17. Coordinates

Given a basis $\alpha = \left\{ \vec{a}_1, \ \vec{a}_2, \ \cdots, \ \vec{a}_n \right\}$ of a vector space V we can write any vector $\vec{v} \in V$ **uniquely** as a linear combination of this basis:

$$\vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + v_3 \vec{a}_3 + \cdots + v_n \vec{a}_n$$

These unique coefficients v_i, we call the **coordinates** of $\vec{v}$ with respect to the basis α.
We write the coordinates as a coordinate vector as follows:

$$[\vec{v}]_\alpha = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

This, in effect, induces a map φ_α from the n-dimensional vector space V to $\mathbb{F}^n$, which we will denote by

$$\varphi_\alpha : V \to \mathbb{F}^n \quad \text{with} \quad \varphi_\alpha(\vec{v}) = [\vec{v}]_\alpha$$

This map is a **linear** transformation, **one-to-one**, **onto**, and **invertible**.

Theorem: Let $\alpha = \left\{ \vec{a}_1, \ \vec{a}_2, \ \cdots, \ \vec{a}_n \right\}$ be a basis of the vector space V, then $\varphi_\alpha : V \to \mathbb{F}^n$ defined by $\varphi_\alpha(\vec{v}) = [\vec{v}]_\alpha$ is

(1) Linear
(2) One-to-one
(3) Onto
(4) Invertible

Proof: **(1)** We need to show that **(a)** $\varphi_\alpha(\vec{v} + \vec{w}) = \varphi_\alpha(\vec{v}) + \varphi_\alpha(\vec{w})$
(b) $\varphi_\alpha(t\vec{v}) = t\,\varphi_\alpha(\vec{v})$

Express $\vec{v}$ and $\vec{w}$ as linear combinations of the basis $\alpha = \left\{ \vec{a}_1, \ \vec{a}_2, \ \cdots, \ \vec{a}_n \right\}$

$$\vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + v_3 \vec{a}_3 + \cdots + v_n \vec{a}_n$$

$$\vec{w} = w_1 \vec{a}_1 + w_2 \vec{a}_2 + w_3 \vec{a}_3 + \cdots + w_n \vec{a}_n$$

Adding them gives

$$\vec{v}+\vec{w} = (v_1+w_1)\vec{a}_1+(v_2+w_2)\vec{a}_2+(v_3+w_3)\vec{a}_3+ \cdots +(v_n+w_n)\vec{a}_n$$

which implies $\left[\vec{v}+\vec{w}\right]_\alpha = \left[\vec{v}\right]_\alpha + \left[\vec{w}\right]_\alpha$ i.e. $\varphi_\alpha(\vec{v}+\vec{w}) = \varphi_\alpha(\vec{v})+\varphi_\alpha(\vec{w})$.

Similarly $\left[t\vec{v}\right]_\alpha = t\left[\vec{v}\right]_\alpha$ i.e. $\varphi_\alpha(t\vec{v}) = t\,\varphi_\alpha(\vec{v})$.

(2) Suppose $\left[\vec{v}\right]_\alpha = \left[\vec{w}\right]_\alpha = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}$ then $\vec{v} = x_1\vec{a}_1+x_2\vec{a}_2+x_3\vec{a}_3+ \cdots +x_n\vec{a}_n = \vec{w}$

So that $\varphi_\alpha(\vec{v}) = \varphi_\alpha(\vec{w})$ implies that $\vec{v} = \vec{w}$, hence the map φ_α is **one-to-one**.

(3) Given any $\begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} \in \mathbb{F}^n$, then $\varphi_\alpha(\underbrace{x_1\vec{a}_1+x_2\vec{a}_2+x_3\vec{a}_3+ \cdots +x_n\vec{a}_n}_{\in V}) = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}$.

So φ_α is **onto**.

(4) Since the map φ_α is both **one-to-one** and **onto**, it is invertible.

An invertible, linear transformation is called an isomorphism. If there exists an isomorphism between two vector spaces we call them isomorphic.

Hence this shows that

Theorem: An n-dimenional vector space V is isomorphic to $\mathbb{F}^n$: $\boxed{V \cong \mathbb{F}^n}$

Example: $\left\{t^2+t,\ t^2-1,\ t^2+t+1\right\}$ is a basis of $P_2(\mathbb{R})$.

$$\left[2t^2+4t-3\right]_\beta = \begin{bmatrix} 9 \\ -2 \\ -5 \end{bmatrix} \quad \text{since} \quad 2t^2+4t-3 = 9(t^2+t)-2(t^2-1)-5(t^2+t+1).$$

To compute these coordinates, write all vectors with respect to the standard basis and rref:

$$\operatorname{rref}\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ -5 \end{bmatrix}$$

Example: $\left\{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix}\right\}$ is a basis of $M_{2\times 2}(\mathbb{R})$.

$$\left[\begin{bmatrix} 5 & -2 \\ 3 & -6 \end{bmatrix}\right]_\beta = \begin{bmatrix} 10 \\ -5 \\ -1 \\ 4 \end{bmatrix} \quad \text{since} \quad \begin{bmatrix} 5 & -2 \\ 3 & -6 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - 1\begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix}$$

To compute these coordinates, write all vectors with respect to the standard basis and rref:

$$\operatorname{rref}\begin{bmatrix} 1 & 1 & 4 & 1 & 5 \\ 0 & 1 & 1 & 1 & -2 \\ 2 & 1 & 0 & -3 & 3 \\ 1 & 3 & 1 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & -3 \\ 1 & 3 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ -1 \\ 4 \end{bmatrix}$$

Example: $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$. Express $\vec{v} = \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix}$ in terms of β.

Let $[\vec{v}]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, i.e

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix} \quad \text{i.e.}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Check: $5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix}$