Friday 12/13/19.

1. a. The perpendicular bisector of the line segment PQ is the locus of points R such that $|PR| = |QR|$.

$P = (1,2)$, $Q = (3,4)$, $R = (x,y)$:-

$$|PR| = |QR|$$

$$\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y-4)^2}$$

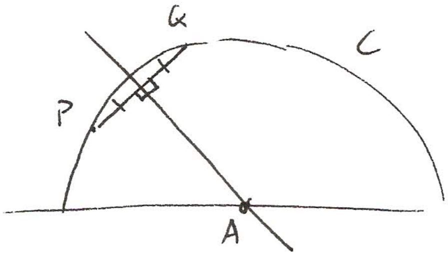
$$(x-1)^2 + (y-2)^2 = (x-3)^2 + (y-4)^2$$

$$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 - 8y + 16$$

$$4x + 4y = 20$$

$$x + y = 5$$ — equation of perp. bisector of PQ. □.

b.



The center A of C is the intersection point of the perpendicular bisector of PQ with the x-axis.

$x + y = 5$ & $y = 0$ $\Rightarrow$ $(x,y) = (5,0)$. center of C.

The radius of C is $r = |AP| = |AQ| = \sqrt{(5-1)^2 + (0-2)^2}$

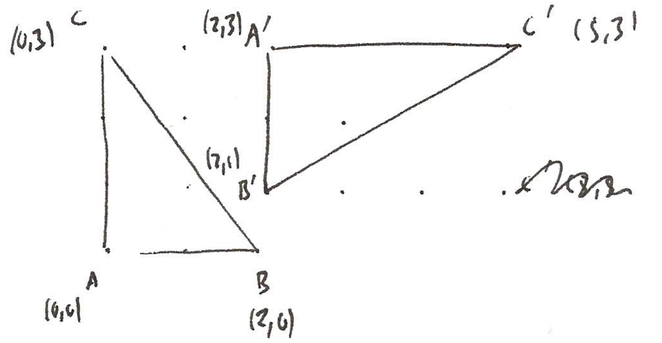
$$= \sqrt{4^2 + 2^2} = \sqrt{20}.$$

The equation of C is $(x-a)^2 + (y-b)^2 = r^2$

where $(a,b) = (5,0)$ is the center of C & $r = \sqrt{20}$ is the radius of C, i.e.,

$$(x-5)^2 + y^2 = 20.$$

2.



Rotate ccw through angle $3\pi/2$ about $A = (0,0)$

(equivalently, cw through angle $\pi/2$)

then translate by $\underline{v} = (2,3)$.

matrix for rotation through angle θ about $(0,0)$.

Algebraic formula:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\theta = 3\pi/2$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} y \\ -x \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} y+2 \\ -x+3 \end{pmatrix}.$$

$\mathrm{Fix}(T) = \{ P \in \mathbb{R}^2 \mid T(P) = P \}.$

$$\begin{pmatrix} y+2 \\ -x+3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

1. $-x+y = -2$

2. $-x-y = -3$

1.+2. $\quad -2x = -5, \quad x = 5/2 \overset{1.}{\Rightarrow} y = 1/2.$

$\mathrm{Fix}(T) = \{ (5/2, 1/2) \}. \Rightarrow T$ is a rotation about $(5/2, 1/2)$

through angle $3\pi/2$ ccw.

3. $T: \mathbb{R}^2 \to \mathbb{R}^2$ an isometry

$\Rightarrow$ T is 1. the identity, 2. a translation by a vector $\underline{v}$, 3. a rotation about a point p through angle θ ccw

4. a reflection in a line L, or 5. a glide reflection (a reflection in a line L followed by a translation by a vector $\underline{v}$ parallel to L)

$\Rightarrow$ $T^2 = T \circ T$ is
1. the identity
2. translation by $2\underline{v}$
3. rotation about P through angle 2θ ccw
4. the identity
5. translation by $2\underline{v}$ (cf. HW5 Q4)

So T^2 is the identity, a translation, or a rotation. $\square$.

4. a. $T(x,y) = (x+5, y+7) = (x,y) + (5,7)$

Translation by $\underline{v} = (5,7)$.

b. $T(x,y) = (1-y, 3-x)$.

