Embracing the quantum future: fundamental training for UPT students

Multiple qubits and quantum circuits

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NAND NOR

Universal Quantum Gates

n-qubit state

2^n- elements

- Deutsch

Universal gate

$$D(\theta):|x,y,z\rangle \Rightarrow \begin{cases} i\cos\theta|x,y,z\rangle + \sin\theta|x,y,1-z\rangle & x=y=1 \\ |x,y,z\rangle & \text{otherwise} \end{cases}$$

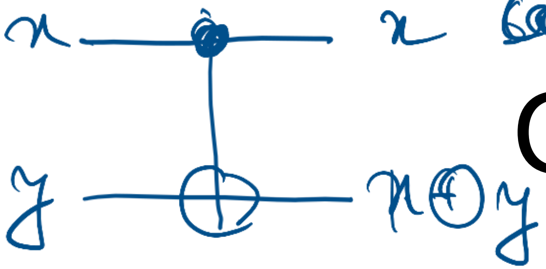
$2^n \times 2^n$ matrix

* • Di Vincenzo

$$U_2(\omega,\alpha,\beta,\phi) = e^{-i\phi}\begin{bmatrix} e^{i\alpha}\cos\omega & -e^{-i\phi}\sin\omega \\ e^{i\phi}\sin\omega & e^{-i\alpha}\cos\omega \end{bmatrix}$$

$$XOR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Universal gate set



control qubit

target qubit

Quantum Gates

John Preskill

- Pseudo-classical operators

$$PC|\psi\rangle = PC\sum_{i=0}^{2^n-1} a_i|i\rangle = \sum_{i=0}^{2^n-1} a_i|p(i)\rangle$$

p(*k*) is a permutation over the eigenvectors

Toffoli

- $XOR : |x, y\rangle \rightarrow |x, x \oplus y\rangle$

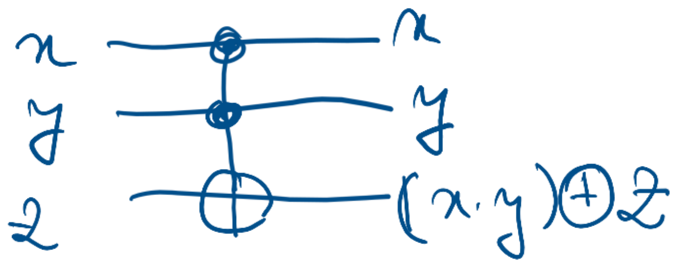
00	→	00
01	→	01
10	→	11
11	→	10

$$XOR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

000	→	000
001	→	001
010	→	010
011	→	011
100	→	100
101	→	101
110	→	111
111	→	110

Quantum Gates

- Toffoli Gate:



TOFF : $|x, y, z\rangle \rightarrow |x, y, (x{\cdot}y) \oplus z\rangle$

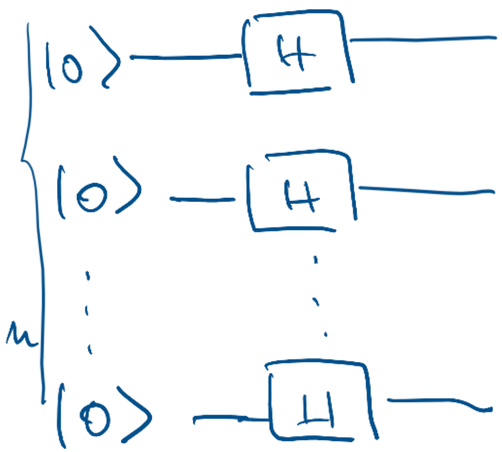
$$TOFF|\psi\rangle = \begin{array}{c} \begin{array}{cccccccc} 000 & 001 & \; & \; & \; & \; & 110 & 111 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_7 \\ a_6 \end{bmatrix} \begin{array}{l} 000 \\ 001 \\ \\ \\ \\ \\ 110 \\ 111 \end{array}$$

Useful Unitary Gates

- Hadamard gate

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{aligned}
|\Psi\rangle &= H \otimes H \otimes \ldots H |00\ldots 0\rangle = H^{\otimes n} |0\rangle^{\otimes n} \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \ldots \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
&= \frac{1}{2^{\frac{n}{2}}}\left(|00\ldots 0\rangle + |00\ldots 0\rangle + \ldots |11\ldots 1\rangle\right) \\
&= \frac{1}{2^{\frac{n}{2}}} \sum_{i=0}^{2^n - 1} |i\rangle
\end{aligned}$$



$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\vdots$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

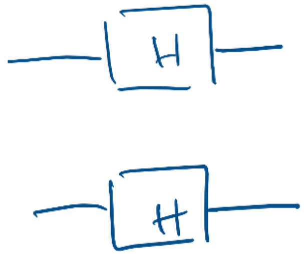
$$|0\rangle^{\otimes n} \qquad H^{\otimes n}$$

$$\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]^{\otimes n}$$

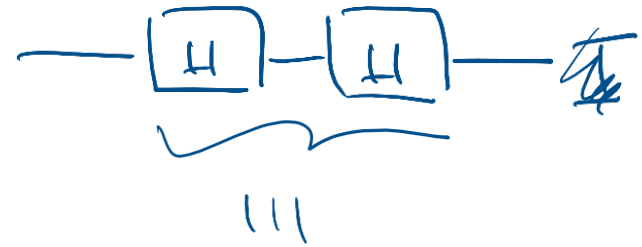
$$|\psi\rangle = \frac{1}{2^{\frac{n}{2}}} \sum_{i=1}^{2^n-1} |i\rangle$$

idQuantique

$H^{\otimes 2}$



$H \cdot H$



$$H \cdot H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

Useful Unitary Gates

- Qubit rotation

around axis *y*

$$R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

around axis *z*

$$R_z(\phi) = \begin{bmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{bmatrix}$$

Useful Unitary Gates

- Conditional Phase Shift

$$P(\varepsilon)|c_0 c_1 \dots c_{n-1}\rangle \rightarrow \begin{cases} e^{i\varepsilon}|c_0 c_1 \dots c_{n-1}\rangle, if\ c_0 c_1 \dots c_{n-1} = 11\cdots 1 \\ |c_0 c_1 \dots c_{n-1}\rangle, otherwise \end{cases}$$

$$P_{n-qubit}(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & 1 & 0 \\ 0 & & & \cdots & 0 & e^{i\varepsilon} \end{bmatrix}$$

Useful Unitary Gates

- Conditional NOT Gate

$$CNOT_{(n+1)-qubit}\colon |x_0, x_1 \ldots, x_{n-1}, z\rangle \rightarrow \left|x_0, x_1 \ldots, x_{n-1}, \left(\Lambda_{i=0}^{n-1} x_i\right) \otimes z\right\rangle$$

$$CNOT = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \\ 0 & & \cdots & 0 & 1 & 0 \end{bmatrix}$$

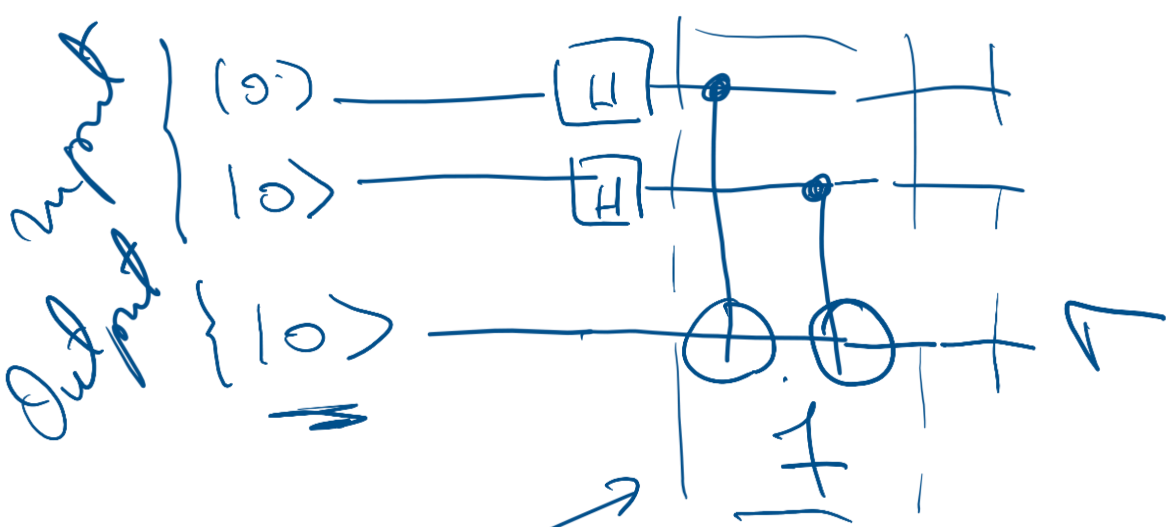
Quantum Functions

- Pseudo-classic operator *f* over *n* input qubits and *m* output qubits:

$$F_{n,m} : |k\rangle \otimes |0\rangle = |k,0\rangle \to |k, f(k)\rangle = |k\rangle \otimes |f(k)\rangle$$

