#### **Matrices of Linear Transformations**

A linear transformation  $T: \mathbb{F}^n \to \mathbb{F}^m$  can always be written as a matrix multiplication.

$$T(\vec{x}) = M \cdot \vec{x}$$

where  $M \in M_{m \times n}(\mathbb{F})$ .

# Example 1

Let  $T: \mathbb{R}^2 \to \mathbb{R}^3$  be a linear transformation defined by

$$T\begin{bmatrix} 1\\0 \end{bmatrix} = \begin{bmatrix} 3\\-1\\2 \end{bmatrix} \text{ and } T\begin{bmatrix} 0\\1 \end{bmatrix} = \begin{bmatrix} 2\\1\\-5 \end{bmatrix}$$

Since  $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$  is a basis, the transformation is completely determined by the above:

$$T \begin{bmatrix} x \\ y \end{bmatrix} = T \left( x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = x \cdot T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= x \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3x + 3y \\ -x + y \\ 2x - 5y \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 2 & -5 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence any image is computed by left multiplication with the matrix M, e.g.

$$T\begin{bmatrix} 3\\2 \end{bmatrix} = \begin{bmatrix} 3 & 2\\-1 & 1\\2 & -5 \end{bmatrix} \begin{bmatrix} 3\\2 \end{bmatrix} = \begin{bmatrix} 13\\-1\\-4 \end{bmatrix}$$

so that

$$T: \begin{bmatrix} 3\\2 \end{bmatrix} \mapsto \begin{bmatrix} 13\\-1\\-4 \end{bmatrix}$$

Even though  $T: \mathbb{F}^n \to \mathbb{F}^m$  is just one type of linear transformations, it is crucial in dealing with  $T: \mathbb{V} \to \mathbb{W}$  in general.

Let  $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$  be a linear transformation defined by

$$T(at^{2} + bt + c) = \begin{bmatrix} a+b & b-2c \\ 3a+2b & a+b+c \end{bmatrix}$$

For example:

$$t^2 + 3t - 2 \longmapsto \begin{bmatrix} 4 & 7 \\ 9 & 2 \end{bmatrix}$$

Clearly there is no matrix M such that:  $M \cdot (t^2 + 3t - 2) = \begin{bmatrix} 4 & 7 \\ 9 & 2 \end{bmatrix}$ .

We can however rewrite both vectors with respect to their standard bases:

$$\begin{bmatrix} 1\\3\\-2 \end{bmatrix} \qquad \begin{bmatrix} 4\\7\\9\\2 \end{bmatrix}$$

$$t^2 + 3t - 2 \xrightarrow{T} \qquad \begin{bmatrix} 4&7\\9&2 \end{bmatrix}$$

We basically reduced it to a transformation of the previous type:  $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ . In general for this transformation we have:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \begin{bmatrix} a+b \\ b-2c \\ 3a+2b \\ a+b+c \end{bmatrix}$$

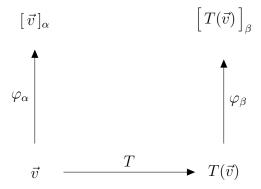
$$\uparrow \qquad \qquad \uparrow$$

$$a t^2 + b t + c \qquad T \qquad \qquad \begin{bmatrix} a+b & b-2c \\ 3a+2b & a+b+c \end{bmatrix}$$

A matrix transforms 
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \begin{bmatrix} a+b \\ b-2c \\ 3a+2b \\ a+b+c \end{bmatrix} \text{ as follows } \begin{bmatrix} a+b \\ b-2c \\ 3a+2b \\ a+b+c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This procedure we can use in general for any linear transformations  $T: \mathbb{V} \to \mathbb{W}$ . We can even use any basis  $\alpha$  for  $\mathbb{V}$  and any basis  $\beta$  for  $\mathbb{W}$ .