Compute Fix(T). $(1-y, 3-x) = (x,y)$

$$\begin{aligned} 1 &= x+y \\ 3 &= x+y \end{aligned} \quad \#. \qquad \text{i.e., } \mathrm{Fix}(T) = \emptyset.$$

So T is a glide reflection (cf. HW5 Q3)

$$T^2(x,y) = T(1-y, 3-x) = (1-(3-x), 3-(1-y)) = (x-2, y+2)$$
$$= (x,y) + (-2,2)$$

translation by $(-2,2)$

So $\quad T = \mathrm{Trans}(\underline{v}) \circ \mathrm{Refl}_L \qquad$ (reflection in L followed by translation by $\underline{v}$)

where $\underline{v} = \frac{1}{2}(-2,2) = (-1,1)$ & L is parallel to $\underline{v}$.

To find L : $\quad \mathrm{Refl}_L = \left(\mathrm{Trans}(\underline{v})\right)^{-1} \circ T$

$$\mathrm{Refl}_L(x,y) = T(x,y) - \underline{v} = (1-y, 3-x) - (-1,1)$$

$$= (2-y, 2-x)$$

$$L = \mathrm{Fix}(\mathrm{Refl}_L) = \{(x,y) \mid (2-y, 2-x) = (x,y)\}$$

$$= \{(x,y) \mid x+y=2\}.$$

So T is the glide reflection give by reflection in $L: x+y=2$ followed by translation by $\underline{v} = (-1,1)$ (parallel to L). $\square$

c. $\quad T(x,y) = (y+2, 8-x)$.

$\mathrm{Fix}(T)$: $\quad (y+2, 8-x) = (x,y)$

1. $\quad -x+y = -2$
2. $\quad -x-y = -8$

$1. + 2. \quad -2x = -10, \quad x = 5 \overset{1.}{\Rightarrow} y = x-2 = 3.$

i.e. $\mathrm{Fix}(T) = \{(5,3)\} \Rightarrow \quad T$ is a rotation about $(5,3)$ through some angle θ ccw.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+2 \\ 8-x \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\theta = 3\pi/2}.$$

d. $T(x,y) = \frac{1}{13}(5x+12y+4,\ 12x-5y-6)$

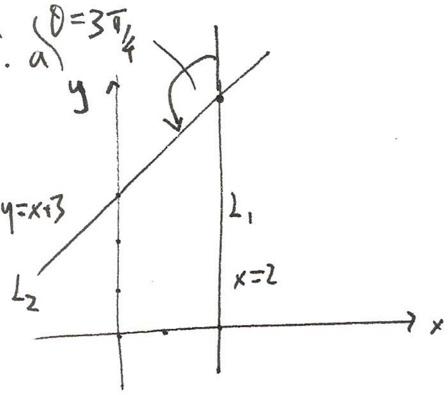
Fix T: $\quad \frac{1}{13}(5x+12y+4,\ 12x-5y-6) = (x,y)$

$$\begin{aligned} 5x+12y+4 &= 13x \\ 12x-5y-6 &= 13y \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} -8x+12y &= -4,\ \div 4 \\ 12x-18y &= 6,\ \div 6 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} &1.\quad -2x+3y = -1 \\ &2.\quad 2x-3y = 1,\quad 2. = (-1)\times 1. \end{aligned}$$

i.e. $\operatorname{Fix} T = \{(x,y) \mid -2x+3y=-1\}$, line $L \subset \mathbb{R}^2$ w/ equation $-2x+3y=-1$.

So $T = \operatorname{Refl}_L$ is reflection in L. $\square$.

5. a)



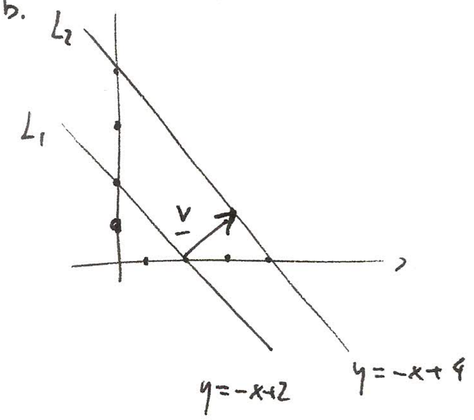
$T = \operatorname{Refl}_{L_2} \circ \operatorname{Refl}_{L_1} = \operatorname{Rot}(P, 2\theta)$ (rotation about point P through angle 2θ ccw)

where P is the intersection point of L_1 & L_2

& θ is the ccw angle from L_1 to L_2.

