Quantum Black Holes and 2 Dimensional Conformal Field Theory

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Timeline of Black Hole Physics

- 1973: Bekenstein's paper on black hole entropy.
- 1996: Strominger and Vafa's paper *Microscopic Origin of the Bekenstein-Hawking Entropy.*
- A review of $\frac{1}{8}$ BPS Black Holes in String theory.
- A look at the Microscopic Interpretation and the emergance of automorphic forms.
- Gravitational Path integral and localisation within supergravity.
- The case of $\mathcal{N} = 4$ what is known and what is not?
- Current work

Bekenstein's Formula

Classically Bekenstein derived a formula for the entropy of a black hole [1]

$$S_{BH} = \frac{A}{4}\frac{k}{\ell_p^2} \qquad \ell_p \equiv \sqrt{\frac{\hbar G}{c^3}}$$

where k is Boltzmann's constant and ℓ_p is the Planck length.

A Triumph of String theory

In 1996, for the case of 5 d black holes an expression for the Microscopic entropy was obtained in the context of a $D1 - D5$ system [2].
This lead to the discovery in 1997 of the infamous AdS/CFT Correspondence.

1/8 BPS Black Holes in String Theory
subtitle

The unique quartic invariant of the U-duality group is [3]

$$\Delta\left(\mathcal{Q}\right) = C^{abcd}\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_c\mathcal{Q}_d$$

The Bekenstein-Hawking entropy of these black holes is

$$S_{BH}\left(\mathcal{Q}\right) = \pi\sqrt{\Delta\left(\mathcal{Q}\right)}$$

A Type IIB compactification on $T^6 = T^4 \times S^1 \times \tilde{S}^1$ with an $E_{7(7)}\left(\mathbb{Z}\right)$ charge invariant of $\Delta = 4n - \ell^2$

Automorphic Forms in the Microscopic Interpretation

Generating functions of BPS states.

$$q := e^{2\pi i\tau},\ \zeta := e^{2\pi i z},\ p := e^{2\pi i\rho} \qquad \tau \in \mathbb{H},\ z, \rho \in \mathbb{C}$$

$\frac{1}{8}$ BPS in $\mathcal{N} = 8$ $\quad \varphi_{-2,1}(\tau, z) = \dfrac{\vartheta_1(\tau, z)^2}{\eta^6(\tau)} = \displaystyle\sum_{n,\ell\in\mathbb{Z}} c_{-2,1}(n, \ell)\, q^n \zeta^\ell$

$\frac{1}{4}$ BPS in $\mathcal{N} = 4$ $\quad \dfrac{1}{\Phi_{10}(\rho, \tau, z)} = \displaystyle\sum_{\substack{m,n,\ell\in\mathbb{Z},\\ m,n\geq -1}} g(m, n, \ell)\, p^m q^n \zeta^\ell$

$\frac{1}{2}$ BPS in $\mathcal{N} = 4$ $\quad \dfrac{1}{\eta^{24}(\tau)} = \displaystyle\sum_{n\geq -1} d(n)\, q^n.$

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta^6(\tau)}$$

$$\vartheta_1(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (-1)^n q^{\frac{n^2}{2}} \zeta^n \qquad \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

We have a theta expansion for ϑ_1

$$\varphi_{k,m}(\tau, z) = \sum_{\mu \bmod 2m} h_\mu \theta_{m,\mu}(\tau, z) \qquad \mu \in \frac{\mathbb{Z}}{2m\mathbb{Z}}$$

Generalising [4, Thm. 5.10]

$$C(\Delta) = \frac{1}{2\pi i} \oint_{|q|=r} \frac{h_\mu(\tau)}{q^{\frac{\Delta}{4}+1}} dq \qquad \Delta \equiv 4n - \ell^2$$

Microscopic Result for $\mathcal{N} = 8$

A replication of B

$$\boxed{C\left(\Delta\right) = 2\pi\left(\frac{\pi}{2}\right)^{\frac{7}{2}}\sum_{c=1}^{\infty} c^{-\frac{9}{2}} K_c\left(\Delta\right)\tilde{\mathcal{I}}_{\frac{7}{2}}\left(\frac{\pi\sqrt{\Delta}}{c}\right)} \tag{1}$$

$$W_{\text{micro}}\left(\Delta\right) = \left(-1\right)^{\Delta+1} C\left(\Delta\right)$$

Taking the asymptotic expansion as ($\Delta \to \infty$, $\mathcal{I}_P \to e^{\frac{\pi\sqrt{\Delta}}{c}}$)

$$S = \log W_{\text{micro}}\left(\Delta\right) \sim \pi\sqrt{\Delta} + \ldots$$

The Gravitational Path Integral

Maximally supersymmetric $AdS_2 \times S^2$ with constant (q, p)

