# Math 462: Homework 4

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In the problems below $S^2$ denotes the sphere of radius 1 in $\mathbb{R}^3$ with center the origin $O$.

(1) A *spherical circle* with center a point $P$ on $S^2$ and radius $r$ is the locus of points $Q$ on $S^2$ such that the spherical distance $d(P, Q)$ equals $r$.

  (a) Show that the circumference of a spherical circle of radius $r$ equals $2\pi \sin r$. [Hint: A spherical circle with center $P$ is a Euclidean circle in $\mathbb{R}^3$ obtained by intersecting the sphere $S^2$ with a plane normal to the line $OP$. Notice that the Euclidean circumference is equal to the spherical circumference, but the Euclidean center and radius of the circle are different from the spherical center $P$ and radius $r$.]

  (b) Recall that the spherical distance between two points $P$ and $Q$ on $S^2$ is at most $\pi$. So it only makes sense to talk about spherical circles of radius $r \leq \pi$. What happens to the circumference of a spherical circle of radius $r$ as $r$ approaches $\pi$? Explain your answer geometrically.

  (c) Show that the circumference of a spherical circle of radius $r$ is less than the circumference of a Euclidean circle of the same radius.

(2) A *spherical disc* is the region on $S^2$ enclosed by a spherical circle.

  (a) Show that the area of a spherical disc of radius $r$ equals $2\pi(1 - \cos r)$. [Hint: Use Q1(a) and integration.]

  (b) What happens to the area of a spherical disc of radius $r$ as $r$ approaches $\pi$? Explain your answer geometrically.

  (c) Show that the area of a spherical disc of radius $r$ is less than the area of a Euclidean disc in $\mathbb{R}^2$ of the same radius.

(3) Show that if $R \subset S^2$ is any region, then there is no map $T\colon R \to \mathbb{R}^2$ from $R$ to the plane which preserves distances, that is $d(T(P), T(Q)) = d(P, Q)$ (here we are using the spherical distance on $S^2$ and the usual Euclidean distance on $\mathbb{R}^2$). [Hint: Use Q1(c).] Note: It follows that any map of a portion of the earth's surface distorts distances, that is, distances are not exactly to scale.

(4) Let $L$ be a spherical line (great circle) on $S^2$ and $P$ a point on $S^2$ not lying on $L$. Show how to construct a spherical line $M$ through $P$ and perpendicular to $L$. [Hint: Give a construction in terms of planes in $\mathbb{R}^3$]. Is the line $M$ uniquely determined by $P$ and $L$?

(5) In class we proved that the sum of the angles of a Euclidean triangle in $\mathbb{R}^2$ equals $\pi$ radians. [See p. 19–20 of the textbook.] What goes wrong when you try to prove that the sum of the angles of a spherical triangle equals $\pi$ by the same method? More precisely, let $ABC$ be a spherical triangle, and assume (by choosing coordinates appropriately) that the edge $AC$ lies on the equator of the sphere. Now let $T$ be the rotation about the axis $NS$ joining the north and south poles through an angle $\theta$ chosen so that $T(A) = C$. Write $A', B', C'$ for $T(A), T(B), T(C)$. Is the spherical triangle $B'A'B$ congruent to the spherical triangle $ABC$?