Math 461 Lecture 16 10/10

Homework 4 due today

Homework 3 returned

Last time:

Algebraic descriptions of isometries

$$\mathrm{Rot}(0,\theta)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\mathrm{Rot}(P,\theta) = \mathrm{Trans}_{\overrightarrow{OP}} \circ \mathrm{Rot}(0,\theta) \circ \mathrm{Trans}_{-\overrightarrow{OP}}$$

$$\mathrm{Refl}_L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \text{ if } 0 \in L$$

What if $0 \notin L$?

Use the same trick as rotation: translate to the origin, reflect, then translate back.

Today:

Compositions of Isometries

Compositions:

Ex 0: Composition of two translations

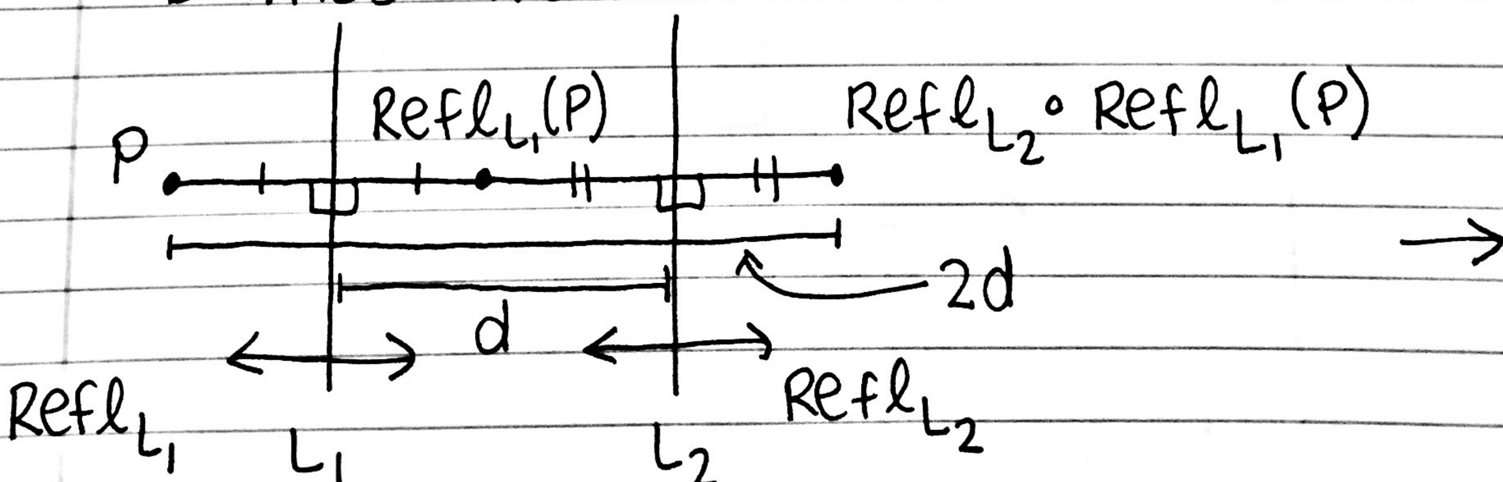
$$\mathrm{Trans}_{\vec{w}} \circ \mathrm{Trans}_{\vec{v}} = \mathrm{Trans}_{\vec{v}+\vec{w}}$$

Ex 1: Composition of two reflections

$L_1, L_2 \subset \mathbb{R}^2$

a lines are parallel

b lines intersect



Does $\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$ = Translation?

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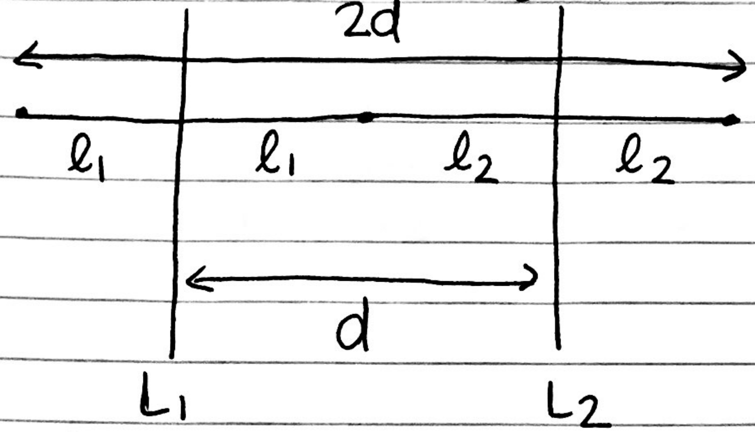


