697B Final Exam

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3 December 2010

This is a take home exam, due Friday December 10 at 4PM in my mailbox. You are allowed to consult your notes and textbooks. Please do not *discuss the exam with other students. There are* 5 *questions worth* 10 *points each. Show all your work and justify your answers carefully.*

(1) Let X be the plane curve $(x^5 + xy^3 + 1 = 0) \subset \mathbb{C}^2_{x,y}$ and $\overline{X}$ its closure in $\mathbb{P}^2$. Find all the singular points of $\overline{X}$ and compute the genus of its normalization.

(2) Let X be the plane curve $(y^3 - 3y + 2x^4 = 0) \subset \mathbb{C}^2_{x,y}$ and $\overline{X}$ its closure in $\mathbb{P}^2$. The curve $\overline{X}$ is smooth and the map $X \to \mathbb{C}$ given by $(x, y) \mapsto x$ extends to a holomorphic map $F \colon \overline{X} \to \mathbb{C} \cup \{\infty\}$. Find the degree of F, the ramification points, and the ramification indices. Check your answer using the Riemann–Hurwitz formula and the genus formula for plane curves.

(3) Let X be the plane curve $(y^3 + x^4 + x = 0) \subset \mathbb{C}^2_{x,y}$ and $\overline{X}$ its closure in $\mathbb{P}^2$. The curve $\overline{X}$ is smooth. Let ω be the meromorphic differential on $\overline{X}$ given by dx/y^2. Find the zeroes and poles of ω and their orders. Find a basis of the complex vector space $\Omega(\overline{X})$ of holomorphic differentials on $\overline{X}$. (Justify your answer carefully. You may assume that the dimension of $\Omega(\overline{X})$ is equal to the genus of $\overline{X}$.)

(4) Let $\overline{X} \subset \mathbb{P}^2$ be a smooth curve of degree 3. Let $P, Q \in \overline{X}$ be two points and $D = P + Q$. Show that $L(D)$ determines a morphism $F \colon \overline{X} \to \mathbb{P}^1$ of degree 2. Give a geometric description of the map F in terms of lines in $\mathbb{P}^2$.

(5) Let X be a compact Riemann surface of genus $g > 0$, and let $P \in X$ be a point.

(a) Use the Riemann-Roch theorem to prove the following assertion: There are exactly g integers $1 = n_1 < n_2 < \cdots < n_g < 2g$ such that there does *not* exist a meromorphic function f on X such that f has a pole of order n_i at P and is holomorphic on $X \setminus \{P\}$.

(b) Show that if $n_2 > 2$ then $n_i = 2i - 1$ for each $i = 1, \ldots, g$. Give a geometric description of X in this case.