$P = (2,5)$, $\theta = 3\pi/4$, so $T = \operatorname{Rot}((2,5), 3\pi/2)$. $\square$.

b.

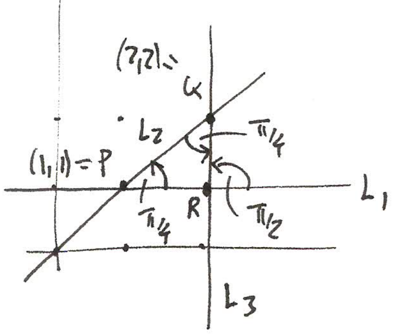


$T = \operatorname{Refl}_{L_2} \circ \operatorname{Refl}_{L_1} = \operatorname{Trans}_{2\underline{v}}$ (translation by vector $2\underline{v}$)

for L_1 & L_2 parallel, where $\underline{v}$ is the perpendicular vector from L_1 to L_2

Here, $\underline{v} = \begin{pmatrix}1\\1\end{pmatrix}$, $T = \operatorname{Trans}_{\begin{pmatrix}2\\2\end{pmatrix}}$. $\square$.

c.



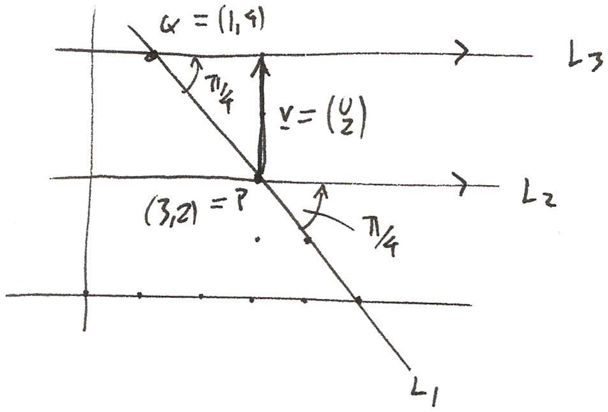
$T = \operatorname{Rot}(Q, \pi/2) \circ \operatorname{Rot}(P, \pi/2)$

$= (\operatorname{Refl}_{L_3} \circ \cancel{\operatorname{Refl}_{L_2}}) \circ (\cancel{\operatorname{Refl}_{L_2}} \circ \operatorname{Refl}_{L_1})$

cf. a.

$= \operatorname{Refl}_{L_3} \circ \operatorname{Refl}_{L_1} = \operatorname{Rot}(R, 2\cdot\pi/2) = \operatorname{Rot}((2,1), \pi)$. $\square$.

d.

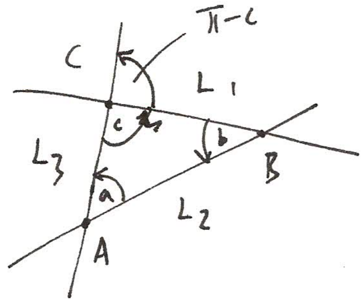


$$\text{Trans}_{\binom{0}{4}} \circ \text{Rot}(P, \pi/2) = (\text{Refl}_{L_3} \circ \cancel{\text{Refl}_{L_2}}) \circ (\cancel{\text{Refl}_{L_2}} \circ \text{Refl}_{L_1})$$

$$= \text{Refl}_{L_3} \circ \text{Refl}_{L_1} = \text{Rot}(Q, 2 \cdot \pi/4) = \text{Rot}(Q, \pi/2),$$

$Q = (1,4)$. $\square$.

6.



$$\text{Rot}(A, 2a) \circ \text{Rot}(B, 2b)$$

$$= (\text{Refl}_{L_3} \circ \cancel{\text{Refl}_{L_2}}) \circ (\cancel{\text{Refl}_{L_2}} \circ \text{Refl}_{L_1})$$

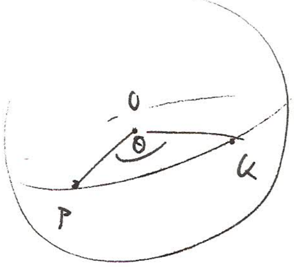
$$= \text{Refl}_{L_3} \circ \text{Refl}_{L_1}$$

$$= \text{Rot}(C, 2 \cdot (\pi - c)) = \text{Rot}(C, 2\pi - 2c)$$

Rotation about C through angle $2\pi - 2c$ ccw

(equivalently, angle $2c$ cw.) $\square$.

