Matrices

A matrix is a rectangular array of numbers. For example $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix}$ is a 2×3 matrix whose entries are integers. The entries could be from any number system. In this section we mainly deal with matrices with real entries, such as the following 2×3 matrix

$$\begin{bmatrix} \sqrt{3} & 3 + \ln 4 & \pi \\ e^3 - 4 & 4/7 & \sin(2.5) \end{bmatrix}$$

In general an $m \times n$ matrix looks like

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where the entries $a_{ij} \in \mathbb{R}$. Sometimes we write $A = [\, a_{i\,j} \,]_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}}$.

Note that an $m \times n$ matrix has m rows and n columns:

$$\begin{array}{cc} & \longleftarrow n \longrightarrow \\ \updownarrow m & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

The entry in row i and column j is indicated by a_{ij}, or more precisely $a_{i,j}$.

Note 1

If i or $j > 9$, we need a comma in $a_{i,j}$. For example a_{213} would be ambiguous. The comma is needed to distinguish between $a_{21,3}$ and $a_{2,13}$. For small i and j no comma is needed: a_{34} can only mean $a_{3,4}$.

Definition 1

$M_{m \times n}(\mathbb{R})$ is the set of all $m \times n$ matrices with real entries.

Example 1

$\begin{bmatrix} 5 & \pi \\ -1 & 6 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$

Note 2 The vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we are basically matrices:

$$\begin{bmatrix} x \\ y \end{bmatrix} \in M_{2\times 1}(\mathbb{R}) \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in M_{3\times 1}(\mathbb{R})$$

In the previous section we defined an addition and a scalar multiplication for vectors. We will generalize these operations to matrices.

Matrix Operations

We will introduce three basic matrix operations:

(a) Matrix Addition
(b) Scalar Multiplication
(c) Matrix Multiplication

(a) Matrix addition

We *only* add matrices with the *same* dimensions, i.e. the same number of rows and the same number of columns. The addition is performed component wise, e.g.

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 8 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 7 \\ 5 & -3 & -1 \end{bmatrix}$$

In general we have

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \end{aligned}$$

or more condensed

$$A + B = [\,a_{ij}\,] + [\,b_{ij}\,] = [\,c_{ij}\,] \quad \text{where} \quad c_{ij} = a_{ij} + b_{ij} \quad \text{for} \quad \left\{ \begin{smallmatrix} i=1,\cdots,m \\ j=1,\cdots,n \end{smallmatrix} \right. .$$

Addition of real numbers is commutative, hence this is true for matrices as well, since addition is performed component-wise $A+B=B+A$

Example 2 $\begin{bmatrix} 3 & 4 \\ 2 & -7 \end{bmatrix} + \begin{bmatrix} 5 & -1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 2 & -7 \end{bmatrix}$

Addition of real numbers is also associative, hence so is matrix addition

$$A+(B+C)=(A+B)+C$$

In other words, matrix addition inherits commutativity and associativity, because addition is defined component-wise.

Definition 2

The **zero matrix** in $M_{m\times n}(\mathbb{R})$ is the $m\times n$ matrix whose entries are all zero.

$$O = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

It is usually denoted by O, or if needed $O_{m\times n}$.

Clearly if $A \in M_{m\times n}(\mathbb{R})$ and $O = O_{m\times n}$ then $A+O=A$.

The zero matrix O is the 'neutral' element in matrix addition, just as the real number 0 is the neutral element in the addition of real numbers: $x+0=x$.

Example 3 $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix}$

Definition 3

The **opposite** of a matrix $A \in M_{m\times n}(\mathbb{R})$, denoted $-A$, is the $m\times n$ matrix whose entries are the opposites of the entries of A

$$-A = [\,-a_{ij}\,]_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}} \qquad \text{when} \qquad A = [\,a_{ij}\,]_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}}$$

So if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then $-A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}$, and hence

it is clear that each matrix has a unique opposite, and that $A+(-A)=O$.

Example 4

If $A = \begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix}$ then $-A = \begin{bmatrix} -3 & -4 & 1 \\ -2 & 7 & 0 \end{bmatrix}$, and

$$A + (-A) = \begin{bmatrix} 3 & 4 & -1 \\ 2 & -7 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -4 & 1 \\ -2 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note 3 These definitions are in sync with vector addition:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

where we also saw the properties:

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

$$\vec{x} + \vec{0} = \vec{x}$$

$$\vec{x} + (-\vec{x}) = \vec{0}$$

which makes sense when we realize that vectors are matrices (with just one column).