$d = \ell_1 + \ell_2 \quad 2\ell_1 + 2\ell_2 = 2d$

$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} = \mathrm{Trans}_{2\bar{v}}$ where $\bar{v}$ is the vector perpendicular to lines, in direction from L_1 to L_2, with length the distance between the lines

Notice: $\mathrm{Refl}_{L_1} \circ \mathrm{Refl}_{L_2} = \mathrm{Trans}_{-2\bar{v}}$

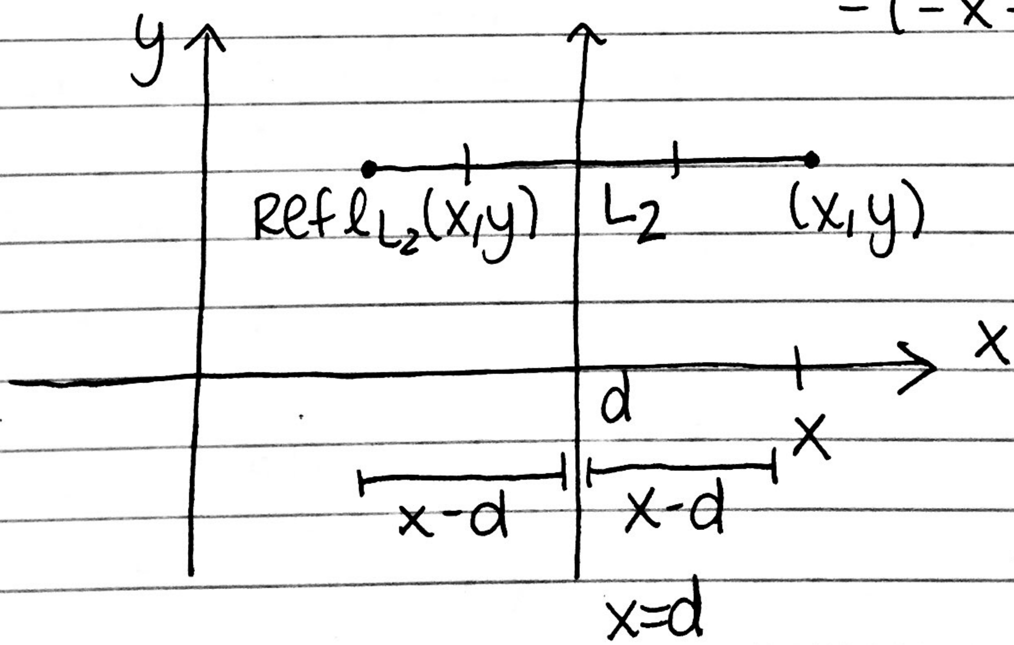
Algebraic Calculation:

