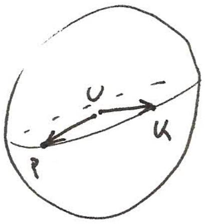
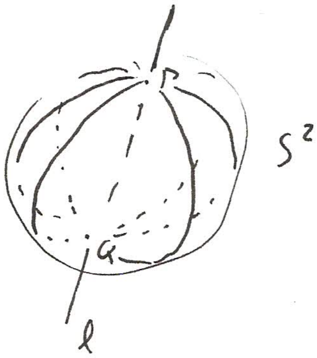
Wednesday 11/20/14.

1.

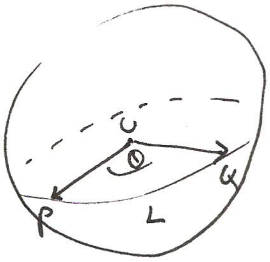


If P & Q are not antipodal then the vectors $\vec{OP}$ & $\vec{OQ}$ are linearly independent (not multiples of one another) so span a plane $\Pi \subset \mathbb{R}^3$ through O. Then $L = \Pi \cap S^2$ is the unique spherical line through P & Q.

If P & Q are antipodal then let l be the line through O containing P & Q. Then any plane Π through O containing l gives a spherical line $L = \Pi \cap S^2$ through P & Q



2.



Let L be the spherical line through P & Q.

Then L is a circle of radius 1 contained in the plane Π through the origin such that $\Pi \cap S^2 = L$.

So the length of the shorter arc of L connecting P & Q is given by

$$\underbrace{2\pi R}_{\text{circumference of } L} \cdot \frac{\theta}{2\pi} = R\theta \underbrace{=}_{R=1} \theta,$$

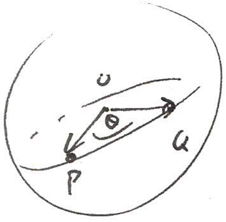
where θ is the angle between $\vec{OP}$ & $\vec{OQ}$.

In particular $d_{S^2}(P,Q) = \theta \leq \pi$, with equality if and only if $\vec{OP}$ & $\vec{OQ}$ are in opposite directions, i.e. P & Q are antipodal. $\square$

3. $P = \frac{1}{3}(1,2,2)$, $Q = \frac{1}{3}(2,2,1) \in S^2$.

a) $d_{S^2}(P,Q) = \theta = \cos^{-1}(\vec{OP}\cdot\vec{OQ}) = \cos^{-1}\left(\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}\right)$

$= \cos^{-1}\left(\frac{1}{9}\cdot(1\cdot2+2\cdot2+2\cdot1)\right) = \cos^{-1}(8/9)$.



using

$\vec{OP}\cdot\vec{OQ} = \|\vec{OP}\|\cdot\|\vec{OQ}\|\cdot\cos\theta$

$= 1\cdot1\cdot\cos\theta = \cos\theta$.

b) $L = \Pi \cap S^2$ where Π is the plane through O containing $\vec{OP}$ & $\vec{OQ}$.

Π has equation $\underline{x}\cdot\underline{n} = 0$, where $\underline{x} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ & $\underline{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is a normal vector to Π, i.e., $ax+by+cz=0$.

To find $\underline{n}$, we can take $\underline{n} = \vec{OP}\times\vec{OQ} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}$

$= \frac{1}{9}\begin{pmatrix}2\cdot1-2\cdot2\\2\cdot2-1\cdot1\\1\cdot2-2\cdot2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-2\\3\\-2\end{pmatrix}$

So Π: $\frac{1}{9}(-2x+3y-2z)=0$, or $-2x+3y-2z=0$.

& $L = \{(x,y,z)\in S^2 \mid -2x+3y-2z=0\}$. □.

4 a. Solve $\begin{array}{l}x+y+z=0\\x+2y+3z=0.\end{array}$ $\begin{array}{c}\\-R1\end{array}\left(\begin{array}{ccc|c}1&1&1&0\\1&2&3&0\end{array}\right) \sim \begin{array}{c}-R2\\ \\\end{array}\left(\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\end{array}\right)$

"augmented matrix"

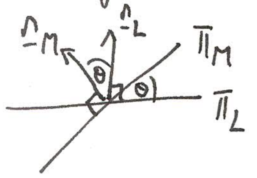
$x-z=0$
$y+2z=0$
z free.

$\rightsquigarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\-2z\\z\end{pmatrix} = z\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $z\in\mathbb{R}$. (alternatively, use cross product: $\begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$.)

$\begin{pmatrix}x\\y\\z\end{pmatrix}\in S^2 \Rightarrow 1 = \left\|\begin{pmatrix}x\\y\\z\end{pmatrix}\right\| = |z|\cdot\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\| = |z|\cdot\sqrt{6} \Rightarrow z = \pm 1/\sqrt{6}$,

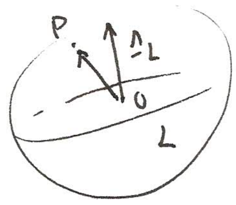
$$L \cap M = \left\{ \pm 1/\sqrt{6} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

b. Angle between L & M = angle between normal vectors to corresponding planes



$$= \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\|} \right) = \cos^{-1}\left(\frac{6}{\sqrt{3} \cdot \sqrt{14}} \right) = \cos^{-1}\left(\frac{6}{\sqrt{42}} \right)$$

5.