- \* Rewrite every vector  $\vec{v} \in \mathbb{V}$  with respect to the basis  $\alpha$  using  $\varphi_{\alpha}$ :  $\varphi_{\alpha}(\vec{v}) = [\vec{v}]_{\alpha}$
- \* Rewrite every vector  $\vec{w} \in \mathbb{W}$  with respect to the basis  $\beta$  using  $\varphi_{\beta}$ :  $\varphi_{\beta}(\vec{w}) = [\vec{w}]_{\beta}$

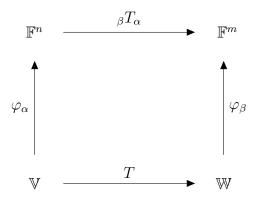


Note that if  $\dim(\mathbb{V}) = n$  and  $\dim(\mathbb{W}) = m$  then

$$\varphi_{\alpha}: \mathbb{V} \to \mathbb{F}^n$$

$$\varphi_{\beta}: \mathbb{W} \to \mathbb{F}^m$$

so that we have the following maps between the spaces  $\mathbb{V}$ ,  $\mathbb{W}$ ,  $\mathbb{F}^n$  and  $\mathbb{F}^m$ :



Note that we named the map between  $\mathbb{F}^n$  and  $\mathbb{F}^m$ :

The map  $_{\beta}T_{\alpha}$  mirrors the map T:

$$\vec{v} \vdash T \rightarrow T(\vec{v})$$

$$[\vec{v}]_{\alpha} \stackrel{\beta}{\longmapsto} [T(\vec{v})]_{\beta}$$

The maps T and  $_{\beta}T_{\alpha}$  are describing the same transformation, from different perspectives:

\* T maps  $\vec{v} \in \mathbb{V}$  to  $T(\vec{v}) \in \mathbb{W}$ ,

$$\vec{v} \xrightarrow{T} T(\vec{v})$$

\*  $_{\beta}T_{\alpha}$  maps  $[\vec{v}]_{\alpha} \in \mathbb{F}^n$  to  $[T(\vec{v})]_{\beta} \in \mathbb{F}^m$ 

$$[\vec{v}]_{\alpha} \xrightarrow{\beta T_{\alpha}} [T(\vec{v})]_{\beta}$$

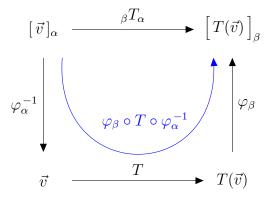
But:  $\vec{v}$  and  $[\vec{v}]_{\alpha}$  are referring to the same vector, except  $\vec{v}$  is in the original space  $\mathbb{V}$  and  $[\vec{v}]_{\alpha}$  is in  $\mathbb{F}^n$ , and is the coordinate vector of  $\vec{v}$  with respect to the basis  $\alpha$ .

Similarly,  $T(\vec{v})$  and  $[T(\vec{v})]_{\beta}$  are referring to the same vector, except  $T(\vec{v})$  is in the original space  $\mathbb{W}$  and  $[T(\vec{v})]_{\beta}$  is in  $\mathbb{F}^m$ , and is the coordinate vector of  $T(\vec{v})$  with respect to the basis  $\beta$ .

This link defines the transformation  $_{\beta}T_{\alpha}: \mathbb{F}^{n} \to \mathbb{F}^{m}: _{\beta}T_{\alpha}([\vec{v}]_{\alpha}) = [T(\vec{v})]_{\beta}$  or if you like:

$$_{\beta}T_{\alpha}=\varphi_{\beta}\circ T\circ\varphi_{\alpha}^{-1}$$

taking the route indicated in the next diagram



or equivalently:  $\beta T_{\alpha} \circ \varphi_{\alpha} = \varphi_{\beta} \circ T$  which indeed gives us

$${}_{\beta}T_{\alpha}\Big(\,[\,\vec{v}\,]_{\alpha}\Big)\,=\,{}_{\beta}T_{\alpha}\Big(\varphi_{\alpha}(\vec{v})\Big)\,=\,\Big({}_{\beta}T_{\alpha}\circ\varphi_{\alpha}\Big)(\vec{v})\,=\,\Big(\varphi_{\beta}\circ T\Big)(\vec{v})\,=\,\varphi_{\beta}\Big(T(\vec{v})\Big)\,=\,\Big[\,T(\vec{v})\,\Big]_{\beta}$$

#### Definition 1

Let  $T: \mathbb{V} \to \mathbb{W}$  be a linear transformation, with  $\dim(\mathbb{V}) = n$  and  $\dim(\mathbb{W}) = m$ . Let  $\alpha$  be a basis of  $\mathbb{V}$ , and  $\beta$  a basis of  $\mathbb{W}$ .