Consider $\mathcal{N} = 2$ sugra in 4d (8 supercharges) coupled to n_v vector multiplets $I = 1, \ldots, n_v$. The on-shell graviton multiplet contains a vector eld (the graviphoton), so that we have a total of $n_v + 1$ gauge fields $\Lambda = 0, \ldots, n_v$. This theory has black hole solutions carrying electric and magnetic charges $\left(q^\Lambda, p^\Lambda\right)$ preseving 4 of 8 charges (1/2 BPS) after lifting to $\mathcal{N} = 8$ they preserve $4/32$ supercharges.

$$\mathcal{W}\left(q, p\right) = \int \left[\mathcal{D}g\mathcal{D}\Psi\mathcal{D}A\mathcal{D}\Phi\right] e^{-S_{\text{sugra}}^{\text{bulk}} - S_{\text{sugra}}^{\text{bdry}}}$$

Localisation in Supergravity

In 4 d BPS black holes in $\mathcal{N} = 2$ supergravity the gravitational path integral reduces via supersymmetric localisation to the quantum entropy function [5]

$$\mathcal{W}^c = \int \left[d^{n_\nu+1}\phi\right]_c \exp\left[-\frac{\pi q_\Lambda \phi^\Lambda}{c} + \frac{4\pi}{c}\Im F\left(\frac{\phi^\Lambda + ip^\Lambda}{2}\right)\right] \mathcal{Z}_{\text{1-loop}}^{Q_{\text{eq}}\mathcal{V}} \mathcal{Z}_c^{\text{top}}$$

After localisation we obtain a contribution from the orbifold saddle points to get

$$W^c(q,p) = \sqrt{c} K_c(\Delta) \int d^{16}\phi \sqrt{\frac{-2C_{IJK}p^Ip^Jp^K \,\det\,(3C_{IJ})}{c^{16}\left(\phi^0\right)^{18}}} \frac{1}{c}\left(\frac{\phi^0}{-2C_{IJK}p^Ip^Jp^K}\right)^4$$

$$\cdot\exp\left[-\frac{\pi C_{IJK}p^Ip^Jp^K}{c\phi^0} + \frac{3\pi C_{IJK}\phi^I\phi^Jp^K}{c\phi^0} - \frac{\pi q_0\phi^0}{c} - \frac{\pi q_I\phi^I}{c}\right]$$

Heterotic String Theory Compactified T^6 with unit torsion

$$d\left(m, n, \ell\right) = \left(-1\right)^{\ell+1} \int_C d\sigma dv d\rho \frac{1}{\Phi_{10}\left(\sigma, v, \rho\right)} e^{-i\pi\left(\sigma n + 2v\ell + \rho m\right)}$$

Microscopic Result for $\mathcal{N} = 4$

subtitle

$$d(m,n,\ell) = (-1)^{\ell} i^{1/2} \sum_{\gamma\geq 1} \sum_{\tilde\ell \pmod{2m}} \Bigg[2\pi \sum_{\substack{\tilde n \geq -1 \\ \tilde\Delta < 0}} c_m^F(\tilde n, \tilde\ell) \frac{Kl(\frac{\Delta}{4m}, \frac{\tilde\Delta}{4m}; \gamma, \psi)_{\ell\tilde\ell}}{\gamma} \left(\frac{|\tilde\Delta|}{\Delta}\right)^{23/4}$$

$$I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde\Delta|}\right) - \delta_{\tilde\ell,0}\sqrt{2m}\, d(m) \frac{Kl(\frac{\Delta}{4m}, -1; \gamma, \psi)_{\ell 0}}{\sqrt{\gamma}} \left(\frac{4m}{\Delta}\right)^6 I_{12}\left(\frac{2\pi}{\gamma}\sqrt{\frac{\Delta}{m}}\right)$$

$$+ \frac{d(m)}{2\pi} \sum_{\substack{g \pmod{2m\gamma} \\ g = \tilde\ell\,(2m)}} Kl\left(\frac{\Delta}{4m}, -1 - \frac{g^2}{4m}; \gamma, \psi\right)_{\ell\tilde\ell} \gamma^2 \left(\frac{4m}{\Delta}\right)^{25/4}$$

$$\times \int_{-1/\sqrt{m}}^{1/\sqrt{m}} dx'\, f_{\gamma,g,m}(x')(1 - mx'^2)^{25/4} I_{25/2}\left(\frac{2\pi}{\gamma}\sqrt{m\Delta(1-mx'^2)}\right) \Bigg]$$

'The gravitational path integral for $N = 4$ *BPS black holes from black hole microstate counting'*- Lopes Cardoso, et al.

