Math 461 Lecture 38 12/10
Homework 8 returned (out of 43)
Final exam this Friday 12/14
8:00-10:00 AM LGRT 141
Final review problems available
(check email and website)
will post solutions later today
Final review session:
Thursday 12/13 3-4 PM room TBA
Syllabus for final exam:
spherical geometry including
isometries of $\mathbb{R}^3$
see class log and lecture notes
on webpage
no hyperbolic geometry
Euclidean geometry will not be
explicitly tested
Last time:
hyperbolic lines give shortest paths
inversion in $\mathcal{C}$: $x^2+y^2=1$
takes circles and lines to circles
and lines
(and circle/line $C \rightsquigarrow$ line $\Leftrightarrow 0 \in C$)
preserves angles
given ~~xxx~~ L hyperbolic line, $P \in H$,
there's a unique hyperbolic line
M through P and $\perp$ to L
pictures $\rightarrow$

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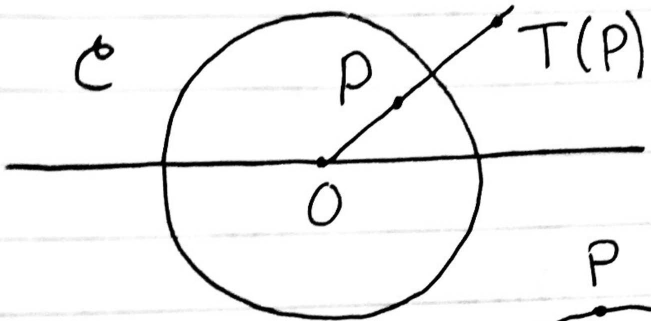
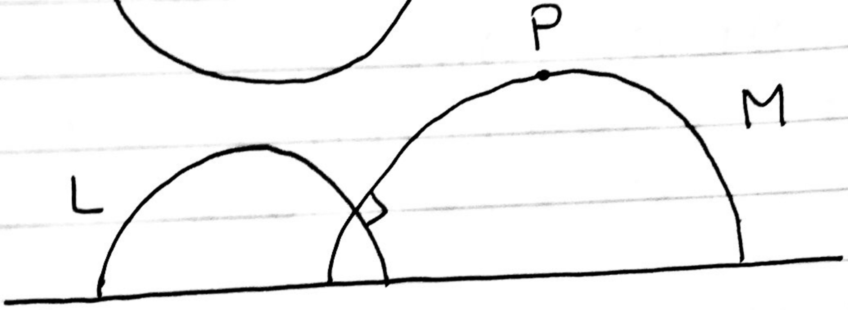


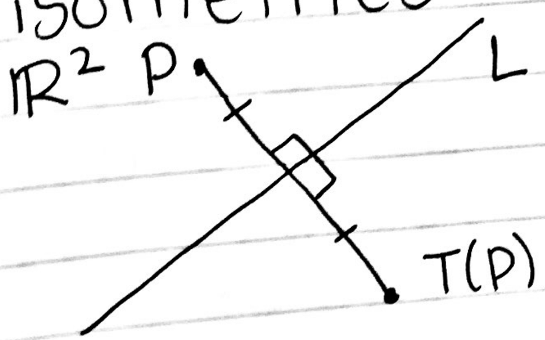
$T(P) = Q$

$T: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$

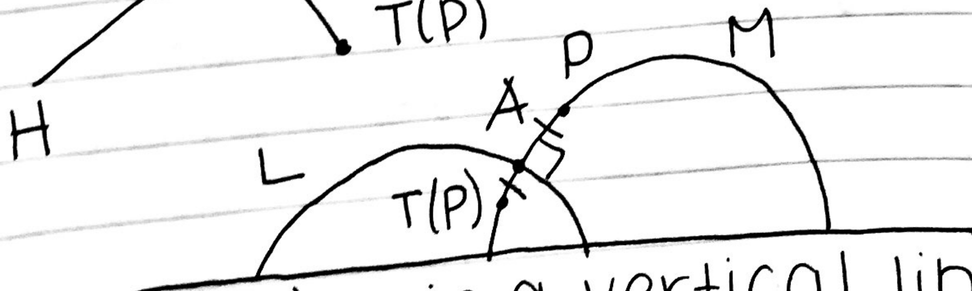
$|OP| \cdot |OQ| = 1$



Today:
inversion is hyperbolic reflection
3 reflections theorem $\rightsquigarrow$
classification of hyperbolic isometries



$T: \mathbb{R}^2 \to \mathbb{R}^2$
(Euclidean)
reflection in L



$T: H \to H$

claim: (1) L is a vertical line through
T is Euclidean reflection in L
(2) If L is a semicircle, then T is

