Math 462: Midterm review questions

Paul Hacking

3/4/10

In the spherical geometry problems below we work on the sphere S^2 with center the origin and radius 1 in $\mathbb{R}^3$, unless otherwise stated.

(1) Describe the following motions of $\mathbb{R}^2$ geometrically.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where $A = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+3 \\ x+4 \end{pmatrix}$.

(2) Describe the following compositions of motions of $\mathbb{R}^2$ geometrically (with full details: for example, if it is a rotation, give the center and angle of rotation).

(a) Reflection in the line $y = x + 1$ followed by reflection in the line $y = 3x + 2$.

(b) Reflection in the line $x + 2y = 3$ followed by reflection in the line $2x + 4y = 9$.

(3) Let S be the motion of $\mathbb{R}^2$ given by rotation by $\pi/2$ radians anticlockwise about the point $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and T be the motion of $\mathbb{R}^2$ given by rotation by $\pi/2$ radians anticlockwise about the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Describe the composite $T \circ S$ geometrically.

(4) Let T be the motion of $\mathbb{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$. Describe T geometrically in the following cases.

(a)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

(b)

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -\sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(5) Let A, B, C, D be 4 points in $\mathbb{R}^2$. Let L be the line joining the midpoints of AB and CD, M the line joining the midpoints of AC and BD, and N the line joining the midpoints of AD and BC. Show that the lines L, M, N are concurrent, that is, they all pass through a common point P. What is the physical significance of P? [Hint: Modify the argument we used to prove that the medians of a triangle are concurrent.]

(6) Let ABC be a triangle in the plane with angles a, b, c and side lengths α, β, γ (where a is the angle at the vertex A, and α is the length of the side BC opposite angle a, etc.) Prove the *sine rule*:

$$\frac{\sin a}{\alpha} = \frac{\sin b}{\beta} = \frac{\sin c}{\gamma}.$$

[Hint: Draw a perpendicular line from one of the vertices to the opposite side and compute its length in two ways.]

(7) Find the spherical distance $d(P, Q)$ between the points $P = \frac{1}{\sqrt{2}}(1, 0, 1)$ and $Q = \frac{1}{\sqrt{3}}(1, 1, 1)$ on S^2. [Hint: Use the dot product.]

(8) Let C be the spherical circle given by the intersection of the sphere S^2 with the plane

$$\Pi = (x + 2y + 2z = 1) \subset \mathbb{R}^3.$$

What is the center and radius of C (in the sense of spherical geometry)?

(9) Let L, M, N be the great circles on the sphere S^2 obtained by intersecting the sphere with the planes

$$\Pi_L = (x = 0), \quad \Pi_M = (y = 0), \quad \Pi_N = (x + 2y + 3z = 0).$$

What is the area of the spherical triangle with sides given by arcs of the great circles L, M, N and all angles less than or equal to $\pi/2$

radians? [Hint: The area of a spherical triangle with angles a, b, c is $a+b+c-\pi$. The angle between two great circles is the dihedral angle between the corresponding planes, which is equal to the angle between the normal vectors. This can be computed using the dot product.]

(10) In this question we consider measurements in spherical geometry on a sphere of radius R (we do not assume $R = 1$).

(a) Show that a spherical circle of radius r drawn on a sphere of radius R has circumference $2\pi R \sin(r/R)$.

(b) Suppose that r is small in comparison to R. Show that the circumference of a spherical circle of radius r drawn on a sphere of radius R is approximately $(1 - \frac{1}{6}(\frac{r}{R})^2)$ times the circumference of a Euclidean circle of radius r. [Hint: Use the power series for the sine function.]

(c) Suppose a spherical circle of radius 4 on a sphere of unknown radius R has circumference 25. Use part (b) to find an approximate value for R.

Answers are on the next page.

Answers.

(1) (a) Rotation about the point $\frac{1}{2}\begin{pmatrix}-1\\7\end{pmatrix}$ through an angle of $\cos^{-1}(4/5) =$ 0.643 radians anticlockwise.

(b) A glide given by reflection in the line through the point $\frac{1}{4}\begin{pmatrix}-1\\1\end{pmatrix}$ in the direction $\begin{pmatrix}1\\1\end{pmatrix}$ followed by translation by $\frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix}$ (parallel to the line).

(2) (a) Rotation about the point $(-\frac{1}{2}, \frac{1}{2})$ through an angle of $2\cos^{-1}(\frac{4}{\sqrt{20}}) =$ 0.927 radians anticlockwise.

(b) Translation by $\frac{3}{5}\begin{pmatrix}1\\2\end{pmatrix}$.

(3) Rotation about the point $\begin{pmatrix}1\\2\end{pmatrix}$ through an angle of π radians.

(4) (a) Rotation about the line through the origin in the direction $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ through an angle of $2\pi/3$ radians clockwise.

(b) Rotary reflection given by reflection in the xz-plane followed by rotation about the y-axis through an angle of $\pi/4$ radians clockwise.

(5) Write $\mathbf{a} = \overrightarrow{OA}$ for the position vector of A, etc. Then P is the point with position vector

$$\mathbf{p} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$$

It is the center of mass of a collection of 4 equal weights positioned at the points A, B, C, D.

(6) See the hint.

(7) $d(P, Q) = \cos^{-1}(\frac{2}{\sqrt{6}}) = 0.615$.

(8) The center of C is $\frac{1}{3}(1, 2, 2)$ and the radius of C is $\cos^{-1}(\frac{1}{3}) = 1.23$.

(9) The area of the triangle is $\frac{\pi}{2} + \cos^{-1}(\frac{1}{\sqrt{14}}) + \cos^{-1}(\frac{2}{\sqrt{14}}) - \pi = 0.736$.

(10) (a) Use the same argument as on HW4 Q1 applied to a sphere of radius R.

(b) A further hint: the series expansion of $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, so $\sin x$ is approximately equal to $x - \frac{x^3}{6}$ for small values of x.

(c) R is approximately 22.47.