12/9/19

HW8 solutions available

Final review available

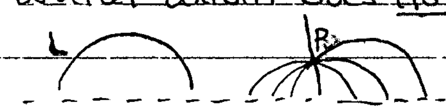
Final Exam Thurs 12/19/19 8AM – 10AM, Goessman Lab Add. Rm 51

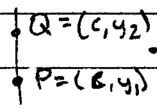
Syllabus for final: Coordinates, isometries of $\mathbb{R}^2$,
spherical geometry

* class log & lecture notes

people.math.umass.edu/~hacking/461F19/class log

<u>Last Time</u>: • Parallel axiom does <u>not</u> hold in hyperbolic plane





• Vertical lines give shortest path in $\mathcal{H}$

$d_{\mathcal{H}}(P,Q) = \ln(y_2/y_1)$

• Hyperbolic isometries: must preserve $\frac{\sqrt{x'^2+y'^2}}{y}$

$\left(\text{hyperbolic length}(\gamma) = \int_a^b \frac{\sqrt{x'^2+y'^2}}{y}\,dt \quad \gamma:[a,b] \to \mathcal{H},\ \gamma(t) = (x(t),y(t)),\ \text{parametrized curve}\right)$

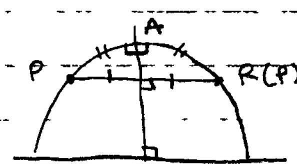
<u>Exs</u>: 1. Horizontal translation

2. Reflection in vertical line

<u>Today</u>: • More hyperbolic isometries

• Proof that semicircles give shortest paths (with center on x-axis)

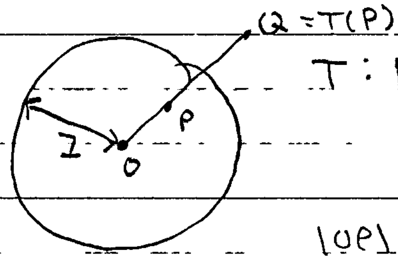
From before: Reflection



$d_{\mathcal{H}}(A,P) = d_{\mathcal{H}}(A,R(P))$

*Euclidean reflection in vertical line coincides with hyperbolic reflection (determined using hyperbolic distance)

HW8 Q6 $\longrightarrow$ Inversion in the unit circle $\mathcal{C}$ (center $(0,0)$, radius 1)



$T : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ bijection

$$T(x,y) = \frac{1}{x^2+y^2}(x,y)$$

$$|OP| \cdot |OQ| = 1$$

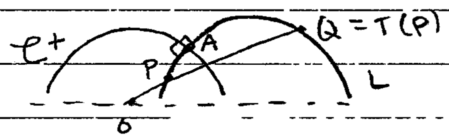
<u>Basic facts</u>: 1. T fixes $\mathcal{C}$ pointwise

2. T exchanges inside & outside of $\mathcal{C}$

3. $T^2 = T \circ T$ = identity

$$\begin{array}{ccc} \mathbb{R}^2 \setminus \{0\} & \xrightarrow{T} & \mathbb{R}^2 \setminus \{0\} \\ \cup & & \cup \\ \mathcal{H} & \xrightarrow{T} & \mathcal{H} \end{array}$$

<u>Guess</u>: $T : \mathcal{H} \to \mathcal{H}$, is the $\boxed{\text{hyperbolic reflection}}$ in the hyperbolic line $\mathcal{C}^+ = \mathcal{C} \cap \mathcal{H}$



i.e. If L is the hyperbolic line passing through P & $T(P) = Q$ then L is perp. to $\mathcal{C}^+$ and, writing $A = \mathcal{C}^+ \cap L$, $d_{\mathcal{H}}(A,P) = d_{\mathcal{H}}(A,Q)$.

This is <u>correct</u>!

To prove this need to show using formula $T(x,y) = \frac{1}{x^2+y^2}(x,y)$ that preserves hyperbolic distance.

Need to show $\frac{\sqrt{x'^2+y'^2}}{y}$ is preserved.

$T(x,y) = \frac{1}{x^2+y^2}(x,y)$ Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

$$\left(\frac{x}{x^2+y^2}\right)' \underset{\text{QR + Chain Rule}}{=} \frac{x'(x^2+y^2) - x(2x \cdot x' + 2y \cdot y')}{(x^2+y^2)^2}$$

$$= \frac{(y^2-x^2) \cdot x' - 2xy \cdot y'}{(x^2+y^2)^2}$$

Similarly $\left(\frac{y}{x^2+y^2}\right)' = \frac{(x^2-y^2)\cdot y' - 2xy\cdot x'}{(x^2+y^2)^2}$

$$\left(\frac{x}{x^2+y^2}\right)'^2 + \left(\frac{y}{x^2+y^2}\right)'^2 = \frac{\left((y^2-x^2)x' - 2xyy'\right)^2 + \left((x^2+y^2)y' - 2xyx'\right)^2}{(x^2+y^2)^4}$$

