An example of change of coordinate matrices in a subspace of $P_3(\mathbb{C})$

Let $\begin{cases} \vec{p}_1 = (1+2i)t^3 + (-1+2i)t^2 + (2+2i)t + i \\ \vec{p}_2 = (2-i)t^3 + (1+2i)t^2 + (1-i)t + (1+2i) \\ \vec{p}_3 = i\,t^3 + (1+i)t^2 + t + i \end{cases}$ so that when $S = \{t^3, t^2, t, 1\}$:

$$[\vec{p}_1]_S = \begin{bmatrix} 1+2i \\ -1+2i \\ 2+2i \\ i \end{bmatrix}, \quad [\vec{p}_2]_S = \begin{bmatrix} 2-i \\ 1+2i \\ 1-i \\ 1+2i \end{bmatrix}, \quad [\vec{p}_3]_S = \begin{bmatrix} i \\ 1+i \\ 1 \\ i \end{bmatrix}.$$

We define $W = \operatorname{span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} \sqsubseteq P_3(\mathbb{C})$.

Note that W is a 3 dimensional **subspace** of $P_3(\mathbb{C})$. This is easily checked with a rref:

$$\operatorname{rref}\begin{bmatrix} 1+2i & 2-i & i \\ -1+2i & 1+2i & 1+i \\ 2+2i & 1-i & 1 \\ i & 1+2i & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We checked their independence by expressing them with respect to the standard basis of $P_3(\mathbb{C})$!

Although $S = \{t^3, t^2, t, 1\}$ is the standard basis of $P_3(\mathbb{C})$, it is NOT a basis of W. In fact **not a single** elements of S is even in W! (Check).

S is the standard basis of the **ambient** space $P_3(\mathbb{C})$ that W lives in.

A standard basis of W would be $\sigma = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$, since this is a linearly independent set that spans W, by definition!

Let $\vec{w} = (2+10i)t^3 + (-3+5i)t^2 + (5+5i)t + (-4+6i) \in P_3(\mathbb{C})$

We'll answer the following questions

(1) Show that $\vec{w} \in W$.

(2) Compute $[\vec{w}]_S$ and $[\vec{w}]_\sigma$.

Before we proceed let's define two more bases of W:

Let $\begin{cases} \vec{\beta}_1 = t^3 + (1+i)t^2 + (1-i)t + i \\ \vec{\beta}_2 = (1+i)t^3 + t \\ \vec{\beta}_3 = i t^2 + i t + (1+i) \end{cases}$ and $\begin{cases} \vec{\alpha}_1 = i t^3 + (-1+2i)t^2 + (1+2i)t + i \\ \vec{\alpha}_2 = -i t^3 + (1+2i)t^2 + (1+2i) \\ \vec{\alpha}_3 = (-1+i)t^3 + i t^2 + 2i t + (1+i) \end{cases}$

so that

$$[\vec{\beta}_1]_S = \begin{bmatrix} 1 \\ 1+i \\ 1-i \\ i \end{bmatrix}, \quad [\vec{\beta}_2]_S = \begin{bmatrix} 1+i \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\vec{\beta}_3]_S = \begin{bmatrix} 0 \\ i \\ i \\ 1+i \end{bmatrix}$$

and

$$[\vec{\alpha}_1]_S = \begin{bmatrix} i \\ -1+2i \\ 1+2i \\ i \end{bmatrix}, \quad [\vec{\alpha}_2]_S = \begin{bmatrix} -i \\ 1+2i \\ 0 \\ 1+2i \end{bmatrix}, \quad [\vec{\alpha}_3]_S = \begin{bmatrix} -1+i \\ i \\ 2i \\ 1+i \end{bmatrix}$$

(3) Check that both $\beta = \{ \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \}$ and $\alpha = \{ \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \}$ are bases of W.

(4) Compute $[\vec{w}]_\beta$ and $[\vec{w}]_\alpha$ [for the same $\vec{w}$ as defined earlier]

(5) Compute ${}_\alpha C_\beta$.

