Matrices of Linear Transformations

A linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ can always be written as a matrix multiplication.

$$\boxed{T(\vec{x}) = M \cdot \vec{x}}$$

where $M \in M_{m \times n}(\mathbb{F})$.

Example 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\-1\\2\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\-5\end{bmatrix}$$

Since $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis, the transformation is completely determined by the above:

$$\begin{aligned} T\begin{bmatrix}x\\y\end{bmatrix} &= T\left(x \cdot \begin{bmatrix}1\\0\end{bmatrix} + y \cdot \begin{bmatrix}0\\1\end{bmatrix}\right) = x \cdot T\begin{bmatrix}1\\0\end{bmatrix} + y \cdot T\begin{bmatrix}0\\1\end{bmatrix} \\ &= x \cdot \begin{bmatrix}3\\-1\\2\end{bmatrix} + y \cdot \begin{bmatrix}2\\1\\-5\end{bmatrix} = \begin{bmatrix}3x+3y\\-x+y\\2x-5y\end{bmatrix} \\ &= \underbrace{\begin{bmatrix}3 & 2\\-1 & 1\\2 & -5\end{bmatrix}}_{M}\begin{bmatrix}x\\y\end{bmatrix} \end{aligned}$$

Hence any image is computed by left multiplication with the matrix M, e.g.

$$T\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3 & 2\\-1 & 1\\2 & -5\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}13\\-1\\-4\end{bmatrix}$$

so that

$$T : \begin{bmatrix}3\\2\end{bmatrix} \mapsto \begin{bmatrix}13\\-1\\-4\end{bmatrix}$$

Even though $T : \mathbb{F}^n \to \mathbb{F}^m$ is just one type of linear transformations, it is crucial in dealing with $T : \mathbb{V} \to \mathbb{W}$ in general.

Example 2

Let $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(a\,t^2 + b\,t + c) = \begin{bmatrix} a+b & b-2c \\ 3a+2b & a+b+c \end{bmatrix}$$

For example: $$t^2 + 3t - 2 \xmapsto{\;T\;} \begin{bmatrix} 4 & 7 \\ 9 & 2 \end{bmatrix}$$

Clearly there is no matrix M such that: $M \cdot (t^2 + 3t - 2) = \begin{bmatrix} 4 & 7 \\ 9 & 2 \end{bmatrix}$.

We can however rewrite both vectors with respect to their standard bases:

$$\begin{tikzcd} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} & \begin{bmatrix} 4 \\ 7 \\ 9 \\ 2 \end{bmatrix} \\ t^2 + 3t - 2 \arrow[u] \arrow[r, "T"] & \begin{bmatrix} 4 & 7 \\ 9 & 2 \end{bmatrix} \arrow[u] \end{tikzcd}$$

We basically reduced it to a transformation of the previous type: $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$.
In general for this transformation we have:

$$\begin{tikzcd} \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \begin{bmatrix} a+b \\ b-2c \\ 3a+2b \\ a+b+c \end{bmatrix} \\ a\,t^2 + b\,t + c \arrow[u] \arrow[r, "T"] & \begin{bmatrix} a+b & b-2c \\ 3a+2b & a+b+c \end{bmatrix} \arrow[u] \end{tikzcd}$$

A matrix transforms $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \begin{bmatrix} a+b \\ b-2c \\ 3a+2b \\ a+b+c \end{bmatrix}$ as follows $\begin{bmatrix} a+b \\ b-2c \\ 3a+2b \\ a+b+c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

This procedure we can use in general for any linear transformations $T : \mathbb{V} \to \mathbb{W}$.
We can even use any basis α for $\mathbb{V}$ and any basis β for $\mathbb{W}$.

$*$ Rewrite every vector $\vec{v} \in \mathbb{V}$ with respect to the basis α using φ_α: $\quad \varphi_\alpha(\vec{v}) = [\,\vec{v}\,]_\alpha$

$*$ Rewrite every vector $\vec{w} \in \mathbb{W}$ with respect to the basis β using φ_β: $\quad \varphi_\beta(\vec{w}) = [\,\vec{w}\,]_\beta$

$$\begin{tikzcd}
[\,\vec{v}\,]_\alpha & \Big[\,T(\vec{v})\,\Big]_\beta \\
\vec{v} \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] & T(\vec{v}) \arrow[u, "\varphi_\beta"']
\end{tikzcd}$$

Note that if $\dim(\mathbb{V}) = n$ and $\dim(\mathbb{W}) = m$ then

$$\varphi_\alpha : \mathbb{V} \to \mathbb{F}^n$$

$$\varphi_\beta : \mathbb{W} \to \mathbb{F}^m$$

so that we have the following maps between the spaces $\mathbb{V}$, $\mathbb{W}$, $\mathbb{F}^n$ and $\mathbb{F}^m$:

