Math 300.3 Homework 2

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Reading: Sundstrom, Sections 2.3 and 2.4.

Justify your answers carefully.

(1) In each of the following cases, list the elements of the set. (You may use the " ... " notation if the set is infinite, but you should list enough elements so that the pattern is clear.)

(a) $\{n \in \mathbb{Z} \mid n^2 < 3\}$.

(b) $\{x \in \mathbb{R} \mid x^3 + 5x^2 + 4x = 0\}$.

(c) $\{x \in \mathbb{R} \mid x^2 + x + 1 = 0\}$.

(d) $\{n \in \mathbb{N} \mid n$ is not a multiple of 2 or 3 $\}$.

(e) $\{x \in \mathbb{R} \mid \sin x = 0\}$.

(2) For each of the following sets, describe the set using the set-builder notation

$$S = \{x \in U \mid P(x)\}$$

where U is one of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $P(x)$ is an open sentence involving the variable x. (You should try to make $P(x)$ as simple as possible.)

(a) $\{1, 3, 5, 7, 9, \ldots\}$.

(b) $\{\ldots, -6, -3, 0, 3, 6, \ldots\}$.

(c) $[2, 5]$. (Notation as in MATH 131 and 132.)

(d) $\{\ldots, -5\pi/2, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots\}$.

(3) For each of the following true statements, write the statement in symbolic form using quantifiers.

(a) For every real number x, if $x > 3$ then $e^x > 20$.

(b) There is a real number x such that $x^3 = 13$.

(c) There is a positive real number δ such that for all real numbers x if $|x - 1| < \delta$ then $|x^3 - 1| < 0.1$.

[You may use the notation $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ for the set of positive real numbers.]

(d) For every positive integer n, $n^3 + 1$ is a multiple of $n + 1$.

(4) For each of the following true statements, translate the statement into an english sentence.

(a) $(\exists x \in \mathbb{R})(x^3 + 5x + 3 = 0)$.

(b) $(\forall n \in \mathbb{Z})((n \text{ is odd}) \operatorname{OR} (n \text{ is even}))$.

(c) $(\forall m, n \in \mathbb{Z})(mn = nm)$.

(d) $(\forall x \in \mathbb{R})((x \geq 0) \Rightarrow ((\exists y \in \mathbb{R})(y^2 = x)))$.

(e) $(\forall x \in \mathbb{R})((x = 0) \operatorname{OR} ((\exists y \in \mathbb{R})(xy = 1)))$.

(5) For each of the following false statements, express the negation of the statement (a true statement) in symbolic form using quantifiers, and translate into an english sentence.

(a) $(\exists x \in \mathbb{Z})(x^3 + x + 1 = 0)$.

(b) $(\forall x \in \mathbb{R})((1.01)^x < x^2)$.

(c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 + y^2 = 1)$.

(d) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(y \neq x^3 + x + 1)$.

(6) Determine whether the following statements are true or false (justify your answers carefully).

(a) $(\forall x \in \mathbb{R})(x^2 + 2x + 3 > 0)$.

(b) $(\exists x \in \mathbb{Z})(x^3 + x = 10)$.

(c) $(\forall x \in \mathbb{Z})(\exists y, z \in \mathbb{Z})(x = 4y + 7z)$.

(d) $(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{Z})(n = x^2 + y^2)$.

(7) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and a a real number. Recall that we say that f *is continuous at* $x = a$ (or "$f(x) \to f(a)$ as $x \to a$") if the following statement is true:

For every positive real number ϵ there is a positive real number δ such that for every real number x if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

(a) Translate the above statement into symbolic form using quantifiers. (We did this in class on Tuesday 1/31.)

(b) Express the negation of the statement in part (a) in symbolic form using quantifiers, and translate into an english sentence.
[Hint: Recall from HW1Q5c that NOT($P \Rightarrow Q$) $\equiv$ (P AND NOT(Q)).]

(c) Now consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Using your answer to part (b) show carefully that f is *not* continuous at $x = 0$.

(8) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers and l a real number. Recall that we say *the limit of the sequence a_n as n approaches ∞ equals l* and write $\lim_{n\to\infty} a_n = l$ if the following statement is true:

For every positive real number ϵ there is a positive integer N such that for all positive integers n if $n \geq N$ then $|a_n - l| < \epsilon$.

(a) Translate the above statement into symbolic form using quantifiers.

(b) Express the negation of the statement in part (a) in symbolic form using quantifiers, and translate into an english sentence.

(c) Consider the sequence a_n defined by

$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is not a multiple of } 1000 \\ 0.001 & \text{if } n \text{ is a multiple of } 1000 \end{cases}$$

Is the statement $\lim_{n\to\infty} a_n = 0$ true or false? Justify your answer carefully.

(9) Let A and B be sets and $f\colon A \to B$ a function with domain A and codomain B (that is, f is a rule which associates to each element a of A an element $f(a)$ of B.)

(a) We say that f is *onto* if the following statement is true:

For every element b of B there is an element a of A such that $f(a) = b$.

(i) Translate the above statement into symbolic form using quantifiers.

(ii) Express the negation of the statement from part (i) in symbolic form, and translate into an english sentence.

(b) We say that f is *one-to-one* if the following statement is true:

For all elements x and y of A, if $x \neq y$ then $f(x) \neq f(y)$.

(i) Translate the above statement into symbolic form using quantifiers.

(ii) Use the contrapositive $(P \Rightarrow Q) \equiv (\mathrm{NOT}(Q) \Rightarrow \mathrm{NOT}(P))$ to give an equivalent statement to the statement in part (i), and translate into an english sentence.

(iii) Express the negation of the statement from part (i) in symbolic form, and translate into an english sentence.