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Threefolds and deformations of surface singularities

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§ 1. Introduction

The central theme of this article is the study of deformations of surface singularities using recent advances in three dimensional geometry. The basic idea is the following. Let X_0 be a surface singularity and consider a one parameter deformation $\{X_0: t \in \Delta\}$. Then the total space $X = \bigcup X_t$ is a three dimensional object. One can attempt to use the geometry of X to get information about the surface X_t.

In general X is very singular and so one can try to study it via a suitable resolution of singularities $f: X' \to X$. The existence of a resolution was established by Zariski; the problem is that there are too many of them, none particularly simple.

Mori and Reid discovered that the best one can hope for is a partial resolution $f: X' \to X$ where X' possesses certain mild singularities but otherwise is a good analog of the minimal resolution of surface singularities.

The search for such a resolution is known as Mori's program (see e.g. [Ko 3, KMM]). After substantial contributions by several mathematicians (Benveniste, Kawamata, Kollár, Mori, Reid, Shokurov, Viehweg) this was recently completed by Mori [Mo 3].

A special case, which is nonetheless sufficient for the applications presented here, was settled by several persons. A proof was first announced by Tsunoda [TsM], later followed by Shokurov [Sh], Mori [Mo 2] and Kawamata [Kaw 2]. A precise formulation of the result we need will be provided at the end of the introduction.

In certain situations X_0 will impose very strong restrictions on X' and one can use this to obtain information about X and X_t for $t \neq 0$.

The first application is in chapter two. Teissier [Te] posed the following problem. Let $\{X_s: s \in S\}$ be a flat family of surfaces parameterized by the connected space S. Let $\bar{X}_s$ be the minimal resolution of X_s. In general $\{\bar{X}_s: s \in S\}$ is not a flat family of surfaces, and it is of interest to find necessary and sufficient conditions for this to hold.

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If $\{\bar{X}_s: s\in S\}$ is a flat family, then $K^2_{\bar{X}_s}$ is locally constant on S. One can hope that the converse is also true (at least after a finite and surjective base change). This was proved by Laufer [La, 5.7] under the additional assumption that the X_s have isolated Gorenstein singularities. The main result of chapter two (2.10) generalizes this to the case where the X_s have arbitrary isolated singularities. In the presence of nonisolated singularities the converse is usually false.

Chapter three provides a new approach to the study of the deformation space of a quotient singularity. The main result (Theorem 3.9) relates the irreducible components of the deformation space to certain partial resolutions of the singularity. This gives an algorithm to compute the number of components of the deformation space and the dimension of the components in terms of the dual graph of the minimal resolution. It is our belief that this method will lead eventually to the understanding of all small deformations of a quotient singularity, but so far our attempts have been frustrated.

The next two chapters contain results pertaining to the problem of compactifying the moduli of surfaces. This involves the study of certain singular surfaces that appear as limits of smooth ones. A good class of such singular surfaces is suggested by the above-mentioned results of three dimensional geometry. The possible singularities of such surfaces are introduced and classified in chapter four. This generalizes earlier results of Kawamata [Kaw1, Kaw2]. In chapter five the deformations of these singularities are analyzed. The main conclusion is that the proposed compactification of the moduli of surfaces is a *separated* algebraic space.

The methods of the last two chapters are largely unrelated to the rest of the article. In chapter six some missing points of the classification of three dimensional terminal singularities are settled. The proofs are fairly elementary and independent of the theory of minimal models.

The results of the last chapter imply that any small deformation of a cyclic quotient singularity is again a cyclic quotient. This was conjectured by Riemenschneider.

Now we turn to give a precise formulation of the result from three dimensional geometry that we will repeatedly use.

Minimal models for semi-stable reductions

First we have to define the class of singularities that we will allow on the partial resolution.

Let Y be a normal variety and let $g: Y'\to Y$ be a resolution of singularities of Y. Let K_Y be a canonical divisor on Y and assume that mK_Y is Cartier for some $m>0$. Then we can write $mK'_Y \sim g^*(mK_Y)+\sum a_i E_i$, where the E_i are g-exceptional divisors. Y is said to have only terminal (resp. canonical) singularities if all the a_i are positive (resp. nonnegative). The complete list of three dimensional terminal singularities is reproduced in 6.3–4.

Let $f: X\to T$ be a map of an algebraic (or complex analytic) threefold on to a smooth curve T. We say that f admits a semi-stable resolution if there exists a bimeromorphic and projective morphism $g: X' \to X$ such that X' is smooth and $(g \circ f)^{-1}(t)$ is reduced and has at worst normal crossing singularities. By a result of Knudsen and Mumford [KKMS], if $h: Y \to S$ is arbitrary then there is a finite surjective map $p: T \to S$ such that $f = h \times p: Y \times_S T \to T$ admits a semi-stable resolution.

Now one can formulate the following result which was proved by Tsunoda, Shokurov, Mori and Kawamata.

Theorem. *Let $f: X \to T$ be a morphism of an algebraic (complex analytic) three dimensional space onto a smooth curve T. Assume that f admits a semi-stable resolution. Then there is a projective birational (bimeromorphic) morphism g: $\bar{X} \to X$ such that $\bar{X}$ has only terminal singularities and such that for every compact curve $C \subset \bar{X}$ such that $g(C) = \text{point}$ we have $C \cdot K_{\bar{X}} \geqq 0$. By [Ben, Kaw3] one can also find a $\tilde{g}$: $\tilde{X} \to X$ such that $\tilde{X}$ has canonical singularities and such that $C \cdot K_{\tilde{X}} > 0$ for every compact curve $C \subset \tilde{X}$ such that $\tilde{g}(C) = \text{point}$. Such an $\tilde{X}$ is unique, and is called the relative canonical model of X.*

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§ 2. Simultaneous resolution of surface singularities

It is well understood how to resolve the singularities of an algebraic surface. For families of surfaces however, the situation is very complicated. One could simply desingularize the total space of a family, but usually the resulting family will not even be flat. Frequently this is the best one can hope for. Therefore, it is of interest to understand the cases when some particularly "nice" desingularization is possible. There are several notions of "nice" (cf [Te]); we recall some of them here.

Definition 2.1. Let $f: X \to Y$ be a flat family of reduced algebraic surfaces. Let $g: \bar{X} \to X$ be a projective morphism such that $f \circ g: \bar{X} \to Y$ is flat too. The fibre of f (resp. $f \circ g$) above $y \in Y$ will be denoted by X_y (resp. $\bar{X}_y$). We will always assume that Y is reduced.

(i) g is called a simultaneous DuVal (or rational double point) resolution of f if each $\bar{X}_y \to X_y$ is the minimal resolution with DuVal singularities.

(ii) g is called a very weak simultaneous resolution if each $\bar{X}_y \to X_y$ is the minimal resolution of X_y.

(iii) If $p: Y \to X$ is a section of f, then f is called a weak simultaneous resolution along p if it is a very weak simultaneous resolution near $f(Y)$ and the map $g: (g^{-1}(p(Y))) \to p(Y)$ is simple (i.e. locally in the Euclidean topology on $g^{-1}(p(Y))$ it is the projection of a direct product.)

(iv) f is said to admit a simultaneous DuVal resolution (respectively a very weak or weak simultaneous resolution) if there exists a $g: \bar{X} \to X$ satisfying (i) (resp. (ii) or (iii)).

(v) It is important to note that very weak simultaneous resolutions are not unique in general, despite the fact that the minimal resolution of the individual surfaces are unique.

One advantage of the DuVal resolution is that it is unique. If $g: \bar{X} \to X$ is a simultaneous DuVal resolution then $K_{\bar{X}/X}$ is g-ample. This implies uniqueness essentially by a result of Matsusaka-Mumford [MM].

One of the early results relating these notions is the following.

Theorem 2.2. (Brieskorn [Br2]) *Assume that $f: X \to Y$ as above admits a simultaneous DuVal resolution. Then there exists a finite and surjective map $f: Y' \to Y$ such that $f': X \times_Y Y' \to Y'$ admits a very weak simultaneous resolution.*

Very weak simultaneous resolution without a base change seems a very delicate property; therefore we will not consider it.

Definition 2.3. (i) Let Z be an algebraic surface with isolated singularities and $h: Z' \to Z$ be a resolution of singularities, $E_i \subset Z'$ the exceptional curves. There exists a formal linear combination $\sum a_i E_i$ such that $E_j \cdot \sum a_i E_i = E_j \cdot K_{Z'}$ for every j. The rational number $\sum a_i E_i \cdot \sum a_i E_i$ will be called the self intersection number of the relative canonical class. If Z' is the minimal resolution of Z, then it will be denoted by $K^2_{\mathrm{rel},Z}$.

(ii) If Z is projective and $\bar{Z}$ is the minimal resolution, then we shall write $\bar{K}^2_Z$ instead of $K^2_{\bar{Z}}$.

(iii) Mumford [M] defined the intersection number of any two Weil divisors on a normal projective surface Z. In particular K^2_Z is defined for such surfaces.

Note the relationship $\bar{K}^2_Z = K^2_Z + K^2_{\mathrm{rel},Z}$.

(iv) If X_y, $y \in Y$, is a family of surfaces we shall write K^2_y instead of $K^2_{X_y}$ (similarly $\bar{K}^2_y$, $K^2_{\mathrm{rel},y}$) if no confusion is likely.

The following theorems were the strongest known results concerning simultaneous resolution of surface singularities.

Theorem 2.4. (i) (Laufer, [La, 5.7]) *Let $f: X \to Y$ be a flat family of normal, Gorenstein surface singularities. Then $K^2_{\mathrm{rel},y}$ is lower semi-continuous. $K^2_{\mathrm{rel},y}$ is locally constant $\Leftrightarrow f$ admits a very weak simultaneous resolution after a finite and surjective base change.*

(ii) (Vaquié, [V, 2.4]) *If f is a flat family of normal surface singularities, then $K^2_{\mathrm{rel},y}$ is lower semi-continuous.*

(iii) (Vaquié, [V, 2.5]) *If $f: X \to D$ is a flat of rational surface singularities, $K^2_{\mathrm{rel},y}$ is constant and K_X is $\mathbb{Q}$-Cartier, then f admits a simultaneous DuVal resolution.*

Theorem 2.5. (Laufer, [La, 6.4]) *Let $f: X \to Y$ be a family of normal, Gorenstein surfaces over $\mathbb{C}$, and let $p: Y \to X$ be a section. Assume that Y is reduced and connected. Then f admits a weak simultaneous resolution along p if and only if the following condition is satisfied:*

For every $y \in Y$ there exists a Euclidean neighbourhood $V_y \subset X_y$ of $p(y)$ such that the pairs $(V_y, p(y))$ are all homeomorphic to each other. (we assume that the homeomorphism preserves the natural orientation).

Definition 2.6. If this condition is satisfied we shall say that the germs of X_y along p are pairwise homeomorphic.

Remark 2.7. The previous theorems are valid for complex analytic maps as well. In fact, one can claim this to be their natural setting since the assumptions are local in the Euclidean topology.

Example 2.8. The following example was noticed by several people (see e.g. [Pi1]) and it shows that one cannot omit the Gorenstein assumption in 2.4.

Let $V \subset \mathbb{P}^5$ be the Veronese surface and $C_V \subset \mathbb{P}_6$ the cone over it. Let $F \subset \mathbb{P}^5$ be the scroll $\mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O}(2)+\mathcal{O}(2))$ embedded with the relative $\mathcal{O}(1)$, and $C_F \subset \mathbb{P}^6$ be the cone over it. Finally, let $C_4 \subset \mathbb{P}^5$ be the cone over the rational normal curve of degree four. It we cut C_V (resp. C_F) with a pencil of hyperplanes, then the hyperplane through the vertex gives a copy of C_4; the general member will be isomorphic to V (resp. F). Obviously $K^2_{\mathrm{rel},V} = K^2_{\mathrm{rel},F} = 0$. The family obtained from C_F admits a simultaneous DuVal resolution: blow up C_l, the cone over a line on F. The family obtained from C_V does not admit a simultaneous DuVal resolution: the singularity at the vertex is $\mathbb{C}^3/(x \sim -x)$; hence for any birational modification the exceptional locus must be of pure dimension two. Therefore any modification of C_V would introduce a new component to the central fibre.

Note however that $\bar{K}^2_{C_4} = \bar{K}^2_F = 8$, whereas $\bar{K}^2_V = 9$. The global invariant detects the existence of simultaneous resolution. This is the case in general (see 2.10).

Definition 2.9. Recall that a scheme Y is called seminormal [Tr] if the following holds:

Every map $p: Y' \to Y$ which is a homeomorphism (in the Zariski topology) is in fact an isomorphism.

Thus a normal scheme is seminormal, and so are varieties with normal crossings only. For every variety Z there is an $s: Z' \to Z$ such that Z' is seminormal and s is a homeomorphism; in particular s is finite. This Z' is unique and is called the seminormalization of Z (see [Tr]).

Theorem 2.10. *Let $f: X \to Y$ be a flat family of projective surfaces with isolated singularities only such that Y is seminormal. Then*

(i) $\bar{K}^2_y$ is lower semi-continuous;

(ii) $\bar{K}^2_y$ is locally constant if and only if f admits a simultaneous DuVal resolution;

(iii) If $p: Y \to X$ is a section and Y is connected; then X admits a weak simultaneous resolution along p if and only if the germs of X_y along p are pairwise homeomorphic.

Remark 2.11. Karras informed us that he also proved part (iii) of the theorem. He considers only the case of normal singularities, but he is able to treat the analytic case as well. His proof is a generalization of Laufer's ideas [Kar2].

Remark 2.12. There is some hope that the present methods will work in the analytic case as well. (See 2.25).

Proposition 2.13. *Let $g\colon \tilde{U} \to V$ be an arbitrary resolution of the projective surface V, and let $\bar{U}$ be the minimal resolution. Assume that $K_{\tilde{U}} = H - E$, where H is g-nef, E is effective and supported on the g-exceptional locus. Then $K_{\bar{U}}^2 \geqq K_{\tilde{U}}^2 + E^2$, and equality holds iff $E = 0$.*

Proof. If $K_{\tilde{U}}$ is g-nef, then $\tilde{U} = \bar{U}$. By [M] we have $E^2 \leqq 0$, with $=$ only if $E = 0$. Hence in this case we are done.

Otherwise, let $C \subset \tilde{U}$ be an exceptional curve such that $C \cdot K_{\tilde{U}} < 0$ and let $h\colon \tilde{U} \to U'$ be the contraction. $H = h^* H' + aC$, $E = h^* E' + bC$ for some H', E', a and b. $0 \leqq H \cdot C = aC^2$ implies $a \leqq 0$. Since $0 \geqq K_{\tilde{U}} \cdot C = (a-b)\, C^2$ we have $a - b \geqq 0$. Hence $-b \geqq a - b \geqq 0$.

$K_{U'} = H' - E'$, and E' is clearly effective. If $F \neq C$ is an exceptional curve, then

$$h(F) \cdot H' = F \cdot h^* H' = F \cdot H + (-a)\, F \cdot C \geqq 0$$

and therefore H' is g'-nef for $g'\colon U' \to V$. By induction $K_{U'}^2 \geqq K_{\bar{U}}^2 + E'^2$ with equality iff $E' = 0$. On the other hand,

$$\begin{aligned} K_{\tilde{U}}^2 - E^2 &= (h^* K_{U'} + (a-b)\, C)^2 - (h^* E' + bC)^2 \\ &= K_{U'}^2 - E'^2 + [b^2 - (b-a)^2](-C^2) \\ &\geqq K_{\bar{U}}^2 + [b^2 - (b-a)^2](-C^2). \end{aligned}$$

As we saw, $-b \geqq a - b \geqq 0$. Thus the last term is positive and this proves the inequality. If equality holds, then $E' = 0$, and $E = bC$. Since $b \leqq 0$, this can be effective only for $b = 0$.

Corollary 2.14. *Let $g\colon U \to V$ be a regular birational map between projective surfaces and assume that U has only quotient singularities. Assume that $K_U \equiv H - E$, where H is g-nef, E is effective and supported on the g-exceptional locus. Finally, let $\bar{U}$ be the minimal resolution of V. Then $K_{\bar{U}}^2 \geqq K_{\tilde{U}}^2 + E^2$ with equality iff $E = 0$ and U has only DuVal singularities.*

Proof. Let $j\colon \tilde{U} \to U$ be the minimal resolution of U. Then $K_{\tilde{U}} = j^* K_U - E'$ where E' is effective, j-exceptional and $E' = 0$ iff U has only DuVal singularities (cf. 4.12). Let $\tilde{H} = j^* H$, $\tilde{E} = E' + j^* E$. We can apply 2.13 for $g \circ j\colon \tilde{U} \to V$ and $K_{\tilde{U}} = \tilde{H} - \tilde{E}$. We obtain that $K_{\tilde{U}}^2 + E'^2 = K_{\bar{U}}^2 \geqq K_{\tilde{U}}^2 + E'^2 + E^2$, and this gives the inequality. Equality holds if $E = 0$ and $E' = 0$, hence U has only DuVal singularities.

