The Algebra of Complex Numbers

Definition: The field of complex numbers is defined as follows:

Let $\mathbb{C} = \{\ a + b\,\boldsymbol{i} \mid a, b \in \mathbb{R}\ \}$, and

(a) $(a + b\,\boldsymbol{i}) + (c + d\,\boldsymbol{i}) \;=\; (a + c) + (b + d)\,\boldsymbol{i}$

(b) $(a + b\,\boldsymbol{i}) \cdot (c + d\,\boldsymbol{i}) \;=\; (ac - bd) + (ad + bc)\,\boldsymbol{i}$

With these two operations $\mathbb{C}$ is a field.

Notation:

* $0 + 0\,\boldsymbol{i} = 0$
* $1 + 0\,\boldsymbol{i} = 1$
* $a + 0\,\boldsymbol{i} = a$ [This way: $\mathbb{R} \subseteq \mathbb{C}$]
* $0 + 1\,\boldsymbol{i} = \boldsymbol{i}$
* $0 + b\,\boldsymbol{i} = b\,\boldsymbol{i}$
* $a + 1\,\boldsymbol{i} = a + \boldsymbol{i}$
* $a + (-b)\,\boldsymbol{i} = a - b\,\boldsymbol{i}$

Property of $\boldsymbol{i}$: Note that $\boldsymbol{i}$ is just a symbol, just e.g. π, but it has a special property

$$\boxed{\boldsymbol{i}^2 = -1}$$

Proof: $\boldsymbol{i}^2 = \boldsymbol{i} \cdot \boldsymbol{i} = (0 + 1 \cdot \boldsymbol{i}) \cdot (0 + 1 \cdot \boldsymbol{i}) = (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0)\boldsymbol{i} = -1 + 0\,\boldsymbol{i} = -1.$

With this property, the definition of multiplication can be seen as the 'usual' algebraic "multiplying out"

$$\begin{aligned}
(a + b\,\boldsymbol{i}) \cdot (c + d\,\boldsymbol{i}) &= a \cdot c + a \cdot (d\,\boldsymbol{i}) + (b\,\boldsymbol{i}) \cdot c + (b\,\boldsymbol{i}) \cdot (d\,\boldsymbol{i}) \\
&= a\,c + a\,d\,\boldsymbol{i} + b\,c\,\boldsymbol{i} + b\,d\,\boldsymbol{i}^2 \\
&= a\,c + (a\,d + b\,c)\,\boldsymbol{i} + b\,d(-1) \\
&= a\,c - b\,d + (a\,d + b\,c)\,\boldsymbol{i}
\end{aligned}$$

Real and Imaginary parts

We call a the **real** part of $z = a + b\boldsymbol{i}$: $\boxed{\mathrm{Re}(a + b\boldsymbol{i}) = a}$ and

b the **imaginary** part of $z = a + b\boldsymbol{i}$: $\boxed{\mathrm{Im}(a + b\boldsymbol{i}) = b}$

A complex number is completely determined by its real and imaginary parts, i.e. two complex numbers are the same if their real and imaginary parts are the same

$$a + b\boldsymbol{i} \;=\; A + B\boldsymbol{i} \qquad \Leftrightarrow \qquad a = A \quad \text{and} \quad b = B$$

Note that we say $z \in \mathbb{R}$ when $\mathrm{Im}(z) = 0$: i.e. $z = x + 0\boldsymbol{i} \in \mathbb{R}$, where $\mathrm{Re}(z) = x$.
This way $\mathbb{R}$ is 'embedded' in $\mathbb{C}$: $\boxed{\mathbb{R} \subseteq \mathbb{C}}$

Field properties: $\mathbb{C}$ satisfies the following field properties: $[\forall\, z, w, u \in \mathbb{C}\,]$

Addition and multiplication are **commutative**

(1) $\boxed{z + w = w + z}$ **(5)** $\boxed{z \cdot w = w \cdot z}$

Addition and multiplication are **associative**

(2) $\boxed{z + (w + u) = (z + w) + u}$ **(6)** $\boxed{z \cdot (w \cdot u) = (z \cdot w) \cdot u}$

There are 'neutral' elements 0 and 1 with respect to addition and multiplication. $[\,1 \neq 0\,]$

(3) $\boxed{z + 0 = z}$ **(7)** $\boxed{z \cdot 1 = z}$

There are **opposites** and **inverses**

(4) $\boxed{z + (-z) = 0}$ **(8)** $\boxed{z \cdot (z^{-1}) = 1}$ when $z \neq 0$

Multiplication is **distributive** over addition

(9) $\boxed{z \cdot (w + u) = z \cdot w + z \cdot u}$

We'll prove everything at the end of these notes

Note that $\boxed{0 \cdot z = 0}$ for $\forall z \in \mathbb{C}$

Uniqueness

Although it is not explicitly mentioned above, but 0 and 1 are unique. So are opposites and (multiplicative) inverses, in the sense that each z has only one opposite, and each $z \neq 0$ has only one inverse. [See proofs at the end]

Convention: The usual order of operations is used: e.g. multiplication takes **precedence** over addition

Note that in writing the right hand side of (9) we used this convention, so that we didn't have to write: $(z \cdot w) + (z \cdot u)$

Opposites and inverses

If $z = a + b\boldsymbol{i}$ then $\boxed{-z = -a - b\boldsymbol{i}}$ and $\boxed{z^{-1} = \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2}\boldsymbol{i}}$

Another notation for the inverse is $z^{-1} = \dfrac{1}{z}$. We can use these to define subtraction and division.

Subtraction and Division

Eventhough a field is defined with just two operations, addition and multiplication, we can also define subtraction and division, using opposites and inverses as follows:

$$\boxed{\begin{array}{c} z - w = z + (-w) \\ \dfrac{z}{w} = z \cdot \dfrac{1}{w} \end{array}} \quad \Rightarrow \quad \boxed{\begin{array}{c} (a+b\boldsymbol{i}) - (A+B\boldsymbol{i}) = (a-A) + (b-B)\boldsymbol{i} \\ \dfrac{a+b\boldsymbol{i}}{A+B\boldsymbol{i}} = \dfrac{aA+bB}{A^2+B^2} - \dfrac{bA-aB}{A^2+B^2}\boldsymbol{i} \end{array}}$$

Don't try to remember this last equation. We'll give you an easier one, using conjugates and moduli, later.

