1. Recall that $|G| = p^n \Rightarrow Z(G) \neq \{e\}$

Also, if G is any group, $G/Z(G)$ cyclic $\xRightarrow{\text{HW3 Q7}}$ $\underbrace{G = Z(G)}_{G \text{ abelian.}}$, †

So, in our case $|G| = p^3$, G non-abelian

$\Rightarrow |Z(G)| = p$

(because $|Z(G)| \mid |G| = p^3$, $|Z(G)| \neq 1$

& $|Z(G)| \neq p^2, p^3$ because G non-abelian, using †)

Now let $x \in G \setminus Z(G)$.

Consider the centralizer $Z(x)$ of x.

We have $Z(G) \underset{\neq}{<} Z(x) \underset{\neq}{<} G$

because $x \in Z(x)$ because $x \notin Z(G)$

$\Rightarrow |Z(x)| = p^2$.

$\Rightarrow |C(x)| = |G| / |Z(x)| = p$.

So, the class equation of G is

$$p^3 = \underbrace{(1 + \cdots + 1)}_{p} + \underbrace{(p + \cdots + p)}_{p^2 - 1}$$

2. a Again recall if $G/Z(G)$ is cyclic, then G is abelian.

But $|G| = 21 = 3 \cdot 7$, product of 2 primes.

So if $Z(G) \neq \{e\}$ see $G/Z(G)$ is cyclic ※

$\therefore Z(G) = \{e\}$.

b. The class equation is $21 = 1 + \underbrace{(3 + \cdots + 3)}_{a} + \underbrace{(7 + \cdots + 7)}_{b}$

for some $a, b \in \mathbb{Z}$, $a, b \geq 0$.
The only solution is $21 = 1 + 3 + 3 + 7 + 7$

3. a. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

if $H \leq Q_8$, $H \neq \{e\} = \{1\}$, let $e \neq h \in H$.

The either $h = -1$ or $h^2 = -1$, so $-1 \in H$.

b. Q_8 is isomorphic to a subgroup of S_8 by Cayley's theorem

(consider action of $G = Q_8$ on itself by left mult.)

Suppose Q_8 is isomorphic to a subgroup of S_n, for $n \leq 7$.
Say $Q_8 \xrightarrow[\varphi]{\sim} \varphi(Q_8) \leq S_n$
Consider the associated action of Q_8 on $X = \{1, 2, \ldots, n\}$.
For $x \in X$, we have $|Q_8| = \underbrace{|(Q_8)_x|}_{\text{stabilizer}} \cdot \underbrace{|O_x|}_{\text{orbit}}$

$|Q_8| = 8$, $|O_x| \leq |X| \leq 7 \Rightarrow |(Q_8)_x| > 1$, $(Q_8)_x \neq \{e\}$.

By part a, see $-1 \in (Q_8)_x \quad \forall x \in X$.

Then $-1 \in \ker \varphi$ ※ (φ assumed injective). □.

4. $|G|$ odd, $x \in G$. $\exists g \in G$ s.t. $gxg^{-1} = x^{-1}$.

Consider $H = \langle x \rangle \leq G$ (cyclic group generated by x)
& $H \triangleleft N(H) \leq G$, the normalizer of H.

We have a group hom

$$N(H) \xrightarrow{\varphi} \mathrm{Aut}(H)$$

$$y \longmapsto (h \mapsto yhy^{-1})$$

Note that $g \in N(H)$ and $\varphi(g)$ has order 2 if $x \neq e$.

But G has odd order $\Rightarrow$ $N(H)$ has odd order
$\Rightarrow$ $\varphi(N(H))$ has odd order
$\Rightarrow$ $\varphi(g)$ has odd order

So $x = e$. $\square$.

5. $|G| = 60$, $4 \mid |Z(G)|$

$\Rightarrow$ $|G/Z(G)| \mid 60/4 = 15$

$\Rightarrow$ $|G/Z(G)| = 1, 3, 5, 15$

A group of order $15 = 3 \cdot 5$ is cyclic (using $5 \not\equiv 1 \mod 3$ & Sylow theorems)

So in each case $G/Z(G)$ is cyclic $\Rightarrow$ G abelian.

6. $|G| = 168$, G simple.

elements of order 7 ?

$|G| = 2^3 \cdot 3 \cdot 7$

$s :=$ # Sylow 7-subgroups, $s \equiv 1 \mod 7$, $s \mid 2^3 \cdot 3 = 24$

$\Rightarrow$ $s = 1, 8$.