- Could be applied over a superposition:

$$F : \frac{1}{2^{\frac{n}{2}}} \sum_{i=0}^{2^n-1} |i\rangle|0\rangle \to \frac{1}{2^{\frac{n}{2}}} \sum_{i=0}^{2^n-1} |i\rangle|f(i)\rangle$$



			Parity
000	$\rightarrow$	00	0
010	$\rightarrow$	01	1
100	$\rightarrow$	10	1
110	$\rightarrow$	11	0
		Input	Output

$$\frac{1}{2}\left(|000\rangle + |010\rangle + |100\rangle + |110\rangle\right) =$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \otimes |0\rangle \xrightarrow{f}$$

input register

$$\frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right)$$

output register

$$|q_1 q_2 q_3\rangle \qquad \underbrace{|q_1 q_2\rangle}_{\text{input}} \otimes \underbrace{|q_3\rangle}_{\text{output}}$$

$$\frac{1}{2}\left(|00\underline{0}\rangle + |01\underline{1}\rangle + |101\rangle + |11\underline{0}\rangle\right) \xrightarrow[\text{measurement}]{q_1 q_2 \to 01} |011\rangle$$

$$\frac{1}{2}\left(|000\rangle + \underline{|011\rangle} + \underline{|101\rangle} + |110\rangle\right) \xrightarrow[\text{measurement}]{q_3 \to 1}$$

$$\longrightarrow \frac{1}{\sqrt{2}}\left(|011\rangle + |101\rangle\right)$$

Quantum Functions

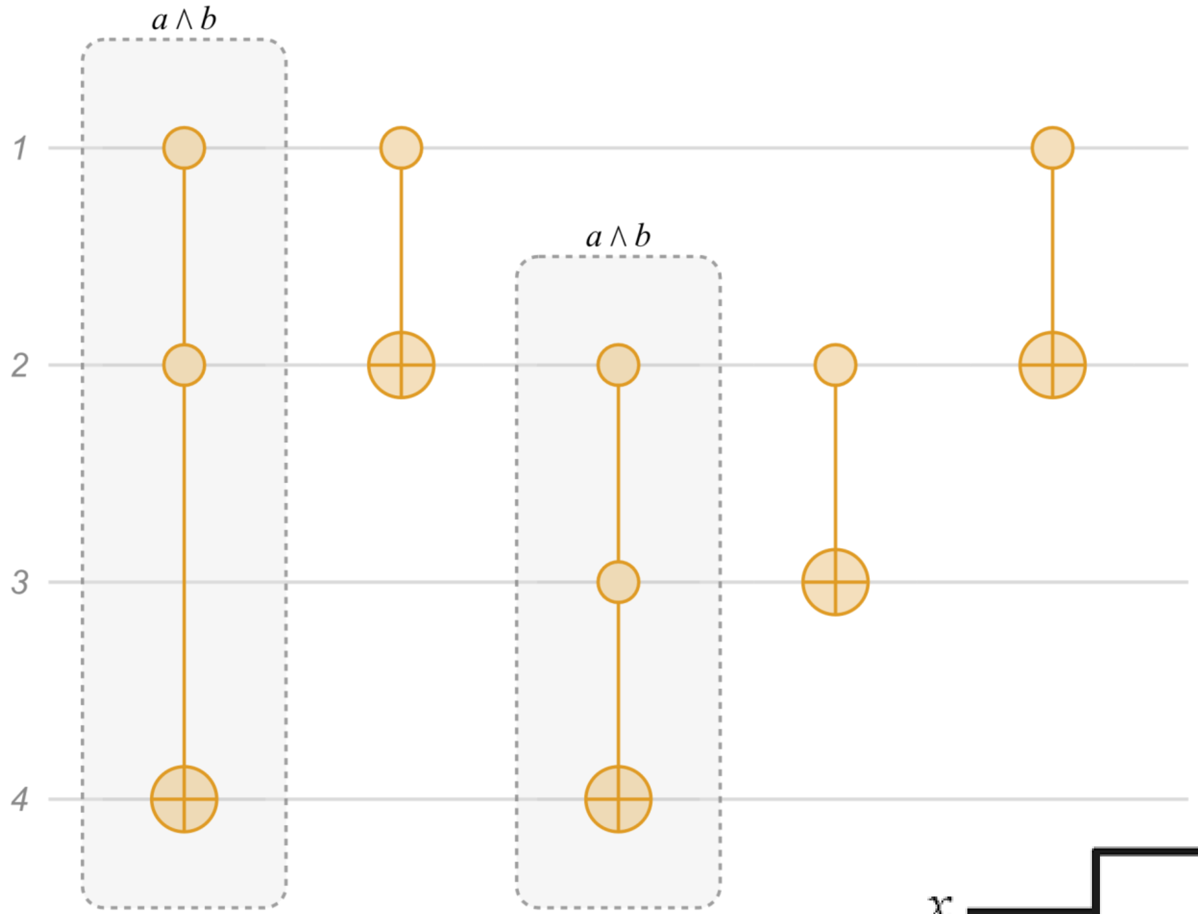
- Measurement

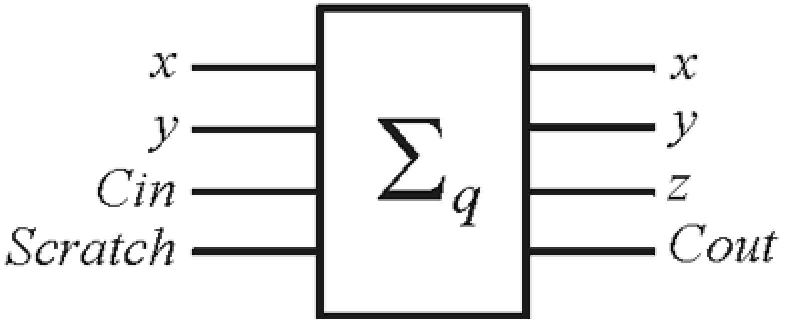
$$\frac{1}{2^{\frac{n}{2}}}\sum_{i=0}^{2^n-1}|i\rangle|f(i)\rangle \xrightarrow{\text{measurement}} |q\rangle|f(q)\rangle$$

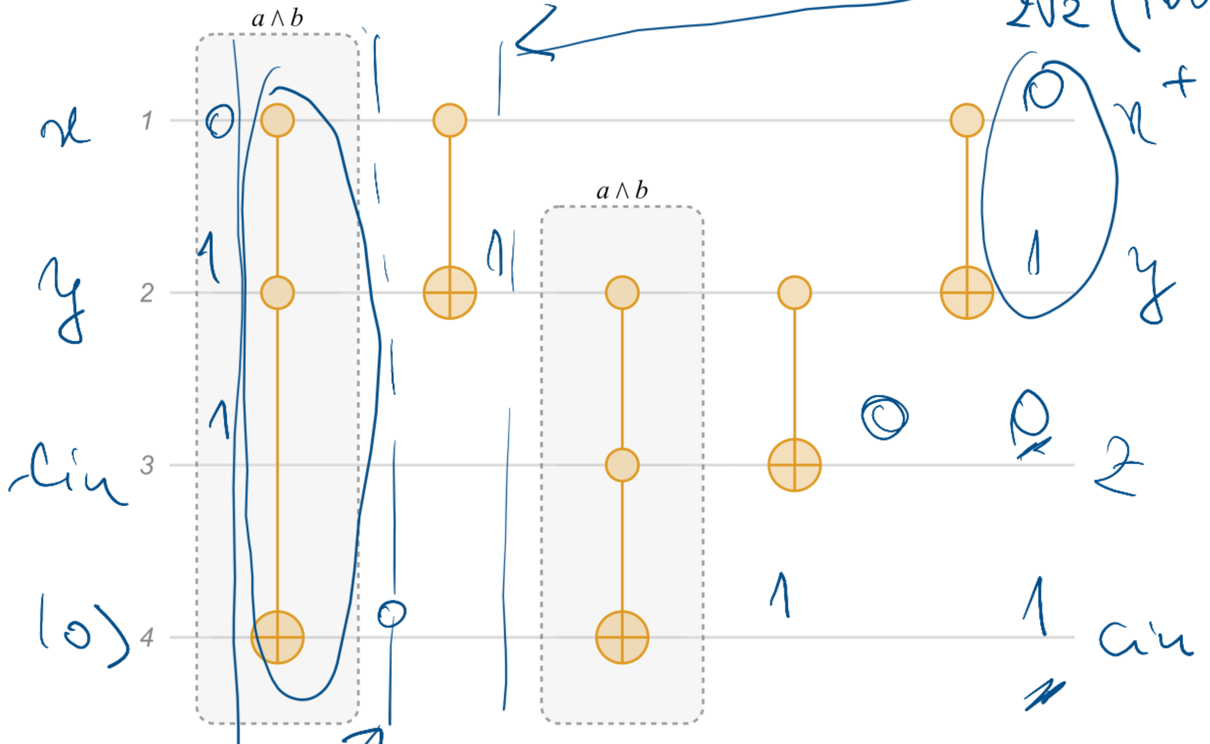
- $f(j_0) = f(j_1) = \ldots f(j_{w-1}) = q$

$$\frac{1}{2^{\frac{n}{2}}}\sum_{i=0}^{2^n-1}|i\rangle|f(i)\rangle \xrightarrow{\text{measurement}} \frac{1}{\sqrt{w}}\sum_{i=0}^{w-1}|q\rangle|f(j_i)\rangle.$$

