Bases

Definition : Let β be a set of vectors of a vector space V. We say β is a **basis** of V if

(a) span$(\beta) = V$, and

(b) β is linearly independent, and

(c) the elements of β have a fixed order

hence: a **basis** is an **ordered, linearly independent, spanning set**.

Many of the familiar vector spaces have a 'natural' basis called the **standard** basis: e.g.

- $\mathbb{F}^n$: $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$
- $P_n(\mathbb{F})$: $S = \left\{ t^n, t^{n-1}, \cdots, t^2, t, 1 \right\}$
- $M_{2\times 2}(\mathbb{F})$: $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ etc.

Theorem: Let $\beta = \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3, \cdots, \vec{b}_n \right\}$ be a basis of the vector space V, then every vector $\vec{v}$ in V can be written **uniquely** as a linear combination of $\vec{b}_1, \vec{b}_2, \vec{b}_3, \cdots, \vec{b}_n$.

Proof: Since β is a spanning set, every vector *can* be written as a linear combination

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3 + \cdots + a_n \vec{b}_n .$$

Suppose that also $\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 + \cdots + c_n \vec{b}_n$, then by subtracting we find that

$$\vec{0} = (a_1 - c_1)\vec{b}_1 + (a_2 - c_2)\vec{b}_2 + (a_3 - c_3)\vec{b}_3 + \cdots + (a_n - c_n)\vec{b}_n$$

But since $\beta = \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3, \cdots, \vec{b}_n \right\}$ is linearly independent, it follows that

$$a_1 - c_1 = a_2 - c_2 = a_3 - c_3 = \cdots = a_n - c_n = 0$$

so that

$$a_1 = c_1 , \quad a_2 = c_2 , \quad a_3 = c_3 , \quad \cdots , \quad a_n = c_n$$

which proves uniqueness. $\square$

Coordinates

Since we can write any vector $\vec{v}$ in a vector space V, **uniquely** as a linear combination of a given basis $\beta = \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3, \cdots, \vec{b}_n \right\}$:

$$\vec{v} = v_1 \vec{b}_1 + v_2 \vec{b}_2 + v_3 \vec{b}_3 + \cdots + v_n \vec{b}_n$$

we will report these unique coefficients v_i, which we call the **coordinates** of $\vec{v}$ with respect to the basis β, as the coordinate vector:

$$[\vec{v}]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Example 1: $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$ (check!) Let $\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ then since

$$\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ we find that } \boxed{[\vec{v}]_\beta = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}}$$

Of course: $\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ with respect to the standard basis, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is

$$\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = 6 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 8 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ so that } \boxed{[\vec{v}]_S = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}}$$

And $\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ with respect to the basis, $\alpha = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \right\}$ is

$$\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = -11 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 57 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 8 \cdot \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \text{ so that } \boxed{[\vec{v}]_\alpha = \begin{bmatrix} -11 \\ 57 \\ -8 \end{bmatrix}}$$

Example 2: $\vec{v} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ can be written with respect to the standard basis

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ as } \boxed{[\vec{v}]_S = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}} \text{ since}$$

$$\vec{v} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 3\cdot\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

And $\vec{v} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ with respect to the basis $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 4 & -9 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \right\}$ is

$$\boxed{[\vec{v}]_\beta = \begin{bmatrix} -5 \\ -11 \\ -6 \\ 18 \end{bmatrix}} \text{ since } \vec{v} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = -5\cdot\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} - 11\cdot\begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix} - 6\cdot\begin{bmatrix} 2 & 0 \\ 4 & -9 \end{bmatrix} + 18\cdot\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Example 3: $\vec{v} = t^2 + bt + a \in P_2(\mathbb{F}_4)$ can be written with respect to the standard basis

$$S = \{ t^2, t, 1 \} \text{ as } \boxed{[\vec{v}]_S = \begin{bmatrix} 1 \\ b \\ a \end{bmatrix}} \text{ since } \vec{v} = t^2 + bt + a = 1\cdot t^2 + b\cdot t + a\cdot 1.$$

But $\vec{v} = t^2 + bt + a$ with respect to the basis $\beta = \{ bt^2 + t + a,\ t + b,\ at^2 + 1 \}$ is

$$\boxed{[\vec{v}]_\beta = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}} \text{ since } \vec{v} = t^2 + bt + a = 1\cdot(bt^2 + t + a) + a\cdot(t + b) + 1\cdot(at^2 + 1).$$

Next we will use coordinates and row reduction to examine if a given set is a basis or not, i.e. if the vectors are linear independent and spanning:

Example 4: $\{t^2+t,\ t^2-1,\ t^2+t+1\}$ in $P_2(\mathbb{R})$ is a basis.

