Math 462 Homework 4

Paul Hacking

February 18, 2015

(1) Determine algebraic formulas $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for the following isometries T. (Here A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector.)

   (a) Rotation about the point $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ through angle $\pi/4$ counterclockwise.

   (b) Reflection in the line $y = 4$.

   (c) Reflection in the line $y = x + 2$ followed by a translation parallel to the line through distance $3\sqrt{2}$ in the direction of increasing x. (This is a glide reflection.)

(2) Describe the following isometries of $\mathbb{R}^2$ geometrically.

   (a) $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + 2 \\ -x + 3 \end{pmatrix}$.

   (b) $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + 4 \\ -x - 3 \end{pmatrix}$.

(3) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be rotation about a point P through angle θ counterclockwise and $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ be rotation about a point Q through angle φ counterclockwise. What can you say about the composition $U \circ T$?

   [Hint: Recall that a rotation T about a point P is given by an algebraic formula $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where $S(\mathbf{x}) = A\mathbf{x}$ is the rotation about the origin through the same angle and $\mathbf{b} \in \mathbb{R}^2$ is a vector (which is determined by the point P).]

(4) Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry of $\mathbb{R}^2$ given by rotation about a point P through angle θ counterclockwise. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry given by rotation about a point Q through angle θ clockwise.

(a) Show that the composition $S \circ R$ is a translation, i.e., $S(R(\mathbf{x})) = \mathbf{x} + \mathbf{b}$ for some vector $\mathbf{b} \in \mathbb{R}^2$.

(b) Show that if the angle θ is small, then the translation vector $\mathbf{b}$ has length approximately $\theta \cdot d(P,Q)$ and is approximately perpendicular to the vector $\overrightarrow{PQ}$. (Here $d(P,Q)$ denotes the distance from P to Q.) [This fact is sometimes useful when moving furniture.]

(c) What happens for $\theta = \pi$?

[Hint: By choosing coordinates appropriately we can assume P is the origin and Q lies on the x-axis. Now compute by expressing R and S in the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector. For part *(b)* use the approximations $\cos(\theta) \simeq 1$ and $\sin(\theta) \simeq \theta$ for θ small.]

(5) Let S be the square $S = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x, y \leq 1\} \subset \mathbb{R}^2$. We say an isometry $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetry of S if $\mathbf{x} \in S \Rightarrow T(\mathbf{x}) \in S$.

(a) Explain why any symmetry T of S must fix the origin.

(b) List all the symmetries of S, giving both a geometric and algebraic description of each symmetry. [Note: Each symmetry has the algebraic form $T(\mathbf{x}) = A\mathbf{x}$ where A is a 2×2 orthogonal matrix (why?).]

(6) Let Π be a plane through the origin in $\mathbb{R}^3$ with normal vector $\mathbf{n}$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be reflection in the plane Π.

(a) Show that

$$T(\mathbf{x}) = \mathbf{x} - 2\left(\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}.$$

(b) Now suppose $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Use the formula in part (a) to write T in the form $T(\mathbf{x}) = A\mathbf{x}$ where A is a 3×3 matrix, and check that A is orthogonal.