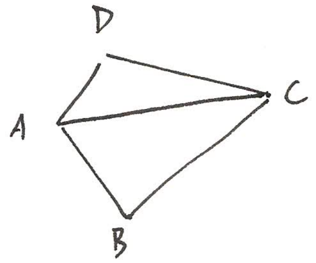
MATH 461 HW2 SOLUTIONS

Thursday 9/26/19.

1. a



Angle sum of quadrilateral ABCD = angle sum of ΔABC + angle sum of ΔADC

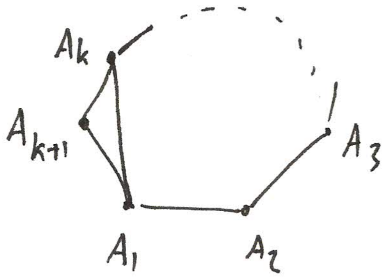
$= \pi + \pi = 2\pi$ □

b. Proof by induction

Base case $n=3$: Angle sum of triangle $= (3-2)\cdot\pi = \pi$

– this was proved in class.

Induction step: Suppose true for $n=k$, & show true for $n=k+1$.



Consider a convex $(k+1)$-gon.

Label the vertices $A_1, A_2, \dots, A_{k+1}$ in ccw order.

Draw the line A_1A_k.

This divides the $(k+1)$-gon $A_1A_2 \cdots A_{k+1}$ into the triangle $\Delta A_1 A_k A_{k+1}$ and the k-gon $A_1A_2 \cdots A_k$.

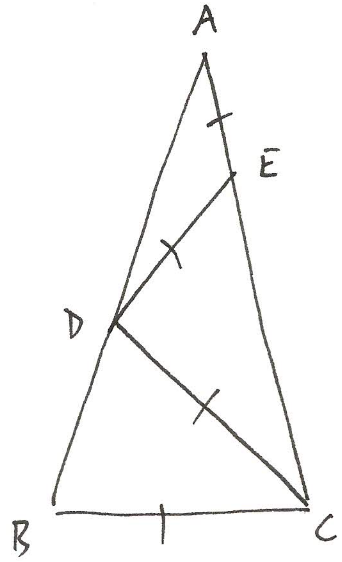
We see that Angle sum of $(k+1)$-gon $A_1A_2 \cdots A_{k+1}$

= Angle sum of triangle $A_1A_kA_{k+1}$

+ Angle sum of k-gon $A_1A_2\cdots A_k$

$= \pi + \underbrace{(k-2)\pi}_{\text{by induction hypothesis}} = ((k+1)-2)\pi$ □.

2.



Let $\angle BAC = \alpha$.

Then $\angle ADE = \angle DAE = \angle BAC = \alpha$ by isosceles triangle theorem.

$\angle AED = \pi - 2\alpha$ by "angle sum of $\Delta = \pi$"

$\angle DEC = 2\alpha$ (angle on straight line $= \pi$)

$\angle DCE = 2\alpha$ by isosceles triangle thm

$\angle EDC = \pi - 4\alpha$ by "angle sum of $\Delta = \pi$"

$\angle BDC = \pi - \angle EDC - \angle ADE$ (angle on straight line $= \pi$)

$= \pi - (\pi - 4\alpha) - \alpha = 3\alpha$

$\angle DBC = \angle BDC = 3\alpha$ by isosceles triangle thm

$\angle ACB = \angle ABC = \angle DBC = 3\alpha$ by isosceles triangle thm (recall ΔABC is isosceles: $|AB| = |AC|$)

So $3\alpha = \angle ACB = \angle BCD + \angle DCE$

$= (\pi - (\angle BDC + \angle DBC)) + 2\alpha$ by "angle sum of $\Delta = \pi$"

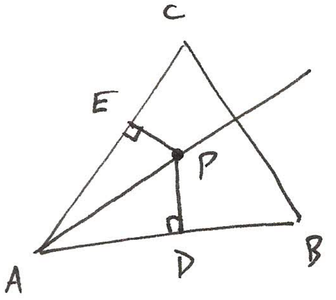
$= (\pi - (3\alpha + 3\alpha)) + 2\alpha$

$= \pi - 4\alpha$

$7\alpha = \pi$, $\alpha = \pi/7$.

