Linear Combinations

The only two operations in a vector space $\langle \mathbb{V}, +, \cdot, \mathbb{F} \rangle$ are vector addition, and scalar multiplication. Vectors can be added and multiplied with scalars, that is it! Hence the only objects that we can create in a vector space use those two operations, e.g.

$$2 \cdot \vec{u} - 5 \cdot \vec{v} + 4 \cdot \vec{w}$$

We call these linear combinations.

Theorem 1

Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n$ be vectors in $\langle \mathbb{V}, +, \cdot, \mathbb{F} \rangle$ and let $a_1, a_2, a_3, \cdots, a_n$ be scalars from $\mathbb{F}$.

A **finite** sum

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 + \cdots + a_n \cdot \vec{v}_n$$

is called a **linear combination**.

Note 1

We do **not** allow infinite sums as linear combinations. Recall that the number π is an irrational number

$$\pi = 3.14159265358979323846264338327950288\cdots$$

Hence the infinite sum $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.04 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.001 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.0005 \\ 1 \end{bmatrix} + \cdots = \begin{bmatrix} \pi \\ \infty \end{bmatrix}$

is not in $\mathbb{Q}^2$, even though all summands are vectors in $\mathbb{Q}^2$. Adding infinitely many vectors together might result in an object that is no longer in our vector space.

Example 1

In $\mathbb{R}^2$: $\quad 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 2 \\ -4 \end{bmatrix} + 2 \cdot \begin{bmatrix} -6 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

In $P_2(\mathbb{R})$: $\quad 8 \cdot \left(t^2 + 1\right) - 3 \cdot \left(3t^2 + t + 1\right) + \left(t^2 + 3t - 13\right)$

In $M_{2\times 2}(\mathbb{R})$: $\quad 5 \cdot \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} + 3 \cdot \begin{bmatrix} 3 & -4 \\ -5 & 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} -7 & 1 \\ 0 & 7 \end{bmatrix}$

Of course any linear combinations represents another vector. Each of the linear combinations above represent the zero vector: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\ 0 \cdot t^2 + 0 \cdot t + 0$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ respectively.

There are some important questions we will address in the coming chapters:

(a) Given a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n\} \subset \mathbb{V}$ can any vectors $\vec{w}$ be written as a linear combination of these vectors? This amounts to solving

$$\vec{w} = a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 + \cdots + a_n \cdot \vec{v}_n$$

Which $\vec{w}$ can be so represented, and which ones cannot?

(b) If $\vec{w} = a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 + \cdots + a_n \cdot \vec{v}_n$ has a solution, is it unique? What are all solutions?

The first question has to do with the notation of a **span** of vectors, which we will discuss in chapter 12. The second question has to do with the notion of **linear independence** of vectors, which we will be discussed in chapter 13. Both notions converge in the concept of a **basis**, which in turn will induce **coordinates**, the concepts we will be discussing in chapters 14 and 17. Once bases and coordinates come to the front, one realizes we can do row reduction in essentially any vector space, and that matrices become a tool for describing linear transformations between between vector spaces. These notions are absolutely central in Linear Algebra.