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{}_\beta T_\alpha"] & \mathbb{F}^m \\
\mathbb{V} \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] & \mathbb{W} \arrow[u, "\varphi_\beta"']
\end{tikzcd}$$

Note that we named the map between $\mathbb{F}^n$ and $\mathbb{F}^m$: $\quad \boxed{{}_\beta T_\alpha}$

The map ${}_\beta T_\alpha$ mirrors the map T:

$$\vec{v} \overset{T}{\longmapsto} T(\vec{v})$$

$$[\,\vec{v}\,]_\alpha \overset{{}_\beta T_\alpha}{\longmapsto} [\,T(\vec{v})\,]_\beta$$

The maps T and ${}_\beta T_\alpha$ are describing the same transformation, from different perspectives:

$*$ T maps $\vec{v} \in \mathbb{V}$ to $T(\vec{v}) \in \mathbb{W}$,

$$\vec{v} \xrightarrow{\quad T \quad} T(\vec{v})$$

$*$ ${}_\beta T_\alpha$ maps $[\,\vec{v}\,]_\alpha \in \mathbb{F}^n$ to $[\,T(\vec{v})\,]_\beta \in \mathbb{F}^m$

$$[\,\vec{v}\,]_\alpha \xrightarrow{\quad {}_\beta T_\alpha \quad} [\,T(\vec{v})\,]_\beta$$

But: $\vec{v}$ and $[\,\vec{v}\,]_\alpha$ are referring to the same vector, except $\vec{v}$ is in the original space $\mathbb{V}$ and $[\,\vec{v}\,]_\alpha$ is in $\mathbb{F}^n$, and is the coordinate vector of $\vec{v}$ with respect to the basis α.

Similarly, $T(\vec{v})$ and $[\,T(\vec{v})\,]_\beta$ are referring to the same vector, except $T(\vec{v})$ is in the original space $\mathbb{W}$ and $[\,T(\vec{v})\,]_\beta$ is in $\mathbb{F}^m$, and is the coordinate vector of $T(\vec{v})$ with respect to the basis β.

This link defines the transformation ${}_\beta T_\alpha : \mathbb{F}^n \to \mathbb{F}^m$: $\quad {}_\beta T_\alpha\Big(\,[\,\vec{v}\,]_\alpha\Big) \;=\; \Big[\,T(\vec{v})\,\Big]_\beta$
or if you like:

$$\boxed{{}_\beta T_\alpha = \varphi_\beta \circ T \circ \varphi_\alpha^{-1}}$$

taking the route indicated in the next diagram

$$\begin{tikzcd}
{[\,\vec{v}\,]_\alpha} \arrow[r, "{}_\beta T_\alpha"] \arrow[d, "\varphi_\alpha^{-1}"'] \arrow[r, bend right=90, looseness=2, "\varphi_\beta \circ T \circ \varphi_\alpha^{-1}"'] & {\Big[\,T(\vec{v})\,\Big]_\beta} \\
\vec{v} \arrow[r, "T"] & T(\vec{v}) \arrow[u, "\varphi_\beta"']
\end{tikzcd}$$

or equivalently: $\boxed{{}_\beta T_\alpha \circ \varphi_\alpha = \varphi_\beta \circ T}$ which indeed gives us

$${}_\beta T_\alpha\Big(\,[\,\vec{v}\,]_\alpha\Big) \;=\; {}_\beta T_\alpha\Big(\varphi_\alpha(\vec{v})\Big) \;=\; \Big({}_\beta T_\alpha \circ \varphi_\alpha\Big)(\vec{v}) \;=\; \Big(\varphi_\beta \circ T\Big)(\vec{v}) \;=\; \varphi_\beta\Big(T(\vec{v})\Big) \;=\; \Big[\,T(\vec{v})\,\Big]_\beta$$

Definition 1

Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation, with $\dim(\mathbb{V}) = n$ and $\dim(\mathbb{W}) = m$.
Let α be a basis of $\mathbb{V}$, and β a basis of $\mathbb{W}$.

The transformation ${}_{\beta}T_{\alpha} : \mathbb{F}^n \to \mathbb{F}^m$ is defined by any of the following

$$ {}_{\beta}T_{\alpha}([\vec{v}]_{\alpha}) = \Big[T(\vec{v})\Big]_{\beta} $$

$$ {}_{\beta}T_{\alpha} \circ \varphi_{\alpha} = \varphi_{\beta} \circ T $$

$$ {}_{\beta}T_{\alpha} = \varphi_{\beta} \circ T \circ \varphi_{\alpha}^{-1} $$

Since ${}_{\beta}T_{\alpha} : \mathbb{F}^n \to \mathbb{F}^m$ this transformation can be performed by a matrix multiplication:

$$ {}_{\beta}T_{\alpha}(\vec{x}) = M \cdot \vec{x} $$

This matrix is called ${}_{\beta}[\,T\,]_{\alpha}$, so that

$$ {}_{\beta}T_{\alpha}(\vec{x}) = {}_{\beta}[\,T\,]_{\alpha} \cdot \vec{x} $$

We'll first define the matrix ${}_{\beta}[\,T\,]_{\alpha}$, and then show that indeed ${}_{\beta}T_{\alpha}(\vec{x}) = {}_{\beta}[\,T\,]_{\alpha} \cdot \vec{x}$.