Quantum adders







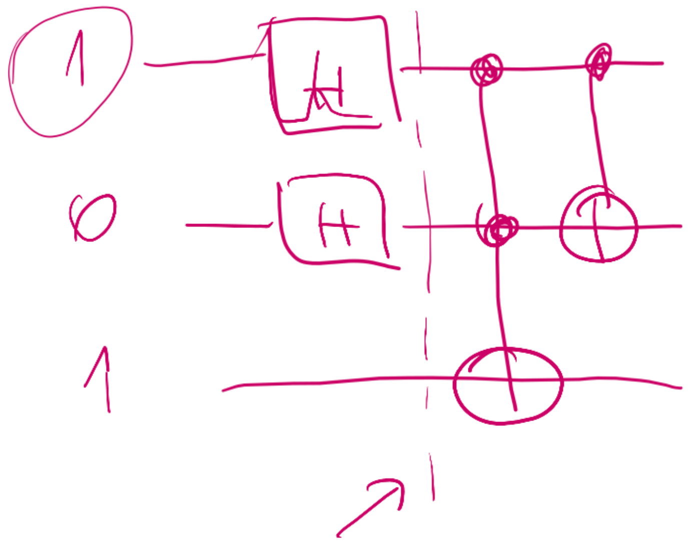
$$\frac{1}{2\sqrt{2}}\left(|0000\rangle + |0010\rangle + |0100\rangle + |0110\rangle + |1100\rangle + |1110\rangle + |1001\rangle + |1011\rangle\right)$$

$\cancel{|0110\rangle}$

$$\frac{1}{2\sqrt{2}}\left(|0000\rangle + |0010\rangle + |0100\rangle + |0110\rangle + |1000\rangle + |1010\rangle + |1100\rangle + |1110\rangle\right) \longmapsto$$

$$\longmapsto \frac{1}{2\sqrt{2}}\left(|0000\rangle + |0010\rangle + |0100\rangle + |0110\rangle + |1000\rangle + |1010\rangle + |1101\rangle + |1111\rangle\right)$$

$|1,2,3\rangle = |101\rangle$



$$\frac{1}{2}\left(|001\rangle + |011\rangle - |101\rangle - |111\rangle\right)$$

$$\frac{1}{2}\left(|001\rangle + |011\rangle - |101\rangle - |110\rangle\right)$$

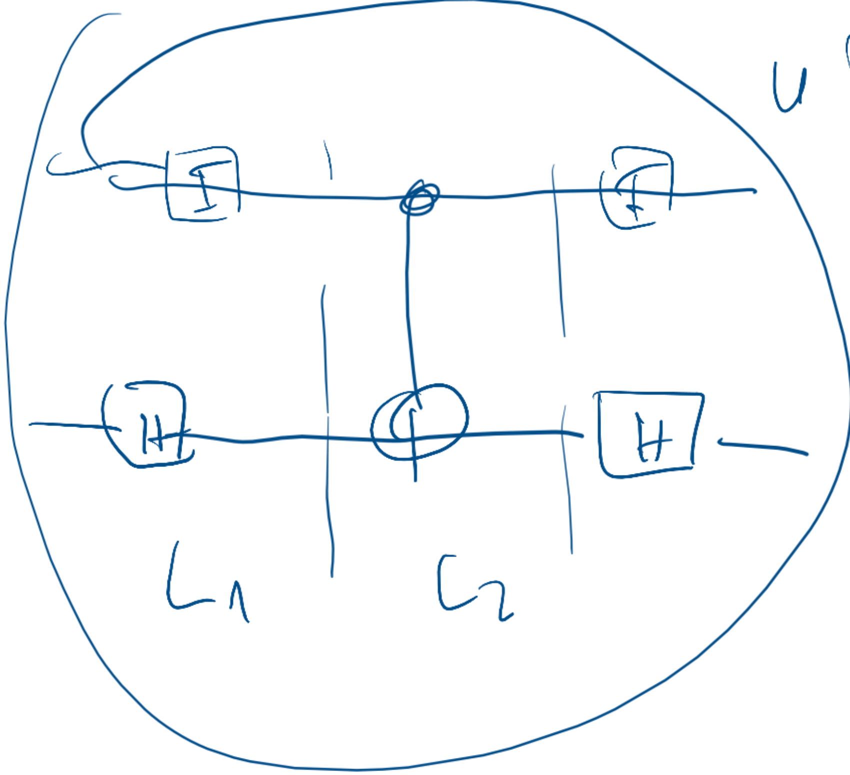
$$\boxed{\frac{1}{2}\left(|001\rangle + |011\rangle - |111\rangle - |100\rangle\right)}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\0\\1\\0\\-1\\0\\-1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}0\\1\\0\\1\\-1\\0\\0\\-1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

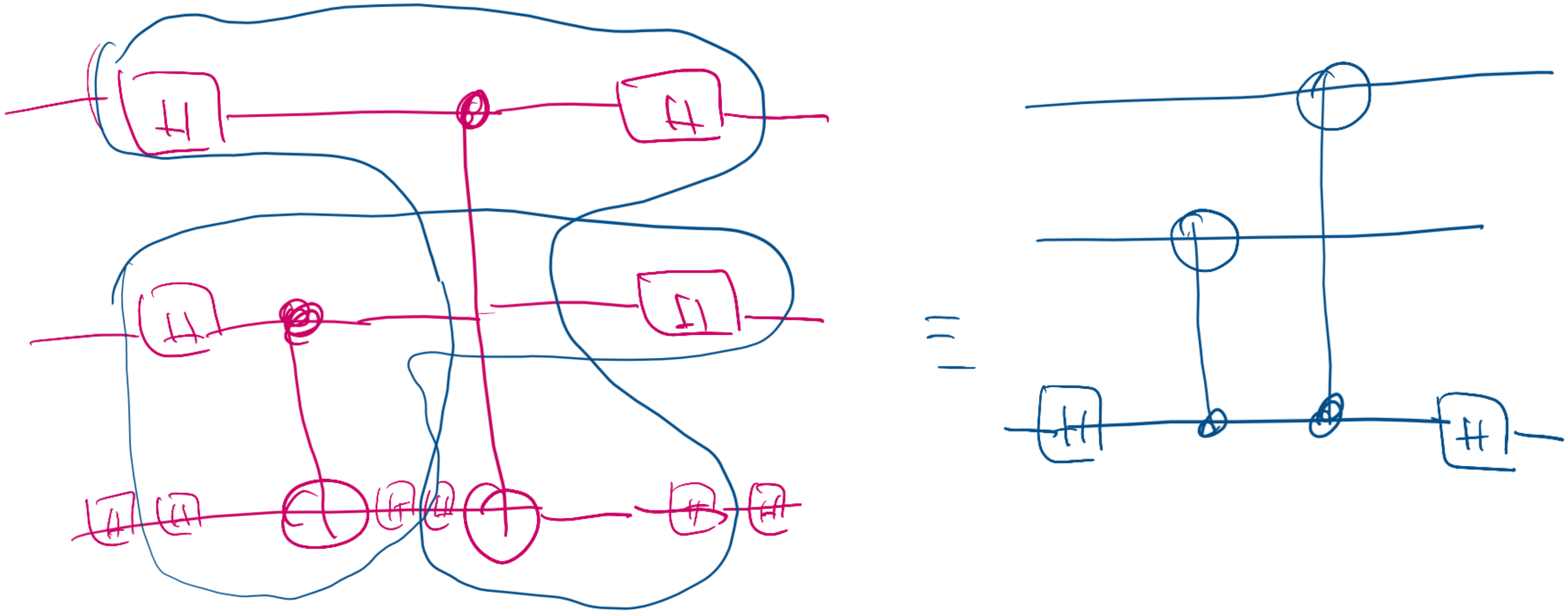


U ?

