Math 300.2 Homework 8

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March 23, 2012

Reading: Gilbert and Vanstone, Chapter 6.

(1) Determine the range (or image) of the following functions.

  (a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$.
  (b) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 5$.
  (c) $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, $f(x, y) = 12x + 57y$.
  (d) $f\colon \mathbb{Z} \to \{0, 1, 2, 3\}$, $f(x) = x^2 \bmod 4$ (that is, $f(x)$ is the remainder on dividing x^2 by 4).

(2) For each of the following pairs of functions $f\colon X \to Y$ and $g\colon Y \to Z$ describe the composite function $g \circ f\colon X \to Z$ explicitly.

  (a) $f\colon \{1, 2, 3\} \to \{A, B, C, D\}$, $f(1) = B$, $f(2) = C$, $f(3) = A$; $g\colon \{A, B, C, D\} \to \{\alpha, \beta, \gamma\}$, $g(A) = \gamma$, $g(B) = \alpha$, $g(C) = \beta$, $g(D) = \alpha$.
  (b) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$; $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^3 + 4$.
  (c) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + y, 2x + y)$; $g\colon \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (3x + 4y, 2x + 5y)$.

(3) Which of the following functions have an inverse? If the inverse exists, describe it explicitly. Otherwise explain carefully why the inverse does not exist.

  (a) $f\colon \{1, 2, 3, 4\} \to \{A, B, C, D\}$, $1 \mapsto C$, $2 \mapsto D$, $3 \mapsto A$, $4 \mapsto B$.
  (b) $f\colon \mathbb{R} \to (0, \infty)$, $f(x) = e^x$.
  (c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 4x + 3$.

(d) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 7$.

(e) $f\colon \mathbb{N}^2 \to \mathbb{N}$, $f(x, y) = 2^x \cdot 3^y$.

(f) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 2x$.

(g) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + 4y, 3x + 6y)$.

(h) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + 5y, 3x + 7y)$.

(4) Which of the following functions are injective? Justify your answer carefully.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$

(b) $f\colon [0, 2\pi) \to \mathbb{R}^2$, $f(t) = (\cos t, \sin t)$.

(c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 2x + 1$. [Hint: Use Q8(a) below]

(d) $f\colon \mathbb{N}^2 \to \mathbb{N}$, $f(x, y) = 3^x \cdot 5^y$.

(e) $f\colon X \to Y$, X and Y are finite sets, and $|X| > |Y|$.

(5) Let $X = \{1, 2, 3\}$ and $Y = \{A, B, C, D, E\}$. How many functions $f\colon X \to Y$ are there? How many of these functions are injective?

(6) Describe a bijective function $f\colon \mathbb{N} \to \mathbb{Z}$. (Recall $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}$ is the set of all integers.)

(7) Give an example of a pair of functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $g(f(x)) = x$ for all $x \in X$ but $f(g(y)) \neq y$ for some $y \in Y$.

(8) (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose that $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Show that f is injective. [Hint: Use the mean value theorem].

(b) Give an example of an injective differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 0$ for some $x \in \mathbb{R}$.

(9) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function which preserves distances. That is, for a pair of points $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{R}^2$, define the distance $d(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Then the function f satisfies $d(f(p_1), f(p_2)) = d(p_1, p_2)$ for all $p_1, p_2 \in \mathbb{R}^2$. Show that f is a bijection, so has an inverse.

[Hint: To show f is injective, prove that $d(p_1, p_2) = 0 \iff p_1 = p_2$. To show that f is surjective, fix two distinct points $p_1, p_2 \in \mathbb{R}^2$, say

$p_1 = (1, 0)$ and $p_2 = (0, 1)$, and consider $f(p_1), f(p_2) \in \mathbb{R}^2$. Given a point $q \in \mathbb{R}^2$, we want to show that there is a point $p \in \mathbb{R}^2$ such that $f(p) = q$. If $f(p) = q$ then we must have $d(p_1, p) = d(f(p_1), q)$ and $d(p_2, p) = d(f(p_2), q)$. Now draw circles with centers at p_1 and p_2 to find 1 or 2 possibilities for p, and show that one of them works.]