Monday 10/14/13
235 Midterm 1
Solutions.

Q1.
$$\begin{aligned} x + y + 4z &= -1 \\ 2x + 5y + 2z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

augmented matrix
$$\begin{array}{c} \\ -2R1 \\ -R1 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 4 & -1 \\ 2 & 5 & 2 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right) \underset{\div 3}{\rightsquigarrow} \left(\begin{array}{cccc} 1 & 1 & 4 & -1 \\ 0 & 3 & -6 & 12 \\ 0 & 2 & -2 & 6 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ \\ -2R2 \end{array}\left(\begin{array}{cccc} 1 & 1 & 4 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 2 & -2 & 6 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \\ \\ \div 2 \end{array}\left(\begin{array}{cccc} 1 & 1 & 4 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & -2 \end{array}\right) \overset{\substack{-4R3 \\ +2R3}}{\rightsquigarrow} \left(\begin{array}{cccc} 1 & 1 & 4 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right) \overset{-R2}{\rightsquigarrow} \left(\begin{array}{cccc} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \qquad \boxed{\begin{aligned} x &= 1 \\ y &= 2 \\ z &= -1. \end{aligned}}$$

Q2. a)
$$\left.\begin{aligned} x + 2y + 4z &= 7 \\ 2x + 3y - 5z &= 1 \end{aligned}\right\} \quad (*)$$

augmented matrix
$$\begin{array}{c} \\ -2R1 \end{array}\left(\begin{array}{ccc|c} 1 & 2 & 4 & 7 \\ 2 & 3 & -5 & 1 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ \div(-1) \end{array}\left(\begin{array}{cccc} 1 & 2 & 4 & 7 \\ 0 & -1 & -13 & -13 \end{array}\right) \overset{-2R2}{\rightsquigarrow} \left(\begin{array}{cccc} 1 & 2 & 4 & 7 \\ 0 & 1 & 13 & 13 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc} 1 & 0 & -22 & -19 \\ 0 & 1 & 13 & 13 \end{array}\right) \qquad \begin{aligned} x - 22z &= -19 \\ y + 13z &= 13 \end{aligned} \qquad \boxed{\begin{aligned} x &= -19 + 22z \\ y &= 13 - 13z \\ z &\text{ is free} \end{aligned}}$$

Equivalently,
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -19 + 22z \\ 13 - 13z \\ z \end{pmatrix} = \begin{pmatrix} -19 \\ 13 \\ 0 \end{pmatrix} + z\begin{pmatrix} 22 \\ -13 \\ 1 \end{pmatrix}, \quad z \in \mathbb{R} \text{ is arbitrary.}$$

b) Geometric interpretation: The set of solutions is the line in $\mathbb{R}^3$ through the point $\begin{pmatrix} -19 \\ 13 \\ 0 \end{pmatrix}$ in the direction $\begin{pmatrix} 22 \\ -13 \\ 1 \end{pmatrix}$.

It is the intersection of the two planes in $\mathbb{R}^3$ given by the equations $(*)$

Q3.

augmented Matrix for $A\underline{x}=\underline{b}$

$$(A\ \underline{b}) = \left(\begin{array}{cccccc|c} 1 & 1 & 2 & 3 & 4 & 2 & 1 \\ 1 & 1 & 1 & 3 & 5 & 3 & 2 \\ 2 & 2 & 3 & 3 & 6 & 2 & 3 \\ 2 & 2 & 4 & 3 & 5 & 1 & 2 \end{array}\right) \xrightarrow{G.E.} \begin{array}{c} \begin{array}{cccccc} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \end{array} \\ \left(\begin{array}{cccccc|c} \boxed{1} & 1 & 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & \boxed{1} & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \quad \text{RREF}$$

a) All solutions of $A\underline{x}=\underline{b}$:-

From RREF of augmented matrix :

$$\begin{aligned} x_1 + x_2 + 3x_5 + x_6 &= 3 \\ x_3 - x_5 - x_6 &= -1 \\ x_4 + x_5 + x_6 &= 0 \end{aligned} \quad \leadsto \quad \begin{aligned} x_1 &= 3 - x_2 - 3x_5 - x_6 \\ x_3 &= -1 + x_5 + x_6 \\ x_4 &= -x_5 - x_6 \end{aligned}$$

x_2, x_5, x_6 are free $\qquad$ x_2, x_5, x_6 free.

