Quantum Black Holes and Two Dimensional Conformal Field Theory

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Section 1

Abstract

This report focuses on the details of black hole microstate counting from a number of frameworks. We consider $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity in 4 dimensions. We also take the string theory in a $\mathcal{N} = 2$ truncation of the $\mathcal{N} = 8$ setup, specified by a prepotential and implement the Hardy-Ramanujan-Rademacher formula for the modular/Jacobi partition function to obtain the indexed degeneracy of BPS states in terms of modified Bessel functions and the Kloosterman sum. The same structure is also obtained via the implementation of the gravitational path integral (GPI) using the technique of equivariant localisation. The results from the microscopic and macroscopic sides match. Analogous structures are briefly explored in $\mathcal{N} = 4$ compactifications.

Section 2

Notations, Convensions, and Units

Throughout this report, unless stated otherwise, we use natural units where $c = \hbar = G = k_B = 1$. Greek letters $\mu, \nu, \rho, \ldots$ denote spacetime indices while Latin letters $i, j, k, \ldots$ denote spatial indices. We use the mostly plus metric signature $(-, +, +, +, \ldots)$. Einstein summation convention is assumed throughout this report.

Section 3

Introduction and Background

Since the discovery of a black hole solution to the field equations [1] we have sought to understand not only their properties but also their tension within a quantum mechanical framework.
We have to ask not only the phenomenological questions of their nature, but also how we can reconcile them within a quantum framework.
Classically, the laws of black hole mechanics closely parallel the laws of thermodynamics [2]. In the 1970's, building on Hawking's area theorem,[3] and this analogy, Bekenstein argued by information theoretic and dimensional reasons: that a black hole carries an entropy proportional to the area A of its event horizon [4].

$$S_{BH} = \frac{A}{4}\frac{k}{\ell_p^2} \qquad \ell_p \equiv \sqrt{\frac{\hbar G}{c^3}} \tag{1}$$

where k is Boltzmann's constant and ℓ_p is the Planck length.
Since then, there have been various corrections and parallels to the Bekenstein-Hawking entropy [5]. In 1993, Wald generalized the Einstein–Hilbert theory to an arbitrary diffeomorphism-invariant Lagrangian (including possible higher-curvature terms), and derived the corresponding entropy formula

$$S_{\text{Wald}} = -2\pi \int_C \sqrt{|g|}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\alpha\beta}} dA$$

In principle, this should agree with other entropic interpretations of quantum mechanical systems [6].

Subsection 3.1

Corrections to Black Hole Entropy

In 2009, Sen developed the quantum entropy function formalism [7] which provides a method to compute the exact quantum degeneracy of black hole microstates including higher derivative and quantum corrections. This is achieved by evaluating a Euclidean path integral in the near-horizon AdS_2 geometry of extremal black holes with appropriate boundary conditions. The leading saddle point of this path integral reproduces the classical Bekenstein-Hawking entropy, while subleading contributions account for quantum corrections. This formalism has been successfully applied to various supersymmetric black holes in string theory, yielding results that match microscopic counting of states [8].

$$d_{\text{ micro }}(\vec{q}) = \langle \exp\left[-iq_i \oint d\theta A_\theta^{(i)}\right]\rangle_{AdS_2}^{\text{finite}}$$

This idea of matching a microscopic and macroscopic result will be a key theme throughout this report and may be seen as in the AdS/CFT correspondence.

Subsection 3.2

AdS/CFT

The AdS/CFT correspondence [9] provided a broader holographic framework: loosely speaking, a gravitational theory on AdS_{d+1} is dual to a CFT_d defined on its boundary [10].
In the precise formulation developed by (GKPW) [11, 12], the boundary value of a bulk field acts as a source for the corresponding operator in the CFT. Correlation functions in the boundary theory are then obtained by functionally differentiating the bulk on-shell action with respect to these boundary conditions.
Within this framework, the supersymmetric (indexed) degeneracy of extremal black holes can be computed directly from a gravitational path integral in Euclidean $AdS_2 \times S^2$ near horizon region with appropriate boundary insertions.

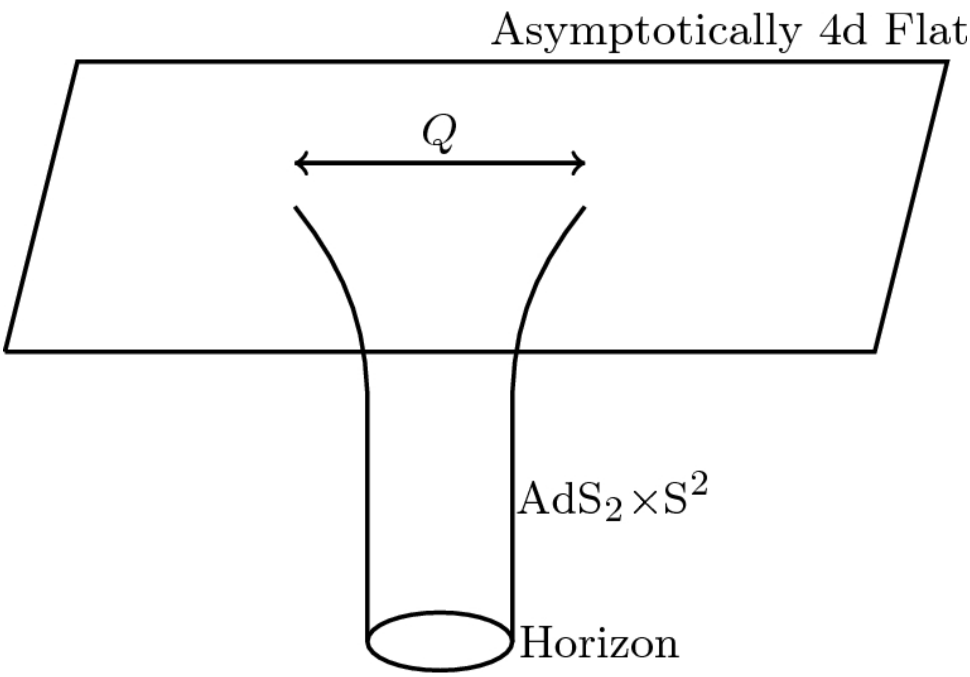

Figure 1: Diagram of the Euclidean $AdS_2 \times S^2$ geometry. This is often known as the AdS_2 'throat'. We have near-horizon geometry at the bottom which is topologically a disk. The boundary circle is the asymptotic boundary. Orbifolds act as $\mathbb{Z}_c$ rotations about the center [13].

In this report we will focus on extremal Reissner-Nordstrom black holes in four-dimensional supergravity theories. The near-horizon geometry of these black holes is given by the product space $AdS_2 \times S^2$. This geometry plays a crucial role in the study of black hole entropy and quantum gravity due to its enhanced symmetries and simplified structure.

Mathematically the $AdS_2 \times S^2$ metric is given in complex coordinates by

$$ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2 + d\theta^2 + \sin^2 \theta d\phi^2 = \frac{4dwd\overline{w}}{(1 - w\overline{w})^2} + \frac{4dzd\overline{z}}{(1 + z\overline{z})^2} \tag{2}$$

Subsection 3.3

JT Gravity

The macroscopic described of near-extremal black holes simplifies due to the emergence of an AdS_2 throat region in their near-horizon geometry. In approaches such as the quantum entropy function the GPI of interest is AdS_2 with specific boundary conditions. A toy model for AdS_2 dynamics is Jackiw–Teitelboim JT gravity. JT gravity is a two-dimensional dilaton Φ model of gravity that captures essential features of black hole thermodynamics and quantum gravity in a simplified setting [14, 15]. The dilaton encodes the size of the transverse sphere. The action for JT gravity with boundary term(s) is given by:

$$\mathcal{S}_{JT} = \frac{1}{16\pi G_N} \left[\int_M d^2x \sqrt{-g} \Phi (R + 2) + 2 \int_{\partial M} dt \sqrt{h} \Phi_b (K - 1) \right] \tag{3}$$

where Φ_b is the value of the dilaton on the boundary and K is the extrinsic curvature of the boundary.

Subsection 3.4

Reissner-Nordstrom Black Holes

The Reissner-Nordstrom (RN) black hole is a spherically symmetric static solution to the Einstein-Maxwell equations. It is a specific case of the Kerr-Newman metric where we have finite electric q and magnetic charge p but the angular momentum vanishes. The metric is given by:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)$$

which gives us inner ($-$) and outer ($+$) horizons at

$$r_\pm = M \pm \sqrt{M^2 - q^2 - p^2} \quad \text{Extremal limit: } r_+ \longrightarrow r_- \quad M^2 = q^2 + p^2 \tag{4}$$

In the 1970's the BPS bound was found in the context of gauge theory with gravity turned off [16, 17]. This bound was derived for the stability against the emission of vector mesons. Extremal RN black holes don't lose energy via the penrose process or by thermal radiation [1]. Thus we can claim that they are stable if we neglect other processes such as Schwinger pair production and under certain conditions, namely black holes that are supersymmetric obey the BPS bound. The class of black holes satisfying eq. (4) are at the stable endpoint of Hawking evaporation and for the case uncharged black holes this regime leads to the open problem known as the information paradox [18].

[1]This is due to the fact that their temperature goes to zero see [A.2]

Subsection 3.5

The Gravitational Path Integral (GPI)

Generally, the path integral was introduced by Feynman [19]. Typically the path integral is taken over fields with a static background but this is not the case when we consider the gravitational path integral. Hence, for a gravitational path integral we need to fix our boundary conditions not only for the fields but also for the geometry that we integrate over. This leads us to need to define a boundary term to have a well defined variational principle [2]. One such term is the Gibbons-Hawking-York Term denoted by I_{GHY} [20, 21, 22]. We modify the Einstein-Hilbert action I_{EH} to obtain the action

$$\mathcal{S} = I_{\text{EH}} + I_{\text{GHY}} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}R + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3y\epsilon\sqrt{h}K \tag{5}$$

where $\mathcal{M}$ is the manifold with boundary in which we integrate over, h is the determinant of the induced metric, K is the trace of the second fundamental form $\epsilon = \pm 1$ for $\partial\mathcal{M}$ spacelike and timelike respectively and y^a is the coordinates on the boundary.
Just as in standard QFT we can define the partition function Z as the path integral over all field configurations weighted by the action where we have the boundary condition that Euclidean time is periodic with period $\beta = \frac{1}{T}$ where T is the temperature.

The gravitational path integral is a functional integral over the space of metrics (and matter fields) modulo diffeomorphisms, so it is inherently infinite-dimensional and its measure/contour are formal rather than mathematically rigorous [20, 23]. Nevertheless, one can often make progress by expanding about semiclassical saddle points. In supersymmetric settings, equivariant localisation can further reduce certain observables to a finite-dimensional integral. , as discussed in Sec. [4.2].

Subsection 3.6

Supergravity and Black Holes

Despite the lack of experimental evidence for supersymmetry, supergravity remains a compelling framework for exploring quantum aspects of gravity and black holes. Despite the unease this fact may cause, I will give some reasons why we should consider such a theory and the powerful framework it provides.
Supergravity provides a consistent way to incorporate fermions into a gravitational theory, addressing issues that arise in purely bosonic theories such as tachyons. In supersymmetry we can define the Witten index [24] which counts the difference between the number of bosonic and fermionic zero energy states.

$$\mathcal{I} \equiv Tr_{\mathcal{H}}\left[(-1)^F e^{-\beta H}\right] \tag{6}$$

[2]The intorduction of the GHY term is also necessary to obtain the correct form of the ADM energy

where F is the fermion number operator, $\beta = \frac{1}{kT}$ is the inverse temperature, H is the Hamiltonian and we take the trace over the Hilbert space $\mathcal{H}$.

