14. Q-topia

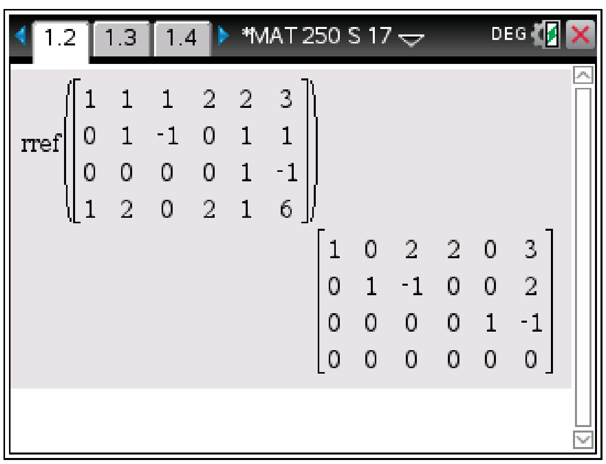
Let $M = \text{rref}(A)$ then there exists an **invertible** matrix Q such that $QA = M = \text{rref}(A)$.

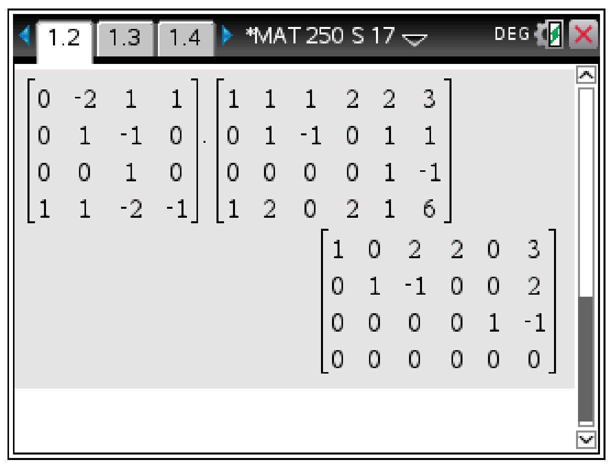
For example:

$$\text{rref}\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 0 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 0 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$





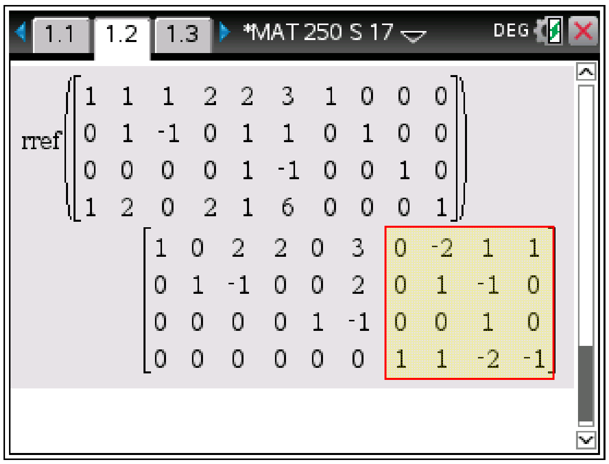
What Q actually is, is usually not important. Just the fact that such a Q exists allows us to understand the structure of $M = \text{rref}(A)$ and how it is related to A.

But there are situations where it is useful to know Q. Fortunately, it is not too hard to find Q. Simply augment the identity matrix and row reduce:

$$\text{rref}\,[\,A \,|\, I\,] = [\,\text{rref}(A) \,|\, Q\,]$$

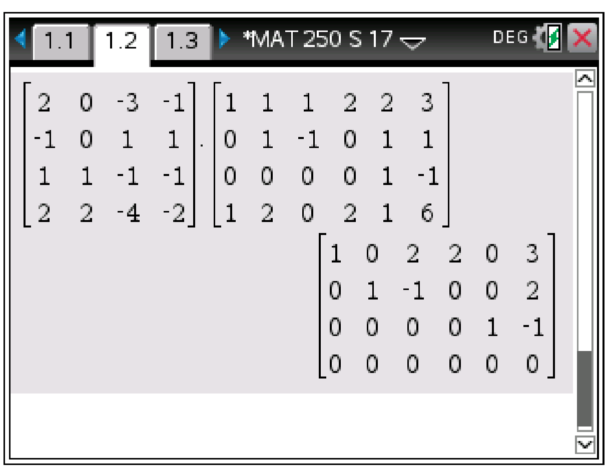
For example, to get the Q we saw above:

$$\operatorname{rref}\left[\begin{array}{cccccc|cccc} 1 & 1 & 1 & 2 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 1 & 6 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccccc|cccc} 1 & 0 & 2 & 2 & 0 & 3 & 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$$



