Math 462 Homework 9

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(1) Let

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

be the upper half plane and

$$D = \{w \in \mathbb{C} \mid |w| < 1\}$$

be the interior of the disc with center the origin and radius 1. The Mobius transformation

$$F(z) = \frac{i - z}{i + z}$$

defines a bijection $F\colon \mathcal{H} \to D$ with inverse $F^{-1}\colon D \to \mathcal{H}$ given by

$$F^{-1}(w) = i\frac{1 - w}{1 + w}.$$

For a parametrized curve

$$\gamma\colon [a, b] \to D, \quad \gamma(t) = u(t) + iv(t)$$

in D, the hyperbolic length of γ is defined by the integral

$$\int_a^b \frac{2\sqrt{u'(t)^2 + v'(t)^2}}{1 - u^2 - v^2} dt.$$

(Then the bijection F preserves hyperbolic length.)

Let $w_1, w_2 \in \mathbb{R}$, $-1 < w_1 < w_2 < 1$.

(a) Compute the hyperbolic length of the line segment $[w_1, w_2]$ in D joining the two points w_1, w_2 using the integral formula above.

(b) Let $I = D \cap \mathbb{R} = (-1,1)$. Describe the image of I under F^{-1}.

(c) Use part (b) and a result about hyperbolic lengths in $\mathcal{H}$ to compute the hyperbolic length of the line segment in part (a) without integration.

(2) Let L be a hyperbolic line in $\mathcal{H}$ passing through the point $i \in \mathcal{H}$.

(a) Explain why the image of L under the bijection $F \colon \mathcal{H} \to D$ is a diameter of the disc D given by a line through the origin.

(b) Let $T \colon \mathcal{H} \to \mathcal{H}$ denote the hyperbolic reflection in the hyperbolic line L. What is the corresponding hyperbolic isometry $S = F \circ T \circ F^{-1}$ of the disc D?

(c) Now suppose L_1 and L_2 are two hyperbolic lines in $\mathcal{H}$ passing through $i \in \mathcal{H}$ which meet at an angle θ measured counterclockwise from L_1 to L_2. For each $i = 1,2$ let T_i be the hyperbolic reflection in L_i, and $S_i = F \circ T \circ F^{-1}$ the corresponding hyperbolic isometry of D as in part (b). Describe the composition $S_2 \circ S_1$, and use your answer to describe the composition $T_2 \circ T_1$.

(3) (a) Let $R \in \mathbb{R}$, $0 < R < 1$. Let $C = \{w \in \mathbb{C} \mid |w| = R\}$ be the circle with center the origin and radius R. (Then C is contained in the disc D.) Show that the image of C under the transformation F^{-1} is a circle with center the point $i\frac{1+R^2}{1-R^2}$ and radius $\frac{2R}{1-R^2}$. (Here the radius and center of the circles are defined using the ordinary (Euclidean) notion of distance.)

(b) Let $S \in \mathbb{R}$, $S > 0$, and $P \in D$. The *hyperbolic circle* in D with radius S is the set

$$\{Q \in D \mid d(P,Q) = S\}$$

where $d(P,Q)$ denotes the hyperbolic distance from P to Q. Explain why the set C from part (a) is a hyperbolic circle with center 0 and some hyperbolic radius S.

(c) We can define hyperbolic circles in $\mathcal{H}$ in the same way as we did in part (b) for D. Explain why the image $F^{-1}(C)$ of C computed in part (a) is a hyperbolic circle in $\mathcal{H}$ with center i. (In particular, in this case the hyperbolic center is different from the Euclidean center.)

(4) Let $A, B, C \in D$ be points in the disc D. Consider the hyperbolic triangle T with vertices A, B, C and the Euclidean triangle T' with vertices A, B, C. (That is, the sides of T are the segments of hyperbolic lines joining the points A, B, C and the sides of T' are the segments of ordinary (Euclidean) lines joining the points A, B, C.) Let α, β, γ be the angles of T at the vertices A, B, C and α', β', γ' the angles of T' at A, B, C.

(a) Explain carefully why we have $\alpha < \alpha'$, $\beta < \beta'$, and $\gamma < \gamma'$.

(b) Use part (b) to give another proof that $\alpha + \beta + \gamma < \pi$. (We showed in class that the area of T is given by $\pi - (\alpha + \beta + \gamma)$, so in particular $\alpha + \beta + \gamma < \pi$; here we give another more direct proof of this inequality.)

(5) Let $n \in \mathbb{N}$ and $r \in \mathbb{R}$, $0 < r < 1$. For $k = 0, 1, 2, \ldots, n-1$ let $w_k = re^{2\pi i k/n} \in D$. Let P be the hyperbolic polygon with vertices $w_0, w_1, \ldots, w_{n-1}$ (that is, P is the plane region bounded by the hyperbolic line segments $w_0w_1, w_1w_2, \ldots, w_{n-1}w_0$).

(a) Explain why P is a regular polygon (that is, the angles of P are all equal and the hyperbolic lengths of the sides of P are all equal).

(b) Now consider varying the parameter r. Let $\theta = \theta(r)$ denote the angle of P at a vertex. Justify the following assertions (a complete proof is not required but you should try to explain why they are true).

(i) $\theta(r)$ is a continuous function of r for $0 < r < 1$.

(ii) $\lim_{r \to 1} \theta(r) = 0$.

(iii) $\lim_{r \to 0} \theta(r) = \frac{(n-2)}{n}\pi$.

(c) Using part (b) deduce that there is a regular hyperbolic polygon with n sides and each angle equal to θ for any $\theta \in \mathbb{R}$ such that $0 < \theta < \frac{(n-2)}{n}\pi$. (For example, there is a regular hyperbolic pentagon with all angles equal to $\pi/2$.)

[Hint: (b)(iii) First show that for a regular Euclidean polygon with n sides the angles φ are given by $\varphi = \frac{(n-2)}{n}\pi$.]