Excercise: Show that $\boxed{\dfrac{z_1}{z_2} \cdot \dfrac{w_1}{w_2} = \dfrac{z_1 \cdot w_1}{z_2 \cdot w_2}}$

Cancellation laws: **(a)** If $z + u = w + u$ then $z = w$.

(b) If $z \cdot u = w \cdot u$, and $u \neq 0$, then $z = w$.

Modulus or length

If $z = a + b\boldsymbol{i}$ then $|\,z\,| = \sqrt{a^2 + b^2}$ is called the **modulus** or **length** of z.

Complex conjugates

If $z = a + b\boldsymbol{i}$ then $\bar{z} = a - b\boldsymbol{i}$ is called the **complex conjugate** of z.

Complex numbers behave as we would expect. Here are some of the main properties:

Theorem: **(1)** $\boxed{\overline{z+w} = \overline{z} + \overline{w}}$ and $\boxed{\overline{z-w} = \overline{z} - \overline{w}}$

(2) $\boxed{\overline{z \cdot w} = \overline{z} \cdot \overline{w}}$ and $\boxed{\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}}$

(3) $\boxed{\overline{\overline{z}} = z}$

(4) $\boxed{\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}}$ and $\boxed{\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2\boldsymbol{i}}}$

(5) $\boxed{z \in \mathbb{R} \;\Leftrightarrow\; z = \overline{z}}$ and $\boxed{z \in \mathbb{R} \;\Leftrightarrow\; \mathrm{Im}(z) = 0}$

(6) $\boxed{|\,\overline{z}\,| = |\,z\,|}$

(7) $\boxed{z \cdot \overline{z} = |\,z\,|^2}$

(8) $\boxed{\dfrac{1}{z} = \dfrac{\overline{z}}{|\,z\,|^2}}$ and $\boxed{\dfrac{z}{w} = \dfrac{z \cdot \overline{w}}{|\,w\,|^2}}$

(9) $\boxed{|\,z \cdot w\,| = |\,z\,| \cdot |\,w\,|}$ and $\boxed{\left|\dfrac{z}{w}\right| = \dfrac{|\,z\,|}{|\,w\,|}}$

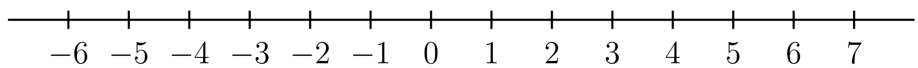
Note that $|\,z\,|$ is actually an extension of the absolute value for real numbers. If $z \in \mathbb{R}$ then $\mathrm{Im}(z) = 0$. Let $\mathrm{Re}(z) = x$ then $|\,z\,| = |\,x\,|$ since

$$|\,z\,| = |\,x + 0\boldsymbol{i}\,| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |\,x\,|$$

where $|\,x\,|$ is the usual absolute on $\mathbb{R}$.

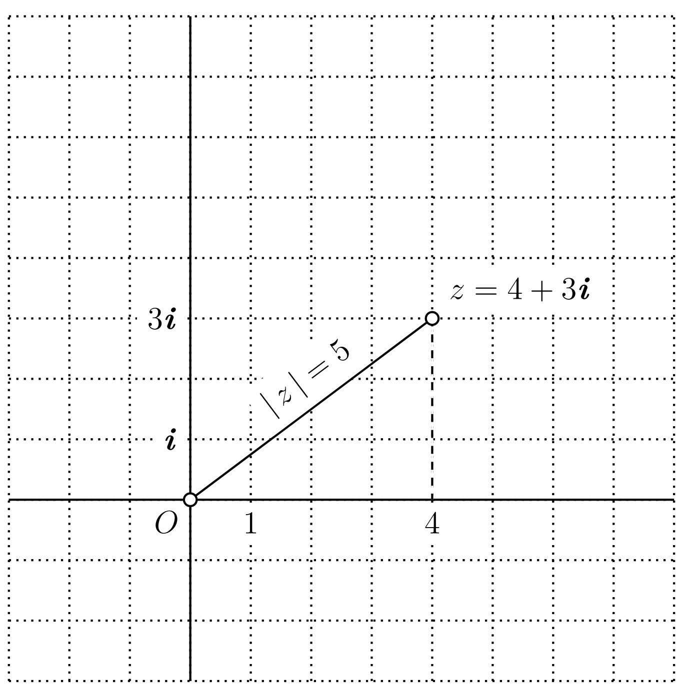
The Geometry of Complex Numbers

In high school you learned about the geometric representation of the real numbers in the form of a number line:



The real number line

We also have a geometric representation of the complex numbers: a two dimensional plane with an origin, and two perpendicular axes, the **real** axis and the **imaginary** axis. The real axis is labelled, as usual, with e.g.$\cdots -2, -1, 0, 1, 2, 3\cdots$, whereas the imaginary axis is labelled with e.g. $\cdots -2\boldsymbol{i}, -\boldsymbol{i}, 0, \boldsymbol{i}, 2\boldsymbol{i}, 3\boldsymbol{i}\cdots$. Both 1 and $\boldsymbol{i}$ are one unit away from the origin (they are on a unit circle centered at the origin).



