# Answers for Midterm 1

Paul Hacking

10/8/09

**1**.

(a) We apply the row reduction algorithm to the augmented matrix $(A\ \mathbf{b})$. This gives reduced row echelon form (RREF)

$$\begin{pmatrix} 1 & 0 & 0 & -10 & 17 \\ 0 & 1 & 0 & 5 & -8 \\ 0 & 0 & 1 & -2 & 3 \end{pmatrix}$$

So the solutions of the equation are

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 17 + 10t \\ -8 - 5t \\ 3 + 2t \\ t \end{pmatrix}$$

where $t$ is arbitrary ($t$ is a free variable).

(b) Yes because the RREF of the matrix $A$ has a pivot in every row. (In more detail: the RREF of the augmented matrix $(A\ \mathbf{c})$ of the equation $A\mathbf{x} = \mathbf{c}$ is given by the RREF of $A$ together with a last column (which depends on $\mathbf{c}$). We know from part (a) that the RREF of $A$ has a pivot in every row. So there are no rows of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & d \end{pmatrix}$ in the RREF of $(A\ \mathbf{c})$. It follows that the equation $A\mathbf{x} = \mathbf{c}$ has a solution.)

**2**.

(a) We apply the row reduction algorithm to the augmented matrix

$$\begin{pmatrix} 1 & -3 & 2 & 3 \\ 4 & -9 & 17 & 6 \\ 1 & -1 & 8 & c \end{pmatrix}$$

of the given system of linear equations. After the "forward phase" of the algorithm, we have matrix

$$\begin{pmatrix} 1 & -3 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & c+1 \end{pmatrix}.$$

So the system of equations has a solution if $c+1=0$, that is, $c=-1$, and does not have a solution if $c \neq -1$.

Now set $c=-1$ and finish the algorithm (doing the "backward phase"). We obtain the matrix

$$\begin{pmatrix} 1 & 0 & 11 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, when $c=-1$, the system of equations has solutions

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3-11z \\ -2-3z \\ z \end{pmatrix}$$

where $z$ is arbitrary.

(b) Geometrically, each of the three equations defines a plane in $\mathbb{R}^3$. If $c=-1$, the intersection of the 3 planes is a line. If $c \neq -1$, the intersection of the 3 planes is the empty set. (In more detail: The intersection of the planes corresponding to the first two equations is a line in $\mathbb{R}^3$. This line is parallel to the plane defined by the third equation $x-y+8z=c$, and is contained in this plane when $c=-1$. Notice that changing the value of $c$ corresponds to replacing the third plane with a parallel plane.)

Alternatively, you can describe the results of part (a) in terms of the column vectors of the augmented matrix of the system of linear equations as follows. The three vectors

$$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -9 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 17 \\ 8 \end{pmatrix}$$

lie in a plane in $\mathbb{R}^3$, and the vector

$$\begin{pmatrix} 3 \\ 6 \\ c \end{pmatrix}$$

lies in this plane when $c = -1$ (and does not lie in the plane if $c \neq -1$). So the equation

$$x \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + y \begin{pmatrix} -3 \\ -9 \\ -1 \end{pmatrix} + z \begin{pmatrix} 2 \\ 17 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ c \end{pmatrix}$$

(which is just another way of writing the system of equations in part (a)) has a solution if $c = -1$, and has no solutions if $c \neq -1$.

**3**.

(1) We apply the row reduction algorithm to the $3 \times 6$ matrix $(A\ I)$. The result is a matrix of the form $(I\ B)$, and then $B = A^{-1}$ is the inverse of $A$. (This is the standard algorithm to find the inverse of a matrix.) In our case we find

$$A^{-1} = \begin{pmatrix} -23 & 6 & 4 \\ 19 & -5 & -3 \\ -7 & 2 & 1 \end{pmatrix}.$$

(2) The equation $A\mathbf{x} = \mathbf{b}$ has solution $\mathbf{x} = A^{-1}\mathbf{b}$ (when $A$ is invertible). In our case we compute

$$A^{-1}\mathbf{b} = \begin{pmatrix} -23 & 6 & 4 \\ 19 & -5 & -3 \\ -7 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -9 \\ 7 \\ -2 \end{pmatrix}.$$

**4**.

(a) The matrix of a rotation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ through an angle $\theta$ anticlockwise about the origin is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In our case $\theta = 3\pi/4$ so we obtain

$$\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(b) (4 points) The map $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by projection onto the line through the origin in the direction $\mathbf{v}$ is given by

$$U(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}.$$

In our case $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ so we find

$$U(\mathbf{x}) = \frac{1}{10} \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{x}.$$

Alternatively, we can use the formula

$$U(\mathbf{x}) = \mathbf{x} - \left( \frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n}$$

where $\mathbf{n}$ is a vector perpendicular to the line. In our case we can use $\mathbf{n} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (this is perpendicular to the line because $\mathbf{n} \cdot \mathbf{v} = 0$).

(c) The reflection $V : \mathbb{R}^3 \to \mathbb{R}^3$ in a plane through the origin with normal vector $\mathbf{n}$ is given by the formula

$$V(\mathbf{x}) = \mathbf{x} - 2 \left( \frac{\mathbf{x} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n}.$$

In our case, a normal vector to the plane $2x - y + z = 0$ is $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ (given by the coefficients of the equation) and we find

$$V(\mathbf{x}) = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} \mathbf{x}.$$

**5.**

(a) $T$ is rotation about the origin through an angle of $\pi/2$ radians anti-clockwise. $U$ is projection onto the line with direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $V$ is a horizontal shear (to the left).

(b) If $S \colon \mathbb{R}^m \to \mathbb{R}^l$ and $T \colon \mathbb{R}^n \to \mathbb{R}^m$ are linear maps with matrices $A$ and $B$, then the matrix of the composition $S \circ T$ is the product $AB$. Using this, in our case we find that the matrices of $T \circ T$, $U \circ U$, and $T^{-1} \circ V \circ T$ are

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(Note: To find the last matrix, we need to compute the matrix of $T^{-1}$, that is, the inverse of the matrix of $T$. To do this, we can use the formula for the inverse of a $2\times 2$ matrix: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$, then $A$ is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.) Geometrically, $T \circ T$ is rotation about the origin through $\pi$ radians, $U \circ U = U$ is projection onto the line with direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $T^{-1} \circ V \circ T$ is a vertical shear (upwards).