Math 797W Homework 2

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Reading: David Mumford, The red book of varieties and schemes, Chapter I, Sections 6–10.

Justify your answers carefully.

(1) Let $f\colon X \to Y$ be a morphism of affine varieties. Let $S = k[Y]$, $R = k[X]$, and $\varphi = f^*\colon S \to R$. Prove the following statements.

  (a) $\overline{f(X)} = V(\ker \varphi) \subset Y$.

  (b) Let $p \in X$ be a point corresponding to a maximal ideal $\mathfrak{m} \subset R$. Then $\varphi^{-1}(\mathfrak{m}) \subset S$ is the maximal ideal corresponding to the point $f(p) \in Y$.

  (c) Similarly, let $Z \subset X$ be an irreducible closed subset corresponding to a prime ideal $\mathfrak{p} \subset R$. Then $\varphi^{-1}(\mathfrak{p}) \subset S$ is the prime ideal corresponding to the irreducible closed subset $\overline{f(Z)} \subset Y$.

(2) Let $k \subset K$ be a field extension. Let $\alpha_1, \ldots, \alpha_r \in K$ be such that the field extension $k(\alpha_1, \ldots, \alpha_r) \subset K$ is algebraic, and let $\beta_1, \ldots, \beta_s \in K$ be algebraically independent over k. Prove the following analogue of the Steinitz exchange lemma of linear algebra: We have $s \leq r$ and, after reordering the α_i, we may assume that the field extension $k(\beta_1, \ldots, \beta_s, \alpha_{s+1}, \ldots, \alpha_r) \subset K$ is algebraic. [Hint: Argue by induction, exchanging one element at a time.] Deduce that the transcendence degree of a finitely generated field extension $k \subset K$ is well defined.

(3) Let X and Y be varieties. Show directly from the definition of dimension that $\dim(X \times Y) = \dim X + \dim Y$.

(4) Let $f\colon X \to Y$ be a finite morphism. Show directly from the definition of dimension that $\dim X = \dim f(X) \leq \dim Y$ (with equality iff $f(X) = Y$).

(5) Assume $\mathrm{char}(k) \neq 2$. Let $X = \mathbb{A}^1_x$, $Y = \mathbb{A}^1_y$, and $f\colon X \to Y$ the morphism of affine varieties given by $x \mapsto x^2$. Now let $U = X \setminus \{p\}$ for some $p \in X$, $p \neq 0$. Show that the restriction of f to U is surjective and has finite fibers but is not a finite morphism.

[You may assume the following result, stated without proof in class: A morphism $f\colon X \to Y$ is finite iff for all open affine $U \subset Y$ the open subvariety $f^{-1}(U) \subset X$ is affine and the map $f^*\colon k[U] \to k[f^{-1}U]$ is an integral extension of rings. (See www.jmilne.org/math/CourseNotes/ag.html, Prop. 8.21, p. 178, for a proof of this result.)]

(6) Let $g(x,y) \in k[x,y]$ be an irreducible polynomial,

$$X = V(z^3 - g(x,y)) \subset \mathbb{A}^3_{x,y,z},$$

$Y = \mathbb{A}^2_{x,y}$ and $f\colon X \to Y$ the morphism of affine varieties given by $(x,y,z) \mapsto (x,y)$. Let $h \in k[X]$ and suppose $Z = V(h) \subset X$ is irreducible. Compute the equation of $f(Z) \subset Y$ in terms of h.

[Hint: Use the method from the proof of Krull's principal ideal theorem.]

(7) Assume $\mathrm{char}(k) \neq 2$. Let $F \in k[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree 2 and $X = V(F) \subset \mathbb{P}^n$, a quadric hypersurface.

   (a) State a standard result on bilinear forms and use it to show that after a linear change of homogeneous coordinates on $\mathbb{P}^n$ we may assume $F = X_0^2 + X_1^2 + \cdots + X_m^2$, for some $m \leq n$.