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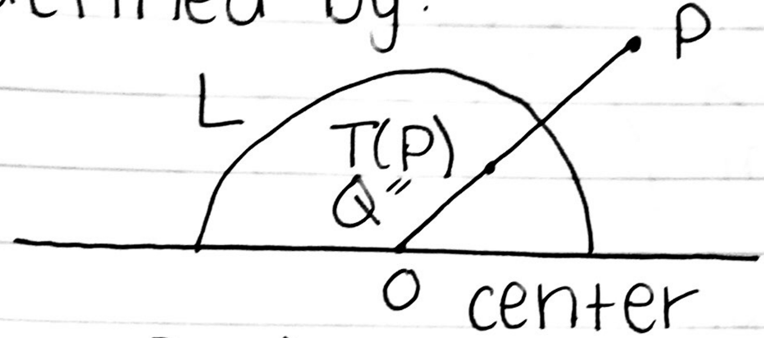
Inversion in L

defined by:



$|OP| \cdot |OQ| = r^2$

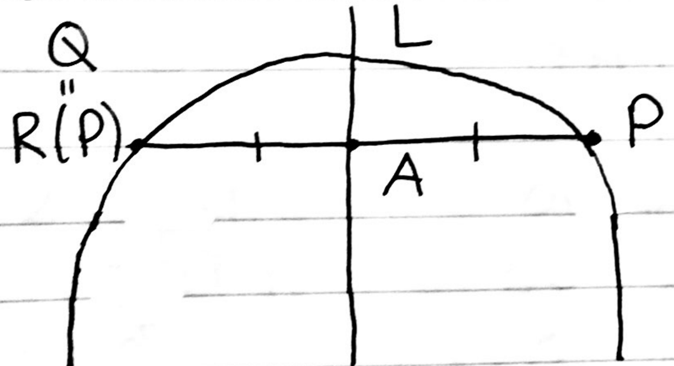
r = radius of L

proof: (1)

$R: H \to H$ Euclidean reflection

$T: H \to H$ hyperbolic reflection

want to show $T = R$



i.e. $|AP| = |AQ|$ (Euclidean distances)

draw hyperbolic line M through P perpendicular to L: semicircle, center $O = L \cap (x\text{-axis})$, radius $|OP|$

passes through Q

need to show 1. $M \perp L$ ✓

2. $d_H(P,B) = d_H(Q,B)$?

then $Q = T(P)$, true for all $P \Rightarrow T = R$

2: know R (Euclidean reflection in L) is hyperbolic isometry

and $Q = R(P)$ $B = R(B)$

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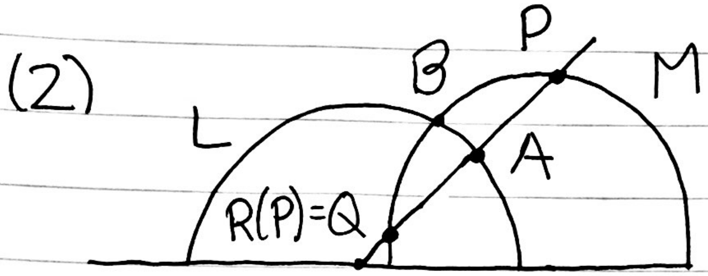
so $d_H(B,Q) = d_H(R(B), R(P)) = d_H(B,P)$ ✓
isometry

(2)


R: H → H
inversion in L
T: H → H
hyperbolic reflection in L

goal: R=T

there's a unique hyperbolic line M through P and Q

need to show 1) $M \perp L$

2) $d_H(P,B) = d_H(Q,B)$

then $Q = T(P)$, true for all P ⇒ T=R

2 similar to before: know inversion R is a hyperbolic isometry and $Q = R(P)$, $B = R(B)$ ⇒ $d_H(P,B) = d_H(Q,B)$

Remark: only proved inversion in $\mathcal{C}: x^2+y^2=1$ is isometry

to get inversion in any circle $\mathcal{C}'$ with center an x-axis: move $\mathcal{C}'$ to $\mathcal{C}$ by horizontal translation and scaling invert in $\mathcal{C}$, then move $\mathcal{C}$ back to $\mathcal{C}'$
⇒ isometry

$M \perp L$? notice M is preserved by inversion R

why? R interchanges P and Q, R isometry and M is the unique

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Shortest path from P to Q

R preserves angles $\Rightarrow \theta = \pi - \theta$,

i.e. $\theta = \frac{\pi}{2}$ ✓ □

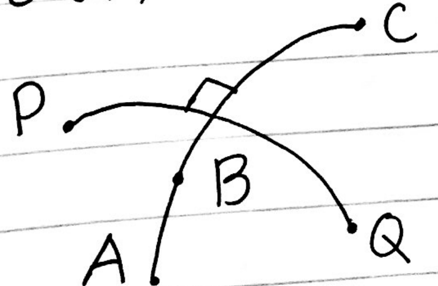
3 reflections theorem in H:
any hyperbolic isometry is a composite of less than or equal to 3 reflections

$\Uparrow$

GPS Theorem: any $P \in H$ is determined by $d(P,A)$, $d(P,B)$, $d(P,C)$ for A, B, C 3 points not lying on a hyperbolic line