Now we are ready to prove a seemingly very special case of 2.10. The rest, however, turns out to be a formal consequence of it.

Theorem 2.15. *Let $f\colon X \to D$ be a flat family of complete surfaces with isolated singularities over the unit disc D. Assume that f admits a semi-stable resolution of singularities. Then*

(i) $\bar{K}_t^2$ is lower semi-continuous;

(ii) $\bar{K}_t^2$ is constant iff f admits a simultaneous DuVal resolution.

Proof. Let $\tilde{g}\colon \tilde{X} \to X$ be a relative minimal model (cf. chapter 1.). Let $\tilde{X}_t \to X_t$ be the general fibre, $X'_0 + \sum E_i$ the central fibre of $\tilde{X}$, and $n\colon \tilde{X}_0 \to X'_0$ the normalization. Let $F \subset \tilde{X}_0$ be the conductor of n. Thus

$$K_{X'_0} = K_{\tilde{X}} + X'_0 \,|\, X'_0 = K_{\tilde{X}} - \textstyle\sum E_i \,|\, X'_0.$$

If we set $E=F+n^*\sum E_i$, then $K_{\tilde{X}_0}=n^*K_{\bar{X}}-E$. Thus

$$K_{\bar{X}}^2\cdot X'_0=n^*K_{\bar{X}}^2=K_{\tilde{X}_0}^2+2E\cdot n^*K_{\bar{X}}-E^2.$$

Substituting this into

$$\bar{K}_t^2=K_{\bar{X}}^2\cdot X_t=K_{\bar{X}}^2\cdot X'_0+K_{\bar{X}}^2\cdot\sum E_i,$$

we get

$$\bar{K}_t^2=K_{\tilde{X}_0}^2-E^2+2E\cdot n^*K_{\bar{X}}+K_{\bar{X}}^2\cdot\sum E_i.$$

Since E is effective and exceptional with respect to $\tilde{X}_0\to X_0$, and $n^*K_{\bar{X}}$ is relatively nef the last two terms are non-negative.

This gives $\bar{K}_t^2\geqq K_{\tilde{X}_0}^2-E^2$. Using 5.1 and taking $H=n^*K_{\bar{X}}$ the conditions of 2.14 are satisfied for $(\tilde{X}_0\to X_0,\ K_{\tilde{X}_0}=H-E)$. Therefore, if $\bar{X}_0$ is the minimal desingularization of X_0, then $K_{\tilde{X}_0}^2\geqq\bar{K}_0^2+E^2$. These two inequalities give that $\bar{K}_t^2\geqq\bar{K}_0^2$, which is (i). If we have equality, then $E=0$. Hence $\sum E_i=0$ and X'_0 is normal and has only DuVal singularities.

This is nearly what we want. Let h: $\tilde{X}\to\bar{X}$ be the relative canonical model, g: $\bar{X}\to X$. This is the required simultaneous DuVal resolution. To see this, first note that h is an isomorphism in codimension one since the central fibre of $\tilde{X}$ is irreducible. Therefore, $K_{\tilde{X}}=h^*K_{\bar{X}}$. This in turn gives that $K_{\tilde{X}_0}=h^*K_{\bar{X}_0}$; thus $\bar{X}_0$ has DuVal singularities only. Since $K_{\bar{X}}$ is g-ample, each fibre of $\bar{X}\to D$ is the minimal DuVal resolution of the corresponding fibre of $X\to D$. This proves one direction of (ii), and the other one is trivial.

2.16 *Proof of* 2.10 *(i).* Y can be written as the union of locally closed subsets Y_i such that f admits a simultaneous DuVal resolution over each Y_i. This implies that $\bar{K}_y^2$ is a constructible function on Y. By 2.15 it is lower semi-continuous along curves, and therefore it is lower semi-continuous.

2.17 *Proof of* 2.10 *(ii).* If there is a simultaneous DuVal resolution, then $\bar{K}_y^2$ is clearly locally constant. The proof of the converse is intuitively clear. We take the minimal DuVal resolutions and by 2.15 these glue together along any curve. This should imply that they all glue together. The actual proof is unfortunately rather cumbersome.

Pick a relatively ample divisor H on X/Y, and let H_y denote its restriction to X_y. Let r_y: $\bar{X}_y\to X_y$ be the minimal DuVal resolution. There are n, $m>0$ such that $n(K_y+mr_y^*H_y)=\bar{H}_y$ is very ample for every y and $H^0(\bar{X}_y,H_y)=\chi(\bar{X}_y,\bar{H}_y)$. This latter is a polynomial in $\bar{K}_y^2$, $\bar{K}_y\cdot r^*H=K_y\cdot H$ and $\chi(\mathcal{O}_{\bar{X}_y})$. The first of these is assumed to be constant, the second one is automatically constant, and the third one is constant along any curve by 2.15, hence constant. Therefore $h^0(\bar{X}_y,\bar{H}_y)$ is independent of $y\in Y$. Let $\mathbb{P}$ be the projective space of dimension $h^0(\bar{X}_y,\bar{H}_y)-1$.

In $\mathbb{P}\times X$ let the coordinate projections be p and q. We consider subchemes $Z\subset\mathbb{P}\times X$ satisfying the following conditions.

(i) Z has only DuVal singularities
(ii) q: $Z\to X$ is birational onto a fibre of f;
(iii) K_Z is relatively q-ample;

(iv) $p: Z \to \mathbb{P}$ is a non-degenerate embedding and

$$p^* \mathcal{O}(1) = \mathcal{O}(n(K_Z + mq^* H)).$$

(i.e. Z is the graph of an embedding of the minimal DuVal resolution of a fibre X_y given by $\bar{H}_y$.)

These subschemes are parametrized by some subscheme R of the relative Hilbert scheme of $\mathbb{P} \times X/Y$. Let u: $U \to R$ be the universal family. We have a natural map c: $R \to Y$ given by $r \to f(q(u^{-1}(r)))$. R parametrizes pairs (minimal resolution of some X_y; an embedding of it given by $\bar{H}_y$). The different embeddings differ by an element of Aut $\mathbb{P}$, so the natural action of Aut $\mathbb{P}$ on R is transitive on the fibers of c. Aut $\mathbb{P}$ clearly operates without fixed points. Hence the universal family U descends to $\bar{Y} = R/\mathrm{Aut}\,\mathbb{P}$ (cf. [Po, Lecture 3]) and we get a map $\bar{u}$: $\bar{X} = U/\mathrm{Aut}\,\mathbb{P} \to \bar{Y}$ and a natural map $\bar{c}$: $\bar{Y} \to Y$ which is 1:1 on closed points. If $\bar{c}$ is an isomorphism, then $\bar{X} \to \bar{Y} = Y$ is the sought after simultaneous DuVal resolution.

We claim that $\bar{c}$ is finite. By the valuative criterion it is sufficient to check this along curves. 2.15 provides a simultaneous resolution along any smooth curve. This in turn can be embedded into $\mathbb{P} \times X$, and hence we get a lifting of our curve to R which descends to $\bar{Y}$. Therefore c: $\bar{Y} \to Y$ is finite. It is 1:1 on closed points, and hence it is an isomorphism since Y is assumed to be seminormal. This proves 2.10 (ii).

Remark 2.18. The assumption in 2.10 that Y be seminormal is somewhat unnatural. One certainly has to assume that Y is reduced, since the assumption about K^2 makes sense for closed points only and we have no control over the infinitesimal part. If Y is reduced, then the previous proof shows that c: $\bar{Y} \to Y$ is finite and 1:1 on closed points. Therefore, it is an isomorphism iff it is an isomorphism on the tangent spaces. This is a question about the infinitesimal deformation of a surface and of its minimal resolution. This was studied by Wahl [Wa 1], and the situation seems quite complicated. It is not clear whether 2.10 (ii) is true for arbitrary reduced Y or not.

Before we turn to the proof of 2.10 (iii), some technical results are needed.

Lemma 2.19. *Let Z be a reduced complete surface, Gorenstein in codimension one. Let $S \subset Z$ be a finite set of normal points of Z. Then there exists a constant $c(Z)$ such that for every Weil divisor D which is Cartier outside S, we have*

$$|\chi(Z, \mathcal{O}(D)) - \tfrac{1}{2} D(D - K_Z)| < c(Z).$$

Proof. Let f: $Z' \to Z$ be a resolution of the singularities in S. As in [M] one can define the pull-back $f^* D$, which is a divisor with rational coefficients. Let $[f^* D]$ be the divisor obtained by rounding up the coefficients. $f_* \mathcal{O}([f^* D]) = \mathcal{O}(D)$ by [Sak], and one can easily prove that length $R^1 f_* \mathcal{O}([f^* D])$ is bounded by a constant depending on f only.

$|[f^* D] \cdot ([f^* D] - K_{Z'}) - D(D - K_Z)| <$ (some constant depending on f) and

$$\chi(Z', \mathcal{O}([f^* D])) - \tfrac{1}{2}[f^* D] \cdot ([f^* D] - K_{Z'}) = \chi(\mathcal{O}_{Z'}).$$

Putting these together, we get the required estimate.

Lemma 2.20. *Let $f: X \to D$ be a family of reduced surfaces which are Gorenstein except at finitely many normal points. Then $K^2_{\bar{X}_t}$ is upper semi-continuous.*

Proof. Consider the sheaf $\mathcal{O}(mK_{X/D})$. For a generic point t we have $\mathcal{O}(mK_{X/D}) \otimes \mathcal{O}_{X_t} \cong \mathcal{O}(mK_{X_t})$, and for 0 we still have an injection $\mathcal{O}(mK_{X/D}) \otimes \mathcal{O}_{X_0} \to \mathcal{O}(mK_{X_0})$ which is an isomorphism outside the non-Gorenstein locus. Therefore,

$$\begin{aligned}\chi(X_0, \mathcal{O}(mK_{X_0})) &\geqq \chi(X_0, \mathcal{O}(mK_{X/D}) \otimes \mathcal{O}_{X_0}) \\ &= \chi(X_t, \mathcal{O}(mK_{X/D}) \otimes \mathcal{O}_{X_t}) = \chi(X_t, \mathcal{O}(mK_{X_t})).\end{aligned}$$

Applying 2.19 for $D = K_Z$ gives that the l.h.s. is asymptotic to $\frac{1}{2}m^2 K^2_{\bar{X}_0}$, the r.h.s. to $\frac{1}{2}m^2 K^2_{\bar{X}_t}$. Dividing by $\frac{1}{2}m^2$ gives the required inequality.

We shall also need the following result.

Theorem 2.21. (Neumann, [Ne]) *Let (p_i, V_i) be isolated surface singularities, f_i: $\bar{V}_i \to V_i$ be the minimal resolutions, $E_i \subset \bar{V}_i$ the exceptional sets. Assume that the pairs (p_i, V_i) are homeomorphic (orientation preserved). Then the pairs $(E_i, \bar{V}_i)$ are also homeomorphic. In particular K^2_{rel, V_i} is independent of i.*

Now we can prove the following special case of 2.10 (iii). As earlier, the general case will follow easily.

Theorem 2.22. *Let $f: X \to D$ be a flat family of surfaces with isolated singularities. Let p: $D \to X$ be a section. Assume that the germs of X_t along p are pairwise homeomorphic. Then, possibly after a base change, there exists a simultaneous DuVal resolution along p.*

Proof. Let $S \subset X$ be the union of those irreducible components of Sing X that contain $p(0)$. Shrinking D, making a base change and applying semi-stable reduction for Sing $X - S$, we can assume that $S = \mathrm{Sing}\, X$ (for simplicity the new f, X, D are denoted by the same symbols). Let $\bar{X}_t \to X_t$ be the minimal resolution of the singularities $S \cap X_t$. For $t \neq 0$ these are all the singularities, but X_0 has other singularities, namely normal crossing points. Although X_0 is reducible, the arguments of 2.15 apply verbatim and we get that $\bar{K}^2_0 \leqq \bar{K}^2_t$.

By 2.3 (iii) $\bar{K}^2_0 = K^2_0 + K^2_{\mathrm{rel},0}$ and $\bar{K}^2_t = K^2_t + K^2_{\mathrm{rel},t}$. $K^2_{\mathrm{rel},t}$ is a sum over all singularities of X_t. One of them is $p(t)$, and $K^2_{\mathrm{rel},p(t)} = K^2_{\mathrm{rel},p(0)} = K^2_{\mathrm{rel},0}$ by 2.21. Since for any singularity Z we have $K^2_{\mathrm{rel},Z} \leqq 0$, this yields that $K^2_{\mathrm{rel},t} \leqq K^2_{\mathrm{rel},0}$. By 2.20 $K^2_t \leqq K^2_0$ and hence $\bar{K}^2_t \leqq \bar{K}^2_0$. These imply that $\bar{K}^2_t = \bar{K}^2_0$. Thus again by 2.15 there is a simultaneous DuVal resolution. As in [La, 6.2] this yields that in fact $S = p(D)$ and the exceptional divisors of the DuVal resolutions are homeomorphic.

Now we can blow down the part we blew up at the very beginning, and this gives a simultaneous DuVal resolution along p.

2.23. *Proof of* 2.10 *(iii)*. We proceed as in 2.17. For each $y \in Y$ let r_y: $\bar{X}_y \to X_y$ be the DuVal resolution of the singularity $p(y)$. If $E_{i,y}$ are the exceptional curves on $\bar{X}_y$, then for some $a_i \in \mathbb{Q}$ we have $E_{j,y} \cdot \sum a_i E_{i,y} = E_{j,y} \cdot \bar{K}_y$ for every j. By 2.20 the a_i are independent of y. Let $K'_y = \sum a_i E_{i,y}$. Then we define $\bar{H}_y = n(K'_y + m r^*_y H_y)$ to be the polarizing divisor.

Continuing as in 2.17, in $\mathbb{P} \times X$ we consider subschemes Z with the following properties.

(i) $q\colon Z \to X$ is birational onto a fibre of f and is an isomorphism off $p(y)$;

(ii) above $p(y)$ it has only DuVal singularities;

(iii) the exceptional divisor is homeomorphic to the above $\sum E_{i,y}$ and $\sum a_i E_{i,y}$ is q-ample;

(iv) $p\colon Z \to \mathbb{P}$ is a non-degenerate embedding and $p^* \mathcal{O}(1) = \mathcal{O}(n(\sum a_i E_i + m q^* H))$.

As there we obtain that $f\colon X \to Y$ admits a simultaneous DuVal resolution along p.

As was already remarked by Laufer [La, 6.2], this can be blown up to a weak simultaneous resolution. This completes the proof of 2.10 (iii).

Remark 2.24. (i) Working in a similar manner, but using certain ideas of Laufer [La] and Wahl [Wa 1], one can see that 2.9 (iii) is true if Y is only assumed to be reduced, and the fibres are all normal.

(ii) As was pointed out by Laufer [La] non-normal isolated singularities do frequently occur in families with weak simultaneous resolution.

(iii) If (z, Z) is an isolated surface singularity, $\bar{Z}$ the normalization, and $r\colon Z' \to Z$ a resolution, then both $R^1 r_* \mathcal{O}'_Z$ and $\mathcal{O}_{\bar{Z}}/\mathcal{O}_Z$ are of finite length. Let h_z denote the difference of these lengths. One can prove that in a flat family $K^2_{\mathrm{rel},z}$ constant implies that h_z is constant.

Remark 2.25. As was already pointed out in 2.8, the straightforward local version of 2.10 is false. Wahl pointed out to us that in certain cases one can define a more local version of K^2. The idea is as follows [LW].

Let $(M, \partial M)$ be a smooth complex surface with boundary. If $a_i \in H^2(M)$ such that $a_i | \partial M = 0$ in $H^2(\partial M)$, then one can define $a_1 \cap a_2 \in H^4(M, \partial M) \cong \mathbb{Q}$ as follows. Lift a_1 to $H^2(M, \partial M)$ and then cap it with a_2. The resulting class is independent of the lifting.

Now let $f\colon X \to D$ be a flat deformation of the isolated surface singularity $(0, X_0)$. Let $(B, \partial B)$ be a small ball around 0, and let $(X'_t, \partial X_t) = (X_t \cap B, X_t \cap \partial B)$. For t small X'_t has only isolated singularities and is non-singular along ∂X_t. Let $(M_t, \partial M_t) \to (X'_t, \partial X_t)$ be the minimal resolution of the singularities. Since im $[H^2(M_0, \mathbb{Q}) \to H^2(\partial M, \mathbb{Q})] = 0$, $c_1(K_{M_0}) | \partial M_0 = 0$ in $H^2(\partial M_0)$; hence by continuity this holds for every t. This way one can define $K^2_{M_0}$ for every nearby t.

As in 2.15 one can prove that $K^2_{M_t}$ is lower semicontinuous, and is constant iff f admits a simultaneous DaVal resolution.

