Reduced Row Echelon Form

Except for the zero matrix, any matrix will have non-zero entries. There may be rows and columns with just zeros in them. For example, some of the rows in the following 6×9 matrix contain just zeros:

$$\begin{bmatrix} 0 & 4 & 6 & 3 & 2 & 1 & 0 & 7 & 9 \\ \boxed{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0} \\ 0 & 0 & 0 & 5 & 3 & 0 & 0 & 1 & 0 \\ 8 & 1 & 0 & 7 & 3 & 0 & 9 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We begin with some terminology

Definition 1

A row whose entries are all zeros is called a **zero row**

A row with at least one non-zero entry is called a **non-zero row**

Since by definition, any non-zero row has at least one non-zero entry, when we scan through that non-zero row from left to right there will be a **first** non-zero entry. That entry is called a pivot.

Definition 2

The first non-zero entry in a non-zero row, scanning from left to right, is called a **pivot**

The pivots of the matrix below are as indicated

$$\begin{bmatrix} 0 & 4 & 6 & 3 & 2 & 1 & 0 & 7 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 3 & 0 & 0 & 1 & 0 \\ 8 & 1 & 0 & 7 & 3 & 0 & 9 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following 5×9 matrix is in what we call **reduced row echelon form**

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 2 & 4 & 0 & 7 & 9 \\ 0 & 0 & 0 & 1 & 6 & 5 & 0 & 8 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 3

We say a matrix is in **Reduced Row Echelon Form** if

(1) All zero rows—if any—are at the bottom of the matrix

(2) All pivots are 1

(3) If row i and row $i+1$ are two consecutive non-zero rows, the pivot in row $i+1$ is to the right of the pivot in row i

(4) Each column containing a pivot contains—besides that pivot—only zeros

The following matrices are **not** in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 3 & 4 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 4 & 0 & 2 & 1 & \boxed{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) Not all zero rows are at the bottom
(2) Not all pivots are 1
(3) The pivot of row 4 is to the left of the pivot in row 3
(4) There are non-zero entries above and below the pivot in row 3

$$\begin{bmatrix} 0 & 1 & 0 & 5 & \boxed{4} \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) ✓
(2) ✓
(3) ✓
(4) There is a non-zero entry above the pivot in row 3

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) ✓
(2) ✓
(3) The pivot in row 3 is to the left of the pivot in row 2
(4) ✓

Note 1 We say a matrix is in **row echelon form** when only properties **(1)–(3)** are satisfied.

Compare the following matrices:

$$\begin{bmatrix} 0 & 1 & 8 & 2 & 3 & 5 & 4 & 8 \\ 0 & 0 & 1 & 5 & 7 & 2 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Both are in **row echelon form**, but the one on the right is in **reduced** row echelon form, i.e. property **(4)** is also satisfied.

$$\begin{bmatrix} 0 & 1 & \mathbf{8} & 2 & 3 & \mathbf{5} & 4 & \mathbf{8} \\ 0 & 0 & 1 & 5 & 7 & \mathbf{2} & 1 & \mathbf{9} \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & \mathbf{0} & 2 & 3 & \mathbf{0} & 4 & \mathbf{0} \\ 0 & 0 & 1 & 5 & 7 & \mathbf{0} & 1 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Any matrix can be transformed into reduced row echelon form in a finite number of steps, using only the following three row operations:

Type I	Swapping two rows
Type II	Multiplying a row with a **non-zero** constant
Type III	Adding a multiple of one row to another row

Note 2 All these operations are invertible! [How can each be un-done?]

The procedure of transforming a matrix to reduced row echelon form is called **row reduction** (or Gaussian elimination or Gauss-Jordan elimination etc.)

We will begin with an example to show how we can row reduce the matrix on the left to its reduced row echelon form on the right:

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The sequence of steps is not unique. There are many ways to do it. But it will always produce the same reduced row echelon form.

