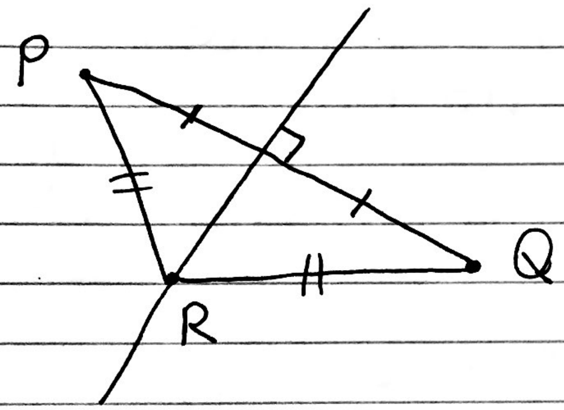
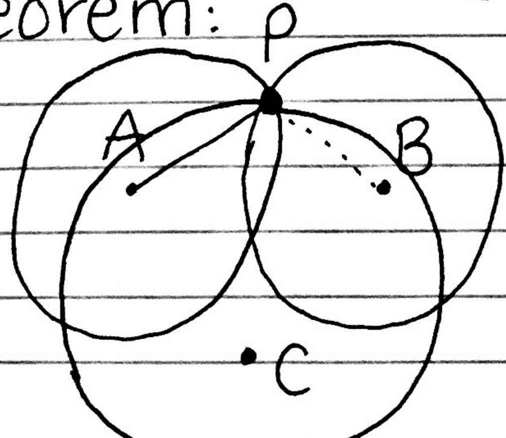
Math 461 Lecture 18 10/15
Guest Professor this week:
Kuan-Wen Lai

GPS Theorem:
$A, B, C \in \mathbb{R}^2$ not collinear
Then a point $P \in \mathbb{R}^2$ is uniquely determined by $|PA|, |PB|, |PC|$.
In other words, if $Q \in \mathbb{R}^2$ such that $|QA| = |PA|, |QB| = |PB|, |QC| = |PC|$ then $Q = P$ (the same point).
Proof by contradiction:
Assume to the contrary, $P \neq Q$.



All points $R \in \mathbb{R}^2$ such that $|RP| = |RQ|$ are all on this line (perpendicular bisector)
then A, B, C are on the same line which means they are collinear which is a contradictor ※ or $\rightarrow\leftarrow$ $\square$

Alternative way to see GPS Theorem:



P is the only intersection of the three circle
$P = Q$

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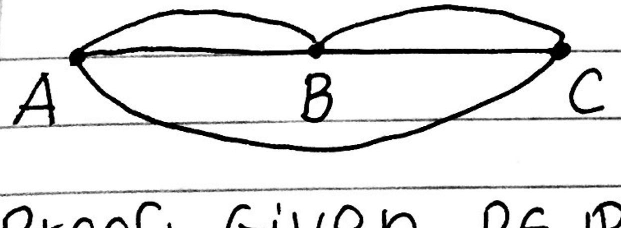
Remark: GPS = Global Positioning System uses the theorem to determine your position given your distances from 4 satellites. (3+1 for adjustment)

Corollary: Given $A, B, C \in \mathbb{R}^2$ that are noncollinear.
Then an isometry $T: \mathbb{R}^2 \to \mathbb{R}^2$ is uniquely determined by $T(A)$, $T(B)$, and $T(C)$.
In other words, if there is an isometry $T': \mathbb{R}^2 \to \mathbb{R}^2$, $T'(A) = T(A)$, $T'(B) = T(B)$, $T'(C) = T(C)$, then $T' = T$

Note that: $T(A), T(B), T(C)$ are not collinear.
If $T(A), T(B), T(C)$ are collinear:

then $|T(A)T(B)| + |T(B)T(C)| = |T(A)T(C)|$
$|T(A)T(B)| = |AB|$ because T is an isometry.
Similiarly, $|T(B)T(C)| = |BC|$ and $|T(A)T(C)| = |AC|$.
Therefore $|AB| + |BC| = |AC|$ and A, B, C are collinear which is a contradiction.



Proof: Given $P \in \mathbb{R}^2$ consider $T(P) \in \mathbb{R}^2$
then $|T(P)T(A)| = |PA|$.
$|T(P)T(B)| = |PB|$
$|T(P)T(C)| = |PC|$

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then if $T': \mathbb{R}^2 \to \mathbb{R}^2$