Is Q **unique**? **No**. For example, here is another Q that does the same rref as before:

$$\begin{bmatrix} 2 & 0 & -3 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & 2 & -4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 0 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Of course when we compute $\text{rref}\left[\,A\,|\,I\,\right] = \left[\,\text{rref}(A)\,|\,Q\,\right]$ using our calculator it will always produce just one Q. Since the calculator follows a prescribed algorithm, it will always produce exactly the same Q. [Unless there are randomized steps in the algorithm.] But that doesn't mean there is only one Q such that $Q \cdot A = \text{rref}(A)$. As we have seen there may be others, in fact there may be infinitely many. We could ask:

Are all these Qs invertible? Again the answer is: **No**. Here is an example of a non-invertible Q that does the same row reduction:

$$\begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 0 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly this Q is not invertible (with all zeros in its last row its determinant is zero).

The important thing though, is not whether there exists only one or many, or whether there exist Qs that are not invertible, but rather that there exists **at least** one that **is** invertible:

> There exists a Q , that is invertible, such that $Q\,A = \text{rref}(A)$

That Q allows us to understand the relations between the columns of A and $\text{rref}(A)$, independent vectors and dependencies, spanning sets and linearly independent set. All this can be deduced from row reduction … the soul of linear algebra.

We'll use Q to demonstrate the following fact

> The relationships between the column vectors of A and the relationships between the column vectors of $\text{rref}(A)$ are the same.

This when coupled with

> It is easy to read off relationships between the column vectors of $\text{rref}(A)$

this yields a treasure trove of information.

Relationship between columns

Let $QA = M = \operatorname{rref}(A)$, where Q is an invertible matrix. Let

$$Q\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_{k-1} & \vec{a}_k \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ \vec{m}_1 & \vec{m}_2 & \vec{m}_3 & \cdots & \vec{m}_{k-1} & \vec{m}_k \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \end{bmatrix}$$

Then clearly: $Q\vec{a}_1 = \vec{m}_1, \quad Q\vec{a}_2 = \vec{m}_2, \quad \cdots \quad, Q\vec{a}_k = \vec{m}_k.$

Suppose there exists a relationship between the columns of A: e.g.

$$\boxed{3\vec{a}_1 + 5\vec{a}_3 + 2\vec{a}_7 + 8\vec{a}_{10} = \vec{a}_{11}}$$

then

$$Q\cdot\left(3\vec{a}_1 + 5\vec{a}_3 + 2\vec{a}_7 + 8\vec{a}_{10}\right) = Q\cdot\vec{a}_{11}$$

so that

$$3Q\cdot\vec{a}_1 + 5Q\cdot\vec{a}_3 + 2Q\cdot\vec{a}_7 + 8Q\cdot\vec{a}_{10} = Q\cdot\vec{a}_{11}$$

which means that

$$\boxed{3\vec{m}_1 + 5\vec{m}_3 + 2\vec{m}_7 + 8\vec{m}_{10} = \vec{m}_{11}}$$

Hence there is exactly the same relation between the corresponding columns of $M = \operatorname{rref}(A)$.

And, since Q is invertible, it goes both ways. Any relationship between the columns of $M = \operatorname{rref}(A)$ mirrors the same relationship between the corresponding columns of A.
To show this you just reverse the above steps using Q^{-1}.

