Linear Transformation Matrices

Two Examples:

(1) Let $T: P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be defined by

$$T(at^2+bt+c) = (a+2b-c)\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} + (2a-b)\begin{bmatrix}1 & 0\\ 2 & 5\end{bmatrix} + (5b-2c)\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$$

Define the following bases:

$$\alpha = \left\{ t^2+2t+1,\ t^2+3t+1,\ t^2+2 \right\} \text{ and } \beta = \left\{ \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}, \begin{bmatrix}2 & 2\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} \right\}$$

and the standard bases:

$$s = \left\{ t^2,\ t,\ 1 \right\} \text{ and } S = \left\{ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \right\}$$

(a) Find ${}_S[\,T\,]_s$ **(b)** Find ${}_\beta[\,T\,]_\alpha$

If $[\,\vec{v}\,]_\alpha = \begin{bmatrix}-1\\ 0\\ 2\end{bmatrix}$ **(c)** Find $[\,T(\vec{v})\,]_\beta$ **(d)** Find $T(\vec{v})$ and $[\,T(\vec{v})\,]_S$

(2) $T: M_{2\times2}(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by

$$T\begin{bmatrix}a & b\\ c & d\end{bmatrix} = (4a-3b-5c+3d)\,t^3 + (a+3b-c)\,t^2 + (-3b-c+3d)\,t + (a+b-c+2d)$$

Define the following bases:

$$\alpha = \left\{ \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} \right\} \text{ and } \beta = \left\{ t^3+t+1,\ t^2+1,\ t^3+t^2+1,\ t^3+t \right\}$$

and the standard bases:

$$s = \left\{ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \right\} \text{ and } S = \left\{ t^3,\ t^2,\ t,\ 1 \right\}$$

(a) Find ${}_S[\,T\,]_s$ **(b)** Find ${}_\beta[\,T\,]_\alpha$

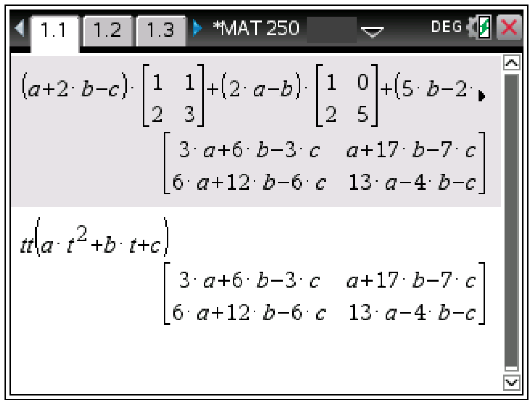
If $\vec{v} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ **(c)** Find $[\,T(\vec{v})\,]_\beta$ **(d)** Find $T(\vec{v})$ and $[\,T(\vec{v})\,]_S$

Complete Solution of Example (1):

(a) Let $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by

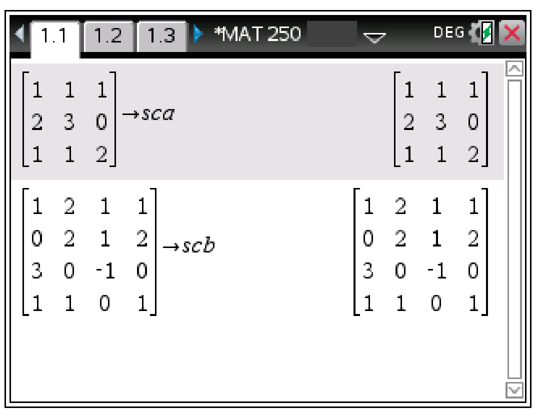
$$T(at^2+bt+c) = (a+2b-c)\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + (2a-b)\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} + (5b-2c)\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

Note: $T(at^2+bt+c) = \begin{bmatrix} 3a+6b-3c & a+17b-7c \\ 6a+12b-6c & 13a-4b-c \end{bmatrix}$



Define the following bases:

$$\alpha = \left\{ t^2+2t+1,\ t^2+3t+1,\ t^2+2 \right\} \text{ and } \beta = \left\{ \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$$

