Lectures on Flips and Minimal Models

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This document contains notes from the lectures of Corti, Kollár, Lazarsfeld, and Mustaţă at the workshop "Minimal and canonical models in algebraic geometry" at the Mathematical Sciences Research Institute, Berkeley, CA, April 16-20, 2007. The lectures give an overview of the recent advances on canonical and minimal models of algebraic varieties obtained in the papers [HM], [BCHM].

Lecture 1, by Lazarsfeld, explains the lifting theorems used in the proof. Lectures 2 and 3 describe the proof of the existence of flips in dimension n assuming the MMP in dimension $n - 1$ from [HM]. Lecture 2, by Mustaţă, gives a geometric description of the restriction of the log canonical algebra to a boundary divisor (it is a so called adjoint algebra). Lecture 3, by Corti, proves that this algebra is finitely generated, following ideas of Shokurov. Lecture 4, by Kollár, gives an overview of the paper [BCHM] on the existence of minimal models for varieties of log general type.

Videos of the lectures are available at

www.msri.org/calendar/workshops/WorkshopInfo/418/show_workshop

Chapter 1

Extension theorems

Theorem 1.0.1. *(Hacon–McKernan) Let X be a smooth projective variety over $\mathbb{C}$. Let $T + \Delta$ be an effective $\mathbb{Q}$-divisor with snc support, with T irreducible and $\lfloor \Delta \rfloor = 0$. Assume $\Delta \sim_{\mathbb{Q}} A + B$, where A is ample, B is effective, and $T \not\subset \mathrm{Supp}\, B$. Assume that the stable base locus of $K_X + T + \Delta$ does not contain any intersection of irreducible components of $T + \Delta$. Choose k such that $k\Delta$ is integral and set $L = k(K_X + T + \Delta)$. Then the restriction map*

$$H^0(X, mL) \to H^0(T, mL_T)$$

is surjective for all $m \geq 1$.

Here and in what follows we write D_T for the restriction of a divisor D to a subvariety T.

Remark 1.0.2. Hacon and McKernan prove an analogous result for a projective morphism $X \to Z$ with Z affine.

History: The proof uses an idea of Siu which first appeared in his work on deformation invariance of plurigenera. There are also related works by Kawamata, Takayama, and others.

1.1 Multiplier and adjoint ideals

For an effective $\mathbb{Q}$-divisor D on X, we have the multiplier ideal $\mathcal{J}(X, D) \subseteq \mathcal{O}_X$. Roughly speaking it "measures" the singularities of the pair (X, D) — worse singularities correspond to deeper ideals.

Let L be a big line bundle on X. Set

$$\mathcal{J}(X, \|L\|) = \mathcal{J}(X, \frac{1}{p}D_p)$$

where $D_p \in |pL|$ is general and $p \gg 0$.

Proposition 1.1.1. *(1)* $H^0(L \otimes \mathcal{J}(X, \|L\|)) \overset{\sim}{\longrightarrow} H^0(L)$, *i.e., every section of L vanishes on $\mathcal{J}(\|L\|)$.*

(2) *If $M = K_X + L + P$ where P is nef then*

$$H^i(X, M \otimes \mathcal{J}(\|L\|)) = 0$$

for all $i > 0$ (this is a restatement of Kawamata–Viehweg–Nadel vanishing).

(3) *If $M = K_X + L + (\dim X + 1)H$ where H is very ample then $M \otimes \mathcal{J}(\|L\|)$ is globally generated (this follows from (2) plus Castelnuovo–Mumford regularity: if a sheaf $\mathcal{F}$ on projective space satisfies $H^i(\mathcal{F}(-i)) = 0$ for all $i > 0$ then $\mathcal{F}$ is globally generated).*

Here, given an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, line bundle M, and section $s \in \Gamma(X, M)$, we say *s vanishes on $\mathfrak{a}$* if $s \in \mathrm{Im}(\Gamma(M \otimes \mathfrak{a}) \to \Gamma(M))$.

Consider a smooth divisor $T \subset X$, $T \not\subset \mathrm{Supp}\, D$. We can define the *adjoint ideal* $\mathrm{Adj}_T(X, D) \subseteq \mathcal{O}_X$, such that

$$0 \to \mathcal{J}(X, D) \otimes \mathcal{O}_X(-T) \to \mathrm{Adj} \to \mathcal{J}(T, D_T) \to 0.$$

Assume that T is not contained in the stable base locus of L. We have

$$\mathcal{J}(T, \|L\|_T) := \mathcal{J}(T, \frac{1}{p}D_p|_T) \subseteq \mathcal{J}(T, \|L_T\|)$$

where $D_p \in |pL|$ is general and $p \gg 0$. We get $\mathrm{Adj}_T(X, \|L\|) \subseteq \mathcal{O}_X$, with

$$0 \to \mathcal{J}(X, \|L\|) \otimes \mathcal{O}_X(-T) \to \mathrm{Adj} \to \mathcal{J}(T, \|L\|_T) \to 0.$$

Now apply $(\cdot) \otimes M$. Idea: If $s \in \Gamma(T, M_T)$ vanishes on $\mathcal{J}(T, \|L\|_T)$ and if $M - (K_X + L + T)$ is nef, then s extends to a section of $\mathcal{O}_X(M)$ (we get vanishing of H^1 by Kawamata–Viehweg vanishing).

Consider as above $L = k(K_X + T + \Delta)$, $\Delta = A + B$, etc.

Lemma 1.1.2. *(Main Lemma) There exists a very ample divisor H (independent of p) such that for every $p \geq 0$, every $\sigma \in \Gamma(T, \mathcal{O}(pL_T + H_T) \otimes \mathcal{J}(\|pL_T\|))$ extends to $\hat{\sigma} \in \Gamma(X, \mathcal{O}_X(pL + H))$.*

Assuming the lemma, we prove the theorem:

(0) We may assume (T, B_T) is klt. Take $h \in \Gamma(X, H)$ general.

(1) Let $s \in \Gamma(T, mL_T)$. Consider $\sigma = s^l \cdot h \in \Gamma(T, lmL_T + H_T)$. The section σ vanishes on $\mathcal{J}(T, \|lmL_T\|)$. So there exists $\hat{\sigma} \in \Gamma(X, lmL + H)$ such that $\hat{\sigma}|_T = \sigma$ (by the Main Lemma).

(2) Let $F = \frac{mk-1}{mlk} \operatorname{div}(\hat{\sigma}) + B$. We find (using $\Delta = A + B$) that $mL - F - T = K_X + (\text{ample})$ if $l \gg 0$.

(3) Using (T, B_T) klt, we check that $\mathcal{O}_T(-\operatorname{div}(s)) \subseteq \mathcal{J}(T, F_T)$. So s vanishes on $\mathcal{J}(T, F_T)$.

(4) Finally, consider the sequence

$$0 \to \mathcal{J}(X, F) \otimes \mathcal{O}_X(-T) \to \operatorname{Adj} \to \mathcal{J}(T, F_T) \to 0$$

tensored by mL. We have $s \in H^0(\mathcal{O}_T(mL_T) \otimes \mathcal{J}(T, F_T))$ and $H^1(\mathcal{O}_X(mL - T) \otimes \mathcal{J}(X, F)) = 0$ by vanishing. So s extends.

