Math 612 Homework 1

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Reading: Dummit and Foote, 13.1,13.2,13.4.
Justify your answers carefully.

(1) Let $\alpha = \sqrt[3]{2}$. Compute the minimal polynomial for $\beta := 1 + \alpha^2$ over $\mathbb{Q}$.

(2) Let $\alpha = \sqrt{2} + i$. Compute the minimal polynomial for α over (a) $\mathbb{Q}$, (b) $\mathbb{Q}(\sqrt{2})$, (c) $\mathbb{Q}(i)$, (d) $\mathbb{Q}(\sqrt{-2})$.

(3) Let $\zeta_n = e^{2\pi i/n}$ for $n \in \mathbb{N}$. Compute the minimal polynomial for ζ_n over $\mathbb{Q}$ for $n = 4, 6, 8, 9, 10, 12$.

(4) Let $f = x^3 - x + 1$.

   (a) Prove that f is irreducible over $\mathbb{Q}$.

   (b) Let $\alpha \in \mathbb{C}$ be a root of f and let $K = \mathbb{Q}(\alpha)$. Then $1, \alpha, \alpha^2$ is a basis for K as a vector space over $\mathbb{Q}$ (why?). Express $(1+\alpha+\alpha^2)^{-1}$ in the form $c_0 + c_1\alpha + c_2\alpha^2$ for $c_0, c_1, c_2 \in \mathbb{Q}$.

(5) Determine whether i is in the following fields (a) $\mathbb{Q}(\sqrt{-2})$, (b) $\mathbb{Q}(\sqrt[4]{-2})$, (c) $\mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{C}$ is a root of $x^3 + x + 1$.

(6) Let $F \subset K$ be a field extension and $\alpha, \beta \in K$. Let $m = [F(\alpha) \colon F]$, $n = [F(\beta) \colon F]$ and suppose $\gcd(m, n) = 1$. Show that $[F(\alpha, \beta) : F] = mn$ and write down a basis for $F(\alpha, \beta)$ as a vector space over F.

(7) For each of the following polynomials, determine the degree of the splitting field over $\mathbb{Q}$.

   (a) $x^4 + x^3 + x^2 + x + 1$.

(b) $x^4 - 2$.

(8) Let $\omega = e^{2\pi i/3}$, $\alpha = \sqrt[3]{2}$ and $\beta = \omega\sqrt[3]{2}$.

(a) Describe an isomorphism $\varphi\colon \mathbb{Q}(\alpha) \xrightarrow{\sim} \mathbb{Q}(\beta)$.

(b) Prove that -1 cannot be written as a sum of squares in $\mathbb{Q}(\beta)$.

(9) Let $F \subset K$ be a field extension such that $[K : F] = 2$. Assume that $\mathrm{char}(F) \neq 2$. Prove that $K = F(\alpha)$ for some $\alpha \in K$ such that $\alpha^2 \in F$.

(10) Let $F \subset K$ be a field extension such that $K = F(\alpha)$ for some $\alpha \in K$ and $[K : F]$ is odd. Prove that $K = F(\alpha^2)$.

(11) (a) Let $F \subset K$ be a field extension and $\alpha \in K$ an element such that $K = F(\alpha)$, $\alpha^2 \in F$, and $\alpha \notin F$. Determine all elements $\beta \in K$ such that $\beta^2 \in F$.

(b) Let $p_1, p_2, \ldots, p_n \in \mathbb{N}$ be distinct primes. Prove that

$$[\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}) : \mathbb{Q}] = 2^n.$$

(12) Let $\alpha, \beta \in \mathbb{C}$ be such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\beta) : \mathbb{Q}] = 3$. What are the possible values of $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$? Give examples to show that each case occurs.