It is quite likely that the rest of 2.10 is true in this localized version, but the proofs given here rely heavily on global techniques. Substantial changes seem to be required.

There is an alternative global approach to 2.15 via the follow result:

Proposition 2.26. *Let $f\colon X \to D$ be a flat family of reduced surfaces. Assume that $m_0 K_X$ is Cartier for some $m_0 > 0$ and K_X is f-ample. Let $X_0 = \cup E_i$ be the central fiber. Let $k^2(Z) = \lim (2/m^2)\, h^0(Z', \omega_{Z'}^m)$ for any surface Z with desingularization Z'. Then*

$$\sum k^2(E_i) \leqq k^2(X_t) \quad \text{for} \quad t \neq 0.$$

Equality holds iff X_0 is irreducible and has DuVal singularities only.

Proof. First note that if Z is a smooth minimal surface, then $k^2(Z)=K_Z^2$ if Z is of general type and $k^2(Z)=0$ otherwise.

If m is sufficiently large and divisible by m_0, then

$$h^0(X_0, \mathcal{O}(mK_{X_0}))=h^0(X_t, \mathcal{O}(mK_{X_t}))$$

since f is flat and K_X is f-ample. If p: $E_i' \to X_0$ is the natural map then one has a natural map (called trace map)

$$\sum p_* \mathcal{O}(K_{E_i'}) \to \mathcal{O}(K_{X_0})$$

This gives ups maps

$$t_m: \sum p_* \mathcal{O}(mK_{E_i'}) \to \mathcal{O}(mK_{X_0}), \quad \text{and}$$

every t_m is an injection. If m is divisible by m_0, then $\mathcal{O}(mK_{X_0})$ is invertible; hence one can write the image of t_m as $I_m \cdot \mathcal{O}(mK_{X_0})$ for an ideal sheaf I_m. We have

$$\begin{aligned}\sum h^0(E_1', \mathcal{O}(mK_{E_i'}))&=h^0(X_0, I_m \cdot \mathcal{O}(mK_{X_0}))\\ &\leqq h^0(X_0, \mathcal{O}(mK_{X_0})).\end{aligned}$$

This gives the required inequality.

To analyze the case when equality holds, note that since the canonical ring of a surface is finitely generated we get that for some m_1, if $m=km_1$ then

$$h^0(X_0, I_m \cdot \mathcal{O}(mK_{X_0}))=h^0(X_0, I_{m_1}^k \cdot \mathcal{O}(mK_{X_0})).$$

Now we use the following easy

Lemma 2.27. *Let $Z \subset \mathbb{P}^N$ be a projective surface and $I \subset \mathcal{O}_Z$ an ideal sheaf, $I \neq \mathcal{O}_Z$. Then there exists $\varepsilon > 0$ such that for every large n*

$$h^0(Z, I^n \mathcal{O}(n)) < (1-\varepsilon)\, h^0(Z, \mathcal{O}(n)).$$

Applying this for $Z=X_0$, $\mathcal{O}(1)=\mathcal{O}(m_1 K_{X_0})$ and $I=I_{m_1}$, we get that t_{m_1} is an isomorphism. Therefore, X_0 is irreducible and has DuVal singularities only. This completes the proof.

Remark 2.27. (i) This proposition is very closely related to some results of Nakayama [Na, 11] and Laufer [La, 4.4].

(ii) If all the surfaces E_i are such that their minimal desingularizations are minimal surfaces of general type, then 2.26 implies 2.15. The general case can be reduced to this one via a suitable cyclic covering.

(iii) Nakayama [Na, 11] proves that $\sum h^0(E_i', \mathcal{O}(mK_{E_i'})) \leqq h^0(X_t, \mathcal{O}(mK_{X_t}))$ holds for every $m \geqq 1$. It is possible that equality for a few small m already implies that X_0 is irreducible with DuVal singularities only.

§ 3. Deformations of quotient singularities

It is well-known that for a rational surface singularity (X_0, P), a one-parameter deformation $X \to \Delta$ of X_0 need have no simultaneous resolution, even in the very weak sense or even after making a ramified covering of the base Δ. In fact, there is a unique irreducible component Z of the local moduli space $\mathrm{Def}(X_0)$ of X_0, known as the Artin component [A2], such that a deformation as above admits a simultaneous resolution if and only if the image of Δ in $\mathrm{Def}(X_0)$ lies in Z. We aim in this section to show, however, that a weakened analogue of simultaneous resolution holds for any one-parameter deformation of a quotient singularity X_0, and to use this result to show how the number of components in $\mathrm{Def}(X_0)$ may be computed. Some analogous results may be proved for a rather wider class of singularities; for the sake of clarity and simplicity these generalizations are postponed to section 5.

We begin by recalling some terminology and a theorem of Kawamata.

Definition 3.1. (i) A normal variety Y is $\mathbb{Q}$-*Gorenstein* if some non-zero integral multiple mK_Y of the canonical divisor K_Y is Cartier and Y is Cohen-Macaulay.

(ii) A germ Y of a normal threefold is *pseudo terminal* if it is canonical and its canonical cover has only cDV singularities. Equivalently, Y is pseudo-terminal if it is $\mathbb{Q}$-Gorenstein and for some (or equivalently, every) resolution $f\colon \tilde{Y} \to Y$ whose exceptional prime divisors are $E_1, \ldots, E_r$, we have $K_{\tilde{Y}} \sim f^* K_Y + \sum a_i E_i$, where $a_i \geqq 0$ for all i and $a_j > 0$ if $f(E_j)$ is a point.

(iii) A normal surface singularity (X, P) is *log terminal* if for some (or equivalently, every) resolution $f\colon \tilde{X} \to X$ whose exceptional locus $E = \bigcup_{i=1}^{r} E_i$ has normal crossings, there are rational numbers $a_1, \ldots, a_r$ such that $K_{\tilde{X}} \equiv -\sum a_i E_i$ and for all i, $a_i < 1$.

Theorem 3.2. (Kawamata [Kaw 2]). *The normal surface singularity (X, P) is log terminal if and only if it is a quotient singularity.*

Lemma 3.3. *Suppose that $X \to \Delta$ is a one-parameter deformation of a normal surface singularity (X_0, P), that $\Delta' \to \Delta$ is a finite base change and that $X' = X \times_\Delta \Delta'$. Then if X' has just terminal (resp. canonical, resp. pseudo-terminal) singularities the same holds for X.*

Proof. We first deal with the case where X' is terminal (resp. canonical). There are uniformizing parameters t, t' on Δ, Δ' respectively such that $t = t'^m$, some m. Then the Galois group $G = \mathrm{Gal}(\Delta'/\Delta)$ is cyclic of order m; say $G = \langle \tau \rangle$. Suppose that X' has index M; choose a generator $\phi \in \mathcal{O}(M \cdot K_{X'})$. Then $\omega = \bigotimes_{i=0}^{m-1} (\tau^i)^* \phi$ is a G-invariant generator of $\mathcal{O}(mMK_{X'})$, and so $t^{M(m-1)} \cdot \omega$ is a generator of $\mathcal{O}(mM \cdot K_X)$. I.e. X is $\mathbb{Q}$-Gorenstein. Put $r = m \cdot M$, so that $r \cdot K_X$ and $r \cdot K_{X'}$ are both Cartier. Put $\sigma = t^{M(m-1)} \cdot \omega$. Let v be a discrete rank one valuation of the function field $\mathbb{C}(X)$ that is centred at P (resp. centred at a point or component of the singular locus of X) and let v' be an extension of v to $\mathbb{C}(X')$. Let $x' \in \mathbb{C}(X')$ be a uniformizing parameter for v'; then for some $C \in \mathbb{C}(X')$ with $v'(C) = 0$ and some integer e with $e|m$, $C\cdot x'^e$, say, is a uniformizing parameter for v. We have $t'=D\cdot x'^{\alpha}$, where $v'(D)=0$ and $\alpha>0$. Choose y, $z\in\mathcal{O}_v$, the valuation ring of v, that induce a transcendence basis of the residue field of $\mathcal{O}_v$ over $\mathbb{C}$. We can write $\sigma=B\cdot x^{\delta}\cdot(dx\wedge dy\wedge dz)^{\otimes r}$, where $v(B)=0$, and we must show that $\delta>0$ (resp. $\delta\geqq 0$). Now $\sigma/t'^{r(m-1)}=\omega$ generates $\mathcal{O}(r\cdot K_{X'})$, and so we can write $\omega=A\cdot x'^{\gamma}(dx'\wedge dy\wedge dz)^{\otimes r}$, where $v'(A)=0$ and by hypothesis $\gamma>0$ (resp. $\gamma\geqq 0$). Now $dx=dc\cdot x'^e+e\cdot C\cdot x'^{e-1}dx'=C'\cdot x'^{e-1}(dx'+C_2\,dy+C_3\,dz)$, where $v'(C')=0$, and so $\sigma=B\cdot C^{\delta}\cdot x'^{\delta e}\cdot C'^r\cdot x'^{r(e-1)}\cdot(dx'\wedge dy\wedge dz)^{\otimes r}$; comparing the powers of x' shows that $\delta e+r(e-1)=\alpha r(m-1)+\gamma$. Since $\alpha>0$ and $e\leqq m$ it follows that $r(e-1)\leqq\alpha r(m-1)$, and so $\delta e\geqq\gamma$. Hence $\delta>0$ (resp. $\delta\geqq 0$). So, since the geometric generic fibre of X is isomorphic to that of X', it must be smooth (resp. have only RDP's) and the Lemma is proved in this case.

Finally, to deal with the case where X' is pseudo-terminal, we must show that if v is a valuation of $\mathbb{C}(X)$ centred at the generic point of a curve in the singular locus of X, then (in the same notation as above), $\delta\geqq 0$, and that if v is centred at P, then $\delta>0$. Then arguments, however, are the same as those above. QED

A proof of the next lemma (which is in any case very easy) is implicit in the preceding argument.

Lemma 3.4. *Suppose that $X\to\Delta$ is a normal one-parameter deformation of the reduced surface X_0, $\Delta_1\to\Delta$ is finite base change and $X_1=X\times_{\Delta}\Delta_1$. Then X is $\mathbb{Q}$-Gorenstein if and only if X_1 is so.*

Theorem 3.5(a). *Suppose that $X\to\Delta$ is a one-parameter deformation of the quotient singularity (X_0,P). Then, after making a finite base change if necessary, there is a proper birational morphism f: $Y\to X$ with the following properties*:

(i) *Y has only terminal singularities.*

(ii) *For all complete curves C contracted by f, we have $K_Y\cdot C\geqq 0$.*

(iii) *For $t\in\Delta^*=\Delta-\{0\}$, the morphism $Y_t\to X_t$ is the minimal resolution.*

(iv) *The special fibre Y_0 is normal, with only quotient singularities.*

(b) *Suppose that $X\to\Delta$ is as above. Then, even without base-change, there is a birational model g: $Z\to X$ with following properties*:

(i) *Z has only pseudo-terminal singularities*

(ii) *For all complete curves C contracted by g, we have $K_Z\cdot C>0$.*

(iii) *For $t\in\Delta^*$, the morphism $Z_t\to X_t$ is the canonical model.*

(iv) *The special fibre Z_0 is normal, with only quotient singularities.*

Notice that these results are non-vacuous even if the general fibre X_t is smooth, since X need not be $\mathbb{Q}$-Gorenstein.

Proof (a): After making a finite base change if necessary, there is by Knudsen's and Mumford's semi-stable reduction theorem [KKMS] a projective birational map g: $\tilde{X}\to X$ such that $\tilde{X}$ is smooth and the special fibre $\tilde{X}_0$ is reduced. Now as explained in the introduction, there is a birationally equivalent model Y with only terminal singularities such that the rational map f: $Y\to X$ is a morphism, and K_Y is numerically effective relative to f. Hence conditions (i)–(iii) hold. Let V denote the strict transform of X_0 in Y_0 and $\tilde{V}$ the normalization of V. We can write $Y_0=\sum V_i$, with $V_1=V$. Denote the exceptional locus of the morphism $h\colon \tilde{V} \to X_0$ by $\bigcup C_i$, where each C_i is an integral curve. We have $K_Y|_{\tilde{V}} \sim D - E + h^* F$, where D, E are effective $\mathbb{Q}$-divisors supported on $\bigcup C_i$ with no common component and F is a $\mathbb{Q}$-divisor on X_0. By the negative definiteness of an exceptional locus, if $D \neq 0$, then there is a component C_j of D such that $C_j \cdot D < 0$; this would imply that $K_Y \cdot C_j < 0$, which is absurd. So $K_Y|_{\tilde{V}} \sim -E + h^* F$. Say $E = \sum \nu_i C_i$, $\nu_i \in \mathbb{Q}$ and $\nu_i \geqq 0$.

If $Y_0 \neq \tilde{V}$ (i.e. if Y_0 is not normal), then by the adjunction formula $K_{\tilde{V}} \sim -\sum \mu_i C_i$ in a neighbourhood of $\bigcup C_i$, where for all i, $\nu_i - \mu_i$ is a non-positive integer, and for some value j, $\mu_j \geqq \nu_j + 1$. However, this would contradict the log terminal property of X_0, and so Y_0 is normal.

Consider the canonical model $Z \to \Delta$ of X mentioned in the introduction. This is obtained via a morphism $h\colon Y \to Z$ that contracts every complete curve C for which $K_Y \cdot C = 0$. Since Y_0 is irreducible, it follows that every surface contracted by h is flat over Δ, and so Z has only pseudo-terminal singularities.

Since K_Z is ample, $Z \cong \operatorname{Proj} \sum g_*(\omega_{\tilde{X}}^n)$. For any finite map $\Delta' \to \Delta$ let $\tilde{X}' = \tilde{X} \times_\Delta \Delta'$, $X' = X \times_\Delta \Delta'$, $g' = G \times_\Delta \Delta'$. Since $\tilde{X}/\Delta$ is semistable, $\tilde{X}'$ has canonical singularities. Thus if we put $Z' = \operatorname{Proj} \sum g'_*(\omega_{\tilde{X}'}^n)$ then $Z' = Z \times_\Delta \Delta'$. By the above consideration Z' has only pseudo-terminal singularities, hence by 5.1 (b), Z_0 has only log-terminal ($\equiv$quotient) singularities. Since $K_{Y_0} \equiv h^* K_{Z_0}$, the same holds for Y_0. This completes the proof of part (a).

In order to complete the proof of (b), suppose that $X \to \Delta$ is a one-parameter deformation of the quotient singularity X_0, and that $\Delta_1 \to \Delta$ is a finite base change, with Galois group G, such that $X_1 = X \times_\Delta \Delta_1$ admits a semi-stable resolution $\tilde{X}_1 \to X_1$. Let $Z_1 \to \Delta$ be the canonical model of $\tilde{X}_1$; then the birational action of G on Z_1 is in fact biregular. Put $Z = Z_1/G$; then $Z_1 \cong Z \times_\Delta \Delta_1$, and so by Lemma 3.4 Z is $\mathbb{Q}$-Gorenstein. Since the special fibres of Z and Z_1 are the same and that of Z_1 is log terminal, it follows from Lemma 3.3 that Z has just pseudo-terminal singularities. Finally, the ampleness of K_Z can be checked on the special fibre, and the geometric generic fibre of Z is isomorphic to that of Z_1, and so has only RDP's. Q.E.D.

Corollary 3.6. *If $X \to \Delta$ is a $\mathbb{Q}$-Gorenstein one-parameter deformation of the quotient singularity X_0, then X has just a terminal singularity if the general fibre X_t is smooth, and pseudo-terminal singularities if X_t has only RDP's.*

Proof. By Lemma 3.3, we may ignore any base change. Then by Theorem 3.5, there is a proper birational morphism $f\colon Y \to X$ that has at most one-dimensional fibres and is an isomorphism over the smooth locus of X; furthermore Y has only terminal singularities. Since X is $\mathbb{Q}$-Gorenstein, it has just pseudo-terminal singularities, and terminal singularities if the general fibre X_t is smooth.

Now fix a quotient singularity (X_0, P). Our aim is to analyze the versal deformation space $\operatorname{Def} X_0$ in terms of certain partial resolutions of X_0.

Definition 3.7. A normal surface singularity is *of class T* if it is a quotient singularity and admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing. By Corollary 3.6 such a smoothing must terminal.

These singularities will be given an explicit description below.

Definition 3.8. A *P-resolution* of X_0 is a partial resolution $f\colon Z_0 \to X_0$ such that Z_0 has only singularities of class T and K_{Z_0} is ample relative to f.

Next, we recall three things that are well-known to hold for any rational surface singularity Y_0.

(i) Every component of Def Y_0 contains smoothings (cf [A2, 3.1]).

(ii) If $f\colon Z \to Y_0$ is a normal partial resolution, then there is an induced map $F\colon \operatorname{Def} Z \to \operatorname{Def} Y_0$ of formal deformation spaces [Wa1, 1.4].