But G is simple so $s \neq 1$ ($s = 1 \Leftrightarrow$ Sylow 7-subgroup is normal by Sylow Thm 2)

Now $\#$ elements of order $7 = s \cdot (7-1) = 8 \cdot 6 = 48$.

7. $|G| = 20$. $\exists\ x \in G$ of order 4. $Z(G) = \{e\}$.

$|G| = 2^2 \cdot 5$

$s := \#$ Sylow 2-subgroups $\equiv 1 \bmod 2$, $s \mid 5 \Rightarrow s = 1$ or 5.
$t := \#$ Sylow 5-subgroups $\equiv 1 \bmod 5$, $t \mid 4 \Rightarrow t = 1$.

$\therefore$ Let H be Sylow 5-subgroup, K Sylow 2-subgroup

then $H = \mathbb{Z}/5\mathbb{Z}$, $H \triangleleft G$ $(t=1)$.

Also $K \simeq \mathbb{Z}/4\mathbb{Z}$. (because $\langle x \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ is Sylow 2-subgp
& all Sylow 2-subgps are conjugate
(Sylow Thm 2)).

Now $\gcd(|H|, |K|) = 1$ & $|H| \cdot |K| = |G|$

$$\Rightarrow \quad G \cong H \rtimes_\varphi K, \qquad K \xrightarrow{\varphi} \mathrm{Aut}(H)$$
$$k \mapsto (h \mapsto khk^{-1})$$

$$\cong \mathbb{Z}/5\mathbb{Z} \rtimes_\psi \mathbb{Z}/4\mathbb{Z}$$

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{\psi} \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z}) \simeq (\mathbb{Z}/5\mathbb{Z})^\times = \mathbb{Z}/4\mathbb{Z}$$
$$1 \longmapsto l$$

$$\Rightarrow \quad G \cong \langle a, b \mid a^5 = b^4 = e,\ bab^{-1} = a^l \rangle.$$

Finally, use the condition that $Z(G) = \{e\}$ to see that $l \in (\mathbb{Z}/5\mathbb{Z})^\times$ must have order 4 (equivalently, ψ is an isom.) – otherwise b^2 commutes with $a \Rightarrow e \neq b^2 \in Z(G)$ ※.
Changing the choice of isom $K \simeq \mathbb{Z}/4\mathbb{Z}$, we may assume l is

any given element of order 4, e.g. $l = 2 \in (\mathbb{Z}/5\mathbb{Z})^\times$.

Thus $G \cong \langle a, b \mid a^5 = b^4 = e,\ bab^{-1} = a^2 \rangle$.

8. Recall the bijective correspondence

$$\{\text{subgroups } H \text{ of } G \text{ containing } N\} \longleftrightarrow \{\text{subgroups } \bar{K} \text{ of } G/N\}$$

$$H \longmapsto H/N$$

$$q^{-1}K \longleftarrow K$$

where $q : G \twoheadrightarrow G/N$ is the quotient hom,
and $q^{-1}K = \{x \in G \mid q(x) \in K\}$ is the preimage of K.

Under this correspondence, Sylow p-subgroups of G/N correspond to Sylow p-subgroups of G containing N.
This proves the assertion. $\square$

9. $|G| = p^2 q$, p, q distinct primes.

$s = \#$ Sylow p-subgroups $\equiv 1 \bmod p$, $s \mid q$. $\Rightarrow s = 1$ or q
$t = \#$ Sylow q-subgroups $\equiv 1 \bmod q$, $t \mid p^2 \Rightarrow t = 1, p,$ or p^2.

Suppose $s, t \neq 1$.
Then $s = q$, $q \equiv 1 \bmod p$, in particular $p < q$.

Now $p \not\equiv 1 \bmod q$ so $t = p^2$ (& $p^2 \equiv 1 \bmod q$)

But now $\#$ elements of order $q = t \cdot (q-1) = p^2(q-1)$

This leaves p^2 elements of G — these must form a unique Sylow p-subgroup

(a Sylow p-subgroup is contained in this set, & has the same size).
This is a contradiction. $\square$.

10. $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$.

$H \leq A_5$ Sylow 2-subgroup, i.e., $|H| = 2^2 = 4$.

