Math 461 Lecture 29 11/9

homework 6 solutions available

homework 7 available

no class monday (Veteran's Day)

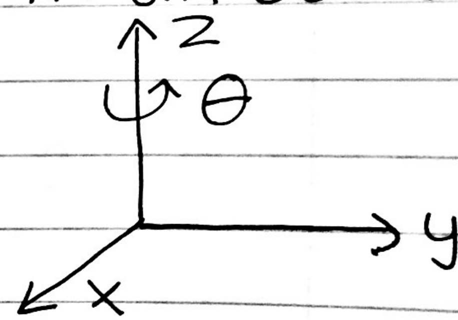
last time:

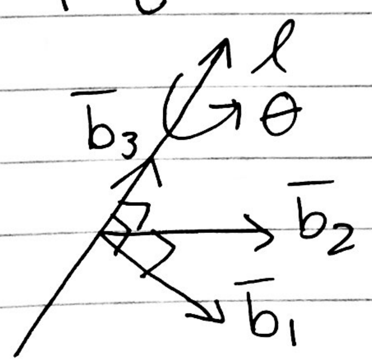
algebraic formulas for 3D rotations

$T: \mathbb{R}^3 \to \mathbb{R}^3 \quad T(\bar{x}) = A\bar{x} \quad A = PMP^T$

$P = (\bar{b}_1\ \bar{b}_2\ \bar{b}_3)$ right handed orthonormal basis $\bar{b}_3$ in direction of ℓ

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$





Today:

3D isometries

algebraic description $\rightsquigarrow$ geometric description

example: $T(\bar{x}) = \frac{1}{7}\begin{pmatrix} -2 & -3 & -6 \\ -3 & 6 & -2 \\ -6 & -2 & 3 \end{pmatrix}\bar{x}$

i.e. $T(x,y,z) = \frac{1}{7}(-2x-3y-6z, -3x+6y-2z, -6x-2y+3z)$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ isometry of $\mathbb{R}^3$

reflection in plane π

question: find the plane?

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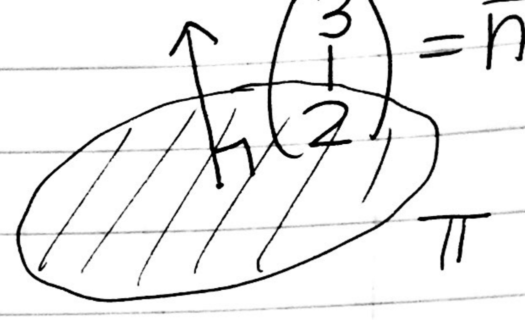
Find $\text{Fix}(T) = \{\bar{x} \in \mathbb{R}^3 \mid T(\bar{x}) = \bar{x}\}$

$$x = \frac{1}{7}(-2x - 3y - 6z)$$

$$y = \frac{1}{7}(-3x + 6y - 2z)$$

$$z = \frac{1}{7}(-6x - 2y + 3z)$$



Simplify:
$$9x + 3y + 6z = 0$$
$$3x + y + 2z = 0$$
$$6x + 2y + 4z = 0$$

π: $3x + y + 2z = 0$

could have put them in a matrix and row reduced to find the solution

$$3x + y + 2z = 0 \iff \bar{x} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 0$$

Example: $T(\bar{x}) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \bar{x}$ is a rotation

Find the axis and angle of rotation

$T(x, y, z) = (-y, z, -x)$

$\text{Fix}(T)$ = axis of rotation ℓ, a line in $\mathbb{R}^3$

$\bar{x} = T(\bar{x})$ $\quad x = -y \quad y = z \quad z = -x$

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$x = -y \rightsquigarrow x + y = 0$

$y = z \rightsquigarrow y - z = 0$

$z = -x \rightsquigarrow z + x = 0$

augmented matrix:

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ \\ -R1 \end{array} \rightsquigarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \begin{array}{c} \\ \\ +R2 \end{array} \rightsquigarrow \begin{array}{c} -R2 \\ \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

reduced echelon

z free variable

$x + z = 0 \qquad x = -z$

$y - z = 0 \qquad y = z$

z arbitrary

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

ℓ (axis) is line through origin in direction $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

angle of rotation?

we know $T(\bar{x}) = A\bar{x}$ $A = PMP^T$ where

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$P = (\bar{b}_1, \bar{b}_2\ \bar{b}_3)$

$P^T = P^{-1}$

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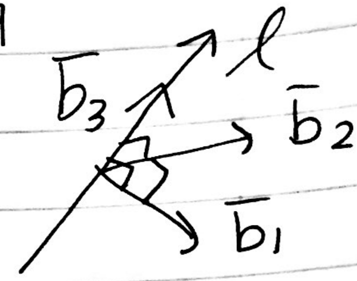
$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = A = PMP^T$$



recall: $\text{trace}(PMP^{-1}) = \text{trace}(M)$

$\text{trace}(M) =$ sum of diagonal entries

so $0 = \text{trace}(A) = \text{trace}(M) = 2\cos\theta + 1$

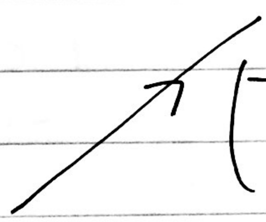
$2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

