# Span

**Definition 1**

Let $T$ be a set of vectors of a vector space $\langle \mathbb{V}, +, \cdot, \mathbb{F} \rangle$. The **span** of $T$, denoted $\text{span}(T)$, is the set of all linear combinations of vectors from $T$.

that means

$$\text{span}(T) = \left\{ a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + \cdots + a_n \cdot \vec{v}_n \;\middle|\; n \in \mathbb{Z}^+;\ a_i \in \mathbb{F},\ \vec{v}_i \in \mathbb{V} \text{ for } i = 1, 2, \cdots, n \right\}$$

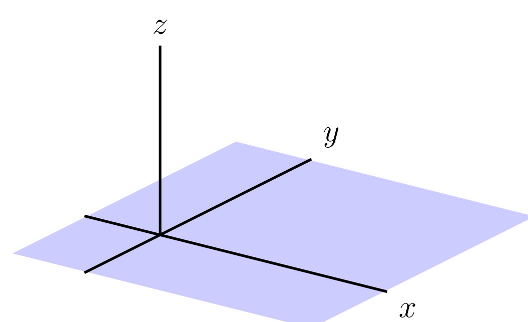
**Example 1**

**(a)** $$\text{Span}\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \left\{ x \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

This is the set of all vectors in the $xy-$plane.

We say that the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **span** this plane.

(As you see we use span both as a noun and a verb.)




**(b)** The $xy-$plane can be spanned by other vectors too. Here is an example:

$$\text{span}\left( \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \left\{ x \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

It is not to hard to see that both the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are in this span:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = - \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

These spans are the same, both span the $xy-$plane.

**(c)** Here is yet another span of the $xy-$plane

$$\text{span}\left(\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}5\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right) = \left\{ x\cdot \begin{bmatrix}2\\1\\0\end{bmatrix} + y\cdot \begin{bmatrix}5\\2\\0\end{bmatrix} + z\cdot \begin{bmatrix}3\\1\\0\end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

It is easy to see that all vectors in this span lie in the $xy-$plane. But are **all** vectors in the $xy-$plane in this span? It could be that we are missing some? Again it is not too hard to show that both the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ are in this span:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix} + 0\cdot \begin{bmatrix}5\\2\\0\end{bmatrix} - \begin{bmatrix}2\\1\\0\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} = 2\begin{bmatrix}3\\1\\0\end{bmatrix} - \begin{bmatrix}5\\2\\0\end{bmatrix} + 0\cdot \begin{bmatrix}2\\1\\0\end{bmatrix}$$

And if we can create these two vectors we can create any vector in the $xy-$plane. Hence this again is the same span as in (a) and (b).

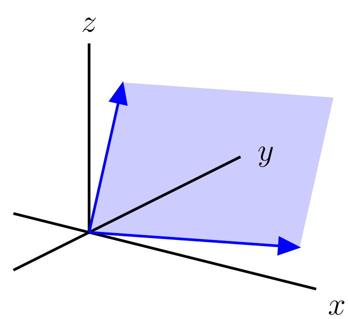
**(d)** The following span is different

$$\text{span}\left(\begin{bmatrix}4\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right) = \left\{ x\cdot \begin{bmatrix}4\\1\\0\end{bmatrix} + y\cdot \begin{bmatrix}0\\1\\3\end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

This is the set of all vectors in another plane.

We say that the vectors $\begin{bmatrix}4\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\3\end{bmatrix}$

**span** the plane illustrated on the right.




We have to be able to answer the following Shakespearean question:

$$\boxed{\vec{w} \in \text{span}(T) \qquad \text{or} \qquad \vec{w} \notin \text{span}(T)}$$

"To be in or not to be in, that's the question" [From the famous play: *Spanlet*].

