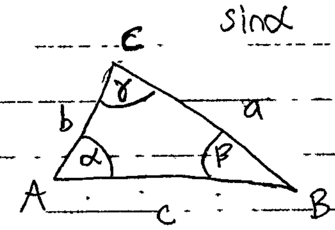
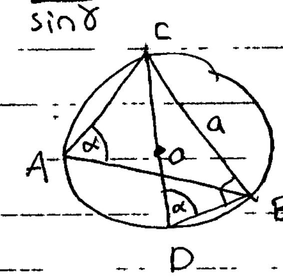
10/4/19

HW3 solutions available (check email/webpage)

Last Time: Gauss' theorem on constructibility of regular polygons.

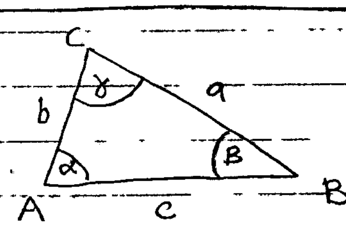
Sine rule: $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} = 2R$





Today: - Cosine rule
- Center of mass
- Coordinates

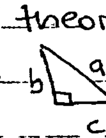
Cosine Rule



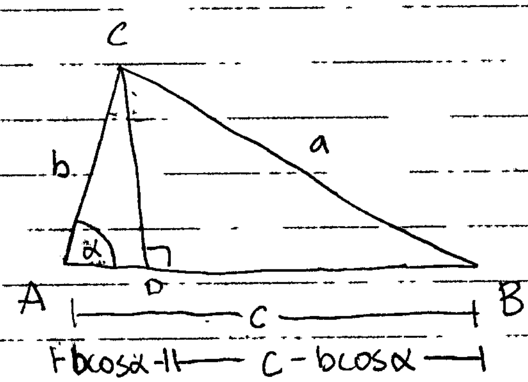
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

similar to Pythagoras' theorem

$a^2 = b^2 + c^2$ if $\alpha = \pi/2$



Proof



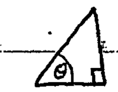
$\cos\alpha = \dfrac{|AD|}{b} \Rightarrow |AD| = b\cos\alpha$

$\sin\alpha = \dfrac{|CD|}{b} \Rightarrow |CD| = b\sin\alpha$

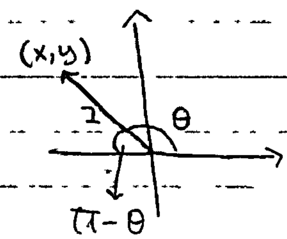
$$\begin{aligned} a^2 &= (c - b\cos\alpha)^2 + (b\sin\alpha)^2 \\ &= c^2 - 2bc\cos\alpha + b^2(\cos\alpha)^2 + b^2(\sin\alpha)^2 \\ &= c^2 - 2bc\cos\alpha + b^2 \end{aligned}$$

$(\cos\alpha)^2 + (\sin\alpha)^2 = 1$

Aside: $\cos \pi/2 = 0$, $\sin \pi/2 = 1$
$\cos 0 = 1$, $\sin 0 = 0$
$\cos \pi = -1$, $\sin \pi = 0$

Remark When we defined sine & cosine, assumed $0 < \theta < \pi/2$
So have right angled triangle .

In sine & cosine rules, need sin & cosine for $0 < \theta < \pi$.

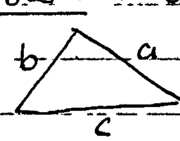


$x = \cos\theta$
$y = \sin\theta$

Equivalently, for $\pi/2 < \theta < \pi$,
$\sin\theta = \sin(\pi - \theta)$
$\cos\theta = -\cos(\pi - \theta)$.

Proofs we gave still work with minor modifications.

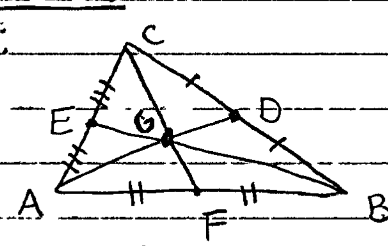
HW problem: Use the cosine rule to prove the triangle inequality.



$\boxed{a < b + c}$ "easier to cut corners"

Center of Mass

Theorem:

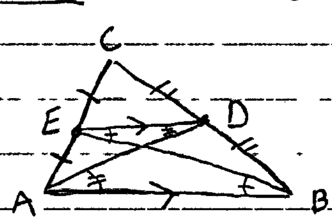


Let D be the midpoint of BC
" E " CA
" F " AB

The lines AD, BE, CF ("medians") all meet at a point G, and

$$\frac{|AG|}{|GD|} = \frac{|BG|}{|GE|} = \frac{|CG|}{|GF|} = 2.$$

Proof. First show $\frac{|AG|}{|GD|} = \frac{|BG|}{|GE|} = 2$, where G is the intersection point of AD & BE.



Converse of Thales' theorem:

$\frac{|CE|}{|EA|} = \frac{|CD|}{|DB|}$ ($= 1$ here) $\Rightarrow$ ED is parallel to AB.

$\longrightarrow$

$\triangle AGB \sim \triangle DGE$ (alternate angles)

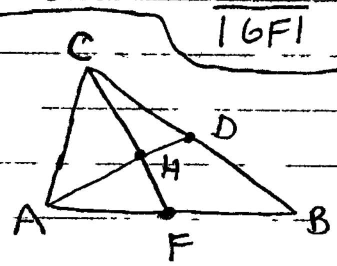
$$\frac{|AG|}{|DG|} = \frac{|BG|}{|EG|} = \frac{|AB|}{|DE|}.$$

$\triangle CED \sim \triangle CAB$ (corresponding angles)

$$\frac{|AB|}{|DE|} = \frac{|CA|}{|CE|} = \frac{2}{1} = 2$$ Combining, $\frac{|AG|}{|DG|} = \frac{|BG|}{|EG|} = 2$

where $G = AD \cap BE$.

Want: The 3rd median CF also passes through G & $\frac{|CG|}{|GF|} = 2$.



Let $H = AD \cap CF$.

$\frac{|AH|}{|DH|} = \frac{|CH|}{|HF|} = 2$ (from same logic as before)

But, we know $\frac{|AG|}{|DG|} = 2 = \frac{|AH|}{|DH|}$; so $G = H$, so medians meet at a point G.

G & H are two points in the interior of the line segment AD and $|AG| = |AH| = 2/3\,|AD| \Rightarrow G = H$

↳ $|AG| = 2 \cdot |DG|$

$|AD| = |AG| + |DG| = 3|DG|$ $\Rightarrow |AG| = 2/3\,|AD|$.

■

G = center of mass; can balance triangle on the head of a pin at this point.

Sir Michael Atiyah → "Geometry vs. Algebra"