(b) Scalar multiplication

Next we define a scalar multiplication for matrices which generalizes the scalar multiplication of vectors:

Definition 4

The **scalar multiplication** $t \cdot A$ where $t \in \mathbb{R}$ and the matrix $A = [\, a_{ij} \,]_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}}$ is defined by

$$t \cdot A = [\, t \cdot a_{ij} \,]_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}}$$

Explicitly, if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then $t \cdot A = \begin{bmatrix} t \cdot a_{11} & t \cdot a_{12} & \cdots & t \cdot a_{1n} \\ t \cdot a_{21} & t \cdot a_{22} & \cdots & t \cdot a_{2n} \\ \vdots & \vdots & & \vdots \\ t \cdot a_{m1} & t \cdot a_{m2} & \cdots & t \cdot a_{mn} \end{bmatrix}$

It follows, for example, that

$$1 \cdot A = A$$

$$-1 \cdot A = -A$$

$$0 \cdot A = O$$

In fact let's list the basic properties of vector addition and scalar multiplication in one theorem:

Theorem 1

If $A, B, C \in M_{m\times n}(\mathbb{R})$ and $s, t \in \mathbb{R}$ then

(a) $A + B = B + A$ [Commutativity]

(b) $A + (B + C) = (A + B) + C$ [Associativity]

(c) $A + O = A$

(d) $A + (-A) = O$

(e) $1 \cdot A = A$

(f) $s \cdot (t \cdot A) = (st) \cdot A$

(g) $t \cdot (A + B) = t \cdot A + t \cdot B$ [Distributive property 1]

(h) $(s + t) \cdot A = s \cdot A + t \cdot A$ [Distributive property 2]

Furthermore

(i) $0 \cdot A = O$

(j) $-1 \cdot A = -A$

Properties **(a)–(h)** are the defining properties of what is called a **Vector Space** in Linear Algebra. In fact $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ and $M_{m\times n}(\mathbb{R})$ are very important examples of vector spaces.

(c) Matrix Multiplication

The definition of matrix multiplication, at first, seems strange. But later when we talk about compositions of linear transformations it becomes clear why this is precisely the definition we would want.

First of all, to compute the product AB of two matrices A and B, we need that the number of columns of A is equal the number of rows of B.

So let A be an $m \times n$ matrix, and B be an $n \times p$ matrix.

$$\begin{array}{c} \xleftarrow{\qquad} n \xrightarrow{\qquad} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \begin{array}{cc} \updownarrow n & \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \end{array}$$

The product of an $m \times n$ matrix and an $n \times p$ matrix will be an $m \times p$ matrix.

$$\begin{tikzcd}[column sep=small] m \times n \arrow[dr] & & n \times p \arrow[dl] \\ & m \times p & \end{tikzcd}$$

We will refer to n as the 'inner' dimensions, and m and p as the 'outer' dimensions. Hence we need the inner dimensions to be equal. The outer dimensions give us the dimensions of the matrix of the product.

We'll begin with an example of how to compute the following matrix multiplication:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -1 & 3 & 0 \\ -2 & 1 & -2 & 2 \\ 6 & 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 8 & -1 & 4 & 2 \end{bmatrix}$$

Notice that indeed the product of this 2×3 matrix and 3×4 matrix is a 2×4 matrix.

The actual definition of the product uses a multiplication of rows and columns.

(1) To get the entry in row 1, column 1 we take the entire row 1 of the first matrix and the entire column 1 of the second matrix and **add** the **products** of the **corresponding entries**, as follows:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -1 & 3 & 0 \\ -2 & 1 & -2 & 2 \\ 6 & 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} \boxed{3 \cdot 5 + 4 \cdot (-2) + (-1) \cdot 6} & 0 & 2 & 3 \\ 8 & -1 & 4 & 2 \end{bmatrix}$$

(2) To get the entry in row 1, column 2 we take the entire row 1 of the first matrix and the entire column 2 of the second matrix and again **add** the **products** of the **corresponding entries**:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -1 & 3 & 0 \\ -2 & 1 & -2 & 2 \\ 6 & 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & \boxed{3 \cdot (-1) + 4 \cdot 1 + (-1) \cdot 1} & 2 & 3 \\ 8 & -1 & 4 & 2 \end{bmatrix}$$

(3) To get the entry in row 2, column 4 we take the entire row 2 of the first matrix and the entire column 4 of the second matrix and and again **add** the **products** of the **corresponding entries**:

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -1 & 3 & 0 \\ -2 & 1 & -2 & 2 \\ 6 & 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 8 & -1 & 4 & \boxed{2 \cdot 0 + 1 \cdot 2 + 0 \cdot 5} \end{bmatrix}$$

In general if A is an $m \times n$ matrix and B is an $n \times p$ matrix then

$$A\,B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \color{red}{a_{i1}} & \color{red}{a_{i2}} & \color{red}{\cdots} & \color{red}{a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & \color{blue}{b_{1j}} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & \color{blue}{b_{2j}} & \cdots & b_{2p} \\ \vdots & \vdots & & \color{blue}{\vdots} & & \vdots \\ b_{n1} & b_{n2} & \cdots & \color{blue}{b_{nj}} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \\ c_{i1} & c_{i2} & \cdots & \boxed{\color{red}{c_{ij}}} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

where $c_{ij} = \color{red}{a_{i1}}\,\color{blue}{b_{1j}} + \color{red}{a_{i2}}\,\color{blue}{b_{2j}} + \color{red}{a_{i3}}\,\color{blue}{b_{3j}} + \cdots + \color{red}{a_{in}}\,\color{blue}{b_{nj}} = \sum_{t=1}^{n} \color{red}{a_{it}}\,\color{blue}{b_{tj}}$ or in compact form

$$A\,B = [\,a_{ij}\,]\,[\,b_{ij}\,] = \left[\sum_{t=1}^{n} \color{red}{a_{it}}\,\color{blue}{b_{tj}}\right]$$

The TI-Nspire has as default, exactly this multiplication of matrices:

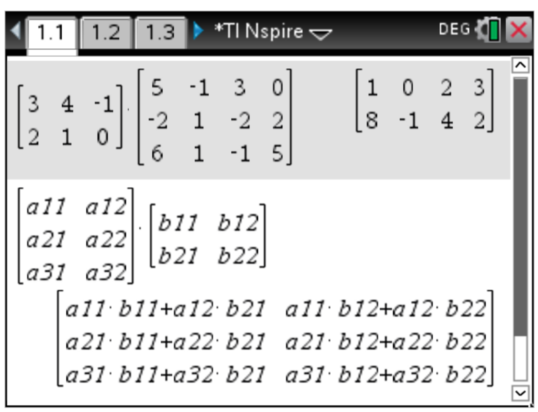


Figure 1: Matrix Multiplication

Two special matrix multiplications, of column and row vectors, are worth noting:

$$\begin{bmatrix} 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -17 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 9 \\ 5 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -36 & 27 \\ 5 & -20 & 15 \\ -2 & 8 & -6 \end{bmatrix}$$

Definition 5

The $n \times n$ **identity matrix** I_n is defined as

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

where all entries are 0, except on the main diagonal where they are all 1.

The identity matrix plays a similar ‘neutral’ role in matrix multiplication as 1 does in ordinary multiplication $(\,1 \cdot x = x \cdot 1 = x\,)$

If A is an $m \times n$ matrix, then $\quad A\, I_n = A = I_m\, A$.

$$\overset{\longleftarrow\; n \;\longrightarrow}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \; \underset{\downarrow}{\overset{\uparrow}{n}} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \; = \; A \; = \; \overset{\longleftarrow\; m \;\longrightarrow}{\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}} \; \underset{\downarrow}{\overset{\uparrow}{m}} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Here are the basic properties we need to know

Theorem 2

Let A, B, C be matrices of the appropriate dimensions so that the matrix operations below are properly defined, and let $s, t \in \mathbb{R}$, then

(a)	$A(BC) = (AB)C$	[Associative]
(b)	$A\,(B + C) = AB + AC$	[Left Distributive]
(c)	$(B + C)\,A = BA + CA$	[Right Distributive]
(d)	$t \cdot (AB) = (t \cdot A)\,B = A\,(t \cdot B)$	
(e)	$A\,I_n = A = I_m\,A$	[A an $m \times n$ matrix]
(f)	$A\,O = O$ and $O\,A = O$	

Note in **(f)** the various dimensions of the Os: $A\,O_{n\times p} = O_{m\times p}$ and $O_{q\times m}\,A = O_{q\times n}$ when A is and $m \times n$ matrix. The proofs of these are pretty straight forward. Only **(a)** is a bit challenging: it is quite doable, but algebraically messy.

All these properties look like familiar properties from the real numbers. But there are some important properties we know about real numbers that are not true, in general, for matrices.