We can check independence **and** spanning with one row reduction (rref).

(1) First write every vector with respect to the standard basis:

$$[t^2+t]_S = \begin{bmatrix}1\\1\\0\end{bmatrix},\ [t^2-1]_S = \begin{bmatrix}1\\0\\-1\end{bmatrix} \text{ and } [t^2+t+1]_S = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

(2) Then row reduce: $\operatorname{rref}\begin{bmatrix}1 & 1 & 1 & x\\1 & 0 & 1 & y\\0 & -1 & 1 & z\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & -x+2y-z\\0 & 1 & 0 & x-y\\0 & 0 & 1 & x-y+z\end{bmatrix}$ which tells us that

- The vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ are linearly independent (see the pivots), and that
- Every vector $\begin{bmatrix}x\\y\\z\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$:

  In fact $\begin{bmatrix}x\\y\\z\end{bmatrix} = (-x+2y-z)\begin{bmatrix}1\\1\\0\end{bmatrix} + (x-y)\begin{bmatrix}1\\0\\-1\end{bmatrix} + (x-y+z)\begin{bmatrix}1\\1\\1\end{bmatrix}$

(3) Consequently

- The vectors $t^2+t,\ \ t^2-1,\ \ t^2+t+1$ are linearly independent, and
- Every vector $xt^2+yt+z \in \operatorname{span}\{t^2+t,\ t^2-1,\ t^2+t+1\}$; explicitly

  $xt^2+yt+z = (-x+2y-z)(t^2+t) + (x-y)(t^2-1) + (x-y+z)(t^2+t+1)$

  [Notice *uniqueness*: this is the only way to write xt^2+yt+z as a linear combination of the basis vectors $t^2+t,\ \ t^2-1,\ \ t^2+t+1$]

Example 5:

$\left\{\begin{bmatrix}1&0\\5&1\end{bmatrix},\begin{bmatrix}1&1\\1&6\end{bmatrix},\begin{bmatrix}7&1\\0&1\end{bmatrix}\right\}$ is **not** a basis of $M_{2\times2}(\mathbb{R})$. One could simply point out that these three vectors in a four dimensional vector space cannot possibly be spanning, and therefore could not form a basis. Or we can show that they are linearly independent, **but** are **not** spanning, using just one row reduction (rref):

(1) First write every vector (and one more) with respect to the standard basis:

$$\left[\begin{bmatrix}1&0\\5&1\end{bmatrix}\right]_S=\begin{bmatrix}1\\0\\5\\1\end{bmatrix},\quad \left[\begin{bmatrix}1&1\\1&6\end{bmatrix}\right]_S=\begin{bmatrix}1\\1\\1\\6\end{bmatrix},\quad \left[\begin{bmatrix}7&1\\0&1\end{bmatrix}\right]_S=\begin{bmatrix}7\\1\\0\\1\end{bmatrix}\quad\text{and}\quad \left[\begin{bmatrix}1&2\\3&4\end{bmatrix}\right]_S=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

(2) Then row reduce: $\operatorname{rref}\begin{bmatrix}1&1&7&1\\0&1&1&2\\5&1&0&3\\1&6&1&4\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ which tells us that

- All these vectors are linearly independent (see the pivots)
- So in particular that $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\notin\operatorname{span}\left\{\begin{bmatrix}1\\0\\5\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\6\end{bmatrix},\begin{bmatrix}7\\1\\0\\1\end{bmatrix}\right\}$ which means that

  $\begin{bmatrix}1&2\\3&4\end{bmatrix}\notin\operatorname{span}\left\{\begin{bmatrix}1&0\\5&1\end{bmatrix},\begin{bmatrix}1&1\\1&6\end{bmatrix},\begin{bmatrix}7&1\\0&1\end{bmatrix}\right\}$.

Hence $\left\{\begin{bmatrix}1&0\\5&1\end{bmatrix},\begin{bmatrix}1&1\\1&6\end{bmatrix},\begin{bmatrix}7&1\\0&1\end{bmatrix}\right\}$ is **not** a basis of $M_{2\times2}(\mathbb{R})$. They do, however, span a three dimensional **sub**space of $M_{2\times2}(\mathbb{R})$.