Row Reduction

(1) $\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 2 & 0 & 1 & -1 \end{bmatrix}$

Rows 1 and 2 were swapped. We now have a pivot 1 in the first row. We could also have accomplished this by multiplying the first row by $\frac{1}{3}$. Notice the notation we used for rowswaps: $R_1 \leftrightarrow R_2$. Next we use the pivot in row 1 to eliminate the non-zero entries below it, using the third type operation:

(2) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 2 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -4 & -17 \\ 2 & 0 & 1 & -1 \end{bmatrix}$

Here we added -3 times row 1 to row 2. The notation of this is $-3R_1 + R_2$, where the last entry, R_2, indicates the row that is changing. We have now created a 0 below the pivot in the first column. We now use the same pivot to eliminate the 2 below it.

(3) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -4 & -17 \\ 2 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{-1R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 2 & 0 & 1 & -1 \end{bmatrix}$

Row 2 was multiplied with -1. The notation for this is $-1R_2$ or $-R_2$. Only row 2 was changed. The pivot of row 2 is now 1. Next we'll eliminate the 2 in row 3.

(4) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 2 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & -2 & -1 & -13 \end{bmatrix}$

Here we added -2 times row 1 to row 3. The notation of this is $-2R_1 + R_3$, where the last entry, R_3, indicates the row that is changing. We have now created all zeros below the pivot in the first column. We use the pivot of row 2 to create a zero below it.

(5) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & -2 & -1 & -13 \end{bmatrix} \xrightarrow{2R_2 + R_3} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 7 & 21 \end{bmatrix}$

Here 2 times row 2 was added to row 3. The notation for this is $2R_2 + R_3$ Only row 3 was changed. The pivot of row 3 is now 7. We'll produce a 1 there next.

(6) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 7 & 21 \end{bmatrix} \xrightarrow{\frac{1}{7}R_3} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Row 3 was multiplied with $\frac{1}{7}$. The notation for this is $\frac{1}{7}R_2$. Only row 3 was changed. The pivot of row 3 is now 1. Notice that we have a matrix in row echelon form.

(7) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-4R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Here -4 times row 3 was added to row 2. The notation for this is $-4R_3 + R_2$. Only row 2 was changed. We have created a 0 above the pivot of row 3. Notice that the new matrix is also in row echelon form. **Row echelon forms are not unique!**

(8) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-R_3 + R_1} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Here -1 times row 3 was added to row 1. The notation for this is $-R_3 + R_1$. Only row 1 was changed. We have created all zeros above the pivot of row 3. Again, the new matrix is in row echelon form. Row echelon forms are not unique.

(9) $\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Finally, -1 times row 2 was added to row 1. The notation for this is $-R_2 + R_1$. Only row 1 was changed. We have created all zeros above the pivots. We are done. This is the reduced row echelon form of the matrix we started with.

We can report this result as follows:

$$\boxed{\operatorname{rref}\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}}$$

Note 3 The **reduced row echelon form** of a matrix **is** unique!

We will use **rref** both as a verb and a noun. We will talk about '**to rref** a matrix', and '**the rref** of a matrix'.

Row Reduction using the TI-Nspire

The TI-Nspire can do these three row operations. It has them built-in. We will do the same steps (in the same order) which we performed by hand earlier, but now with the TI-Nspire:

(1) rowSwap

(2) rRowAdd

(3) mRow

$$\mathrm{rowSwap}\left(\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix},1,2\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathrm{mRowAdd}\left(-3,\begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 2 & 0 & 1 & -1 \end{bmatrix},1,2\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -4 & -17 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathrm{mRow}\left(-1,\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -4 & -17 \\ 2 & 0 & 1 & -1 \end{bmatrix},2\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathrm{mRowAdd}\left(-2,\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 2 & 0 & 1 & -1 \end{bmatrix},1,3\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & -2 & -1 & -13 \end{bmatrix}$$

$$\mathrm{mRowAdd}\left(2,\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & -2 & -1 & -13 \end{bmatrix},2,3\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 7 & 21 \end{bmatrix}$$

$$\mathrm{mRow}\left(\frac{1}{7},\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 7 & 21 \end{bmatrix},3\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\mathrm{mRowAdd}\left(-4,\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix},3,2\right) \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\mathrm{mRowAdd}\left(-1,\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix},3,1\right) \qquad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\mathrm{mRowAdd}\left(-1,\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix},2,1\right) \qquad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Of course the TI-Nspire can do all these steps using one command: **rref**

$$\mathrm{rref}\left(\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

But is good to know the entire procedure. So we encourage you to do things by hand for the first couple of times. And of course check your work, all steps, with the calculator.