Thank You

$$K_c\left(\Delta\right) := e^{\frac{5\pi i}{4}} \sum_{\substack{-c\leq d<0 \\ (d,c)=1}} e^{2\pi \frac{d}{c}\left(\frac{\Delta}{4}\right)} M\left(\gamma_{c,d}\right)^{-1}_{\nu 1} e^{2\pi i \frac{a}{c}\left(-\frac{1}{4}\right)} \qquad \nu = \Delta \text{mod } 2$$

$$M^{-1}\left(\gamma_{c,d}\right)_{\nu 1} = e^{\frac{\pi i}{4}} \frac{1}{\sqrt{6c}} e^{-\frac{i\pi}{6}\Phi(\gamma)} \cdot \sum_{\epsilon=\pm 1} \sum_{n=0}^{c-1} \epsilon \exp\left[\frac{i\pi}{6c}\left(d(\nu+1)^2 - 4(\nu+1)(3n+\epsilon) + 4a(3n+\epsilon)^2\right)\right].$$

Where Φ is the Rademacher function

$$\Phi\left(\gamma\right) = \frac{a+d}{c} - 12 \text{ sign}\left(c\right) s\left(a, |c|\right)$$

for $c > 0$, the Dedekind sum is

$$s\left(a, c\right) = \frac{1}{4c} \sum_{i=1}^{c-1} \cot\left(\frac{\pi i}{c}\right) \cot\left(\frac{\pi i a}{c}\right)$$

We define the modified Bessel function $\tilde{\mathcal{I}}_p$ with weight ρ

$$\tilde{\mathcal{I}}_\rho\left(z\right)=\left(\frac{z}{2}\right)^{-\rho}\mathcal{I}_\rho\left(z\right)=\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\sigma}{\sigma^{p+1}}\exp\left[\sigma+\frac{z^2}{4\sigma}\right]$$

where $\mathcal{I}_{\rho(z)}$ is the standard bessel function of the first kind, which satisfies the asymptotic expansion for $z\rightarrow\infty$

$$\mathcal{I}_\rho=\frac{e^z}{\sqrt{2\pi z}}\left(1-\frac{\left(4\rho^2-1\right)}{8z}+\frac{\left(4\rho^2-1\right)\left(4\rho^2-3^2\right)}{2!(8z)^3}-\frac{\left(4\rho^2-1\right)\left(4\rho^2-3^2\right)\left(4\rho^2-5^2\right)}{3!(8z)^5}+\ldots\right)$$

$$c_m^F(\tilde{n},\tilde{\ell}) = \sum_{\substack{a>0,\, c<0 \\ b\in\mathbb{Z}/a\mathbb{Z},\, ad-bc=1 \\ 0\le \frac{b}{a}+\frac{\tilde{\ell}}{2m}<-\frac{1}{ac}}} \Big((ad+bc)\tilde{\ell}+2ac\tilde{n}+2bdm\Big) \times d\left(c^2\tilde{n}+d^2m+cd\tilde{\ell}\right) d\left(a^2\tilde{n}+b^2m+ab\tilde{\ell}\right),$$

$$\frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i\tau n},\quad Kl\left(\frac{\Delta}{4m},\frac{\tilde{\Delta}}{4m};\gamma,\psi\right)_{\ell\tilde{\ell}} = \sum_{\substack{0\le-\delta<\gamma \\ (\delta,\gamma)=1,\ \alpha\delta\equiv 1 \pmod{\gamma}}} e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{\tilde{\Delta}}{4m}+\frac{\delta}{\gamma}\frac{\Delta}{4m}\right)}\psi(\Gamma)_{\ell\tilde{\ell}}$$

$$\psi(\Gamma)_{\ell\tilde{\ell}} = \frac{1}{\sqrt{2m\gamma i}} \sum_{T\in\mathbb{Z}/\gamma\mathbb{Z}} e^{2\pi i\left(\frac{\alpha}{\gamma}\frac{(\tilde{\ell}-2mT)^2}{4m}-\frac{\ell(\tilde{\ell}-2mT)}{2m\gamma}+\frac{\delta}{\gamma}\frac{\ell^2}{4m}\right)},$$

$$f_{\gamma,g,m}(x') = \sum_{\substack{p\in\mathbb{Z} \\ 2m\gamma p+g\neq 0}} \frac{\gamma^2}{\left(x'-i\gamma p-i\frac{g}{2m}\right)^2} = \begin{cases} \dfrac{\pi^2}{\sinh^2\left(\frac{\pi x'}{\gamma}-\frac{\pi i g}{2m\gamma}\right)} & \text{if } g\not\equiv 0 \pmod{2m\gamma}, \\ \dfrac{\pi^2}{\sinh^2\left(\frac{\pi x'}{\gamma}\right)}-\dfrac{\gamma^2}{x'^2} & \text{if } g\equiv 0 \pmod{2m\gamma}. \end{cases}$$

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