Change of Coordinates Matrices

In this chapter we will learn how to find matrices that will change coordinates of vectors from one basis to another. We might as well define these matrices immediately.

Definition 1

Let α and β be two bases of a vector space $\mathbb{V}$. If $\alpha = \{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \cdots, \vec{\alpha}_n\}$ we define the matrix ${}_{\beta}C_{\alpha}$ by

$$ {}_{\beta}C_{\alpha} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [\,\vec{\alpha}_1\,]_{\beta} & [\,\vec{\alpha}_2\,]_{\beta} & \cdots & [\,\vec{\alpha}_n\,]_{\beta} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} $$

Note that in the columns of the matrix we have the coordinate vectors of the basis vectors $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \cdots, \vec{\alpha}_n$ of the basis α, each of them with respect to the basis β.

$$ \boxed{ {}_{\beta}C_{\alpha} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [\,\vec{\alpha}_1\,]_{\beta} & [\,\vec{\alpha}_2\,]_{\beta} & \cdots & [\,\vec{\alpha}_n\,]_{\beta} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} } $$

$\cdots$

$[\,\vec{\alpha}_1\,]_{\beta}$ in first column

$[\,\vec{\alpha}_n\,]_{\beta}$ in last column

$[\,\vec{\alpha}_2\,]_{\beta}$ in second column

To find the columns $[\,\vec{\alpha}_i\,]_{\beta}$ of this matrix, we use row reduction, which is *the* tool to express vectors with respect to other vectors. As will become clear after the following examples the row reduction that produces ${}_{\beta}C_{\alpha}$ is

$$ \boxed{ \operatorname{rref}\left[\, \beta_S \,\middle|\, \alpha_S \,\right] } $$

Example 1 Let $\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ and $\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

be two bases in $\mathbb{R}^4$

To find $[\,\vec{\alpha}_1\,]_\beta$, $[\,\vec{\alpha}_2\,]_\beta$, $[\,\vec{\alpha}_3\,]_\beta$ and $[\,\vec{\alpha}_4\,]_\beta$ we compute $\operatorname{rref}\left[\,\beta_S \,\middle|\, \alpha_S\,\right]$

$$\operatorname{rref}\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 1 & 1 & -3 \\ 0 & 1 & 0 & 0 & -3 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 4 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 6 & 1 & 0 & 6 \end{bmatrix}$$

$$\begin{array}{cccccccc} \vec{\beta}_1 & \vec{\beta}_2 & \vec{\beta}_3 & \vec{\beta}_4 & \vec{\alpha}_1 & \vec{\alpha}_2 & \vec{\alpha}_3 & \vec{\alpha}_4 \end{array} \qquad {}_\beta C_\alpha$$

which gives us: $[\,\vec{\alpha}_1\,]_\beta = \begin{bmatrix} -3 \\ -3 \\ 4 \\ 6 \end{bmatrix}$, $[\,\vec{\alpha}_2\,]_\beta = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $[\,\vec{\alpha}_3\,]_\beta = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $[\,\vec{\alpha}_4\,]_\beta = \begin{bmatrix} -3 \\ -2 \\ 2 \\ 6 \end{bmatrix}$

so that

$${}_\beta C_\alpha = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [\,\vec{\alpha}_1\,]_\beta & [\,\vec{\alpha}_2\,]_\beta & \cdots & [\,\vec{\alpha}_n\,]_\beta \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 0 & -2 \\ 4 & 0 & -1 & 2 \\ 6 & 1 & 0 & 6 \end{bmatrix}$$

Example 2 In $P_2(\mathbb{R})$ the following bases are given

$$\alpha = \left\{ t^2+t,\ t^2-1,\ t^2+t+1 \right\} \qquad \text{and} \qquad \beta = \left\{ 2t^2+t+1,\ t^2+t-3,\ 4t^2+2t+1 \right\}$$

When we write all vectors with respect to the standard basis we get

$$[\,\vec{\alpha}_1\,]_S = [\,t^2+t\,]_S = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad [\,\vec{\alpha}_2\,]_S = [\,t^2-1\,]_S = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad [\,\vec{\alpha}_3\,]_S = [\,t^2+t+1\,]_S = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$[\,\vec{\beta}_1\,]_S = [\,2t^2+t+1\,]_S = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad [\,\vec{\beta}_2\,]_S = [\,t^2+t-3\,]_S = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \qquad [\,\vec{\beta}_3\,]_S = [\,4t^2+2t+1\,]_S = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

To find ${}_\beta C_\alpha$ find $[\,\vec{\alpha}_1\,]_\beta$, $[\,\vec{\alpha}_2\,]_\beta$ and $[\,\vec{\alpha}_3\,]_\beta$, we compute $\mathrm{rref}\left[\,\beta_S \,\middle|\, \alpha_S\,\right]$

$$\mathrm{rref}\begin{array}{c}\left[\begin{array}{cccccc} 2 & 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 1 \\ 1 & -3 & 1 & 0 & -1 & 1 \end{array}\right] \\ \begin{array}{cccccc} [\,\vec{\beta}_1\,]_S & [\,\vec{\beta}_2\,]_S & [\,\vec{\beta}_3\,]_S & [\,\vec{\alpha}_1\,]_S & [\,\vec{\alpha}_2\,]_S & [\,\vec{\alpha}_3\,]_S \end{array}\end{array} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 6 & -9 & 8 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \multicolumn{3}{c}{\underbrace{-3 \quad\; 5 \quad\; -4}_{{}_\beta C_\alpha}} \end{array}\right]$$

Hence $[\,\vec{\alpha}_1\,]_\beta = \begin{bmatrix} 6 \\ 1 \\ -3 \end{bmatrix}$, $[\,\vec{\alpha}_2\,]_\beta = \begin{bmatrix} -9 \\ -1 \\ 5 \end{bmatrix}$, $[\,\vec{\alpha}_3\,]_\beta = \begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix}$, so that ${}_\beta C_\alpha = \begin{bmatrix} 6 & -9 & 8 \\ 1 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$

