Saturday 11/7/15 421 HW7 solutions

1. GCIF : $f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$

Also $\left|\int_C g(z)\,dz\right| \leqslant \text{length}(C)\cdot M$

where $|g(z)| \leq M$ for $z \in C$.

So $f'(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^2}\,dz.$

$\left|\frac{f(z)}{z^2}\right| = \frac{|f(z)|}{|z|^2}$ $\quad \left.\begin{array}{l} |f(z)| \leq 5 \\ |z|^2 = 3^2 \end{array}\right\}$ for $z \in C$.

$\Rightarrow \quad |f'(0)| = \frac{1}{2\pi}\cdot\left|\int_C \frac{f(z)}{z^2}\,dz\right| \leq \frac{1}{2\pi}\cdot \text{length}(C)\cdot\frac{5}{3^2}$

$= \frac{1}{2\pi}\cdot(2\pi\cdot 3)\cdot\frac{5}{3^2} = 5/3$

2. $f^{(3)}(2i) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-2i)^4}\,dz$

$\Rightarrow \quad |f^{(3)}(2i)| = \frac{6}{2\pi}\left|\int_C \frac{f(z)}{(z-2i)^4}\,dz\right|$

$\left|\frac{f(z)}{(z-2i)^4}\right| = \frac{|f(z)|}{|z-2i|^4}$ $\quad \left.\begin{array}{l} |f(z)| \leq 7 \\ |z-2i| \geqslant \big||z|-2\big| = |5-2| = 3 \end{array}\right\}$ for $z \in C$

$\Rightarrow \quad |f^{(3)}(2i)| \leqslant \frac{6}{2\pi}\cdot\text{length}(C)\cdot\frac{7}{3^4}$

$= \frac{6}{2\pi}\cdot(2\pi\cdot 5)\cdot\frac{7}{3^4} = \frac{70}{27}$

3. a $f'(x) = 0$ for $x<0$

& $f'(x) = 2x$ for $x>0$.

At $x=0$.

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$$

Compute one-sided limits:

$$\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0.$$

$$\lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0.$$

$$\therefore \lim_{h\to 0} \frac{f(h)}{h} = 0, \quad f'(0) = 0.$$

b. $$f'(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 2x & \text{for } x>0. \end{cases}$$

$$f''(0) = \lim_{h\to 0} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0} \frac{f'(h)}{h}$$ this limit does not exist :-

$$\lim_{h\to 0^-} \frac{f'(h)}{h} = 0, \quad \lim_{h\to 0^+} \frac{f'(h)}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

$$\lim_{h\to 0^-} \ne \lim_{h\to 0^+} \Rightarrow \lim_{h\to 0}$$ does not exist.

c. $$f(x) = \begin{cases} 0 & x<0 \\ x^{n+1} & x \geq 0 \end{cases}$$

4. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ (Cauchy Riemann eqs.)

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \underset{CR}{=} \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

So f' cx differentiable $\Rightarrow$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are real differentiable

$\Rightarrow$ 2nd partial derivatives of u & v exist.

Now $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Combining, $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

& $$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2}.$$

5. a. i. $$f(\alpha) = \int_0^{2\pi} \cos(\alpha x)\,dx = \left[\frac{1}{\alpha}\sin(\alpha x)\right]_0^{2\pi}$$

$$= \frac{1}{\alpha}\sin(2\pi\alpha)$$

$$\therefore f'(\alpha) = \frac{-1}{\alpha^2}\sin(2\pi\alpha) + \frac{1}{\alpha}\cdot 2\pi\cos(2\pi\alpha)$$

ii. $$f'(\alpha) = \int_0^{2\pi} \frac{\partial}{\partial \alpha}(\cos\alpha x)\,dx = \int_0^{2\pi} x\cdot(-\sin\alpha x)\,dx$$

b.

$$= -\int_0^{2\pi} x\sin\alpha x\,dx$$

$$\int_0^{2\pi} x\sin\alpha x\,dx = \left[-\frac{x}{\alpha}\cos(\alpha x)\right]_0^{2\pi} - \int_0^{2\pi} -\frac{1}{\alpha}\cos(\alpha x)\,dx$$

$$\left(\int u\,dv = uv - \int v\,du\right)$$

$$= -\frac{2\pi}{\alpha}\cos(2\pi\alpha) + \left[\frac{1}{\alpha^2}\sin(\alpha x)\right]_0^{2\pi}$$

