Math 461 Lecture 31 11/16

Last time:

GPS theorem $\Rightarrow$ 3 reflections theorem $\Rightarrow$ classification of isometries of S^2

0 reflections - identity

1 reflection - reflection

2 reflections - rotation

3 reflections - rotary reflection

Final step:

composition of 3 reflections = rotary reflection

$\mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$

rotate L_1, L_2 about $P \in L_1 \cap L_2 \rightsquigarrow$

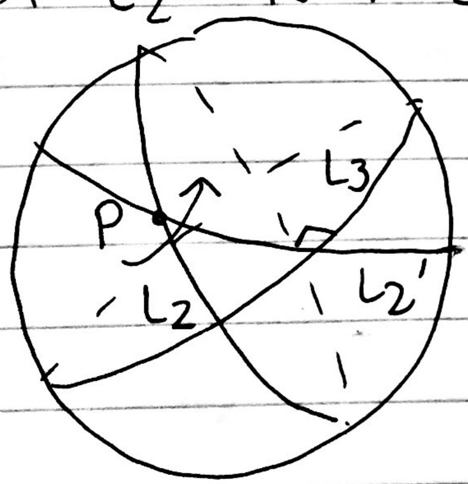
L_1', L_2', $L_2' \perp L_3$

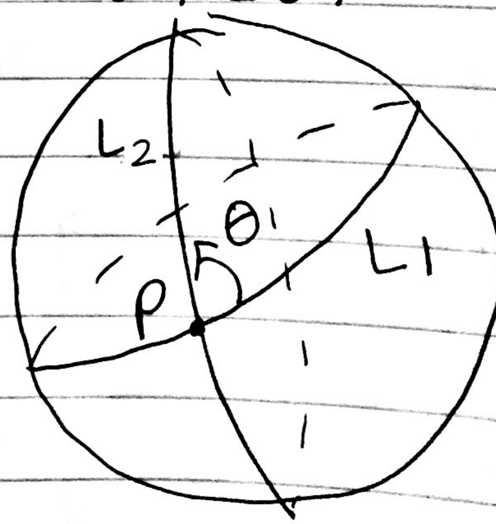
rotate L_2', L_3 about $Q \in L_2' \cap L_3 \rightsquigarrow$

L_2'', L_3', $L_1 \perp L_2''$ $L_2'' \perp L_3'$

here we're using:

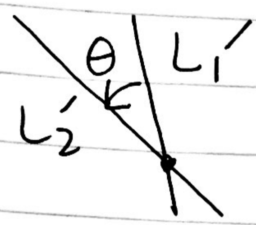
$\mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1} = \mathrm{Rot}(P, 2\theta)$

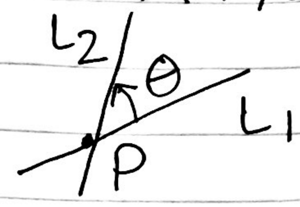


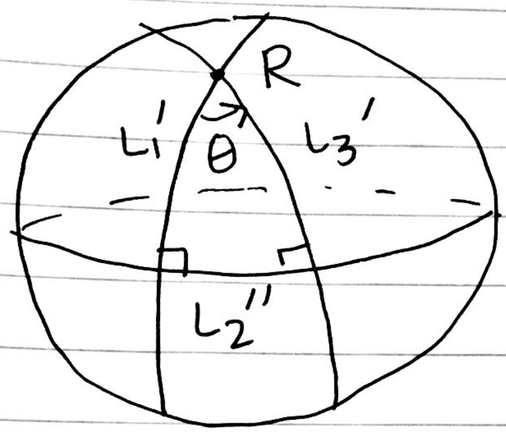


Math 461 Lecture 31 11/16

$Refl_{L_2} \circ Refl_{L_1} = Refl_{L_2'} \circ Refl_{L_1'}$
$= Rot(P, 2\theta)$