Example 6:

Here is another way on looking at "spanning" by a given set. For example let

$$\alpha=\left\{at^2+bt+1,\ t^2+t+a,\ bt+1\right\}\quad\text{and}\quad\beta=\left\{t^2+at+b,\ bt^2+1,\ bt+1\right\}$$

To determine which of these sets are "spanning", i.e. which of these two sets generate every vector of the vector space $P_2(\mathbb{F}_4)$, we could ask:

do they generate all vectors of the standard basis $S = \{t^2, t, 1\}$?

i.e. in case of the set α we need to solve

1. $x(at^2+bt+1) + y(t^2+t+a) + z(bt+1) = t^2$

2. $x(at^2+bt+1) + y(t^2+t+a) + z(bt+1) = t$

3. $x(at^2+bt+1) + y(t^2+t+a) + z(bt+1) = 1$

which in terms of the standard basis becomes:

1. $x\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} + z\begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ i.e. $\begin{bmatrix} a & 1 & 0 \\ b & 1 & b \\ 1 & a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

2. $x\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} + z\begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ i.e. $\begin{bmatrix} a & 1 & 0 \\ b & 1 & b \\ 1 & a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

3. $x\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} + z\begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ i.e. $\begin{bmatrix} a & 1 & 0 \\ b & 1 & b \\ 1 & a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

When we solve these separately we get:

1. $\text{rref4}\begin{bmatrix} a & 1 & 0 & 1 \\ b & 1 & b & 0 \\ 1 & a & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b & 1 \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow t^2 \in \text{span}(\alpha)$

2. $\text{rref4}\begin{bmatrix} a & 1 & 0 & 0 \\ b & 1 & b & 1 \\ 1 & a & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow t \notin \text{span}(\alpha)$

3. $\text{rref4}\begin{bmatrix} a & 1 & 0 & 0 \\ b & 1 & b & 0 \\ 1 & a & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow 1 \notin \text{span}(\alpha)$

So that clearly α doesn't span all of $P_2(\mathbb{F}_4)$.

In case of the set β we need to solve

1. $x(t^2+at+b) + y(bt^2+1) + z(bt+1) = t^2$

2. $x(t^2+at+b) + y(bt^2+1) + z(bt+1) = t$

3. $x(t^2+at+b) + y(bt^2+1) + z(bt+1) = 1$

which in terms of the standard basis becomes:

$$\textbf{1.}\quad x\begin{bmatrix}1\\a\\b\end{bmatrix} + y\begin{bmatrix}b\\0\\1\end{bmatrix} + z\begin{bmatrix}0\\b\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix}1 & b & 0\\a & 0 & b\\b & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\textbf{2.}\quad x\begin{bmatrix}1\\a\\b\end{bmatrix} + y\begin{bmatrix}b\\0\\1\end{bmatrix} + z\begin{bmatrix}0\\b\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix}1 & b & 0\\a & 0 & b\\b & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\textbf{3.}\quad x\begin{bmatrix}1\\a\\b\end{bmatrix} + y\begin{bmatrix}b\\0\\1\end{bmatrix} + z\begin{bmatrix}0\\b\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix}1 & b & 0\\a & 0 & b\\b & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

When we solve these separately we get:

$$\textbf{1.}\quad \text{rref}\,4\begin{bmatrix}1 & b & 0 & 1\\a & 0 & b & 0\\b & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & b\end{bmatrix} \quad\Rightarrow\quad t^2 \in \text{span}(\beta)$$

$$\textbf{2.}\quad \text{rref}\,4\begin{bmatrix}1 & b & 0 & 0\\a & 0 & b & 1\\b & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & a\\0 & 0 & 1 & 1\end{bmatrix} \quad\Rightarrow\quad t \in \text{span}(\beta)$$

$$\textbf{3.}\quad \text{rref}\,4\begin{bmatrix}1 & b & 0 & 0\\a & 0 & b & 0\\b & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & b\\0 & 1 & 0 & 1\\0 & 0 & 1 & a\end{bmatrix} \quad\Rightarrow\quad 1 \in \text{span}(\beta)$$

so that β indeed spans the entire space $P_2(\mathbb{F}_4)$. An important observation in this last case is that, when we combine the rrefs into one we get the following:

$$\text{rref}\,4\begin{bmatrix}1 & b & 0 & 1 & 0 & 0\\a & 0 & b & 0 & 1 & 0\\b & 1 & 1 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1 & 1 & b\\0 & 1 & 0 & 0 & a & 1\\0 & 0 & 1 & b & 1 & a\end{bmatrix}$$

i.e. $\begin{bmatrix} 1 & b & 0 \\ a & 0 & b \\ b & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & b \\ 0 & a & 1 \\ b & 1 & a \end{bmatrix}$ or $\det\begin{vmatrix} 1 & b & 0 \\ a & 0 & b \\ b & 1 & 1 \end{vmatrix} \neq 0$

This leads to a simple way of checking if a set is a basis (linearly independent and spanning), by looking at one determinant!

