Vector Spaces

A Vector Space is a non-empty set $\mathbb{V}$ over a field of scalars $\mathbb{F}$, with two binary operations satisfying 8 properties. These are the spaces studied in Linear Algebra.

Definition

A **Vector Space** $\langle \mathbb{V}, \mathbb{F}, +, \cdot \rangle$ is a non-empty set $\mathbb{V}$, together with a **field** of scalars $\mathbb{F}$ and two binary operations called **vector addition** $(+)$ and **scalar multiplication** $(\cdot)$ such that

(a) For any $\vec{v}, \vec{w} \in \mathbb{V}$ we have $\vec{v} + \vec{w} \in \mathbb{V}$, and (+ is closed)

(b) For any $s \in \mathbb{F}$ and $\vec{v} \in \mathbb{V}$ we have $s \cdot \vec{v} \in \mathbb{V}$ ($\cdot$ is closed)

and such that the following properties hold:

(1) $\boxed{\vec{v} + \vec{w} = \vec{w} + \vec{v}}$ for all $\vec{v}, \vec{w} \in \mathbb{V}$ (+ is commutative)

(2) $\boxed{\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}}$ for all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$ (+ is associative)

(3) There exists a unique $\vec{0} \in \mathbb{V}$ such that (zero vector)
$\boxed{\vec{v} + \vec{0} = \vec{v}}$ for all $\vec{v} \in \mathbb{V}$

(4) For each $\vec{v} \in \mathbb{V}$ there exists an element $-\vec{v} \in \mathbb{V}$ (opposites)
such that $\boxed{\vec{v} + (-\vec{v}) = \vec{0}}$

(5) $\boxed{1 \cdot \vec{v} = \vec{v}}$ for all $\vec{v} \in \mathbb{V}$ (1, identity)

(6) $\boxed{r \cdot (s \cdot \vec{v}) = (r \cdot s) \cdot \vec{v}}$ for all $r, s \in \mathbb{F}$ and $\vec{v} \in \mathbb{V}$

(7) $\boxed{s \cdot (\vec{v} + \vec{w}) = s \cdot \vec{v} + s \cdot \vec{w}}$ for all $s \in \mathbb{F}$ and $\vec{v}, \vec{w} \in \mathbb{V}$ (distributive prop 1)

(8) $\boxed{(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}}$ for all $r, s \in \mathbb{F}$ and $\vec{v} \in \mathbb{V}$ (distributive prop 2)

If the underlying field of scalars is known as well as the two binary operations, we usual refer to the vector space as $\mathbb{V}$. Otherwise we indicate all 4 components:

$$\boxed{\langle \mathbb{V},\, \mathbb{F},\, +,\, \cdot\, \rangle}$$

We usually call the elements of the set 'vectors', but the set could be a set of e.g. matrices, functions, polynomials, sequences. The binary operations we usually call 'vector addition' and 'scalar multiplication', and they could be any well-defined binary operation. We also usually use $\vec{v} + \vec{w}$ and $t \cdot \vec{v}$ to indicate these operations. Now let's look at the three most important examples of Vector Spaces we'll use this course:

(1) $\boxed{\mathbb{F}^n}$ ***n*-tuples over the field** $\mathbb{F}$

(2) $\boxed{M_{n\times m}(\mathbb{F})}$ $\boldsymbol{n \times m}$ **matrices over the field** $\mathbb{F}$

(3) $\boxed{P_n(\mathbb{F})}$ **polynomials of degree *n* or less over the field** $\mathbb{F}$

Example 1

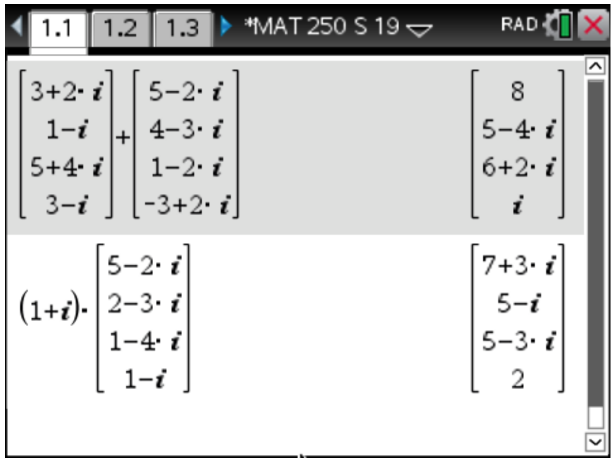
$$\mathbb{F}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_1,\, x_2, \cdots,\, x_n \in \mathbb{F} \right\}$$

with vector addition defined by:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and scalar multiplication defined by:

$$s \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s \cdot x_1 \\ s \cdot x_2 \\ \vdots \\ s \cdot x_n \end{bmatrix}$$