Theorem 1

$${}_\beta C_\alpha \cdot [\,\vec{v}\,]_\alpha = [\,\vec{v}\,]_\beta$$

Proof: Suppose we have the two bases α and β in a vector space $\mathbb{V}$, with

$$\alpha = \{\,\vec{\alpha}_1,\ \vec{\alpha}_2,\ \vec{\alpha}_3,\ \cdots\ ,\ \vec{\alpha}_n\,\}$$

We can express any vector $\vec{v} \in V$ with respect to the basis α: $\quad [\,\vec{v}\,]_\alpha = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$

Hence $\quad \vec{v} = a_1 \cdot \vec{\alpha}_1 + a_2 \cdot \vec{\alpha}_2 + a_3 \cdot \vec{\alpha}_3 + \cdots + a_n \cdot \vec{\alpha}_n$

$$\Rightarrow \quad [\,\vec{v}\,]_\beta = [\,a_1 \cdot \vec{\alpha}_1 + a_2 \cdot \vec{\alpha}_2 + a_3 \cdot \vec{\alpha}_3 + \cdots + a_n \cdot \vec{\alpha}_n\,]_\beta$$

$$\Rightarrow \quad [\,\vec{v}\,]_\beta = a_1 \cdot [\,\vec{\alpha}_1\,]_\beta + a_2 \cdot [\,\vec{\alpha}_2\,]_\beta + a_3 \cdot [\,\vec{\alpha}_3\,]_\beta + \cdots + a_n \cdot [\,\vec{\alpha}_n\,]_\beta$$

$$\Rightarrow \quad [\,\vec{v}\,]_\beta = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ [\,\vec{\alpha}_1\,]_\beta & [\,\vec{\alpha}_2\,]_\beta & \cdots & [\,\vec{\alpha}_n\,]_\beta \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

$$\Rightarrow \quad [\,\vec{v}\,]_\beta = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ [\,\vec{\alpha}_1\,]_\beta & [\,\vec{\alpha}_2\,]_\beta & \cdots & [\,\vec{\alpha}_n\,]_\beta \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix} \cdot [\,\vec{v}\,]_\alpha \quad \Rightarrow \quad {}_\beta C_\alpha \cdot [\,\vec{v}\,]_\alpha = [\,\vec{v}\,]_\beta \qquad \square$$

To continue the previous two examples:

Example 3 In example 1 we found that ${}_\beta C_\alpha = \begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 0 & -2 \\ 4 & 0 & -1 & 2 \\ 6 & 1 & 0 & 6 \end{bmatrix}$ when

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\} \text{ and } \beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

If for example $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ then $[\vec{v}]_\alpha = \begin{bmatrix} 7 \\ -12 \\ 18 \\ -4 \end{bmatrix}$ and $[\vec{v}]_\beta = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 6 \end{bmatrix}$ and clearly

$$\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 0 & -2 \\ 4 & 0 & -1 & 2 \\ 6 & 1 & 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -12 \\ 18 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 6 \end{bmatrix}$$

so that indeed: ${}_\beta C_\alpha \cdot [\vec{v}]_\alpha = [\vec{v}]_\beta$

Example 4 In example 2 we found that ${}_\beta C_\alpha = \begin{bmatrix} 6 & -9 & 8 \\ 1 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$ when

$$\alpha = \left\{ t^2 + t,\ t^2 - 1,\ t^2 + t + 1 \right\} \qquad \text{and} \qquad \beta = \left\{ 2t^2 + t + 1,\ t^2 + t - 3,\ 4t^2 + 2t + 1 \right\}$$

So, for example, if $\vec{v} = 3t^2 + 2t - 4$ then $[\vec{v}]_\alpha = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$ [Check!] and hence

$$[\vec{v}]_\beta = \begin{bmatrix} 6 & -9 & 8 \\ 1 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Theorem 2 ${}_\alpha C_\beta = {}_\beta C_\alpha^{-1}$

Proof: ${}_\beta C_\alpha \cdot [\vec{v}]_\alpha = [\vec{v}]_\beta \quad \Leftrightarrow \quad [\vec{v}]_\alpha = {}_\beta C_\alpha^{-1} \cdot [\vec{v}]_\beta$ ◻

A Special Case: $\boxed{{}_S C_\alpha}$

Let S be the standard basis, then

$$\boxed{{}_S C_\alpha = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ [\,\vec{\alpha}_1\,]_S & [\,\vec{\alpha}_2\,]_S & \cdots & [\,\vec{\alpha}_n\,]_S \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}}$$

$\cdots$

$[\,\vec{\alpha}_1\,]_S$ in first column

$[\,\vec{\alpha}_n\,]_S$ in last column

$[\,\vec{\alpha}_2\,]_S$ in second column

It turns out that most of the time it is trivial to write down ${}_S C_\alpha$.

Example 5 Let $\alpha = \{\, t^2 + t,\ t^2 - 1,\ t^2 + t + 1 \,\}$ and $S = \{\, t^2,\ t,\ 1 \,\}$ in $P_2(\mathbb{R})$ then

$${}_S C_\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Example 6 Let $\beta = \{\, 2t^2 + t + 1,\ t^2 + t - 3,\ 4t^2 + 2t + 1 \,\}$ and $S = \{\, t^2,\ t,\ 1 \,\}$ in $P_2(\mathbb{R})$ then

$${}_S C_\beta = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

Example 7 Let $\alpha = \left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 5 \\ 4 \end{bmatrix} \right\}$ in $\mathbb{R}^4$ with the usual standard basis, then

$${}_S C_\alpha = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 5 & 3 & 1 & 0 \\ 2 & 6 & 3 & 5 \\ 4 & 0 & 1 & 4 \end{bmatrix}$$

Example 8 Let $\alpha = \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$ with the standard basis $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, then

$$ {}_S C_\alpha = \begin{bmatrix} 1 & 5 & 2 & 0 \\ 4 & 3 & 6 & 3 \\ 2 & 0 & 1 & 5 \\ 6 & 3 & 4 & 1 \end{bmatrix} $$

