Math 462: Homework 6 solutions

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4/3/10

1.

(a) Suppose we have a set of tiles each in the shape of a regular polygon with n sides. For which values of n can we tile the plane $\mathbb{R}^2$ (with no gaps)? [Hint: Think about the angles meeting at a vertex of the tiling.]

(b) Now suppose we have two different types of tiles. Can you find a tiling of the plane using both squares and regular octagons? Can you find a tiling using both equilateral triangles and squares such that no two squares meet along an edge?

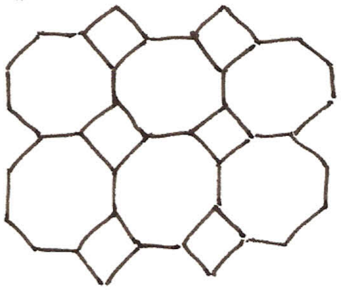
(c) Now consider the same problem in 3-dimensions: can we "tile" 3-dimensional space $\mathbb{R}^3$ with regular polyhedra (with no gaps)? Give one example of such a tiling.

(d) Can we tile 3-dimensional space with octahedra? [Hint: What is the dihedral angle between two faces of the octahedron which meet along an edge?]

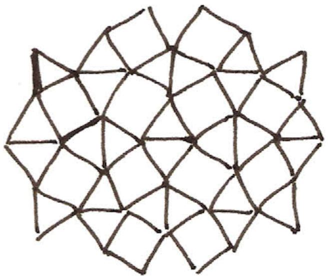
(e) Can you find a tiling of 3-dimensional space by octahedra and tetrahedra?

(a) Suppose we have a tiling of the plane by regular polygons with n sides. Let θ be the angle of a regular polygon with n sides at a vertex. If k polygons meet at a vertex of the tiling then $k\theta = 2\pi$ radians. Now the angle sum of a polygon with n sides is $(n-2)\pi$ (see HW3, Q2) so $\theta = (n-2)\pi/n$. So the equation $k\theta = 2\pi$ gives $k = 2n/(n-2)$. We find $k = 6$ if $n = 3$, $k = 4$ if $n = 4$, $k = 3$ if $n = 6$, and k is not an integer for any other value of $n \geq 3$. In the cases $n = 3, 4, 6$ it is easy to draw the corresponding tilings of the plane (by equilateral triangles, squares, or hexagons). So the plane can be tiled by regular n-gons if $n = 3, 4, 6$.

(b) Similarly to part (a) we consider the angles meeting at a vertex of the tiling. The angle of a regular octagon is $3\pi/4$ (using the formula $\theta = (n-2)\pi/n$ from part (a)) and the angle of a square is $\pi/2$. Notice that $3\pi/4 + 3\pi/4 + \pi/2 = 2\pi$ so this suggests there is a tiling where 2 octagons and 1 square meet at each vertex. Here is a picture of such a tiling:



For the second case, the angle of an equilateral triangle is $\pi/3$ and the angle of a square is $\pi/2$, and $\pi/2+\pi/2+\pi/3+\pi/3+\pi/3 = 2\pi$. This suggests there is a tiling with 2 squares and 3 triangles meeting at each vertex. Note that we are asked to use both types of tiles and ensure that the squares should not meet along an edge. Here is a picture of such a tiling:



(c) We can tile 3-dimensional space by stacking cubes.

(d) Suppose we are given a tiling of 3-space by octahedra. Then, similarly to part (a), looking at an edge in the tiling, and slicing by a plane perpendicular to that edge, we find $k\theta = 2\pi$ where k is the number of octahedra which contain the edge and θ is the dihedral angle between the faces of an octahedron. We can compute θ using the description of the octahedron in coordinates that we gave in class: the vertices of the octahedron are the points $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ in $\mathbb{R}^3$. The midpoint M of the edge e joining $(1, 0, 0)$ and $(0, 1, 0)$ is the average $\frac{1}{2}(1, 1, 0)$. The dihedral angle between the faces meeting along e is the angle between the lines joining M to the other vertices $(0, 0, 1)$ and $(0, 0, -1)$ of these faces. That is, we need to compute the angle between the vectors

