697B Example Sheet 4

Paul Hacking

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(1) Consider the map $F\colon \mathbb{P}^1_{(X_0:X_1)} \to \mathbb{P}^n_{(Y_0:\cdots:Y_n)}$ given by

$$(X_0 : X_1) \mapsto (X_0^n : X_0^{n-1}X_1 : X_0^{n-2}X_1 : \cdots : X_1^n).$$

The image X of F is called the *rational normal curve of degree n*.

(a) Show that F is well defined and holomorphic.

(b) Show that F is an isomorphism onto its image X. [Hint: Show first that F is injective. Then describe charts for X, and check that the inverse of $F\colon \mathbb{P}^1 \to X$ is holomorphic.]

(c) Let $P_0, \ldots, P_n$ be $n+1$ distinct points on $\mathbb{P}^1$. Show that their images $F(P_0), \ldots, F(P_n) \in \mathbb{P}^n$ are not contained in a hyperplane. [Hint: Use the Vandermonde determinant.]

(d) Let $H \subset \mathbb{P}^n$ be a general hyperplane. What is the size of $H \cap X$?

(e) Show that X is contained in the quadric hypersurfaces $Q_{ij} = (Y_{i-1}Y_j - Y_iY_{j-1} = 0) \subset \mathbb{P}^n$, $1 \le i < j \le n$, and that $X = \bigcap_{i,j} Q_{ij}$.

(f) Now suppose $n = 3$. Because X has dimension 1 and $\mathbb{P}^3$ has dimension 3, we might expect that $X \subset \mathbb{P}^3$ can be defined by $2 = 3 - 1$ homogeneous equations. We know from part (d) that $X = Q_{12} \cap Q_{23} \cap Q_{13}$. What is $Q_{12} \cap Q_{23}$? Can X be defined by 2 equations (this is harder and may be omitted)?

(2) Consider the map $F\colon \mathbb{P}^1_{(X_0:X_1)} \times \mathbb{P}^1_{(Y_0:Y_1)} \to \mathbb{P}^3_{(Z_0:Z_1:Z_2:Z_3)}$ given by

$$((X_0 : X_1), (Y_0 : Y_1)) \mapsto (X_0Y_0 : X_0Y_1 : X_1Y_0 : X_1Y_1).$$

(a) Show that F is well defined and holomorphic, and is an isomorphism onto its image $X \subset \mathbb{P}^3$.

(b) Find the homogeneous equation of the hypersurface $X \subset \mathbb{P}^3$.

(c) Recall that a *line* $L \subset \mathbb{P}^3$ is the locus $L \simeq \mathbb{P}^1$ corresponding to a 2-dimensional subspace $V \subset \mathbb{C}^4$ under the quotient map $(\mathbb{C}^4 \setminus \{0\}) \to \mathbb{P}^3$. Equivalently, L is the closure of an affine line $\mathbb{C} \to \mathbb{C}^3$, $t \mapsto \mathbf{a} + t\mathbf{b}$ in some chart $\mathbb{C}^3 \subset \mathbb{P}^3$. Find all the lines $L \subset \mathbb{P}^3$ which are contained in X. Explain the origin of these lines in terms of the map F. [Hint: Use the equation of X from part (b).]

(3) Consider the map $F\colon \mathbb{P}^2_{(X_0:X_1:X_2)} \to \mathbb{P}^5_{(Y_0:\cdots:Y_5)}$ given by

$$(X_0 : X_1 : X_2) \mapsto (X_0^2 : X_1^2 : X_2^2 : X_0X_1 : X_1X_2 : X_0X_2).$$

The image X of F is called the *Veronese surface*.

(a) Show that F is well defined and holomorphic, and is an isomorphism onto its image.

(b) Show that $X \subset \mathbb{P}^5$ is defined by the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} Y_0 & Y_3 & Y_5 \\ Y_3 & Y_1 & Y_4 \\ Y_5 & Y_4 & Y_2 \end{pmatrix}$$

(c) Let $H \subset \mathbb{P}^5$ be a hyperplane and consider the locus $H \cap X$. What does $H \cap X$ correspond to under the isomorphism $F\colon \mathbb{P}^2 \xrightarrow{\sim} X$?

(4) Let $X = (F = G = 0) \subset \mathbb{P}^3$ where

$$F = X^2 + Y^2 + Z^2 + T^2, \quad G = aX^2 + bY^2 + cZ^2 + dT^2,$$

and G is not a multiple of F. Find necessary and sufficient conditions on a, b, c, d for X to be smooth.