Theorem 3 $\boxed{{}_\beta C_\alpha = {}_S C_\beta^{-1} \cdot {}_S C_\alpha}$ i.e. $\boxed{{}_\beta C_\alpha = {}_\beta C_S \cdot {}_S C_\alpha}$

Proof:

$$ \left. \begin{array}{l} {}_S C_\alpha \cdot [\vec{v}]_\alpha = [\vec{v}]_S \\ {}_S C_\beta \cdot [\vec{v}]_\beta = [\vec{v}]_S \end{array} \right\} \Rightarrow \quad {}_S C_\beta \cdot [\vec{v}]_\beta = {}_S C_\alpha \cdot [\vec{v}]_\alpha $$

$$ \Rightarrow \quad [\vec{v}]_\beta = {}_S C_\beta^{-1} {}_S C_\alpha \cdot [\vec{v}]_\alpha $$

$$ \Rightarrow \quad \boxed{{}_\beta C_\alpha = {}_S C_\beta^{-1} \cdot {}_S C_\alpha} \quad \text{i.e.} \quad {}_\beta C_\alpha = {}_\beta C_S \cdot {}_S C_\alpha $$

Example 9 In examples 4, 5 and 6 we found

$$ {}_\beta C_\alpha = \begin{bmatrix} 6 & -9 & 8 \\ 1 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix}, \quad {}_S C_\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad {}_S C_\beta = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix} $$

when $\alpha = \{ t^2 + t,\ t^2 - 1,\ t^2 + t + 1 \}$, $\beta = \{ 2t^2 + t + 1,\ t^2 + t - 3,\ 4t^2 + 2t + 1 \}$ and $S = \{ t^2, t, 1 \}$. It is easy to check that ${}_\beta C_\alpha = {}_S C_\beta^{-1} \cdot {}_S C_\alpha$

$$ \begin{bmatrix} 6 & -9 & 8 \\ 1 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \checkmark $$

Example 10 In $M_{2\times 2}(\mathbb{F}_7)$ the following bases are given

$$\alpha = \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix} \right\}$$

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Clearly

$$_SC_\alpha = \begin{bmatrix} 1 & 5 & 2 & 0 \\ 4 & 3 & 6 & 3 \\ 2 & 0 & 1 & 5 \\ 6 & 3 & 4 & 1 \end{bmatrix} \quad \text{and} \quad _SC_\beta = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 4 & 1 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

so that

$$_\beta C_\alpha = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 4 & 1 \\ 3 & 1 & 0 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 5 & 2 & 0 \\ 4 & 3 & 6 & 3 \\ 2 & 0 & 1 & 5 \\ 6 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 3 \\ 1 & 3 & 4 & 6 \\ 5 & 2 & 4 & 0 \\ 6 & 6 & 6 & 2 \end{bmatrix}$$

we can check our work with a row reduction: $\text{rref}\left[\,\beta_S \,\middle|\, \alpha_S\,\right]$

$$\text{rref7} \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 5 & 2 & 0 \\ 0 & 1 & 1 & 2 & 4 & 3 & 6 & 3 \\ 1 & 0 & 4 & 1 & 2 & 0 & 1 & 5 \\ 3 & 1 & 0 & 0 & 6 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 3 & 4 & 6 \\ 0 & 0 & 1 & 0 & 5 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 6 & 6 & 6 & 2 \end{bmatrix}$$

As an illustration, let's go through an explicit proof of theorem 3, using the above example:

Suppose $\vec{v} \in \mathbb{V} = M_{2\times 2}(\mathbb{F}_7)$ and $[\vec{v}]_\alpha = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $[\vec{v}]_\beta = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ then

$$\vec{v} = a_1 \cdot \vec{\alpha}_1 + a_2 \cdot \vec{\alpha}_2 + a_3 \cdot \vec{\alpha}_3 + a_4 \cdot \vec{\alpha}_4$$

and

$$\vec{v} = b_1 \cdot \vec{\beta}_1 + b_2 \cdot \vec{\beta}_2 + b_3 \cdot \vec{\beta}_3 + b_4 \cdot \vec{\beta}_4$$

so that

$$a_1 \cdot \vec{\alpha}_1 + a_2 \cdot \vec{\alpha}_2 + a_3 \cdot \vec{\alpha}_3 + a_4 \cdot \vec{\alpha}_4 = b_1 \cdot \vec{\beta}_1 + b_2 \cdot \vec{\beta}_2 + b_3 \cdot \vec{\beta}_3 + b_4 \cdot \vec{\beta}_4$$

i.e.

$$a_1\begin{bmatrix}1&4\\2&6\end{bmatrix}+a_2\begin{bmatrix}5&3\\0&3\end{bmatrix}+a_3\begin{bmatrix}2&6\\1&4\end{bmatrix}+a_4\begin{bmatrix}0&3\\5&1\end{bmatrix}=b_1\begin{bmatrix}1&0\\1&3\end{bmatrix}+b_2\begin{bmatrix}2&1\\0&1\end{bmatrix}+b_3\begin{bmatrix}1&1\\4&0\end{bmatrix}+b_4\begin{bmatrix}3&2\\1&0\end{bmatrix}$$

writing all basis elements in terms of the **standard** basis we get

$$a_1\begin{bmatrix}1\\4\\2\\6\end{bmatrix}+a_2\begin{bmatrix}5\\3\\0\\3\end{bmatrix}+a_3\begin{bmatrix}2\\6\\1\\4\end{bmatrix}+a_4\begin{bmatrix}0\\3\\5\\1\end{bmatrix}=b_1\begin{bmatrix}1\\0\\1\\3\end{bmatrix}+b_2\begin{bmatrix}2\\1\\0\\1\end{bmatrix}+b_3\begin{bmatrix}1\\1\\4\\0\end{bmatrix}+b_4\begin{bmatrix}3\\2\\1\\0\end{bmatrix}$$

and rewriting this as matrix multiplications we get

$$\begin{bmatrix}1&5&2&0\\4&3&6&3\\2&0&1&5\\6&3&4&1\end{bmatrix}\cdot\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix}=\begin{bmatrix}1&2&1&3\\0&1&1&2\\1&0&4&1\\3&1&0&0\end{bmatrix}\cdot\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

so that

$$\underbrace{\begin{bmatrix}1&2&1&3\\0&1&1&2\\1&0&4&1\\3&1&0&0\end{bmatrix}^{-1}\cdot\begin{bmatrix}1&5&2&0\\4&3&6&3\\2&0&1&5\\6&3&4&1\end{bmatrix}}_{{}_{\beta}C_{\alpha}}\cdot\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