(6) Compute ${}_\sigma C_\beta$ and ${}_\sigma C_\alpha$, and check that ${}_\sigma C_\alpha^{-1} \cdot {}_\sigma C_\beta = {}_\alpha C_\beta$.

(7) Verify that $[\vec{w}]_\alpha = {}_\alpha C_\beta \cdot [\vec{w}]_\beta$.

Solutions:

(1) To show that $\vec{w} \in W$: $\quad \text{rref} \begin{bmatrix} 1+2i & 2-i & i & 2+10i \\ -1+2i & 1+2i & 1+i & -3+5i \\ 2+2i & 1-i & 1 & 5+5i \\ i & 1+2i & i & -4+6i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1+i \\ 0 & 1 & 0 & 2i \\ 0 & 0 & 1 & 3-i \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So that $\vec{w} = (1+i)\,\vec{p}_1, + (2i)\,\vec{p}_2, + (3-i)\,\vec{p}_3$ and hence in W.

(2) To compute $[\vec{w}]_S$ we see $\vec{w}$ as an element of $P_3(\mathbb{C})$ and use $S = \left\{ t^3, t^2, t, 1 \right\}$:

$$[\vec{w}]_S = \begin{bmatrix} 2+10i \\ -3+5i \\ 5+5i \\ -4+6i \end{bmatrix}. \qquad \text{We already computed } [\vec{w}]_\sigma = \begin{bmatrix} 1+i \\ 2i \\ 3-i \end{bmatrix} \text{ in the rref of } \textbf{(1)}$$

Don't be alarmed that the expressions have a different number of coordinates. It is all a matter of perspective: expressing $\vec{w}$ in terms of the standard basis of the 4 dimensional ambient space $P_3(\mathbb{C})$, or with respect to its own internal reference frame σ. Of course the four dimensional expression doesn't make any sense to the 3 dimensional inhabitants of W, who do not see their world as 4 dimensional, who may not even be aware of the fact that they appear to us as embedded as a subspace of a 4 dimensional world. This is just our perspective of looking in on the world within a world: $W \sqsubseteq P_3(\mathbb{C})$. But this perspective does allow us to compute using the standard basis of $P_3(\mathbb{C})$ when needed.

(3) To check that $\beta = \left\{ \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \right\}$ is a basis first we verify that each element of β is indeed in W, and then that they are linearly independent.

We do the same for $\alpha = \left\{ \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \right\}$.

Infact **one** rref checks that both $\beta = \left\{ \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \right\}$ and $\alpha = \left\{ \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \right\}$ are in W:

$$\begin{array}{ccccccccc} [\vec{p}_1]_S & [\vec{p}_2]_S & [\vec{p}_3]_S & [\vec{\beta}_1]_S & [\vec{\beta}_2]_S & [\vec{\beta}_3]_S & [\vec{\alpha}_1]_S & [\vec{\alpha}_2]_S & [\vec{\alpha}_3]_S \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\text{rref} \begin{bmatrix} 1+2i & 2-i & i & 1 & 1+i & 0 & i & -i & -1+i \\ -1+2i & 1+2i & 1+i & 1+i & 0 & i & -1+2i & 1+2i & i \\ 2+2i & 1-i & 1 & 1-i & 1 & i & 1+2i & 0 & 2i \\ i & 1+2i & i & i & 0 & 1+i & i & 1+2i & 1+i \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4}(-1-i) & \frac{1}{4}(1-i) & \frac{1}{2}(1+i) & \frac{1}{4}(3+i) & \frac{1}{2}i & \frac{1}{4}(3+3i) \\ 0 & 1 & 0 & \frac{1}{4}(1+i) & \frac{1}{4}(-1+i) & \frac{1}{2}(1-i) & \frac{1}{4}(1-i) & \frac{1}{2}(2-i) & \frac{1}{4}(1-3i) \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2}i & 0 & \frac{1}{2}i & \frac{1}{2}(1+i) & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And two more rrefs check independence:

$$\text{rref}\begin{bmatrix} 1 & 1+i & 0 \\ 1+i & 0 & i \\ 1-i & 1 & i \\ i & 0 & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{rref}\begin{bmatrix} i & -i & -1+i \\ -1+2i & 1+2i & i \\ 1+2i & 0 & 2i \\ i & 1+2i & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So that indeed both α and β are bases of W.