Example:

$$\operatorname{rref}\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 0 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3\cdot\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + 2\cdot\begin{bmatrix}1\\1\\0\\2\end{bmatrix} + (-1)\cdot\begin{bmatrix}2\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\1\\-1\\6\end{bmatrix} \quad\Leftrightarrow\quad 3\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 2\cdot\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + (-1)\cdot\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\2\\-1\\0\end{bmatrix}$$

Relations between the column vectors of $\mathrm{rref}(A)$

It is easier to “see” relationships in a reduced row echelon form: $\mathrm{rref}(A)$.

Example over $\mathbb{F}_7$:

$$\mathrm{rref}\begin{bmatrix} \boxed{1} & \boxed{2} & 0 & \boxed{3} & 4 & \boxed{0} & 4 & 6 \\ \boxed{1} & \boxed{2} & 0 & \boxed{1} & 1 & \boxed{2} & 3 & 3 \\ \boxed{4} & \boxed{5} & 5 & \boxed{1} & 5 & \boxed{1} & 2 & 3 \\ \boxed{6} & \boxed{0} & 6 & \boxed{1} & 3 & \boxed{0} & 5 & 4 \\ \boxed{3} & \boxed{1} & 6 & \boxed{2} & 6 & \boxed{0} & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

First lets look at the pivots.

Pivots and independent columns

It is clear that the columns of $\mathrm{rref}(A)$ with pivots are **linearly independent**:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

If these are linearly **in**dependent, i.e. there are **NO** dependencies amongst them, this means there can be **NO** dependencies amongst the corresponding columns of A either, i.e. these columns of A are also **linearly independent**:

$$\begin{bmatrix} 1 \\ 1 \\ 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Next: The non-pivot columns are dependent on the pivot columns. In fact each non-pivot column depends on the pivot columns before it (to the **left** of it). The non-pivot columns of $\mathrm{rref}(A)$ make it clear how the corresponding columns of A depend on its columns to the left:

Using the above example lets look at some of the relationships between the columns of $\mathrm{rref}(A)$ and A.

$$\operatorname{rref}\begin{bmatrix} 1 & 2 & \boxed{0} & 3 & 4 & 0 & 4 & 6 \\ 1 & 2 & 0 & 1 & 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & 1 & 5 & 1 & 2 & 3 \\ 6 & 0 & 6 & 1 & 3 & 0 & 5 & 4 \\ 3 & 1 & 6 & 2 & 6 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$1\cdot\begin{bmatrix}1\\1\\4\\6\\3\end{bmatrix} + 3\cdot\begin{bmatrix}2\\2\\5\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\5\\6\\6\end{bmatrix} \qquad (\text{ over } \mathbb{F}_7\ )$$

$$\operatorname{rref}\begin{bmatrix} 1 & 2 & 0 & 3 & \boxed{4} & 0 & 4 & 6 \\ 1 & 2 & 0 & 1 & 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & 1 & 5 & 1 & 2 & 3 \\ 6 & 0 & 6 & 1 & 3 & 0 & 5 & 4 \\ 3 & 1 & 6 & 2 & 6 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2\cdot\begin{bmatrix}1\\1\\4\\6\\3\end{bmatrix} + 4\cdot\begin{bmatrix}2\\2\\5\\0\\1\end{bmatrix} + 5\cdot\begin{bmatrix}3\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}4\\1\\5\\3\\6\end{bmatrix} \qquad (\text{ over } \mathbb{F}_7\ )$$

$$\operatorname{rref}\begin{bmatrix} 1 & 2 & 0 & 3 & 4 & 0 & 4 & \boxed{6} \\ 1 & 2 & 0 & 1 & 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & 1 & 5 & 1 & 2 & 3 \\ 6 & 0 & 6 & 1 & 3 & 0 & 5 & 4 \\ 3 & 1 & 6 & 2 & 6 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$5\cdot\begin{bmatrix}1\\1\\4\\6\\3\end{bmatrix} + 1\cdot\begin{bmatrix}2\\2\\5\\0\\1\end{bmatrix} + 2\cdot\begin{bmatrix}3\\1\\1\\1\\2\end{bmatrix} + 4\cdot\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix} = \begin{bmatrix}6\\3\\3\\4\\6\end{bmatrix} \qquad (\text{ over } \mathbb{F}_7\ )$$