$$(I \otimes H)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}(I \otimes H)$$

$$I \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$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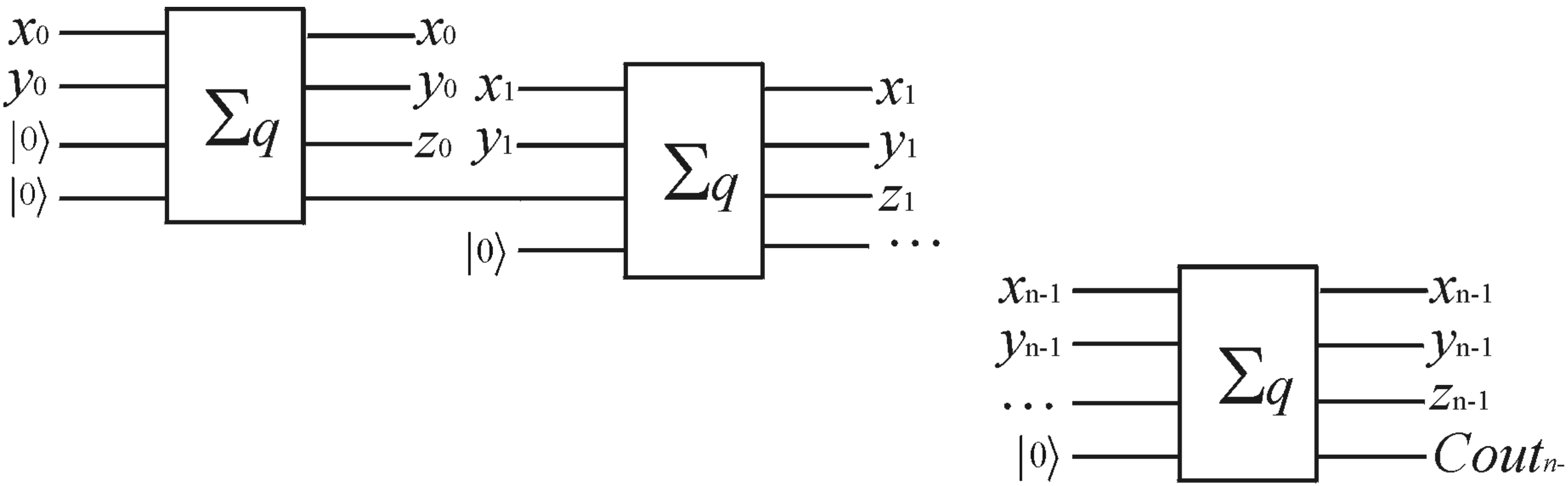
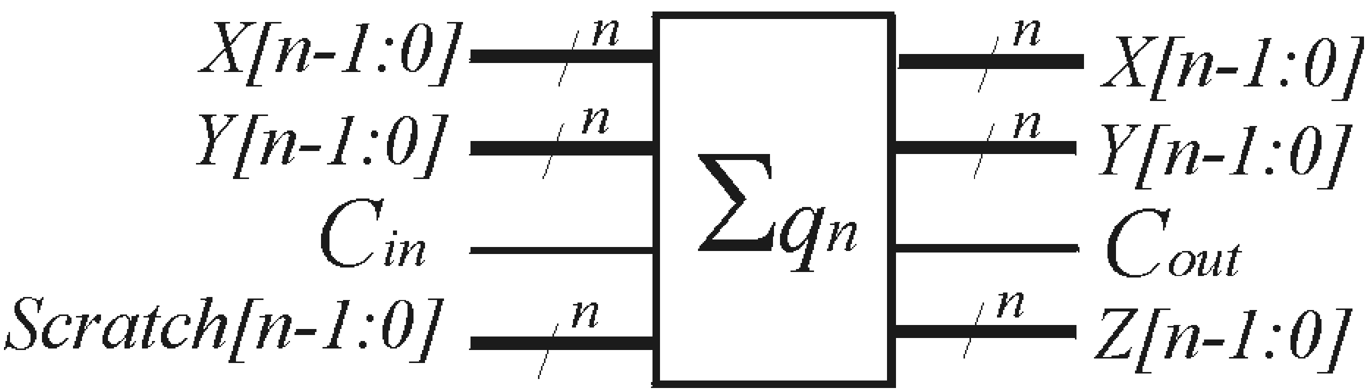
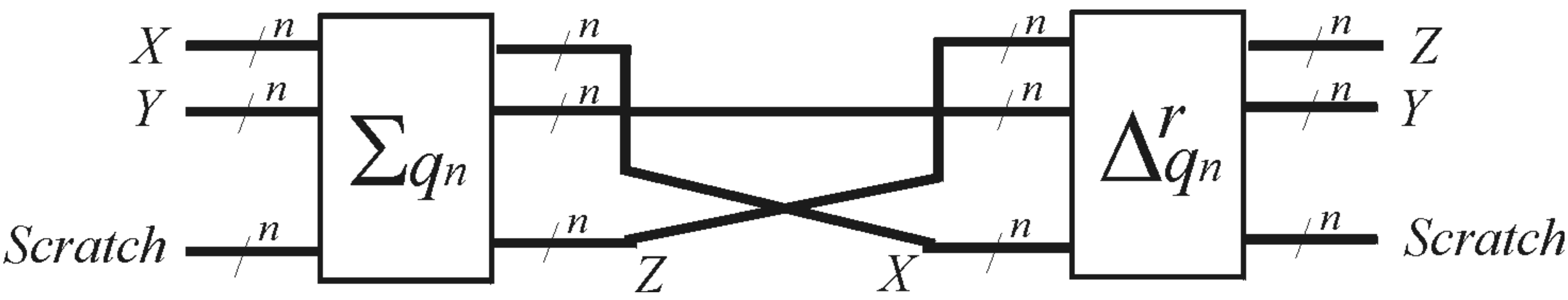
Quantum adders



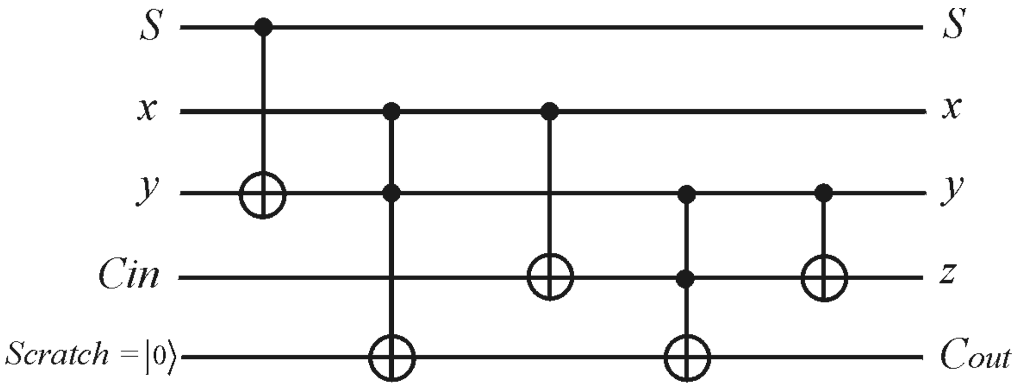


Quantum adders

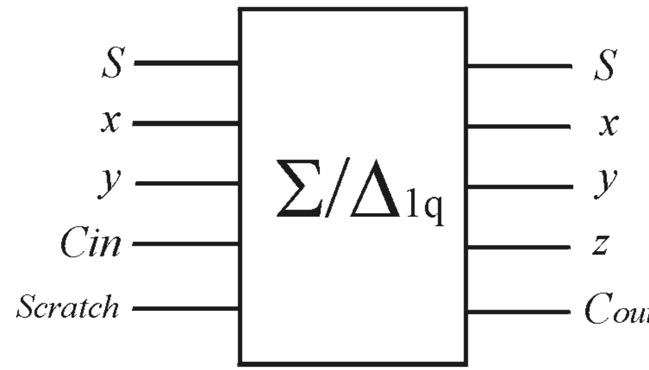
- Space saving



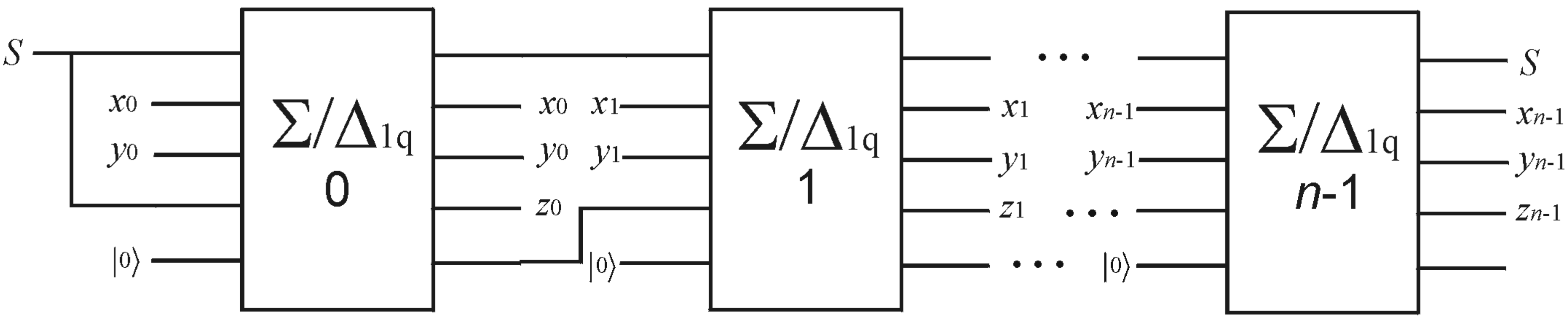
Subtracting devices



a)



b)



c)

