Math 300.3 Homework 9

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April 13, 2017

Reading: Sundstrom, Sections 5.1, 5.2, and 5.3.

Note: I did not follow the textbook closely in class for these sections. In particular I covered the binomial theorem, the binomial coefficients $\binom{n}{k}$ (pronounced "n choose k") and the inclusion-exclusion priniciple, which are not discussed in our textbook and are needed for some of the problems below (but you can find this material on Wikipedia for example).

Justify your answers carefully.

(1) Let A, B, and C be sets, and let P, Q, and R be the statements $(x \in A)$, $(x \in B)$, and $(x \in C)$.

(a) Express the statements $(x \in A \backslash (B \cap C))$ and $(x \in (A \backslash B) \cup (A \backslash C))$ in terms of the statements P, Q, R and logical operators.

(b) Using either truth tables or a sequence of known logical equivalences, show that the statements are logically equivalent, so that $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

(c) Give an alternative proof that $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ using Venn diagrams.

[Question: What is the relation between the Venn diagrams in part (c) and the truth tables in part (b)?]

(2) Determine whether each of the following equalities hold for all sets A, B, and C. (Give a proof or a counterexample using the method of Q1ab or Q1c.)

(a) $(A \backslash B) \backslash C = A \backslash (B \cup C)$.

(b) $(A \cup B)\backslash C = A \cup (B\backslash C)$.

(3) The downtown portion of a city is a rectangular grid. How many ways are there to travel from one street corner to another, a total distance of 4 blocks south and 7 blocks west, which are as short as possible?

(4) A coin is tossed 8 times. What is the probability of getting exactly 4 heads?

(5) Compute $(2x^2 - yz)^4$. Simplify your answer as much as possible.

(6) Recall in class we proved the *binomial theorem*: For all $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of subsets with k elements of a set with n elements.

Now prove the *trinomial theorem*: For all $a, b, c \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b+c)^n = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!l!(n-k-l)!} a^k b^l c^{n-k-l}.$$

[Hint: Write $(a+b+c) = (a+(b+c))$ and use the binomial theorem twice.]

[Question: What is the analogue of Pascal's triangle for the trinomial theorem?]

(7) Recall the product rule from MATH 131: If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are differentiable functions, then the product $f \cdot g$ is differentiable and $(f \cdot g)' = f \cdot g' + f' \cdot g$.

For $f\colon \mathbb{R} \to \mathbb{R}$ a function and $k \in \mathbb{N}$ we write $f^{(k)}$ for the kth derivative of f (when it exists). And we define $f^{(0)} = f$.

(a) Suppose $n \in \mathbb{N}$ and $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are functions which can be differentiated n times. Show that $f \cdot g$ can be differentiated n times and

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

[Hint: Proof by induction on n using the product rule. This is similar to the proof by induction of the binomial theorem given in class and uses the property

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

].

(b) Using part (a) or otherwise, for each $n \in \mathbb{N}$ compute the nth derivative of $h(x) = x^2 e^x$.

(8) You are dealt two cards from a standard deck. What is the probability that at least one of the cards is an ace?

[Hint: What is the inclusion-exclusion principle? Let U be the set of possible hands and A, B, C, and D the subsets of U where one of the cards in the hand is the ace of hearts, clubs, diamonds, and spades respectively.]

(9) You roll 8 dice. What is the probability that all 6 numbers appear?

[Hint: Let $U = \{(a_1, \ldots, a_8) \mid a_i \in \{1, 2, \ldots, 6\}$ for each $i = 1, 2, \ldots, 8\}$ be the set of possible outcomes, and for each $1 \leqslant j \leqslant 6$, let $A_j \subset U$ be the subset of outcomes where the number j does *not* occur.]