Time is Money: Cash-Flow Risk and Product Market Behavior

Technical appendix

This technical appendix first recalls the techniques of resolution introduced by DeMarzo and Sannikov (2006) and Biais et al. (2007) for this type of continuous-time contracts; it then adapts a comparative statics lemma from DeMarzo and Sannikov (2006) to the case of Poisson-distributed cash flows.

Resolution of the model

Preliminary results

The resolution of the model starts by introducing the lifetime expected utility of the agent evaluated at time t

$$U_t^\theta = E^\theta\left[\int_0^{\tau_L} e^{-rt} dc_s + R_{\tau_L\wedge\tau_E} e^{-r(\tau_L\wedge\tau_E)} | \mathscr{F}_t^N\right] = \int_0^t e^{-rt} dc_s + e^{-rt} W_t^\theta$$

where W_t^θ is the *continuation* utility of the agent defined as

$$W_t^\theta = E^\theta\left[\int_t^{\tau_L} e^{-r(s-t)} dc_s + R_{\tau_L\wedge\tau_E} e^{-r(\tau_L\wedge\tau_E)} | \mathscr{F}_t^N\right] \tag{1}$$

The lifetime expected utility is a $\mathscr{F}^N$-martingale under the probability measure $\mathbf{P}^\theta$ given by θ. It is useful in this context to introduce the compensated process $M^\theta = \{M_t^\theta\}_{t\geq 0}$ given by

$$M_t^\theta = \int_0^t (dN_s - \lambda_s\theta_s) ds.$$

The process M^θ can be interpreted as the difference between the *realized* and *expected* number of payments at time t; in line with the Girsanov theorem for Brownian motions, it can be proved that M^θ is a $\mathscr{F}^N$-martingale under $\mathbf{P}^\theta$. In other terms, once the change of probability induced by the process θ is accounted for, the expectation of the number of cash payments should not change over time. The martingale representation theorem for point processes then allows to state that:

Lemma 0.1. *The martingale U^θ satisfies*

$$U_t^\theta = U_0^\theta + \int_0^{t\wedge\tau} e^{-rs} H_s^\theta dM_s^\theta$$

for all $t \geq 0$, $\mathbf{P}^\theta$-almost surely, for some $\mathscr{F}^N$-predictable process $H^\theta = \{H_t^\theta\}_{t\geq 0}$.

Proof: see Brémaud (1981). ■

This result shows that the lifetime utility of the agent evolves directly with the compensated process M^{θ}. It follows that the evolution of the continuation utility of the agent W_t^{θ} can be computed as

$$dW_t^{\theta} = rW_t^{\theta}dt + H_t^{\theta}X_t dM_t^{\theta} - dc_t$$

This representation of the continuation utility allows to characterize precisely incentive-compatible contracts in delays of conditions on the process H_t^{θ}. For the agent to prefer to maintain a high rate of cash-flows arrival by choosing $\theta = 1$, it must be that shirking entails higher future losses in terms of W_t than immediate gains:

$$(1-\theta_t)\lambda_t X_t \leq H_t^{\theta}(1-\theta_t)\lambda_t X_t$$

which boils down to $H_t^{\theta} \geq 1$. This result can be summarized in the following proposition (see Cvitanic and Zhang (2012) for a very detailed proof):

Proposition 0.1. *A necessary and sufficient condition for the diverting process θ to be incentive-compatible given the contract (i, τ_E, τ_L) is that*

$$\theta_t = 0 \textit{ if and only if } H_t^{\theta} \geq 1$$

for all $t \in [0, \tau_L[$, $\mathbf{P}^{\theta}$-almost surely.

As a corollary, it can be deduced that $\hat{\tau} = \inf\{t \geq 0 \,|\, W_t = R_t\} = \tau_L$. Suppose on the contrary that $\hat{\tau} < \tau_L$. Then by proposition 0.1 with $\delta > 0$

$$W_{t+\delta} \leq R_t + \int_{\hat{\tau}}^{\hat{\tau}+\delta} ((rW_s - \lambda_s X_s)ds + X_s dN_s)$$

For the condition (1) to hold, the integral should be negative $\mathbf{P}^{\theta}$-almost surely, which is not the case.

