Math 461 Homework 8

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(1) Let

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

be the sphere with center the origin and radius 1. Let $N = (0, 0, 1) \in S^2$ be the north pole. Let $F \colon S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection of the sphere from the north pole onto the xy-plane. Recall that in class we derived the formulas

$$F \colon S^2 \setminus \{N\} \to \mathbb{R}^2, \quad F(x, y, z) = \frac{1}{1 - z}(x, y)$$

and

$$F^{-1} \colon \mathbb{R}^2 \to S^2 \setminus \{N\}, \quad F^{-1}(u, v) = \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1)$$

for F and its inverse F^{-1}.

(a) Check the formulas by showing that

(i) $F^{-1}(F(x, y, z)) = (x, y, z)$ for all $(x, y, z) \in S^2 \setminus \{N\}$, and

(ii) $F(F^{-1}(u, v)) = (u, v)$ for all $(u, v) \in \mathbb{R}^2$.

[Hint: For (i), use the equation $x^2 + y^2 + z^2 = 1$ of the sphere S^2 to simplify the expression (replace $x^2 + y^2$ by $1 - z^2$).]

(b) Check the formula for F^{-1} by showing that the vector $(x, y, z) = F^{-1}(u, v)$ satisfies the equation $x^2 + y^2 + z^2 = 1$ of the sphere S^2 for all $(u, v) \in \mathbb{R}^2$.

(c) Using part (b) or otherwise, find a solution of the equation

$$a^2 + b^2 + c^2 = d^2$$

such that a, b, c, d are positive integers.

(2) In class we showed that the image of a spherical circle C on S^2 under stereographic projection is either a circle or a line in the plane. Describe the image precisely in the following cases.

(a) $C_1 = \Pi_1 \cap S^2$ where $\Pi_1 \subset \mathbb{R}^3$ is the plane with equation

$$x + 2y + 3z = 3.$$

(b) $C_2 = \Pi_2 \cap S^2$ where $\Pi_2 \subset \mathbb{R}^3$ is the plane with equation

$$3x + 4y + 5z = 6.$$

[Hint: Recall that the image of C is a line if C contains the north pole N and a circle otherwise. In the first case the line is just the intersection of the plane Π containing C with the xy-plane. In the second case we can find the equation of the image circle using the algebraic formula for the inverse of F: $F^{-1}(u,v) = (2u, 2v, u^2+v^2-1)/(u^2+v^2+1)$.]

(3) Let $F\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection. Let $P = (x,y,z) \in S^2 \setminus \{N\}$ and $Q = F(P)$.

(a) Compute the distance $d_{\mathbb{R}^2}(O,Q)$ from the origin O to Q as a function of z.

(b) Deduce from your formula in part (a) that $d_{\mathbb{R}^2}(O,Q) \to \infty$ as $Q \to N$ (equivalently, as $z \to 1$).

(4) Let $R\colon S^2 \to S^2$ be the reflection in the xy-plane. Let $F\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection. Notice that R interchanges the north pole $N = (0,0,1)$ and the south pole $S = (0,0,-1)$, and $F(S) = (0,0)$. It follows that the composition $T = F \circ R \circ F^{-1}$ defines a bijection

$$T\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}.$$

This is the transformation of the plane (with the origin removed) corresponding to the reflection R of S^2 in the equator under stereographic projection.

(a) Determine a formula for $R(x,y,z)$.

(b) Determine a formula for $T(u,v)$.

(c) Show that T fixes the circle

$$\mathcal{C} = \{(u,v) \mid u^2 + v^2 = 1\} \subset \mathbb{R}^2$$

with center the origin and radius 1 and interchanges the inside and the outside of $\mathcal{C}$ (that is, if $P \in \mathbb{R}^2 \setminus \{(0,0)\}$ is inside $\mathcal{C}$ then $T(P)$ is outside $\mathcal{C}$ and vice versa). The transformation T is called *inversion* in the circle C.

(5) In class we showed that if γ is a curve on S^2 (not passing through the north pole N), parametrized by

$$\mathbf{x} \colon [a,b] \to S^2 \subset \mathbb{R}^3, \quad t \mapsto \mathbf{x}(t) = (x(t), y(t), z(t)),$$

and $F(\gamma)$ is the image of γ in $\mathbb{R}^2$ under stereographic projection, parametrized by

$$(u,v) \colon [a,b] \to \mathbb{R}^2, \quad t \mapsto (u(t), v(t)) = F(x(t), y(t), z(t)),$$

then the length of γ can be computed in the uv-plane by the formula

$$\text{length}(\gamma) = \int_a^b \sqrt{x'^2 + y'^2 + z'^2}\, dt = \int_a^b \frac{2}{u^2 + v^2 + 1} \sqrt{u'^2 + v'^2}\, dt.$$

In this question and the next one we will check this formula in two cases.

