Math 797W Algebraic geometry. Homework 3

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(1) Let X be an irreducible topological space and $U \subset X$ an open subset. Show that U is irreducible.

(2) Let $U = \mathbb{A}^2_{x,y} \setminus \{(0,0)\} \subset X = \mathbb{A}^2_{x,y}$. Show that the prevariety U is not isomorphic to an affine variety. [Hint: First show $\mathcal{O}_U(U) := \mathcal{O}_X(U) = k[x,y]$. So, writing $i \colon U \to X$ for the inclusion, the k-algebra homomorphism $i^* \colon \mathcal{O}_X(X) \to \mathcal{O}_U(U)$ is an isomorphism. Now it follows that U cannot be affine (why?)]

(3) Let $n \in \mathbb{Z}$. Let $X = X(n)$ be the prevariety defined as follows: X is the union of two open affine sets $U_1 = \mathbb{A}^2_{x_1,y_1}$ and $U_2 = \mathbb{A}^2_{x_2,y_2}$ glued via the isomorphism

$$U_1 \supset (x_1 \neq 0) \xrightarrow{\sim} (y_2 \neq 0) \subset U_2, (x_1, y_1) \mapsto (x_1^n y_1, x_1^{-1}).$$

(a) Compute $\mathcal{O}_X(X)$.

(b) Show that $X(0) \simeq \mathbb{P}^1 \times \mathbb{A}^1$ and $X(-1) \simeq \mathbb{P}^2 \setminus \{P\}$, where $P \in \mathbb{P}^2$ is a point.

(c) In class we defined a morphism $p \colon X \to \mathbb{P}^1$, given by

$$U_1 \to V_0 = \mathbb{A}^1_{z_{01}}, \quad (x_1, y_1) \mapsto x_1$$

$$U_2 \to V_1 = \mathbb{A}^1_{z_{10}}, \quad (x_2, y_2) \mapsto y_2$$

where $V_i = (Z_i \neq 0) \subset \mathbb{P}^1_{(Z_0:Z_1)}$ and $z_{ij} = Z_j/Z_i$. The morphism p is a locally trivial fibration with fiber $\mathbb{A}^1$. Describe this morphism geometrically for $n = -1$ in terms of the identification $X(-1) \simeq \mathbb{P}^2 \setminus \{P\}$.

(4) Let $Y = Z(f) \subset \mathbb{A}^n_{x_1,\dots,x_n}$ where $f \in k[x_1,\dots,x_n]$ is irreducible. We say the hypersurface Y is *smooth* at a point $P \in Y$ if $\frac{\partial f}{\partial x_i}(P) \neq 0$ for some i. [This definition relies on an algebraic version of the implicit function theorem. The partial derivatives are defined formally via k-linearity, the Leibniz rule, and $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$.] Now let $Y = Z(y^2 - f(x)) \subset \mathbb{A}^2_{x,y}$ where $f(x) \in k[x]$ is a polynomial of degree m. (We assume $f(x) \in k[x]$ is not a square, so that $y^2 - f(x)$ is irreducible.)

(a) Show that X is smooth at each point iff $f(x)$ has no multiple roots.

(b) Let $\overline{Y} \subset \mathbb{P}^2$ denote the closure of Y in

$$\mathbb{P}^2_{(X:Y:Z)} = (Z \neq 0) \cup (Z = 0) = \mathbb{A}^2_{x,y} \cup \mathbb{P}^1_{(X:Y)},$$

where $x = X/Z$, $y = Y/Z$. Show that $\overline{Y} \setminus Y$ is a single point P for $m \geq 3$ and $\overline{Y}$ is not smooth at P for $m \geq 4$.

(c) Let $n = -(m+\delta)/2 \in \mathbb{Z}$, where $\delta = 0$ or 1, $m \equiv \delta \bmod 2$. Let Z be the isomorphic image of Y in $X(n)$ under the map

$$\mathbb{A}^2_{x,y} \xrightarrow{\sim} U_1 = \mathbb{A}^2_{x_1,y_1} \subset X(n), \quad (x,y) \mapsto (x_1, y_1).$$

Let $\overline{Z}$ denote the closure of Z in $X(n)$ Show that $|\overline{Z} \setminus Z| = 2 - \delta$ and $\overline{Z}$ is smooth at each point of $\overline{Z} \setminus Z$.

(d) Assume $k = \mathbb{C}$. Show that $\overline{Z}$ is compact for the Euclidean topology (although $X(n)$ is not compact).

Remark: In particular, the hyperelliptic curve $\overline{Z}$ with affine open set $Y \simeq Z \subset \overline{Z}$ does *not* embed in $\mathbb{P}^2$ as the closure of Y for $m \geq 4$. (In fact, a smooth projective plane curve $W = Z(F) \subset \mathbb{P}^2$ is never hyperelliptic if $d = \deg F > 3$.)

