Wednesday 12/2/15

MATH 421 HW8 Solutions

1a. $$\frac{e^z}{z^4} = \frac{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots}{z^4}$$

$$= z^{-4} + z^{-3} + \frac{z^{-2}}{2!} + \frac{z^{-1}}{3!} + \cdots$$

Pole of order 4.

$$\operatorname{Res}_{z=0}\left(\frac{e^z}{z^4}\right) = \frac{1}{3!} = \frac{1}{6}.$$

b. $$\frac{\sin z - z}{z^3} = \frac{\left(z - z^3/3! + z^5/5! - \cdots\right) - z}{z^3}$$

$$= \frac{-z^3/3! + z^5/5! - \cdots}{z^3} = -1/3! + z^2/5! - \cdots$$

Removable singularity.

$$\operatorname{Res}_{z=0}\left(\frac{\sin z - z}{z^3}\right) = 0.$$

c. $$\frac{1-\cos(2z)}{z^5} = \frac{1-\left(1 - (2z)^2/2! + (2z)^4/4! - \cdots\right)}{z^5}$$

$$= \frac{2^2/2! \cdot z^2 - 2^4/4!\, z^4 + \cdots}{z^5}$$

$$= 2z^{-3} - 2/3\, z^{-1} + \cdots$$

Pole of order 3

$$\operatorname{Res}_{z=0}\left(\frac{1-\cos(2z)}{z^5}\right) = -2/3$$

d. $$\frac{1}{z^3(z-1)} = \frac{-1}{z^3}\cdot\frac{1}{1-z} = \frac{-1}{z^3}\left(1+z+z^2+\cdots\right)$$

$$= -z^{-3} - z^{-2} - z^{-1} - \cdots$$

Pole of order 3. $\operatorname{Res}_{z=0}\left(\frac{1}{z^3(z-1)}\right) = -1.$

e. $$\sin\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right) - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \cdots$$

$$= z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \cdots$$

Essential singularity.

$\operatorname{Res}_{z=0}\left(\sin\left(\frac{1}{z}\right)\right) = 1.$

2. a $$f(z) = \frac{z}{z^2+1}$$

Singularities: $z^2+1=0 \iff z = \pm i.$

$$\operatorname{Res}_{z=i} \frac{z}{z^2+1} = \operatorname{Res}_{z=i} \frac{z}{(z-i)(z+i)} = \operatorname{Res}_{z=i} \frac{g(z)}{z-i} = g(i) = 1/2.$$

where $g(z) = \frac{z}{z+i}$, cx diff'ble at $z=i$, $g(i) \neq 0$

Similarly, $$\operatorname{Res}_{z=-i} \frac{z}{z^2+1} = \operatorname{Res}_{z=-i} \frac{h(z)}{z+i} = h(-i) = 1/2.$$

$$h(z) = \frac{z}{z-i}$$

b. $$f(z) = \frac{e^z}{z^2-4z+3}$$

Singularities: $z^2-4z+3=0 \iff (z-1)(z-3)=0 \iff z=1,3.$

$$\operatorname{Res}_{z=1} f(z) = \operatorname{Res}_{z=1} \frac{\left(e^z/z-3\right)}{z-1} = \frac{e^1}{1-3} = -1/2\, e$$

$$\operatorname{Res}_{z=3} f(z) = \operatorname{Res}_{z=3} \frac{\left(e^z/z-1\right)}{z-3} = \frac{e^3}{3-1} = 1/2\, e^3$$

c. $f(z) = \dfrac{\mathrm{Log}(z)}{z-e}$

Singularities : $z \in (-\infty, 0]$ and $z = e$.

Only $z = e$ is isolated.

$$\mathrm{Res}_{z=e} \frac{\mathrm{Log}(z)}{z-e} = \mathrm{Log}(e) = 1.$$

d. $f(z) = \dfrac{z^3+1}{(z-i)^2}$

Singularities: $z = i$

$$\mathrm{Res}_{z=i} f(z) = \mathrm{Res}_{z=i} \frac{(z^3+1)}{(z-i)^2} = g'(i) = -3.$$

$g(z) = z^3+1$

$g'(z) = 3z^2$

e $f(z) = \dfrac{e^{3z}}{(z-2)^5}$

Singularities: $z = 2$

$$\mathrm{Res}_{z=2} f(z) = \mathrm{Res}_{z=2} \frac{e^{3z}}{(z-2)^5} = \frac{1}{4!} \cdot g^{(4)}(2) = \frac{3^4 \cdot e^6}{4!}$$

$$= \frac{27}{8} e^6$$

$g(z) = e^{3z}$

$\Rightarrow g^{(4)}(z) = 3^4 \cdot e^{3z}$

f. $f(z) = \dfrac{1}{z^5 - 4z^4 + 4z^3}$

$z^5 - 4z^4 + 4z^3 = z^3(z^2 - 4z + 4) = z^3(z-2)^2.$

$\therefore$ Singularities : $z=0$, $z=2$

$$\operatorname{Res}_{z=0} f(z) = \operatorname{Res}_{z=0} \frac{1/(z-2)^2}{z^3} = \frac{1}{2!} g^{(2)}(0)$$