(iii) If $f\colon Z \to Y_0$ is as above, $Q_1, \ldots, Q_r$ are the singular points of Z and Z_i is a representative of the germ (Z, Q_i), then the natural morphism $\alpha\colon \operatorname{Def} Z \to \Pi_i \operatorname{Def} Z_i$ is smooth.

We can now state our main result; the proof will be given in stages.

Theorem 3.9. (i) *If Z_i is any germ of class T, then there is an irreducible subspace* $\operatorname{Def}' Z_i \subseteq \operatorname{Def} Z_i$ *that corresponds to the $\mathbb{Q}$-Gorenstein deformations. (We shall make this precise later.)*

(ii) *If $f\colon Z \to X_0$ is a P-resolution and $F\colon \operatorname{Def} Z \to \operatorname{Def} X_0$ is the induced map of deformation spaces, then $F(\operatorname{Def}' Z)$ is an irreducible component of $\operatorname{Def} X_0$, where $\operatorname{Def}' Z = \alpha^{-1}(\Pi_i \operatorname{Def}' (Z_i))$.*

(iii) *If Z, $\tilde{Z}$ are two P-resolutions of X_0 that are not isomorphic over X_0 and F, $\tilde{F}$ are the corresponding maps of deformation spaces, then*
$F(\operatorname{Def}' Z) \neq \tilde{F}(\operatorname{Def}' \tilde{Z})$.

(iv) *Every component of* $\operatorname{Def} X_0$ *arises in this way.*

In other words, there is a one-one correspondence between the components of $\operatorname{Def} X_0$ and the P-resolutions of X_0.

Before proving this, we shall determine the singularities of class T and show how to find the P-resolutions of X_0.

Proposition 3.10. *Suppose that X_0 is a quotient singularity having a terminal smoothing $X \to \Delta$. Then either X_0 is an RDP or X_0 is a cyclic quotient singularity of the form* $\operatorname{Spec} \mathbb{C}[[u, v]]/H$, *where $H = \langle \alpha \rangle$ is of order $r^2 s$ for some integers r, s, and there is a primitive $r^2 s$'th root of unity, say η, such that the action of H on* $\operatorname{Spec} \mathbb{C}[[u, v]]$ *is given by $\alpha(u, v) = (\eta u, \eta^{dsr-1} v)$, where d is prime to r. Moreover, every such cyclic quotient singularity admits a terminal smoothing.*

Proof. Let $Y \to X$ be the canonical cover, so that as before the special fibre Y_0 is the canonical cover of X_0. Say $X = Y/G$, where $G = \langle \sigma \rangle$ is cyclic of order r. Assume that X_0 is not an RDP, so that $r \geqq 2$. Then Y is a cDV singularity with a G-invariant function on it whose vanishing defines the RDP Y_0. Then the classification due to Danilov, Morrison and Stevens, and Mori of terminal singularities (completed below, in Theorem 6.5) shows that either Y is smooth and σ acts via $\sigma(x, y, z) = (\omega x, \omega^{-1} y, \omega^c z)$, where $\omega = \exp(2\pi i/r)$, $(c, r) = 1$ and x, y, z are suitable local coordinates on Y, or Y is given by an equation $xy + f(z, u^r) = 0$, where σ acts via $\sigma(x, y, z, u) = (\omega x, \omega^{-1} y, z, \omega^c u)$ and c is prime to r. In either case Y_0 is of type A, and so by [Mo1, Corollary 4] we can suppose that Y_0 is given by the equation $xy - z^{sr} = 0$, and the action of σ is given by $\sigma(x, y, z) = (\omega x, \omega^{-1} y, \omega^c z)$. It follows directly that X_0 is as described.

If conversely X_0 is one of these quotient singularities, then we can write $X_0 = Y_0/\langle \sigma \rangle$, where Y_0 is given by $xy - z^{sr} = 0$ and σ acts via $\sigma(x, y, z) = (\omega x,$

$\omega^{-1}y,\ \omega^c z)$. Thus we can write down a $\mathbb{Q}$-Gorenstein smoothing $X \to \Delta = \operatorname{Spec} \mathbb{C}[[u]]$ by taking $X = Y/\langle\sigma\rangle$, where Y is given by $xy - z^{sr} + u = 0$ and σ acts on Y via $\sigma(x, y, z, u) = (\omega x, \omega^{-1}y, \omega^c z, u)$. By corollary 3.6, this is a terminal smoothing.

Another way of proving this first part of Proposition 3.10 is to note that if X_0 has a terminal smoothing and if Z_0 is a resolution of X_0, then $K^2_{Z_0}$ is an integer; Wahl has classified the quotient singularities with this property [Wa 3], and apart from RDP's they are as described in 3.10.

The next result, due essentially to Wahl [Wa 3], tells us how a cyclic quotient singularity of class T may be recognized from its minimal resolution.

Proposition 3.11 (i). *The singularities* $\overset{-4}{\bullet}$ *and* $\overset{-3}{\bullet}\!-\!\overset{-2}{\bullet}\!-\!\overset{-2}{\bullet}\cdots\overset{-2}{\bullet}\!-\!\overset{-3}{\bullet}$ *are of class T.*

(ii) *If the singularity* $\overset{-b_1}{\bullet}\cdots\overset{-b_r}{\bullet}$ *is of class T, then so are*

$$\overset{-2}{\bullet}\!-\!\overset{-b_1}{\bullet}\cdots\overset{-b_{r-1}}{\bullet}\!-\!\overset{-b_r-1}{\bullet} \quad \textit{and} \quad \overset{-b_1-1}{\bullet}\!-\!\overset{-b_2}{\bullet}\cdots\overset{-b_r}{\bullet}\!-\!\overset{-2}{\bullet}$$

(iii) *Every singularity of class T that is not an RDP can be obtained by starting with one of the singularities described in (i) and iterating the steps described in (ii).*

Proof. Suppose that $X_0 = \operatorname{Spec} \mathbb{C}[[u, v]]/H$, where $H = \langle\alpha\rangle$ is cyclic of order dm^2 and α acts via $\alpha(u, v) = (\eta u, \eta^{adm-1}\cdot v)$, where a is prime to m and $a < m$. Write $\dfrac{dm^2}{adm-1} = b_1 - \dfrac{1}{b_2 - \dots} = [b_1, \dots, b_r]$, where each $b_i \geqq 2$, so that the dual graph for the minimal resolution of X_0 is $\overset{-b_1}{\bullet}\!-\!\overset{-b_2}{\bullet}\cdots\overset{-b_r}{\bullet}$. Then $[b_r, \dots, b_1] = dm^2/(m-a)\,dm - 1$. Suppose first that $b_1 = b_r = 2$. Then $dm^2/\{adm-1\} < 2$ and $dm^2/\{(m-a)\,dm-1\} < 2$, so that $a > \frac{1}{2}m$ and $m - a > \frac{1}{2}m$; this is absurd, and so b_1 and b_r cannot both be equal to 2. Next, suppose that $b_1 = 2$ and $b_r > 2$, so that $\dfrac{dm^2}{adm-1} = 2 - \dfrac{1}{[b_2, \dots, b_r]}$. Then $[b_2, \dots, b_r] = \dfrac{adm-1}{2(adm-1) - dm^2}$. Since $(2(adm-1) - dm^2)(ad(m-a) - 1) \equiv 1 \cdot (\operatorname{mod} adm - 1)$, it follows that $[b_r, \dots, b_2] = \dfrac{adm-1}{ad(m-a)-1}$. Then $[b_r - 1, b_{r-1}, \dots, b_2] = \dfrac{da^2}{(m-a)\,da - 1}$, which corresponds to a singularity of type T.

Finally, suppose that $b_1 > 2$ and $b_r > 2$. In this case $a \leqq \dfrac{m}{2}$ and $m - a \leqq m/2$, so that $a = 1$ and $m = 2$. If $d = 1$, then $\dfrac{dm^2}{adm-1} = [4]$; suppose therefore that $d > 1$. Then

$$\begin{aligned}\frac{dm^2}{adm-1} &= \frac{4d}{2d-1} = 3 - \frac{2d-3}{2d-1}\\ &= [3, 2, 2, \dots, 2, 3],\end{aligned}$$

where 2 appears $d-2$ times. This completes the proof.

We proceed to find the P-resolutions of a fixed quotient singularity (X, P) which is not an RDP.

Definition 3.12. A resolution $f\colon Y \to X$ is *maximal* if $K_Y \sim f^* K_X - \sum a_i E_i$, where $0 < a_i < 1$, and for any proper birational morphism $g\colon Z \to Y$ that is not an isomorphism, we have $K_Z \sim h^* K_X - \sum b_j F_j$, where $h = f \circ g$ and some $b_j \leqq 0$.

Lemma 3.13. *A quotient singularity (X, P) has a unique maximal resolution.*

Proof. for any resolution $\pi_i\colon X_i \to X$, we can write $K_{X_i} \sim \pi_i^* K_X + \sum_j (-1+\alpha_j) E_j$, the E_j being the exceptional curves. If $Q = E_j \cap E_{j+1}$, $\sigma\colon X_{i+1} \to X_i$ is the blow-up along Q and $E_k = \sigma^{-1}(Q)$, then $\alpha_k = \alpha_i + \alpha_{i+1}$. Clearly, we may not blow up along a point lying on only one component. Now if π_i is the minimal resolution, every α_i lies between 0 and 1, and so to construct a maximal resolution one starts with the minimal resolution and successively blows up points where exceptional curves meet until the quantities α_i have the property that $\alpha_i < 1$ for all i, while if $E_i \cap E_j \neq \emptyset$, then $\alpha_i + \alpha_j \geqq 1$. This certainly happens, and so a maximal resolution does exist. Moreover, any two maximal resolutions are isomorphic in codimension one, and so isomorphic, since we are dealing with normal surfaces.

Lemma 3.14. *Suppose that (X, P) is a quotient singularity and that $f\colon Z \to X$ is a partial resolution such that Z has only quotient singularities and K_Z is ample relative to f. Then Z is dominated by the maximal resolution X_m of X.*

Proof. Suppose that X_m does not dominate Z. Then there is a point $Q \in X_m$ corresponding to a curve C in Z. The multiplicity of C in K_Z cannot be negative, by the defining property of X_m. As we saw in the proof of 3.5, we have $K_Z \sim -E$, with E an effective $\mathbb{Q}$-divisor. The multiplicity of C is non-negative, and so C is not a component of E. Hence Supp $E \neq f^{-1}(P)$, so that by the connectedness of $f^{-1}(P)$ there is a curve G in $f^{-1}(P)$ such that $K_Z \cdot G < 0$; this is absurd, and the lemma is proved.

Example 3.15. Consider the quotient $(X, P) = \operatorname{Spec} \mathbb{C}[[u, v]]/\langle\sigma\rangle$, where $\sigma(u, v) = (\eta u, \eta^7 v)$, $\eta = \exp(2\pi i/19)$. The minimal resolution is

$$\begin{array}{ccccc} 11/19 & & 5/19 & & 12/19 \\ \bullet & \text{———} & \bullet & \text{———} & \bullet \\ -3 & & -4 & & -2 \end{array},$$

where the negative integers are self-intersections and the positive numbers are the α_i occurring in the proof of Lemma 3.13. Then the maximal resolution X_m is

$$\begin{array}{ccccccccc} 11/19 & & 16/19 & & 5/19 & & 17/19 & & 12/19 \\ \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet \\ -4 & & -1 & & -6 & & -1 & & -3 \end{array},$$

and it is easy to see that the P-resolutions are Z_1, Z_2 and Z_3, where Z_1 is obtained from the minimal resolution by contracting the (-2) curve (i.e. Z_1 is the RDP model of X), Z_2 is obtained from X_m by contracting all curves except the (-1) curve between the (-4) and the (-6) curve and Z_3 is obtained from the minimal resolution by contracting the (-4) curve. It will follow from Theorem 3.9 that Def X has exactly three components, the Artin component corresponding to Z_1.

Before proceeding with the proof of Theorem 3.9 we need some further preliminary results.

For any deformation $f\colon X \to S$ of X_0, define $j\colon X^0 \to X$ as the locus where f is Gorenstein, and then define $\omega_{X/S}^{[n]} = j_*(\omega_{X^0/S}^{\otimes n})$. Suppose that X_0 is of index r.

Lemma 3.16. *Suppose that $f\colon X \to S$ is a deformation of X_0 over the spectrum S of a complete local ring A and that X is of finite index m (i.e. that $\omega_{X/S}^{[m]}$ is invertible and that m is the least positive integer t such that $\omega_{X/S}^{[t]}$ is invertible). Then $m=r$.*

Proof. For any n there is a natural map $\omega_{X/S}^{[m]} \otimes_{\mathcal{O}_S} k \to \omega_{X_0}^{[n]}$, where k is the residue field of S, which is an isomorphism in codimension one. Hence if $\omega_{X/S}^{[n]}$ is invertible, so is $\omega_{X_0}^{[n]}$, and so $r|m$. Conversely, we wish to prove $m|r$. First suppose that A is Artinian. Let $\pi\colon Y^0 \to X^0$ be the étale cover corresponding to $\omega_{X^0/S}$; then Y^0 is connected. To prove that $m|r$ it is enough to show that the special fibre Y_0^0 of $Y^0 \to S$ is connected. But this is obvious, since Y^0 and Y_0^0 have the same underlying spaces. In the general case, let m denote the maximal ideal of A, put $A_n = A/m^n$, $S_n = \operatorname{Spec} A_n$ and $X_n = X \times_S S_n$. By what we have just shown, we have $\omega_{X_n^0}^{\otimes r}/S_n \cong \mathcal{O}_{X_n^0}$ for every n; it follows that $\omega_{X^0/S}^{\otimes r} \cong \mathcal{O}_{X^0}$, and the Lemma is proved.

Now suppose that X_0 is a singularity of class T of index $r>1$. Suppose that Y_0 is the canonical cover of X_0, and that $X_0 = Y_0/G$, where $G = \langle\sigma\rangle \cong \mathbb{Z}_r$. By 3.10 Y_0 is given by equation $xy - z^{rs} = 0$; in the notation of 3.10 we have $x = u^{rs}$, $y = v^{rs}$, $z = uv$ and $\sigma = \alpha^{rs}$, so that σ acts via $\sigma(x, y, z) = (\varepsilon x, \varepsilon^{-1} y, \varepsilon^d z)$, where $\varepsilon = \eta^{sr}$. Then it is clear that the locus $\operatorname{Fix} G \subseteq \operatorname{Def} Y_0$ is smooth of dimension s.

Our next object is to construct the subspaces Def′ mentioned in 3.9 (i).

Definition 3.17. Suppose that X_0 is a quotient singularity of class T and index r. If $\underline{S}$ is the category of finite local $\mathbb{C}$-algebras we define a functor $\operatorname{Def}' X_0$: $\underline{S} \to \mathit{Sets}$ by $\operatorname{Def}' X_0(S) = \{$isomorphism classes of flat morphism $f\colon X \to S$ such that $X \otimes \mathbb{C} \cong X_0$ and the sheaf $\omega_{X/S}^{[r]}$ is invertible$\}$. This is pro-represented by a subspace of Def X_0, which we denote by $\operatorname{Def}' X_0$.

Lemma 3.18. *In these circumstances, there is a natural morphism* $\operatorname{Fix} G \to (\operatorname{Def}' X_0)_{\text{red}}$ *that is $1-1$. In particular,* $(\operatorname{Def}' X_0)_{\text{red}}$ *is irreducible and has smooth normalization. Moreover,* $\operatorname{Def}' X_0$ *contains smoothings.*

Proof. Put $S = \operatorname{Fix} G$. Let $Y \to S$ be the deformation induced from the versal deformation of Y_0, and put $X = Y/G$. Since G acts trivially on S, it follows from the existence of the Reynolds operator that the induced map $g\colon X \to S$ is flat. Since X is $\mathbb{Q}$-Gorenstein, f is induced from $\operatorname{Def}' X_0$, by Lemma 3.16. So there is a natural morphism $\beta\colon S \to \operatorname{Def}' X_0$.

Now let $\Delta = \operatorname{Spec} \mathbb{C}[[t]]$, and let $p\colon \Delta \to \operatorname{Def}' X_0$ be a map (called a formal arc), and let $X \to \Delta$ the corresponding deformation. By Lemma 3.16, X and

X_0 have the same index, so that if Y is the canonical cover of X then $Y \to \Delta$ is a deformation of Y_0. Hence $\Delta \subseteq \beta(S)$, and so the image of β is just $(\mathrm{Def}' X_0)_{\mathrm{red}} = D$, say and so in particular D is irreducible. Also, we have just shown that any formal arc in D lifts to an arc in S, and so β is of degree one. Finally since β is a map of a local schemes, β is $1-1$

Now suppose that X_0 is an arbitrary quotient singularity and that $f: Z_0 \to X_0$ is a P-resolution. Suppose that $Q_1, \ldots, Q_r$ are the singular points of Z_0, and let Z_i^* denote the germ (Z_0, Q_i). There is a natural smooth morphism ϕ: $\mathrm{Def}\, Z_0 \to \prod_{i=1}^{r} \mathrm{Def}\, Z_i^*$; define $\mathrm{Def}'\, Z_0$ to be $\phi^{-1}\left(\prod_{i=1}^{r} \mathrm{Def}'\, Z_i^*\right)$.

