Tuesday 11/13/18. MATH 461 HW7 SOLUTIONS.

1. a. $T(\underline{x}) = \underline{x} - 2\left(\frac{\underline{x}\cdot\underline{n}}{\underline{n}\cdot\underline{n}}\right)\underline{n}$ where $\underline{n}$ is the normal vector to Π.

Π : $x+2y+z=0$, i.e., $\underline{x}\cdot\underline{n}=0$ where $\underline{n}=\begin{pmatrix}1\\2\\1\end{pmatrix}$.

$$\text{So } T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - 2\left(\frac{x+2y+z}{1^2+2^2+1^2}\right)\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}x\\y\\z\end{pmatrix} - \frac{2}{6}\begin{pmatrix}x+2y+z\\2x+4y+2z\\x+2y+z\end{pmatrix}$$

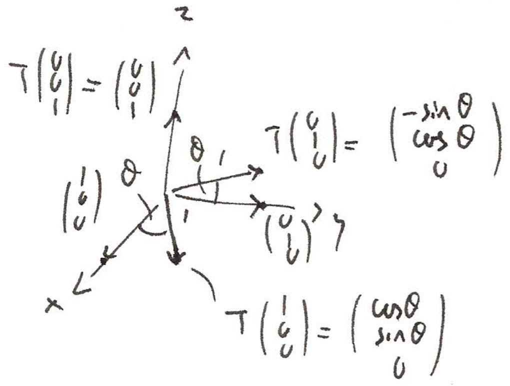
$$= \frac{1}{3}\begin{pmatrix}2x-2y-z\\-2x-y-2z\\-x-2y\ \ 2z\end{pmatrix} = \underbrace{\frac{1}{3}\begin{pmatrix}2&-2&-1\\-2&-1&-2\\-1&-2&2\end{pmatrix}}_{A.}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

b.

$$A\underline{n} = \frac{1}{3}\begin{pmatrix}2&-2&-1\\-2&-1&-2\\-1&-2&2\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-3\\-6\\-3\end{pmatrix} = -\begin{pmatrix}1\\2\\1\end{pmatrix} = -\underline{n} \checkmark$$

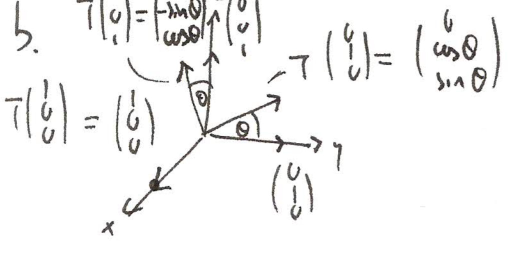
$$A^TA = \left(\frac{1}{3}\right)^2\cdot\begin{pmatrix}2&-2&-1\\-2&-1&-2\\-1&-2&2\end{pmatrix}\begin{pmatrix}2&-2&-1\\-2&-1&-2\\-1&-2&2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}9&0&0\\0&9&0\\0&0&9\end{pmatrix} = I \checkmark.$$

2. a.



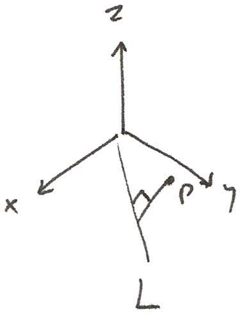
$$\rightsquigarrow \quad T(\underline{x}) = \begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}\cdot\underline{x}$$

b.

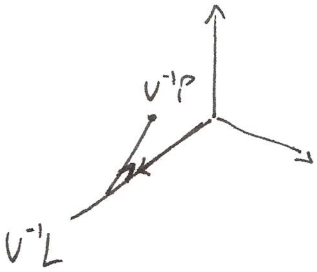


$$\rightsquigarrow \quad T(\underline{x}) = \begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}\cdot\underline{x}$$

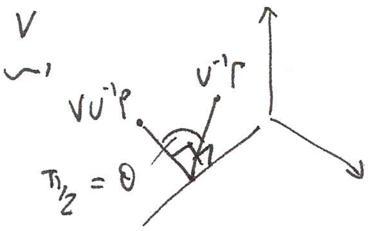
3a.



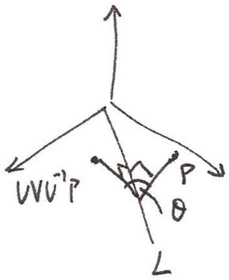
$\overset{U^{-1}}{\rightsquigarrow}$



$\overset{V}{\rightsquigarrow}$



$\overset{U}{\rightsquigarrow}$



See UVU^{-1} = rotation thru angle $\theta = \pi/2$ about the line L.

(U^{-1} moves L to the x axis, V rotates about the x-axis, U moves the x-axis back to L.)

b.

$T(\underline{x}) = A\underline{x}$

where

$$A = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 & 0 \\ \sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi/2 & -\sin\pi/2 \\ 0 & \sin\pi/2 & \cos\pi/2 \end{pmatrix} \cdot \begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) & 0 \\ \sin(-\pi/4) & \cos(-\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

4.

