Math 462 Homework 3

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(1) Consider the following isometries of $\mathbb{R}^2$: R is rotation about the origin through angle θ counterclockwise, S is reflection in the x-axis, and T is translation by a vector $\mathbf{b} \in \mathbb{R}^2$. Give a precise geometric description of the following compositions (as a translation, rotation, reflection, or glide reflection).

(a) $T \circ R \circ T^{-1}$.

(b) $T \circ S \circ T^{-1}$.

(c) $R \circ S \circ R^{-1}$.

(d) $S \circ R \circ S$.

[Here $F \circ G$ denotes the composition of the functions F and G, i.e., $(F \circ G)(\mathbf{x}) = F(G(\mathbf{x}))$. And F^{-1} denotes the inverse of the function F, i.e., $F(\mathbf{x}) = \mathbf{y} \iff F^{-1}(\mathbf{y}) = \mathbf{x}$.]

(2) For $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ an isometry, the fixed locus of T is the set of points $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{x}$. Describe the fixed locus for each type of isometry T (translation, rotation, reflection, and glide reflection).

(3) Describe the following isometries geometrically.

(a) $R \colon \mathbb{R}^2 \to \mathbb{R}^2$, $R\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x+y \\ x-y \end{pmatrix}$.

(b) $S \colon \mathbb{R}^2 \to \mathbb{R}^2$, $S\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3x-4y \\ 4x+3y \end{pmatrix}$.

(4) Determine algebraic formulas $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for the following isometries T. (Here A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector.)

(a) Rotation about the point $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ through angle $\pi/4$ counterclockwise.

(b) Reflection in the line $y = 4$.

(c) Reflection in the line $y = x + 2$ followed by a translation parallel to the line through distance $3\sqrt{2}$ in the direction of increasing x. (This is a glide reflection.)

(5) Describe the following isometries of $\mathbb{R}^2$ geometrically.

(a) $R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + 2 \\ -x + 3 \end{pmatrix}$.

(b) $S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + 4 \\ -x - 3 \end{pmatrix}$.

(6) Let $r_L \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $r_M \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometries given by reflection in lines L and M in $\mathbb{R}^2$. Suppose L and M meet in a point P such that the angle from L to M is θ (measured counterclockwise). Show that the composition $r_M \circ r_L$ is the rotation about P through angle 2θ counterclockwise.

[Hint: One way to do this is to choose coordinates so that the point P is the origin and the line L is the x-axis. Now compute using matrices: writing $r_L(\mathbf{x}) = A\mathbf{x}$ and $r_M(\mathbf{x}) = B\mathbf{x}$, we have $r_M \circ r_L(\mathbf{x}) = BA\mathbf{x}$.]

(7) As in the previous question, let r_L and r_M be the isometries of $\mathbb{R}^2$ given by reflection in lines L and M in $\mathbb{R}^2$. Suppose the lines L and M are parallel. Give a precise geometric description of the composition $r_M \circ r_L$.

(8) Let $R \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry of $\mathbb{R}^2$ given by rotation about a point P through angle θ counterclockwise. Let $S \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry given by rotation about a point Q through angle θ clockwise.

(a) Show that the composition $S \circ R$ is a translation, i.e., $S(R(\mathbf{x})) = \mathbf{x} + \mathbf{b}$ for some vector $\mathbf{b} \in \mathbb{R}^2$.

(b) Show that if the angle θ is small, then the translation vector $\mathbf{b}$ has length approximately $\theta \cdot d(P, Q)$ and is approximately perpendicular to the vector $\overrightarrow{PQ}$. (Here $d(P, Q)$ denotes the distance from P to Q.) [This fact is sometimes useful when moving furniture.]

(c) What happens for $\theta = \pi$?

[Hint: By choosing coordinates appropriately we can assume P is the origin and Q lies on the x-axis. Now compute by expressing R and S in the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector. For part (b) use the approximations $\cos(\theta) \simeq 1$ and $\sin(\theta) \simeq \theta$ for θ small.]