1.2 Proof of the Main Lemma

We will only prove the special case $k = 1$, $L = K_X + T$ (so $\Delta = 0$ — we don't need $\Delta = A + B$ here), T not contained in the stable base locus of L. (Note: $L_T = K_T$). We prove

$(*)_p$: If $H = (\dim X + 1)(\text{very ample})$, then

$$H^0(T, \mathcal{O}(pL_T + H_T) \otimes \mathcal{J}(\|(p-1)L_T\|))$$

lifts to $H^0(X, pL + H)$.

The proof is by induction on p. The case $p = 1$ is OK by vanishing. Assume $(*)_p$ holds.

Claim 1. $\mathcal{J}(T, \|(p-1)L_T\|) \subseteq \mathcal{J}(T, \|pL + H\|_T)$.

Proof. By Castelnuovo–Mumford regularity and vanishing

$$\mathcal{O}_T(pL_T + H_T) \otimes \mathcal{J}(\|(p-1)L_T\|) \qquad (**)$$

is globally generated.

$(*)_p$ implies sections of $(**)$ lift to X. So

$$\mathcal{J}(T, \|(p-1)L_T\|) \subseteq b(X, |pL + H|) \cdot \mathcal{O}_T \subseteq \mathcal{J}(T, \|pL + H\|_T).$$

(here $b(X, |D|)$ denotes the ideal defining the base locus of $|D|$). $\qquad \square$

Consider the adjoint sequence

$$0 \to \mathcal{J}(\|pL + H\|)(-T) \to \mathrm{Adj} \to \mathcal{J}(T, \|pL + H\|_T) \to 0.$$

Apply $(\cdot) \otimes \mathcal{O}_X((p+1)L + H)$. By the Claim

$$H^0(\mathcal{O}_T((p+1)L_T + H_T) \otimes \mathcal{J}(\|pL_T\|)) \subseteq H^0(\mathcal{O}_T((p+1)L_T + H_T) \otimes \mathcal{J}(T, \|pL+H\|_T))$$

and

$$H^1(\mathcal{O}_X((p+1)L + H - T) \otimes \mathcal{J}(\|pL + H\|)) = 0.$$

This gives the desired lifting.

Chapter 2

Existence of flips I

This chapter is (mostly) an exposition of work of Hacon and McKernan.

2.1 The setup

Let $f\colon (X, D) \to Z$ be a birational projective morphism, where X is $\mathbb{Q}$-factorial, D is an effective $\mathbb{Q}$-divisor, Z is normal, $\rho(X/Z) = 1$, and $-(K_X + D)$ is f-ample. Assume also that f is small, i.e., that the exceptional locus has codimension ≥ 2. We consider two cases:

(1) klt flip: (X, D) klt.

(2) pl flip: (X, D) plt, $D = S + \Delta$ with $\lfloor D \rfloor = S$ irreducible, and $-S$ is f-ample.

It is well-known that the flip of f exists iff the $\mathcal{O}_Z$-algebra

$$\oplus_{m \geq 0} f_* \mathcal{O}_X(\lfloor m(K_X + D) \rfloor)$$

is finitely generated. This is a local question on Z, hence we may and will assume that Z is affine.

Shokurov: MMP in dimension $n-1$ plus existence of pl flips in dimension n implies existence of klt flips in dimension n. Therefore we need only consider the case of pl flips.

Remark 2.1.1. Let $R = \oplus_{i \geq 0} R_i$ be a graded domain such that R_0 is a finitely generated $\mathbb{C}$-algebra. Then the algebra R is finitely generated iff the *truncation*

$$R_{(k)} := \oplus_{i \geq 0} R_{ki}$$

is finitely generated. Indeed, we have an obvious action of $\mathbb{Z}/k\mathbb{Z}$ on R such that the ring of invariants is $R_{(k)}$, hence R finitely generated implies $R_{(k)}$ finitely generated. To see the converse, it is enough to note that for every $0 < j < k$, if $s \in \oplus_{i \geq 0} R_{ki+j}$ is a nonzero homogeneous element, then multiplication by s^{k-1} embeds $\oplus_{i \geq 0} R_{ki+j}$ as an ideal of $R_{(k)}$.

From now on, we assume that we are in the pl flip setting.

Remark 2.1.2. Since $\rho(X/Z) = 1$ and since we work locally over Z, it follows from our assumption that we may assume that there are positive integers p and q such that $p(K_X + D) \sim qS$ are linearly equivalent Cartier divisors.

Remark 2.1.3. Since Z is affine and f is small, it follows that S is linearly equivalent to an effective divisor not containing S in its support. In particular, it follows from the previous remark that there is a positive integer k and $G \in |k(K_X + D)|$ such that $S \not\subset \mathrm{Supp}(G)$.

A key remark due to Shokurov is that the algebra

$$\mathcal{R} = \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + S + \Delta) \rfloor))$$

is finitely generated iff the *restricted algebra*

$$\mathcal{R}|_S := \oplus_{m \geq 0} \mathrm{Im}\left(H^0(X, m(K_X + S + \Delta)) \to H^0(S, m(K_X + S + \Delta)|_S) \right)$$

is finitely generated. Sketch of proof: replacing X by a suitable $U \subset X$ such that $\mathrm{cod}(X - U) \geq 2$, we may assume that S is Cartier. Since

$$p(K_X + S + \Delta) \sim qS$$

for some positive integers p and q, it follows from Remark 2.1.1 that it is enough to show that the algebra $\mathcal{R}' = \oplus_{m \geq 0} H^0(X, \mathcal{O}(mS))$ is finitely generated. Using the fact that $\mathcal{R}|_S$ is finitely generated and Remark 2.1.1, we deduce that the quotient $\mathcal{R}'/h\mathcal{R}'$ is finitely generated, where $h \in \mathcal{R}'_1$ is an equation for S. Therefore $\mathcal{R}'$ is finitely generated.

The above discussion shows that existence of pl flips in dimension n (and so by Shokurov's result, existence of klt flips in dimension n) follows from MMP in dimension $(n-1)$ and the following Main Theorem, due to Hacon and McKernan.

Theorem 2.1.4. *Let $f \colon (X, S + \Delta) \to Z$ be a projective birational morphism, where X is $\mathbb{Q}$-factorial, $S + \Delta$ is an effective $\mathbb{Q}$-divisor divisor with $\lfloor S + \Delta \rfloor = S$ irreducible, and Z is a normal affine variety. Let k be a positive integer such that $k(K_X + S + \Delta)$ is Cartier. Suppose*

(1) $(X, S + \Delta)$ is plt.

(2) S is not contained in the base locus of $|k(K_X + S + \Delta)|$.

(3) $\Delta \sim_{\mathbb{Q}} A + B$ where A is ample, B is effective, and $S \not\subset \operatorname{Supp} B$.

(4) $-(K_X + S + \Delta)$ is f-ample.

