MATH 461 HW5 SOLUTIONS.

Saturday 11/2/19.

1. $\mathcal{C}_1 :$ ① $x^2+y^2=1$

$\mathcal{C}_2 :$ ② $(x-1)^2+(y-2)^2 = 2^2$, i.e. $x^2-2x+1+y^2-4y+4=4$

a. ① − ② $\quad 2x-1+4y-4=-3$

$$2x+4y=2$$

$$x+2y=1$$

$$x=1-2y$$

Substitute in ① $\quad (1-2y)^2+y^2=1$

$$4y^2-4y+1+y^2=1$$

$$5y^2-4y=0$$

$$y\cdot(5y-4)=0$$

$$y=0 \text{ or } 4/5. \underset{x=1-2y}{\Longrightarrow} (x,y)=(1,0) \text{ or } (-3/5, 4/5) \quad \square.$$

b. This is the equation of the line passing through the intersection points of the two circles.

2. $A=(0,0)$, $B=(1,2)$, $C=(-1,3)$.

Perpendicular bisector of AB: $\quad x^2+y^2=(x-1)^2+(y-2)^2 \rightsquigarrow 2x-1+4y-4=0$

① $2x+4y=5$

" " " " AC: $\quad x^2+y^2=(x+1)^2+(y-3)^2 \rightsquigarrow -2x-1+6y-9=0$

② $-2x+6y=10.$

① + ② : $\quad 10y=15$, $y=3/2$, $x \overset{①}{=} \dfrac{5-4y}{2} = -1/2$

So, circle has center $P=(-1/2, 3/2)$ & radius $|AP|=\sqrt{(-\tfrac12)^2+(\tfrac32)^2}=\sqrt{\tfrac14+\tfrac94}=\tfrac12\sqrt{10}.$

$\square.$

3.

T	$\mathrm{Fix}(T)$
identity	$\mathbb{R}^2$
translation by $(a,b) \neq (0,0)$	$\emptyset$ (empty set)
rotation about point P through angle θ ccw, $0<\theta<2\pi$	$\{P\}$
reflection in a line L	L
glide reflection (reflection in a line L followed by translation by $(a,b) \neq (0,0)$ parallel to L)	$\emptyset$

4. A glide reflection T is a composition $T = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L$, i.e.,

reflection in a line L followed by translation by a vector $\underline{v}$ parallel to L.

As noted in class, $T = \mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L = \mathrm{Refl}_L \circ \mathrm{Trans}_{\underline{v}}$ (i.e., Refl_L & $\mathrm{Trans}_{\underline{v}}$ commute)

Now
$$\begin{aligned} T^2 = T \circ T &= (\mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L) \circ (\mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L) \\ &= (\mathrm{Trans}_{\underline{v}} \circ \mathrm{Refl}_L) \circ (\mathrm{Refl}_L \circ \mathrm{Trans}_{\underline{v}}) \\ &= \mathrm{Trans}_{\underline{v}} \circ (\mathrm{Refl}_L \circ \mathrm{Refl}_L) \circ \mathrm{Trans}_{\underline{v}} \\ &= \mathrm{Trans}_{\underline{v}} \circ \mathrm{id} \circ \mathrm{Trans}_{\underline{v}} \\ &= \mathrm{Trans}_{\underline{v}} \circ \mathrm{Trans}_{\underline{v}} = \mathrm{Trans}_{2\underline{v}}, \end{aligned}$$
translation by $2\underline{v}$. $\square$.

5. If T = identity then $T^n = \mathrm{id}$ for all n.

If T is translation by a vector $\underline{v} \neq \underline{0}$, then T^n is translation by $n \cdot \underline{v}$ so $T^n \neq \mathrm{id}$ for all $n \in \mathbb{N}$.

If T is rotation about a point P through angle θ ccw, then T^n is rotation about P through angle $n \cdot \theta$ ccw, so $T^n = \mathrm{id}$ iff $n\theta = 2\pi \cdot k$ for some integer k, i.e.
$\theta = \dfrac{2\pi k}{n}$.

If T is a reflection then $T^2 = id$, so $T^n = id$ if n is even & $T^n = T \neq id$ if n is odd.

If T is a glide reflection then, using the notation of Q4, $T^2 = Trans_{2\underline{v}}$, so

$$T^n = (T^2)^k = Trans_{2k\underline{v}} = Trans_{n\underline{v}} \quad \text{if } n = 2k \text{ is even}$$

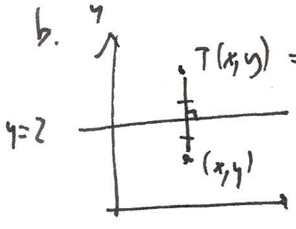
$$\& \ T^n = (T^2)^k \circ T = Trans_{2k\underline{v}} \circ T = Trans_{(2k+1)\underline{v}} \circ Refl_L \quad \text{a glide reflection if } n = 2k+1 \text{ is odd.}$$

So $T^n \neq id$ for all $n \in \mathbb{N}$.

