

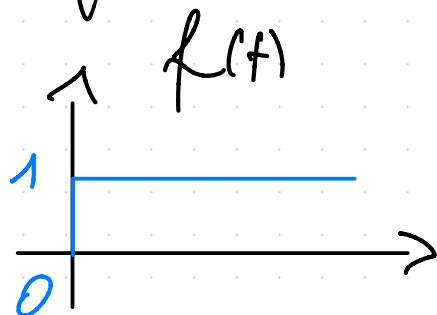
# Transformata Laplace

Tema : ex 1, 2, 3

Def:  $\mathcal{L} \{ f(t) \} = F(s) =$

$$= \int_0^{+\infty} f(t) e^{-st} dt$$

①  $f(t) = \sigma(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$



$$F(s) = \mathcal{L} \{ f(t) \} = \mathcal{L} \{ \sigma(t) \} =$$

$$= \int_0^{\infty} \sigma(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt =$$

$$= -\frac{1}{s} \int_0^{\infty} s e^{-st} dt = -\frac{1}{s} \int_0^{\infty} (e^{-st})' dt$$

$$= -\frac{1}{s} (e^{-st}) \Big|_0^{\infty} = -\frac{1}{s} \left( \lim_{t \rightarrow \infty} e^{-st} - \lim_{t \rightarrow 0} e^{-st} \right)$$

$$= -\frac{1}{s} (0 - 1) = \frac{1}{s}$$

$$\begin{aligned} \textcircled{2}) f(t) = e^{-at} \Rightarrow F(s) &= \mathcal{L}\{e^{-at}\} \\ &= \int_0^\infty e^{-at} \cdot e^{-st} dt \Rightarrow \int_0^\infty e^{-(s+a)t} dt = \\ &= -\frac{1}{(s+a)} \int_0^{+\infty} (e^{-(s+a)t})' dt = \left(-\frac{1}{s+a}\right) \begin{cases} \lim_{t \rightarrow \infty} e^{-(s+a)t} \\ \lim_{t \rightarrow 0} e^{-(s+a)t} \end{cases} \\ &= \frac{1}{s+a} \end{aligned}$$

$$\textcircled{3}) f(t) = f(t-m) \rightarrow$$

$$F(s) = \mathcal{L}\{f(t-m)\} = \int_0^\infty f(t-m)e^{-st} dt$$

Notez  $t-m = \bar{t} \Rightarrow t = \bar{t} + m$   
 $dt = d\bar{t}$

$$t \rightarrow \infty \Rightarrow \bar{t} \rightarrow \infty$$

$$t \rightarrow 0 \Rightarrow \bar{t} \rightarrow -m$$

$$= \int_{-m}^{\infty} f(\bar{t}) e^{-s(\bar{t}+m)} d\bar{t} =$$

$$= \int_{-m}^0 f(\bar{t}) e^{-s\bar{t}} \cdot e^{-sm} d\bar{t} + \int_0^{+\infty} \dots$$

$$= e^{-sm} \int_0^{+\infty} f(z) e^{-sz} dz$$

$F(s)$

$$\rightarrow \mathcal{L}\{f(t-m)\} = e^{-sm} F(s)$$


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$$(4) \quad \mathcal{L}\{f'(t)\} = \int_0^{+\infty} f'(t) e^{-st} dt$$

$$= \left( f(t) e^{-st} \right) \Big|_0^{+\infty} - \int_0^{+\infty} f(t) (-e^{-st}) dt$$

$$= \left( \lim_{t \rightarrow \infty} \frac{f(t)}{e^{-st}} - \lim_{t \rightarrow 0} \frac{f(t)}{e^{-st}} \right) - \int_0^{\infty} -sf(t) e^{-st} dt$$

$$\mathcal{L}(f'(t)) = s \cdot F(s) - f(0)$$


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$$(4') \quad \mathcal{L}(f''(t)) = \int_0^{+\infty} f''(t) e^{-st} dt =$$

$$= \int_0^{+\infty} \left( f'(t) \right)' e^{-st} dt = f'(t) e^{-st} \Big|_0^{\infty} -$$

$$- \int_0^{+\infty} f'(t) (-e^{-st})' =$$

$$= \left( \lim_{t \rightarrow \infty} \frac{f'(t)}{e^{-st}} - \lim_{t \rightarrow -\infty} \frac{f'(t)}{e^{-st}} \right) + s \underbrace{\int_0^{\infty} f'(t) e^{-st} dt}_{\{ \text{L} \{ f'(t) \} \}}$$

