

Math 461 Homework 7

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- (1) (a) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the reflection in the plane $\Pi \subset \mathbb{R}^3$ with equation $x + 2y + z = 0$. Find an algebraic formula for T . Express your formula in the form $T(\mathbf{x}) = A\mathbf{x}$ where A is a 3×3 matrix.
- (b) Check your answer to part (a) by verifying that (i) $A\mathbf{n} = -\mathbf{n}$ where \mathbf{n} is a normal vector to Π and (ii) the matrix A is orthogonal, that is, $A^T A = I$ where the superscript T is used to denote the transpose of the matrix (interchange rows and columns) and I is the identity matrix. (Property (ii) holds iff the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an isometry.)
- (2) (a) Determine the matrix A_θ of a rotation about the z -axis through angle θ counterclockwise as viewed from $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (looking towards the origin).
- (b) Determine the matrix B_φ of a rotation about the x -axis through angle φ counterclockwise as viewed from $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

[Hint: If $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation then the matrix A of T has columns the vectors $T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Use this fact together with geometric reasoning to determine the matrices A_θ and B_φ .]

- (3) Let $L \subset \mathbb{R}^3$ be the line in the direction $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation about the line L through angle $\pi/2$ counterclockwise as viewed from $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

- (a) Explain why T can be expressed as the composition $U \circ V \circ U^{-1}$ where U is the rotation about the z -axis through angle $\pi/4$ and V is the rotation about the x -axis through angle $\pi/2$.
- (b) Now use the result of Q2 to compute the matrix A of T .

[Remark: One could also use the general method described in class on Wednesday 11/7 to compute A , but in this case the method described above is easier.]

- (4) Let $\tilde{T}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an isometry of \mathbb{R}^3 fixing the origin and $T: S^2 \rightarrow S^2$ the induced isometry of S^2 (given by $T(P) = \tilde{T}(P)$ for $P \in S^2$). For each type of \tilde{T} (the identity, reflection, rotation, or rotary reflection) describe the fixed locus of \tilde{T} and the fixed locus of T .
- (5) For each of the following isometries $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, classify the isometry as a reflection, rotation, or rotary reflection, and give a precise geometric description as follows: for a reflection give the plane of reflection, for a rotation give the axis and angle of rotation, for a rotary reflection give the plane of reflection and axis and angle of rotation. (The sense of rotation (counterclockwise or clockwise) may be omitted.)

(a) $T(\mathbf{x}) = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix} \mathbf{x}.$

(b) $T(\mathbf{x}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mathbf{x}.$

(c) $T(\mathbf{x}) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \mathbf{x}.$

- (6) What type of isometry is the transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(\mathbf{x}) = -\mathbf{x}$? Give a precise geometric description as in Q4.

[Hint: In this case the description of the isometry as in Q4 is not uniquely determined — you will need to make a choice.]

- (7) A tetrahedron is a regular polytope with 4 faces. For example, there is a tetrahedron $T \subset \mathbb{R}^3$ with vertices the points

$$(1, 1, 1), (-1, -1, 1), (1, -1, -1), (-1, 1, -1).$$

It may help to visualize the tetrahedron by first drawing the cube with 8 vertices $(\pm 1, \pm 1, \pm 1)$. There are 24 symmetries of the tetrahedron (including the identity transformation) — because there is exactly one symmetry realizing each permutation of the 4 vertices, and $4! = 24$. Give a geometric description of each of the symmetries as a rotation, reflection, or rotary reflection.