697B Example Sheet 5

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(1) Show that the discriminant of $ay^2 + by + c$ is $a(4ac - b^2)$ and the discriminant of $y^3 + ay + b$ is $4a^3 + 27b^2$.

(2) (Weak Bézout theorem) Let $f(x, y)$ and $g(x, y)$ be polynomials of degree n and m with no common factors. Let $X = (f(x, y) = 0) \subset \mathbb{C}^2_{x,y}$ and $Y = (g(x, y) = 0) \subset \mathbb{C}^2_{x,y}$. In this question we will use the resultant to show that $|X \cap Y| \leq mn$.

(a) Show that $X \cap Y$ is finite.

(b) Show that after a linear change of coordinates x, y we may assume that no two points in $X \cap Y$ have the same x coordinate and that $f(x, y) = y^n + a_1 y^{n-1} \cdots + a_n$, $g(x, y) = y^m + b_1 y^{m-1} + \cdots + b_m$, where the a_i, b_i are polynomials in x of degree at most i.

(c) With coordinates as in (b), show that the resultant $r(x) := R(f, g)$ is a nonzero polynomial of degree at most mn, and deduce that $|X \cap Y| \leq mn$.

(3) Let $f(x, y) = y^3 - xy + 2$ and $X = (f(x, y) = 0) \subset \mathbb{C}^2_{x,y}$.

(a) Let $F \colon X \to Y$ be a non constant holomorphic map between Riemann surfaces X and Y, and $P \in X$ a point. Let z be a local coordinate at P and w a local coordinate at $F(P)$ and let $z \mapsto f(z)$ be the map F in the charts given by z, w. Then the *ramification index* of F at P is $e_P := \nu_P(f(z))$, and we say P is a ramification point if $e_P > 1$. (Note: After an analytic change of the local coordinate z, we may assume that F is given by $z \mapsto z^{e_P}$.) Find the ramification points and ramification indices of the map $p \colon X \to \mathbb{C}^1_x$ given by $(x, y) \mapsto x$. [Hint: The locus of ramification points is given by $(f = \frac{\partial f}{\partial y} = 0) \subset \mathbb{C}^2_{x,y}$ (why?).]

(b) Let $\mathbb{C}^2 \subset \mathbb{P}^2$ be the inclusion given by $(x, y) \mapsto (x : y : 1)$ and let $\overline{X} \subset \mathbb{P}^2$ be the closure of X. Find the singular points of $\overline{X}$ and describe their type.

(c) Describe the normalization $n\colon \tilde{X} \to \overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ explicitly. (Hint: Solve $f(x, y) = 0$ for x.)

(d) Show that the map $p\colon X \to \mathbb{C}^1$ extends to a holomorphic map $\tilde{p}\colon \tilde{X} \to \mathbb{C} \cup \{\infty\}$. Describe the ramification of $\tilde{p}$ over ∞.

(4) (Degree) Let $F\colon X \to Y$ be a non constant holomorphic map between compact Riemann surfaces X and Y. Recall the definition of ramification points and ramification indices from Q3(a) above.

(a) Show that F is surjective.

(b) Show that the number of ramification points of F is finite. (Hint: Use compactness of X and argue by contradiction.)

(c) Show that the function

$$f\colon Y \to \mathbb{Z}, \quad Q \mapsto \sum_{P \in F^{-1}(Q)} e_P.$$

is constant, say equal to d. Thus every fiber $F^{-1}(Q)$ of F consists of d points (provided we count with multiplicities), and all but finitely many fibers consist of d distinct points. The number d is called the *degree* of F.(Hint: Show that f is locally constant and use connectedness of Y.)

(d) Let X be a compact Riemann surface and f a meromorphic function on X. In class we showed that $\sum \nu_P(f) = 0$. Use (c) to give a different proof of this fact. (Hint: Consider the holomorphic map $F\colon X \to \mathbb{P}^1$ defined by f.)

(5) (Riemann–Hurwitz formula) Let $F\colon X \to Y$ be a non constant holomorphic map between compact Riemann surfaces X and Y. In this question we will prove the Riemann–Hurwitz formula

$$2g(X) - 2 = d(2g(Y) - 2) + \sum_{P \in X} (e_P - 1)$$

where $g(X)$ and $g(Y)$ are the genus of X and Y, and d is the degree of F (see Q4).

(a) Let ω be a nonzero meromorphic differential on Y. (Actually, the existence of ω is not obvious, but we will assume it here.) We can define the *pullback* $F^*\omega$ of ω, a meromorphic differential on X, as follows: if $P \in X$ is a point, let z be a local coordinate at P and w a local coordinate at $F(P)$, and let $z \mapsto f(z)$ be the map F in the charts given by z, w. Write $\omega = g(w)dw$ in the chart given by w, then $F^*(\omega) = g(f(z))f'(z)dz$ in the chart given by z. Show that $F^*(\omega)$ is well defined.

(b) Show that $\nu_P(F^*(\omega)) = e_P \cdot \nu_{F(P)}(\omega) + (e_P - 1)$.

(c) Deduce the Riemann–Hurwitz formula (Hint: Use Poincaré–Hopf and Q4.)

(d) Use Riemann–Hurwitz to show that $g(X) \geq g(Y)$. Show that if $g(X) = g(Y)$ then either F is an isomorphism ($d = 1$), or $d > 1$ and $g(X) = 0$ or $g(X) = 1$. Give examples to show that these cases really do occur.

(e) Compute the genus of the curve $\tilde{X}$ from HW2Q4 and HW3Q8 using Riemann–Hurwitz (this method is faster than the one described in HW3Q8).

(f) Check your answer to Q3 using Riemann–Hurwitz.

(6) Suppose $F \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is given by $z \mapsto f(z) = p(z)/q(z)$, where p, q are polynomials with no common factors. Then the degree d of F is the maximum of the degrees of p and q by HW1Q2(c). Show that the set of ramification points of F is given by $(p'q - q'p = 0) \subset \mathbb{C}$ together with ∞ if $p'q - q'p$ has degree $< 2d - 2$. What is the maximum possible number of ramification points of F?