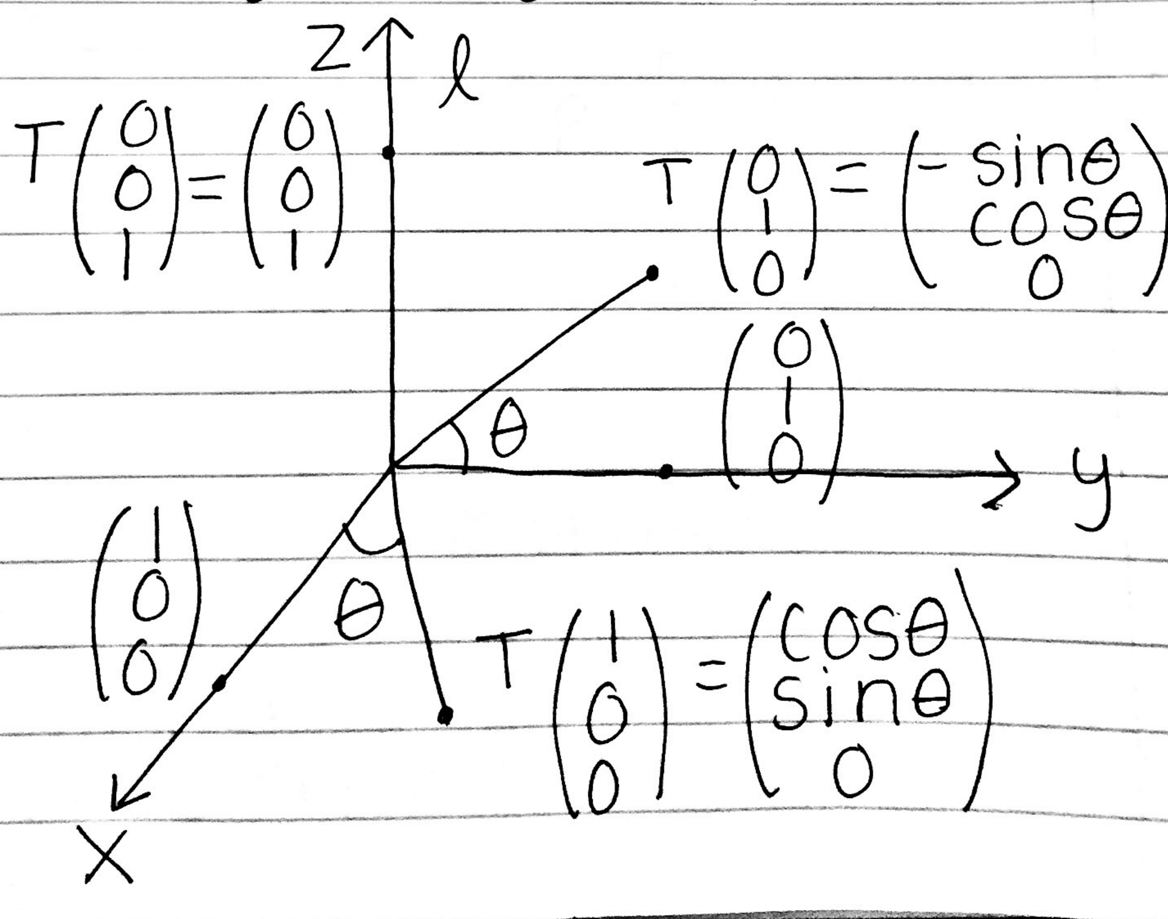
Math 461 Lecture 28 11/7
Homework 6 due now
Homework 5 returned
Homework 7 available later today
Last time:
algebraic formulas for isometries of $\mathbb{R}^3$ (fixing the origin)
reflection in a plane π through the origin

$$T(\bar{x}) = \bar{x} - 2\left(\frac{\bar{x}\cdot\bar{n}}{\bar{n}\cdot\bar{n}}\right)\bar{n}$$, $\bar{n}$ normal vector to π

rotation about an axis ℓ through origin
Ex. ℓ = z-axis

$$T(\bar{x}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \bar{x}$$



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general case:

