235.4 Midterm 1 Review Questions

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(1) Describe the set of solutions of the linear equation $x + 2y + 3z = 4$ geometrically.

(2) Find all (simultaneous) solutions of the linear equations

$$x + y + z = 3, \quad x + 2y + 3z = 7.$$

Intepret your answer geometrically.

(3) Consider a collection of linear equations in several variables. What are the possibilities for the number of (simultaneous) solutions of the equations? What if the number of equations is smaller than the number of variables?

(4) Suppose we have a system of linear equations, and we find two solutions. How many solutions are there altogether?

(5) What is the rank of a matrix A (give a precise definition)? If A is a 3×5 matrix , and the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^5$ for every $\mathbf{b} \in \mathbb{R}^3$, what is the rank of A?

(6) Find all solutions of the following systems of linear equations.

(a)

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + y + 2z &= 1 \\ x - y + 3z &= 2 \end{aligned}$$

(b)

$$\begin{aligned} x+y+z+t &= 2 \\ x+2y+3z+2t &= 4 \\ 2x+4y+4z-t &= 5 \end{aligned}$$

(7) Consider the system of linear equations

$$\begin{aligned} x-3y+2z &= 3 \\ 4x-9y+17z &= 6 \\ x-y+8z &= c \end{aligned}$$

where c is a real number. For what values of c does the system of equations have a solution? Find all solutions in each case.

(8) Find all solutions of the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$$

where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Intepret your answer geometrically.

(9) Find all solutions of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 8 & 3 & 4 \\ 2 & 6 & 5 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}.$$

Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^3$? Explain your answer.

(10) Find all solutions of the equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^3$? Explain your answer.

(11) Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be a function from $\mathbb{R}^5$ to $\mathbb{R}^3$. What does it mean to say that T is a linear transformation? What do we mean by the matrix of T? What size is this matrix (how many rows and columns)?

(12) Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation. What is $T\begin{pmatrix}0\\0\end{pmatrix}$?

(13) Which of the following are linear transformations? If the transformation is linear compute its matrix.

(a) $T\colon \mathbb{R} \to \mathbb{R}$, $T(x) = 2x + 3$.

(b) $U\colon \mathbb{R}^3 \to \mathbb{R}^2$, $U\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x+4y-z\\x+3y+z\end{pmatrix}$.

(c) $V\colon \mathbb{R}^2 \to \mathbb{R}^2$, $V\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1^2\\x_1+x_2^2\end{pmatrix}$.

(d) $W\colon \mathbb{R}^2 \to \mathbb{R}^2$, W is the reflection in the line $y = x + 1$.

(e) $P\colon \mathbb{R}^2 \to \mathbb{R}^2$, P is the rotation about the point $(1, 2)$ through angle $\pi/3$ counter-clockwise.

(f) $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$, Q is the reflection in the line $y = -x$.

(g) $R\colon \mathbb{R}^2 \to \mathbb{R}^2$, R is the rotation about the origin through angle $\pi/2$ clockwise.

(h) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$, S is the orthogonal projection onto the y-axis.

(14) Let L be the line through the origin in $\mathbb{R}^2$ in the direction of a vector $\mathbf{w}$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto L. Recall the formula $T(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}$. Now suppose L is the line $y = 3x$. Use the formula to find the matrix of the linear transformation T.

(15) (a) The linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \frac{1}{13}\begin{pmatrix}4&6\\6&9\end{pmatrix}$ is the orthogonal projection onto a line. Find the line.

(b) The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $B = \frac{1}{3}\begin{pmatrix}1&2&-2\\2&1&2\\-2&2&1\end{pmatrix}$ is the reflection in a plane. Find the plane.

(c) The linear transformation $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$ is a rotation. Find the axis of rotation.

[Hints: (a) For any $\mathbf{v} \in \mathbb{R}^2$ the point $S(\mathbf{v})$ is on the line (why?). (b) If $\mathbf{n}$ is a normal vector for the plane then $T(\mathbf{n}) = -\mathbf{n}$ (why?). Now solve the equation $T(\mathbf{x}) = -\mathbf{x}$ to find a normal vector. (c) If $\mathbf{v}$ is a point on the axis, what is $U(\mathbf{v})$? Now solve an equation to find the direction of the axis (similarly to (b)).]

(16) The unit cube in $\mathbb{R}^3$ has vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$. Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map such that $T(1,0,0) = (2,1)$, $T(0,1,0) = (1,2)$, and $T(0,0,1) = (1,1)$. Write down the matrix of T. Draw the image of the unit cube in $\mathbb{R}^2$ under the map T (draw the image of each edge of the cube).

(17) Let $S\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

and $T\colon \mathbb{R}^2 \to \mathbb{R}^4$ the linear transformation with matrix

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Compute the matrix of the composite function $T \circ S\colon \mathbb{R}^3 \to \mathbb{R}^4$. [Recall that the composite function is defined by $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x}))$.]

(18) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the x-axis and $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $y = x$. Compute the matrix of (a) T, (b) U, and the composite functions (c) $U \circ T$, (d) $T \circ U$, and (e) $T \circ U \circ T$. Interpret your answers to (c), (d), and (e) geometrically.

(19) Give an example of two 2×2 matrices A and B such that $AB \neq BA$.

(20) Let $f\colon A \to B$ be a function from a set A to a set B. What does it mean to say that a function $g\colon B \to A$ from B to A is the inverse of

f (written $g = f^{-1}$)? Under what conditions does a function f have an inverse? For each of the following functions, determine whether the function has an inverse, and if so describe the inverse (either give a precise geometric description or an explicit formula).

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.

(b) $g\colon \mathbb{R} \to [-1, 1]$, $g(x) = \sin(x)$.

(c) $h\colon [0, 1] \to [0, 2]$, $h(x) = 2x^2$.

(d) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, T is the rotation with center the origin through angle θ counterclockwise.

(e) $U\colon \mathbb{R}^2 \to \mathbb{R}^2$, U is the orthogonal projection onto the line $y = 3x$.

(f) $V\colon \mathbb{R}^2 \to \mathbb{R}^2$, V is the reflection in the line $y = 2x$.

(g) $W\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with matrix $\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$.

(21) Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -1 & -1 & 2 \end{pmatrix}.$$

(a) Compute A^{-1}.

(b) Using your result from (a), solve the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 3x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

(22) Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 4 & 1 \end{pmatrix}.$$

(a) Compute the inverse A^{-1} of A.

(b) Using your answer to part (a), find the solution of the system of linear equations

$$\begin{aligned} x + 2y + 2z &= 1 \\ 2x + 5y + 7z &= 3 \\ 3x + 4y + z &= -1 \end{aligned}$$

(23) Suppose A, B, C and D are $n \times n$ matrices such that $BAC = D$ and the matrices B and C are invertible. Solve for A in terms of the other matrices.