Math 462: Review questions

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(1) What is the angle between two interior diagonals of a cube joining opposite vertices? [Hint: It is probably quickest to use a description of the cube in coordinates and use the dot product formula $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$.]

(2) Compute the dihedral angle between two faces of an octahedron that meet along an edge. [Hint: Use the description of the octahedron in coordinates we gave in class.]

(3) Compute the angle between the lines joining the center of a tetrahedron to two of its vertices. [Hint: Use the description of the tetrahedron in coordinates we gave in class.]

(4) What polygon is obtained by intersecting a cube with the plane through its center perpendicular to the line joining two opposite vertices? What polygon is obtained by intersecting a tetrahedron with the plane through its center perpendicular to the line joining the midpoints of two opposite edges?

(5) A polyhedron has a faces with 3 sides and b faces with 4 sides (and no other faces). How many vertices, edges, and faces are there? [Hint: Use Euler's formula $V - E + F = 2$.]

(6) A polyhedron has only triangular faces. There are a vertices where 5 faces meet and b vertices where 6 faces meet (and no other vertices). Find the number of vertices, edges, and faces. Show that $a = 12$. [Hint: Compute the number of faces in two different ways.]

(7) Describe geometrically a symmetry of the tetrahedron of order 4. Describe geometrically a symmetry of the cube of order 6. [Hint: Q4 may help.]

(8) Show carefully that the group G of symmetries of an equilateral triangle is isomorphic to the symmetric group S_3. Describe the correspondence explicitly. [Hint: Consider the permutations of the vertices induced by the symmetries.]

(9) Let G be the group of the symmetries of the cube. Label the vertices $1, 2, \ldots, 8$. By considering the permutations of the vertices induced by the symmetries, we get a group homomorphism $\theta \colon G \to S_8$. Explain in simple geometric terms why θ is *not* surjective.

(10) What are the groups of rotational symmetries of the following objects? [Hint: Recall the classification of finite groups of rotational symmetries we gave in class, or compute directly.]

  (a) A triangular prism.
  (b) A rectangular block with side lengths $a < b < c$.
  (c) A soccer ball (the type with pentagonal and hexagonal patches).

(11) In class we showed that every permutation can be expressed as a composition of transpositions. We also explained that a permutation can be expressed as a composition of disjoint cycles. The list $(l_1, \ldots, l_r)$ of cycle lengths is called the *cycle type* of the permutation.

  (a) Suppose f is a permutation of cycle type $(l_1, \ldots, l_r)$. What is the order of f? If we write f as a product of transpositions, how many transpositions are required?
  (b) Recall that we say f is an *even* permutation if it can be expressed as a composition of an even number of transpositions. The *alternating group* A_n is the subgroup of S_n consisting of even permutations. Show that every element of the alternating group can be written as a composition of 3-cycles. [Hint: It suffices to show that a composition of two transpositions can be expressed as a composition of 3-cycles.]
  (c) We say two permutations f, g are *conjugate* if $g = hfh^{-1}$ for some permutation h. Show that f and g are conjugate precisely when f and g have the same cycle type. [Hint: If $f = (1234)(567)$ for example, what is the permutation hfh^{-1} as a composite of disjoint cycles?]
  (d) Let f be a permutation. Let A be the *permutation matrix* associated to f, that is, the jth column of A has entry 1 in row $f(j)$

and entry 0 in the remaining rows. Show that $\det A = \operatorname{sgn}(f)$ (the *sign* of f), that is, $\det A = +1$ if f is a composite of an even number of transpositions and $\det A = -1$ if f is a composite of an odd number of transpositions. [Hint: What happens to the determinant of a matrix when we switch two rows?]

(12) In class we showed that the group of symmetries of the tetrahedron is isomorphic to S_4. What are the symmetries of the tetrahedron corresponding to the transpositions? Use the correspondence to express each type of symmetry of the tetrahedron as a product of reflections.

(13) Recall that a *motion* of $\mathbb{R}^3$ is a map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that preserves distances. A motion T may be expressed algebraically by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 3×3 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^3$ is a vector. We say T is *direct* if $\det A = 1$ and *opposite* if $\det A = -1$.