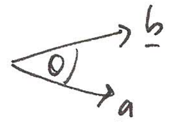
7.



The shortest path from P to Q on S^2 is given by the shorter arc of the great circle connecting P & Q. (Here a <u>great circle</u> or <u>spherical line</u> is the intersection of S^2 with a plane $\Pi \subset \mathbb{R}^3$ passing through the origin).

Since the great circle is a circle of radius $r = 1$, the length of the arc PQ $= r \cdot \theta = 1 \cdot \theta = \theta$.

Recall the dot product $\underline{a} \cdot \underline{b} = \|\underline{a}\| \cdot \|\underline{b}\| \cdot \cos\theta$



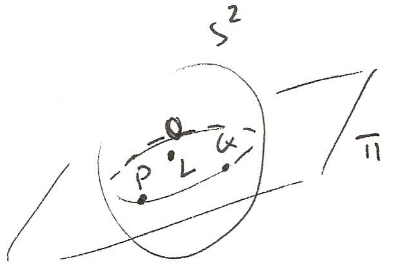
OR $\underline{a} \cdot \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$\vec{OP}\cdot\vec{OQ} = \|\vec{OP}\|\cdot\|\vec{OQ}\|\cdot\cos\theta \underset{P,Q\in S^2}{=} 1\cdot 1\cdot\cos\theta = \cos\theta$$

$$\Rightarrow \quad d_{S^2}(P,Q) = \theta = \cos^{-1}(\vec{OP}\cdot\vec{OQ}).$$

In our case, $d_{S^2}(P,Q) = \cos^{-1}\left(\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\frac{1}{9}\begin{pmatrix}4\\4\\7\end{pmatrix}\right) = \cos^{-1}\left(\frac{1}{27}(4+8+14)\right) = \cos^{-1}\left(\frac{26}{27}\right).$ $\square$.

8.

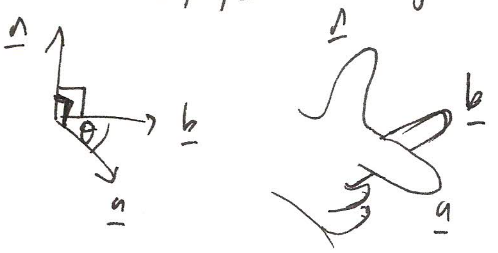


The spherical line L through P & Q is given by $\Pi\cap S^2$ where Π is the plane through O, P, Q.

Recall the cross product: $\underline{a}\times\underline{b} = \|\underline{a}\|\cdot\|\underline{b}\|\cdot\sin\theta\cdot\underline{n}$

where $\underline{n}$ is a vector of length 1 perpendicular to $\underline{a}$ & $\underline{b}$

such that $\underline{a}, \underline{b}, \underline{n}$ is a right handed set of vectors:-



(right hand.)

Also, $\underline{a}\times\underline{b} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{pmatrix}.$

In particular, a normal vector to Π is given by

$\underline{n} = \vec{OP}\times\vec{OQ}$, & then Π has equation $\underline{x}\cdot\underline{n} = 0$

(because $O\in\Pi$, & $\underline{x}\cdot\underline{n} = \underline{0} \Leftrightarrow \underline{x}\perp\underline{n}$)

In our case: $\underline{n} = \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}\times\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix} = \frac{1}{21}\begin{pmatrix}2\cdot6-1\cdot3\\1\cdot2-2\cdot6\\2\cdot3-2\cdot2\end{pmatrix} = \frac{1}{21}\begin{pmatrix}9\\-10\\2\end{pmatrix}$

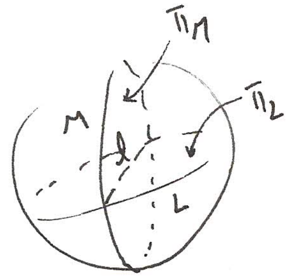
so Π has eqn $\frac{1}{21}\begin{pmatrix}9\\-10\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$, i.e. $\frac{1}{21}(9x-10y+2z) = 0$,

or $\boxed{9x-10y+2z=0}$.

9.

$$\left.\begin{aligned} L:&\ x+y+2z=0.\\ M:&\ x+2y+3z=0.\end{aligned}\right\}(*)$$

a. $L = S^2 \cap \Pi_L$, $M = S^2 \cap \Pi_M$



The planes Π_L & Π_M intersect in a line ℓ through the origin.

