Math 412 Homework 7

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Reading: Saracino, Chapter 23.
Show your work and justify your answers carefully.

(1) Let $\zeta = e^{2\pi i/5}$.

(a) Write down the irreducible polynomial of ζ over $\mathbb{Q}$, determine the degree $[\mathbb{Q}(\zeta) : \mathbb{Q}]$, and write down a basis of $\mathbb{Q}(\zeta)$ as a vector space over $\mathbb{Q}$.

(b) Let $\alpha = \zeta + \zeta^4 = \zeta + \bar{\zeta} = 2\cos(2\pi/5) \in \mathbb{R}$. Show that the degree $[\mathbb{Q}(\zeta) : \mathbb{Q}(\alpha)] = 2$ and determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

(c) Compute the irreducible polynomial of α over $\mathbb{Q}$, and deduce an exact expression for $\cos(2\pi/5)$ using square roots.

[Hint for (c): Let $n = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ from part (b). Write the powers $1, \alpha, \ldots, \alpha^n$ in terms of the basis from part (a), find a linear relation $c_0 + c_1\alpha + \cdots + c_n\alpha^n = 0$ with $c_0, \ldots, c_n \in \mathbb{Q}$, and so determine the irreducible polynomial of α over $\mathbb{Q}$.]

(2) In each of the following cases, determine whether the given real number α is constructible.

(a) $\alpha = \sqrt[4]{5}$.

(b) $\alpha = 2 + \sqrt{3 + \sqrt{5 + \sqrt{7}}}$

(c) $\alpha \in \mathbb{R}$ is a root of the polynomial $f(x) = x^5 + 4x + 2$.

(d) $\alpha = \cos(2\pi/13)$.

(3) Let $f(x) = ax^4 + bx^2 + c \in \mathbb{Q}[x]$ and suppose $\alpha \in \mathbb{R}$ is a root of f. Show that α is constructible.

(4) Find the intersection points of the circle with center the origin and radius 1 and the circle with center $(2, 2)$ and radius 3.

[Remark: As a special case of a result proved in class, given two circles determined by points with coordinates in $\mathbb{Q}$, the coordinates of the intersection points are either in $\mathbb{Q}$ or a degree 2 extension $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}$.]

(5) Compute the intersection points of the ellipses $4x^2 + y^2 = 1$ and $x^2 + 9y^2 = 1$. Show that for each intersection point $P = (\alpha, \beta)$, the field extension $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$ has degree 4.

(6) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible monic polynomial of degree 3. Then $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$.

   (a) Show that one of the roots $\alpha_1, \alpha_2, \alpha_3$ of $f(x)$ must be real.

   (b) Suppose that $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \notin \mathbb{R}$. Show that $[\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] = 6$.

(7) Consider the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

   (a) State the degree $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and write down a basis of $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$. (You may omit the full proof here as we have done similar calculations before.)

   (b) For $\gamma \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$ what are the possible values of the degree $[\mathbb{Q}(\gamma) : \mathbb{Q}]$?

   (c) Prove that $\gamma = \sqrt{2} + 2\sqrt{3}$ satisfies $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 4$ using the following observation and the basis of part (a): if $1, \gamma, \ldots, \gamma^{n-1}$ are linearly independent over $\mathbb{Q}$ then $[\mathbb{Q}(\gamma) : \mathbb{Q}] \geq n$.