If the MMP (with $\mathbb{R}$-coefficients) holds in dimension $\dim X - 1$, then the restricted algebra

$$\mathcal{R}|_S = \oplus_{m \geq 0} \operatorname{Im}(H^0(X, L^m) \to H^0(S, L^m|_S))$$

is finitely generated, where $L = \mathcal{O}_X(k(K_X + S + \Delta))$.

2.2 Adjoint algebras

In the setting of Theorem 2.1.4, we have by adjunction $(K_X + S + \Delta)|_S = K_S + \Delta|_S$. Moreover, the pair $(S, \Delta|_S)$ is klt. The trouble comes from the fact that the maps

$$H^0(X, mk(K_X + S + \Delta)) \to H^0(S, mk(K_S + \Delta|_S)) \qquad (2.1)$$

are not surjective in general.

Goal: find a model $T \to S$ such that (some truncation of) the restricted algebra $\mathcal{R}|_S$ can be written as

$$\oplus_{m \geq 0} H^0(T, B_m),$$

where $\{B_m\}$ is an additive sequence of Cartier divisors on T.

Terminology: an *additive sequence* is a sequence of divisors $\{B_m\}$ on a normal variety T such that $B_i + B_j \leq B_{i+j}$ for all i, j. Note that in this case $\oplus_{m \geq 0} H^0(T, \mathcal{O}(B_m))$ has a natural algebra structure. A typical example of additive sequence: start with a divisor D such that $|D|$ is nonempty, and let $B_m = \operatorname{Mob}(mD) := mD - \operatorname{Fix}|mD|$. More generally, if $\{D_m\}$ is an additive sequence such that $|D_m| \neq \emptyset$ for every m, and if we put $B_m = \operatorname{Mob}(D_m)$, then $\{B_m\}$ forms an additive sequence.

Given an additive sequence $\{B_m\}$, the *associated convex sequence* is given by $\{\frac{1}{m} B_m\}$. If each $\frac{1}{m} B_m$ is bounded above by a fixed divisor, set

$$B := \sup \frac{1}{m} B_m = \lim_{m \to \infty} \frac{1}{m} B_m.$$

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Remark 2.2.1. If B is semiample and if there exists i such that $B = \frac{1}{i}B_i$ (hence, in particular, B is a $\mathbb{Q}$-divisor), then $\oplus_{m \geq 0} H^0(T, \mathcal{O}(B_m))$ is finitely generated. Indeed, in this case $B = \frac{1}{m}B_m$ whenever $i \mid m$, and it is enough to use the fact that if L is a globally generated line bundle, then $\oplus_{m \geq 0} H^0(X, L^m)$ is finitely generated.

Suppose that $f \colon T \to Z$ is a projective morphism, where T is smooth and Z is affine. An *adjoint algebra* on T is an algebra of the form

$$\oplus_{m \geq 0} H^0(T, \mathcal{O}(B_m)),$$

where $\{B_m\}$ is an additive sequence and $B_m = mk(K_T + \Delta_m)$ for some $k \geq 1$ and $\Delta_m \geq 0$ such that $\Delta := \lim_{m \to \infty} \Delta_m$ exists and (T, Δ) is klt.

Our goal in what follows is to show that under the hypothesis of Theorem 2.1.4 (without assuming $-(K_X + S + \Delta)$ ample or MMP in $\dim(X) - 1$), the algebra $\mathcal{R}|_S$ can be written as an adjoint algebra. It is shown in Chapter 3 how one can subsequently use the fact that $-(K_X + S + \Delta)$ is ample to deduce that $\mathcal{R}|_S$ is "saturated", and then use MMP in $\dim(X) - 1$ to reduce to the case when the limit of the above Δ_m is such that $K_T + (\text{this limit})$ is semiample. This is enough to give the finite generation of $\mathcal{R}|_S$ (Shokurov proved this using diophantine approximation).

2.3 The Hacon–McKernan extension theorem

As we have already mentioned, the difficulty comes from the non-surjectivity of the restriction maps (2.1). We want to replace X by higher models on which we get surjectivity of the corresponding maps as an application of the following Extension Theorem, due also to Hacon and McKernan.

Theorem 2.3.1. *Let $(Y, T + \Delta)$ be a pair with Y smooth and $T + \Delta$ an effective $\mathbb{Q}$-divisor with snc support, with $\lfloor T + \Delta \rfloor = T$ irreducible. Let k be a positive integer such that $k\Delta$ is integral and set $L = k(K_Y + T + \Delta)$. Suppose*

(1) $\Delta \sim_{\mathbb{Q}} A + B$ where A is ample, B is effective, and $T \not\subset \text{Supp}(B)$.

(2) No intersection of components of $T + \Delta$ is contained in the base locus of L.

Then the restriction map

$$H^0(Y, L) \to H^0(T, L|_T)$$

is surjective.

This theorem was discussed in Chapter 1.

2.4   The restricted algebra as an adjoint algebra

Let $f\colon (X, S + \Delta) \to Z$ be a projective morphism, where X is $\mathbb{Q}$-factorial, $S + \Delta$ is an effective $\mathbb{Q}$-divisor with $\lfloor S + \Delta \rfloor = S$ irreducible, $(X, S + \Delta)$ is plt, and Z is an affine normal variety. Let k be a positive integer such that $k(K_X + S + \Delta)$ is Cartier and S is not contained in the base locus of $|k(K_X + S + \Delta)|$. Assume

$$\Delta \sim_{\mathbb{Q}} A + B \qquad (*)$$

where A is ample, B is effective, and $S \not\subset \operatorname{Supp} B$.

We will replace $(X, S + \Delta)$ by a log resolution (in fact, a family of resolutions) on which we can apply the extension theorem and use this to exhibit the restricted algebra as an adjoint algebra. Consider a birational morphism $f\colon Y \to X$, and let T be the strict transform of S. Write

$$K_Y + T + \Delta_Y = f^*(K_X + S + \Delta) + E$$

where Δ_Y and E are effective and have no common components, $f_* \Delta_Y = \Delta$, and E is exceptional.

Since E is effective and exceptional, we have $H^0(X, mk(K_X + S + \Delta)) \simeq H^0(Y, mk(K_Y + T + \Delta_Y))$ for every m. Therefore we may "replace" $K_X + S + \Delta$ by $K_Y + T + \Delta_Y$.

Step 1. After replacing Δ by the linearly equivalent divisor $A' + B'$, where $A' = \epsilon A$ and $B' = (1 - \epsilon)\Delta + \epsilon B$, with $0 < \epsilon \ll 1$, we may assume that we have equality of divisors $\Delta = A + B$ in $(*)$. We may also assume (after possibly replacing k by a multiple) that kA is very ample and that

$$A = \frac{1}{k} \cdot (\text{very general member of } |kA|).$$

Since A is general in the above sense, if $f\colon Y \to X$ is a projective birational morphism, we may assume $f^*A = \tilde{A}$ is the strict transform. In this case, $(*)$ will also hold for $(Y, T + \Delta_Y)$: indeed, there exists an effective exceptional divisor E' such that $f^*A - E'$ is ample, hence

$$\Delta_Y = (f^*A - E') + E' + (\cdots),$$

where $(\cdots)$ is an effective divisor that does not involve T.

