Math 300.3 Homework 4

Paul Hacking

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Reading: Sundstrom, Sections 3.2 and 3.3.

Justify your answers carefully.

(1) Let a, b and c be real numbers such that $a \neq 0$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be the quadratic function defined by $g(x) = ax^2 + bx + c$ for all real numbers x.

(a) Show that for all real numbers x

$$g(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

(This technique is sometimes called "completing the square".)

(b) Using part (a) or otherwise, prove that if $a > 0$ and $4ac - b^2 > 0$ then $g(x) > 0$ for all real numbers x.

(2) Recall that, for a conditional statement $P \Rightarrow Q$, the *contrapositive* is the conditional statement $(\text{NOT}\, Q) \Rightarrow (\text{NOT}\, P)$. The conditional statement and its contrapositive are logically equivalent (they are either both true or both false).

For each of the following statements, write down the contrapositive of the conditional statement, and give a direct proof of the contrapositive.

(a) For all positive integers n, if n is prime then either $n = 2$ or n is odd.

[Hint: Recall that we say a positive integer n is *prime* if $n \neq 1$ and the only positive integers which divide n are 1 and n.]

(b) For all real numbers x, if $x^3 - 3x^2 + 5x + 1 > 7$ then $x > 2$.

[Hint: Recall that we say a function $f\colon \mathbb{R} \to \mathbb{R}$ is *increasing* if for all real numbers a and b, $a \leqslant b \Rightarrow f(a) \leqslant f(b)$. The following result is proved using the Mean value theorem: If $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) \geqslant 0$ for all $x \in \mathbb{R}$ then f is increasing. Use this result together with Q1(b).]

(3) Prove the following biconditional statements.

(a) For all real numbers x,
$$x^3 + x^2 = 0 \iff ((x = 0)\,\mathrm{OR}(x = -1)).$$

(b) For all real numbers x and y,
$$x^2 = y^2 \iff ((x = y)\,\mathrm{OR}(x = -y)).$$

(c) For all integers n,
$$n \text{ is odd} \iff n^3 \text{ is odd}.$$

[Hint: Recall that, to prove a biconditional statement $P \iff Q$ (in words "P if and only if Q") we need to prove $P \Rightarrow Q$ *and* $Q \Rightarrow P$.]

(4) Prove the following existence statements.

(a) There exists $x \in \mathbb{R}$ such that $x^2 + 7x + 11 = 0$.

(b) There exists $x \in \mathbb{R}$ such that $x^3 + x^2 + 1 = 0$.

[Hint: Use the *intermediate value theorem*, which is the following result. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let a and b be real numbers such that $a < b$ and let c be a real number. If $f(a) < c < f(b)$ or $f(a) > c > f(b)$ then there exists a real number t such that $a < t < b$ and $f(t) = c$.]

(c) There exists $x \in \mathbb{R}$ such that $e^x \sin(\pi x/2) = 2$.

(5) Prove the following statement: For all positive integers n, if $n > 1$ then $4^n - 1$ is not prime.

[Hint: We proved the following statement in class on Tuesday 2/14/17: For all real numbers x and positive integers n,
$$x^n - 1 = (x - 1)(x^{n-1} + \cdots + x + 1).$$
]

(6) (a) Prove the following statement: For all *odd* positive integers n and real numbers x,

$$(x^n+1) = (x+1)(x^{n-1}-x^{n-2}+x^{n-3}-\cdots-x+1) = (x+1)(\sum_{i=0}^{n-1}(-1)^i x^i).$$

[Hint: Adapt the proof given in class of the statement in the hint for Q5.]

(b) Using part (a) or otherwise, prove the following statement: For all positive integers n, if there is a positive integer d such that $d > 1$, d is odd, and d divides n then $2^n + 1$ is not prime.

[Hint: Adapt the proof given in class of the following statement: For all positive integers n, if n is not prime then $2^n - 1$ is not prime.]

(c) If n is a positive integer and $n > 1$ then n can be written as a product of prime numbers (we will prove this in class later). Use this result, part (b), and Q2(a) to prove the following statement: For all positive integers n, if $2^n + 1$ is prime then $n = 2^m$ for some non-negative integer m.

[Hint: First write down the contrapositive of the conditional statement proved in part (b) and then use the stated result and Q2(a).]