Math 300.2 Homework 1

Paul Hacking

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Reading: Sundstrom, Sections 2.1, 2.2, and 2.3.

Justify your answers carefully.

(1) Compute the truth tables for the following compound statements.

(a) $P \Rightarrow (\text{NOT}\, Q)$.

(b) $(P \,\text{OR}\, Q)\, \text{AND}(\text{NOT}(P \,\text{AND}\, Q))$.
[This compound statement is sometimes called "exclusive or" and denoted $P \,\text{XOR}\, Q$.]

(c) $(P \,\text{AND}\, Q) \Rightarrow R$.

(2) In each of the following cases, show using truth tables that the two compound statements are equivalent.

(a) $\text{NOT}(P \,\text{OR}\, Q)$, $\quad (\text{NOT}\, P)\, \text{AND}(\text{NOT}\, Q)$.

(b) $P \Rightarrow Q$, $\quad (\text{NOT}\, P)\, \text{OR}\, Q$.

(c) $P\, \text{AND}(Q \,\text{OR}\, R)$, $\quad (P \,\text{AND}\, Q)\, \text{OR}(P \,\text{AND}\, R)$.

(3) Recall that the *converse* of the conditional statement $P \Rightarrow Q$ is the conditional statement $Q \Rightarrow P$. WARNING: The statements $P \Rightarrow Q$ and $Q \Rightarrow P$ are *not* equivalent (it can happen that one is true and the other is false).

For each of the following true conditional statements, determine whether the converse is true or false.

[Note: In each case, the statement depends on a variable. I am asking whether the converse is true for all possible values of that variable. Later we will be more precise about this using "quantifiers".]

(a) Let n be an integer. If n is a multiple of 6 then n is even.

(b) Let x be real number. If $x > 1$ then $x^2 > 1$.

(c) Let x be a real number. If $x > 2$ then $x^3 > 8$.

(d) Let T be a triangle. If T has two sides of equal lengths then T has two equal angles.

(4) Recall that the conditional statement $(P \Rightarrow Q)$ is equivalent to the conditional statement $(\mathrm{NOT}\, Q) \Rightarrow (\mathrm{NOT}\, P)$, called its *contrapositive* (we showed this in class using truth tables).

(a) Let x be a real number. Write down the contrapositive of the conditional statement $(x^3 + x < 2) \Rightarrow (x < 1)$. (Simplify the contrapositive statement as much as possible.)

(b) Show that the contrapositive statement is always true (for any value of x), and deduce that the original statement is true as well.

[In general, when we want to show that a conditional statement is true, it is often useful to replace the statement by its contrapositive.]

(5) Recall from class the following logical equivalences:

1 $\mathrm{NOT}(\mathrm{NOT}\, P) \equiv P$.

2 $P \,\mathrm{AND}\, Q \equiv Q \,\mathrm{AND}\, P$.

3 $P \,\mathrm{OR}\, Q \equiv Q \,\mathrm{OR}\, P$.

4 $(P \,\mathrm{AND}\, Q) \,\mathrm{AND}\, R \equiv P \,\mathrm{AND}(Q \,\mathrm{AND}\, R)$.

5 $(P \,\mathrm{OR}\, Q) \,\mathrm{OR}\, R \equiv P \,\mathrm{OR}(Q \,\mathrm{OR}\, R)$.

6 $P \,\mathrm{AND}(Q \,\mathrm{OR}\, R) \equiv (P \,\mathrm{AND}\, Q) \,\mathrm{OR}(P \,\mathrm{AND}\, R)$.

7 $P \,\mathrm{OR}(Q \,\mathrm{AND}\, R) \equiv (P \,\mathrm{OR}\, Q) \,\mathrm{AND}(P \,\mathrm{OR}\, R)$.

8 $\mathrm{NOT}(P \,\mathrm{AND}\, Q) \equiv (\mathrm{NOT}\, P) \,\mathrm{OR}(\mathrm{NOT}\, Q)$.

9 $\mathrm{NOT}(P \,\mathrm{OR}\, Q) \equiv (\mathrm{NOT}\, P) \,\mathrm{AND}(\mathrm{NOT}\, Q)$.

10 $P \Rightarrow Q \equiv (\mathrm{NOT}\, P) \,\mathrm{OR}\, Q$.

11 $\mathrm{NOT}(P \Rightarrow Q) \equiv P \,\mathrm{AND}(\mathrm{NOT}\, Q)$.

In each of the following cases, show using the above logical equivalences that the two compound statements are logically equivalent (do *not* use truth tables).

(a) $(\text{NOT}\, P) \Rightarrow Q, \quad P\, \text{OR}\, Q$.

(b) $(P\, \text{OR}\, Q) \Rightarrow R, \quad ((\text{NOT}\, P)\, \text{AND}(\text{NOT}\, Q))\, \text{OR}\, R$.

(c) $P \Rightarrow (Q\, \text{AND}\, R), \quad (P \Rightarrow Q)\, \text{AND}(P \Rightarrow R)$.

(6) We say a compound statement is a *tautology* if it is true for all truth values of the component statements. Show using truth tables that each of the following statements is a tautology.

(a) $P\, \text{OR}(\text{NOT}\, P)$.

(b) $(P\, \text{AND}(P \Rightarrow Q)) \Rightarrow Q$.

(c) $((P \Rightarrow Q)\, \text{AND}(Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$.

(7) The logical operator NAND is defined by $P\, \text{NAND}\, Q = \text{NOT}(P\, \text{AND}\, Q)$. It has the following useful property: any compound statement is logically equivalent to a compound statement using only the logical operator NAND. (This fact is sometimes used in digital electronic circuits.)

(a) Show that $P\, \text{NAND}\, P \equiv \text{NOT}\, P$.

(b) Find compound statements using only the NAND operator which are logically equivalent to (i) $P\, \text{AND}\, Q$ and (ii) $P\, \text{OR}\, Q$.

(8) In each of the following cases, list the elements of the set. (You may use the " ... " notation if the set is infinite, but you should list enough elements so that the pattern is clear.)

(a) $\{n \in \mathbb{Z} \mid n^2 < 3\}$.

(b) $\{x \in \mathbb{R} \mid x^3 + 5x^2 + 4x = 0\}$.

(c) $\{x \in \mathbb{R} \mid x^2 + x + 1 = 0\}$.

(d) $\{n \in \mathbb{N} \mid n \text{ is not a multiple of 2 or 3 }\}$.

(e) $\{x \in \mathbb{R} \mid \sin x = 0\}$.

(9) For each of the following sets, describe the set using the set-builder notation

$$S = \{x \in U \mid P(x)\}$$

where U is one of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $P(x)$ is an open sentence involving the variable x. (You should try to make $P(x)$ as simple as possible.)

(a) $\{1, 3, 5, 7, 9, \ldots\}$.

(b) $\{\ldots, -6, -3, 0, 3, 6, \ldots\}$.

(c) $[3, 5]$. (Notation as in MATH 131 and 132.)

(d) $\{\ldots, -5\pi/2, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots\}$.