Examples from $\mathbb{C}^4$

We'll check the vectorspace properties for $\mathbb{F}^3$:

(1) $\boxed{\vec{v}+\vec{w}=\vec{w}+\vec{v}}$

$$\vec{v}+\vec{w}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=\begin{bmatrix}v_1+w_1\\v_2+w_2\\v_3+w_3\end{bmatrix}=\begin{bmatrix}w_1+v_1\\w_2+v_2\\w_3+v_3\end{bmatrix}=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\vec{w}+\vec{v}$$

(2) $\boxed{\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}}$

$$\vec{u}+(\vec{v}+\vec{w})=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\left(\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}v_1+w_1\\v_2+w_2\\v_3+w_3\end{bmatrix}=\begin{bmatrix}u_1+(v_1+w_1)\\u_2+(v_2+w_2)\\u_3+(v_3+w_3)\end{bmatrix}$$

$$=\begin{bmatrix}(u_1+v_1)+w_1\\(u_2+v_2)+w_2\\(u_3+v_3)+w_3\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{bmatrix}+\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)+\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$$

$$=(\vec{u}+\vec{v})+\vec{w}$$

(3) $\boxed{\vec{v}+\vec{0}=\vec{v}}$

Note 1 In $\mathbb{F}^n$: $\vec{0}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

$$\vec{v}+\vec{0}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}v_1+0\\v_2+0\\v_3+0\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\vec{v}$$

(4) $\boxed{\vec{v}+(-\vec{v})=\vec{0}}$

Note 2 In $\mathbb{F}^n$: $-\vec{v}=-\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}-v_1\\-v_2\\\vdots\\-v_n\end{bmatrix}$

$$\vec{v}+(-\vec{v})=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}-v_1\\-v_2\\-v_3\end{bmatrix}=\begin{bmatrix}v_1+(-v_1)\\v_2+(-v_2)\\v_3+(-v_3)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\vec{0}$$

(5) $\boxed{1\cdot\vec{v}=\vec{v}}$

$$1\cdot\vec{v}=1\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\cdot v_1\\1\cdot v_2\\1\cdot v_3\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\vec{v}$$

(6) $\boxed{r\cdot(s\cdot\vec{v})=(r\cdot s)\cdot\vec{v}}$

$$r\cdot(s\cdot\vec{v})=r\cdot\left(s\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)=r\cdot\begin{bmatrix}s\cdot v_1\\s\cdot v_2\\s\cdot v_3\end{bmatrix}=\begin{bmatrix}r\cdot(s\cdot v_1)\\r\cdot(s\cdot v_2)\\r\cdot(s\cdot v_3)\end{bmatrix}=\begin{bmatrix}(r\cdot s)\cdot v_1\\(r\cdot s)\cdot v_2\\(r\cdot s)\cdot v_3\end{bmatrix}=(r\cdot s)\cdot\vec{v}$$

(7) $\boxed{s\cdot(\vec{v}+\vec{w})=s\cdot\vec{v}+s\cdot\vec{w}}$

$$s\cdot(\vec{v}+\vec{w})=s\cdot\left(\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\right)=s\cdot\begin{bmatrix}v_1+w_1\\v_2+w_2\\v_3+w_3\end{bmatrix}=\begin{bmatrix}s\cdot(v_1+w_1)\\s\cdot(v_2+w_2)\\s\cdot(v_3+w_3)\end{bmatrix}=\begin{bmatrix}s\cdot v_1+s\cdot w_1\\s\cdot v_2+s\cdot w_2\\s\cdot v_3+s\cdot w_3\end{bmatrix}$$

$$=\begin{bmatrix}s\cdot v_1\\s\cdot v_2\\s\cdot v_3\end{bmatrix}+\begin{bmatrix}s\cdot w_1\\s\cdot w_2\\s\cdot w_3\end{bmatrix}=s\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+s\cdot\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=s\cdot\vec{w}+s\cdot\vec{v}$$

(8) $\boxed{(r+s)\cdot\vec{v}=r\cdot\vec{v}+s\cdot\vec{v}}$

$$(r+s)\cdot\vec{v}=(r+s)\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}(r+s)\cdot v_1\\(r+s)\cdot v_2\\(r+s)\cdot v_3\end{bmatrix}=\begin{bmatrix}r\cdot v_1+s\cdot v_1\\r\cdot v_2+s\cdot v_2\\r\cdot v_3+s\cdot v_3\end{bmatrix}=\begin{bmatrix}r\cdot v_1\\r\cdot v_2\\r\cdot v_3\end{bmatrix}+\begin{bmatrix}s\cdot v_1\\s\cdot v_2\\s\cdot v_3\end{bmatrix}$$

$$=r\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+s\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=r\cdot\vec{v}+s\cdot\vec{v}$$

