Examples of Vector Spaces

In this section we'll give examples of Vector Spaces

$$\boxed{\langle \mathbb{V}, \mathbb{F}, +, \cdot \rangle}$$

We start again with the three most important examples, which we will use a lot in this course:

(1) $\mathbb{F}^n$ ***n*-tuples over the field** $\mathbb{F}$

(2) $M_{n\times m}(\mathbb{F})$ $\boldsymbol{n \times m}$ **matrices over the field** $\mathbb{F}$

(3) $P_n(\mathbb{F})$ **polynomials of degree** $\boldsymbol{n}$ **or less over the field** $\mathbb{F}$

Eventhough we discussed them in the previous section, for completeness, we'll give their definitions again:

(1)
$$\mathbb{F}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_1,\, x_2, \cdots,\, x_n \in \mathbb{F} \right\}$$

with vector addition and scalar multiplication defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad s \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s \cdot x_1 \\ s \cdot x_2 \\ \vdots \\ s \cdot x_n \end{bmatrix}$$

(2)

$$M_{n\times m}(\mathbb{F}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \;\middle|\; a_{ij} \in \mathbb{F} \right\}$$

with vector addition and scalar multiplication defined by:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm1}+b_{nm} \end{bmatrix}$$

and

$$s \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} s\cdot a_{11} & s\cdot a_{12} & \cdots & s\cdot a_{1m} \\ s\cdot a_{21} & s\cdot a_{22} & \cdots & s\cdot a_{2m} \\ \vdots & \vdots & & \vdots \\ s\cdot a_{n1} & s\cdot a_{n2} & \cdots & s\cdot a_{nm} \end{bmatrix}$$

(3)

$$P_n(\mathbb{F}) = \left\{ a_n t^n + \cdots + a_1 t + a_0 \;\middle|\; a_i \in \mathbb{F} \right\}$$

with vector addition and scalar multiplication defined by:

$$(a_n t^n + \cdots + a_1 t + a_0) + (b_n t^n + \cdots + b_1 t + b_0) = (a_n + b_n)t^n + \cdots + (a_1 + b_1)t + (a_0 + b_0)$$

and

$$s \cdot (a_n t^n + \cdots + a_1 t + a_0) = (s \cdot a_n)t^n + \cdots + (s \cdot a_1)t + (s \cdot a_0)$$

In this course we will primarily us the fields: $\mathbb{R}$ $\mathbb{C}$ $\mathbb{F}_2$ $\mathbb{F}_4$ $\mathbb{F}_7$

When the underlying field is $\mathbb{R}$ we call the vector space a **real** vector space.

More Examples of Vector Spaces

(4) $\boxed{\mathbb{F}_n}$ **row vectors (row n-tuples) over the field $\mathbb{F}$**

$$\mathbb{F}_n = \left\{ \, [\, x_1 \; x_2 \; \cdots \; x_n \,] \;\middle|\; x_i \in \mathbb{F} \, \right\}$$

With the vector addition:

$$[\, x_1 \quad x_2 \quad \cdots \quad x_n \,] + [\, y_1 \quad y_2 \quad \cdots \quad y_n \,] = [\, x_1 + y_1 \quad x_2 + y_2 \quad \cdots \quad x_n + y_n \,]$$

and the scalar multiplication:

$$s \cdot [\, x_1 \quad x_2 \quad \cdots \quad x_n \,] = [\, s \cdot x_1 \quad s \cdot x_2 \quad \cdots \quad s \cdot x_n \,]$$

(5) $\boxed{P(\mathbb{F})}$ **polynomials (of any degree) over the field $\mathbb{F}$**

With the usual (term-wise) addition of polynomials: $p(t) + q(t)$

and the usual scalar multiplication: $s \cdot p(t)$

(6) $\boxed{\mathcal{F}(A, \mathbb{R})}$ **functions with domain $A \subseteq \mathbb{R}$ and codomain $\mathbb{R}$, over the field $\mathbb{R}$** (where A is a compact subset of $\mathbb{R}$)

$$\mathcal{F}(A, \mathbb{R}) = \{ \; f : \; A \to \mathbb{R} \; \}$$

With the usual addition of functions: $f(x) + g(x)$

and the usual scalar multiplication: $s \cdot f(x)$ (with $s \in \mathbb{R}$)

