Solving Systems of Linear Equations

Consider the following three problems:

Problem 1

Find the intersection of the three planes $\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \\ 2x + z = -1 \end{cases}$ over $\mathbb{R}$, i.e. find all the points (x, y, z) with $x,\ y,\ z \in \mathbb{R}$ that satisfy all three equations.

Problem 2

Write $\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, i.e. find $x,\ y,\ z \in \mathbb{R}$ such that $x \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

Problem 3

Find all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

Each of these problems asks a different question in a different arena:

(1) A geometric question about points in 3D Euclidean space, that lie simultaneously on three planes.

(2) An algebraic question about dependence of vectors in the vector space $\mathbb{R}^3$.

(3) A question about a transformation $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ between two vector spaces, defined by $T(\vec{x}) = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \vec{x}$, and the pre-images of the vector $\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$.

Nevertheless, all three questions can be answered with the same row reduction!
Note that all these are related as follows

$$\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \\ 2x + z = -1 \end{cases} \Leftrightarrow \begin{bmatrix} 3x + 2y - z \\ x + y + z \\ 2x + z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix} \Leftrightarrow x \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Example 1 Let's solve $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

Look at the **augmented** matrix: $A = \begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix}$.

We know there exists a Q such that $Q \cdot A = \text{rref}(A)$ in fact $Q = \frac{1}{7} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & -4 \\ -2 & 4 & 1 \end{bmatrix}$.

Hence

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \text{rref}(A)$$

which means that

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q \cdot \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

Hence if we left multiply each side of $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$ by Q we get

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q \cdot \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

We have solved the three problems simultaneously:

(1) $(-2, 5, 3)$ is the only point on all three planes: $\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \\ 2x + z = -1 \end{cases}$

(2) There is only one linear combination: $-2 \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

(3) $\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ is the only pre-image of $\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$, which means $\vec{x} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ is the only vector such that $T(\vec{x}) = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$, i.e. for which $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

Example 2 Let's solve $\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \end{cases}$

This is equivalent to solving $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

Row reduce the augmented matrix $A = \begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix}$.

We know there exists a Q such that $Q \cdot A = \text{rref}(A)$ in fact $Q = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. Hence

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 4 & 17 \end{bmatrix} = \text{rref}(A)$$

which means that

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix} \qquad \text{and} \qquad Q \cdot \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \end{bmatrix}$$

Hence if we left multiply each side of $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ by Q we get

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q \cdot \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

hence we get

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \end{bmatrix}$$

so that

$$\begin{cases} x \qquad - 3z = -11 \\ \qquad y + 4z = \ \ 17 \end{cases}$$

At first it looks like we have made no progress. We have simply replaced the system of equations

$$\begin{cases} 3x + 2y - z = 1 \\ \ \ x + \ \ y + z = 6 \end{cases} \qquad \text{with} \qquad \begin{cases} x \qquad - 3z = -11 \\ \qquad y + 4z = \ \ 17 \end{cases}$$

Note that both systems of equations have the same solution set. This is guaranteed by the fact that we used an **invertible** matrix Q. And it is easy to read off the solution from the new set of equations. You just have to learn how. Rewrite it as follows:

$$\begin{cases} x = -11 + 3z \\ y = \ \ 17 - 4z \end{cases}$$

and add a trivial equation

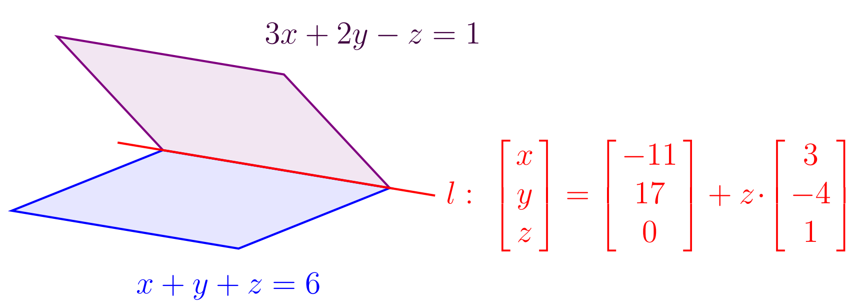
$$\begin{cases} x = -11 + 3z \\ y = \ \ 17 - 4z \\ z = \qquad\quad z \end{cases}$$