But even relationships that are harder to find are still shared by both matrices:

Here is a relationship between columns **3**, **7**, **8** and **5**: $\quad 2\cdot C_3 + 6\cdot C_7 + 2\cdot C_8 = C_5$

$$\begin{array}{r c c}
\text{Columns:} & \begin{array}{cccccccc} & & 3 & & 5 & & 7 & 8 \end{array} & \begin{array}{cccccccc} & & 3 & & 5 & & 7 & 8 \end{array} \\
\text{rref} & \left[\begin{array}{cccccccc} 1 & 2 & 0 & 3 & 4 & 0 & 4 & 6 \\ 1 & 2 & 0 & 1 & 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & 1 & 5 & 1 & 2 & 3 \\ 6 & 0 & 6 & 1 & 3 & 0 & 5 & 4 \\ 3 & 1 & 6 & 2 & 6 & 0 & 0 & 6 \end{array}\right] = & \left[\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{array}$$

$$(\text{ over } \mathbb{F}_7) \quad 2\cdot\begin{bmatrix}0\\0\\5\\6\\6\end{bmatrix} + 6\cdot\begin{bmatrix}4\\3\\2\\5\\0\end{bmatrix} + 6\cdot\begin{bmatrix}6\\3\\3\\4\\6\end{bmatrix} = \begin{bmatrix}4\\1\\5\\3\\6\end{bmatrix} \quad \text{and} \quad 2\cdot\begin{bmatrix}1\\3\\0\\0\\0\end{bmatrix} + 6\cdot\begin{bmatrix}2\\1\\0\\3\\0\end{bmatrix} + 6\cdot\begin{bmatrix}5\\1\\2\\4\\0\end{bmatrix} = \begin{bmatrix}2\\4\\5\\0\\0\end{bmatrix}$$

And a relationship between columns **3**, **6**, **7** and **1**: $\quad 4\cdot C_3 + 1\cdot C_6 + 2\cdot C_7 = C_1$

$$\begin{array}{r c c}
\text{Columns:} & \begin{array}{cccccccc} 1 & & 3 & & & 6 & 7 & \end{array} & \begin{array}{cccccccc} 1 & & 3 & & & 6 & 7 & \end{array} \\
\text{rref} & \left[\begin{array}{cccccccc} 1 & 2 & 0 & 3 & 4 & 0 & 4 & 6 \\ 1 & 2 & 0 & 1 & 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & 1 & 5 & 1 & 2 & 3 \\ 6 & 0 & 6 & 1 & 3 & 0 & 5 & 4 \\ 3 & 1 & 6 & 2 & 6 & 0 & 0 & 6 \end{array}\right] = & \left[\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 3 & 0 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{array}$$

$$(\text{ over } \mathbb{F}_7) \quad \begin{bmatrix}1\\1\\4\\6\\3\end{bmatrix} = 4\cdot\begin{bmatrix}0\\0\\5\\6\\6\end{bmatrix} + 1\cdot\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix} + 2\cdot\begin{bmatrix}4\\3\\2\\5\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = 4\cdot\begin{bmatrix}1\\3\\0\\0\\0\end{bmatrix} + 1\cdot\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} + 2\cdot\begin{bmatrix}2\\1\\0\\3\\0\end{bmatrix}$$

An example over $\mathbb{F}_4$:

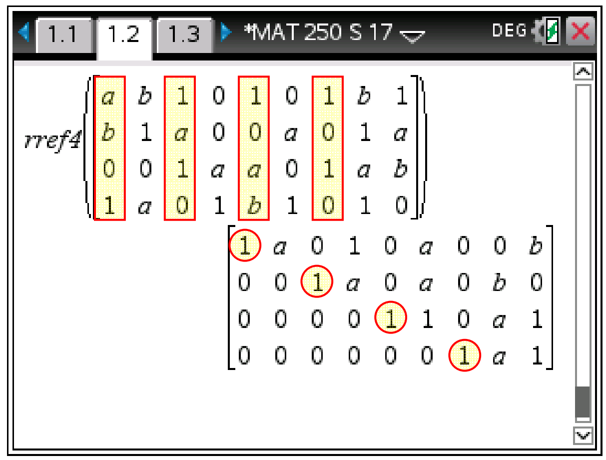
$$\text{Let}\quad S = \left\{ \begin{bmatrix} a \\ b \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ 0 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ a \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ a \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ a \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ a \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ b \\ 0 \end{bmatrix} \right\}$$

(a) Is S a linearly independent set?