Example 2

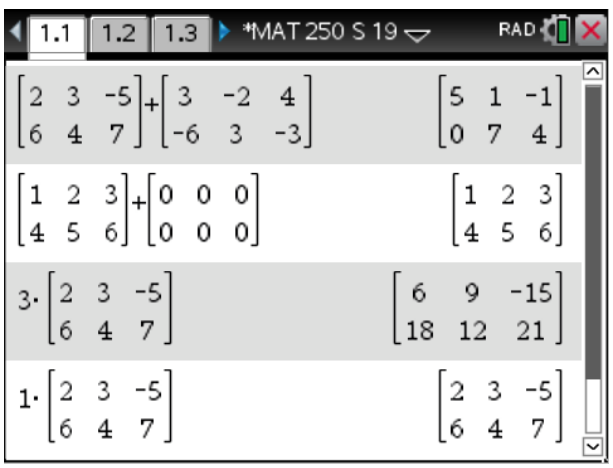
$$M_{n\times m}(\mathbb{F}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \;\middle|\; a_{ij} \in \mathbb{F} \right\}$$

with vector addition defined by:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm1}+b_{nm} \end{bmatrix}$$

and scalar multiplication defined by:

$$s\cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} s\cdot a_{11} & s\cdot a_{12} & \cdots & s\cdot a_{1m} \\ s\cdot a_{21} & s\cdot a_{22} & \cdots & s\cdot a_{2m} \\ \vdots & \vdots & & \vdots \\ s\cdot a_{n1} & s\cdot a_{n2} & \cdots & s\cdot a_{nm} \end{bmatrix}$$



Examples from $M_{2\times 3}(\mathbb{R})$

Note 3 In $M_{n\times m}(\mathbb{F})$: $\quad \vec{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ and

$$-\vec{v} = -\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1m} \\ -a_{21} & -a_{22} & \cdots & -a_{2m} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nm} \end{bmatrix}$$

We'll just **illustrate some** of the vectorspace properties for $M_{2\times 3}(\mathbb{F})$, and leave the proofs in their full generality to the readers;

(1) $\boxed{\vec{v}+\vec{w}=\vec{w}+\vec{v}}$

$$\begin{aligned}\vec{v}+\vec{w} &= \begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 4 \\ -6 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -1 \\ 0 & 7 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 & 4 \\ -6 & 3 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = \vec{w}+\vec{v}\end{aligned}$$

(3) $\boxed{\vec{v}+\vec{0}=\vec{v}}$

$$\vec{v}+\vec{0} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 & 3+0 \\ 4+0 & 5+0 & 6+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \vec{v}$$

(5) $\boxed{1\cdot\vec{v}=\vec{v}}$

$$1\cdot\vec{v} = 1\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 & 1\cdot 3 & 1\cdot(-5) \\ 1\cdot 6 & 1\cdot 4 & 1\cdot 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = \vec{v}$$

(6) $\boxed{r\cdot(s\cdot\vec{v})=(r\cdot s)\cdot\vec{v}}$

$$2\cdot\left(3\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix}\right) = 2\cdot\begin{bmatrix} 6 & 9 & -15 \\ 18 & 12 & 21 \end{bmatrix} = \begin{bmatrix} 12 & 18 & -30 \\ 36 & 24 & 42 \end{bmatrix} = 6\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix}$$

(8) $\boxed{(r+s)\cdot\vec{v}=r\cdot\vec{v}+s\cdot\vec{v}}$

$$(2+3)\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = 5\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 15 & -25 \\ 30 & 20 & 35 \end{bmatrix} \quad \text{and}$$

$$2\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} + 3\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 6 & -10 \\ 12 & 8 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 9 & -15 \\ 18 & 12 & 21 \end{bmatrix} = \begin{bmatrix} 10 & 15 & -25 \\ 30 & 20 & 35 \end{bmatrix}$$

so that $(2+3)\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} = 2\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix} + 3\cdot\begin{bmatrix} 2 & 3 & -5 \\ 6 & 4 & 7 \end{bmatrix}$

