Math 611 Homework 4

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All rings are assumed to be commutative with 1 unless explicitly stated otherwise.

Reading: Dummit and Foote, Section 6.3 and Chapter 7.

Justify your answers carefully.

(1) Let $G = \langle x, y, z \mid yz^2xy \rangle$ be the group generated by x, y, z subject to the relation $yz^2xy = e$. Prove that G is isomorphic to the free group generated by two elements.

(2) Let $F = \langle x, y \rangle$ be the free group generated by x and y. Consider the subgroup G of F generated by x^2, xy, y^2. Let $F' = \langle u, v, w \rangle$ be the free group generated by u, v, w. Show that the homomorphism $F' \to G$ given by

$$u \mapsto x^2, \quad v \mapsto xy, \quad w \mapsto y^2$$

is an isomorphism.

(3) Identify the following groups with a standard group.

(a) $\langle a, b \mid a^5, b^2, abab \rangle$.

(b) $\langle a, b \mid a^4, a^2b^2, abab^{-1} \rangle$.

(c) $\langle a, b \mid a^2, b^3, (ba)^3 \rangle$.

[Hint: First guess the standard group G and a set of two generators $x, y \in G$. Then show that the surjective homomorphism $\phi\colon F/N \to G$ from the abstractly defined group to G given by $\phi(a) = x$, $\phi(b) = y$ (using the universal property of the free group) is injective and so an isomorphism.]

(4) Let A be a ring, F a field, and $\varphi\colon F \to A$ a ring homomorphism. Show that if $A \neq \{0\}$ then φ is injective.

(5) Let A be a ring.

(a) Show that there is a unique homomorphism $\varphi\colon \mathbb{Z} \to A$ and describe it explicitly. Write $\ker(\varphi) = (n)$, $n \in \mathbb{Z}$, $n \geq 0$. (The number n is called the *characteristic* of A.)

(b) Show that if A is an integral domain then $n = 0$ or $n = p$, a prime.

(6) Suppose A is a ring of characteristic p, a prime. Show that the map

$$\psi\colon A \to A, \quad \psi(x) = x^p$$

is a ring homomorphism. (The homomorphism ψ is called the *Fröbenius* homomorphism.) Describe ψ explicitly in the case $A = \mathbb{F}_p[x]$.

(7) Identify the quotient rings as explicitly as possible. (Describe an isomorphism with a product of standard rings using the first isomorphism theorem and/or the Chinese remainder theorem.)

(a) $\mathbb{Q}[x]/(x-3)$.

(b) $\mathbb{R}[x]/(x^2-4x-5)$.

(c) $\mathbb{R}[x]/(x^2+4)$.

(8) Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, the ring of *Gaussian integers*. Identify the quotient $\mathbb{Z}[i]/(3+4i)$ with a standard ring.

[Hint: Show that the map $\mathbb{Z} \to \mathbb{Z}[i]/(3+4i)$ is surjective and identify the kernel.]

(9) Identify the kernel and the image of each of the following homomorphisms explicitly (in each case the kernel is a principal ideal).

(a) $\varphi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$ given by $x \mapsto t^2$, $y \mapsto t^3$.

(b) $\psi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$ given by $x \mapsto t^2-1$, $y \mapsto t(t^2-1)$.

[Hint: Find an element of the kernel, and prove it generates the kernel by using $\mathbb{C}[x,y] = (\mathbb{C}[x])[y] = (\mathbb{C}[y])[x]$ and the following division algorithm: Let A be a ring and $a, b \in A[x]$. If b is a monic polynomial (that is, the leading coefficient of b equals 1) then there exist unique $q, r \in A[x]$ such that $a = qb + r$ where $r = 0$ or $\deg(r) < \deg(b)$.]

(10) Let A be a ring. We say an element $a \in A$ is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$.

   (a) Prove that the set $N \subset A$ of nilpotent elements is an ideal of A. (N is called the *nilradical* of A.)

   (b) Show that in the quotient ring A/N there are no nonzero nilpotent elements.

(11) We say a ring A is *local* if there is a unique maximal ideal $m \subset A$.

   (a) Show that if A is local with maximal ideal m then the units $A^\times$ of A are given by $A^\times = A \setminus m$.

   (b) Conversely, show that if A is a ring and $I \subset A$ is an ideal such that every element of $A \setminus I$ is a unit, then A is local with maximal ideal $m = I$.

(12) The *formal power series ring* $\mathbb{C}[[x]]$ has elements

$$f = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_i \in \mathbb{C}$ for each i. (The adjective "formal" indicates that we do not require that the series converges for any nonzero value of x in $\mathbb{C}$. That is, using the terminology of complex analysis, the radius of convergence may be equal to zero.) Addition and multiplication of formal power series are defined in the obvious way, e.g.,

$$(\sum a_i x^i) \cdot (\sum b_i x^i) = \sum_k (\sum_{i+j=k} a_i b_j) x^k = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \cdots .$$

(Notice that the coefficients of the product are finite sums, so the product is well-defined.) Show that $\mathbb{C}[[x]]$ is a local ring with maximal ideal $m = (x)$.

(13) Let $A = \mathbb{Z}[\sqrt{3}] := \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$, a subring of $\mathbb{R}$. Identify the fraction field $F = \mathrm{ff}(A) \subset \mathbb{R}$ of A explicitly (determine a basis of F as a $\mathbb{Q}$-vector space).

(14) Let A be a noncommutative ring with 1. We say $I \subset A$ is a *two-sided ideal* if $I \subset A$ is an additive subgroup and

$$a \in A, x \in I \Rightarrow ax \in I \text{ and } xa \in I$$

Then we can define the quotient ring A/I as in the commutative case. (However, in the noncommutative case, there are typically very few two-sided ideals.)

(a) Let $A = \mathbb{C}\langle x, y\rangle$ be the polynomial ring in noncommuting variables x and y. (Thus an element f of A is a finite sum $\sum a_i w_i$ where $a_i \in \mathbb{C}$ and w_i is a word in x and y, e.g. $xyyxxxy = xy^2x^3y$.) Define a surjective homomorphism $\varphi\colon A \to \mathbb{C}[x, y]$ from A to the usual commutative polynomial ring. (Then $I = \ker(\varphi) \subset A$ is a two-sided ideal and $A/I \simeq \mathbb{C}[x, y]$.)

(b) Let $A = \mathbb{C}^{n\times n}$ be the ring of $n \times n$ complex matrices. Show that the only two-sided ideals of A are $\{0\}$ and A.

(c) Let A be the *Weyl algebra* given by the polynomial ring $\mathbb{C}\langle x, y\rangle$ in noncommuting variables x and y modulo the two sided ideal generated by the relation $yx - xy - 1$. (Thus an element f of A may be written uniquely as a finite sum $f = \sum a_{ij}x^iy^j$, $a_{ij} \in \mathbb{C}$, with the usual addition and the multiplication determined by $yx = xy + 1$ together with the associative and distributive laws.) Show that the only two-sided ideals of A are $\{0\}$ and A.

[Hint: Given $f \in A$ consider the commutators $[x, f] = xf - fx$ and $[y, f] = yf - fy$.]