Vectors in $\mathbb{R}^n$

We will begin with a definition of the most important class of vector spaces. We will define them in general over any field $\mathbb{F}$, but in this section we will focus on $\mathbb{F} = \mathbb{R}$. In future sections we will introduce other fields, such as $\mathbb{C}$, $\mathbb{F}_4$ and $\mathbb{F}_7$. In this section we will primarily look at the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 1

$$\mathbb{F}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{F} \right\}$$

These spaces are called vector spaces, and the n-tuples $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ are called vectors.

Example 1

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} \quad \text{and} \quad \mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

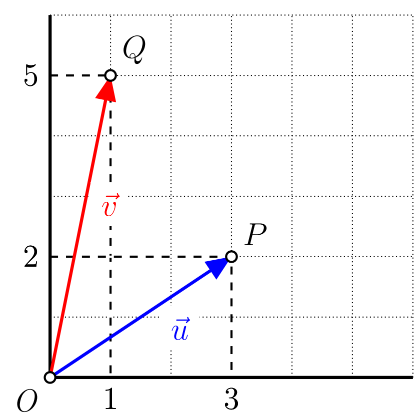
$$\text{e.g.} \quad \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} \pi \\ \ln(2) \end{bmatrix} \in \mathbb{R}^2 \quad \text{and} \quad \begin{bmatrix} 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.5 \\ e^2 \\ \sqrt{3} \end{bmatrix} \in \mathbb{R}^3$$

Although we defined these spaces purely algebraically, we can use Euclidean geometry to visualize the vectors, as arrows anchored at the origin:

Example 2

The vectors $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ are shown.

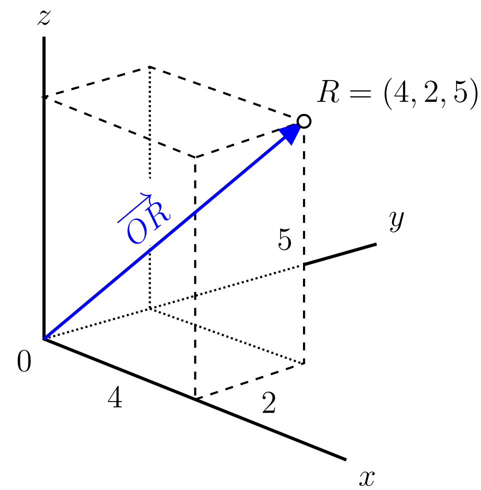
Notice the relationship between the vectors $\vec{u}$ and $\vec{v}$ and the points $P = (3, 2)$ and $Q = (1, 5)$. The vectors are arrows that start at the origin and point at the indicated points.



Example 3

The vectors $\vec{r} = \overrightarrow{OR} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$ is shown.

It is a vectors that starts at the origin and points at the point $R = (4, 2, 5)$.



In any vector space there are two operations defined on vectors: vector addition and scalar multiplications. We will define these in the next two sections, and list their usual properties.

Vector Addition

We can add two vectors provided they have the same dimensions. **Algebraically** this is defined as follows:

Definition 2

The **addition** of two vectors is defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Notice that the corresponding coordinates were simply added.

Example 4

If $\vec{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ then $\vec{u} + \vec{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

If $\vec{u} = \begin{bmatrix} 8 \\ -3 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}$ then $\vec{u} + \vec{v} = \begin{bmatrix} 8 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$.

Geometrically the addition of two vectors follows a parallelogram construction as seen in the following pictures. Strictly speaking vectors are always anchored at the origin in $\mathbb{R}^n$—by definition—as is seen in the picture on the left, but sometimes it is useful to see a vector as just having a direction and a length and ... **floating** freely.

Using “floating” vectors, putting them head to tail, we have another way of looking at vector addition (see the Fig. 1 and 2).

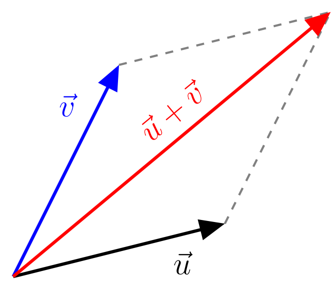


Figure 1: Vector Addition: “parallelogram rule” and “head-to-tail rule”

Notice that the sum of the two vectors forms the main diagonal of the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides.

Example 5 The sum $\vec{u}+\vec{v}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}4\\7\end{bmatrix}$ is shown in Fig.2.

