Ex 03

1)

a) $E(x) = \sum_{i \in V} \psi_i(x_i) + \sum_{(i,j) \in E} \psi_{ij}(x_i, x_j)$

$V = \{0,1,2\} \quad E = \{0,1,2\}$

$\Rightarrow$ Non Linear due to random variable x

b) unary indicator variables:

$\Rightarrow \mu_u = \sum_{i \in V} \sum_{k \in L} \mu_i(k) \quad V = \{0,1,2\} \quad L = \{0,1,2\}$

↳ integer encoding for all classes

with: $x_i = \sum_{k \in L} \mu_i(k) \cdot k$ and: $\mu_i(k, x_i) := I(x_i = k)$

$I(\text{true}) = 1$
$I(\text{false}) = 0$

unary cost vector c:

$c_u = \sum_{i \in V} \sum_{k \in L} \psi_i(k)$

pairwise indicator variables:

$\mu_p = \sum_{(i,j) \in E} \sum_{k \in L} \sum_{l \in L} \mu_{ij}(k,l)$ with: $\mu_i(k,l,x_i,x_j) := I(x_i = k, x_j = l)$

$I(\text{true}) = 1$
$I(\text{false}) = 0$

pairwise cost vector c:

$c_p = \sum_{k \in L} \sum_{l \in L} \psi_p(k,l)$

c) $E(\mu, \psi) = \sum_{i \in V} \sum_{k \in L} \mu_i(k) \cdot \psi_i(k) + \sum_{(i,j) \in E} \sum_{k \in L} \sum_{l \in L} \mu_{ij}(k,l) \cdot \psi_p(k,l)$

$\Rightarrow$ With appropriate constraints, this is now linear

d) unary Expression of cost and indicator vector:

$$\begin{aligned}
\psi(0,0,0) &= 1.1 + 0 \\
\psi(0,0,1) &= 0.3 + 2\beta \\
\psi(0,1,1) &= 1.1 + 2\beta \\
\psi(1,1,1) &= 1.1 + 0 \\
\psi(1,1,0) &= 1.9 + \beta \\
\psi(1,0,0) &= 1.1 + \beta \\
\psi(1,0,1) &= 0.3 + 2\beta \\
\psi(0,1,0) &= \underbrace{1.9}_{\text{unary term}} + \underbrace{2\beta}_{\text{pairwise term}}
\end{aligned}$$

$\Rightarrow$ Due to the restriction $\sum_k \mu(k) = 1$ the vector $\mu(k) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is not accepted

$$\mu_u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

pairwise vectors:

for $\beta > 0$ $\quad \mu_p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

because: $\sum_{k \in L} \sum_{l \in L} \mu(k,l) = 1$

$\Rightarrow$ since $\mu_u \neq \mu_p$, this solution does not lead to a valid state for X

e)

It depends on β. If $\beta \leq 0$ then $\mu_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is one indicator vector that minimizes the cost-function. In that case, we have $\mu_u = \mu_p$ which proves consistency with valid x-states.

(1) $\beta \leq 0$

f)

(1) $\|\mu\| = 1$ => Since we want each vertex to commit to exactly one state.

(2) $\mu = \{0, 1\}$ => The indicator variable is supposed to find a combination of x that minimizes the energy-function. Thus say no or yes $\{0, 1\}$ for each compo.