Unitary Synthesis

- Gate Family

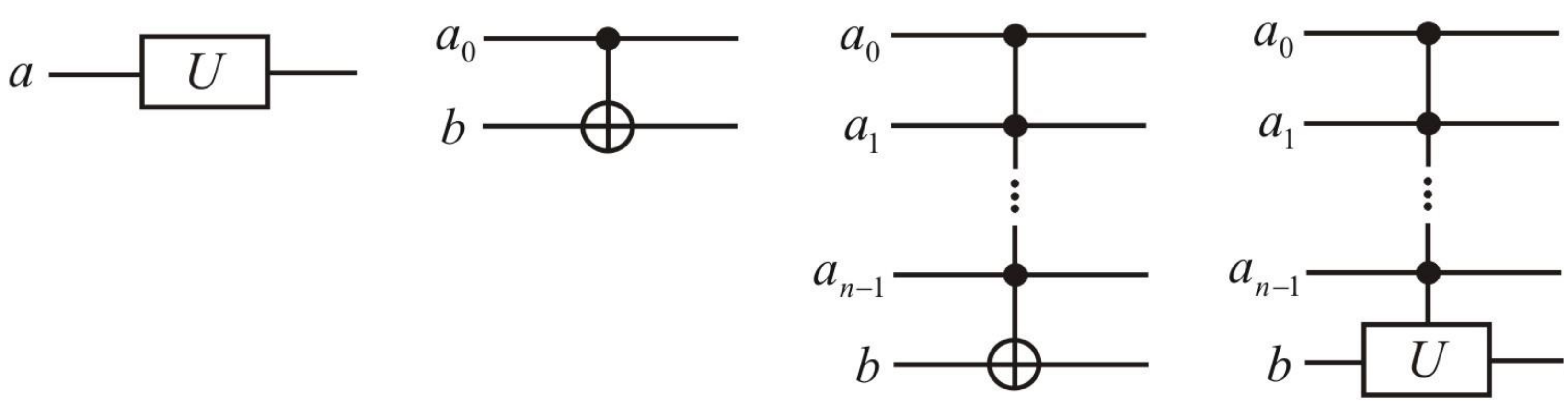
$$\wedge_n(U) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & u_{00} & u_{01} \\ & & & & u_{10} & u_{11} \end{pmatrix}$$

- Universal Set in QC

$$\left\{\wedge_1(\sigma_x), \wedge_0(U)\right\}$$

Unitary Level Design

- $\wedge_n(U)$ circuits



$\wedge_0(U)$ $\wedge_1(\sigma_x)$ $\wedge_n(\sigma_x)$ $\wedge_n(U)$

Unitary Level Design

- Pauli Matrices

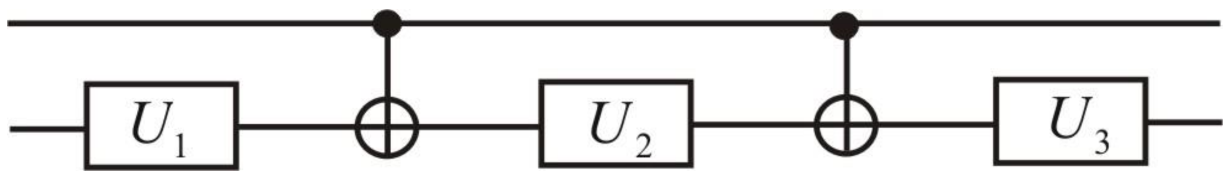
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Unitary Synthesis Rules

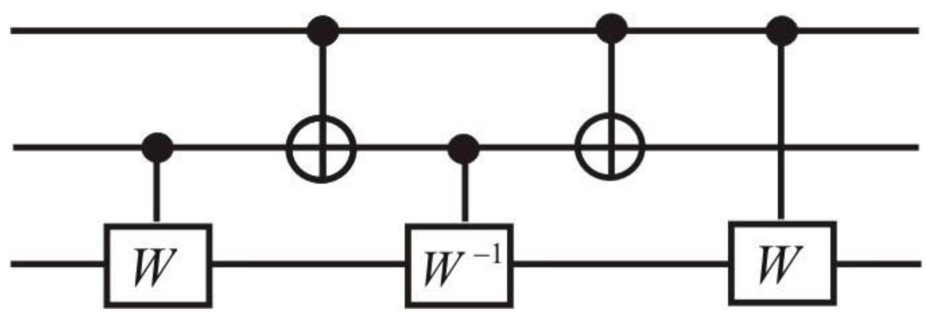
$\wedge_1(U)$

$U = U_1 \cdot U_2 \cdot U_3$



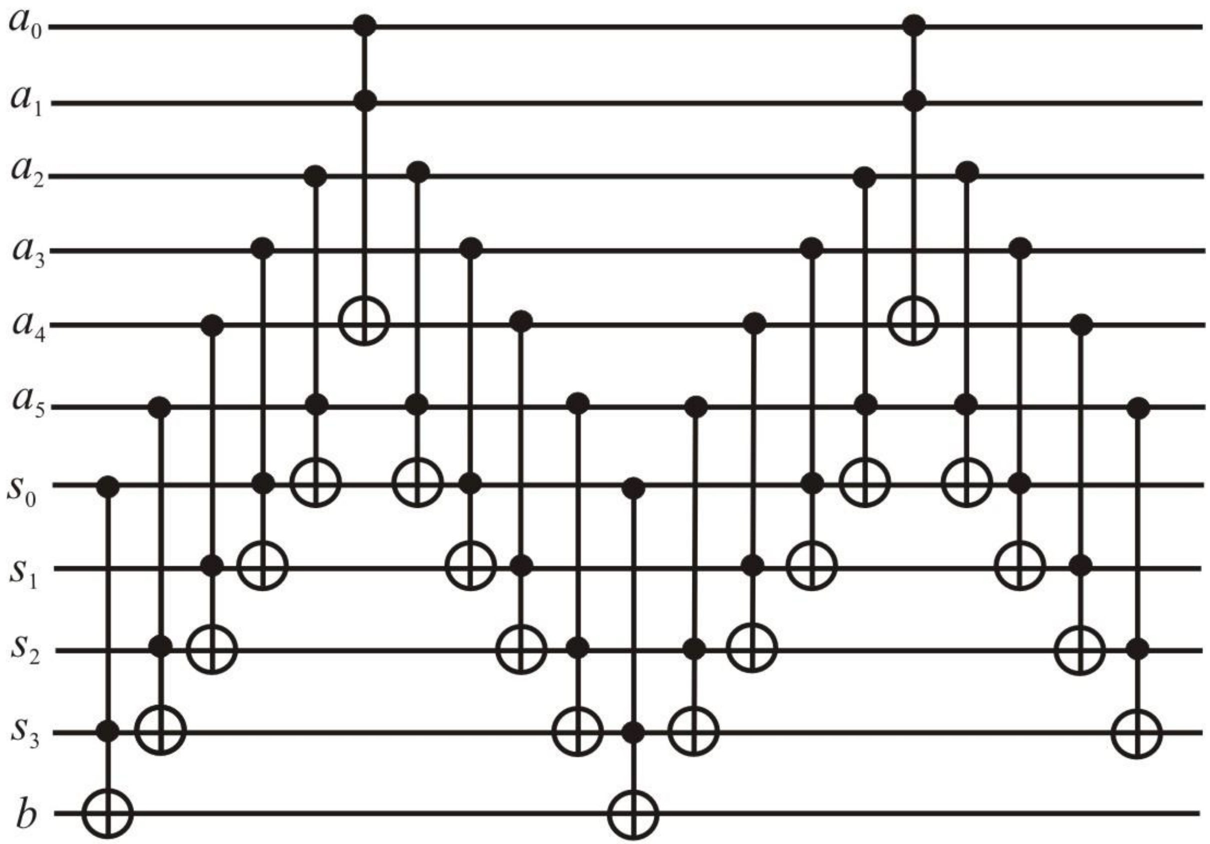
$\wedge_2(U)$

$U = W^2$



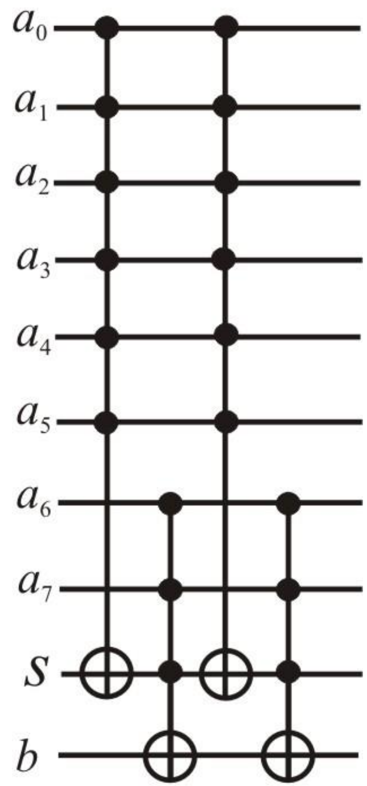
Unitary Synthesis Rules

$\wedge_n(\sigma_x)$ from $4(n-2)\times\wedge_2(\sigma_x)$ and $(n-2)\times\text{scratch}$



