Math 462: Homework 1 Solutions

Paul Hacking

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1. Suppose a, b, c are positive real numbers satisfying the inequalities

$$a < b + c, \quad b < c + a, \quad c < a + b.$$

Give a geometric construction using a ruler and compass for a triangle with side lengths a, b, c. What happens if one of the inequalities becomes an equality, say $a = b + c$?

Draw a line segment PQ of length a. Draw a circle with center P and radius b and a circle with center Q and radius c. These circles intersect in two points R and R' (this is where we need to use the inequalities). Now PQR (or PQR') is a triangle with sides PQ, PR, QR of lengths a, b, c.

If $a = b + c$ then in the construction above the circles touch at a single point R on the line PQ. Then the line segments PQ, PR, QR have lengths a, b, c and R lies on the line PQ between P and Q (we do not get an honest triangle).

2. Can you find 4 points A, B, C, D in $\mathbb{R}^2$ such that the distance between A and D equals 2, but the distance between every other pair of points equals 1? Explain. Can you find 4 such points on a sphere?

There are many ways to do this. For example, suppose we have 4 points A, B, C, D in the plane with distances as described in the question. Since $d(A, B) + d(B, D) = d(A, D)$ it follows that B lies on the line AD, and it is the midpoint of the line segment AD. For the same reason C is the midpoint of AD. But then $B = C$ so the distance $d(B, C) = 0$. This is a contradiction. So there do not exist 4 points A, B, C, D in the plane with the desired distances.

We can find 4 points on a sphere with the given distances: Let A be the north pole, D the south pole, and B, C two points on the equator, such that B and C are related by a rotation of $\pi/2$ about the axis AD.

Then all the distances $d(A,B), d(A,C), d(B,D), d(C,D), d(B,C)$ are equal to $\frac{1}{2}\pi r$, where r is the radius of the sphere, and $d(A,D) = 2d(A,B)$. We need $\frac{1}{2}\pi r = 1$, so $r = \frac{2}{\pi}$.

3. Find the eigenvectors of the matrix $A = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$. Describe the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{x}) = A\mathbf{x}$ geometrically.

The eigenvalues are $\lambda = 1, -1$ and the corresponding eigenvectors are $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. So $T(\mathbf{v}) = \mathbf{v}$ and $T(\mathbf{w}) = -\mathbf{w}$. Notice that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. It follows that T is a reflection in the line through the origin in the direction $\mathbf{v}$.

4. Describe the following motions of $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ geometrically (as a translation, rotation, reflection, or glide reflection).

(a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+2 \\ y+1 \end{pmatrix}$.

(b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+3 \\ -x+1 \end{pmatrix}$.

(c) $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is as in Q3 and $\mathbf{b} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$.

(d) $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where $A = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Here is a general procedure for solving this type of problem. Write the motion T in the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector. There are two cases: $\det A = 1$ (T is *direct*) or $\det A = -1$ (T is *opposite*).

If $\det A = 1$ then T is either a translation or a rotation. A translation is $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$, this corresponds to $A = I$. Otherwise, the matrix A is a rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The motion $U(\mathbf{x}) = A\mathbf{x}$ is a rotation about the origin through an angle θ anticlockwise. The motion $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is a rotation about some point $\mathbf{c}$ through the same angle. To find the center $\mathbf{c}$ of rotation, we observe that it is the solution of the equation $T(\mathbf{x}) = \mathbf{x}$. This is a linear equation we can solve for $\mathbf{c}$.

If $\det A = -1$ then T is either a reflection or a glide. The motion $U(\mathbf{x}) = A\mathbf{x}$ is reflection in a line through the origin in some direction $\mathbf{v}$. To find the direction $\mathbf{v}$ we observe that it is a solution of the equation $U(\mathbf{x}) = \mathbf{x}$ (in other words, $\mathbf{v}$ is an eigenvector of A). To describe the motion $T(\mathbf{x}) = \mathbf{x}+\mathbf{b}$, write $\mathbf{b} = c\cdot\mathbf{v}+\mathbf{w}$ where $c \in \mathbb{R}$ is a scalar and $\mathbf{w}$ is perpendicular to $\mathbf{v}$. (Note that c is given by the formula $c = (\mathbf{b}\cdot\mathbf{v})/(\mathbf{v}\cdot\mathbf{v})$ (this is the Gram-Schmidt process from Math 235).) The motion $V(\mathbf{x}) = A\mathbf{x}+\mathbf{w}$ is the reflection in the line L through the point $\frac{1}{2}\mathbf{w}$ in the direction $\mathbf{v}$. Finally the motion $T(\mathbf{x}) = A\mathbf{x}+\mathbf{w}+c\mathbf{v}$ is the glide given by reflection in the line L followed by translation by $c\mathbf{v}$ (the translation is parallel to L).

(a) T is a glide given by reflection in the line $x = 1$ followed by translation by $\begin{pmatrix}0\\1\end{pmatrix}$.

(b) T is a rotation about the point $\begin{pmatrix}2\\-1\end{pmatrix}$ through $\pi/2$ radians clockwise.

(c) T is a glide given by reflection in the line L through the point $\begin{pmatrix}1\\-2\end{pmatrix}$ in the direction $\begin{pmatrix}2\\1\end{pmatrix}$ followed by translation by $4\begin{pmatrix}2\\1\end{pmatrix}$.

(d) T is a rotation about the point $\frac{1}{2}\begin{pmatrix}-\sqrt{3}\\5+2\sqrt{3}\end{pmatrix}$ through $\pi/6$ radians anticlockwise.

5.

(a) Two lines l and m in $\mathbb{R}^2$ intersect at a point P and meet at an angle θ, $0 < \theta \leq \pi/2$. Describe geometrically the composition of the reflections in l and m.

(b) Suppose $\theta = \pi/m$ for some integer $m \geq 2$. What is the group G generated by reflections in l and m? Describe a fundamental domain for the action of G on $\mathbb{R}^2$, that is, a region $R \subset \mathbb{R}^2$ such that each orbit of G contains exactly one point of R.

(a) The composition is a rotation about P through an angle 2θ. It is easiest to prove this by drawing a diagram showing the image of an arbitrary point Q under the composition of reflections. Alternatively we can work in coordinates as follows. Choose coordinates so that P is the origin, l is the x-axis and the angle from l to m is θ (anticlockwise). Then reflection in l is given by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and reflection in m is given by the matrix

$$B = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

(here we use that the columns of the matrix B are $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $T(\mathbf{x}) = B\mathbf{x}$ is the reflection in m). So the matrix of reflection in l followed by reflection in m is given by the product

$$BA = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

This is a rotation about the origin through angle 2θ anticlockwise.

(b) The group G is the *dihedral group* of order $2m$. It is the group of symmetries of a regular polygon with m sides. It consists of the m rotations about P through the angles $2\pi k/m$, $k = 0, 1, \ldots, m-1$, and m reflections. A fundamental domain R is given by the sector of the plane lying between the two lines l and m (with angle $\theta = \pi/m$ at P).