Math 300.2 Homework 5

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Reading: Sundstrom, Sections 4.1 and 4.2.

Justify your answers carefully.

(1) Guess a formula for

$$\sum_{r=1}^{n}(-1)^{r-1}(2r-1) = 1 - 3 + 5 - \cdots + (-1)^{n-1}(2n-1)$$

and prove your formula is correct using induction.

(2) Prove the following statements by induction:

(a) For all $n \in \mathbb{N}$,

$$\sum_{r=1}^{n}(r+1)2^r = 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}.$$

(b) For all $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} r(r+1) = 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{1}{3}n(n+1)(n+2).$$

(c) For all $n \in \mathbb{N}$,

$$\sum_{r=1}^{n}(2r-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1).$$

(3) Use Q2(b) and the formula

$$\sum_{r=1}^{n} r = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

proved in class to deduce a formula for

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \cdots + n^2$$

without using induction.

(4) Prove the following statement by induction: For all $x \in \mathbb{R}$ such that $x \geqslant -1$ and all $n \in \mathbb{N}$,

$$(1+x)^n \geqslant 1 + nx.$$

(5) Suppose you are given 3^n coins and a pair of balancing scales. All but one of the coins have exactly the same weight, and the remaining coin is slightly heavier. All the coins look and feel identical. The balancing scales can be used to compare the weights of two collections of coins, showing whether the two collections have the same weight or, if not, which is heavier. Prove by induction that it is possible to find the heavy coin using n weighings.

(6) Consider n lines in the plane such that no two lines are parallel and no three lines are concurrent (meet at a point). In class we proved by induction that the lines divide the plane into $\frac{1}{2}(n^2 + n + 2)$ regions. Prove by induction that we can color the regions red and blue so that regions which meet along an edge have different colors.

(7) For $a, b \in \mathbb{N}$, consider an $a \times b$ rectangular chocolate bar made up of ab unit squares. If $ab \geqslant 2$, we can break the chocolate bar along any horizontal or vertical line, dividing it into two smaller bars. Doing this repeatedly, we eventually divide the bar into single squares. Prove by induction that the number of breaks required to do this is $ab - 1$.

[Hint: One way to formulate this is as a proof by strong induction on $n = ab \in \mathbb{N}$. (In the text, strong induction is called the second principle of mathematical induction.)]