i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

This is the parametric equation of a line. The parameter is z which can run through all possible real numbers. There are infinitely many solutions to our system of equations, an entire line, which shouldn't come as a surprise, since we are intersecting two planes.
We can easily check our solution by substitution:

$$\begin{cases} 3(-11 + 3z) + 2(17 - 4z) - z = 1 \quad \Leftrightarrow \quad 1 = 1 \ \checkmark \\ \ \ (-11 + 3z) + \ \ (17 - 4z) + z = 6 \quad \Leftrightarrow \quad 6 = 6 \ \checkmark \end{cases}$$



We need to point out that the method outlined above, with the use of Q, and all steps and explanations, is basically intended to show why it works, and has much more detail than we would usually include. Let's show the way we actually solve a system of equations:

To solve: $\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \end{cases}$ we compute, the rref of the augmented matrix

$$\operatorname{rref}\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 4 & 17 \end{bmatrix}$$

and conclude that

$$\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = -11 + 3z \\ y = 17 - 4z \end{cases}$$

so that

$$\begin{cases} x = -11 + 3z \\ y = 17 - 4z \\ z = z \end{cases}$$

i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

Example 3 Let's solve $\begin{cases} 2w - 4x + y + 2z = 17 \\ -w + 2x + y - 7z = 2 \\ 3w - 6x + y + 5z = 22 \\ y - 4z = 7 \end{cases}$ in $\mathbb{R}^4$

First we row reduce the augmented matrix

$$\operatorname{rref}\begin{bmatrix} 2 & -4 & 1 & 2 & 17 \\ -1 & 2 & 1 & -7 & 2 \\ 3 & -6 & 1 & 5 & 22 \\ 0 & 0 & 1 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 3 & 5 \\ 0 & 0 & 1 & -4 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{cases} 2w - 4x + y + 2z = 17 \\ -w + 2x + y - 7z = 2 \\ 3w - 6x + y + 5z = 22 \\ y - 4z = 7 \end{cases} \quad \Leftrightarrow \quad \begin{cases} w - 2x + 2z = 17 \\ y - 4z = 7 \end{cases}$$

which means $\begin{cases} w = 17 + 2x - 2z \\ x = x \\ y = 7 + 4z \\ z = z \end{cases} \quad \Rightarrow \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$

Example 4 Solve the system of equations: $\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \\ 2x + y - 2z = 5 \end{cases}$

Row reduction gives us: $\quad \operatorname{rref}\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Hence $\quad \begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \\ 2x + y - 2z = 5 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x - 3z = 0 \\ y + 4z = 0 \\ 0 = 1 \end{cases}$

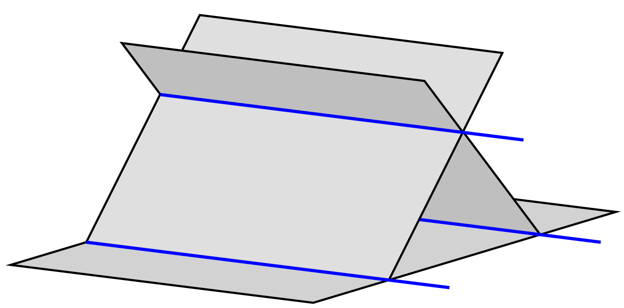
The solution sets of these systems of equations are the same! The new system of equations (the one on the right) clearly does NOT have solutions, since there are no x, y and z for which $0 = 1$ becomes true. Hence the original system doesn't have solutions either.

Note 1 When, while solving systems of equations, the row reduction of the augmented matrix gives us a row of zeros and one single 1 at the end,

$$[\,0\ 0\ 0 \cdots 0\ 0\ 1\,]$$

we know that there are no solutions, since $0 \neq 1$.