(7) $\boxed{\mathcal{C}(A, \mathbb{R})}$ **continuous functions with domain $A \subseteq \mathbb{R}$ and codomain $\mathbb{R}$, over the field $\mathbb{R}$**
(where A is a compact subset of $\mathbb{R}$)

$$\mathcal{C}(A, \mathbb{R}) = \left\{ \; f : \; A \to \mathbb{R} \;\middle|\; f \text{ continuous} \; \right\}$$

With the usual addition of functions: $f(x) + g(x)$

and the usual scalar multiplication: $s \cdot f(x)$ (with $s \in \mathbb{R}$)

(8) $\boxed{\mathcal{D}(A,\mathbb{R})}$ **differentiable functions with domain $A \subseteq \mathbb{R}$ and codomain $\mathbb{R}$, over the field $\mathbb{R}$**

(where A is a compact subset of $\mathbb{R}$)

$$\mathcal{D}(A,\mathbb{R}) = \left\{ \, f : \; A \to \mathbb{R} \;\middle|\; f \text{ differentiable} \, \right\}$$

With the usual addition of functions: $f(x) + g(x)$

and the usual scalar multiplication: $s \cdot f(x)$ (with $s \in \mathbb{R}$)

(9) $\boxed{\mathcal{I}(A,\mathbb{R})}$ **integrable functions with domain $A \subseteq \mathbb{R}$ and codomain $\mathbb{R}$, over the field $\mathbb{R}$**

(where A is a compact subset of $\mathbb{R}$)

$$\mathcal{I}(A,\mathbb{R}) = \left\{ \, f : \; A \to \mathbb{R} \;\middle|\; f \text{ integrable} \, \right\}$$

With the usual addition of functions: $f(x) + g(x)$

and the usual scalar multiplication: $s \cdot f(x)$ (with $s \in \mathbb{R}$)

(10) $\boxed{\{\vec{0}\}}$ **the trivial or zero vector space (over any field)**

[This is the smallest possible vector space]

With the addition: $\vec{0} + \vec{0} = \vec{0}$

and the scalar multiplication: $s \cdot \vec{0} = \vec{0}$ (with $s \in \mathbb{F}$)

(11) $\boxed{U_{n \times n}(\mathbb{F})}$ **upper triangular matrices over the filed $\mathbb{F}$**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \mathbf{0} & a_{22} & a_{23} & \dots & a_{2n} \\ \mathbf{0} & \mathbf{0} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & a_{nn} \end{bmatrix}$$

With the usual addition of matrices

and the usual scalar multiplication of matrices

(12) $\boxed{L_{n\times n}(\mathbb{F})}$ **lower triangular matrices over the field** $\mathbb{F}$

$$\begin{bmatrix} a_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ a_{21} & a_{22} & \mathbf{0} & \dots & \mathbf{0} \\ a_{31} & a_{32} & a_{33} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

With the usual addition of matrices
and the usual scalar multiplication of matrices

(13) $\boxed{\mathcal{S}(\mathbb{F})}$ **infinite sequences over the field** $\mathbb{F}$

$$\mathcal{S}(\mathbb{F}) = \left\{ \, (a_1, \ a_2, \ a_3, \ \cdots) \, \middle| \, a_i \in \mathbb{F} \, \right\}$$

With the usual (term by term) addition of sequences:

$$(a_1, \ a_2, \ a_3, \ \cdots) + (b_1, \ b_2, \ b_3, \ \cdots) = (a_1+b_1, \ a_2+b_2, \ a_3+b_3, \ \cdots)$$

and the usual scalar multiplication of sequences:

$$s \cdot (a_1, \ a_2, \ a_3, \ \cdots) = (s \cdot a_1, \ s \cdot a_2, \ s \cdot a_3, \ \cdots)$$