Example: $H = \{e, (12)(34), (13)(24), (14)(23)\} \leq A_4 \leq A_5$

All Sylow p-subgroups are conjugate, so see
Sylow 2-subgroups = # conjugates of H $= \binom{5}{4} = 5$.

11. a. $|G| = 40. = 2^3 \cdot 5$

$s =$ # Sylow 2-subgroups $\equiv 1 \bmod 2$, $s \mid 5 \Rightarrow s = 1$ or 5
$t =$ # Sylow 5-subgroups $\equiv 1 \bmod 5$, $t \mid 8 \Rightarrow t = 1$.

So, let $H \trianglelefteq G$ be Sylow 5-subgroup, $H \cong \mathbb{Z}/5\mathbb{Z}$.
It's enough to show G/H is solvable.

But $|G/H| = 2^3$, and finite p-groups are solvable. $\square$.

b. $|G| = 48 = 2^4 \cdot 3$

$s =$ # Sylow 2-subgroups $\equiv 1 \bmod 2$, $s \mid 3 \Rightarrow s = 1$ or 3.
$t =$ # Sylow 3-subgroups $\equiv 1 \bmod 3$, $t \mid 16 \Rightarrow t = 1, 4,$ or 16.
If $s = 1$, have $H \trianglelefteq G$, $|H| = 2^4$, $G/H \cong \mathbb{Z}/3\mathbb{Z}$.
Then H, G/H solvable $\Rightarrow$ G solvable.
If $s = 3$, have group hom $\varphi: G \longrightarrow S_3$ given by action of G on Sylow 2-subgroups by conjugation.

φ is transitive $\Rightarrow 3 \mid |\varphi(G)| \Rightarrow H := \ker\varphi \trianglelefteq G$ is a 2-group
$\Rightarrow H$ solvable

Also $G/H \xrightarrow{\sim} \varphi(G) \leq S_3$, & S_3 solvable $\Rightarrow G/H$ solvable.
So G is solvable.

12. $|G| = 120 = 2^3 \cdot 3 \cdot 5$. Required to prove G NOT simple
Let $s = \#$ Sylow 5-subgroups } i.e. $\exists \{e\} \neq H \underset{\neq}{\trianglelefteq} G$.
$s \equiv 1 \bmod 5$, $s \mid 24 \Rightarrow s = 1$ or 6.
If $s = 1$, OK (Sylow 5-subgroup is normal).
If $s = 6$, consider action of G on Sylow 5-subgroups by conjugation.
$\leadsto \varphi : G \longrightarrow S_6$ non trivial hom (action is transitive)

If φ not injective, OK (have $\ker\varphi \underset{\neq}{\trianglelefteq} G$)
If $\varphi(G) \not\leq A_6$ OK (consider $G \xrightarrow{\varphi} S_6 \xrightarrow{\mathrm{sgn}} \{\pm 1\}$, $\ker\psi \underset{\neq}{\trianglelefteq} G$)
ψ

So, WMA $G \xrightarrow{\sim} \varphi(G) \leq A_6$.

Now use fact: A_6 is simple.
But $H := \varphi(G) \leq A_6$ has index $\dfrac{\frac{1}{2}(6!)}{120} = 3$

$\#$ conjugate subgroups of $H = |A_6| / |N(H)| \leq |A_6| / |H| = 3$

So we get either H normal ※
or non trivial hom $\psi : A_6 \to S_3$, $\{e\} \neq \ker\psi \trianglelefteq A_6$ ※

(Here we are using a special case of a general fact: if a group G has a subgroup H of index n, then it has a normal subgroup N of index $\leq n!$)

13. G finite, $H \leq G$, $[G:H] = p$, prime.

(a) # conjugate subgroups $= |G|/|N(H)|$ $\Big|$ $|G|/|H| = p$

$\Rightarrow$ # $= 1$ or p.

(b) Assume p is the smallest prime dividing $|G|$. †
Required to prove H normal.
Otherwise, # conjugate subgroups $= p$.
Consider action of G on {conjugate subgroups} by conjugation.
$\rightsquigarrow$ $\varphi : G \longrightarrow S_p$ hom, nontrivial.

$\gcd(|G|, |S_p|) = \gcd(|G|, p!) = p$ by assumption †

So $|\varphi(G)| = p$, $\ker\varphi \lhd G$ has index p.
But $\ker\varphi \leq N(H) = H$ (stabilizer of H)

$\Rightarrow$ $\ker\varphi = H$, H normal #. □

14. $G = \langle x, y, z \mid yz^2xy = e \rangle$

$yz^2xy = e \Rightarrow x = z^{-2}y^{-1}y^{-1} = z^{-2}y^{-2}$.