$T(\bar{x}) = A\bar{x} \quad A = PMP^{-1} = PMP^T$

$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$

$\bar{b}_1, \bar{b}_2, \bar{b}_3$ right handed orthonormal basis (pairwise orthogonal, length 1)

$\bar{b}_3$ in direction of ℓ

Today:

rotation example

given algebraic description of isometry, describe geometrically

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "A"] \arrow[d, shift right, "P^{-1}"'] \arrow[rru, phantom] & \mathbb{R}^3 \\ \mathbb{R}^3 \arrow[r, "M"'] \arrow[u, shift right, "P"'] & \mathbb{R}^3 \arrow[u, "P"'] \end{tikzcd}$$

T in usual coordinates

T in "$\mathcal{B}$-coordinates"

$P = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$

P sends $\begin{pmatrix}1\\0\\0\end{pmatrix}$ to $\bar{b}_1$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ to $\bar{b}_2$, $\begin{pmatrix}0\\0\\1\end{pmatrix}$ to $\bar{b}_3$

$A = PMP^{-1}$

$\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$

new basis

$A = PMP^{-1} = PMP^T$

$\mathcal{B}$ orthonormal

$P = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$

$\bar{b}_1, \bar{b}_2, \bar{b}_3$ orthonormal basis:

pairwise orthogonal, length 1

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equivalently, $\bar{b}_i \cdot \bar{b}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

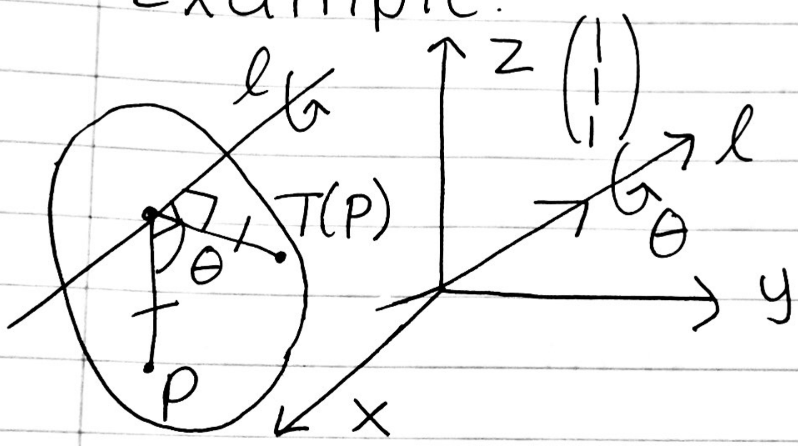
$(\bar{x} \cdot \bar{x} = \|x\|^2)$

$\rightsquigarrow P^T P = \begin{pmatrix} \bar{b}_1^T \\ \bar{b}_2^T \\ \bar{b}_3^T \end{pmatrix} (\bar{b}_1\ \bar{b}_2\ \bar{b}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

(i,j) entry is $\bar{b}_i^T \bar{b}_j = \bar{b}_i \cdot \bar{b}_j$

$\rightsquigarrow P^T = P^{-1}$

Example:



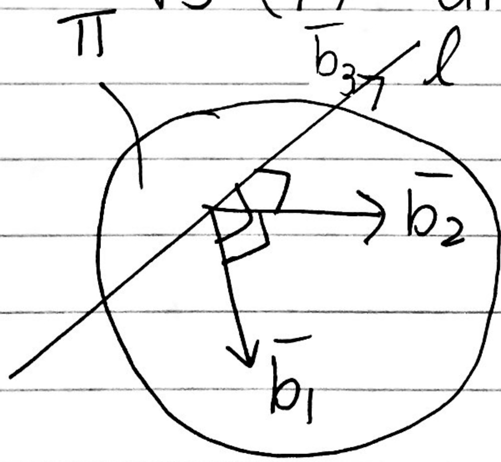
$\theta = \frac{2\pi}{3}$

formula $T(\bar{x}) = A\bar{x}$ for this rotation?