(b) Does S span all of $\mathbb{F}_4^4$, i.e. $\operatorname{span}(S) = \mathbb{F}_4^4$?

(c) Find a basis of $\mathbb{F}_4^4$, using vectors from S.

Solutions:



Row reduction answers all these questions:

(a) No. We can read off several dependencies:

$$\text{e.g.}\quad a\cdot\begin{bmatrix} a \\ b \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ 1 \\ 0 \\ a \end{bmatrix}$$

$$\text{and}\quad 1\cdot\begin{bmatrix} a \\ b \\ 0 \\ 1 \end{bmatrix} + a\cdot\begin{bmatrix} 1 \\ a \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a \\ 1 \end{bmatrix}$$

(b) Yes. There are four pivots, hence the corresponding columns

$$\left\{ \begin{bmatrix} a \\ b \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ a \\ b \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{are linearly independent too.}$$

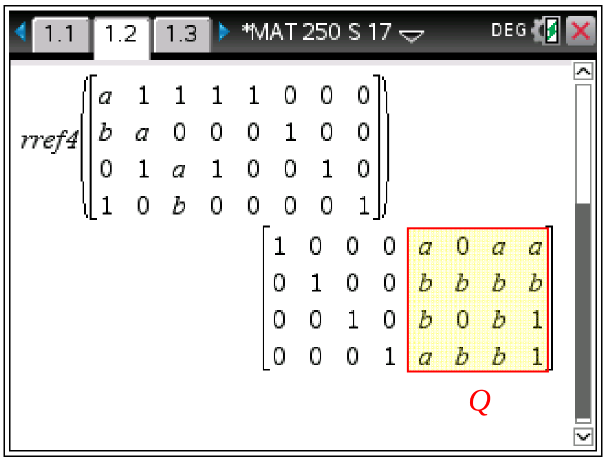
We could stop here since this is a 4 dimensional vector space. These 4 linearly independent vectors form a basis. But let's prove that it is a spanning set 'the long way': i.e. solve

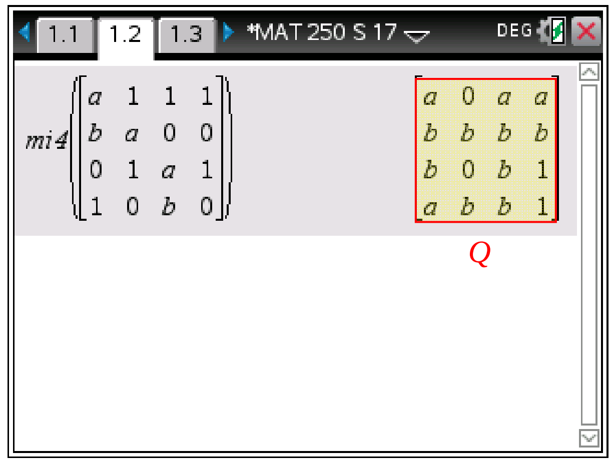
$$k\begin{bmatrix} a \\ b \\ 0 \\ 1 \end{bmatrix} + l\begin{bmatrix} 1 \\ a \\ 1 \\ 0 \end{bmatrix} + m\begin{bmatrix} 1 \\ 0 \\ a \\ b \end{bmatrix} + n\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

For this we need to use row reduction: $\operatorname{rref4}\begin{bmatrix} a & 1 & 1 & 1 & w \\ b & a & 0 & 0 & x \\ 0 & 1 & a & 1 & y \\ 1 & 0 & b & 0 & z \end{bmatrix}$

Currently the TI-Nspire programs don't handle the variables w, x, y and z. But we can use Q to do the work for us.