i.e.
$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 - x_2 - 3x_5 - x_6 \\ x_2 \\ -1 + x_5 + x_6 \\ -x_5 - x_6 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$x_2, x_5, x_6 \in \mathbb{R}$ are arbitrary.

b) Does the equation $A\underline{x}=\underline{c}$ have a solution for every vector $\underline{c} \in \mathbb{R}^4$?

NO :-

$$(A\ \underline{c}) = \left(\begin{array}{cccccc|c} 1 & 1 & 2 & 3 & 4 & 2 & c_1 \\ 1 & 1 & 1 & 3 & 5 & 3 & c_2 \\ 2 & 2 & 3 & 3 & 6 & 2 & c_3 \\ 2 & 2 & 4 & 3 & 5 & 1 & c_4 \end{array}\right) \xrightarrow{G.E.} \overbrace{\left(\begin{array}{cccccc|c} \boxed{1} & 1 & 0 & 0 & 3 & 1 & d_1 \\ 0 & 0 & \boxed{1} & 0 & -1 & -1 & d_2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_4 \end{array}\right)}^{\text{RREF}(A\ \underline{c})}$$

RREF(A), same as above.

$A\underline{x}=\underline{c}$ has a solution precisely when the entry $d_4 = 0$.

The condition $d_4 = 0$ is only satisfied for some (not all) choices of $\underline{c}$.

Q4. $S : \mathbb{R}^2 \to \mathbb{R}^3$, $S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

$$\Rightarrow \quad S\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 1\\1 & 2\\1 & 3\end{pmatrix}}_{A = \text{Matrix of } S.}\begin{pmatrix}x\\y\end{pmatrix}$$

$T : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

$$\Rightarrow \quad T\begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 1 & 3\\2 & 1 & 1\end{pmatrix}}_{B = \text{matrix of } T.}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & \mathbb{R}^3 \arrow[r, "T"] & \mathbb{R}^2 \end{tikzcd}$$

$$T \circ S(\underline{x}) = T(S(\underline{x})) = B \cdot (A \cdot \underline{x}) = (BA) \cdot \underline{x}.$$

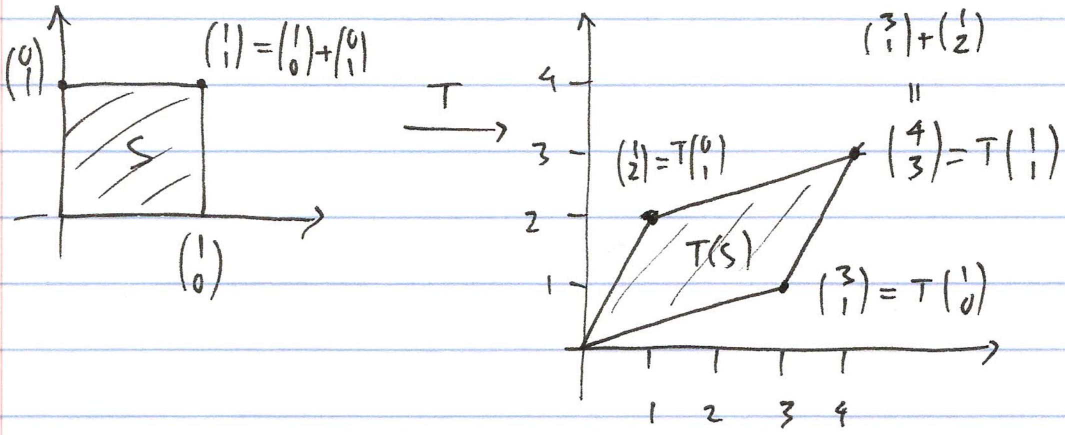
$$\text{Matrix of } T\circ S = BA = \begin{pmatrix}1 & 1 & 3\\2 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\\1 & 3\end{pmatrix}$$

$$= \begin{pmatrix}1\cdot1+1\cdot1+3\cdot1 & 1\cdot1+1\cdot2+3\cdot3\\2\cdot1+1\cdot1+1\cdot1 & 2\cdot1+1\cdot2+1\cdot3\end{pmatrix} = \begin{pmatrix}5 & 12\\4 & 7\end{pmatrix}.$$

