Math 797W Algebraic geometry. Homework 1

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September 10, 2012

Throughout we work over an algebraically closed field k.

(1) Let $J_1, J_2 \subset k[x_1, \ldots, x_n]$ be ideals. Show that

$$Z(J_1 \cap J_2) = Z(J_1 J_2) = Z(J_1) \cup Z(J_2).$$

(2) Let $J \subset k[x_1, \ldots, x_n]$ be a radical ideal. Show that $Z(J)$ is irreducible iff J is prime.

(3) Let $J = (z, x^2 + y^2 + z^3) \subset k[x, y, z]$.

  (a) Find the irreducible components of $X := Z(J) \subset \mathbb{A}^3_{x,y,z}$.

  (b) Write J as an intersection of prime ideals.

(4) Let $X \subset \mathbb{A}^3$ denote the union of the coordinate axes. Find $I(X)$. Show that $I(X)$ cannot be generated by 2 elements.

(5) Let $k = \mathbb{C}$. Show that

$$X := \{(x, y) \in \mathbb{A}^2 \mid y = e^x\} \subset \mathbb{A}^2 = \mathbb{C}^2$$

is not an algebraic set.

(6) Let $X \subset \mathbb{A}^n$ be an algebraic set. By the Zariski topology on X we mean the subspace topology induced by the Zariski topology on $\mathbb{A}^n$. Show that X is irreducible iff for every pair of non-empty open subsets $U_1, U_2 \subset X$ we have $U_1 \cap U_2 \neq \emptyset$.

(7) Let $f\colon X \to Y$ be a morphism of affine varieties. Let

$$f^*\colon k[Y] \to k[X], \quad g \mapsto g \circ f$$

be the corresponding k-algebra homomorphism.

(a) Show that $\overline{f(X)} = Z(\ker(f^*))$ (where the bar denotes closure in the Zariski topology).

(b) We say f is *dominant* if $\overline{f(X)} = Y$. Show that f is dominant iff f^* is injective, and in this case f^* extends to a homomorphism of fields $f^*\colon k(Y) \to k(X)$.

(8) Let $X = \mathbb{A}^1_t$ and $Y = Z(y^2 - x^2(x+1)) \subset \mathbb{A}^2_{x,y}$.

(a) Show that the assignment $t \mapsto (t^2-1, t(t^2-1))$ defines a morphism of affine varieties $f\colon X \to Y$.

(b) Show that for $p \in Y$ we have $|f^{-1}p| = 1$ for $p \neq (0,0)$ and $|f^{-1}(0,0)| = 2$.

(c) Describe the homomorphism $f^*\colon k[Y] \to k[X]$ explicitly. Show that f^* is injective and the induced homomorphism of fields $f^*\colon k(Y) \to k(X)$ is an isomorphism. Deduce that $k[X]$ is the integral closure of $f^*(k[Y])$ in its field of fractions.

(9) Let $f\colon X = \mathbb{A}^2_{x_1,x_2} \to \mathbb{A}^3_{y_1,y_2,y_3}$ be the morphism $(x_1,x_2) \mapsto (x_1^2, x_1x_2, x_2^2)$.

(a) Show that $Y := f(X)$ is an algebraic set and determine $I(Y)$. Describe the homomorphism $f^*\colon k[Y] \to k[X]$ explicitly.

(b) Let $G = \mathbb{Z}/2\mathbb{Z}$ and consider the action of G on X given by $(x_1,x_2) \mapsto (-x_1,-x_2)$. Show that f^* maps $k[Y]$ isomorphically onto the invariant subring $k[X]^G \subset k[X]$.

(c) Show that, as maps of sets, $f = g \circ q$ where $q\colon X \to X/G$ is the quotient map and $g\colon X/G \to Y$ is a bijection.

(10) Let $J = (x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3) \subset k[x_1,x_2,x_3,x_4]$. Show that the map

$$k[x_1,\ldots,x_4]/J \to k[s,t], \quad x_1,x_2,x_3,x_4 \mapsto s^3, s^2t, st^2, t^3$$

is injective. Deduce that J is a prime ideal. The affine variety $X = Z(J) \subset \mathbb{A}^4$ is the surface studied in class (the "cone over the twisted cubic"). [Hint: We can use the generators of J to write an element of $k[x_1, \dots, x_4]/J$ as a linear combination $a+bx_2+cx_3$ for $a, b, c \in k[x_1, x_4]$ (why?)]

(11) Show that the prime ideals of $k[x, y]$ are (0), (f) for f an irreducible polynomial, and $(x-a, y-b)$ for $a, b \in k$. [Hint: If $\mathfrak{p}$ is a prime ideal containing two elements $f, g \in k[x, y]$ with no common factors, show that $\mathfrak{p} \cap k[x] \neq (0)$. (Use the Gauss lemma to show f, g are coprime in $k(x)[y]$ and the Euclidean algorithm in $k(x)[y]$.)]

(12) Let $f \colon X \to Y$ be a morphism of affine varieties. Then the ring homomorphism $f^* \colon k[Y] \to k[X]$ gives the structure of a $k[Y]$-module on $k[X]$. We say f is a *finite morphism* if $k[X]$ is a finitely generated $k[Y]$-module.

(a) Show that if f is finite then the fiber $f^{-1}(p)$ is finite for all $p \in Y$.

(b) Give a counterexample to the converse of the statement in (a).