Since the principal has the possibility to pay the agent for every W_t, the profit function must verify $b(W_t + \Delta i) \geq b(W_t) - \Delta i$ for all nonnegative payment Δi: hence, $b'(W_t) \geq -1$ for all W_t. On the other hand, writing that for all $W_t \geq R_t$, total surplus $TS(W_t)$ can not be superior to the perfect information case yields $TS(W_t) = b(W_t) + W_t \leq \Pi/r$. These conditions impose that for $W_t \geq R^* = \Pi/r$, $b'(W_t) = -1$.

Proof of propositions 1 and 2

Proposition 1. *In the optimal contract, the continuation value W_t evolves according to*

$$dW_t = rW_t dt + X_t(dN_t - \lambda_t dt) - di_t$$

starting with value W_0. When $W_t \in [R_E, R^*[$ with $R^* = \Pi_E/r$, the agent receives no payment: $di_t = 0$. When W_t reaches R^*, payments di_t cause W_t to reflect at R^*. The profit function b^E follows

$$rb(W_t) = \Pi_t + (rW_t - \Pi_t)b'(W_t) + \lambda_t(b(W_t + X_t) - b(W_t))$$

on the interval $[R_E, R^*]$ *with the boundary conditions* $b^E(R_E) = L_E$ *and* $b^E(R^* + \omega) = -\omega$ *for all* $\omega \geq 0$. *The principal gets the liquidation value* L_E *at time* τ_L *when* W_t *reaches* R_E.

Proposition 2. *In the optimal contract, the continuation value* $W_t \in [R_I, W_E[$ *evolves according to the same equation as in proposition 1 starting with value* W_0 *with* $di_t = 0$ *(no payment). The profit function* b_I *follows the same differential equation as in proposition 1 on the interval* $[R_I, W_E]$ *with the boundary conditions* $b^I(R_I) = L_I$ *and* $b^I(W_E + \omega) = b^E(W_E + \omega) - K$ *for all* $\omega \geq 0$. *If* W_t *reaches* W_E *before* R_I *at time* τ_E, *the firm starts exporting and the optimal contract is given by Proposition 1. If* W_t *reaches* R_I *first at time* τ_L, *the principal gets the liquidation value* L_I.

Proof: Recall first that the value function b_E given by

$$b^E(W_0) = \max_{(i,\tau_L)\in P} \max_{\theta \in A(i,\tau_L)} E^\theta \left[\int_0^{\tau_L} e^{-rt}(dY_t^R - di_t) + L_E e^{-r\tau_L} \right]$$

The Hamilton-Jacobi-Bellman equation verified by b^E is

$$rb_E(W_t^\theta) = \max_{H_t^\theta} \Pi_t + (rW_t^\theta - H_t^\theta \Pi_t) b_E'(W_t^\theta) + \lambda_E(b_E(W_t^\theta + H_t^\theta X_t) - b_E(W_t^\theta))$$

Assuming that b_E is a concave function, we get

$$(b_E(W_t^\theta + H_t^\theta X_t) - b_E(W_t^\theta)) \leq H_t^\theta (b_E(W_t^\theta + X_t) - b_E(W_t^\theta))$$

and

$$\lambda_E(b_E(W_t^\theta + X_t) - b_E(W_t^\theta)) - \Pi_E b_E'(W_t^\theta) \leq 0$$

which shows that it is optimal for the investor to set H_t^θ as low as possible, that is by taking $H_t^\theta = 1$. Define then G_t^E as

$$G_t^E = \int_0^t e^{-rt}(dY_t - di_t) + e^{-rt} b_E(W_t^\theta)$$

Ito calculus shows that G_t^E evolves according to

$$\begin{aligned} e^{rt} dG_t^E =& (X_t + b_E(W_t^\theta + H_t^\theta X_t) - b_E(W_t^\theta))(dN_t - \lambda_t dt) - (b_E'(W_t^\theta) + 1) di_t \\ &+ (\Pi_t - rb_E(W_t^\theta) + (rW_t^\theta - H_t^\theta \Pi_t) b_E'(W_t^\theta) + \lambda_t (b_E(W_t^\theta + H_t^\theta X) - b_E(W_t^\theta)) \\ &+ b_E'(W_t^\theta)(H_t^\theta - 1)(1 - \theta_t)\lambda_t X_t) dt \end{aligned}$$