$$\begin{aligned} \mathcal{L} \{ f''(t) \} &= s[sF(s) - f(0)] - f'(0) \\ &= s^2 F(s) - s f(0) - f'(0) \end{aligned}$$

$$\mathcal{L} \{ f^{(n)}(t) \} = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$$

$$(5) \quad y'(t) + b y(t) = 1, \quad y(0) = y_0 = 0$$

$$\mathcal{L} \{ y'(t) + b y(t) \} = \mathcal{L} \{ 1 \}$$

$$\begin{aligned} \mathcal{L} \{ a f(t) + b g(t) \} &= a \mathcal{L} \{ f(t) \} \\ &\quad + b \mathcal{L} \{ g(t) \} \\ &= a F(s) + b G(s) \end{aligned}$$

$$= \mathcal{L} \{ y'(t) \} + b \mathcal{L} \{ y(t) \} = \frac{1}{s}$$

$$1 \quad Y(s) - y(0) + b \quad Y(s) = \frac{1}{s}$$

$$Y(s)(s+b) = \frac{1}{s} + \underbrace{y(0)}_0.$$

$$Y(s) = \frac{1}{s(s+b)} =$$

$$\mathcal{L}^{-1} \{ Y(s) \} = y(t)$$

$$\frac{1}{s(s+b)} = \frac{A}{s} + \frac{B}{s+b} \Rightarrow A(s+b) + Bs = 1$$

$$As + Ab + Bs = 1$$

$$\underbrace{(A+B)s + Ab}_1 = 1$$

$$\left\{ \begin{array}{l} A = -B \\ Ab = 1 \end{array} \right.$$

$$\left\{ \begin{array}{l} A = \frac{1}{b} \\ B = -\frac{1}{b} \end{array} \right.$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{b \cdot s} - \frac{1}{b(s+b)} \right\} =$$

$$= \frac{1}{b} \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s+b} \right\} = \frac{1}{b} \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} -$$

$$- \frac{1}{b} \mathcal{L}^{-1} \left\{ \frac{1}{s+b} \right\} \quad y(t) = \frac{1}{b} \left( 1 - e^{-bt} \right)$$

$$⑥ \quad y''(t) - 2y'(t) + y(t) = 3e^t,$$

$y(0) = 1$   
 $y'(0) = 1$

$$\mathcal{L}\{y''(t)\} - 2\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = 3\mathcal{L}\{e^t\}$$

$$s^2 Y(s) - s y(0) - y'(0) - 2[s Y(s) - y(0)] + y(s) = 3 \frac{1}{s-1}$$

$$s^2 Y(s) - s - 1 - 2s Y(s) + 2 + Y(s) = \frac{3}{s-1}$$

$$(s^2 - 2s + 1) Y(s) = \frac{3}{s-1} - \frac{s-1}{1+s} =$$

$$= \frac{3}{s-1} - \frac{(s-1)}{(s-1)^2} + \frac{s^2 - s}{s-1} = \frac{s^2 - 2s + 4}{s-1}$$

$$Y(s) = \frac{s^2 - 2s + 4}{(s-1)^3}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{s^2 - 2s + 4}{(s-1)^3} \right\}$$

$$\frac{s^2 - 2s + 3}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \Rightarrow$$

$$\Leftrightarrow A(s-1)^2 + B(s-1) + C = s^2 - 2s + 4$$

$$As^2 - 2As + A + Bs - B + C = s^2 - 2s + 4$$

$$\begin{cases} A = 1 \\ -2A + B = -2 \quad \rightarrow B = 0 \\ A - B + C = 4 \quad \rightarrow C = 3 \end{cases}$$

$$= \frac{1}{s-1} + \frac{3}{(s-1)^3}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s-1} + \frac{3}{(s-1)^3} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} + 3 \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^3} \right\}$$

$$= e^t + \frac{3}{2} t^2 e^t = e^{t + \left(1 + \frac{3}{2} t^2\right)}$$

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$$\mathcal{L}^{-1} \left\{ \frac{2a^3}{(s^2 + a^2)^2} \right\} = \sin(at) - at \cos(at)$$