Theorem 4

If α, β and γ are three bases in the vector space $\mathbb{V}$ then

$${}_{\gamma}C_{\alpha}={}_{\gamma}C_{\beta}\cdot{}_{\beta}C_{\alpha}$$

The proof should be obvious by now. And, of course, this can be extended to more bases:

$${}_{\delta}C_{\alpha}={}_{\delta}C_{\gamma}\cdot{}_{\gamma}C_{\beta}\cdot{}_{\beta}C_{\alpha}$$

$${}_{\epsilon}C_{\alpha}={}_{\epsilon}C_{\delta}\cdot{}_{\delta}C_{\gamma}\cdot{}_{\gamma}C_{\beta}\cdot{}_{\beta}C_{\alpha}$$

etc.

Change of Coordinates Matrices of Subspaces

The spaces $\mathbb{F}^n$, $M_{n\times m}(\mathbb{F})$ and $P_n(\mathbb{F})$ all have nice **standard** bases. But sometimes it is not at all clear what a standard basis would look like. Let's look at some examples. We'll start with some subspaces of the vector spaces $\mathbb{F}^n$, $M_{n\times m}(\mathbb{F})$ and $P_n(\mathbb{F})$.

Example 11 Let $\vec{a}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix}1\\0\\1\\2\end{bmatrix}$ and $\vec{a}_3 = \begin{bmatrix}3\\5\\6\\1\end{bmatrix} \in \mathbb{F}_7^4$ and

$$\mathbb{W} = \operatorname{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3) = \operatorname{span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}3\\5\\6\\1\end{bmatrix}\right) \sqsubseteq \mathbb{V} = \mathbb{F}_7^4.$$

It is easy to check that the set $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is linearly independent, so that

$$\alpha = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \quad \text{is a basis of } \mathbb{W}$$

Hence $\mathbb{W}$ is a 3 dimensional subspace of $\mathbb{V}$, and any basis of $\mathbb{W}$ has 3 elements.

Let $\vec{b}_1 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}1\\6\\3\\2\end{bmatrix}$ and $\vec{b}_3 = \begin{bmatrix}4\\5\\0\\3\end{bmatrix}$. The following row reduction

$$\operatorname{rref7}\begin{bmatrix}1 & 1 & 4 & 1 & 1 & 3\\ 2 & 6 & 5 & 2 & 0 & 5\\ 1 & 3 & 0 & 3 & 1 & 6\\ 1 & 2 & 3 & 4 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 2 & 2 & 5\\ 0 & 1 & 0 & 5 & 2 & 5\\ 0 & 0 & 1 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

reveals a lot of information

(1) $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ is linearly independent

(2) $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \operatorname{span}\left(\vec{b}_1, \vec{b}_2, \vec{b}_3\right)$, so that $\beta = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ is a basis of $\mathbb{W}$

(3) ${}_{\beta}C_{\alpha} = \begin{bmatrix}2 & 2 & 5\\ 5 & 2 & 5\\ 2 & 1 & 0\end{bmatrix}$

That ${}_{\beta}C_{\alpha}$ is a 3×3 matrix should not come as a surprise, since the bases of $\mathbb{W}$ all have three elements.

Note that $\vec{w} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 6 \end{bmatrix} \in \mathbb{W}$, in fact $[\,\vec{w}\,]_\alpha = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ and $[\,\vec{w}\,]_\beta = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$, and indeed

$$ {}_\beta C_\alpha \cdot [\,\vec{v}\,]_\alpha = [\,\vec{v}\,]_\beta : \qquad \begin{bmatrix} 2 & 2 & 5 \\ 5 & 2 & 5 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \quad \checkmark $$

Note that ${}_\beta C_\alpha = {}_S C_\beta^{-1} \cdot {}_S C_\alpha$ doesn't work here, since it is not clear what the standard basis of $\mathbb{W}$ is. We certainly can**not** use the standard basis of $\mathbb{V}$

$$ \sigma = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} $$

which has four elements, while $\dim(\mathbb{W}) = 3$, but even worse

$$ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ are } \textbf{\textit{not in}} \ \mathbb{W} $$

as is clear from

$$ \text{rref7} \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 & 0 \\ 3 & 1 & 6 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 5 & 1 \\ 0 & 1 & 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 1 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 6 & 3 & 5 \end{bmatrix} $$

So **none** of the standard basis vectors of $\mathbb{V}$ can be part of a basis of $\mathbb{W}$.

One might contemplate using ${}_\sigma C_\beta$ in ${}_\beta C_\alpha = {}_\sigma C_\beta^{-1} \cdot {}_\sigma C_\alpha$, but ${}_\sigma C_\beta = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 5 \\ 1 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$.

So this wouldn't make sense since clearly a 4×3 matrix is not invertible. We would be mixing a basis of $\mathbb{V}$ which is 4 dimensional, with a basis of $\mathbb{W}$ which is 3 dimensional. Note that the rrefs we performed where all done in the ambient space $\mathbb{V}$.

What would be ***a*** or ***the*** standard basis of $\mathbb{W}$?

