A Review of Black Hole Entropy

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description
reference

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Timeline of GR and Black Hole Physics

- 1915: Einstein's General Relativity paper
- 1915: Schartzschild solution to the Einstein equations
- 1919: First observational evidence of black holes from Mercury's orbit
- 1921: Reissner-Nordstrom solution, first electromagnetic black hole
- 1963: Kerr solution, first rotating black hole
- 1970: Christodoulou's work on black hole thermodynamics
- 1972 Bekenstein's, Ph. D. thesis, (unpublished).
- **1973: Bekenstein's seminal paper on black hole entropy**
- 1975: Hawking's discovery of black hole radiation

Einstein's Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

"Spacetime tells matter how to move; matter tells spacetime how to curve."- J.A. Wheeler

Kerr Black holes

Kerr Metric

$$ds^2 = -\left(1 - \frac{2GMr^2}{\rho}\right)dt^2 - \left(\frac{2Gmar\sin^2\theta}{\rho^2}\right)(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

with $\Delta(r) = r^2 - 2GMr + a^2$ and $\rho^2(r,\theta) = r^2 + a^2 cos^2\theta$.

Kerr-Newman

$$2GMr \to 2GMr - G\left(Q^2 + P^2\right).$$

Essential and Coordinate Singularities

$$\rho \to 0 \implies |r| = a\cos(\theta) \quad r = 0, \quad \theta = \frac{\pi}{2}.$$

$$\Delta \to 0 \implies \quad r_{\pm} = \frac{-2Gm \pm \sqrt{4G^2m^2 - 4a^2}}{2} = -Gm \pm \sqrt{G^2m^2 - a^2}.$$

It is impossible to observe a singularity from null infinity.
Formally: *The causal structure is such that the maximal Cauchy development possesses a complete future null infinity.*

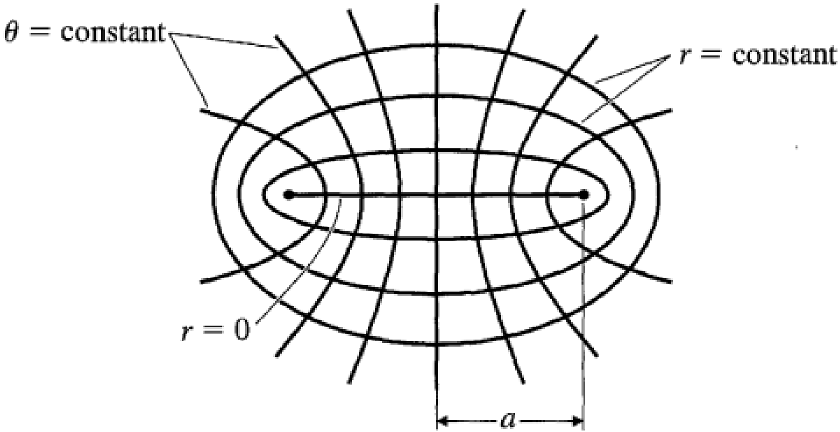
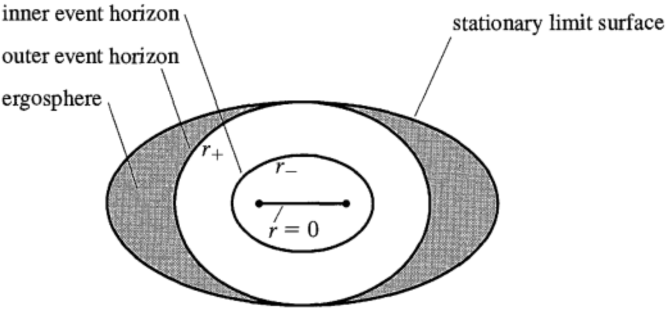


Figure: Side view of Kerr Metric in Boyer - Lindquist Coordinates (r, θ).

Horizons

Video by Alessandro Roussel



Side view of the horizons.