minor problem:

we don't determine the sense of rotation by this method

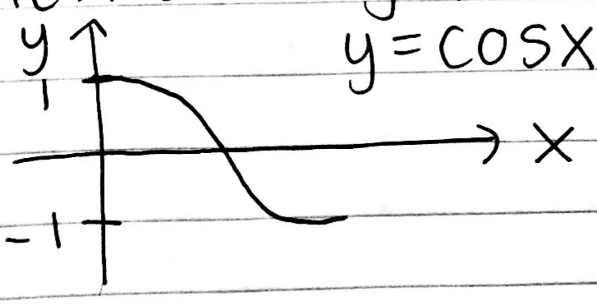
sense: ccw or cw

axis



$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \theta = \frac{2\pi}{3}$

when solving for θ take $0 \le \theta \le \pi$



example: rotary reflection

$$T(\bar{x}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bar{x}$$

find the plane of reflection, axis of rotation and angle

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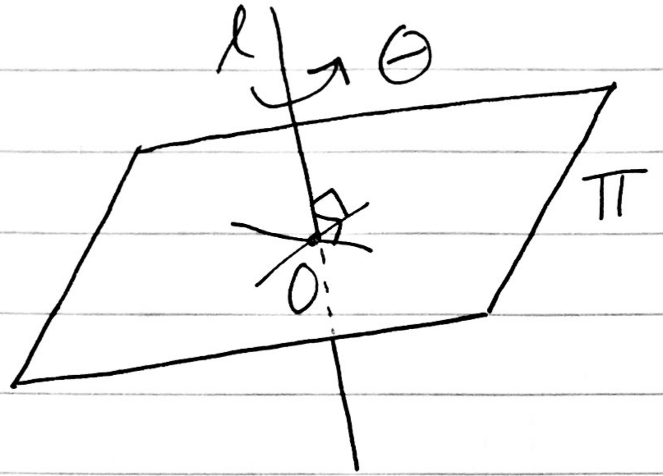

reflection in Π, rotate about ℓ

observe T^2:

$$T^2 = (Rot(\ell,\theta) \circ Refl_\pi) \circ (Rot(\ell,\theta) \circ Refl_\pi)$$
$$= (Rot(\ell,\theta) \circ \cancel{Refl_\pi}) \circ (\cancel{Refl_\pi} \circ Rot(\ell,\theta))$$
$$= Rot(\ell, 2\theta)$$

similar to case of $\mathbb{R}^2$ - almost works but lose information about θ

$Fix(T) = ?$ $\quad \{\bar{x} \mid T(\bar{x}) = \bar{x}\}$

$T(\bar{x}) = -\bar{x}$ satisfied by $\bar{x} \in \ell$

$\{\bar{x} \mid T(\bar{x}) = -\bar{x}\} = \ell \qquad T(\bar{x}) = \lambda \bar{x}$

eigenspace with eigenvalue $\lambda = -1$

$T(x,y,z) = (-z,-y,x)$

$T(\bar{x}) = -\bar{x} \quad (-z,-y,x) = (-x,-y,-z)$

$-z = -x \qquad x-z=0 \qquad \ell: (x=z=0)$, y-axis

$-y = -y \qquad x+z=0 \qquad \Pi$: xz-plane

$x = -z \qquad \Rightarrow x=z=0$

angle of rotation:

$$T(\bar{x}) = \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A} \bar{x}$$

general method:

$A = PMP^T$

$P = (\bar{b}_1\ \bar{b}_2\ \bar{b}_3)$

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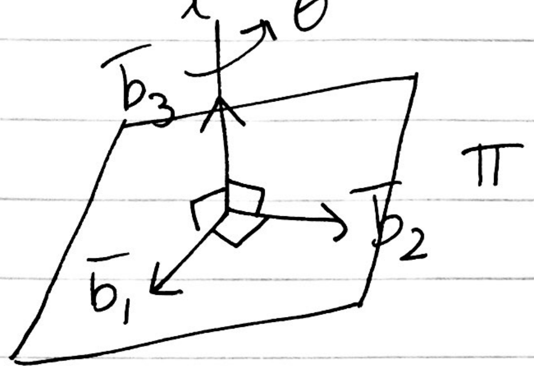

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

matrix for rotary reflection with axis z-axis, plane of reflection xy-plane

$\mathrm{trace}(A) = \mathrm{trace}(M) = 2\cos\theta - 1 = -1$

$-1 = 2\cos\theta - 1 \Rightarrow 2\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

summarize:

classify isometries

1. compute $\mathrm{Fix}(T)$
   $\mathbb{R}^3$: isometry
   plane π: reflection in π
   line ℓ: rotation about ℓ
   origin 0: rotary reflection
2. for rotation
   $2\cos\theta + 1 = \mathrm{trace}\, A \rightsquigarrow \theta$
3. for rotary reflection
   $T(\bar{x}) = -x \underset{\text{solve}}{\rightsquigarrow}$ find axis ℓ

and so π: plane normal to ℓ

$2\cos\theta - 1 = \mathrm{trace}\, A \rightsquigarrow \theta$