It can also give you just a row echelon form: **ref**

$$\mathrm{ref}\left(\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & -1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{-5}{4} & \frac{5}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

We can formalize this procedure as follows:

Row reduction Input: matrix $M \neq O$

Phase I

(a) Move all zero rows to the bottom (by rowSwaps)

(b) Select the left most pivot. (If there is more than one, select the one closest to the top.) Move the row of this pivot to the top. Use mRow to make the pivot 1.

(c) If there are any nonzero entries below the pivot, use mRowAdd to remove them

(d) If the sub-matrix consisting of all the entries to the right and below the pivot is non-zero, repeat steps (a) – (d) on this sub-matrix. Else phase I ends.

After phase I the matrix is in row echelon form

Phase II

(a) When the entries (if any) above all pivots are zero phase II terminates.

(b) Using mRowAdd create zeros above the pivots

After phase II the matrix is in reduced row echelon form

Example 1 Phase I (a)–(d)

$$\begin{bmatrix} 0&0&3&6&0&3\\ 0&2&4&6&2&0\\ 0&0&0&1&3&4\\ 0&0&0&0&0&0\\ 0&1&3&6&5&3\\ 0&0&0&0&0&0\\ 0&2&4&6&2&0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_7} \begin{bmatrix} 0&0&3&6&0&3\\ 0&2&4&6&2&0\\ 0&0&0&1&3&4\\ 0&2&4&6&2&0\\ 0&1&3&6&5&3\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 0&2&4&6&2&0\\ 0&0&3&6&0&3\\ 0&0&0&1&3&4\\ 0&2&4&6&2&0\\ 0&1&3&6&5&3\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0&1&2&3&1&0\\ 0&0&3&6&0&3\\ 0&0&0&1&3&4\\ 0&2&4&6&2&0\\ 0&1&3&6&5&3\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \xrightarrow[-R_1+R_5]{-2R_1+R_4} \begin{bmatrix} 0&1&2&3&1&0\\ 0&0&3&6&0&3\\ 0&0&0&1&3&4\\ 0&0&0&0&0&0\\ 0&0&1&3&4&3\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}$$

Phase I (a)–(d) on sub-matrix

$$\begin{bmatrix} 3&6&0&3\\ 0&1&3&4\\ 0&0&0&0\\ 1&3&4&3\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_5} \begin{bmatrix} 3&6&0&3\\ 0&1&3&4\\ 1&3&4&3\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1&2&0&1\\ 0&1&3&4\\ 1&3&4&3\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$$

$$\xrightarrow{-R_1+R_3} \begin{bmatrix} 1&2&0&1\\ 0&1&3&4\\ 0&1&4&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1&2&0&1\\ 0&1&3&4\\ 0&0&1&-2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$$

Phase I (a)–(d) on sub-matrix

$$\begin{bmatrix} 1&-2\\ 0&0\\ 0&0\\ 0&0 \end{bmatrix}$$ no steps needed

Phase I (a)–(d) on sub-matrix:

$$\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$ Phase I terminates

We have produced the row echelon form $\begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Phase II starts.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_4+R_1]{-3R_4+R_3} \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-3R_3+R_1]{-2R_3+R_2} \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & -28 \\ 0 & 0 & 1 & 0 & 0 & -19 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & -19 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here is a TI-Nspire screen shot with the same end result:

$$\operatorname{rref}\left(\begin{bmatrix} 0 & 0 & 3 & 6 & 0 & 3 \\ 0 & 2 & 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 2 & 0 \end{bmatrix}\right) \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & -19 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Next we'll introduce elementary row matrices to do the row reduction.

Elementary Row Matrices

Definition 4

An **Elementary Row Matrix** is a matrix we get by performing **one** elementary row operation on the identity matrix

Type I Swapping rows

Example 2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

Type II Multiplying a row by a non-zero constant

Example 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

Type III Adding a multiple of one row to another row

Example 4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_3$$

Note 4

Elementary Row Matrices are invertible, since the elementary row operations are invertible: they can be un-done.

Example 5

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Each row operation can be achieved by left multiplication with an elementary row matrix.

