Examples and Solutions: bases, subspaces, lin. indep, spanning, dimensions etc.

Example 1

The standard basis of $M_{2\times 2}(\mathbb{R})$ is $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

1. Is $\alpha = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

2. Is $\gamma = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

3. Is $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

4. Extend the set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ to a basis (call it $\tilde{\beta}$).

5. If $\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ what is $[\vec{v}]_{\tilde{\beta}}$?

6. Let $\vec{b}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, the second basis vector of $\tilde{\beta}$. What is $[\vec{b}_2]_{\tilde{\beta}}$?

Example 2

Let $W = \left\{ \begin{bmatrix} w+2x+y & w+x+2y+z \\ x-y-z & w+3x-z \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$

1. When we take $w=4,\ x=1,\ y=2$ and $z=-3$, what element of W do we get?

2. Is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in W$?

3. Is $\begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix} \in W$?

4. Is W all of $M_{2\times 2}(\mathbb{R})$?

5. Explain why $W = \left\{ w\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$.

6. Explain why $W = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\} \right)$.

7. Is W a subspace of $M_{2\times 2}(\mathbb{R})$? (i.e. $W \sqsubseteq M_{2\times 2}(\mathbb{R})$?)

8. Find a basis for W.

9. What is $\dim(W)$?

10. Let $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right\}$. Check that β is a basis for W.

11. Let $\vec{w} = \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix}$. Find $[\vec{w}]_\beta$.

12. Let $\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$. Check that γ is also a basis for W.

13. Let $\vec{w} = \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix}$. Find $[\vec{w}]_\gamma$.

Example 3

The standard basis of $P_2(\mathbb{F}_4)$ is $S = \{ t^2, t, 1 \}$

1. Is $\alpha = \{ t^2 + at + b, \quad at^2 + t + 1, \quad at^2 + 1, \quad at^2 + t + 1, \quad t^2 + t + 1 \}$ a basis of $P_2(\mathbb{F}_4)$?

2. If not, prune the set to get a basis for $P_2(\mathbb{F}_4)$.

3. Let $\beta = \{ t^2 + at + b, \quad at^2 + t + 1, \quad t^2 + t + 1 \}$. Check that β is a basis for $P_2(\mathbb{F}_4)$.

4. Let $\vec{v} = at^2 + bt + 1$. Find $[\vec{v}]_S$ and $[\vec{v}]_\beta$.

5. Let $\vec{w} = t^2 + bt + 1$. Find $[\vec{w}]_S$ and $[\vec{w}]_\beta$.

6. Is $\gamma = \{ t^2 + at + 1, \quad t^2 + b \}$ a basis of $P_2(\mathbb{F}_4)$?

7. If not extend it to obtain a basis for $P_2(\mathbb{F}_4)$. Call it δ.

8. Find $[\vec{v}]_\delta$ and $[\vec{w}]_\delta$ where $\vec{v} = at^2 + bt + 1$ and $\vec{w} = t^2 + bt + 1$.

Example 4

Let $W = \operatorname{span}\{ t^2 + t + a, \quad bt^2 + 1, \quad at^2 + t + b, \quad at \}$.

1. Is $t \in W$?

2. Is $at^2 + bt + b \in W$?

3. Is $at^2 + at + a \in W$?

4. Is $W \sqsubseteq P_2(\mathbb{F}_4)$?

5. What is $\dim(W)$?

6. Is $\beta = \{ t^2+t+a, \quad bt^2+1 \}$ a basis for W?

7. Is $\gamma = \{ t, \quad t^2+a \}$ a basis for W?

8. Let $\vec{v} = t^2+at+a$. Compute $[\vec{v}]_\beta$ and $[\vec{v}]_\gamma$.

9. Let $\vec{w} = at^2+bt+b$. Compute $[\vec{w}]_\beta$ and $[\vec{w}]_\gamma$

Example 5

Let $W = \left\{ \begin{bmatrix} v+2w+2x+y+2z \\ 2v+4w+4x+2y+4z \\ 3v+3x+6y+4z \\ v+5w+3y+3z \\ w+4x+3y+5z \end{bmatrix} : v,w,x,y,z \in \mathbb{F}_7 \right\}$

1. When we take $v=5$, $w=4$, $x=1$, $y=2$ and $z=3$, what element of W do we get?

2. Is $\begin{bmatrix} 6 \\ 5 \\ 2 \\ 0 \\ 5 \end{bmatrix} \in W$?

3. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \in W$?