choose coordinates

we may assume $L_1 = (x=0)$, the y-axis and $L_2 = (x=d)$

$\mathrm{Refl}_{L_1}(x,y) = (-x,y)$

$\mathrm{Refl}_{L_2}(x,y) = (x-2(x-d),y) = (x-2x+2d,y)$
$= (-x+2d,y)$



CHECK: if $x=d$ then $\mathrm{Refl}_{L_2}(x,y) = (x,y)$

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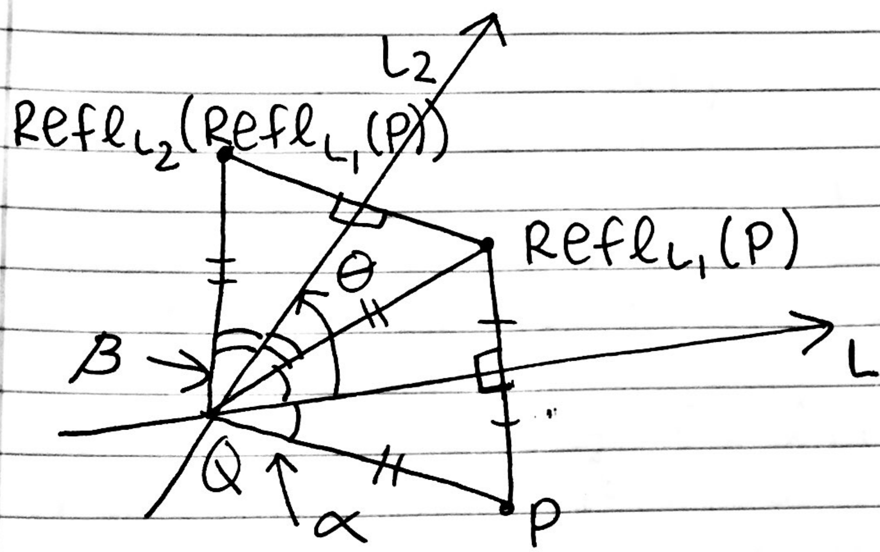
$Refl_{L_1}(x,y) = (-x,y)$ $\quad Refl_{L_2}(x,y) = (-x+2d,y)$

$Refl_{L_2}(Refl_{L_1}(x,y)) = Refl_{L_2}(-x,y) =$
$((-x)+2d,y) = (x+2d,y) = (x,y) + (2d,0)$

what is $Refl_{L_2} \circ Refl_{L_1}$?



$Refl_{L_2} \circ Refl_{L_1} = Rot(Q, 2\theta)$ where
$Q = L_1 \cap L_2$ and θ is angle between the lines from L_1 to L_2 counterclockwise

Algebraic Proof:
we may assume Q = origin and L_1 = x-axis

$Refl_{L_1}(x,y) = (x,-y)$

$$Refl_{L_1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$Refl_{L_2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$Refl_{L_2} \circ Refl_{L_1}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ Rotation by 2θ counterclockwise about the origin

picture →

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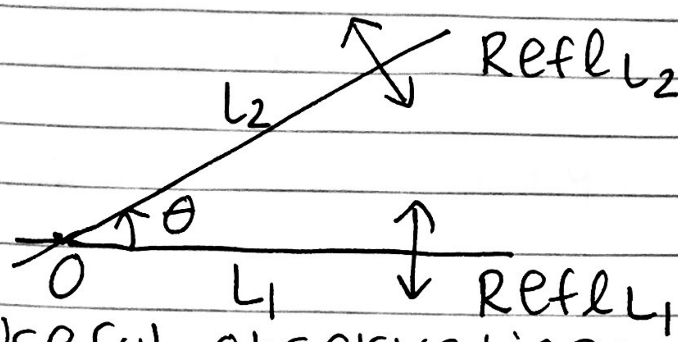
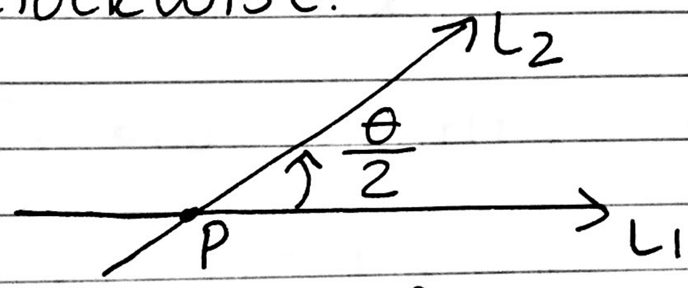
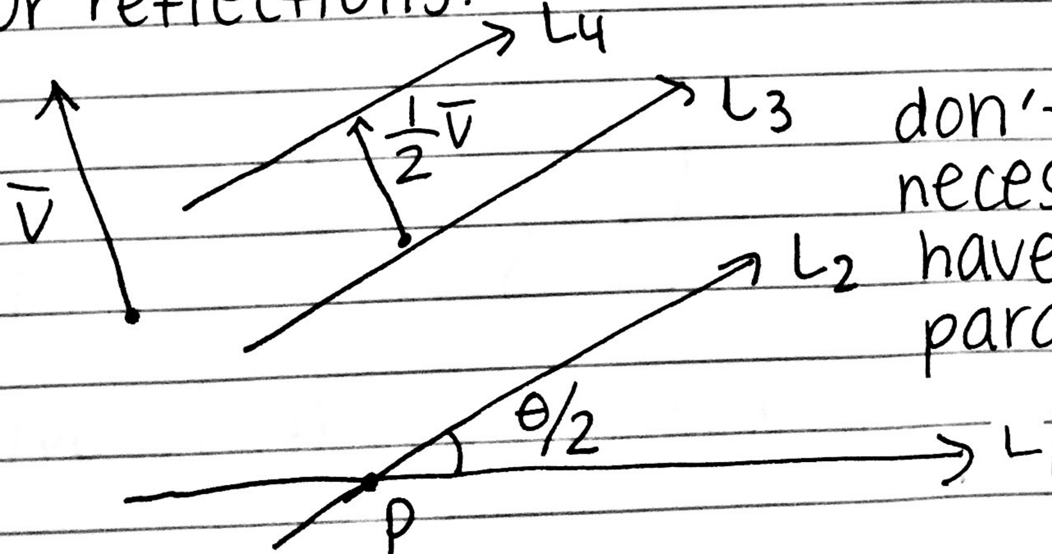