Geometrically what is going on in this case is that the system of equations represents three planes in 3D Euclidean space, which pairwise intersect in lines, but do not all intersect simultaneously: the lines of intersection are parallel



Let's discuss why a system of equations and its row reduced system have the same solution set. We'll use the example

$$\begin{cases} 3x + 2y - z = 1 \\ x + y + z = 6 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x - 3z = -11 \\ y + 4z = 17 \end{cases}$$

Row reduction gives us:

$$\operatorname{rref}\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 4 & 17 \end{bmatrix}$$

Hence we know there exists a Q such that

$$Q \cdot \begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 4 & 17 \end{bmatrix}$$

which means $Q \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix}$ and $Q \cdot \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \end{bmatrix}$

(a) Suppose (x_0, y_0, z_0) is a solution of the first system, then

$$\begin{cases} 3x_0 + 2y_0 - z_0 = 1 \\ x_0 + y_0 + z_0 = 6 \end{cases} \Rightarrow \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\Rightarrow Q \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = Q \cdot \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_0 \qquad - 3z_0 = -11 \\ \qquad y_0 + 4z_0 = 17 \end{cases}$$

So if (x_0, y_0, z_0) is a solution of the first system, then (x_0, y_0, z_0) is also a solution of the second system.

(b) Suppose (x_0, y_0, z_0) is a solution of the second system, then

$$\begin{cases} x_0 \qquad - 3z_0 = -11 \\ \qquad y_0 + 4z_0 = 17 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \end{bmatrix}$$

$$\Rightarrow Q^{-1} \cdot \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = Q^{-1} \cdot \begin{bmatrix} -11 \\ 17 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3x_0 + 2y_0 - z_0 = 1 \\ x_0 + y_0 + z_0 = 6 \end{cases}$$

So if (x_0, y_0, z_0) is a solution of the second system, then (x_0, y_0, z_0) is also a solution of the first system.

Notice that we used Q^{-1} here. That is why it was so important to know that there exists an **invertible** Q. Of course this derives from the fact that row reduction is done with elementary row operations, and each of those is invertible.

(a) and **(b)** tell us that the system of equations and it row reduced system have the same solution set!

Most Linear Algebra books will actually introduce row reduction in the setting of equations, rather than matrices or augmented matrices. The operations they use are

Type I	Swapping two equations
Type II	Multiplying an equation with a **non-zero** constant
Type III	Adding a multiple of one equation to another equation

They will proceed with solving e.g. $\begin{cases} 3x+2y-z=1\\ x+y+z=6\\ 2x+z=-1\end{cases}$ as follows:

$$\begin{cases} 3x+2y-z=1\\ x+y+z=6\\ 2x+z=-1\end{cases} \xrightarrow{\mathrm{Eq}_1\leftrightarrow\mathrm{Eq}_2} \begin{cases} x+y+z=6\\ 3x+2y-z=1\\ 2x+z=-1\end{cases}$$

$$\xrightarrow{-3\mathrm{Eq}_1+\mathrm{Eq}_2} \begin{cases} x+y+z=6\\ -y-4z=-17\\ 2x+z=-1\end{cases} \xrightarrow{-2\mathrm{Eq}_1+\mathrm{Eq}_3} \begin{cases} x+y+z=6\\ -y-4z=-17\\ -2y-z=-13\end{cases}$$

$$\xrightarrow[-\mathrm{Eq}_3]{-\mathrm{Eq}_2} \begin{cases} x+y+z=6\\ y+4z=17\\ 2y+z=13\end{cases} \xrightarrow{-2\mathrm{Eq}_2+\mathrm{Eq}_3} \begin{cases} x+y+z=6\\ y+4z=17\\ -7z=-21\end{cases}$$

$$\xrightarrow{-\frac{1}{7}\mathrm{Eq}_3} \begin{cases} x+y+z=6\\ y+4z=17\\ z=3\end{cases} \xrightarrow{-4\mathrm{Eq}_3+\mathrm{Eq}_2} \begin{cases} x+y+z=6\\ y=5\\ z=3\end{cases}$$

$$\xrightarrow{-\mathrm{Eq}_3+\mathrm{Eq}_1} \begin{cases} x+y=3\\ y=5\\ z=3\end{cases} \xrightarrow{-3\mathrm{Eq}_1+\mathrm{Eq}_2} \begin{cases} x=-2\\ y=5\\ z=3\end{cases}$$

Only after such a tedious discussion are matrices introduced. Clearly what is happening is exactly the same as what we did, except we didn't have to go through the torture of having to copy x, y and z in every line. After all, what is changing at every step are only the coefficients. That's why we introduced row reduction with the augmented matrix of the coefficients. This makes the entire procedure simpler and more elegant.

