Math 300.2 Homework 8

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November 6, 2017

Reading: Sundstrom, 6.1, 6.2, 6.3, 6.4.

Justify your answers carefully.

(1) Determine the range (or image) of the following functions.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$.

(b) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 5$.

(c) $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, $f(x, y) = 12x + 57y$.

(d) $f\colon \mathbb{Z} \to \{0, 1, 2, 3\}$, $f(x)$ is the remainder on dividing x^2 by 4.

(2) For each of the following pairs of functions $f\colon A \to B$ and $g\colon B \to C$ describe the composite function $g \circ f\colon A \to C$ explicitly.

(a)

$$f\colon \{1, 2, 3\} \to \{a, b, c, d\}, \quad f(1) = b,\ f(2) = d,\ f(3) = a;$$

$$g\colon \{a, b, c, d\} \to \{\alpha, \beta, \gamma\}, \quad g(a) = \gamma,\ g(b) = \alpha,\ g(c) = \beta,\ g(d) = \alpha.$$

(b)

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 + 1;$$
$$g\colon \mathbb{R} \to \mathbb{R}, \quad g(x) = x^3 + 4.$$

(c)

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (x + y, 2x + y);$$
$$g\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x, y) = (3x + 4y, 2x + 5y).$$

(3) Which of the following functions are injective? Justify your answer carefully.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$

(b) $f\colon [0, 2\pi) \to \mathbb{R}^2$, $f(t) = (\cos t, \sin t)$.

(c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 2x + 1$.
[Hint: Use Q6(a) below]

(d) $f\colon \mathbb{N}^2 \to \mathbb{N}$, $f(x, y) = 3^x \cdot 5^y$.

(e) $f\colon A \to B$, where A and B are finite sets and $|A| > |B|$.

(4) Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$. How many functions f from A to B are there? How many of these functions are injective?

(5) Describe a bijective function $f\colon \mathbb{N} \to \mathbb{Z}$. (Recall $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}$ is the set of all integers.)

(6) (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose that $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Show that f is injective.
[Hint: Use the mean value theorem].

(b) Give an example of an injective differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 0$ for some $x \in \mathbb{R}$.

(7) Which of the following functions have an inverse? If the inverse exists, describe it explicitly. Otherwise explain carefully why the inverse does not exist.

(a)
$$f\colon \{1, 2, 3, 4\} \to \{a, b, c, d\},$$
$$f(1) = c,\ f(2) = d,\ f(3) = a,\ f(4) = b.$$

(b) $f\colon \mathbb{R} \to (0, \infty)$, $f(x) = e^x$.

(c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 4x + 3$.

(d) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 7$.

(e) $f\colon \mathbb{N}^2 \to \mathbb{N}$, $f(x, y) = 2^x \cdot 3^y$.

(f) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 2x$.

(g) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + 4y, 3x + 6y)$.

(h) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + 5y, 3x + 7y)$.

(8) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function which preserves distances. That is, for a pair of points $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{R}^2$, define

$$d(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

the distance between p_1 and p_2. Then the function f satisfies

$$d(f(p_1), f(p_2)) = d(p_1, p_2) \text{ for all } p_1, p_2 \in \mathbb{R}^2.$$

Show that f is a bijection, so it has an inverse.

[Hint: To show f is injective, use $d(p_1, p_2) = 0 \iff p_1 = p_2$. To show that f is surjective, fix two distinct points $p_1, p_2 \in \mathbb{R}^2$, say $p_1 = (1, 0)$ and $p_2 = (0, 1)$, and consider $f(p_1), f(p_2) \in \mathbb{R}^2$. Given a point $q \in \mathbb{R}^2$, we want to show that there is a point $p \in \mathbb{R}^2$ such that $f(p) = q$. If $f(p) = q$ then we must have $d(p_1, p) = d(f(p_1), q)$ and $d(p_2, p) = d(f(p_2), q)$. Now draw circles with centers at p_1 and p_2 to find 1 or 2 possibilities for p, and show that one of them works.]