$\ell = \{c \cdot \underline{v} \mid c \in \mathbb{R}\} = \mathrm{Span}(\underline{v})$

where $\underline{v} = \underline{n}_L \times \underline{n}_M$, $\underline{n}_L$ & $\underline{n}_M$ the normal vectors to Π_L & Π_M.

So $\underline{v} = \begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\cdot3-2\cdot2\\2\cdot1-1\cdot3\\1\cdot2-1\cdot1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$ (because $\underline{v}$ is perpendicular to $\underline{n}_L$ & $\underline{n}_M$)

(Alternatively, solve the equations $(*)$ using the row reduction algorithm from 235)

Now the two intersection points of L & M are given by $\ell \cap S^2$.

Note $\ell \cap S^2 = \{c \cdot \underline{v} \mid c \in \mathbb{R},\ \|c\underline{v}\| = 1\} = \{\pm \underline{v}/\|\underline{v}\|\}$

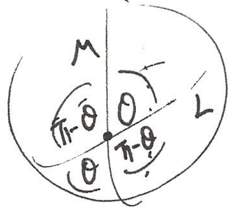
$$= \left\{ \pm \frac{1}{\sqrt{3}} \begin{pmatrix}-1\\-1\\1\end{pmatrix} \right\}. \quad \square.$$

b. The angle between L & M is the same as the angle between the normal vectors $\underline{n}_L$ & $\underline{n}_M$ of the associated planes Π_L & Π_M.

$\underline{n}_L \cdot \underline{n}_M = \|\underline{n}_L\| \cdot \|\underline{n}_M\| \cdot \cos\theta$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\underline{n}_L \cdot \underline{n}_M}{\|\underline{n}_L\| \cdot \|\underline{n}_M\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{6}\cdot\sqrt{14}}\right) = \cos^{-1}\left(\frac{9}{\sqrt{84}}\right) \quad \square.$$

c. The sphere is divided into 4 "lunes", 2 with angle θ & 2 with angle $\pi - \theta$ (for θ angle from b.)

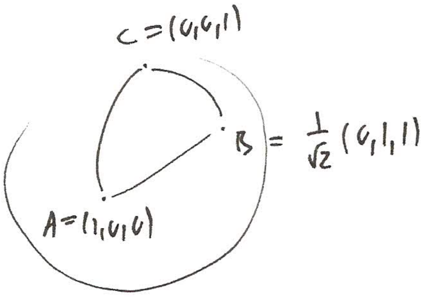


The area of a lune with angle α equals $4\pi \cdot \dfrac{\alpha}{2\pi} = 2\alpha$

area of $S^2 = 4\pi r^2 = 4\pi \cdot 1^2 = 4\pi$.

So, the 4 regions have areas $2\theta, 2\theta, 2\pi - 2\theta, 2\pi - 2\theta$. $\square$.

10.



a) $\vec{OA} \times \vec{OB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\Rightarrow$ spherical line through A, B has equation $\frac{1}{\sqrt{2}}(-y+z) = 0$

i.e. $-y+z=0$.

$\vec{OB} \times \vec{OC} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ BC has eq. $x=0$.

$\vec{OC} \times \vec{OA} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ CA has eq. $y=0$.

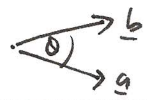
b) $b = \cos^{-1}\left(\dfrac{\Delta_{AB} \cdot \Delta_{BC}}{\|\Delta_{AB}\| \cdot \|\Delta_{BC}\|}\right) = \cos^{-1}\left(\dfrac{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{2} \cdot 1}\right) = \cos^{-1}(0) = \pi/2$

$c = \cos^{-1}\left(\dfrac{\Delta_{BC} \cdot \Delta_{CA}}{\|\Delta_{BC}\| \cdot \|\Delta_{CA}\|}\right) = \cos^{-1}\left(\dfrac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{1 \cdot 1}\right) = \cos^{-1}(0) = \pi/2$

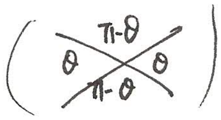
$a = \cos^{-1}\left(\dfrac{\Delta_{CA} \cdot \Delta_{AB}}{\|\Delta_{CA}\| \cdot \|\Delta_{AB}\|}\right) = \cos^{-1}\left(\dfrac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}{1 \cdot \sqrt{2}}\right) = \cos^{-1}(-1/\sqrt{2}) = 3\pi/4$