Maximal Extension

Analytical Continuation

$$r_* = \int \frac{r^2 + a^2}{\Delta}\, dr = r + \frac{2Mr_+}{r_+ - r_-}\ln|r - r_+| - \frac{2Mr_-}{r_+ - r_-}\ln|r - r_-|,$$

$$\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}.$$
$$v = t + r_*,$$

$$\tilde{\phi} = \phi + \int \frac{a}{\Delta}\, dr = \phi + \frac{a}{r_+ - r_-}\ln\left|\frac{r - r_+}{r - r_-}\right|.$$

Near $r = r_+$, we can define

$$U_+ = -e^{\kappa_+ u}, \qquad V_+ = e^{\kappa_+ v}.$$

with $u = t - r_*$, $v = t + r_*$, and $\kappa_+ = \frac{r_+ - r_-}{2\left(r_+^2 + a^2\right)}$

Conformal Diagram

Extended Schwarzschild Metric

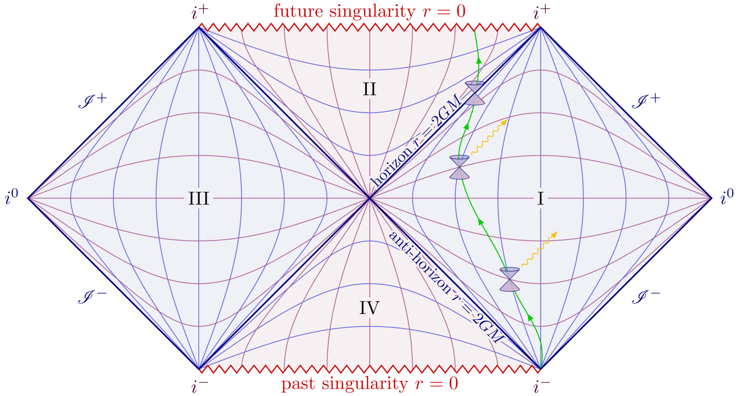


Figure: The Physical Picture.

Conformal Diagram

Kerr Metric Extended

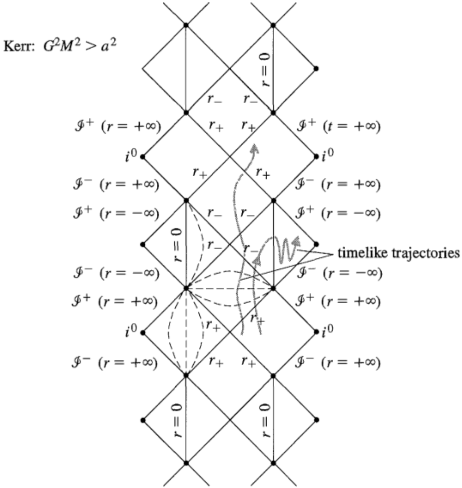


Figure: The Physical Picture.

Killing Vectors and Gauge Symmetries

Dr Killing

Horizon

Killing Vectors and Gauge Symmetries

$$\nabla_{(\mu}K_{\nu)} = 0.$$

Transformation by

$$x_\mu \to x_\mu + \varepsilon_\mu.$$

Gauge Transformation

$$g'\left(x'\right)_{\mu\nu} = g'_{\mu\nu}\left(x+\varepsilon'\right) = g'_{\mu\nu}\left(x\right) + \varepsilon^{\alpha}\partial_{\alpha}g_{\mu\nu} + \mathcal{O}\left(\varepsilon^{2}\right)$$

$$g'\left(x'\right)_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}\partial_{\mu}\varepsilon^{\alpha} - g_{\alpha}\mu\partial_{\nu}\varepsilon^{\alpha} + \mathcal{O}\left(\varepsilon^{2}\right)$$

$$\textbf{(1)-(2)} \implies \delta g_{\mu\nu} = -2\nabla_{(\mu}\varepsilon_{\nu)}.$$

Killing Horizons

Conditions on surfaces being Killing horizons:

- $K^{\mu} K_{\mu} = 0$
- The surface must also be null

Killing versus Event Horizons

- **Stationary**: Every event horizon is a Killing horizon.
- **Stationary and Static**: $K^{\mu} = (\partial_t)^{\mu}$
- **Stationary but not Static**: $R^{\mu} = (\partial_{\phi})^{\mu}$ the other surface corresponds to a linear combination.