$\boldsymbol{z = 4 + 3i}$ **in the Complex Plane**

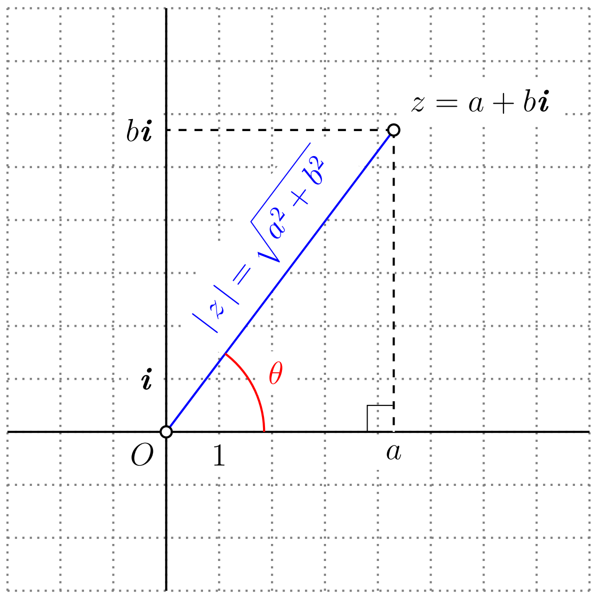
Each complex number $z = a + b\boldsymbol{i}$ corresponds to a point $(a, b\boldsymbol{i})$ in this plane. Note that the length $|z| = \sqrt{a^2 + b^2}$ corresponds to the distance of that point to the origin.

Not only does a point have Euclidean/Cartesian coordinates $(a, b\boldsymbol{i})$, it also has polar coordinates: $r = |z|$ and θ. In this world θ is usually called the argument of $z = a + b\boldsymbol{i}$.

Polar coordinates: Any complex number $z = a + b\boldsymbol{i} \neq 0$, has a certain length r, its distance from the origin,

$$r = |\,z\,| = \sqrt{a^2 + b^2}$$

and an angle θ associated with it



$z = a + b\boldsymbol{i}$ in the Complex Plane

θ is the angle the line segment Oz makes with the positive x-axis. In fact there are multiple angles to choose from. These angles are called 'arguments' of z: $\arg(z)$. When $-\pi < \theta \leq \pi$ we call θ the principal argument of z, which we capatilize

$$\operatorname{Arg}(z) = \theta$$

r and θ are the polar coordinates of the complex number z. The only point without an argument is 0. The relations between polar coordinates and Cartesian coordinates are:

(1) From polar coordinates to Cartesian coordinates $[\,z = a + b\boldsymbol{i}\,]$

$$\boxed{\begin{cases} a = r\cos(\theta) \\ b = r\sin(\theta) \end{cases}}$$

(2) From Cartesian coordinates to polar coordinates $[\,z = a + b\boldsymbol{i}\,]$

$$\boxed{\begin{cases} |\,z\,| = \sqrt{a^2+b^2} \\ \arg(z) = \tan^{-1}(b/a) \end{cases}} \quad \text{if } a \neq 0$$

$$\boxed{\begin{cases} |\,z\,| = \sqrt{a^2+b^2} \\ \operatorname{Arg}(z) = \pm\dfrac{\pi}{2} \end{cases}} \quad \text{if } a = 0, \text{ but } b \neq 0$$

Addition of complex numbers

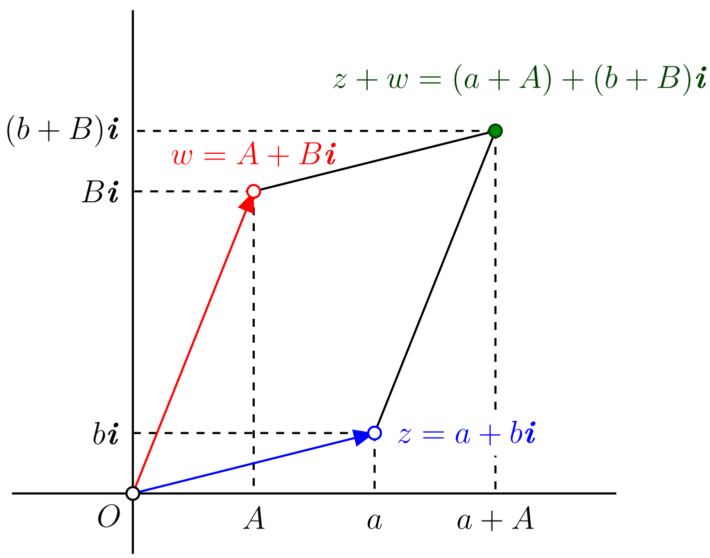
When adding complex numbers we add real parts together and imaginary parts together

$$z + w = (a + b\boldsymbol{i}) + (A + B\boldsymbol{i}) = (a + A) + (b + B)\boldsymbol{i}$$

i.e.

$$\boxed{\begin{cases} \mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w) \\ \mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w) \end{cases}}$$

which is basically a vector addition:



Addition in the Complex Plane

Multiplication of complex numbers

Multiplication in the complex plane has a twist: it involves a rotation.

Given two complex numbers z and w, with polar coordinates

$$\begin{cases} \theta = \mathrm{Arg}(z) \\ r = |\,z\,| \end{cases} \quad \text{and} \quad \begin{cases} \varphi = \mathrm{Arg}(w) \\ R = |\,w\,| \end{cases}$$

so that

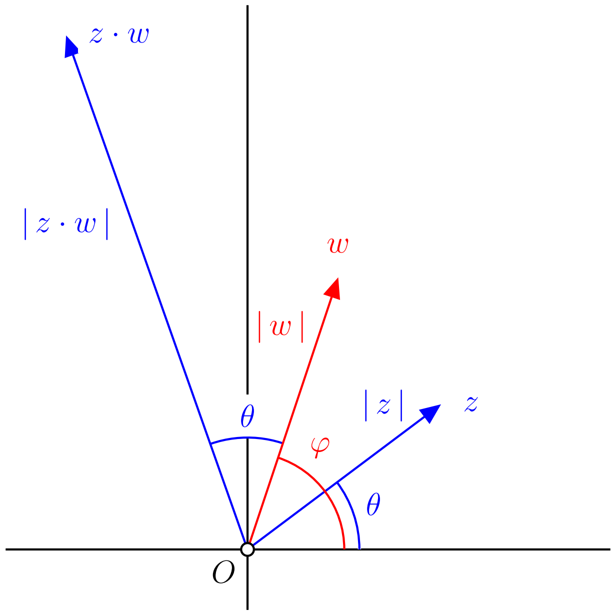
$$\begin{cases} z = r\Big(\cos(\theta) + \sin(\theta)\boldsymbol{i}\Big) \\ w = R\Big(\cos(\varphi) + \sin(\varphi)\boldsymbol{i}\Big) \end{cases}$$