The transformation  $_{\beta}T_{\alpha}: \mathbb{F}^n \to \mathbb{F}^m$  is defined by any of the following

$$_{\beta}T_{\alpha}([\vec{v}]_{\alpha}) = [T(\vec{v})]_{\beta}$$
$$_{\beta}T_{\alpha} \circ \varphi_{\alpha} = \varphi_{\beta} \circ T$$
$$_{\beta}T_{\alpha} = \varphi_{\beta} \circ T \circ \varphi_{\alpha}^{-1}$$

Since  $_{\beta}T_{\alpha}:\mathbb{F}^{n}\to\mathbb{F}^{m}$  this transformation can be performed by a matrix multiplication:

$$_{\beta}T_{\alpha}(\vec{x}) = M \cdot \vec{x}$$

This matrix is called  $_{\beta}[T]_{\alpha}$ , so that

$$_{\beta}T_{\alpha}(\vec{x}) = _{\beta}[T]_{\alpha} \cdot \vec{x}$$

We'll first define the matrix  $_{\beta}[T]_{\alpha}$ , and then show that indeed  $_{\beta}T_{\alpha}(\vec{x}) = _{\beta}[T]_{\alpha} \cdot \vec{x}$ .

#### Definition 2

Let  $T: \mathbb{V} \to \mathbb{W}$  be a linear transformation. Let  $\alpha = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$  be a basis of  $\mathbb{V}$ , and  $\beta$  a basis of  $\mathbb{W}$ . The matrix  $\beta T_{\alpha}$  is defined by

$$_{eta}[T]_{lpha} = \left[egin{array}{cccc} \uparrow & & & \uparrow & & \uparrow \\ ig[T(ec{a}_1)ig]_{eta} & [T(ec{a}_2)ig]_{eta} & \cdots & ig[T(ec{a}_n)ig]_{eta} \\ \downarrow & & \downarrow & & \downarrow \end{array}
ight]$$

#### Example 3

Let  $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$  be defined by

$$T(at^{2} + bt + c) = \begin{bmatrix} a+b & -5a-2c \\ a+b+c & a-9c \end{bmatrix}$$

Let  $\alpha = \{t^2 + 3t - 1, t^2 + t + 2, t^2 + 4t - 3\}$  be a basis of  $P_2(\mathbb{R})$ , and

Let 
$$\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \right\}$$
 be a basis of  $M_{2\times 2}(\mathbb{R})$ .

We'll compute  $_{\beta}[T]_{\alpha}$  the matrix of  $_{\beta}T_{\alpha}$ .

(a) 
$$T(\vec{a}_1) = T(t^2 + 3t - 1) = \begin{bmatrix} 4 & -3 \\ 3 & 10 \end{bmatrix}$$
 and therefore  $[T(\vec{a}_1)]_{\beta} = \begin{bmatrix} -3 \\ 17 \\ 2 \\ -10 \end{bmatrix}$ 

**(b)** 
$$T(\vec{a}_2) = T(t^2 + t + 2) = \begin{bmatrix} 2 & -9 \\ 4 & -17 \end{bmatrix}$$
 and therefore  $[T(\vec{a}_2)]_{\beta} = \begin{bmatrix} -6 \\ 51 \\ 8 \\ -43 \end{bmatrix}$ 

(c) 
$$T(\vec{a}_3) = T(t^2 + 4t - 3) = \begin{bmatrix} 5 & 1 \\ 2 & 28 \end{bmatrix}$$
 and therefore  $\begin{bmatrix} T(\vec{a}_3) \end{bmatrix}_{\beta} = \begin{bmatrix} 0 \\ -11 \\ -3 \\ 16 \end{bmatrix}$ 

so that 
$$\beta[T]_{\alpha} = \begin{bmatrix} -3 & -6 & 0 \\ 17 & 51 & -11 \\ 2 & 8 & -3 \\ 10 & 42 & 16 \end{bmatrix}$$

For example let  $\vec{v} = 3t^2 + 4t + 5$  then

And indeed 
$$\begin{bmatrix} -3 & -6 & 0 \\ 17 & 51 & -11 \\ 2 & 8 & -3 \\ -10 & -43 & 16 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -18 \\ 147 \\ 23 \\ -122 \end{bmatrix}$$