Useful observation:
If I have a rotation or a translation then it can be expressed as a composite of two reflections in several ways.
$Rot(P,\theta) = Refl_{L_2} \circ Refl_{L_1}$ for any two lines meeting at P such that the angle from L_1 to L_2 in $\theta/2$ counter clockwise.



(we have freedom to choose slope of L_1)

Composition of 1st rotation and 2nd translation:
Each is a composite of two reflections.
So the composition is a composite of four reflections.



don't necessarily have to be parallel

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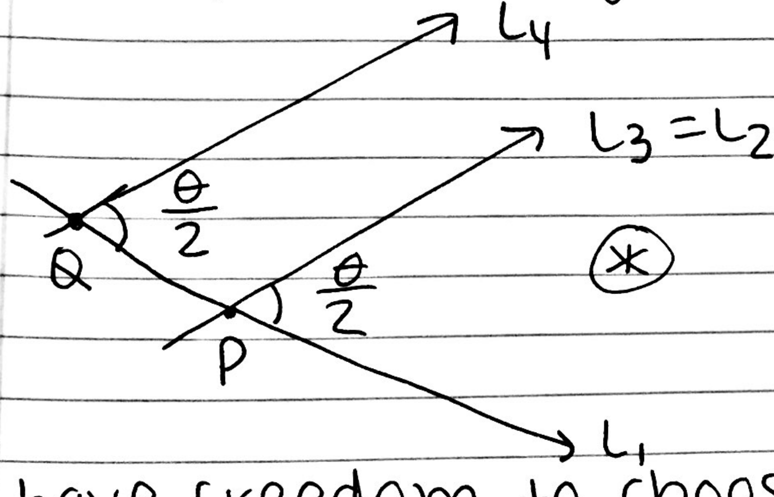
$$\underbrace{Refl_{L_4} \circ Refl_{L_3}}_{Trans_{\vec{v}}} \circ \underbrace{Refl_{L_2} \circ Refl_{L_1}}_{Rot(P,\theta)}$$

$Refl_L^2 = \text{identity}$



have freedom to choose the lines so make L_3 and L_2 the same line

$Refl_{L_4} \circ Refl_{L_3} \circ Refl_{L_2} \circ Refl_{L_1} =$

$Refl_{L_4} \circ \text{identity} \circ Refl_{L_1} = Refl_{L_4} \circ Refl_{L_1}$
$= Rot(Q, \theta)$

Summarize:

$Rot(Q, \theta) = Trans_{\vec{v}} \circ Rot(P, \theta)$
← determined by diagram (*)

Question: How to find Q algebraically? (Given algebraic formulas for $Rot(P,\theta)$ & $Trans_{\vec{v}}$)

e.g. $Rot(0, \pi/2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

$Trans_{\binom{2}{3}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2 \\ y+3 \end{pmatrix}$

$Trans_{\binom{2}{3}} \circ Rot(0, \pi/2) = \begin{pmatrix} -y+2 \\ x+3 \end{pmatrix} = Rot(Q, \pi/2)$

Solve $T(x,y)$ and only solution is Q.