Example 6 **A rowSwap**

$$\begin{bmatrix} 0 & 3 & 3 & 9 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

The following matrix multiplication performs this operation:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 3 & 9 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the matrix we need, we get by taking the identity matrix and performing exactly the operation on it we want to perform on the actual matrix.

Example 7 **An mRow**

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 3 & 8 \end{bmatrix} \xrightarrow{\frac{1}{3} R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

The following matrix multiplication performs this operation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3} R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the matrix we need, we get by taking the identity matrix and performing exactly the operation on it we want to perform on the actual matrix.

Example 8 An mRowAdd

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix} \xrightarrow{\quad -2R_2 + R_3 \quad} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The following matrix multiplication performs this operation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\quad -2R_2 + R_3 \quad} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

So the matrix we need, we get by taking the identity matrix and performing exactly the operation on it we want to perform on the actual matrix.

Let's put these three consecutive transformations together, with the elementary row matrices for each

$$\begin{bmatrix} 0 & 3 & 3 & 9 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 3 & 8 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Then

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_3} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \cdot \begin{bmatrix} 0 & 3 & 3 & 9 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Notice the **order** of the matrix multiplications. Also, since $E_3 \cdot E_2 \cdot E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 0 \\ -2/3 & 0 & 1 \end{bmatrix}$, we can do all three operations with one matrix multiplication

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 0 \\ -2/3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 3 & 9 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Note 5

When we left multiply a matrix A by an elementary row matrix E it performs a row operation on that matrix: precisely the elementary row operation that was performed on I to get E.

$$\boxed{\left.\begin{array}{l} A \xrightarrow{\text{row op}} B \\ I \xrightarrow{\text{row op}} E \end{array}\right\} \quad \Rightarrow \quad E \cdot A = B}$$

[Here the same row operation (**row op**) is performed on A and I.]

Hence row reduction can be done by left multiplication with a finite number of elementary row matrices:

$$\boxed{\operatorname{rref}(A) = M \quad \Rightarrow \quad E_k \cdots E_3 \cdot E_2 \cdot E_1 \cdot A = M}$$

where each E_i is an elementary row matrix. We can multiply out all these matrices to get a matrix Q

$$E_k \cdots E_3 \cdot E_2 \cdot E_1 = Q$$

Note that this Q is invertible! (Since all the E_i are invertible.)

Theorem 1

Let $\operatorname{rref}(A) = M$. There exists an invertible matrix Q such that

$$Q \cdot A = M$$

Example 9

$$\underbrace{\begin{bmatrix} -5 & 3 & 1 & -1 \\ -6 & 3 & 1 & -1 \\ -2 & 1 & 0 & 0 \\ 13 & -7 & -1 & 2 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M}$$

$$Q^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 \end{bmatrix} \quad \text{so that} \quad \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 \end{bmatrix}}_{Q^{-1}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M}$$

So it is possible to go back from $M = \operatorname{rref}(A)$ to the original matrix A, with Q^{-1}.

Note 6 The matrix Q is **not** unique!

Example 10

$$\underbrace{\begin{bmatrix} 8 & -4 & 0 & 1 \\ 7 & -4 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ -13 & 7 & 1 & -2 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M}$$

Note 7 There also exist **non-invertible** matrices Q with $Q \cdot A = M$

Example 11

$$\underbrace{\begin{bmatrix} -5 & 3 & 1 & -1 \\ -6 & 3 & 1 & -1 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M} \quad \text{and}$$

$$\underbrace{\begin{bmatrix} 8 & -4 & 0 & 1 \\ 7 & -4 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M}$$

BUT there exists **at least one invertible** Q such that $Q \cdot A = \text{rref}(A)$.

In the next section we are going to use this matrix Q. Strangely enough we only need to know that it exists, and is invertible, but most of the time we do not explicitly need to know what it looks like. However in case we do want to find Q, we could find the elementary row matrices and multiply them out as we showed earlier: $E_k \cdots E_3 \cdot E_2 \cdot E_1 = Q$, or we could use an augmented matrix to record the operations while we perform them.

