Math 462 Homework 8

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(1) Let $f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be the Mobius transformation given by $f(z) = -\frac{1}{z}$.

   (a) Show directly that $f(\mathcal{H}) = \mathcal{H}$, where $\mathcal{H} = \{z = x+iy \in \mathbb{C} \mid y > 0\}$ is the upper half plane.

   (b) Prove that f is given by inversion in the circle with center the origin and radius 1 followed by reflection in the y-axis.

(2) Let $f\colon \mathcal{H} \to \mathcal{H}$ be the isometry of the hyperbolic plane

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

given by

$$f(z) = \frac{2z+1}{z+2}.$$

Express f as a composite of isometries of the following types:

   (a) $f_1(z) = z + b$, $b \in \mathbb{R}$ (translation parallel to the x-axis).

   (b) $f_2(z) = az$, $a \in \mathbb{R}$, $a > 0$ (scaling with factor a, center the origin).

   (c) $f_3(z) = -\frac{1}{z}$ (inversion in the circle center the origin and radius 1 followed by reflection in the y-axis).

Use this expression to describe the effect of f geometrically.

[Hint: Adapt the approach used in HW7Q5 (the difference here is that we only consider Mobius transformations preserving $\mathcal{H}$).]

(3) Prove carefully that for any two points $P, Q \in \mathcal{H}$, there is a unique hyperbolic line L passing through P and Q.

[Hint: Recall that a hyperbolic line is by definition either a vertical line or a circle with center on the x-axis. These are the curves which give the shortest paths between points in the hyperbolic plane $\mathcal{H}$.]

(4) Let L be the hyperbolic line given by the circle with center the origin and radius 1. Let P be the point $(1,2) \in \mathcal{H}$. Determine the unique hyperbolic line M such that M passes through P and is perpendicular to L.

[Hint: We can find M algebraically as we did in class: If the center of M is the point $(c, 0)$ and the radius of M is r then $c^2 = 1 + r^2$ (why?). Write down another equation satisfied by r and c using $P \in M$ and solve for r and c.]

(5) Let T be the hyperbolic triangle with vertices $-1+i, 2i, 1+i$.

(a) Determine the hyperbolic lines defining the sides of T.

(b) Show that the area of T equals $4\theta - \pi/2$ where $\theta = \tan^{-1}(1/2)$.

(6) In plane Euclidean geometry we have the following property: Given a point $P \in \mathbb{R}^2$ and a line L not passing through P, there is a unique line M passing through P such that L and M do not intersect. (M is the line through P parallel to L.)

Is the corresponding assertion true in the hyperbolic plane $\mathcal{H}$? That is, given a point $P \in \mathcal{H}$ and a hyperbolic line L not passing through P, is there a unique hyperbolic line M passing through P such that L and M do not intersect? Give a proof or a counterexample.