697B Example Sheet 2

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15 September 2010

(1) Let $X = (y^2 = x^3) \subset \mathbb{C}^2$. Consider the inclusion

$$\mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}, \quad (x, y) \mapsto (X : Y : Z) = (x : y : 1).$$

Let $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ be the closure of X.

(a) Find all the singular points of $\overline{X}$.

(b) Describe the normalization $f \colon \tilde{X} \to \overline{X}$ explicitly. [Hint: The method used in class to construct a rational parametrization of the nodal cubic $y^2 = x^2(x+1)$ works in this case too.]

(2) Let $X = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and f a meromorphic function on X. Show explicitly that

$$\sum_{P \in X} \nu_P(f) = 0$$

where $\nu_P(f)$ is the order of f at P. Informally, this formula says that the number of zeroes of f is equal to the number of poles of f (provided we count with multiplicities).

(3) Let $X = \mathbb{C}/\Lambda$ be a complex torus, where $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$ (linearly independent over $\mathbb{R}$). Let $n \in \mathbb{N}$ be a positive integer. Show that the map

$$\mathbb{C} \to \mathbb{C}, \quad z \mapsto n \cdot z$$

induces a holomorphic map

$$f_n \colon X \to X.$$

What is the size of the fiber $f_n^{-1}(P)$ of f_n over a point $P \in X$?

(4) Let X be a Riemann surface. Show that a meromorphic function $f\colon X \dashrightarrow \mathbb{C}$ on X extends to a holomorphic map $F\colon X \to \mathbb{C} \cup \{\infty\}$ from X to the Riemann sphere, and conversely if $F\colon X \to \mathbb{C}\cup\{\infty\}$ is a holomorphic map, its restriction $f\colon X\backslash F^{-1}(\infty) \to \mathbb{C}$ is a meromorphic function on X.

(5) An automorphism of a Riemann surface is by definiton a bijective holomorphic map $f\colon X \to X$ with holomorphic inverse. Show that the automorphism group of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is the group of Mobius transformations

$$z \mapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$.[Hint: Use the previous question and the description of meromorphic functions on the Riemann sphere as rational functions of the coordinate z.]

(6) We recall the Casorati–Weierstrass theorem: Let $f\colon \Delta^{\times} \to \mathbb{C}$ be a holomorphic function on the punctured disc

$$\Delta^{\times} := \{z \in \mathbb{C} \mid 0 < |z| < r\}$$

of some radius r. Suppose f has an essential singularity at $z = 0$. Then the image of f is dense in $\mathbb{C}$. Use the CW theorem to show that the automorphism group of the complex plane $\mathbb{C}$ is the group of affine linear transformations

$$z \mapsto az + b.$$

[Hint: Given an automorphism $F\colon \mathbb{C} \to \mathbb{C}$, show that F extends to an automorphism of $\mathbb{C} \cup \{\infty\}$. To do this, you need to show that F cannot have an essential singularity at infinity.]

(7) In class we considered algebraic curves $X = (f(x, y) = 0) \subset \mathbb{C}^2_{x,y}$ defined as the zero locus of a polynomial in complex variables x, y. We observed that the closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ is given by $\overline{X} = (F(X, Y, Z) = 0)$ where $F(X, Y, Z)$ is a homogeneous polynomial determined by f. In this question we see that if we replace $f(x, y)$ by a transcendental function then we should not expect the closure to be well-behaved. Let $X = (y = e^x) \subset \mathbb{C}^2_{x,y}$. What is the closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$? Describe what happens near L_∞ (the line at infinity). It may help to also consider $X = (y = x \sin x) \subset \mathbb{C}^2_{x,y}$ and draw the real locus. [Hint: Change coordinates in $\mathbb{P}^2_{\mathbb{C}}$ to see a neighbourhood of L_∞ and use the Casorati–Weierstrass theorem stated in the previous question.]

(8) In class we showed that the *hyperelliptic curve* $X = (y^2 = p(x)) \subset \mathbb{C}^2_{x,y}$, where $p(x)$ is a polynomial of degree n with distinct roots, is smooth, but the closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ is singular if $n \geq 4$. Here we describe explicitly the normalization $f \colon \tilde{X} \to \overline{X}$ of $\overline{X}$ (so $\tilde{X}$ is a compact Riemann surface and $f \colon \tilde{X} \to \mathbb{P}^2_{\mathbb{C}}$ is a holomorphic map with image $\overline{X}$). We will assume $n \geq 4$ (otherwise $\overline{X}$ is already smooth).

(a) Write $p(x) = c(x - \alpha_1) \cdots (x - \alpha_n)$. Write $n = 2m - \delta$, $\delta = 0$ or 1. Define $\tilde{X} = X \cup Y$ where

$$Y = (w^2 = cz^\delta(1 - \alpha_1 z) \cdots (1 - \alpha_n z)) \subset \mathbb{C}^2_{w,z}$$

and the gluing of the two sets is given by

$$\mathbb{C}^2_{x,y} \supset (x \neq 0) \xrightarrow{\sim} (z \neq 0) \subset \mathbb{C}^2_{z,w}, \quad (x, y) \mapsto (x^{-1}, x^{-m}y).$$

Show that $\tilde{X}$ is a compact Riemann surface. Show that the map

$$X \to \mathbb{C}, \quad (x, y) \mapsto x$$

extends to a holomorphic map

$$\tilde{X} \to \mathbb{C} \cup \{\infty\}.$$

(b) Show that the inclusion $X \subset \overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ extends to a holomorphic map $f \colon \tilde{X} \to \mathbb{P}^2_{\mathbb{C}}$ with image $\overline{X}$.

(c) What is the genus of $\tilde{X}$?