Killing Surfaces for the Kerr Blackhole

For a Kerr Black Hole (axisymetric) the killing vectors will be

$$R^\mu = (\partial_\phi)^\mu \quad \text{and} \quad K^\mu + \Omega_H R^\mu$$

$$\text{with} \quad \Omega_H \equiv \left(\frac{d\phi}{dt}\right)_- (r_+) = \frac{a}{r_+^2 + a^2}.$$

Note: ∂_t no longer corresponds to its own killing horizon.

Penrose Process

$$E = -K_\mu p^\mu$$

$$L = R_\mu p^\mu$$

Inside ergosphere K_μ becomes spacelike.

$$E = -K_\mu p^\mu < 0$$

Penrose Process

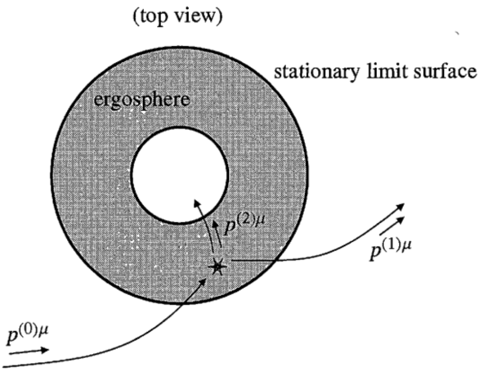


Figure: The Penrose Process

Irriducible Mass and Chrisodulou

Given in terms of area, Irreducible mass is:

$$M_{ir} = \sqrt{\frac{A}{16\pi}}$$

Consider triangle inequality if the following:

$$E_d = (\Sigma M_{ir}{}^2)^{1/2}$$

Schwartzschild Merger GWs

Bekenstein (1973) and Black Hole Entropy

$$\delta M_{ir} > 0$$

Similar to entropy?

Is there a relation to

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V$$

Analogy To Thermodynamics

$$\alpha = \frac{A}{4\pi} \quad \text{and} \quad \alpha = r_+^2 + a^2 = 2Mr_+ - Q^2$$

for Kerr black hole

$$\mathrm{d}M = \Theta \mathrm{d}\alpha + \vec{\Omega} \cdot \mathrm{d}\vec{L} + \Phi \mathrm{d}Q$$

where

$$\Theta \equiv \frac{1}{4}\frac{(r_+ - r_-)}{\alpha} \leftrightarrow T = \left.\frac{\partial E}{\partial S}\right|_V,$$

$$\left\{ \begin{array}{l} \vec{\Omega} \equiv \frac{\vec{a}}{\alpha} \\ \phi = \frac{Qr_+}{\alpha} \end{array} \right\} \leftrightarrow -P = \left.\frac{\partial E}{\partial V}\right|_S,$$

More on units later ...

Redefining Entropy

Brillouin's relation for Information and entropy

$$\Delta I = -\Delta S.$$

Elementary particles as bits

Stationary, asymptotically flat, vacuum solution of the Einstein equations coupled to electromagnetism that are nonsingular outside the event horizon are fully characterised by their mass , angular momentum , and electromagnetic charge.

Information and Bits

$$S = -\sum_n p_n \ln p_n$$

Minimum unit of information is the answer to a yes or no question. Entropy for this is maximized when $p_{yes} = p_{no}$. So one bit corresponds to $\ln 2$ of information.

Form of S

Restriction: Monotically increasing

$$S_{BH} = f(\alpha).$$

Anzatz with counter example

$$f(\alpha) \propto \alpha^{1/2}.$$

Dimensional Argument

Final ansatz:

$$f(\alpha) = \gamma\alpha.$$

Extracting dimension:

$$S_{BH} = \eta\hbar^{-1}\alpha.$$

Can we find a value for $\eta\hbar^{-1}$?