4. Is W all of $\mathbb{F}_7^5$?

5. Explain why $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ 3 \\ 5 \end{bmatrix} \right)$.

6. Is W a subspace of $\mathbb{F}_7^5$? (i.e. $W \sqsubseteq \mathbb{F}_7^5$?)

7. Find a basis for W.

8. What is $\dim(W)$?

9. Let $\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$. Check that β is a basis of W.

10. let $\gamma = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$. Check that γ is a basis for W.

11. Let $\vec{v} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 0 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$ Compute $[\vec{v}]_\beta$, $[\vec{v}]_\gamma$, $[\vec{w}]_\beta$ and $[\vec{w}]_\gamma$

12. If $[\vec{w}]_\gamma = \begin{bmatrix} x \\ y \end{bmatrix}$ then what is $\vec{w}$?

13. In part **3** we found that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \notin W$. What element $\begin{bmatrix} ?? \\ ?? \\ ?? \\ 4 \\ 5 \end{bmatrix} \in W$? [Use part **9**]

14. Notice that $\delta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ is also a basis of W. What element $\begin{bmatrix} 2 \\ ?? \\ ?? \\ ?? \\ 3 \end{bmatrix} \in W$?

15. Can you find and element $\begin{bmatrix} 5 \\ 4 \\ ?? \\ ?? \\ ?? \end{bmatrix} \in W$?

Solutions

Example 1

The standard basis of $M_{2\times 2}(\mathbb{R})$ is $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

1. Is $\alpha = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

No. $M_{2\times 2}(\mathbb{R})$ is 4 dimensional, hence this set is too small to be a basis.
[Also see part **3**. This set is not even linearly independent]

2. Is $\gamma = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

No. $M_{2\times 2}(\mathbb{R})$ is 4 dimensional, hence this set is too large to be a basis.
[Also see part **3**. This set is not linearly independent]

3. Is $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

No. This set is **not** linearly independent.

e.g. $3\cdot\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

and $2\cdot\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\Leftarrow$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 3 & 0 & -1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Extend the set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ to a basis (call it $\tilde{\beta}$).

Take e.g. $\tilde{\beta} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

5. If $\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ what is $[\,\vec{v}\,]_{\tilde{\beta}}$?

$$[\,\vec{v}\,]_{\tilde{\beta}} = \begin{bmatrix} -13 \\ 3 \\ 8 \\ 12 \end{bmatrix} \quad \Leftarrow \quad \operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 3 & 0 & 0 & -4 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & -13 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 12 \end{bmatrix}$$

6. Let $\vec{b}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, the second basis vector of $\tilde{\beta}$. What is $\left[\vec{b}_2\right]_{\tilde{\beta}}$?

$$\left[\vec{b}_2\right]_{\tilde{\beta}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Example 2

Let $W = \left\{ \begin{bmatrix} w+2x+y & w+x+2y+z \\ x-y-z & w+3x-z \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$

1. When we take $w = 4,\ x = 1,\ y = 2$ and $z = -3$, what element of W do we get?

$$\begin{bmatrix} w+2\cdot x+y & w+x+2\cdot y+z \\ x-y-z & w+3\cdot x-z \end{bmatrix} \qquad \begin{bmatrix} w+2\cdot x+y & w+x+2\cdot y+z \\ x-y-z & w+3\cdot x-z \end{bmatrix}$$

$$\begin{bmatrix} w+2\cdot x+y & w+x+2\cdot y+z \\ x-y-z & w+3\cdot x-z \end{bmatrix} \Big| w=4 \text{ and } x=1 \text{ and } y=2 \text{ and } z=-3 \qquad \boxed{\begin{bmatrix} 8 & 6 \\ 2 & 10 \end{bmatrix}}$$

