Canonical Divisor Class on Algebraic Variety

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Let X be a smooth complex projective variety (or a compact complex manifold). Let Ω be a meromorphic top form on X. If $\dim_{\mathbb{C}} X = n$, then Ω is an n-form.

So locally Ω is given by $\Omega = f dz_1 \wedge \cdots \wedge dz_n$, where $z_1, \ldots, z_n$ are local coordinates on X and f is a meromorphic function.

If there is a change of coordinates, then

$$dw_1 \wedge \cdots \wedge dw_n = \det \left(\frac{\partial w_i}{\partial z_j} \right) dz_1 \wedge \cdots \wedge dz_n.$$

Canonical Divisor

Definition 1.1. Define the Canonical Divisor K_X to be
$K_X := (\Omega) =$ "zeroes" $-$ "poles"of Ω.

We have $K_X = \sum n_i Y_i$ where $Y_i \subset X$ are codimension 1 subvarieties and $n_i \in \mathbb{Z}$.

If $n_i > 0$, then Ω has a zero of order n_i along Y_i, if $n_i < 0$, then Ω has a pole of order n_i along Y_i.

Locally $\Omega = f dz_1 \wedge \cdots \wedge dz_n$, $Y = (g = 0) \in X$. Then we can write f as $f = g^n h$, where h is meromorphic, nonzero and holomorphic at a general point of Y, n is the order.

If Ω' is another choice of meromorphic n-form on X, then $(\Omega') = (\Omega) + (f)$, where $(f) =$ "zeroes"$-$ "poles"of f is the principal divisor associated to f. So K_X is well defined up to a principal divisor.

Example 1.1. Let X be a compact Riemann surface of genus g. Then $K_X = \sum n_i P_i$, where $P_i \in X$ are points. Define the degree of the canonical divisor to be $\deg K_X = \sum n_i$. Nice fact: $\deg K_X = 2g - 2$.

Example 1.2. Let $X = \mathbb{P}^1 = \mathbb{C}^1_z \cup \{\infty\}$. Take $\Omega = dz$. Then at ∞ the local coordinate is $w = 1/z$ and $\Omega = d(1/w) = -1/w^2 dw$. Therefore the form Ω has a pole of order 2 at ∞ and no zeroes, so $(\Omega) = -2(\infty)$.

Example 1.3. Let $X = \mathbb{P}^n = \mathbb{C}^n_{z_1,\dots z_n} \cup H_\infty$. Take $\Omega = dz_1 \wedge \cdots \wedge dz_n$, where $z_i = x_i/x_0$. If we consider coordinates $w_j = x_j/x_1$, then the coordinate change $z_i = w_i/w_0$.

Therefore $\Omega = \wedge \frac{1}{w_0^2}(w_0 dw_i - w_i dw_0) = \frac{1}{w_0^{n+1}} dw_0 \wedge \cdots \wedge dw_n$. So $(\Omega) = -(n+1)H_\infty$. We have $K_{\mathbb{P}^n} = -(n+1)H$, where H is a hyperplane class.

Adjunction Formula

"The most important way to compute K_X"

Theorem 1.1 (Adjunction formula)**.** *Let X be a complex manifold, $Y \subset X$ a submanifold of codimension 1, then*

$$K_Y = (K_X + Y)\big|_Y.$$

Generally, if D is a divisor on X, $Y \subset X$ we can define restriction $D|_Y$ provided we allow linear equivalences: if $Y \subset supp(D)$, then we must replace D by $D' = D + (f)$ so that it is transverse, and then restrict.

Example 1.4. Let $Y \subset X = \mathbb{P}^2$ be a plane curve of degree d. Then $K_Y = (K_X + Y)\big|_Y = (-3H + dH)|_Y = (d-3)H|_Y$. Note that this allows us to compute the genus of Y. We have $2g - 2 = \deg K_Y = (d-3)d$, so $g = 1/2(d-1)(d-2)$. More generally, if X is a smooth surface and $Y \subset X$ is a curve, then $2g(Y) - 2 = (K_X + Y) \cdot Y$.

Proof of **Theorem 1.**
Let $\dim X = n+1$, $\dim Y = n$. Let Ω be $(n+1)$-form on X. Let f be a meromorphic function such that $(f) = Y + D$, D is a divisor, $Y \not\subset Supp(D)$. Define η by $\eta := Res_Y(\Omega/f)$. Then η is an n-form on Y.

Locally, say $Y = (z_0 = 0) \subset X = \mathbb{C}^n_{z_0,\dots z_n}$. Then $\Omega/f = dz_0 \wedge \cdots \wedge dz_n(a_{-1}z_0^{-1} + a_0 + a_1 z_0 + \dots)$, then $Res(\Omega/f) = a_{-1} dz_1 \wedge \cdots \wedge dz_n$ on Y.