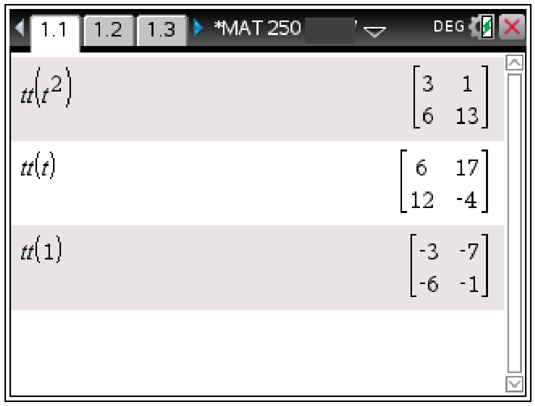


and the standard bases:

$$s = \left\{ t^2,\ t,\ 1 \right\} \quad \text{and} \quad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Note that $T(at^2+bt+c) = \begin{bmatrix} 3a+6b-3c & a+17b-7c \\ 6a+12b-6c & 13a-4b-c \end{bmatrix}$

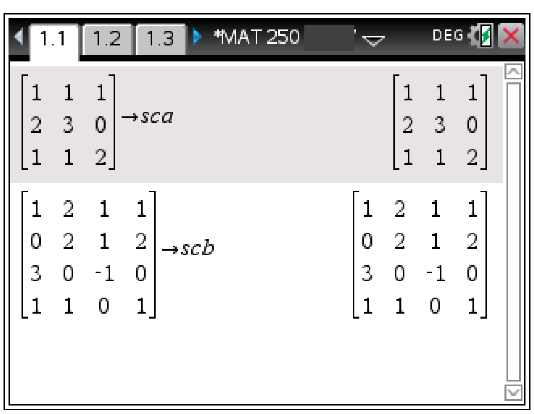
$$= a\begin{bmatrix} 3 & 1 \\ 6 & 13 \end{bmatrix} + b\begin{bmatrix} 6 & 17 \\ 12 & -4 \end{bmatrix} + c\begin{bmatrix} -3 & -7 \\ -6 & -1 \end{bmatrix}$$

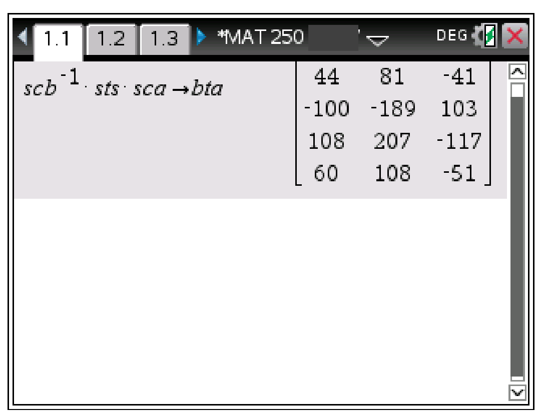


So that ${}_S[\,T\,]_S = \begin{bmatrix} 3 & 6 & -3 \\ 1 & 17 & -7 \\ 6 & 12 & -6 \\ 13 & -4 & -1 \end{bmatrix}$ and hence

(b) ${}_\beta[\,T\,]_\alpha = {}_\beta C_S \cdot {}_S[\,T\,]_S \cdot {}_S C_\alpha = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 6 & -3 \\ 1 & 17 & -7 \\ 6 & 12 & -6 \\ 13 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$${}_\beta[\,T\,]_\alpha = \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix}$$

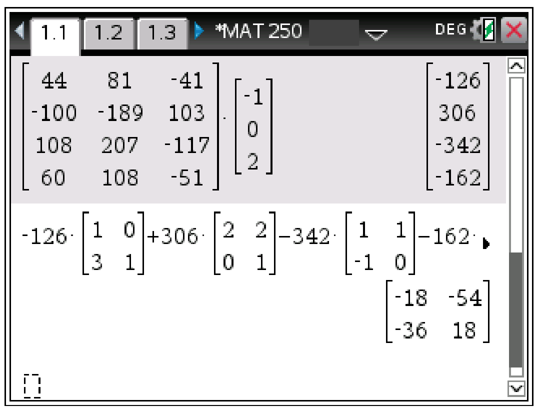




Let $\vec{v} \in P_2(\mathbb{R})$ such that $[\,\vec{v}\,]_\alpha = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ then

