Ex 03

1)

a) $E(x) = \sum_{i \in V} \psi_i(x_i) + \sum_{(i,j) \in E} \psi_{ij}(x_i, x_j)$

$V = \{0,1,2\} \quad E = \{0,1,2\}$

$\Rightarrow$ Non Linear due to random variable x

b) unary indicator variables:

$\Rightarrow \mu_u = \sum_{i \in V} \sum_{k \in L} \mu_i(k) \qquad V = \{0,1,2\} \qquad L = \{0,1,2\}$

$\hookrightarrow$ integer encoding for all classes

with: $x_i = \sum_{k \in L} \mu_i(k) \cdot k$ and: $\mu_i(k, x_i) := I(x_i = k)$

$I(true) = 1$
$I(false) = 0$

unary cost vector c:

$c_u = \sum_{i \in V} \sum_{k \in L} \psi_i(k)$

pairwise indicator variables:

$\mu_p = \sum_{(i,j) \in E} \sum_{k \in L} \sum_{l \in L} \mu_{i,j}(k,l)$ with: $\mu_i(k,l,x_i,x_j) := I(x_i = k, x_j = l)$

and: $\sum_{l \in L} \mu_{i,j}(k,l) = \mu_i(k)$

$I(true) = 1$
$I(false) = 0$

pairwise cost vector c:

$c_p = \sum_{k \in L} \sum_{l \in L} \psi_p(k,l)$

c) $E(\mu, \psi) = \sum_{i \in V} \sum_{k \in L} \mu_i(k) \cdot \psi_i(k) + \sum_{(i,j) \in E} \sum_{k \in L} \sum_{l \in L} \mu_{ij}(k,l) \cdot \psi_p(k,l)$

$\Rightarrow$ With appropriate constraints, this is now linear

d) unary Expression of cost and indicator vector:

$$\begin{aligned}
\psi(0,0,0) &= 1.1 + 0 \\
\psi(0,0,1) &= 0.3 + 2\beta \\
\psi(0,1,1) &= 1.1 + 2\beta \\
\psi(1,1,1) &= 1.1 + 0 \\
\psi(1,1,0) &= 1.9 + \beta \\
\psi(1,0,0) &= 1.1 + \beta \\
\psi(1,0,1) &= 0.3 + 2\beta \\
\psi(0,1,0) &= \underbrace{1.9}_{\text{unary term}} \underbrace{+2\beta}_{\text{pairwise term}}
\end{aligned}$$

$$\mu_u: \quad \mu_0 = \begin{pmatrix}1\\0\end{pmatrix}, \ \mu_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \mu_2 = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{or} \quad \mu_0 = \begin{pmatrix}0\\1\end{pmatrix}, \ \mu_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \mu_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$\Rightarrow$ Two Solutions for μ_u

pairwise vectors:

for $\beta > 0$

because: $\sum_{k \in L} \sum_{l \in L} \mu(k,l) = 1$

$$\mu_p: \quad \mu_0 = \begin{pmatrix}1\\0\end{pmatrix}, \ \mu_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \mu_2 = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{or} \quad \mu_0 = \begin{pmatrix}0\\1\end{pmatrix}, \ \mu_1 = \begin{pmatrix}0\\1\end{pmatrix}, \ \mu_2 = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \text{Two Solutions}$$

constraint:

$$\sum_{l \in L} \mu_{i,j}(k,l) = \mu_i(k)$$

$\Rightarrow$ Sum across one row of the quadratic pairwise indicator variables, should result in the unary-indicator vector. Thus: $\mu_p = \mu_u$

$\Rightarrow$ since $\mu_u \neq \mu_p$ for any combination of Solutions, this Solution does not lead to a valid state for x

e)

It depends on β. If $\beta \leq 0$ then μ_u is one indicator vector that minimizes the cost-function. In that case, we have $\mu_u = \mu_p$ which proves consistency with valid x-States.

(1) $\beta \leq 0$

(2)

$$\sum_{l \in L} \mu_{i,l}(k,l) = \mu_i(k)$$