For example: $\begin{bmatrix}8\\1\\-3\end{bmatrix} \in \text{span}\left(\begin{bmatrix}4\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right)$ or $\begin{bmatrix}8\\1\\-3\end{bmatrix} \notin \text{span}\left(\begin{bmatrix}4\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right)$

i.e. do there exist $x$ and $y$ such that

$$x\cdot \begin{bmatrix}4\\1\\0\end{bmatrix} + y\cdot \begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}4\\2\\3\end{bmatrix}$$

Of course a row reduction answers this question:

$$\operatorname{rref}\begin{bmatrix} 4 & 0 & 8 \\ 1 & 1 & 1 \\ 0 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$2 \cdot \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix} \in \operatorname{span}\left( \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right)$$

Similarly

$$\operatorname{rref}\begin{bmatrix} 4 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives us

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin \operatorname{span}\left( \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right)$$

Exactly this tool (row reduction) we can use to answer a bigger question:

$$\text{Are two spans equal:} \quad \boxed{\operatorname{span}(T) = \operatorname{span}(S)} \; ?$$

Let's show that the span in example **(d)** can be spanned using different vectors

$$\operatorname{span}\left( \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right) = \operatorname{span}\left( \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \right)$$

Note that $\begin{bmatrix} 4 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 4 \\ 1 & 0 & 2 \\ -3 & 3 & 3 \end{bmatrix}$ and

$$\begin{bmatrix} 8 & -4 & 4 \\ 1 & 0 & 2 \\ -3 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$$

This shows that every vector in the second span is in the first span and every vector in the first span is in the second span.

Let's see that is does, and then show how we got it (two row reductions).

$$\begin{bmatrix} 4 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 4 \\ 1 & 0 & 2 \\ -3 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix} \qquad - \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

Similarly the other way around:

$$\begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix} + 2 \cdot \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

How did we find these relations? We solved seven systems of equations: namely

$$x \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix}, \quad \text{and} \quad x \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} \quad \text{etc.}$$

using row reductions

$$\text{rref} \begin{bmatrix} 4 & 0 & 8 \\ 1 & 1 & 1 \\ 0 & 3 & -4 \end{bmatrix}, \quad \text{and} \quad \text{rref} \begin{bmatrix} 4 & 0 & -4 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix} \quad \text{etc.}$$

Which we can all do with just two row reductions

$$\text{rref} \begin{bmatrix} 4 & 0 & 8 & -4 & 4 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 3 & -3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\text{rref} \begin{bmatrix} 8 & -4 & 4 & 4 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ -3 & 3 & 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{etc.}$$

As you can see, row reduction at the heart of finding linear combinations.

Here is an important theorem:

**Theorem 1**

If $T$ is a nonempty subset of the vector space $\mathbb{V}$, then $\text{span}(T)$ is a **subspace** of $\mathbb{V}$:

$$\boxed{\text{span}(T) \sqsubseteq \mathbb{V}}$$

**Proof**: We'll use the main theorem of subspaces which we covered in chapter 10:

(1) If $\vec{x}$ is any vector in $\text{span}(T)$ than $0 \cdot \vec{x} = \vec{0}$ is in $\text{span}(T)$.

(2) The sum of two linear combinations of vectors from $T$, is also a linear combination of vectors from $T$.

(3) A scalar multiple of a linear combination of vectors from $T$, is again a linear combination of vectors from $T$.

Hence $\boxed{\text{span}(T) \sqsubseteq \mathbb{V}}$ ◻

**Example 2**

**(a)** $\text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$ is a subspace of $\mathbb{R}^3$: a line in $\mathbb{R}^3$.

**(b)** $\text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right)$ is a subspace of $\mathbb{R}^3$: a plane in $\mathbb{R}^3$.

We will adopt the following convention:

**Convention**

$$\boxed{\text{span}(\emptyset) = \left\{\vec{0}\right\}}$$

This might look a bit weird, but it is something you see quite frequently in math: the definition of an "extreme" case:

- For factorials we define: $0! = 1$
- In power series we define: $x^0 \equiv 1$ ( even when $x = 0$ )
- For products we define the "empty" product: $\prod(\ ) = 1$
- For sums we define the "empty" sum: $\sum(\ ) = 0$
- In sets we talk about "empty" set: $\{\ \} = \emptyset$
- In combinations we have "$n$ choose nothing": $\binom{n}{0} = 1$