Caveat: we will construct various morphisms f as above starting from $(X, S + \Delta)$. We will then modify Δ to satisfy $f^*A = \tilde{A}$, and therefore we need to check how this affects the properties of f.

Step 2. Let $f\colon Y \to X$ be a log resolution of $(X, S+\Delta)$. After modifying Δ as explained in Step 1, we write $\Delta_Y = \tilde{A} + \sum a_i D_i$ (note that f remains a log resolution for the new Δ).

Claim 2. After blowing up intersections of the D_i (and of their strict transforms) we may assume $D_i \cap D_j = \emptyset$ for all $i \neq j$.

The proof of the claim is standard, by induction first on the number of intersecting components and then on the sum of the coefficients of intersecting components. Note also that as long as we blowup loci that have snc with $\tilde{A}$, the condition $f^*(A) = \tilde{A}$ is preserved.

Step 3. We need to satisfy condition 2) in Theorem 2.3.1. The hypothesis implies that $T \not\subset \mathrm{Bs}|k(K_Y + T + \Delta_Y)|$. Since A is general, it follows that we need to worry only about the components D_i or about the intersections $D_i \cap T$ that are contained in $\mathrm{Bs}|k(K_Y + T + \Delta_Y)|$.

Canceling common components, we may replace Δ_Y by $0 \leq \Delta_Y' \leq \Delta_Y$ such that no component of Δ_Y' appears in the fixed part of $|k(K_Y + T + \Delta_Y')|$. We put $\Delta_Y' = \tilde{A} + \sum_i a_i' D_i$.

Step 4. Now we deal with intersections $T \cap D_i$ which are contained in the base locus of $|k(K_Y + T + \Delta_Y')|$. Let $h \colon \overline{Y} \to Y$ be the blowup of $T \cap D_i$. Note that because T is smooth and $T \cap D_i \subset T$ is a divisor the strict transform of T maps isomorphically to T. Let $F \subset \overline{Y}$ be the exceptional divisor. We write

$$K_{\overline{Y}} + (T + h^* \Delta_Y' - F) = h^*(K_Y + T + \Delta_Y').$$

Note that $T + h^* \Delta_Y' - F$ is effective, the coefficient of F being a_i'.

On $\overline{Y}$ the divisors T and $\tilde{D}_i$ are disjoint. We need to blowup again along $F \cap \tilde{D}_i$, but this gives an isomorphism around T. We repeat this process; however, we can continue only finitely many times because

$$\mathrm{ord}_{T \cap F}|k(K_{\overline{Y}} + T + h^* \Delta_Y' - F)|_T \leq \mathrm{ord}_{T \cap D_i}|k(K_X + T + \Delta_Y')|_T - 1.$$

Step 5. We change notation to denote by $(Y, T + \Delta_Y')$ the resulting pair (we emphasize that T hasn't changed starting with Step 3). We can now apply the extension theorem for $(Y, T + \Delta_Y')$. Consider the commutative diagram

$$\begin{array}{ccc} H^0(X, k(K_X + S + \Delta)) & \to & H^0(S, k(K_X + S + \Delta)|_S) \\ \downarrow & & \downarrow \\ H^0(Y, k(K_Y + T + \Delta_Y')) & \to & H^0(T, k(K_Y + T + \Delta_Y')|_T) \end{array}$$

The left arrow is an isomorphism and the right arrow is injective by construction, and the bottom arrow is surjective by the extension theorem. Hence writing $\Theta_1 = \Delta_Y'|_T$, we see that the component of degree k in $\mathcal{R}|_S$ is isomorphic to $H^0(T, k(K_T + \Theta_1))$.

Step 6. In order to prove further properties of the restricted algebra, one needs to combine the previous construction with "taking log mobile parts". For every $m \geq 1$ write

$$mk(K_Y + T + \Delta_Y) = M_m + \text{(fixed part)}.$$

Let $0 \leq \Delta_m \leq \Delta_Y$ be such that Δ_m has no common component with the above fixed part. After possibly replacing k by a multiple, $\{mk(K_Y + T + \Delta_m)\}$ is an additive sequence. We now apply Steps 3-5 for each $(Y, T + \Delta_m)$. Note that T remains unchanged. We get models $Y_m \to Y$ and divisors Θ_m on T such that the kth truncation of $\mathcal{R}|_S$ is isomorphic to

$$\oplus_{m \geq 0} H^0(T, km(K_T + \Theta_m)).$$

Moreover $\Theta_m \leq \Theta := \Delta_Y|_T$ implies $\Theta' := \lim_{m \to \infty} \Theta_m$ exists and (T, Θ') is klt. This proves that the restricted algebra is an adjoint algebra.

Chapter 3

Existence of flips II

This chapter is an exposition of work of Shokurov.

Recall (from Chapter 2)

Definition 3.0.1. Let $Y \to Z$ be a projective morphism with Y smooth and Z affine. An *adjoint algebra* is an algebra of the form

$$\mathcal{R} = \oplus_{m \geq 0} H^0(Y, N_m)$$

where $N_m = mk(K_Y + \Delta_m)$, the limit $\Delta := \lim_{m \to \infty} \Delta_m$ exists, and (Y, Δ) is klt.

Remark 3.0.2. Note that Δ can be an $\mathbb{R}$-divisor.

In Chapter 2, it was shown that the "restricted algebra" is an adjoint algebra.

Definition 3.0.3. Given an adjoint algebra $\mathcal{R}(Y, N_\bullet)$, set $M_i = \operatorname{Mob} N_i$, the mobile part of N_i, and $D_i = \frac{1}{i} M_i$. We say $\mathcal{R}$ is *a-saturated* if there exists a $\mathbb{Q}$-divisor F on Y, $\lceil F \rceil \geq 0$, such that $\operatorname{Mob}\lceil jD_i + F \rceil \leq jD_j$ for all $i \geq j > 0$.

Remark 3.0.4. In applications, F is always the discrepancy of some klt pair (X, Δ), $f \colon Y \to X$. That is

$$K_Y = f^*(K_X + \Delta) + F.$$

We sometimes write $F = \mathbb{A}(X, \Delta)_Y$.

Example 3.0.5. An *a*-saturated adjoint algebra on an affine curve is finitely generated.

Let $Y = \mathbb{C}$ and $P = 0 \in \mathbb{C}$. Let $N_i = m_i \cdot 0$, where $m_i + m_j \leq m_{i+j}$, and $D_i = \frac{1}{i} m_i \cdot 0 = d_i \cdot 0$. By assumption $d = \lim_{i \to \infty} d_i \in \mathbb{R}$ exists. In this context, a-saturation means there exists $b < 1$, $F = -b \cdot 0$, such that

$$\lceil jd_i - b \rceil \leq jd_j$$

for all $i \geq j > 0$.