Properties that are not true for Matrices

Here are three properties we know to be true for real numbers that fail to be true, in general, for matrices:

(a) If $x, y \in \mathbb{R}$ then $x \cdot y = y \cdot x$

(b) If $x, y \in \mathbb{R}$ then $x \cdot y = 0$ implies $x = 0$ or $y = 0$

(c) If $a, x, y \in \mathbb{R}$ and $a \neq 0$ then $a\,x = a\,y$ implies $x = y$ [The cancellation law]

In general, these do not hold for matrices:

(a) If A and B are matrices then, in general, $\boxed{AB \neq BA}$

Here are three levels at which this can go wrong.

(1) If $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$ then $A\,B$ exists since the inner dimensions match, but $B\,A$ doesn't exist since the inner dimensions don't match when $m \neq p$, hence $A\,B$ cannot possibly equal $B\,A$.

(2) If $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$ then both $A\,B$ and $B\,A$ exist since the their inner dimensions match, but $A\,B$ is an $m \times m$ matrix, and $B\,A$ is an $n \times n$ matrix, so that $AB \neq BA$ when $m \neq n$.

Finally

(3) If $A \in M_{m\times m}(\mathbb{R})$ and $B \in M_{m\times m}(\mathbb{R})$ i.e. they are both square matrices of the same dimensions, then both $A\,B$ and $B\,A$ exist *and* both are $m \times m$ matrices, but, even then $AB \neq BA$, in general. Here is a counter example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ -1 & -2 \end{bmatrix}$$

Of course it could happen to be true occasionally

$$\begin{bmatrix} 13 & 16 \\ -9 & -11 \end{bmatrix}\begin{bmatrix} 25 & 32 \\ -18 & -23 \end{bmatrix} = \begin{bmatrix} 37 & 48 \\ -27 & -35 \end{bmatrix} = \begin{bmatrix} 25 & 32 \\ -18 & -23 \end{bmatrix}\begin{bmatrix} 13 & 16 \\ -9 & -11 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & 4 & -4 \\ 1 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}\begin{bmatrix} 5 & 8 & -8 \\ -1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 12 & -12 \\ 0 & 1 & 0 \\ 3 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 8 & -8 \\ -1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 & -4 \\ 1 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$$

(b) If $A \in M_{m\times n}(\mathbb{R}),\ B \in M_{n\times p}(\mathbb{R})$ then $A{\cdot}B = 0$ does **not** imply $A = 0$ or $B = 0$

A few counter examples dispel this dream

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & -4 & 6 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) If $A\,B = A\,C$, then even if $A \neq O$, it does **not** follow that $B = C$. A few counter examples will suffice

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 3 & 4 \end{bmatrix} \quad \text{yet} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 7 & 5 \\ 3 & 4 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 16 & 17 \\ -6 & 32 & 34 \\ 6 & -32 & -34 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 & 1 \\ 6 & 6 & 9 \\ 8 & 0 & 1 \end{bmatrix}$$

yet

$$\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 & 1 \\ 6 & 6 & 9 \\ 8 & 0 & 1 \end{bmatrix}$$

We will discover later that when A is invertible this cancellation law *does* work for matrices.

Let's introduce three more fundamental concepts: the transpose, the determinant and the inverse of a matrix.

The Transpose of a Matrix

Definition 6

The **transpose** of a matrix $A = [\,a_{ij}\,]$ is defined by $A^\top = [\,a_{ji}\,]$

Example 5 $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ As we see the first row has become the first column and the second row had become the second column. But also if we look at it from the perspective of columns,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

every column has become the corresponding row (1^{st} row has become 1^{st} column etc.)

Columns become rows:

If $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then $\quad A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$ or

Rows become columns:

If $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then $\quad A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$

Theorem 3

If A, B are matrices of the appropriate dimensions so that the matrix operations below are properly defined, and $s, t \in \mathbb{R}$, then

(a) $(A^\top)^\top = A$

(b) $(A + B)^\top = A^\top + B^\top$

(c) $(t \cdot A)^\top = t \cdot A^\top$

(d) $(t \cdot A + s \cdot B)^\top = t \cdot A^\top + s \cdot B^\top$

(e) $(A\,B)^\top = B^\top A^\top$

The proofs of **(a)** – **(d)** are pretty straight forward. The only non-trivial case is **(e)**, which is not hard, but algebraically messy.

Example 6 $\left(\begin{bmatrix} 3 & 7 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}\right)^\top = \begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}^\top \begin{bmatrix} 3 & 7 \\ 4 & 9 \end{bmatrix}^\top$ can be verified by direct computation: $\begin{bmatrix} 1 & -5 \\ 2 & -7 \end{bmatrix}^\top = \begin{bmatrix} 5 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$.