#### Theorem 1

If  $T: \mathbb{V} \to \mathbb{W}$  is a linear transformation then

$$_{\beta}T_{\alpha}(\vec{x}) = _{\beta}[T]_{\alpha} \cdot \bar{x}$$

$$_{\beta}T_{\alpha}(\vec{x})=_{\beta}[\,T\,]_{\alpha}\cdot\vec{x}$$
 so that 
$$\left[\,T(\vec{v})\,\right]_{\beta}=_{\beta}[\,T\,]_{\alpha}\cdot[\,\vec{v}\,]_{\alpha}$$

### **Proof**:

Let  $T: \mathbb{V} \to \mathbb{W}$  be a linear transformation, and let

$$\alpha = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$$
 be a basis of  $\mathbb{V}$ 

and

 $\beta$  be a basis of W

then if 
$$[\vec{v}]_{\alpha} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$
 we can compute  $T(\vec{v})$  as follows:

$$T(\vec{v}) = T(v_1 \cdot \vec{a}_1 + v_2 \cdot \vec{a}_2 + \cdots + v_n \cdot \vec{a}_n)$$
  
=  $v_1 \cdot T(\vec{a}_1) + v_2 \cdot T(\vec{a}_2) + \cdots + v_n \cdot T(\vec{a}_n)$ 

so that expressed with respect to  $\beta$  we get

$$\begin{bmatrix} T(\vec{v}) \end{bmatrix}_{\beta} = \begin{bmatrix} v_1 \cdot T(\vec{a}_1) + v_2 \cdot T(\vec{a}_2) + \cdots + v_n \cdot T(\vec{a}_n) \end{bmatrix}_{\beta} 
= v_1 \cdot \begin{bmatrix} T(\vec{a}_1) \end{bmatrix}_{\beta} + v_2 \cdot \begin{bmatrix} T(\vec{a}_2) \end{bmatrix}_{\beta} + \cdots + v_n \cdot \begin{bmatrix} T(\vec{a}_n) \end{bmatrix}_{\beta}$$

Hence

$$\begin{bmatrix} T(\vec{v}) \end{bmatrix}_{\beta} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T(\vec{a}_1) \end{bmatrix}_{\beta} \begin{bmatrix} T(\vec{a}_2) \end{bmatrix}_{\beta} & \cdots & \begin{bmatrix} T(\vec{a}_n) \end{bmatrix}_{\beta} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

so that

$$\left[T(\vec{v})\right]_{\beta} = {}_{\beta}[T]_{\alpha} \cdot [\vec{v}]_{\alpha}$$

and since

$$[\vec{v}]_{\alpha} \xrightarrow{\beta T_{\alpha}} [T(\vec{v})]_{\beta}$$

 $_{\beta}[T]_{\alpha}$  is exactly the matrix that performs this operation:

$$[\vec{v}]_{\alpha} \xrightarrow{\beta T_{\alpha}} \beta [T]_{\alpha} \cdot [\vec{v}]_{\alpha} \qquad \Box$$

As we mentioned before, it often happens that when we write everything with respect to the standard bases, things become easier.

In particular  $_{S}[T]_{s}$  is usually easier to find than  $_{\beta}[T]_{\alpha}$ .

#### Example 4

Let  $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$  be a linear transformation defined by

$$T(t^2) = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}, \quad T(t) = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad T(1) = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

The transformation is completely determined by the above:

$$T(xt^{2} + yt + z) = x \cdot T(t^{2}) + y \cdot T(t) + z \cdot T(1)$$

$$= x \cdot \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} + z \cdot \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} x + y - z & 3x - y + 2z \\ 5x - 2y + z & 4x + 3y - 5z \end{bmatrix}$$

Here is the corresponding diagram

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{ST_s} \begin{bmatrix} x+y-z \\ 3x-y+2z \\ 5x-2y+z \\ 4x+3y-5z \end{bmatrix}$$

$$\varphi_s \qquad \qquad \varphi_s$$

$$x t^2 + y t + z \qquad \qquad T \qquad \qquad \begin{bmatrix} x+y-z & 3x-y+2z \\ 5x-2y+z & 4x+3y-5z \end{bmatrix}$$

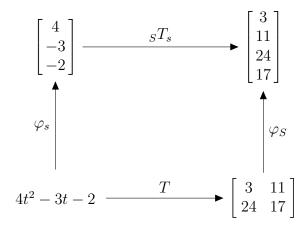
Hence

$$\begin{bmatrix} x+y-z \\ 3x-y+2z \\ 5x-2y+z \\ 4x+3y-5z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 5 & -2 & 1 \\ 4 & 3 & -5 \end{bmatrix}}_{S[T]_s} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and as you can see, its columns are precisely:  $[T(t^2)]_S$ ,  $[T(t)]_S$  and  $[T(t^2)]_S$  since

$$T(t^2) = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}, \quad T(t) = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad T(1) = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

