Math 412 Homework 9

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April 24, 2013

Show your work and justify your answers carefully. All fields are assumed to have characteristic 0.

(1) Let $f(x) = x^2 + bx + c \in K[x]$ be a monic quadratic polynomial with coefficients in a field K. Let $D = b^2 - 4c$ be the *discriminant* of $f(x)$, and $\alpha_1, \alpha_2 \in L$ the roots of f in some field extension $K \subset L$. Show that $D = (\alpha_1 - \alpha_2)^2$.

(2) Let $L = \mathbb{Q}(\sqrt[4]{3})$.

(a) Compute the Galois group $G(L/\mathbb{Q})$ explicitly. Is the extension $\mathbb{Q} \subset L$ Galois?

(b) Find an intermediate field $\mathbb{Q} \subset M \subset L$ such that both $\mathbb{Q} \subset M$ and $M \subset L$ are Galois.

(c) Find a field $L \subset N$ such that $\mathbb{Q} \subset N$ is Galois.

(3) For each of the following polynomials $f(x) \in \mathbb{Q}[x]$: (i) Determine the splitting field L of $f(x)$ over $\mathbb{Q}$ and the Galois group $G(L/\mathbb{Q})$; (ii) Draw the subgroup diagram of $G(L/\mathbb{Q})$, the diagram of intermediate fields $\mathbb{Q} \subset M \subset L$, and identify explicitly the correspondence between the diagrams given by the Main Theorem of Galois theory.

(a) $x^3 + 1$.

(b) $x^3 - 7$.

(c) $x^4 - 8x^2 + 15$.

(d) $x^4 + x^3 + x^2 + x + 1$.

[Hint: (d) Review HW8Q5b.]

(4) Let K be a field and $L = K(\sqrt{d}, \sqrt{e})$ for some $d, e \in K$. Assume $[L : K] = 4$.

  (a) List the intermediate fields $K \subset M \subset L$.

  (b) Let $\gamma \in L$, then we have

$$\gamma = a_1 + a_2\sqrt{d} + a_3\sqrt{e} + a_4\sqrt{de}$$

  for some $a_1, a_2, a_3, a_4 \in K$. Using part (a) or otherwise, show that $K(\gamma) = L$ iff at least 2 of a_2, a_3, a_4 are nonzero.

(5) Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$. Let L be the splitting field for $f(x)$ over $\mathbb{Q}$

  (a) Describe L explicitly in the form $\mathbb{Q}(\alpha, \beta)$ for some $\alpha, \beta \in \mathbb{C}$. (Note: The complex numbers α, β are not uniquely determined, but a simple choice of α, β may be useful for the rest of this problem.)

  (b) Compute the degree $[L : \mathbb{Q}]$.

  (c) The Galois group $G(L/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group D_4 (the group of symmetries of a square) by HW8Q7. State a general theorem and use it to deduce that $G(L/\mathbb{Q})$ is isomorphic to D_4.

  (d) Give a direct proof that $G(L/\mathbb{Q})$ is isomorphic to D_4 by exhibiting elements of $G(L/\mathbb{Q})$ corresponding to generators σ and τ of D_4 given by a rotation of order 4 and a reflection respectively.

(6) Let $f(x) \in K[x]$ be a monic polynomial of degree n and suppose there is a factorization $f(x) = g(x)h(x)$ with $g(x), h(x) \in K[x]$. Let L be the splitting field of $f(x)$ over K, and assume that f has no repeated roots in L. Show that the Galois group $G(L/K)$ is isomorphic to a subgroup of the direct product $S_m \times S_{n-m}$, where m is the degree of $g(x)$.

(7) Let $\mathbb{C}(t)$ denote the field of rational functions in the variable t with complex coefficients. Consider the automorphism $\varphi\colon \mathbb{C}(t) \to \mathbb{C}(t)$ given by $\varphi(t) = \frac{1}{t}$ and $\varphi(a) = a$ for $a \in \mathbb{C}$.

  (a) Show that φ is an automorphism of order 2, that is $\varphi^2(f) := \varphi(\varphi(f)) = f$ for all $f \in \mathbb{C}(t)$. Let $G = \langle\varphi\rangle = \{1, \varphi\} \simeq \mathbb{Z}/2\mathbb{Z}$ denote the group generated by φ.

(b) Show that the function $u(t) := t + \frac{1}{t}$ is fixed by φ, that is, $\varphi(u) = u$.

(c) State a general theorem and use it to determine the degree $[\mathbb{C}(t) : \mathbb{C}(t)^G]$ of $\mathbb{C}(t)$ over the fixed field

$$\mathbb{C}(t)^G := \{f \in \mathbb{C}(t) \mid g(f) = f \,\forall\, g \in G\} = \{f \in \mathbb{C}(t) \mid \varphi(f) = f\}.$$

(d) Compute the irreducible polynomial $f(x) \in \mathbb{C}(u)[x]$ of t over the field $\mathbb{C}(u)$.

(e) Using part (d) compute the degree $[\mathbb{C}(t) : \mathbb{C}(u)]$ and, combining with part (c), deduce that $\mathbb{C}(t)^G = \mathbb{C}(u)$.