We want to show $d \in \mathbb{Q}$, and $d = d_j$ for j divisible. Passing to the limit as $i \to \infty$ we get

$$\lceil jd - b \rceil \leq jd_j.$$

Assume $d \notin \mathbb{Q}$. Then

$$\{\langle jd \rangle \mid j \in \mathbb{N}\} \subset [0,1]$$

is dense (here $\langle \cdot \rangle$ denotes the fractional part). So, there exists j such that $\langle jd \rangle > b$, and then

$$jd_j \leq jd < \lceil jd - b \rceil \leq jd_j,$$

a contradiction. So $d \in \mathbb{Q}$. The same argument shows that $d_j = d$ if $j \cdot d \in \mathbb{Z}$.

Definition 3.0.6. An adjoint algebra $\mathcal{R}(Y, N_\bullet)$ is *semiample* if the limit $D = \lim_{i \to \infty} \frac{1}{i} M_i$ is semiample, where $M_i := \mathrm{Mob}(N_i)$.

Remark 3.0.7. We say that an $\mathbb{R}$-divisor D on Y is *semiample* if there exists a morphism $f \colon Y \to W$ with W quasiprojective such that D is the pullback of an ample $\mathbb{R}$-divisor on W. We say an $\mathbb{R}$-divisor is *ample* if it is positive on the Kleiman–Mori cone of curves.

Theorem 3.0.8. *If an adjoint algebra $\mathcal{R} = \mathcal{R}(Y, N_\bullet)$ is a-saturated and semiample then it is finitely generated.*

The proof is a modification of the one dimensional case, based on the following

Lemma 3.0.9. *Let $Y \to Z$ be projective with Y smooth and Z affine and normal. Let D be a semiample $\mathbb{R}$-divisor on Y, and assume that D is not a $\mathbb{Q}$-divisor. Fix $\epsilon > 0$. There exists a $\mathbb{Z}$-divisor M and $j > 0$ such that*

   (1) M is free.

   (2) $\|jD - M\|_{\sup} < \epsilon$

   (3) $jD - M$ is not effective.

Theorem 3.0.10. *Assume the MMP in dimension n (precisely, MMP with scaling for klt pairs with $\mathbb{R}$-coefficients. For the definition of MMP with scaling see Chapter 4). Let $\mathcal{R} = \mathcal{R}(Y, N_\bullet)$ be an adjoint algebra. Let $N_i = ik(K_Y + \Delta_i)$, $\Delta = \lim_{i\to\infty} \Delta_i$, and assume $K_Y + \Delta$ big. Then there exists a modification $\pi\colon Y' \to Y$ and $N'_\bullet$ such that $\mathcal{R} = \mathcal{R}(Y', N'_\bullet)$ is semiample.*

Remark 3.0.11. In the context of flips, the condition $K_Y + \Delta$ big is not an issue.

The proof of the theorem is based on the following

Lemma 3.0.12. *Let (X, Δ) be a klt pair, where X is $\mathbb{Q}$-factorial, Δ is an $\mathbb{R}$-divisor, and $K_X + \Delta$ is big. Assume the MMP in dimension n. Let $\Delta \in V \subset \mathrm{Div}_\mathbb{R} X$ be a finite dimensional vector space. There exist $\epsilon > 0$ and finitely many $g\colon X \dashrightarrow W_i$ (birational maps) such that if $D \in V$, $\|D - \Delta\| < \epsilon$, then for some i the pair $(W_i, g_{i*} D)$ is a log minimal model of (X, D).*

For the proof of the lemma see Chapter 4.

That the restricted algebra is *a*-saturated can be proved by a straightforward application of Kawamata–Viehweg vanishing as follows.

Theorem 3.0.13. *If $-(K_X + S + \Delta)$ is big and nef, then the restricted algebra is a-saturated.*

Proof. Let $(X, S + \Delta) \to Z$ be a pl-flipping contraction, and consider a birational morphism $f\colon Y \to X$. Let T denote the strict transform of S. Write

$$K_Y + T + \Delta_Y = f^*(K_X + S + \Delta) + E,$$

where Δ_Y and E are effective (with no common components) and E is exceptional. Assume $k(K_X + S + \Delta)$ is integral and Cartier. Let Δ_m be the largest divisor such that $0 \le \Delta_m \le \Delta$ and

$$M_m := \mathrm{Mob}(mk(K_Y + T + \Delta_m)) = \mathrm{Mob}(mk(K_Y + T + \Delta)).$$

We may assume M_m is free (for some f). Write $M_m^0 = M_m|_T$. For simplicity we only consider the case $i = j$, i.e., we prove

$$\mathrm{Mob}\lceil M_j^0 + F \rceil \le M_j^0,$$

where $F = \mathbb{A}(S, \Delta|_S)_T$.

If $\lceil E \rceil$ is an f-exceptional divisor, then $\mathrm{Mob}\lceil M_j + E \rceil \le M_j$. Choose $E = \mathbb{A}(X, S + \Delta)_Y + T$, so $E|_T = F$, and consider the short exact sequence

$$0 \to \mathcal{O}_Y(M_j + \mathbb{A}(X, S + \Delta)_Y) \to \mathcal{O}_Y(M_j + E) \to \mathcal{O}_T(M_j^0 + F) \to 0.$$

By the Kawamata-Viehweg vanishing theorem and our assumption, we know $H^1(\mathcal{O}_Y(M_j + \mathbb{A}(X, S + \Delta)_Y)) = 0$ (because $\mathbb{A}(X, S + \Delta)_Y = K_Y - f^*(K_X + S + \Delta)$). Now the a-saturated condition $\text{Mob}\lceil M_j^0 + F \rceil \leq M_j^0$ is implied by $\text{Mob}\lceil M_j + E \rceil \leq M_j$ and this extension result.

$\square$

Chapter 4

Notes on Birkar-Cascini-Hacon-McKernan

By the previous 3 lectures, we can start with the:

Assumption: We proved the existence of flips in dimension n, using minimal models in dimension $n-1$.

The main result is the following:

Theorem 4.0.14 (Minimal models in dimension n). *Assume that Δ is a big $\mathbb{R}$-divisor, (X, Δ) is klt and $K + \Delta$ is pseudo-effective. Then (X, Δ) has a minimal model.*

Let us first see some corollaries.

Corollary 4.0.15. *Δ is a big $\mathbb{Q}$-divisor, (X, Δ) is klt and $K + \Delta$ is pseudo-effective. Then the canonical ring*

$$\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)) \quad \text{is finitely generated.}$$

Proof. Get minimal model for (X, Δ), then use base point freeness. $\square$

Corollary 4.0.16. *X smooth, projective and K_X is big. Then the canonical ring*

$$\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)) \quad \text{is finitely generated.}$$

Proof. Pick some effective $D \sim mK_X$. Then $(X, \epsilon D)$ is klt (even terminal) for $0 < \epsilon \ll 1$ and ϵD is big. So $(X, \epsilon D)$ has a minimal model. It is automatically a minimal model for X. $\square$

Corollary 4.0.17. *X smooth, projective. Then the canonical ring*

$$\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)) \quad \text{is finitely generated.}$$

Proof. As in Kodaira's canonical bundle formula for elliptic surfaces, Fujino-Mori reduces the ring to a general type situation in lower dimension. $\square$

Corollary 4.0.18. *Let X be a Fano variety. Then the Cox ring*

$$\sum_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \quad \text{is finitely generated.}$$

Proof. See original. $\square$

Corollary 4.0.19. *If $K + \Delta$ is not pseudo-effective, then there exists a birational map $X \dashrightarrow X'$ and a Mori fiber space $X' \to Z'$.*

Proof. Fix H ample and the smallest $c > 0$ such that $K + \Delta + cH$ is pseudo-effective. (Note that a priori, c may not be rational.) After MMP, we get $X \dashrightarrow X'$ such that $K + \Delta + cH$ is nef on X'. It cannot be big since then $K + \Delta + (c - \epsilon)H$ would still be effective. So base point freeness gives $X' \to Z'$. $\square$

4.1 Comparison of 3 MMP's

In a minimal model program (or MMP) we start with a pair (X, Δ) and the goal is to construct a minimal model $(X, \Delta)^{min}$. (Note that minimal models are usually not unique, so $(X, \Delta)^{min}$ is a not well defined notational convenience.)