$$\text{Then}\quad z \cdot w = r\Big(\cos(\theta) + \sin(\theta)\boldsymbol{i}\Big) \cdot R\Big(\cos(\varphi) + \sin(\varphi)\boldsymbol{i}\Big)$$

$$= r \cdot R\left(\Big(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)\Big) + \Big(\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)\Big)\boldsymbol{i}\right)$$

$$= r \cdot R\Big(\cos(\theta + \varphi) + \sin(\theta + \varphi)\boldsymbol{i}\Big)$$

Hence: $\boxed{\begin{array}{c} |\, z \cdot w \,| = r \cdot R \\ \arg(z \cdot w) = \mathrm{Arg}(z) + \mathrm{Arg}(w) \end{array}}$ i.e. moduli are multiplied, and arguments added!



Multiplication in the Complex Plane

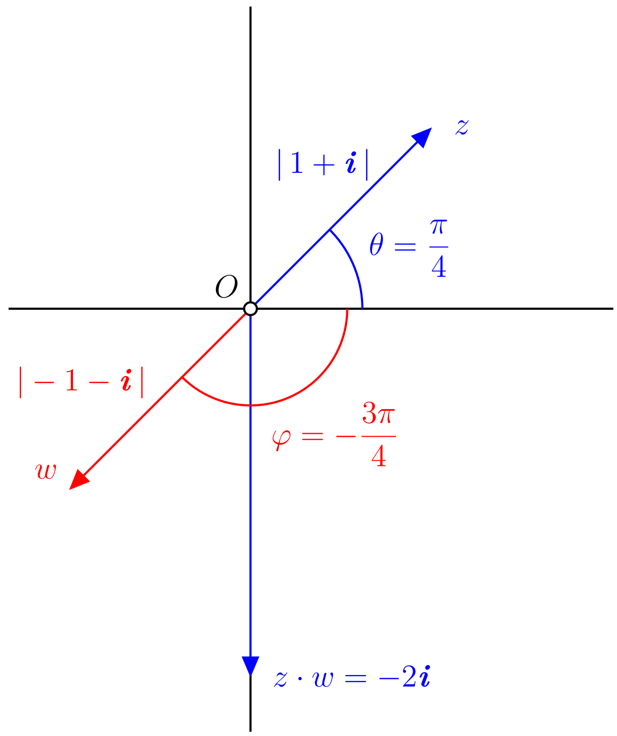
So for example: $(1 + \boldsymbol{i}) \cdot (-1 - \boldsymbol{i}) = -2\boldsymbol{i}$

$$\left.\begin{array}{l} |\, 1 + \boldsymbol{i} \,| = \sqrt{2} \\ |\, -1 - \boldsymbol{i} \,| = \sqrt{2} \end{array}\right\} \quad \Rightarrow \quad \Big|(1 + \boldsymbol{i}) \cdot (-1 - \boldsymbol{i})\Big| = \sqrt{2} \cdot \sqrt{2} = 2$$

and

$$\left.\begin{array}{l} \mathrm{Arg}(1 + \boldsymbol{i}) = \dfrac{\pi}{4} \\ \mathrm{Arg}(-1 - \boldsymbol{i}) = -\dfrac{3\pi}{4} \end{array}\right\} \quad \Rightarrow \quad \mathrm{Arg}\Big((1 + \boldsymbol{i}) \cdot (-1 - \boldsymbol{i})\Big) = \frac{\pi}{4} + (-\frac{3\pi}{4}) = -\frac{\pi}{2}$$

and indeed: $\Big| -2\boldsymbol{i} \Big| = 2$ and $\mathrm{Arg}(-2\boldsymbol{i}) = -\dfrac{\pi}{2}$



The geometry of $(1+\boldsymbol{i}) \cdot (-1-\boldsymbol{i}) = -2\boldsymbol{i}$

There is a lot more that can be said, but for this course the algebra of the complex numbers is the most important feature we need. We'll end (without proof) with

The Fundamental Theorem of Algebra

$$\boxed{\text{Any non-constant polynomial with coefficients in } \mathbb{C}, \text{ has a root (a zero) in } \mathbb{C}.}$$

As a consequence all polynomials over $\mathbb{C}$ completely factor in linear factors. Hence a polynomial of degree n has n complex roots, counting multiplicities.

Example: $z^2+1 = (z-\boldsymbol{i})(z+\boldsymbol{i})$ and

$$\begin{aligned} z^6+64 &= (z^2+4)(z^2-4z^2+16) \\ &= (z-2\boldsymbol{i})(z+2\boldsymbol{i})\Big(z-(\sqrt{3}+\boldsymbol{i})\Big)\Big(z-(\sqrt{3}+\boldsymbol{i})\Big)\Big(z+(\sqrt{3}-\boldsymbol{i})\Big)\Big(z+(\sqrt{3}-\boldsymbol{i})\Big) \end{aligned}$$

The remainder of these notes are the proofs of most statements.