(c) $[\,T(\vec{v})\,]_\beta = {}_\beta[\,T\,]_\alpha \cdot [\,\vec{v}\,]_\alpha = \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -126 \\ 306 \\ -342 \\ -162 \end{bmatrix}$ so that

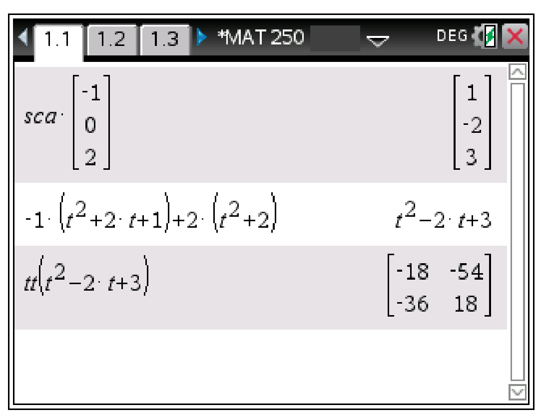
(d) $$T(\vec{v}) = -126 \cdot \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + 306 \cdot \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} - 342 \cdot \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - 162 \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -18 & -54 \\ -36 & 18 \end{bmatrix}$$



To check this another way: note that $[\,\vec{v}\,]_\alpha = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ implies that

$\vec{v} = -1 \cdot (t^2 + 2t + 1) + 0 \cdot (t^2 + 3t + 1) + 2 \cdot (t^2 + 2)$ so that $\vec{v} = t^2 - 2t + 3$ and hence

$T(\vec{v}) = \begin{bmatrix} -18 & -54 \\ -36 & 18 \end{bmatrix}$ i.e. $[\,T(\vec{v})\,]_S = \begin{bmatrix} -18 \\ -54 \\ -36 \\ 18 \end{bmatrix}$.



The various vector spaces involved, and the linear maps between them are illustrated:

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "{_S[T]_s}"] & \mathbb{R}^4 \\
P_2(\mathbb{R}) \arrow[u, "\varphi_s"] \arrow[d, "\varphi_\alpha"] \arrow[r, "T"] & M_{2\times 2}(\mathbb{R}) \arrow[u, "\varphi_S"] \arrow[d, "\varphi_\beta"] \\
\mathbb{R}^3 \arrow[r, "{_\beta[T]_\alpha}"] \arrow[uu, bend left=60, "{_sC_\alpha}"] & \mathbb{R}^4 \arrow[uu, bend right=60, "{_SC_\beta}"']
\end{tikzcd}
$$

Seen from the perspective of vectors:

$$
\begin{tikzcd}
{[\vec{v}\,]_s} \arrow[r, "{_S[T]_s}"] & {[T(\vec{v})]_S} \\
\vec{v} \arrow[u, "\varphi_s"] \arrow[d, "\varphi_\alpha"] \arrow[r, "T"] & T(\vec{v}) \arrow[u, "\varphi_S"] \arrow[d, "\varphi_\beta"] \\
{[\vec{v}\,]_\alpha} \arrow[r, "{_\beta[T]_\alpha}"] \arrow[uu, bend left=60, "{_sC_\alpha}"] & {[T(\vec{v})]_\beta} \arrow[uu, bend right=60, "{_SC_\beta}"']
\end{tikzcd}
$$

e.g. $\vec{v} = t^2 - 2t + 3$:

$$
\begin{tikzcd}
\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \arrow[r, "{_S[T]_s}"] & \begin{bmatrix} -18 \\ -54 \\ -36 \\ 18 \end{bmatrix} \\
t^2 - 2t + 3 \arrow[u, "\varphi_s"] \arrow[d, "\varphi_\alpha"] \arrow[r, "T"] & \begin{bmatrix} -18 & -54 \\ -36 & 18 \end{bmatrix} \arrow[u, "\varphi_S"] \arrow[d, "\varphi_\beta"] \\
\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \arrow[r, "{_\beta[T]_\alpha}"] \arrow[uu, bend left=60, "{_sC_\alpha}"] & \begin{bmatrix} -126 \\ 306 \\ -342 \\ -162 \end{bmatrix} \arrow[uu, bend right=60, "{_SC_\beta}"']
\end{tikzcd}
$$