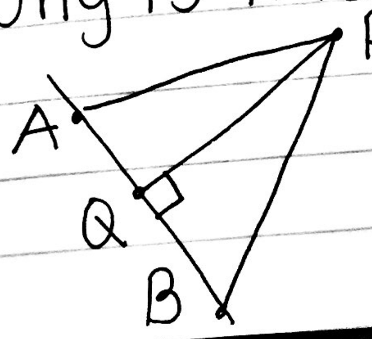
$\Uparrow$

perpendicular bisector of a line segment AB of a hyperbolic line is the of points P such that $d(P,A) = d(P,B)$



proof of GPS $*$

why is this true in hyperbolic geometry



$d(P,A) = d(P,B) \Longleftrightarrow d(Q,A) = d(Q,B)$ $*$

$\mathbb{R}^2$: use pythagorean theorem

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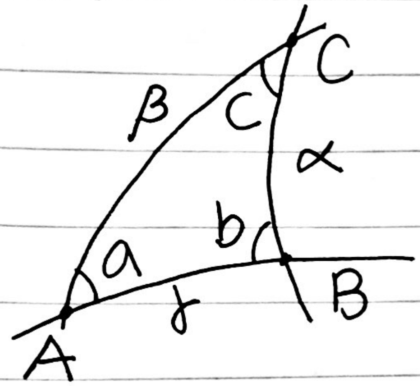
H: analogue: hyperbolic cosine rule

$\sinh(t) = \dfrac{e^t - e^{-t}}{2}$ $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$

$\cosh\alpha = \cosh\beta\cosh\gamma - \sinh\beta\sinh\gamma\cos a$



$a = \frac{\pi}{2} \rightsquigarrow \cosh\alpha = \cosh\beta\cosh\gamma$ hyperbolic pythagoras $\Rightarrow$ (*)

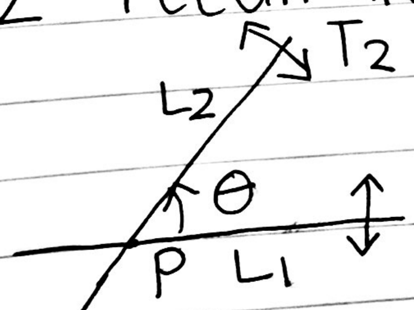
classification of hyperbolic isometries:

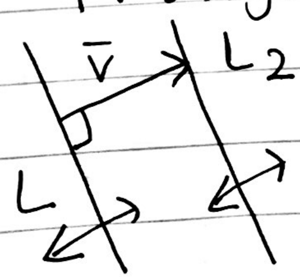
≤ 3 reflections

0 identity

1 hyperbolic reflection (reflection in vertical line OR inversion in semicircle)

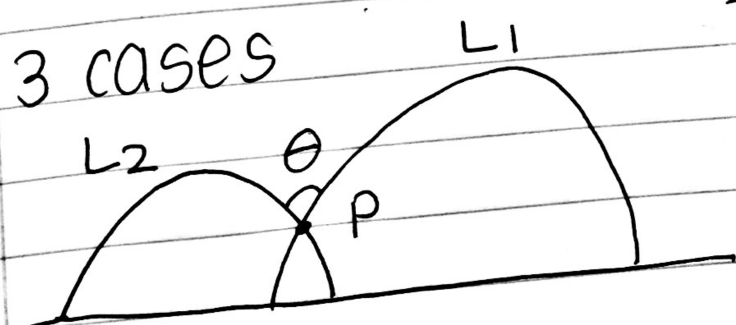
2 recall $\mathbb{R}^2$: $T_2 \circ T_1$ = rotation about P, angle 2θ





$T_2 \circ T_1$ = translation by $2\bar{v}$

3 cases



$T_2 \circ T_1$ = hyperbolic rotation, center P, angle 2θ

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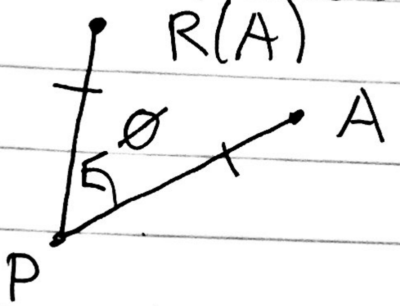
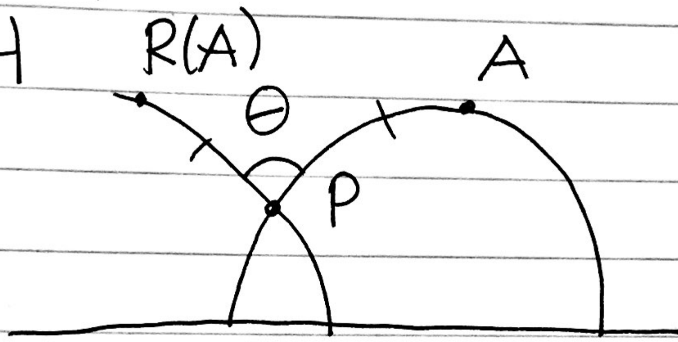
$\mathbb{R}^2$



R = rotation about P through angle ϕ

H



lemma: given a point $P \in H$ and a tangent direction $\bar{v}$ at P, there's a unique hyperbolic line through P in direction $\bar{v}$