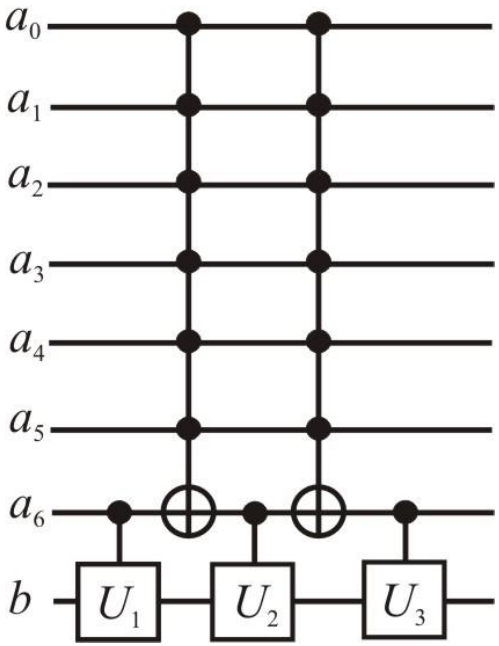
Unitary Synthesis Rules

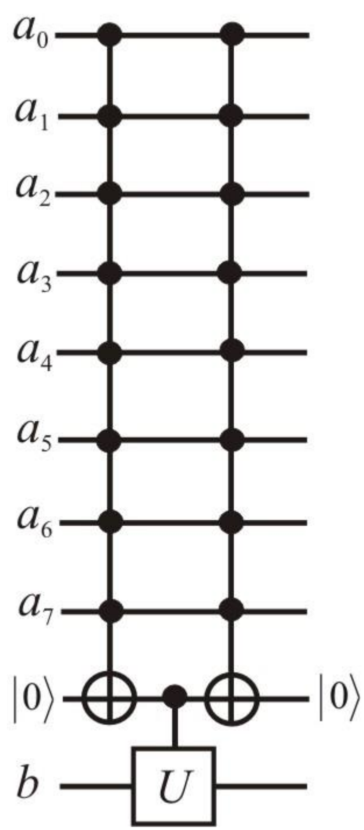
$\wedge_n(\sigma_x)$ from $2\times\wedge_m(\sigma_x)$ and $2\times\wedge_{n-m+1}(\sigma_x)$

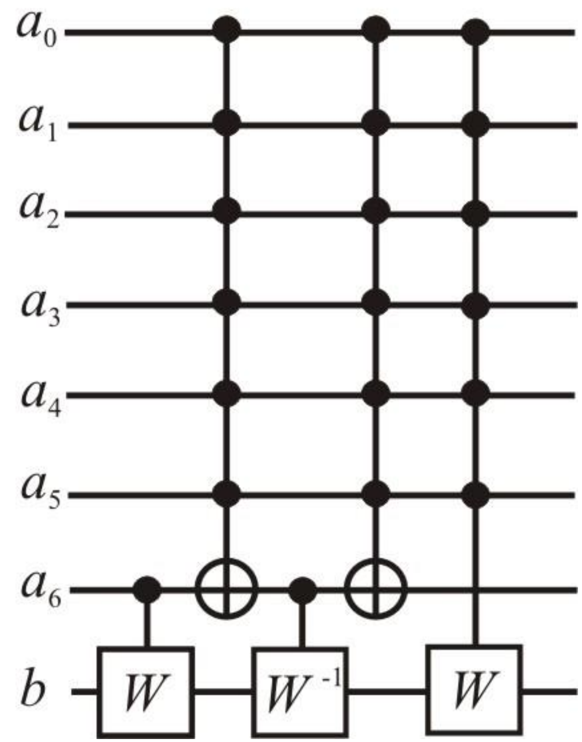


Unitary Synthesis Rules

$$\wedge_n(U)$$



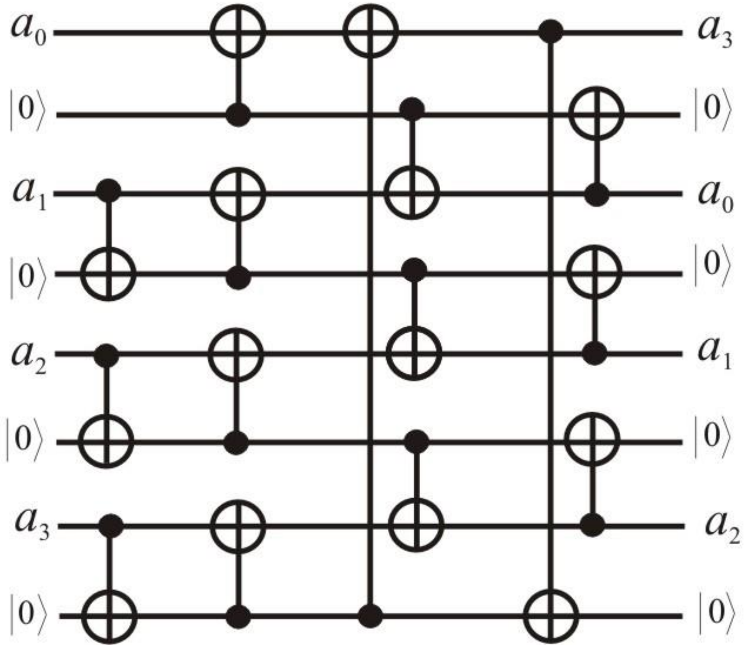




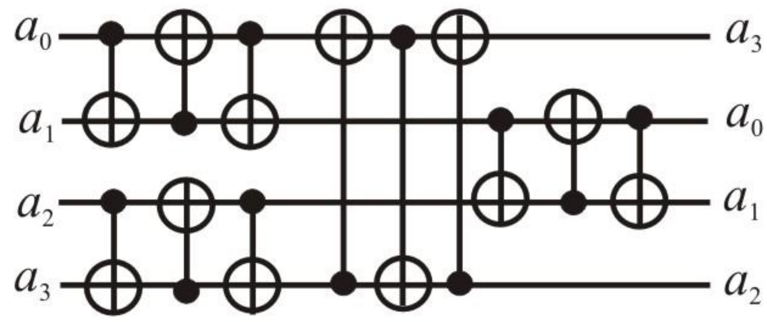
Unitary Synthesis Refinements

- Qubit permutation

With scratch

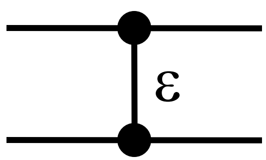


No scratch

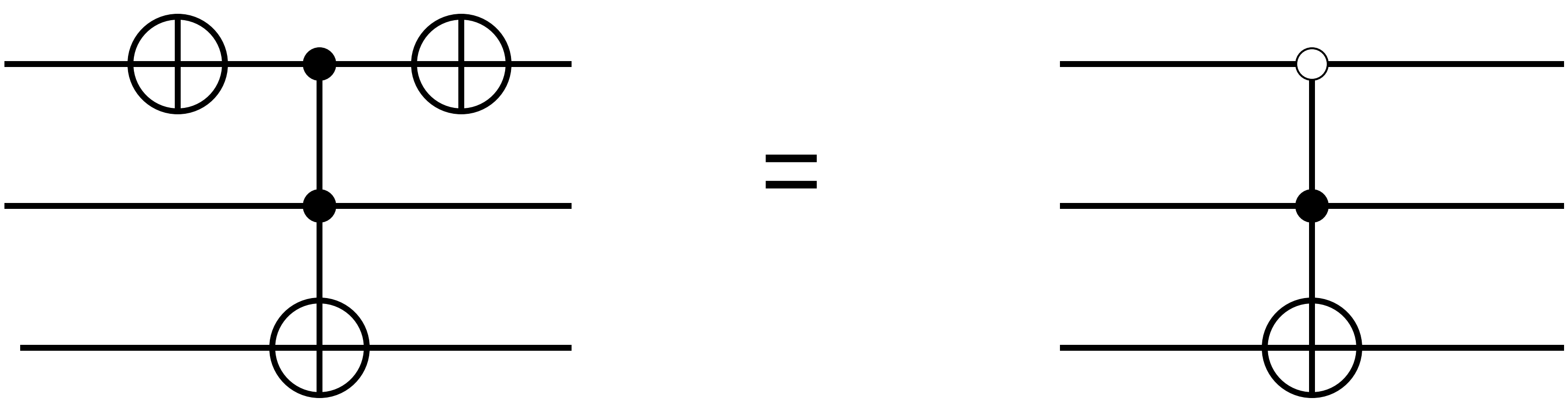


Other Gates

Symetric phase shift

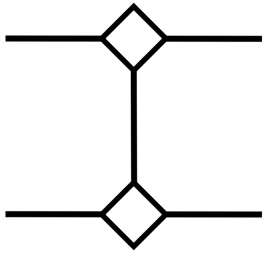


$$\wedge_1 \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varepsilon} \end{pmatrix} \qquad \varepsilon = \pi \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



