Midterm 2 Review

Paul Hacking

11/9/09

1 Vector space

A *vector space* V is a set with operations of addition and scalar multiplication satisfying "the same properties as for $\mathbb{R}^n$". See Definition 4.1.1 on p. 154 of the text for the precise statement.

Example 1.1. (1) $\mathbb{R}^n$.

(2) $\mathcal{P}_n = \{$ polynomials of degree $\leq n$ $\}$
$= \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \cdots, a_n \in \mathbb{R}\}$.

(3) $\mathbb{R}^{m \times n} = \{$ $m \times n$-matrices $\}$.

(4) $F(\mathbb{R}, \mathbb{R}) = \{$ functions $f \colon \mathbb{R} \to \mathbb{R}$ $\}$.

(5) $C^\infty(\mathbb{R}, \mathbb{R}) = \{f \colon \mathbb{R} \to \mathbb{R} \mid f$ has derivatives of all orders $\}$.

2 Subspace

A *subspace* W of a vector space V is a subset $W \subset V$ such that

(1) $\mathbf{0} \in W$,

(2) If $\mathbf{x}, \mathbf{y} \in W$ then $\mathbf{x} + \mathbf{y} \in W$, and

(3) If $\mathbf{x} \in W$ and $c \in \mathbb{R}$ then $c\mathbf{x} \in W$.

Note: By using (2) and (3) repeatedly, we find: If $\mathbf{x}_1, \ldots, \mathbf{x}_n \in W$ and $c_1, \ldots, c_n \in \mathbb{R}$ then $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n \in W$.

Example 2.1. (1) $W = (x + 2y + 3z = 0) \subset V = \mathbb{R}^3$, a plane through the origin in $\mathbb{R}^3$.

(2) $W = \{f \in \mathcal{P}_3 \mid f(2) = f'(2) = 0\} \subset V = \mathcal{P}_3$.

(3) $W = \mathcal{P}_n \subset V = C^\infty(\mathbb{R}, \mathbb{R})$.

(4) $W = \{X \in \mathbb{R}^{2\times 2} \mid AX = XA\} \subset V = \mathbb{R}^{2\times 2}$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (for example).

3 Linear map

A *linear map* $T\colon V \to W$ between vector spaces V and W is a map (function) from V to W such that

(1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in V$.

(2) $T(c\mathbf{x}) = cT(\mathbf{x})$ for $\mathbf{x} \in V$ and $c \in \mathbb{R}$.

Example 3.1. (1) If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix. The columns of A are the vectors $\mathbf{v}_1 = T(\mathbf{e}_1), \ldots, \mathbf{v}_n = T(\mathbf{e}_n)$, where $\mathbf{e}_j \in \mathbb{R}^n$ is the column vector with a 1 in the jth row and 0 in the other rows. With this notation, the map T is given by

$$T\mathbf{x} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

(2) $T\colon \mathcal{P}_3 \to \mathbb{R}^2$, $T(f) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$.

(3) $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $T(X) = AX + XB$ where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ (for example).

(4) $T\colon C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$, $T(f) = f'$.

4 Kernel and Image

Let $T\colon V \to W$ be a linear map between vector spaces V and W.

The *kernel* of T is defined by

$$\ker(T) = \{\mathbf{x} \in V \mid T(\mathbf{x}) = \mathbf{0}\} \subset V.$$

The kernel of T is a subspace of the domain V of T.

The *image* of T is defined by

$$\operatorname{Im}(T) = \{\mathbf{y} \in W \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in V\} \subset W.$$

The image of T is a subspace of the target (or range) W of T.

Example 4.1. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane

$$W = (x + 2y + 7z = 0) \subset \mathbb{R}^3.$$

Then

$$\ker(T) = \operatorname{Span}\left(\begin{pmatrix}1\\2\\7\end{pmatrix}\right),$$

the line through the origin spanned by the normal vector of W, and $\operatorname{Im}(T) = W$.

5 Linear independence, bases, and dimension

Let V be a vector space. We say elements $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are *linearly independent* if

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \Rightarrow c_1 = \cdots = c_n = 0.$$

We say $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a *basis* of V if

(1) $V = \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, and

(2) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

(Note: Equivalently, a basis is a spanning set of smallest possible size.)

The *dimension* of V is the size of any basis $\mathcal{B}$ of V. (WARNING: Some vector spaces do not have a basis because they are "too big", for example $F(\mathbb{R}, \mathbb{R})$. We say that they are *infinite dimensional*.)

Example 5.1. Which of the following sets are linearly independent?