There are 3 ways of doing it.

4.1.1 Mori-MMP (libertarian)

This is the by now classical approach. Pick *any* extremal ray, contract/flip as needed. The hope is that eventually we get $(X, \Delta)^{min}$. This is known only in dimension ≤ 3 and almost known in dimension 4.

Note that even if (X, Δ) is known to have a minimal model, it is not at all clear that every Mori-MMP starting with (X, Δ) has to end (and thus yield a minimal model).

4.1.2   MMP with scaling (dictatorial)

Fix H and $t_0 > 0$ such that $K + \Delta + t_0 H$ is nef.

Let $t \to 0$. For a while $K + \Delta + tH$ is nef, but then we reach a critical value $t_1 \leq t_0$. That is, there exists an extremal ray R_1 such that $R_1 \cdot \left(K + \Delta + (t_1 - \eta)H\right) < 0$ for $\eta > 0$. Contract/flip this R_1 and continue.
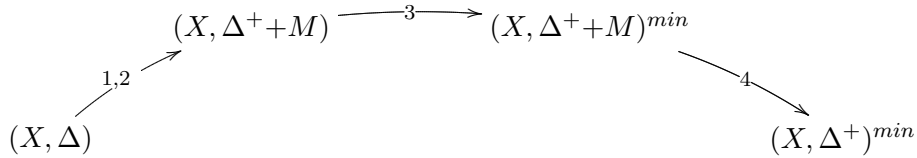
We will show that MMP with scaling works in any dimension, provided that Δ is big and $K + \Delta$ is pseudo-effective.

4.1.3   Roundabout MMP of BCHM

Instead of directly going for a minimal model, we start by steps that seem to make things more complicated. Then we aim for the minimal model on a carefully chosen path. There are 5 stages:

(1) Increase Δ in a mild manner to Δ^+.

(2) Increase Δ^+ wildly to $\Delta^+ + M$.

(3) Construct $(X, \Delta^+ + M)^{min}$.

(4) Remove the excess M to get $(X, \Delta^+)^{min}$.

(5) Prove that $(X, \Delta^+)^{min}$ is also $(X, \Delta)^{min}$.

You can imagine the process as in the picture:

$$(X, \Delta^+ + M) \xrightarrow{\quad 3 \quad} (X, \Delta^+ + M)^{min}$$

$$(X, \Delta) \xrightarrow{\quad 1,2 \quad} \qquad\qquad \xrightarrow{\quad 4 \quad} (X, \Delta^+)^{min}$$

Bending it like BCHM

(Note that the "C" of Cascini is pronounced as a "K".) We will show that bending works in any dimension, provided that Δ is big and $K + \Delta$ is pseudo effective.

(Major side issues). My presentation below ignores 3 important points.

(1) Difference between klt/dlt/lc. The main results are for klt pairs but many intermediate steps are needed for dlt or lc. These are purely technical points but in the past proofs collapsed on such technicalities.

(2) Relative setting. The induction and applications sometimes need the relative case: dealing with morphisms $X \to S$ instead of projective varieties. The relevant technical issues are well understood.

(3) I do not explain how to prove the *non-vanishing theorem*: If $K + \Delta$ is pseudo-effective (that is, a limit of effective divisors) then it is in fact effective.

4.1.4 Spiraling induction

We prove 4 results together as follows:

$$\boxed{\text{MMP with scaling in dim. } n - 1}$$

$$\Downarrow \text{Section 3}$$

$$\boxed{\text{Termination with scaling in dim. } n \text{ near } \lfloor \Delta \rfloor}$$

$$\Downarrow \text{Section 4}$$

$$\boxed{\text{Existence of } (X, \Delta)^{min} \text{ in dim. } n}$$

$$\Downarrow \text{Section 5}$$

$$\boxed{\text{Finiteness of } (X, \Delta + \sum t_i D_i)^{min} \text{ in dim. } n \text{ for } 0 \le t_i \le 1}$$

$$\Downarrow \text{Section 2}$$

$$\boxed{\text{MMP with scaling in dim. } n}$$

(Finiteness questions). While we expect all 3 versions of the MMP to work for any (X, Δ), the above finiteness is quite subtle. Even a smooth surface can contain infinitely many extremal rays, thus the very first step of the Mori-MMP sometimes offers an infinite number of choices.

By contrast, if Δ is big, then there are only finitely many possible models reached by Mori-MMP. This is, however, much stronger than what is needed for the proof.

It would be very useful to pin down what kind of finiteness to expect in general.

4.2   MMP with scaling

Begin with $K + \Delta + t_0 H$ nef, $t_0 > 0$.

(1) Set $t = t_0$ and decrease it.

(2) We hit a first *critical value* $t_1 \leq t_0$. Here $K + \Delta + t_1 H$ nef but $K + \Delta + (t_1 - \eta)H$ is not nef for $\eta > 0$.

(3) This means that there exists an extremal ray $R \subset \overline{NE}(X)$ such that

$$R \cdot (K + \Delta + t_1 H) = 0 \quad \text{and} \quad R \cdot H > 0.$$

Thus $R \cdot (K + \Delta) < 0$ and R is a "usual" extremal ray.

(4) Contract/flip R to get $X_0 \dashrightarrow X_1$ and continue.

The problem is that we could get an infinite sequence

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots .$$

*A*dvantage of scaling:

In the MMP with scaling, each X_i is a minimal model for some $K + \Delta + tH$. So, if we know finiteness of models as t varies then there is no infinite sequence. Thus we have proved the implication

$$\boxed{\text{Finiteness of } (X, \Delta + tH)^{min} \text{ in dim. } n \text{ for } 0 \leq t \leq t_0}$$
$$\Downarrow$$
$$\boxed{\text{MMP with scaling in dim. } n}$$

Remark 4.2.1. In the Mori-MMP, the X_i are *not* minimal models of anything predictable. This makes it quite hard to prove termination as above since we would need to control all possible models in advance.

4.3   MMP with scaling near $\lfloor \Delta \rfloor$

Start with $(X, S + \Delta)$, S integral. Here $(X, S + \Delta)$ is assumed dlt and in all interesting cases $S \neq 0$, so $(X, S + \Delta)$ is not klt.

