12/4/19

HW 8 due Friday at start of class

HW 7 returned 2 4ab 5abc 6abcd

5 5+5 5+5+5 5+5+5+5 +5 = 55.

Office hours tomorrow, 5–6PM, LGRT 1235H

Last Time: • Stereographic projection preserves angles
→ algebraic/calculus proof

• $\gamma : [a,b] \to S^2$ parametrized curve on S^2,
$F : S^2 \setminus \{N\} \to \mathbb{R}^2$ stereographic projection
$\gamma(t) = (x,y,z)(t)$, $F(\gamma(t)) = (u,v)(t)$
$\rightsquigarrow$ length$(\gamma) = \int_a^b \sqrt{x'^2+y'^2+z'^2}\,dt$
$= \int_a^b \frac{2}{u^2+v^2+1} \sqrt{u'^2+v'^2}\cdot dt$

compare w/ length $(F(\gamma)) = \int_a^b \sqrt{u'^2+v'^2}\,dt$

stereographic projection sends spherical circles to circles & lines in $\mathbb{R}^2$:

$C = \Pi \cap S^2$, $\Pi \not\ni N$

$\Rightarrow F(C): \left(u+\frac{a}{c-d}\right)^2 + \left(v+\frac{b}{c-d}\right)^2 = \frac{a^2+b^2+c^2-d}{(c-d)^2}$

$(u-\alpha)^2 + (v-\beta)^2 = r^2$

circle, center $(\alpha,\beta) = \left(\frac{-a}{c-d}, \frac{-b}{c-d}\right)$,

radius $r = \frac{\sqrt{a^2+b^2+c^2-d^2}}{|c-d|}$ HW8Q5

there was a typo in last time's notes

Today: Hyperbolic geometry

Hyperbolic geometry

The Euclidean plane $\mathbb{R}^2$ & the sphere have lots of symmetry: – they look the same at every point and in every direction.

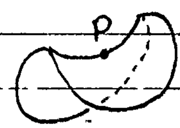
— more precisely, given two points $P, Q \in X = \mathbb{R}^2$ or S^2 & two tangent vectors $\underline{v}$ at P & $\underline{w}$ at Q, there is an isometry of X sending P to Q & $\underline{v}$ to $c \cdot \underline{w}$, for some $c > 0$.

(Ex: in $\mathbb{R}^2$, translate P to Q, then apply rotation at Q)

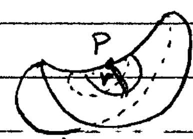
There is another space X with this property, the <u>hyperbolic plane</u>, $\mathbb{H}^2$.

Hard to visualize (does not "fit" inside $\mathbb{R}^3$ although a small portion does)

Roughly, near a point $P \in \mathbb{H}^2$, looks like a saddle or Pringles chip.

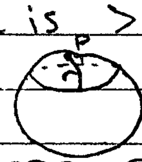


In particular, if we consider a circle, center P & radius r on $\mathbb{H}^2$



$C = \{ Q \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(P,Q) = r \}$

See circumference of C is $> 2\pi r$

(Contrast with S^2  circumference $< 2\pi r$)

"negative curvature" Math 563H

In fact, circumference of $C(P,r)$ grows exponentially with r!

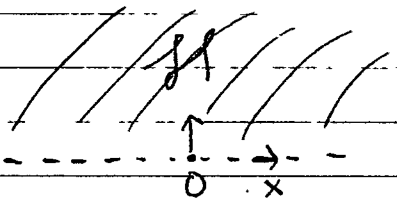
How can we study $\mathbb{H}^2$?

<u>Use a "map" of $\mathbb{H}^2$!</u>

"Upper half plane model"

$$F: \mathbb{H}^2 \longrightarrow \mathcal{H} \subset \mathbb{R}^2$$

bijection $\quad \mathcal{H} = \{(x,y) \mid y > 0\}$



Compare w/ Euclidean length: $\int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt$

hyperbolic distance on $\mathcal{H}$ given by :-

$\gamma : [a,b] \to \mathcal{H}$

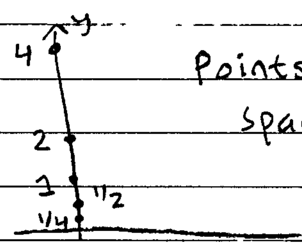
$\gamma(t) = (x(t), y(t))$

hyperbolic length of $\gamma := \int_a^b \frac{1}{y(t)} \cdot \sqrt{x'(t)^2+y'(t)^2}\,dt$

hyperbolic distance $d_{\mathcal{H}}(P,Q)$ = length of shortest path from P to Q (for hyperbolic length)

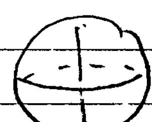
Note: hyperbolic distance >> Euclidean distance when P & Q are close to x-axis

Ex:



Points $(0, 2^n)$, n integer, are equally spaced for hyperbolic distance (distance $= \ln 2$)

Aside [



$$\int_a^b \frac{2}{u^2+v^2+1} \sqrt{u'^2+v'^2}\,dt$$

↳ no distortion when $u^2+v^2=1$

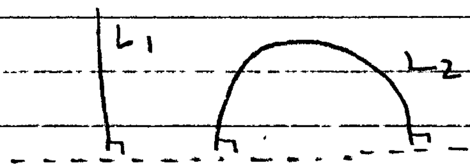
↳ this happens on the equator

* for hyperbolic geometry this occurs at $y=1$.]

Q: What do the shortest paths look like in the upper half plane model $\mathcal{H}$ of $\mathbb{H}^2$?

A: Define a $\boxed{\text{hyperbolic line}}$ L to be either:

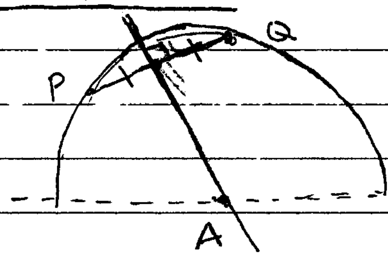
1. A vertical half line or
2. A semicircle with center on the x-axis



Theorem 1. Given $P, Q \in \mathcal{H}$ there's a unique hyperbolic line through P & Q.

2. The shortest path from P to Q is given by the segment of this hyperbolic line from P to Q.

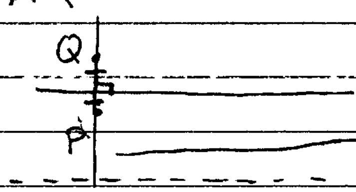
Proof of 1.



Take circle, center A; radius $|AP| = |AQ|$

$\rightarrow$ this is the hyperbolic line

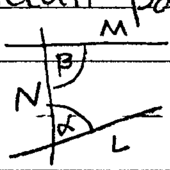
OR



$\rightarrow$ then this is the hyperbolic line

Proof of 2. later

Recall parallel axiom of Euclidean geometry:-



$\alpha + \beta < \pi \Rightarrow$ L & M must meet on this side of N

$\rightarrow$ This axiom fails in hyperbolic geometry but all the other axioms hold $\rightsquigarrow$ finally shows parallel axiom is needed in Euclidean geometry.