Proofs

(1) Addition is **commutative**: $\boxed{z+w=w+z}$

Proof:
$$\begin{aligned} z+w &= (z_1+z_2\boldsymbol{i})+(w_1+w_2\boldsymbol{i}) \\ &= (z_1+w_1)+(z_2+w_2)\boldsymbol{i} \\ &= (w_1+z_1)+(w_2+z_2)\boldsymbol{i} \\ &= (w_1+w_2\boldsymbol{i})+(z_1+z_2\boldsymbol{i}) \;=\; w+z \end{aligned}$$

(2) Addition is **associative**: $\boxed{z+(w+u)=(z+w)+u}$

Proof:
$$\begin{aligned} z+(w+u) &= (z_1+z_2\boldsymbol{i})+\big((w_1+w_2\boldsymbol{i})+(u_1+u_2\boldsymbol{i})\big) \\ &= (z_1+z_2\boldsymbol{i})+\big((w_1+u_1)+(w_2+u_2)\boldsymbol{i}\big) \\ &= \big(z_1+(w_1+u_1)\big)+\big(z_2+(w_2+u_2)\big)\boldsymbol{i} \\ &= \big((z_1+w_1)+u_1\big)+\big((z_2+w_2)+u_2\big)\boldsymbol{i} \\ &= \big((z_1+w_1)+(z_2+w_2)\boldsymbol{i}\big)+(u_1+u_2\boldsymbol{i}) \\ &= \big((z_1+z_2\boldsymbol{i})+(w_1+w_2\boldsymbol{i})\big)+(u_1+u_2\boldsymbol{i}) \;=(z+w)+u \end{aligned}$$

(3) There exists a $0\in\mathbb{C}$ such that for all $z\in\mathbb{C}$: $\boxed{z+0=z}$

Clearly, $0=0+0\boldsymbol{i}$ has this property.

Proof: Let $z=z_1+z_2\boldsymbol{i}$ then

$$z+0=(z_1+z_2\boldsymbol{i})+(0+0\boldsymbol{i})=(z_1+0)+(z_2+0)\boldsymbol{i}=z_1+z_2\boldsymbol{i}=z$$

(4) For each $z\in\mathbb{C}$ there exists an element $-z\in\mathbb{C}$ such that $\boxed{z+(-z)=0}$

Clearly, If $z=z_1+z_2\boldsymbol{i}$ then $-z=-z_1+(-z_2)\boldsymbol{i}$ has this property.

Proof: Let $z=z_1+z_2\boldsymbol{i}$ and $-z=-z_1+(-z_2)\boldsymbol{i}$ then

$$z+(-z)=(z_1+z_2\boldsymbol{i})+\big(-z_1+(-z_2)\boldsymbol{i}\big)=\big(z_1+(-z_1)\big)+\big(z_2+(-z_2)\big)\boldsymbol{i}=0+0\boldsymbol{i}=0$$

(5) Multiplication is **commutative**: $\boxed{z \cdot w = w \cdot z}$

Proof:
$$\begin{aligned} z \cdot w &= (z_1 + z_2\boldsymbol{i}) \cdot (w_1 + w_2\boldsymbol{i}) \\ &= (z_1 \cdot w_1 - z_2 \cdot w_2) + (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i} \\ &= (w_1 \cdot z_1 - w_2 \cdot z_2) + (w_1 \cdot z_2 + w_2 \cdot z_1)\boldsymbol{i} \\ &= (w_1 + w_2\boldsymbol{i}) \cdot (z_1 + z_2\boldsymbol{i}) \\ &= w \cdot z \end{aligned}$$
$\square$

(6) Multiplication is **associative**: $\boxed{z \cdot (w \cdot u) = (z \cdot w) \cdot u}$

Proof: First we compute the left-hand side
$$\begin{aligned} z \cdot (w \cdot u) &= (z_1 + z_2\boldsymbol{i}) \cdot \Big((w_1 + w_2\boldsymbol{i}) \cdot (u_1 + u_2\boldsymbol{i})\Big) \\ &= (z_1 + z_2\boldsymbol{i}) \cdot \Big((w_1 \cdot u_1 - w_2 \cdot u_2) + (w_1 \cdot u_2 + w_2 \cdot u_1)\boldsymbol{i}\Big) \\ &= z_1(w_1 \cdot u_1 - w_2 \cdot u_2) - z_2(w_1 \cdot u_2 + w_2 \cdot u_1) \\ &\qquad + \Big(z_1(w_1 \cdot u_2 + w_2 \cdot u_1) + z_2(w_1 \cdot u_1 - w_2 \cdot u_2)\Big)\boldsymbol{i} \end{aligned}$$

Next compute the right-hand side
$$\begin{aligned} (z \cdot w) \cdot u &= \Big((z_1 + z_2\boldsymbol{i}) \cdot (w_1 + w_2\boldsymbol{i})\Big) \cdot (u_1 + u_2\boldsymbol{i}) \\ &= \Big((z_1 \cdot w_1 - z_2 \cdot w_2) + (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i}\Big) \cdot (u_1 + u_2\boldsymbol{i}) \\ &= \Big((z_1 \cdot w_1 - z_2 \cdot w_2)u_1 - (z_1 \cdot w_2 + z_2 \cdot w_1)u_2\Big) \\ &\qquad + \Big((z_1 \cdot w_1 - z_2 \cdot w_2)u_2 + (z_1 \cdot w_2 + z_2 \cdot w_1)u_1\Big)\boldsymbol{i} \end{aligned}$$

comparing terms we find: $z \cdot (w \cdot u) = (z \cdot w) \cdot u$ $\square$

(7) Axiom (7) says: there exists a $1 \in \mathbb{C}$, $1 \neq 0$, such that $\boxed{1 \cdot z = z}$ for all $z \in \mathbb{C}$

Clearly $1 = 1 + 0\boldsymbol{i}$ satisfies this.