2. Is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in W$? **No**

3. Is $\begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix} \in W$? **Yes**

$\Leftarrow$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 1 & 4 \\ 1 & 3 & 0 & -1 & 1 & 6 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 3 & 2 & 0 & -6 \\ 0 & 1 & -1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Is W all of $M_{2\times2}(\mathbb{R})$? **No**

5. Explain why $W = \left\{ w\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$.

$$w\cdot\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + x\cdot\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + y\cdot\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + z\cdot\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \begin{bmatrix} w+2\cdot x+y & w+x+2\cdot y+z \\ x-y-z & w+3\cdot x-z \end{bmatrix}$$

6. Explain why $W = \operatorname{span}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right)$. This follows from **5.**

7. Is W a subspace of $M_{2\times2}(\mathbb{R})$? (i.e. $W \sqsubseteq M_{2\times2}(\mathbb{R})$?) **Yes**, since it is a span.

8. Find a basis for W.

e.g. $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right\}$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 3 & 0 & -1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

9. What is $\dim(W)$? $\dim(W) = 2$

10. Let $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right\}$. Check that β is a basis for W.

Then rref in **8**. shows they are linearly independent (and they are spanning all of W).

11. Let $\vec{w} = \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix}$. Find $[\vec{w}]_\beta$. $\quad [\vec{w}]_\beta = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & 4 \\ 1 & 3 & 6 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-6\cdot\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 4\cdot\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix}$$

12. Let $\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$. Check that γ is also a basis for W.

Both $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$ are in W $\Leftarrow$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$ linearly indep. $\Leftarrow$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A linearly independent set of two vectors in a 2 dimensional space is a basis of that space.

13. Let $\vec{w} = \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix}$. Find $[\vec{w}]_\gamma$.

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 1 & 1 & 4 \\ 2 & 1 & 6 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2\cdot\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + 2\cdot\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix}$$

$[\vec{w}]_\gamma = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \Leftarrow$

Example 3

The standard basis of $P_2(\mathbb{F}_4)$ is $S = \{ t^2,\ t,\ 1 \}$

1. Is $\alpha = \{ t^2 + at + b,\quad at^2 + t + 1,\quad at^2 + 1,\quad at^2 + t + 1,\quad t^2 + t + 1 \}$ a basis of $P_2(\mathbb{F}_4)$?

No, $P_2(\mathbb{F}_4)$ is 3 dimensional. This set contains too many vectors. There will be dependencies.

2. If not, prune the set to get a basis for $P_2(\mathbb{F}_4)$.

e.g. $\{ t^2 + at + b,\ at^2 + t + 1,\ at^2 + t + 1 \}$

$$rref4\left(\begin{bmatrix} 1 & a & a & a & 1 \\ a & 1 & 0 & 1 & 1 \\ b & 1 & 1 & b & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & a & 0 & b \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

3. Let $\beta = \{ t^2 + at + b,\quad at^2 + t + 1,\quad t^2 + t + 1 \}$. Check that β is a basis for $P_2(\mathbb{F}_4)$.

Since this is a subset of α with three elements (note that any basis must have 3 elements, since the basis we found in **2.** has 3 elements), hence we only have to check if they are linearly independent:

$$rref4\left(\begin{bmatrix} 1 & a & 1 \\ a & 1 & 1 \\ b & 1 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Let $\vec{v} = at^2 + bt + 1$. Find $[\vec{v}]_S$ and $[\vec{v}]_\beta$.

$$[\vec{v}]_S = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \qquad [\vec{v}]_\beta = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

5. Let $\vec{w} = t^2 + bt + 1$. Find $[\vec{w}]_S$ and $[\vec{w}]_\beta$.

$$[\vec{w}]_S = \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix} \qquad [\vec{w}]_\beta = \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix}$$

$$rref4\left(\begin{bmatrix} 1 & a & 1 & a & 1 \\ a & 1 & 1 & b & b \\ b & 1 & 1 & 1 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 & a & a \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

