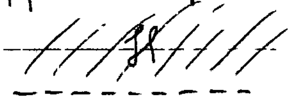
12/16/19

HW 8 due now

Last Time • Hyperbolic plane $\mathbb{H}^2$, $F: \mathbb{H}^2 \xrightarrow{\sim} \mathcal{H} \subset \mathbb{R}^2$

Upper half plane model $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

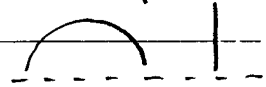


* distances distorted
* angles preserves

$\gamma : [a,b] \to \mathcal{H}$; $\gamma(t) = (x(t), y(t))$ parametrized curve

hyperbolic length $(\gamma) = \int_a^b \frac{\sqrt{x'^2 + y'^2}}{y}\, dt$ | compare with Euclidean length

• Shortest paths are given by hyperbolic lines:-



* semicircles, center on x-axis
* vertical half lines

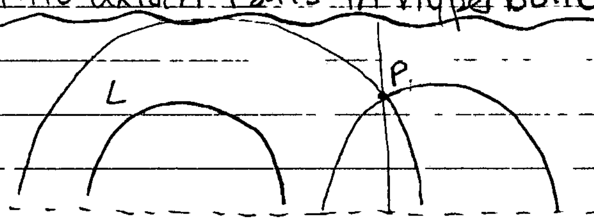
(Proof later) $\boxed{\text{Compare with } S^2 \setminus \{N\} \xrightarrow{F} \mathbb{R}^2}$

Shortest path given by great circles $\longrightarrow$ circles or lines in $\mathbb{R}^2$, satisfying an additional property $\to$ intersection with unit circle is two antipodal points

Today • Parallel axiom fails in hyperbolic plane
• Computation, vertical lines give shortest paths
• Hyperbolic isometries

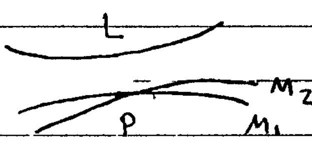
Equivalent to Euclid's parallel axiom:- Playfair's axiom:
Given a line L & a point $P \notin L$, there is a unique line M through P $\underbrace{\text{parallel to L}}_{\text{i.e. L\&M don't intersect.}}$

This axiom fails in hyperbolic plane:-



Many choices for M!

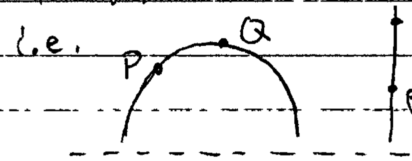
In $\mathbb{H}^2$:

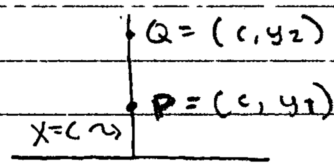


~ seemingly parallel lines diverge from each other

Computation: Show a vertical line gives shortest paths in $\mathcal{H}$.
↳ the easier case.

i.e.





$y_1 < y_2$, $\boxed{d_{\mathcal{H}}(P,Q) = ?}$ (= hyperbolic length of shortest path from P to Q)

Note: $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$

* $\gamma : [a,b] \to \mathcal{H}$, $\gamma(t) = (x(t), y(t))$, $\gamma(a) = P$, $\gamma(b) = Q$

$$\text{h-length}(\gamma) = \int_a^b \frac{\sqrt{x'^2 + y'^2}}{y}\,dt \geq \int_a^b \frac{\sqrt{y'^2}}{y}\,dt = \int_a^b \frac{|y'|}{y}\,dt$$

equal $\Leftrightarrow x' = 0 \Leftrightarrow x = c$ (constant)

$$\geq \int_a^b \frac{y'}{y}\,dt$$

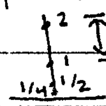
equal $\Leftrightarrow y' \geq 0$ (no back tracking)

$$= \int_a^b \frac{1}{y}\frac{dy}{dt}\cdot dt = \int_{y_1}^{y_2} \frac{1}{y}\,dy = \left[\ln y\right]_{y_1}^{y_2} = \ln y_2 - \ln y_1 = \boxed{\ln(y_2/y_1)}$$

substitution rule

Conclusion: Vertical line segment gives shortest path & $d_{\mathcal{H}}(P,Q) = \ln(y_2/y_1)$.

ex:

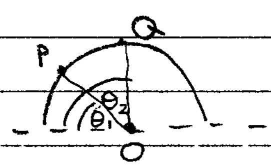


$d_{\mathcal{H}} = \ln 2$

Other case: hyperbolic line is a semicircle.

Idea: Find a hyperbolic isometry which sends L to a vertical line, & so reduce to previous case!

Aside:



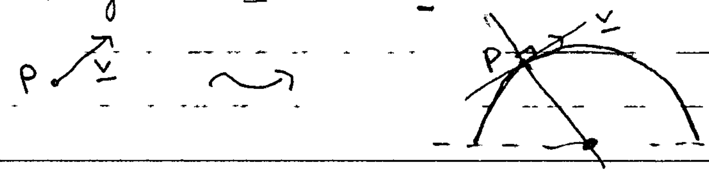
$\gamma(t) = (R\cos t, R\sin t)$ $t \in [\theta_1, \theta_2]$

$x' = -R\sin t$, $y' = R\cos t$

$$\text{length}(\gamma) = \int_{\theta_1}^{\theta_2} \frac{R}{R\sin t}\,dt$$

Alternative approach, but still need to prove that this is the shortest path.

Another Aside: Fact: Given $P \in \mathcal{H}$ & a tangent vector $\underline{v}$ at P, there's a unique hyperbolic line through P with tangent direction $\underline{v}$.



Hyperbolic Isometries:

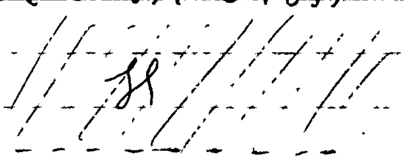
Examples?

$$h\text{-length}(\gamma) = \int_a^b \frac{\sqrt{x'^2+y'^2}}{y}\,dt$$

$$d_{\mathcal{H}}(P,Q) = h\text{-length of shortest path}$$

So $T: \mathcal{H} \to \mathcal{H}$ will be an isometry provided it preserves the quantity $\dfrac{\sqrt{x'^2+y'^2}}{y}$ $\left(\begin{array}{l}\rightsquigarrow \text{preserve lengths of paths} \\ \rightsquigarrow \text{preserve } d_{\mathcal{H}}\end{array}\right)$

$T(a(x,y), b(x,y))$



Ex 1: Translation parallel to x-axis (Horizontal translation)

$$T(x,y) = (x+a, y)$$

$$\frac{\sqrt{(x+a)'^2+y'^2}}{y} = \frac{\sqrt{x'^2+y'^2}}{y} \checkmark$$

{because a is a constant}

Ex 2: Reflection in a vertical line

e.g. y-axis $\rightsquigarrow T(x,y) = (-x,y)$

(more generally: in vertical line $x=c \rightsquigarrow T(x,y) = (2c-x,y)$)

$$\frac{\sqrt{(-x)'^2+y'^2}}{y} = \frac{\sqrt{x'^2+y'^2}}{y}$$

{because $-x$ is squared}

To be continued...