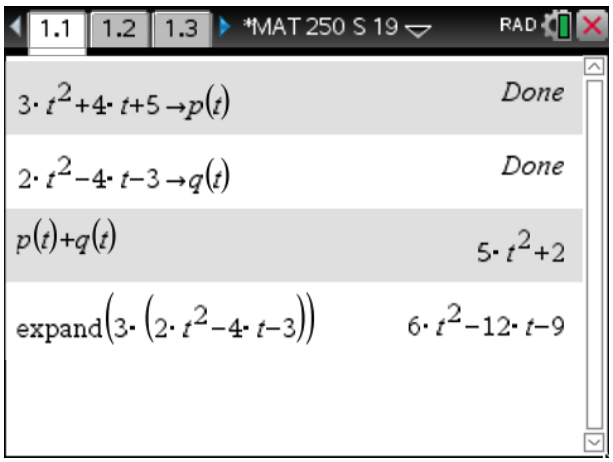
Example 3 $\boxed{P_n(\mathbb{F}) = \left\{ a_n t^n + \cdots + a_1 t + a_0 \;\middle|\; a_i \in \mathbb{F} \right\}}$

with vector addition defined by:

$$\boxed{(a_n t^n + \cdots + a_1 t + a_0) + (b_n t^n + \cdots + b_1 t + b_0) = (a_n + b_n)t^n + \cdots + (a_1 + b_1)t + (a_0 + b_0)}$$

and scalar multiplication defined by:

$$\boxed{s \cdot (a_n t^n + \cdots + a_1 t + a_0) = (s \cdot a_n)t^n + \cdots + (s \cdot a_1)t + (s \cdot a_0)}$$



Examples from $P_2(\mathbb{R})$

Note 4 In $P_n(\mathbb{F})$ the zero vector is $\vec{0} = 0\,t^n + \cdots 0\,t + 0$, and the opposite of a vector $\vec{v}$ is

$-\vec{v} = -(a_n t^n + \cdots + a_1 t + a_0) = (-a_n)t^n + \cdots + (-a_1)t + (-a_0)$

We'll check the vectorspace properties for $P_2(\mathbb{F})$:

(1) $\boxed{\vec{v} + \vec{w} = \vec{w} + \vec{v}}$

$$\begin{aligned}
\vec{v} + \vec{w} &= (a_2\,t^2 + a_1\,t + a_0) + (b_2\,t^2 + b_3\,t + b_0) \\
&= (a_2 + b_2)\,t^2 + (a_1 + b_1)\,t + (a_0 + b_0) \\
&= (b_2 + a_2)\,t^2 + (b_1 + a_1)\,t + (b_0 + a_0) \\
&= (b_2\,t^2 + b_1\,t + b_0) + (a_2\,t^2 + a_1\,t + a_0) \\
&= \vec{w} + \vec{v}
\end{aligned}$$

(2) $\boxed{\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}}$

$$\begin{aligned}
\vec{u}+(\vec{v}+\vec{w}) &= (a_2\,t^2+a_1\,t+a_0)+\Big(\,(b_2\,t^2+b_1\,t+b_0)+(c_2\,t^2+c_1\,t+c_0)\,\Big)\\
&= (a_2\,t^2+a_1\,t+a_0)+\Big(\,(b_2+c_2)\,t^2+(b_1+c_1)\,t+(b_0+c_0)\,\Big)\\
&= \Big(a_2+(b_2+c_2)\Big)\,t^2+\Big(a_1+(b_1+c_1)\Big)\,t+\Big(a_0+(b_0+c_0)\Big)\\
&= \Big((a_2+b_2)+c_2\Big)\,t^2+\Big((a_1+b_1)+c_1\Big)\,t+\Big((a_0+b_0)+c_0\Big)\\
&= \Big((a_2+b_2)\,t^2+(a_1+b_1)\,t+(a_0+b_0)\Big)+(c_2\,t^2+c_1\,t+c_0)\,\Big)\\
&= \Big(\,(a_2\,t^2+a_1\,t+a_0)+(b_2\,t^2+b_1\,t+b_0)\,\Big)+(c_2\,t^2+c_1\,t+c_0)\\
&= (\vec{u}+\vec{v})+\vec{w}
\end{aligned}$$

(3) $\boxed{\vec{v}+\vec{0}=\vec{v}}$

$$\begin{aligned}
\vec{v}+\vec{0} &= (a_2\,t^2+a_1\,t+a_0)+(0\,t^2+0\,t+0)\\
&= (a_2+0)\,t^2+(a_1+0)\,t+(a_0+0)\\
&= a_2\,t^2+a_1\,t+a_0\\
&= \vec{v}
\end{aligned}$$