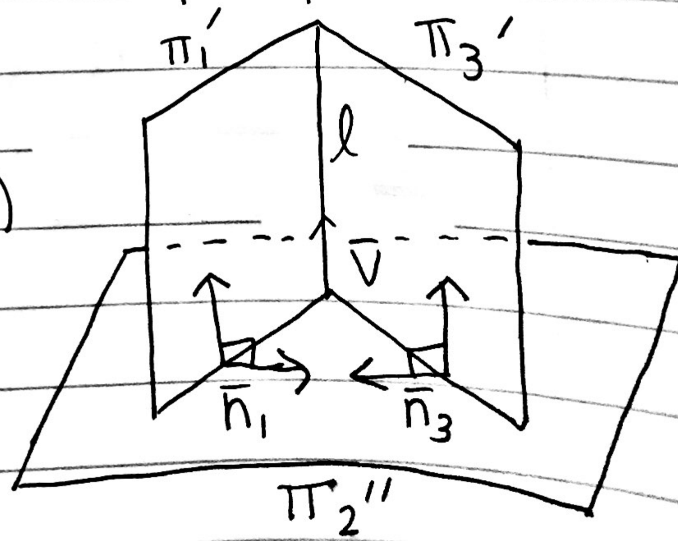


$T = Refl_{L_3} \circ (Refl_{L_2} \circ Refl_{L_1})$
$= Refl_{L_3} \circ (Refl_{L_2'} \circ Refl_{L_1'})$
$= (Refl_{L_3} \circ Refl_{L_2'}) \circ Refl_{L_1'}$
$= Refl_{L_3'} \circ Refl_{L_2''} \circ Refl_{L_1'}$
$= (Refl_{L_3'} \circ Refl_{L_1'}) \circ Refl_{L_2''}$
$= Rot(R, 2\theta) \circ Refl_{L_2''}$

rotatary reflection
axis of rotation is perpendicular to plane of reflection
recall if two lines meet at right angles the reflections commute
$Refl_{L_2} \circ Refl_{L_1} = Rot(P, \pi) = Rot(P, -\pi) = Refl_{L_1} \circ Refl_{L_2}$

claim: $\ell = \Pi_1' \cap \Pi_3'$ is perpendicular to Π_2''

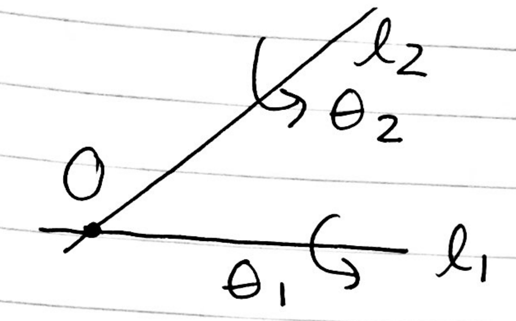
$\bar{v} \perp \bar{n}_1$, $\bar{v} \perp \bar{n}_3$
$\Pi_2'' = span(\bar{n}_1, \bar{n}_3)$
$\rightsquigarrow \bar{v} \perp \Pi_2''$