Then clearly 3.9 (i) follows from 3.18.

Proof of 3.9 (iv). Let B be a component of $\mathrm{Def}\, X_0$, and suppose that Δ is a general arc in B; we know that this corresponds to a smoothing $X \to \Delta$ of X_0. By Theorem 3.5, the canonical model $Z \to \Delta$ of X is a $\mathbb{Q}$-Gorenstein smoothing of a P-resolution Z_0 of X_0. So if F: $\mathrm{Def}\, Z_0 \to \mathrm{Def}\, X_0$ is the natural map, we have $\Delta \subseteq F(\mathrm{Def}'\, Z_0)$. Since X_0 has only finitely many P-resolutions, by 3.14, there is a P-resolution Z_0 such that $F(\mathrm{Def}'\, Z_0) = B$. This proves 3.9 (iv).

Proof of 3.9 (ii), (iii). Suppose that Z_0, Z_0^* are two P-resolutions of X_0 that are not isomorphic over X_0. let F: $\mathrm{Def}\, Z_0 \to \mathrm{Def}\, X_0$, F^*: $\mathrm{Def}\, Z_0^* \to \mathrm{Def}\, X_0$ be the map of versal deformation spaces. Assume that $F(\mathrm{Def}'\, Z_0) \subseteq F^*(\mathrm{Def}'\, Z_0^*)$. Let $Z \to \Delta$ be a $\mathbb{Q}$-Gorenstein smoothing of Z_0; these smoothings exist because each singularity of Z_0 admits such a smoothing, and through a general point of $\mathrm{Def}'\, Z_0$ there passes an arc corresponding to a smoothing. Consider the surface Y obtained by contracting all the complete curves on Z_{gen} (the general fibre), and take a smoothing of Y that lies on the Artin component of each singularity of Y. The general fibre of this is smooth and contains no complete curves. So by the openness of versality, there is a one-parameter smoothing $\tilde{Z} \to \Delta$ of Z_0 whose general fibre contains no complete curves, and whose total space $\tilde{Z}$ is a deformation of Z. Since Z is $\mathbb{Q}$-Gorenstein, so is $\tilde{Z}$, by [Ko2]. Hence we may assume that Z_{gen} contains no complete curves. Thus if $X \to \Delta$ is the deformation of X_0 obtained from $Z \to \Delta$ via F, the general fibre X_{gen} is smooth. We also know that X can be blown up to give a terminal smoothing $Z^* \to \Delta$ of Z_0^*; i.e., Z and Z are both canonical models of $X \to \Delta$, and so $Z_0 \cong Z_0^*$ over X_0. This contradiction completes the proof of Theorem 3.9.

We now indicate briefly how to compute the dimensions of the various components of $\mathrm{Def}\, X_0$. Suppose that Z is P-resolution of X_0, $Q_1, \ldots, Q_r$ its singularities and B the component of $\mathrm{Def}\, X_0$ corresponding to Z. We first refine 3.9 (ii) slightly.

Lemma 3.19. *The map* $\mathrm{Def}'\, Z \to B$ *is one to one.*

Proof. This follows from the fact that a one-parameter smoothing $X \to \Delta$ of X_0 that lies in B can be blown up to give a unique smoothing $\tilde{Z} \to \Delta$ of Z, by taking the canonical model.

Corollary 3.20. $\mathrm{Dim}\, B = \sum_{i=1}^{r} \dim \mathrm{Def}'(Z, Q_i) + d$, *where d is the dimension of the space D of locally trivial deformations of Z.*

Lemma 3.21. *If (X, P) is the singularity* Spec $\mathbb{C}[[u, v]]/G$, *where* $G = \langle\alpha\rangle \cong Z_{r^2 s}$ *and α acts via* $\alpha(u, v) = (\eta u, \eta^{dsr-1} v)$, *where* $\eta = \exp(2\pi i/r^2 s)$ *and* $(d, r) = 1$, *then* dim Def′ $X = s$.

Proof. This follows directly from Proposition 3.10 and Lemma 3.18.

Remark 3.22. The dimension d of the space D occurring in Corollary 3.20 can be computed in terms of the dimensions of various Artin components, as follows. Let $g\colon Y \to Z$ be the minimal resolution, and put $g^{-1}(Q_i) = C_i$, $C = \bigcup C_i$. Put $h = f \circ g\colon Y \to X_0$ and $h^{-1}(P) = E$; let $F_1, \ldots, F_s (S \geqq 0)$ denote the connected components of $\overline{E - C}$, let X_i denote the germ of the singularity obtained by contracting F_i and let A_i denote the Artin component of Def X_i. Since D can be identified with the sublocus of Def Y corresponding to deformations that preserve every component of C so by 3.8 (iii) $D \cong \prod_i A_i$, thus $d = \sum \dim A_i$.

Remark 3.23. Return now to Example 3.15; we shall compute the dimensions of the components B_1, B_2, B_3 corresponding respectively to the partial resolutions Z_1, Z_2, Z_3. Recall [Ri] that for a cyclic quotient of type $[b_1, \ldots, b_r]$, the Artin component has dimension $\sum (b_i - 1)$. So we find

$$\dim B_1 = 1 + (3-1) + (4-1) = 6,$$
$$\dim B_2 = 1 + 1 = 2,$$
$$\dim B_3 = 1 + (3-1) + (2-1) = 4.$$

Remark 3.24. In summary, we have shown, for any quotient singularity X_0, how to find the components of Def(X_0), that the components have smooth normalization and how to compute their dimensions, but we have said nothing of how the components intersect, or of what adjacencies occur over the various components. We know only that singularities deform to quotient singularities [EV] and that the class of cyclic quotient singularities is closed under deformation (see ch. 7 for a direct geometrical proof of this).

§ 4. Semi-log-canonical singularities

Normal surfaces can be very effectively studied via their desingularizations. One of the reasons is that if $f\colon Y \to X$ is a desingularization of a normal surface, then $f_* \mathcal{O}_Y = \mathcal{O}_X$. Therefore, most cohomological questions on X can be treated on Y instead. However, if X is singular in codimension one, the desingularization does not carry any information about the singularities in codimension one. One might try to remedy the problem by considering some conductor of $f\colon Y \to X$, but this is very cumbersome at best.

An alternate approach is to try to resolve singularities in codimension two only. This seems to work much better, although we treat just the special case when the singularities in codimension one are normal crossing only. As we shall see the results parallel very closely the case of normal surfaces.

An analogous theory can be developed for higher dimensional varieties. Since for the purposes of the present article the surface case is sufficient, the general problem will be treated elsewhere.

Definition 4.1. A surface singularity (x, X) is called a normal crossing point (resp. pinch point) if it is analytically isomorphic to $(0, xy=0) \subset (0, \mathbb{C}^3)$ (resp. $(0, x^2=zy^2) \subset (0, \mathbb{C}^3)$. Normal crossing will usually be abbreviated to n.c.. Note that we will not call the triple normal crossing point $(xyz=0)$ an n.c. point.

Definition 4.2. A surface X will be called semi-smooth if every closed point of X is either smooth or n.c. or a pinch point.

The singular locus of a semi-smooth surface is a smooth curve D_X, which will be called the double curve of X. The normalization $\pi: \bar{X} \to X$ is smooth. $\bar{D}_X = \pi^{-1}(D_X)$ is again smooth; $\pi: \bar{D}_X \to D_X$ is generically 2:1 and ramifies exactly at the pinch points.

Definition 4.3. A map $f: Y \to X$ is called a semi-resolution of X if the following conditions are satisfied:

(i) f is proper;

(ii) Y is semi-smooth;

(iii) if D_Y is the double curve of Y, then no component of D_Y is mapped to a point;

(iv) there is a finite set $S \subset X$ such that $f: f^{-1}(X-S) \to X-S$ is an isomorphism.

Definition 4.4. (i) If $f: Y \to X$ is a semi-resolution, then a curve $E_i \subset Y$ is called exceptional if $f(E_i)$ is a point. Let $E = \bigcup E_i$ be all the exceptional curves.

(ii) $f: Y \to X$ is called a good semi-resolution if it is a semi-resolution and $E \cup D_Y$ has smooth components and transverse intersections. This implies that E has at most double points and $E \cup D_Y$ at most triple points.

It is important to note that E is not a Cartier divisor in general. The only points where E is non-Cartier are the pinch points; here E is necessarily smooth.

Notions akin to these were introduced independently by the Dutch school. They informed us that 4.5–4.15 are closely related to some of their results, especially to forthcoming works of van Straten [vS]. The following result is a very special case of a theorem of van Straten [vS].

Proposition 4.5. *Let X be a surface such that outside finitely many points $S \subset X$ it is either smooth or has normal crossing. Then X has a good semi-resolution.*

Proposition-Definition 4.6. *Notation as in 4.2. Let $E \subset X$ be a curve which is not contained in D_X and let $\bar{E} = \pi^{-1} E$. Since $\omega_{\bar{X}} = \pi^* \omega_X(-\bar{D}_X)$, the usual adjunction formula gives*

$$2g(\bar{E}) - 2 = (\omega_{\bar{X}} + \bar{E}) \cdot \bar{E} = \bar{E}^2 + \deg_E \omega_X - \bar{E} \cdot \bar{D}_X.$$

Therefore we get:

(i) *$\bar{E}^2 < 0$ and $\deg_E \omega_X < 0$ iff $\bar{E}$ is exceptional of the first kind and does not intersect $\bar{D}_X$.*

(ii) *$\bar{E}^2 < 0$ and $\deg_E \omega_X = 0$ iff either $\bar{E}$ is exceptional of the first kind and intersects $\bar{D}_X$ once or $\bar{E}^2 = -2$, $\bar{E} \cong \mathbb{P}^1$ and $\bar{E}$ does not intersect $\bar{D}_X$.*

In the first two cases (i.e. when $\bar{E}^2=-1$) we shall call E a -1 curve on X. It is clear that one can contract $E\subset X$ and the resulting surface X' is again semi-smooth.

Corollary 4.7. *Let X be semi-smooth and $f\colon Y\to X$ be semi-resolution. Then X can be obtained from Y by repeatedly contracting -1 curves.*

Proof. Consider $\bar{f}\colon \bar{Y}\to\bar{X}$. If this is not an isomorphism, then there is an exceptional curve of the first kind $\bar{E}\subset\bar{Y}$, contracted by $\bar{f}$. $\bar{E}\cdot\bar{D}_Y\leqq 1$ since otherwise $\bar{D}_X$ would be singular. Therefore, E is a -1 curve and so it can be contracted.

Corollary 4.8. *With the above notation let $E_i\subset Y$ be the exceptional curves. Then $\omega_Y=f^*\omega_X(\sum a_i E_i)$ and $a_i\geqq 0$.*

Proof. It is easy to check this for a single contraction; thus the claim follows from 4.7. We note for 4.6. (ii) we get $a_1=0$.

Definition 4.9. A semi-resolution $f\colon Y\to X$ is called minimal if no -1 curve is contracted by f.

Proposition 4.10. *Let X be a surface such that for a finite set of points $S\subset X$, $X-S$ is semi-smooth. Then X has a unique minimal semi-resolution.*

Proof. Let $f\colon Y\to X$ be a semi-resolution. Now contract all -1 curves on Y that are contracted by f. This way we get a minimal semi-resolution. Uniqueness can be proved the same way as for normal surfaces.

Definition 4.11. (i) Let $f\colon Y\to X$ be a good semi-resolution of X, and let E_i be the exceptional curves. It is called a minimal good semi-resolution if for every -1 curve E_i, contracted by f, we have $\bar{E}_i\cdot\sum_{j\neq i}\bar{E}_j+\bar{E}_i\cdot\bar{D}_Y>2$. If one starts with a good semi-resolution, then one can repeatedly contract -1 curves that do not satisfy the above requirement and obtain a minimal good semi-resolution.

Proposition 4.12. (Reid [Re1, 2.6]) *Let $f\colon Y\to X$ be a semi-resolution of X and let E_i be the exceptional curves. Then*

(i) *There is a unique cycle $Z=-\sum a_i E_i$ such that $\bar{Z}\cdot\bar{E}_i=-\deg_{E_i}\omega_Y$. (The a_i are rational in general.)*

(ii) *If f is the minimal semi-resolution then $Z\geqq 0$.*

(iii) *If X is Cohen-Macaulay, f is minimal and $Z=0$, then X is either a smooth, an n.c. or a pinch point or a DuVal singularity.*

(iv) *If X is Cohen-Macaulay and f is minimal, then $Z=0$ or $-a_i>0$ for every i.*

(v) *If X is Gorenstein and f is a minimal good semi-resolution, then either $Z=0$ or $-a_i>0$ for every i.*

Proof. The intersection matrix $\bar{E}_i\cdot\bar{E}_j$ is negative definite; this shows the existence of Z. If f is minimal, then $\deg_{E_i}\omega_Y\geqq 0$, and (ii) is a formal consequence of this as in [Z, 7.1]

Assume that $a_i=0$ for some i. Then $\bar{Z}\cdot\bar{E}_i\geqq 0$, with equality only if $a_j=0$ for every E_j such that $\bar{E}_j\cdot\bar{E}_i>0$. If f is minimal, then $-\deg_{E_i}\omega_Y\leqq 0$; hence we have $\deg_{E_i}\omega_Y=0$. By 4.6 (ii) E_i is therefore a -2 curve that does not intersect

D_Y. If $\bar{E}_j \cdot \bar{E}_i > 0$, then again we get that E_j is such a -2 curve. E_i lies on an irreducible component Y' of Y and by the above Y' contains no double curve. Hence it is a connected component. If X is Cohen-Macaulay, then by [Hart] this implies that $Y' = Y$. Hence X is a DuVal singularity. This proves both (iii) and (iv).

(v) follows from (iv) as in [Z, 7.1].

Proposition 4.13. *Let $f\colon Y \to X$ be a semi-resolution of a Cohen-Macaulay surface. Then*

(i) $f_* \mathcal{O}_Y = \mathcal{O}_X$,

(ii) $R^1 f_* \omega_Y = 0$,

(iii) *$R^1 f_* \mathcal{O}_Y$ and $\omega_X / f_* \omega_Y$ are dual to each other.*

Proof. (i) $\mathcal{O}_X \subseteq f_* \mathcal{O}_Y$ and they agree in codimension one. Since $\mathcal{O}_X$ is S_2 they are equal.

(ii) We have an exact sequence $0 \to \pi_* \omega_{\bar{Y}} \to \omega_Y \to Q \to 0$, where Q is supported on D_Y. $R^1 f_* \pi_* \omega_{\bar{Y}} = 0$ by Grauert-Riemenschneider vanishing [G-R], and $R^1 f_* Q = 0$ since $\pi | D_Y$ is finite. This proves (ii).

(iii) This can be proved the same way as for normal X. See e.g. [Ko 1, 3.3.3].

Definition 4.14. Let $f\colon Y \to X$ be a semi-resolution of a Cohen-Macaulay surface singularity. X is called semi-rational if $R^1 f_* \mathcal{O}_Y = 0$.

Remark 4.15. (i) It is easy to check that this definition is independent of the semi-resolution.

(ii) This definition declares n.c. and pinch points to be "rational". This is a reasonable thing to do. We hope that everyone agrees that a n.c. point should be considered "rational". A pinch point is a quotient of a n.c. point so it should be rational.

(iii) It is true that the deformation of a semi-rational singularity is semi-rational again [vS].

Definition 4.16. (i) If F is a rank one sheaf, then $F^{[s]}$ denotes the double dual of the s^{th} tensor power of F.

(ii) A singularity (x, X) is called $\mathbb{Q}$-Gorenstein if $\omega_X^{[s]}$ is locally free for some $s > 0$ and X is Cohen-Macaulay. The smallest such s is called the index of (x, X).

Definition 4.17. Let (x, X) be a $\mathbb{Q}$-Gorenstein surface singularity such that $X - x$ is semi-smooth. Let $f\colon Y \to X$ be a good semi-resolution of X. We can write $\omega_Y^s \cong f^* \omega_X^{[s]} \otimes \mathcal{O}(\sum s a_i E_i)$, where the E_i are exceptional divisors and the a_i are rational. With the notation of 4.11, we have that $Z = \sum a_i E_i$. (x, X) is called

(i) semi-canonical if $a_i \geqq 0$,

(ii) semi-log-terminal if $a_i > -1$,

(iii) semi-log-canonical if $a_i \geqq -1$.

Remark 4.18. (i) Using 4.8 one can easily see that these notions are independent of the good semi-resolution chosen.

