A Strange Example of a Vector Space

Let $\mathbb{V} = \mathbb{R}^+$ and $\mathbb{F} = \mathbb{R}$ and let's define the following operations: $\oplus$ and $\odot$

(1) $a \oplus b = a \cdot b$ called the vector addition.

(2) $t \odot a = a^t$ called the scalar multiplication.

Example 1

$$3 \oplus 2 = 6, \qquad 3 \oplus 5 = 15, \qquad 1 \oplus 3 = 3$$

$$3 \odot 2 = 8, \qquad 3 \odot 5 = 125, \qquad -2 \odot 3 = \frac{1}{9}, \qquad 0 \odot 2 = 1,$$

Theorem 1

$\langle \mathbb{V} = \mathbb{R}^+, \ \oplus, \ \odot, \ \mathbb{F} = \mathbb{R} \rangle$ with

(1) $a \oplus b = a \cdot b$ [vector addition]

(2) $t \odot a = a^t$ [scalar multiplication]

is a vector space.

We'll have to check if all 8 defining properties of a vector space are satisfied:

(1) $a \oplus b = b \oplus a$ is true, since $a \cdot b = b \cdot a$.

(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ is true, since $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) The zero vector is: $\vec{0} = 1$ since $a \oplus 1 = a$.

(4) The opposite of a vector: $-a = \dfrac{1}{a}$ since $a \oplus \dfrac{1}{a} = 1 = \vec{0}$.

(5) $1 \odot a = a^1 = a$.

(6) $s \odot (t \odot a) = (a^t)^s = a^{st} = (st) \odot a$.

(7) $t \odot (a \oplus b) = (t \odot a) \oplus (t \odot b)$

since $t \odot (a \oplus b) = t \odot (a \cdot b) = (a \cdot b)^t = a^t \cdot b^t = (t \odot a) \oplus (t \odot b)$

(8) $(s + t) \odot a = (s \odot a) \oplus (t \odot a)$

since $(s + t) \odot a = a^{s+t} = a^s \cdot a^t = (s \odot a) \oplus (t \odot a)$