Linear Independence

In this chapter we will study the very important concepts of linear dependence and linear independence. We begin with some definitions.

Definition 1

A relation
$$\vec{u} = a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + \cdots + a_m \cdot v_m$$
where one vector $\vec{u}$ is written as a linear combination of some given set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_m\}$ is called a **dependency**.

Example 1

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = 3 \cdot \begin{bmatrix}1\\1\\2\\1\end{bmatrix} - 2 \cdot \begin{bmatrix}4\\2\\1\\3\end{bmatrix} + 0 \cdot \begin{bmatrix}3\\1\\-1\\2\end{bmatrix} + \begin{bmatrix}6\\3\\-1\\7\end{bmatrix} \quad \text{is a dependency.}$$

A dependency is just **one** way of expressing a vector in terms of other vectors. There may be many other ways, other dependencies, for example:

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}1\\1\\2\\1\end{bmatrix} + 0 \cdot \begin{bmatrix}4\\2\\1\\3\end{bmatrix} - 2 \cdot \begin{bmatrix}3\\1\\-1\\2\end{bmatrix} + \begin{bmatrix}6\\3\\-1\\7\end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = 2 \cdot \begin{bmatrix}1\\1\\2\\1\end{bmatrix} - \begin{bmatrix}4\\2\\1\\3\end{bmatrix} - \begin{bmatrix}3\\1\\-1\\2\end{bmatrix} + \begin{bmatrix}6\\3\\-1\\7\end{bmatrix}$$

Note that at times we cannot express a vector in terms of given vectors; for example there is no way to write $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ in terms of the vectors $\begin{bmatrix}1\\5\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\0\end{bmatrix}, \begin{bmatrix}1\\0\\4\\0\end{bmatrix}$

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \neq a \cdot \begin{bmatrix}1\\5\\1\\0\end{bmatrix} + b \cdot \begin{bmatrix}2\\-1\\3\\0\end{bmatrix} + c \cdot \begin{bmatrix}1\\0\\4\\0\end{bmatrix}$$

In fact there are no dependencies between these four vectors.

Definition 2

A set of vectors is called **linearly dependent** if (at least) **one** of the vectors can be written as a linear combination of the others.

Example 2 The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$ is linearly dependent since the third vector can be written as a linear combination of the first two

$$\begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

Of course there may be other dependencies, other combinations, e.g.

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix}$$

As soon as there is just one dependency among a set of vectors, we call the set linearly dependent. On the other hand if no dependencies exist then, we call the set linearly independent:

Definition 3

A set of vectors is called **linearly independent** if **none** of the vectors can be written as a linear combination of the others.

Note 1 Any set that contains the **zero vector** is linearly **de**pendent. After all it is easy to write down a dependency for the zero vector using any given set of vectors: $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_m$

Example 3 The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$ is linearly dependent since the zero vector can be written as a linear combination of the others in a trivial way as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Example 4 However if we look at the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$ we cannot find any dependencies. None of the vectors is dependent on the other two.

Even though these definitions make the concepts crystal clear, they are computationally not extremely useful. For example if we have a set of 108 vectors we would have to check if any of these 108 vectors can be written as a linear combination of the others. So we might have to do as many as 108 computations, i.e. 108 row reductions. In case of example 4 we would have to do 3 checks, i.e. 3 row reductions.

To wit:

$$\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \neq a \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{since} \quad \text{rref} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} \neq a \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{since} \quad \text{rref} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 3 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \neq a \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} \quad \text{since} \quad \text{rref} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

With 108 vectors we would certainly like to avoid all this work. Fortunately we can do just one computation to resolve the issue.

Theorem 1

A set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is **linearly independent** iff

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 + \cdots + a_n \cdot \vec{v}_n = \vec{0}$$

implies that

$$a_1 = a_2 = a_3 = \cdots = a_n = 0$$

Note 2 Clearly if all $a_1 = a_2 = a_3 = \cdots = a_n = 0$ then

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 + \cdots + a_n \cdot \vec{v}_n = \vec{0}$$

We'll refer to this as the **trivial** solution, the trivial way to write the zero vector as a linear combination of the vectors $\vec{v}_1,\, \vec{v}_2, \cdots, \vec{v}_n$

So the theorem boils down to the question: Can we write the zero vector as a linear combination of the vectors $\vec{v}_1,\, \vec{v}_2, \cdots, \vec{v}_n$ in a non-trivial way?

(1) If the zero vector can **only** be written as a linear combination of the vectors in the **trivial** way, then the vectors have to be **linearly independent**.

(2) If there is a **non-trivial** way to write the zero vector we automatically have a dependency, and hence the set of vectors is dependent.

Example 5 Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 4 \\ 9 \end{bmatrix} \right\}$ linearly independent?

No: Here is a dependency $\begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Hence we have a nontrivial way of writing the zero vector:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ 5 \\ 4 \\ 9 \end{bmatrix}$$

Here is another nontrivial way of writing the zero vector:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 4 \\ 9 \end{bmatrix}$$

Hence we have another dependency (for example)

$$\begin{bmatrix} 4 \\ 5 \\ 4 \\ 9 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix}$$

Any dependency gives a non-trivial way to write the zero vector as a linear combination of these vectors, and vice versa, every nontrivial way to write the zero vector as a linear combination of these vectors gives a dependency.

This makes the proof of the theorem rather obvious.