(1) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix} \in \mathbb{R}^2$.

(2) $\begin{pmatrix}2\\6\end{pmatrix}, \begin{pmatrix}3\\9\end{pmatrix} \in \mathbb{R}^2$.

(3) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}5\\7\end{pmatrix} \in \mathbb{R}^2$.

(4)
$$\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \in \mathbb{R}^3.$$

(5)
$$\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-3\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix} \in \mathbb{R}^3.$$

Answer: Sets (1) and (5) are linearly independent. For (3) use the fact below. For (5) form the matrix A with columns the given vectors. Use row reduction to show that the only solution $\mathbf{c}$ of $A\mathbf{c} = \mathbf{0}$ is $\mathbf{c} = \mathbf{0}$. This is equivalent to the columns of A being linearly independent.

Fact: If $\dim V = n$, and $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are linearly independent, then $m \leq n$. If $m = n$, then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis of V.

6 Coordinates

Let V be a vector space and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a basis of V. The basis $\mathcal{B}$ defines an invertible linear map

$$L_{\mathcal{B}} \colon V \to \mathbb{R}^n, \quad \mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$$

with inverse

$$L_{\mathcal{B}}^{-1} \colon \mathbb{R}^n \to V, \quad \begin{pmatrix}c_1\\ \vdots \\ c_n\end{pmatrix} \mapsto c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Example 6.1. (1) $\mathcal{P}_n$ has basis $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$, and

$$L_{\mathcal{B}} \colon \mathcal{P}_n \to \mathbb{R}^{n+1}, \quad a_0 + a_1 t + \cdots + a_n t^n \mapsto \begin{pmatrix}a_0\\ \vdots \\ a_n\end{pmatrix}.$$

(2) $\mathbb{R}^{2\times 2}$ has basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

and

$$L_{\mathcal{B}} \colon \mathbb{R}^{2\times 2} \to \mathbb{R}^4, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

7 Matrix of a linear map

Let $T \colon V \to W$ be a linear map between vector spaces V and W. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V and $\mathcal{C} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis of W. The $(\mathcal{B}, \mathcal{C})$-matrix M of T is defined by

$$[T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}}.$$

So we have a "commutative diagram"

$$\begin{array}{ccccccc} V & \xrightarrow{T} & W & & \mathbf{x} & \longrightarrow & T(\mathbf{x}) \\ \big\downarrow{\scriptstyle L_{\mathcal{B}}} & & \big\downarrow{\scriptstyle L_{\mathcal{C}}} & & \big\downarrow & & \big\downarrow \\ \mathbb{R}^n & \xrightarrow{M} & \mathbb{R}^m & & [\mathbf{x}]_{\mathcal{B}} & \longrightarrow & [T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}} \end{array}$$

Alternatively, working column by column, $M = ([T\mathbf{v}_1]_{\mathcal{C}} \cdots [T\mathbf{v}_n]_{\mathcal{C}})$.

Note: If $V = W$ then we can take $\mathcal{B} = \mathcal{C}$ above. In this case we call M the $\mathcal{B}$-matrix of T. (This is the only case described in the text).

Example 7.1. Let $\mathcal{B}$ be the usual basis of $\mathbb{R}^{2\times 2}$ (see Example 6.1(2)). We find the $\mathcal{B}$-matrix M of the linear map

$$T \colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, \quad T(X) = AX + XB,$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$. We compute

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 2a + 2b + c & 3a + 6b + d \\ 2a + 2c + 2d & 2b + 3c + 6d \end{pmatrix} = a\begin{pmatrix} 2 & 3 \\ 2 & 0 \end{pmatrix} + b\begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 2 & 6 \end{pmatrix}.$$

We deduce that

$$M = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 3 & 6 \end{pmatrix}.$$

Example 7.2. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the usual basis of $\mathcal{P}_3$. We find the $\mathcal{B}$-matrix M of the linear map

$$T\colon \mathcal{P}_3 \to \mathcal{P}_3, \quad f(t) \mapsto 2f''(t) + f(3t).$$

We compute

$$\begin{aligned} T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) &= (a_0 + 4a_2) + (3a_1 + 12a_3)t + 9a_2 t^2 + 27a_3 t^3 \\ &= a_0(1) + a_1(3t) + a_2(4 + 9t^2) + a_3(12t + 27t^3). \end{aligned}$$

We deduce that

$$M = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 3 & 0 & 12 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 27 \end{pmatrix}.$$

Q: Is T invertible? A: Yes, because the matrix M is invertible. (Note that M is invertible by 2.4 Exercise 36 on p. 89 of the text.)