Beyond the First Law and Quantisation of Entropy

Minimize the increase in entropy as a particle enters a black hole
Christodolou Result:

$$(\Delta\alpha)_{min} = 0$$

If we give the particle a rest mass μ and a proper radius b it becomes

$$(\Delta\alpha)_{min} = 2\mu b$$

And

$$\ln 2 = (\Delta S)_{min} = (\Delta\alpha)_{min} \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}$$

Appendix A Bekenstein

Quantisation of Entropy

Take the fact that the proper radius b is greater than both the Compton wavelength $\hbar\mu^{-1}$ and gravitational radius 2μ of the particle.

$$b > 2\mu, \qquad\qquad b > \hbar\mu^{-1}$$

Solving both of these bounds for μ and plugging back into the previous gives

$$(\Delta\alpha)_{min} = 2\hbar$$

Then from our expression relating S to α

$$S_{bh} = \frac{\hbar}{2}\ln(2)\alpha$$

Black Hole Temperature

From the first law we saw an analogy between T and Θ. Can we use our expression for S_{bh} to find T_{bh}?

Let

$$T_{bh}^{-1} = \left(\frac{\partial S_{bh}}{\partial M} \right),$$

Similar to the standard thermodynamic .....word here
We get

$$T_{bh} = \frac{2\hbar}{\ln 2} \Theta$$

Generalized Second Law

Can we combine 'common' entropy S_c with S_{bh}?

Throw a system with entropy S into a black hole.
We get

$$\Delta S_{bh} + \Delta S_c = \Delta\left(S_{bh} + S_c\right) > 0$$

How can we test this?

Throw a box with an oscillator into a black hole and minimize the increase in entropy.
We know from Statistical Physics that

$$p_n = \frac{e^{-nx}}{1 - e^{-x}}$$

with $x = \frac{\hbar\omega}{T}$ and

$$\langle E \rangle \equiv \Sigma p_n (n + \frac{1}{2})\hbar\omega = \left[\frac{1}{e^x - 1} + \frac{1}{2}\right]\hbar\omega$$

Using our previous equation for common entropy we get

$$S = \frac{x}{e^x - 1} - \ln(1 - e^{-x})$$

Examples in Bekenstein

Harmonic Oscillator Continued

We can now find our ΔS_{bh} using the radius and total rest mass of the box in the formula from before with $\xi = m \langle E \rangle^{-1}$.

$$\Delta(S_{bh} + S_c) > \xi^{-\frac{1}{2}}(1+\xi)[\frac{1}{2} + (e^x - 1)^{-1}] \ln 2$$

$$-x(e^x - 1) - 1 + ln(1 - e^{-1})$$

Find the minimum of this as a function of x to be

$$\frac{1}{2}\xi^{-\frac{1}{2}}(1+\xi)\ln 2 + \ln[1 - e^{-\xi^{-\frac{1}{2}}(1+\xi)\ln 2}]$$

Which is greater than 0 for all $\frac{m}{\langle E \rangle} \geq 1$

Geroch proposed a perpetual motion machine by lowering a box from infinity into a black hole.
This does not actually violate the second law since the box must have a nonzero radius and thus the energy at infinity is never zero.
We can calculate the actual efficiency of this machine to be.

$$\epsilon = 1 - 2b\Theta < 1$$

Fluctuations in Black Hole Entropy

The second law is statistical so we can see random decreases in entropy due to fluctuations.

Hawkings theorem prevents the black hole area and thus the entropy from decreasing classically.