$u = x \quad dv = \sin(\alpha x)\,dx$

$du = dx \quad v = -\frac{1}{\alpha}\cos(\alpha x).$

$$= -\frac{2\pi}{\alpha}\cos(2\pi\alpha) + \frac{1}{\alpha^2}\sin(2\pi\alpha)$$

✓

6. $f : \mathbb{R} \longrightarrow \mathbb{R}$ $\qquad f(\alpha) = \int_1^e x^\alpha \, dx$

a. i. $$f(\alpha) = \left[\frac{x^{\alpha+1}}{\alpha+1} \right]_1^e = \frac{e^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} \qquad (\alpha \neq -1)$$

$$\therefore \quad f'(\alpha) = \frac{-e^{\alpha+1}}{(\alpha+1)^2} + \frac{e^{\alpha+1}}{\alpha+1} + \frac{1}{(\alpha+1)^2}$$

ii. $$f'(\alpha) = \int_1^e \frac{\partial}{\partial\alpha}(x^\alpha)\, dx$$

$$= \int_1^e \frac{\partial}{\partial\alpha}\left(e^{\alpha \log x}\right) dx$$

$$= \int_1^e \log x \cdot e^{\alpha \log x} \, dx$$

$$= \int_1^e \log x \cdot x^\alpha \, dx.$$

b. $$\int_1^e \log x \cdot x^\alpha \, dx = \left[\log x \cdot \frac{x^{\alpha+1}}{\alpha+1} \right]_1^e - \int_1^e \frac{x^{\alpha+1}}{\alpha+1} \cdot \frac{1}{x} \, dx$$

$\left(\int u \, dv = uv - \int v \, du \right)$

$u = \log x \qquad dv = x^\alpha dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^{\alpha+1}}{\alpha+1}$

$$= \frac{e^{\alpha+1}}{\alpha+1} - \int_1^e \frac{x^\alpha}{\alpha+1} \, dx$$

$$= \frac{e^{\alpha+1}}{\alpha+1} - \left[\frac{x^{\alpha+1}}{(\alpha+1)^2} \right]_1^e$$

$$= \frac{e^{\alpha+1}}{\alpha+1} - \frac{e^{\alpha+1}}{(\alpha+1)^2} + \frac{1}{(\alpha+1)^2} \quad \checkmark$$

7. $f : \mathbb{C} \to \mathbb{C}$ cx diff'ble, $f = u + iv$.

$u(x,y) \leq M$ for all $x, y \in \mathbb{R}$, some $M \in \mathbb{R}$.

$g(z) = e^{f(z)}$, $g : \mathbb{C} \longrightarrow \mathbb{C}$ cx diff'ble.

$$|g(z)| = |e^{u+iv}| = e^u \leq e^M \quad \text{for all } z.$$

i.e. $g: \mathbb{C} \to \mathbb{C}$ cx diff'ble, g bounded $\Rightarrow$ g constant

Liouville's theorem.

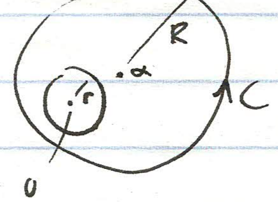
Now $g(z) = e^{f(z)}$, & f continuous $\Rightarrow$ f constant.

[In more detail : $e^{f(z)}$ constant $\Rightarrow$ $f(z_1) - f(z_2)$ is a multiple of $2\pi i$ for each $z_1, z_2 \in \mathbb{C}$.

But f is continuous (because it's differentiable) so value of f can't jump, & so f is constant.]

8. a. Let $\alpha \in \mathbb{C}$.

Let C be a circle, center α, radius R sufficiently large so that $|z-\alpha| = R \Rightarrow |z| \geq r$.



(i.e. $R \geq |\alpha| + r$)

Orient C ccw.

GCIF $\quad f''(\alpha) = \dfrac{2!}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-\alpha)^3}\, dz$

$$|f''(\alpha)| = \frac{1}{\pi} \cdot \left| \int_C \frac{f(z)}{(z-\alpha)^3}\, dz \right|$$

$|f(z)| \leq M \cdot |z| \leq M \cdot (R + |\alpha|)$ for $z \in C$

$|z - \alpha| = R$ for $z \in C$

$$\therefore \quad |f''(\alpha)| \leq \frac{1}{\pi} \cdot \operatorname{length} C \cdot \frac{M(R+|\alpha|)}{R^3}$$

$$= \frac{1}{\pi} \cdot 2\pi R \cdot \frac{M(R+|\alpha|)}{R^3} = \frac{2M(R+|\alpha|)}{R^2}$$

Let $R \to \infty$

$$\lim_{R\to\infty} \frac{2M(R+|\alpha|)}{R^2} = \lim_{R\to\infty} 2M\left(\frac{1}{R}+\frac{|\alpha|}{R^2}\right) = 0.$$

$\Rightarrow \; f''(\alpha) = 0.$

b) $f''(z) = 0 \;\Rightarrow\; f'(z) = A, \quad A \in \mathbb{C}$ constant.

$\Rightarrow \; f(z) = Az + B, \quad A, B \in \mathbb{C}$ constants

c) Same argument shows $f^{(n+1)}(z) = 0$

$\Rightarrow \; f(z) = A_n z^n + \dots + A_1 z + A_0,$

$A_n, \dots, A_0 \in \mathbb{C}$ constants.

i.e. f is a polynomial of degree $\leq n$.