Here we are using: the angle between two spherical lines is equal to the angle between the normal vectors of the associated planes, and $\underline{a} \cdot \underline{b} = \|\underline{a}\| \cdot \|\underline{b}\| \cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{\underline{a} \cdot \underline{b}}{\|\underline{a}\| \cdot \|\underline{b}\|}\right)$



but, given $a \leq \pi/2$ & angle between lines only determined up to $\theta \rightsquigarrow \pi - \theta$



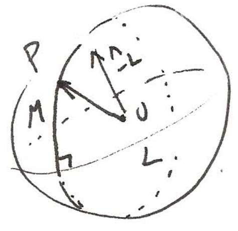
so $a = \pi - 3\pi/4 = \pi/4$.

i.e. $a = \pi/4,\ b = \pi/2,\ c = \pi/2$.

c). $a+b+c = \pi + \text{Area}(\Delta ABC)$

$\Rightarrow \text{Area}(\Delta ABC) = a+b+c-\pi = \pi/4 + \pi/2 + \pi/2 - \pi = \pi/4.$

11.



The plane Π_M must contain P & the normal vector $\underline{n}_L$ of Π_L (because L & M are perpendicular).

So $\underline{n}_M = \overrightarrow{OP} \times \underline{n}_L$ is a normal vector to Π_M,

& Π_M has equation $\underline{x} \cdot \underline{n}_M = 0$.

In our case, $\overrightarrow{OP} = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\underline{n}_L = \begin{pmatrix}1\\2\\3\end{pmatrix}$,

so $\underline{n}_M = \overrightarrow{OP} \times \underline{n}_L = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\cdot 3 - 2\cdot 2\\ 2\cdot 1 - 2\cdot 3\\ 2\cdot 2 - 1\cdot 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1\\-4\\3\end{pmatrix}$

Π_M: $\frac{1}{3}(-x-4y+3z) = 0$

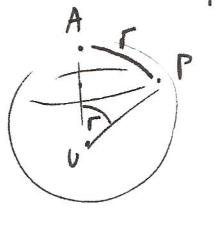
OR $\boxed{-x-4y+3z = 0.}$

12. The spherical center A of the spherical circle $C = \Pi \cap S^2$ is given by $\pm \frac{\underline{n}}{\|\underline{n}\|} \in S^2$,

where $\underline{n}$ is a normal vector for the plane Π. (see HW7 Q6).

In our case, $\underline{n} = (3,4,5)$, $A = \frac{\underline{n}}{\|\underline{n}\|} = \frac{(3,4,5)}{\sqrt{50}}$

The spherical radius r



of C is given by $r = \cos^{-1}(\overrightarrow{OA} \cdot \overrightarrow{OP})$

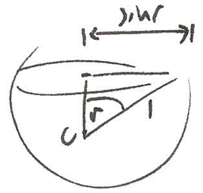
$= \cos^{-1}\left(\frac{\underline{n} \cdot \overrightarrow{OP}}{\|\underline{n}\|}\right)$

for $P \in C$.

Now $\overrightarrow{OP} \cdot \underline{n} = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\4\\5\end{pmatrix} = \underbrace{3x+4y+5z = 6}_{\text{eq. of } \Pi}$ for $P \in C$,

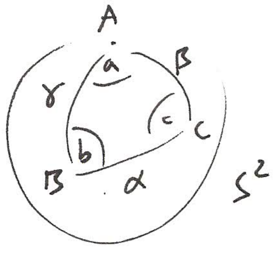
so $r = \cos^{-1}\left(\frac{6}{\sqrt{50}}\right)$.

To compute the circumference of C, observe that C is a Euclidean circle of radius $\sin r$ in the plane Π



so the circumference of $C = 2\pi \sin r = 2\pi\sqrt{1-(\cos r)^2} = 2\pi\cdot\sqrt{1-\frac{36}{50}} = 2\pi\cdot\sqrt{\frac{14}{50}}$ $\square$.