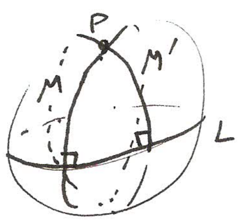
a. Let Π_M be the plane through 0 containing the vectors $\vec{OP}$ & $\underline{n}_L$ (a normal vector to the plane Π_L such that $L = \Pi_L \cap S^2$).

Then $M = \Pi_M \cap S^2$ is a spherical line through P perpendicular to L

(note $\underline{n}_L \in \Pi_M \Rightarrow \underline{n}_L \cdot \underline{n}_M = 0 \Rightarrow L$ & M are perpendicular)

b. Π_M is uniquely determined unless $\vec{OP}$ & $\underline{n}_L$ are linearly dependent, equivalently, $\vec{OP}$ is normal to Π_L. In that case there are infinitely many choices for M given by the planes through 0 containing $\vec{OP}$.

(if L is the equator, the choices for M are the lines of longitude)
the P is the north or south pole &



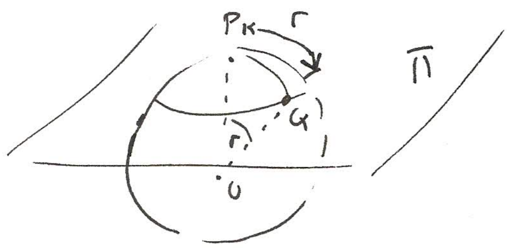
c. $P = \frac{1}{\sqrt{3}}(1,1,1)$, $L: 2x+4y+z=0$. $\Rightarrow$ $\underline{n}_M = \frac{1}{\sqrt{3}} \overset{\vec{OP}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \times \overset{\underline{n}_L}{\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\cdot1-1\cdot4 \\ 1\cdot2-1\cdot1 \\ 1\cdot4-2\cdot1 \end{pmatrix}$

$= 1/\sqrt{3} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

So M has equation $\frac{1}{\sqrt{3}}(-3x+y+2z)=0$

or $-3x+y+2z=0$. $\square$

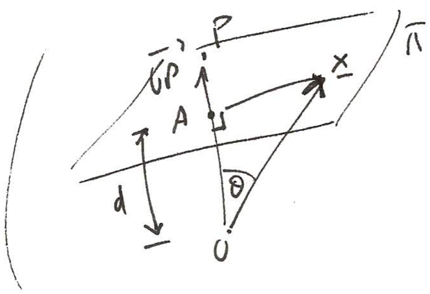
6a



$$C(P,r) = \{Q \in S^2 \mid d_{S^2}(P,Q) = r\}$$
$$= \{Q \in S^2 \mid \vec{OP} \cdot \vec{OQ} = \cos r\}$$
$$= S^2 \cap \Pi$$

where $\Pi = \{\underline{x} \in \mathbb{R}^3 \mid \underline{x} \cdot \vec{OP} = \cos r\}$

this is a plane in $\mathbb{R}^3$, with normal vector $\vec{OP}$ and perpendicular distance $d = \cos r$ from O.



$$\underline{x} \cdot \vec{OP} = \|\underline{x}\| \cdot \|\vec{OP}\| \cdot \cos\theta$$
$$= \|\underline{x}\| \cdot 1 \cdot \cos\theta \qquad (P \in S^2)$$
$$= \|\underline{x}\| \cdot \cos\theta = d.$$

b. Let A be the intersection point of the plane Π & the line OP.

The $|OA| = d = \cos r$ (see Q6a above)

Now, for $Q \in \Pi$, $OA \perp AQ$,

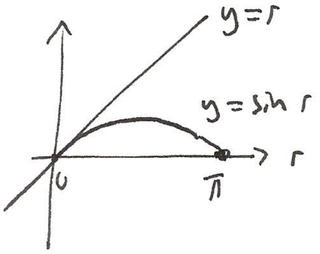
so $Q \in S^2 \iff |OQ| = 1 \overset{\text{P.T.}}{\iff} |OA|^2 + |AQ|^2 = 1 \iff |AQ| = \sqrt{1-(\cos r)^2}$ ($|OA| = \cos r$) $= \sin r$

i.e. $C(P,r) \underset{a.}{=} \Pi \cap S^2 = \{Q \in \Pi \mid |AQ| = \sin r\}$,

Euclidean circle in Π, center A, radius $\sin r$.

