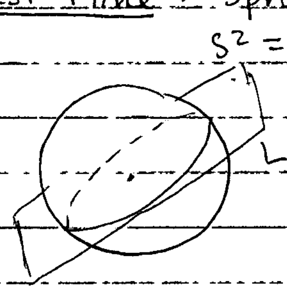
11/6/19

HW 5 due at start of Friday's class
Office hours Thurs. 5-6 PM LGRT 1235H

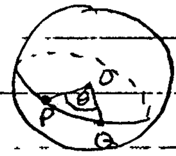
Last Time : Spherical Geometry

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ sphere, center the origin, radius 1.



spherical line : $L = \Pi \cap S^2$,
(great circle)
Π plane through the origin

- Given P & $Q \in S^2$ there's a spherical line through P & Q, unique unless P & Q antipodal
- 2 spherical lines intersect in a pair of antipodal points



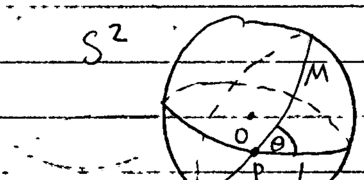
Fact : The shortest path from P to Q on S^2 is a segment of a spherical line.

Spherical distance $d_{S^2}(P,Q)$ = length of shortest path $= \theta = \cos^{-1}(\vec{OP} \cdot \vec{OQ})$

Today : - Angles between spherical lines
- Spherical triangles
- Angle sum of sph. triangle (spherical cosine rule)

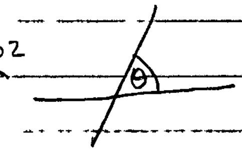
Angle between spherical lines.

S^2



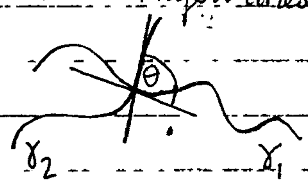
L, M spherical lines

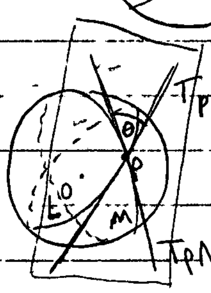
$\mathbb{R}^2$



Angle between two curves is defined as the angle between their tangent lines.

consider :





T_pL = tangent line to L at $P \subset \mathbb{R}^3$

$T_PL, T_PM \subset T_pS^2$ = tangent plane to sphere at P $\subset \mathbb{R}^3$

T_PM

How to compute the angle θ?

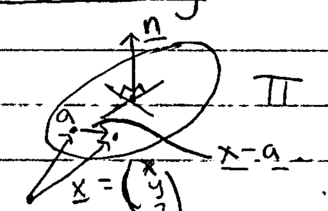
Spherical line $L = \Pi \cap S^2$; Π plane through origin in $\mathbb{R}^3$.

What's the "equation" of a spherical line? — What's the equation of Π?

A plane in $\mathbb{R}^3$ is given by a linear equation $ax+by+cz=d$
$a,b,c,d \in \mathbb{R}$ constants.

Π passes through the origin $\Leftrightarrow d=0$. (plug in $x=y=z=0$)

Geometrically



$\underline{n}$ – "normal vector to Π"
(uniquely determined up to $\underline{n} \rightsquigarrow c \cdot \underline{n}$, $0 \neq c \in \mathbb{R}$)

$(\underline{x}-\underline{a}) \cdot \underline{n} = 0$ (because $\underline{x}-\underline{a} \perp \underline{n}$, because $\underline{x}-\underline{a}$ is a vector in the plane)

$\underline{x} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

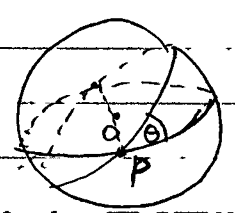
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$ $\longrightarrow$ say $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ & $\underline{a} \cdot \underline{n} = d$

$ax+by+cz=d$

In particular, if $\Pi: ax+by+cz=d$ then $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a normal vector to Π.

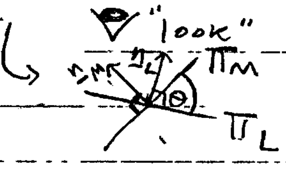
Ex: $L = \Pi_L \cap S^2$ $\quad \Pi_L: x+y+z=0$
$M = \Pi_M \cap S^2$ $\quad \Pi_M: x+2y+3z=0$

Angle between Π_L & Π_M?
$\theta = ?$



So, $\underline{n}_L \cdot \underline{n}_M = \|\underline{n}_L\| \cdot \|\underline{n}_M\| \cdot \cos\theta$

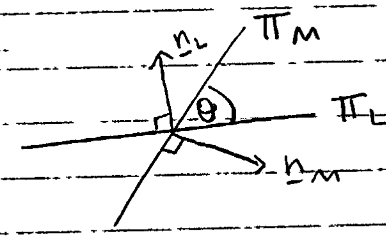
$$\theta = \cos^{-1}\left(\frac{\underline{n}_L \cdot \underline{n}_M}{\|\underline{n}_L\| \cdot \|\underline{n}_M\|}\right)$$



Know: $\underline{n}_L = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\underline{n}_M = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

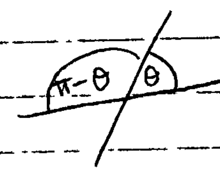
$$= \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{1^2+1^2+1^2} \cdot \sqrt{1^2+2^2+3^2}}\right) = \cos^{-1}\left(\frac{6}{\sqrt{42}}\right)$$

Remark Could have:



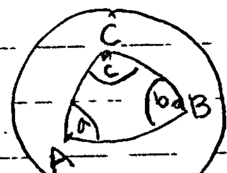
angle between n_L & n_M is $\pi - \theta$

BUT

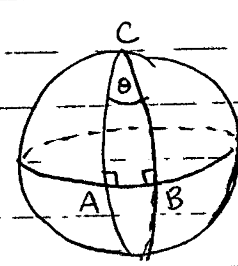


A $\boxed{\text{spherical triangle}}$ $\triangle ABC$. Let $A, B, C \in S^2$ such that they do not line on a spherical line (equivalently, don't lie on a plane Π through origin).

A spherical triangle is defined to be the figure in S^2 formed by the shortest paths from A to B, B to C, & C to A.



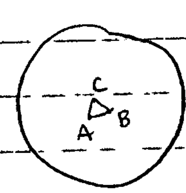
Q: What is the angle sum of a spherical triangle?



2 lines of longitude & the equator } great circles / spherical lines

Angle sum of $\triangle ABC = \frac{\pi}{2} + \frac{\pi}{2} + \theta = \pi + \theta$

In particular, not constant independent of the triangle. (as it is in $\mathbb{R}^2$)



angle sum of $\triangle ABC \approx \pi$
if $\triangle ABC$ is small relative to radius of S^2. (= 1 for us)

Theorem $\triangle ABC$ is a spherical triangle, then the angle sum
$a + b + c = \pi + \text{Area}(\triangle ABC)$.