So in particular for  $\vec{v} = 4t^2 - 3t - 2$ 



and indeed

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 5 & -2 & 1 \\ 4 & 3 & -5 \end{bmatrix}}_{S[T]_{2}} \cdot \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 24 \\ 17 \end{bmatrix}$$

# Example 5

Let  $T: U_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$  be a linear transformation defined by

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 3t^2 - 2t + 4, \quad T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2t^3 + 5t - 6 \text{ and } T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 6t^3 + 7t$$

Then

$$s[T]_{s} = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 0 & 0 \\ -2 & 5 & 7 \\ 4 & -6 & 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 3t^{2} - 2t + 4$$

$$\varphi_{s}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 6t^{3} + 7t$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2t^{3} + 5t - 6$$

where s and S are the standard bases of  $U_{2\times 2}(\mathbb{R})$  and  $P_3(\mathbb{R})$ .

Alternatively we could have computed

$$T\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = T\left(x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= x \cdot T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \cdot T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \cdot T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= x \cdot (3t^2 - 2t + 4) + y \cdot (2t^3 + 5t - 6) + z \cdot (6t^3 + 7t)$$

$$= (2y + 6z) \cdot t^3 + 3x \cdot t^2 + (-2x + 5y + 7z) \cdot t + (4x - 6y)$$

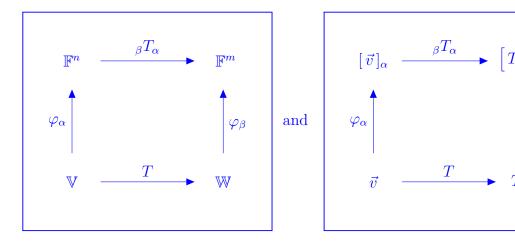
Hence 
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
  $\xrightarrow{\beta T_{\alpha}}$   $\begin{bmatrix} 2y + 6z \\ 3x \\ -2x + 5y + 7z \\ 4x - 6y \end{bmatrix}$   $=$   $\underbrace{\begin{bmatrix} 0 & 2 & 6 \\ 3 & 0 & 0 \\ -2 & 5 & 7 \\ 4 & -6 & 0 \end{bmatrix}}_{S[T]_{s}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 

**Summary**:  $_{\beta}T_{\alpha}$  and how to compute  $_{\beta}[T]_{\alpha}$ 

$$T: \mathbb{V} \to \mathbb{W}$$

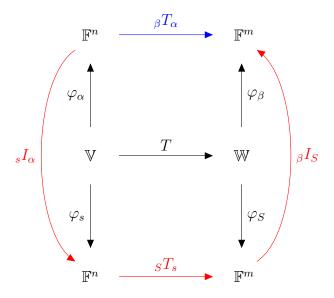
$$egin{aligned} eta[T]_{oldsymbol{lpha}} &= egin{bmatrix} igwedge & ig$$

#### Corresponding diagrams



But since it is often easier to compute  ${}_{S}[T]_{s}$  we can compute  ${}_{\beta}[T]_{\alpha}$  using this matrix as follows:

$$_{\beta}[T]_{\alpha} = _{\beta}C_{S} \cdot _{S}[T]_{s} \cdot _{s}C_{\alpha}$$



since

$$_{\beta}T_{\alpha} = _{\beta}I_{S} \cdot _{S}T_{s} \cdot _{s}I_{\alpha}$$

where  $_{s}I_{\alpha}$  is the map defined by

$$_{s}I_{\alpha}: \quad [\vec{v}]_{\alpha} \longmapsto [\vec{v}]_{s}$$

i.e.  ${}_sI_{\alpha}$  is the transformation that takes a vector  $[\vec{v}]_{\alpha}$ , which is the vector  $\vec{v} \in \mathbb{V}$  expressed with respect to the basis  $\alpha$ , and maps it to  $[\vec{v}]_s$ , which is the same vector but now expressed with respect to the standard basis s. [Here we use I because it is basically the identity map]