We run MMP with scaling to get a series of contractions/flips

$$\cdots \dashrightarrow (X_i, S_i + \Delta_i) \xrightarrow{\phi_i} (X_{i+1}, S_{i+1} + \Delta_{i+1}) \dashrightarrow \cdots$$

Instead of termination, we claim only a much weaker result, traditionally known as *special termination*.

Proposition 4.3.1. *There are only finitely many i such that $S_i \cap \mathrm{Ex}(\phi_i) \neq \emptyset$.*

Proof. This is all old knowledge, relying on 6 basic observations.

(1) We can concentrate on 1 component of S and by a perturbation trick we can assume that S is irreducible.

(2) The discrepancy $a(E, X_i, S_i + \Delta_i)$ weakly increases with i and strictly increases iff $\mathrm{center}_{X_i}(E) \subset \mathrm{Ex}(\phi_i)$.

(3) There are very few E with $a(E, X_i, S_i + \Delta_i) < 0$.

(4) If $S_i \dashrightarrow S_{i+1}$ creates a *new* divisor $F_{i+1} \subset S_{i+1}$, then there exists E_{i+1} with $a(E_{i+1}, X_i, S_i + \Delta_i) < a(E_{i+1}, X_{i+1}, S_{i+1} + \Delta_{i+1}) \leq 0$.

(5) Combining these we see that $S_i \dashrightarrow S_{i+1}$ creates no new divisors for $i \gg 1$.

(6) Only finitely many $S_i \dashrightarrow S_{i+1}$ contracts a divisor, since at such a step the Picard number of S_i drops.

Thus, for $i \gg 1$, each $S_i \dashrightarrow S_{i+1}$ is an isomorphism in codimension 1. (Maybe such a map could be called a *traverse*.)

Therefore, if $X_0 \dashrightarrow X_1 \cdots$ is an MMP with scaling, then

$$S_N \xrightarrow{\phi_N|_{S_N}} S_{N+1} \dashrightarrow S_{N+1} \dashrightarrow \cdots$$

is also an MMP with scaling for $N \gg 1$, except that $\phi_i|_{S_i}$ is an isomorphism when $S_i \cap \mathrm{Ex}(\phi_i) \neq \emptyset$. By induction, $\phi_i|_{S_i}$ is an isomorphism for $i \gg 1$, hence $S_i \cap \mathrm{Ex}(\phi_i) = \emptyset$. $\qquad\square$

Thus we have shown that

$$\boxed{\text{MMP with scaling in dim. } n-1}$$
$$\Downarrow$$
$$\boxed{\text{Termination with scaling in dim. } n \text{ near } \lfloor \Delta \rfloor}$$

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4.4 Bending it like BCHM

This is the hardest part. We assume termination with scaling near $\lfloor \Delta \rfloor$ (for many different X and Δ), and we prove that the roundabout MMP also works.

The proof uses 2 basic lemmas. The first one shows that under certain conditions, every flip we have to do involves $\lfloor \Delta \rfloor$. The second one shows how to increase $\lfloor \Delta \rfloor$ without changing $(X, \Delta)^{min}$.

Lemma 4.4.1 (Scaling to the boundary). *Assume that*

(1) $K + \Delta \sim cH + F$ for some $c \geq 0$ and $F \geq 0$.

(2) $K + \Delta + H$ is nef.

(3) $\mathrm{Supp}(F) \subset \lfloor \Delta \rfloor$.

Then $(X, \Delta + t \cdot H)^{min}$ exists for every $0 \leq t \leq 1$.

Proof. (To accomodate $\mathbb{R}$-divisors, $\sim$ can stand for $\mathbb{R}$-linear equivalence.) Start scaling. At the critical value, get a ray R such that $R \cdot H > 0$ and $R \cdot (K + \Delta) < 0$. Thus $R \cdot (cH + F) < 0$ and $R \cdot F < 0$, so $\mathrm{locus}(R) \subset \mathrm{Supp}(F) \subset \lfloor \Delta \rfloor$.

Thus every flip encountered in the scaling MMP intersects the boundary. Thus we have termination. $\qquad \square$

If we start with a klt pair (X, Δ), then $\lfloor \Delta \rfloor = 0$. From condition (3) thus $F = 0$ and hence $K + \Delta \sim cH$. Therefore $K + \Delta \sim \frac{c}{c+1}(K + \Delta + H)$ is nef and we have nothing to do. Conclusion: *We need a way to increase Δ without changing the minimal model!*

Lemma 4.4.2 (Useless divisor lemma). *If $\Delta' \subset$ stable base locus of $(K + \Delta)$, then $(X, \Delta + \Delta')^{min} = (X, \Delta)^{min}$.*

Proof. Note that

$$(\Delta')^{min} \quad \subset \quad \begin{array}{c} \text{stable base locus of } (K + \Delta)^{min} \\ \| \\ \text{stable base locus of } (K + \Delta + \Delta')^{min} \end{array}$$

However, $(K + \Delta + \Delta')^{min}$ is base point free, thus $(\Delta')^{min} = 0$, and so $(X, \Delta + \Delta')^{min} = (X, \Delta)^{min}$. $\qquad \square$

Corollary 4.4.3. *We can always reduce to the case when the stable base locus of $K + \Delta^+$ is in $\lfloor \Delta^+ \rfloor$.*

Proof. First we can take a log resolution to assume that (X, Δ) has simple normal crossings only. Write the stable base locus as $\sum_{i=1}^{r} D_i$ and $\Delta = \sum_{i=1}^{r} d_i D_i + \text{(other divisors)}$, where $d_i = 0$ is allowed. Set $\Delta' := \sum_{i=1}^{r}(1 - d_i)D_i$. Then $\Delta^+ := \Delta + \Delta' = \sum_{i=1}^{r} D_i + \text{(other divisors)}$ and $\lfloor \Delta + \Delta' \rfloor \supset \sum_{i=1}^{r} D_i$. $\qquad\square$

4.4.4 (Bending I: $K + \Delta$ is a $\mathbb{Q}$-divisor).

We proceed in 6 steps:

(0) Write $K + \Delta \sim rM + F$ where M is mobile, irreducible and F is in the stable base locus.

(1) Take log resolution. With Δ' as above, set $\Delta^+ := \Delta + \Delta'$ and $F^+ := F + \Delta'$. Then $K + \Delta^+ \sim rM + F^+$ and $\mathrm{Supp}(F^+) \subset \lfloor \Delta^+ \rfloor$.

(2) Add M to Δ^+ and pick an ample H to get

    (a) $K + \Delta^+ + M \sim 0 \cdot H + \big((r+1)M + F^+\big)$

    (b) $K + \Delta^+ + M + H$ is nef, and

    (c) $\mathrm{Supp}(M + F^+) \subset \lfloor \Delta^+ + M \rfloor$.

(3) Scale by H to get $(X, \Delta^+ + M)^{min}$. Now we have:

    (a) $K + \Delta^+ \sim rM + F^+$.

    (b) $K + \Delta^+ + M$ is nef.

    (c) $\mathrm{Supp}(F^+) \subset \lfloor \Delta^+ \rfloor$.

(4) Scale by M to get $(X, \Delta^+)^{min}$.

