Math 132.5. Series (11.2–11.6)

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1 Section 11.2

1.1 Definition of series

If $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, the associated *series* is the sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

which is defined as the limit of the *partial sums* $a_1 + a_2 + \cdots + a_n$ as $n \to \infty$.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i = \lim_{n \to \infty} (a_1 + a_2 + \cdots + a_n).$$

If the limit exists (and is finite) we say the series is *convergent*. If the limit does not exist or is infinite, we say the series is *divergent*.

1.2 Geometric series

The series

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is called a *geometric series* with *initial term* a and *common ratio* r. It is convergent if $|r| < 1$ and divergent if $|r| \geq 1$.

If $|r| < 1$ then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

1.3 The divergence test

If $\lim_{n\to\infty} a_n \neq 0$ or the limit does not exist then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

2 Section 11.3

2.1 The integral test

Let $f(x)$ be a positive, decreasing, continuous function which is defined for $x \geq 1$. The improper integral $\int_1^\infty f(x)dx$ and the series $\sum_{n=1}^{\infty} f(n)$ are either both convergent or both divergent.

2.2 The p-series convergence criterion

The *p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

3 Section 11.4

3.1 The comparison test

Let a_n and b_n be two sequences such that $0 \leq a_n \leq b_n$ for n sufficiently large.

(1) If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is divergent.

3.2 The limit comparison test

Let a_n and b_n be two sequences such that $a_n, b_n > 0$ for all n sufficiently large. If the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is nonzero and finite then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent or both divergent.

4 Section 11.5

4.1 The alternating series test

Let b_n be a sequence such that $b_n \geq 0$ and

(1) b_n is decreasing, that is, $b_n \geq b_{n+1}$ for all n.

(2) $\lim_{n\to\infty} b_n = 0$.

Then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

is convergent.

5 Section 11.6

5.1 Absolute convergence implies convergence

If $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

5.2 Absolute and conditional convergence

(1) We say $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(2) We say $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

5.3 Ratio test

Let $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.

(1) If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $L = 1$ the ratio test is inconclusive.

5.4 Root test

Let $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

(1) If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $L = 1$ the root test is inconclusive.

Note that the conclusions of the ratio and root test are identical; the difference is the definition of the limit L.