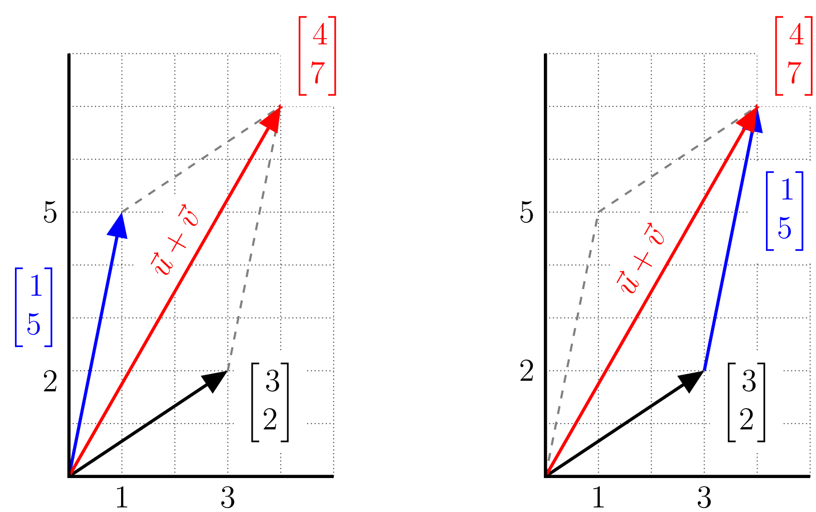


Figure 2: Vector Addition: “parallelogram rule” and “head-to-tail rule”

The vector addition geometrically works like this in general (not just 2D, 3D): the sum $\vec{u}+\vec{v}$ of the two vectors $\vec{u}$ and $\vec{v}$ forms the main diagonal of the parallelogram with $\vec{u}$ and $\vec{v}$ as two of its sides; it is the vector starting at the “origin” and ending at the other corner of the parallelogram. Another example:

Example 6

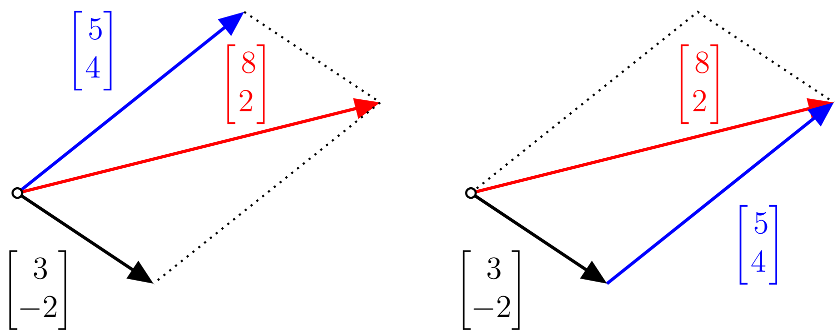


Figure 3: Vector Addition: "parallelogram rule" and "head-to-tail rule"

There is one vector that plays a special role in the world of vectors: the zero vector. Whereas all other vectors have a direction, the zero vector has no direction.

Definition 3

The n-dimensional **zero vector** $\vec{0}$ is defined as: $\vec{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

Example 7 $\begin{bmatrix}0\\0\end{bmatrix} \in \mathbb{R}^2$ and $\begin{bmatrix}0\\0\\0\end{bmatrix} \in \mathbb{R}^3$. Note that: $\boxed{\vec{a} + \vec{0} = \vec{a}}$

There is a special, unique vector associated to each vector: its opposite. In a sense it is its annihilator.

Definition 4

The **opposite** of a vector $\vec{a}$, denoted by $-\vec{a}$, is defined by:

$$-\vec{a} = \begin{bmatrix}-a_1\\-a_2\\\vdots\\-a_n\end{bmatrix} \quad \text{when} \quad \vec{a} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$$

Clearly this means that: $\boxed{\vec{a} + (-\vec{a}) = \vec{0}}$

Scalar Multiplication

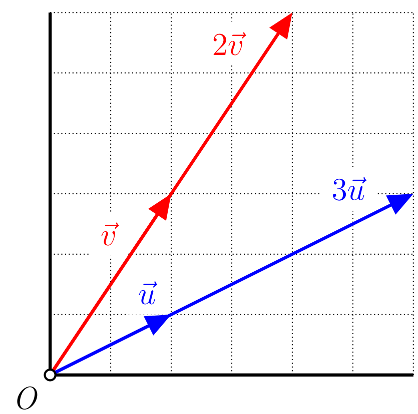
We can multiply (scale) a vector by any real number.

Algebraically we have

Definition 5

$$t \cdot \begin{bmatrix} a_n \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ta_n \\ \vdots \\ ta_n \end{bmatrix}$$

Example 8



The vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are shown.