then $|T'(P)T'(A)| = |PA|$

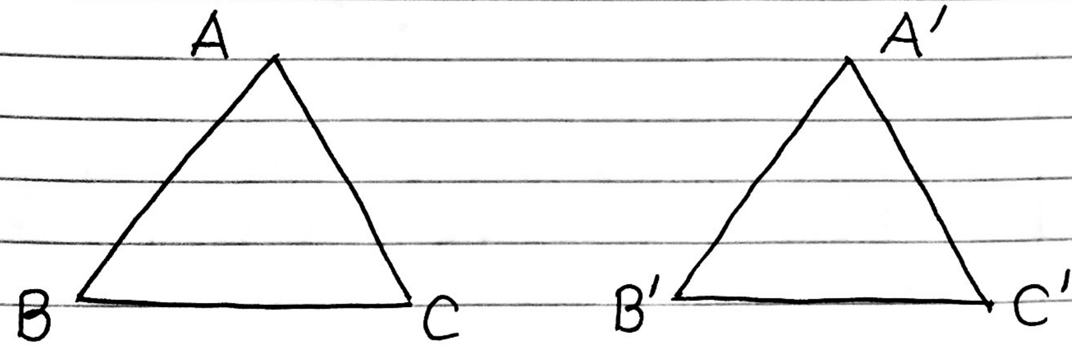
$|T'(P)T'(B)| = |PB|$

$|T'(P)T'(C)| = |PC|$

then $T(P) = T'(P)$ by GPS Theorem

and $T = T'$ $\square$

Theorem:

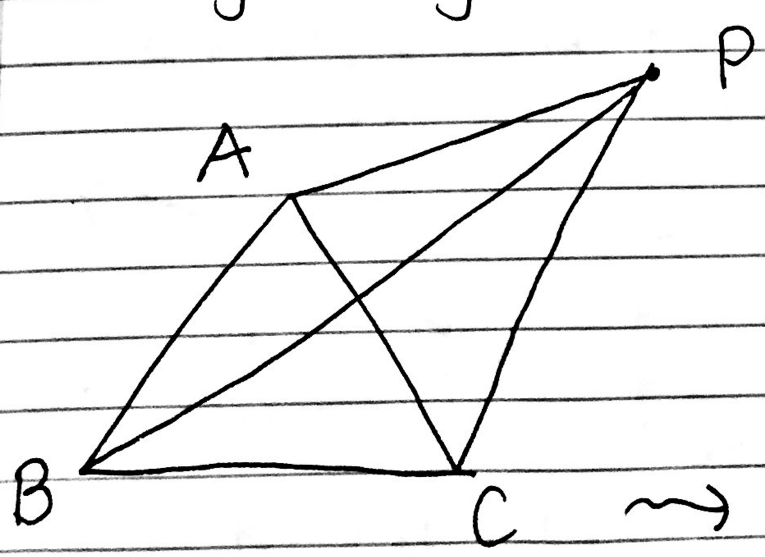


$\Delta ABC \cong \Delta A'B'C' \iff$ if there exists a unique isometry $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(A) = A'$, $T(B) = B'$, $T(C) = C'$.

$\Leftarrow$ direction of proof is easier because of the S-S-S congruence criterion and definition of an isometry.

$\Rightarrow$ direction of proof:

(taught by Dan)



$|AP| = |T(A)T(P)|$

$|BP| = |T(B)T(P)|$

$|CP| = |T(C)T(P)|$

$A' = T(A)$

$B' = T(B)$

$C' = T(C)$ $\rightsquigarrow$ uniqueness

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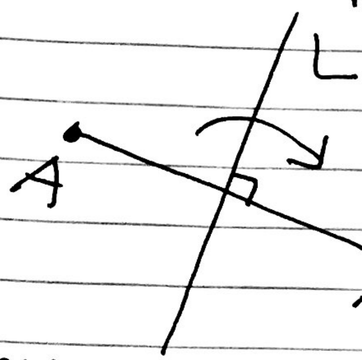
Existence of T?

Construction:

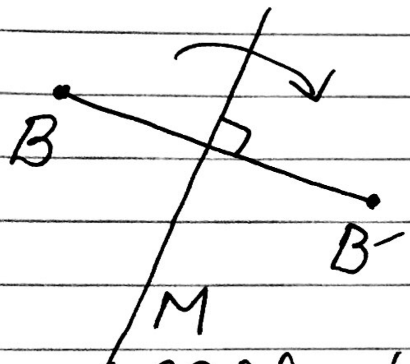
If $A \neq A'$ apply reflection $Refl_L(A) = A'$.



Next, assume $A = A'$.

If $B \neq B'$, apply reflection $Refl_M(B) = B'$

$Refl_M(A) = A$
(fix A)



why $Refl_M(A) = A$?

because $|AB| = |A'B'| = |AB'|$

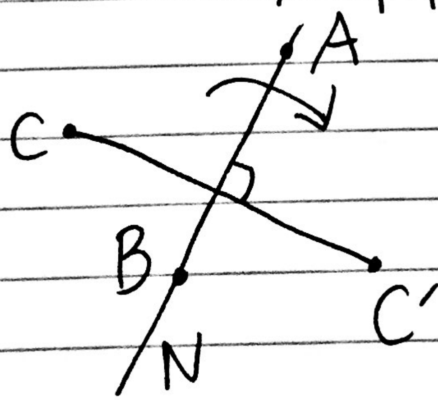
by congruence $\quad A = A'$

$\Rightarrow A \in M$, $Refl_M(A) = A$ (fixed point on M)

Now, $A = A'$ and $B = B'$.

If $C \neq C'$, apply reflection

$Refl_N(C) = C'$



$=$ means sends

$Refl_N(A) = A$

$Refl_N(B) = B$

A and B are on line N

$\Rightarrow T = Refl_N \circ Refl_M \circ Refl_L$

this isometry is unique!