8 Algorithm to compute basis for kernel and image of linear map

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, $T(\mathbf{x}) = A\mathbf{x}$. To compute a basis for the kernel and image of T:

(1) Compute the RREF (reduced row echelon form) B of A.

(2) A basis of $\mathrm{Im}(T)$ is given by the columns of A corresponding to the pivot columns of B.

(3) A basis of $\ker(T)$ is given by writing the solutions of $B\mathbf{x} = \mathbf{0}$ in terms of the free variables (as usual when solving $A\mathbf{x} = \mathbf{0}$ by Gaussian elimination) — there is one basis element for each free variable.

Example 8.1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 3 & 5 & 6 \end{pmatrix}.$$

Then the reduced row echelon form of A is the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, a basis of the image of T is

$$\begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}1\\3\\5\end{pmatrix}.$$

The solutions of $B\mathbf{x} = 0$ are

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-x_2 + \frac{1}{2}x_4\\x_2\\-\frac{3}{2}x_4\\x_4\end{pmatrix} = x_2\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}\frac{1}{2}\\0\\-\frac{3}{2}\\1\end{pmatrix}$$

(the free variables are x_2 and x_4). So, a basis of the kernel of T is

$$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\0\\-\frac{3}{2}\\1\end{pmatrix}.$$

9 Change of basis

Let V be a vector space and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ two bases of V. The change of basis matrix $S = S_{\mathcal{B}\to\mathcal{C}}$ is defined by $S[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$. Working column by column, we have $S = ([\mathbf{v}_1]_{\mathcal{C}} \cdots [\mathbf{v}_n]_{\mathcal{C}})$.

Example 9.1. Let $V = \mathbb{R}^2$, $\mathcal{B} = \{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\}$, $\mathcal{C} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$. If $[\mathbf{x}]_{\mathcal{C}} = \begin{pmatrix}5\\7\end{pmatrix}$, what is $[\mathbf{x}]_{\mathcal{B}}$? Notice that

$$S := S_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}.$$

So

$$[\mathbf{x}]_{\mathcal{B}} = S_{\mathcal{C}\to\mathcal{B}}[\mathbf{x}]_{\mathcal{C}} = S^{-1}[\mathbf{x}]_{\mathcal{C}} = \frac{1}{-2}\begin{pmatrix}4 & -2\\-3 & 1\end{pmatrix}\begin{pmatrix}5\\7\end{pmatrix} = \begin{pmatrix}-3\\4\end{pmatrix}.$$

Now suppose $T\colon V \to V$ is a linear map. Let M be the $\mathcal{B}$-matrix of T and N the $\mathcal{C}$-matrix of T. Then

$$N = S_{\mathcal{B}\to\mathcal{C}} M S_{\mathcal{C}\to\mathcal{B}} = SMS^{-1}$$

where $S = S_{\mathcal{B}\to\mathcal{C}}$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Example 9.2. Let $\mathcal{B} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ and $\mathcal{C} = \{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\5\end{pmatrix}\}$ be two bases of $\mathbb{R}^2$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map with $\mathcal{B}$-matrix $M = \begin{pmatrix}2 & 3\\5 & 1\end{pmatrix}$. We find the $\mathcal{C}$-matrix N of T. Notice that

$$S_{\mathcal{C}\to\mathcal{B}} = \begin{pmatrix}2 & 4\\3 & 5\end{pmatrix},$$

so

$$S_{\mathcal{B}\to\mathcal{C}} = S_{\mathcal{C}\to\mathcal{B}}^{-1} = \frac{1}{-2}\begin{pmatrix}5 & -4\\-3 & 2\end{pmatrix}$$

and

$$N = S_{\mathcal{B}\to\mathcal{C}} M S_{\mathcal{C}\to\mathcal{B}} = \frac{1}{2}\begin{pmatrix}-13 & -15\\13 & 19\end{pmatrix}.$$

10 Isomorphisms

Let $T\colon V \to W$ be a linear map between vector spaces V and W. We say T is *invertible* if there is a map $T^{-1}\colon W \to V$ such that

$$T(\mathbf{x}) = \mathbf{y} \iff \mathbf{x} = T^{-1}(\mathbf{y}).$$

An *isomorphism* is an invertible linear map.

Fact: If $T\colon V \to W$ is invertible, then $\dim V = \dim W$. Conversely, if $\dim V = \dim W$, then

$$T \text{ is invertible} \iff \ker(T) = \{\mathbf{0}\} \iff \operatorname{Im}(T) = W.$$