* $1 \neq 0$ since $1 = 1 + 0\boldsymbol{i}$ and $0 = 0 + 0\boldsymbol{i}$
* $1 \cdot z = (1 + 0\boldsymbol{i}) \cdot (z_1 + z_2\boldsymbol{i}) = (1 \cdot z_1 - 0 \cdot z_2) + (1 \cdot z_2 + 0 \cdot z_1)\boldsymbol{i} = z_1 + z_2\boldsymbol{i} = z$

(8) Each $z \in \mathbb{C}$, provided $z \neq 0$, has a (multiplicative) inverse: $\boxed{z \cdot \left(z^{-1}\right) = 1}$

If $z = z_1 + z_2\boldsymbol{i}$ then $\boxed{z^{-1} = \dfrac{z_1}{z_1^2 + z_2^2} - \dfrac{z_2}{z_1^2 + z_2^2}\boldsymbol{i}}$

Proof: $(z_1 + z_2\boldsymbol{i}) \cdot \left(\dfrac{z_1}{z_1^2 + z_2^2} - \dfrac{z_2}{z_1^2 + z_2^2}\boldsymbol{i}\right)$

$$= \left(z_1 \cdot \frac{z_1}{z_1^2 + z_2^2} + z_2 \cdot \frac{z_2}{z_1^2 + z_2^2}\right) + \left(z_1 \cdot \frac{z_2}{z_1^2 + z_2^2} - z_2 \cdot \frac{z_1}{z_1^2 + z_2^2}\right)\boldsymbol{i} = 1 + 0\boldsymbol{i} = 1 \qquad \square$$

(9) Multiplication distributes over addition: $\boxed{z \cdot (w + u) = z \cdot w + z \cdot u}$

Proof: First we compute the left-hand side

$$\begin{aligned} z \cdot (w + u) &= (z_1 + z_2\boldsymbol{i}) \cdot \big((w_1 + w_2\boldsymbol{i}) + (u_1 + u_2\boldsymbol{i})\big) \\ &= (z_1 + z_2\boldsymbol{i}) \cdot \big((w_1 + u_1) + (w_2 + u_2)\boldsymbol{i}\big) \\ &= \big((z_1 \cdot (w_1 + u_1) - z_2 \cdot (w_2 + u_2)\big) + \big((z_1 \cdot (w_2 + u_2) + z_2 \cdot (w_1 + u_1)\big)\boldsymbol{i} \end{aligned}$$

Next we compute the right-hand side

$$\begin{aligned} z \cdot w + z \cdot u &= (z_1 + z_2\boldsymbol{i}) \cdot (w_1 + w_2\boldsymbol{i}) + (z_1 + z_2\boldsymbol{i}) \cdot (u_1 + u_2\boldsymbol{i}) \\ &= \big((z_1 \cdot w_1 - z_2 \cdot w_2) + (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i}\big) \\ &\qquad + \big((z_1 \cdot u_1 - z_2 \cdot u_2) + (z_1 \cdot u_2 + z_2 \cdot u_1)\big)\boldsymbol{i} \\ &= \big((z_1 \cdot w_1 - z_2 \cdot w_2) + (z_1 \cdot u_1 - z_2 \cdot u_2)\big) \\ &\qquad + \big((z_1 \cdot w_2 + z_2 \cdot w_1) + (z_1 \cdot u_2 + z_2 \cdot u_1)\big)\boldsymbol{i} \end{aligned}$$

Comparing both sides we find that $z \cdot (w + u) = z \cdot w + z \cdot u$.

Uniqueness

Uniqueness of 0 and 1 are usually taken for granted. We are so used to it in our familiar number field $\mathbb{R}$. But $\mathbb{C}$ is a new world. For example the equation $x^4 = 1$ has only two solutions in $\mathbb{R}$, namely $1, -1$, but $z^4 = 1$ has four solutions in $\mathbb{C}$: $1, -1, \boldsymbol{i}, -\boldsymbol{i}$. So everything that we know to be true over $\mathbb{R}$, we have to check to see if it is still true over $\mathbb{C}$.

0 is unique

Proof: Field axiom 3 states that there is a zero 0 such that $z + 0 = z$ for all $z \in \mathbb{C}$. Suppose there are two zeros, 0_1 and 0_2, with this property, then

$$\left.\begin{array}{lll} \forall z \in \mathbb{C}: \ z + 0_1 = z & \Rightarrow & 0_2 + 0_1 = 0_2 \\ \forall z \in \mathbb{C}: \ z + 0_2 = z & \Rightarrow & 0_1 + 0_2 = 0_2 \end{array}\right\} \Rightarrow \quad 0_1 = 0_2 \qquad \square$$

1 is unique

Proof: Field axiom 7 states that there is an identity 1 such that $1 \cdot z = z$ for all $z \in \mathbb{C}$. Suppose there are two identities, 1_1 and 1_2, with this property, then

$$\left.\begin{array}{lll} \forall z \in \mathbb{C}: \ 1_1 \cdot z = z & \Rightarrow & 1_1 \cdot 1_2 = 1_2 \\ \forall z \in \mathbb{C}: \ 1_2 \cdot z = z & \Rightarrow & 1_2 \cdot 1_1 = 1_2 \end{array}\right\} \Rightarrow \quad 1_1 = 1_2 \qquad \square$$

The cancellation laws: **(a)** If $z + u = w + u$ then $z = w$.

(b) If $z \cdot u = w \cdot u$, and $u \neq 0$, then $z = w$.

Proof:

(a) Field axiom 4 states that for any $u \in \mathbb{C}$ there exists an opposite $-u$, with $u + (-u) = 0$, hence

$$\begin{aligned} z + u = w + u \quad &\Rightarrow \quad (z + u) + (-u) = (w + u) + (-u) \\ &\Rightarrow \quad z + \big(u + (-u)\big) = w + \big(u) + (-u)\big) \\ &\Rightarrow \quad z + 0 = w + 0 \\ &\Rightarrow \quad z = w \end{aligned}$$

(b) Field axiom 7 states that for any $u \in \mathbb{C}$, $u \neq 0$ there exists an inverse u^{-1}, with $u \cdot (u^{-1}) = 1$, hence

$$\begin{aligned} z \cdot u = w \cdot u \quad &\Rightarrow \quad (z \cdot u) \cdot (u^{-1}) = (w \cdot u) \cdot (u^{-1}) \\ &\Rightarrow \quad z \cdot \big(u \cdot (u^{-1})\big) = w \cdot \big(u \cdot (u^{-1})\big) \\ &\Rightarrow \quad z \cdot 1 = w \cdot 1 \\ &\Rightarrow \quad z = w \end{aligned}$$

$\boxed{\text{Opposites are unique}}$

Proof: Field axiom 4 states that for each $z \in \mathbb{C}$ there is an opposite $-z$ such that $z + (-z) = 0$. Suppose there are two opposites, $-z_1$ and $-z_2$, with this property, then

$$\left.\begin{aligned} z + (-z_1) &= 0 \\ z + (-z_2) &= 0 \end{aligned}\right\} \quad \Rightarrow \quad z + (-z_1) = z + (-z_2) \quad \Rightarrow \quad -z_1 = -z_2 \qquad \square$$

by the first cancellation law.

