Math 621 Midterm review questions

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(1) Find the power series expansion of $\frac{1}{z+i}$ centered at $z = 1$. What is its radius of convergence?

(2) Find the poles and residues of the function $\frac{1}{1-e^z}$.

(3) Show that $f(z) = \cos z - \sin z$ has an essential singularity at ∞.

(4) Find the Laurent series expansion for $f(z) = \frac{1}{(z-1)(z-2)}$ about $z = 0$ in the regions $|z| < 1$, $1 < |z| < 2$, and $|z| > 2$.

(5) What is the radius of convergence of the power series expansion of $f(z) = \frac{1}{\sin z}$ about $z = i$?

(6) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that the real part of f is bounded above. Prove that f is constant.

(7) Compute the integral of $f(z) = z^2 + 3z + 2$ along a path γ from 1 to i.

(8) Compute the following integrals.

   (a) $\int_\gamma \frac{z^4}{e^z+1} dz$ where $\gamma = \{z \mid |z| = 4\}$ with its positive orientation.

   (b) $\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} dx$.

   (c) $\int_0^{2\pi} \frac{1}{5-4\cos\theta} d\theta$.

(9) (a) Find the number of zeroes of $f(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$ in the unit disc $D = \{z \mid |z| < 1\}$.

   (b) Show that the equation $ze^{\lambda - z} = 1$ has exactly one solution in the unit disc for $\lambda \in \mathbb{R}$, $\lambda > 1$. Show also that this solution is a real number.

(10) Let $\Omega \subset \mathbb{C}$ be an open set and $f\colon \Omega \to \mathbb{C}$ a holomorphic function. Let $z_0 \in \Omega$ be a point such that $f'(z_0) \neq 0$. Show that

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{1}{f(z) - f(z_0)} dz$$

where γ is a small circle centered at z_0 traversed counterclockwise.

(11) Let $\Omega \subset \mathbb{C}$ be a connected open set, $f\colon \Omega \to \mathbb{C}$ a non-constant holomorphic function, and $z_0 \in \Omega$.

(a) Show that there exists $\epsilon > 0$ such that $f(z) \neq 0$ for $0 < |z - z_0| < \epsilon$.

(b) What does it mean to say that f has a zero of order m at z_0? Show that in this case there exist $\delta, \epsilon > 0$ such that for $0 < |\lambda| < \delta$ the equation $f(z) = \lambda$ has exactly m solutions satisfying $|z - z_0| < \epsilon$.