Math 461 Lecture 23 10/26

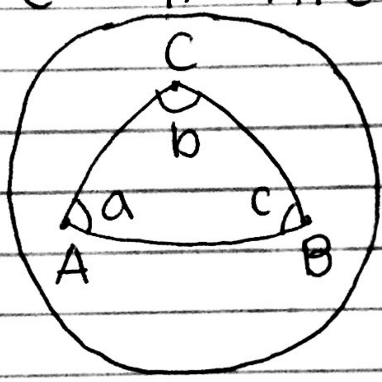
Announcements:

Homework 5 available.
Due at start of Wednesday's (10/31) class.

Midterm exams will be returned on Monday.

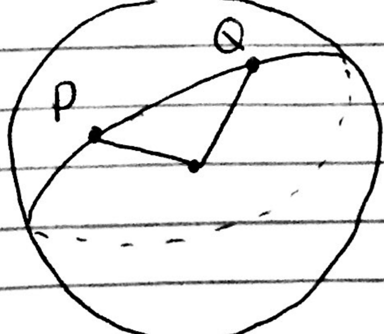
Last time:

$a+b+c = \pi + \text{Area}(\Delta ABC)$



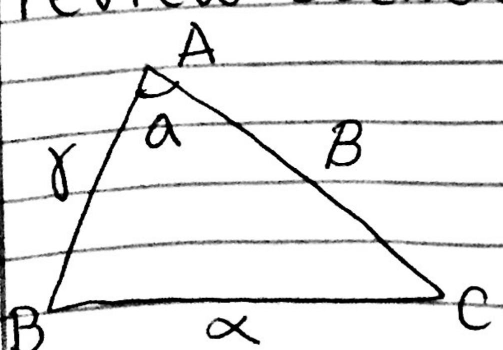
Today:

spherical cosine rule $\Rightarrow$
spherical triangle inequality $\Rightarrow$
spherical lines give shortest paths



spherical cosine rule:

review Euclidean cosine rule:



$$\alpha^2 = \beta^2 + \gamma^2 - 2\beta\gamma \cos a$$

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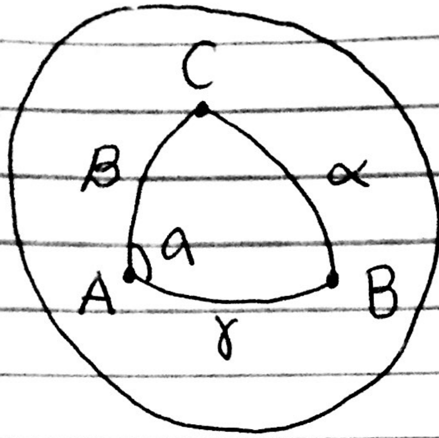
Spherical cosine rule:

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

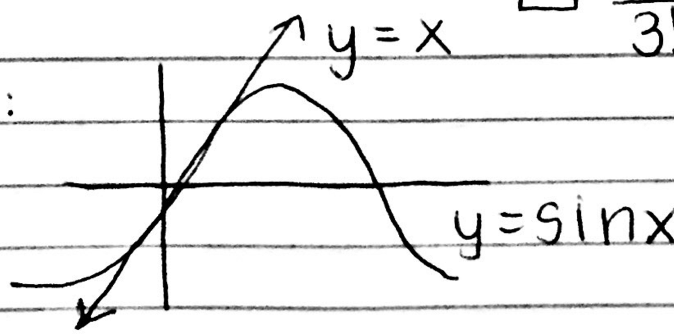


for α, β, γ small $\rightsquigarrow$ can reduce to the Euclidean cosine rule

approximate $\sin x \approx x$

taylor series: $\sin x = \boxed{x} - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$

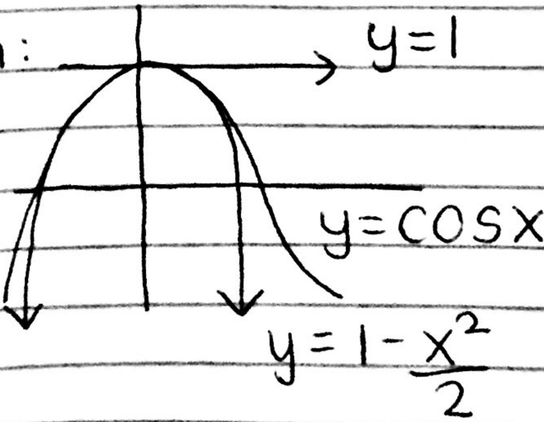
graph:



approximate $\cos x \approx 1 - \frac{x^2}{2}$

taylor series: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$

graph:



