can have cheat sheet

12/11/19

Final Exam: Th 12/19/19 8AM–10AM Goessman Lab Add. Rm 51

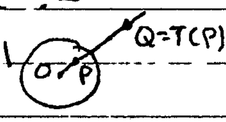
Final Review: Office hrs next week M & Tu 4–5PM LGRT 1235H

Course survey: owl.umass.edu/partners/courseEvalSurvey/umal

Last Time $T: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ $T(x,y) = \frac{1}{x^2+y^2}(x,y)$

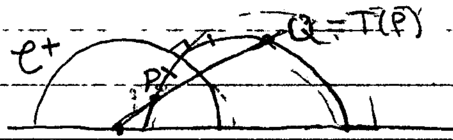
→ inversion in circle $\mathcal{C}$ center $(0,0)$ & radius 1 :–

$|OP| \cdot |OQ| = 1$



$T: \mathcal{H} \to \mathcal{H}$ hyperbolic isometry

(= hyperbolic reflection in $\mathcal{C}^+ = \mathcal{C} \cap \mathcal{H}$

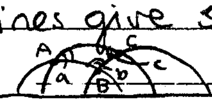


. T preserves angles & sends circles & lines to circles & lines,

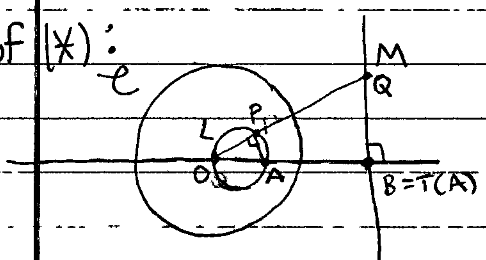
circles & lines through O ⟶ lines

(*) In particular, T sends semicircle in $\mathcal{H}$ through O to vertical line.

Today
- Euclidean geometry proof of (*)
- Semicircle hyperbolic lines give shortest paths
- Hyperbolic triangles



Proof of (*):



Want to show $T(L \setminus \{0\}) = M$

$|OA| \cdot |OB| = 1$

Equivalently, for every point $P \in L$, $P \neq O$, Q (defined in diagram) $= T(P)$

i.e. $|OP| \cdot |OQ| = 1$.

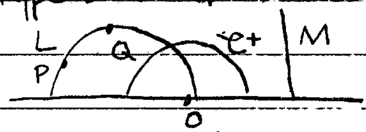
Similar triangles? $\triangle OAP \sim \triangle OQB$

$\Rightarrow \frac{|OA|}{|OQ|} = \frac{|OP|}{|OB|} \Rightarrow |OP| \cdot |OQ| = |OA| \cdot |OB|$

$= 1$ ✓

Consequence

All hyperbolic lines give shortest paths for the hyperbolic distances



Ok for vertical lines (checked earlier)

If instead L is a semicircle, find a hyperbolic isometry that sends L to a vertical line :—

1. Translate $(x,y) \overset{T_1}{\mapsto} (x+a, y)$ (Horizontal) so $T_1(L)$ goes through O.
2. Perform inversion T_2 (= hyperbolic reflection in $\mathcal{C}^+$)

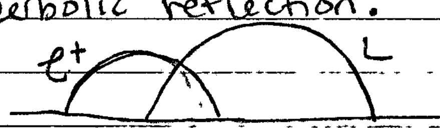
$T = T_2 \circ T_1$

Now, $T(L)$ gives shortest path from $T(P)$ to $T(Q)$ (vertical line)

$\Rightarrow$ L gives shortest path from P to Q. ▨

↳ T preserves lengths of paths (isometry)

Remark: We have described hyperbolic reflection in unit semicircle $\mathcal{C}^+$. (& also in vertical line — usual Euclidean reflection). Given any hyperbolic line, get description of hyperbolic reflection:



Find hyperbolic isometry F sending L to $\mathcal{C}^+$ & then $\mathrm{Refl}_L = T^{-1} \circ \mathrm{Refl}_{\mathcal{C}^+} \circ T$ (same as in $\mathbb{R}^2$)

What if we want to send $\mathcal{C}^+$ to L?