There have been various extensions to the Witten index such as the helicity supertrace [25] and the generalised prepotential index [8] which we will focus on in this report also there has been work on the twisted index [26, 27]

Supersymmetric indices such as the elliptic genus and the helicity supertrace are invariant under continuous deformations of the theory (away from walls of marginal stability), making them a powerful tool for counting BPS states associated with extremal black holes. This allows for exact calculations of BPS black hole's entropy in certain supersymmetric settings. Since indices such as an elliptic genus are invariant under continuous changes of the coupling, we can count the BPS microstates in a weakly coupled D-brane description and extrapolate to strong coupling where the same charges form an extremal black hole. Strominger and Vafa famously carried out this program and found agreement with Bekenstein-Hawking entropy to leading order [28].

The index can be computed in both weakly and strongly coupled regimes, allowing for a comparison between microscopic state counting and macroscopic entropy calculations. This duality is particularly useful in string theory contexts where black holes can be described by D-brane configurations.
As we will see later, we explicitly observe this in the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ cases where we match the GPI results to the microscopic counting.

Supersymmetry also allows for the GPI to be computed exactly using equivariant localisation techniques [29] [30]. Typically, the GPI is intractable due to the infinite-dimensional nature of the space of metrics and fields.
Supersymmetric black holes often exhibit enhanced symmetries in their near-horizon geometries, such as $AdS_2 \times S^2$ for extremal Reissner-Nordstrom black holes. These symmetries facilitate the application of the AdS/CFT correspondence, allowing for a holographic interpretation of black hole entropy and microstates. This leads to properties such as superconformal symmetry, fixed attractor values for scalars, no dependence on asymptotic moduli. This greatly simplifies the analysis as the entropy is determined entirely by charges, not by UV details (avoiding divergencies). The path integral can be defined purely in the near-horizon region, avoiding full quantum gravity in asymptotically flat space.
For the case of $\frac{1}{8}$ BPS black holes in $\mathcal{N} = 8$ sugra and $\frac{1}{4}$ BPS black holes in $\mathcal{N} = 4$ sugra have sufficient supersymmetry to reduce the GPI to an integral where contributions from multi-centered black holes and non-BPS configurations can be avoided.

The supergravity framework obeys the supersymmetry algebra generated by the spinors [31]

$$\left\{Q^{\Lambda}_{\alpha}, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu} \qquad \left\{Q^{I}_{\alpha}, Q^{J}_{\beta}\right\} = \epsilon_{\alpha\beta}Z^{IJ} \tag{7}$$

where the generators (spinors) are Q_α and $\overline{Q}_{\dot{\alpha}}$, the momentum is P_μ, Z^{IJ} is a complex tensor $\epsilon_{\alpha\beta}$ is the Levi-Civita two tensor and $\sigma^{\mu}_{\alpha\dot{\alpha}}$ is the Pauli matrices (ϵ, Z and σ are antisymmetric).

Subsection 3.7

$\mathcal{N}=2$ Supergravity

A pertinent formulation of supergravity is the $\mathcal{N}=2$ Poincare theory which arises from gauge fixing the superconformal gravity.

We have the Lagrangian density for $\mathcal{N}=2$ supergravity coupled to n_v vector multiplets and a single hypermultiplet is given below: [8]

$$
\begin{aligned}
8\pi e^{-1}\mathcal{L} = &\Big(-i\big(X^I\bar{F}_I - F_I\bar{X}^I\big)\Big)\left(-\tfrac{1}{2}R\right) \\
&+\Big[i\,\nabla_\mu F_I\nabla^\mu\bar{X}^I + \tfrac{1}{4}i\,F_{IJ}\Big(F^{-I}_{ab} - \tfrac{1}{4}\bar{X}^I T^{ij}_{ab}\varepsilon_{ij}\Big)\Big(F^{-ab\,J} - \tfrac{1}{4}\bar{X}^J T^{ij\,ab}\varepsilon_{ij}\Big) \\
&\quad - \tfrac{1}{8}i\,F_I\Big(F^{+I}_{ab} - \tfrac{1}{4}X^I T_{ab\,ij}\varepsilon^{ij}\Big)T^{ij\,ab}\varepsilon_{ij} - \tfrac{1}{8}i\,F_{IJ}Y^I_{ij}Y^{J\,ij} \\
&\quad - \tfrac{i}{32}F\,(T_{ab\,ij}\varepsilon^{ij})^2 + \tfrac{1}{2}i\,F_{\hat{A}}\,\hat{C} - \tfrac{1}{8}i\,F_{\hat{A}\hat{A}}\Big(\varepsilon^{ik}\varepsilon^{jl}\hat{B}_{ij}\hat{B}_{kl} - 2\,\hat{F}^{-}_{ab}\hat{F}^{-ab}\Big) \\
&\quad + \tfrac{1}{2}i\,\hat{F}^{-ab}F_{\hat{A}I}\Big(F^{-I}_{ab} - \tfrac{1}{4}\bar{X}^I T^{ij}_{ab}\varepsilon_{ij}\Big) - \tfrac{1}{4}i\,\hat{B}_{ij}F_{\hat{A}I}Y^{I\,ij} + \text{h.c.}\Big] \\
&- i\big(X^I\bar{F}_I - F_I\bar{X}^I\big)\Big(\nabla^a V_a - \tfrac{1}{2}V^a V_a - \tfrac{1}{4}|M_{ij}|^2 + D^a\Phi^i{}_\alpha D_a\Phi^\alpha{}_i\Big)\,..
\end{aligned}
$$

where we have the covariant derivatives in terms of one another

$$D^a V_a = \nabla^a V_a - 2f_a^a + \text{ fermionic terms} \tag{8}$$

$I = 1,\dots,n_v$ labels the vector multiplets with: X^I is a complex scalar, A^I_μ is a gauge field, $F^I_{\mu\nu}$ is the field strength with $F_I \equiv \frac{\partial F}{\partial X^I}$, Y^I_{ij} is an auxiliary field and $F(X)$ is the holomorphic prepotential. Φ are the hypermultiplet scalars and ψ are the gravitini. We also have the Weyl multiplet fields $g_{\mu\nu}$ is the metric, R the Ricci scalar, $T^{-}_{\mu\nu}$ is an auxiliary antisymmetric tensor and D is an auxiliary scalar.
If we evaluate the full supergravity Lagrangian on the BPS configuration we get a contribution to our localisation integrand as we will see in Section [4.2].

Subsection 3.8

Conformal Field Theory and Modularity

As we will see later, conformal field theories (CFTs) play a crucial role in understanding the microscopic structure of black holes, particularly in the context of the AdS/CFT correspondence. Thus, it is pertinent to give a brief overview of some key concepts in CFT relevant to black hole physics.

We consider a $2D$ CFT defined on a torus. The torus is characterised by its modular parameter τ, which encodes the shape of the torus. The modular group $SL(2,\mathbb{Z})$ acts on τ via Möbius transformations:

$$\tau \longrightarrow \frac{a\tau+b}{c\tau+d} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \quad ad-bc=1 \tag{9}$$

where we have the generators of $SL(2,\mathbb{Z})$ given by a shift to the parallelogram $T:\tau\longrightarrow\tau+1$ and a swap of the fundamental cycles $S:\tau\longrightarrow-\frac{1}{\tau}$.

The torus has large coordinate re-identifications that aren't continuously connected to the identity, and physics can't depend on which coordinates you chose.

We have the $2D$ partition function on a torus given by [32]

$$Z(\tau,\overline{\tau}) = Tr_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\overline{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right) \qquad q=e^{2\pi i\tau} \tag{10}$$

where $L_0, \tilde{L}_0$ are the zero modes of the Virasoro algebra, $c,\tilde{c}$ are the central charges and τ is the modular parameter of the torus.
For an ordinary, non-chiral, modular-invariant CFT one has

$$Z\left(\frac{a\tau+b}{c\tau+d},\frac{a\overline{\tau}+b}{c\overline{\tau}+d}\right) = Z(\tau,\overline{\tau}) \tag{11}$$

We can generalise this to the case of chiral CFTs where we have modular forms of weight k

$$Z(\tau)\longrightarrow(c\tau+d)^k Z(\tau) \tag{12}$$

Subsection 3.9

The role of Automorphic Forms in the Microscopic Interpretation

The effectiveness of automorphic forms in counting BPS states in superstring theory is hard to overstate. Specifically for toroidal compactifications of heterotic and Type II string theory in 4D we can express them in terms of the generating functions of the degeneracies [33]

$$\frac{1}{8}\text{ BPS in }\mathcal{N}=8 \quad \varphi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta^6(\tau)} = \sum_{n,\ell\in\mathbb{Z}} c_{-2,1}(n,\ell)\,q^n\zeta^\ell \quad q=e^{2\pi i\tau},\ \zeta=e^{2\pi i z}$$

$$\frac{1}{4}\text{ BPS in }\mathcal{N}=4 \quad \frac{1}{\Phi_{10}(\rho,\tau,z)} = \sum_{\substack{m,n,\ell\in\mathbb{Z},\\ m,n\geq-1}} g(m,n,\ell)\,p^m q^n\zeta^\ell \qquad p=e^{2\pi i\rho}$$

$$\frac{1}{2}\text{ BPS in }\mathcal{N}=4 \quad \frac{1}{\eta^{24}(\tau)} = \sum_{n\geq-1} d(n)\,q^n.$$

where η is the Dedekind eta function, Φ_{10} is the Siegel modular form of weight 10, ϑ_1 is the odd Jacobi theta function. For the different levels of supersymmetry, corresponding to different compactification manifolds, we have to modify our target space

$$\begin{aligned} \mathcal{N} &= 8 \qquad CY_3 = T^6 = T^2 \times T^2 \times T^2 \\ \mathcal{N} &= 4 \qquad CY_3 = K3 \times T^2 \\ &. \end{aligned}$$

For the $\mathcal{N} = 2$ we no longer have a unique CY_3 rather our target space is given by

$$F\left(X^I\right) = \frac{C_{IJK} X^I X^J X^K}{X^0} + \text{ higher derivative terms}$$

It is noteworthy that in [34] it was shown that all full superspace integrals vanish on the localization locus.

Section 4

The Frameworks for $\mathcal{N} = 8$ and $\mathcal{N} = 4$.

In this section we will describe how to obtain an exact expression for the Fourier coefficients of Jacobi forms using the Hardy-Ramanujan-Rademacher method. We will first describe the microscopic counting of BPS states in $\mathcal{N} = 8$ supergravity and then generalise to $\mathcal{N} = 4$ supergravity.

Subsection 4.1

Microscopic Interpretation in $\mathcal{N} = 8$

The counting of microscopic BPS states can be computed in the duality frame described by Type IIB string theory compactified on $T^6 = T^4 \times S^1 \times \tilde{S}^1$.

The generating function for the BPS degeneracies is given by

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6} \qquad \tau \in \mathbb{H}, z \in \mathbb{C} \tag{13}$$

where $\vartheta_1(\tau, z)$ is the odd Jacobi theta function and $\eta(\tau)$ is the Dedekind eta function see Appendix [A.7] for definitions.

The Hardy-Ramanujan-Rademacher formula gives an exact expression for the Fourier coefficients of modular forms. In particular, the Fourier coefficients $C(\Delta)$ of $\varphi_{-2,1}(\tau, z)$ count the number of BPS states with given charges.
We have

$$C(\Delta) = 2\pi \left(\frac{\pi}{2}\right)^{\frac{7}{2}} \sum_{c=1}^{\infty} c^{-\frac{9}{2}} K_c(\Delta) \tilde{I}_{\frac{7}{2}} \left(\frac{\pi\sqrt{\Delta}}{c}\right) \tag{14}$$

where $\tilde{I}$ is the modified Bessel function of the first kind and $K_c(\Delta)$ is the Kloosterman sum.