As are the vectors $3 \cdot \vec{u} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $2 \cdot \vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

Geometrically it is a true scaling: a vector $3\vec{v}$ is the vector in the same direction as $\vec{v}$ but 3 times as long. The vector $-2\vec{v}$ is the vector in the opposite direction of $\vec{v}$ (i.e. in the direction of $-\vec{v}$) but 2 times as long. (see picture)

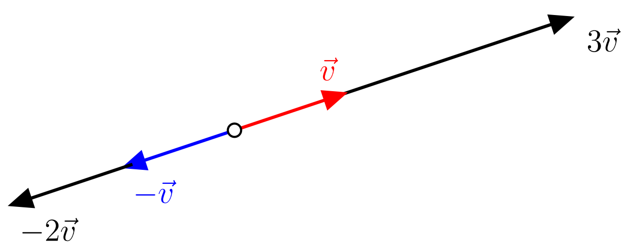


Figure 4: Scalar multiplication of a vector

It should not come as a surprise that $\boxed{(-1)\cdot\vec{a} = -\vec{a}}$ and $\boxed{1 \cdot \vec{a} = \vec{a}}$

Let's list the basic properties of vectors. In fact, any vector space, not just $\mathbb{R}^n$, has these properties.

Theorem 1

For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, and any scalars $s,\ t \in \mathbb{R}$ we have that

(1) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(2) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(3) $\vec{a}+\vec{0} = \vec{a}$

(4) $\vec{a}+(-\vec{a}) = \vec{0}$

(5) $1\cdot\vec{a} = \vec{a}$

(6) $s\cdot(t\cdot\vec{a}) = (s\cdot t)\vec{a}$

(7) $t\cdot(\vec{a}+\vec{b}) = t\cdot\vec{a}+t\cdot\vec{b}$

(8) $(s+t)\cdot\vec{a} = s\cdot\vec{a}+t\cdot\vec{a}$

Furthermore:

(9) $(-1)\cdot\vec{a} = -\vec{a}$

(10) $0\cdot\vec{a} = \vec{0}$

Most of the proofs we will leave to you as useful exercises. We'll do a couple of them in $\mathbb{R}^3$.

Proofs:

(1) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} a+A \\ b+B \\ c+C \end{bmatrix} = \begin{bmatrix} A+a \\ B+b \\ C+c \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(3) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+0 \\ b+0 \\ c+0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(7) $$t\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} A \\ B \\ C \end{bmatrix}\right) = t\begin{bmatrix} a+A \\ b+B \\ c+C \end{bmatrix} = \begin{bmatrix} t(a+A) \\ t(b+B) \\ t(c+C) \end{bmatrix} = \begin{bmatrix} ta+tA \\ tb+tB \\ tc+tC \end{bmatrix} = \begin{bmatrix} ta \\ tb \\ tc \end{bmatrix} + \begin{bmatrix} tA \\ tB \\ tC \end{bmatrix}$$
$$= t\begin{bmatrix} a \\ b \\ c \end{bmatrix} + t\begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

The length of a vector

The **length** of a vector $\vec{v}$ is indicated by $\|\vec{v}\|$.

Theorem 2

If $\vec{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $\|\vec{v}\| = \sqrt{a_1^2 + \cdots + a_n^2}$

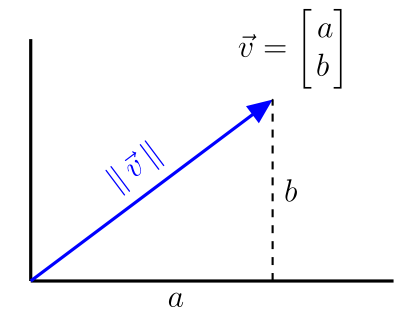


Figure 5: $\|\vec{v}\|$ in $\mathbb{R}^2$

Example 9

2D: if $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ then $\|\vec{v}\| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$.

3D: if $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ then $\|\vec{v}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

The proof of the 2D case follows immediately from the theorem of Pythagoras. (See Fig. 5)

The proof of the 3D case uses **two** applications of the theorem of Pythagoras (see Fig. 6):

Proof:

Let d be the length of the diagonal of the base triangle, then

$$a^2 + b^2 = d^2$$

In the vertical triangle we have

$$\|\vec{v}\|^2 = d^2 + c^2$$

which combined gives us indeed

$$\|\vec{v}\|^2 = a^2 + b^2 + c^2. \qquad \square$$

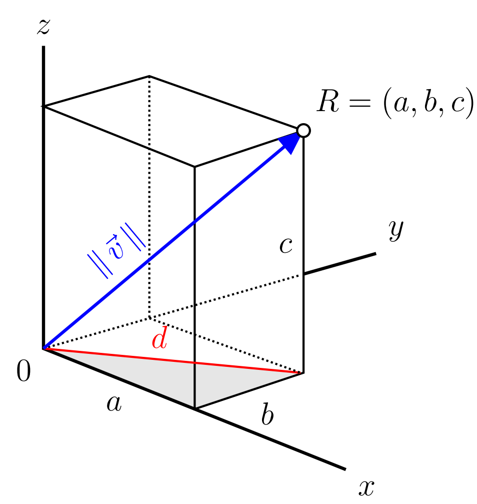


Figure 6: $\|\vec{v}\|$ in $\mathbb{R}^3$

The length of a scaled vector

The length of a scalar multiple of a vector is easily found by

Theorem 3 $\| t\,\vec{v}\, \| = |\, t\,| \cdot \|\, \vec{v}\, \|$

Note that the absolute values are essential when t is a negative scalar, e.g.

