Math 612 Homework 1

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January 31, 2020

Reading: Dummit and Foote, Sections 10.1, 10.2, 10.3, 11.1, 11.2, 11.3, 11.4, 12.1, 12.2, 12.3.

Justify your answers carefully (complete proofs are expected). All rings are assumed commutative with 1.

Let F be a field. Let $A, B \in F^{n\times n}$ be $n \times n$ matrices with entries in F. We say A *is similar to* B if there exists an invertible $n \times n$ matrix $P \in \mathrm{GL}_n(F)$ such that $B = P^{-1}AP$.

Note: If $F \subset K$ is an inclusion of fields then there exists $Q \in \mathrm{GL}_n(K)$ such that $Q^{-1}AQ = B$ iff there exists $P \in GL_n(F)$ such that $P^{-1}AP = B$. This follows from the uniqueness of the rational canonical form, see e.g. DF, p. 477, Corollary 18. So similarity is well defined independent of the choice of the field F.

(1) Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \in \mathbb{Q}^{3\times 3}.$$

(a) Determine the rational canonical form B and Jordan normal form C of A.

(b) Determine invertible matrices R and S in $\mathrm{GL}_3(\mathbb{Q})$ such that $R^{-1}AR = B$ and $S^{-1}AS = C$.

(2) Let F be a field and A a square matrix with entries in F. Suppose $A^2 = A$. What are the possible Jordan normal forms of A?

(3) Let A be a square matrix with entries in a field F. Let

$$c_A(x) = \det(xI - A) \in F[x],$$

the *characteristic polynomial* of A. Let $m_A \in F[x]$ denote the minimal polynomial of A, that is, the monic polynomial $f(x) \in F[x]$ of smallest degree such that $f(A) = 0$.

(a) Show that if A and B are similar matrices then $c_A = c_B$ and $m_A = m_B$.

(b) Show that if $c_A = c_B$ and A and B are diagonalizable, then A and B are similar. Give an example of matrices A and B such that $c_A = c_B$ but A and B are not similar.

(c) Suppose that A and B are $n \times n$ matrices such that $c_A = c_B$ and $m_A = m_B$. Show that if $n \leq 3$ then A and B are similar, but that A and B need not be similar if $n \geq 4$.

(4) Let F be a field. We say a square matrix A with entries in F is *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$. Show that the similarity classes of $n \times n$ nilpotent matrices with entries in F are in bijection with partitions of n, that is, expressions

$$n = n_1 + n_2 + \cdots + n_r$$

where r and $n_1, \ldots, n_r$ are positive integers and $n_1 \leq n_2 \leq \cdots \leq n_r$.

(5) Let $A \in \mathbb{C}^{n \times n}$. Show that there exists a decomposition $A = D + N$ where D is diagonalizable, N is nilpotent, and $DN = ND$.

(6) Let

$$J = J(n, \lambda) = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \lambda \end{pmatrix}$$

be a $n \times n$ Jordan block with eigenvalue λ. Determine a formula for J^k for $k \in \mathbb{N}$.

(7) Let $J = J(n, \lambda)$ be a $n \times n$ Jordan block with eigenvalue λ. Determine all subspaces $W \subset F^n$ such that $J \cdot W \subset W$.

(8) Suppose V is a vector space over $\mathbb{Q}$ and $T\colon V \to V$ is a linear transformation such that $T^5 = \mathrm{id}$ and there does *not* exist a nonzero vector $v \in V$ such that $T(v) = v$. What can you say about the dimension of V? (Justify your answer carefully.)

(9) Classify conjugacy classes of matrices $A \in GL_3(\mathbb{Q})$ of order 6.

(10) (Optional) Let F be a field and A a square matrix with entries in F. Prove that A is similar to its transpose.

(11) (Optional) Let p be a prime and $n \in \mathbb{N}$ a positive integer. Let $\mathbb{F}_{p^n}$ be a finite field of order p^n. Let $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be the *Frobenius automorphism* defined by $F(\alpha) = \alpha^p$. We have an inclusion $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \subset \mathbb{F}_{p^n}$ (why?) so that $\mathbb{F}_{p^n}$ has the structure of a vector space over $\mathbb{F}_p$.

  (a) Show that, regarding $\mathbb{F}_{p^n}$ as an $\mathbb{F}_p$-vector space, the Frobenius automorphism F is a linear transformation.
  (b) Determine the minimal polynomial of F.

(12) (a) (Optional) Consider the space $\mathbb{C}^{n\times n}$ of $n \times n$ complex matrices. Show that there is a polynomial $F(\{z_{ij}\})$ in the entries z_{ij} of the matrix such that the characteristic polynomial c_A of $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ has n distinct roots iff $F(\{a_{ij}\}) \neq 0$. Note that the locus

$$U := (F \neq 0) \subset \mathbb{C}^{n\times n}$$

is a dense open subset.

  (b) What is the (i) rational canonical form and (ii) Jordan normal form of a matrix $A \in U$?

(13) (Optional) Let R be an integral domain and M an R-module. Define

$$\operatorname{Tors} M = \{m \in M \mid \text{ There exists } 0 \neq r \in M \text{ such that } rm = 0\ \}.$$

Then $\operatorname{Tors} M \subset M$ is a submodule of M (why?).

Suppose R is a PID and M is a finitely generated R-module. It follows from the structure theorem that if $\operatorname{Tors} M = \{0\}$ then M is a free module (why?).

Now suppose R is a Noetherian integral domain but is not a PID. Show that there exists a finitely generated R-module M such that $\operatorname{Tors} M = \{0\}$ but M is not a free module.

Hints:

1 Compute as in the class notes or DF 12.2,12.3.

2 What is the minimal polynomial of A?

3 (c) What is the relation between the rational canonical form of A and its minimal and characteristic polynomials?

4 What is the Jordan normal form of a nilpotent matrix?

5 It's enough to consider a single Jordan block (why?).

6 Write $J = D + N$ as in Q3. Note that for *commuting* matrices A and B, the binomial formula $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k$ is valid.

7 If we regard F^n together with the linear transformation $v \mapsto Jv$ as a $\mathbb{C}[T]$-module M, what is the corresponding interpretation of a subspace $W \subset F^n$ such that $J \cdot W \subset W$? (Alternatively, one can compute directly.)

8 What is the minimal polynomial of T? What are the possible rational canonical forms of T?

9 What is the factorization of $x^6 - 1 \in \mathbb{Q}[x]$ into irreducibles? What are the possible minimal polynomials of A? What are the possible primary rational canonical forms of A?

10 Recall that the rational canonical form of A is determined by the Smith normal form of $xI - A$. What is the relation between the Smith normal form of $xI - A$ and $xI - A^T$ (where A^T denotes the transpose of A)?

11 For F a field, a polynomial $p(x) \in F[x]$ of degree n has at most n roots in F.

12 (a) What is the discriminant of a polynomial (DF p. 610 and p. 621, Ex 32)?

13 Consider an ideal $I \subset R$ which is not principal.