Math 235.4 Midterm 1, Wednesday 10/9/13, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 9 questions for a total of 65 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully.

Q1 (8 points). Find all solutions of the system of linear equations

$$\begin{aligned} x + y + 4z &= -1 \\ 2x + 5y + 2z &= 10 \\ x + 3y + 2z &= 5. \end{aligned}$$

Q2 (8 points).

(a) (6 points) Find all solutions of the system of linear equations

$$\begin{aligned} x + 2y + 4z &= 7 \\ 2x + 3y - 5z &= 1. \end{aligned}$$

(b) (2 points) Interpret your answer geometrically.

Q3 (8 points). Let

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 2 \\ 1 & 1 & 1 & 3 & 5 & 3 \\ 2 & 2 & 3 & 3 & 6 & 2 \\ 2 & 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}.$$

The row reduced echelon form (RREF) of the augmented matrix $(A\,\mathbf{b})$ for the system of linear equations $A\mathbf{x} = \mathbf{b}$ is the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(You do NOT need to check this!!)

(a) (4 points) Use the RREF above to give an explicit formula for all solutions $\mathbf{x} \in \mathbb{R}^6$ of the equation $A\mathbf{x} = \mathbf{b}$.

(b) (4 points) Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c} \in \mathbb{R}^4$? Explain your answer carefully in terms of the Gaussian elimination algorithm.

Q4 (5 points). Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \text{ and } S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \text{ and } T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}.$$

Compute the matrix of S, the matrix of T, and the matrix of the composite transformation $T \circ S$ defined by $T \circ S(\mathbf{x}) = T(S(\mathbf{x}))$.

Q5 (5 points). Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix

$$\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}.$$

Let $S \subset \mathbb{R}^2$ be the unit square (the set of points $(x, y) \in \mathbb{R}^2$ with $0 \le x \le 1$ and $0 \le y \le 1$). Draw a careful sketch of the image of S under the transformation T.

Q6 (9 points). Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be rotation about the origin through angle $\pi/4$ counter-clockwise. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the y-axis.

(a) (6 points) Compute the matrices of S, T, and the composite transformation $T \circ S$ defined by $T \circ S(\mathbf{x}) = T(S(\mathbf{x}))$.

(b) (3 points) Give a precise geometric description of the composite transformation $T \circ S$.

Q7 (9 points). For each of the following linear transformations, determine whether the transformation is invertible, and if so describe the inverse (give a precise geometric description or an explicit formula).

(a) (2 points) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation about the origin through angle $\pi/7$ counter-clockwise.

(b) (2 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the line $y = 5x$.

(c) (2 points) $U\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation with matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

(d) (3 points) $V\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with matrix $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Q8 (8 points).

(a) (6 points) Compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

(b) (2 points) Using your answer to part (a) or otherwise, solve the system of linear equations

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 5 \\ x + y + 2z &= 7 \end{aligned}$$

Q9 (5 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The linear transformation T is a rotation.

(a) (3 points) Determine the axis of rotation. (Note: The axis of rotation is the line of points $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{x}$.)

(b) (2 points) What is the angle of rotation?