Here is a good candidate for the standardbasis of $\mathbb{W}$:

$$ S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right\} $$

These vectors clearly form a linearly independent set: $\operatorname{rref7}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

and all of them are **in** $\mathbb{W}$, since $\operatorname{rref7}\begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 3 & 1 & 6 & 0 & 0 & 1 \\ 4 & 2 & 1 & 4 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 4 & 1 & 3 \\ 0 & 1 & 0 & 6 & 1 & 2 \\ 0 & 0 & 1 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

One of the features of a standard basis should be that the coordinates of a vector with respect to this standard basis are immediately clear:

(a) In $\mathbb{R}^3$ if $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then $[\,\vec{v}\,]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(b) In $P_2(\mathbb{R})$ if $\vec{v} = a\,t^2 + b\,t + c$, then $[\,\vec{v}\,]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(c) In $M_{2\times 2}(\mathbb{R})$ if $\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $[\,\vec{v}\,]_S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

The basis $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right\}$ has this property: If $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{W}$ then $[\,\vec{w}\,]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

For example:

$$[\,\vec{a}_1\,]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ [\,\vec{a}_2\,]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ [\,\vec{a}_3\,]_S = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix},\ [\,\vec{b}_1\,]_S = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ [\,\vec{b}_2\,]_S = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \text{ and } [\,\vec{b}_3\,]_S = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

In fact a criteria for being ***in*** $\mathbb{W}$ is thrown in for free: $d = 4a + 3b + 5c$, since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 4a + 3b + 5c \end{bmatrix}$$

Now that we have a standard basis of $\mathbb{W}$ we could use ${}_{\beta}C_{\alpha} = {}_{S}C_{\beta}^{-1} \cdot {}_{S}C_{\alpha}$.

Note that ${}_{S}C_{\alpha} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix}$ and ${}_{S}C_{\beta} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 5 \\ 1 & 3 & 0 \end{bmatrix}$ so that

$$ {}_{\beta}C_{\alpha} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 5 \\ 1 & 3 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 5 & 2 & 5 \\ 2 & 1 & 0 \end{bmatrix} $$

Let's compare this with our previous computation:

$$ \text{rref7} \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 1 & 3 \\ 2 & 6 & 5 & 2 & 0 & 5 \\ 1 & 3 & 0 & 3 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 1 \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 5 \\ 0 & 1 & 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] $$

${}_{\beta}C_{\alpha}$

$$ \beta_{\sigma} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \\ 3 \end{bmatrix} \right\} \qquad \alpha_{\sigma} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \\ 1 \end{bmatrix} \right\} $$

Here we bypassed trying to find a standard basis of $\mathbb{W}$ all together, and just used the standard base of the ambient space $\mathbb{V}$, and expressed the vectors of α with respect to the vectors of β, using our main tool: row reduction.

Example 12 Let $\mathbb{W} = \operatorname{span}(a\,t^3 + b\,t,\ b\,t^3 + 1) \sqsubseteq P_3(\mathbb{F}_4)$. Note $\dim(\mathbb{W}) = 2$

Here are two bases of $\mathbb{W}$: $\alpha = \{\, a\,t^3 + b\,t,\ b\,t^3 + 1 \,\}$ and $\beta = \{\, t^3 + b\,t + 1,\ t^3 + t + b \}$

$$ \text{rref} \left[\begin{array}{cc|cc} 1 & 1 & a & b \\ 0 & 0 & 0 & 0 \\ b & 1 & b & 0 \\ 1 & b & 0 & 1 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & b & a \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] $$

${}_{\beta}C_{\alpha}$

$$ \beta_{\sigma} = \left\{ \begin{bmatrix} 1 \\ 0 \\ b \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ b \end{bmatrix} \right\} \qquad \alpha_{\sigma} = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} $$

Hence $\quad {}_{\beta}C_{\alpha} = \begin{bmatrix} b & b \\ 1 & a \end{bmatrix}$.

Again we didn't even bother to look for a standard basis of $\mathbb{W}$.

But if we wanted to have gone that route: here is a candidate for "standard basis" of $\mathbb{W}$:

$$S = \left\{ t^3 + a, \quad t + 1 \right\}$$

This means that: if $\vec{w} \in \mathbb{W}$ and $[\,\vec{w}\,]_S = \begin{bmatrix} x \\ y \end{bmatrix}$ then $\vec{w} = x\,t^3 + y\,t + (a\,x + y)$.

Hence the coordinates with respect to this "standard basis" of any vector in $\mathbb{W}$ are just the coefficients of t^3 and t. This makes it easy to find:

$$[\,\vec{\alpha}_1\,]_S = [\,a\,t^3 + b\,t\,]_S = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad [\,\vec{\alpha}_2\,]_S = [\,b\,t^3 + 1\,]_S = \begin{bmatrix} b \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad {}_{S}C_{\alpha} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$$

$$[\,\vec{\beta}_1\,]_S = [\,t^3 + b\,t + 1\,]_S = \begin{bmatrix} 1 \\ b \end{bmatrix}, \qquad [\,\vec{\beta}_2\,]_S = [\,t^3 + t + b\,]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad {}_{S}C_{\beta} = \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

Hence: $\quad {}_{\beta}C_{\alpha} = {}_{S}C_{\beta}^{-1} \cdot {}_{S}C_{\alpha} = \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ 1 & 1 \end{bmatrix}$ as we found before.

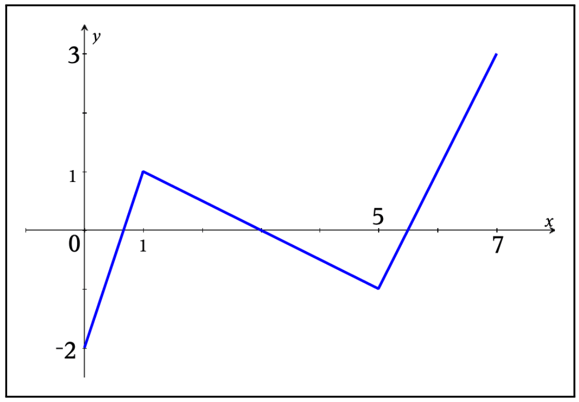
We'll end with a more complicated example, to underscore that it is sometimes not easy to find a "standard basis".

(Optional) Here is a more complicated example

Example 13 Let $\mathbb{V}$ be the vector space of continuous, piece-wise linear functions on the intervals $[0,\,1]$, $[1,\,5]$ and $[3,\,7]$.

Here is an example of such a function and its graph:

$$f(x) = \begin{cases} 3x-2 & \text{if } 0 \le x < 1 \\ -\frac{1}{2}x + \frac{3}{2} & \text{if } 1 \le x < 5 \\ 2x - 11 & \text{if } 5 \le x \le 1 \end{cases}$$



This turns out to be a four dimensional vector space.