Q5. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$S \subset \mathbb{R}^2$, $\quad S = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ AND } 0 \leq y \leq 1\}$



Q6. $S: \mathbb{R}^2 \to \mathbb{R}^2$ rotation about the origin through angle $\pi/4$ counterclockwise.

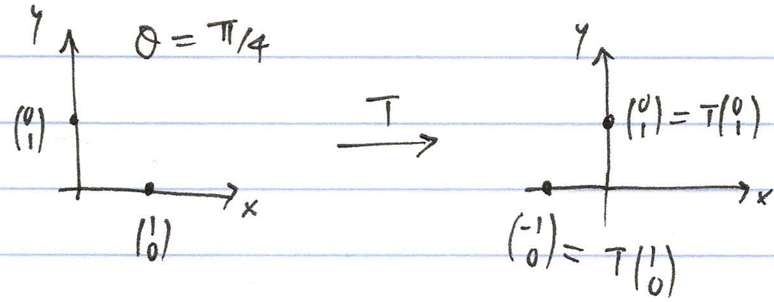
$T: \mathbb{R}^2 \to \mathbb{R}^2$ reflection in the y-axis.

a) Compute the matrices of S, T, and the composite $T \circ S$.

Matrix of S, $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

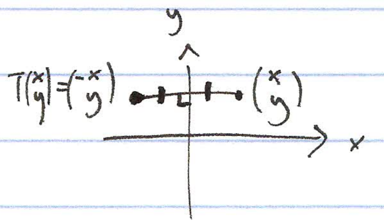
$\theta = \pi/4$

Matrix of T :



$\Rightarrow$ matrix of T, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(OR: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

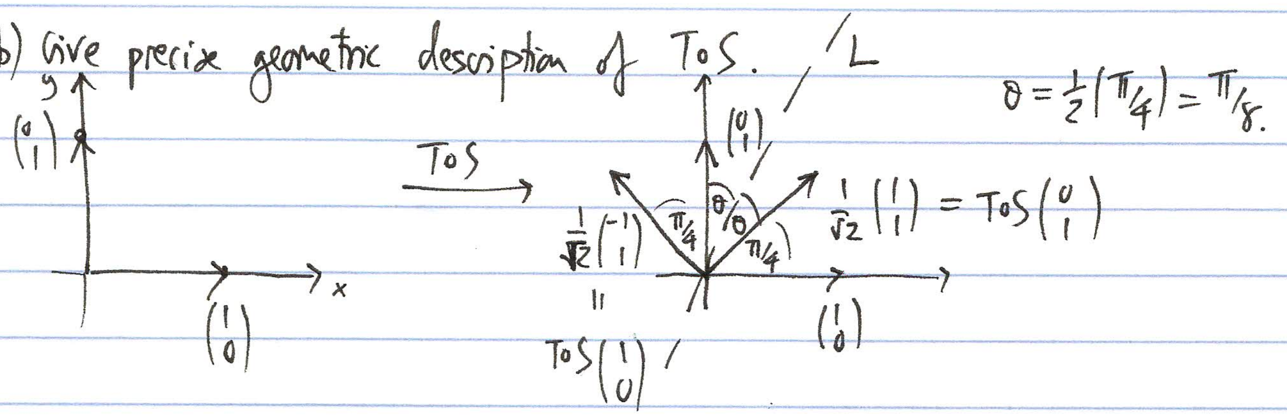


)

Matrix of $T \circ S = B \cdot A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

b) Give precise geometric description of $T \circ S$.



