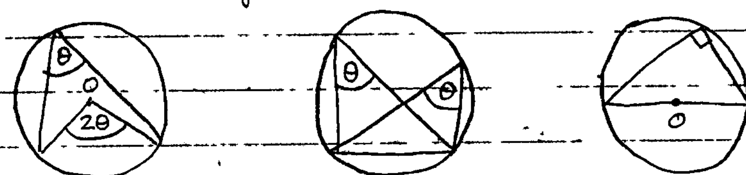
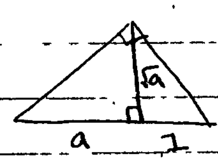
9/30/19

Office Hours 3-5pm tomorrow (no office hours today)
HW 3 due Wednesday at start of class.

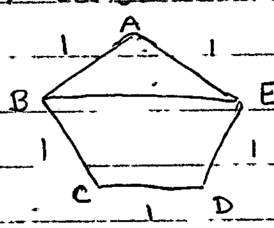
Last time: - Angles in a circle



- Construction of square root



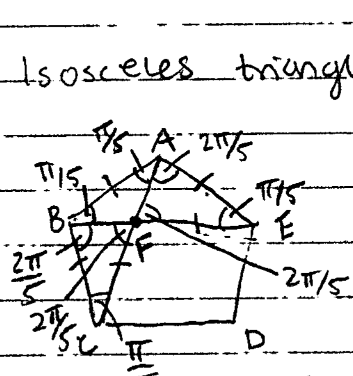
Today: - Construction of regular pentagon.



First, understand ratio $|BE|/|AB|$ of the length of the diagonal to the length of a side.
WLOG, say side length = 1
diagonal length = x, $x = ??$

Q1: What are the interior angles of a regular pentagon?
Sum of angles for n-gon $= (n-2)\pi = 3\pi \rightsquigarrow$ each angle $= 3\pi/5$
(by definition, all angles in a regular n-gon are equal)

Isosceles triangle theorem $\Rightarrow \angle ABE = \angle AEB = \underset{\substack{\uparrow \\ \text{angle sum of} \\ \triangle ABE = \pi}}{\tfrac{1}{2}(\pi - 3\pi/5)} = \pi/5$



See, using angle sum of a triangle, (and the above calculation applied to $\triangle ABE$ & $\triangle BCA$)

use these two facts

- $\triangle AEF$ is isosceles $\Rightarrow |FE| = |AE| = 1$
($\triangle BEC \cong \triangle AFE$ by ASA)
- $\triangle AFB \sim \triangle BAE \Rightarrow |BE|/|AB| = |AB|/|BF|$

$|BF| = |BE| - |FE|$, $|BE| = x$
$= x - 1$ and $|AB| = 1$

Want to find BE, so....

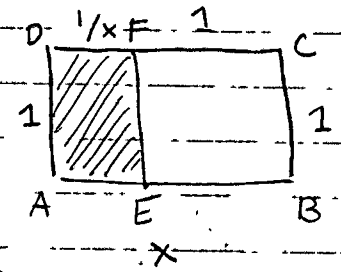
$$\frac{|BE|}{|AB|} = \frac{|AB|}{|BF|} \Rightarrow \frac{x}{1} = \frac{1}{x-1}$$

$$x^2 - x = 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$x > 0 \Rightarrow$ $\boxed{x = \frac{1+\sqrt{5}}{2} = 1.618...}$ = "golden ratio" by Greeks

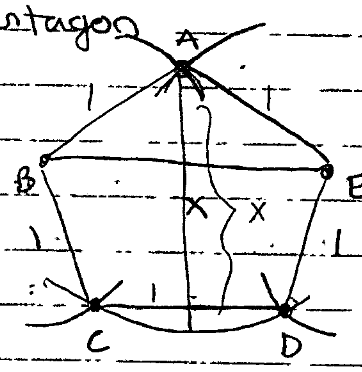


Aside:
ABCD ~ EFDA

In particular x can be constructed using ruler & compass.
(only need $+, -, \times, \div, \sqrt{\ }$)