One more (useful) hyperbolic isometry :

$\boxed{T(x,y) = (cx, cy) \quad c \in \mathbb{R},\ c > 0}$

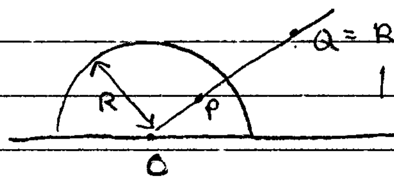
scaling by c, center origin

hyperbolic isometry?

$$\frac{\sqrt{x'^2 + y'^2}}{y} \rightsquigarrow \frac{\sqrt{(cx)'^2 + (cy)'^2}}{(cy)} = \frac{\sqrt{(c \cdot x')^2 + (c \cdot y')^2}}{cy} = \frac{\sqrt{x'^2 + y'^2}}{y} \quad \checkmark$$

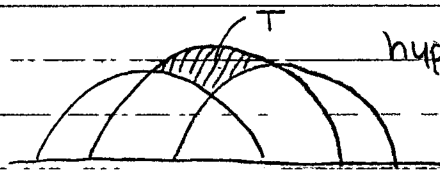
Now, isometry sending C^+ to L:
- scale so same radius, translate. ▨

Result



$Q = \mathrm{Refl}_L(P)$

$|CP| \cdot |CQ| = R^2$



Math 132

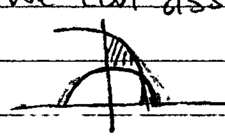
$$\text{hyperbolic-Area } T = \iint_T \frac{1}{y^2}\, dx\, dy$$

$(x+\Delta x, y+\Delta y)$

(x,y) length $= \dfrac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{y}$

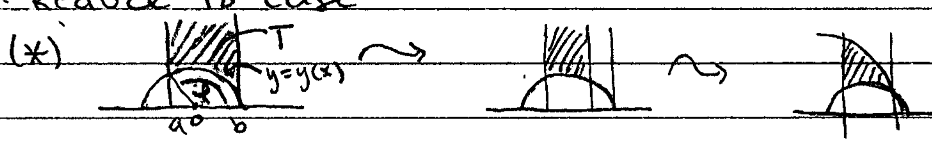
Now need to change coordinates to simplify calculation.

1. We can assume 1 side of T is vertical



2. Reduce to case

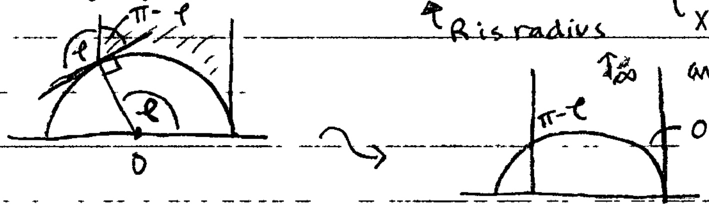
(*)



$$A = \iint_T \frac{1}{y^2}\, dx\, dy = \int_a^b \left(\int_{y(x)}^{\infty} \frac{1}{y^2}\, dy \right) dx = \int_a^b \left[\frac{-1}{y} \right]_{y(x)}^{\infty}$$

$$= \int_a^b \frac{1}{y(x)}\, dx = \int_a^b \frac{1}{\sqrt{R^2 - x^2}}\, dx = \int_0^{\varphi} \frac{R\cos\theta\, d\theta}{R\cos\theta} = \varphi .$$

R is radius $\quad x = R\sin\theta$



Area $= \varphi = \pi - \left((\pi - \varphi) + 0 + 0\right)$

In general, get Area $= \pi - (a+b+c)$ using reduction described earlier.

$a, b, c =$ angles of triangles