So $\angle BAC = \pi/7$. $\square$

3a. <u>Claim</u>:



$$\angle PAD = \angle PAE \iff |PD| = |PE|$$

<u>Proof</u>: $\underline{\Rightarrow}$ $\quad \angle PAD = \angle PAE \Rightarrow \angle APD = \pi - (\angle PAD + \pi/2)$

$$= \pi - (\angle PAE + \pi/2) = \angle APE$$

by "angle sum of triangle $= \pi$".

Now $\quad \triangle APD \cong \triangle APE \quad (ASA)$

$\because \quad \angle PAD = \angle PAE, \quad |AP| = |AP|, \quad \angle APD = \angle APE.$

So ~~$\angle PAD$~~ $|PD| = |PE|$.

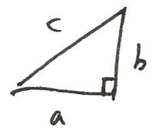
$\underline{\Leftarrow}$. $\quad |PD| = |PE|$ & $|AP| = |AP| \Rightarrow |AD| = \sqrt{|AP|^2 - |PD|^2}$

$$= \sqrt{|AP|^2 - |PE|^2}$$

$$= |AE|$$

by Pythagoras' thm

(NB.



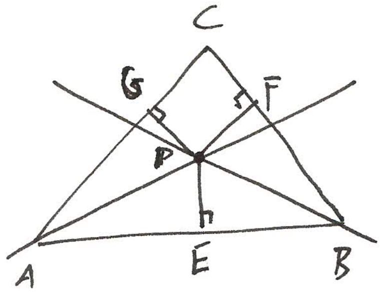
$a^2 + b^2 = c^2$

$\iff a = \sqrt{c^2 - b^2}$ (<u>positive</u> square root))

Now $\quad \triangle APD \cong \triangle APE \quad (SSS)$

So $\quad \angle PAD = \angle PAE$. $\square$

3b.



Let P be the intersection point of the bisectors of the angles $\angle BAC$ and $\angle ABC$. Let the perpendicular line through P to AB intersect AB at E, the perpendicular line through P to BC intersect BC at F, & the perpendicular line through P to CA intersect CA at G.

By part a $|PE| = |PF|$ and $|PE| = |PG|$.

So $|PF| = |PG|$.

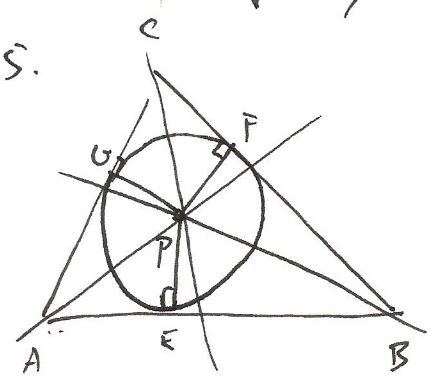
By part a again, P lies on the bisector of the angle $\angle BCA$.

So the angle bisectors of ΔABC all pass through P.

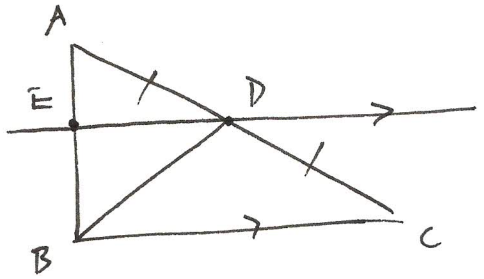
3c. Let $\mathcal{C}$ be the circle with center P and radius $|PE| = |PF| = |PG|$ (using the notation of 3b above).

Then $\mathcal{C}$ passes through $E, F,$ & G and at each of these points the side of the triangle is perpendicular to the radius of $\mathcal{C}$, so the side is tangent to $\mathcal{C}$ by HWIGS.

So $\mathcal{C}$ is an inscribed circle for ΔABC



4.



Draw the line through D parallel to BC, and let E be its intersection point with AB.