Complete Solution of Example (2) :

$T: M_{2\times2}(\mathbb{R}) \to P_3(\mathbb{R})$ defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (4a-3b-5c+3d)\,t^3 + (a+3b-c)\,t^2 + (-3b-c+3d)\,t + (a+b-c+2d)$$

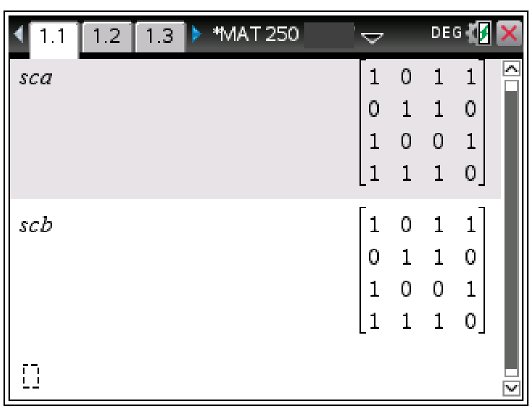
Define the following bases:

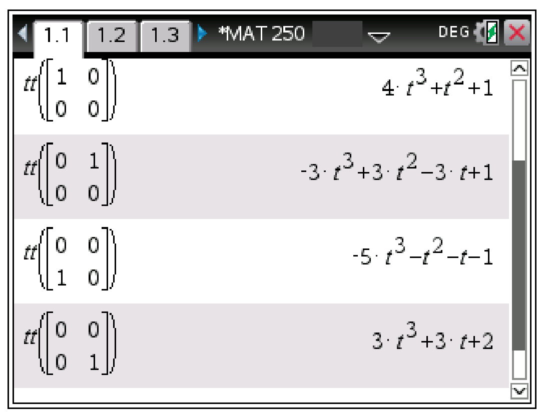
$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\} \text{ and } \beta = \left\{ t^3+t+1,\ t^2+1,\ t^3+t^2+1,\ t^3+t \right\}$$

and the standard bases:

$$s = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } S = \left\{ t^3,\ t^2,\ t,\ 1 \right\}$$

then note that ${}_sC_\alpha = {}_SC_\beta$



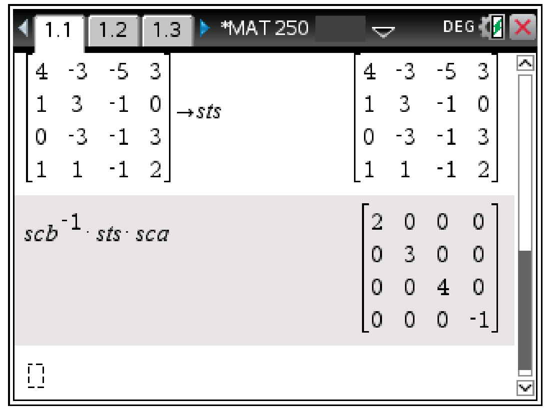


(a) $${}_S[T]_s = \begin{bmatrix} 4 & -3 & -5 & 3 \\ 1 & 3 & -1 & 0 \\ 0 & -3 & -1 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$ so that

(b) $${}_\beta[T]_\alpha = {}_\beta C_S \cdot {}_S[T]_s \cdot {}_sC_\alpha = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -3 & -5 & 3 \\ 1 & 3 & -1 & 0 \\ 0 & -3 & -1 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

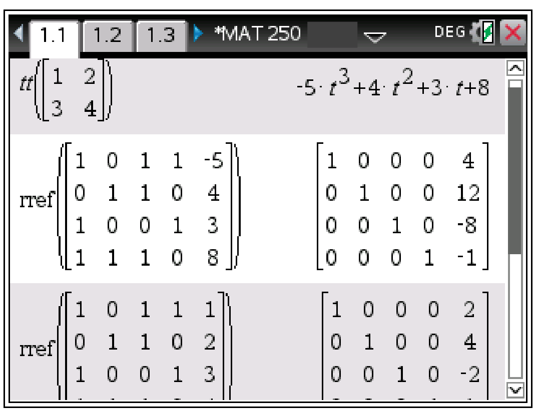
$$\Rightarrow \quad {}_{\beta}[\,T\,]_{\alpha} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

