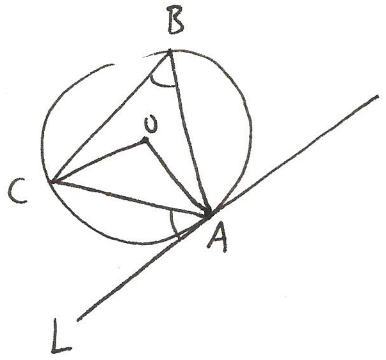
MATH 461 HW3 Solutions

1.



Let O be the center of the circle.

$\angle COA = 2 \cdot \angle CBA$ *

(the angle subtended by a chord at the centre of the circle equals twice the angle subtended at the circumference).

ΔCOA is isosceles ($|CO| = |OA| =$ radius of circle)

Thus $\angle OAC = \angle OCA = \frac{1}{2}(\pi - \angle COA) \overset{*}{=} \pi/2 - \angle CBA$

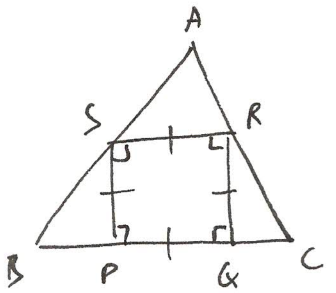
(using angle sum of a triangle $= \pi$).

The tangent L is perpendicular to the radius OA.

So the angle between L and AC equals

$\pi/2 - \angle OAC = \pi/2 - (\pi/2 - \angle CBA) = \angle CBA$. $\square$.

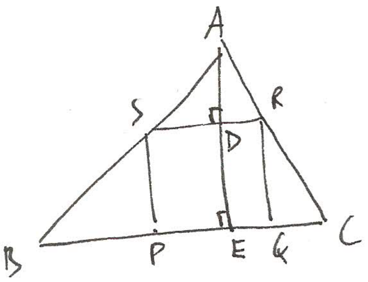
2.



SR is parallel to BC (because they are both perpendicular to PS).

So $\angle ASR = \angle ABC$, & $\Delta ASR \sim \Delta ABC$.

Therefore $\dfrac{|SR|}{|BC|} = \dfrac{|AS|}{|AB|}$ ①



Drop a perpendicular line from A to BC, meeting SR at D and BC at E.

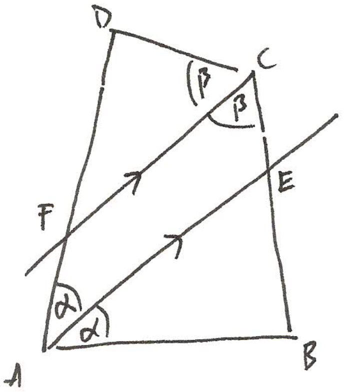
Then $\Delta ASD \sim \Delta ABE$, so $\dfrac{|AS|}{|AB|} = \dfrac{|AD|}{|AE|}$. ②

Combining, writing l for the side length of the square $PQRS$

$$\frac{l}{a} = \frac{|SR|}{|BC|} \overset{①}{=} \frac{|AS|}{|AB|} \overset{②}{=} \frac{|AD|}{|AE|} = \frac{h-l}{h}$$

$$\Rightarrow \quad l \cdot h = (h-l) \cdot a, \quad l \cdot (h+a) = ha, \quad l = \frac{ha}{h+a}. \ \square.$$

3.



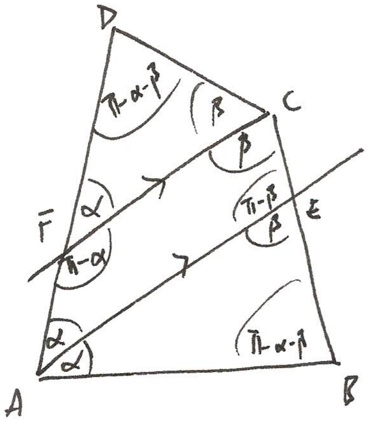
$\angle BAE = \angle EAD = \alpha$

$\angle DCF = \angle FCB = \beta$

AE is parallel to FC

$\Rightarrow \angle AFC = \pi - \angle EAD = \pi - \alpha$

$\angle AEC = \pi - \angle FCB = \pi - \beta$



$\Rightarrow \angle DFC = \alpha$

$\angle AEB = \beta$ (angle on straight line $= \pi$)

