Math 461 Lecture 33 11/28

Homework 8 available later today

Due next Friday 12/7

Professor Hacking will be away next Monday and Tuesday

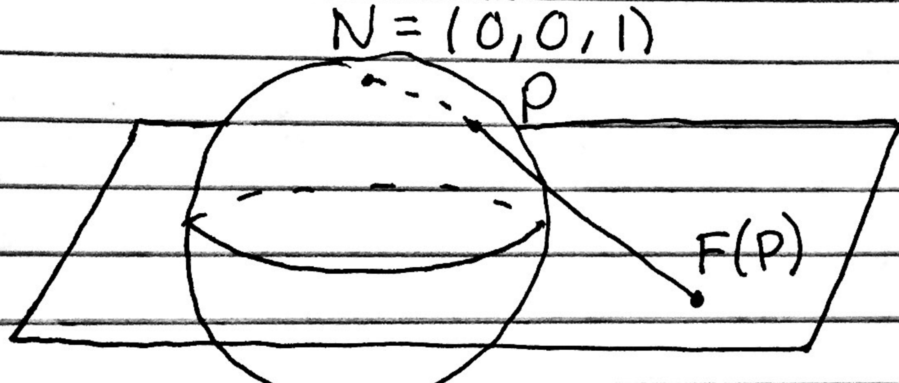
Professor Lai will teach Monday's class

Office hours next Thursday 4-5 pm LGRT 1235H (no office hours Monday and Tuesday)

Last time:

stereographic projection:

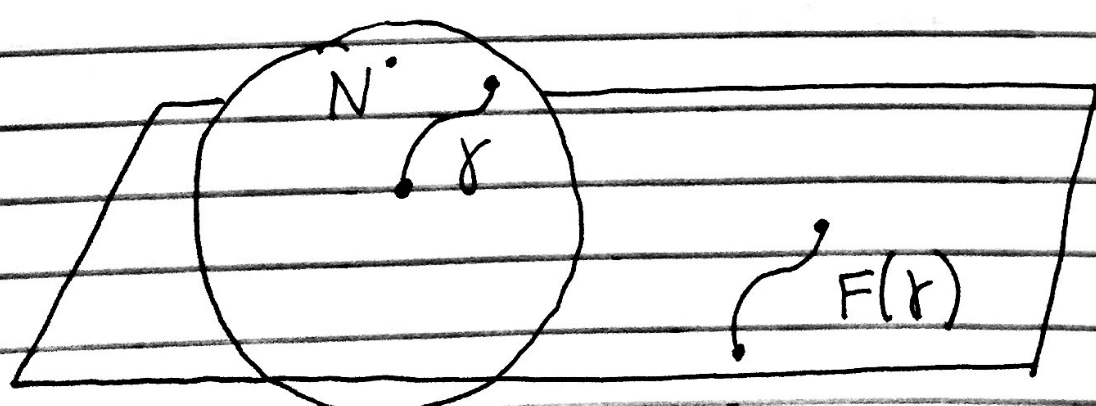
$F: S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$



algebraic formulas for F and F^{-1}:

$$F(x,y,z) = \frac{1}{1-z}(x,y)$$

$$F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$$



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distances:

γ curve on S^2, parameterization

$\bar{x}: [a,b] \to S^2 \subset \mathbb{R}^3$

$t \longmapsto \bar{x}(t) = (x(t), y(t), z(t))$

$F(\gamma)$ image in $\mathbb{R}^2$, parameterization

$(u,v): [a,b] \to \mathbb{R}^2$

$t \longmapsto (u(t), v(t)) = F(\bar{x}(t))$

then $\text{length}(\gamma) = \int_a^b \sqrt{x'^2 + y'^2 + z'^2}\, dt$

$$* \quad = \int_a^b \frac{2}{u^2+v^2+1} \sqrt{u'^2 + v'^2}\, dt$$

$\text{length}(F(\gamma)) = \int_a^b \sqrt{u'^2 + v'^2}\, dt$

Today: proof of formula *

stereographic projection preserves angles

stereographic projection sends spherical circles to circles and lines in $\mathbb{R}^2$