In order to obtain (14) we will use theta decomposition on (13) as described in [35]

$$\varphi_{k,m}(\tau,z) = \sum_{\mu \bmod 2m} h_\mu \theta_{m,\mu}(\tau,z) \qquad \mu \in \frac{\mathbb{Z}}{2m\mathbb{Z}}$$

$$\text{where } \theta_{m,\mu}(\tau,z) = \sum_{r\in\mathbb{Z}, r\equiv\mu \bmod 2m} q^{\frac{r^2}{4m}}\zeta^r \qquad q = e^{2\pi i\tau}, \zeta = e^{2\pi iz}$$

$$\text{and } h_\mu(\tau) = \sum_{n\in\mathbb{Z}} a_\mu(n) q^n$$

$$\theta_{1,0} = \sum_{r\in 2\mathbb{Z}} q^{\frac{r^2}{4}}\zeta^r \qquad \theta_{1,1} = \sum_{r\in 2\mathbb{Z}+1} q^{\frac{r^2}{4}}\zeta^r \qquad \mu \in \{0,1\}.$$

Separating by weight we have

$$\varphi_{k,1}(\tau,z) = h_0(\tau)\theta_{1,0}(\tau,z) + h_1(\tau)\theta_{1,1}(\tau,z)$$
$$\varphi_{k,1}(\tau,z) = \left(\sum_{n,r\in 2\mathbb{Z}} a_0(n) q^{\frac{r^2}{4}+n}\zeta^r\right) + \left(\sum_{n,r\in 2\mathbb{Z}+1} a_1(n) q^{\frac{r^2}{4}+n}\zeta^r\right)$$
.

Since φ is a weak Jacobi form it has a double Fourier expansion

$$\varphi_{-2,1}(\tau,z) = \sum_{n,\ell\in\mathbb{Z}} c_{-2,1}(n,\ell) q^n \zeta^\ell \tag{15}$$

Using the elliptic property of φ (shift in z) we have $c(n,\ell) = c_\nu(\Delta)$, $\Delta = 4mn - \ell^2$ and $\nu = \ell \bmod 2m$. So for $m = 1$, $\Delta = 4n - \ell^2$.

$$c(N,\ell) = a_0\left(N - \frac{\ell^2}{4}\right) \qquad \forall \ell \in 2\mathbb{Z}$$
$$c(N,\ell) = a_1\left(N - \frac{\ell^2}{4}\right) \qquad \forall \ell \in 2\mathbb{Z}+1$$
$$\implies h_\mu(\tau) = \sum_{n\in\mathbb{Z}} a_\mu(n) q^n = \sum_{N,\ell\equiv\mu \bmod 2} c(N,\ell) q^{N-\frac{\ell^2}{4}}$$
$$h_\mu(\tau) = \sum_{\Delta\equiv-\mu^2 \bmod 4} C(\Delta) q^{\frac{\Delta}{4}}.$$

We have the generators of $SL(2,\mathbb{Z})$ namely T and S given by

$$T(\tau\to\tau+1): \quad \theta_{1,\mu}(\tau+1,z) = e^{\pi i\frac{\mu^2}{2}}\theta_{1,\mu}(\tau,z) \implies \begin{cases}\theta_{1,0}(\tau+1,z) = \theta_{1,0}(\tau,z)\\ \theta_{1,1}(\tau+1,z) = i\theta_{1,1}(\tau,z)\end{cases}$$
.

$$S\left(\tau \to -\frac{1}{\tau}, z \to \frac{z}{\tau}\right): \quad \theta_{1,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sqrt{-i\tau}e^{2\pi i \frac{z^2}{\tau}} \frac{1}{\sqrt{2}} \sum_{\nu \bmod 2} e^{-\pi i \frac{\mu\nu}{1}} \theta_{1,\nu}(\tau, z)$$

.

$$\implies \begin{cases} \theta_{1,0}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) & = \sqrt{-i\tau}e^{2\pi i \frac{z^2}{\tau}} \frac{1}{\sqrt{2}} \left(\theta_{1,0}(\tau, z) + \theta_{1,1}(\tau, z)\right) \\ \theta_{1,1}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) & = \sqrt{-i\tau}e^{2\pi i \frac{z^2}{\tau}} \frac{1}{\sqrt{2}} \left(\theta_{1,0}(\tau, z) - \theta_{1,1}(\tau, z)\right) \end{cases}.$$

$$\begin{aligned}
\underbrace{\theta_{1,\mu}(\tau, z)}_{r=2k+\mu} &= \sum_{k\in\mathbb{Z}} e^{2\pi i\left(\frac{(2k+\mu)^2}{4}\tau + (2k+\mu)z\right)} \\
\theta_{1,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sum_{k\in\mathbb{Z}} e^{2\pi i\left(\frac{(2k+\mu)^2}{4}\left(-\frac{1}{\tau}\right) + (2k+\mu)\frac{z}{\tau}\right)} \\
\theta_{1,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi i}{\tau}\left(+\frac{(2k+\mu)^2}{4} - (2k+\mu)z\right)} \\
&= e^{\frac{2\pi i z^2}{\tau}} \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi i}{\tau}\left(+\frac{2k+\mu-2z}{2}\right)^2} \\
&= e^{\frac{2\pi i z^2}{\tau}} \sum_{k\in\mathbb{Z}} e^{\frac{-2\pi i}{\tau} t(k)^2} \text{ where } t(k) = \frac{2k+\mu-2z}{2}
\end{aligned}$$

.

We can use the standard Poisson summation for Gaussians formula

$$\sum_{k\in\mathbb{Z}} e^{-\pi a(k+\beta)^2} = \frac{1}{\sqrt{a}} \sum_{n\in\mathbb{Z}} e^{\frac{-\pi n^2}{(a)}} e^{2\pi i n\beta} \tag{16}$$

Poisson with $a = \frac{2i}{\tau}$ and $\beta = \frac{\mu}{2} - z$

$$\begin{aligned}
\implies &= \frac{1}{\sqrt{2}} (-i\tau)^{\frac{1}{2}} e^{2\pi i \frac{z^2}{\tau}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2 \tau}{2i} + 2\pi i n\left(\frac{\mu}{2} - z\right)} \\
\theta_{1,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \frac{1}{\sqrt{2}} (-i\tau)^{\frac{1}{2}} e^{2\pi i \frac{z^2}{\tau}} \sum_{n\in\mathbb{Z}} e^{\frac{\pi i\tau n^2}{2}} e^{\pi i n\mu} e^{-2\pi i n z} \\
\theta_{1,\mu}(\tau, z) &= \sum_{r\in\mathbb{Z}, r\equiv\mu \bmod 2} e^{\frac{\pi i r^2}{2}} e^{2\pi i r z} \\
\theta_{1,\mu}(\tau, -z) &= \theta_{1,\mu}(\tau, z) \quad n = 2k \iff \mu = 0 \quad n = 2k+1 \iff \mu = 1 \quad n \to -n \\
S_\mu(\tau, z) &= \sum_{k\in\mathbb{Z}} \left[e^{\frac{\pi i\tau(2k)^2}{2}} e^{-2\pi i(2k)z} e^{\pi i\mu(2k)} + e^{\frac{\pi i\tau(2k+1)^2}{2}} e^{-2\pi i(2k+1)z} e^{\pi i\mu(2k+1)} \right] \\
S_\mu(\tau, z) &= \sum_{v=0,1} e^{-\pi i\mu\nu} \theta_{1,\nu}(\tau, z)
\end{aligned}$$

.

Thus

$$\theta_{1,\mu}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=(-i\tau)^{\frac{1}{2}}\,e^{2\pi i\frac{z^2}{\tau}}\,\frac{1}{\sqrt{2}}\sum_{\nu=0,1}e^{-\pi i\mu\nu}\theta_{1,\nu}(\tau,z) \tag{17}$$

Jacobi

$$\varphi_{k_J,1}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^{k_J}\,e^{2\pi i\frac{cz^2}{c\tau+d}}\varphi_{k_J,1}(\tau,z) \tag{18}$$

Theta decomposition:

$$\varphi_{k_J,1}(\tau,z)=h_0(\tau)\,\theta_{1,0}(\tau,z)+h_1(\tau)\,\theta_{1,1}(\tau,z) \tag{19}$$

If we define $\gamma=\frac{a\tau+b}{c\tau+d}$ and look at the theta transformation

$$\theta_{1,\mu}\left(\gamma\tau,\frac{z}{c\tau+d}\right)=(c\tau+d)^{\frac{1}{2}}\,e^{2\pi i\frac{cz^2}{c\tau+d}}\sum_{\nu=0,1}M^{-1}_{\nu\mu}\theta_{1,\nu}(\tau,z) \tag{20}$$

For the left hand side we can substitute : (19) into (18) to obtain

$$\varphi_{k_J,1}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\sum_{\mu=0,1}h_\mu(\gamma\tau)\,\theta_{1,\mu}\left(\gamma\tau,\frac{z}{c\tau+d}\right) \tag{21}$$

Likewise (20) into (21) gives

$$\varphi_{k_J,1}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\sum_{\mu=0,1}h_\mu(\gamma\tau)\left[(c\tau+d)^{\frac{1}{2}}\,e^{2\pi i\frac{cz^2}{c\tau+d}}\sum_{\nu=0,1}{M_{\nu\mu}}^{-1}(\gamma)\,\theta_{1,\nu}(\tau,z)\right] \tag{22}$$

The right hand side is given by

$$(c\tau+d)^{k_J}\,e^{2\pi i\frac{cz^2}{c\tau+d}}\left[\sum_{\mu=0,1}h_\mu(\tau)\,\theta_{1,\mu}(\tau,z)\right] \tag{23}$$

$$(c\tau+d)^{k_J}\left[\sum_{\mu=0,1}h_\mu(\tau)\,\theta_{1,\mu}(\tau,z)\right]=\sum_{\mu=0,1}h_\mu(\gamma\tau)\left[(c\tau+d)^{\frac{1}{2}}\sum_{\nu=0,1}M^{-1}_{\nu\mu}(\gamma)\,\theta_{1,\nu}(\tau,z)\right] \tag{24}$$

Thus we have for $\nu=0,1$

$$\sum_\mu h_\mu(\gamma\tau)\,M^{-1}_{\nu\mu}=(c\tau+d)^{k_J-\frac{1}{2}}\,h_\nu(\tau) \tag{25}$$

$$\mathbf{h}(\tau)=\begin{bmatrix}h_0(\tau)\\ h_1(\tau)\end{bmatrix} \tag{26}$$

$$M^{-1}(\gamma)\,\mathbf{h}(\gamma\tau)=\sum_\mu M^{-1}(\gamma)_{\nu\mu}\,h_\mu(\gamma\tau)$$

$$M^{-1}(\gamma)\,\mathbf{h}(\gamma\tau)=(c\tau+d)^{k_J-\frac{1}{2}}\,h_\nu(\tau)\,.$$

We can generalise how [36, Thm. 5.10] derives the Ramanujan partition function for a scalar into a vector-valued modular form which we shall call $h_\mu(\tau)$.

$$C(\Delta) = \frac{1}{2\pi i}\oint_{|q|=r} \frac{h_\mu(\tau)}{q^{\frac{\Delta}{4}+1}} dq$$
$$C(\Delta) = \oint_{C_\tau} h_\mu(\tau)\, e^{-2\pi i\tau\left(\frac{\Delta}{4}\right)} d\tau$$
.