Example 7 If $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ compute $M = \vec{n}\,\vec{n}^\top$.

$$M = \vec{n}\,\vec{n}^\top = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

Definition 7 A matrix A is called **symmetric** if $A^\top = A$.

Example 8 $\begin{bmatrix} 6 & 7 & 8 \\ 7 & 9 & 1 \\ 8 & 1 & 2 \end{bmatrix}$ is symmetric since $\begin{bmatrix} 6 & 7 & 8 \\ 7 & 9 & 1 \\ 8 & 1 & 2 \end{bmatrix}^\top = \begin{bmatrix} 6 & 7 & 8 \\ 7 & 9 & 1 \\ 8 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 3 & 7 & 8 \\ 4 & 9 & 1 \\ 5 & 6 & 2 \end{bmatrix}$ is not symmetric $\begin{bmatrix} 3 & 7 & 8 \\ 4 & 9 & 1 \\ 5 & 6 & 2 \end{bmatrix}^\top = \begin{bmatrix} 3 & 4 & 5 \\ 7 & 9 & 6 \\ 8 & 1 & 2 \end{bmatrix}$

$$\neq \begin{bmatrix} 3 & 7 & 8 \\ 4 & 9 & 1 \\ 5 & 6 & 2 \end{bmatrix}.$$

Example 9 Show that if $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $M = \vec{n}\,\vec{n}^\top$ is a symmetric matrix.

We can do this in two ways:

(1) Check that the matrix we computed in example 11.7 is symmetric, or

(2) Check that $M^\top = M$ as follows $M^\top = (\vec{n}\,\vec{n}^\top)^\top = (\vec{n}^\top)^\top \vec{n}^\top = \vec{n}\,\vec{n}^\top = M$.

The Determinant of a 2×2 Matrix

We will discuss determinants of square matrices in general in a later chapter. Here we'll give the definition of the determinant of a 2×2 matrix.

Definition 8

The **determinant** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = a\,d - b\,c.$

Example 10

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1, \ \det\left(\begin{bmatrix} 3 & 6 \\ -2 & -5 \end{bmatrix}\right) = 3 \cdot (-5) - 6 \cdot (-2) = -3.$$

and $\det\left(\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}\right) = 3 \cdot 4 - 6 \cdot 2 = 0$

We'll usually leave out the extra parenthesis and simply write e.g. $\det\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

The following theorem is actually true in general, but here we'll prove it for 2×2 matrices. The general proof requires some knowledge such as row reduction, which we will discuss later. It is useful to see the proof of the 2×2 case, which makes you appreciate how non-trivial this theorem is for larger matrices.

Theorem 4

If $A, B \in M_{2\times 2}(\mathbb{R})$ then $\det(AB) = \det(A)\det(B)$

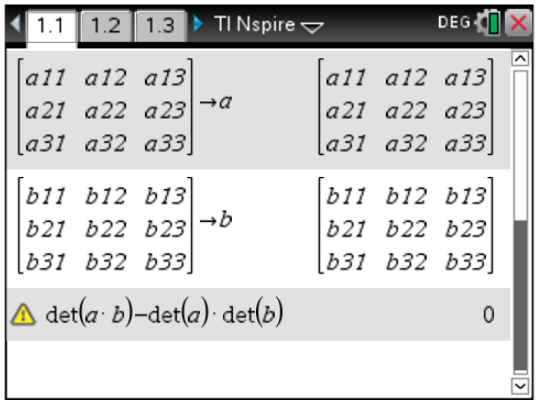
Proof: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ then $A\,B = \begin{bmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{bmatrix}$ hence

$$\begin{aligned} \det(AB) &= (ar+bt)(cs+du) - (as+bu)(cr+dt) \\ &= acrs + adru + bcst + bdtu - acrs - adst - bcru - bdtu \\ &= adru + bcst - adst - bcru \\ &= (ad-bc)(ru-st) \\ &= \det(A)\det(B) \end{aligned}$$

$\square$

You can only imagine the algebra needed to prove the 3×3 or 4×4 case, let alone the general $n \times n$ case!

Here is a proof of $\det(AB) = \det(A)\det(B)$ for 3×3 matrices using the TI-Nspire:



The Inverse of a 2×2 Matrix

We will discuss inverses of matrices in general in a later chapter. Here we'll give a formal definition, show uniqueness and give a formula for the inverse of a 2×2 matrix.