(5) By the useless divisor lemma, $(X, \Delta^+)^{min} = (X, \Delta)^{min}$.

4.4.5 (Bending II: $K + \Delta$ is an $\mathbb{R}$-divisor).

We follow the same 6 steps, but there are extra complications.

(0) Write $K + \Delta \sim r_i M_i + F$, where the M_i are mobile, irreducible and F is in the stable base locus. (This is actually not obvious and the Oct. 2006 version has mistakes.)

(1–3) goes as before and we get

    (a) $K + \Delta^+ \sim \sum r_i M_i + F^+$

    (b) $K + \Delta^+ + \sum M_i$ is nef, and

(c) $\mathrm{Supp}(F^+) \subset \lfloor\Delta^+\rfloor$.

(4) Let me first describe 2 attempts that do not work.

(First try): Scale $\sum M_i$. The problem is that $\sum r_i M_i \neq c \sum M_i$.

(Second try): Scale M_1. This works, but the support condition (c) fails at next step when we try to scale M_2.

(Third try): Do not take *all* of M_1 away at the first step.

Reorder the indices so that $r_1 \leq r_2 \leq \cdots \leq r_k$. We then construct inductively minimal models for

$$\left(X, \Delta^+ + \tfrac{1}{r_j}(r_1 M_1 + \cdots + r_{j-1} M_{j-1}) + M_j + \cdots + M_k\right)$$

We already have the $j=1$ case. Let us see how to do $j \to j+1$.

By assumption $K + \Delta^+ + \tfrac{1}{r_j}(r_1 M_1 + \cdots + r_{j-1} M_{j-1}) + (M_j + \cdots + M_k)$ is nef. Move M_j from the right sum to the left. Thus now we have:

(a) $K + \Delta^+ \sim (r_1 M_1 + \cdots + r_j M_j) + (r_{j+1} M_{j+1} + \cdots + r_k M_k + F^+)$

(b) $K + \Delta^+ + \tfrac{1}{r_j}(r_1 M_1 + \cdots + r_j M_j) + (M_{j+1} + \cdots + M_k)$ is nef, and

(c) $\mathrm{Supp}(r_{j+1} M_{j+1} + \cdots + r_k M_k + F^+) \subset \lfloor M_{j+1} + \cdots + M_k + \Delta^+ \rfloor$.

Scale $r_1 M_1 + \cdots + r_j M_j$ with scale factor r_j/r_{j+1}. This gives the $j+1$ case.

At the end (and after moving M_k into the sum) we have:

(a) $K + \Delta^+ \sim \sum r_i M_i + F^+$

(b) $K + \Delta^+ + \tfrac{1}{r_k}(r_1 M_1 + \cdots + r_k M_k)$ is nef.

(c) $\mathrm{Supp}(F^+) \subset \lfloor\Delta^+\rfloor$.

Now we can scale by $r_1 M_1 + \cdots + r_k M_k$ to get $(X, \Delta^+)^{min}$.

(5) By the useless divisor lemma, $(X, \Delta^+)^{min} = (X, \Delta)^{min}$.

Thus we have proved that

$$\boxed{\text{Termination with scaling in dim. } n \text{ near } \lfloor\Delta\rfloor}$$
$$\Downarrow$$
$$\boxed{\text{Existence of } (X, \Delta)^{min} \text{ in dim. } n}$$

4.5 Finiteness of models

More generally, we claim that the set of models $(X, \Delta_w)^{min}$ is finite as Δ_w moves in a compact set of $\mathbb{R}$-divisors satisfying 3 conditions:

(1) Every Δ_w is big. Note that being big is not a closed condition and it would be very good to remove this restriction. Without it one could get minimal models for non-general type (X, Δ) by getting $(X, \Delta + \epsilon(\text{ample}))^{min}$ and then letting $\epsilon \to 0$.

(2) Every $K + \Delta_w$ is effective. By definition, being pseudo-effective is a closed condition. It is here that non-vanishing comes in: it ensures that being effective is also a closed condition.

(3) $K + \Delta_w$ is klt. This is preserved by making Δ_w smaller, which is what we care about.

By compactness, it is enough to prove finiteness locally, that is, finiteness of the models $(X, \Delta' + \sum t_i D_i)^{min}$ for $|t_i| \leq \epsilon$ (depending on Δ'). (Important point: Even if we only care about $\mathbb{Q}$-divisors, we need this for $\mathbb{R}$-divisors Δ'!)

Proof. Induction on r for $D_1, \ldots, D_r$. Let $(X^m, \Delta^m) := (X, \Delta')^{min}$. By the base point free theorem, we have $g : X^m \to X^c$ such that $K_{X^m} + \Delta^m \sim g^*(\text{ample})$.

So, for $|t_i| \ll 1$, $g^*(\text{ample}) \gg \sum t_i D_i$, *except* on the fibers of g. (This is more delicate than it sounds, but not hard to prove for extremal contractions.)

Thus the MMP to get $(X^m, \Delta^m + \sum t_i D_i)^{min}$ is relative to X^c. We can switch to working locally over X^c, we thus assume that $K_{X^m} + \Delta^m \sim 0$. So

$$K_{X^m} + \Delta^m + c\sum t_i D_i \sim c\big(K_{X^m} + \Delta^m + \sum t_i D_i\big)$$

Therefore $(X^m, \Delta^m + \sum t_i D_i)^{min} = (X^m, \Delta^m + c\sum t_i D_i)^{min}$. For $(t_1, \ldots, t_r)$ choose c such that $\max_i |ct_i| = \epsilon$, that is, $(ct_1, \ldots, ct_r)$ is on a face of $[-\epsilon, \epsilon]^r$. Ths shows that we get all possible $(X^m, \Delta^m + \sum t_i D_i)^{min}$ by computing $(X^m, \Delta^m + \sum t_i D_i)^{min}$ only for those $(t_1, \ldots, t_r)$ which are on a face of the r-cube $[-\epsilon, \epsilon]^r$. The faces are $2r$ copies of the $(r-1)$-cube. So we are done by induction on r.

The notation $(X, \Delta)^{min}$ helped us skirt the issue whether there may be infinitely many minimal models for a given (X, Δ). (This happens in the non-general type cases, even for smooth minimal models.) This is, however,

no problem here. We found one $\mathbb{Q}$-factorial model $g : X^m \to X^c$ such that K_{X^m} is numerically g-trivial.

Let $D_1, \ldots, D_r$ be a basis of the Néron-Severi group. Every other model where $K + \Delta$ is nef lives over X^c and some $\sum t_i H_i$ is g-ample on it. We can thus find every such model as a canonical model for $(X, \Delta + \epsilon \sum t_i H_i)$ for $0 < \epsilon \ll 1$. $\qquad\square$

We have now proved that

$$\boxed{\text{Existence of } (X, \Delta)^{min} \text{ in dim. } n}$$
$$\Downarrow$$
$$\boxed{\text{Finiteness of } (X, \Delta + \sum t_i D_i)^{min} \text{ in dim. } n \text{ for } 0 \le t_i \le 1}$$

and the spiraling induction is complete.

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