Math 462 Homework 9

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April 15, 2015

Recall that
$$\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$
is the upper half plane model of the hyperbolic plane. The hyperbolic distance $d_{\mathcal{H}}(P,Q)$ between points $P, Q \in \mathcal{H}$ is defined as follows. First, if
$$\gamma \colon [a,b] \to \mathcal{H}, \quad \gamma(t) = x(t) + iy(t)$$
is a path in $\mathcal{H}$ then the *hyperbolic length* of γ is defined by
$$\text{length}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$
The hyperbolic distance $d_{\mathcal{H}}(P,Q)$ is the length of the shortest path from P to Q for the hyperbolic length.

(1) Write the Mobius transformation $f(z) = \frac{z+i}{z-i}$ as a composition of Mobius transformations of the following 3 types:

  (a) $f_1(z) = z + b$, some $b \in \mathbb{C}$ (translation by b).
  (b) $f_2(z) = az$, some $a = re^{i\theta} = r(\cos\theta + i\sin\theta) \in \mathbb{C}$, $a \neq 0$ (scaling by r followed by counter-clockwise rotation through angle θ, with center the origin).
  (c) $f_3(z) = \frac{1}{z}$ (inversionin the circle with center the origin and radius 1, followed by reflection in the x-axis).

Using this expression describe the effect of the Mobius transformation geometrically.

[Hint: First write down a translation f_1 such that $f_1(f(\infty)) = 0$, then $f_3(f_1(f(\infty))) = \infty$. Now it follows that $f_3(f_1(f(z))) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$ (why?), so that $f_3 \circ f_1 \circ f = g_1 \circ g_2$ for some Mobius transformations g_1 and g_2 of types (a) and (b). Solving for f gives $f = f_1^{-1} \circ f_3^{-1} \circ g_1 \circ g_2 = f_1^{-1} \circ f_3 \circ g_1 \circ g_2$.]

(2) Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be the Mobius transformation given by $f(z) = -1/z$.

(a) Show directly that $f(\mathcal{H}) = \mathcal{H}$, where

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

is the upper half plane.

(b) Prove that f is given by inversion $g(z) = z/|z|^2$ in the circle with center the origin and radius 1 followed by reflection in the y-axis.

(3) Recall that Mobius transformations of the form

$$f(z) = \frac{az + b}{cz + d}, \text{ where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0$$

define isometries of the hyperbolic plane $\mathcal{H}$. That is, f defines a function from $\mathcal{H}$ to itself, and $d_{\mathcal{H}}(f(P), f(Q)) = d_{\mathcal{H}}(P, Q)$ for all $P, Q \in \mathcal{H}$.

Let $f : \mathcal{H} \to \mathcal{H}$ be the isometry of the hyperbolic plane given by

$$f(z) = \frac{2z + 1}{z + 2}.$$

Express f as a composite of isometries of the following types:

(a) $f_1(z) = z + b$, some $b \in \mathbb{R}$ (translation parallel to the x-axis).

(b) $f_2(z) = az$, some $a \in \mathbb{R}$, $a > 0$ (scaling with factor a, center the origin).

(c) $f_3(z) = -1/z$ (inversion in the circle center the origin and radius 1 followed by reflection in the y-axis).

Use this expression to describe the effect of f geometrically.

[Hint: Adapt the approach used in Q1 (the difference here is that we only consider Mobius transformations preserving the upper half plane $\mathcal{H} \subset \mathbb{C} \cup \{\infty\}$).]

(4) Recall that a *hyperbolic line* in $\mathcal{H}$ is either a vertical line or a semicircle with center on the x-axis. Segments of hyperbolic lines define the shortest paths in $\mathcal{H}$ for the hyperbolic distance.

Find the hyperbolic line in $\mathcal{H}$ through the following pairs of points.

(a) $2+i$, $2+5i$.

(b) $1+2i$, $3+2i$.

(c) $i, 2+3i$.

(5) Find the hyperbolic line in the upper half plane $\mathcal{H}$ passing through the point $3+4i$ and having tangent direction $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ at that point.

(6) Let L be the hyperbolic line in the upper half plane given by a semicircle with center the point 5 and radius 2. Find a hyperbolic isometry $f : \mathcal{H} \to \mathcal{H}$ such that $f(L)$ is the hyperbolic line given by the y-axis.

(7) Find a hyperbolic isometry $f \colon \mathcal{H} \to \mathcal{H}$ such that $f(1+2i) = 6+4i$ [Hint: We can use an isometry of the form $f(z) = az + b$, $a, b \in \mathbb{R}$, $a > 0$ (a composition of a scaling and a translation).]

(8) Compute the hyperbolic length of the segment of the Euclidean line connecting the points i and $4+2i$. [Hint: Describe a parametrization γ of the line segment and compute the integral defining the hyperbolic length.]

(9) (a) Compute the hyperbolic distance $d_{\mathcal{H}}(z_1, z_2)$ between the points $z_1 = -3+4i$ and $z_2 = 3+4i$ and describe the shortest path from z_1 to z_2 geometrically.

(b) Check that the hyperbolic length of the Euclidean line segment joining z_1 and z_2 is strictly larger than $d_{\mathcal{H}}(z_1, z_2)$.

[Hint: First compute the circle C passing through z_1 and z_2 with center on the x-axis. Let $a, b \in \mathbb{R}$, $a < b$ be the intersection points of C with the x-axis. The Mobius transformation $f(z) = -\frac{z-a}{z-b}$ sends the upper half plane $\mathcal{H}$ to itself, is a hyperbolic isometry, and sends C to the y-axis (why?). Now use the formula $d_{\mathcal{H}}(P, Q) = \ln(y_2/y_1)$ for the hyperbolic distance between two points P,Q on a vertical line with y-coordinates $y_1 < y_2$.]

(10) (a) Find all the hyperbolic isometries of the form

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad-bc>0$$

such that $f(i) = i$.

(b) Now consider the stereographic projection $\bar{F}\colon S^2 \to \mathbb{C} \cup \{\infty\}$. Let f be one of the hyperbolic isometries found in part (a). What is the transformation T of the sphere corresponding to the Mobius transformation f?

[Hint for part (b): Recall we showed that some Mobius transformations correspond to rotations of the sphere using quaternions, see HW7Q5.]