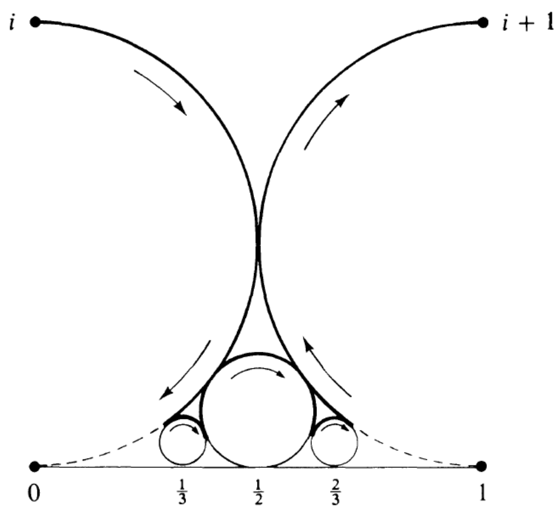

Figure 2: We have Rademacher's path of integration namely our contour of integration $P(N)$. The diagram is of $P(3)$ [36]. To be precise it is the union of upper arcs of the Ford circles of Farey fractions (endpoints at i and $i+1$). Integrating along $P(N)$ allows us to write it as a sum of integrals over each arc, indexed by coprime (h, k)

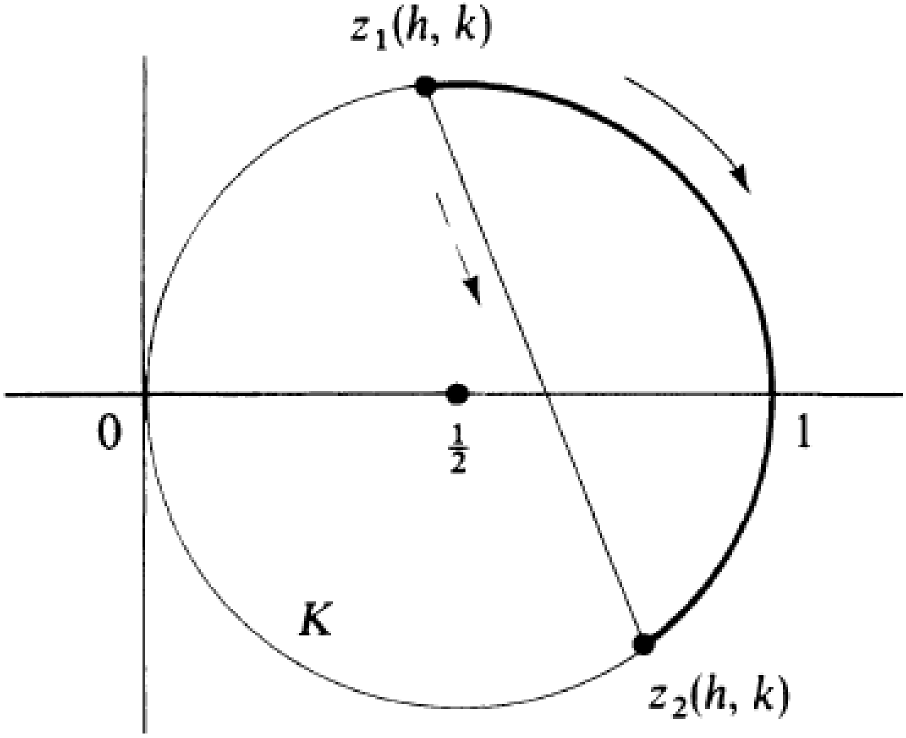

Figure 3: We have the chord joining z_1 and z_2 and we know that the length of this chord is bounded above by $2\sqrt{2}\frac{k}{N}$. [36]

$h(\tau)$
$h_\mu(\tau) = (c\tau+d)^{-\left(k_J-\frac{1}{2}\right)} \sum_\nu \left[M_{\mu\nu}^{-1}(\gamma)\, h(\gamma\tau)_\nu\right]$
We can make the following transformations:
$\tau_{\text{old}} = \gamma\tau' \implies d\tau_{\text{old}} = d(\gamma\tau') = \frac{1}{(c\tau'+d)^2} d\tau' \qquad \tau_{\text{old}} = \gamma^{-1}\tau$
$h_\mu(\gamma\tau) = (c\tau+d)^{k_J-\frac{1}{2}} \sum_\nu \left[M_{\mu\nu}(\gamma)\, h_\nu(\tau)\right]$
$h_\mu(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} = (c\tau+d)^{-\left(k_J-\frac{1}{2}\right)} \sum_\nu \left[M_{\mu\nu}^{-1}(\gamma)\, h_\nu(\gamma\tau)\right] e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau}$
$h_\mu(\gamma\tau) = \sum_{\tilde\Delta} C_\mu(\tilde\Delta)\, e^{2\pi i\left(\frac{\tilde\Delta}{4}\gamma\tau\right)}$
$h_\mu(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau = (c\tau+d)^{-\left(k_J-\frac{1}{2}\right)} \left[M_{\mu\nu}^{-1}(\gamma) \sum_\Delta C_\nu(\tilde\Delta)\, e^{2\pi i\left(\frac{\tilde\Delta}{4}\gamma\tau\right)}\right] e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau$
.

We have $\kappa = 2 - w = 2 - \left(k_J - \frac{1}{2}\right) = \frac{9}{2}$.
Substituting this into the arc integral and using the Fourier expansion of h_ν we get

$$C_\mu(\Delta) = \frac{1}{2\pi i} \sum_{c\le N} \sum_{\substack{0\le h<c \\ (h,c)=1}} \int_{\text{arc}(h/c)} (c\tau+d)^{-\kappa} \sum_{\nu=0,1} M(\gamma)^{-1}_{\mu\nu} \sum_{\tilde\Delta} C_\nu(\tilde\Delta)\, e^{2\pi i(\tilde\Delta/4)(\gamma\tau)}\, e^{-2\pi i(\Delta/4)\tau}\, d\tau.$$

If we split the vector-valued modular form into a polar $\Delta < 0$ term and a reg term $\Delta \ge 0$

$$h_\mu(\tau) = h_\mu^{\text{Pol}}(\tau) + h_\mu^{\text{reg}}(\tau)$$

.

So we get two contributions

$$C(\Delta) = C_\mu^{\text{Pol}}(\Delta) + C_\mu^{\text{Reg}}(\Delta)$$
$$C(\Delta) = \frac{1}{2\pi i} \oint h_\mu^{\text{Pol}}(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau + \frac{1}{2\pi i} \oint h_\mu^{\text{Reg}}(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau$$

.

In components we have

$$h_0(\tau) = \sum_{\substack{\Delta\ge0 \\ \Delta=0 \bmod 4}} C_0(\Delta)\, q^{\frac{\Delta}{4}} \quad h_1(\tau) = q^{\frac{-1}{4}} + \sum_{\substack{\Delta\ge0, \\ \Delta=-1 \bmod 4}} C_1(\Delta)\, q^{\frac{\Delta}{4}} \tag{27}$$

If we work with the polar part:
We notice $h_0^{\text{Pol}} = 0$ and $h_1^{\text{Pol}} = q^{-\frac{1}{4}}$.

$$h_\mu(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau = (c\tau+d)^{-\frac{9}{2}} \left[M_{\mu 1}^{-1}(\gamma) \underbrace{C_1(\tilde\Delta = -1)}_{=1} e^{2\pi i\left(\frac{-1}{4}\right)\gamma\tau} e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau}\right] d\tau \tag{28}$$

and we have $c = k$, $d = -h$, and we already subbed $\tilde{\Delta} = -1$.

$$h_\mu(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\tau\right)} d\tau = (k\tau - h)^{-\frac{9}{2}} \left[M^{-1}_{\mu 1}(\gamma)\, e^{2\pi i(-\frac{1}{4})\gamma\tau} e^{2\pi i\left(\frac{-\Delta}{4}\right)\tau} \right] d\tau$$

.

We have the following parameterisation for z

$$\gamma\tau = \frac{a\tau + b}{k\tau - h} \qquad \tau = \frac{h}{k} + i\frac{z}{2\pi k^2} = \frac{h}{k} + \epsilon \tag{29}$$

$$\gamma\tau = \frac{a\left(\frac{h}{k} + \epsilon\right)}{k\left(\frac{h}{k} + \epsilon\right) - h} \qquad \det\gamma = 1 = -ah - bk \implies \frac{ah}{k} + b = -\frac{1}{k} \tag{30}$$

$$\gamma\tau = \frac{-\frac{1}{k} + a\epsilon}{k\epsilon} = -\frac{1}{k^2\epsilon} + \frac{a}{k} = \frac{2\pi i}{z} + \frac{a}{k} \tag{31}$$

Thus our integrand becomes

$$h^{\mathrm{Pol}}_\mu(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau = \left(\frac{i}{2\pi k} z\right)^{-\frac{9}{2}} \left[M^{-1}_{\mu 1}(\gamma)\, e^{2\pi i(-\frac{1}{4})\left(\frac{2\pi i}{z} + \frac{a}{k}\right)} e^{-2\pi i\left(\frac{\Delta}{4}\right)\left(\frac{h}{k} + i\frac{z}{2\pi k^2}\right)} \right] \left(\frac{i}{2\pi k^2}\right) dz$$

$$h^{\mathrm{Pol}}_\mu(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau = z^{-\frac{9}{2}} \frac{1}{k} \left(\frac{i}{2\pi k}\right)^{-\frac{7}{2}} \left[M^{-1}_{\mu 1}(\gamma)\, e^{-\frac{a\pi i}{2k}} e^{\frac{\pi^2}{z}} e^{\frac{-\pi i\Delta h}{2k}} e^{\frac{z\Delta}{4k^2}} \right] dz$$

$$C(\Delta) = \sum_{1 \leqslant k \leqslant N} \sum_{\substack{0 \leqslant h < k \\ (h,k)=1}} \frac{1}{k} \left(\frac{i}{2\pi k}\right)^{-\frac{7}{2}} \left[M^{-1}_{\mu 1}(\gamma)\, e^{-\frac{a\pi i}{2k}} e^{\frac{-\pi i\Delta h}{2k}} \right] \cdot \underbrace{\frac{1}{2\pi i} \int_{z_1(h,k)}^{z_2(h,k)} z^{-\frac{9}{2}} e^{\frac{z\Delta}{4k^2}} e^{\frac{\pi^2}{z}} dz}_{\Lambda}$$

We can notice the factor Λ resembles a modified Bessel function [A.7].
Now we can evaluate the integral.

$$\sigma = Bz = \left(\frac{\Delta}{4k^2}\right) z \implies z = \frac{\sigma}{B} \qquad A = \pi^2$$

$$\Lambda = \frac{1}{2\pi i} B^{\frac{7}{2}} \int_{\sigma_1(h,k)}^{\sigma_2(h,k)} \sigma^{-\frac{9}{2}} e^{\sigma + \frac{AB}{\sigma}} d\sigma = \frac{1}{2\pi i} B^{\frac{7}{2}} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \sigma^{-\frac{9}{2}} e^{\sigma + \frac{\Delta\pi^2}{4k^2\sigma}} d\sigma + \mathcal{O}(\ldots)$$

$$\Lambda = B^{\frac{7}{2}} \tilde{\mathcal{I}}_{\frac{7}{2}} \left(\frac{\pi}{k}\sqrt{\Delta}\right)$$

$$C(\Delta) = \sum_{1 \leqslant k < N} \sum_{0 < h \leqslant k} \frac{1}{k} \left(\frac{i}{2\pi k}\right)^{-\frac{7}{2}} \left[M^{-1}_{\mu 1} e^{-\frac{a\pi i}{2k}} e^{\frac{-\pi i\Delta h}{2k}} \right] \cdot B^{\frac{7}{2}} \tilde{\mathcal{I}}_{\frac{7}{2}} \left(\frac{\pi}{k}\sqrt{\Delta}\right)$$

$$C(\Delta) = \sum_{1 \leqslant k < N} \sum_{\substack{0 < h \leqslant k \\ (h,k)=1}} k^{-\frac{9}{2}} \left(\frac{2\pi}{i}\right)^{\frac{7}{2}} 2^{-7} \Delta^{\frac{7}{2}} M^{-1}_{\mu 1} e^{-\frac{a\pi i}{2k}} e^{\frac{-\pi i\Delta h}{2k}} \tilde{\mathcal{I}}_{\frac{7}{2}} \left(\frac{\pi}{k}\sqrt{\Delta}\right)$$

$$\left(\frac{2\pi}{i}\right)^{\frac{7}{2}} = (2\pi)^{\frac{7}{2}} e^{\frac{-7\pi i}{4}} \qquad e^{\frac{-7\pi i}{4}} = e^{\frac{i\pi}{4}} \bmod 2\pi$$

$$C(\Delta) = \left(\frac{\pi}{2}\right)^{\frac{7}{2}} \sum_{k=1}^{\infty} k^{-\frac{9}{2}} \left[\sum_{0 < h \leqslant k} e^{-\pi i \frac{h}{2k}\Delta} M^{-1}_{\mu 1} e^{2\pi i \frac{a}{k}(-\frac{1}{4})} \right] \cdot \tilde{\mathcal{I}}_{\frac{7}{2}} \left(\frac{\pi}{k}\sqrt{\Delta}\right).$$