But using the incentive constraint $H_t^\theta \geq 1$ and the fact that $b_E'(W_t^\theta) \geq -1$, one gets the inequality

$$\begin{aligned} e^{rt} dG_t^E \leq& (X_t + b_E(W_t^\theta + H_t^\theta X_t) - b_E(W_t^\theta))(dN_t - \lambda_t dt) \\ &+ (\Pi_t - rb_E(W_t^\theta) + (rW_t^\theta - H_t^\theta \Pi_t) b_E'(W_t^\theta) + \lambda_t (b_E(W_t^\theta + H_t^\theta X) - b_E(W_t^\theta)) \end{aligned}$$

Using the HJB equation, we see that the second line of the equation is strictly negative if $H_t^\theta > 1$ and therefore

$$e^{rt} dG_t^E \leq (X_t + b_E(W_t^\theta + H_t^\theta X_t) - b_E(W_t^\theta))(dN_t - \lambda_t dt)$$

By concavity of b_E, $b_E(W_t^\theta + H_t^\theta X_t) - b_E(W_t^\theta)$ is bounded: G_t^E is therefore a $\mathbf{P}^\theta$-supermartingale, which yields

$$E^\theta\left[\int_0^{\tau_L} e^{-rt}(dY_t^R - di_t) + L_E e^{-r\tau_L}\right] = E^\theta[G_\tau^E] \leq E^\theta[G_0^E] = b^E(W_0)$$

with equality if and only if b_E if $H_t^\theta = 1$, if $di_t = 0$ for $W_t^\theta \in [R_E, R^*]$ and if $di_t = W_t^\theta - R^*$ for $W_t^\theta \geq R^*$.

A similar reasoning shows $E^\theta_{\tau_E \wedge \tau_L}[G^I_{\tau_E\wedge\tau_L}] \leq E^\theta[G_0^I] = b^I(W_0)$ with equality if and only if $H_t^\theta = 1$, if $di_t = 0$ for $W_t^\theta \in [R_I, W_E]$ and if the contract follows proposition 1 after starting to export, which shows proposition 2. ■

A comparative statics lemma

Lemma 0.2. *Suppose that W_t evolves according to $dW_t = rW_t - dZ_t + \lambda X dM_t$ in the interval $[R, W^*]$ until W_t reaches R (time τ_L) or W^* (time τ^*). R is a stopping value and W^* an optimally chosen reflecting barrier. Z_t is a reflecting process which makes W_t stay in the interval $[R, W^*]$.*

Take h_{x,W^*} a function of parameter x and of the threshold W^* such that the application $(x, W^*, W) :\rightarrow h_{x,W^*}(W)$ is C^2 and such that $W \rightarrow \partial h_{x,W^*}(W)/\partial x$ is bounded. If h_{x,W^*} verifies the equation

$$rh_{x,W^*}(W_t) = k + rW_t h'_{x,W^*}(W_t) + \lambda\left(h_{x,W^*}(W_t + X) - h_{x,W^*}(W_t)\right)$$

with boundary conditions $h_{x,W^*}(R) = \alpha$ and $h'_{x,W^*}(W^*) = \beta$, then noting

$$\Delta : W_t \rightarrow \lambda\left(h_{x,W^*}(W_t + X) - h_{x,W^*}(W_t)\right)$$

we get that $\partial h_{x,W^*}(W_t)/\partial x$ is equal to

$$E\left[\int_0^{\tau_L\wedge\tau_E} e^{-rs}\left(\frac{\partial k}{\partial x} + \frac{\partial r}{\partial x}W_t h'_{x,W^*}(W_s) + \frac{\partial \Delta(W_s)}{\partial x}\right)ds + \frac{\partial \beta}{\partial x}\int_0^{\tau_L\wedge\tau_E} dZ_t + e^{-r(\tau_L\wedge\tau_E)}\frac{\partial \alpha}{\partial x} \mid W_0 = W_t\right]$$