Proof: Suppose there is a **non-trivial** way to write the zero vector as

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 + \cdots + a_n \cdot \vec{v}_n = \vec{0}$$

Let a_i be the non-zero coefficient with smallest index i [the "first" non-zero coefficient] then in fact

$$a_i \cdot \vec{v}_i + a_{i+1} \cdot \vec{v}_{i+1} + \cdots + a_n \cdot \vec{v}_n = \vec{0}$$

so that

$$\vec{v}_i = -\frac{a_{i+1}}{a_i} \cdot \vec{v}_{i+1} - \frac{a_{i+2}}{a_i} \cdot \vec{v}_{i+2} - \cdots - \frac{a_n}{a_i} \cdot \vec{v}_n$$

That means that the set is dependent, there is a dependency: $\vec{v}_i$ can be written as a linear combination of the other vectors.

And vice versa, if there is a dependency

$$\vec{v}_i = a_1 \cdot \vec{v}_1 + \cdots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \cdots + a_n \cdot \vec{v}_n$$

then there is a non trivial way to write the zero vector as a linear combination

$$a_1 \cdot \vec{v}_1 + \cdots + a_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i + a_{i+1}\vec{v}_{i+1} + \cdots + a_n \cdot \vec{v}_n = \vec{0}$$

Hence: There exists a dependency $\Leftrightarrow$ There is a non-trivial way to write the zero vector.

So that: No dependencies $\Leftrightarrow$ Only the trivial way to write zero vector.

A set is linearly independent (i.e. there are no dependencies) if and only if the only way to write the zero vector as a linear combination of the vectors $\vec{v}_1, \ \vec{v}_2, \ \cdots, \ \vec{v}_n$ is the trivial way. □

Example 6 We can check if the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$ is linearly independent by one row reduction:

$$\text{rref} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which solves the equation

$$a \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

so that by Theorem 1 the set is linearly independent, since there is only the trivial solution.

As we have seen row reduction gives us an easy way to find dependencies. It is *the* tool to analyze linear (in)dependence.

Example 7 Is $S = \left\{ \begin{bmatrix} 6\\3\\2\\5\\4\\1 \end{bmatrix}, \begin{bmatrix} 3\\5\\1\\6\\2\\4 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 2\\0\\2\\2\\0\\2 \end{bmatrix}, \begin{bmatrix} 5\\3\\1\\4\\4\\0 \end{bmatrix}, \begin{bmatrix} 2\\2\\4\\6\\5\\1 \end{bmatrix}, \begin{bmatrix} 0\\4\\5\\1\\2\\0 \end{bmatrix}, \begin{bmatrix} 1\\5\\3\\4\\5\\3 \end{bmatrix} \right\}$ a linearly independent set in $\mathbb{F}_7^6$?

No, since the rref7 $\begin{bmatrix} 6&3&1&2&5&2&0&1\\ 3&5&0&0&3&2&4&5\\ 2&1&1&2&1&4&5&3\\ 5&6&1&2&4&6&1&4\\ 4&2&0&0&4&5&2&5\\ 1&4&1&2&0&1&0&3 \end{bmatrix} = \begin{bmatrix} 1&4&0&0&1&3&0&1\\ 0&0&1&2&6&5&0&2\\ 0&0&0&0&0&0&1&4\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$ gives us many dependencies: e.g.

$$\begin{bmatrix} 3\\5\\1\\6\\2\\4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 6\\3\\2\\5\\4\\1 \end{bmatrix}, \quad \begin{bmatrix} 2\\2\\4\\6\\5\\1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 6\\3\\2\\5\\4\\1 \end{bmatrix} + 5 \cdot \begin{bmatrix} 1\\0\\1\\1\\0\\1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1\\5\\3\\4\\5\\3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 6\\3\\2\\5\\4\\1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1\\0\\1\\1\\0\\1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0\\4\\5\\1\\2\\0 \end{bmatrix}$$

The pivots in columns 1, 3 and 7 tell us that the corresponding columns in the first matrix are linearly **in**dependent:

The subset $\left\{ \begin{bmatrix} 6\\3\\2\\5\\4\\1 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\4\\5\\1\\2\\0 \end{bmatrix} \right\}$ is linearly independent.

Of course there are other linearly independent subsets of S: e.g. $\left\{ \begin{bmatrix} 3\\5\\1\\6\\2\\4 \end{bmatrix}, \begin{bmatrix} 2\\0\\2\\2\\0\\2 \end{bmatrix}, \begin{bmatrix} 1\\5\\3\\4\\5\\3 \end{bmatrix} \right\}$

[Check!]

Example 8 $S = \left\{ \begin{bmatrix} 1 \\ a \\ b \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a \\ a \\ b \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ b \\ b \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ 1 \\ b \\ a \end{bmatrix} \right\}$ is linearly **in**dependent in $\mathbb{F}_4^5$,

Since solving

$$w \cdot \begin{bmatrix} 1 \\ a \\ b \\ 1 \\ 0 \end{bmatrix} + x \cdot \begin{bmatrix} 0 \\ a \\ a \\ b \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ b \\ b \\ a \\ b \end{bmatrix} + z \cdot \begin{bmatrix} a \\ b \\ 1 \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

using

$$\operatorname{rref4} \begin{bmatrix} 1 & 0 & 1 & a & 0 \\ a & a & b & b & 0 \\ b & a & b & 1 & 0 \\ 1 & b & a & b & 0 \\ 0 & 1 & b & a & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

gives us only the trivial solution $\begin{cases} w = 0 \\ x = 0 \\ y = 0 \\ z = 0 \end{cases}$, and we then invoke Theorem 1.