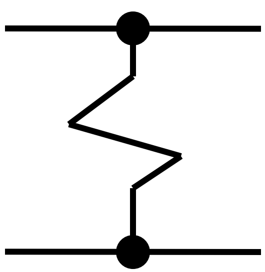
Other Gates

Arbitrary diagonal gate



$$\begin{bmatrix} e^{i\xi_0} & 0 & 0 & 0 \\ 0 & e^{i\xi_1} & 0 & 0 \\ 0 & 0 & e^{i\xi_2} & 0 \\ 0 & 0 & 0 & e^{i\xi_3} \end{bmatrix}$$

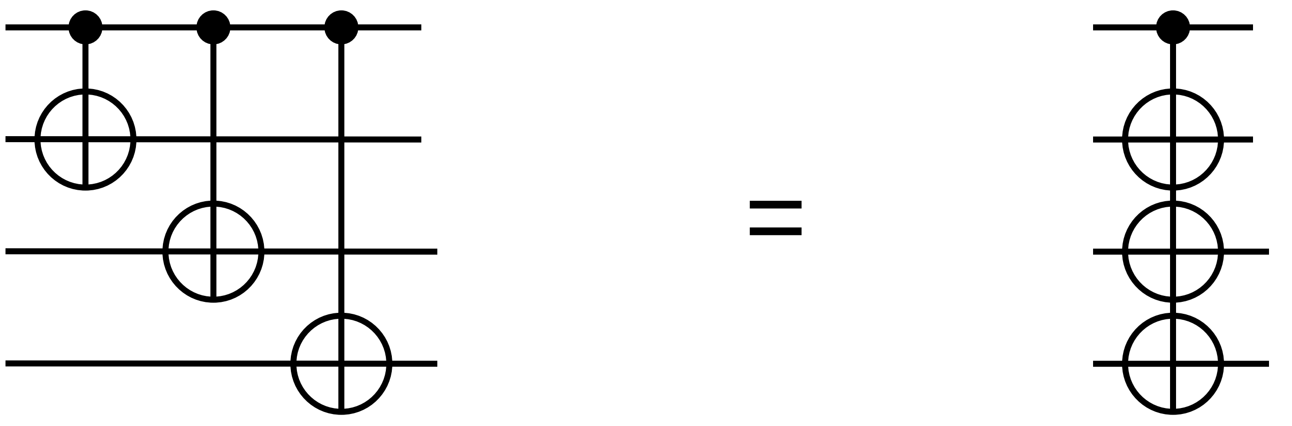
Wiggle gate



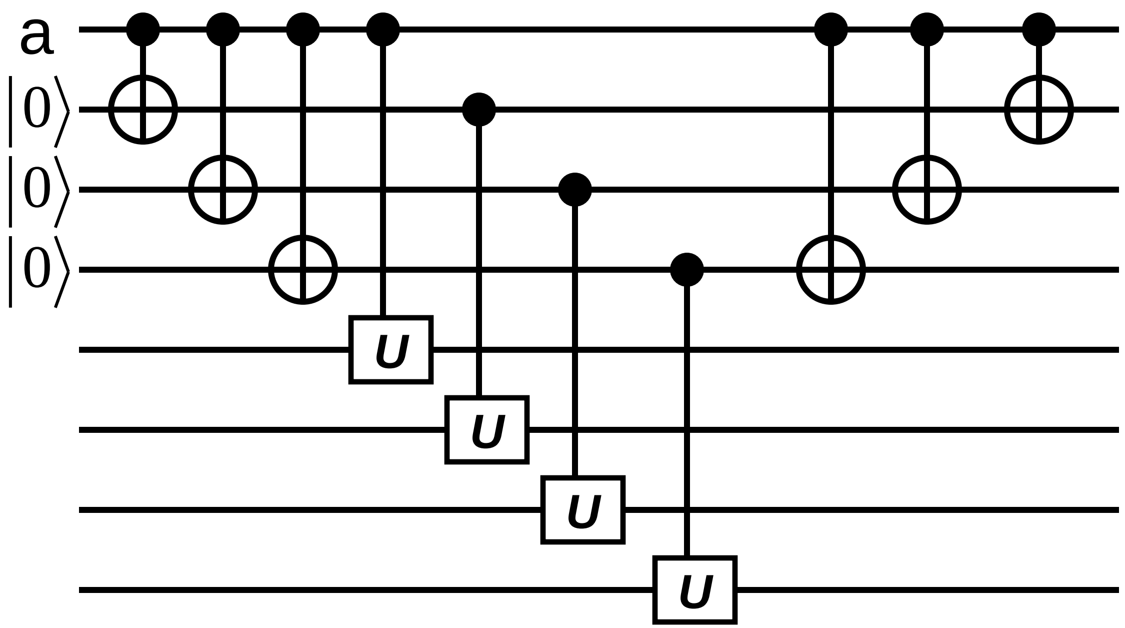
$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Other Gates