(5) Let $X = Z(f) \subset \mathbb{A}^2_{x,y}$, where $f \in k[x,y]$ is irreducible. Let $P \in X$ be a point and suppose that X is smooth at P. Suppose WLOG that $P = (0,0)$ and $\frac{\partial f}{\partial y}(P) \neq 0$.

(a) Show that the maximal ideal $m_{X,P} \subset \mathcal{O}_{X,P}$ is generated by x. [Hint: Clearly the maximal ideal is generated by x and y. Use the equation f to eliminate y.]

(b) Show that $\bigcap_{n\geq 1} m_{X,P}^n = (0)$. [Hint: Suppose $0 \neq g \in \mathcal{O}_{X,P}$ and $g \in m_{X,P}^n$. Thus $g = x^n \cdot h$, $h \in \mathcal{O}_{X,P}$. Write $g = \frac{a}{b}$, $h = \frac{c}{d}$, where $a, b, c, d \in k[x, y]$ and $b(P), d(P) \neq 0$. There are $\alpha, \beta \in k[x, y]$ and $\gamma \in k[x]$ such that $\alpha f + \beta a = \gamma$ (because f, a are coprime in $k(x)[y]$, cf. HW1 Q11). Write $\gamma = x^l \cdot \delta$, $\delta(0) \neq 0$. Then $(\beta a - \gamma)d \equiv (\beta b c x^n - \delta d x^l) \equiv 0 \bmod f$. Now show by contradiction that $n \leq l$.]

(c) Using (a) and (b), deduce that every nonzero element $0 \neq g \in k(X)$ can be written uniquely in the form $g = x^n \cdot u$, where $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{X,P}$, $u(P) \neq 0$. (In particular, $g \in \mathcal{O}_{X,P}$ if $n \geq 0$ and $g^{-1} \in \mathcal{O}_{X,P}$ if $n \leq 0$.)

(d) Let $0 \neq g_0, \ldots, g_n \in k(X)$. Show that the assignment

$$Q \mapsto (g_0(Q) : g_1(Q) : \ldots : g_n(Q))$$

extends to a morphism

$$g \colon U \to \mathbb{P}^n$$

defined on some open neighborhood U of $P \in X$. [Hint: Use (c) and "clear denominators".]

(e) Give an example of a nonzero element $0 \neq g \in k(\mathbb{A}^2)$ such that $g \notin \mathcal{O}_{\mathbb{A}^2,0}$ and $g^{-1} \notin \mathcal{O}_{\mathbb{A}^2,0}$.

(6) For $n = 1$ there is a morphism $\pi \colon X(1) \to \mathbb{A}^2$ given by

$$U_1 \to \mathbb{A}^2, \quad (x_1, y_1) \mapsto (x_1 y_1, y_1)$$

$$U_2 \to \mathbb{A}^2, \quad (x_2, y_2) \mapsto (x_2, x_2 y_2).$$

The morphism π is called the *blowup* of the point $0 \in \mathbb{A}^2$. The closed set $C = \pi^{-1}(0) \subset X$ is irreducible and isomorphic to $\mathbb{P}^1$, and π restricts to an isomorphism $X \setminus C \xrightarrow{\sim} \mathbb{A}^2 \setminus \{0\}$.

Now suppose $n \geq 1$. Let $\pi \colon X(n) \to \mathbb{A}^{n+1}$ be the morphism defined by

$$U_1 \to \mathbb{A}^{n+1}, \quad (x_1, y_1) \mapsto (x_1^n y_1, x_1^{n-1} y_1, \ldots, y_1)$$

$$U_2 \to \mathbb{A}^{n+1}, \quad (x_2, y_2) \mapsto (x_2, x_2 y_2, \ldots, x_2 y_2^n).$$

(a) Check that π is a well defined morphism.

(b) Show that $C := \pi^{-1}(0)$ is irreducible and isomorphic to $\mathbb{P}^1$.

(c) Let $Y = \pi(X) \subset \mathbb{A}^{n+1}_{z_0,\ldots,z_n}$. Show that Y is the affine variety defined by the prime ideal $J \subset k[z_0,\ldots,z_n]$ generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}$$

In particular, in the case $n = 2$, Y is the hypersurface

$$Z(z_0 z_2 - z_1^2) \subset \mathbb{A}^3_{z_0,z_1,z_2}$$

studied in HW1 Q9. [Hint: Compare HW2 Q8. $Y \subset \mathbb{A}^{n+1}$ is the affine cone over the rational normal curve of degree n in $\mathbb{P}^n$.]

(d) Show that $X \setminus C \to Y \setminus \{0\}$ is an isomorphism.

(e) Show that $\pi^* \colon k[Y] = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is an isomorphism. [Hint: $\overline{\pi(X)} = Y$ implies π^* is injective (why?). Now use your explicit computation of $\mathcal{O}_X(X)$ from Q3(a) to show π^* is surjective.]

Remark: Note that $0 \in Y$ is a singular point for $n > 1$ (singular means not smooth). The morphism $\pi \colon X \to Y$ is a *resolution* of the singularity $0 \in Y$.