Another way to say it: Write $\Omega/f = \frac{dz_0}{z_0} \wedge \zeta$, then $Res(\Omega/f) = \zeta|_Y$.

So $(\eta) = (\Omega)|_Y - D|_Y$, therefore $K_Y = K_X + Y|_Y$ since $Y + d \sim 0$. $\square$

Example 1.5. Let $Y \subset X = \mathbb{P}^2 = \mathbb{C}^2_{z_1,z_2} \cup H_\infty$. In the chart $\mathbb{C}^2_{z_1,z_2}$ write $Y = (f = 0)$, $\Omega = dz_1 \wedge dz_2$. Then $\eta = Res(\Omega/f) = dz_2/\dfrac{\partial f}{\partial z_1} = -dz_1/\dfrac{\partial f}{\partial z_2}$. So $(f) = Y - dH_\infty$, $f = f(z_1, z_2) = f(X_1/X_0, X_2/X_0) = \frac{F(X_0,X_1,X_2)}{X_0^d}$.

Example 1.6. Let $Y \subset X = \mathbb{P}^3$, Y is a smooth surface. Then

$$K_Y = K_X + Y|_Y = -4H + dH|_Y = (d-4)H|_Y.$$

The possibilities for d are:

- $d < 4$; $d = 1, 2, 3$ correspond to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $Bl^6\mathbb{P}^2$, $K_Y < 0$;
- $d = 4$; corresponds to K3 surface, $K_Y = 0$;
- $d > 4$; correspond to surfaces of general type, $K_Y > 0$.

Blowup

Let X be a smooth surface, $P \in X$, $\pi : \tilde{X} \to X$ a blowup, let E be the exceptional divisor. Choose coordinates x, y at $P \in X$, then
$(u = 0) = E \cap U_1$, $U_1 = \mathbb{C}^2_{u,y'}$, $(u, y') \to (u, uy') = (x, y)$,
$(v = 0) = E \cap U_2$, $U_2 = \mathbb{C}^2_{x',v}$, $(x', v) \to (vx', v) = (x, y)$.

Theorem 1.2. *$K_{\tilde{X}} = \pi^* K_X + E$.*

Proof. Work locally at $P \in X$, then $\Omega = dx \wedge dy$.

On U_1 we have $\pi^* dx \wedge dy = d(u) \wedge d(uy') = udu \wedge dy'$. So $(\pi^*\Omega) = E$, $(\Omega) = 0$. □

ALso $E \simeq \mathbb{P}^1$, so $(K_{\tilde{X}} + E) \cdot E = 2g - 2 = -2$. Since $K_{\tilde{X}} = \pi^* K_X + E$, we get $K_{\tilde{X}} \cdot E = E \cdot E = -1$. Therefore E is called the (-1)-curve.

Corollary 1.3. *If $X \subset \mathbb{P}^3$ is a smooth surface of degree ≥ 4, then X is not a blowup.*

Proof. $K_X = (d-4)H \geq 0$. So $K_X \cdot C \geq 0$ for all $C \subset X$. □

Riemann–Hurwitz

If $f : X \to Y$ is a map of compact RS, then

$$K_X = f^* K_Y + \sum_{P \in X} (e_P - 1) P,$$

where e_P is the ramification index of $P \in X$.

Sometimes we only care about the degrees:

$$2g(X) - 2 = \deg f (2g(Y) - 2) + \sum_{P \in X} (e_P - 1).$$

If $\dim X > 1$, let $f : X \to Y$ be a map of compact complex manifolds, finite:1. Then

$$K_X = f^* K_Y + \sum_{Z \in X} (e_Z - 1) Z,$$

where $Z \in X$ is a codimension 1 irreducible subvariety in X, e_Z is the ramification index.

Locally, $(x_1, \dots, x_n) \to (x_1^e, x_2, \dots, x_n)$, where $Z = (x_1 = 0) \subset X$.

Example 1.7. Let $f : X \to Y = \mathbb{P}^2$ be the $2 : 1$ map, let $Z \subset X$ be the ramification locus, and let $B \subset Y$ be the branch locus. B is a plane curve of degree $2n = d$. Here

$$K_X = f^* K_Y + Z = f^*(K_Y + (1/2)B) = (n-3) f^* H,$$

where Z is the ramification locus ($Z = f^{-1}B$) and H is the class of a line on $Y = \mathbb{P}^2$.

Then if

- $d = 2$, then $X = \mathbb{P}^1 \times \mathbb{P}^1$;
- $d = 4$, then $X = Bl^7 \mathbb{P}^2$;
- $d = 6$, then X is K3 ($K_X = 0$);
- $d \geq 8$, then X is of general type.