Math 412 Midterm 2 review questions

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April 2, 2013

Reading: Saracino, Chapters 19,20,22.

(1) Compute the gcd of the polynomials $f(x) = x^5 + 2x^3 + x^2 + x + 1$ and $g(x) = x^4 - x^3 + 2x^2 - x + 1$.

(2) Let $f(x) = x^3 + 3x^2 - 2x - 3$ and $g(x) = x^2 - 2$. Compute the ideal (f, g) generated by f and g in (a) $\mathbb{Q}[x]$ and (b) $\mathbb{Z}[x]$.

(3) Compute the factorization of the following polynomials into irreducibles.

(a) $x^3 - x^2 - x - 2 \in \mathbb{Q}[x]$.

(b) $x^3 + 5x + 1 \in \mathbb{Q}[x]$.

(c) $x^3 + 2 \in (\mathbb{Z}/7\mathbb{Z})[x]$.

(d) $x^6 - 1 \in \mathbb{Q}[x]$.

(e) $x^5 + 6x^3 + 9x + 12 \in \mathbb{Q}[x]$.

(f) $x^4 + 4x^3 + x^2 + 4 \in (\mathbb{Z}/5\mathbb{Z})[x]$.

(4) Let $f(x) = x^4 - 4x^3 + 6x^2 + 7$. Using the substitution $x = y + 1$ or otherwise, prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(5) Using reduction modulo 2 as a guide, factor the following polynomials into irreducibles in $\mathbb{Q}[x]$.

(a) $x^2 + 2013x + 173$.

(b) $x^3 + 7x^2 + 12x + 35$.

(c) $x^3 + 6x^2 + 5x + 21$.

(d) $x^4 + 2x^3 + 2x^2 + 2x + 2$.

(e) $x^4 + 2x^3 + 3x^2 + 2x + 1$.

(f) $x^4 + 2x^3 - x^2 + 2x + 1$.

(6) Let $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{R}[x]$, $a_2 \neq 0$. Show that f is irreducible in $\mathbb{R}[x]$ iff $a_1^2 - 4a_0a_2 < 0$.

(7) Let $f(x) = x^3 + x^2 + 1 \in \mathbb{Q}[x]$.

(a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.

(b) Let $\alpha \in \mathbb{C}$ be a root of $f(x)$. Describe an isomorphism

$$\psi\colon \mathbb{Q}[x]/(f(x)) \to \mathbb{Q}(\alpha)$$

explicitly.

(c) Using part (b) or otherwise, explain why $1,\alpha,\alpha^2$ forms a basis for $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$.

(d) Compute an expression for $(1+\alpha^2)^{-1}$ of the form $a + b\alpha + c\alpha^2$ for some $a, b, c \in \mathbb{Q}$ (to be determined).

(8) Let $\alpha = \sqrt[3]{2}$ be the real cube root of 2. Compute the irreducible polynomial of $\beta = 1 + \alpha^2$ over $\mathbb{Q}$.

(9) Find the irreducible polynomial for $\alpha = \sqrt{2} + \sqrt{7}$ over each of the following fields

(a) $\mathbb{Q}$

(b) $\mathbb{Q}(\sqrt{2})$

(c) $\mathbb{Q}(\sqrt{7})$

(d) $\mathbb{Q}(\sqrt{14})$

(10) Which of the following quotient rings are fields?

(a) $\mathbb{Q}[x]/(x^3 + 2x + 1)$

(b) $\mathbb{R}[x]/(x^3 + 2x + 1)$

[Hint: If K is a field and $f(x) \in K[x]$ then the quotient ring $K[x]/(f(x))$ is a field iff f is irreducible over K (why?)]

(11) Give a construction of a finite field L as a quotient ring in the following cases.

   (a) $|L| = 9$
   (b) $|L| = 8$

   [Hint: If K is a field and $f(x) \in K[x]$ is an irreducible polynomial then the quotient ring $L := K[x]/(f)$ is a field containing K as a subring. Moreover the dimension of L as a vector space over K equals the degree n of f (and L has a basis given by the elements $1, x, \ldots, x^{n-1}$ modulo f).]

(12) Let $K \subset L$ be a field extension and $\alpha \in L$ an algebraic element with irreducible polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

   over K. Express α^{-1} as a sum $b_{n-1}\alpha^{n-1} + \cdots + b_1\alpha + b_0$ for some $b_i \in K$ (to be determined).

(13) Let $\alpha = e^{2\pi i/5}$ and $\beta = e^{2\pi i/7}$. Prove that $\alpha \notin \mathbb{Q}(\beta)$.

   [Hint: Recall that the polynomial $(x^p - 1)/(x - 1) = x^{p-1} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ for p prime.]

(14) For each of the following fields $L \supset \mathbb{Q}$, compute the degree $[L : \mathbb{Q}]$ and find a basis for L as a vector space over $\mathbb{Q}$.

   (a) $\mathbb{Q}(\sqrt[3]{5})$
   (b) $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$.
   (c) $\mathbb{Q}(\sqrt{3}, \sqrt[3]{7})$.
   (d) $\mathbb{Q}(\sqrt{2}, \sqrt{5})$.

   [Hint: The result of Q17 may be used for (c) and (d).]

(15) Let $K \subset L \subset M$ be a tower of fields. Let $\alpha \in M$ be algebraic over K. Show that $[K(\alpha) : K] \geq [L(\alpha) : L]$.

   [Hint: Recall that the degree $[K(\alpha) : K]$ equals the degree of the irreducible polynomial of α over K. How are the irreducible polynomial of α over K and the irreducible polynomial of α over L related?]

(16) Let $a \in \mathbb{Q}$, $a > 0$. Suppose $\sqrt{a} \notin \mathbb{Q}$. Show that $\sqrt[4]{a}$ has degree 4 over $\mathbb{Q}$.

[Hint: Consider the tower of fields $\mathbb{Q} \subset \mathbb{Q}(\sqrt{a}) \subset \mathbb{Q}(\sqrt[4]{a})$. First show that $\sqrt[4]{a} \notin \mathbb{Q}(\sqrt{a})$, so that $\mathbb{Q}(\sqrt{a}) \neq \mathbb{Q}(\sqrt[4]{a})$. Now deduce that the degree $[\mathbb{Q}(\sqrt[4]{a}) : \mathbb{Q}] = 4$.]

(17) Let $K \subset L$ be a field extension and $\alpha, \beta \in L$ be algebraic over K of degrees d and e.

(a) Show that $[K(\alpha, \beta) : K] \leq de$.

(b) Show that we have equality in (a) if $\gcd(d, e) = 1$.

(c) Show that if we have equality in (a) then the set

$$\{\alpha^i \beta^j \mid 0 \leq i \leq d-1, 0 \leq j \leq e-1\}$$

is a basis of $K(\alpha, \beta)$ as a vector space over K.

[Hint: For part (a) consider the tower

$$K \subset K(\alpha) \subset K(\alpha, \beta) = K(\alpha)(\beta).$$

For part (c) review the proof of the equality

$$[M : K] = [M : L][L : K]$$

for a tower of fields $K \subset L \subset M$.]