Every function in $\mathbb{W}$ looks like: $f(x) = \begin{cases} a\,x+b & \text{if } 0 \le x < 1 \\ c\,x+d & \text{if } 1 \le x < 5 \\ e\,x+f & \text{if } 5 \le x \le 1 \end{cases}$

But since f is continuous, we have six variables with two relations:

$$a+b = c+d \quad \text{and} \quad 5c+d = 5e+f$$

.

Row reduction shows 4 free variables (two dependent). Hence the space is 4 dimensional.

For example a function in $\mathbb{W}$ is completely determined by the following 4 bits of information

(1) The y-values at $x=0$, $x=1$, $x=5$ and $x=7$,

(2) The y-value at $x=0$, and the slopes on the intervals $[0,\,1]$, $[1,\,5]$ and $[5,\,7]$.

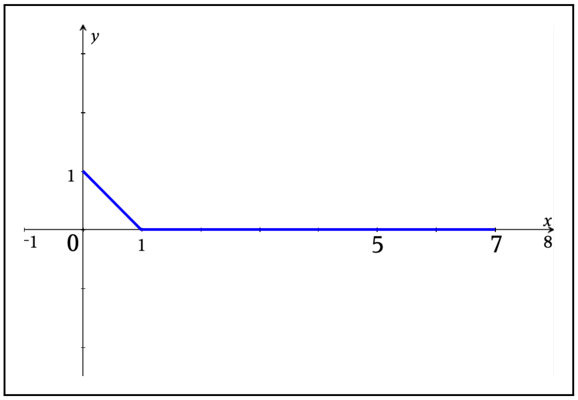
(3) The y-value at $x=0$, and the first slope on the interval $[0,\,1]$, and the *increases* in slope from the first to the second interval, and the second to third interval.

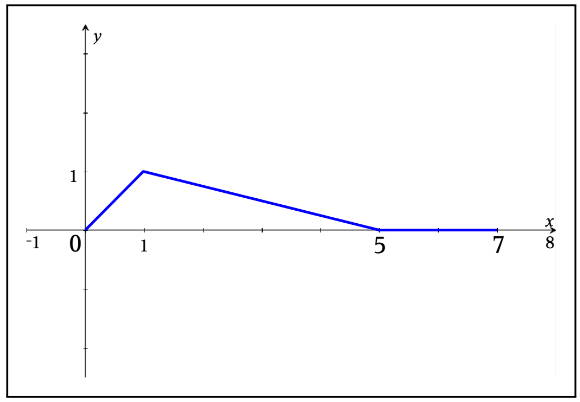
(4) The y-values at $x=0$ and $x=7$, and the first and last slope (on the intervals $[0,\,1]$ and $[5,\,7]$ respectively.)

etc.

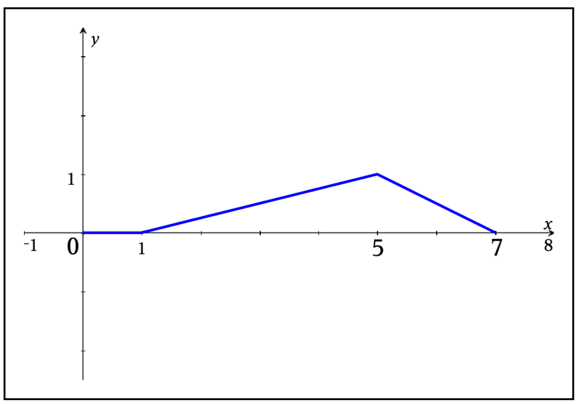
Here are three bases for this world:

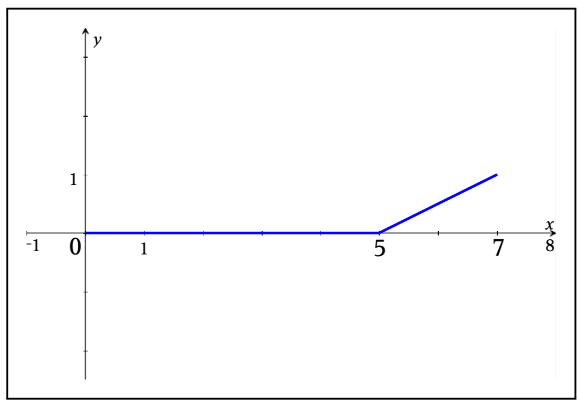
(1) $\boxed{\alpha = \{\vec{\alpha}_1,\ \vec{\alpha}_2,\ \vec{\alpha}_3,\ \vec{\alpha}_4\}}$ where the graphs of $\vec{\alpha}_i$ are as follows (in order):





$$\vec{\alpha}_1 = \alpha_1(x) = \begin{cases} 1-x & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 5 \\ 0 & \text{if } 5 \le x \le 1 \end{cases} \qquad \vec{\alpha}_2 = \alpha_2(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ -\frac{1}{4}x + \frac{5}{4} & \text{if } 1 \le x < 5 \\ 0 & \text{if } 5 \le x \le 1 \end{cases}$$





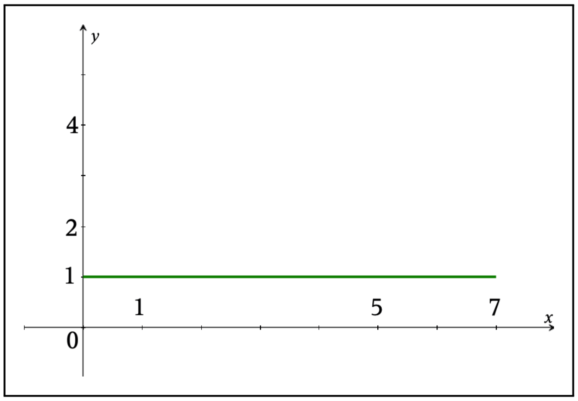
$$\vec{\alpha}_3 = \alpha_3(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ \frac{1}{4}x - \frac{1}{4} & \text{if } 1 \le x < 5 \\ -\frac{1}{2}x + \frac{7}{2} & \text{if } 5 \le x \le 1 \end{cases} \qquad \vec{\alpha}_4 = \alpha_4(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 5 \\ \frac{1}{2}x - \frac{5}{2} & \text{if } 5 \le x \le 1 \end{cases}$$