   (b) Show that F is irreducible iff $m \geq 2$.

   (c) Show that X is smooth iff $m = n$ or $m = 0$ (a degenerate case).

   (d) Observe that the map $f\colon \mathbb{P}^1 \to \mathbb{P}^2$, given by $(Y_0 \colon Y_1) \mapsto (Y_0^2 \colon Y_0Y_1 \colon Y_1^2)$ is an isomorphism onto $Y = V(X_0X_2 - X_1^2) \subset \mathbb{P}^2$. (You are not required to prove this as it is similar to and easier than HW1Q10.) Deduce that if $m = n = 2$ then $X \simeq \mathbb{P}^1$.

(e) Similarly, by considering the Segre embedding $f\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ show that if $m = n = 3$ then $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(8) Assume $\mathrm{char}(k) \neq 2$. Let $X = V(y^2 - f(x)) \subset \mathbb{A}^2_{x,y}$ where $f \in k[x]$ is a polynomial of degree $d \geq 1$ with distinct roots.

(a) Show that X is smooth.

(b) Let $\overline{X}$ denote the closure of X in $\mathbb{P}^2$. Compute the homogeneous equation of $\overline{X}$

(c) Determine the set $\overline{X} \setminus X$ for $d \geq 3$.

(d) Show that $\overline{X}$ is not smooth if $d \geq 4$.

(9) Assume $\mathrm{char}(k) \neq 2$. Let $X = V(y^2 - f(x)) \subset \mathbb{A}^2_{x,y}$ where $f \in k[x]$ is a polynomial of degree $d \geq 1$ with distinct roots. Let $Y = \mathbb{A}^2_{x,y} \cup \mathbb{A}^2_{z,t}$ with glueing given by

$$\mathbb{A}^2_{x,y} \supset (x \neq 0) \xrightarrow{\sim} (z \neq 0) \subset \mathbb{A}^2_{z,t}, \quad (x, y) \mapsto (x^{-1}, x^{-l}y)$$

where $l = \lceil d/2 \rceil$. (Then Y is isomorphic to the variety $X(n)$, $n = -l$, of HW1Q11.) Let $\overline{X}$ be the closure of X in Y. Prove the following:

(a) The variety $\overline{X}$ is smooth.

(b) The morphism $Y \to \mathbb{P}^1$ of HW1Q11(b) restricts to a finite morphism $f\colon \overline{X} \to \mathbb{P}^1$.

(c) For each $p \in \mathbb{P}^1$ we have $|f^{-1}(p)| = 1$ or 2, and $|f^{-1}(p)| = 1$ for exactly $2l$ points $p \in \mathbb{P}^1$ (these points are called the *branch points* of f).

(d) Now assume $k = \mathbb{C}$. Then $\overline{X}$ is compact for the analytic topology and is a Riemann surface of genus $g = l - 1$.

(10) Let $F_1, \ldots, F_m$ be nonconstant homogeneous polynomials in n variables with coefficients in an algebraically closed field k. Prove that if $m < n$ then the equations

$$F_1 = \ldots = F_m = 0$$

have a nonzero solution.

[Remark: In the special case that each F_i has degree 1 this is a basic result of linear algebra.]

(11) (a) Let $X \subset \mathbb{P}^n$ be a closed subset, not necessarily irreducible. Let Y be a variety and $f\colon X \to Y$ a morphism, that is, f is a map of sets such that the restriction to each irreducible component of X is a morphism. Suppose $f(X)$ is irreducible and every fiber of f is irreducible of the same dimension r. Prove that X is irreducible.

(b) Using part (a) or otherwise, prove that the closed subset

$$X := V(X_0Y_0 + X_1Y_1 + \cdots + X_nY_n) \subset \mathbb{P}^n_{(X_0\,:\,\cdots\,:\,X_n)} \times \mathbb{P}^n_{(Y_0\,:\,\cdots\,:\,Y_n)}$$

is irreducible.