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For α, β, γ small:

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

$1-\frac{\alpha^2}{2} \approx \left(1-\frac{\beta^2}{2}\right)\left(1-\frac{\gamma^2}{2}\right) + \beta\gamma\cos a$

$\cancel{1}-\frac{\alpha^2}{2} \approx \cancel{1}-\frac{\beta^2}{2}-\frac{\gamma^2}{2}+\cancel{\frac{\beta^2\gamma^2}{4}}+\beta\gamma\cos a$

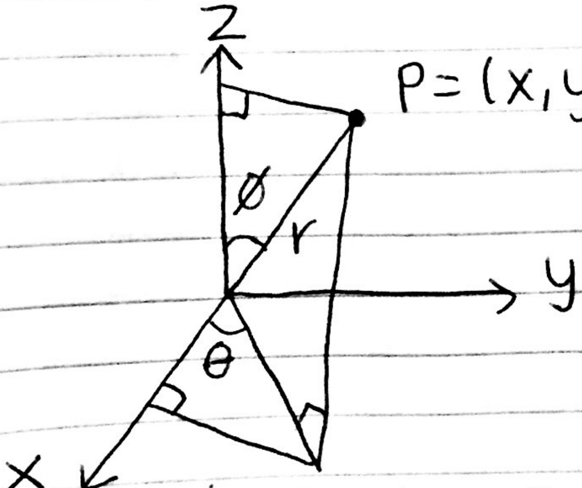
0 very small

$-\frac{\alpha^2}{2} \approx -\frac{\beta^2}{2}-\frac{\gamma^2}{2}+\beta\gamma\cos a$

multiply by -2

$\alpha^2 \approx \beta^2+\gamma^2-2\beta\gamma\cos a$ ✓

Spherical polor coordinates: (233?)



$r \in \mathbb{R}$ $r \geq 0$

$\theta \in [0, 2\pi]$

$\phi \in [0, \pi]$

$(x,y,z) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$

we are working on S^2

$x^2+y^2+z^2=1$, $r=1$

so for $P \in S^2$

$P=(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

Proof of SCR (spherical cosine rule):

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

choose spherical coordinates

we may assume while choosing coordinates appropriately that $A=(0,0,1)$ and B lies in the xz plane

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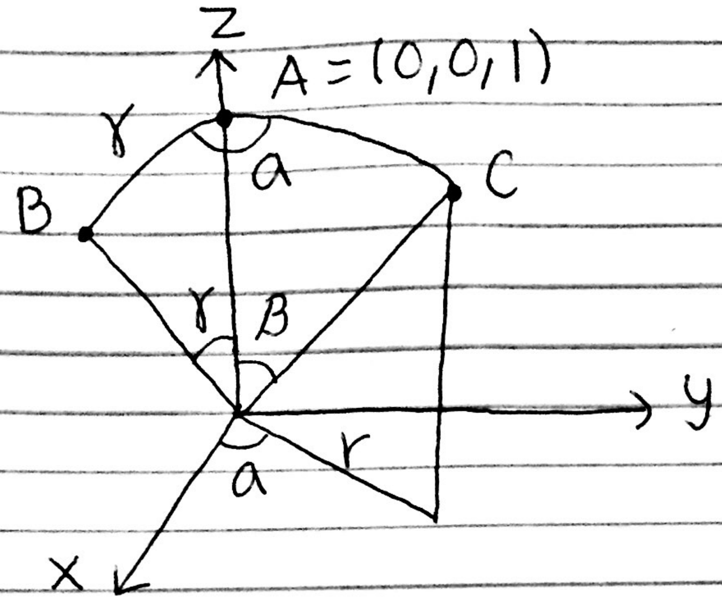
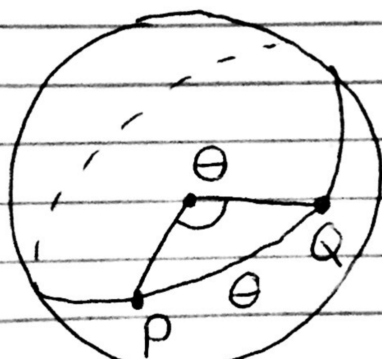
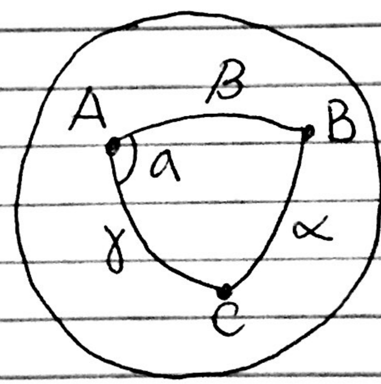