The final expression in the expanded form is

$$C(\Delta) = 2\pi\left(\frac{\pi}{2}\right)^{\frac{7}{2}}\sum_{c=1}^{\infty} c^{-\frac{9}{2}}\left[e^{\frac{5\pi i}{4}}\sum_{-c\leqslant d\leqslant 0} e^{2\pi i\frac{d}{c}\left(\frac{\Delta}{4}\right)} M\left(\gamma_{c,d}\right)^{-1}_{v_1} e^{2\pi i\frac{a}{c}\left(-\frac{1}{4}\right)}\right]\cdot\left[\frac{1}{2\pi i}\int_{\epsilon-\infty}^{\epsilon+\infty}\frac{d\sigma}{\sigma^{p+1}}\exp\left(\sigma+\frac{z^2}{4\sigma}\right)\right] \tag{32}$$

$$\begin{aligned} h_\mu(\tau) &= h_\mu^{\mathrm{Pol}}(\tau) + h_\mu^{\mathrm{reg}}(\tau) \\ &= C(-1)\, q^{\frac{-1}{4}} + \sum_{\Delta\geq 0} C(\Delta)\, q^{\frac{\Delta}{4}}. \end{aligned}$$

Now we will consider the Regular contribution which turns out to vanish as $N \longrightarrow \infty$.

$$C_\mu^{\mathrm{Reg}}(\Delta) = \frac{1}{2\pi i}\sum_{c\leqslant N}\sum_{\substack{0\leqslant h<c \\ (h,c)=1}}\int_{\mathrm{arc}\left(\frac{h}{c}\right)} h_\mu^{\mathrm{Reg}}(\tau)\, e^{-2\pi i\left(\frac{\Delta}{4}\right)\tau} d\tau$$

.

Now we can take the same transformations as done above to obtain the integrand

$$C_\mu^{\mathrm{Reg}}(\Delta) = \frac{1}{2\pi i}\sum_{c\leqslant N}\sum_{(h,c)=1}\int_{\mathrm{arc}\left(\frac{h}{c}\right)} (c\tau+d)^{-\kappa}\sum_{\nu} M(\gamma)^{-1}_{\mu\nu}\sum_{\tilde{\Delta}\geq 0} C_\nu(\tilde{\Delta})\, e^{2\pi i\left(\frac{\tilde{\Delta}}{4}\right)(\gamma\tau)} e^{-2\pi i\left(\frac{\Delta}{4}\tau\right)} d\tau$$

We can use our parameterisation for z (29) above to get

$$\begin{aligned} C_\mu^{\mathrm{Reg}}(\Delta) = \frac{1}{2\pi i}\sum_{c\leqslant N}\sum_{(h,c)=1}\int_{\mathrm{arc}\left(\frac{h}{c}\right)} (c\tau+d)^{-\kappa}\sum_{\nu} M(\gamma)^{-1}_{\mu\nu}\sum_{\tilde{\Delta}\geq 0} C_\nu(\tilde{\Delta})\, e^{2\pi i\left(\frac{\tilde{\Delta}}{4}\right)\left(\frac{a}{c}+\frac{2\pi i}{z}\right)} e^{-2\pi i\left(\frac{\Delta}{4}\left(\frac{h}{c}+i\frac{z}{2\pi c^2}\right)\right)} \\ \cdot\left(\frac{i}{2\pi c^2}\right) dz. \end{aligned}$$

Inspecting our $\tilde{\Delta}$ dependence we notice we have the form

$$\sum_{\tilde{\Delta}\geq 0} C_\nu(\tilde{\Delta})\, e^{2\pi i\frac{\tilde{\Delta}a}{4c}} e^{-\frac{\pi^2\tilde{\Delta}}{z}}$$

where on the Rademacher contour $\Re z > 0, \implies \Re\frac{1}{z} \geq \frac{N}{2c^2}\ \forall z$ on the chord. We know that the measure $|d\tau| \leqslant \ell_C$ satisfies $\ell_C \ll \frac{c}{N}$ where ℓ_C is the length of the chord. We also have $|c\tau+d| \geqslant \frac{1}{2\pi c}$ where $d \equiv -h \bmod c$ [36]. So we have a term which acts as a exponential dampning term due to the positivity of $\tilde{\Delta}$.
Thus we have that $\exists\, \epsilon > 0$ such that

$$|C_\mu^{\mathrm{Reg}}| \leqslant \sum_{c\leqslant N}\sum_{(h,c)=1}\frac{\epsilon}{N}c^{\kappa+1}\sum_{\nu}\sum_{\tilde{\Delta}\geq 1}|C_\nu(\tilde{\Delta})|\exp\left[-\frac{\pi^2\tilde{\Delta}}{2}\frac{N}{c^2}\right] \tag{33}$$

We see that we have

$$\lim_{N\to\infty} C_\mu^{\mathrm{Reg}}(\Delta) = 0 \tag{34}$$

So we only receive the contribution from the polar aspect.

Subsection 4.2

Macroscopic Interpretation in $\mathcal{N} = 8$

Subsubsection 4.2.0

Localisation Setup and the Localising Manifold

In what follows, we will follow the treatment of [8], [37] and [38]. The quantum entropy function is defined as the gravitational path integral on the near-horizon $AdS_2 \times S^2$ geometry with fixed magnetic charges p^Λ and fixed electric charges implemented by a Wilson line insertion:

$$W(q,p) = \int [\mathcal{D}g\mathcal{D}A\mathcal{D}\Phi] \exp\left[-S_{\text{Bulk}} + S_{\text{Bndry}} - iq_\Lambda \int_0^{2\pi} A_\theta^\Lambda d\theta\right] \tag{35}$$

We have the indices $\Lambda, \Sigma = 0, 1, \ldots, n_v$ with $n_v = 15$ is the number of vector multiplets. Note that $\Lambda = (0, I)$, and $I, J, K = 1, \ldots, 15$.
We choose an equivariantly closed supercharge Q_{eq} satisfying

$$Q_{\text{eq}}^2 = H \equiv L_0 - J_0 \tag{36}$$

We will deform the action by a Q exact term

$$S \to S + \lambda Q_{\text{eq}} V \quad V = \int d^4x \sqrt{g^\circ} \sum_\psi \overline{\psi} Q_{eq} \psi \tag{37}$$

where ψ are fermion fields. Take $\lambda \to \infty$ thus we localise the path integral about the critical points. The critical points are given by $Q_{eq}\psi = 0$.

Since QV is Q-exact, $W(q,p)$ is independent of λ, and taking $\lambda \to \infty$ localises the functional integral onto the critical set

$$Q_{\text{eq}}\psi = 0, \tag{38}$$

i.e. the BPS configurations invariant under Q_{eq}.

$$X^I(\eta) = X_*^I + \frac{C^I}{\cosh \eta}, \qquad \overline{X}^I(\eta) = \overline{X}_*^I + \frac{C^I}{\cosh \eta}, \tag{39}$$

Let $r = \cosh(\eta)$ thus we have $X^I(r) = X_*^I + \Sigma^I(r)$, and $\Sigma^I(r) = \frac{C^I}{r}$, We can express the integral in terms of ϕ^Λ using $\phi^I := e_*^I + 2C^I$

$$S_{\text{ren}} \equiv S_{\text{bulk}} + S_{\text{bdry}} - iq_\Lambda \int_0^{2\pi} A_\theta^\Lambda \, d\theta. \tag{40}$$

Evaluating the action on a cutoff surface at $r = r_0$ gives us divergences. These divergences can be cancelled by our boundary term.

$$S_{\text{Bulk}} = 2\pi (r_0 - 1) v\mathcal{L} \qquad S_{\text{Bndry}} = -2\pi r_0 \left(v\mathcal{L} - q_\Lambda e^\Lambda\right) + \mathcal{O}\left(\frac{1}{r_0}\right)$$

$$-iq_i \int_0^{2\pi} A_\theta^\Lambda d\theta = -2\pi q_\Lambda e^\Lambda (r_0 - 1) \qquad \Longrightarrow W(q,p) \sim \exp\left[2\pi \left(q_\Lambda e^\Lambda - v\mathcal{L}\right)\right] = \exp\left[S_{\text{Wald}}\right]$$

.

Here v is the radius parameter of the near-horizon $AdS_2 \times S^2$ metric, and $\mathcal{L}$ denotes the bulk Lagrangian density evaluated on the homogeneous near-horizon saddle (so $v\mathcal{L}$ is a constant).

Subsubsection 4.2.0

Evaluation of the GPI and Comparison with Microscopic Degeneracy

We can obtain the contribution from localisation around the leading saddle point to get

$$W(q,p) = \int_{\mathcal{M}_Q} \left[d^{n_v+1}\phi\right] \exp\left[-\pi q_\Lambda \phi^\Lambda + 4\pi Im F\left(\frac{\phi^\Lambda + ip^\Lambda}{2}\right)\right] Z_{\text{1-loop}}^{Q_{eq}V}\left(\phi^\Lambda\right) \tag{41}$$

where n_v is the number of vector multiplets and $F(X)$ is the prepotential. For the $\mathcal{N} = 2$ supergravity coming from type IIA on a Calabi-Yau threefold the prepotential is given by

$$F\left(X^I\right) = C_{IJK}\frac{X^I X^J X^K}{X^0} + \ldots \tag{42}$$

Where in the case of $\mathcal{N} = 8$ there are no correction terms since we are on T^6 and this fixes our intersection matrix to be of the form

$$C_{IJK} \equiv \frac{1}{6}\int_{T^6} \alpha_I \wedge \alpha_J \wedge \alpha_k \qquad C_{IJ} \equiv C_{IJK}p^K \tag{43}$$

where α_I is an integer basis for the two form cohomology class on T^6. We get the contribution from the orbifold saddle points to get

$$W^c(q,p) = \sqrt{c}K_c(\Delta) \int d^{16}\phi \sqrt{\frac{-2C_{IJK}p^Ip^Jp^K \det\ (3C_{IJ})}{c^{16}\left(\phi^0\right)^{18}}} \frac{1}{c}\left(\frac{\phi^0}{-2C_{IJ}p^Ip^J}\right)^4 \tag{44}$$

We have the GPI for maximum supersymmetric black holes in $AdS_2 \times S^2$ giving

$$W(q,p) = \int [\mathcal{D}g\mathcal{D}\psi\mathcal{D}A\mathcal{D}\Phi]\, e^{-S^{\text{bulk}}_{\text{sugra}} - S^{\text{bdry}}_{\text{sugra}}} \tag{45}$$

Localisation can be applied to evaluate the functional integral.
For $\mathcal{N} = 8$ the field content is as follows: we have one graviton multiplet, 6 gravitino multiplets and 15 vector multiplets and 10 hypermultiplets.