$$\| -3\,\vec{v}\, \| = |-3| \cdot \|\, \vec{v}\, \| = 3 \cdot \|\, \vec{v}\, \|$$

It would be a mistake to leave out the absolute values: $\| -3\,\vec{v}\, \| \neq -3 \cdot \|\, \vec{v}\, \|$

Proof: We'll do the 3D case (the 2D is done similarly)

$$\| t\,\vec{v}\, \| = \sqrt{(ta)^2 + (tb)^2 + (tc)^2} = \sqrt{t^2(a^2 + b^2 + c^2)} = |\, t\,|\sqrt{a^2 + b^2 + c^2} = |\, t\,| \cdot \|\, \vec{v}\, \| \qquad \square$$

At times it is useful to work with a vector with length 1. Such a vector is called a **unit vector**.

Given a vector $\vec{v} \neq \vec{0}$ it is always possible to create a vector $\hat{v}$ that points in the same direction as $\vec{v}$ but has unit length, by scaling the vector $\vec{v}$ as follows

Definition 6 $\hat{v} = \dfrac{\vec{v}}{\|\, \vec{v}\, \|}$

That the scaled vector $\hat{v}$ has unit length is clear from

$$\|\, \hat{v}\, \| = \left\| \frac{1}{\|\, \vec{v}\, \|}\, \vec{v} \right\| = \left| \frac{1}{\|\, \vec{v}\, \|} \right| \|\vec{v}\| = \frac{1}{\|\, \vec{v}\, \|}\|\vec{v}\| = 1$$

Example 10 If $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ then $\hat{v} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

Example 11 Find a unit vector perpendicular to $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$: $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is perpendicular to $\vec{v}$ with $\|\vec{w}\| = 5$, so take $\hat{w} = \dfrac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$.

The Difference of two Vectors

We first define the difference of two vectors **algebraically**

Definition 7

The **difference** of two vectors $\vec{u}-\vec{v}$ is defined by

$$\vec{u}-\vec{v} := \vec{u}+(-\vec{v})$$

Example 12

2D $$\begin{bmatrix}7\\3\end{bmatrix}-\begin{bmatrix}4\\5\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}+\begin{bmatrix}-4\\-5\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix}$$

3D $$\begin{bmatrix}3\\2\\4\end{bmatrix}-\begin{bmatrix}1\\5\\-2\end{bmatrix}=\begin{bmatrix}3\\2\\4\end{bmatrix}+\begin{bmatrix}-1\\-5\\2\end{bmatrix}=\begin{bmatrix}2\\-3\\6\end{bmatrix}$$

The difference of two vectors $\vec{u}-\vec{v}$ also has a nice **geometric** interpretation. In particular when we consider "floating" vectors. Figure 7 on the left illustrates $\vec{u}-\vec{v}$ as the sum $\vec{u}+(-\vec{v})$, whereas the picture on the right shows the same vector, but now sitting between the end points of the two vectors $\vec{u}$ and $\vec{v}$ (notice its direction: from $\vec{v}$ to $\vec{u}$!)

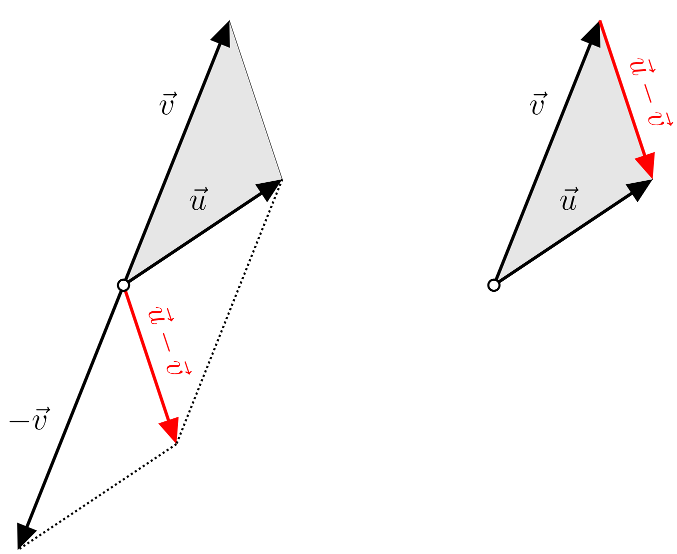


Figure 7: The difference of two vectors