697B Example Sheet 7

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(1) (a) Let $X, Y \subset \mathbb{P}^2_{\mathbb{C}}$ be two curves of degree d such that $X \cap Y$ is a finite set of size d^2. Suppose there exists an irreducible curve Z of degree e through de of the points $X \cap Y$. Show that there exists a curve W of degree $d - e$ through the remaining points. [Hint: Write $X = (F = 0)$, $Y = (G = 0)$, $Z = (H = 0)$. Let $P \in Z \setminus X \cap Y$. Show that there exist $\lambda, \mu \in \mathbb{C}$, not both zero, such that $P \in (\lambda F + \mu G = 0)$, and use Bézout's theorem to deduce that H divides $\lambda F + \mu G$.]

(b) (Pascal's mystic hexagon) Show that the vertices of a hexagon lie on an irreducible conic iff the intersection points of the opposite sides are collinear. More precisely: given 6 distinct points $A, B, \ldots, F$ in $\mathbb{R}^2$, let AB denote the line through A and B, etc. Show that $A, \ldots, F$ lie on an irreducible conic iff the points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ are collinear (if some of the sides are parallel, the result is still true provided we work in $\mathbb{P}^2_{\mathbb{R}}$). [Hint: Check part (a) works over $\mathbb{R}$ and apply it to two reducible cubics X and Y.]

(2) Let $X = (xy^2 = p(x)) \subset \mathbb{C}^2_{x,y}$ where $p(x)$ is a polynomial of degree 4 with distinct roots and $p(0) \neq 0$.

(a) Show that the closure $\overline{X} \subset \mathbb{P}^2$ of X in $\mathbb{P}^2$ is a curve of degree 4.

(b) Show that $\overline{X} \cap L_\infty$ is a single point P.

(c) Show that $X = \overline{X} \setminus \{P\}$ is smooth and $P \in \overline{X}$ is a node.

(d) Let $n \colon \tilde{X} \to \overline{X}$ be the normalization of X. Use the genus formula to determine the genus of $\tilde{X}$.

(e) The map $X \to \mathbb{C}$ given by $(x, y) \mapsto x$ extends to a holomorphic map $F \colon \tilde{X} \to \mathbb{P}^1$ of degree 2. Find the ramification points of F. Check your answer to part (d) using the Riemann-Hurwitz formula.

(f) Give a geometric description of the map F using HW6Q6c.

(3) Let X be a compact Riemann surface. Recall that a *divisor* D on X is a finite formal sum $\sum_{i=1}^{r} n_i P_i$, where $P_i \in X$ and $n_i \in \mathbb{Z}$. The *degree* of D equals $\sum n_i$. For f a nonzero meromorphic function on X the *principal divisor* associated to f is defined by $(f) = \sum \nu_P(f) P$. Note that a principal divisor has degree 0. We say divisors D and E are *linearly equivalent* and write $D \sim E$ if $D - E$ is principal.

(a) Suppose $X = \mathbb{P}^1_{\mathbb{C}}$. Show that every divisor of degree 0 is principal.

(b) Show that if X is not isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and $P, Q \in X$ are distinct points then $D = P - Q$ is *not* principal.

(c) Let X be a curve of genus 1 and $P \in X$ a point. Let D be a divisor of degree 0 on X. Show that there exists a unique point $Q \in X$ such that $D + P \sim Q$. [Hint: Use the Riemann-Roch theorem.] Let $J(X)$ denote the group of divisors of degree 0 on X modulo linear equivalence. Deduce that the map

$$X \to J(X), \quad Q \mapsto [Q - P]$$

is a bijection of sets.

(4) Let $X = \mathbb{C}^1_x \cup \{\infty\}$ and $P = \infty \in X$.

(a) Find a basis for $L(nP)$.

(b) Show that for $n \geq 1$ the map $F\colon X \to \mathbb{P}^n$ determined by $L(nP)$ is an embedding.

(c) What is the degree of the curve $F(X) \subset \mathbb{P}^n$?

(d) Describe the divisors F^*H where $H \subset \mathbb{P}^n$ is a hyperplane.

(5) Let $X = \mathbb{C}^1_z/\Lambda$ be a complex torus and

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

the Weierstrass $\wp$-function. You may assume that $\wp$ defines a meromorphic function on X with a unique pole at $P = [0] \in X$ of order 2.

(a) Find a basis for $L(nP)$.

(b) Show that the map $F\colon X \to \mathbb{P}^1$ determined by $L(2P)$ has degree 2. Show that P is a ramification point of F. Use Riemann-Hurwitz to determine the number of ramification points.

(c) Show that for $n \geq 3$ the map $F\colon X \to \mathbb{P}^{n-1}$ determined by $L(nP)$ is an embedding. What is the degree of $F(X) \subset \mathbb{P}^{n-1}$? Show that for $n = 3$ there exists a line $L \subset \mathbb{P}^2$ such that $L \cap X = \{P\}$ and $(L \cdot X)_P = 3$.

(6) Let X be a compact Riemann surface and $F\colon X \to \mathbb{P}^1_{\mathbb{C}}$ be a map of degree 2. Let $P \in X$ be a ramification point. Choose coordinate x on $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^1_x \cup \{\infty\}$ so that $P = F^{-1}(\infty)$. Then $U := F^{-1}(\mathbb{C}^1_x)$ is given by

$$(y^2 = (x - \alpha_1) \cdots (x - \alpha_{2g+1})) \subset \mathbb{C}^2_{x,y}$$

for some $\alpha_1, \ldots, \alpha_{2g+1} \in \mathbb{C}^1_x$ (note that F has $2g+2$ ramification points by Riemann–Hurwitz).

(a) Show that

$$L(nP) = \begin{cases} \langle 1, x, \ldots, x^{\lfloor n/2 \rfloor} \rangle & \text{if } 0 \leq n \leq 2g \\ \langle 1, x, \ldots x^{\lfloor n/2 \rfloor}, y, xy, \ldots, x^{\lfloor (n-2g-1)/2) \rfloor} y \rangle & \text{if } n > 2g \end{cases}$$

[Hint: First determine $\nu_P(x)$ and $\nu_P(y)$. Second compute the dimension $l(nP)$ for $n > 2g - 2$ using Riemann–Roch.]

(b) Let $F_n\colon X \to \mathbb{P}^{r(n)}$ denote the map determined by $L(nP)$. Show that F_n is an embedding if $n \geq 2g + 1$.

(c) Describe the map F_n for $n \leq 2g$.