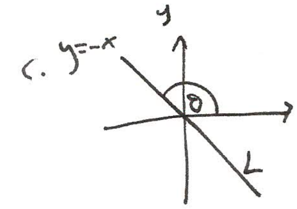
These are all the types of isometries of $\mathbb{R}^2$. □

6. a. $$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\left(\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}\right) + \begin{pmatrix}1\\2\end{pmatrix} \qquad \theta = \pi/2$$

$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x-1\\y-2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-y+2\\x-1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-y+3\\x+1\end{pmatrix}. \ \square$$

b. 

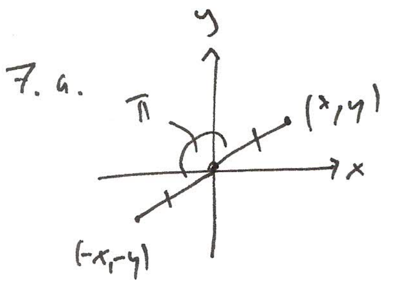
$T(x,y) = (x, y+2\cdot(2-y)) = (x, 4-y)$

c. 

$$Refl_L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-y\\-x\end{pmatrix}$$

$\theta = 3\pi/4$, $2\theta = 3\pi/2$

$$\therefore T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-y\\-x\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}-y+2\\-x-2\end{pmatrix}. \ \square$$

7. a. 

$\Rightarrow$ T = rotation about origin through angle π ccw.

b. $T(x,y) = \frac{1}{5}(4x+3y+2, 3x-4y-6)$

Solve for $Fix(T)$ & use Q3:-

Solve $T(x,y) = (x,y)$:
$$\tfrac{1}{5}(4x+3y+2) = x \quad \rightsquigarrow \quad ① \ -x+3y = -2$$
$$\tfrac{1}{5}(3x-4y-6) = y \qquad ② \ 3x-9y = 6$$

$② + 3\cdot ①$: $0 = 0$.

$\Rightarrow \mathrm{Fix}(T) = \{(x,y) \in \mathbb{R}^2 \mid -x+3y = -2\}$.

$\overset{Q3}{\Rightarrow}$ T is reflection in line L : $-x+3y=-2$

(or $y = \tfrac{1}{3}x - \tfrac{2}{3}$) $\square$.

c. $T(x,y) = \tfrac{1}{5}(3x-4y+8,\ 4x+3y+4)$

Solve $T(x,y) = (x,y)$:
$$\tfrac{1}{5}(3x-4y+8) = x \quad \rightsquigarrow \quad -2x-4y = -8 \quad \div -2$$
$$\tfrac{1}{5}(4x+3y+4) = y \qquad 4x-2y = -4 \quad \div 2$$

$\rightsquigarrow$ ① $x+2y = 4$
② $2x-y = -2$

$② - 2\cdot ①$ $-5y = -10 \Rightarrow y = 2 \underset{①}{\Rightarrow} x = 4-2y = 0$.

$\mathrm{Fix}(T) = \{(0,2)\}$.

So, by Q3, T is a rotation about $(0,2)$.

$$T\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\tfrac{1}{5}\begin{pmatrix}3&-4\\4&3\end{pmatrix}}_{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix} + \tfrac{1}{5}\begin{pmatrix}8\\4\end{pmatrix}$$

$\Rightarrow \theta = \tan^{-1}(4/3)$, ccw.

T is a rotation about $(0,2)$ thru angle $\tan^{-1}(4/3)$ ccw. $\square$.

d. $T(x,y) = (y+4,\ x+8)$.

Solve $T(x,y) = (x,y)$
$$y+4 = x \quad \rightsquigarrow \quad ① \ -x+y = -4$$
$$x+8 = y \qquad ② \ x-y = -8$$

$①+②$: $0 = -12$ #

So $\mathrm{Fix}(T) = \emptyset$.

Clearly, T is not a translation (because then $T(x,y) = (x,y) + (a,b) = (x+a, y+b)$, some $a, b \in \mathbb{R}$ $\#$)

So, by Q3, T is a glide reflection.

Now, by Q4, $T^2 = \text{Trans}_{2\underline{v}}$ where $T = \text{Trans}_{\underline{v}} \circ \text{Refl}_L$.

Compute: $T^2(x,y) = T(y+4, x+8) = ((x+8)+4, (y+4)+8) = (x+12, y+12)$

$\Rightarrow 2\underline{v} = (12, 12)$, $\underline{v} = (6,6)$.

$\Rightarrow \text{Refl}_L(x,y) = T(x,y) - (6,6) = (y-2, x+2)$

Fix (Refl_L) : $\begin{array}{l} x = y-2 \\ y = x+2 \end{array}$ $\left.\begin{array}{l} x-y = -2 \\ -x+y = 2 \end{array}\right\}$ $x - y = -2$, or $y = x+2$

So, L is the line $y = x+2$.

T is the glide reflection given by reflection in the line $L : y = x+2$ followed by translation by $\underline{v} = (6,6)$ parallel to L. $\square$.