6. Is $\gamma = \{ t^2 + at + 1,\quad t^2 + b \}$ a basis of $P_2(\mathbb{F}_4)$? **No**, only two elements.

7. If not extend it to obtain a basis for $P_2(\mathbb{F}_4)$. Call it δ.

E.g. $\delta = \{ t^2 + at + 1,\quad t^2 + b,\quad t^2 + t + 1 \}$

8. Find $[\vec{v}]_\delta$ and $[\vec{w}]_\delta$ where $\vec{v} = at^2 + bt + 1$ and $\vec{w} = t^2 + bt + 1$.

$$[\vec{v}]_\delta = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix} \qquad [\vec{w}]_\delta = \begin{bmatrix} b \\ 0 \\ a \end{bmatrix}$$

$$rref4\left(\begin{bmatrix} 1 & 1 & 1 & a & 1 \\ a & 0 & 1 & b & b \\ 1 & b & 1 & 1 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 & 1 & b \\ 0 & 1 & 0 & a & 0 \\ 0 & 0 & 1 & 1 & a \end{bmatrix}$$

Example 4

Let $W = \text{span}\{\, t^2 + t + a, \quad bt^2 + 1, \quad at^2 + t + b, \quad at \,\}$.

One **rref4** will answer questions **1 — 6**

$$rref4\left(\left[\begin{array}{cccc|ccc} 1 & b & a & 0 & 0 & a & a \\ 1 & 0 & 1 & a & 1 & b & a \\ a & 1 & b & 0 & 0 & b & a \end{array}\right]\right) \qquad \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & a & 1 & b & 0 \\ 0 & 1 & 1 & b & a & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

1. Is $t \in W$? **Yes**.

2. Is $at^2 + bt + b \in W$? **Yes**.

3. Is $at^2 + at + a \in W$? **No**.

4. Is $W \sqsubseteq P_2(\mathbb{F}_4)$? **No**. (only 2 dimensional)

5. What is $\dim(W)$? $\dim(W) = 2$

6. Is $\beta = \{\, t^2 + t + a, \quad bt^2 + 1 \,\}$ a basis for W? **Yes**.

7. Is $\gamma = \{\, t, \quad t^2 + a \,\}$ a basis for W?

Both $\{\, t, \quad t^2 + a \,\}$ are in $W \Leftarrow$

$$rref4\left(\begin{bmatrix} 1 & b & 0 & 1 \\ 1 & 0 & 1 & 0 \\ a & 1 & 0 & a \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & a & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{\, t, \quad t^2 + a \,\}$ linearly indep. $\Leftarrow$

$$rref4\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & a \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A linearly independent set of two vectors in a 2 dimensional space is a basis of that space.

The next two questions can be answered with the following rref4

$rref4\left(\begin{bmatrix} 1 & b & 1 & a \\ 1 & 0 & a & b \\ a & 1 & a & b \end{bmatrix}\right)$	$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$rref4\left(\begin{bmatrix} 0 & 1 & 1 & a \\ 1 & 0 & a & b \\ 0 & a & a & b \end{bmatrix}\right)$	$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$

8. Let $\vec{v} = t^2 + at + a$. Compute $[\vec{v}]_\beta$ and $[\vec{v}]_\gamma$. $[\vec{v}]_\beta = \begin{bmatrix} a \\ 1 \end{bmatrix}$ $[\vec{v}]_\gamma = \begin{bmatrix} a \\ 1 \end{bmatrix}$

9. Let $\vec{w} = at^2 + bt + b$. Compute $[\vec{w}]_\beta$ and $[\vec{w}]_\gamma$ $[\vec{w}]_\beta = \begin{bmatrix} b \\ a \end{bmatrix}$ $[\vec{w}]_\gamma = \begin{bmatrix} b \\ a \end{bmatrix}$

