697B Example Sheet 3

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(1) Let $X = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let ω be a meromorphic differential on X. Show by explicit computation that the sum of the residues of ω equals zero. [Hint: We showed in class that $\omega = f(z)dz$ where $f(z)$ is a rational function of the coordinate z on $\mathbb{C} \subset X$. We can write $f(z) = \sum_{i=1}^{N} p_i(z) + q(z)$ where $q(z)$ is a polynomial and $p_i(z) = \sum_{j=1}^{m_i} a_{ij}/(z - \alpha_i)^j$ is the "tail" of the Laurent expansion of f at a pole $\alpha_i \in \mathbb{C}$ of f of order m_i. Now compute the residue of ω at ∞.]

(2) Let $X = (f(x,y) = 0) \subset \mathbb{C}^2_{x,y}$ be a smooth algebraic curve. Show that $\omega := dx/\frac{\partial f}{\partial y}$ is a holomorphic differential on X with no zeroes (that is, $\nu_P(\omega) = 0$ for each $P \in X$). [Hint: Show first that $dx/\frac{\partial f}{\partial y} = -dy/\frac{\partial f}{\partial x}$ on X.]

(3) Let $X = (y^2 = p(x)) \subset \mathbb{C}^2_{x,y}$ where $p(x)$ is a polynomial of degree 3 with distinct roots, and $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ its closure. Note that $\overline{X}$ is smooth by HW1 Q4(a).

   (a) Show that $\omega = dx/y$ defines a holomorphic differential on $\overline{X}$ with no zeroes. [Hint: Use Q2 above. You will also need to consider a chart containing $\overline{X} \cap L_\infty$.]

   (b) Deduce that $\Omega(\overline{X}) = \mathbb{C} \cdot \omega$.

(4) Let $G \colon \tilde{X} \to \mathbb{C} \cup \{\infty\}$ be the hyperelliptic Riemann surface described in HW2 Q8.

   (a) Show that $G^{-1}(\infty)$ is a single point P_∞ if n is odd and two points P_∞, Q_∞ if n is even.

   (b) Let ω be the meromorphic differential dx/y on $\tilde{X}$. Find the zeroes and poles of ω and compute their orders.

(c) Use the Poincaré-Hopf theorem to compute the genus g of X.

(d) Show that $x^i\omega$ is a holomorphic differential on $\tilde{X}$ for $i = 0, 1, \ldots, g-1$. (We will see later that this is a basis of the complex vector space $\Omega(\tilde{X})$ of holomorphic differentials on $\tilde{X}$.)

(5) We use the notation of Q2. Let $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$ be the closure of X in $\mathbb{P}^2_{\mathbb{C}}$, and assume that $\overline{X}$ is smooth. Let d be the degree of f. Show that $g \cdot \omega$ defines a holomorphic differential on $\overline{X}$ for $g = g(x, y)$ a polynomial of degree $\leq d - 3$ in x and y. [Hint: Write $x = X/Z$, $y = Y/Z$ as usual. Consider another chart $\mathbb{C}^2_{u,v} \subset \mathbb{P}^2_{\mathbb{C}}$ given by $u = Y/X$, $v = Z/X$. Show that $dx/\frac{\partial f}{\partial y} = -v^{d-3}(dv/\frac{\partial h}{\partial u})$ where $\overline{X} \cap \mathbb{C}^2_{u,v} = (h(u, v) = 0)$.]

(6) (a) Let $X = \mathbb{C}_z \cup \{\infty\}$ and $\gamma = (|z| = R) \subset X$, for some $R \gg 0$, with the anticlockwise orientation. Let ω be the meromorphic differential on X given by $\omega = f(z)dz = ((z^5 + 1)/(z^6 + 1))dz$. Compute $\int_\gamma \omega$.

(b) Let $X = \mathbb{C}_z/\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ be a complex torus, $\omega = dz$ and $\gamma \subset X$ a closed curve. What are the possible values of $\int_\gamma \omega$?