Replication of Derived Results

R. M. Wald, “Black hole entropy is the Noether charge,” Phys. Rev. D 48, R3427 (1993).

Microscopic Origin of the Bekenstein-Hawking Entropy, A. Strominger (A string theory perspective)

Implications

$$N(V) = 2^n \qquad \ln(N(V)) = n \ln(2) = \frac{V}{\ell_p^3} \ln(2)\,.$$

$$S \propto A.$$

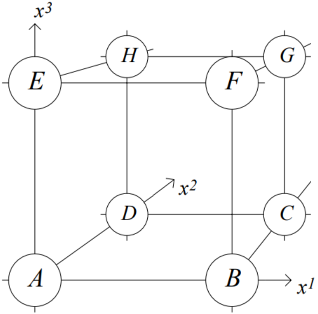


Figure: The Physical Picture.

Thermodynamics vs BH Thermodynamics

Zeroth law – A system in thermal equilibrium has uniform temperature T everywhere.	**Zeroth law (BH)** – For any stationary (Killing) horizon, the surface gravity κ is constant over the entire horizon.
First law – Energy conservation with heat and work: $dE = T\,dS - P\,dV + \sum_i \mu_i\,dN_i$.	**First law (BH)** – For a small perturbation of a stationary black hole: $dM = \frac{\kappa}{8\pi G}\,dA + \Omega_H\,dJ + \Phi_H\,dQ$.
Second law – Total entropy of an isolated system never decreases: $\Delta S_{\text{tot}} \geq 0$.	**Second Law (BH)** – Bekenstein's *generalized second law* says $S_{\text{outside}} + S_{\text{BH}}$ never decreases.
Third law – Absolute zero ($T = 0$) cannot be reached in a finite sequence of operations.	**Third law (BH)** – One cannot reduce κ (and hence T_{BH}) to zero via any finite physical process; extremal black holes ($\kappa = 0$) are unattainable limits.

Table: Correspondence between the laws of thermodynamics and the laws of BH Thermodynamics.

Thank You

Appendix A

Wald's Reproduction of the Entropy-Area Formula

$$S = 2\pi \int_\Sigma \tilde{Q}$$

diff-invariant $\implies S = -2\pi \int_\Sigma E^{abcd}\, \epsilon_{ab}\, \epsilon_{cd}$

with $E^{abcd} = \dfrac{\partial L}{\partial R_{abcd}}, \quad \epsilon_{ab}\epsilon^{ab} = -2$

$$L_{\mathrm{EH}} = \frac{1}{16\pi G} R\, \epsilon, \; R = g^{ac} g^{bd} R_{abcd}$$

$$\frac{\partial R}{\partial R_{abcd}} = \tfrac{1}{2}\left(g^{ac} g^{bd} - g^{ad} g^{bc}\right)$$

$$\Rightarrow E^{abcd} = \frac{1}{32\pi G}\left(g^{ac} g^{bd} - g^{ad} g^{bc}\right)$$

binormal spans dirs. orthogonal to Σ

$$\epsilon_{mnpq} = 2\,\epsilon_{mn}\,\epsilon^{(\Sigma)}_{pq}, \; \epsilon^{(\Sigma)} = \tfrac{1}{2}\epsilon^{(\Sigma)}_{pq}\, dx^p \wedge dx^q$$

$$\epsilon_{abcd}\,\epsilon^{ab} = -2\,\epsilon^{(\Sigma)}_{cd}$$

$$\Rightarrow E^{abcd}\epsilon_{ab}\epsilon_{cd} = \frac{1}{32\pi G}\left(g^{ac} g^{bd} - g^{ad} g^{bc}\right)\epsilon_{mnpq}\epsilon_{ab}\epsilon_{cd}$$

$$= \frac{1}{32\pi G}\,(2\epsilon_{ab}\epsilon^{ab})\,\epsilon^{(\Sigma)}$$

$$= \frac{1}{32\pi G}\,(-4)\,\epsilon^{(\Sigma)} = -\frac{1}{8\pi G}\,\epsilon^{(\Sigma)}$$

$$\therefore\; S = -2\pi \int_\Sigma \left(-\tfrac{1}{8\pi G}\right)\epsilon_{(\Sigma)} = \tfrac{1}{4G}\int_\Sigma \epsilon_{(\Sigma)} = \tfrac{A}{4G}.$$

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