Example 5

Let $W = \left\{ \begin{bmatrix} v+2w+2x+y+2z \\ 2v+4w+4x+2y+4z \\ 3v+3x+6y+4z \\ v+5w+3y+3z \\ w+4x+3y+5z \end{bmatrix} : v,w,x,y,z \in \mathbb{F}_7 \right\}$

1. When we take $v=5$, $w=4$, $x=1$, $y=2$ and $z=3$, what element of W do we get?

Solution: $\begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 4 & 2 & 4 \\ 3 & 0 & 3 & 6 & 4 \\ 1 & 5 & 0 & 3 & 3 \\ 0 & 1 & 4 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} v+2\cdot w+2\cdot x+y+2\cdot z \\ 2\cdot v+4\cdot w+4\cdot x+2\cdot y+4\cdot z \\ 3\cdot v+3\cdot x+6\cdot y+4\cdot z \\ v+5\cdot w+3\cdot y+3\cdot z \\ w+4\cdot x+3\cdot y+5\cdot z \end{bmatrix}$
$s7\left(\begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 4 & 2 & 4 \\ 3 & 0 & 3 & 6 & 4 \\ 1 & 5 & 0 & 3 & 3 \\ 0 & 1 & 4 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}\right)$	$\begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \\ 1 \end{bmatrix}$

2. Is $\begin{bmatrix}6\\5\\2\\0\\5\end{bmatrix} \in W$? **Yes**

3. Is $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \in W$? **No**

$$rref7\left(\left[\begin{array}{ccccc|cc}1&2&2&1&2&6&1\\2&4&4&2&4&5&2\\3&0&3&6&4&2&3\\1&5&0&3&3&0&4\\0&1&4&3&5&5&5\end{array}\right]\right) \qquad \left[\begin{array}{ccccc|cc}1&0&1&2&6&3&0\\0&1&4&3&5&5&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{array}\right]$$

4. Is W all of $\mathbb{F}_7^5$? **No**, W is only 2 dimensional.

5. Explain why $W = \text{span}\left(\begin{bmatrix}1\\2\\3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\4\\0\\5\\1\end{bmatrix}, \begin{bmatrix}2\\4\\3\\0\\4\end{bmatrix}, \begin{bmatrix}1\\2\\6\\3\\3\end{bmatrix}, \begin{bmatrix}2\\4\\4\\3\\5\end{bmatrix}\right)$.

$$v\cdot\begin{bmatrix}1\\2\\3\\1\\0\end{bmatrix}+w\cdot\begin{bmatrix}2\\4\\0\\5\\1\end{bmatrix}+x\cdot\begin{bmatrix}2\\4\\3\\0\\4\end{bmatrix}+y\cdot\begin{bmatrix}1\\2\\6\\3\\3\end{bmatrix}+z\cdot\begin{bmatrix}2\\4\\4\\3\\5\end{bmatrix} = \begin{bmatrix}v+2\cdot w+2\cdot x+y+2\cdot z\\2\cdot v+4\cdot w+4\cdot x+2\cdot y+4\cdot z\\3\cdot v+3\cdot x+6\cdot y+4\cdot z\\v+5\cdot w+3\cdot y+3\cdot z\\w+4\cdot x+3\cdot y+5\cdot z\end{bmatrix}$$

6. Is W a subspace of $\mathbb{F}_7^5$? (i.e. $W \sqsubseteq \mathbb{F}_7^5$?) **Yes**, since $W = \text{span}\left(\begin{bmatrix}1\\2\\3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\4\\0\\5\\1\end{bmatrix}\right)$

7. Find a basis for W. e.g. $\beta = \left\{\begin{bmatrix}1\\2\\3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\4\\0\\5\\1\end{bmatrix}\right\}$

8. What is $\dim(W)$? $\dim(W) = 2$

9. Let $\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$. Check that β is a basis of W.

$$\mathit{rref7}\left(\begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 4 & 2 & 4 \\ 3 & 0 & 3 & 6 & 4 \\ 1 & 5 & 0 & 3 & 3 \\ 0 & 1 & 4 & 3 & 5 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & 4 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. let $\gamma = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$. Check that γ is a basis for W.

