Math 300.3 Homework 6

Paul Hacking

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Reading: Sundstrom, Sections 4.2 and 4.3.

Justify your answers carefully.

(1) Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively by $a_1 = 5$, $a_2 = 13$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geqslant 3$. Prove using strong induction that $a_n = 2^n + 3^n$ for all $n \in \mathbb{N}$.

(2) Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively by $a_0 = 1$ and $a_n = 1 + a_0 + a_1 + \ldots + a_{n-1}$ for $n \geqslant 1$. Guess a formula for a_n and prove your formula is correct using strong induction.

[Hint: For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, if $x \neq 1$ then $1 + x + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$.]

(3) Let $f_1, f_2, f_3, \ldots$ be the Fibonacci numbers defined recursively by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ be the two solutions of the quadratic equation $x^2 = x + 1$. Prove by strong induction that $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$.

(4) Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geqslant 1$.

(a) Let $f_1, f_2, \ldots$ be the Fibonacci numbers defined in Q3. Prove by induction that $a_n = \frac{f_{n+1}}{f_n}$ for all $n \in \mathbb{N}$.

(b) What is the limit of a_n as $n \to \infty$?

[Hint: Use the result of Q3 and the fact that $|\frac{\beta}{\alpha}| < 1$.]

(5) Let $f_1, f_2, \ldots$ be the Fibonacci numbers defined in Q3. Prove by induction that $f_{n+1}^2 - f_n f_{n+2} = (-1)^n$ for all $n \in \mathbb{N}$.

(6) Suppose you are given 3^n coins and a pair of balancing scales. All but one of the coins have exactly the same weight, and the remaining coin is slightly heavier. All the coins look and feel identical. The balancing scales can be used to compare the weights of two collections of coins, showing whether the two collections have the same weight or, if not, which is heavier. Prove by induction that it is possible to find the heavy coin using n weighings.