13 a. The spherical cosine rule states



$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

$a = \frac{\pi}{2} \Rightarrow \cos\alpha = \cos\beta\cos\gamma.$

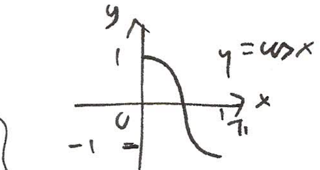
$\beta < \pi/2 \Rightarrow \cos\beta > 0.$

Also $0 < \gamma < \pi \Rightarrow 1 > \cos\gamma > -1.$

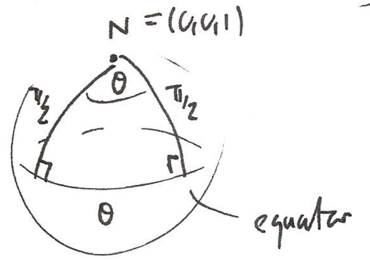
So $\cos\alpha = \cos\beta\cos\gamma < \cos\beta\cdot 1 = \cos\beta \Rightarrow \alpha > \beta$ $\square$.

$\cos\beta > 0$ & $\cos\gamma < 1$

$\cos x$ strictly decreasing for $0 < x < \pi$.



b. No, counter example:-



For $\theta > \pi/2$, largest side is not hypotenuse.

14. SCR:
$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a \rightsquigarrow \cos a = \dfrac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$

Similarly $\cos b = \dfrac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}$

So, if $\alpha = \beta$, find $\cos a = \cos b \Rightarrow a = b$ (because $\cos x$ is one-to-one for $0 \le x \le \pi$)

$\square$.

15 a. Area(triangle ABC) $= a+b+c-\pi$

$$= \pi/3 + \pi/3 + 2\pi/5 - \pi = \pi \cdot \left(\frac{5+5+6-15}{15} \right) = \pi/15$$

Area $(S^2) = 4\pi r^2 = 4\pi \cdot 1^2 = 4\pi$.

$\therefore$ # triangles $= 4\pi \Big/ \pi/15 = 60$. $\square$.

b. By Q14, equilateral spherical triangles have all angles equal.

Area (triangle) $= a+a+a-\pi = 3a-\pi = 4\pi/20 = \pi/5$

(Area $S^2/20$)

$\Rightarrow 3a = 6\pi/5$, $\boxed{a = 2\pi/5}$. $\square$.

16. a.

$$F(C) = \{ (u,v) \in \mathbb{R}^2 \mid F^{-1}(u,v) \in C \}$$

$$= \{ (u,v) \in \mathbb{R}^2 \mid F^{-1}(u,v) \in \Pi = (x = 1/2) \}$$

$$= \{ (u,v) \in \mathbb{R}^2 \mid \frac{2u}{u^2+v^2+1} = 1/2 \}$$ using $F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1)$

$\wr$

$4u = u^2+v^2+1$

$0 = (u-2)^2 + v^2 + 1 - 4$ "complete the square"

$(u-2)^2 + v^2 = 3$.

circle, center $(2,0)$, radius $\sqrt{3}$.

b. The equator on S^2 corresponds to the unit circle in the xy-plane $\mathbb{R}^2$ (the circle center $(0,0)$ & radius 1)

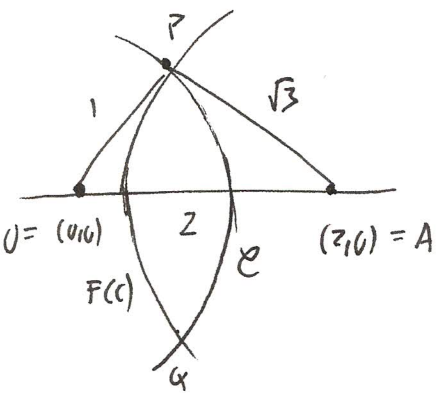
(in fact, this circle "doesn't move" under stereographic projection.

The equator is perpendicular to the spherical circle $x = 1/2$ (because the tangent line

to this spherical circle is vertical at the two intersection points with the xy-plane

(because it's contained in the plane $x = 1/2$ and is perpendicular to the Euclidean radius of the circle which is horizontal at these points) whereas the tangent line to the equator is horizontal).

So we need to check that the unit circle $\mathcal{C}$ & the circle $F(C)$ in $\mathbb{R}^2$ are perpendicular:



$|OP|^2 + |PA|^2 = |OA|^2$

$\Rightarrow \angle OPA = \pi/2$

$\Rightarrow$ radii at P are perpendicular

$\Rightarrow$ tangents at P are perpendicular

(because radius $\perp$ tangent).

Similarly, tangents at Q are perpendicular.

$\square$.