(a) What are the 6 possible types of motions of $\mathbb{R}^3$?

(b) Which types have a fixed point? What is the dimension of the locus of fixed points in each case?

(c) Which types of motion are direct motions and which are opposite motions?

(14) Suppose G is a finite group of motions of $\mathbb{R}^3$ (equivalently, G is a finite set of motions of $\mathbb{R}^3$ such that if g, h are motions in G then the composite gh is also in G). First explain why there is a point which is fixed by every motion $g \in G$. [Hint: Consider the "average" of the points in an orbit of G.] Second, show that either every motion in G is a rotation or exactly half the motions in G are rotations. (Here we are counting the identity as a rotation.) [Hint: Consider the group homomorphism $\det\colon G \to \{\pm 1\}$ given by the determinant.]

(15) In class we showed that the points $(\pm\tau, \pm 1, 0)$, $(\pm 1, 0, \pm\tau)$, $(0, \pm\tau, \pm 1)$ are the vertices of an icosahedron. Here τ is the golden ratio, $\tau = \frac{1+\sqrt{5}}{2} = 1.618\ldots$, which satisfies the equation $\tau^2 = \tau + 1$.

(a) Draw a picture showing that the 12 points are the vertices of a set of 3 rectangles with side lengths 2 and 2τ, meeting at right angles at the origin.

(b) What is the side length of the icosahedron? What is the distance between two opposite vertices?

(c) What is the side length of the dodecahedron whose vertices are the centers of the faces of the icosahedron (the *dual* polyhedron)?

(d) We explained in class that a dodecahedron contains 5 inscribed cubes. The edges of the cubes are the diagonals of the faces of the dodecahedron. One of the cubes has faces parallel to the rectangles from part (a). Find its vertices.

(16) Use the orbit-stabilizer theorem to check that the sizes of the groups of rotational symmetries of the tetrahedron, cube, and dodecahedron are 12, 24, and 60.

(17) (a) Show that there are two tetrahedra inscribed in a cube whose edges are diagonals of the faces of the cube.

(b) What is the size of the stabilizer S of one of the inscribed tetrahedra in the group of symmetries of the cube? (Here the stabilizer is the subgroup of symmetries of the cube which map the tetrahedron to itself.) [Hint: Use the orbit-stabilizer theorem.]

(c) In class we identified the group H of symmetries of the tetrahedron with S_4 and the group G of symmetries of the cube with $S_4 \times \{\pm 1\}$. Use these identifications to describe the stabilizer S of part (b) as a subgroup of $S_4 \times \{\pm 1\}$.

(18) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the motion given by reflection in the plane with equation $x+2y+3z=5$. Express T in the form $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$. [Hint: The plane through a point $\mathbf{a} \in \mathbb{R}^3$ with normal vector $\mathbf{n}$ has equation $(\mathbf{x}-\mathbf{a})\cdot\mathbf{n} = 0$. The reflection in this plane is given by the formula $T(\mathbf{x}) = \mathbf{x} - 2\left(\frac{(\mathbf{x}-\mathbf{a})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}$. (Draw a picture to see why this formula is true.)]

(19) (a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the z-axis through angle θ anticlockwise. What is the matrix A of T (that is, the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$)?

(b) Let $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the y-axis through angle ϕ anticlockwise. What is the matrix B of U?

(c) The composite $U \circ T$ is another rotation (why?). If $\theta = \pi/2$ and $\phi = \pi/2$, find the axis and angle of rotation for $U \circ T$.

(20) (a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the xy-plane. What is the matrix A of T?

(b) Let $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane with equation $x + y + z = 0$. What is the matrix B of U? [Hint: Use the formula in Q18 above].

(c) What type of motion is $U \circ T$? What is its matrix? Give a precise geometric description (for example, if it is a rotation, give the axis, angle, and direction of rotation). [Hint: It is easiest to argue geometrically (it is not necessary to use the matrix of $U \circ T$).]