By Thales' theorem, $\frac{|AE|}{|EB|} = \frac{|AD|}{|DC|} = 1$, that is, $|AE| = |EB|$

Also $\angle AED = \angle ABC = \pi/2$ (corresponding angles for the parallel lines ED & BC and transversal line AB)

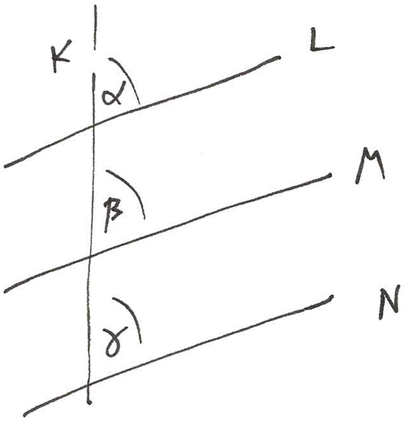
So $\angle BED = \pi - \angle AED = \pi/2$ (angle on a straight line)

Now $\triangle AED \cong \triangle BED$ (SAS)

:— $|AE| = |EB|$, $|ED| = |ED|$, $\angle AED = \angle BED$

So $|BD| = |AD| = \frac{1}{2}|AC|$. □.

5. a Draw a transversal line K to L, M, and N



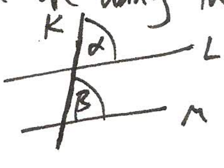
& consider the corresponding angles α, β, γ as shown.

L parallel to $M \Rightarrow \alpha = \beta$

M parallel to $N \Rightarrow \beta = \gamma$

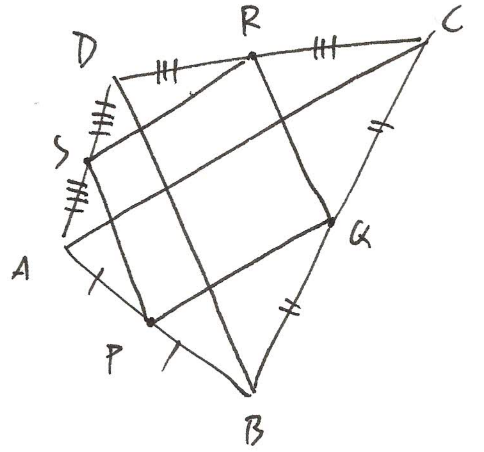
So $\alpha = \gamma \Rightarrow L$ parallel to N. □

(Here we are using the following fact proved in class:



L parallel to $M \iff \alpha = \beta$)

5b.



Draw the diagonals AC and BD of quadrilateral $ABCD$.

By the converse of Thales' theorem applied to ΔABC

$$\frac{|BP|}{|PA|} = \frac{|BQ|}{|QC|} = 1 \implies PQ \text{ is parallel to } AC.$$

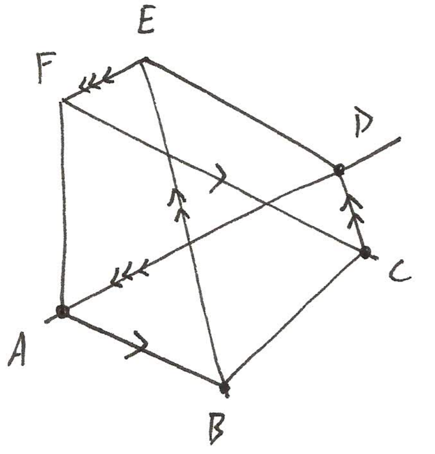
Similarly, considering ΔADC, RS is parallel to AC

Now by part a PQ is parallel to RS.

The same argument shows that SP is parallel to RQ.

So $PQRS$ is a parallelogram (opposite sides are parallel) $\square$

6.



$\text{Area}(\Delta ACE) + \text{Area}(\Delta ABC) + \text{Area}(\Delta CDE) + \text{Area}(\Delta EFA)$

$= \text{Area}(\text{hexagon } ABCDEF)$ $\quad (*)$

$= \text{Area}(\Delta BDF) + \text{Area}(\Delta BCD) + \text{Area}(\Delta DEF) + \text{Area}(\Delta FAB)$

But note that $\text{Area}(\Delta ABC) = \text{Area}(\Delta FAB)$ because they have the same base $|AB|$ and perpendicular height (the perpendicular distance between the parallel lines AB & FC). Similarly, $\text{Area}(\Delta CDE) = \text{Area}(\Delta BCD)$ and $\text{Area}(\Delta EFA) = \text{Area}(\Delta DEF)$. Now cancelling these terms in $(*)$ gives $\text{Area}(\Delta ACE) = \text{Area}(\Delta BDF)$ $\square$