Math 462 Homework 4

Paul Hacking

February 19, 2013

(1) Determine algebraic formulas $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for the following isometries T. (Here A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector.)

(a) Rotation about the point $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ through angle $\pi/4$ counterclockwise.

(b) Reflection in the line $y = 4$.

(c) Reflection in the line $y = x + 2$ followed by a translation parallel to the line through distance $3\sqrt{2}$ in the direction of increasing x. (This is a glide reflection.)

(2) Describe the following isometries of $\mathbb{R}^2$ geometrically.

(a) $R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + 2 \\ -x + 3 \end{pmatrix}$.

(b) $S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + 4 \\ -x - 3 \end{pmatrix}$.

(3) For $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ an isometry, the fixed locus of T is the set of points $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{x}$. Describe the fixed locus for each type of isometry T (identity, rotation, translation, reflection, and glide reflection).

(4) Let $R \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry of $\mathbb{R}^2$ given by rotation about a point P through angle θ counterclockwise. Let $S \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry given by rotation about a point Q through angle θ clockwise.

(a) Show that the composition $S \circ R$ is a translation, i.e., $S(T(\mathbf{x})) = \mathbf{x} + \mathbf{b}$ for some vector $\mathbf{b} \in \mathbb{R}^2$.

(b) Show that if the angle θ is small, then the translation vector $\mathbf{b}$ has length approximately $\theta \cdot d(P,Q)$ and is approximately perpendicular to the vector $\overrightarrow{PQ}$. (Here $d(P,Q)$ denotes the distance from P to Q.) [This fact is sometimes useful when moving furniture.]

(c) What happens for $\theta = \pi$?

[Hint: By choosing coordinates appropriately we can assume P is the origin and Q lies on the x-axis. Now compute by expressing R and S in the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector. For part (b) use the approximations $\cos(\theta) \simeq 1$ and $\sin(\theta) \simeq \theta$ for θ small.]

(5) Let T be the isometry of $\mathbb{R}^3$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ z+1 \end{pmatrix}$. Describe T geometrically.

[Hint: This is type of an isometry of $\mathbb{R}^3$ we have not discussed earlier, but it is a composition of two simpler isometries (similar to the case of a glide reflection in $\mathbb{R}^2$).]

(6) Consider the following isometries of $\mathbb{R}^2$: R is rotation about the origin through angle θ counterclockwise, S is reflection in the x-axis, and T is translation by a vector $\mathbf{b} \in \mathbb{R}^2$. Describe the following compositions geometrically.

(a) $T \circ R \circ T^{-1}$.

(b) $T \circ S \circ T^{-1}$.

(c) $R \circ S \circ R^{-1}$.

(d) $S \circ R \circ S$.

(7) Recall that we can identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ via $(x,y) \mapsto z = x + iy$. From this point of view show that the isometries T of $\mathbb{R}^2 = \mathbb{C}$ are the maps given by $T(z) = e^{i\theta}z + c$ or $T(z) = e^{i\theta}\bar{z} + c$ for some $0 \le \theta < 2\pi$ and $c \in \mathbb{C}$. (Here $\bar{z}$ denotes the complex conjugate of z given by $\bar{z} = x - iy$ for $z = x + iy$, and $e^{i\theta} = \cos\theta + i\sin\theta$.) Describe the isometry T geometrically in terms of the data θ, c, and the choice of z or $\bar{z}$.