10/23/19

HW 5 available : Due next Wednesday at start of class
(check webpage)

<u>Last Time</u> • $T : \mathbb{R}^2 \to \mathbb{R}^2$ isometry $\Leftrightarrow T(\underline{x}) = A\underline{x} + \underline{b}$

$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, some a, b with $a^2+b^2=1$

"orthogonal matrix" , $\underline{b} \in \mathbb{R}^2$ vector

• direct/opposite isometries ; preserve/reverse the sense of angles. (ccw or cw)

direct $\Leftrightarrow \det A = +1$

opposite $\Leftrightarrow \det A = -1$

• Algebraic Formula $\rightsquigarrow$ geometric description.

First compute $\mathrm{Fix}(T) = \{ P \in \mathbb{R}^2 \mid T(P) = P \}$

T	$\mathrm{Fix}(T)$
identity	$\mathbb{R}^2$
translation	$\emptyset$
rotation	P, center of rotation
reflection	L, line of reflection
glide reflection	$\emptyset$

<u>Today</u> • Examples + Compositions

<u>Ex 1</u> : $T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3x+4y+8 \\ 4x+3y-4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \end{pmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2 \rightsquigarrow$ isometry. Find geometric description of T?

First compute $\mathrm{Fix}(T) = \{ P \in \mathbb{R}^2 \mid T(P) = P \}$

i.e. solve $\frac{1}{5}\begin{pmatrix} -3x+4y+8 \\ 4x+3y-4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{matrix} -3x+4y+8 = x \\ 4x+3y-4 = y \end{matrix}$

$\rightsquigarrow$ ① $-8x+4y = -8$

② $4x - 2y = 4$

cases: $\mathrm{Fix}(T) = P$ $\quad$ $\mathrm{Fix}(T) = \emptyset$ $\quad$ $\mathrm{Fix}(T) = L$

$\longrightarrow$

$$\div 8\begin{pmatrix} -8 & 4 & \vdots & -8 \\ 4 & -2 & \vdots & 4 \end{pmatrix} \quad \text{row reductions}$$

$$-4R1\begin{pmatrix} 1 & -1/2 & \vdots & 1 \\ 4 & -2 & \vdots & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1/2 & \vdots & 1 \\ 0 & 0 & \vdots & 0 \end{pmatrix} \rightsquigarrow x - 1/2y = 1$$

$$\rightsquigarrow Fix(T) = L = \{(x,y) \mid x - 1/2y = 1\}$$
$$= \{(x,y) \mid y = 2x - 2\}$$

Conclude T is a reflection in L. ■

<u>Ex 2</u>: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+3 \\ x+7 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$Fix(T)$: $\begin{matrix} y+3 = x \\ x+7 = y \end{matrix} \rightsquigarrow \begin{matrix} -x + y = -3 \\ x - y = -7 \end{matrix}$

$$\rightsquigarrow \begin{pmatrix} -1 & 1 & \vdots & -3 \\ 1 & -1 & \vdots & -7 \end{pmatrix} {+R1} \rightsquigarrow \begin{pmatrix} -1 & 1 & \vdots & -3 \\ 0 & 0 & \vdots & -10 \end{pmatrix}$$

$0x + 0y = -10$

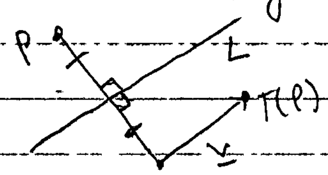
$0 = -10$ ※

$Fix(T) = \emptyset \Rightarrow T$ is either a glide reflection or a translation.

$\boxed{\text{Aside: Translation by } \underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}, \ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \end{pmatrix}}$

$\Rightarrow$ Cannot be a translation (not in correct form), must be a glide reflection.

<u>Recall</u>: a glide reflection is given by reflection in a line L followed by translation by a vector $\underline{v}$ parallel to L.



What is L? $\underline{v}$?

On HW5.

Compositions of Isometries

Notation:

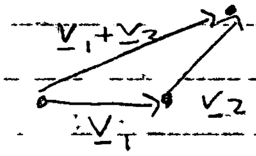
$\mathrm{Trans}_{\underline{v}}$ = translation by $\underline{v}$

Refl_L = reflection in L

$\mathrm{Rot}(P,\theta)$ = rotation about P through θ

Ex 0: Composite of two translations.

