# Basic Facts of Vector Spaces

Here we list and prove some basic facts of vector spaces, that most people find obvious or take for granted. They follow immediately from the axioms of vector spaces:

**Theorem 1**

Let $\vec{v}$, $\vec{w}$, $\vec{a} \in \mathbb{V}$.

**(1)** The zero vector $(\vec{0})$ is unique.

**(2)** The cancellation law: $\vec{v} + \vec{a} = \vec{w} + \vec{a} \quad \Rightarrow \quad \vec{v} = \vec{w}$.

**(3)** Opposites $(-\vec{v})$ are unique.

**(4)** $0 \cdot \vec{v} = \vec{0}$

**(5)** $-\vec{v} = -1 \cdot \vec{v}$

**(1)** The third axiom of a vector space says there exists a vector $\vec{0}$ such that

$$\forall \vec{v} \in \mathbb{V}: \quad \vec{v} + \vec{0} = \vec{v}$$

But it doesn't rule out the possibility that there are other vectors with this property. We will prove that in fact there is only ***one*** vector with this property.

Suppose there are two vectors with this property, $\vec{0}_1$ and $\vec{0}_2$, then

$$\left.\begin{array}{lcl} \forall \vec{v} \in \mathbb{V}: \quad \vec{v} + \vec{0}_1 = \vec{v} & \Rightarrow & \vec{0}_2 + \vec{0}_1 = \vec{0}_2 \\ \forall \vec{v} \in \mathbb{V}: \quad \vec{v} + \vec{0}_2 = \vec{v} & \Rightarrow & \vec{0}_1 + \vec{0}_2 = \vec{0}_1 \end{array}\right\} \quad \Rightarrow \quad \vec{0}_1 = \vec{0}_2 \qquad \square$$

**(2)** The fourth vector space axiom says that for **any** $\vec{v} \in \mathbb{V}$ there exists a $-\vec{v}$ such that

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Hence
$$\begin{aligned} \vec{v} + \vec{a} = \vec{w} + \vec{a} \quad & \Rightarrow \quad (\vec{v} + \vec{a}) + (-\vec{a}) = (\vec{w} + \vec{a}) + (-\vec{a}) \\ & \Rightarrow \quad \vec{v} + \big(\vec{a} + (-\vec{a})\big) = \vec{w} + \big(\vec{a} + (-\vec{a})\big) \\ & \Rightarrow \quad \vec{v} + \vec{0} = \vec{w} + \vec{0} \\ & \Rightarrow \quad \vec{v} = \vec{w} \qquad \square \end{aligned}$$

We'll use this cancellation law in the next three proofs.

**(3)** Even though the fourth vector space axiom says that for **any** $\vec{v} \in \mathbb{V}$ there exists an opposite $-\vec{v}$ with the property that $\vec{v} + (-\vec{v}) = \vec{0}$, it doesn't rule out the possibility that there are multiple opposites. In fact we'll prove that for each $v$, there is only ***one*** opposite.

Suppose there is a vector $\vec{v} \in \mathbb{V}$, with two opposites, $\vec{w}_1$ and $\vec{w}_2$, then

$$\begin{matrix} \vec{v} + \vec{w}_1 = \vec{0} \\ \vec{v} + \vec{w}_2 = \vec{0} \end{matrix} \quad \Rightarrow \quad \vec{v} + \vec{w}_1 = \vec{v} + \vec{w}_2 \quad \overset{(2)}{\Longrightarrow} \quad \vec{w}_1 = \vec{w}_2 \qquad \square$$

**(4)** Note that $\vec{v} + 0 \cdot \vec{v} = 1 \cdot \vec{v} + 0 \cdot \vec{v} = (1+0) \cdot \vec{v} = 1 \cdot \vec{v} = \vec{v} = \vec{v} + \vec{0}$ which means that

$$\vec{v} + 0 \cdot \vec{v} = \vec{v} + \vec{0} \quad \overset{(2)}{\Longrightarrow} \quad 0 \cdot \vec{v} = \vec{0} \qquad \square$$

Here is another proof this:

Note that $0 \cdot \vec{v} + \vec{0} = 0 \cdot \vec{v} = (0+0) \cdot \vec{v} = 0 \cdot \vec{v} + 0 \cdot \vec{v}$ which means that

$$0 \cdot \vec{v} + \vec{0} = 0 \cdot \vec{v} + 0 \cdot \vec{v} \quad \overset{(2)}{\Longrightarrow} \quad 0 \cdot \vec{v} = \vec{0} \qquad \square$$

**(5)** Again we can use the cancellation law to prove this.

Note that $\vec{v} + (-\vec{v}) = \vec{0} \overset{(4)}{=} 0 \cdot \vec{v} = \big(1 + (-1)\big) \cdot \vec{v} = 1 \cdot \vec{v} + (-1) \cdot \vec{v} = \vec{v} + (-1) \cdot \vec{v}$ which means that

$$\vec{v} + (-\vec{v}) = \vec{v} + (-1) \cdot \vec{v} \quad \overset{(2)}{\Longrightarrow} \quad -\vec{v} = (-1) \cdot \vec{v} \qquad \square$$