$\tilde{T}$	$\mathrm{Fix}(\tilde{T})$	$\mathrm{Fix}(T)$
identity	$\mathbb{R}^3$	S^2
reflection in plane Π thru $\underline{0}$	Π	$L = \Pi \cap S^2$, spherical line
rotation about line l thru $\underline{0}$	l	$\{P, P'\} = l \cap S^2$, pair of antipodal points
rotary reflection	$\{\underline{0}\}$	$\emptyset$ (empty set)

5a. $$T(\underline{x}) = \frac{1}{3}\begin{pmatrix}1&2&-2\\2&1&2\\-2&2&1\end{pmatrix}\underline{x}$$

Fix(T):
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{3}\begin{pmatrix}x+2y-2z\\2x+y+2z\\-2x+2y+z\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}-2x+2y-2z\\2x-2y+2z\\-2x+2y-2z\end{pmatrix}$$

$\rightsquigarrow$ $2x-2y+2z=0.$ plane Π, normal $\underline{n} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

OR $x-y+z=0.$

T is reflection in the plane Π w/ equation $x-y+z=0.$

b. $$T(\underline{x}) = \begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\underline{x}$$

Fix(T)
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-z\\-y\\-x\end{pmatrix} \qquad \begin{matrix}x+z=0\\2y=0\\x+z=0\end{matrix} \quad \rightsquigarrow \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-z\\0\\z\end{pmatrix} = z\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix},$$

$z \in \mathbb{R}$ arbitrary

Fix(T) $= l$, line in direction $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ thru $\underline{0}$.

So T is a rotation about axis l, thru angle θ given by

$$\text{trace}\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix} = 2\cos\theta+1$$

$$0+(-1)+0 = 2\cos\theta+1$$

$$\cos\theta = -1, \quad \underline{\theta = \pi}.$$

c. $$T(\underline{x}) = \begin{pmatrix}0&-1&0\\0&0&-1\\-1&0&0\end{pmatrix}\underline{x}$$

Fix(T):
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y\\-z\\-x\end{pmatrix} \qquad \begin{matrix}x+y=0\\y+z=0\\x+z=0\end{matrix}$$

$$\left(\begin{array}{ccc:c}1&1&0&0\\0&1&1&0\\1&0&1&0\end{array}\right)\begin{matrix}\\ \\ -R1\end{matrix}$$ augmented matrix

$$\rightsquigarrow \left(\begin{array}{ccc:c}1&1&0&0\\0&1&1&0\\0&-1&1&0\end{array}\right)\begin{matrix}\\ \\ +R2\end{matrix} \rightsquigarrow \left(\begin{array}{ccc:c}1&1&0&0\\0&1&1&0\\0&0&2&0\end{array}\right)$$

Pivot in every col. of coefficient matrix $\Rightarrow$ no free variables $\Rightarrow$ only solution is $x=y=z=0$.

So T is a rotary reflection.

To find the axis of rotation, solve $T(\underline{x}) = -\underline{x}$.

$$-\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y\\-z\\-x\end{pmatrix} \qquad \begin{matrix}x-y=0\\y-z=0\\z-x=0\end{matrix} \qquad \rightsquigarrow \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} = z\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad z\in\mathbb{R} \text{ arbitrary.}$$

So axis l is line thru $\underline{0}$ in direction $\begin{pmatrix}1\\1\\1\end{pmatrix}$,

& plane Π of reflection is the plane thru $\underline{0}$ perpendicular to l, i.e. $\Pi: x+y+z=0$.

Angle of rotation:
$$\operatorname{trace}\begin{pmatrix}0&-1&0\\0&0&-1\\-1&0&0\end{pmatrix} = 2\cos\theta - 1$$

$$0 = 2\cos\theta - 1, \qquad \theta = \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3.$$

6. $T(\underline{x}) = -\underline{x} \quad \Rightarrow \quad \mathrm{Fix}(T) = \{\underline{0}\} \quad \Rightarrow \quad T$ is a rotary reflection.

$(\underline{x} = -\underline{x} \Rightarrow 2\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0})$

In this case the axis is not uniquely determined

(every vector $\underline{x} \in \mathbb{R}^3$ satisfies $T(\underline{x}) = -\underline{x}$).

Let l be the z-axis (for example).

$$\text{Then } T(\underline{x}) = \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\cdot\underline{x} = \overset{\substack{\text{rotation by } \pi \\ \text{about z-axis}}}{\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}}\cdot\overset{\substack{\text{reflection in} \\ \text{xy plane}}}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}}\cdot\underline{x}$$

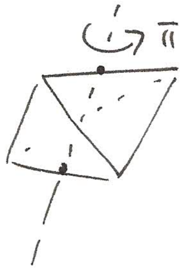
□.

7. 1: Identity

6: Reflection in plane passing through 2 vertices & the midpoint of the opposite edge.

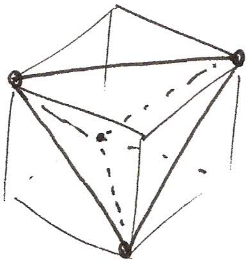
8: Rotation about line through center of tetrahedron & 1 vertex through angle $\pm\frac{2\pi}{3}$

3 : Rotation about line joining midpoints of two opposite edges thru angle π.



6 : Rotation about line joining midpts of two opposite edges thru angle $\pm\pi/2$, followed by reflection in plane perpendicular to this line passing thru center of tetrahedron.

– a rotary reflection.



Ex:

tetrahedron drawn in cube w/ vertices $(\pm1,\pm1,\pm1)$.

Rotate thru $\pi/2$ about z-axis , Reflect in xy-plane.

Total: $1+6+8+3+6 = 24 = 4!$.