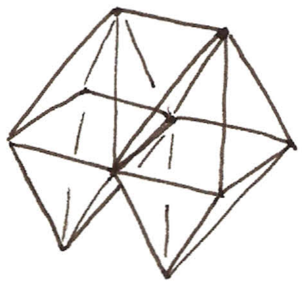
$$(0,0,1)-\frac{1}{2}(1,1,0) = \frac{1}{2}(-1,-1,2), \quad (0,0,-1)-\frac{1}{2}(1,1,0) = \frac{1}{2}(-1,-1,-2).$$

We use the formula for the dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

to obtain $\cos\theta = -1/3$, $\theta = 1.91$ radians. So $k = 2\pi/\theta = 3.29$ is not an integer. This is a contradiction, so there does not exist a tiling of 3-dimensional space by octahedra.

(e) Yes there is a tiling of 3-dimensional space by octahedra and tetrahedra. First note that an octahedron can be thought of as two square base pyramids glued along their bases. Now position octahedra so these square bases give a tiling of the plane $\mathbb{R}^2 = (z = 0) \subset \mathbb{R}^3$. Now consider two adjacent octahedra in this tiling, say for definiteness the vertices of the first octahedron are $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ and the vertices of the second octahedron are $(1, 0, 0)$, $(0, 1, 0)$, $(2, 1, 0)$, $(1, 2, 0)$, $(1, 1, \pm 1)$. Then observe that the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ are the vertices of a tetrahedron (all the edge lengths equal $\sqrt{2}$). Adding tetrahedra to the top of the layer of octahedra in this way we get a sequence of parallel ridges in the direction $(1, 1, 0)$. We can do the same thing on the bottom of the layer of octahedra. Now we can stack layers of this form to obtain a tiling of space by octahedra and tetrahedra.



2. Suppose we draw a graph (a network of vertices and edges) on the surface of the sphere S^2 which subdivides it into a number of faces. Let V, E, F be the number of vertices, edges, and faces. We showed in class that

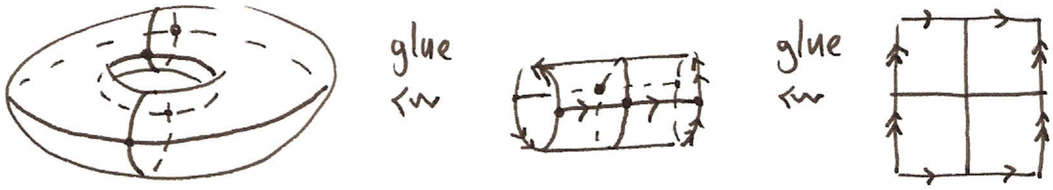
$$V - E + F = 2.$$

Now suppose we draw a graph on the surface of a donut (the type with a hole — in mathematics it is usually called a *torus*). We must assume here that each face of the subdivision looks like a polygon (we don't allow faces that wrap around the donut to form a ring). Then there is a formula

$$V - E + F = C$$

where C is a constant (it does not depend on the subdivision). Find C by computing V, E, F for one example.

We just have to compute one example. A simple subdivision of the torus is given by 4 squares as shown. We find $V = 4$, $E = 8$, $F = 4$, so $V - E + F = 0$.



3. Suppose we subdivide the surface of S^2 into triangles. Let V, E, F be the numbers of vertices, edges and faces in the subdivision.

(a) Find a relation between the numbers E of edges and F of faces. [Hint: Use the fact that each face is a triangle.]

(b) If you've done part (a), you'll see that the number F of faces has to be an even integer, say $F = 2n$. Now find the numbers V of vertices and E of edges in terms of n. [Hint: Use Euler's formula $V - E + F = 2$.]

(c) Check your results for the tetrahedron, octahedron, and icosahedron (project them onto the sphere). Can you find a symmetric example with $F = 60$? [Hint: Use the dodecahedron.]