Note that: $E \cdot \left[\boxed{A} \quad \boxed{B} \right] = \left[\boxed{E \cdot A} \quad \boxed{E \cdot B} \right]$

Example 12

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 2 \\ 2 & 7 & 8 & 3 & 5 & 8 \\ 1 & 1 & 1 & 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 2 \\ 0 & 3 & 2 & 1 & 5 & 4 \\ 1 & 1 & 1 & 4 & 3 & 1 \end{bmatrix}$$

So if we take $B = I$, and perform row operations on the augmented matrix $\left[\boxed{A} \quad \boxed{I} \right]$, we get

$$E_1 \cdot \left[\boxed{A} \quad \boxed{I} \right] = \left[\boxed{E_1 \cdot A} \quad \boxed{E_1} \right]$$

$$E_2 \cdot E_1 \cdot \left[\boxed{A} \quad \boxed{I} \right] = \left[\boxed{E_2 \cdot E_1 \cdot A} \quad \boxed{E_2 \cdot E_1} \right]$$

$$E_3 \cdot E_2 \cdot E_1 \cdot \left[\boxed{A} \quad \boxed{I} \right] = \left[\boxed{E_3 \cdot E_2 \cdot E_1 \cdot A} \quad \boxed{E_3 \cdot E_2 \cdot E_1} \right]$$

etc.

$$Q \cdot \left[\boxed{A} \quad \boxed{I} \right] = \left[\boxed{Q \cdot A} \quad \boxed{Q} \right]$$

To find a Q using the TI-Nspire, let it perform the following rref

$$\boxed{\mathrm{rref}\left[\boxed{A} \; \middle| \; \boxed{I} \right] = \left[\boxed{M} \; \middle| \; \boxed{Q} \right]}$$

The Q appears automatically on the ‘augmented’ side.

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \to m \qquad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{mRowAdd}(-2,m,1,2) \to m \qquad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{mRowAdd}(-1,m,1,3) \to m \qquad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{mRowAdd}(-1,m,1,4) \to m \qquad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 4 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{rowSwap}(m,2,4) \to m \qquad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & -1 & 0 & 0 & 1 \\ 0 & 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathrm{mRowAdd}(-2,m,2,3) \to m \qquad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -7 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{mRowAdd}(7,m,4,3) \to m \quad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -13 & 7 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{rowSwap}(m,3,4) \to m \quad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -13 & 7 & 1 & -2 \end{bmatrix}$$

$$\text{mRowAdd}(-4,m,3,2) \to m \quad \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 7 & -4 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -13 & 7 & 1 & -2 \end{bmatrix}$$

$$\text{mRowAdd}(1,m,2,1) \to m \quad \begin{bmatrix} 1 & 0 & 3 & 0 & 8 & -4 & 0 & 1 \\ 0 & 1 & 2 & 0 & 7 & -4 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -13 & 7 & 1 & -2 \end{bmatrix}$$

The 'augmented' side kept track of all the elementary row operations. And since the left side of the matrix $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form, the augmented side is Q.

$$\underbrace{\begin{bmatrix} 8 & -4 & 0 & 1 \\ 7 & -4 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ -13 & 7 & 1 & -2 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M}$$

We get another $Q = \dfrac{1}{13}\begin{bmatrix} 0 & 4 & 8 & -3 \\ 0 & -3 & 7 & -1 \\ 0 & -1 & -2 & 4 \\ 13 & -7 & -1 & 2 \end{bmatrix}$ by rref-ing $\left[\ \boxed{A} \quad \boxed{I}\ \right]$ using the TI-Nspire

$$\text{rref}\left(\begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \quad \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & \frac{4}{13} & \frac{8}{13} & \frac{-3}{13} \\ 0 & 1 & 2 & 0 & 0 & \frac{-3}{13} & \frac{7}{13} & \frac{-1}{13} \\ 0 & 0 & 0 & 1 & 0 & \frac{-1}{13} & \frac{-2}{13} & \frac{4}{13} \\ 0 & 0 & 0 & 0 & 1 & \frac{-7}{13} & \frac{-1}{13} & \frac{2}{13} \end{bmatrix}$$

Row Reduction over any Field $\mathbb{F}$

Row reduction can be done over any Field $\mathbb{F}$. Swapping rows (RowSwap) is certainly not Field specific. Multiplying a row by a non-zero constant (mRow) is possible over any Field. In particular any pivot can be made 1, since in any field we have multiplicative inverses (Pivots are non-zero by definition). So if a pivot is b over $\mathbb{F}_4$, it becomes 1 when we multiply the row by a, or if a pivot is 3 over $\mathbb{F}_7$, it becomes 1 when we multiply the row by 5. Finally we can add a multiple of a row to another row over any field. We'll do some examples of row reductions over $\mathbb{F}_7$ and $\mathbb{F}_4$.