Both $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$ are in W $\Leftarrow$

$$\mathit{rref7}\left(\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 2 & 1 \\ 3 & 0 & 3 & 6 \\ 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent $\Leftarrow$

$$\mathit{rref7}\left(\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 3 & 6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A linearly independent set of two vectors in a 2 dimensional space is a basis of that space.

11. Let $\vec{v} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 0 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$ Compute $[\vec{v}]_\beta$, $[\vec{v}]_\gamma$, $[\vec{w}]_\beta$ and $[\vec{w}]_\gamma$

$$[\vec{v}]_\beta = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad [\vec{w}]_\beta = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \Leftarrow$$

$$[\vec{v}]_\gamma = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad [\vec{w}]_\gamma = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Leftarrow$$

$$rref\left(\begin{bmatrix} 1 & 2 & 6 & 4 \\ 2 & 4 & 5 & 1 \\ 3 & 0 & 2 & 0 \\ 1 & 5 & 0 & 3 \\ 0 & 1 & 5 & 2 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$rref\left(\begin{bmatrix} 1 & 4 & 6 & 4 \\ 2 & 1 & 5 & 1 \\ 3 & 6 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 5 & 2 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

12. If $[\vec{w}]_\gamma = \begin{bmatrix} x \\ y \end{bmatrix}$ then what is $\vec{w}$?

$$\vec{w} = \begin{bmatrix} x+4y \\ 2x+y \\ 3x+6y \\ x \\ y \end{bmatrix}$$

$$x \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 4 \\ 1 \\ 6 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x+4\cdot y \\ 2\cdot x+y \\ 3\cdot x+6\cdot y \\ x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 3 & 6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x+4\cdot y \\ 2\cdot x+y \\ 3\cdot x+6\cdot y \\ x \\ y \end{bmatrix}$$

13. In part **3** we found that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \notin W$. What element $\begin{bmatrix} ?? \\ ?? \\ ?? \\ 4 \\ 5 \end{bmatrix} \in W$? [Use part **9**]

Note that if $\vec{w} = \begin{bmatrix} ?? \\ ?? \\ ?? \\ 4 \\ 5 \end{bmatrix} \in W$ then $[\vec{w}]_\gamma = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ i.e. $\vec{w} = \left.\begin{bmatrix} x+4y \\ 2x+y \\ 3x+6y \\ x \\ y \end{bmatrix}\right|_{\substack{x=4 \\ y=5}} = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 4 \\ 5 \end{bmatrix}$

$$s7\left(\begin{bmatrix} x+4\cdot y \\ 2\cdot x+y \\ 3\cdot x+6\cdot y \\ x \\ y \end{bmatrix} | x=4 \text{ and } y=5\right) \qquad \begin{bmatrix} 3 \\ 6 \\ 0 \\ 4 \\ 5 \end{bmatrix}$$

14. Notice that $\delta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ is also a basis of W. What element $\begin{bmatrix} 2 \\ ?? \\ ?? \\ ?? \\ 3 \end{bmatrix} \in W$?

Note that if $[\vec{w}]_\delta = \begin{bmatrix} x \\ y \end{bmatrix}$

then $\vec{w} = \begin{bmatrix} x \\ 2x \\ 3x+y \\ x+3y \\ y \end{bmatrix}$ so that

if $[\vec{w}]_\delta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then $\vec{w} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ 2\cdot x \\ 3\cdot x+y \\ x+3\cdot y \\ y \end{bmatrix}$$

$$s7\left(\begin{bmatrix} x \\ 2\cdot x \\ 3\cdot x+y \\ x+3\cdot y \\ y \end{bmatrix} | x=2 \text{ and } y=3\right) \qquad \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

15. Can you find and element $\begin{bmatrix} 5 \\ 4 \\ ?? \\ ?? \\ ?? \end{bmatrix} \in W$? **No**, the second coordinate is always twice the first

in W (See **14.**)