We can choose charges such that the attractor values of the scalars in the gravitino multiplet are zero thus we can consistently truncate them out leaving us with $\mathcal{N} = 2$ sugra coupled to 15 vector multiplets. The coupling of the vector multiplet scalars to gravity is governed by the prepotential

$$F(X) = C_{IJK}\frac{X^I X^J X^K}{X^0} \tag{46}$$

where we can restrict to $p^0 = 0$ and on our localisation locus we have

$$X^\Lambda = \frac{\phi^\Lambda + ip^\Lambda}{2} \tag{47}$$

so that the prepotential becomes

$$F\left(\frac{\phi + ip}{2}\right) = \frac{C_{IJK}}{4\phi^0}\left(\phi^I + ip^I\right)\left(\phi^J + ip^J\right)\left(\phi^K + ip^K\right) \tag{48}$$

taking the imaginary part (considering odd powers of i) we have

$$\Im F = \frac{3}{4\phi^0}C_{IJK}\phi^I\phi^J p^K - \frac{C_{IJK}}{4\phi^0}p^I p^J p^K \tag{49}$$

where we used symmetricity of C_{IJK}.

$$\Im F = \frac{3}{4\phi^0}C_{IJK}\phi^I\phi^J p^K - \frac{C_{IJK}}{4\phi^0}p^I p^J p^K \tag{50}$$

We have the action on our localising manifold as a function of the charges

$$\mathcal{S} = \frac{\pi q_\Lambda \phi^\Lambda}{c} - \frac{4\pi}{c}\Im F\left(\frac{\phi^\Lambda + ip^\Lambda}{2}\right) = \frac{\pi}{c}\left[q_I\phi^I + \frac{C_{IJK}p^I p^J p^K}{\phi^0} - 3C_{IJK}\frac{\phi^I\phi^J p^K}{\phi^0} + q_0\phi^0\right] \tag{51}$$

From [8] we have the following expression for the Wilson line

$$W^1(q,p) = \int_{\mathcal{M}_Q}\left[d^{n_v+1}\phi\right]\exp\left[-\pi q_\Lambda\phi^\Lambda + 4\pi\Im F\left(\frac{\phi^\Lambda + ip^\Lambda}{2}\right)\right] Z^{Q_{eq}V}_{\text{1-loop}}\left(\phi^\Lambda\right) \tag{52}$$

Subsubsection 4.2.0
One Loop Determinant

We have two contributions to the one loop determinant: from the non-zero modes and from the zero modes.
These contributions are

$$Z^{Q_{eq}\mathcal{V}}_{\text{non-zero}} = \begin{cases} e^{5\kappa}e^{-\frac{17}{2}\kappa} & \text{if } c = 1 \\ e^{5\kappa}e^{-\frac{15}{2}\kappa} & c > 1 \end{cases}, \quad Z^{Q_{eq}\mathcal{V}}_{\text{zero},c} = \begin{cases} e^{\frac{15}{2}\kappa} & \text{if } c = 1 \\ \frac{1}{c}e^{\frac{13}{2}\kappa} & c > 1 \end{cases} \tag{53}$$

$$Z^{Q_{eq}\mathcal{V}}_{\text{1-loop},c}\left(\phi\right) = Z^{Q_{eq}\mathcal{V}}_{\text{non-zero},c}\left(\phi\right) \cdot Z^{Q_{eq}\mathcal{V}}_{\text{zero-modes},c}\left(\phi\right) = \frac{1}{c}e^{4\kappa} \tag{54}$$

where $Z^{Q_{eq}\mathcal{V}}_{\text{zero-modes},c}\left(\phi\right)$ is comprised of the zero mode integrals over the super-diffeomorphisms [39] and $Z^{Q_{eq}\mathcal{V}}_{\text{non-zero},c}\left(\phi\right)$ is the contribution from the complement and the excited boundary modes [40]. The factor of $\frac{1}{c}$ comes from the path integral over the zero modes on the orbifolded geometries.

Subsubsection 4.2.0

Measure for $\mathcal{N} = 8$

For the measure we have the requirements that

$$\int [\mathcal{D}\Phi^m]\, e^{-\int \sqrt{g} G_{mn}\delta\Phi^m\delta\Phi^n} = 1 \quad \Phi \text{ is the fields}$$
$$\partial_\Lambda \overline{\partial}_\Sigma e^{-K} = \Im F_{\Lambda\Sigma} \quad \text{is the Kahler metric}$$
$$.$$

where G_{mn} defines the inner product on the field space.
We can propose the ultra-localised measure that obeys these conditions as

$$\left[d^{16}\phi\right]_c \equiv d^{16}\phi\sqrt{2 \det\left(\frac{1}{2c}\Im F_{\Lambda\Sigma}\right)} \tag{55}$$

where c is the orbifold number and we have the decomposition below with $I, J, K = 1, 2, \ldots 15$

$$\Im F_{00} = -\frac{2C_{IJ}p^I p^J}{(\phi^0)^3} + \frac{6C_{IJ}\phi^I\phi^J}{(\phi^0)^3} \qquad \Im F_{I0} = \Im F_{0I} = -\frac{6C_{IJ}\phi^J}{(\phi^0)^2} \qquad \Im F_{IJ} = \frac{6C_{IJ}}{\phi^0}.$$

We can use Schur's complement to calculate the determinant

$$\det \Im F_{\Lambda\Sigma} = \det\begin{bmatrix} \Im F_{00} & \Im F_{0J} \\ \Im F_{I0} & \Im F_{IJ}\end{bmatrix} = \det \Im F_{IJ}\cdot\left(\Im F_{00} - \Im F_{0I}\left(\Im F_{IJ}\right)^{-1}\Im F_{J0}\right)$$
$$\text{we have } \det \Im F_{IJ} = \left(\frac{6}{\phi^0}\right)^{15} \det C_{IJ},$$
$$\Im F_{00} - \Im F_{0I}\left(\Im F_{IJ}\right)^{-1}\Im F_{J0} = \left[-\frac{2C_{IJ}p^I p^J}{(\phi^0)^3} + \frac{6C_{IJ}\phi^I\phi^J}{(\phi^0)^3}\right] - \left[\left(-\frac{6C_{IJ}\phi^J}{(\phi^0)^2}\right)\left(\frac{6C_{IJ}}{\phi^0}\right)^{-1}\left(-\frac{6C_{IJ}\phi^J}{(\phi^0)^2}\right)\right]$$
$$\Im F_{00} - \Im F_{0I}\left(\Im F_{IJ}\right)^{-1}\Im F_{J0} = \frac{1}{(\phi^0)^3}\left[6C_{IJ}\phi^I\phi^J - 2C_{IJ}p^I p^J\right] - \frac{1}{(\phi^0)^3}\left[6C_{IJ}\phi^I\phi^J\right]$$
$$\det \Im F_{\Lambda\Sigma} = \frac{-2^{16}}{(\phi^0)^{18}} C_{IJ}p^I p^J \det 3C_{IJ}.$$

Thus our measure becomes

$$\left[d^{16}\phi\right]_c = d^{16}\phi\sqrt{\frac{-2C_{IJK}p^I p^J p^K \det 3C_{IJ}}{c^{16}\left(\phi^0\right)^{18}}} \tag{56}$$

Subsubsection 4.2.0

Evaluating the Integral

$$W^c(q,p) = \sqrt{c}K_c(\Delta)\int d^{16}\phi\sqrt{\frac{-2C_{IJK}p^Ip^Jp^K\ \det\ (3C_{IJ})}{c^{16}\left(\phi^0\right)^{18}}}\frac{1}{c}\left(\frac{\phi^0}{-2C_{IJK}p^Ip^Jp^K}\right)^4$$
$$\cdot\exp\left[-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0}+\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_0\phi^0}{c}-\frac{\pi q_I\phi^I}{c}\right]$$

$C_{IJ}\equiv C_{IJK}p^K$ Simplifying our exponential:

$$\begin{aligned}
\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_I\phi^I}{c} &\to 3\frac{\pi}{c\phi^0}\left[C_{IJ}\phi^I\phi^J-\frac{\phi^0}{3}q_I\phi^I\right]\\
&= \frac{3\pi}{c\phi^0}\left[\phi^TC\phi-\frac{\phi^0q^T\phi}{3}\right] \qquad \phi\to\tilde{\phi}+\delta\\
&= \frac{3\pi}{c\phi^0}\left[\left(\tilde{\phi}+\delta\right)^TC\left(\tilde{\phi}+\delta\right)-\frac{\phi^0}{3}q^T\left(\tilde{\phi}+\delta\right)\right]\\
&= \frac{3\pi}{c\phi^0}\left[\tilde{\phi}^TC\tilde{\phi}+\delta^TC\delta+\tilde{\phi}^TC\delta+\delta^TC\tilde{\phi}-\frac{\phi^0}{3}q^T\tilde{\phi}-\frac{\phi^0}{3}q^T\delta\right]\\
&= \frac{3\pi}{c\phi^0}\left[\tilde{\phi}^TC\tilde{\phi}+\delta^TC\delta+2\delta^TC\tilde{\phi}-\frac{\phi^0}{3}q^T\tilde{\phi}-\frac{\phi^0}{3}q^T\delta\right]
\end{aligned}$$

We can pick a δ such that $2\delta^TC\tilde{\phi}-\frac{\phi^0}{3}q^T\tilde{\phi}=0 \implies \delta=\frac{\phi^0C^{-1}q}{6}$.

Thus we have

$$\begin{aligned}
\frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0}-\frac{\pi q_I\phi^I}{c} &= \frac{3\pi}{c\phi^0}\left[\tilde{\phi}^TC\tilde{\phi}+\delta^TC\delta-\frac{\phi^0}{3}q^T\delta\right]\\
&= \frac{3\pi}{c\phi^0}\left[\tilde{\phi}^TC\tilde{\phi}+\left(\frac{\phi^0C^{-1}q}{6}\right)^TC\left(\frac{\phi^0C^{-1}q}{6}\right)-\frac{\phi^0}{3}q^T\left(\frac{\phi^0C^{-1}q}{6}\right)\right]\\
&= \frac{3\pi}{c\phi^0}\left[\tilde{\phi}^TC\tilde{\phi}+\frac{(\phi^0)^2}{36}q^TC^{-1}q-\frac{(\phi^0)^2}{18}q^TC^{-1}q\right]\\
&= \frac{3\pi}{c\phi^0}\left[\tilde{\phi}^TC\tilde{\phi}-\frac{(\phi^0)^2}{36}q^TC^{-1}q\right].
\end{aligned}$$

So if we take the 15 Gaussian integrals out we have the form

$$\int d^{15}\phi e^{\frac{3\pi}{c\phi^0}\phi^TC\phi-\frac{\pi}{c}q^T\phi}=e^{-\frac{\pi}{12c}\phi^0q_IC^{IJ}q_J}\int d^{15}\tilde{\phi}e^{\frac{3\pi}{c\phi^0}\tilde{\phi}^TC\tilde{\phi}}$$

$C_{IJ}=C_{JI} \implies C=O^T\Lambda O, \qquad \Lambda=\ \text{diag}\ (\lambda_1,\dots,\lambda_{15})$ and let $\tilde{\phi}=O^Ty$

$$\int d^{15}\phi e^{\frac{3\pi}{c\phi^0}\phi^TC\phi-\frac{\pi}{c}q^T\phi}\to\prod_{i=1}^{15}\int dy_ie^{\frac{3\pi}{c\phi^0}\lambda_iy_i^2}.$$

So we have

$$
\begin{aligned}
W^c(q,p) &= K_c(\Delta)\int d\phi^0 \frac{1}{c}\left(\phi^0\right)^{\frac{5}{2}} i^{-1}\left(-2C_{IJK}p^Ip^Jp^K\right)^{-\frac{7}{2}} \\
&\cdot \exp\left[-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0} - \frac{\pi\phi^0}{c}\left(q^0 + \frac{1}{12}q_I C^{IJ} q_J\right)\right] \\
P^3 &:= C_{IJK}p^Ip^Jp^K \qquad Q := q_0 + \frac{1}{12}q_I c^{IJ} q_J \\
\exp &\to \exp\left[-\frac{\pi P^3}{c\phi^0} - \frac{\pi Q}{c}\phi^0\right] \qquad \text{and let } \sigma = -\frac{\pi P^3}{c\phi^0} \\
\implies d\phi^0 &= \frac{\pi P^3}{c\sigma^2}d\sigma \qquad \left(\phi^0\right)^{\frac{5}{2}} = \left(-\frac{\pi P^3}{c}\right)^{\frac{5}{2}}\sigma^{-\frac{5}{2}} \qquad \Delta := 4P^3Q
\end{aligned}
$$

.

Thus we have:

$$
W^c(q,p) = -K_c(\Delta)\int d\sigma \frac{1}{c}\int\left(-\frac{\pi P^3}{c}\right)^{\frac{7}{2}}\sigma^{-\frac{9}{2}}\left(i\left(-2P^3\right)^{-\frac{7}{2}}\right)\exp\left[\sigma + \frac{\pi^2 Q P^3}{c^2\sigma}\right]
$$

$$
W^c(q,p) = K_c(\Delta)\frac{2\pi}{c^{\frac{9}{2}}}(\pi)^{\frac{7}{2}}\cdot\frac{1}{2\pi i}\int\frac{d\sigma}{\sigma^{\frac{9}{2}}}\exp\left[\sigma + \frac{\pi^2\Delta}{4c^2\sigma}\right]
$$

Using [A.7] gives $\tilde{\mathcal{I}}_p(z) = \int_{\epsilon+i\infty}^{\epsilon-i\infty}\frac{d\sigma}{\sigma^{p+1}}\exp\left[\sigma + \frac{z^2}{4\sigma}\right] \qquad \tilde{\mathcal{I}}_p(z) = \left(\frac{z}{2}\right)^{-p}\mathcal{I}_p(z)$

$$
W^c(p,q) = K_c(\Delta)\frac{2\pi}{c^{\frac{9}{2}}}\left(\frac{\pi}{2}\right)^{\frac{7}{2}}\cdot\tilde{\mathcal{I}}_{\frac{7}{2}}\left(\frac{\pi}{c}\sqrt{\Delta}\right).
$$

Combining everything we have

$$
W(q,p) = \sum_c W^c(q,p) = \sum_{c=1}^{\infty}\frac{2\pi}{c^{\frac{9}{2}}}\left(\frac{\pi}{2}\right)^{\frac{7}{2}}K_c(\Delta)\tilde{\mathcal{I}}_{\frac{7}{2}}\left(\frac{\pi\sqrt{\Delta}}{c}\right) = W_{\text{micro}}\,.
$$

This matches the microscopic answer.

Subsection 4.3

$\mathcal{N} = 4$ Macroscopic Interpretation

From [41] we have the following expression for the degeneracy of $\frac{1}{4}$ BPS states $d(m,n,\ell)$ with heterotic string theory compactified on T^6:

$$
d(m,n,\ell) = (-1)^{\ell+1}\int_C d\sigma dv d\rho \frac{1}{\Phi_{10}(\sigma,v,\rho)}e^{-i\pi(\sigma n + 2v\ell + \rho m)} \tag{57}
$$

where $\frac{1}{\Phi_{10}}$ is a meromorphic Siegel modular form of weight 10 [A.6], thus it depends on the contour C. Since the entropy should be independent of the multi centered contributions, we choose the contour C such that it picks only the single-centered contributions

($\Delta := 4mn - \ell^2 > 0$) namely the $\mathcal{R}$ chamber given below.

$$\frac{\rho_2}{\sigma_2} \gg 1, \qquad \frac{v_2}{\sigma_2} = -\frac{\ell}{2m}, \qquad \frac{\ell}{2m} \in [0,1) \tag{58}$$

The R-chamber is a charge-dependent choice of contour that isolates single-centred black hole degeneracies by excluding wall-crossing contributions from multi-centred BPS states, while aligning the integral with the black-hole attractor saddle.

Near $v = 0$ we have the following behavior for ϕ_{10}^{-1}

$$\Phi_{10}^{-1}(\sigma, v, \rho) = -\frac{1}{4\pi^2} \frac{1}{v^2 \eta^{24}(\rho)\, \eta^{24}(\sigma)} + \mathcal{O}\left(v^0\right) \tag{59}$$

Consider the set of transformations $SL(2,\mathbb{Z})_\sigma \subset Sp(2,\mathbb{Z})$ where elements are given by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{60}$$

After integrating over ρ see [A.6] we obtain the following expression for the degeneracy

$$d(m,n,\ell) = \sum_{P} \sum_{\Sigma \in \mathbb{Z}/|ac|\mathbb{Z}} \frac{(-1)^{\ell+1}}{\gamma^2 (ac)^{13}} \int_{\hat{C}} \frac{d\tau \wedge d\overline{\tau}}{(\tau - \overline{\tau})^{13}} \cdot \left(\frac{m}{ac} + \frac{a}{c} E_2\left(\rho_0'\right) + \frac{c}{a} E_2\left(\sigma_0'\right) \right) \cdot \frac{1}{\eta^{24}\left(\rho_0'\right)} \frac{1}{\eta^{24}\left(\sigma_0'\right)} e^{-2\pi i \Lambda}$$

where P is a subset of $SL(2,\mathbb{Z})$ with the following restrictions

$$P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z}) \,|\, a, \gamma > 0,\ c < 0,\ \alpha, \delta \in \mathbb{Z}/\gamma\mathbb{Z},\ b \in \mathbb{Z}/a\gamma\mathbb{Z} \right\}$$

Section 5

Results

Throughout this paper we took a chronological review of some of the literature regarding quantum black holes and two dimensional conformal field theory.
Various results and derivations were computed and interpreted in particular the results of [42] were verified in detail and a start was made on replicating the results of [41].
In the process we have matched results from string theory and the microscopic description of black holes to results from the macroscopic description of black holes using general relativity.
This has been a non-trivial test of the AdS/CFT correspondence and holography in general.

Section 6

Conclusions

Throughout this report, we have given an overview of some recent and historical developments in quantum black holes and their thermodynamic properties. The matching of the string theory microstate counting with the Bekenstein-Hawking entropy formula has been a significant milestone in our understanding of black hole thermodynamics. We have also discussed the implications of the AdS/CFT correspondence in providing a holographic description of black holes, which has opened new avenues for research in quantum gravity. These developments have relied on deep mathematical structures such as CFTs, automorphic forms and the Hardy-Ramanujan Rademacher expansion.

The results of [42] have been verified and depending on the progress of ongoing research, the remaining results of [41] may also be verified in the near future.
There are several areas both described in the literature and areas which are yet to be explored in detail which warrant further investigation. If we reduce the level of supersymmetry to $\mathcal{N} = 4$ and $\mathcal{N} = 2$, despite the black hole solution being known the description of microstate counting is less understood For the case of $\mathcal{N} = 4$, the phenomenon of wall-crossing, the role of mock-modular forms and its effects on the degeneracy of states could be studied in more detail. Additionally, the extension of these results to non-supersymmetric black holes remains an open question.

Section A

Appendices

Subsection A.1

BH Entropy from Rationalised Horizon Area

We take the entropy to be an increasing monotonic function of the rationalised horizon area, $S = f(\alpha)$. We can make the ansatz that $S = \eta\hbar^{-1}\alpha$ where η is a dimensionless constant which is determined by the Compton wavelength of a particle falling into the black hole. Follwoing Christodoulou we get the lower bound of $\eta = \frac{1}{2}\ln 2$, thus if we use SI units we obtain (1)

Subsection A.2

Extremal Black holes

Given the Lorentzian RN solution

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{61}$$

we can perform a Wick rotation $t \to i\tau$ and expand about $r = r_+$:

$$ds^2 = f'(r_+)\,\delta r d\tau^2 + \frac{(d(\delta r))^2}{f'(r_+)\,\delta r} \qquad \delta r \equiv r - r_+ \tag{62}$$

Introduce the coordinate change

$$\delta r = \frac{f'(r_+)}{4}\rho^2 \qquad \tilde{\tau} = \frac{2\pi}{\beta}\tau$$

$$ds^2 = \frac{\beta^2 f'(r_+)}{16\pi^2}\rho^2 d\tilde{\tau}^2 + d\rho^2 + \ldots$$

.

We notice this looks like polar in 2D flat space so we can make the following identifications

$$\tilde{\tau} \sim \tilde{\tau} + 2\pi \qquad \Longrightarrow \tau \sim \tau + \beta.$$

Since spacetime is smooth and there is no conical singularity (suppressed) we have

$$\beta = \frac{4\pi}{f'(r_+)} \qquad \Longrightarrow T_{RN} = \frac{r_+ - r_-}{4\pi r_+} \tag{63}$$

This in the extremal limit $r_+ \to r_-$ is equivalent to taking the limit as the temperature vanishes.

Subsection A.3

The Partition Function

To find the partition function we can plug our solution into the action to find the on-shell action.

$$I_E = -\frac{1}{16\pi G}\int d^4x\sqrt{+g}\left(R - F_{ij}F^{ij}\right) + I_{\text{GHY}} + I_{\text{EM. bndry}} \tag{64}$$

$$I_{\text{GHY}} = \frac{1}{8\pi G}\int_{M\times S_1^\beta}\sqrt{h}K$$

$$I_{\text{EM. bndry}} = -\frac{1}{8\pi G}\int_{M\times S_1^\beta}\sqrt{h}F^{ij}\hat{h}_i A_j.$$

Using $\nabla_\mu F^{\mu\nu} = 0$ and the fact that we find $R = 0$ we can use IBP to obtain an on-shell action

$$I_E^{\text{On-Shell}} \approx -\log Z(\beta) = -\left[\frac{A}{4G} - \beta M_{\text{adm}}(-\beta\mu Q)\right] \tag{65}$$

Subsection A.4

Verification of Wald Entropy from Sen Formalism

First we consider a $SO(2,1)$ invariant so that the near horizon geometry is $AdS_2 \times S^2$. The metric is given by

$$ds^2 = v\left(-\left(r^2-1\right)dt^2 + \frac{dr^2}{r^2-1}\right) \qquad F^i_{rt} = e_i \tag{66}$$

We have $\mathcal{L}^{(2)}(v, e_i, \ldots)$ (where the parameters have no dependence on t, r) as the two dimensional lagrangian density obtained from the four dimensional one by integrating over the S^2.

$$
\begin{aligned}
g_{tt} &= -v\left(r^2-1\right) \qquad g_{rr} = \frac{v}{r^2-1} \\
g_{rr} &= \frac{v}{r^2-1} \qquad \sqrt{-\det g} = v \\
&.
\end{aligned}
$$

Sen defines

$$
E\left(q_i, v, e_i, \ldots\right) = 2\pi\left(e_i q_i - v\mathcal{L}^{(2)}\right) \tag{67}
$$

Thus $v\mathcal{L}^{(2)} \equiv \int d^{D-2}y\sqrt{g_K}\sqrt{-g_{(2)}}\mathcal{L}_D$

Now we need to show that the wald entropy is $2\pi\left(e_i q_i - f\right)$ when we evaluate in on shell.