Example 13 Row reduction of $\begin{bmatrix} 1 & 3 & 1 & 1 \\ 6 & 4 & 6 & 6 \\ 1 & 3 & 2 & 0 \end{bmatrix}$ over $\mathbb{F}_7$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 6 & 4 & 6 & 6 \\ 1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{6R_1 + R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{6R_2 + R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 \cdot E_3 \cdot E_2 \cdot E_1 = \begin{bmatrix} 2 & 0 & 6 \\ 6 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} 2 & 0 & 6 \\ 6 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & 3 & 1 & 1 \\ 6 & 4 & 6 & 6 \\ 1 & 3 & 2 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rref}(A)}$$

Here is the TI-Nspire's output, using **rref7**

$$rref7\left(\begin{bmatrix} 1 & 3 & 1 & 1 \\ 6 & 4 & 6 & 6 \\ 1 & 3 & 2 & 0 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 14 Row reduction of $\begin{bmatrix} 1 & a & b & 0 \\ a & 1 & 0 & 1 \\ b & b & b & 1 \end{bmatrix}$ over $\mathbb{F}_4$

Here is the TI-Nspire doing the steps and by hand on this side

$$\begin{bmatrix} 1 & a & b & 0 \\ a & 1 & 0 & 1 \\ b & b & b & 1 \end{bmatrix} \to m \qquad \begin{bmatrix} 1 & a & b & 0 \\ a & 1 & 0 & 1 \\ b & b & b & 1 \end{bmatrix}$$

$$\mathit{mrowadd4}(a,m,1,2) \to m \qquad \begin{bmatrix} 1 & a & b & 0 \\ 0 & a & 1 & 1 \\ b & b & b & 1 \end{bmatrix}$$

$$\mathit{mrowadd4}(b,m,1,3) \to m \qquad \begin{bmatrix} 1 & a & b & 0 \\ 0 & a & 1 & 1 \\ 0 & a & 1 & 1 \end{bmatrix}$$

$$\mathit{mrowadd4}(1,m,2,3) \to m \qquad \begin{bmatrix} 1 & a & b & 0 \\ 0 & a & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathit{mrowadd4}(1,m,2,1) \to m \qquad \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & a & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathit{mrow4}(b,m,2) \to m \qquad \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & b & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a & b & 0 \\ a & 1 & 0 & 1 \\ b & b & b & 1 \end{bmatrix} \xrightarrow{a\,R_1 + R_2} \begin{bmatrix} 1 & a & b & 0 \\ 0 & a & 1 & 1 \\ b & b & b & 1 \end{bmatrix}$$

$$\xrightarrow{b\,R_1 + R_3} \begin{bmatrix} 1 & a & b & 0 \\ 0 & a & 1 & 1 \\ 0 & a & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & a & b & 0 \\ 0 & a & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & a & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{b\,R_2} \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & b & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

You can check that the Q corresponding to these steps is

$$\underbrace{\begin{bmatrix} b & 1 & 0 \\ 1 & b & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & a & b & 0 \\ a & 1 & 0 & 1 \\ b & b & b & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & b & b \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathrm{rref}(A)}$$

rref-ing the augmented matrix $\left[\, \boxed{A} \;\; \boxed{I} \,\right]$ gives us another Q

$$\mathit{rref4}\left(\begin{bmatrix} 1 & a & b & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 1 & 0 & 1 & 0 \\ b & b & b & 1 & 0 & 0 & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & a & 1 & 0 & a & b \\ 0 & 1 & b & b & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathit{s4}\left(\begin{bmatrix} 0 & a & b \\ 0 & a & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & b & 0 \\ a & 1 & 0 & 1 \\ b & b & b & 1 \end{bmatrix}\right) \qquad \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & b & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$