10/20/19

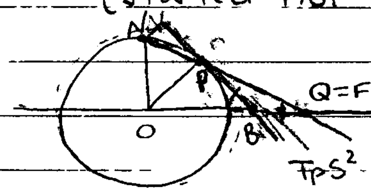
HW 6 returned

HW 7 due now

HW 8 available, due Wednesday 12/4/19 at start of class

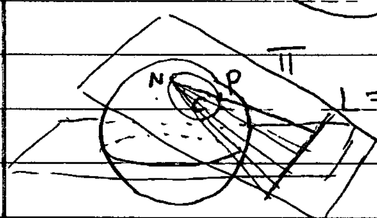
Last Time
- Spherical lines give shortest paths
- Stereographic projection preserves angles (started not finished)

Warm Up



$|BP| = |BQ|$

"vertical slice" through $O, N, \& P$



$L = \Pi \cap (z=0)$

$N \in \Pi$, $C = \Pi \cap S^2$

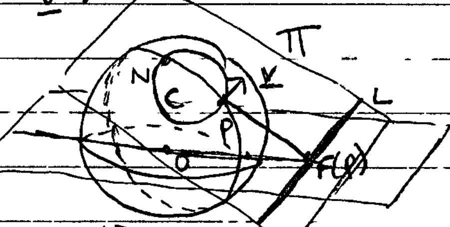
$C = \Pi \cap S^2 \ni N$

$\Rightarrow F(C \setminus \{N\}) = L = \Pi \cap (z=0)$, line in $\mathbb{R}^2$

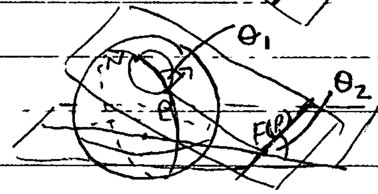
Today
- Finish geometric proof that S.P. preserves angles
- Alternative (algebraic) proof
- Notion of distance on $\mathbb{R}^2$ corresponding to distance on S^2 under S.P.

Continuing proof that S.P. preserves angles:

Suppose given $P \in S^2$ & a tangent vector $\underline{v}$ at P to S^2. ($P \neq N$)
Let Π be the plane in $\mathbb{R}^3$ containing N, P, & tangent vector $\underline{v}$.



Also consider great circle D passing through N & P.
($D = \Pi_D \cap S^2$, Π_D plane through $O, N \& P$)



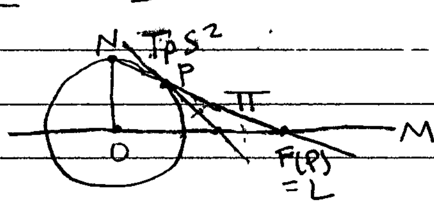
$F(D)$ passes through O & $F(P)$, and is a line on the plane. Want to show $\theta_1 = \theta_2$, the angle between D and $\underline{v}$ is the same as the angle between $F(D)$ and L.

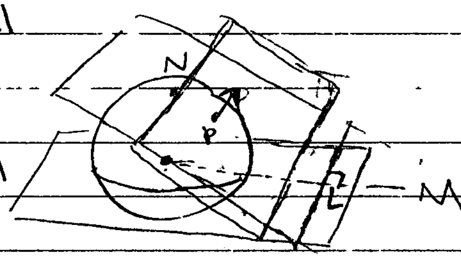
Note : Once we prove this, we will deduce that the angle between any two curves on S^2 is preserved by S.P.
(because if curves γ_1 & γ_2 on S^2 have some tangent at P then $F(\gamma_1)$ & $F(\gamma_2)$ have some tangent at $F(P)$ in $\mathbb{R}^2$ — F is differentiable)

So given γ_1, γ_2 consider circles C_1 & C_2 as above with some tangents : Our calc shows angle between C_1 & C_2 is preserved $\Rightarrow$ OK.

"Easy" case : $\underline{v}$ is horizontal

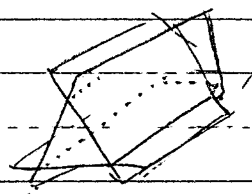
Picture



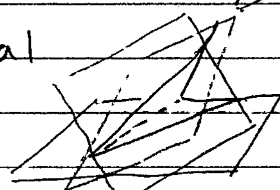


Angle is proved $(= \pi/2)$

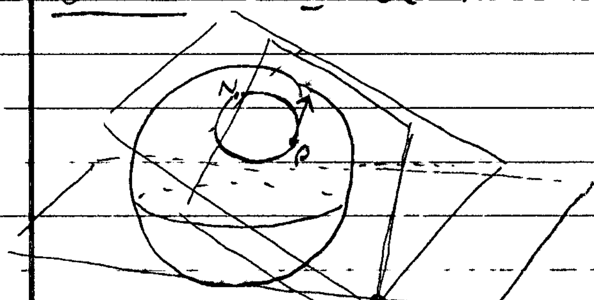
$\rightsquigarrow$ three planes

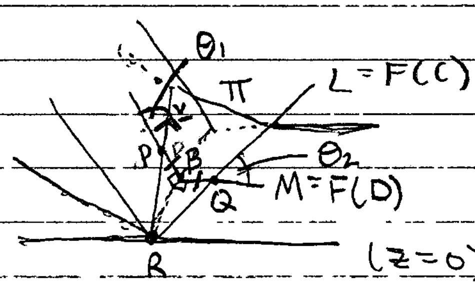


$\rightsquigarrow$ in general



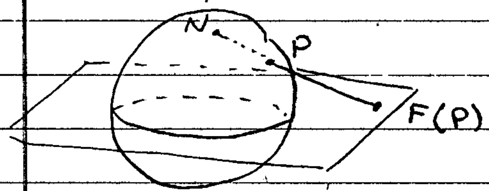
Case 2 : $\underline{v}$ not horizontal





$\triangle RBQ \cong \triangle RBP$ (SAS)
$\hookrightarrow |PB| = |QB|,\ \angle PBR = \angle QBR,\ |BR| = |BR|$
$\Rightarrow \theta_1 = \theta_2$.

Alternative approach: algebra/calculus



$F: S^2 \setminus \{N\} \to \mathbb{R}^2$

$F^{-1}: \mathbb{R}^2 \to S^2 \setminus \{N\} \subset \mathbb{R}^3$

$(u,v) \mapsto \underline{x}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2+1)$

F^{-1} is a differentiable function
means we have an induced map ("the derivative of F^{-1}")

$D(F^{-1}): T_{F(P)}\mathbb{R}^2 \longrightarrow T_P S^2 \subset \mathbb{R}^3$

linear map
given by matrix
$$\left.\begin{pmatrix} \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \end{pmatrix}\right|_{F(P)}$$

By the chain rule, $D(F^{-1})$ has property that is sends the tangent vector to a parametrized curve in $\mathbb{R}^2$ to tangent vector to the corresponding curve in S^2.
So, just need to check that linear map $D(F^{-1})$ preserves angles.