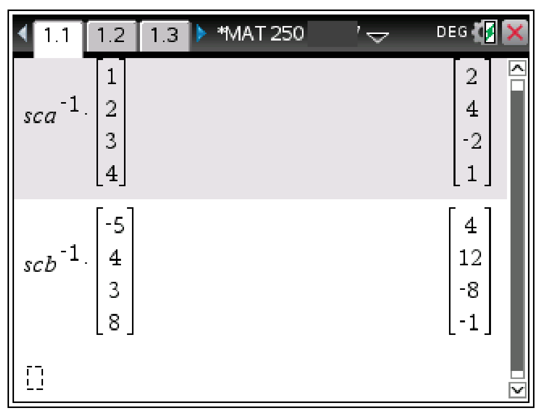
So with respect to the bases α and β the transformation matrix is a **diagonal matrix**! Images are easy to compute!



(c) If we take $\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $[\,\vec{v}\,]_s = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $[\,\vec{v}\,]_{\alpha} = {}_{\alpha}C_s \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}$

(d) $T(\vec{v}) = -5t^3 + 4t^2 + 3t + 8$, $\quad [\,T(\vec{v})\,]_S = \begin{bmatrix} -5 \\ 4 \\ 3 \\ 8 \end{bmatrix}$ and $[\,T(\vec{v})\,]_{\beta} = {}_{\beta}C_S \cdot \begin{bmatrix} -5 \\ 4 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -8 \\ -1 \end{bmatrix}$





so that $[\,T(\vec{v})\,]_{\beta} = {}_{\beta}[\,T\,]_{\alpha} \cdot [\,\vec{v}\,]_{\alpha} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -8 \\ -1 \end{bmatrix}$

Here are the relationships between the vectors and their images using matrices:

$$
\begin{tikzcd}[row sep=large, column sep=huge]
{[\vec{v}\,]_S} \arrow[r, "{}_S[T]_S"] & {[\,T(\vec{v})\,]_S} \\
\vec{v} \arrow[u, "\varphi_S"'] \arrow[r, "T"] \arrow[d, "\varphi_\alpha"] & T(\vec{v}) \arrow[u, "\varphi_S"'] \arrow[d, "\varphi_\beta"] \\
{[\vec{v}\,]_\alpha} \arrow[r, "{}_\beta[T]_\alpha"] \arrow[uu, bend left=60, "{}_S C_\alpha"] & {[\,T(\vec{v})\,]_\beta} \arrow[uu, bend right=60, "{}_S C_\beta"']
\end{tikzcd}
$$

e.g. $\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$
\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = {}_S C_\alpha
\qquad\qquad
{}_S C_\beta = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
$$

$$
\begin{tikzcd}[row sep=large, column sep=huge]
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \arrow[r, "{}_S[T]_S"] \arrow[r, bend left=50, "{\begin{bmatrix} 4 & -3 & -5 & 3 \\ 1 & 3 & -1 & 0 \\ 0 & -3 & -1 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix}}"] & \begin{bmatrix} -2 \\ 4 \\ 3 \\ 8 \end{bmatrix} \\
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \arrow[u, "\varphi_S"'] \arrow[r, "T"] \arrow[d, "\varphi_\alpha"] & -5t^3 + 4t^2 + 3t + 8 \arrow[u, "\varphi_S"'] \arrow[d, "\varphi_\beta"] \\
\begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix} \arrow[r, "{}_\beta[T]_\alpha"] \arrow[r, bend right=50, "{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}"'] \arrow[uu, bend left=60, "{}_S C_\alpha"] & \begin{bmatrix} 4 \\ 12 \\ -8 \\ 1 \end{bmatrix} \arrow[uu, bend right=60, "{}_S C_\beta"']
\end{tikzcd}
$$