Math 621 Homework 3

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Reading: Stein and Shakarchi, Chapter 3, Sections 3–6.

(1) Show that $f(z) = \frac{2\cos z - 2 + z^2}{z^4}$ has a removable singularity at $z = 0$.

(2) Show that $\sin(z)$ has an essential singularity at ∞.

(3) Let $f \colon \mathbb{C} \to \mathbb{C}$ and $g \colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f = c \cdot g$ for some constant $c \in \mathbb{C}$.

(4) Let $f \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function which is injective.

   (a) Using the Casorati–Weierstrass theorem, show that f does not have an essential singularity at ∞.

   (b) Deduce that f is a polynomial.

   (c) Deduce that $f(z) = az + b$, some $a, b \in \mathbb{C}$, $a \neq 0$. (In particular, f is bijective.)

(5) Let $\Omega \subset \mathbb{C}$ be an open set containing the closure of the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $f \colon \Omega \to \mathbb{C}$ be a non-constant holomorphic function. Suppose that $|f(z)| = 1$ for all z such that $|z| = 1$. Show that $f(\Omega) \supset D$. [Hint: First show that there exists $z \in \Omega$ such that $f(z) = 0$ (argue by contradiction using the maximum principle). Second deduce the result using Rouché's theorem]

(6) Compute the Laurent expansion of $\frac{1}{z}$ about $z = 1$ in the following regions:

   (a) $\Omega = \{z \in \mathbb{C} \mid |z - 1| > 1\}$.

(b) $\Omega' = \{z \in \mathbb{C} \mid |z-1| < 1\}$.

[Hint: Use the identity $\frac{1}{1-w} = \sum_{n\geq 0} w^n$ for $|w| < 1$.]

(7) (From the qualifying exam, Fall 2011) Consider the meromorphic function

$$f(z) = \frac{1}{z^2 - (2+5i)z + 10i}.$$

In each of the following cases, compute the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$$

of f centered at a which is valid in a neighbourhood of b, and determine its domain of convergence

(a) $a = b = 2$.

(b) $a = 0$, $b = 3$.

(8) Find the number of zeroes (counted with multiplicities) of the polynomial $p(z) = z^5 + 5z^2 + z + 2$ in the annulus $A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$. Are all the zeroes simple (i.e. multiplicity 1)?[Hint: If $f\colon \Omega \to \mathbb{C}$ is a holomorphic function and $z_0 \in \Omega$ is a zero of f, then it is a simple zero iff $f'(z_0) \neq 0$.]

(9) Find the number of zeros (counted with multiplicities) of $f(z) = e^z + 4z^5 + 1$ in the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

(10) We say an open set $\Omega \subset \mathbb{C}$ is *star-shaped* if there exists $z_0 \in \Omega$ such that for all $z \in \Omega$ the line segment connecting z_0 and z is contained in Ω. Show that a star-shaped region is simply connected.

(11) Recall that if $\Omega \subset \mathbb{C}$ is a simply connected open subset and $f\colon \Omega \to \mathbb{C}$ is a holomorphic function such that $f(z) \neq 0$ for all $z \in \Omega$, then we may define a holomorphic function $\log f\colon \Omega \to \mathbb{C}$ by

$$(\log f)(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)} dw + c,$$

where γ_z is a path in Ω from a fixed basepoint $z_0 \in \Omega$ to z, and $c \in \mathbb{C}$ is a constant chosen so that $e^c = f(z_0)$ (thus c is determined up to an integer multiple of $2\pi i$). Now let $\Omega = \mathbb{C} \setminus (-\infty, 0]$, $f(z) = z^n$, $z_0 = 1$, and $c = 0$. Compute $(\log f)(re^{i\theta})$ for $r \in \mathbb{R}$, $r > 0$, and $\theta \in (-\pi, \pi)$.

(12) Let $\Omega \subset \mathbb{C}$ be an open set and $f \colon \Omega \to \mathbb{C}$ a holomorphic function. Let $z_0 \in \Omega$ be a point such that $f(z_0) \neq 0$. Let $n \in \mathbb{N}$ be a positive integer. Show that there exists an open neighbourhood U of z_0 in Ω and a holomorphic function $g \colon U \to \mathbb{C}$ such that $f = g^n$ on U.

(13) Let $\Omega \subset \mathbb{C}$ be an open set, $f \colon \Omega \to \mathbb{C}$ a holomorphic function, and $z_0 \in \Omega$ a point such that $f'(z_0) \neq 0$. Show that there exists an open neighbourhood U of z_0 in Ω and an open neighbourhood V of $f(z_0)$ in $\mathbb{C}$ such that f restricts to a bijection $g \colon U \to V$, and moreover g^{-1} is also holomorphic. [Hint: First recall the inverse function theorem for differentiable maps $F \colon U \to \mathbb{R}^2$, $U \subset \mathbb{R}^2$ (see any textbook on real analysis). Now use the relation between complex differentiability and real differentiability for a function $f \colon \Omega \to \mathbb{C}$.]