In class we will further investigate bases. In particular we will discuss the following facts about finitely generated vector spaces [we call a vector space **finitely generated** if there exists a finite set T such that $V = \text{span}(T)$]:

- **(a)** Any **linearly independent** set can be **extended to a basis**.
- **(b)** Any **spanning** set can be **reduced** (pruned) to a basis.
- **(c)** The number of elements of a linear independent set is less or equal to the number of elements of a spanning set.
- **(d)** All bases have the same number of elements, called the **dimension** of the vector space.

Example 7:

(a) As we saw $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linear independent, but does not span $\mathbb{R}^3$. We can extend it to a basis though: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Here is another linearly independent set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ that doesn't span $\mathbb{R}^3$ but we can extend it to a basis as well $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) The set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$ is a spanning set but not a basis, since it is not linearly independent. To create a basis from it we can just toss out the vectors that are not needed, i.e. prune the set and end up with a linearly independent spanning set, e.g.

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$$

Of course we need to be careful not to toss out too many.

(c) The following sets $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} \right\}$ are three bases of $\mathbb{R}^3$. Each of them has 3 vectors. *Any* other basis of $\mathbb{R}^3$ will have also three vectors in it. A set with just two vectors in it, even if they are linearly independent, cannot be a basis. A set with four vectors in it, even if it is spanning, cannot be a basis. But we can extend a set with two linearly independent vectors to a basis by adding another vector, and we can prune a spanning set with, say, 5 vectors, i.e. too many vectors, to create a basis.

Definition: The number of elements in a basis of a vector space is called the **dimension** of that vector space.

Notation: $\dim(V)$

This definition makes perfect sense since all bases of a given vector space have the same number of elements. It doesn't make any difference which one we pick.

Example 8:

$*$ $\dim(\mathbb{R}^3) = 3$ since $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} \right\}$ is a basis, and it has 3 elements.

$*$ $\dim\left(P_2(\mathbb{R})\right) = 3$ since $\left\{1, x, x^2\right\}$ is a basis, and it has 3 elements.

$*$ $\dim(\mathbb{R}^2) = 2$ since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis, and it has 2 elements.

Example 9:

(a) The subspace $W = \text{span}\left(\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}\right)$ consists of all vectors that are linear combinations of $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\\0\end{bmatrix}$. It spans the plane through those two vectors. Note that the set $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$ is linearly independent, and it (clearly) spans W, hence it is a basis of W, so that $\dim(W) = 2$.

(b) $W = \text{span}\left(\begin{bmatrix}1\\2\\5\end{bmatrix}\right)$ is a one dimensional subspace of $\mathbb{R}^3$.

(c) $W = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^3$. It is the smallest subspace. It has only one element, the zero vector. Note that $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is not a basis of W, since it is linearly **de**pendent.

Here is a non trivial linear combination for $\vec{0}$: $1 \cdot \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. As we noted before any set containing the zero vector is dependent.

Since a vectors space spanned by two linearly independent vectors is two dimensional, and a vector space spanned by one linearly independent vector is 1 dimensional, we would like to say that this zero space, which is spanned by no linearly independent vectors is a zero dimensional vector space. That would mean that we would need a basis with no vectors: the empty set ϕ. We will actually adopt this as a convention:

Convention: ϕ is the basis for $W = \{\vec{0}\}$.

Recall we defined $span(\phi) = \{\vec{0}\}$

It may look a bit weird, surreal even, but it is in sync with our the definition of dimensions. It is also in sync with our definition of a basis as an linearly independent spanning set: after all the empty set is linearly independent [there is no way to write the zero vector as a linear combination of vectors from ϕ, since it is empty] and it spans [by definition: $span(\phi)=\{\vec{0}\}$] ; hence ϕ is a perfect basis for $\{\vec{0}\}$.

(d) The subspaces of $\mathbb{R}^3$ are either 0, 1, 2 or 3 dimensional:

- $\mathbb{R}^3$, the space itself, is the only 3 dimensional subspace.
- The span of any two linearly independent vectors of $\mathbb{R}^3$, spanning a plane, is a two dimensional subspace of $\mathbb{R}^3$.
- The span of any nonzero vector of $\mathbb{R}^3$, spanning a line, is a one dimensional subspace of $\mathbb{R}^3$.
- $\{\vec{0}\}$ is the only zero dimensional subspace of $\mathbb{R}^3$.