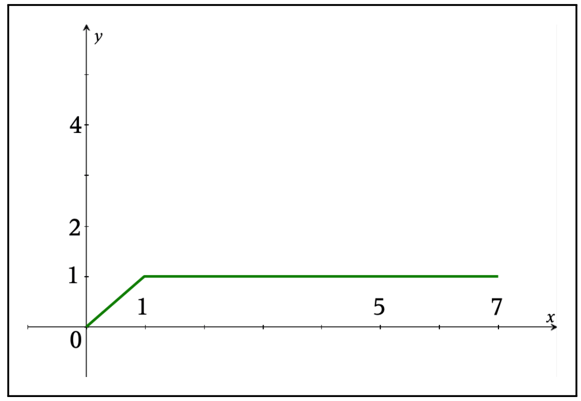
This basis corresponds to **(1)**. For example if $f(x) = \begin{cases} 3x-2 & \text{if } 0 \le x < 1 \\ -\frac{1}{2}x + \frac{3}{2} & \text{if } 1 \le x < 5 \\ 2x - 11 & \text{if } 5 \le x \le 1 \end{cases}$

Then the y-values at $x = 0, 1, 5, 7$ are $f(0) = -2$, $f(1) = 1$, $f(5) = -1$ and $f(7) = 3$

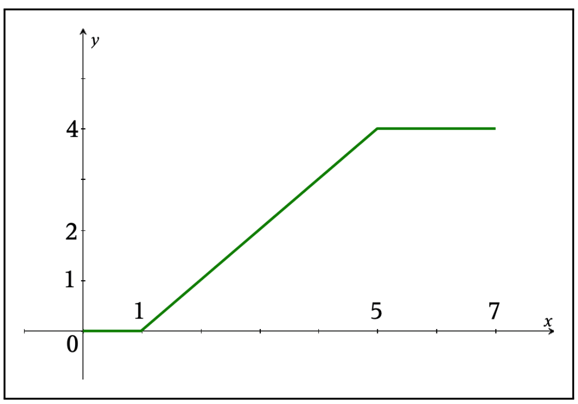
so that: $f(x) = -2 \cdot \vec{\alpha}_1 + 1 \cdot \vec{\alpha}_2 - 1 \cdot \vec{\alpha}_3 + 3 \cdot \vec{\alpha}_4 \quad \Rightarrow \quad [\, f(x)\,]_\alpha = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

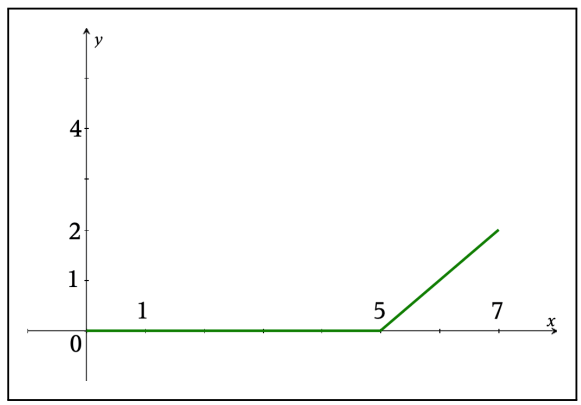
(2) $\boxed{\beta = \left\{\vec{\beta}_1,\ \vec{\beta}_2,\ \vec{\beta}_3,\ \vec{\beta}_4\right\}}$ where the graphs of $\vec{\beta}_i$ are as follows (in order):





$$\vec{\beta}_1 = \beta_1(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 5 \\ 1 & \text{if } 5 \le x \le 1 \end{cases} \qquad \vec{\beta}_2 = \beta_2(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 5 \\ 1 & \text{if } 5 \le x \le 1 \end{cases}$$





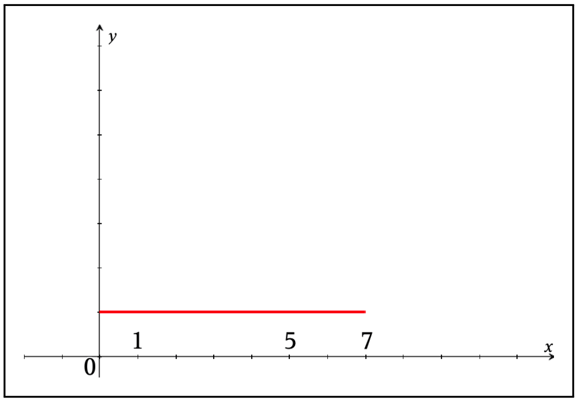
$$\vec{\beta}_3 = \beta_3(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ x-1 & \text{if } 1 \le x < 5 \\ 4 & \text{if } 5 \le x \le 1 \end{cases} \qquad \vec{\beta}_4 = \beta_4(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 5 \\ x-5 & \text{if } 5 \le x \le 1 \end{cases}$$

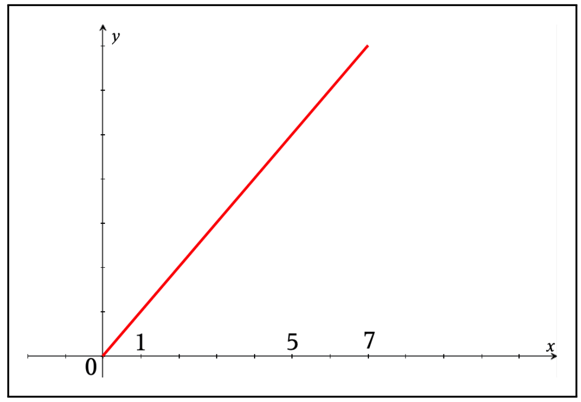
This basis corresponds to **(2)**. For example if $f(x) = \begin{cases} 3x-2 & \text{if } 0 \le x < 1 \\ -\frac{1}{2}x + \frac{3}{2} & \text{if } 1 \le x < 5 \\ 2x - 11 & \text{if } 5 \le x \le 1 \end{cases}$

