Math 300.3 Homework 3

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Reading: Sundstrom, Sections 3.1 and 3.2.

Justify your answers carefully.

(1) Let a be a non-zero integer and b an integer. We say a *divides* b and write $a \mid b$ if $b = qa$ for some integer q. Prove the following statements carefully.

   (a) For all non-zero integers a and b and integers c and d, if $a \mid c$ and $b \mid d$ then $ab \mid cd$.

   (b) For all non-zero integers a and integers b, c and d, if $a \mid b$, $a \mid c$ and $a \mid d$ then $a^2 \mid bc - d^2$.

(2) (a) For a real number x the *absolute value* $|x|$ of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0. \\ -x & \text{if } x < 0. \end{cases}$$

   Prove the following statement: For all non-zero integers a and b, if $a \mid b$ then $|a| \leq |b|$.

   [Hint: For all real numbers x and y, $|xy| = |x| \cdot |y|$. For all integers x, either $x = 0$ or $|x| \geq 1$ (why ?).]

   (b) Using part (a) or otherwise, prove the following statement: For all non-zero integers a and b, if $a \mid b$ and $b \mid a$ then $a = \pm b$.

(3) Prove the following statement: For all real numbers a and b,

$$\frac{a^2 + b^2}{2} \geq ab.$$

[Hint: For all real numbers x, $x^2 \geq 0$. Show that the inequality above can be rewritten in this form (where x depends on a and b).]

(4) Prove or give a counterexample for each of the following statements.

(a) For all integers a, if $a^2 \equiv 1 \bmod 8$ then $a \equiv 1 \bmod 8$ or $a \equiv -1 \bmod 8$.

(b) For all positive integers n, $2n^2 + 5$ is prime.

(c) For all positive integers n, there exist integers x and y such that $n = x^2 + 2y^2$.

(5) Prove the following statement: For all non-zero integers a and c and integers b, $ac \mid bc$ if and only if $a \mid b$.

[Reminder: To prove a biconditional statement "P if and only if Q" (also written $P \iff Q$) we must show $P \Rightarrow Q$ *and* $Q \Rightarrow P$.]

(6) (a) Prove the following statement: For all integers a, the last digit of a^2 is either 0,1,4,5,6, or 9.

(b) We say an integer n is a *perfect square* if $n = m^2$ for some integer m. Is the integer 18446744073709551617 a perfect square?

[Hint: For a non-negative integer a, the last digit of a is equal to the remainder r when we divide a by 10, so $a \equiv r \bmod 10$. In class we showed that for all integers a, b, c, d and positive integers n, if $a \equiv b \bmod n$ and $c \equiv d \bmod n$ then $ac \equiv bd \bmod n$. As the special case $a = c$ and $b = d$ of this result, we deduce that for all integers a, b and positive integers n, if $a \equiv b \bmod n$ then $a^2 \equiv b^2 \bmod n$.]

(7) Prove the following statement: For all integers a, $a^2 \equiv 0, 1$ or $4 \bmod 8$.

(8) Let n be a positive integer.

(a) Prove the following statement: For all integers x,

$$(n - x)^2 \equiv x^2 \bmod n.$$

(b) We say an integer r is a *quadratic residue* modulo n if $0 \leq r < n$ and there exists an integer x such that $x^2 \equiv r \bmod n$. Using part (a) or otherwise, show that the number of quadratic residues modulo n is at most $(n+1)/2$ if n is odd and at most $n/2 + 1$ if n is even.

(c) Give a proof or a counterexample for the following statement: For every positive integer n, the number of quadratic residues modulo n is equal to $(n+1)/2$ if n is odd and is equal to $n/2+1$ if n is even.