(a) 2 faces of the subdivision meet at each edge, and each face has 3 edges. So the sum over all the faces of the number of edges of that face can be computed in two ways to give $3F = 2E$.

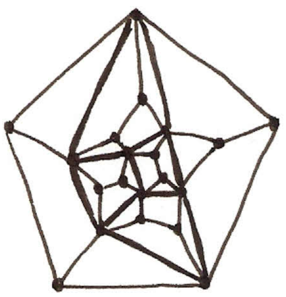
(b) From part (a) we see that F is even (because $3F = 2E$ is even and 3 is odd). Write $F = 2n$ where n is an integer, then $E = \frac{3}{2}F = 3n$, and Euler's formula $V - E + F = 2$ gives $V = n + 2$. So $V = n + 2$, $E = 3n$, and $F = 2n$.

(c) Tetrahedron: $V = 4$, $E = 6$, $F = 4$ ($n = 2$), Octahedron: $V = 6$, $E = 12$, $F = 8$ ($n = 4$), Icosahedron: $V = 12$, $E = 30$, $F = 20$ ($n = 10$). A symmetric example with $F = 60$ is obtained from the dodecahedron by dividing each of its 12 pentagonal faces into 5 equilateral triangles with apex the center of the face.

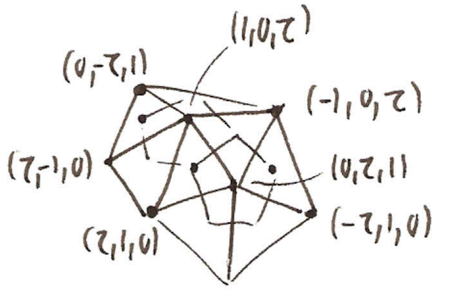
4. Find 12 diagonals on the faces of the dodecahedron which are the edges of a cube. You should check that the edges meeting at each vertex are perpendicular (by symmetry you only need to do it for one vertex and one pair of edges). One way to do this is to write down coordinates for (some of) the vertices of the dodecahedron using the coordinate description of the icosahedron we gave in class and the fact that the dodecahedron is dual to the icosahedron. It might be helpful to make a cardboard model of the dodecahedron.

Note: Another way to explain this observation is that the dodecahedron can be built from a cube by gluing a polyhedron shaped like a ridge tent onto each face.

First draw the Schlegel diagram of the dodecahedron and draw one diagonal on each face so that the diagonals have the same combinatorics as the edges of the cube. (The rule for drawing the diagonals is the following: a diagonal divides a pentagonal face into a triangle and a quadrilateral. When two faces meet along an edge e the diagonals should be such that e is an edge of the quadrilateral on one face and the triangle on the other.) Note that one face of the dodecahedron is missing in the diagram (the face we are looking through) so one edge of the cube is also missing.



Next we need to check that the diagonals we have drawn on the Schelgel diagram correspond to perpendicular lines in $\mathbb{R}^3$. To do this we can use the description of the dodecahedron in coordinates we gave in class. We first showed that the points $(\pm\tau, \pm 1, 0), (\pm 1, 0, \pm\tau), (0, \pm\tau, \pm 1)$ are the vertices of an icosahedron in $\mathbb{R}^3$, where $\tau = (1+\sqrt{5})/2$ is the *golden ratio* which satisfies the equation $\tau^2 = \tau + 1$. Then we used the fact that the dodecahedron



