Math 462 Homework 5

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(1) Let
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
a 3×3 orthogonal matrix, and let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the associated isometry of $\mathbb{R}^3$. Give a precise geometric description of T as a rotation, reflection, or rotary reflection. That is, in the case of a rotation, determine the direction of the axis of rotation (the fixed line) and the angle of rotation; in the case of a reflection, determine the equation of the plane of reflection; and in the case of a rotary reflection, determine the axis and angle of rotation and the plane of reflection.

(2) Repeat Q1 for
$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

(3) Repeat Q1 for
$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & -1 \end{pmatrix}.$$

(4) (a) Determine the matrix A_θ of a rotation about the z-axis through angle θ counterclockwise as viewed from $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (looking towards the origin).

(b) Determine the matrix B_φ of a rotation about the x-axis through angle φ counterclockwise as viewed from $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

[Hint: If $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation then the matrix A of T has columns the vectors $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Use this fact together with geometric reasoning to determine the matrices A_θ and B_φ.]

(5) Let $L \subset \mathbb{R}^3$ be the line in the direction $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the line L through angle $\pi/2$ counterclockwise as viewed from $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(a) Explain why T can be expressed as the composition $U \circ V \circ U^{-1}$ where U is the rotation about the z-axis through angle $\pi/4$ and V is the rotation about the x-axis through angle $\pi/2$.

(b) Now use the result of Q4 to compute the matrix A of T.

(6) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about some axis $L \subset \mathbb{R}^3$ passing through the origin, through some angle θ; and $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation about another axis $M \subset \mathbb{R}^3$ passing through the origin, through some angle φ.

(a) What can you say about the composition $U \circ T$?

(b) Give a precise geometric description of the composition $U \circ T$ in the following case: T is rotation about the x-axis through angle $\pi/2$ counterclockwise as viewed from $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and U is rotation about the z-axis through angle $\pi/2$ counterclockwise as viewed from $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(7) A tetrahedron is a regular polyhedron with 4 faces. For example, there is a regular tetrahedron $T \subset \mathbb{R}^3$ with vertices the points

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

It may help to visualize the tetrahedron by first drawing the cube with 8 vertices $\begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix}$. There are 24 symmetries of the tetrahedron (including the identity transformation) — because there is exactly one symmetry realizing each permutation of the 4 vertices, and $4! = 24$. Give a geometric description of each of the symmetries as a rotation, reflection, or rotary reflection.