# Math 132.5. Representations of functions as power series (11.9); Taylor series and Maclaurin Series (11.10)

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## 1 Section 11.9

### 1.1 Geometric series

The formula for the geometric series gives the basic formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \text{ for } |x| < 1. \tag{$\star$}$$

### 1.2 Differentiation and Integration of power series

Let

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

be a power series centered at $x = a$ with radius of convergence $R > 0$. Then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on the interval $(a-R, a+R)$ is continuous and differentiable, and

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$\int f(x)dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\right) + c = c + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

That is, the derivative and integral of $f$ can be computed term-by-term. Moreover, these power series have the same radius of convergence $R$ as the original series.

*Example* 1.1.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1+x+x^2+\cdots) = 1+2x+3x^2+\cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

valid for $|x| < 1$, where we have used $(\star)$.

*Example* 1.2.

$$\ln(1+x) = \int \frac{1}{1+x}dx = \int \frac{1}{1-(-x)}dx = \int \sum_{n=0}^{\infty}(-x)^n dx$$

$$= \int 1 - x + x^2 - x^3 + \cdots dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},$$

valid for $|-x| < 1$, that is, $|x| < 1$. Here we have used $(\star)$ with $x$ replaced by $-x$, and the constant of integration $c = 0$ because $\ln(1+0) = \ln 1 = 0$.

*Example* 1.3.

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}dx = \int \frac{1}{1-(-x^2)}dx = \int \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \int 1 - x^2 + x^4 - \cdots dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

valid for $|-x^2| < 1$, that is, $|x| < 1$. Here we have used $(\star)$ with $x$ replaced by $-x^2$, and the constant of integration $c = 0$ because $\tan^{-1}(0) = 0$.

## 1.3 Substitution

Note that we can get power series expansions of new functions by substitution.

*Example* 1.4.

$$\frac{x^7}{(5+x^3)^2} = \frac{x^7}{5^2} \cdot \frac{1}{(1-(-x^3/5))^2} = \frac{x^7}{5^2}\sum_{n=0}^{\infty}(n+1)\left(\frac{-x^3}{5}\right)^n$$

$$= \frac{x^7}{5^2} \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{x^{3n}}{5^n} = \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{x^{3n+7}}{5^{n+2}}$$

valid for $|-x^3/5| < 1$, that is $|x| < \sqrt[3]{5}$. Here we used Example 1.1 with $x$ replaced by $-x^3/5$.

# 2 Section 11.10

Suppose $f(x)$ is a function that has a power series expansion centered at $x = a$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

with radius of convergence $R > 0$. Then the coefficients $c_n$ of the power series are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

where $f^{(n)}(x)$ denotes the $n$th derivative of $f(x)$. (This is proved by repeatedly differentiating the power series expansion as in section 1.2.) So the power series expansion is given by the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{6}(x-a)^3+\cdots$$

This is the *Taylor series* for $f(x)$ (also called the *Maclaurin series* when $a = 0$).

*Example* 2.1.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

These power series expansions are valid for all $x$.

*Example* 2.2. Let $f(x) = \frac{1}{\sqrt{1+x}}$. We assume $f$ has a power series expansion and compute it using Taylor's formula. We have

$$f(x) = (1+x)^{-1/2}$$

$$f'(x) = \frac{-1}{2}(1+x)^{-3/2}$$

$$f''(x) = \frac{-1}{2}\frac{-3}{2}(1+x)^{-5/2}$$

etc. So

$$f^{(n)}(x) = \frac{-1}{2}\frac{-3}{2}\frac{-5}{2}\cdots\frac{-(2n-1)}{2}(1+x)^{-(2n+1)/2} = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}(1+x)^{-(2n+1)/2}$$

for all $n \geq 0$. So

$$f^{(n)}(0) = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}$$

and

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n n!}x^n = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{15x^3}{48} + \cdots$$

by Taylor's formula. Using the ratio test, we find that this power series expansion has radius of convergence $R = 1$.