$T: \mathbb{R}^3 \to \mathbb{R}^3$

first find orthonormal basis $\bar{b}_1, \bar{b}_2, \bar{b}_3$ as described earlier

$\bar{b}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ length 1 vector in direction of ℓ



plane Π perpendicular ℓ, through origin has equation:

$\bar{x} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$

$x + y + z = 0$

first write down some basis of this plane

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$x = -y - z$ $\quad y, z$ arbitrary ← free variables

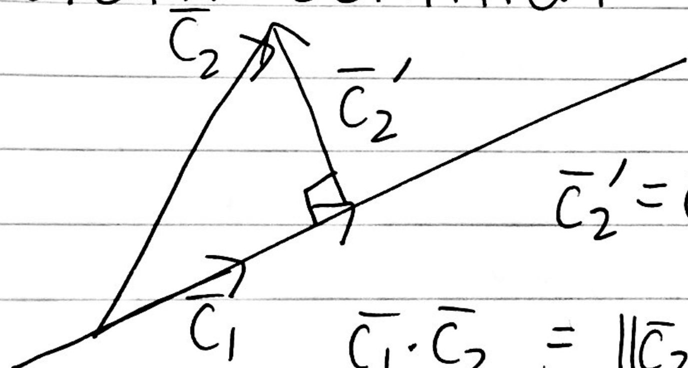
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

basis Π $\quad \bar{c}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \bar{c}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\bar{c}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\bar{c}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are not orthogonal: $\bar{c}_1 \cdot \bar{c}_2 = 1$

there's a way to replace $\bar{c}_1, \bar{c}_2$ by orthogonal basis:-

"Gram-Schmidt"



$$\bar{c}_2' = \bar{c}_2 - \left(\frac{\bar{c}_1 \cdot \bar{c}_2}{\bar{c}_1 \cdot \bar{c}_1}\right) \bar{c}_1$$

$$\frac{\bar{c}_1 \cdot \bar{c}_2}{\bar{c}_1 \cdot \bar{c}_1} = \|\bar{c}_2\| \cos\theta \cdot \frac{\bar{c}_1}{\|\bar{c}_1\|}$$

$$\bar{c}_2' = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{1}{2}\right) \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

check: $\bar{c}_1 \cdot \bar{c}_2' = 0$ ✓

$\rightsquigarrow$ basis $\bar{b}_1, \bar{b}_2, \bar{b}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

scale $\bar{c}_1, \bar{c}_2', \bar{b}_3$ so length 1

$$\bar{c}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \rightsquigarrow \bar{b}_1 = \frac{\bar{c}_1}{\|\bar{c}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

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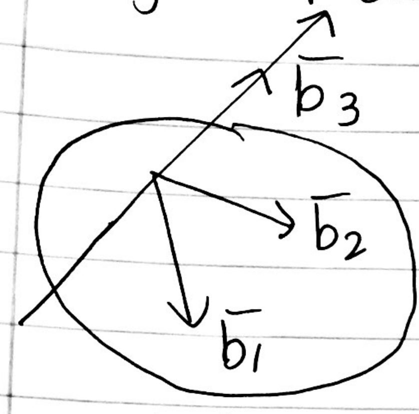
final check: want $\bar{b}_1, \bar{b}_2, \bar{b}_3$ to be right handed



equivalently, $\bar{b}_1 \times \bar{b}_2 = +\bar{b}_3$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} \times \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\2\\2\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \bar{b}_3 \quad \checkmark$$

if $\bar{b}_1, \bar{b}_2, \bar{b}_3$ left handed, get $\bar{b}_1 \times \bar{b}_2 = -\bar{b}_3$, replace $\bar{b}_2$ by $-\bar{b}_2$ to fix this

can now write down matrix:

$T(\bar{x}) = A\bar{x} \quad A = PMP^{-1} = PMP^T$

$$P = (\bar{b}_1\ \bar{b}_2\ \bar{b}_3) = \frac{1}{\sqrt{6}}\begin{pmatrix}-\sqrt{3} & -1 & \sqrt{2}\\ \sqrt{3} & -1 & \sqrt{2}\\ 0 & 2 & \sqrt{2}\end{pmatrix}$$

$\theta = \frac{2\pi}{3}$

$$M = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-1/2 & -\sqrt{3}/2 & 0\\ \sqrt{3}/2 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A = PMP^T$$
$$= P \cdot \frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3} & 0\\ \sqrt{3} & -1 & 0\\ 0 & 0 & 2\end{pmatrix} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix}-\sqrt{3} & \sqrt{3} & 0\\ -1 & -1 & 2\\ \sqrt{2} & \sqrt{2} & \sqrt{2}\end{pmatrix}$$

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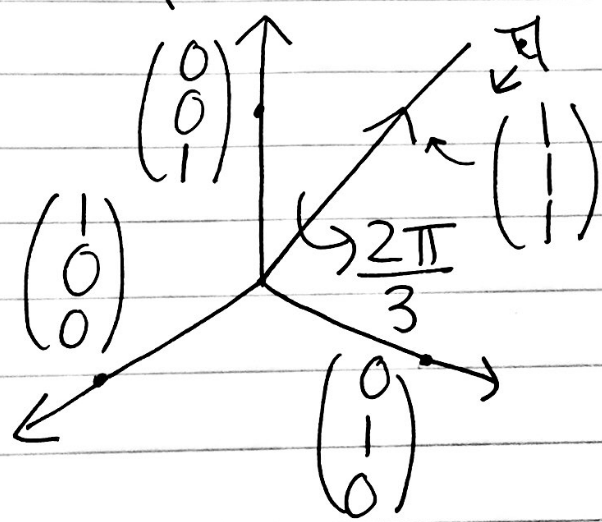
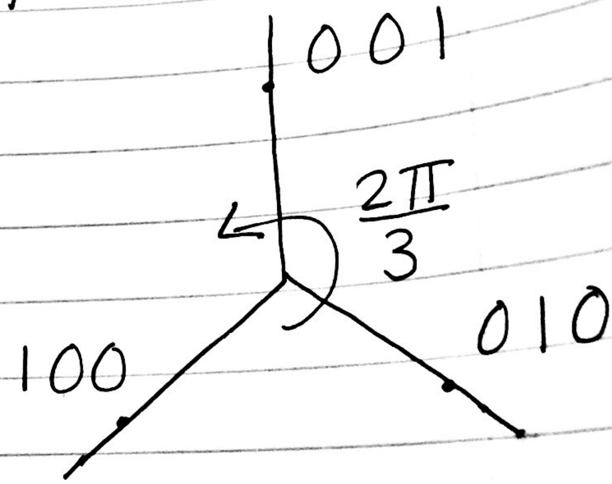
$$= P \cdot \frac{1}{2\sqrt{6}} \begin{pmatrix} 2\sqrt{3} & 0 & -2\sqrt{3} \\ -2 & 4 & -2 \\ 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} -\sqrt{3} & -1 & \sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \\ 0 & 2 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 0 & -2\sqrt{3} \\ -2 & 4 & -2 \\ 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 0 & 0 & 12 \\ 12 & 0 & 0 \\ 0 & 12 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$





given a formula $T: \mathbb{R}^3 \to \mathbb{R}^3$ $T(\bar{x}) = A\bar{x}$ for an isometry, how to give a geometric description?

$$T(\bar{x}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \bar{x} \qquad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$