Math 462 Homework 7

Paul Hacking

April 3, 2013

(1) Determine the Mobius transformation

$$f \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad f(z) = \frac{az+b}{cz+d}$$

(where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$) such that $f(i) = 0$, $f(2) = \infty$, and $f(3) = 1$.

(2) Determine the Mobius transformation f such that $f(i) = i$, $f(-i) = -i$ and $f(0) = -1$.

[Hint: Use $\{0, \infty, 1\}$ as an intermediate set of 3 points. Also, recall that a Mobius transformation $f(z) = \frac{az+b}{cz+d}$ corresponds to the 2×2 invertible complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (determined up to a scalar factor $0 \neq \lambda \in \mathbb{C}$), and composition of Mobius transformations corresponds to multiplication of matrices (this is the fastest way to compute compositions and inverses of Mobius transformations).]

(3) Show that the inverse of the Mobius transformation $f(z) = \frac{az+b}{cz+d}$ is given by $f^{-1}(w) = \frac{dw-b}{-cw+a}$ in two ways: (a) by direct calculation of the inverse, and (b) using the formula for the inverse of a 2×2 matrix.

(4) Find a Mobius transformation $f(z) = \frac{az+b}{cz+d}$ which maps the circle C with center the point $i \in \mathbb{C}$ and radius 1 to the x-axis.

[Hint: Since Mobius transformations take circles and lines to circles and lines, and a circle or line is uniquely determined by 3 points, it suffices to pick 3 points z_1, z_2, z_3 on C and 3 points w_1, w_2, w_3 on the x-axis and write down a Mobius transformation f such that $f(z_i) = w_i$

for each $i = 1, 2, 3$. (Also, picking the points carefully will make the calculation easier.)]

(5) Write the Mobius transformation $f(z) = \frac{z+i}{z-i}$ as a composition of Mobius transformations of the following 3 types:

   (1) $f_1(z) = z + b$, $b \in \mathbb{C}$ (translation by b).

   (2) $f_2(z) = az$, $a = re^{i\theta} \in \mathbb{C}$, $a \neq 0$ (scaling by r followed by counterclockwise rotation through angle θ, with center the origin).

   (3) $f_3(z) = \frac{1}{z}$ (inversion in the circle with center the origin and radius 1, followed by reflection in the x-axis).

   Using this expression describe the effect of the Mobius transformation geometrically.

   [Hint: First write down a translation f_1 such that $f_1(f(\infty)) = 0$, then $f_3(f_1(f(\infty))) = \infty$. Now it follows that $f_3(f_1(f(z))) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$ (why?), so that $f_3 \circ f_1 \circ f = g_1 \circ g_2$ for some Mobius transformations g_1 and g_2 of types (1) and (2). Finally we deduce that $f = f_1^{-1} \circ f_3^{-1} \circ g_1 \circ g_2 = f_1^{-1} \circ f_3 \circ g_1 \circ g_2$.]

(6) Compute the hyperbolic distance $d(z_1, z_2)$ between the points $z_1 = -3 + 4i$ and $z_2 = 3 + 4i$ and describe the shortest path from z_1 to z_2 geometrically.

   [Hint: First compute the circle C passing through z_1 and z_2 with center on the x-axis. Let $a, b \in \mathbb{R}$, $a < b$ be the intersection points of C with the x-axis. The Mobius transformation $f(z) = -\frac{z-a}{z-b}$ sends the upper half plane $\mathcal{H}$ to itself, is a hyperbolic isometry, and sends C to the y-axis (why?). Now use the formula $d(P, Q) = \ln(y_2/y_1)$ for the hyperbolic distance between two points P, Q on a vertical line with y-coordinates $y_1 < y_2$.]

(7) (Optional) In this question we will describe an alternative approach to the extended complex plane $\mathbb{C} \cup \{\infty\}$ which explains the relation between Mobius transformations and matrices. (It also provides a more conceptual way of dealing with the point at infinity.) We consider the set

$$\mathbb{C}^2 \setminus \{(0,0)\} = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C},\ (z_1, z_2) \neq (0,0)\}$$

with the equivalence relation

$$(z_1, z_2) \sim (w_1, w_2) \iff (z_1, z_2) = \lambda(w_1, w_2) \text{ for some } 0 \neq \lambda \in \mathbb{C}.$$

The set of equivalence classes $[(z_1, z_2)]$ is called the complex projective space of (complex) dimension 1 and denoted $\mathbb{CP}^1$.

(a) Show that there is a bijection

$$A\colon \mathbb{C} \cup \{\infty\} \to \mathbb{CP}^1, \quad A(z) = [(z, 1)] \text{ for } z \in \mathbb{C},\ A(\infty) = [(1, 0)].$$

with inverse

$$B\colon \mathbb{CP}^1 \to \mathbb{C} \cup \{\infty\}, \quad B([(z_1, z_2)]) = z_1/z_2 \text{ if } z_2 \neq 0,\ B([(z_1, z_2)]) = \infty \text{ if } z_2 = 0.$$

(b) Show that, under the bijection A, a Mobius transformation

$$f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad f(z) = \frac{az + b}{cz + d}$$

corresponds to the map

$$g\colon \mathbb{CP}^1 \to \mathbb{CP}^1$$

given by regarding a point $(z_1, z_2) \in \mathbb{C}^2$ as a column vector and multiplying by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$g\left(\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]\right) = \left[M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \left[\begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}\right].$$

That is, $g = A \circ f \circ A^{-1}$, equivalently $g \circ A = A \circ f$, or $g(A(z)) = A(f(z))$ for all $z \in \mathbb{C} \cup \{\infty\}$.