Subspaces

We call a nonempty subset $\mathbb{W}$ of a vector space $\langle \mathbb{V}, +, \cdot, \mathbb{F} \rangle$, a subspace if $\mathbb{W}$ with the same field $\mathbb{F}$ and the same operations $+$ and $\cdot$ is a vector space in its own right, i.e. $\langle \mathbb{W}, +, \cdot, \mathbb{F} \rangle$ is a vector space. Notation: $\boxed{\mathbb{W} \sqsubseteq \mathbb{V}}$

Example 1

(a) $\mathbb{W} = \left\{ a \cdot t^2 + a \cdot t + a \;\middle|\; a \in \mathbb{R} \right\} \sqsubseteq P_2(\mathbb{R})$

(b) $\mathbb{W} = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \sqsubseteq \mathbb{R}^3$

(c) $\mathbb{W} = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \;\middle|\; a, b, c, d, e, f \in \mathbb{R} \right\} = \mathbb{U}_{3\times 3}(\mathbb{R}) \sqsubseteq \mathbb{M}_{3\times 3}(\mathbb{R})$

Of course we could just simply check if all 8 properties of a vector space are true for W, but this is not really necessary, since most properties are inherited by virtue of $\mathbb{W} \subseteq \mathbb{V}$. The following theorem is sufficient:

Theorem 1

Let $\mathbb{W} \subseteq \mathbb{V}$. If $\langle \mathbb{V}, +, \cdot, \mathbb{F} \rangle$ is a vector space, and

(a) $\vec{0}_{\mathbb{V}} \in \mathbb{W}$

(b) $\forall\, \vec{a}, \vec{b} \in \mathbb{W}$: $\vec{a} + \vec{b} \in \mathbb{W}$

(c) $\forall\, t \in \mathbb{F},\ \vec{a} \in \mathbb{W}$: $t \cdot \vec{a} \in \mathbb{W}$

then $\langle \mathbb{W}, +, \cdot, \mathbb{F} \rangle$ is a vector space, i.e. $\mathbb{W} \sqsubseteq \mathbb{V}$.

Proof: We will check that all properties of a vector space are true for $\langle \mathbb{W}, +, \cdot, \mathbb{F} \rangle$

Clearly the operations are closed: $\begin{cases} \forall\, \vec{a}, \vec{b} \in \mathbb{W}: & \vec{a} + \vec{b} \in \mathbb{W} \\ \forall\, t \in \mathbb{F},\ \vec{a} \in \mathbb{W}: & t \cdot \vec{a} \in \mathbb{W} \end{cases}$

Next we'll check all 8 properties. We'll see most properties are inherited from $\langle \mathbb{V}, +, \cdot, \mathbb{F} \rangle$

(1) $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ for all $\vec{v},\vec{w}\in\mathbb{W}$, since it is true for all $\vec{v},\vec{w}\in\mathbb{V}$, i.e. it is inherited.

(2) $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ for all $\vec{u},\ \vec{v},\ \vec{w}\in\mathbb{W}$, is inherited from $\mathbb{V}$.

(3) There exists a unique $\vec{0}\in\mathbb{W}$ such that $\vec{w}+\vec{0}=\vec{w}$ for all $\vec{v}\in\mathbb{W}$: $\vec{0}=\vec{0}_{\mathbb{V}}\in\mathbb{W}$

(4) Since scalar multiplication is closed: if $w\in\mathbb{W}$ then $-w=-1\cdot w\in\mathbb{W}$. Hence $\mathbb{W}$ contains all opposites, and $\vec{w}+(-\vec{w})=\vec{0}$ for all $\vec{w}\in\mathbb{W}$ is inherited.

(5) $1\cdot\vec{w}=\vec{w}$ is inherited

(6) $r\cdot(s\cdot\vec{w})=(r\cdot s)\cdot\vec{w}$ for all $r,s\in\mathbb{F}$ and $\vec{w}\in\mathbb{W}$ is inherited.

(7) $s\cdot(\vec{u}+\vec{w})=s\cdot\vec{u}+s\cdot\vec{w}$ for all $s\in\mathbb{F}$ and $\vec{u},\vec{w}\in\mathbb{W}$ is inherited.

(8) $(r+s)\cdot\vec{w}=r\cdot\vec{w}+s\cdot\vec{w}$ for all $r,s\in\mathbb{F}$ and $\vec{w}\in\mathbb{W}$ is inherited. □

Example 2 $\mathbb{W}=\left\{p(t)\in P_2(\mathbb{R})\ \middle|\ p(0)=0\right\}\sqsubseteq P_2(\mathbb{R})$

$\mathbb{W}$ is the subspace of all polynomials of degree 2 or less, with zero constant term, i.e.

$$\mathbb{W}=\left\{a\,t^2+b\,t\ \middle|\ a,\ b\in\mathbb{R}\right\}$$

It is clearly a sub**space**, since

- **(a)** $\vec{0}_{P_2(\mathbb{R})}=0t^2+0t+0\in\mathbb{W}$
- **(b)** $\forall\ p_1(t),\ p_2(t)\in\mathbb{W}$: $p_1(t)+p_2(t)\in\mathbb{W}$, since $p_1(0)+p_2(0)=0$.
- **(c)** $\forall\ s\in\mathbb{R},\ p(t)\in\mathbb{W}$: $s\cdot p(t)\in\mathbb{W}$, since $s\cdot p(0)=s\cdot 0=0$.

Example 3 The subset of all **differentiable** real valued functions

$$D\ =\ \left\{f:\mathbb{R}\to\mathbb{R}\ \middle|\ f\text{ is differentiable}\right\}$$

is a sub**space** of the vector space of all real valued functions

$$\left\{f:\mathbb{R}\to\mathbb{R}\ \middle|\ f\text{ is differentiable}\right\}\ \sqsubseteq\ \{f:\mathbb{R}\to\mathbb{R}\}$$

It is clearly a sub**space**, since

- **(a)** $f(x)\equiv 0\in D$ since $f'(x)=0$, i.e. $f(x)\equiv 0$ is differentiable.
- **(b)** $\forall\ f(x),\ g(x)\in D$: $f(x)+g(x)\in D$, since $(f(x)+g(x))'=f'(x)+g'(x)$.
- **(c)** $\forall\ t\in\mathbb{R},\ f(x)\in D$: $t\cdot f(x)\in D$, since $(t\cdot f(x))'=t\cdot f'(x)$.