Row Reduction is the most important tool of Linear Algebra

We'll end with some examples over other fields:

Example 5 Solve in $\mathbb{C}^5$ the following system of equations

$$\begin{cases} \boldsymbol{i}\,v + w + (1+\boldsymbol{i})x + y + (3-2\boldsymbol{i})z = 0 \\ (1-\boldsymbol{i})v + (-1-\boldsymbol{i})w + (2-\boldsymbol{i})x + (2-2\boldsymbol{i})y + (1+3\boldsymbol{i})z = \boldsymbol{i} \\ 2\boldsymbol{i}\,v + 2\,w + -\boldsymbol{i}\,x + (-3-\boldsymbol{i})y + 2\,z = 2+13\boldsymbol{i} \\ (1+\boldsymbol{i})v + (1-\boldsymbol{i})w + 3\,x + (2-2\boldsymbol{i})y + (1+\boldsymbol{i})z = -3+6\boldsymbol{i} \end{cases}$$

Row reduction gives us

$$\operatorname{rref}\begin{bmatrix} \boldsymbol{i} & 1 & 1+\boldsymbol{i} & 1 & 3-2\boldsymbol{i} & 0 \\ 1-\boldsymbol{i} & -1-\boldsymbol{i} & 2-\boldsymbol{i} & 2-2\boldsymbol{i} & 1+3\boldsymbol{i} & \boldsymbol{i} \\ 2\boldsymbol{i} & 2 & -\boldsymbol{i} & -3-\boldsymbol{i} & 2 & 2+13\boldsymbol{i} \\ 1+\boldsymbol{i} & 1-\boldsymbol{i} & 3 & 2-2\boldsymbol{i} & 1+\boldsymbol{i} & -3+6\boldsymbol{i} \end{bmatrix} = \begin{bmatrix} 1 & -\boldsymbol{i} & 0 & \boldsymbol{i} & 0 & 5 \\ 0 & 0 & 1 & 1-\boldsymbol{i} & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that

$$\begin{cases} v - \boldsymbol{i}\,w + \boldsymbol{i}\,y = 5 \\ x + (1-\boldsymbol{i})y = -3 \\ z = 1 \end{cases}$$

i.e.

$$\begin{cases} v = 5 + \boldsymbol{i}\,w - \boldsymbol{i}\,y \\ w = w \\ x = -3 - (1-\boldsymbol{i})y \\ y = y \\ z = 1 \end{cases}$$

so that

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + w \cdot \begin{bmatrix} \boldsymbol{i} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} -\boldsymbol{i} \\ 0 \\ -1+\boldsymbol{i} \\ 1 \\ 0 \end{bmatrix}$$

This is basically a two-dimensional affine space in $\mathbb{C}^5$.

Example 6 In $\mathbb{F}_4^3$ write $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as a linear combination of $\begin{bmatrix}a\\b\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\b\end{bmatrix}$ and $\begin{bmatrix}a\\1\\b\end{bmatrix}$.

Note this is equivalent to

$$x\cdot\begin{bmatrix}a\\b\\1\end{bmatrix}+y\cdot\begin{bmatrix}1\\1\\b\end{bmatrix}+z\cdot\begin{bmatrix}a\\1\\b\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix} \quad\Leftrightarrow\quad \begin{cases} a\,x+\ \ y+a\,z=1\\ b\,x+\ \ y+\ \ z=1\\ \ \ x+b\,y+b\,z=1\end{cases} \quad\Leftrightarrow\quad \begin{bmatrix}a&1&a\\b&1&1\\1&b&b\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Row reduction gives us:

$$\operatorname{rref}\begin{bmatrix}a&1&a&1\\b&1&1&1\\1&b&b&1\end{bmatrix}=\begin{bmatrix}1&0&0&b\\0&1&0&a\\0&0&1&1\end{bmatrix}$$

so that

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}b\\a\\1\end{bmatrix}$$

i.e.

