10/30/19

H.W. 5 due at start of Friday's class
(Q3 will not be graded)
Office hours: Thursday 3-4PM LGRT 1235H

<u>Last Time</u>:
A composition of two rotations about points P & Q through angles θ & ϕ is another rotation about a point R through angle $\theta+\phi$ OR a translation if $\phi = 2\pi - \theta$

<u>GPS Theorem</u>: Given 3 points A, B, C in the plane, not lying on a line, any point P in the plane is determined by the distances $|PA|, |PB|, |PC|$.

<u>Today</u>:
- Isometries & Congruence
- 3 reflections theorem
- Classification of isometries

<u>Corollary</u> A, B, C 3 points in the plane, not lying on a line
$T: \mathbb{R}^2 \to \mathbb{R}^2$ an isometry of the plane
then T is uniquely determined by $T(A), T(B), T(C)$.

<u>Proof</u> $P \in \mathbb{R}^2$ a point. We'd like to determine $T(P)$
We know (because T is an isometry):

$$|T(P)T(A)| = |PA|,$$
$$|T(P)T(B)| = |PB|,$$
$$|T(P)T(C)| = |PC|,$$

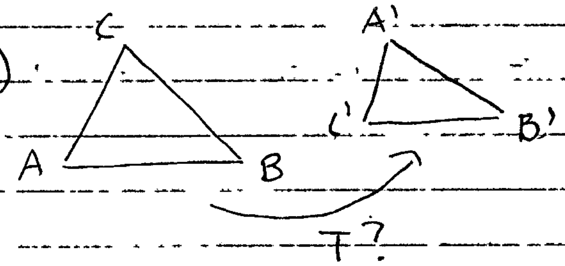
so we know distances of $T(P)$ from 3 points $T(A), T(B), T(C)$
$\underset{\text{GPS theorem}}{\Longrightarrow}$ determines $T(P)$. ▨ (Note: $T(A), T(B), T(C)$ don't lie on a line, otherwise, using T isometry, find A, B, C also lie on line #)

→ Argument used to show isometries send lines to lines.
Alternatively, if $T: \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, have inverse $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ which is also an isometry. So, if $T(A), T(B), T(C)$ lie on a line, apply T^{-1}, conclude A, B, C lie on a line.

__Theorem__ $\triangle ABC \cong \triangle A'B'C'$
$\iff$ there is an isometry T such that $T(A)=A'$, $T(B)=B'$, & $T(C)=C'$ (Moreover (by the corollary) T is uniquely determined)

__Proof__ ($\Leftarrow$) Isometries preserve distances (& angles).
So $AB=A'B'$, $BC=B'C'$, $AC=A'C'$ &
$\triangle ABC \cong \triangle A'B'C'$ by SSS. ✓

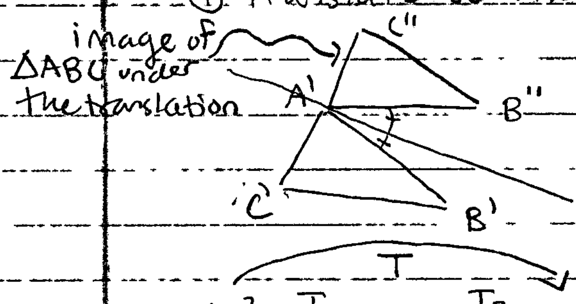
($\Rightarrow$)



$\triangle ABC \cong \triangle A'B'C'$

(Corollary only says if T exists, then it's unique.)

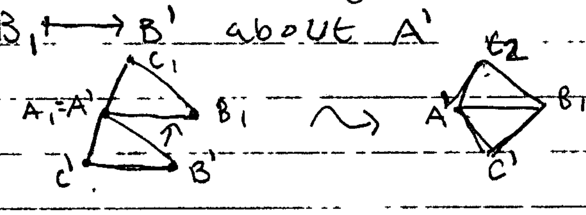
Try to write T as a composition of several simple isometries.
① Translate so that $A \mapsto A'$
② Reflect in the line bisecting $\angle B''A'B'$.
} in our example



$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_1"] \arrow[rr, bend left, "T"] & \mathbb{R}^2 \arrow[r, "T_2"] & \mathbb{R}^2 \end{tikzcd}$$
translation reflection

Then $T = T_2 \circ T_1$ is an isometry sending $\triangle ABC$ to $\triangle A'B'C'$.

__General Argument__:
① Translate so $A \longmapsto A'$ $(A, B, C \longrightarrow A_1, B_1, C_1)$
② Rotate so $B_1 \longmapsto B'$ about A'



③ Either done (if $C' = C_2$) or reflect in $A'B'$.
Composite sends $\triangle ABC$ to $\triangle A'B'C'$. ▨

Three reflections theorem

Any isometry T of the plane is a composite of at most 3 reflections.

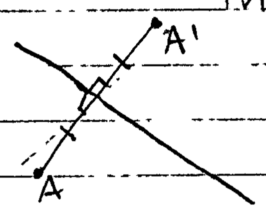
Proof. We could try to "optimize" previous proof using earlier results about compositions to show ≤ 3 reflections suffice.

Translation followed by rotation = rotation = 2 reflection + additional reflection = 3 reflections, and we are done. ▨

Alternative proof

Fix a triangle A, B, C & consider $T(A), T(B), T(C) =: A', B', C'$. We want to show T is a composite of ≤ 3 reflections, or equivalently, write down a composite of ≤ 3 reflections which sends A, B, C to A', B', C' (then $= T$ by corollary).

Idea: if $A \neq A'$, reflect in perpendicular bisector of AA'.
— this sends $A \to A'$.



$$A, B, C \to \underset{\overset{\shortparallel}{A'}}{A_1}, B_1, C_1$$

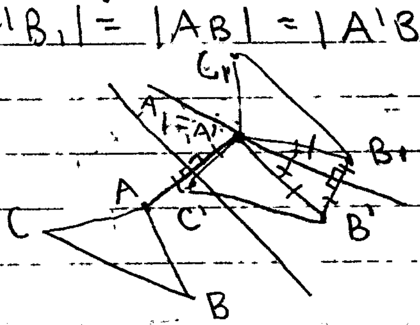
If $B_1 \neq B'$, then reflect in perp. bisector of $B_1 B'$

$$\underset{\overset{\shortparallel}{A'}}{A_1}, B_1, C_1 \longrightarrow \underset{\overset{\shortparallel}{A'}}{A_2}, \underset{\overset{\shortparallel}{B'}}{B_2}, C_2$$

Why? must have A' lies on perp. bisector of $B_1 B'$ because $A \to A'$, $B \to B_1$

reflection is isometry T is isometry

$$|A'B_1| = |AB| = |A'B'|$$



A' lies on perp. bisector of $B_1 B'$.

$$\Delta ABC \rightsquigarrow \Delta \underset{\overset{\shortparallel}{A'}}{A_1} B_1 C_1 \rightsquigarrow \Delta \underset{\overset{\shortparallel}{A'}}{A_2} \underset{\overset{\shortparallel}{B'}}{B_2} C_2 .$$

Finally, if $C_2 \neq C'$ reflect in perp. bisector of $C_2 C' \rightsquigarrow \Delta A_3 B_3 C_3 = \Delta A'B'C'$. ▨

Now use 3 reflections theorem to classify isometries.

# reflections	isometry
0	identity
1	reflection
2	translation or rotation
3	glide reflection ↳ will show next time.