(ii) One could try to define semi-terminal by $a_i > 0$. This, however, would not be independent of the resolution chosen.

(iii) If f': $Y' \to X$ is any semi-resolution of X and if $Z' = \sum a'_i E'_i$, then X is semi-canonical iff $a'_i \geq 0$. This follows easily from 4.8.

(iv) With the above notation, if X is semi-log-terminal (resp. canonical), then $a_i > -1$ (resp. ≥ -1).

Definition 4.19. The rest of this chapter will be devoted to the classification of the above singularities. For normal surfaces this was done by Kawamata [Kaw 1]; see also [Kaw 2]. The general case runs very much along the same lines. Using 4.12 the Gorenstein case can be handled very efficiently. Therefore we shall give full proofs for this. First we have to recall the definition of certain singularities.

Definition 4.20. (i) [Sai] A normal Gorenstein surface singularity is called simple elliptic if the exceptional divisor of the minimal resolution is a smooth elliptic curve.

(ii) [Kar 1] A normal Gorenstein surface singularity is called a cusp if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.

(iii) [S-B 2] Let X be a Gorenstein surface singularity which has a minimal semi-resolution f: $Y \to X$. X is called a degenerate cusp if X is not normal and the exceptional divisor is a cycle of smooth rational curves or a rational nodal curve. In this case Y has no pinch points and the irreducible components of X have cyclic quotient singularities.

Theorem 4.21. *Let (x, X) be a Gorenstein surface singularity such that $X - x$ is semi-smooth. Then*

(i) *X is semi-canonical iff $x \in X$ is either a smooth, a n.c. point, a pinch point or a Du Val singularity.*

(ii) *X is semi-log-canonical iff X is either a simple elliptic singularity, a cusp, a degenerate cusp or semi-canonical.*

Proof. (i) Assume that X is semi-canonical and let f: $Y \to X$ be the minimal good semi-resolution. Let $\omega_Y = f^* \omega_X \otimes \mathcal{O}(-Z)$. Since X is semi-canonical we have $-Z \geq 0$. By 4.12 $Z \geq 0$. So $Z = 0$ and by 4.12 X is one of those listed.

(ii) With the above notation we have $Z = \sum a_i E_i$. If $Z = 0$, then X is semi-canonical, so assume that $Z \neq 0$. Then $a_i > 0$ by 4.12 and $a_i \leq 1$ by definition. Thus $Z = \sum E_i$, the reduced exceptional divisor.

$$\omega_Z \cong \omega_Y \otimes \mathcal{O}(Z)|Z \cong f^* \omega_X \otimes \mathcal{O}(-Z+Z)|Z \cong \mathcal{O}_Z.$$

Hence $h^0(\mathcal{O}_Z) = h^1(\mathcal{O}_Z) = 1$ (since Z is reduced and connected).

The adjunction formula now gives

$$2g(\bar{E}_i) - 2 = \bar{E}_i^2 - Z \cdot \bar{E}_i - \bar{E}_i \cdot \bar{D}_Y = -\bar{E}_i \cdot \sum_{j \neq i} \bar{E}_j - \bar{E}_i \cdot \bar{D}_Y.$$

Therefore $g(\bar{E}_i) \leq 1$. If $g(\bar{E}_i) = 1$, then $\bar{E}_i$ intersects neither $\bar{D}_Y$ nor any other $\bar{E}_j$. Hence E_i is the only exceptional curve and $\bar{D}_Y = 0$. Consequently, X is normal and either simple elliptic or a cusp and E_i is a nodal cubic. Otherwise, $g(\bar{E}_j) = 0$ for every j and $\bar{E}_j$ has two intersections with the rest of $\bar{E}_i$'s and $\bar{D}_Y$.

If X is normal, then $D_Y=0$. Thus each E_i intersects two others, and hence they form a cycle. X is a cusp.

If X is not normal, then on each component of $\bar{Y}$ the exceptional curves form a chain and the two curves at the end intersect $\bar{D}_Y$. On Y the E_i's form either a cycle, in which case X is a degenerate cusp, or a chain. The latter is impossible since $h^1(\mathcal{O}_Z)=1$, and this completes the proof.

The classification of the non-Gorenstein case is reduced to the Gorenstein one by the usual Reid-Wahl cyclic covering trick. The proof is the same as in the normal case [Re2, 1.7–1.9]; therefore we state only the result in the form best suited to our purpose.

Proposition 4.22. *Let (z, Z) be a semi-log-canonical surface singularity. Then there is a Gorenstein semi-log-canonical singularity (x, X) and an action of a cyclic group $\mathbb{Z}_r$ on (x, X) such that*

(i) *The action is free on $X-x$.*

(ii) *The action of $\mathbb{Z}_r$ on $\omega_X/m_x\omega_X$ is faithful (m_x = the ideal of $x \in X$)*

(iii) $(z, Z) = x, X)/\mathbb{Z}_r$.

(iv) *r = index of Z.*

If Z is semi-log-terminal, then X is in fact semi-canonical.

Theorem 4.23. *The semi-log-terminal surface singularities are exactly the following.*

(i) *Quotients of $\mathbb{C}^2$, enumerated by Brieskorn* [Br1],

(ii) *Normal crossing or pinch points,*

(iii) *$xy=0$ modulo the group action $x\to\varepsilon^a x$; $y\to\varepsilon^b y$; $z\to\varepsilon z$, where ε is a primitive r^{th} root of unity; $(a,r)=1$, and $(b,r)=1$.*

(iv) *$xy=0$ modulo the group action $x\to\varepsilon^a y$, $y\to x$, $z\to\varepsilon z$ where ε is a primitive r'th root of unity, $4|r$ and $(a,r)=2$.*

(v) *$x^2=zy^2$ modulo the group action $x\to\varepsilon^{1+a}x$, $y\to\varepsilon^a y$, $z\to\varepsilon^2 z$, r odd, $(a,r)=1$.*

Proof. Quotients of DuVal points give the first case (cf. [Br1]).

A $\mathbb{Z}_r$-action on $xy=0$ can be assumed to be linear on $\mathbb{C}^3$ and it is easy to enumerate them. (iii) is the case where the two branches are not interchanged, and (iv) is the case where they are.

The normalization of $(x^2=zy^2)$ is $\mathbb{C}^2$, where the possible actions are understood. It is easy to decide which ones descend to an action on the pinch point.

Theorem 4.24. *The semi-log-canonical surface singularities are exactly the following:*

(i) *The semi-log-terminal ones enumerated in* 4.23,

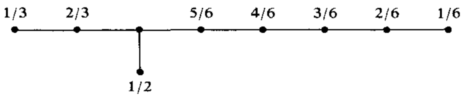
(ii) *The Gorenstein ones enumerated in* 4.21 *(ii)*,

(iii) *$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_6$ quotients of simiple elliptic ones enumerated in* (Kaw1, pp. 226–227],

(iv) *$\mathbb{Z}_2$ quotients of cusp and degenerate cusps.*

Proof. The only new case is that of quotient of degenerate cusps; the argument of [Kaw2, 9.6] yields that only $\mathbb{Z}_2$ quotients should be considered.

Remark 4.25. [Kaw1, pp. 226–227] lists the dual graph of the minimal resolution of the singularity. The first one is a $\mathbb{Z}_2$ quotient of a cusp; this we consider in (iv). Note also that the last graph is incorrect. It should be



It is not difficult, and very instructive, to work out the minimal semi-resolution of these singularities. In the nonnormal case there are three basic building blocks.

Definition 4.26. We define three types of objects that can be components of the pair $(\bar{Y}, \bar{D}_Y)$ for a minimal resolution of a semi-log-canonical singularity.

(i) $(A, \mathbb{Z}_n, a, \Delta)$ or (A, Δ) denotes the minimal resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_n(x \to \varepsilon x, y \to \varepsilon^a y)$ and Δ the proper transform of the image of the x-axis.

(ii) $(A, \mathbb{Z}_n, a, 2\Delta)$ or $(A, 2\Delta)$ the same as if $n>2$ above except that 2Δ is the proper transform of the image of the x and y axes. For $n=1$ we blow up the origin of $\mathbb{C}^2$ once.

(iii) (C, Δ) denotes the minimal resolution of a dihedral quotient of $\mathbb{C}^2$, Δ is the proper transform of the coordinate axes. (Δ will have only one component.)

These are the same objects as given in [Kaw2, 9.6 (5), (7) and (6)].

(iv) If (X, Δ) is one of the above objects, then Δ is a small disc which intersects the exceptional divisor in one point, which we can take as the origin of some coordinatization of Δ. By the expression "(X, Δ) with Δ pinched together" we mean the object we obtain from X by identifying $d \in \Delta$ with $-d \in \Delta$. This way we get a pinch point at the origin.

Proposition 4.27. *The minimal semi-resolutions of the semi-log-canonical singularities are given as follows:*

4.23 (i) *See Brieskorn* [Br 1].

(ii) *identity resolution.*

(iii) $(A, \mathbb{Z}_r, a, \Delta)$ *and* $(A, \mathbb{Z}_r, b, \Delta)$ *attached along the curves* Δ.

(iv) $(A, \mathbb{Z}_{r/2}, \frac{1}{2}a, \Delta)$ *pinched along* Δ.

(v) $(A, \mathbb{Z}_r, a, \Delta)$ *pinched along* Δ.

4.24 (iii) *see Kawamata* [Kaw 2,9] *and* 4.25.

(vi) *The* $\mathbb{Z}_2$ *quotient of a cusp is in* [Kaw1, p. 266] (*the first graph*). *The quotient of a degenerate cusp is constructed as follows. There are some components of type* $(A, 2\Delta)$ *arranged in a chain. At the ends we have either a* (C, Δ) *component or an* $(A, 2\Delta)$ *component with one component of* Δ *pinched. We can also have only one* $(A, 2\Delta)$ *with both components of* Δ *pinched.*

Remark 4.28. Both (iii) and (iv) are of the form $(A, \mathbb{Z}_n, \Delta)$ with Δ pinched. However, if n is even, then the index is $2n$, while if n is odd, then the index is n. This is somewhat surprising.

Remark 4.29. Let (x, X) be a singularity such that $X-x$ is semi-smooth and let $D \subset X$ be the double curve. Let $g: \bar{X} \to X$ be the normalization and $\Delta = g^* D$. Kawamata [Kaw2, 9] defines the notion of the pair $(\bar{X}, \Delta)$ being log-canonical. One can easily prove the following.

Proposition 4.30. *If X is $\mathbb{Q}$-Gorenstein, then X is semi-log-canonical iff $(\bar{X}, \Delta)$ is log-canonical.*

However, there are many cases when $(\bar{X}, \Delta)$ is log-canonical but X is not $\mathbb{Q}$-Gorenstein. For example, this is the case when the minimal semi-resolution has two components $(A, \mathbb{Z}_n, \Delta)$ and $(A, \mathbb{Z}_m, \Delta)$ attached along Δ and $n \neq m$. This follows at once from 4.24, but it is not easy to get this directly. This phenomenon, namely that information is lost after normalization or becomes less accessible, was our main reason for considering non-normal surfaces directly.

§ 5. Deformations of semi-log-canonical singularities and surfaces at the boundary of moduli

In the introduction, we mentioned that one of our motivating problems is that of compactifying moduli spaces for surfaces of general type. In this section we shall first show that the appropriate singularities to permit on the surfaces at the boundaries of moduli spaces are semi-log-canonical (s.l.c.), and secondly we shall make precise the moduli problem to consider.

The first part amounts to showing tht if $\tilde{X} \to \Delta$ is a semi-stable family of surfaces of general type and $X \to \Delta$ is its canonical model, then the special fibre X_0 has just s.l.c. singularities, and that conversely if $X \to \Delta$ is a one-parameter deformation of the surface X_0 that has just s.l.c. singularities, if the general fibre X_t has only RDP singularities and if X is $\mathbb{Q}$-Gorenstein, then X has just canonical singularities. This is made precise in Theorem 5.1, which is a generalization of Corollary 3.6.

Theorem 5.1. *Suppose that $X \to \Delta$ is a $\mathbb{Q}$-Gorenstein one-parameter deformation of the surface X_0.*

(a) *Suppose that X admits a resolution $\tilde{X} \to X$ such that the composite $\tilde{X} \to \Delta$ is semi-stable. Then X has canonical singularities if and only if X_0 has s.l.c. singularities and for $t \neq 0$, X_t has only RDP's.*

(b) *The product $X \times_\Delta \Delta_1$ has canonical (resp. pseudo-terminal, resp. terminal) singularities for all finite base changes $\Delta_1 \to \Delta$ if and only if X_0 has s.l.c. (resp. s.l.t., resp. quotient) singularities and for $t \neq 0$. X_t has only RDP's (resp. RDP's, resp. no singularities). In particular, if X_0 has s.l.c. (resp. s.l.t., resp. quotient) singularities and for $t \neq 0$, X_t has RDP's (resp. RDP's, resp. no singularities), then X has only canonical (resp. pseudo-terminal, resp. terminal) singularities.*

Proof. (a) By localizing, we can replace X_0 by the germ (X_0, P). Suppose that X has canonical singularities. Let V be a component of X_0, D its double curve with reduced structure and W the strict transform of V in $\tilde{X}$. Let μ: $W \to V$ denote the induced map. We can write $K_W \sim \mu^*(K_V + D) + \sum a_i F_i$, where $\cup F_i$ is the union of the exceptional locus of μ with the strict transform of D and $a_i = -1$ if $\dim \mu(F_i) = 1$. Now $K_{\tilde{X}} \sim \pi^* K_X + \sum v_i E_i$, where the E_i are the exceptional divisors in $\tilde{X}$ and every $v_i \geqq 0$, then by the adjunction formula $a_j \geqq -1$ for every j. I.e. the pair (V, D) is log canonical in the sense of Kawamata [Kaw 2] and so by 4.30, X_0 is s.l.c. That X_t has only RDP's is clear.

Conversely, suppose that X_0 has s.l.c. singularities and that X_t has RDP's for $t \neq 0$. Let X' denote a minimal model of $\tilde{X}$, so that there is a factorization $\tilde{X} \to X' \to X$ over Δ. Let V, D be as above and W' the strict transform of V in X'. Let v: $W' \to V$ denote the induced map. Since by 4.30 X_0 has log canonical singularities, we can write $K_{W'} \sim v^*(K_V + D) + \sum a_i F_i$, where $a_i \geqq -1$ and $a_i = -1$ if $\dim v(F_i) = 1$. Since by the adjunction formula $K_{X'}|W' \sim K_{W'} + G$, where G is the reduced sum of the double curves on W', we get $K_{X'}|W' \sim v^*(K_V + D) + \sum b_j H_j$, where the H_j are the exceptional curves of v, $b_j = a_j + 1$ if H_j is a double curve in X'_0 and $b_j = a_j$ otherwise. Then since $\bigcup_{H_j}$ is connected and $K_{X'}$ is numerically effective, if follows that if some H_j is a double curve in X'_0, then every H_j is double and every $a_i = -1$. In this case, $K_{X'} \cdot H_j = 0$ for all j. Now let X'' denote the canonical model of X' and g: $X' \to X''$ the corresponding contraction. Then for every component V of X_0 with strict transform W'' in X'', no exceptional curve of the contraction $W'' \to V$ is a double curve in X''_0. Hence for every exceptional divisor E of the morphism h: $X'' \to X$, the centre of E on X is not P. Since X certainly has canonical (in fact cDV) singularities outside P, it follows that $K_{X''} \sim h^* K_X + B$, where $B \geqq 0$, and so X has canonical singularities.

The proof of (b) is very similar, and so omitted.

5.2. We pause to consider which s.l.c. singularities admit one-parameter $\mathbb{Q}$-Gorenstein deformations to RDP's. We refer to the classification given in Theorems 4.21, 4.23, 4.24.

4.21. (i): All

(ii) The simple elliptic singularities of multiplicity at most 9 [Pi2]. For cusps the problem is unsolved, and we have nothing to contribute. Recently J. Stevens showed that all degenerate cusps are smoothable (unpublished).

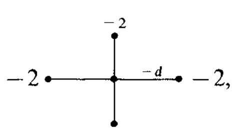
4.23. (i): See Proposition 3.10

(iii) Only those with $a + b = r$.

(iv), (v): None

All these can be proved as in 3.10.

4.24. (iii) The $\mathbb{Z}_2$-quotients of simple elliptic points have dual graph



and admit deformations as above if and only if $3 \leqq d \leqq 6$. There are examples of $\mathbb{Z}_3$- and $\mathbb{Z}_4$-quotients possessing the deformations we see.

(iv) This is at least as hard as determining which cusps are smoothable, but there are examples of arbitrarily high multiplicity.