(4) $\boxed{\vec{v}+(-\vec{v}\,)=\vec{0}}$

$$\begin{aligned}
\vec{v}+(-\vec{v}\,) &= (a_2\,t^2+a_1\,t+a_0)+\Big(\,(-a_2)\,t^2+(-a_1)\,t+(-a_0)\Big)\\
&= (a_2+(-a_2))\,t^2+(a_1+(-a_1))\,t+(a_0+(-a_0))\\
&= 0\,t^2+0\,t+0\\
&= \vec{0}
\end{aligned}$$

(5) $\boxed{1\cdot\vec{v}=\vec{v}}$

$$\begin{aligned}
1\cdot(a_2\,t^2+a_1\,t+a_0) &= (1\cdot a_2)\,t^2+(1\cdot a_1)\,t+(1\cdot a_0)\\
&= a_2\,t^2+a_1\,t+a_0\\
&= \vec{v}
\end{aligned}$$

(6) $\boxed{r\cdot(s\cdot\vec{v})=(r\cdot s)\cdot\vec{v}}$

$$r\cdot(s\cdot\vec{v})=r\cdot\Big(s\cdot(a_2\,t^2+a_1\,t+a_0)\Big)$$

$$
\begin{aligned}
&= s \cdot \Big((r \cdot a_2)\, t^2 + (r \cdot a_1)\, t + (r \cdot a_0) \Big) \\
&= \Big(s \cdot (r \cdot a_2) \Big)\, t^2 + \Big(s \cdot (r \cdot a_1) \Big)\, t + \Big(s \cdot (r \cdot a_0) \Big) \\
&= \Big((s \cdot r) \cdot a_2 \Big)\, t^2 + \Big((s \cdot r) \cdot a_1 \Big)\, t + \Big((s \cdot r) \cdot a_0 \Big) \\
&= (s \cdot r) \cdot (a_2\, t^2 + a_1\, t + a_0) \\
&= (s \cdot r) \cdot \vec{v}
\end{aligned}
$$

(7) $\boxed{s \cdot (\vec{v} + \vec{w}) = s \cdot \vec{v} + s \cdot \vec{w}}$

$$
\begin{aligned}
s \cdot (\vec{v} + \vec{w}) &= s \cdot \Big((a_2\, t^2 + a_1\, t + a_0) + (b_2\, t^2 + b_3\, t + b_0) \Big) \\
&= s \cdot \Big((a_2 + b_2)\, t^2 + (a_1 + b_1)\, t + (a_0 + b_0) \Big) \\
&= \Big(s \cdot (a_2 + b_2) \Big)\, t^2 + \Big(s \cdot (a_1 + b_1) \Big)\, t + \Big(s \cdot (a_0 + b_0) \Big) \\
&= \Big(s \cdot a_2 + s \cdot b_2 \Big)\, t^2 + \Big(s \cdot a_1 + s \cdot b_1 \Big)\, t + \Big(s \cdot a_0 + s \cdot b_0 \Big) \\
&= \Big((s \cdot a_2)\, t^2 + (s \cdot a_1)\, t + (s \cdot a_0) \Big) + \Big((s \cdot b_2)\, t^2 + (s \cdot b_1)\, t + (s \cdot b_0) \Big) \\
&= s \cdot (a_2\, t^2 + a_1\, t + a_0) + s \cdot (b_2\, t^2 + b_1\, t + b_0) \\
&= s \cdot \vec{v} + s \cdot \vec{w}
\end{aligned}
$$

(8) $\boxed{(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}}$

$$
\begin{aligned}
(r + s) \cdot \vec{v} &= (r + s) \cdot (a_2\, t^2 + a_1\, t + a_0) \\
&= \Big((r + s) \cdot a_2 \Big)\, t^2 + \Big((r + s) \cdot a_1 \Big)\, t + \Big((r + s) \cdot a_0 \Big) \\
&= \Big(r \cdot a_2 + s \cdot a_2 \Big)\, t^2 + \Big(r \cdot a_1 + s \cdot a_1 \Big)\, t + \Big(r \cdot a_0 + s \cdot a_0 \Big) \\
&= \Big((r \cdot a_2)\, t^2 + (r \cdot a_1)\, t + (r \cdot a_0) \Big) + \Big((s \cdot a_2)\, t^2 + (s \cdot a_1)\, t + (s \cdot a_0) \Big) \\
&= r \cdot (a_2\, t^2 + a_1\, t + a_0) + s \cdot (a_2\, t^2 + a_1\, t + a_0) \\
&= s \cdot \vec{v} + t \cdot \vec{v}
\end{aligned}
$$