Simpler notation

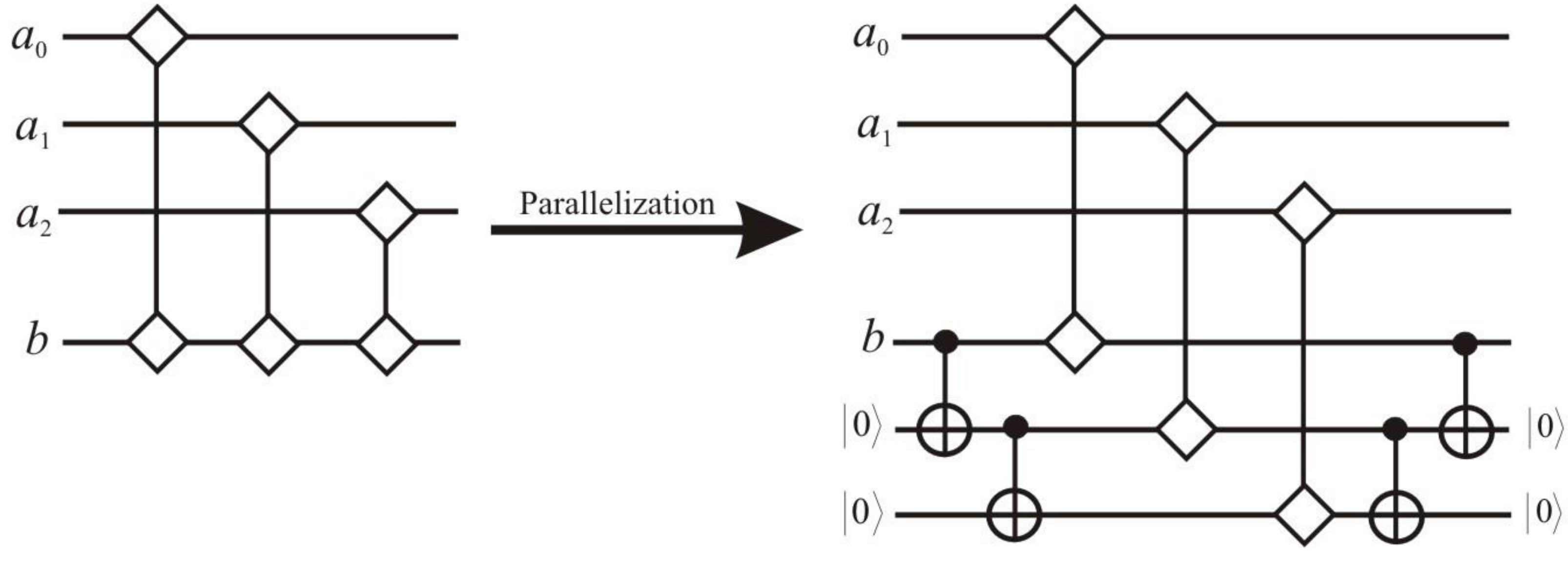


Multiple-qubit U



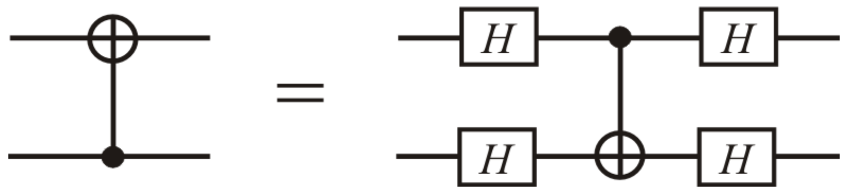
Unitary Synthesis Refinements

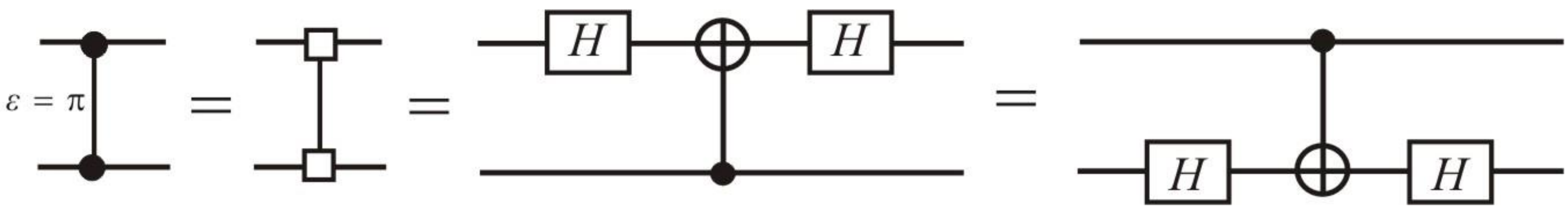
- Parallelization



Unitary Synthesis Refinements

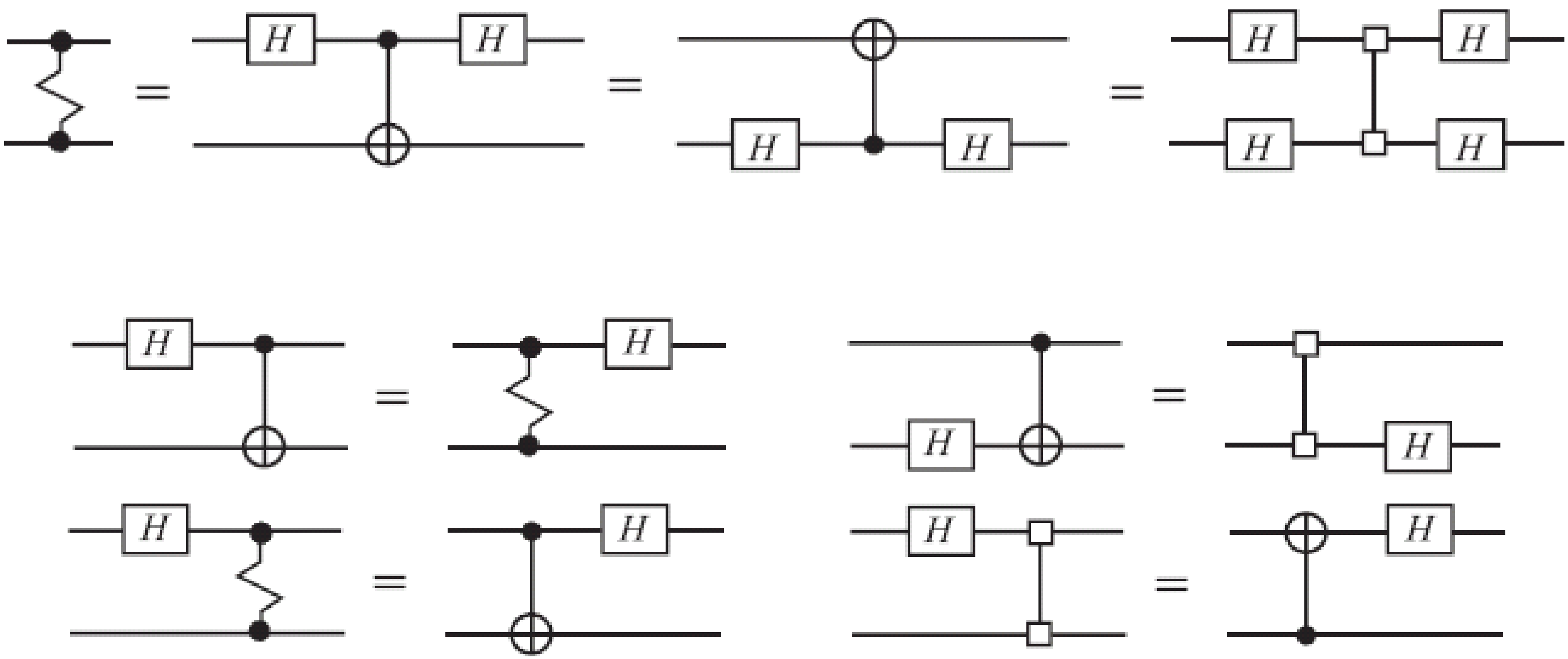
- Quantum circuit coding





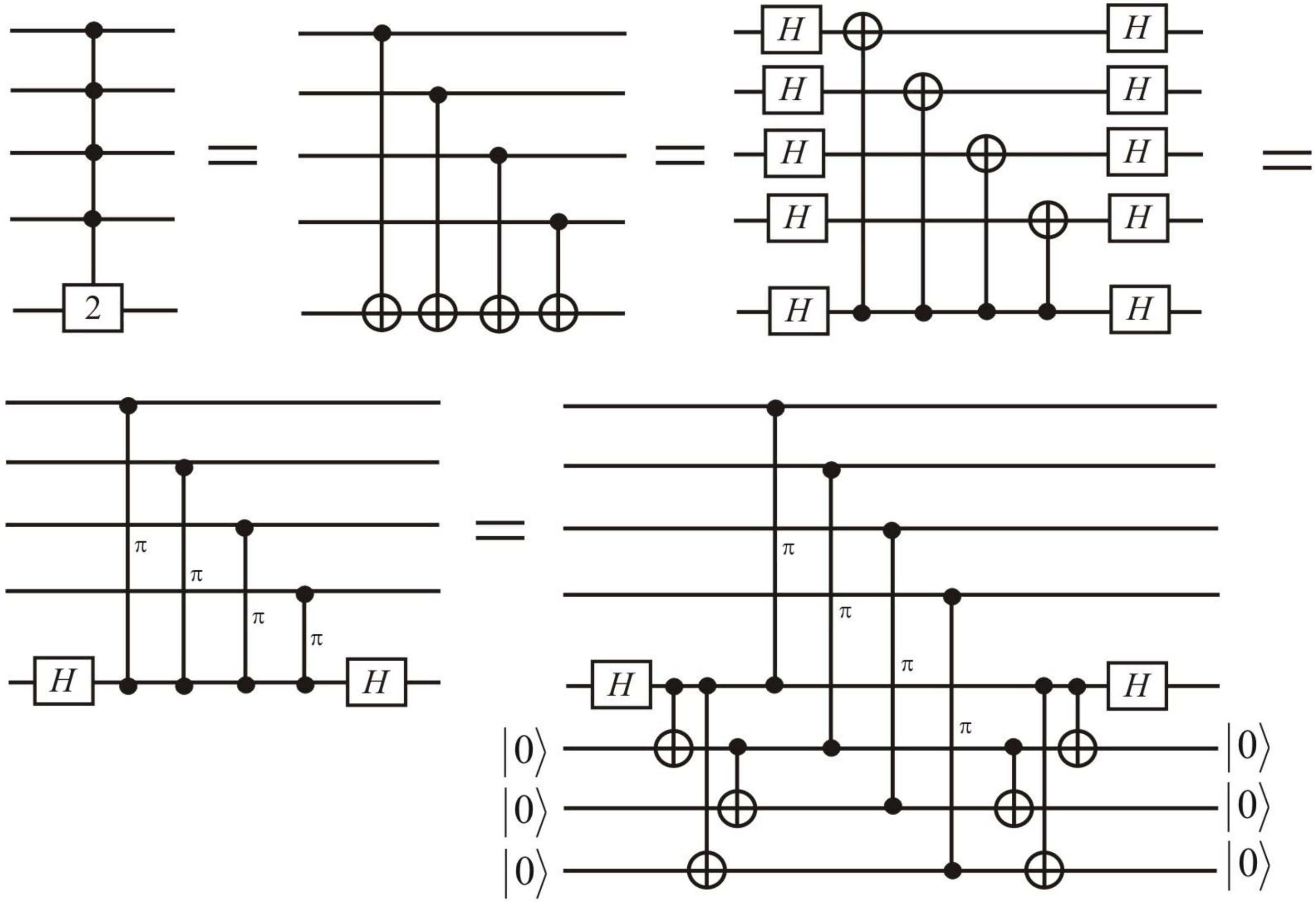
Unitary Synthesis Refinements

- Quantum circuit coding



Unitary Synthesis Refinements

- Example: MOD_2^4



Unitary Synthesis Refinements

- Example: Coppersmith's staircase circuit