Then the y-value at $x = 0$ is $f(0) = -2$, and the slopes are 3, $-\frac{1}{2}$ and 2, on the intervals $[0, 1]$, $[1, 5]$ and $[3, 7]$ respectively. Hence

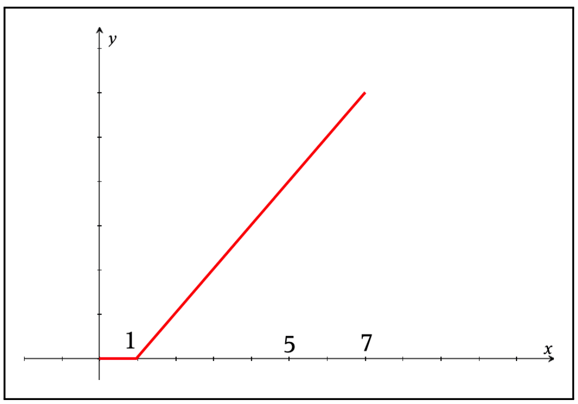
$$f(x) = -2 \cdot \vec{\beta}_1 + 3 \cdot \vec{\beta}_2 - \frac{1}{2} \cdot \vec{\beta}_3 + 2 \cdot \vec{\beta}_4 \quad \Rightarrow \quad [\,f(x)\,]_\beta = \begin{bmatrix} -2 \\ 3 \\ -1/2 \\ 2 \end{bmatrix}$$

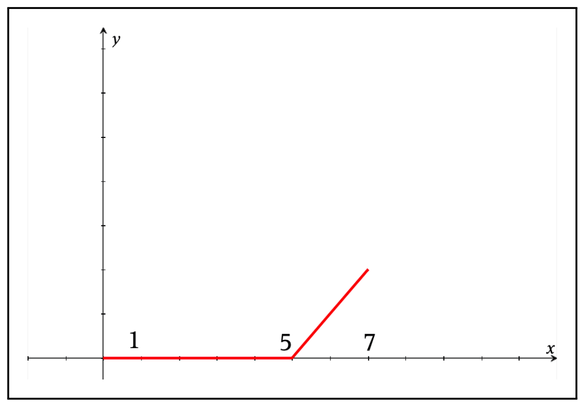
(3) $\boxed{\gamma = \{\vec{\gamma}_1,\ \vec{\gamma}_2,\ \vec{\gamma}_3,\ \vec{\gamma}_4\}}$ where the graphs of $\vec{\gamma}_i$ are as follows (in order):





$$\vec{\gamma}_1 = \gamma_1(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 5 \\ 1 & \text{if } 5 \le x \le 1 \end{cases} \qquad \vec{\gamma}_2 = \gamma_2(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x < 5 \\ x & \text{if } 5 \le x \le 1 \end{cases}$$





$$\vec{\gamma}_3 = \gamma_3(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ x-1 & \text{if } 1 \le x < 5 \\ x-1 & \text{if } 5 \le x \le 1 \end{cases} \qquad \vec{\gamma}_4 = \gamma_4(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 5 \\ x-5 & \text{if } 5 \le x \le 1 \end{cases}$$

This basis corresponds to **(3)**. For example if $f(x) = \begin{cases} 3x-2 & \text{if } 0 \le x < 1 \\ -\frac{1}{2}x + \frac{3}{2} & \text{if } 1 \le x < 5 \\ 2x - 11 & \text{if } 5 \le x \le 1 \end{cases}$

Then the y-value at $x = 0$ are $f(0) = -2$, the first slope is 3, the next slope is $-\frac{7}{2}$ **more** $(3 - \frac{7}{2} = -\frac{1}{2})$, and the next one is $\frac{5}{2}$ **more** $(-\frac{1}{2} + \frac{5}{2} = 2)$, so that:

$$f(x) = -2 \cdot \vec{\gamma}_1 + 3 \cdot \vec{\gamma}_2 - \frac{7}{2} \cdot \vec{\gamma}_3 + \frac{5}{2} \cdot \vec{\gamma}_4 \quad \Rightarrow \quad [\,f(x)\,]_\gamma = \begin{bmatrix} -2 \\ 3 \\ -7/2 \\ 5/2 \end{bmatrix}$$

As an exercise: Find the basis that corresponds to **(4)**

Which of these bases (if any) would be a candidate for "standard basis"? I think a case can be made for any of these.

Find ${}_{\beta}C_{\alpha}$, ${}_{\gamma}C_{\beta}$ and ${}_{\gamma}C_{\alpha}$

First: $[\vec{\alpha}_1]_{\beta} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $[\vec{\alpha}_2]_{\beta} = \begin{bmatrix} 0 \\ 1 \\ -1/4 \\ 0 \end{bmatrix}$, $[\vec{\alpha}_3]_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 1/4 \\ -1/2 \end{bmatrix}$, $[\vec{\alpha}_4]_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$

Hence

$$\boxed{{}_{\beta}C_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1/4 & 1/4 & 0 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}}$$

so that e.g. ${}_{\beta}C_{\alpha} \cdot [f(x)]_{\alpha} = [f(x)]_{\beta}$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1/4 & 1/4 & 0 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1/2 \\ 2 \end{bmatrix}$ ✓

Next: $[\vec{\beta}_1]_{\gamma} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $[\vec{\beta}_2]_{\gamma} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $[\vec{\beta}_3]_{\gamma} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $[\vec{\beta}_4]_{\gamma} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Hence

$$\boxed{{}_{\gamma}C_{\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}$$

and indeed ${}_{\gamma}C_{\beta} \cdot [f(x)]_{\beta} = [f(x)]_{\gamma}$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ -1/2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -7/2 \\ 5/2 \end{bmatrix}$ ✓

And finally:

$$\boxed{{}_{\gamma}C_{\alpha} = {}_{\gamma}C_{\beta} \cdot {}_{\beta}C_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -5/4 & 1/4 & 0 \\ 0 & 1/4 & -3/4 & 1/2 \end{bmatrix}}$$