Math 462: Homework 6

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(1) (a) Suppose we have a set of tiles each in the shape of a regular polygon with n sides. For which values of n can we tile the plane $\mathbb{R}^2$ (with no gaps)? [Hint: Think about the angles meeting at a vertex of the tiling.]

(b) Now suppose we have two different types of tiles. Can you find a tiling of the plane using both squares and regular octagons? Can you find a tiling using both equilateral triangles and squares such that no two squares meet along an edge?

(c) Now consider the same problem in 3-dimensions: can we "tile" 3-dimensional space $\mathbb{R}^3$ with regular polyhedra (with no gaps)? Give one example of such a tiling.

(d) Can we tile 3-dimensional space with octahedra? [Hint: What is the dihedral angle between two faces of the octahedron which meet along an edge?]

(e) Can you find a tiling of 3-dimensional space by octahedra and tetrahedra?

(2) Suppose we draw a graph (a network of vertices and edges) on the surface of the sphere S^2 which subdivides it into a number of faces. Let V, E, F be the number of vertices, edges, and faces. We showed in class that

$$V - E + F = 2.$$

Now suppose we draw a graph on the surface of a donut (the type with a hole — in mathematics it is usually called a *torus*). We must assume here that each face of the subdivision looks like a polygon (we don't allow faces that wrap around the donut to form a ring). Then there is a formula

$$V - E + F = C$$

where C is a constant (it does not depend on the subdivision). Find C by computing V, E, F for one example.

(3) Suppose we subdivide the surface of S^2 into triangles. Let V, E, F be the numbers of vertices, edges and faces in the subdivision.

  (a) Find a relation between the numbers E of edges and F of faces. [Hint: Use the fact that each face is a triangle.]

  (b) If you've done part (a), you'll see that the number F of faces has to be an even integer, say $F = 2n$. Now find the numbers V of vertices and E of edges in terms of n. [Hint: Use Euler's formula $V - E + F = 2$.]

  (c) Check your results for the tetrahedron, octahedron, and icosahedron (project them onto the sphere). Can you find a symmetric example with $F = 60$? [Hint: Use the dodecahedron.]

(4) Find 12 diagonals on the faces of the dodecahedron which are the edges of a cube. You should check that the edges meeting at each vertex are perpendicular (by symmetry you only need to do it for one vertex and one pair of edges). One way to do this is to write down coordinates for (some of) the vertices of the dodecahedron using the coordinate description of the icosahedron we gave in class and the fact that the dodecahedron is dual to the icosahedron. It might be helpful to make a cardboard model of the dodecahedron.

Note: Another way to explain this observation is that the dodecahedron can be built from a cube by gluing a polyhedron shaped like a ridge tent onto each face.

(5) Describe the group G of symmetries of the tetrahedron as follows:

  (a) Find all the symmetries given by reflection in a plane.

  (b) Show that any permutation of the vertices can be realized by a symmetry of the tetrahedron. [Hint: Use the reflections.]

  (c) From part (b) it follows that the symmetry group G of the tetrahedron can be identified with the symmetric group S_4 of permutations of 4 objects. Describe each of the elements of G geometrically. [Hint: There are 6 reflections, 11 rotations, 6 rotary reflections, and the identity.]