First we find Q: (two ways shown)

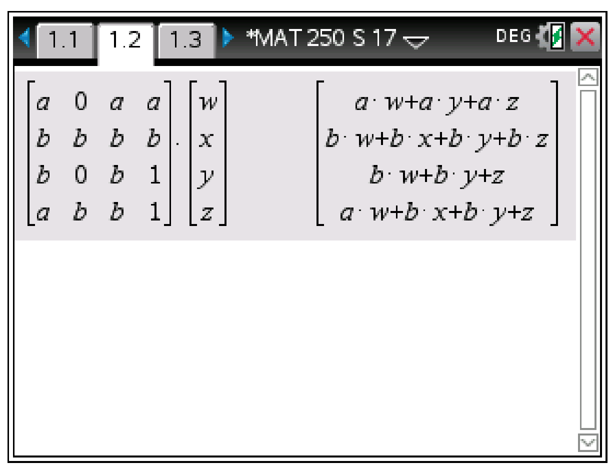




Hence

$$\operatorname{rref4}\begin{bmatrix} a & 1 & 1 & 1 & w \\ b & a & 0 & 0 & x \\ 0 & 1 & a & 1 & y \\ 1 & 0 & b & 0 & z \end{bmatrix} = Q\begin{bmatrix} a & 1 & 1 & 1 & w \\ b & a & 0 & 0 & x \\ 0 & 1 & a & 1 & y \\ 1 & 0 & b & 0 & z \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & Q\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & aw+ay+az \\ 0 & 1 & 0 & 0 & bw+bx+by+bz \\ 0 & 0 & 1 & 0 & bw+by+z \\ 0 & 0 & 0 & 1 & aw+bx+by+z \end{array}\right]$$

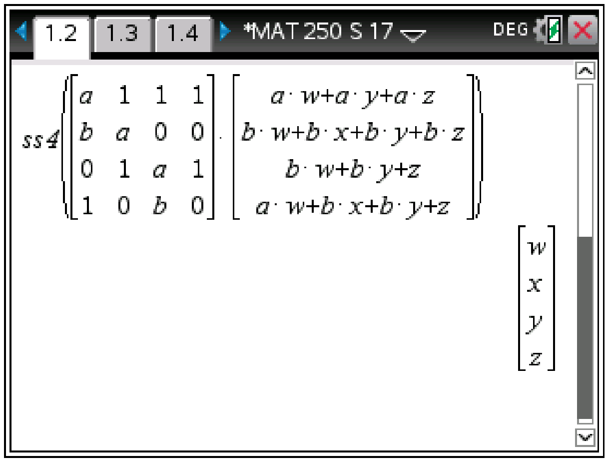


Hence $\begin{cases} k = aw+ay+az \\ l = bw+bx+by+bz \\ m = bw+by+z \\ n = aw+bx+by+z \end{cases}$

which means

$$(aw+ay+az)\begin{bmatrix}a\\b\\0\\1\end{bmatrix}+(bw+bx+by+bz)\begin{bmatrix}1\\a\\1\\0\end{bmatrix}+(bw+by+z)\begin{bmatrix}1\\0\\a\\b\end{bmatrix}+(aw+bx+by+z)\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}w\\x\\y\\z\end{bmatrix}$$

This is easy to check, in fact a little adjustment to the **s4** program— **ss4 –** does just that:



Now that we showed how to produce an arbitrary vector $\begin{bmatrix}w\\x\\y\\z\end{bmatrix}$, we can conclude that the given set spans all of $\mathbb{F}_4^4$.