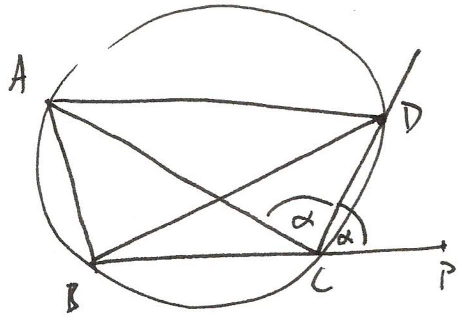
$\Rightarrow \angle ABC = \pi - \angle BAE - \angle AEB = \pi - \alpha - \beta$

$\angle ADC = \angle FDC = \pi - \angle DFC - \angle DCF = \pi - \alpha - \beta$

(angle sum of triangle $= \pi$.

So $\angle ABC = \angle ADC$. $\square$.

4.



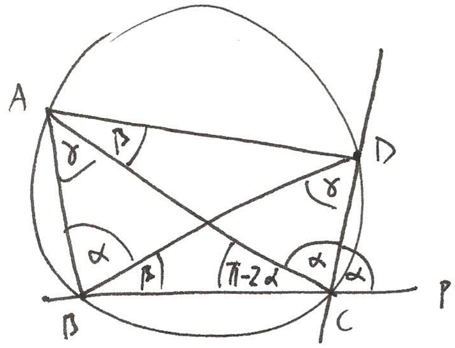
$\angle ACD = \angle DCP = \alpha$.

$\angle DAC = \angle DBC = \beta$ (angles subtended by a chord at the circumference are equal)

$\angle BAC = \angle BDC = \gamma$ (")

$\angle ABD = \angle ACD = \alpha$ (")

$\angle ACB = \pi - 2\alpha$ (angle on straight line)



$\Rightarrow \pi = \angle BAC + \angle ABC + \angle ACB$ (angle sum $\triangle ABC$)

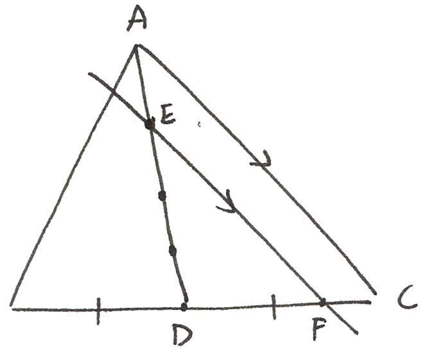
$= \gamma + (\alpha + \beta) + \pi - 2\alpha$,

$\Rightarrow \alpha = \beta + \gamma$. So $\angle ABD = \angle BAD$.

& $|AD| = |BD|$ $\square$.

(isosceles triangle thm: two equal angles $\Leftrightarrow$ two equal sides)

5.



$$\frac{|DF|}{|DC|} = \frac{|DE|}{|DA|} = \frac{3}{4}$$ by Thales' theorem

(EF & AC are parallel)

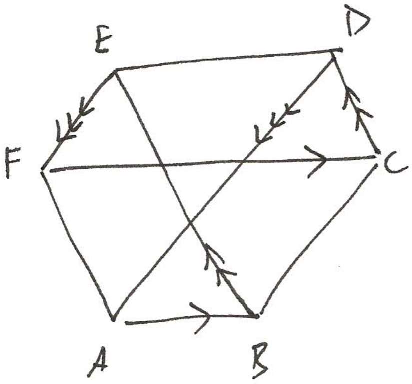
So $$\frac{|FC|}{|DC|} = \frac{1}{4} \qquad (|DF| + |FC| = |DC|)$$

$$\frac{|FC|}{|BC|} = \frac{1}{8} \qquad (|DC| = \tfrac{1}{2}|BC|)$$

$$\frac{|BF|}{|FC|} = \frac{|BC|}{|FC|} - 1 = 8 - 1 = 7. \qquad \square.$$

$|BF| + |FC| = |BC|$.