Definition 2

Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation. Let $\alpha = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ be a basis of $\mathbb{V}$, and β a basis of $\mathbb{W}$.
The matrix ${}_{\beta}[\,T\,]_{\alpha}$ is defined by

$$ {}_{\beta}[\,T\,]_{\alpha} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \Big[T(\vec{a}_1)\Big]_{\beta} & \Big[T(\vec{a}_2)\Big]_{\beta} & \cdots & \Big[T(\vec{a}_n)\Big]_{\beta} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} $$

Example 3

Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by

$$ T(a\,t^2 + b\,t + c) = \begin{bmatrix} a+b & -5a-2c \\ a+b+c & a-9c \end{bmatrix} $$

Let $\alpha = \Big\{ t^2 + 3t - 1, \;\; t^2 + t + 2, \;\; t^2 + 4t - 3 \Big\}$ be a basis of $P_2(\mathbb{R})$, and

Let $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \right\}$ be a basis of $M_{2\times 2}(\mathbb{R})$.

We'll compute ${}_\beta[\,T\,]_\alpha$ the matrix of ${}_\beta T_\alpha$.

(a) $T(\vec{a}_1) = T(t^2+3t-1) = \begin{bmatrix} 4 & -3 \\ 3 & 10 \end{bmatrix}$ and therefore $\big[T(\vec{a}_1)\big]_\beta = \begin{bmatrix} -3 \\ 17 \\ 2 \\ -10 \end{bmatrix}$

(b) $T(\vec{a}_2) = T(t^2+t+2) = \begin{bmatrix} 2 & -9 \\ 4 & -17 \end{bmatrix}$ and therefore $\big[T(\vec{a}_2)\big]_\beta = \begin{bmatrix} -6 \\ 51 \\ 8 \\ -43 \end{bmatrix}$

(c) $T(\vec{a}_3) = T(t^2+4t-3) = \begin{bmatrix} 5 & 1 \\ 2 & 28 \end{bmatrix}$ and therefore $\big[T(\vec{a}_3)\big]_\beta = \begin{bmatrix} 0 \\ -11 \\ -3 \\ 16 \end{bmatrix}$

so that
$${}_\beta[\,T\,]_\alpha = \begin{bmatrix} -3 & -6 & 0 \\ 17 & 51 & -11 \\ 2 & 8 & -3 \\ -10 & -43 & 16 \end{bmatrix}$$

For example let $\vec{v} = 3t^2+4t+5$ then

$$\begin{tikzcd}
\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \arrow[r, "{}_\beta T_\alpha"] & \begin{bmatrix} -18 \\ 147 \\ 23 \\ -122 \end{bmatrix} \\
3t^2+4t+5 \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] & \begin{bmatrix} 7 & -25 \\ 12 & -42 \end{bmatrix} \arrow[u, "\varphi_\beta"']
\end{tikzcd}$$

And indeed
$$\begin{bmatrix} -3 & -6 & 0 \\ 17 & 51 & -11 \\ 2 & 8 & -3 \\ -10 & -43 & 16 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -18 \\ 147 \\ 23 \\ -122 \end{bmatrix}$$

Theorem 1

If $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation then

$$_{\beta}T_{\alpha}(\vec{x}) = {}_{\beta}[\,T\,]_{\alpha} \cdot \vec{x}$$

so that

$$\Big[T(\vec{v})\Big]_{\beta} = {}_{\beta}[\,T\,]_{\alpha} \cdot [\,\vec{v}\,]_{\alpha}$$

Proof:

Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation, and let

$$\alpha = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} \text{ be a basis of } \mathbb{V}$$

and

$$\beta \text{ be a basis of } \mathbb{W}$$

then if $[\,\vec{v}\,]_{\alpha} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ we can compute $T(\vec{v})$ as follows:

$$\begin{aligned} T(\vec{v}) &= T(\,v_1 \cdot \vec{a}_1 + v_2 \cdot \vec{a}_2 + \cdots + v_n \cdot \vec{a}_n) \\ &= v_1 \cdot T(\vec{a}_1) + v_2 \cdot T(\vec{a}_2) + \cdots + v_n \cdot T(\vec{a}_n) \end{aligned}$$

so that expressed with respect to β we get

$$\begin{aligned} \Big[T(\vec{v})\Big]_{\beta} &= \Big[v_1 \cdot T(\vec{a}_1) + v_2 \cdot T(\vec{a}_2) + \cdots + v_n \cdot T(\vec{a}_n)\Big]_{\beta} \\ &= v_1 \cdot \Big[T(\vec{a}_1)\Big]_{\beta} + v_2 \cdot \Big[T(\vec{a}_2)\Big]_{\beta} + \cdots + v_n \cdot \Big[T(\vec{a}_n)\Big]_{\beta} \end{aligned}$$

Hence

$$\Big[T(\vec{v})\Big]_{\beta} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \Big[T(\vec{a}_1)\Big]_{\beta} & \Big[T(\vec{a}_2)\Big]_{\beta} & \cdots & \Big[T(\vec{a}_n)\Big]_{\beta} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

so that

$$\Big[T(\vec{v})\Big]_{\beta} = {}_{\beta}[\,T\,]_{\alpha} \cdot [\,\vec{v}\,]_{\alpha}$$

and since

$$[\,\vec{v}\,]_{\alpha} \xrightarrow{\;_{\beta}T_{\alpha}\;} [\,T(\vec{v})\,]_{\beta}$$

${}_{\beta}[\,T\,]_{\alpha}$ is exactly the matrix that performs this operation:

$$[\,\vec{v}\,]_{\alpha} \xrightarrow{\;_{\beta}T_{\alpha}\;} {}_{\beta}[\,T\,]_{\alpha} \cdot [\,\vec{v}\,]_{\alpha} \qquad \square$$

As we mentioned before, it often happens that when we write everything with respect to the standard bases, things become easier.

In particular ${}_S[\,T\,]_s$ is usually easier to find than ${}_\beta[\,T\,]_\alpha$.

Example 4

Let $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(t^2) = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}, \qquad T(t) = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad T(1) = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

The transformation is completely determined by the above:

$$\begin{aligned} T(\,x\,t^2 + y\,t + z\,) &= x \cdot T(t^2) + y \cdot T(t) + z \cdot T(1) \\ &= x \cdot \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} + z \cdot \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix} \\ &= \begin{bmatrix} x+y-z & 3x-y+2z \\ 5x-2y+z & 4x+3y-5z \end{bmatrix} \end{aligned}$$

Here is the corresponding diagram

$$\begin{tikzcd}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} \arrow[rr, "{}_S T_s"] & & \begin{bmatrix} x+y-z \\ 3x-y+2z \\ 5x-2y+z \\ 4x+3y-5z \end{bmatrix} \\
x\,t^2 + y\,t + z \arrow[u, "\varphi_s"] \arrow[rr, "T"] & & \begin{bmatrix} x+y-z & 3x-y+2z \\ 5x-2y+z & 4x+3y-5z \end{bmatrix} \arrow[u, "\varphi_S"']
\end{tikzcd}$$

Hence

$$\begin{bmatrix} x+y-z \\ 3x-y+2z \\ 5x-2y+z \\ 4x+3y-5z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 5 & -2 & 1 \\ 4 & 3 & -5 \end{bmatrix}}_{{}_S[\,T\,]_s} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and as you can see, its columns are precisely: $[\,T(t^2)\,]_S$, $[\,T(t)\,]_S$ and $[\,T(t^2)\,]_S$ since

$$T(t^2) = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}, \qquad T(t) = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad T(1) = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

So in particular for $\vec{v} = 4t^2 - 3t - 2$

$$
\begin{tikzcd}
\begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} \arrow[rr, "{}_S T_s"] & & \begin{bmatrix} 3 \\ 11 \\ 24 \\ 17 \end{bmatrix} \\
4t^2 - 3t - 2 \arrow[u, "\varphi_s"] \arrow[rr, "T"] & & \begin{bmatrix} 3 & 11 \\ 24 & 17 \end{bmatrix} \arrow[u, "\varphi_S"']
\end{tikzcd}
$$

and indeed

$$
\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 5 & -2 & 1 \\ 4 & 3 & -5 \end{bmatrix}}_{{}_S[\,T\,]_s} \cdot \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 24 \\ 17 \end{bmatrix}
$$

Example 5

Let $T : U_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation defined by

$$
T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 3t^2 - 2t + 4, \qquad T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2t^3 + 5t - 6 \quad \text{and} \quad T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 6t^3 + 7t
$$

Then

$$
{}_S[\,T\,]_s = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 0 & 0 \\ -2 & 5 & 7 \\ 4 & -6 & 0 \end{bmatrix}
$$

$$
T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 3t^2 - 2t + 4 \quad \xrightarrow{\varphi_S} \quad \text{first column}
$$

$$
T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2t^3 + 5t - 6 \quad \xrightarrow{\varphi_S} \quad \text{second column}
$$

$$
T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 6t^3 + 7t \quad \xrightarrow{\varphi_S} \quad \text{third column}
$$

where s and S are the standard bases of $U_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$.