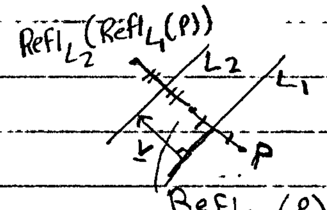
$$\mathrm{Trans}_{\underline{v}_2} \circ \mathrm{Trans}_{\underline{v}_1} = \mathrm{Trans}_{\underline{v}_1+\underline{v}_2}$$



Ex 1: Composite of two reflections.

$\boxed{1a}$ Parallel lines

$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$



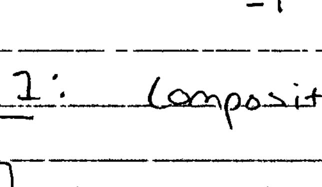
$\rightsquigarrow$ Translation

$$\mathrm{Refl}_{L_2}(\mathrm{Refl}_{L_1}(P)) = \mathrm{Trans}_{2\underline{v}}(P)$$

i.e. translation by twice the perpendicular distance between L_1 & L_2, perpendicular to the lines in the direction from L_1 to L_2.

Algebraic proof:

choose coordinates so that L_1 = x-axis



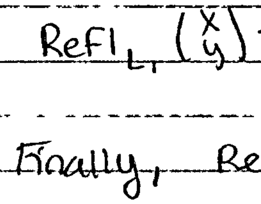
L_2 $(y=d)$, some $d>0$ (distance between L_1 & L_2)

$$\begin{pmatrix}x\\y\end{pmatrix} \xmapsto{\mathrm{Trans}\begin{pmatrix}0\\-d\end{pmatrix}} \begin{pmatrix}x\\y-d\end{pmatrix} \xmapsto{\mathrm{Refl}_{L_1}} \begin{pmatrix}x\\-(y-d)\end{pmatrix} \xmapsto{\mathrm{Trans}\begin{pmatrix}0\\d\end{pmatrix}} \begin{pmatrix}x\\-(y-d)+d\end{pmatrix}$$

$\mathrm{Refl}_{L_1}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix}$ $\mathrm{Refl}_{L_2}\begin{pmatrix}x\\y\end{pmatrix} = ?$ $= \begin{pmatrix}x\\2d-y\end{pmatrix}$ ✓

Finally, $\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}\begin{pmatrix}x\\y\end{pmatrix} = \mathrm{Refl}_{L_2}\begin{pmatrix}x\\-y\end{pmatrix} = \begin{pmatrix}x\\2d-(-y)\end{pmatrix} = \begin{pmatrix}x\\y+2d\end{pmatrix}$ ∎

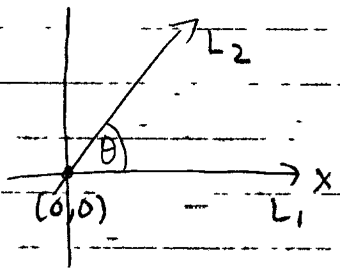
$\boxed{1b}$



$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} \overset{\text{claim}}{=} \mathrm{Rot}(P, 2\theta)$

Algebraic proof: choose coordinates so that L_1 is the x-axis and P is the origin, so that L_2 is the line through the origin making angle θ ccw with the x-axis.

$\longrightarrow$



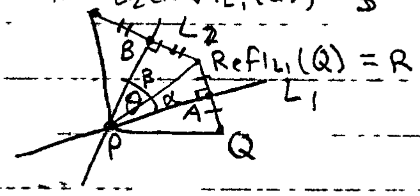
computed earlier

$$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} \rightsquigarrow \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

↳ rotation about the origin through angle 2θ ccw ▨

Geometric proof:

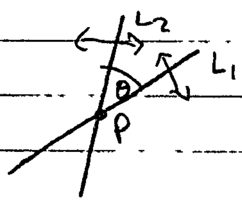


$\Delta RPA \cong \Delta QPA$ (SAS)

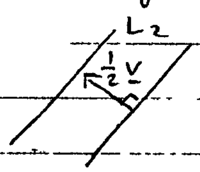
$\Delta SPB \cong \Delta RPB$, $\alpha + \beta = \theta$, $\angle SPR = 2\beta$, $\angle QPR = 2\alpha$

$\angle QPS = \angle SPR + \angle QPR = 2\beta + 2\alpha = 2\theta$. ▨

See a rotation or a translation can be written as a composition of two reflections in different ways.



$\mathrm{Refl}(P, \theta)$



Trans $\underline{v}$