$$g(z) = 1/(z-2)^2$$

$$g'(z) = -2/(z-2)^3 \qquad g''(z) = \frac{-2 \cdot -3}{(z-2)^4} = \frac{6}{(z-2)^4}$$

$$\therefore \quad \operatorname{Res}_{z=0} f(z) = \frac{1}{2} \cdot \frac{6}{(-2)^4} = \frac{3}{16}$$

$$\operatorname{Res}_{z=2} f(z) = \operatorname{Res}_{z=2} \frac{1/z^3}{(z-2)^2} = h'(2) = -\frac{3}{16}.$$

$$h(z) = 1/z^3$$
$$h'(z) = -3/z^4$$

3. a $\quad f(z) = \tan(z) = \dfrac{\sin(z)}{\cos(z)}$

Singularities: $\cos(z) = 0 \iff z = \pi/2 + k\pi$, k integer

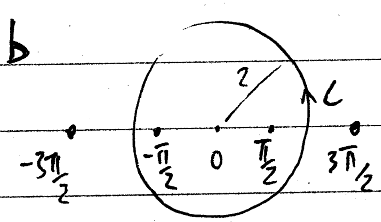
$$\sin(\pi/2 + k\pi) = (-1)^k \neq 0.$$
$$\cos'(\pi/2 + k\pi) = -\sin(\pi/2 + k\pi) = -(-1)^k \neq 0.$$

So have simple pole at $z = \pi/2 + k\pi$ for each k,

and

$$\operatorname{Res}_{z=\pi/2+k\pi} \tan(z) = \frac{\sin(\pi/2 + k\pi)}{\cos'(\pi/2+k\pi)} = -1 \text{ for each } k$$

b



$$\int_C \tan z \, dz \underset{RT}{=} 2\pi i \left(\operatorname{Res}_{z=-\pi/2} \tan z + \operatorname{Res}_{z=\pi/2} \tan z \right)$$
$$= 2\pi i (-1 + -1) = -4\pi i$$

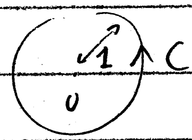
4. $f(z) = z^3 e^{1/z}$ has one singularity at $z=0$.

Laurent series $f(z) = z^3\left(1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots\right)$

$$= z^3 + z^2 + \frac{z}{2} + \frac{1}{6} + \frac{z^{-1}}{24} + \cdots$$

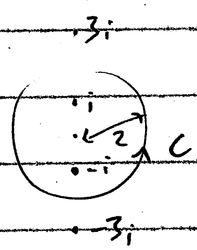
Essential singularity, $\operatorname{Res}_{z=0} z^3 e^{1/z} = 1/24.$

$$\therefore \int_C f(z)\,dz \underset{RT}{=} 2\pi i \cdot \operatorname{Res}_{z=0} f(z) = \frac{\pi i}{12}$$



5. $f(z) = \dfrac{z+1}{(z^2+1)(z^2+9)}$

Singularities : $(z^2+1)(z^2+9) = 0 \iff z = \pm i, \pm 3i$



$$\int_C f(z)\,dz \overset{RT}{=} 2\pi i\left(\operatorname{Res}_{z=i} f(z) + \operatorname{Res}_{z=-i} f(z)\right)$$

$\pm i$ inside C, $\pm 3i$ outside C

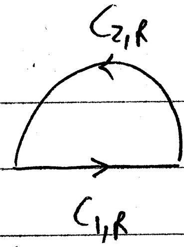
$$\operatorname{Res}_{z=i} f(z) = \operatorname{Res}_{z=i} \frac{z+1/(z+i)(z^2+9)}{z-i}$$

$$= \frac{i+1}{2i\cdot(-1+9)} = \frac{1-i}{16}$$

$$\operatorname{Res}_{z=-i} f(z) = \operatorname{Res}_{z=-i} \frac{z+1/(z-i)(z^2+9)}{z+i} = \frac{-i+1}{-2i(-1+9)} = \frac{+1+i}{16}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left(\frac{1-i}{16} + \frac{1+i}{16}\right) = 2\pi i \cdot \frac{2}{16} = \frac{\pi i}{4}.$$

6.a.



$$C_R = C_{1,R} + C_{2,R}$$

$R > 1.$ $\quad f(z) = \dfrac{e^{iz}}{z^2+1}$ $\quad$ Singularities $z = \pm i.$

$z = i$ is inside C_R, $\quad z = -i$ is outside C_R.