Alternatively we could have computed

$$\begin{aligned}
T\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} &= T\left(x\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= x\cdot T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\cdot T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\cdot T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= x\cdot(3t^2-2t+4) + y\cdot(2t^3+5t-6) + z\cdot(6t^3+7t) \\
&= (2y+6z)\cdot t^3 + 3x\cdot t^2 + (-2x+5y+7z)\cdot t + (4x-6y)
\end{aligned}$$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\;{}_\beta T_\alpha\;} \begin{bmatrix} 2y+6z \\ 3x \\ -2x+5y+7z \\ 4x-6y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2 & 6 \\ 3 & 0 & 0 \\ -2 & 5 & 7 \\ 4 & -6 & 0 \end{bmatrix}}_{{}_S[\,T\,]_s} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Summary: ${}_\beta T_\alpha$ and how to compute ${}_\beta[\,T\,]_\alpha$

$$T : \mathbb{V} \to \mathbb{W}$$

$$\boxed{{}_\beta[\,T\,]_\alpha = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \big[T(\vec{a}_1)\big]_\beta & \big[T(\vec{a}_2)\big]_\beta & \cdots & \big[T(\vec{a}_n)\big]_\beta \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}}$$

Corresponding diagrams

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "{}_\beta T_\alpha"] & \mathbb{F}^m \\ \mathbb{V} \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] & \mathbb{W} \arrow[u, "\varphi_\beta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {[\,\vec{v}\,]_\alpha} \arrow[r, "{}_\beta T_\alpha"] & \big[T(\vec{v})\big]_\beta \\ \vec{v} \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] & T(\vec{v}) \arrow[u, "\varphi_\beta"'] \end{tikzcd}$$

But since it is often easier to compute ${}_S[\,T\,]_s$ we can compute ${}_\beta[\,T\,]_\alpha$ using this matrix as follows:

$$\boxed{\;{}_\beta[\,T\,]_\alpha \;=\; {}_\beta C_S \cdot {}_S[\,T\,]_s \cdot {}_s C_\alpha\;}$$

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{}_\beta T_\alpha"] \arrow[dd, bend right=60, "{}_s I_\alpha"'] & \mathbb{F}^m \\
\mathbb{V} \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] \arrow[d, "\varphi_s"'] & \mathbb{W} \arrow[u, "\varphi_\beta"'] \arrow[d, "\varphi_S"] \\
\mathbb{F}^n \arrow[r, "{}_S T_s"] & \mathbb{F}^m \arrow[uu, bend right=60, "{}_\beta I_S"']
\end{tikzcd}$$

since

$${}_\beta T_\alpha \;=\; {}_\beta I_S \cdot {}_S T_s \cdot {}_s I_\alpha$$

where ${}_s I_\alpha$ is the map defined by

$${}_s I_\alpha : \quad [\,\vec{v}\,]_\alpha \;\longmapsto\; [\,\vec{v}\,]_s$$

i.e. ${}_s I_\alpha$ is the transformation that takes a vector $[\,\vec{v}\,]_\alpha$, which is the vector $\vec{v} \in \mathbb{V}$ expressed with respect to the basis α, and maps it to $[\,\vec{v}\,]_s$, which is the same vector but now expressed with respect to the standard basis s. [Here we use I because it is basically the identity map]

This we can do with the matrix ${}_s C_\alpha$

$$[\,\vec{v}\,]_s \;=\; {}_s C_\alpha \cdot [\,\vec{v}\,]_\alpha$$

Similarly for ${}_\beta I_S$ which is defined by

$${}_\beta I_S : \quad [\,\vec{w}\,]_S \;\longmapsto\; [\,\vec{w}\,]_\beta$$

a transformation we can perform using the matrix ${}_\beta C_S \left(= {}_S C_\beta^{-1}\right)$

$$[\,\vec{w}\,]_\beta \;=\; {}_\beta C_S \cdot [\,\vec{w}\,]_S$$

Let's look at the following diagram where instead of the names of the functions, we indicate all the **matrices** used for each map, and what vector is being mapped to what image vector

$$\begin{tikzcd}
{[\,\vec{v}\,]_\alpha} \arrow[r, "{{}_\beta[\,T\,]_\alpha}"] \arrow[dd, bend right=60, "{{}_sC_\alpha}"'] & {[\,T(\vec{v})\,]_\beta} \\
\vec{v} \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] \arrow[d, "\varphi_s"'] & T(\vec{v}) \arrow[u, "\varphi_\beta"'] \arrow[d, "\varphi_S"] \\
{[\,\vec{v}\,]_s} \arrow[r, "{{}_S[\,T\,]_s}"] & {[\,T(\vec{v})\,]_S} \arrow[uu, bend right=60, "{{}_\beta C_S}"']
\end{tikzcd}$$