is the dual polyhedron of the icosahedron to write down the vertices of a dodecahedron. By symmetry, we only need to check that two of the diagonals meeting at one vertex of the dodecahedron are perpendicular. For example take the 3 faces of the icosahedron with vertices $\{(1,0,\tau),(\tau,1,0),(0,\tau,1)\}$, $\{(1,0,\tau),(0,-\tau,1),(\tau,-1,0)\}$, $\{(-1,0,\tau),(0,\tau,1),(-\tau,1,0)\}$. The vertices of the dodecahedron given by the centers of these faces are $P_1=\frac{1}{3}(\tau+1,\tau+1,\tau+1)$, $P_2=\frac{1}{3}(\tau+1,-(\tau+1),\tau+1)$, $P_3=\frac{1}{3}(-(\tau+1),\tau+1,\tau+1)$. The diagonals P_1P_2 and P_1P_3 of the faces of the dodecahedron have directions $(0,1,0)$ and $(1,0,0)$, so they are perpendicular. (In fact, as we mentioned in class, in these coordinates the vertices of a cube with edges being diagonals of the faces of the dodecahedron are given by $\frac{1}{3}(\tau+1)(\pm1,\pm1,\pm1)$.)

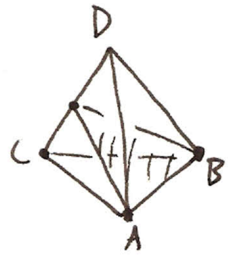
5. Describe the group G of symmetries of the tetrahedron as follows:

(a) Find all the symmetries given by reflection in a plane.

(b) Show that any permutation of the vertices can be realized by a symmetry of the tetrahedron. [Hint: Use the reflections.]

(c) From part (b) it follows that the symmetry group G of the tetrahedron can be identified with the symmetric group S_4 of permutations of 4 objects. Describe each of the elements of G geometrically. [Hint: There are 6 reflections, 11 rotations, 6 rotary reflections, and the identity.]

(a) Given an edge AB of the tetrahedron consider the plane containing this edge and the midpoint of the opposite edge CD. Then reflection in this plane is a symmetry of the tetrahedron which fixes the vertices A and B and interchanges the vertices C and D. All the symmetries given by reflection are of this form (there are 6 of them corresponding to the choice of the edge AB).



(b) Recall: A *permutation* of $\{1,2,\ldots,n\}$ is a function $f\colon\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ such that $f(i)\neq f(j)$ for $i\neq j$. The *symmetric group* S_n is the group of all permutations of $\{1,2,\ldots,n\}$ with the group law being composition of functions. A *transposition* is a permutation f that switches two elements i and j, and fixes the remaining elements (that is, $f(i)=j$, $f(j)=i$ and $f(k)=k$ for $k\neq i,j$).

Label the vertices of the tetrahedron $1,2,3,4$. A symmetry of the tetrahedron induces a permutation of its vertices. This gives a map (a group homomorphism) from the group G of symmetries of the tetrahedron to the symmetric group S_4. It is a basic fact that the symmetric group S_n is generated by transpositions (that is, every permutation can be expressed as a composition of transpositions). From part (a) we see that the transpositions of S_4 correspond to symmetries of the tetrahedron given by reflection in a plane. It follows that every permutation is induced by a symmetry of the tetrahedron.

(c) Recall we have the group homomorphism $\theta\colon G \to S_4$ from the group of symmetries of the tetrahedron to the group of permutations of $\{1, 2, 3, 4\}$. We have shown in (b) that θ is surjective (that is, every permutation is induced by a symmetry of the tetrahedron). Also, θ is injective (that is, a symmetry of the tetrahedron is determined by its effect on the vertices) because for example any motion of $\mathbb{R}^3$ is determined by its effect on 4 points that do not lie in a plane. So $\theta\colon G \to S_4$ is an isomorphism of groups. In particular, G has the same number of elements as S_4, that is, $4! = 24$ elements. We can describe them geometrically as follows. There are 6 symmetries given by reflection in a plane (see part (a)), $8 = 4 \cdot 2$ symmetries given by rotation about the axis joining a vertex to the center of the opposite face through angle $\pm 2\pi/3$, 3 symmetries given by rotation about the axis joining the midpoints of two opposite edges through angle π, $6 = 3 \cdot 2$ symmetries given by rotation about the axis joining the midpoints of two opposite edges through angle $\pm\pi/2$ followed by reflection in the plane through the center of the tetrahedron perpendicular to the axis, and the identity transformation. Since $24 = 6 + 8 + 3 + 6 + 1$ we have enumerated all the symmetries.