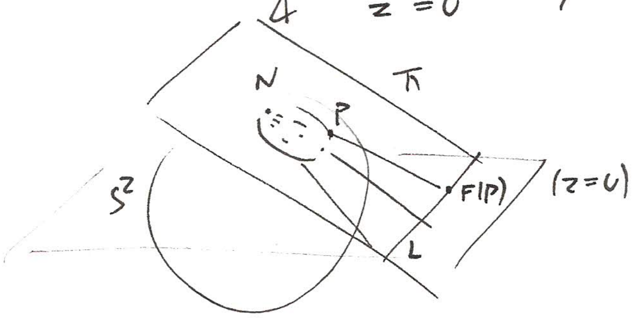
17. $C = \Pi \cap S^2$, $\quad \Pi: \ x+y+z=1$.

Note that $N = (0,0,1) \in C$: $\quad 0+0+1 = 1$ ✓.

So $F(C \setminus \{N\}) = L = \Pi \cap \mathbb{R}^2$, a line.

$\uparrow$ xy-plane, with equation $z=0$.

Eqn of L: $\left.\begin{array}{l} x+y+z=1 \\ z=0 \end{array}\right\} \Rightarrow x+y=1$, or $y=-x+1$. $\square$.



18. A spherical circle $C \subset S^2$ is given by $C = \Pi \cap S^2$ for $\Pi \subset \mathbb{R}^3$ a plane.

C is a great circle (or spherical line) if Π passes through the origin.

Note that C is a great circle if and only if it intersects the equator in two antipodal points.

It follows that the images of the great circles on S^2 under stereographic projection are the circles & lines in $\mathbb{R}^2$ intersecting the unit circle $\mathcal{C}$ in two antipodal points $\pm(x,y)$.

$\square$.

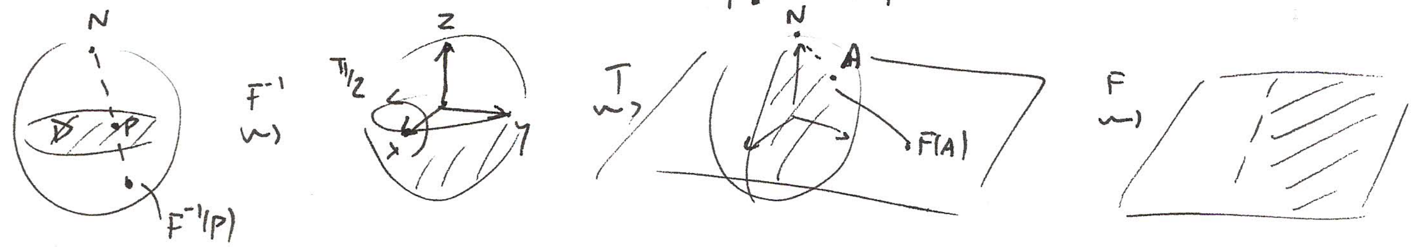
19.

F^{-1} sends the disc D to the southern hemisphere.

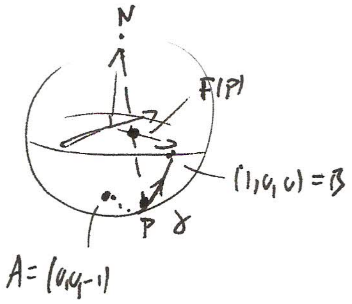
The rotation T sends the southern hemisphere to the hemisphere $y > 0$.

The stereographic projection sends this hemisphere to the upper half plane $\{(u,v) \in \mathbb{R}^2 \mid v > 0\}$.

So $G = F \circ T \circ F^{-1}$ sends the disc D to the upper half plane.



20.



Under stereographic projection,
γ corresponds to the line segment from the origin to $(1,0)$ in the uv-plane. We take parametrization $(u(t), v(t)) = (t, 0),\ 0 \le t \le 1$

$$\therefore \text{length}(\gamma) = \int_0^1 \frac{2}{u^2+v^2+1} \sqrt{u'^2+v'^2}\, dt$$

$$= \int_0^1 \frac{2}{t^2+1} \cdot dt$$

$$= 2 \cdot \left[\tan^{-1} t\right]_0^1 = 2 \cdot \left(\pi/4 - 0\right) = \pi/2.$$

This checks with the calculation on S^2.

$$\left(d_{S^2}(A,B) = \cos^{-1}(\vec{OA} \cdot \vec{OB}) = \cos^{-1}(0) = \pi/2. \right) \quad \square.$$