Instead of performing the transformation

$$[\,\vec{v}\,]_\alpha \longmapsto [\,T(\vec{v})\,]_\beta$$

using the matrix ${}_\beta[\,T\,]_\alpha$ as follows

$${}_\beta[\,T\,]_\alpha \cdot [\,\vec{v}\,]_\alpha = [\,T(\vec{v})\,]_\beta$$

we take the longer, but often simpler route

$$[\,\vec{v}\,]_\alpha \longmapsto [\,\vec{v}\,]_s \longmapsto [\,T(\vec{v})\,]_S \longmapsto [\,T(\vec{v})\,]_\beta$$

using three matrices

$${}_\beta C_S \cdot {}_S[\,T\,]_s \cdot {}_sC_\alpha \cdot [\,\vec{v}\,]_\alpha = [\,T(\vec{v})\,]_\beta$$

where each of these matrices can be found easily. This shows that

$${}_\beta[\,T\,]_\alpha = {}_\beta C_S \cdot {}_S[\,T\,]_s \cdot {}_sC_\alpha$$

or if you like

$${}_\beta[\,T\,]_\alpha = {}_\beta[\,I\,]_S \cdot {}_S[\,T\,]_s \cdot {}_s[\,I\,]_\alpha$$

This sounds all much more abstract than it actually is. Let's do a bunch of examples.

Example 6

Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a\,t^2 + b\,t + c) = \begin{bmatrix} 3a+6b-3c & a+17b-7c \\ 6a+12b-6c & 13a-4b-c \end{bmatrix}$$

The following bases are given

$$\alpha = \left\{ t^2+2t+1, \quad t^2+3t+1, \quad t^2+2 \right\}$$

and

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$$

and of course the usual standard bases.

Since $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xmapsto{\ {}_S T_s\ } \begin{bmatrix} 3a+6b-3c \\ a+17b-7c \\ 6a+12b-6c \\ 13a-4b-c \end{bmatrix}$$

$${}_S[\,T\,]_s = \begin{bmatrix} 3 & 6 & -3 \\ 1 & 17 & -7 \\ 6 & 12 & -6 \\ 13 & -4 & -1 \end{bmatrix}$$

and

$${}_sC_\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad {}_sC_\beta = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

These three matrices where easy to find, and now give us

$$\begin{aligned}
{}_\beta[\,T\,]_\alpha &= {}_\beta C_S \cdot {}_S[\,T\,]_s \cdot {}_sC_\alpha \\
&= \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 & 6 & -3 \\ 1 & 17 & -7 \\ 6 & 12 & -6 \\ 13 & -4 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix}
\end{aligned}$$

Let's also find ${}_\beta[\,T\,]_\alpha$ the other way using $\;{}_\beta[\,T\,]_\alpha = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \big[T(\vec{a}_1)\big]_\beta & \big[T(\vec{a}_2)\big]_\beta & \cdots & \big[T(\vec{a}_n)\big]_\beta \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

(a) $\quad T(\vec{\alpha}_1) = T(t^2+2t+1) = \begin{bmatrix} 12 & 28 \\ 24 & 4 \end{bmatrix} \quad \Rightarrow \quad [\,T(\vec{\alpha}_1)\,]_\beta = \begin{bmatrix} 44 \\ -100 \\ 108 \\ 60 \end{bmatrix}$

(b) $\quad T(\vec{\alpha}_2) = T(t^2+3t+1) = \begin{bmatrix} 18 & 45 \\ 36 & 0 \end{bmatrix} \quad \Rightarrow \quad [\,T(\vec{\alpha}_2)\,]_\beta = \begin{bmatrix} 81 \\ -189 \\ 207 \\ 108 \end{bmatrix}$

(c) $\quad T(\vec{\alpha}_3) = T(t^2+2) = \begin{bmatrix} -3 & -13 \\ -6 & 11 \end{bmatrix} \quad \Rightarrow \quad [\,T(\vec{\alpha}_3)\,]_\beta = \begin{bmatrix} -41 \\ 103 \\ -117 \\ -51 \end{bmatrix}$

where the last parts were computed with one row reduction

$$\text{rref}\begin{bmatrix} 1 & 2 & 1 & 1 & 12 & 18 & -3 \\ 0 & 2 & 1 & 2 & 28 & 45 & -13 \\ 3 & 0 & -1 & 0 & 24 & 36 & -6 \\ 1 & 1 & 0 & 1 & 4 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 44 & 81 & -41 \\ 0 & 1 & 0 & 0 & -100 & -189 & 103 \\ 0 & 0 & 1 & 0 & 108 & 207 & -117 \\ 0 & 0 & 0 & 1 & 60 & 108 & -51 \end{bmatrix}$$