We can look at Sen's attractor equations for extremisation of E wrt e_i.

$$
\begin{aligned}
\frac{\partial E}{\partial e_i} &= 2\pi\left(q_i - \frac{\partial}{\partial e_i}\left(v\mathcal{L}^{(2)}\right)\right) = 0 \\
\Rightarrow q_i &= \frac{\partial}{\partial e_i}\left(v\mathcal{L}^{(2)}\right) = \int d^{D-2}y\sqrt{g_K}\frac{\partial}{\partial e_i}\left(\sqrt{-g_{(2)}}\mathcal{L}_D\right) \\
&= \int d^{D-2}y\sqrt{g_K}\sqrt{-g_{(2)}}\frac{\partial\mathcal{L}_D}{\partial F^{(i)}_{rt}} = \int d^{D-2}y\frac{\partial v\mathcal{L}_D}{\partial e_i} \\
&\implies v\mathcal{L}_D \text{ is a constant wrt } e_i.
\end{aligned}
$$

Sen 2005 shows that for any $SO\left(2,1\right)$ invariant extremal near horizon solution the Wald entropy is given by

$$
S_{\text{Wald}} = 2\pi\left(e_i\frac{\partial f}{\partial e_i} - f\right) \tag{68}
$$

$$
\begin{aligned}
S_{\text{Wald}} &= 2\pi\left(e_i q_i - f\right) \\
S_{\text{Wald}} &= E\Big|_{\text{Extremised}}.
\end{aligned}
$$

Subsection A.5

Eisenstein Series

The Eisenstein series for $SL\left(2,\mathbb{Z}\right)$ is given by

$$
E_k = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}}\frac{1}{\left(c\tau+d\right)^k} \tag{69}
$$

Subsection A.6

Siegel Modular Forms and Igusa cusp form Φ_{10}

A Siegel modular form of weight k is a holomorphic function $F : \mathbb{H}_2 \longrightarrow \mathbb{C}$ satisfying

$$f(M \cdot \tau) = \det\,(c\tau + d)^k f(\tau) \tag{70}$$

$\forall M = (a, b; c, d) \in Sp(2g, \mathbb{Z})$, $\tau \in \mathbb{H}_g$ and if $g = 1$ then f is holomorphic at the cusps. Here $\mathbb{H}_g$ is the Siegel upper half space of genus g defined as

$$\mathbb{H}_g = \left\{\tau \in M_{g\times g}(\mathbb{C})\,|\tau^T = \tau,\ \Im(\tau) > 0\right\} \tag{71}$$

One such Siegel modular form is the Igusa cusp form Φ_{10} which obeys the following transformation properties

$$\Phi_{10}\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det\,(C\Omega + D)^{10}\,\Phi_{10}(\Omega)$$
$$\text{where } \Omega = \begin{bmatrix} \rho & v \\ v & \sigma \end{bmatrix} \quad \Lambda = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2, \mathbb{Z})$$
$$\forall\, \Lambda \in Sp(2, \mathbb{Z}) \text{ satisfies}$$
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Φ_{10} is invariant under S-duality transformations ($SL(2, \mathbb{Z})$) transformations operating on Ω as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_S = \begin{bmatrix} a & -b & 0 & 0 \\ -c & d & 0 & 0 \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{bmatrix} \tag{72}$$

Thus we have

$$\rho' = \rho - \frac{\gamma v^2}{\gamma\sigma + \delta} \quad \sigma' = \frac{\alpha\sigma + \beta}{\gamma\sigma + \delta} \quad v' = \frac{v}{\gamma\sigma + \delta}.$$

$\forall\, \lambda, \mu, \nu \in \mathbb{Z}$

$$\Phi_{10}(\rho + \lambda, \sigma + \mu, v + \nu) = \Phi_{10}(\rho, \sigma, v) \tag{73}$$

Thus Φ_{10} has a period of an integer, therefore there exists a fourier expansion. So we have a net transformation

$$\begin{aligned} \Phi_{10}(\rho', \sigma', v') &= (\gamma\sigma + \delta)^{10}\,\Phi_{10}(\rho, \sigma, v) \\ \rho' &= a^2(\rho - \chi) + b^2\kappa - 2ab\psi \\ \sigma' &= c^2(\rho - \chi) + d^2\kappa - 2cd\psi \\ v' &= -ac(\rho - \chi) - bd\kappa + (ad + bc)\,\psi - \Sigma \\ \chi &= \frac{\gamma v^2}{\gamma\sigma + \delta}\ \kappa = \frac{a\sigma + \beta}{\gamma\sigma + \delta}\ \Psi = \frac{v}{\gamma\sigma + \delta}. \end{aligned}$$

Since $v = 0$ we get the poles given by

$$n_2\left(\rho\sigma - v^2\right) + jvn_1\sigma - m_1\rho + m_2 = 0 \tag{74}$$

$\forall\, z := \{m_1, m_2, j, n_1, n_2\} \in \mathbb{Z}$, where we have the constraint that

$$m_1 n_1 m_2 n_2 = \frac{1}{4}\left(1 - j^2\right) \tag{75}$$

and the poles with the constraint is invariant under $z \to -z$ thus we strictly consider $n_2 \geq 0$. [43].
As proven in [44], poles 74 with $n_2 \geq 1$ can be parametrised by nine integers

$$\begin{aligned} n_2 &= -ac\gamma, \\ j &= ad + bc \\ n_1 &= -bd\alpha - \gamma\Sigma \\ m_1 &= ac\delta \\ m_2 &= -bd\beta - \delta\Sigma \end{aligned}$$

where eight of these can be arranged into $\Lambda_i \in SL(2, \mathbb{Z})\ i = 1, 2$

$$\Lambda_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\ \Lambda_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

with $a > 0$, $c < 0$, $\gamma > 0$, $\alpha \in \mathbb{Z}/\gamma\mathbb{Z}$ and $\Sigma \in \mathbb{Z}$.
One can use the new coordinates for all of the poles restricted to $n_2 \geq 1$ we have a corresponding element of the set $P \cup \{\Sigma \in \mathbb{Z}\}$,

$$P := \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{Z})\, |\alpha, \gamma > 0, c < 0, \alpha \in \mathbb{Z}/\gamma\mathbb{Z}\right\} \tag{76}$$

We have the sets of matrices

$$S_\Gamma = \left\{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{Z}), \gamma > 0\right\}\ S_G = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})\, a > 0, c < 0\right\} \tag{77}$$

$$\Gamma_\infty = \left\{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \in SL(2, \mathbb{Z}), k \in \mathbb{Z}\right\} \tag{78}$$

We sum over the elements of Γ_∞/S_Γ, S_G and $\{\Sigma \in \mathbb{Z}\}$.
Now if we look at the residues at $v' = 0$

we can let

$$\Lambda(\sigma, v) = \frac{\gamma v^2}{\gamma\sigma + \delta} - \frac{bd}{ac}\left(\frac{a\sigma + \beta}{\gamma\sigma + \delta}\right) + \frac{(ad + bc)}{ac}\frac{v}{\gamma\sigma + \delta} - \frac{1}{ac}\Sigma$$

.

by **??** we have $v' = -ac\left(\rho - \Lambda(\sigma, v)\right)$
Should $\Lambda = \rho$ then if $\Sigma \to \Sigma + ac, \implies \Lambda \to \Lambda - 1$ then ρ_1 is shifted by an integer which then is no longer in its fundamental domain $[0, 1)$ integral domain. So we add the restriction that $\Sigma \in \mathbb{Z}/|ac|\mathbb{Z}$.

$$\Phi_{10}^{-1}\Big|_{v\to 0} \to -\frac{1}{4\pi^2}\frac{1}{v^2}\frac{1}{\eta^{24}(\rho)}\frac{1}{\eta^{24}(\sigma)} \tag{79}$$

We have

$$\frac{d}{d\tau}\log\eta^{24}(\tau) = 2\pi i E_2(\tau) \quad Res_{\rho=\Lambda} f(\rho) = \lim_{\rho\to\Lambda}\frac{\partial}{\partial\rho}\left((\rho-\Lambda)^2 f(\rho)\right) \tag{80}$$

$$(-1)^{\ell+1}\frac{(\gamma\sigma+\delta)^{10}}{ac}\left(\frac{m}{ac}+\frac{a}{c}E_2(\rho'_*)+\frac{c}{a}E_2(\sigma'_*)\right)\frac{1}{\eta^{24}(\rho'_*)}\frac{1}{\eta^{24}(\sigma'_*)}e^{-2\pi i(m\Lambda+n\sigma+\ell v)} \tag{81}$$

with

$$\begin{aligned}
\rho'_* &= -\frac{b}{c}\left(\frac{a\sigma+\beta}{\gamma\sigma+\delta}\right)+\frac{a}{c}\left(\frac{v}{\gamma\sigma+\delta}\right)-\frac{a}{c}\Sigma \\
\sigma'_* &= \frac{d}{a}\left(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}\right)-\frac{c}{a}\left(\frac{v}{\gamma\sigma+\delta}\right)-\frac{c}{a}\Sigma \\
v'_* &= 0.
\end{aligned}$$

Thus we get

$$-2\pi i \lim_{\rho\to\Lambda(\sigma,v)}(-1)^{\ell+1}\frac{\partial}{\partial\rho}\left((\rho-\Lambda(\sigma,v))^2(\gamma\sigma+\delta)^{10}\frac{-1}{4\pi^2}\frac{1}{v'^2}\frac{1}{\eta^{24}(\rho')}\frac{1}{\eta^{24}(\sigma')}e^{-2\pi i(m\rho+n\sigma+\ell v)}\right)$$

which may be written in the form

$$d(m,n,\ell)_{\Delta>0} = \sum_P \sum_{\Sigma\in\mathbb{Z}/|ac|\mathbb{Z}}\frac{(-1)^{\ell+1}}{\gamma^2(ac)^{13}}\int_{\hat{C}}\frac{d\tau\wedge d\overline{\tau}}{(\tau-\overline{\tau})^{13}}\left(\frac{m}{ac}+\frac{a}{c}E_2(\rho'_0)+\frac{c}{a}E_2(\sigma'_0)\right)\cdot\frac{1}{\eta^{24}(\rho'_0)}\frac{1}{\eta^{24}(\sigma'_0)}e^{-2\pi i\Lambda$$

Subsection A.7

Definitions of Functions

We define $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$, $\tau\in\mathbb{H}$

The Dedekind eta function is given by

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{82}$$

The odd Jacobi theta function is given by

$$\vartheta_1(\tau,z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}}(-1)^n q^{\frac{n^2}{2}}\zeta^n \tag{83}$$

We have the modified Bessel functions of the first kind given by

$$\begin{aligned}
\tilde{\mathcal{I}}_\rho &= \left(\frac{z}{2}\right)^{-\rho}\mathcal{I}_\rho(\rho) \\
\tilde{\mathcal{I}}_\rho(z) &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\sigma}{\sigma^{\rho+1}}\exp\left[\sigma+\frac{z^2}{4\sigma}\right].
\end{aligned}$$

where ρ is the index and $\mathcal{I}$ is the standard Bessel function of the first kind.

Subsection A.8

Background on Gauge Theories

Given a gauge transformation $A \to A + d\Lambda$ on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. We can classify the gauge transformations into two types, small gauge transformations which vanish at the boundary $\Lambda|_{\partial\mathcal{M}} = 0$ and large gauge transformations which do not vanish at the boundary $\Lambda|_{\partial\mathcal{M}} \neq 0$.

An An isometry diffeomorphism is generated by a Killing vector ξ of the background. Acting with it maps the configuration to itself (it's a symmetry of the saddle). A global gauge symmetry (a gauge transformation that leaves the connection/bundle data unchanged) similarly maps the configuration to itself.

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