Math 621 Homework 2

Paul Hacking

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Reading: Stein and Shakarchi, Chapter 2, Sections 3 and 4, and Chapter 3, Sections 1 and 2.

(1) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function on $\mathbb{C}$ such that, for some $C, R \in \mathbb{R}$ and non-negative integer N,

$$|f(z)| \leq C|z|^N$$

for all $z \in \mathbb{C}$ such that $|z| \geq R$. Show that f is a polynomial of degree $\leq N$.

(2) (a) Compute the set of zeroes of $f(z) = \cos z$.

(b) Determine the set of zeroes of $g(z) = \cos(\frac{1+z}{1-z})$. Find any accumulation points.

(c) Explain why your answer to part (b) does not contradict Theorem 4.8 on p. 52 of the course text ("zeroes of holomorphic functions are isolated").

(3) Let $\Omega \subset \mathbb{C}$ be an open set and $z_0 \in \Omega$. Let $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function.

(a) What does it mean to say f has a pole of order m at z_0? Given that f has a pole at z_0, what is meant by the *residue* of f at z_0?

(b) If f has a pole of order m at z_0, show that

$$\operatorname{res}_{z_0} f(z) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz}\right)^{m-1} (z - z_0)^m f(z).$$

(4) Compute the zeroes and poles (including the orders) of the following functions. Also compute the residue at each pole.

  (a) $f(z) = \frac{z^2+1}{z^3-1}$
  (b) $g(z) = \frac{\sin(z)}{z^3}$.
  (c) $h(z) = \frac{e^z}{z^3-4z^2+4z}$.

(5) Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n with complex coefficients $a_i \in \mathbb{C}$ $(a_n \neq 0)$.

  (a) Show that given $\epsilon > 0$, there exists $r > 0$ such that for $|z| > r$ we have
$$(|a_n| - \epsilon)|z|^n < |P(z)| < (|a_n| + \epsilon)|z|^n.$$
  (b) Now let $F(z) = P(z)/Q(z)$ be a quotient of two polynomials $P(z)$ and $Q(z)$ of degrees n and m and suppose $n+2 \leq m$. Let C_R denote the circle center the origin with radius R, with the positive orientation (i.e., traversed counterclockwise). Show that
$$\lim_{R\to\infty} \int_{C_R} F(z)dz = 0.$$

(6) Compute the following integrals.

  (a) $\int_\gamma \frac{z^2}{z-1}dz$, where γ is the circle of radius 3, center the origin, with positive orientation.
  (b) $\int_\gamma \frac{e^z}{(z+4)(z-1+i)}dz$, where γ is the circle of radius 1, center the origin, with positive orientation.
  (c) $\int_\gamma \frac{z(z+3)}{(z+i)(z-8)}dz$, where γ is the circle of radius 3, center $2+i$, with positive orientation.

(7) Compute the following real integrals using the residue formula.

  (a) $\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$.
  (b) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx$, where $a > 0$.
  (c) $\int_0^{2\pi} \frac{1}{1+(\sin\theta)^2}d\theta$