$$b\cdot\begin{bmatrix}a\\b\\1\end{bmatrix}+a\cdot\begin{bmatrix}1\\1\\b\end{bmatrix}+1\cdot\begin{bmatrix}a\\1\\b\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Example 7 Transformation $T:\ \mathbb{F}_7^4\ \to\ \mathbb{F}_7^4$ is defined by $T(\vec{x})=\begin{bmatrix}1&2&4&6\\4&1&2&3\\2&4&1&5\\3&6&5&4\end{bmatrix}\cdot\vec{x}$

Find all $\vec{x}\in\mathbb{F}_7^4$ such that $T(\vec{x})=\begin{bmatrix}3\\5\\6\\2\end{bmatrix}$

If we let $\vec{x}=\begin{bmatrix}w\\x\\y\\z\end{bmatrix}$ then we want to solve: $\begin{bmatrix}1&2&4&6\\4&1&2&3\\2&4&1&5\\3&6&5&4\end{bmatrix}\cdot\begin{bmatrix}w\\x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\5\\6\\2\end{bmatrix}$

Row reduction gives us: $\operatorname{rref}\begin{bmatrix}1&2&4&6&3\\4&1&2&3&5\\2&4&1&5&6\\3&6&5&4&2\end{bmatrix}=\begin{bmatrix}1&2&4&6&3\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

which means that:

$$w + 2x + 4y + 6z = 3$$

Hence

$$w = 3 + 5x + 3y + z$$

which means that

$$\begin{cases} w &= 3 + 5x + 3y + z \\ x &= x \\ y &= y \\ z &= z \end{cases}$$

so that

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Since in $\mathbb{F}_7$ there are 7 choices for each of the parameters x, y and z, there are $7^3 = 343$ solutions:

For example:

$$\begin{bmatrix} 0 \\ 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's check that this $\vec{x} = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 6 \end{bmatrix}$ is indeed a solution of $T(\vec{x}) = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 2 \end{bmatrix}$:

$$T(\vec{x}) = T\left(\begin{bmatrix} 0 \\ 4 \\ 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 4 & 1 & 2 & 3 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 2 \end{bmatrix} \qquad \checkmark$$

This is just 1 of the 343 solutions.

Warning Some people mistakenly assume that to solve a system of equations over $\mathbb{F}_7$, you can just solve it over $\mathbb{R}$ and reinterpret the answer over $\mathbb{F}_7$.

Example 8

$$\begin{cases} x + 3y + z = 1 \\ 6x + 4y + 6z = 6 \\ x + 3y + 2z = 0 \end{cases}$$

If we row reduce over $\mathbb{R}$ we get: $\text{rref}\begin{bmatrix} 1 & 3 & 1 & 1 \\ 6 & 4 & 6 & 6 \\ 1 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

but it would be wrong to conclude that over $\mathbb{F}_7$ the solution would be: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$, where we just replaced the -1 with 6.

In fact: $\text{rref7}\begin{bmatrix} 1 & 3 & 1 & 1 \\ 6 & 4 & 6 & 6 \\ 1 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + y\cdot \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$.

Hence there are 7 solutions: $\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}$, corresponding to $y = 0, 1, \cdots, 6$.

Example 9

$$\begin{cases} x + 6y + 5z = 5 \\ 3x + y + 3z = 0 \\ 2x + 3y + 2z = 1 \end{cases}$$

Row reduction over $\mathbb{R}$: $\text{rref}\begin{bmatrix} 1 & 6 & 5 & 5 \\ 3 & 1 & 3 & 0 \\ 2 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\frac{11}{14} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{9}{14} \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -11 \\ 6 \\ 9 \end{bmatrix}$

Row reduction over $\mathbb{F}_7$: $\text{rref7}\begin{bmatrix} 1 & 6 & 5 & 5 \\ 3 & 1 & 3 & 0 \\ 2 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$ no solutions

Example 10

$$\begin{cases} x + 2y + z = 5 \\ x + y + 2z = 0 \\ 2x + 4y + 2z = 3 \end{cases}$$

Row reduction over $\mathbb{R}$: $\quad \operatorname{rref}\begin{bmatrix} 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad$ no solutions

Row reduction over $\mathbb{F}_7$: $\quad \operatorname{rref7}\begin{bmatrix} 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + z \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$