$A = (0,0,1)$

$B = (\sin\alpha, 0, \cos\gamma)$

$C = (\sin\beta\cos a, \sin\beta\sin a, \cos\beta)$

spherical coordinates for A, B, C

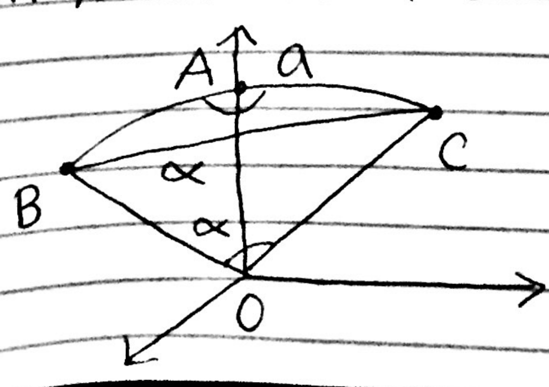
A: θ undefined and $\phi = 0$

B: $\theta = 0$ and $\phi = \gamma$

$B = (\sin\gamma, 0, \cos\gamma)$

C: $\theta = a$ and $\phi = \beta$

$C = (\sin\beta\cos a, \sin\beta\sin a, \cos\beta)$



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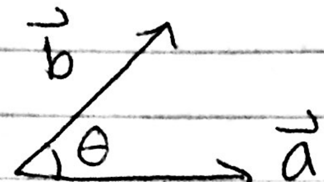
$B = (\sin\gamma, 0, \cos\gamma)$

$C = (\sin\beta\cos a, \sin\beta\sin a, \cos\beta)$

there is some formula for $\cos\alpha$

recall: $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$



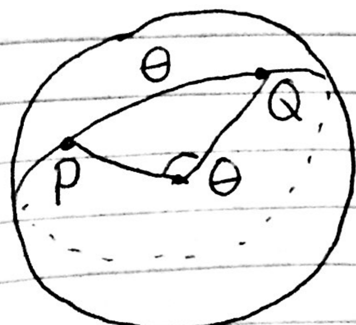
$\vec{OB}\cdot\vec{OC} = |\vec{OB}|\cdot|\vec{OC}|\cos\alpha = (1)(1)\cos\alpha = \cos\alpha$

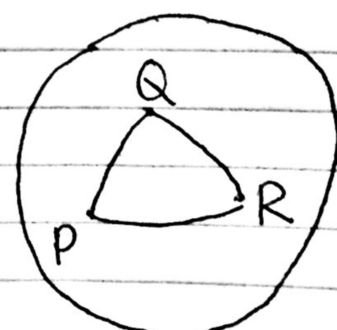
$$\cos\alpha = \begin{pmatrix}\sin\alpha\\0\\\cos\gamma\end{pmatrix}\cdot\begin{pmatrix}\sin\beta\cos a\\\sin\beta\sin a\\\cos\beta\end{pmatrix}$$

$$= \sin\gamma\sin\beta\cos a + \cos\gamma\cos\beta \quad \square$$

Spherical Triangle Inequality:

Spherical distance $d(P,Q)$ is the length of shorter arc of spherical line through P and Q which is θ





$d(P,R) \leq d(P,Q) + d(Q,R)$

switch notation:

$d(B,C) \leq d(B,A) + d(A,C)$

spherical cosine rule:

$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$

i.e. $\alpha \leq \gamma + \beta$

question: what is $\cos(\beta+\gamma)$?

$\cos(\beta+\gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma$