Math 797W Homework 4

Paul Hacking

December 1, 2014

Justify your answers carefully.

(1) Let $f, g \in k[x, y]$ be irreducible polynomials and $X = V(f) \subset \mathbb{A}^2$, $Y = V(g) \subset \mathbb{A}^2$ be the corresponding affine plane curves. Compute the intersection multiplicity $(X \cdot Y)_p := \dim_k \mathcal{O}_{\mathbb{A}^2,p}/(f, g)$ of X and Y at $p = (0, 0)$ in the following cases.

(a) $f = y^2 - x^3$, $g = y - \lambda x$, $\lambda \in k$.

(b) $f = y^2 - x^3$, $g = y^2 - x^5$.

(2) (a) Let $X, Y \subset \mathbb{A}^2_{x,y}$ be affine plane curves with equations f, g as in Q1. Show that if $p \in X \cap Y$ and X is singular at p then $(X \cdot Y)_p \geq 2$

(b) Let $X \subset \mathbb{P}^2$ be an irreducible plane curve of degree 3. Show that X has at most one singular point.

(c) Let $p_1, \ldots, p_5 \in \mathbb{P}^2$ be 5 points in $\mathbb{P}^2$. Show that there is a homogeneous polynomial $F \in k[X_0, X_1, X_2]$ of degree 2 such that $F(p_i) = 0$ for all i.

(d) Let $X \subset \mathbb{P}^2$ be an irreducible plane curve of degree 4. Show that X has at most three singular points.

[Hint: Use the Bezout theorem to deduce (b) and (d) from (a) and (c). Establish part (c) using linear algebra.]

[Remark: In general an irreducible plane curve $X \subset \mathbb{P}^2$ of degree d has at most $\frac{1}{2}(d-1)(d-2)$ singularities. This can be proved by understanding the difference between the genus of the normalization $\tilde{X}$ of X

and the genus $\frac{1}{2}(d-1)(d-2)$ of a smooth plane curve of degree d in terms of the singularities of X.]

(3) Let X be a smooth projective curve and D a divisor on X. Recall

$$L(D) := \{f \in k(X)^{\times} \mid (f) + D \geq 0\} \cup \{0\}$$

and $l(D) := \dim_k L(D)$. Assume that $\deg D \geq 0$. Show that $l(D) \leq \deg D + 1$, with equality iff X is isomorphic to $\mathbb{P}^1$.

(4) Let X be a smooth projective curve of genus 1 and D a divisor on X. Prove that if $n := \deg D \geq 3$ then $L(D)$ defines a closed embedding $f \colon X \to \mathbb{P}^{n-1}$ as a curve of degree n. Deduce that if $n = 3$ then $X \simeq V(F) \subset \mathbb{P}^2$ where $F \in k[X_0, X_1, X_2]$ is an irreducible homogeneous polynomial of degree 3, and if $n = 4$ then $X \simeq V(F, G) \subset \mathbb{P}^3$ where $F, G \in k[X_0, X_1, X_2, X_3]$ are irreducible homogeneous polynomials of degree 2.

(5) Let X be a smooth projective curve of genus 1. Let $p_0 \in X$ be a point.

(a) Use the Riemann-Roch theorem to prove that the map

$$X \longrightarrow \ker(\deg \colon \operatorname{Cl}(X) \to \mathbb{Z}), \quad p \mapsto [p - p_0]$$

is a bijection of sets. Thus X inherits the structure of an abelian group (with identity element $0 := p_0 \in X$) from $\operatorname{Cl}(X)$.

(b) Let $f \colon X \to \mathbb{P}^2$ be the embedding of X as a plane cubic given by $L(3p_0)$ (see Q4). Show that for three points $p, q, r \in X$ we have $p + q + r = 0$ in the group law on X iff the divisor $p + q + r$ is equal to f^*L for some line $L \subset \mathbb{P}^2$. In particular, if p, q, r are distinct then $p + q + r = 0$ in the group law iff $f(p), f(q), f(r)$ are collinear in $\mathbb{P}^2$.

(c) Use part (b) to give a geometric description of the group operation $X \times X \to X$ in terms of lines in $\mathbb{P}^2$.

[Remark: This can be used to show that the group operation $X \times X \to X$ and the inverse map $X \to X$ are morphisms of varieties. We say X is an *algebraic group*.]

(6) Let X be a smooth projective curve and D a divisor on X. For $p \in X$ let $\nu_p(D)$ denote the coefficient of p in D. Define a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules on X by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X)^\times \mid \nu_p(f) + \nu_p(D) \geq 0 \, \forall \, p \in U\} \cup \{0\}$$

Thus the k-vector space of global sections $\Gamma(X, \mathcal{O}_X(D))$ equals $L(D)$, the Riemann–Roch space of D.

(a) Show that $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module of rank 1.

(b) Let $p \colon L \to X$ be a line bundle. Let $s \colon U \to p^{-1}U$ be a nonzero section of L over some Zariski open set $U \subset X$. Define the divisor (s) of zeroes and poles of s (using local trivializations of L).

(c) Now suppose L is the line bundle with sheaf of sections $\mathcal{L} = \mathcal{O}_X(D)$. Show that (s) is linearly equivalent to D.

[Remark: Let $\operatorname{Pic} X$ denote the set of isomorphism classes of line bundles on X (or locally free $\mathcal{O}_X$-modules of rank 1). Note $\operatorname{Pic}(X)$ is an abelian group with group law $(L, M) \mapsto L \otimes M$ and inverse $L \mapsto L^*$. Then we have an isomorphism of abelian groups $\operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Pic}(X)$ given by $[D] \mapsto \mathcal{O}_X(D)$, with inverse $L \mapsto [(s)]$ where s is a nonzero section of L over some open set $U \subset X$. (The same is true for a smooth variety X of arbitrary dimension.)]

(7) (a) Let X be a variety and $p \colon L \to X$ a line bundle (i.e. an algebraic vector bundle of rank 1). Suppose $s_i \colon X \to L$, $i = 0, \ldots, N$ are global sections of L such that for all $q \in X$ we have $s_i(q) \neq 0 \in L_q := p^{-1}(q)$ for some i. Show that the s_i define a morphism

$$f \colon X \to \mathbb{P}^N$$

given by $f = (s_0 : s_1 : \cdots : s_N)$.

(b) Now suppose X is a smooth curve, D is a divisor on X, and L is the line bundle with sheaf of sections $\mathcal{L} = \mathcal{O}_X(D)$ (as defined in Q6. Show that if D satisfies $l(D-p) = l(D)-1$ for all $p \in X$, and $s_0, \ldots, s_N \in \Gamma(X, \mathcal{L}) = L(D)$ correspond to a basis of $L(D)$, then the s_i satisfy the above condition and the morphism f coincides with the morphism defined earlier using the corresponding basis of $L(D)$.

(8) Let $n \in \mathbb{N}$ and $d \in \mathbb{Z}$. Recall the definition of the sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ of $\mathcal{O}_{\mathbb{P}^n}$-modules: for $U \subset \mathbb{P}^n$ an open subset, $\Gamma(U, \mathcal{O}_{\mathbb{P}^n}(d))$ is the $\mathcal{O}_{\mathbb{P}^n}(U)$-module consisting of elements $f \in k(X_0, \ldots, X_n)$ such that $f = F/G$ for homogeneous polynomials $F, G \in k[X_0, \ldots, X_n]$ with $\deg F - \deg G = d$ and $G(p) \neq 0$ for all $p \in U$.

(a) Show that $\mathcal{O}_{\mathbb{P}^n}(d)$ is a locally free sheaf of rank 1 on $\mathbb{P}^n$ as follows: Write $U_i = (X_i \neq 0) \subset \mathbb{P}^n$ and describe a local trivialization

$$\psi_i \colon \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$$

for each i.

(b) Compute the transition functions $g_{ij} = \psi_j \circ \psi_i^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$.

(c) Determine the k-vector space of global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ of $\mathcal{O}_{\mathbb{P}^n}(d)$ for each d.

(d) Let $p \colon L \to \mathbb{P}^n$ be the tautological line bundle over $\mathbb{P}^n$. That is,

$$L = V(\{x_i X_j - x_j X_i \mid 0 \leq i < j \leq n\}) \subset \mathbb{P}^n_{(X_0:\cdots:X_n)} \times \mathbb{A}^{n+1}_{x_0,\ldots,x_n}$$

and $p \colon L \to \mathbb{P}^n$ is the restriction of the first projection

$$\mathrm{pr}_1 \colon \mathbb{P}^n \times \mathbb{A}^{n+1} \to \mathbb{P}^n.$$

(So, the fiber $p^{-1}(q)$ of p over a point $q = (a_0 : \ldots : a_n) \in \mathbb{P}^n$ is the corresponding line

$$l = \{\lambda \cdot (a_0, \ldots, a_n) \mid \lambda \in k\} \subset \mathbb{A}^{n+1}.)$$

Let $\mathcal{L}$ be the sheaf of sections of L. Show that the sheaf $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

(9) Let $X = \mathbb{C}$ with the analytic (or Euclidean) topology. Let $\underline{\mathbb{Z}}$ be the constant sheaf on X with stalk $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions on X (with group operation addition), and $\mathcal{O}_X^\times$ the sheaf of nowhere zero holomorphic functions on X (with group operation multiplication).

(a) Show that there is a short exact sequence of sheaves on X

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

where the first map α is the inclusion and the second map β is given by $f \mapsto e^{2\pi i f}$.

(b) Show that the map $\beta_U \colon \mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ is not surjective for some open set $U \subset X$.

(10) Show that there is an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1} \to k(p_1) \oplus k(p_2) \to 0$$

where $p_1, p_2 \in \mathbb{P}^1$ are distinct points and $k(p)$ denotes the *skyscraper sheaf at p with stalk k*, that is, $\Gamma(U, k(p)) = k$ if $p \in U$ and $\Gamma(U, k(p)) = 0$ if $p \notin U$. Deduce that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$.

(11) Show that there is an exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules (the *Euler sequence*)

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

Here $\mathcal{T}_{\mathbb{P}^n}$ denotes the (sheaf of sections of the) tangent bundle of $\mathbb{P}^n$ (the dual of $\Omega_{\mathbb{P}^n}$), the first map is given by

$$s \mapsto s \cdot (X_0, \ldots, X_n),$$

and the second map is given by

$$(s_0, .., s_n) \mapsto \sum s_i \frac{\partial}{\partial X_i}.$$

Deduce that the *canonical line bundle* $\omega_{\mathbb{P}^n} := \wedge^n \Omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

[Hint: If E is a vector bundle of rank r with transition functions g_{ij} then $\det E := \wedge^r E$ is a line bundle with transition functions $\det(g_{ij})$. If

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles then the identity

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = (\det A) \cdot (\det C).$$

shows that $\det E \simeq \det E' \otimes \det E''$.]