Theorem 5.3. *Suppose that $X \to \Delta$ is a one-parameter deformation of the normal s.l.c. singularity (X_0, P), and that X has a semistable $\tilde{X}$. Let $X' \to \Delta$ be a minimal model of $\tilde{X} \to \Delta$. Then either the special fibre X'_0 has only RDP's or quotient* *singularities of class T (i.e. normal log terminal singularities) or X has just canonical singularities.*

Proof. Write $X'_0 = \cup V_i$, where $V_1 = V$ is the strict transform of X_0. Let $\tilde{V}$ be the normalization of V and let $f\colon \tilde{V} \to X_0$ be the contraction. Write $K_X|\tilde{W} \sim -E + f^*F$, where F is a $\mathbb{Q}$-divisor on X_0 and $E = \sum v_i C_i$, where the C_i are exceptional curves of f, where $v_i \in \mathbb{Q}$, and $v_i \geqq 0$ since K_X is numerically effective and the exceptional locus of f is negative definite. So in a neighbourhood of $f^{-1}(p) = \cup C_i$, we have $K_{X'}|\tilde{W} \sim -\sum v_i C_i$. By the adjunction formula, $K_{\tilde{V}} \sim -\sum \mu_i C_i$, where $\mu_i = v_i$ if C_i is not a double curve and $\mu_i = v_i + 1$ if C_i is a double curve. The condition that X_0 be log canonical implies that $\mu_i \leqq 1$ for all i, so that if we assume C_r to be double, then $\mu_r = 1$ and $v_r = 0$. Since $K_{X'}$ is numerically effective and $\cup C_i$ is connected, it follows that $v_i = 0$ for all i. Then $K_{X'}$ is numerically trivial on $\cup C_i$, so that if $\pi\colon X' \to X''$ is the contraction to the canonical model, then π contracts $\cup C_i$ to a point. So either no curve C_i is double, in which case X'_0 is normal, or $X'' = X$, and the theorem follows from the classification of log-terminal singularities.

5.4. We shall now make precise the moduli problem that we want to consider. Fix positive integers A, B, N; then we should like to construct a coarse moduli space (at least in the category of algebraic spaces) for those schemes S satisfying the following properties:

(i) S is a reduced projective surface;

(ii) S is connected with only s.l.c. singularities;

(iii) the sheaf $\omega_S^{[N]}$, defined by $\omega_S^{[N]} = j_*(\omega_{S_0}^{\otimes N})$, where $j\colon S_0 \to S$ is the locus of Gorenstein points of S is an ample line bundle;

(iv) the self-intersection K_S^2, defined to be $\frac{1}{N^2}(\omega_S^{[N]} \cdot \omega_S^{[N]})$, equals A;

(v) $\chi(\mathcal{O}_S) = B$.

The functor $\mathcal{M}$: (Schemes) $\to$ (Sets) under consideration is given by $\mathcal{M}(T) = \{$isomorphism classes of flat projective morphisms $f\colon \mathcal{S} \to T$ such that (i)–(v) above hold for every geometric fibre of f and for every geometric point $t \in T$, the natural map $\omega_{\mathcal{S}/T}^{[N]} \otimes k(t) \to \omega_{\mathcal{S}_t}^{[N]}$ is an isomorphism, where $\omega_{\mathcal{S}/T}^{[N]} = j_*(\omega_{\mathcal{S}_0/T}^{\otimes N})$, $j\colon \mathcal{S}_0 \to \mathcal{S}$ being the inclusion of the locus where f is a Gorenstein morphism$\}$. More briefly, we should be concerned with families of semi-log canonical surfaces in which the formation of $\omega^{[N]}$ commutes with specialization; this is required to ensure that any resulting moduli space will be separated.

This approach raises several questions:

(1) Given positive integers A, B, does there exist an integer N, depending only on A and B, and such that whenever S is the special fibre of a relative canonical model $X \to \Delta$ whose general fibre satisfies $K^2 = A$ and $\chi(\mathcal{O}) = B$, the index of S divides N?

(2) Given positive integers A, B, N, is there a number M such that for every s.l.c. surface S with $K_S^2 = A$, $\chi(\mathcal{O}_S) = B$ whose index divides N, the sheaf $(\omega_S^{[N]})^{\otimes M}$ is very ample?

(3) Given an s.l.c. surface S with ω_S ample, is Aut S finite?

(4) If $X \to \Delta$ is a one-parameter family of surfaces such that the special fibre S is an s.l.c. surface with $K_S^2 = A$, $\chi(\mathcal{O}_S) = B$ whose index divides N, and if $\omega_{X/\Delta}^{[N]}$ is invertible, then is the geometric generic fibre also an s.l.c. surface? Of course,

it would be of interest to know the answer to this without the hypothesis that $\omega_{X/\Delta}^{[N]}$ be invertible, but for our moduli problem this generality would be superfluous.

We shall (Corollary 5.5 and Remark 5.6) answer question (4) affirmatively. Question (1) seems rather hard, and we leave it completely open. However, the answer to both (2) and (3) is yes. The first by [Ko 1, 2.1.2] and the second by Iitaka [I], who has shown that a surface of log general type (and so every component of S) has finite automorphism group.

Corollary 5.5. *Suppose that $X \to \Delta$ is a one-parameter deformation of the (germ of the) normal s.l.c. singularity (X_0, P) and that for some $N > 0$, the sheaf $\omega_{X/\Delta}^{[N]}$ is invertible. Then the general fibre X_t has just s.l.c. singularities.*

Proof. One may assume that $X \to \Delta$ admits a semistable resolution. By Theorem 5.3, either X is cannonical, in which case the general fibre has only RDP'S, or every relative minimal model X' has normal special fibre X'_0. In the first case there is nothing to prove, while in the second we argue as follows. Let $f\colon X' \to X$ be a minimal model of X. We can write $K_{X'} \sim f^* K_X + \sum a_i E_i$ as $\mathbb{Q}$-divisors, where the exceptional prime divisors E_i are flat over Δ. So $K_{X'_0} \sim f^* K_{X0} + \sum a_i (E_i|_{X'_0})$; since X_0 has only s.l.c. singularities and $E_i|_{X'_0}$ is an effective integral divisor, it follows that $a_i \geqq -1$ for all i, and so, since X'_t is smooth for $t \neq 0$, we see that X_t has only s.l.c. singularities.

Remark 5.6. If X_0 is not normal, then Corollary 5.5 still holds. For by the classification of ch. 4, we may assume that X_0 is a degenerate cusp or a $\mathbb{Z}_2$-quotient of one; in the first case the techniques of [S-B 1] prove the result, and in the second case one can apply the same approach to the $\mathbb{Z}_2$ cover. (In outline, the proof depends on analyzing the tangent cone at the origin of X if X_0 is not a double point, and by considering the branch locus of a 2:1 projection if X_0 is a double point.)

Corollary 5.7. *The functor $\mathscr{M}$ is coarsely represented by a separated algebraic space M of finite type.*

Proof. Since questions (2)–(4) have affirmative answers, we can apply the results described in ch. 4 of Popp's book [Po] to prove the result.

Remark 5.8. Of course, if we knew how to bound the index in terms of the Chern numbers, as asked in question (1), then for sufficiently divisible N, M would be proper.

Finally, we can sometimes analyze the versal deformation space of a normal s.l.c. singularity, as we did for quotient singularities in ch. 3.

Theorem 5.9. *Suppose that (X, P) is a normal s.l.c. singularity whose canonical cover (Y, Q) has a smooth versal deformation space. Then the irreducible components of* Def(X) *are in $1-1$ correspondence with the partial resolutions $f\colon Z \to X$ such that K_Z is ample and either*

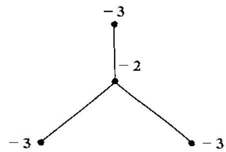
(i) Z has only singularities of class T, or

(ii) f is an isomorphism and X admits a $\mathbb{Q}$-Gorenstein one-parameter deformation to RDP's.

Proof. This is proved in much the same way as Theorem 3.9. The hypothesis that Def(Y) be smooth is used to ensure that if $X = Y/H$, then the locus Fix $(H) \subset$ Def(Y) is irreducible.

Remark 5.10. If we drop the hypothesis that Def(Y) be smooth, then the same result holds except that in (ii) we must count one component for each V of Fix(H) for which the deformation $\mathcal{Y} \to V$ induced from a versal deformation of Y gives, upon taking quotients by H, a deformation of X whose generic fibre has only RDP's.

Example 5.11. Suppose that X is the singularity with dual graph



Then X is s.l.c. and its canonical cover Y is the singularity $x^3+y^3+z^3=0$ on which $H=\langle\sigma\rangle$ acts via $\sigma(x, y, z)=(\omega x, \omega y, \omega^2 z)$, where $\omega=\exp(2\pi i/3)$. Then Def X has five components. One corresponding to Fix $H \subset$ Def Y, one (the Artin component) corresponding to the canonical model of X and one for each of the partial resolutions obrtained by contracting one of [3, 2, 3] configurations in the diagram above. To check that these are all is a little tedious, especially as here there is no maximal resolution.

Example 5.12. We give an example to show that our partial compactifications can lead to disconnected components of the moduli space becoming joined together. To see this, let $Y_0 \subset \mathbb{P}^8$ be the projective cone over an octic elliptic curve in $\mathbb{P}^7$. Then there are (projective) smoothings $Y \to \Delta$ and $Y' \to \Delta$ of Y_0, where for $t \neq 0$, Y_t and Y_t' are octic Del Pezzo surfaces, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the rational ruled surface $\mathbb{F}_1$, respectively. Let $X \to \Delta$ and $X' \to \Delta$ be the double covers of Y and Y', respectively, branched in a general quartic hypersurface section, both of which cut out the same smooth curve on Y_0. Then for $t \neq 0$, both X_t and X_t' have $K^2=16$, $\chi=10$, while K_{X_t} is even and $K_{X_t'}$ is not. Hence X_t and X_t' are not homeomorphic, and so lie on different components of the moduli space; introducing the moduli point of $X_0(\cong X_0')$ connects these two components together.

Notice also that the singularities of X_0 are of multiplicity eight, so that X_0 is asymptotically unstable [M 2].

§ 6. Three dimensional terminal singularities

The aim of this chapter is to settle some questions that were left open in Mori [Mo 1]. This will complete the classification of three dimensional terminal singularities.

Most of the work was done earlier by various persons, we now recall their relevant results.

Theorem 6.1. (Reid [Re3])

(i) *A three dimensional terminal Gorenstein singularity is either smooth or an isolated cDV point.*

(ii) *if* (y, Y) *is a 3-dimensional terminal singularity then there exists a 3-dimensional terminal Gorenstein singularity* (x, X) *and an action of some cyclic group* $\mathbb{Z}_m$ *on* (x, X) *such that* $(y, Y)=(x, X)/\mathbb{Z}_m$. *Moreover the action is free outside* $x \in X$.

Convention 6.2. In order to simplify notation we shall use the following system. If it is understood that $\mathbb{Z}_m$ acts on certain objects, then we fix a primitive m^{th} root of unity ε, and a generator σ of $\mathbb{Z}_m$. If ? is an eigenvector of the action satisfying $\sigma(?)=\varepsilon^a \cdot ?$ then we say that ? has weight a and write $w(?)=a$.

Theorem 6.3. (Danilov [D], Morrison-Stevens [M-S]) *Terminal singularities that are quotients of a smooth point are the following:* $\mathbb{C}^3/\mathbb{Z}_m$ *where* $\mathbb{Z}_m$ *acts with weights* $(1, a, m-a)$ *on the coordinates and* $(a, m)=1$.

Theorem 6.4. (Mori [Mo1]) *Assume that* $\mathbb{Z}_m (m>1)$ *acts on a cDV point, freely outside the origin, and the quotient is terminal. Then one can take a suitable embedding of the cDV point into* $\mathbb{C}^4$ *and suitable coordinates such that the equation g of the cDV point and the action of the group are given by one of the following cases:*

(1) *m arbitrary*, $g=xy+f(z, u^m)$, $w(x)+w(y)=m$, $w(z)=0$, $w(x)$, $w(y)$ *and* $w(u)$ *prime to m.*

(2) $m=2$, $g=x^2+y^2+f(z, u)$, $f \in m^4$, *weight of* x, y, z, u *are* $1, 0, 1, 1$.

(3) $m=2$, $g=u^2+f(x, y, z)$, $f \in m^3$, xyz *or* y^2z *appears in f with non-zero coefficient, weights of variables are* $1, 1, 0, 1$.

(4) $m=2$, $g=u^2+x^3+f(y, z)x+h(y, z)$, $h \notin m^5$, *weights are* $0, 1, 1, 1$.

(5) $m=3$, $g=u^2+f(x, y, z)$, $f \in m^3$, *cubic term of f is* $x^3+y^3+z^3$, x^3+yz^2 *or* x^3+y^3, *weights are* $1, 2, 2, 0$.

(6) $m=4$, $g=x^2+y^2+f(z, u^2)$, *weights are* $1, 3, 2, 1$.

(It is understood that f and h are such that g is an eigenvector of $\mathbb{Z}_m$.*)*

Furthermore, if f (and h) are sufficiently generic, then the quotient is terminal.

The aim of this chapter is to complete the classification by proving the following.

Theorem 6.5. *Assume that in any of the above situations f (and h) are chosen such that* $g=0$ *has an isolated singularity at the origin and the action of* $\mathbb{Z}_m$ *on* $g=0$ *is free outside the origin. Then the quotient is terminal.*

Remark 6.6. (i) (1) should be considered as the main series and (2)–(6) as the exceptional ones. We shall use two different methods to handle these two cases. Neither of the methods seems to work for the other case.

(ii) We shall give two proofs for (1). The first uses 3.6. The second, communicated to us by Mori, does not use any hard result from three dimensional geometry.

6.7. Proof in case of (1). First proof. z is invariant under the group action and therefore $(xy+f(0, u^m)=0)/\mathbb{Z}_m$ is a hyperplane section of the 3-dimensional quotient singularity. $xy+f(0, u^m)=0$ is either a n.c. point ($f \equiv 0$) or a DV point

($f \not\equiv 0$), and hence the quotient is semi-log-terminal by 4.23. Therefore, 3.6 implies that the 3-dimensional quotient is terminal (the n.c. case is the same).

Second proof (by S. Mori). Since $\mathbb{Z}_m$ acts freely on X, z does not divide $f(z, u^m)$. Therefore by Puiseux expansion we can write $f(z, u^m) = \Pi(u^m - h_i(v))$, where $v^n = z$ for some n. By 3.3 it is sufficient to show that the $\mathbb{Z}_m$ quotient of $xy + \Pi(u^m - h_i(v)) = 0$ is terminal. If we blow up the ideal $(x, u^m - h_1(v))$, then we get a small morphism $\pi_1: X_1 \to X$ and easy computation shows that the only singularities of X_1 are the $\mathbb{Z}_m$ quotient of $xy + u^m - h_1(v) = 0$ and the $\mathbb{Z}_m$ quotient of $xy + \Pi_{i \geq 2}(u^m - h_i(v)) = 0$. Repeating this procedure, we get a small morphism $\pi: \bar{X} \to Y$ such that the only singularities of $\bar{X}$ are of the form $(xy + u^m - h_i(v) = 0)/\mathbb{Z}_m$. By a suitable choice of $v_i = v g_i(v)$, this becomes $(xy + u^m - v_i^s = 0)/\mathbb{Z}_m$. By 6.4 and 6.8 (v) $\bar{X}$ has terminal singularities. Since π is small $K_{\bar{X}} = \pi^* K_X$ and consequently X is terminal as well.

Facts 6.8. For convenient references here we gather some facts that will be used in the sequel.

(i) For any f consider the family $f(x_1, \ldots, x_{n-1}, t x_n) = 0$; this is a flat family of hypersurfaces. If $t \neq 0$ then the hypersurface is isomorphic to $f(x_1, \ldots, x_n) = 0$; if $t = 0$, to $f(x_1, \ldots, x_{n-1}, 0) = 0$.

(ii) If $f(t, x_1, \ldots, x_n)$ is a family of hypersurfaces such that the multiplicity of $f(t, x_1, \ldots, x_n) \subset \mathbb{C}^n$ at the origin is independent of t, then the family $B_0 f(t, \underline{x})$ is also flat, where $B_0 f(t, \underline{x})$ is the blow-up of $f(t, \underline{x}) \subset \mathbb{C}^n$ at the origin.

(iii) [MT, Re2] Assume the $\sum a_I x^I$ defines a rational singularity at the origin. Then the following condition is satisfied:

(*) If $0 \leqq v_1, \ldots, v_n$ are such that for any $I = (i_1, \ldots, i_n)$, $a_I \neq 0 \Rightarrow \sum i_j v_j \geqq 0$, then we necessarily have $\sum v_i > 1$.

For any given polynomial it is standard linear algebra to check if (*) is satisfied or not. We shall omit such computations.

(iv) [Re2, §4] Assume that $\sum a_I x^I$ satisfies condition (*). The for generic $\lambda_I \neq 0$ the singularity of $\sum \lambda_I a_I x^I$ is rational at the origin.