(4) Compute $[\vec{w}]_\beta$ and $[\vec{w}]_\alpha$ [for the same $\vec{w}$ as defined earlier]

$$[\vec{w}]_\beta = \begin{bmatrix} 2+2i \\ 4+4i \\ 1+3i \end{bmatrix} \qquad \text{rref}\begin{bmatrix} 1 & 1+i & 0 & 2+10i \\ 1+i & 0 & i & -3+5i \\ 1-i & 1 & i & 5+5i \\ i & 0 & 1+i & -4+6i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2+2i \\ 0 & 1 & 0 & 4+4i \\ 0 & 0 & 1 & 1+3i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\vec{w}]_\alpha = \begin{bmatrix} -7-7i \\ -5+9i \\ 13+i \end{bmatrix} \qquad \text{rref}\begin{bmatrix} i & -i & -1+i & 2+10i \\ -1+2i & 1+2i & i & -3+5i \\ 1+2i & 0 & 2i & 5+5i \\ i & 1+2i & 1+i & -4+6i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -7-7i \\ 0 & 1 & 0 & -5+9i \\ 0 & 0 & 1 & 13+i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(5) Compute ${}_\alpha C_\beta = \begin{bmatrix} -1-i & -1 & i \\ i & i & 1 \\ i & 1-i & -i \end{bmatrix}$.

$$\text{rref}\left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} i & -i & -1+i \\ -1+2i & 1+2i & i \\ 1+2i & 0 & 2i \\ i & 1+2i & 1+i \end{matrix}}_{[\alpha]_S} & \underbrace{\begin{matrix} 1 & 1+i & 0 \\ 1+i & 0 & i \\ 1-i & 1 & i \\ i & 0 & 1+i \end{matrix}}_{[\beta]_S} \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & -1-i & -1 & i \\ 0 & 1 & 0 & i & i & 1 \\ 0 & 0 & 1 & 1 & 1-i & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(6) Note that we already computed ${}_\sigma C_\beta$ and ${}_\sigma C_\alpha$ in the rref in **(3)** :

$${}_\sigma C_\beta = \frac{1}{4}\begin{bmatrix} -1-i & 1-i & 2+2i \\ 1+i & -1+i & 2-2i \\ 2 & -2i & 0 \end{bmatrix} \quad \text{and} \quad {}_\sigma C_\alpha = \frac{1}{4}\begin{bmatrix} 3+i & 2i & 3+3i \\ 1-i & 4-2i & 1-3i \\ 2i & 2+2i & 2 \end{bmatrix}$$

and ineed ${}_\sigma C_\alpha^{-1} \cdot {}_\sigma C_\beta = {}_\alpha C_\beta$.

$$\left(\frac{1}{4}\begin{bmatrix} 3+i & 2i & 3+3i \\ 1-i & 4-2i & 1-3i \\ 2i & 2+2i & 2 \end{bmatrix}\right)^{-1} \cdot \frac{1}{4}\begin{bmatrix} -1-i & 1-i & 2+2i \\ 1+i & -1+i & 2-2i \\ 2 & -2i & 0 \end{bmatrix} = \begin{bmatrix} -1-i & -1 & i \\ i & i & 1 \\ i & 1-i & -i \end{bmatrix} \quad \checkmark$$

(7) Verify that $[\vec{w}]_\alpha = {}_\alpha C_\beta \cdot [\vec{w}]_\beta$.

$${}_\alpha C_\beta \cdot [\vec{w}]_\beta = \begin{bmatrix} -1-i & -1 & i \\ i & i & 1 \\ i & 1-i & -i \end{bmatrix} \begin{bmatrix} 2+2i \\ 4+4i \\ 1+3i \end{bmatrix} = \begin{bmatrix} -7-7i \\ -5+9i \\ 13+i \end{bmatrix} = [\vec{w}]_\alpha \quad \checkmark$$