$\boxed{\text{Inverses are unique}}$

Proof: Field axiom 7 states that for each $z \in \mathbb{C}$, $z \neq 0$, there is an inverse z^{-1} such that $z \cdot (z^{-1}) = 1$. Suppose there are two inverses, z_1^{-1} and z_2^{-1}, with this property, then

$$\left.\begin{aligned} z \cdot (z_1^{-1}) &= 1 \\ z \cdot (z_2^{-1}) &= 1 \end{aligned}\right\} \quad \Rightarrow \quad z \cdot (z_1^{-1}) = z \cdot (-z_2^{-1}) \quad \Rightarrow \quad z_1^{-1} = z_2^{-1} \qquad \square$$

by the second cancellation law.

$\boxed{0 \cdot z = 0}$ for $\forall z \in \mathbb{C}$

Proof: $0 \cdot z = (0 + 0\boldsymbol{i}) \cdot (a + b\boldsymbol{i}) = (0 \cdot a - 0 \cdot b) + (0 \cdot b + 0 \cdot a)\boldsymbol{i} = 0 + 0\boldsymbol{i} = 0$ $\square$

Next we'll prove all parts of the theorem:

(1) $\boxed{\overline{z + w} = \overline{z} + \overline{w}}$

Proof:
$$\begin{aligned} \overline{z + w} &= \overline{(z_1 + z_2\boldsymbol{i}) + (w_1 + w_2\boldsymbol{i})} \\ &= \overline{(z_1 + w_1) + (z_2 + w_2)\boldsymbol{i}} \\ &= (z_1 + w_1) - (z_2 + w_2)\boldsymbol{i} \\ &= (z_1 - z_2\boldsymbol{i}) + (w_1 - w_2\boldsymbol{i}) \\ &= \overline{(z_1 + z_2\boldsymbol{i})} + \overline{(w_1 + w_2\boldsymbol{i})} \; = \; \overline{z} + \overline{w} \qquad \square \end{aligned}$$

The proof of $\overline{z - w} = \overline{z} - \overline{w}$ goes in a similar fashion.

(2) $\boxed{\overline{z \cdot w} = \overline{z} \cdot \overline{w}}$ and $\boxed{\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}}$

Proofs:
$$\begin{aligned}\overline{z \cdot w} &= \overline{(z_1 + z_2\boldsymbol{i}) \cdot (w_1 + w_2\boldsymbol{i})} \\ &= \overline{(z_1 \cdot w_1 - z_2 \cdot w_2) + (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i}} \\ &= (z_1 \cdot w_1 - z_2 \cdot w_2) - (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i}\end{aligned}$$

and
$$\begin{aligned}\overline{z} \cdot \overline{w} &= (z_1 - z_2\boldsymbol{i}) \cdot (w_1 - w_2\boldsymbol{i}) \\ &= (z_1 \cdot w_1 + z_2 \cdot w_2) - (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i}\end{aligned}$$

Hence $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ $\square$

$$\begin{aligned}\overline{\left(\frac{z}{w}\right)} &= \overline{\left(z \cdot \frac{1}{w}\right)} \\ &= \overline{\left((z_1 + z_2\boldsymbol{i}) \cdot \frac{1}{w_1 + w_2\boldsymbol{i}}\right)} \\ &= \overline{\left((z_1 + z_2\boldsymbol{i}) \cdot \left(\frac{w_1}{w_1^2 + w_2^2} - \frac{w_2}{w_1^2 + w_2^2}\boldsymbol{i}\right)\right)} \\ &= \overline{\left(\left(\frac{z_1 \cdot w_1}{w_1^2 + w_2^2} + \frac{z_2 \cdot w_1}{w_1^2 + w_2^2}\right) - \left(\frac{z_1 \cdot w_2}{w_1^2 + w_2^2} - \frac{z_2 \cdot w_1}{w_1^2 + w_2^2}\right)\boldsymbol{i}\right)} \\ &= \left(\frac{z_1 \cdot w_1}{w_1^2 + w_2^2} + \frac{z_2 \cdot w_1}{w_1^2 + w_2^2}\right) + \left(\frac{z_1 \cdot w_2}{w_1^2 + w_2^2} - \frac{z_2 \cdot w_1}{w_1^2 + w_2^2}\right)\boldsymbol{i}\end{aligned}$$

and
$$\begin{aligned}\frac{\overline{z}}{\overline{w}} &= \overline{z} \cdot \frac{1}{\overline{w}} \\ &= \overline{z_1 + z_2\boldsymbol{i}} \cdot \frac{1}{\overline{w_1 + w_2\boldsymbol{i}}} \\ &= z_1 - z_2\boldsymbol{i} \cdot \frac{1}{w_1 - w_2\boldsymbol{i}} \\ &= (z_1 - z_2\boldsymbol{i}) \cdot \left(\frac{w_1}{w_1^2 + w_2^2} + \frac{w_2}{w_1^2 + w_2^2}\boldsymbol{i}\right) \\ &= \left(\frac{z_1 \cdot w_1}{w_1^2 + w_2^2} + \frac{z_2 \cdot w_1}{w_1^2 + w_2^2}\right) + \left(\frac{z_1 \cdot w_2}{w_1^2 + w_2^2} - \frac{z_2 \cdot w_1}{w_1^2 + w_2^2}\right)\boldsymbol{i}\end{aligned}$$

