1.a) i) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j a = a \cdot$

$\cdot \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j = a\left(\sum_{i=1}^{n} c_i\right)\left(\sum_{j=1}^{n} c_j\right) = a\left(\sum_{i=1}^{n} c_i\right)^2 \geq 0$

ii) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \langle x_i, x_j \rangle =$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j x_i x_j = \left(\sum_{i=1}^{n} c_i x_i\right)\left(\sum_{j=1}^{n} c_j x_j\right) =$

$= \left(\sum_{i=1}^{n} c_i x_i\right)^2 \geq 0$

$= \left(\sum_{i=1}^{n} c_i x_i\right)^2 \geq 0$

iii) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i) f(x_j)$

$= \left(\sum_{i=1}^{n} c_i f(x_i)\right)\left(\sum_{i=1}^{n} c_j f(x_j)\right) = \left(\sum_{i=1}^{n} c_i f(x_i)\right)^2 \geq 0$

b) i) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \big(K_1(x_i, x_j)$

$+ K_2(x_i, x_j)\big) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_1(x_i, x_j) +$

$+ \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_2(x_i, x_j) \geq 0$

(2)

, since $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_1(x_i, x_j) \geq 0$ 2

$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_2(x_i, x_j) \geq 0$

ii) $\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(c_i c_j K_1(x_i, x_j)$

$\cdot K_2(x_i, x_j)\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(c_i c_j < \varphi_1(x_i), \varphi_1(x_j) >_{\mathcal{F}}$

$< \varphi_2(x_i), \varphi_2(x_j) >_{\mathcal{F}} \Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(c_i c_j \varphi_1(x_i) \varphi_1(x_j)$

$\cdot \varphi_2(x_i) \varphi_2(x_j) \Big) = \Big(\sum_{i=1}^{n} c_i \varphi_1(x_i) \varphi_2(x_j) \Big) \cdot$

$\cdot K_2(x_i, x_j)\Big) = \sum_{i=1}\sum_{j=1} \Big(c_i c_j < \varphi_1(x_i), \varphi_1(x_j) >_{\mathcal{F}}$

$< \varphi_2(x_i), \varphi_2(x_j) >_{\mathcal{F}} \Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(c_i c_j \varphi_1(x_i) \varphi_1(x_j)$

$\cdot \varphi_2(x_i) \varphi_2(x_j) \Big) = \Big(\sum_{i=1}^{n} c_i \varphi_1(x_i) \varphi_2(x_j) \Big) \cdot$

$\cdot \Big(\sum_{j=1}^{n} c_j \varphi_1(x_j) \varphi_2(x_j) \Big) = \Big(\sum_{i=1}^{n} c_i \varphi_1(x_i) \varphi_2(x_i) \Big)^2$

≥ 0

c) From 1a)ii) $< x, x' >$ is a Mercer kernel.
From 1b)i) $< x, x' > + \gamma$ is a Mercer kernel
since from 1a)i) γ is a M.K.

From 1b)ii) $(\langle x, x' \rangle + v)^d$ is a M. kernel since

$$(\langle x, x' \rangle + v)^d = \prod_{i=1}^{d} (\langle x, x' \rangle + v) \quad \blacksquare$$

d) $\exp() > 0 \Rightarrow$ from 1a)i): $\exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right)$ is a M. kernel $\blacksquare$