This we can do with the matrix  ${}_sC_\alpha$ 

$$[\vec{v}]_s = {}_sC_\alpha \cdot [\vec{v}]_\alpha$$

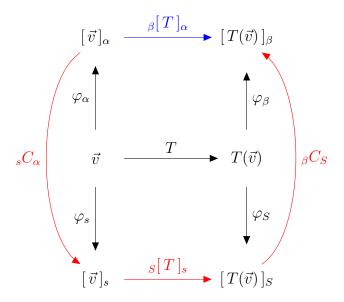
Similarly for  $_{\beta}I_{S}$  which is defined by

$$_{\beta}I_{S}: \quad [\vec{w}]_{S} \longmapsto [\vec{w}]_{\beta}$$

a transformation we can perform using the matrix  $_{\beta}C_{S}$   $\left(={}_{S}C_{\beta}^{-1}\right)$ 

$$[\vec{w}]_{\beta} = {}_{\beta}C_S \cdot [\vec{w}]_S$$

Let's look at the following diagram where instead of the names of the functions, we indicate all the **matrices** used for each map, and what vector is being mapped to what image vector



Instead of performing the transformation

$$[\vec{v}]_{\alpha} \longmapsto [T(\vec{v})]_{\beta}$$

using the matrix  $\beta[T]_{\alpha}$  as follows

$$_{\beta}[T]_{\alpha} \cdot [\vec{v}]_{\alpha} = [T(\vec{v})]_{\beta}$$

we take the longer, but often simpler route

$$[\vec{v}]_{\alpha} \longmapsto [\vec{v}]_s \longmapsto [T(\vec{v})]_S \longmapsto [T(\vec{v})]_{\beta}$$

using three matrices

$$_{\beta}C_{S} \cdot {}_{S}[T]_{s} \cdot {}_{s}C_{\alpha} \cdot [\vec{v}]_{\alpha} = [T(\vec{v})]_{\beta}$$

where each of these matrices can be found easily. This shows that

$$_{\beta}[T]_{\alpha} = _{\beta}C_{S} \cdot _{S}[T]_{s} \cdot _{s}C_{\alpha}$$

or if you like

$$_{\beta}[T]_{\alpha} = _{\beta}[I]_{S} \cdot _{S}[T]_{s} \cdot _{s}[I]_{\alpha}$$

This sounds all much more abstract than it actually is. Let's do a bunch of examples.

Let  $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$  defined by

$$T(at^{2} + bt + c) = \begin{bmatrix} 3a + 6b - 3c & a + 17b - 7c \\ 6a + 12b - 6c & 13a - 4b - c \end{bmatrix}$$

The following bases are given

$$\alpha = \left\{ t^2 + 2t + 1, \quad t^2 + 3t + 1, \quad t^2 + 2 \right\}$$

and

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$$

and of course the usual standard bases.

Since

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xrightarrow{ST_s} \begin{bmatrix} 3a + 6b - 3c \\ a + 17b - 7c \\ 6a + 12b - 6c \\ 13a - 4b - c \end{bmatrix}$$

$$_{S}[T]_{s} = \begin{bmatrix} 3 & 6 & -3 \\ 1 & 17 & -7 \\ 6 & 12 & -6 \\ 13 & -4 & -1 \end{bmatrix}$$

and

$${}_{s}C_{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad {}_{s}C_{\beta} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

These three matrices where easy to find, and now give us

$$\beta[T]_{\alpha} = \beta C_S \cdot S[T]_s \cdot SC_{\alpha}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 6 & -3 \\ 1 & 17 & -7 \\ 6 & 12 & -6 \\ 13 & -4 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix}$$

Let's also find  $_{\beta}[T]_{\alpha}$  the other way

using 
$$_{\beta}[T]_{\alpha} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ [T(\vec{a}_1)]_{\beta} & [T(\vec{a}_2)]_{\beta} & \cdots & [T(\vec{a}_n)]_{\beta} \end{bmatrix}$$

(a) 
$$T(\vec{\alpha}_1) = T(t^2 + 2t + 1) = \begin{bmatrix} 12 & 28 \\ 24 & 4 \end{bmatrix} \Rightarrow [T(\vec{\alpha}_1)]_{\beta} = \begin{bmatrix} 44 \\ -100 \\ 108 \\ 60 \end{bmatrix}$$