Let C be a spherical circle on S^2 with center N and spherical radius r (see HW5Q6).

(a) Show that C is equal to the intersection $S^2 \cap \Pi$ where $\Pi \subset \mathbb{R}^3$ is the horizontal plane with equation $z = \cos r$.

(b) Show that the image $F(C)$ of C is a circle in $\mathbb{R}^2$ with center the origin and determine its radius as a function of r.

(c) Determine the circumference of C by computing the integral in the uv-plane above for $\gamma = C$. Check your answer agrees with HW5Q6.

[Hint: For (b) use Q3a. For (c) note that $\int_a^b \sqrt{u'^2 + v'^2}\, dt$ is the circumference of the circle $F(C)$. Also, the factor $\frac{2}{u^2+v^2+1}$ is constant on the curve $F(C)$. So it is not necessary to parametrize the curve to compute the integral in this case.]

(6) Let L be a spherical line on S^2 passing through $N = (0,0,1)$ and $S = (0,0,-1)$ (so L is a line of longitude). In this question we will compute the length of the shorter arc γ of the spherical line L from S to a point $P \in L \setminus \{N\}$ using the integral formula in the uv-plane of Q5. Choosing coordinates appropriately, we may assume that $L = S^2 \cap \Pi$ where $\Pi \subset \mathbb{R}^2$ is the plane with equation $y = 0$ and the x-coordinate of P is positive.

(a) Show that the image of $L \setminus \{N\}$ under stereographic projection is the u-axis in the uv-plane, and the image of γ is the segment of the u-axis from the origin O to the point $Q = F(P)$.

(b) Let $Q = (b, 0)$, and parameterize the line segment OQ by

$$(u, v)\colon [0, b] \to \mathbb{R}, \quad t \mapsto (u(t), v(t)) = (t, 0).$$

Now compute length(γ) as a function of b using the integral formula in the uv-plane (see Q5).

(c) Finally, show that $b = \tan(\angle ONP)$ and $\angle ONP = \angle SOP/2$ (where O denotes the origin in $\mathbb{R}^3$). Deduce that the formula for length(γ) in (b) agrees with our earlier formula: the length of the shorter arc of the spherical line connecting two points X and Y equals $\angle XOY$.

(7) Let

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

be the sphere with center the origin and radius 1 in $\mathbb{R}^3$. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the north and south poles. Let

$$R = \{(u, v) \mid 0 \leq u < 2\pi, -1 < v < 1\} = [0, 2\pi) \times (-1, 1) \subset \mathbb{R}^2,$$

a rectangular region in the uv-plane. The *Gall–Peters projection* is a bijection

$$G\colon S^2 \setminus \{N, S\} \to R$$

which may be defined geometrically as follows: Consider the cylinder

$$C = \{(x, y, z) \mid x^2 + y^2 = 1 \text{ and } -1 < z < 1\} \subset \mathbb{R}^3$$

with axis the interval $(-1,1)$ on the z-axis and radius 1. So, the sphere S^2 lies inside the cylinder C and touches it along its equator. There is a bijection

$$G_0 \colon S^2 \setminus \{N, S\} \to C$$

given by projecting away from the z-axis along lines perpendicular to the z-axis. We can "roll out" the cylinder C to obtain the rectangular region $R = [0, 2\pi) \times (-1, 1)$, then G_0 gives the Gall–Peters projection $G \colon S^2 \setminus \{N, S\} \to R$.

(a) Using cylindrical polar coordinates or otherwise, show that the inverse G^{-1} of the Gall–Peters projection is given by

$$G^{-1}(u, v) = (\sqrt{1 - v^2}\cos u, \sqrt{1 - v^2}\sin u, v).$$

(b) Show that the Gall–Peters projection preserves areas.

[Hint: Recall from MATH 233 that if $T \subset R$ is a region, then the area of the corresponding region $G^{-1}(T) \subset S^2$ of the sphere is given by

$$\int_T \left\| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right\| du\, dv$$

where we have written $\mathbf{x}(u, v) = G^{-1}(u, v)$. It follows that the area of $G^{-1}(T)$ equals the area of T if $\|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\|$ is constant, equal to 1 (why?).]