11/1/19

HW 5 due now

Last Time: • Two triangles are congruent $\iff$ there is an isometry sending one triangle to the other

• Every isometry is a composite of ≤ 3 reflections

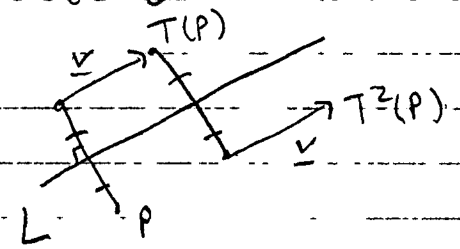
Today: • Finish classification of isometries (identity, translation, rotation, reflection, glide reflection)

• Spherical geometry

Given a glide reflection T, how to find the line of reflection?

Observe $T^2 = T \circ T = \mathrm{Trans}_{2\underline{v}}$ $\quad T = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$

$\rightsquigarrow$ find $\underline{v}$



Compose T with $\mathrm{Trans}_{\underline{v}}^{-1} = \mathrm{Trans}_{-\underline{v}}$

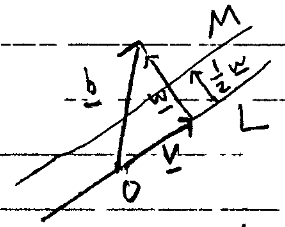
$(\mathrm{Trans}_{\underline{v}})^{-1} \circ T = \mathrm{Refl}_L$:

Then, find L as before ($L = \mathrm{Fix}(\mathrm{Refl}_L)$)

$T(\underline{x}) = A\underline{x} + \underline{b}$ $\quad$ If T is a glide reflection then,

$U(\underline{x}) = A\underline{x}$ is a reflection in a line L (passing through the origin)

So $T = \mathrm{Trans}_{\underline{b}} \circ \mathrm{Refl}_L$ BUT $\underline{b}$ is not in direction of L in general



In fact, there is a line M parallel to L & a vector $\underline{v}$ in the direction of M such that

$T = \mathrm{Trans}_{\underline{b}} \circ \mathrm{Refl}_L = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_M$

(Need to show $\mathrm{Trans}_{\underline{w}} \circ \mathrm{Refl}_L = \mathrm{Refl}_M$).

Classification of Isometries

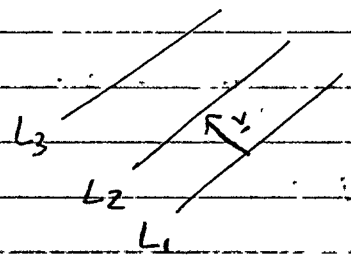
So far: every isometry is a composite of ≤ 3 reflections

# of reflections	Isometry
0	identity
1	reflection
2	rotation or translation (or identity if $L_1 = L_2$)
3	? ← analyze this case

a. reflection...
b. glide reflection

2 cases:

$\boxed{a.}$ L_1, L_2, L_3 are parallel



$T = Refl_{L_3} \circ \underbrace{(Refl_{L_2} \circ Refl_{L_1})}_{Trans_{2v}} ?$

$Trans_{2v}$

$= Refl_{L_2'} \circ Refl_{L_1'}$ where L_2' & L_1' are obtained from L_1, L_2 by translating some distance perp. to the lines

Take $L_2' = L_3$

$\rightsquigarrow = Refl_{L_3} \circ (Refl_{L_2'} \circ Refl_{L_1'})$

$= (\cancel{Refl_{L_3} \circ Refl_{L_2'}}) \circ Refl_{L_1'}$

$= Refl_{L_1'}$

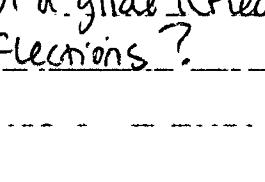
$\boxed{b.}$ L_1, L_2, L_3 not all parallel

subcase (i) L_1, L_2 are not parallel

subcases (ii) L_2, L_3 are not parallel

(Note: if $L_1 \parallel L_2$ & $L_2 \parallel L_3$ then $L_1 \parallel L_3$ & all 3 lines are parallel)

(i)



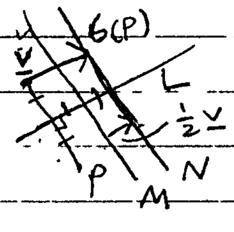
$T = Refl_{L_3} \circ Refl_{L_2} \circ Refl_{L_1}$

"Work backwards". Guess T is a glide reflection.

Given a glide reflection, G, how can we write it as a composite of 3 reflections?

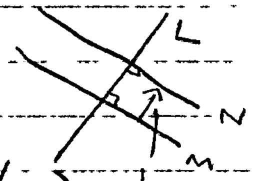
→

$G = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$



$$\overset{\shortparallel}{(\mathrm{Refl}_N \circ \mathrm{Refl}_M) \circ \mathrm{Refl}_L}$$

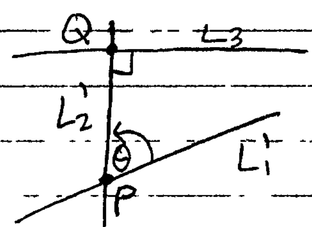
<u>Recap</u>: $\mathrm{Refl}_N \circ \mathrm{Refl}_M \circ \mathrm{Refl}_L$
$= \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$



(*)

$$T = \mathrm{Refl}_{L_3} \circ \underbrace{(\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1})}_{\substack{\shortparallel \\ \mathrm{Rot}(P, 2\theta) \\ \shortparallel \\ \mathrm{Refl}_{L_2'} \circ \mathrm{Refl}_{L_1'}}}$$

where L_1' & L_2' are obtained from L_1 & L_2 by a rotation about P.



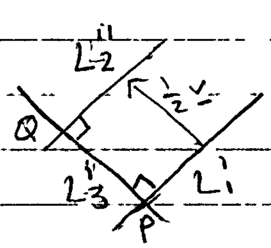
Choose this rotation so $L_2' \perp L_3$.

$T = \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2'} \circ \mathrm{Refl}_{L_1'}$

Trying to get to (*)

Rotate L_2' & L_3 about $Q \rightsquigarrow L_2'', L_3'$ so that $L_3' \perp L_1'$.

$\rightsquigarrow$



$$T = \underbrace{(\mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2'})}_{\shortparallel} \circ \mathrm{Refl}_{L_1'}$$

$\mathrm{Rot}(Q, 2\phi) = \mathrm{Refl}_{L_3'} \circ \mathrm{Refl}_{L_2''}$

Then, $T = \mathrm{Refl}_{L_3'} \circ \underbrace{\mathrm{Refl}_{L_2''} \circ \mathrm{Refl}_{L_1'}}_{\mathrm{Trans}_{2\underline{v}}}$

& L_3' line in direction of $\underline{v}$.
This is a glide reflection.

Subcase (ii) is very similar. ▨

<u>Note</u>: For a glide reflection $G = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$
$= \mathrm{Refl}_L \circ \mathrm{Trans}_{\underline{v}}$
where $\underline{v}$ is in the direction of L.