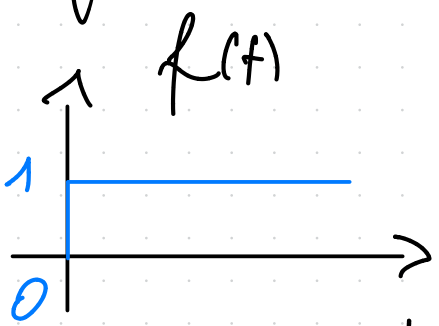
Transformata Laplace

Temă: ex 1, 2, 3

Def: $\mathcal{L}\{f(t)\} = F(s) =$

$$= \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

(1) $f(t) = \sigma(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$



$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sigma(t)\} =$$

$$= \int_0^{\infty} \sigma(t)\, e^{-st}\, dt = \int_0^{\infty} e^{-st}\, dt =$$

$$= -\frac{1}{s}\int_0^{\infty} s\, e^{-st}\, dt = -\frac{1}{s}\int_0^{\infty} \left(e^{-st}\right)' dt$$

$$= -\frac{1}{s}\left(e^{-st}\right)\Big|_0^{+\infty} = -\frac{1}{s}\Big(\underbrace{\lim_{t\to\infty} e^{-st}}_{0} - \underbrace{\lim_{t\to 0} e^{-st}}_{1}\Big)$$

$$= \frac{1}{s}$$

② $f(t) = e^{-at} \Rightarrow F(s) = \mathcal{L}\{e^{-at}\} =$

$$= \int_0^\infty e^{-at} \cdot e^{-st}\, dt \Rightarrow \int_0^\infty e^{-(s+a)t}\, dt =$$

$$= -\frac{1}{(s+a)} \int_0^{+\infty} \left(e^{-(s+a)t}\right)' dt = \left(-\frac{1}{s+a}\right)\Big(\underbrace{\lim_{t\to\infty} e^{-(s+a)t}}_{0} - \underbrace{\lim_{t\to 0} \ldots}_{1}$$

$$= \frac{1}{s+a}$$

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③ $f(t) = f(t-m) \longrightarrow$

$$F(s) = \mathcal{L}\{f(t-m)\} = \int_0^\infty f(t-m) e^{-st} dt$$

Notez $t - m = \tau \to t = \tau + m$

$$dt = d\tau$$

$$t \to \infty \Rightarrow \tau \to \infty$$

$$t \to 0 \Rightarrow \tau \to -m$$

$$= \int_{-m}^{\infty} f(\tau) e^{-s(\tau + m)} d\tau =$$

$$= \int_{-m}^{0} f(\tau) e^{-s\tau} \cdot e^{-sm} d\tau + \int_0^{+\infty} \ldots$$

$$\to 0$$

$$= e^{-sm} \underbrace{\int_0^{+\infty} f(\tau) e^{-s\tau} d\tau}_{F(s)}$$

$$\to \mathcal{L}\{f(t-m)\} = e^{-sm} F(s)$$

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(4) $\mathcal{L}\{f'(t)\} = \int_0^{+\infty} f'(t) e^{-st} dt$

$$= \left(f(t) e^{-st}\right)\Big|_0^{+\infty} - \int_0^{+\infty} f(t) \left(e^{-st}\right)' dt$$

$$= \Big(\underbrace{\lim_{t\to\infty} \frac{f(t)}{e^{st}}}_{0} - \lim_{t\to 0} \frac{f(t)}{e^{st}}\Big) - \int_0^{\infty} -s f(t) e^{-st} dt$$

$$\mathcal{L}(f'(t)) = s \cdot F(s) - f(0)$$

(4') $\mathcal{L}(f''(t)) = \int_0^{+\infty} f''(t) e^{-st} dt =$

$$= \int_0^{+\infty} (f'(t))' e^{-st} dt = f'(t) e^{-st}\Big|_0^{\infty} -$$

$$- \int_0^{+\infty} f'(t) \left(e^{-st}\right)' =$$

$$= \left(\underbrace{\lim_{t\to\infty} \frac{f'(t)}{e^{st}}}_{0} - \lim_{t\to 0} \frac{f'(t)}{e^{st}} \right) + s \underbrace{\int_0^\infty f'(t) e^{-st} dt}_{\mathcal{L}\{f'(t)\}}$$

$$\mathcal{L}\{f''(t)\} = s[sF(s) - f(0)] - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$$

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(5) $y'(t) + b y(t) = 1 \quad , \quad y(0) = y_0 = 0$

$$\mathcal{L}\{ y'(t) + b y(t)\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$$

$$= a F(s) + b G(s)$$

$$= \mathcal{L}\{y'(t)\} + b \mathcal{L}\{y(t)\} = \frac{1}{s}$$

$$sY(s) - y(0) + bY(s) = \frac{1}{s}$$

$$Y(s)(s+b) = \frac{1}{s} + \underbrace{y(0)}_{=0}$$

$$Y(s) = \frac{1}{s(s+b)} =$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t)$$

$$\frac{1}{s(s+b)} = \frac{A}{s} + \frac{B}{s+b} \Rightarrow A(s+b) + Bs = 1$$

$$As + Ab + Bs = 1$$

$$\underbrace{(A+B)}_{0}s + \underbrace{Ab}_{1} = 1$$

$$\begin{cases} A = -B \\ Ab = 1 \Rightarrow A = \frac{1}{b};\ B = -\frac{1}{b} \end{cases}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{b \cdot s} - \frac{1}{b(s+b)}\right\} =$$

$$= \frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+b}\right\} = \frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} -$$

$$- \frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}$$

$$y(t) = \frac{1}{b}\left(1 - e^{-bt}\right)$$

⑥ $y''(t) - 2y'(t) + y(t) = 3e^{t},$

$y(0) = 1$

$y'(0) = 1$

$$\mathcal{L}\{y''(t)\} - 2\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = 3\mathcal{L}\{e^{t}\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\left[s\,Y(0) - y(0)\right] + Y(s) = 3\,\frac{1}{s-1}$$

$$s^2 Y(s) - s - 1 - 2s\,Y(s) + 2 + Y(s) = \frac{3}{s-1}$$

$$(s^2 - 2s + 1)\,Y(s) = \frac{3}{s-1} - 1 + s =$$

$$= \frac{3}{s-1} - \frac{(s-1)}{(s-1)} + \frac{s^2 - s}{s-1} = \frac{s^2 - 2s + 4}{s-1}$$

$$Y(s) = \frac{s^2 - 2s + 4}{(s-1)^3}$$

$$y(s) = \mathcal{L}^{-1}\left\{\frac{s^2 - 2s + 4}{(s-1)^3}\right\}$$

$$\frac{s^2-2s+4}{(s-1)^3} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \Rightarrow$$

$$\Rightarrow A(s-1)^2 + B(s-1) + C = s^2 - 2s + 4$$

$$As^2 - 2As + A + Bs - B + C = s^2 - 2s + 4$$

$$\begin{cases} A = 1 \\ -2A + B = -2 & \rightarrow B = 0 \\ A - B + C = 4 & \rightarrow C = 3 \end{cases}$$

$$= \frac{1}{s-1} + \frac{3}{(s-1)^3}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{3}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= e^t + \frac{3}{2}t^2 e^t = e^t\left(1 + \frac{3}{2}t^2\right)$$

Tema: pag. 9: 1, 2, 3

9 3b) $\mathcal{L}^{-1}\left\{\frac{2a^3}{(s^2+a^2)^2}\right\} = \sin(at) - at\cos(at)$