16. Linear Transformations

Let V and W be two vector spaces over the same field $\mathbb{F}$. A function $T : V \quad W$, is called a **linear transformation** if it satisfies the following two conditions for all $\vec{x},\ \vec{y} \in V$ and $t \in \mathbb{F}$

$$\boxed{\begin{array}{ll} \textbf{(1)} & T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \\ \textbf{(2)} & T(t\cdot\vec{x}) = t\cdot T(\vec{x}) \end{array}}$$

Strictly speaking the vector addition and scalar multiplication in the two vector spaces may be entirely different, and we might use different symbols for them

$$\left(V,\ \oplus,\ \odot,\ \mathbb{F}\right) \text{ and } \left(W,\ \boxplus,\ \boxdot,\ \mathbb{F}\right)$$

Properties (1) and (2) would then become

$$T(\vec{x}\oplus\vec{y}) = T(\vec{x}) \boxplus T(\vec{y})$$

and

$$T(t\odot\vec{x}) = t \boxdot T(\vec{x})$$

But usually the operations are clear and we conveniently use the same symbols for them. The context will provide enough information to know which operation is meant: e.g.

If $T: P_2(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$ then in $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ the additions refer to

an addition of two polynomials   an addition of two matrices

and in $T(t\cdot\vec{x}) = t\cdot T(\vec{x})$ the scalar multiplications refers to

a scalar t times a polynomial   a scalar t times a matrix.

The two properties can be combined into one: $\boxed{T(t\,\vec{x}+\vec{y}) = t\,T(\vec{x}) + T(\vec{y})}$

or

$$\boxed{T(\alpha\,\vec{x}+\beta\vec{y}) = \alpha\,T(\vec{x}) + \beta\,T(\vec{y})}$$

In fact we will often use this property on general linear combinations:

$$\boxed{T(a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + \cdots + a_n\vec{b}_n) = a_1T(\vec{b}_1) + a_2T(\vec{b}_2) + a_3T(\vec{b}_3) + \cdots + a_nT(\vec{b}_n)}$$

Or in short: $\boxed{T\left(\sum_{i=1}^{n} a_i\vec{b}_i\right) = \sum_{i=1}^{n} a_i T(\vec{b}_i)}$

Examples:

- $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(at^2+bt+c) = \begin{bmatrix} a+6b & 2b-c \\ c-2b & a+3c \end{bmatrix}$ is linear.

- $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-3b \\ b+c \\ 3c-a \end{bmatrix}$ is linear.

- $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \int_0^1 \left(at^3+(b-c)t+d\right)dt$ is linear.

- $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(at^3+bt^2+ct+d) = 3at^2+2bt^2+c$, or in calculus terms $T\left(p(t)\right) = \dfrac{d\,p(t)}{dt}$, is linear.

- $\varphi_\beta : V \to \mathbb{R}^n$ defined by $\varphi_\beta(\vec{v}) = [\,\vec{v}\,]_\beta$ is linear, where β is a basis of the n-dimensional vector space V over the field $\mathbb{R}$.

Some properties of linear transformations

(1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(2) $T(t\,\vec{x}) = t\,T(\vec{x})$

(3) $T\left(\sum_{i=1}^{n} a_i\,\vec{b}_i\right) = \sum_{i=1}^{n} a_i\,T(\vec{b}_i)$

(4) $T(\vec{0}_V) = \vec{0}_W$

(5) $T(-\vec{v}) = -T(\vec{v})$

(6) $T(\vec{v}-\vec{w}) = T(\vec{v}) - T(\vec{w})$

Note that if any one of these properties does **not** hold true then T can**not** be linear.

Example: $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ab \\ b+1 \\ a+b+c \end{bmatrix}$ cannot be linear since

e.g. $T\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ so that $T(\vec{0}_V) \neq \vec{0}_W$ (property (4) fails)

or $T\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $T\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -18 \\ -2 \\ 0 \end{bmatrix} \neq 3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ so that

$T\left(3\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}\right) \neq 3\,T\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ (property (2) fails)