$$\therefore \int_{C_R} f(z)\,dz \overset{RT}{=} 2\pi i \cdot \operatorname{Res}_{z=i} f(z)$$

$$= 2\pi i \cdot \operatorname{Res}_{z=i} \frac{\left(e^{iz}/z+i\right)}{z-i}$$

$$= 2\pi i \cdot \frac{e^{i \cdot i}}{i+i} = \frac{2\pi i\, e^{-1}}{2i} = \pi/e$$

b $\quad$ For $|z| = R > 1$, & $z = x+iy$ w/ $y \geq 0$

$$|f(z)| = \left| \frac{e^{iz}}{z^2+1} \right| = \frac{|e^{iz}|}{|z^2+1|}$$

$$|z^2+1| \geq |z^2| - 1 = R^2 - 1 > 0$$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{-y+ix}| = e^{-y} \leq e^0 = 1,$$

because $y \geq 0$.

$$\therefore |f(z)| \leq \frac{1}{R^2-1}$$

$$\therefore \left| \int_{C_{2,R}} f(z)\,dz \right| \leq \operatorname{length}(C_{2,R}) \cdot \frac{1}{R^2-1}$$

$$= \pi R \cdot \frac{1}{R^2-1} = \pi \cdot \frac{1/R}{1 - 1/R^2} \to 0$$

as $R \to \infty$.

So $\int_{C_{2,R}} f(z)\,dz \to 0$ as $R \to \infty$

c. $$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \underbrace{\lim_{R\to\infty} \int_{C_{1,R}} f(z)\,dz}_{} + \underbrace{\lim_{R\to\infty} \int_{C_{2,R}} f(z)\,dz}_{}$$

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz \overset{a.}{=} \pi/e$$

$$\lim_{R\to\infty} \int_{C_{1,R}} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = \left(\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx\right) + i\left(\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx\right)$$

$e^{ix} = \cos x + i \sin x$

$$\lim_{R\to\infty} \int_{C_{2,R}} f(z)\,dz \overset{b.}{=} 0$$

$\therefore$ $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \pi/e$$

and $$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx = 0 \quad \dagger$$

Note that $\dagger$ can be explained by observing that $\frac{\sin x}{x^2+1}$ is an odd function.

7 a. No. $\text{Log}(z)$ is not complex differentiable at any point z on the negative real axis $(-\infty, 0] \subset \mathbb{C}$, so 0 is not isolated singularity.

0

There is no disc D with center 0 such that $D \setminus \{0\}$ is contained in $U = \mathbb{C} \setminus (-\infty, 0]$.

b. $f(z) = \dfrac{1}{e^{1/z} - 1}$ has singularities at $z=0$ and when $e^{1/z} - 1 = 0$.

$e^{1/z} - 1 = 0 \iff e^{1/z} = 1 \iff 1/z = \log 1 = \log 1 + (2\pi i)k$

$= (2\pi i)k$, k an integer

Multivalued complex logarithm

$\iff z = \dfrac{1}{2\pi i k}$, k an integer.

as $k \to \infty$ $\dfrac{1}{2\pi i k} \to 0$

So the singularity $z=0$ is NOT isolated.

8. If f has a pole of order m at α,
then $f(z) = \dfrac{g(z)}{(z-\alpha)^m}$ where $g(\alpha) \neq 0$, g cx diffble near α

So $\dfrac{1}{f(z)} = \dfrac{(z-\alpha)^m}{g(z)}$ for $z \neq \alpha$, z near α.

But the RHS defines a complex differentiable function near α,
so the function $\dfrac{1}{f(z)}$ has a <u>removable singularity</u> at α.

If f has an essential singularity at α, there are two cases.

First, $1/f$ has singularities where $f=0$.
So, if there is a sequence $z_n \to \alpha$ as $n \to \infty$ such that $f(z_n) = 0$, then $1/f$ does not have an isolated singularity at α. (Compare Q7b: $e^{1/z}-1$ has an essential singularity at $z=0$, $1/e^{1/z}-1$ does not have an isolated sing. at $z=0$).

Second, suppose $1/f$ has an isolated singularity at $z=\alpha$.

Then it is either removable, a pole, or an essential singularity

The 3 cases can be distinguished by considering the limit as $z \to \alpha$ of $1/f$ (if g has an isolated singularity at α, then for

$\lim_{z \to \alpha} g(z) = L \in \mathbb{C}$ if α is removable (some L),

$\lim_{z \to \alpha} g(z) = \infty$ if α is a pole, and

$\lim_{z \to \alpha} g(z)$ does not exist if α is essential.)

But $\lim_{z \to \alpha} \frac{1}{f(z)} = L \implies \lim_{z \to \alpha} f(z) = \frac{1}{L}$,

(or ∞ if $L = 0$)

and $\lim_{z \to \alpha} \frac{1}{f(z)} = \infty \implies \lim_{z \to \alpha} f(z) = 0.$

So, since we know $\lim_{z \to \alpha} f(z)$ does not exist, we deduce

$\lim_{z \to \alpha} \frac{1}{f(z)}$ does not exist, and $\frac{1}{f}$ has an essential singularity at $z = \alpha$.