Thus $C(P,r)$ has circumference $2\pi \cdot \sin r$.

c. $2\pi \cdot \sin r < 2\pi r$ (note $0 < r < \pi$) :-



Need to show $\sin r < r$ for $0 < r < \pi$

i.e. $r - \sin r > 0$ for $0 < r < \pi$

$f(r) := r - \sin r$

$f(0) = 0$

$f'(r) = 1 - \cos r > 0$ for $0 < r < \pi$

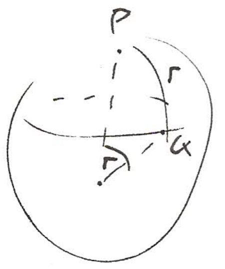
$\Rightarrow f(r)$ strictly increasing for $0 \leq r \leq \pi$ $\Rightarrow f(r) > 0$ for $0 < r < \pi$. $\square$.

d. $\lim\limits_{r \nearrow \pi} 2\pi \cdot \sin r = 2\pi \cdot 0 = 0.$

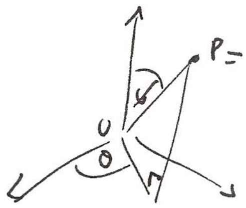
As $r \nearrow \pi$ the spherical circle $C(P, r)$ with center P & radius r shrinks to a point at the antipodal point Q of P.

(In fact $C(P,r) = C(Q, \pi - r)$.)

7 a.



spherical polar coordinates



$P = (x,y,z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

(here $P \in S^2$, i.e. $|OP| = 1$)

$$D(P,r) \setminus \{P\} \xleftarrow[\text{bijection}]{\sim} [0, 2\pi) \times (0, r]$$

$$\underline{x} = (x,y,z) \longleftarrow (\theta, \phi)$$

$$\parallel$$

$$(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$\therefore \text{ Area } D(P,r) = \int_0^r \int_0^{2\pi} \left\| \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \phi} \right\| d\theta \, d\phi$$

$$\frac{\partial \underline{x}}{\partial \theta} = \begin{pmatrix} -\sin\phi \sin\theta \\ \sin\phi \cos\theta \\ 0 \end{pmatrix} \qquad \frac{\partial \underline{x}}{\partial \phi} = \begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ -\sin\phi \end{pmatrix}$$

$$\frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \phi} = \begin{pmatrix} -(\sin\phi)^2 \cos\theta \\ -(\sin\phi)^2 \sin\theta \\ -\sin\phi \cos\phi \left((\sin\theta)^2 + (\cos\theta)^2\right) \end{pmatrix} = -\sin\phi \cdot \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix}$$

$$\left\| \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \phi} \right\| = |\sin\phi| \cdot \left\| \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} \right\|$$

$$= |\sin\phi| \cdot \sqrt{(\sin\phi)^2\left((\cos\theta)^2 + (\sin\theta)^2\right) + (\cos\phi)^2}$$

$$= \underbrace{|\sin\phi| \cdot 1}_{\sin^2+\cos^2=1 \ \times 2.} = \underbrace{\sin\phi}_{0\leq\phi\leq\pi}$$

$$\therefore \ \text{Area}\,(D(P,r)) = \int_0^r \int_0^{2\pi} \sin\phi \, d\theta \, d\phi = 2\pi \cdot \int_0^r \sin\phi \, d\phi$$

$$= 2\pi \cdot \left[-\cos\phi\right]_0^r = 2\pi \cdot (1-\cos r) \quad \square.$$

b. Need to show $\quad 2\pi \cdot (1-\cos r) < \pi r^2 \quad$ for $0 < r < \pi$.

Equiv, $\quad 2 \cdot (1-\cos r) < r^2 \quad$ for $0 < r < \pi$

$g(r) := r^2 - 2\cdot(1-\cos r)$. $\quad g(0) = 0.$

$g'(r) = 2r - 2\sin r = 2(r - \sin r) > 0$ for $0 < r < \pi$ by 6c

$\Rightarrow$ g strictly increasing for $0 \leq r \leq \pi$ $\Rightarrow$ $g(r) > 0$ for $0 < r < \pi$ $\square$.

c. $\lim_{r \to \pi} 2\pi \cdot (1-\cos r) = 2\pi \cdot (1 - \cos\pi) = 2\pi(1-(-1)) = 4\pi$

This is the area of S^2 (Area of sphere of radius $R=1$ is $4\pi \cdot R^2 = 4\pi$)

Q8.

$$f(r) = 2\pi r - 2\pi \sin r$$

$$= 2\pi\,(r - \sin r)$$

$$= 2\pi\left(r - \left(r - \frac{r^3}{3!} + \frac{r^5}{5!} - \dots\right)\right)$$

$$\approx 2\pi\left(r - \left(r - \frac{r^3}{6}\right)\right) \quad \text{for } r \text{ small}$$

$$= 2\pi \cdot \frac{r^3}{6} = \left(\frac{\pi}{3}\right) \cdot r^3$$

$$g(r) = \pi r^2 - 2\pi\,(1 - \cos r)$$

$$= \pi r^2 - 2\pi\left(1 - \left(1 - \frac{r^2}{2!} + \frac{r^4}{4!} - \dots\right)\right)$$

$$\approx \pi r^2 - 2\pi\left(\frac{r^2}{2} - \frac{r^4}{24}\right) \quad \text{for } r \text{ small}$$

$$= \left(\frac{\pi}{12}\right) \cdot r^4$$

$\square$.