(b) 
$$T(\vec{\alpha}_2) = T(t^2 + 3t + 1) = \begin{bmatrix} 18 & 45 \\ 36 & 0 \end{bmatrix} \Rightarrow [T(\vec{\alpha}_2)]_{\beta} = \begin{bmatrix} 81 \\ -189 \\ 207 \\ 108 \end{bmatrix}$$

(c) 
$$T(\vec{\alpha}_3) = T(t^2 + 2) = \begin{bmatrix} -3 & -13 \\ -6 & 11 \end{bmatrix}$$
  $\Rightarrow$   $[T(\vec{\alpha}_3)]_{\beta} = \begin{bmatrix} -41 \\ 103 \\ -117 \\ -51 \end{bmatrix}$ 

where the last parts were computed with one row reduction

$$\operatorname{rref}\begin{bmatrix} 1 & 2 & 1 & 1 & 12 & 18 & -3 \\ 0 & 2 & 1 & 2 & 28 & 45 & -13 \\ 3 & 0 & -1 & 0 & 24 & 36 & -6 \\ 1 & 1 & 0 & 1 & 4 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 44 & 81 & -41 \\ 0 & 1 & 0 & 0 & -100 & -189 & 103 \\ 0 & 0 & 1 & 0 & 108 & 207 & -117 \\ 0 & 0 & 0 & 1 & 60 & 108 & -51 \end{bmatrix}$$

Hence 
$$_{\beta}[T]_{\alpha} = \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix}$$

Both ways are fine. You can do it either way. Once you understand both paths the computations are fairly straightforward in either case. Let's look at one example of a particular  $\vec{v}$ 

$$[\vec{v}]_{\alpha} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \xrightarrow{\beta T_{\alpha}} [T(\vec{v})]_{\beta} = \begin{bmatrix} 7 \\ -11 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 44 & 81 & -41 \\ -100 & -189 & 103 \\ 108 & 207 & -117 \\ 60 & 108 & -51 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$\varphi_{\alpha}$$

$$\vec{v} = t^2 + t + 1 \xrightarrow{T} T(\vec{v}) = \begin{bmatrix} 6 & 11 \\ 12 & 8 \end{bmatrix}$$

Let  $T: \mathbb{R}^4 \to P_3(\mathbb{R})$  defined by

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = (a - b) t^3 + (2a - c + 5d) t^2 + (b - c) t + 3c - 2d)$$

The following bases are given

$$\alpha = \left\{ \begin{bmatrix} 1\\1\\0\\2 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\2\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\1 \end{bmatrix} \right\}$$

and

$$\beta = \left\{ t^3 + 2t^2 + 1, \quad t^3 + 3t - 5, \quad t^2 + 2t + 1, \quad t^3 + 3t^2 + 3 \right\}$$

and of course the usual standard bases.

Since

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto {}_{S}T_{s} \longrightarrow \begin{bmatrix} a-b \\ 2a-c+5d \\ b-c \\ 3c-2d \end{bmatrix}$$

$$_{S}[T]_{s} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix}$$

and

$${}_{s}C_{\alpha} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad {}_{s}C_{\beta} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 2 & 0 \\ 1 & -5 & 1 & 3 \end{bmatrix}$$

These three matrices where easy to find, and now give us

$$\beta[T]_{\alpha} = \beta C_S \cdot S[T]_s \cdot C_{\alpha}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 2 & 0 \\ 1 & -5 & 1 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -259 & 33 & -56 & -97 \\ 55 & -7 & 11 & 20 \\ -82 & 10 & -17 & -30 \\ 204 & -25 & 44 & 76 \end{bmatrix}$$

Let's also find 
$$_{\beta}[T]_{\alpha}$$
 the other way using  $_{\beta}[T]_{\alpha} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ [T(\vec{a}_1)]_{\beta} & [T(\vec{a}_2)]_{\beta} & \cdots & [T(\vec{a}_n)]_{\beta} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ 

(a) 
$$T(\vec{\alpha}_1) = T \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = 12t^2 + t - 4$$
  $\Rightarrow [T(\vec{\alpha}_1)]_{\beta} = \begin{bmatrix} -259 \\ 55 \\ -82 \\ 204 \end{bmatrix}$ 

**(b)** 
$$T(\vec{\alpha}_2) = T \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = t^3 + t^2 - t + 3$$
  $\Rightarrow [T(\vec{\alpha}_2)]_{\beta} = \begin{bmatrix} 33 \\ -7 \\ 10 \\ -25 \end{bmatrix}$ 

