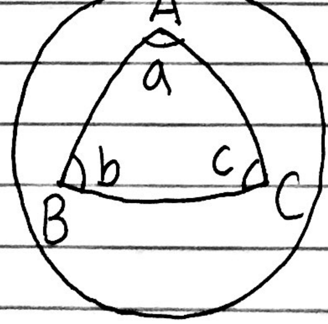
Math 461 Lecture 21 Notes Monday 10/22

Midterm Exam Wednesday 7-9 PM
LGRT 121
You are allowed one sheet of notes
(letter size, both sides).
No calculators, additional notes, or
textbook.
Please try review problems before the
exam. Check email or website.
Will post solutions tomorrow.
Syllabus for exam: Everything we
have covered up to classification of
isometries of $\mathbb{R}^2$. (not including
spherical geometry)

Last time:
spherical geometry
spherical lines
spherical distance
angles between spherical lines

Today:
Theorem: $a+b+c=\pi+\text{Area}(\triangle ABC)$
spherical cosine rule
$\Rightarrow$ spherical triangle inequality
$\Rightarrow$ spherical lines give shortest paths



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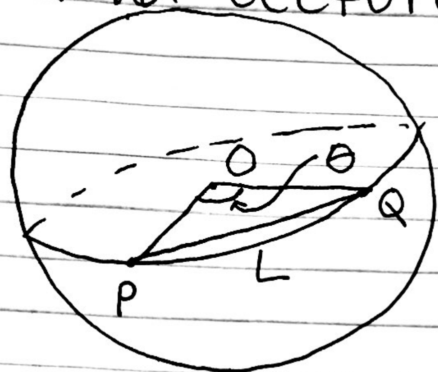

S^2 $x^2+y^2+z^2=1$
sphere center origin
with radius 1

Spherical line: intersection of S^2 with a plane π through origin
$L = \pi \cap S^2$ called the "great circle"
Theorem: (proof later) shortest path on S^2 between two points P and Q is shorter arc of spherical line through P & Q.

$d(P,Q) = (2\pi \cdot 1)\frac{\theta}{2\pi} = \theta = \cos^{-1}\left(\frac{\vec{OP}\cdot\vec{OQ}}{|\vec{OP}|\cdot|\vec{OQ}|}\right) =$
$\cos^{-1}(\vec{OP}\cdot\vec{OQ})$

Angles between two spherical lines := angle between tangent lines = dihedral angle between corresponding planes (angles you see when slice with a plane perpendicular to intersection of the planes) = angle between normal vectors to the planes

picture →

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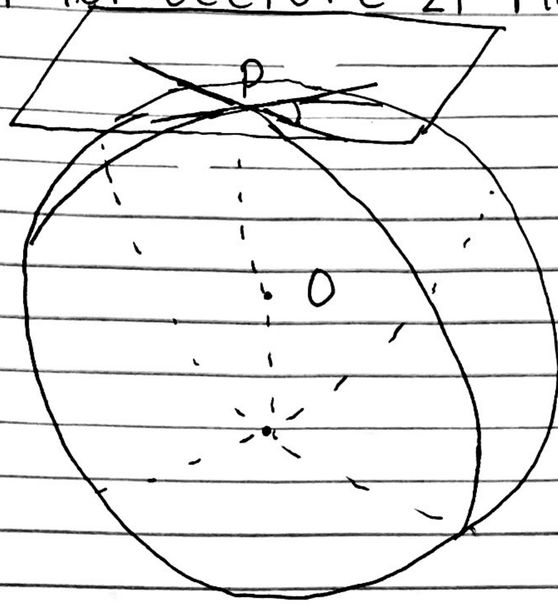
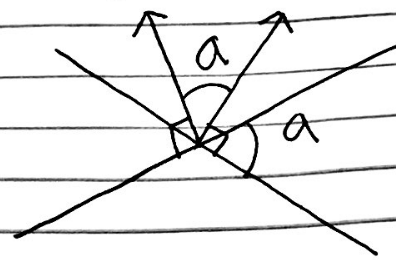
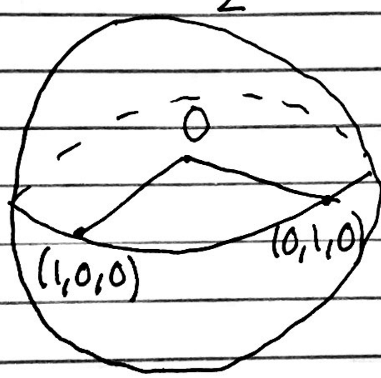