$T \circ S$ = reflection in line L passing through origin at angle $3\pi/8$ to the x-axis

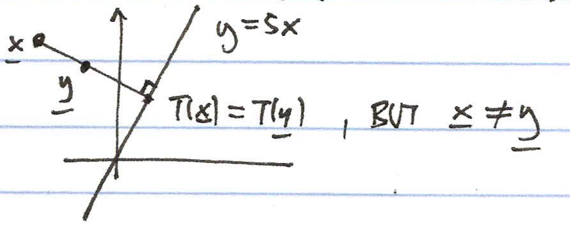
Q7. State whether transformation is invertible & if so describe inverse.

a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ rotation about the origin through angle $\pi/7$ counterclockwise:–

S is invertible, S^{-1} is rotation about the origin through angle $\pi/7$ clockwise.

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ orthogonal projection onto the line $y = 5x$:–

T is NOT invertible because (for example) T is not one-to-one.



(Alternatively, T is not invertible because T is not onto: the image (or range) of T is the line ($y=5x$), not the whole of $\mathbb{R}^2$)

c) $U : \mathbb{R}^3 \to \mathbb{R}^2$, $U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

U is NOT invertible:– In general, if $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation is invertible then we must have $n = m$.

Alternatively, U is NOT invertible because it is not one-to-one.
For example, solving $U(\underline{x}) = \underline{0}$ we find

$$\begin{array}{c} \\ -R1 \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \div(-1) \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{pmatrix} \overset{-2R2}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{array}{ccc} x & y & z \\ \end{array} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{array}{l} x - z = 0 \\ y + 2z = 0 \\ z \text{ free} \end{array} \rightsquigarrow \begin{array}{l} x = z \\ y = -2z \\ z \text{ free} \end{array} \rightsquigarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad z \in \mathbb{R} \text{ arbitrary.}$$

So $U\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = U\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, U not one-to-one

d) $V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible precisely when

$ad - bc \neq 0$, and then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

In our case $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, $\quad 4 \cdot 2 - 1 \cdot 3 = 5 \neq 0$,

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}, \quad V^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Q8. (a) Compute the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$

$$\begin{array}{c} \\ -R1 \\ -R1 \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{c} -R3 \\ -2R3 \\ \\ \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{c} -R2 \\ \\ \\ \end{array}\begin{pmatrix} 1 & 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\left(\underline{\text{Check:}} \quad A^{-1} \cdot A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark \right)$$

(b) Using (a) solve

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 5 \\ x + y + 2z &= 7. \end{aligned}$$

Equations are given by $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$

Solutions : $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 4 \end{pmatrix}$

Q9. $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$

Given : T is a rotation.

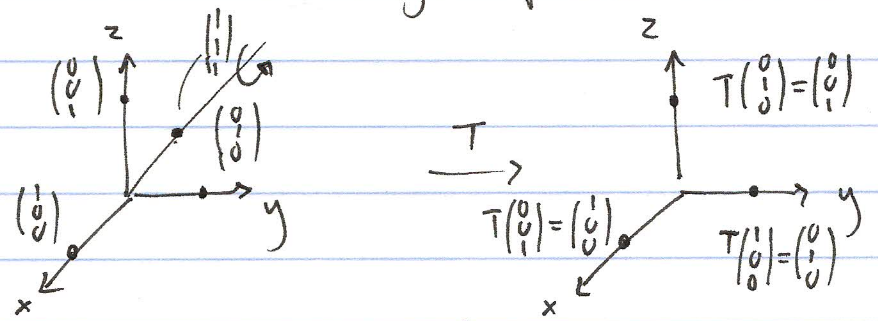
(a) Find axis of rotation.

Solve $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $\begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightsquigarrow x = y = z$,

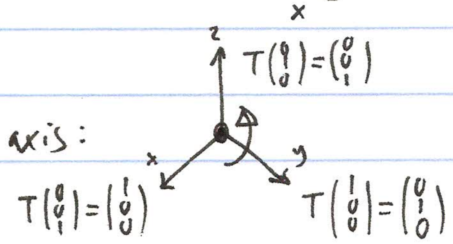
i.e. solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $z \in \mathbb{R}$ arbitrary (OR : use G.E.)

The axis is the line through the origin in the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) What is the angle of rotation.



Looking down the axis:



Angle $= 2\pi/3$, counterclockwise.