Proof: Using the envelope theorem, we have that

$$\frac{\partial h_x(W_t)}{\partial x} = \frac{\partial h_{x,W^*}(W_t)}{\partial x}\Big|_{W^*=W^*(x)}$$

Differentiating the differential equation verified by h with respect to the parameter x at $W^* = W^*(x)$ shows that $\partial h/\partial x$ evolves according to

$$\begin{aligned} r\frac{\partial h_{x,W^*}(W_t)}{\partial x} =& \frac{\partial k}{\partial x} + \frac{\partial r}{\partial x}W_t h'_{x,W^*}(W_t) + \frac{\partial \Delta(W_t)}{\partial x} \\ &+ rW_t\frac{\partial}{\partial W}\frac{\partial h_{x,W^*}(W_t)}{\partial x} + \lambda\left(\frac{\partial h_{x,W^*}(W_t + X)}{\partial x} - \frac{\partial h_{x,W^*}(W_t)}{\partial x}\right) \end{aligned}$$

with boundary conditions $\partial h_{x,W^*}(R_t)/\partial x = \partial\alpha/\partial x$ and $\partial h'_{x,W^*}(W^*)/\partial x = \partial\beta/\partial x$. Define H_t as

$$\int_0^t e^{-rs}\left(\frac{\partial k}{\partial x} + \frac{\partial r}{\partial x}W_s h'_{x,W^*}(W_s) + \frac{\partial \Delta(W_s)}{\partial x}\right)ds + \frac{\partial \beta}{\partial x}\int_0^t e^{-rs}dI_s + e^{-rs}\frac{\partial h_{x,W^*}(W_t)}{\partial x}$$

Using the differential equation verified by $\partial h/\partial x$, we get that

$$dH_t = e^{-rs}\left(\frac{\partial \beta}{\partial x} - \frac{\partial}{\partial W}\frac{\partial h_{x,W^*}(W_t)}{\partial x}\right)dI_t + e^{-rs}\lambda\left(\frac{\partial h_{x,W^*}(W_t + X)}{\partial x} - \frac{\partial h_{x,W^*}(W_t)}{\partial x}\right)dM_t$$

Thanks to Schwartz's theorem,

$$\frac{\partial}{\partial W}\frac{\partial h_{x,W^*}(W_t)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h_{x,W^*}(W_t)}{\partial W}$$

Since Z is a reflecting process that differs only from zero when W_t reaches W^* and using the boundary condition, we get that H is a martingale. But then

$$\begin{aligned}\frac{\partial h_{x,W^*}(W_0)}{\partial x} &= H_0\\ &= E[H_{\tau_L\wedge\tau_E}]\\ &= E\left[\int_0^{\tau_L\wedge\tau_E} e^{-rs}\left(\frac{\partial k}{\partial x}+\frac{\partial r}{\partial x}W_s h'_{x,W^*}(W_s)+\frac{\partial\Delta(W_s)}{\partial x}\right)ds+\frac{\partial\beta}{\partial x}\int_0^{\tau_L\wedge\tau_E} dI_t + e^{-r(\tau_L\wedge\tau_E)}\frac{\partial\alpha}{\partial x}\right]\end{aligned}$$

which proves the lemma. ■

A similar formula can be established for the case where W^* is a stopping value.

Lemma 0.3. *If W^* is a stopping value with the associated boundary condition $h_{x,W^*}(W^*) = \gamma$, then we have that $\partial h_{x,W^*}(W_t)/\partial x$ is equal to*

$$E\left[\int_0^{\tau_L\wedge\tau_E} e^{-rs}\left(\frac{\partial k}{\partial x}+\frac{\partial r}{\partial x}W_s h'_{x,W^*}(W_s)+\frac{\partial\Delta(W_s)}{\partial x}\right)ds + e^{-r(\tau_L\wedge\tau_E)}\left(1_{\tau^*<\tau_L}\frac{\partial\gamma}{\partial x}+1_{\tau^*>\tau_L}\frac{\partial\alpha}{\partial x}\right)\mid W_0 = W_t\right]$$

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