Hence $\;{}_\beta[\,T\,]_\alpha = \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix}$

Both ways are fine. You can do it either way. Once you understand both paths the computations are fairly straightforward in either case. Let's look at one example of a particular $\vec{v}$

$$\begin{tikzcd}
{[\,\vec{v}\,]_\alpha = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} \arrow[r, "{}_\beta T_\alpha"] & {[\;\; T(\vec{v})\,]_\beta = \begin{bmatrix} 7 \\ -11 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} \\
\vec{v} = t^2+t+1 \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] & T(\vec{v}) = \begin{bmatrix} 6 & 11 \\ 12 & 8 \end{bmatrix} \arrow[u, "\varphi_\beta"']
\end{tikzcd}$$

Example 7

Let $T : \mathbb{R}^4 \to P_3(\mathbb{R})$ defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = (a-b)\,t^3 + (2a - c + 5d)\,t^2 + (b-c)\,t + 3c - 2d)$$

The following bases are given

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and

$$\beta = \left\{ t^3 + 2t^2 + 1, \quad t^3 + 3t - 5, \quad t^2 + 2t + 1, \quad t^3 + 3t^2 + 3 \right\}$$

and of course the usual standard bases.

Since
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xmapsto{\ {}_S T_s\ } \begin{bmatrix} a-b \\ 2a - c + 5d \\ b - c \\ 3c - 2d \end{bmatrix}$$

$${}_S[\,T\,]_s = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix}$$

and

$${}_sC_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad {}_sC_\beta = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 2 & 0 \\ 1 & -5 & 1 & 3 \end{bmatrix}$$

These three matrices where easy to find, and now give us

$$\begin{aligned} {}_\beta[\,T\,]_\alpha &= {}_\beta C_S \cdot {}_S[\,T\,]_s \cdot {}_sC_\alpha \\ &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 2 & 0 \\ 1 & -5 & 1 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -259 & 33 & -56 & -97 \\ 55 & -7 & 11 & 20 \\ -82 & 10 & -17 & -30 \\ 204 & -25 & 44 & 76 \end{bmatrix} \end{aligned}$$

Let's also find ${}_\beta[\,T\,]_\alpha$ the other way using ${}_\beta[\,T\,]_\alpha = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \big[T(\vec{a}_1)\big]_\beta & \big[T(\vec{a}_2)\big]_\beta & \cdots & \big[T(\vec{a}_n)\big]_\beta \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

(a) $T(\vec{\alpha}_1) = T\begin{bmatrix}1\\1\\0\\2\end{bmatrix} = 12\,t^2 + t - 4 \qquad \Rightarrow \qquad [\,T(\vec{\alpha}_1)\,]_\beta = \begin{bmatrix}-259\\55\\-82\\204\end{bmatrix}$

(b) $T(\vec{\alpha}_2) = T\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = t^3 + t^2 - t + 3 \qquad \Rightarrow \qquad [\,T(\vec{\alpha}_2)\,]_\beta = \begin{bmatrix}33\\-7\\10\\-25\end{bmatrix}$

(c) $T(\vec{\alpha}_3) = T\begin{bmatrix}0\\1\\2\\1\end{bmatrix} = -t^3 + 3t^2 - t + 4 \qquad \Rightarrow \qquad [\,T(\vec{\alpha}_3)\,]_\beta = \begin{bmatrix}-56\\11\\-17\\44\end{bmatrix}$

(d) $T(\vec{\alpha}_4) = T\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = -t^3 + 4t^2 + 1 \qquad \Rightarrow \qquad [\,T(\vec{\alpha}_4)\,]_\beta = \begin{bmatrix}-97\\20\\-30\\76\end{bmatrix}$

where the last parts were computed with one row reduction

$$\text{rref}\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & -1 & -1 \\ 2 & 0 & 1 & 3 & 12 & 1 & 3 & 4 \\ 0 & 3 & 2 & 1 & 1 & -1 & -1 & 0 \\ 1 & -5 & 1 & 3 & -4 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -259 & 33 & -56 & -97 \\ 0 & 1 & 0 & 0 & 55 & -7 & 11 & 20 \\ 0 & 0 & 1 & 0 & -82 & 10 & -17 & -30 \\ 0 & 0 & 0 & 1 & 204 & -25 & 44 & 76 \end{bmatrix}$$