So G is isomorphic to the free group on $\{y, z\}$
To prove this carefully, let $F = F(\{y,z\})$ denote the free group on $\{y,z\}$, & $F \xrightarrow{\theta} G$ the group hom given by $\theta(y) = y$ & $\theta(z) = z$, $G \xrightarrow{\varphi} F$ the group hom given by $\varphi(x) = z^{-2}y^{-2}$, $\varphi(y) = y$, $\varphi(z) = z$ (this does define a

group hom because the relation $yz^2xy = e$ in G is satisfied by the images of x, y, z in F by construction.)

Now $\varphi \circ \theta(y) = y$ & $\varphi \circ \theta(z) = z \Rightarrow \varphi \circ \theta = id_F$

& $\left.\begin{array}{l} \theta \circ \varphi(y) = y, \ \theta \circ \varphi(z) = z, \\ \theta \circ \varphi(x) = z^{-2}y^{-2} = x \end{array}\right\} \Rightarrow \theta \circ \varphi = id_G.$

So $F \xrightarrow[\sim]{\theta} G$.

15. a. $G = \langle a, b \mid a^5 = b^2 = (ab)^2 = e \rangle$

$= \langle a, b \mid a^5 = b^2 = e, \ ba = a^{-1}b \rangle$

This is (isomorphic to) the dihedral group D_5.

To prove this carefully, let $G \xrightarrow{\varphi} D_5$ be the group hom given by sending a to a rotation by $2\pi/5$ & b to a reflection.

(this defines a group hom because the relations $a^5 = b^2 = e$, $ba = a^{-1}b$ are satisfied by the images of a, b in D_5)

φ is surjective because D_5 is generated by the rotation by $2\pi/5$ & reflection.

φ is injective: we can use the relations $a^5 = b^2 = e$, $ba = a^{-1}b$ to write any element of G in the form a^ib^j, where $0 \le i < 4$ & $0 \le j < 2$.

Thus $|G| \le 5 \cdot 2 = 10$. But $|D_5| = 10$ & φ surjective $\Rightarrow \varphi$ bijective, φ isom.

b. $G = \langle a, b \mid a^4 = e,\ a^2 = b^2,\ ba = a^{-1}b \rangle$

This group is isomorphic to Q_8 :-

Let $\varphi: G \to Q_8$ be hom give by $\varphi(a) = i$, $\varphi(b) = j$

(Note: $i^4 = e = 1$, $i^2 = j^2 = -1$, $ji = k = -ij = i^{-1}j$,
so this does define group hom.)

φ surjective: Q_8 gen'd by i & j ✓
φ injective: Using relations, can write any element of G
as $a^i b^j$ where $0 \le i < 4$, $0 \le j < 2$
So $|G| \le 8$ $\Rightarrow$ φ bijective $\Rightarrow$ φ isom.

16. $G = \langle x, y \mid x^2 = y^2 = e \rangle$

G is generated by xy & y, of order ∞ & 2,
& $y \cdot (xy) \cdot y^{-1} = yx = (xy)^{-1}$ †

$\Rightarrow$ $G \cong H = \langle a, b \mid b^2 = e,\ bab^{-1} = a^{-1} \rangle$

$xy \leftarrow a$ $\qquad \cong$

$y \leftarrow b$ $\qquad \mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$

$a \leftrightarrow (1,0)$, $b \leftrightarrow (0,1)$

where $\varphi: \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$
$1 \mapsto (x \mapsto -x)$

Careful proof: define group hom

$\theta: H \to G$ by $\theta(a) = xy$, $\theta(b) = y$

Well defined: By † and $y^2 = e$.

Note $\theta(ab) = xyy = x$

Now, construct inverse ψ of θ.

Define $\psi: G \longrightarrow H$ by $\psi(x) = ab$
$\psi(y) = b$.

Well defined: $(ab)^2 = abab = a(bab^{-1}) = aa^{-1} = e$ ✓
$b^2 = e$

Now $\psi \circ \theta(a) = \psi(xy) = abb = a$
$\psi \circ \theta(b) = b$ $\Rightarrow \psi \circ \theta = id_H$

& similarly $\theta \circ \psi = id_G$, $\Rightarrow$ θ isom.