1

1.a) i) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j a = a \cdot$

$\cdot \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j = a\left(\sum_{i=1}^{n} c_i\right)\left(\sum_{j=1}^{n} c_j\right) = a\left(\sum_{i=1}^{n} c_i\right)^2 \geq 0$

ii) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \langle x_i, x_j \rangle =$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j x_i x_j = \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{j=1}^{n} c_j x_j\right) =$

$= \left(\sum_{i=1}^{n} c_i x_i\right)^2 \geq 0$

$= \left(\sum_{i=1}^{n} c_i x_i\right)^2 \geq 0$

iii) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i) f(x_j) =$

$= \left(\sum_{i=1}^{n} c_i f(x_i)\right)\left(\sum_{i=1}^{n} c_j f(x_j)\right) = \left(\sum_{i=1}^{n} c_i f(x_i)\right)^2 \geq 0$

b) i) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \big(K_1(x_i, x_j)$

$+ K_2(x_i, x_j)\big) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_1(x_i, x_j) +$

$+ \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_2(x_i, x_j) \geq 0$

, since $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_1(x_i, x_j) \geq 0$ &

$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_2(x_i, x_j) \geq 0$

ii) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i c_j K_1(x_i, x_j)\right.$

$\left.\cdot K_2(x_i, x_j)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i c_j <\varphi_1(x_i), \varphi_1(x_j)>_{\mathcal{F}}\right.$

$\left.<\varphi_2(x_i), \varphi_2(x_j)>_{\mathcal{G}}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(c_i c_j \varphi_1(x_i)\varphi_1(x_j)\right.$

$\left.\cdot \varphi_2(x_i)\varphi_2(x_j)\right) = \left(\sum_{i=1}^{n} c_i \varphi_1(x_i)\varphi_2(x_j)\right)\cdot$


$\cdot K_2(x_i, x_j)) = \sum_{i=1}^{n}\sum_{j=1}^{n}(c_i c_j <\varphi_1(x_i), \varphi_1(x_j)>_{\mathcal{F}}$

$<\varphi_2(x_i), \varphi_2(x_j)>_{\mathcal{G}}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(c_i c_j \varphi_1(x_i)\varphi_1(x_j)$

$\cdot \varphi_2(x_i)\varphi_2(x_j)) = \left(\sum_{i=1}^{n} c_i \varphi_1(x_i)\varphi_2(x_j)\right)\cdot$


$\cdot\left(\sum_{j=1}^{n} c_j \varphi_1(x_j)\varphi_2(x_j)\right) = \left(\sum_{i=1}^{n} c_i \varphi_1(x_i)\varphi_2(x_i)\right)^2$

$\geq 0$

c) From 1a)ii) $<x, x'>$ is a Mercer kernel.
From 1b)i) $<x, x'> + \gamma$ is a Mercer kernel
since from 1a)i) $\gamma$ is a M.K.

From 1b) ii) $(<x,x'>+v)^d$ is a M. kernel since ③

$$(<x,x'>+v)^d = \prod_{i=1}^{d} (<x,x'>+v) \blacksquare$$

d) $\exp() > 0 \Rightarrow$ from 1a) i): $\exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right)$ is a M. kernel $\blacksquare$

# 1 Problem 2

a)
We check if $k(x,y) = \langle \varphi(x), \varphi(y) \rangle_{\mathcal{F}}$ holds:

$$\langle \varphi(x), \varphi(y) \rangle_{\mathcal{F}} = \langle \varphi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \varphi \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle_{\mathcal{F}} = \langle \begin{pmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} y_1^2 \\ \sqrt{2}y_1y_2 \\ y_2^2 \end{pmatrix} \rangle_{\mathcal{F}}$$

$$= x_1^2y_1^2 + 2x_1x_2y_1y_2 + x_2^2y_2^2 = (\sum_{i=1}^{2} x_iy_i) = \langle x, y \rangle^2 = k(x,y)$$

b)
i)

$$\varphi(\mathcal{C}) = \varphi \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix} = \begin{pmatrix} cos(\theta)^2 \\ \sqrt{2}cos(\theta)sin(\theta) \\ sin(\theta)^2 \end{pmatrix}$$

for $0 \leq \theta < 2\pi$
ii)

$$\varphi(\mathcal{A}) = \varphi \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} t^2 \\ \sqrt{2}ts \\ s^2 \end{pmatrix}$$

for s,t $\in \mathbb{R}$
c)
We know that $cos(\theta)^2 + sin(\theta)^2 = 1$, also that $2sin(\theta)cos(\theta) = sin(2\theta)$.
Now we define $v = cos(\theta)^2$ and $w = \frac{\sqrt{2}}{2}sin(2\theta)$

$$\begin{pmatrix} cos(\theta)^2 \\ \sqrt{2}cos(\theta)sin(\theta) \\ sin(\theta)^2 \end{pmatrix} = \begin{pmatrix} cos(\theta)^2 \\ \frac{\sqrt{2}}{2}sin(2\theta) \\ 1 - cos(\theta)^2 \end{pmatrix} = \begin{pmatrix} v \\ w \\ 1-v \end{pmatrix}$$

d)
It is easy to show that:

$$P = \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin \varphi(\mathcal{A})$$

We choose $P_x = t^2 = 1$, so it holds that t=1 and $P_z = s^2 = 1$, where follows that s=1.
In $\varphi(\mathcal{A})$ it should hold that $P_y = \sqrt{2}st$, but it is $\sqrt{2}st = \sqrt{2} \neq 0 = P_y$ instead.