* is an instance of the chain rule

$[a,b] \to \mathbb{R}^2 \xrightarrow[F^{-1}]{} S^2 \subset \mathbb{R}^3$

$$t \longmapsto (u(t), v(t)) \longmapsto (x(t), y(t), z(t))$$
$$= F^{-1}(u(t), v(t))$$
$$= \bar{x}(u(t), v(t))$$

chain rule:

$$\bar{x}' = \frac{\partial \bar{x}}{\partial u} \cdot u' + \frac{\partial \bar{x}}{\partial v} \cdot v'$$

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$\sqrt{x'^2 + y'^2 + z'^2} = \|\bar{x}'\|$

$$\bar{x}(u,v) = \frac{1}{u^2+v^2+1}\begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1 \end{pmatrix}$$

$$\frac{\partial \bar{x}}{\partial u} = \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} 2(u^2+v^2+1) - 2u(2u) \\ 0 - 2v(2u) \\ 2u(u^2+v^2+1) - (u^2+v^2-1)(2u) \end{pmatrix}$$

$$= \frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} -2u^2+2v^2+2 \\ -4uv \\ 4u \end{pmatrix}$$

$$= \frac{2}{(u^2+v^2+1)^2}\begin{pmatrix} -u^2+v^2+1 \\ -2uv \\ 2u \end{pmatrix}$$

similarly $\frac{\partial x}{\partial v} = \frac{2}{(u^2+v^2+1)^2}\begin{pmatrix} -2uv \\ u^2-v^2+1 \\ 2v \end{pmatrix}$

$$\frac{d}{dx}\left(\frac{a}{b}\right) = \frac{\frac{da}{dx}b - a\frac{db}{dx}}{b^2}$$

want to compute: integers of u, v, u', v'

$$\|\bar{x}'\|^2 = \left\|\frac{\partial \bar{x}}{\partial u}\cdot u' + \frac{\partial \bar{x}}{\partial v}\cdot v'\right\|^2$$

$\|\bar{a}+\bar{b}\|^2 = \|\bar{a}\|^2 + \|\bar{b}\|^2 \quad \bar{a}\cdot\bar{b} = 0$

$$\text{①} = \left\|\frac{\partial \bar{x}}{\partial u}\right\|^2 \cdot u'^2 + \left\|\frac{\partial x}{\partial v}\right\| \cdot v'^2 \;\text{②}= \left(\frac{2}{u^2+v^2+1}\right)^2 \cdot (u'^2 + v'^2$$

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① $\frac{\partial \bar{x}}{\partial u} \cdot \frac{\partial \bar{x}}{\partial v} = 0$

② $\left\|\frac{\partial \bar{x}}{\partial u}\right\|^2 = \left\|\frac{\partial \bar{x}}{\partial v}\right\|^2 = \left(\frac{2}{u^2+v^2+1}\right)^2$

this gives *

$$\sqrt{x'^2+y'^2+z'^2} = \frac{2}{u^2+v^2+1}\sqrt{u'^2+v'^2}$$

① compute $\frac{\partial \bar{x}}{\partial u} \cdot \frac{\partial \bar{x}}{\partial v} = 0$

$$\begin{pmatrix} -2uv \\ u^2-v^2+1 \\ 2v \end{pmatrix} \begin{pmatrix} -u^2+v^2+1 \\ -2uv \\ 2u \end{pmatrix} =$$

$$-2uv(-\cancel{u^2}+\cancel{v^2}+1+\cancel{u^2}-\cancel{v^2}+1) + 2u \cdot 2v = 0$$

② $$\left\|\frac{\partial \bar{x}}{\partial u}\right\|^2 = \frac{4}{(u^2+v^2+1)^4}\underbrace{[(-2uv)^2+(u^2-v^2+1^2) + (2v)^2]}$$