6.



Area ΔABC = ~~Area ΔDBC~~ Area ΔFAB

Area ΔCDE = Area ΔBCD

Area ΔEFA = Area ΔDEF

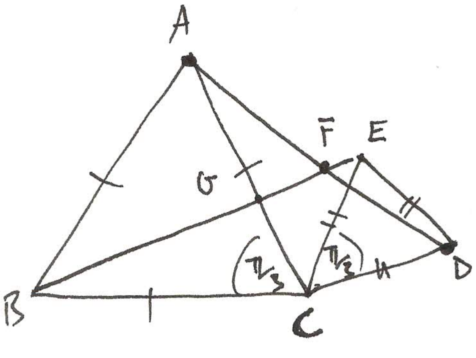
(in each case, triangles have same base & perpendicular height given by distance between parallel lines)

Now Area(ΔACE) = Area$(ABCDEF)$ − (Area ΔABC + Area ΔCDE + Area ΔEFA)

= Area$(ABCDEF)$ − (Area ΔFAB + Area ΔBCD + Area ΔDEF)

= Area(ΔBDF) □.

7.



$\Delta ACD \cong \Delta BCE$ (SAS) :-

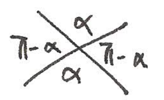
$|AC| = |BC|$, $|CD| = |CE|$,

& $\angle ACD = \pi/3 + \angle ACE = \angle BCE$

(Note: equilateral triangles have equal angles of $\frac{\pi}{3}$)

$\Delta BGC \sim \Delta AGF$: $\angle GBC = \angle EBC = \angle DAC = \angle FAG$

$\Delta BCE \cong \Delta ACD$

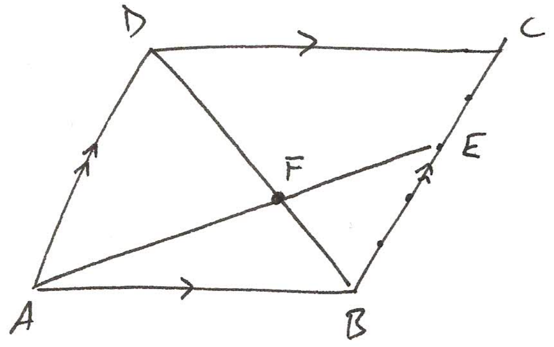
and. $\angle BGC = \angle AGF$ (opposite angles: 

);

so $\angle BCG = \angle AFG$ by angle sum of triangle $= \pi$.

i.e $\angle AFG =$ angle between AD & BE $= \angle BCG = \angle BCA = \pi/3$. $\square$.

8.



$$\text{Area}(\Delta ABE) = \tfrac{1}{2} \cdot |AB| \cdot \text{height} = \tfrac{1}{2}|AB| \tfrac{3}{5} \text{height}(ABCD)$$
$$= \tfrac{3}{10} \text{Area}(ABCD) = 3/10.$$

$\Delta DAF \sim \Delta BEF$:-

$\angle ADF = \angle FBE$ (alternate angles)

$\angle DFA = \angle BFE$ (opposite angles).

& so $\angle DAF = \angle BEF$ (angle sum of Δ)

$$\therefore \frac{|DF|}{|FB|} = \frac{|AD|}{|BE|} = \frac{|BC|}{|BE|} = \frac{5}{3},$$

opposite sides of parallelogram have equal length

so $\dfrac{|DF|}{|DB|} = \dfrac{5}{5+3} = 5/8$

$$\text{Area}(\Delta DAF) = \tfrac{1}{2}|AD| \cdot \text{height}$$
$$= \tfrac{1}{2}|AD| \cdot \tfrac{5}{8} \text{height}(DABC)$$
$$= 5/16 \cdot \text{Area}(DABC)$$
$$= 5/16.$$

Finally, $\text{Area}(CDFE) = \text{Area}(ABCD) - \text{Area}(\Delta ABE) - \text{Area}(\Delta DAF)$

$$= 1 - 3/10 - 5/16 = \frac{80-24-25}{80} = 31/80. \quad \square.$$