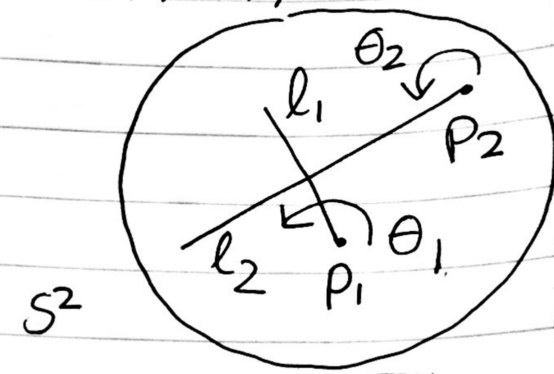


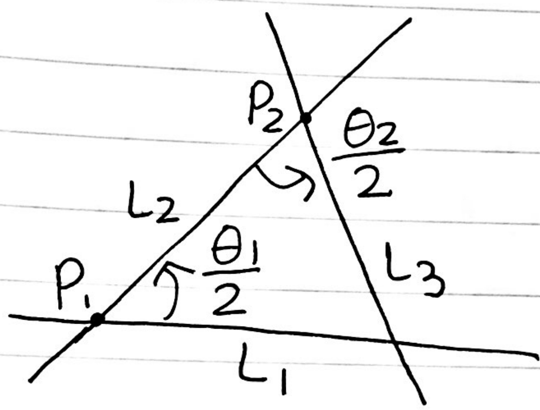
Math 461 Lecture 31 11/16

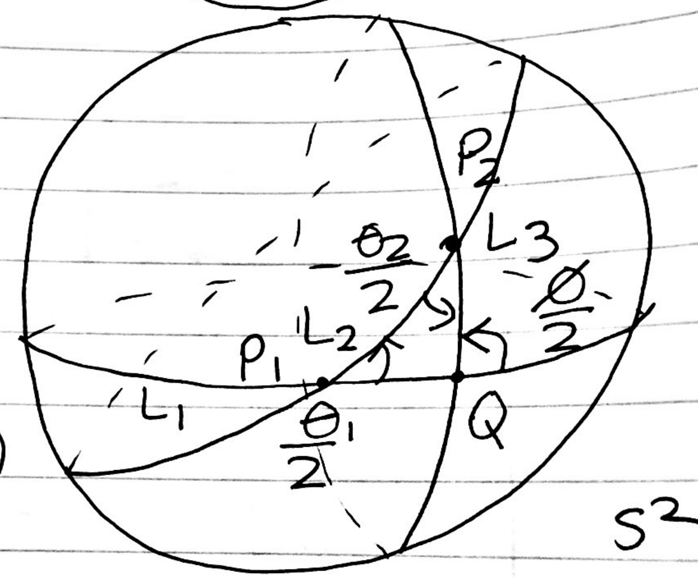
Composition of rotations in $\mathbb{R}^3$ fixing the origin

$\mathrm{Rot}(\ell_2, \theta_2) \circ \mathrm{Rot}(\ell_1, \theta_1) = ?$









$\mathrm{Rot}(P_2, \theta_2) \circ \mathrm{Rot}(P_1, \theta_1)$

$= \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_2} \circ \mathrm{Refl}_{L_1}$

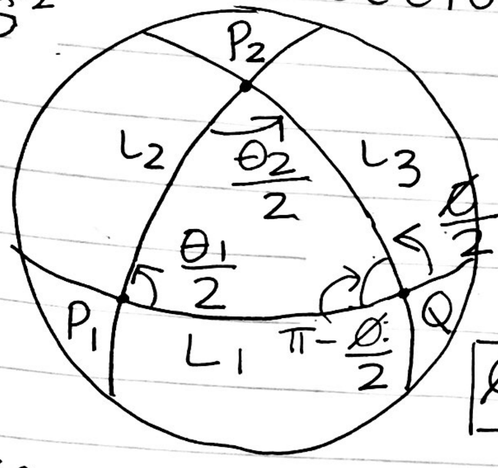
$= \mathrm{Refl}_{L_3} \circ \mathrm{Refl}_{L_1}$ $\quad (\mathrm{Refl}_{L_2})^2 = \text{identity}$

$= \mathrm{Rot}(Q, \phi)$

how to compute ϕ in terms of $\theta_1, \theta_2, P_1, P_2$?

Math 461 Lecture 31 11/16

S^2



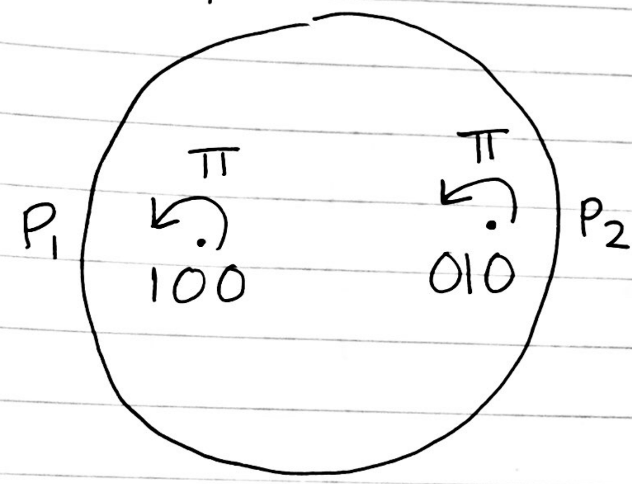
$$\frac{\theta_1}{2}+\frac{\theta_2}{2}+\left(\pi-\frac{\phi}{2}\right)$$

$$=\text{Area}(\Delta P_1P_2Q)+\pi$$

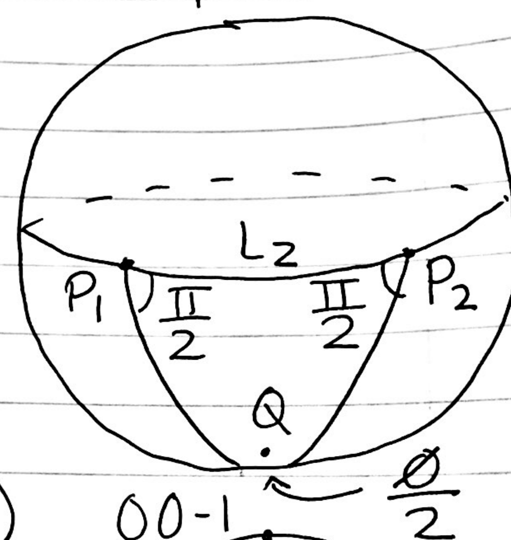
$$\frac{\theta_1}{2}+\frac{\theta}{2}=\frac{\phi}{2}+\text{Area}(\Delta P_1P_2Q)$$