$$(((u^2+1)-v^2)^2 + 4(u^2+1)v^2) = (u^2+v^2+1)^2$$

why? $(a+b)^2 = a^2+2ab+b^2$

$(a-b)^2 = a^2-2ab+b^2$

$(a+b)^2 = (a-b)^2 + 4ab$

$a = u^2+1$, $b = v^2 \Rightarrow +$ ✓

similarly for $\left\|\frac{\partial \bar{x}}{\partial v}\right\|^2$ ✓ □ for *

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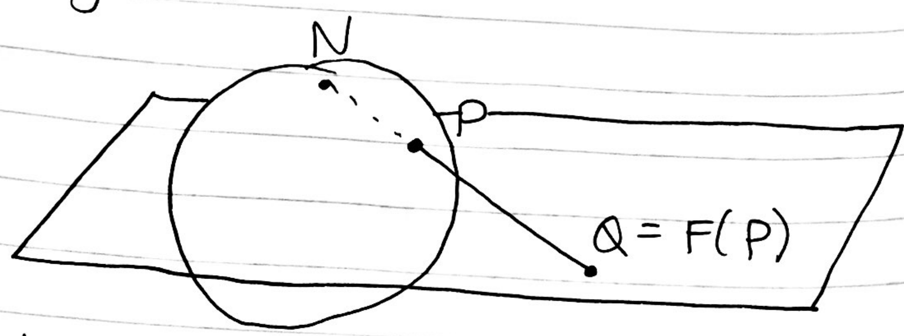
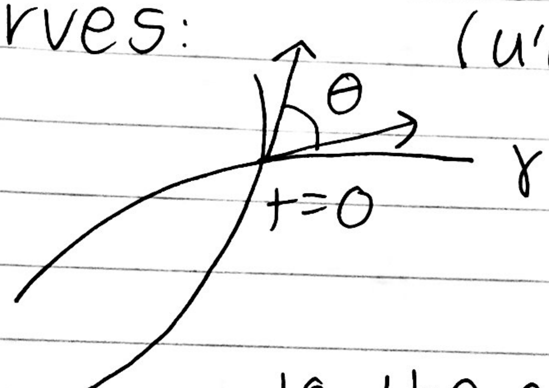
stereographic projection preserves angles:



$F^{-1}: \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\} \subset \mathbb{R}^3$

chain rule: $\bar{x} = (x', y', z') = \frac{\partial \bar{x}}{\partial u} u' + \frac{\partial \bar{x}}{\partial v} v'$

how do we define angle between two curves:



$(u'(0), v'(0))$

$t \mapsto (u(t), v(t))$

answer:
angle between vectors to the curves at the point

another way to say this:
the map $F^{-1}: \mathbb{R}^2 \to S^2 \setminus \{N\} \subset \mathbb{R}^3$ has derivative at the point Q given by the linear transformation

$\mathbb{R}^2 = T_Q \mathbb{R}^2 \longrightarrow T_P S^2 \subset \mathbb{R}^3$

$(\Delta u, \Delta v) \mapsto \left.\frac{\partial \bar{x}}{\partial u}\right|_Q \Delta u + \left.\frac{\partial \bar{x}}{\partial v}\right|_Q \Delta v$

for parametrized curve
$\gamma: t \mapsto (u(t), v(t))$ passing through Q at $t=0$

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$(u'(0), v'(0)) \longmapsto (x'(0), y'(0), z'(0))$

but we showed:

$$\sqrt{x'^2+y'^2+z'^2} = \frac{2}{u^2+v^2+1}\Big|_Q \sqrt{u'^2+v'^2}$$

i.e this linear map scales lengths of vectors by constant factor

$$c = \frac{2}{u^2+v^2+1}\Big|_Q$$

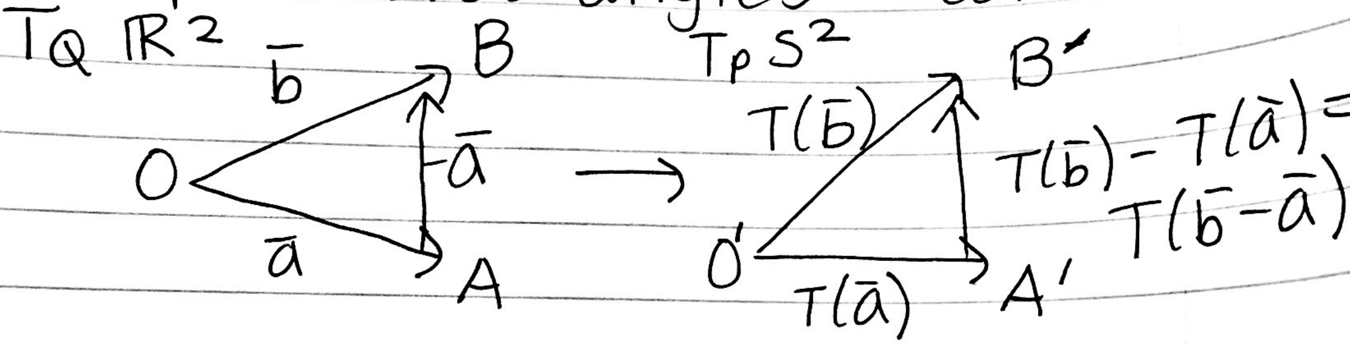
$\Delta O'A'B' \sim \Delta OAB$

lengths scaled by c

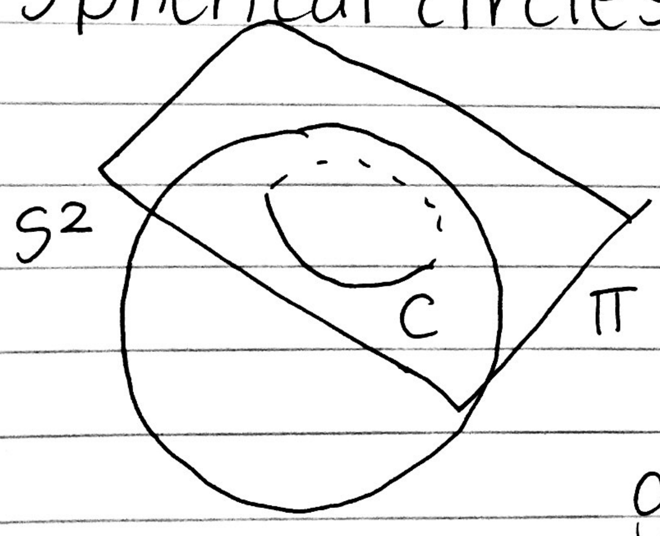
~~=> preserves angles "conformal"~~

$\Rightarrow \angle A'O'B' = \angle AOB$ $\square$

=> preserves angles "conformal"



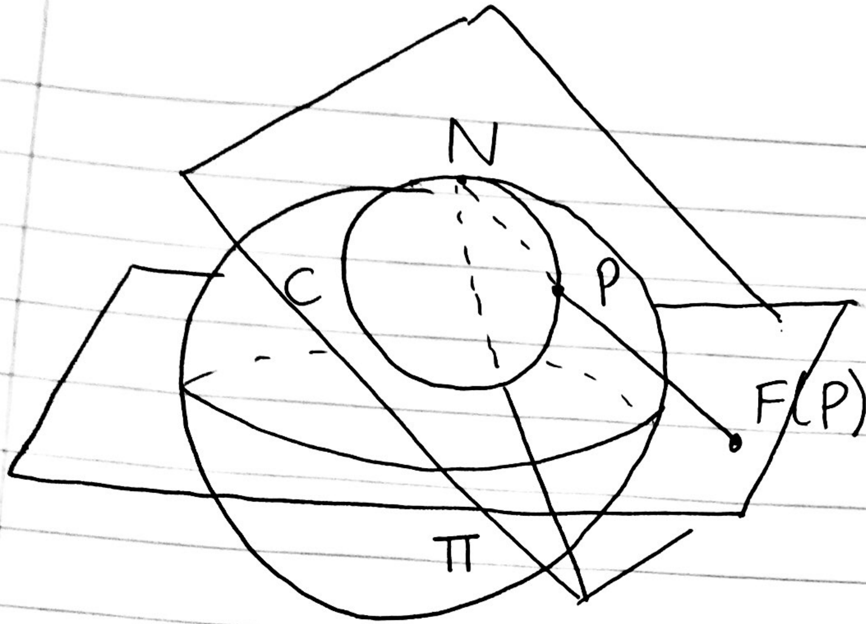
spherical circles: $C = \Pi \cap S^2$



Π plane in $\mathbb{R}^3$ not necessarily through origin

what's the image of C under stereographic projection?

case 1: circle C passes through $N = (0,0,1)$



death star

$C = \pi \cap S^2$

$F(C) = \,?\, \subset \mathbb{R}^2$

$= \text{line } \pi \cap (z=0)$

xy-plane