Definition 9

We call an $n \times n$ matrix A **invertible** if there exits an $n \times n$ matrix B such that $A\,B = I_n = B\,A$

Example 11

Since $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ the matrices $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$ are inverses of each other.

Theorem 5

If A is an invertible matrix, it has a **unique** inverse.

Proof: Suppose A has two inverses B and C, i.e. $\begin{cases} A\,B = I_n = B\,A \\ A\,C = I_n = C\,A \end{cases}$

Hence $B = B\,I_n = B(A\,C) = (B\,A)\,C = I_n\,C = C$ so that $B = C$. ◻

Since the inverse of an invertible matrix A is unique we have a special notation for it:

$$\boxed{A^{-1}}$$

i.e. $A\,A^{-1} = I_n = A^{-1}\,A$

The inverse of an invertible 2×2 matrix is easy to compute:

Theorem 6

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det(A) \neq 0$, in which case its inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof: There are two directions to this statement: A is invertible $\iff$ $\det(A) \neq 0$

$(\Rightarrow)$ Suppose A is invertible, i.e. there exists a matrix B such that $A\,B = I_n = B\,A$. By Theorem 4 this gives us that

$$\det(A)\det(B) = \det(A\,B) = \det(I) = 1$$

which implies that $\det(A) \neq 0$.

$(\Leftarrow)$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that $\det(A) = ad - bc \neq 0$, then the matrix

$B = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ exists. For this B we have $A\,B = I_n = B\,A$ [check!].

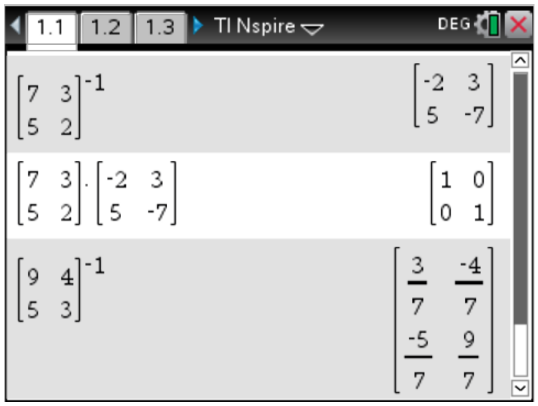
Hence A is invertible. This shows that $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 12 $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}^{-1} = \dfrac{1}{7 \cdot 2 - 3 \cdot 5} \begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}$. Let's check:

$$\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

Example 13 $\begin{bmatrix} 9 & 4 \\ 5 & 3 \end{bmatrix}^{-1} = \dfrac{1}{9\cdot 3 - 4\cdot 5}\begin{bmatrix} 3 & -4 \\ -5 & 9 \end{bmatrix} = \dfrac{1}{7}\begin{bmatrix} 3 & -4 \\ -5 & 9 \end{bmatrix}$. Let's check:

$$\frac{1}{7}\begin{bmatrix} 3 & -4 \\ -5 & 9 \end{bmatrix}\begin{bmatrix} 9 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 5 & 3 \end{bmatrix}\cdot\frac{1}{7}\begin{bmatrix} 3 & -4 \\ -5 & 9 \end{bmatrix}$$



Powers of square Matrices

Let A be a square $m \times m$ matrix. We define A^n for all $n = 0, 1, 2, 3, \cdots$ recursively as follows

(1) $A^0 = I_m$

(2) $A^{n+1} = A\,A^n$ for $n = 0, 1, 2, 3, \cdots$

Thus $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, $A^4 = AAAA$, etc.

If we want to define A^n for all $n \in \mathbb{Z}$, we require A to be invertible. Letting A^{-1} be the inverse of the matrix A, as usual, we then define

(3) $A^{-n} = (A^{-1})^n$

Using induction we can prove that the following power laws also work for matrices:

Theorem 7

(a) $A^nA^m = A^{n+m}$ for all $n, m \in \mathbb{Z}$

(b) $(A^n)^m = A^{nm}$ for all $n, m \in \mathbb{Z}$

Example 14

- $A^3A^4 = (AAA)(AAAA) = A^7 = A^{3+4}$
- $(A^2)^3 = (AA)(AA)(AA) = A^6 = A^{2\cdot 3}$
- $A^{-3}A^5 = (A^{-1}A^{-1}A^{-1})(AAAAA) = A^2 = A^{-3+5}$
- $(A^2)^{-3} = (AA)^{-1}(AA)^{-1}(AA)^{-1} = A^{-6} = A^{2\cdot(-3)}$