Hence ${}_\beta[\,T\,]_\alpha = \begin{bmatrix} -259 & 33 & -56 & -97 \\ 55 & -7 & 11 & 20 \\ -82 & 10 & -17 & -30 \\ 204 & -25 & 44 & 76 \end{bmatrix}$

Both ways are fine. You can do it either way. Once you understand both paths the computations are fairly straightforward in either case. And ... essentially equivalent

Example 8

Let $T : P_2(\mathbb{F}_7) \to M_{2\times 3}(\mathbb{F}_7)$ be defined by

$$T(a\,t^2 + b\,t + c) = \begin{bmatrix} 2a+b & a+b+c & 3b+4c \\ 0 & 2a+5c & 3a+2b \end{bmatrix}$$

Since

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xmapsto{\ {}_S T_s\ } \begin{bmatrix} 2a+b \\ a+b+c \\ 3b+4c \\ 0 \\ 2a+5c \\ 3a+2b \end{bmatrix}$$

we find that

$${}_S[\,T\,]_s = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \\ 3 & 2 & 0 \end{bmatrix}$$

If $\qquad \alpha = \{\, t^2+2t+1, \quad t^2+3t+1, \quad t^2+2 \,\}$

and $\beta = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 5 \\ 6 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 3 \\ 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 6 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} \right\}$

Then

$$\begin{aligned} {}_\beta[\,T\,]_\alpha &= {}_\beta C_S \cdot {}_S[\,T\,]_s \cdot {}_s C_\alpha \\ &= \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 6 \\ 1 & 2 & 0 & 1 & 5 & 5 \\ 1 & 3 & 0 & 5 & 3 & 0 \\ 1 & 1 & 1 & 6 & 1 & 2 \\ 1 & 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 0 & 1 & 2 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \\ 3 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 5 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 5 \\ 6 & 5 & 6 \\ 3 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \end{aligned}$$

Let's look at the diagram for $\vec{v} = 6t^2 + 4t + 5$

$$
\begin{tikzcd}
\begin{bmatrix}3\\4\\6\end{bmatrix} \arrow[r, "{}_\beta[T]_\alpha"] \arrow[dd, bend right, "{}_sC_\alpha"'] & \begin{bmatrix}3\\4\\3\\4\\6\\1\end{bmatrix} \\
6t^2+4t+5 \arrow[u, "\varphi_\alpha"] \arrow[r, "T"] \arrow[d, "\varphi_s"'] & \begin{bmatrix}2 & 1 & 4\\0 & 2 & 5\end{bmatrix} \arrow[u, "\varphi_\beta"'] \arrow[d, "\varphi_S"] \\
\begin{bmatrix}6\\4\\5\end{bmatrix} \arrow[r, "{}_S[T]_s"] & \begin{bmatrix}2\\1\\4\\0\\2\\5\end{bmatrix} \arrow[uu, bend right, "{}_\beta C_S"']
\end{tikzcd}
$$

Here are the TI-Nspire calculations of both the blue and red paths, using the matrices:

$$ {}_sC_\alpha, \quad {}_S[T]_s, \quad {}_sC_\beta \quad \text{and} \quad {}_\beta[T]_\alpha \quad [\text{respectively}] $$

$$ s7\left(\begin{bmatrix}1 & 1 & 1\\2 & 3 & 0\\1 & 1 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\4\\6\end{bmatrix}\right) \qquad \begin{bmatrix}6\\4\\5\end{bmatrix} $$

$$ s7\left(\begin{bmatrix}2 & 1 & 0\\1 & 1 & 1\\0 & 3 & 4\\0 & 0 & 0\\2 & 0 & 5\\3 & 2 & 0\end{bmatrix}\cdot\begin{bmatrix}6\\4\\5\end{bmatrix}\right) \qquad \begin{bmatrix}2\\1\\4\\0\\2\\5\end{bmatrix} $$

$$ s7\left(mi7\left(\begin{bmatrix}1 & 1 & 2 & 3 & 1 & 6\\1 & 2 & 0 & 1 & 5 & 5\\1 & 3 & 0 & 5 & 3 & 0\\1 & 1 & 1 & 6 & 1 & 2\\1 & 0 & 4 & 2 & 4 & 4\\1 & 1 & 0 & 1 & 2 & 3\end{bmatrix}\right)\cdot\begin{bmatrix}2\\1\\4\\0\\2\\5\end{bmatrix}\right) \qquad \begin{bmatrix}3\\4\\3\\4\\6\\1\end{bmatrix} $$

$$ s7\left(\begin{bmatrix}0 & 5 & 3\\2 & 3 & 0\\1 & 3 & 5\\6 & 5 & 6\\3 & 3 & 1\\0 & 6 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\4\\6\end{bmatrix}\right) \qquad \begin{bmatrix}3\\4\\3\\4\\6\\1\end{bmatrix} $$