$P = (1,0,0) \quad Q = (0,1,0)$

$d(P,Q) = \cos^{-1}(\vec{OP} \cdot \vec{OQ}) = \cos^{-1}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix}\right] =$

$\cos^{-1}(0) = \frac{\pi}{2}$



$R = \frac{(7,3,5)}{\sqrt{7^2+3^2+5^2}} \in S^2$

Recall if $P, Q \in S^2$, then provided O, P, Q are not collinear (equivalently, P & Q are not antipodal)
then there's a unique spherical line L through P & Q

How to compute equation of L?

$L = \Pi \cap S^2$ Π plane through origin with equation $ax + by + cz = 0$

$ax + by + cz = d$ is the equation of a plane

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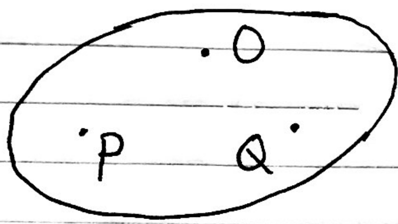
in $\mathbb{R}^3$ where $a, b, c, d \in \mathbb{R}$

π passes through $O = (0,0,0) \iff d = 0$

$P = (1,0,0)$ $\quad Q = \frac{1}{\sqrt{3}}(1,1,1)$

Equation of spherical line L through P & Q?

i.e Equation plane π through O, P, Q then $L = \pi \cap S^2$

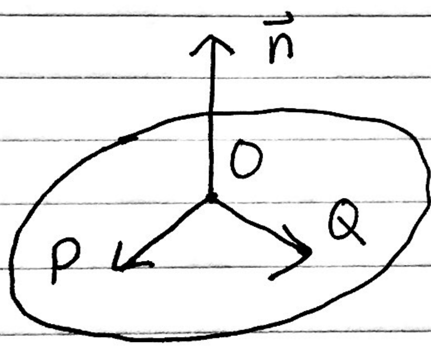


$ax + by + cz = d$

$0 = d \quad a = d$

$\frac{1}{\sqrt{3}}(a+b+c) = d$

expect to get infinitely many solutions (3 equations in 4 variables and $4 > 3$)



compute $\vec{OP} \times \vec{OQ}$

this will be a normal vector $n = (a, b, c)$ to the plane π

the equation of π is $\vec{x} \cdot \vec{n} = 0$

i.e. $ax + by + cz = 0$

Review of cross product:

a, b in $\mathbb{R}^3$ $\quad a \times b = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

$= \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ $\quad \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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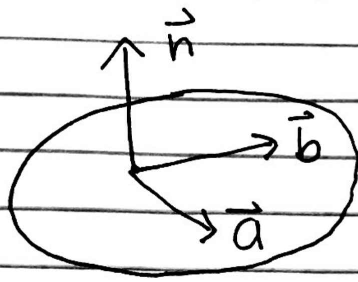
geometric property:

$\vec{a} \times \vec{b} = |a| \cdot |b| \sin\theta \cdot \vec{n}$



where $\vec{n}$ is a vector of length 1 perpendicular to $\vec{a}$ and $\vec{b}$ and $\vec{a}, \vec{b}, \vec{n}$ is "right handed"

back to example:

$P = (1,0,0) \quad Q = \frac{1}{\sqrt{3}}(1,1,1)$

$$\vec{OP} \times \vec{OQ} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\-1\\1\end{pmatrix} \rightsquigarrow n = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$\Pi = \{\vec{x} \cdot \vec{n} = 0\}$
$= \{-y + z = 0\}$

$L, M \subset S^2$

two spherical lines with equations

$x - y = 0 \qquad y - z = 0$

angle between spherical lines at

$P = \frac{1}{\sqrt{3}}(1,1,1)$ point of intersection

(in general, given equations of two spherical lines, solve to find solutions $\lambda(a,b,c)$ where $\lambda \in \mathbb{R}$ is arbitrary, then two points of intersection are

$$\frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$$

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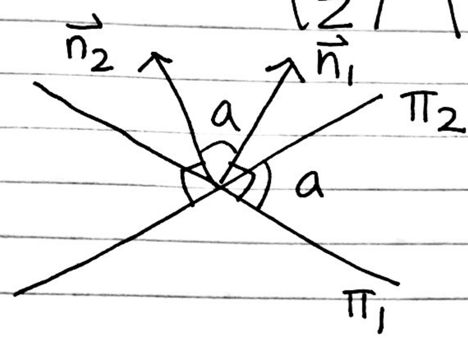
$x - y = 0$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \quad \vec{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