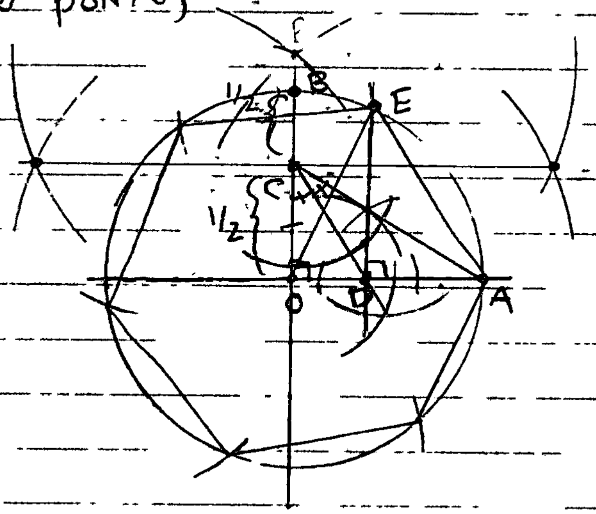
Give a ruler & compass of regular pentagon

1. Construct length x
2. Construct isosceles triangle $\triangle ABE$ w/ base $|BE| = x$ and $|AB| = |AE| = 1$
3. Construct B & C as intersection pts of circle center A radius x & circles center B & E radius 1.



Alternative ruler & compass construction (from scratch)

1. Draw a circle center O, radius 1
2. Draw diameter of the circle, intersecting at a point A (& another point)



3. Construct the perpendicular line to OA through O (the perp. bisector of the diameter) label intesection point B
4. Bisect OB, label midpoint C
5. Draw line CA
6. Bisect $\angle OCA$, label intesection of bisector w/ OA, D
7. Draw perp. line to OA through D, where line intesects with circle (on ΔOCA side) label E.

Claim: $\angle AOE = 2\pi/5$

Now mark off distance AE around the circle
$\rightsquigarrow$ points $A = A_0$, $E = A_1$, A_2, A_3, A_4.
Then $A_0A_1A_2A_3A_4$ is a regular pentagon.

Proof of Claim

Want to show $\angle AOE = 2\pi/5$.

Equivalently $|OD| = \cos 2\pi/5$ (circle has radius 1)

From diagram $\frac{|OD|}{1/2} = \frac{o}{a} = \tan\theta$ ↳ opposite / adjacent

$$\tan 2\theta = \frac{|OA|}{|OC|} = \frac{1}{1/2} = 2$$

Recall double angle formula:

$$\tan 2\theta = \frac{2\tan\theta}{1-(\tan\theta)^2}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\sin 2\theta} = \frac{2\sin\theta\cos\theta}{(\cos\theta)^2-(\sin\theta)^2} = \frac{2\tan\theta}{1-(\tan\theta)^2}$$

↑ divide numerator & denominator by $(\cos\theta)^2$

Write $t = \tan\theta$

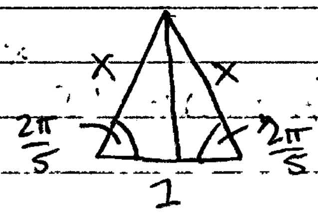
our case $2 = \frac{2t}{1-t^2}$

$\rightsquigarrow 2-2t^2 = 2t \rightsquigarrow 1-t^2 = t \rightsquigarrow t^2+t-1=0$

by quadratic formula: $t = \frac{-1\pm\sqrt{5}}{2}$

when $t>0 \rightsquigarrow t = \frac{\sqrt{5}-1}{2}$

Now $|OD| = 1/2 \tan\theta = \frac{\sqrt{5}-1}{4}$



$\Rightarrow \cos\frac{2\pi}{5} = \frac{a}{h}$ ↱ adjacent / hypotenuse

$$= \frac{1/2}{x} = \frac{1/2}{\frac{1+\sqrt{5}}{2}} = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{5-1} = \frac{\sqrt{5}-1}{4}$$