Let's show this in a different way, which is more constructive, and instructive:

It is easy to show that $\left\{\begin{bmatrix}a\\b\\0\\1\end{bmatrix},\begin{bmatrix}1\\a\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\a\\b\end{bmatrix},\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\}$ can produce $\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$:

We can do this in **one** rref:

$$\text{rref4}\begin{bmatrix}a&1&1&1&1&0&0&0\\b&a&0&0&0&1&0&0\\0&1&a&1&0&0&1&0\\1&0&b&0&0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0&a&0&a&a\\0&1&0&0&b&b&b&b\\0&0&1&0&b&0&b&1\\0&0&0&1&a&b&b&1\end{bmatrix}$$

So that

1. $$a\begin{bmatrix}a\\b\\0\\1\end{bmatrix}+b\begin{bmatrix}1\\a\\1\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\a\\b\end{bmatrix}+a\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

2. $$0\begin{bmatrix}a\\b\\0\\1\end{bmatrix}+b\begin{bmatrix}1\\a\\1\\0\end{bmatrix}+0\begin{bmatrix}1\\0\\a\\b\end{bmatrix}+b\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

2. $$a\begin{bmatrix}a\\b\\0\\1\end{bmatrix}+b\begin{bmatrix}1\\a\\1\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\a\\b\end{bmatrix}+b\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

4. $$a\begin{bmatrix}a\\b\\0\\1\end{bmatrix}+b\begin{bmatrix}1\\a\\1\\0\end{bmatrix}+1\begin{bmatrix}1\\0\\a\\b\end{bmatrix}+1\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Hence, since we can produce the standard basis vectors of $\mathbb{F}_4^4$, we can get all vectors of $\mathbb{F}_4^4$.

Note that this actually amounts to showing that $\begin{bmatrix}a&1&1&1\\b&a&0&0\\0&1&a&1\\1&0&b&0\end{bmatrix}$ is **invertible**!

(c) We did all the heavy lifting for this in part **(b)** and found that

$$\left\{\begin{bmatrix}a\\b\\0\\1\end{bmatrix},\begin{bmatrix}1\\a\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\a\\b\end{bmatrix},\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\}$$

is a basis for $\mathbb{F}_4^4$. [Not the only basis though. We can find many others even using S ! (by just reshuffling the columns)]

One final Note: Suppose we want to determine what kind of subspace W is of $\mathbb{F}_7^4$, where W is given by:

$$W = \operatorname{Span}\left(\left\{\begin{bmatrix}3\\1\\2\\4\end{bmatrix},\begin{bmatrix}2\\3\\1\\5\end{bmatrix},\begin{bmatrix}5\\6\\4\\4\end{bmatrix},\begin{bmatrix}3\\4\\4\\0\end{bmatrix},\begin{bmatrix}6\\5\\0\\4\end{bmatrix},\begin{bmatrix}1\\0\\2\\1\end{bmatrix},\begin{bmatrix}2\\4\\1\\6\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}\right)$$

We could start to **rref7** the vectors as they are presented, and we would get:

$$\text{rref7}\begin{bmatrix} 3 & 2 & 5 & 3 & 6 & 1 & 2 & 0 & 0 \\ 1 & 3 & 6 & 4 & 5 & 0 & 4 & 1 & 0 \\ 2 & 1 & 4 & 4 & 0 & 2 & 1 & 0 & 1 \\ 4 & 5 & 4 & 0 & 4 & 1 & 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 4 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 4 & 1 & 5 & 2 & 1 & 4 \\ 0 & 0 & 1 & 5 & 5 & 1 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that we find that this is a 3-dimensional subspace with basis $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 4 \\ 4 \end{bmatrix} \right\}$.

But we can do better. We could have started by rearranging the vectors and put some “simpler” vectors at the beginning:

$$\text{rref7}\begin{bmatrix} 0 & 0 & 1 & 3 & 2 & 5 & 3 & 6 & 2 \\ 1 & 0 & 0 & 1 & 3 & 6 & 4 & 5 & 4 \\ 0 & 1 & 2 & 2 & 1 & 4 & 4 & 0 & 1 \\ 1 & 0 & 1 & 4 & 5 & 4 & 0 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 6 & 4 & 5 & 4 \\ 0 & 1 & 0 & 3 & 4 & 1 & 5 & 2 & 4 \\ 0 & 0 & 1 & 3 & 2 & 5 & 3 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We still find that this is a 3-dimensional subspace, but we now have a simpler basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

We can even simplify this basis further:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

So that every vector in this subspace looks like: $\begin{bmatrix} a \\ b \\ c \\ a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(Check the original vectors!)