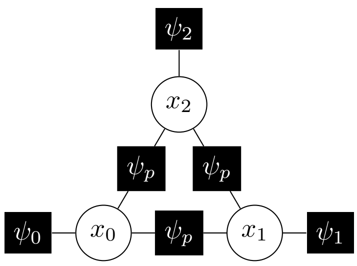
Exercise Sheet 03

Deadline: 09.05.2018, 1pm
Regulations:

- You need $\geq$ **50% of all points** in the weekly exercises and successfully work on a $\geq$ **2 weeks fulltime project** and submit a **written report** in order to pass this class.
- Hand in your solutions in **groups of two or three people**.
- Make sure your code works with python 3.6 or later.
- For exercises that include coding, hand in a **Juypter Notebook** for each exercise containing all your answers, results, plots and code. Separate the different sub-exercises clearly by using headings and describing text. Label each file with the number of the exercise. Before submission, make sure your Jupyter Notebook compiles without any errors when you do Run All. Additionally hand in a PDF version of the Notebook.
- For exercises that can be solved without coding, hand in a **PDF file** for each exercise containing all your answers and results. Separate the different sub-exercises clearly by using headings and describing text. Label each file with the number of the exercise.
- Submit all your results in a **single** .zip archive. The title of the submitted ZIP-file as well as the subject line of your email **must** start with **EX03** followed by the full names of **all group members**. **People not mentioned in the subject will not get credits.**
- Submit all your results to mlcvss18@gmail.com.
- Submissions that do not adhere to the deadline and regulations, or that are chaotic, will not be graded.

Exercise 1. 18 P.

Consider the following graphical model with binary variables $x_i \in \{0, 1\}$.



The potentials are given in the following tables:

x_0	$\psi_0(x_0)$
0	0.1
1	0.1

x_1	$\psi_1(x_1)$
0	0.1
1	0.9

x_2	$\psi_2(x_2)$
0	0.9
1	0.1

x_i	x_j	$\psi_p(x_i, x_j)$
0	0	0
0	1	β
1	0	β
1	1	0

The MAP (maximum a posteriori) solution, i.e. the configuration $\mathbf{x}^*$ that maximizes the posterior, can be found by minimizing the energy function:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} E(\mathbf{x})$$

In this exercise you will rewrite this energy such that it can be represented as an Integer Linear Program (ILP).

An ILP can be written as

$$\begin{aligned} &\text{minimize} && \mathbf{c}^{\mathrm{T}}\mu \\ &\text{subject to} && A\mu \leq \mathbf{b}, \\ & && \mu \geq \mathbf{0}, \\ &\text{and} && \mu \in \mathbb{Z}^n, \end{aligned}$$

In our case μ are indicator variables, $\mathbf{c}$ associated costs, and $\mathbf{A}$ and $\mathbf{b}$ encode the consistency constraints. Additionally we only allow $\mu \in \{0,1\}^n$.

a) Write down the energy as a function of x_0, x_1 and x_2 and the potentials[1].

b) Which indicator variables μ and corresponding entries of the cost vector $\mathbf{c}$ are required to represent this model as an Integer Linear Program? How are they related to the variables and potentials?

c) Write down the energy[1] as a function of these new indicator variables μ and the potentials ψ.

d) What is the solution of this problem (i.e. which μ^* minimizes $\mathbf{c}^{\mathrm{T}}\mu$) if only $\mu \in \{0,1\}^n$ is enforced. Does it lead to a valid state for x_0, x_1 and x_2?

e) Explain which constraints have to be added to enforce that the solution μ^* represents exactly one state for each x_0, x_1 and x_2. Explicitly write down all constraints as equations[1].

f) Explain which constraints are additionally required to enforce consistency between the unary and pairwise indicator variables. Explicitly write down all constraints as equations[1].

g) Use `scipy.optimize.linprog` (see Documentation here) to solve the Linear Program relaxation (i.e. $\mu \in [0,1]^n$ instead of $\mu \in \{0,1\}^n$) for $\beta = 1$ and $\beta = -1$. Do the LP solutions lead to a consistent labeling? Is this solution equivalent to the ILP solution? Explain your answers.

Exercise 2. 2 P.

Consider a graphical model on a grid graph with the size $N \times N$ and only binary variables, i.e. $\mathbf{x} \in \{0,1\}^{N^2}$.

Construct unary ψ_i and pairwise ψ_{ij} potentials such that an arbitrary $\mathbf{x}^*$ is the optimal labeling, i.e. find ψ_i and ψ_{ij} for a given $\mathbf{x}^*$ such that the following equation is true

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\{0,1\}^{N^2}} \sum_i \psi_i(x_i) + \sum_{i\sim j} \psi_{ij}(x_i, x_j).$$

Hint: Yes, it is as easy as you think it is.

[1] Write down all terms without using sum signs ($\sum$) or similar abbreviations