$(A-B)^2 = A^2+B^2-2AB$
Cross terms cancel!

$$\Rightarrow = \frac{(y^2-x^2)^2\cdot(x'^2+y'^2) + (2xy)^2\cdot(y'^2+x'^2)}{(x^2+y^2)^4}$$

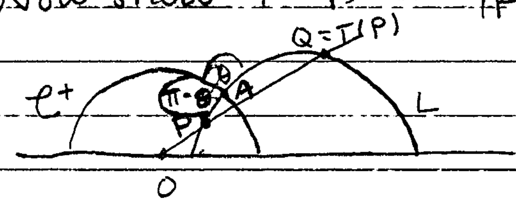
$$= \frac{(x'^2+y'^2)}{(x^2+y^2)^4}\left((y^2-x^2)^2 + 4x^2y^2\right)$$

$(A-B)^2+4AB = (A+B)^2$ $\Rightarrow = \frac{(x'^2+y'^2)}{(x^2+y^2)^4}\cdot(x^2+y^2)^2 = \frac{(x'^2+y'^2)}{(x^2+y^2)^2}$!

Finally, $\frac{\sqrt{\left(\frac{x}{x^2+y^2}\right)'^2 + \left(\frac{y}{x^2+y^2}\right)'^2}}{\frac{y}{x^2+y^2}} = \frac{\sqrt{\frac{x'^2+y'^2}{(x^2+y^2)^2}}}{\frac{y}{x^2+y^2}} = \frac{\sqrt{x'^2+y'^2}}{y}$ ✓

Conclusion: inversion T preserves hyperbolic distance, i.e. it's a hyperbolic isometry.

Now show T is a hyperbolic reflection.



need to show: L is perp. to $\mathcal{C}^+$ & $d_{\mathcal{H}}(A,P) = d_{\mathcal{H}}(A,Q)$.

Under inversion $P \longmapsto Q$ $\Rightarrow T$ sends L to L
$Q \longmapsto P$ (L is shortest path from P to Q)

Want $\theta = \pi/2$. Under T, the angles θ & $\pi-\theta$ are switched. (Note T fixes $\mathcal{C}^+$ pointwise & interchanges inside & outside of $\mathcal{C}^+$). But T is an isometry (& angles are correct in $\mathcal{H}$), so it preserves angles. So $\theta = \pi - \theta$, $\theta = \pi/2$.

Finally, $d_{\mathcal{H}}(A,P) \underset{\substack{\uparrow \\ T \text{ is isometry}}}{=} d_{\mathcal{H}}(T(A),T(P)) = d_{\mathcal{H}}(A,Q)$ ✓.

Review HW8Q6

$T = F \circ R \circ F^{-1}$

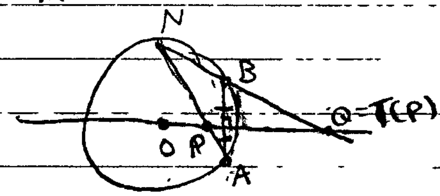
$F : S^2 \setminus \{N\} \to \mathbb{R}^2$ stereographic proj.

$S^2 \xrightarrow{R} S^2$ reflection in xy-plane

$\cap \qquad \cap$

$\mathbb{R}^3 \xrightarrow{R} \mathbb{R}^3$

$$\begin{tikzcd}
S^2 \setminus \{N,S\} \arrow[r, "R"] \arrow[d, "F"'] & S^2 \setminus \{N,S\} \arrow[d, "F"] \\
\mathbb{R}^2 \setminus \{0\} \arrow[r, "T"'] \arrow[r, bend left] & \mathbb{R}^2 \setminus \{0\}
\end{tikzcd}$$



$$\begin{array}{ccc} A & \mapsto & B \\ \downarrow & & \downarrow \\ P & \mapsto & Q \end{array}$$

Consequence:

1. Inversion T sends circles & line in $\mathbb{R}^2$ to circles & lines
2. T preserves angles

Proved earlier spherical circles on S^2 correspond to circles & lines in $\mathbb{R}^2$ under stereographic projection, & R = reflection in xy-plane sends spherical circles to spherical circles.

$\Rightarrow$ 1.

Q: Given a circle or line in $\mathbb{R}^2$, when is the image under inversion a line (not a circle)?

A: When it goes through the origin.

$$\begin{tikzcd}
S \in S^2 \setminus \{N\} \arrow[r, "R"] \arrow[d] \arrow[rr, bend left] & S^2 \ni N \arrow[d] \\
0 \quad \mathbb{R}^2 \ (\downarrow F^{-1}) & \text{"}\infty\text{"}
\end{tikzcd}$$