e. the relations defining G hold in $\mathbb{Z}/4\mathbb{Z}$ when a, b are replaced by $\varphi(a), \varphi(b)$.)

Using the defining relations of G, any element $g \in G$ may be written in the form $g = a^i b^j$ †
for some $0 \le i < 3$ & $0 \le j < 4$.

It is true (but requires proof) that the expression † is unique, so $|G| = 12$.
Probably the best way to prove it is to construct G explicitly as a semidirect product. (I didn't realize this when I wrote the problem!)
One can make some progress as originally suggested:-
$\varphi : G \to \mathbb{Z}/4\mathbb{Z}$ is a surjective hom, & $\varphi(a^i b^j) = j$.
In particular j in † is uniquely determined, and $\ker \varphi = \langle a \rangle$, the cyclic subgroup of G generated by a. Since $a^3 = e$, we have either $\ker\varphi \cong \mathbb{Z}/3\mathbb{Z}$ or $a = e \in G$. It remains to rule out the second possibility.

b) A_4 has 3 elements of order 2 $\{(12)(34), (13)(24), (14)(23)\}$
D_6 has 7 elements of order 2 (rotation by π, 6 reflections)
$\Rightarrow A_4 \not\cong D_6$.

G has an element b of order 4, but A_4 & D_6 have no elements of order 4
$\Rightarrow G \not\cong A_4$ or D_6.

10. Compute: $A = SBS^{-1} \iff AS = SB$. Write $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}, \quad \boxed{b=c,\ d=0}$$

$S = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$ $\det S = -b^2 < 0.$

So A and B are conjugate in $GL_2(\mathbb{R})$ but not in $SL_2(\mathbb{R})$

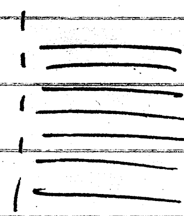
11. a $H = \langle (123) \rangle = \{e, (123), (132)\} \leq A_4$

Left cosets:
$$H = e \cdot H = \{e, (123), (132)\}$$
$$(124) \cdot H = \{(124), (14)(23), (134)\}$$
$$(234) \cdot H = \{(234), (13)(24), (142)\}$$
$$(143) \cdot H = \{(143), (12)(34), (243)\}$$

Right cosets:
$$H = H \cdot e = \{e, (123), (132)\}$$
$$H \cdot (124) = \{(124), (13)(24), (243)\}$$
$$H \cdot (234) = \{(234), (12)(34), (134)\}$$
$$H \cdot (143) = \{(143), (14)(23), (142)\}$$

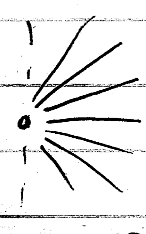
b $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ x > 0 \right\}$

$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\}$


left cosets

$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G, \quad a > 0$

$$g \cdot H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\}$$
$$= \left\{ \begin{pmatrix} ax & b \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\} = \left(\overset{x>0}{y = b} \right) \subset \mathbb{R}^2_{x,y}$$


right cosets

$$H \cdot g = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\}$$
$$= \left\{ \begin{pmatrix} ax & bx \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ x > 0 \right\} = \begin{pmatrix} x > 0 \\ y = b/a \cdot x \end{pmatrix} \subset \mathbb{R}^2_{x,y}$$