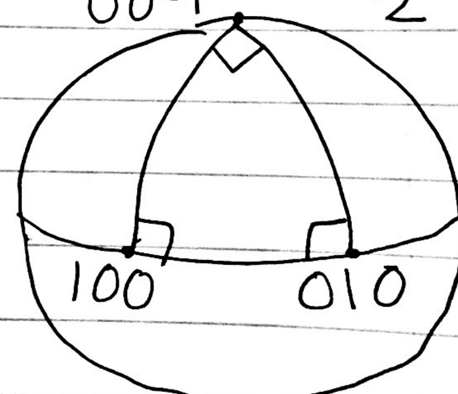
$$\boxed{\phi=\theta_1+\theta_2-2\text{Area}(\Delta P_1P_2Q)}$$

example:



composition?





$\text{Rot}(Q,\phi)$ $Q=(0,0,-1)$

$\frac{\phi}{2}=\frac{\pi}{2} \rightarrow \phi=\pi$

composition:
rotation by π about z-axis

check formula:

$$\phi=\theta_1+\theta_2-2\text{Area}(\Delta P_1P_2Q)$$

$$=\pi+\pi-2\left(\frac{4\pi}{8}\right)$$

$$=\pi+\pi-\pi$$

$$=\pi \quad \checkmark$$

Math 461 Lecture 31 11/16

Remark: Isometries of $\mathbb{R}^3$ fixing $\bar{0}$ correspond to 3×3 orthogonal matrices A ($A^T A = I$)

The classification of isometries:

identity rotation

reflection rotary reflection

corresponds to

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

similar to ($\sim$) i.e if $A = PMP^{-1}$ some P, write $A \sim M$

reflection in xy plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

rotation about z-axis:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotary reflection about z-axis:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

eigenvalues of A:

$1, 1, 1$

$1, 1, -1$

$1, e^{i\theta}, e^{-i\theta}$

$-1, e^{i\theta}, e^{-i\theta}$

notice $\det A = \pm 1$

e.g. reflection $A \sim M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow$

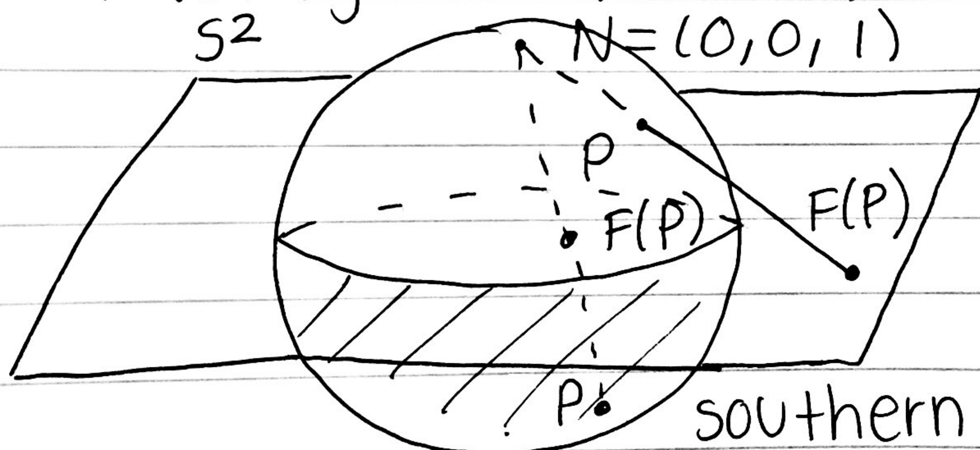
$\det A = \det M = 1 \cdot 1 \cdot (-1) = -1$

$\det AB = \det A \cdot \det B$

Math 461 Lecture 31 11/16

T	det A
identity	+1
reflection	-1
rotation	+1
rotary reflection	-1

stereographic projection:
if you want to express a surface on the plane some distances will be distorted
i.e distances are not preserved
(Ptolemy 100-170 AD)



southern hemisphere

$F: S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ bijection
preserves angles but not distances
as P approaches N F(P) goes to ∞ in $\mathbb{R}^2$
$(z<0) \xrightarrow[F]{} (\text{disc radius } 1 \text{ in } \mathbb{R}^2)$