(c) 
$$T(\vec{\alpha}_3) = T \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = -t^3 + 3t^2 - t + 4$$
  $\Rightarrow [T(\vec{\alpha}_3)]_{\beta} = \begin{bmatrix} -56 \\ 11 \\ -17 \\ 44 \end{bmatrix}$ 

(d) 
$$T(\vec{\alpha}_4) = T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -t^3 + 4t^2 + 1$$
  $\Rightarrow [T(\vec{\alpha}_4)]_{\beta} = \begin{bmatrix} -97 \\ 20 \\ -30 \\ 76 \end{bmatrix}$ 

where the last parts were computed with one row reduction

$$\operatorname{rref}\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & -1 & -1 \\ 2 & 0 & 1 & 3 & 12 & 1 & 3 & 4 \\ 0 & 3 & 2 & 1 & 1 & -1 & -1 & 0 \\ 1 & -5 & 1 & 3 & -4 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -259 & 33 & -56 & -97 \\ 0 & 1 & 0 & 0 & 55 & -7 & 11 & 20 \\ 0 & 0 & 1 & 0 & -82 & 10 & -17 & -30 \\ 0 & 0 & 0 & 1 & 204 & -25 & 44 & 76 \end{bmatrix}$$

Hence 
$$_{\beta}[T]_{\alpha} = \begin{bmatrix} -259 & 33 & -56 & -97 \\ 55 & -7 & 11 & 20 \\ -82 & 10 & -17 & -30 \\ 204 & -25 & 44 & 76 \end{bmatrix}$$

Both ways are fine. You can do it either way. Once you understand both paths the computations are fairly straightforward in either case. And ... essentially equivalent

Let  $T: P_2(\mathbb{F}_7) \to M_{2\times 3}(\mathbb{F}_7)$  be defined by

$$T(at^{2} + bt + c) = \begin{bmatrix} 2a + b & a + b + c & 3b + 4c \\ 0 & 2a + 5c & 3a + 2b \end{bmatrix}$$

Since

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto \begin{array}{c} ST_s \\ ST_s \\$$

we find that

$$s[T]_s = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \\ 3 & 2 & 0 \end{bmatrix}$$

If

$$\alpha = \{ t^2 + 2t + 1, \quad t^2 + 3t + 1, \quad t^2 + 2 \}$$

and 
$$\beta = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 5 \\ 6 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 3 \\ 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 6 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} \right\}$$

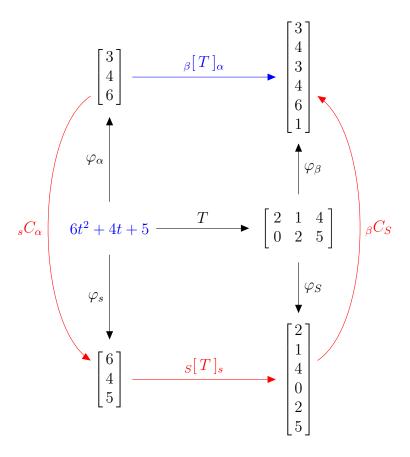
Then

$$\beta[T]_{\alpha} = \beta C_S \cdot S[T]_s \cdot S_c C_{\alpha}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 6 \\ 1 & 2 & 0 & 1 & 5 & 5 \\ 1 & 3 & 0 & 5 & 3 & 0 \\ 1 & 1 & 1 & 6 & 1 & 2 \\ 1 & 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \\ 3 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 5 \\ 6 & 5 & 6 \\ 3 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$$

Let's look at the diagram for  $\vec{v} = 6t^2 + 4t + 5$ 



Here are the TI-Nspire calculations of both the blue and red paths, using the matrices:

$$_{s}C_{\alpha}$$
,  $_{S}[T]_{s}$ ,  $_{s}C_{\beta}$  and  $_{\beta}[T]_{\alpha}$  [respectively]

$$s7 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$s7 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

$$s7 mi7 \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 6 \\ 1 & 2 & 0 & 1 & 5 & 5 \\ 1 & 3 & 0 & 5 & 3 & 0 \\ 1 & 1 & 1 & 6 & 1 & 2 \\ 1 & 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

$$s7 \begin{bmatrix} 0 & 5 & 3 \\ 2 & 3 & 0 \\ 3 & 3 & 1 \\ 6 & 5 & 6 \\ 3 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$