697B Example Sheet 6

Paul Hacking

1 November 2010

(1) Show that $f(x,y) = y^2 - x^2 - x^3$ is irreducible in $\mathbb{C}[x,y]$ but reducible in $\mathbb{C}\{x\}[y]$.

(2) Find the intersection number of the curves $X = (f(x,y) = 0) \subset \mathbb{C}^2_{x,y}$ and $Y = (g(x,y) = 0) \subset \mathbb{C}^2_{x,y}$ at the point $P = (0,0)$ in the following cases. (In cases (a) and (b) $\lambda \in \mathbb{C}$ is a scalar and the answer depends on λ.)

(a) $f(x,y) = y^2 - x^2 - x^3$, $g(x,y) = y - \lambda x$.

(b) $f(x,y) = y^2 - x^3$, $g(x,y) = y - \lambda x$.

(c) $f(x,y) = y^2 - x^3$, $g(x,y) = y^5 - x^2$.

(3) Let $X, Y \subset \mathbb{C}^2_{x,y}$ be plane curves passing through a point P. Suppose that X and Y are smooth at P. Show that $(X \cdot Y)_P = 1$ iff the tangent lines to X and Y at P are distinct. (In this case, we say X and Y intersect transversely at P.) [Reminder: If $P \in X = (f(x,y) = 0) \subset \mathbb{C}^2_{x,y}$ then the tangent line to X at P is given by

$$T_P X = \left(\frac{\partial f}{\partial x}(P)(x - x(P)) + \frac{\partial f}{\partial y}(P)(y - y(P)) = 0 \right) \subset \mathbb{C}^2_{x,y}.]$$

(4) Let $P = (0,0) \in X = (f(x,y) = 0) \subset \mathbb{C}^2_{x,y}$. Write $f = f_m + f_{m+1} + \cdots$ where f_i is homogeneous of degree i in x, y and $f_m \neq 0$. We say that the *multiplicity* $\operatorname{mult}_P X$ of $P \in X$ equals m.

(a) Show that X is smooth at P iff $m = 1$.

(b) Suppose that the multiplicity of X at P is equal to the degree of f. Describe the curve X.

(c) Show that after a linear change of coordinates x, y, we can write $f = u \cdot w$ where $u \in \mathbb{C}\{x, y\}$ is a unit and $w \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial of degree m. (You are allowed to assume the Weierstrass preparation theorem here (see Griffiths II.4), the point is to show that we can choose coordinates so that the Weierstrass polynomial w has minimal degree.)

(d) Let $X = (f(x, y) = 0) \subset \mathbb{C}^2_{x,y}$ and $Y = (g(x, y) = 0) \subset \mathbb{C}^2_{x,y}$ be two plane curves passing through $P = (0, 0)$. Show that

$$(X \cdot Y)_P \geq \operatorname{mult}_P X \cdot \operatorname{mult}_P Y.$$

(5) (a) Show that an irreducible plane curve $X \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d cannot have $\lfloor d/2 \rfloor + 1$ collinear singular points. [Hint: Use Bézout's theorem and Q4.]

(b) Show that there is a conic (not necessarily irreducible) passing through any set of 5 points in $\mathbb{P}^2$. [Hint: This is a linear algebra problem.]

(c) Show that an irreducible plane quartic cannot have 4 singular points. [Hint: Use part (b)]

(d) Give an example of a plane curve of degree d (*not* assumed irreducible) with $d(d-1)/2$ singular points.

(6) (Projection of a plane curve from a point.)

(a) Let $X \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve of degree d. Suppose that $P = (1 : 0 : 0) \notin X$. Show that the assignment

$$(X : Y : Z) \mapsto (Y : Z) \tag{1}$$

defines a holomorphic map $F \colon X \to \mathbb{P}^1_{\mathbb{C}}$ of degree d. Show that the fiber $F^{-1}(\alpha : \beta)$ is equal to $X \cap L$ where L is the line through P and $(0 : \alpha : \beta)$. Show that the intersection number $(X \cdot L)_Q$ equals the ramification index e_Q of F at Q for $Q \in F^{-1}(\alpha : \beta)$.

(b) Now suppose $P \in X$. Show that (1) defines a holomorphic map $F \colon X \to \mathbb{P}^1_{\mathbb{C}}$ of degree $d - 1$. Give a geometric description of the fibers of F analogous to that in part (a). What is $F(P)$?

(c) Finally suppose that X has a node at P and is smooth elsewhere. Let $n \colon \tilde{X} \to X$ be the normalization of X. Show that (1) defines a holomorphic map $F \colon X \setminus \{P\} \to \mathbb{P}^1_{\mathbb{C}}$ that extends to a holomorphic map $\tilde{F} \colon \tilde{X} \to \mathbb{P}^1_{\mathbb{C}}$ of degree $d - 2$.