Math 462 Homework 5

Paul Hacking

February 27, 2013

(1) Let R_1 and R_2 be the isometries of $\mathbb{R}^2$ given by reflection in lines L_1 and L_2. Suppose L_1 and L_2 are parallel, and let $\mathbf{c} \in \mathbb{R}^2$ such that $\mathbf{c}$ is orthogonal to the two lines and L_2 is obtained from L_1 by the translation $T(\mathbf{x}) = \mathbf{x} + \mathbf{c}$. Show that $R_2 \circ R_1$ is the translation $S(\mathbf{x}) = \mathbf{x} + 2\mathbf{c}$.

[Hint: Either choose coordinates carefully and compute the composition algebraically, or argue geometrically.]

(2) Let R_1 and R_2 be the isometries of $\mathbb{R}^3$ given by reflection in planes $\Pi_1 \subset \mathbb{R}^3$ and $\Pi_2 \subset \mathbb{R}^3$. Assume that the planes Π_1 and Π_2 are not parallel, so that they intersect in a line L.

(a) Explain why the composition $R_2 \circ R_1$ is a rotation with axis the line L through angle 2θ, where θ is the dihedral angle from Π_1 to Π_2 (measured in a plane orthogonal to L). [Hint: Choose coordinates carefully and use the analogous result for reflections in $\mathbb{R}^2$.]

(b) Now let $\Pi_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y + 3z = 0\}$ and $\Pi_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$. Describe the composite $R_2 \circ R_1$ explicitly (specify the axis and angle of rotation).

(3) Compute the composition $R_2 \circ R_1$ of the following pairs of rotations R_1,R_2 geometrically (using a triangle in the plane or a triangle on the sphere as we did in class).

(a) $R_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is rotation about the origin through angle $\pi/2$ counterclockwise, $R_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is rotation about the point $(1, 1)$ through angle $\pi/2$ counterclockwise.

(b) $R_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the x-axis through angle π, $R_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the z-axis through angle π.

(c) $R_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the x-axis through angle $\pi/2$ counterclockwise, $R_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the z-axis through angle $\pi/2$ counterclockwise. [Hint: How is the spherical triangle for this case related to the triangle for part (b)?]

(4) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a direct isometry. Thus $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix with determinant 1

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and $\mathbf{b} \in \mathbb{R}^2$ is a vector. Define a continuous family T_t of isometries parametrized by $t \in [0,1]$ such that T_0 equals the identity and $T_1 = T$ by the formulas

$$T_t(\mathbf{x}) = \begin{pmatrix} \cos(2t\theta) & -\sin(2t\theta) \\ \sin(2t\theta) & \cos(2t\theta) \end{pmatrix} \mathbf{x}$$

for $0 \leq t \leq 1/2$ and

$$T_t(\mathbf{x}) = A\mathbf{x} + (2t-1)\mathbf{b}$$

for $1/2 \leq t \leq 1$.

Describe the isometry T_t geometrically for each value of $t \in [0,1]$.

(5) We say a triple of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ is *right handed* if the determinant of the matrix M with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is positive.

(a) Check that the triple $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ given by the unit vectors along the coordinate axes is right handed.

(b) Now suppose A is a 3×3 invertible matrix and consider the corresponding invertible linear map $T \colon \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbf{x}) = A\mathbf{x}$. Show that T sends any right handed triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to another right handed triple $T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)$ iff $\det(A) > 0$.

(6) Let S be the square $S = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x, y \leq 1\} \subset \mathbb{R}^2$. We say an isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetry of S if $\mathbf{x} \in S \Rightarrow T(\mathbf{x}) \in S$.

(a) Explain why any symmetry T of S must fix the origin.

(b) List all the symmetries of S, giving both a geometric and algebraic description of each symmetry. [Note: Each symmetry has the algebraic form $T(\mathbf{x}) = A\mathbf{x}$ where A is a 2×2 orthogonal matrix (why?).]

(7) (Optional) Repeat Q6 for the cube

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x, y, z \leq 1\} \subset \mathbb{R}^3.$$

[Hint: It is helpful to know the order of the group G of symmetries to check your list is complete. This can be computed using the orbit-stabilizer theorem from Math 411.]

(8) (Optional) Suppose that the sphere S^2 is subdivided into spherical polygons. Let V, E, F denote the total number of vertices, edges, and faces in the subdivision. Prove Euler's formula $V - E + F = 2$. [Hint: Compute the sum of all the angles of the spherical polygons in two ways, using HW1Q4(b).]