Hence $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$ $\square$

(3) $\boxed{\overline{\overline{z}} = z}$

Proof: $\overline{\overline{z}} = \overline{\overline{(z_1 + z_2\boldsymbol{i})}} = \overline{(z_1 - z_2\boldsymbol{i})} = z_1 + z_2\boldsymbol{i} = z$ $\square$

(4) $\boxed{\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}}$ and $\boxed{\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2\boldsymbol{i}}}$

Proof: $\mathrm{Re}(z) = \mathrm{Re}(z_1 + z_2\boldsymbol{i}) = z_1$ and $\mathrm{Im}(z) = \mathrm{Im}(z_1 + z_2\boldsymbol{i}) = z_2$ and

$$\left.\begin{aligned} z + \overline{z} &= (z_1 + z_2\boldsymbol{i}) + \overline{(z_1 + z_2\boldsymbol{i})} = (z_1 + z_2\boldsymbol{i}) + (z_1 - z_2\boldsymbol{i}) = 2z_1 \\ z - \overline{z} &= (z_1 + z_2\boldsymbol{i}) - \overline{(z_1 + z_2\boldsymbol{i})} = (z_1 + z_2\boldsymbol{i}) - (z_1 - z_2\boldsymbol{i}) = 2z_2\boldsymbol{i} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \quad \frac{z + \overline{z}}{2} = z_1 = \mathrm{Re}(z) \quad \text{and} \quad \frac{z - \overline{z}}{2\boldsymbol{i}} = z_2 = \mathrm{Im}(z)$$ $\square$

(5) $\boxed{z \in \mathbb{R} \iff z = \overline{z}}$ and $\boxed{z \in \mathbb{R} \iff \mathrm{Im}(z) = 0}$

Proof: Let $z = x + y\boldsymbol{i}$, i.e. $\mathrm{Im}(z) = y$, then

$$z \in \mathbb{R} \quad \iff \quad \mathrm{Im}(z) = y = 0 \quad \iff \quad z = \overline{z}$$ $\square$

(6) $\boxed{|\,\overline{z}\,| = |\,z\,|}$

Proof: $|\,\overline{z}\,| = \left|\,\overline{z_1 + z_2\boldsymbol{i}}\,\right| = |\,z_1 - z_2\boldsymbol{i}\,| = \sqrt{z_1^2 + (-z_2)^2} = \sqrt{z_1^2 + z_2^2} = |\,z\,|$ $\square$

(7) $\boxed{z \cdot \overline{z} = |\,z\,|^2}$

Proof:
$$\begin{aligned} z \cdot \overline{z} &= (z_1 + z_2\boldsymbol{i}) \cdot (\,\overline{z_1 + z_2\boldsymbol{i}}\,) \\ &= (z_1 + z_2\boldsymbol{i}) \cdot (z_1 - z_2\boldsymbol{i}) \\ &= z_1^2 + z_2^2 = |\,z\,|^2 \end{aligned}$$ $\square$

(8) $\boxed{\dfrac{1}{z} = \dfrac{\overline{z}}{|\,z\,|^2}}$ and $\boxed{\dfrac{z}{w} = \dfrac{z \cdot \overline{w}}{|\,w\,|^2}}$

Proof: Since $z \cdot \overline{z} = |\,z\,|^2$ it follows that $\dfrac{1}{z} = \dfrac{\overline{z}}{|\,z\,|^2}$ $\square$

Using this we also find that: $\dfrac{z}{w} = z \cdot \dfrac{1}{w} = z \cdot \dfrac{\overline{w}}{|\,w\,|^2} = \dfrac{z \cdot \overline{w}}{|\,w\,|^2}$ $\square$

(9) $\boxed{|\,z \cdot w\,| = |\,z\,| \cdot |\,w\,|}$ and $\boxed{\left|\dfrac{z}{w}\right| = \dfrac{|\,z\,|}{|\,w\,|}}$

Proof:
$$
\begin{aligned}
|\,z \cdot w\,| &= |\,(z_1 + z_2\boldsymbol{i}) \cdot (w_1 + w_2\boldsymbol{i})\,| \\
&= |\,(z_1 \cdot w_1 - z_2 \cdot w_2) + (z_1 \cdot w_2 + z_2 \cdot w_1)\boldsymbol{i}\,| \\
&= \sqrt{(z_1 \cdot w_1 - z_2 \cdot w_2)^2 + (z_1 \cdot w_2 + z_2 \cdot w_1)^2} \\
&= \sqrt{z_1^2 \cdot w_1^2 + z_2^2 \cdot w_2^2 + z_1^2 \cdot w_2^2 + z_2^2 \cdot w_1^2} \\
&= \sqrt{(z_1^2 + z_2^2) \cdot (w_1^2 + w_2^2)} \\
&= \sqrt{z_1^2 + z_2^2} \cdot \sqrt{w_1^2 + w_2^2} \\
&= |\,z_1 + z_2\boldsymbol{i}\,| \cdot |\,w_1 + w_2\boldsymbol{i}\,| \;=\; |\,z\,| \cdot |\,w\,| \qquad \square
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{z}{w}\right| &= \left|z \cdot \frac{1}{w}\right| \\
&= |\,z\,| \cdot \left|\frac{1}{w}\right| \\
&= |\,z\,| \cdot \left|\frac{w_1}{w_1^2 + w_2^2} - \frac{w_2}{w_1^2 + w_2^2}\boldsymbol{i}\right| \\
&= |\,z\,| \cdot \sqrt{\frac{w_1^2}{(w_1^2 + w_2^2)^2} + \frac{w_2^2}{(w_1^2 + w_2^2)^2}} \\
&= |\,z\,| \cdot \sqrt{\frac{1}{w_1^2 + w_2^2}} \\
&= |\,z\,| \cdot \frac{1}{\sqrt{w_1^2 + w_2^2}} = \frac{|\,z\,|}{|\,w\,|} \qquad \square
\end{aligned}
$$