(v) For certain polynomials one can always find a change of variables $\bar{x}_i = \mu_i x_i$ such that $\sum \lambda_I a_I x^I = \sum a_I \bar{x}^I$ (e.g. $\sum a_i x_i^{b_i}$). We shall call such a polynomial generic. A generic polynomial defines a rational singularity iff (*) is satisfied. It is easy to check whether a polynomial is generic or not, and we shall omit such computations.

(vi) [Hin] Let $\mathbb{Z}_m$ act on $X = (f(x_1, \ldots, x_n) = 0)$ such that the x_i are eigenvectors. Assume that the action is free in codimension one. Then $X/\mathbb{Z}_m$ is Gorenstein iff $\sum w(x_i) - w(f) \equiv 0(m)$.

Proposition 6.9. *Let $X \subset \mathbb{C}^4$ be a cDV point and let $B_0 X$ be the blow-up of the origin. Then $B_0 X$ has rational singularities only.*

Proof. In a generic coordinate system X is given by $f(x_1, \ldots, x_4)$ and by 6.8 (i) is specializes to $f(x_1, x_2, x_3, 0) = 0$, which is the trivial deformation of a DuVal singularity. By a suitable change of coordinates we can assume that $f(x_1, x_2, x_3, 0)$ is in one of the standard forms as in [Re2, 0.2]. In each case one can write down explicit equations for $B_0 f(x_1, x_2, x_3, 0)$, and it is easy to check that all these equations are generic and satisfy (*). Therefore $B_0 f(x_1, x_2, x_3, 0)$ has rational singularities only.

By 6.8 (ii) B_0X is a flat deformation of $B_0f(x_1, x_2, x_3, 0)$. Hence by [E1] it too has rational singularities.

Proposition 6.10. *Let $X \subset \mathbb{C}^4$ be an isolated cDV point with a $\mathbb{Z}_m$-action which is free outside the origin. Let B_0X be the blow-up of the origin and let $E \subset B_0X$ be the exceptional divisor. Assume that*

(i) The induced $\mathbb{Z}_m$-action is faithful on every component of E and

(ii) $B_0X/\mathbb{Z}_m$ has canonical singularities only.

Then $X/\mathbb{Z}_m$ is terminal.

Proof. Let $Y = X/\mathbb{Z}_m$, $q: X \to Y$ the natural map. Let $B_0Y = B_0X/\mathbb{Z}_m$, $g: B_0X \to X$, $h: B_0Y \to Y$, $\bar{g}: B_0X \to B_0Y$ be the natural maps. Finally, let $f: Y' \to B_0Y$ be a resolution of singularities.

Since X has a double point at the origin, an easy computation gives that $K_{BX} = g^*K_X + E$. By (i) the map $\bar{q}: BX \to BY$ is étale in codimension one. Hence $K_{BX} = \bar{q}^*K_{BY}$, and similarly $K_X = q^*K_Y$. This gives that $K_{BY} = h^*K_Y + F$. Since BY has canonical singularities $K_{Y'} = f^*K_{BY} + G$, where G is effective. Therefore $K_{Y'} = f^*K_{BY} + G = (h \circ f)^*K_Y + f^*F + G$.

If $D \subset Y'$ is an $h \circ f$ exceptional divisor, then $f(D) \subseteq F$. Thus D appears in f^*F with a positive coefficient. This shows that Y is terminal.

Remark 6.11. For $m=1$ this gives a simple proof of the terminality of isolated cDV points.

Proposition 6.12. *Let $X \subset \mathbb{C}^4$ be a cDV singularity with a $\mathbb{Z}_2$-action which is free in codimension one. Then $X/\mathbb{Z}_2$ is canonical iff it is Gorenstein outside the origin.*

Proof. The condition is clearly necessary. To prove sufficiency we employ the notation of 6.10. Let $X = (\phi = 0)$.

First we consider the cases when the condition 6.10 (i) is not satisfied. This can happen in three cases:

(i) E is reduced, $w(x_i) = 1$ for every i. Then $w(\phi) = 0$ thus by 6.8 (vi) Y is Gorenstein, hence canonical.

(ii) $\phi = x_1x_2 + \Psi(x_3, x_4)$, $w(x_1) = 0$, $w(x_i) = 1$ for $i > 1$. Then $w(\phi) = 1$ and again Y is Gorenstein.

(iii) $\phi = x_1^2 + \Psi(x_2, x_3, x_4)$, $w(x_i) = 1$ for $i > 1$, and $w(\phi) = 0$. If Ψ contains a degree $\leqq 2$ term, then we are in one of the above situations. By weight considerations Ψ cannot contain a cubic term, but then $\phi = 0$ is not a cDV point.

Therefore we may assume for the rest of the proof that the $\mathbb{Z}_2$-action on any component of E is faithful. Therefore, we obtain that $K_{BY} = h^*K_Y + F$.

Let $D_i \subset Y'$ be the f-exceptional divisors, and let $K_{Y'} = f^*K_{BY} + \sum a_i D_i$. BY has only log-terminal singularities; hence $a_i > -1$. Since $2K_{BY}$ is Cartier, the a_i are half-integers. Therefore $a_i \geqq -\frac{1}{2}$. Let $f^*F = \sum b_i D_i$, where $b_i > 0$ if $f(D_i) \subseteq F$ and $b_i = 0$ otherwise. Furthermore, since $2F$ is Cartier, the b_i are half-integers.

$$K_{Y'} = f^*K_{BY} + \sum a_i D_i = (h \circ f)^*K_Y + \sum (a_i + b_i) D_i.$$

If $f(D_i) \subseteq F$, then $b_i \geqq \frac{1}{2}$, $a_i \geqq -\frac{1}{2}$. Hence $a_i + b_i \geqq 0$. If $f(D_i) \nsubseteq F$, then $b_i = 0$ and $a_i \geqq 0$ since by assumption Y is Gorenstein outside the origin. This proves that Y is canonical.

6.13 Proof of 6.5 in cases (2)–(6). Let $Y=X/\mathbb{Z}_m$ be one of the cases. We want to check the conditions of 6.10. Condition (i) is easy to see. By 6.9 all points of BX are rational. Therefore all the singularities of BY are rational again. Therefore the Gorenstein points of BY are canonical and we have to check the non-Gorenstein points only. These come from points of the $\mathbb{Z}_m$-action on E. These in turn come from fixed lines in $\mathbb{C}^4$. If this line is the x_j axis, 6.8 (vi) and the computation of the blowing-up yield that the corresponding point in BY is Gorenstein iff $\sum w(x_i)-w(\phi)-w(x_j)\equiv 0(m)$.

Armed with this information, we can consider the cases.

(2) All the points of BY are Gorenstein.

(3) The z-axis gives the only non-Gorenstein point. On BX this gives a cDV point (just compute the blow-up); hence by 6.12 the quotient is canonical.

(4) We have to consider the x-axis only. On BX this gives a smooth point; hence the quotient is canonical (even terminal) by 6.3.

(5) Again the x-axis is the only fixed line to be checked. On BX we get a smooth point and again by 6.3 all isolated $\mathbb{Z}_3$ quotients of $\mathbb{C}^3$ are canonical.

(6) The z-axis is the only one to be checked (both for $\mathbb{Z}_4$ and $\mathbb{Z}_2$ fixed points). Here BX is given by the equation $x'^2+y'^2+z'^{-2}f(z', z'^2u'^2)=0$, and the action is given by weights 3, 1, 2, 3, which is the same we started with. The action of $\mathbb{Z}_4$ on the z-axis is not free. Therefore $f(z,0)\not\equiv 0$. Thus we can use induction on the smallest z-power appearing in f. The starting point is the linear case. This is the same as $\mathbb{C}^3/\mathbb{Z}_4(1, 1, 3)$, which is terminal by 6.3. Therefore we are done in this case as well.

This completes the proof of 6.5.

§ 7. Deformation of minimal singularities

The aim of this chapter is to give a simple geometric proof that the deformation of a cyclic quotient singularity is again a cyclic quotient.

The idea of the proof is the following.

7.1. Let $f\colon X\to\Delta$ be a 1-parameter deformation of the singularity $f^{-1}(0)$. Let $s\colon \Delta\to X$ be a section. Let $X_t=f^{-1}(t)$ and $x_t=s(t)$. We say that (x_t, X_t) is a small deformation of (x_0, X_0). We would like to get some information about the tangent cone of (x_t, X_t) in terms of the tangent cone of (x_0, X_0). We proceed as follows.

First assume that X is normally flat along $s(\Delta)$. Then the tangent cone of (x_t, X_t) is a flat deformation of the tangent cone of (x_0, X_0) This is the good case.

Otherwise, let $X^0=X$, $s^0=s$. If we have X^i and s^i, then let X^{i+1} be the blow up of $s^i(0)\in X^i$ and $s^{i+1}\colon \Delta\to X^{i+1}$ the natural map. Let F^{i+1} be the exceptional divisor of $X^{i+1}\to X^i$.

By Hironaka's resolution theorem there is an m such that X^m is normally flat along $s^m(\Delta)$. This means that (x_t, X_t) for $t\neq 0$ is a normally flat deformation of $(s^m(0), F^m)$. If we can control the singularities of F^m in terms of (x_0, X_0), then we get some information about (x_t, X_t).

For the proof the natural context is the class of minimal singularities studied in [Ko 1, 3.4]. We recall some of the definitions and results.

Proposition-Definition 7.2. [Ko 1, 3.4.1–2.] *Let (x, X) be a reduced Cohen-Macaulay singularity. Then* $\operatorname{mult}_x X \geqq \operatorname{emdim}_x X - \dim_x X + 1$. *$(x, X)$ is called minimal if equality holds and tangent cone is reduced.*

Proposition 7.3. ([Ko 1, 3.4], Sally [Sa] and Xambó [X]). *Let (x, X) be a minimal singularity. Then the projectivized tangent cone $F = \cup F_i$ is a subvariety of some $\mathbb{P}^n$ of minimal degree, connected in codimension one. Its irreducible components F_i satisfy* $\deg F_i = \operatorname{emdim} F_i - \dim F_i$ *and any two intersect in a linear subspace. For any $y \in F$ the singularity (y, F) is minimal again.*

The above result implies that the deformation of a minimal singularity is minimal. Here we shall need a more precise result which requires a detailed study of the tangent cones. The following result describes the irreducible components.

Proposition 7.4. ([Ber, Harr])

(i) *A reduced, irreducible, non-degenerate curve $C \subset \mathbb{P}^k$ satisfying* $\deg C = k$ *is a rational normal curve.*

(ii) *A reduced, irreducible, non-degenerate surface $F \subset \mathbb{P}^k$ satisfying* $\deg F = k - 1$ *is one of the following;*

(a) *Veronese in $\mathbb{P}^5$;*

(b) *a cone over a rational normal curve;*

(c) *a ruled surface* $\operatorname{Proj}_{\mathbb{P}^1}(\mathcal{O}(a) \otimes \mathcal{O}(b))$ *embedded by $\mathcal{O}(1)$ for $a, b \geqq 1$.*

Definition 7.5. Let $C = \cup C_i \subset \mathbb{P}^n$ be a reduced, connected curve of degree n. We associated to it a labeled graph as follows. To each curve C_i we associate a vertex and label it $\deg C_i$. We connect two vertices if the corresponding curves intersect. We shall call this the graph of the curve and denote it by $G(C)$.

Definition 7.6. Let (x, X) be a minimal surface singularity, and let C be the projectivized tangent cone of (x, X). Then by 7.3 C satisfies the conditions of 7.5. $G(C)$ will be denoted by $G(x, X)$ and called the graph of (x, X).

Definition 7.7. Assume that C and C' both satisfy the conditions of 7.5. We shall say that $G(C')$ is a simplification of $G(C)$ if it can be obtained from $G(C)$ by repeated application of the following procedures:

(i) Replace $G(C)$ with a connected subgraph, keeping the labeling fixed; or

(ii) change the labeling of a vertex to 1; or

(iii) if a vertex v is connected with vertices $v_1, \ldots, v_2$ and v_1 is labeled 1, then erase v and connect v_1 with each of $v_2, \ldots, v_k$; or

(iv) if $v_1, \ldots, v_k$ are vertices such that any two of them are connected, then replace $G(C)$ with some $G(C'')$ where $\deg C'' \leqq k$.

The reason for this definition is the following.

Lemma 7.8. *Let $F = \cup F_i \subset \mathbb{P}^n$ be a reduced surface of degree $n - 1$, connected in codimension one. Let $\cup C_j = C = F \cap H$ be a hyperplane section of F. Let $x \in F$ be a closed point. Then (x, F) is a minimal surface singularity and $G(x, F)$ is a simplification of $G(C)$.*

Proof. The first statement is a special case of 7.3. As for the second part, let $X \subseteq F$ be the union of those components that contain x. Then $G(x, G) = G(X, F)$, and $G(H\cap X)$ is a connected component of $G(C)$. Therefore, we may assume that $X=F$.

If $x\in H$, then $\mathrm{mult}_x X \leqq \#$ (components of C through x); this corresponds to 7.7 (iv).

Otherwise, we have to check the possible configurations of x, F_i, $F_i\cap H$; using 7.4 it is easy to see that the procedures 7.7 (ii) and (iii) account for all the possibilities.

This in turn implies the following

Proposition 7.9. *Let $(x_0, X)\supset(x_0, X_0)$ be a 1-parameter deformation. Let $BX\to X$ be the blow-up of $x_0\in X$ and let $F\subset BX$ be the exceptional divisor. If (x_0, X_0) is a minimal singularity then so is (y, F) for any y and $G(y, F)$ is a simplification of $G(x_0, X_0)$.*

Proof. The statement is trivial if X is smooth. Otherwise 7.2 easily gives that $\mathrm{mult}_{x_0} X=\mathrm{mult}_{x_0} X_0$. Let C (resp. F) be the projectivized tangent cone of (x_0, X_0) (resp. (x_0, X)). Then C is a hyperplane section of F. This implies that F is reduced. Hence (x_0, X) is minimal. The rest follows from 7.8.

Using the above result the argument given in 7.1 leads at once to the main result of this section:

Theorem 7.10. *Let (x_0, X_0) be a minimal surface singularity and let (x_t, X_t) be a small deformation. Then there exists a minimal singularity (y, F) such that $G(y, F)$ is a simplification of $G(x_0, X_0)$ and the tangent cone of (x_t, X_t) is a flat deformation of the tangent cone of (y, F).*

In order to use this theorem we should know which surface singularities are minimal and how to compare $G(x, X)$ with more customary invariants.

Proposition 7.11. ([Ko 1, 3.4.9–10]) *A normal surface singularity is minimal iff it is rational and its fundamental cycle is reduced.*

Remark 7.12. Let (x, X) be a rational surface singularity with minimal resolution $f\colon Y\to X$. Let $E_i\subset Y$ be the exceptional curves. Let $b_i=-E_i^2$ and $a_i=\#\{j\neq i\colon E_j\cap E_i\neq\emptyset\}$. The fundamental cycle is reduced iff $b_i\geqq a_i$ for every i. By a result of Artin [A 1], in this case $G(x, X)$ can be constructed as follows.

The vertices correspond to the those i such that $b_i>a_i$. They are labeled b_i-a_i. The vertices corresponding to E_i and E_j are connected iff there is a chain of curves $E_i=E_{k_0}, \ldots, E_{k_m}=E_j$ such that $E_{k_s}\cap E_{k_{s+1}}\neq\emptyset$ and $b_{k_s}-a_{k_s}=0$ for all $0<s<m$.

Definition 7.13. With notation as in 7.12 we define $\mathrm{end}(X)$ to be the number of those curves E_i that intersect at most one other exceptional curve E_j.

Remark 7.14. (i) From [Br 1] it follows that (x, X) is a cyclic quotient iff $\mathrm{end}(x, X)\leqq 2$.

(ii) Using 7.12 we can compute $\mathrm{end}(X)$ from $G(x, X)$ as follows: $\mathrm{end}(X)$ is the number of vertices V_i such that two vertices V_j and V_k are connected with V_i only if V_j and V_k are connected.

(iii) It is quite easy to see from this that if $G(C')$ is a simplification of $G(C)$, then $\mathrm{end}\, G(C')\leqq \mathrm{end}\, G(C)$.

This easily yields the following:

Corollary 7.15. *Let (x_t, X_t) be a small deformation of a minimal surface singularity (x, X). Then* $\operatorname{end}(x_t, X_t) \leq \operatorname{end}(x, X)$. *In particular if (x, X) is a cyclic quotient then so is (x_t, X_t).*

This latter result was conjectured by Riemenschneider. It should be compared with the result of [EV].

Remark 7.16. Let (x_0, X_0) be a quotient singularity. From 7.12 it is clear that the tangent cone carries no information about the -2 curves of the minimal resolution. Therefore the above results do not give complete information about the possible deformations of (x_0, X_0).

(ii) Most of the non-cyclic quotient singularities are minimal; they have three ends. Our results however do not imply that their deformations are again quotient singularities.

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