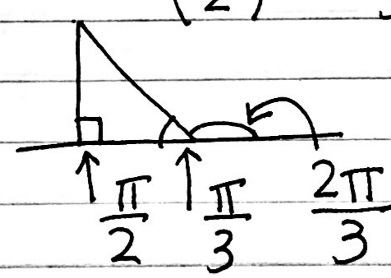
$y - z = 0$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0 \quad \vec{n}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

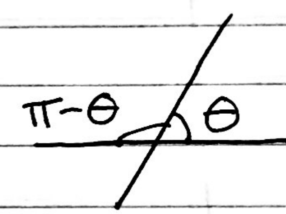


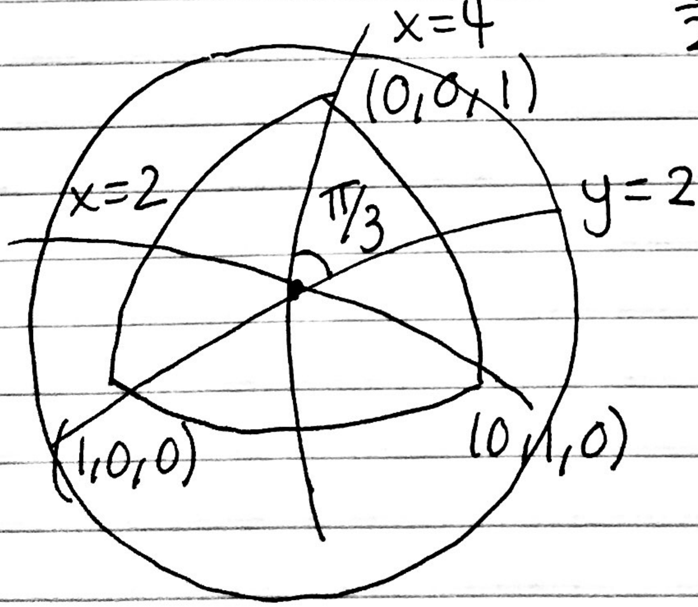
angle between L_1 and L_2
= dihedral angles between planes π_1 & π_2
= angle between normal vectors $\vec{n}_1$ & $\vec{n}_2$

$$= \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|n_1| \cdot |n_2|}\right)$$

$$= \cos^{-1}\left[\frac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{2} \cdot \sqrt{2}}\right] = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$







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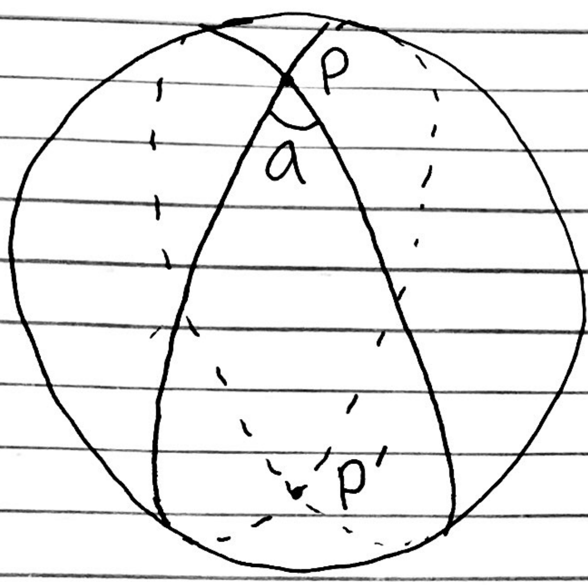
Theorem: ΔABC spherical triangle with angles a, b, c then $a + b + c = \pi + \text{Area}(\Delta ABC)$

called a "lune"

what is the area of a "lune" ?

proof:

$$\text{Area}(S^2) = 4\pi \cdot 1^2 = 4\pi$$



what is the surface area of a sphere?

$A = 4\pi R^2 \quad \left(V = \frac{4}{3}\pi R^3 \right)$

$$\text{area(lune)} = 4\pi \cdot \frac{a}{2\pi} = 2a$$