Math 462 Midterm 1 review questions

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(1) Let S^2 be a sphere in $\mathbb{R}^3$ with center the origin, and $P = (1, 2, 2)$ and $Q = (-2, -2, 1)$ two points on S^2.

(a) Compute the radius R of S^2.

(b) Describe the great circle L passing through P and Q explicitly by equations.

[Hint: Since $L = \Pi \cap S^2$ where Π is a plane passing through the origin, the great circle $L \subset \mathbb{R}^3$ is defined by two equations : the equation of $\Pi \subset \mathbb{R}^3$ and the equation of $S^2 \subset \mathbb{R}^3$.]

(c) Compute the length of the shortest path from P to Q along the surface of the sphere S^2 and describe the shortest path geometrically.

(2) Consider the sphere S^2 in $\mathbb{R}^3$ of radius $R = 1$ with center the origin. Let T be the spherical triangle on S^2 with vertices

$$P = (1, 0, 0), \quad Q = \frac{1}{\sqrt{2}}(1, 1, 0), \quad R = \frac{1}{\sqrt{3}}(1, 1, 1).$$

(a) Compute the equations of the great circles given by the sides of the triangle T.

(b) Compute the angles of T.

(c) Deduce the area of T.

(3) Let S^2 be the sphere in $\mathbb{R}^3$ of radius $R = 1$ with center the origin. Let Π be the plane

$$\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y + 2z = 1\} \subset \mathbb{R}^3$$

in $\mathbb{R}^3$.

(a) Explain why the intersection $C = S^2 \cap \Pi$ is a spherical circle and identify its spherical center. (See HW1Q5 for the definition of a spherical circle.)

(b) Compute the Euclidean radius and spherical radius of C.

[Note: Recall that for a spherical circle there are two antipodal points on the sphere which can be regarded as its center. Here we assume that the center of the spherical circle is chosen so that the spherical radius is $\leq \pi$.]

(4) Give an algebraic formula $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for the following isometries of $\mathbb{R}^n$, where A is a $n \times n$ orthogonal matrix and $\mathbf{b} \in \mathbb{R}^n$ is a vector.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is rotation about the point $(2,3)$ through angle $\pi/2$ counterclockwise.

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = 3 - x$ followed by translation through distance $2\sqrt{2}$ parallel to the line in the direction of increasing x. (This is a glide reflection.)

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the plane

$$\Pi = \{(x,y,z) \in \mathbb{R}^3 \mid 2x + 3y + z = 0\} \subset \mathbb{R}^3.$$

(d) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the y-axis through angle θ counterclockwise.

(5) Give a precise geometric description of the following isometries.

(a)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x + \sqrt{3}y \\ \sqrt{3}x - y \end{pmatrix}.$$

(b)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x + y \\ y - x \end{pmatrix}.$$

(c)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - y \\ 7 - x \end{pmatrix}.$$

(d)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3x - 4y + 2 \\ 4x + 3y + 6 \end{pmatrix}.$$

(6) Give a precise geometric description of the following isometries of $\mathbb{R}^3$. (In the case of a rotation or rotary reflection, the direction (counterclockwise/clockwise) of rotation may be omitted.)

(a)

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -z \\ x \end{pmatrix}$$

(b)

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -y \\ -x \end{pmatrix}$$

(c)

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ y \\ -x \end{pmatrix}$$

[Hint: First compute the eigenvalues of the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.]

(7) Give a precise geometric description of the compositions $T_2 \circ T_1$ of the following isometries.

(a) $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the plane

$$\Pi_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x = y\}$$

and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the plane

$$\Pi_1 = \{(x, y, z) \in \mathbb{R}^3 \mid y = z\}.$$

(b) $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the y-axis through angle $\pi/2$ counterclockwise and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the z-axis through angle $\pi/2$ counterclockwise.

(c) $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the line L_1 through the origin in the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ through angle $2\pi/3$ counterclockwise and $T_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the line L_2 in direction $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ through angle π counterclockwise.

[Hint: Parts (b) and (c) can be computed efficiently using quaternions.]

(8) Let $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in a line L_1 and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in a line L_2. What can you say about the composite $T_2 \circ T_1$?

[Treat the case where L_1 and L_2 are parallel separately.]

(9) Let $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation about a point P_1 through angle θ_1 counterclockwise and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation about a point P_2 through angle θ_2 counterclockwise, where $0 < \theta_1, \theta_2 < 2\pi$. What can you say about the composite $T_2 \circ T_1$?

[Hint: Treat the case $\theta_1 + \theta_2 = 2\pi$ separately.]

(10) Let $\mathbf{v} \in \mathbb{R}^3$ be a vector of length $\|\mathbf{v}\| = 1$.

(a) Show that the quaternions q of the form $a + b\mathbf{v}$ for some $a, b \in \mathbb{R}$ multiply in the same way as complex numbers $z = a + bi$, i.e.,

$$(a + b\mathbf{v}) \cdot (c + d\mathbf{v}) = (ac - bd) + (ad + bc)\mathbf{v}.$$

(b) Now suppose in addition that $q = a + b\mathbf{v}$ with $a^2 + b^2 = 1$ (equivalently, $|q| = 1$). Describe the isometry $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\mathbf{x}) = q\mathbf{x}\bar{q}$. Explain why this is consistent with your answer to part (a).

(11) Give a description of the isometries of the sphere S^2 and identify the fixed locus in each case. (The fixed locus of an isometry $T \colon S^2 \to S^2$ of the sphere is by definition the subset $\{\mathbf{x} \in S^2 \mid T(\mathbf{x}) = \mathbf{x}\} \subset S^2$.)

(12) Let $S \colon \mathbb{R}^n \to \mathbb{R}^n$ and $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be the isometries of $\mathbb{R}^n$ given by $S(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and $T(\mathbf{x}) = C\mathbf{x} + \mathbf{d}$ where A, C are orthogonal $n \times n$ matrices and $\mathbf{c}$ and $\mathbf{d}$ are vectors in $\mathbb{R}^n$.

(a) Compute an algebraic formula for the composite isometry $T \circ S$.

(b) Compute an algebraic formula for the inverse isometry S^{-1}.