Monte Carlo simulation of a QML estimation of an ARCH(1) model

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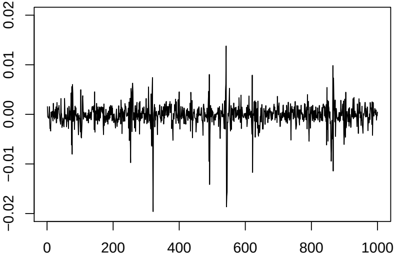
https://github.com/QuantLet/SFE_class_2017

Outline

1. ARCH(1) model
2. Maximum Likelihood estimation
3. Quasi Maximum Likelihood estimation
4. Monte Carlo Set up
5. Monte Carlo Simulation
6. Simulation Results
7. Further information

Simulated ARCH(1) process

Simulated ARCH(1) process with 1000 observations, $\alpha < 1$



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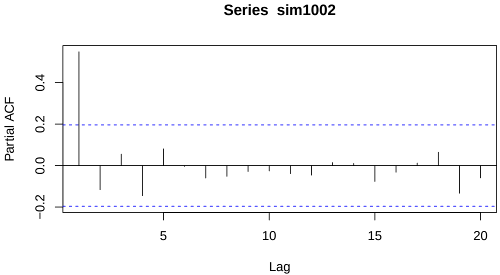
Estimation of ARCH(1) Models

ARCH(1) process ε_t has an AR(1) representation:

$$\varepsilon_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \eta_t, \eta_t = \sigma_t^2(Z_t^2 - 1)$$



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Maximum Likelihood Estimation (MLE)

- Assume pdf of $\varepsilon_t|\mathcal{F}_{t-1}$ normal

$$p(\varepsilon_t|\mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left\{-\frac{\varepsilon_t^2}{2\sigma_t^2}\right\}$$

- Conditional log-likelihood function $l^b(\omega, \alpha)$:

$$l^b(\omega, \alpha) = \sum_{t=2}^{n} \log p\left(\varepsilon_t|\mathcal{F}_{t-1}\right)$$

where $l^b = \log p\left(\varepsilon_n, \ldots, \varepsilon_2|\varepsilon_1\right)$

Maximum Likelihood Estimation (MLE)

- First order conditions

$$\sum_{t=2}^{n} \frac{\partial l_t^b}{\partial \omega} = 0$$

$$\sum_{t=q+1}^{n} \frac{\partial l_t^b}{\partial \alpha} = 0$$

- Under the conditions
  1. $\mathrm{E}[Z_t|\mathcal{F}_{t-1}] = 0$ and $\mathrm{E}[Z_t^2|\mathcal{F}_{t-1}] = 1$
  2. $\mathrm{E}[\log(\alpha Z_t^2)|\mathcal{F}_{t-1}] < 0$(strong stationarity)

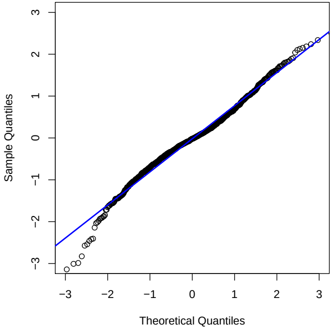
  ML estimators are consistent

Quasi Maximum likelihood Estimation (QMLE)

- Quasi ML (QML): In the case Z_t is not normally distributed, $\hat{\theta}$ is consistent, and asymptotically normally distrubuted.
- However, the estimates are not efficient and "the efficiency loss can be marked under asymmetric ... distributions" (Bollerslev and Wooldridge (1992), p. 166). The robust variance-covariance matrix of the estimates equals the (Eicker-White) sandwich estimator, i.e.

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} N(0, J^{-1} I J^{-1}) \tag{1}$$

QQ Plot of standardized residuals



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Requirements

- Control over the random variables and parameters
- Reproducibility
- Efficiency
- Documented steps

Monte Carlo Set up

- R-Package: fGarch
- Testing for different RNG: Mersenne Twister,Knuth-TAOCP-2002, Wichmann-Hill with seed=123.
- Custom function called 'simulation' which takes as input size of the dataset and returns the string of:
    1. averaged value of estimated $\hat{\alpha}$
    2. standard deviation of estimated $\hat{\alpha}$ from true parameter $\alpha = 0.9$
    3. number of $\hat{\alpha}$ which are larger or equal to 1 (evidence of the stationarity violation)

over $k = 1000$ replications for datasets of size $n = 100, 250, 500, 1000$.

Monte Carlo Set up continued

Functions used:

- fitGarch() with specified conditional distribution to be calculated with the Quasi Maximum Likelihood Estimation;
- garchSim() for simulation of ARCH(1) models.

Results of Monte Carlo Simulation

n	$k^{-1}\sum_{j=1}^{k}\hat{\alpha}_j$	$\sqrt{k^{-1}\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$	$\#(\alpha_j \geq 1)$
100	0.824	0.209	34.5%
250	0.868	0.130	23.2%
500	0.880	0.095	17.3%
1000	0.890	0.073	7.4%

Table 1: RNG: Mersenne Twister

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Results with other RNG

n	$k^{-1}\sum_{j=1}^{k}\hat{\alpha}_j$	$\sqrt{k^{-1}\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$	$\#(\alpha_j \geq 1)$
100	0.820	0.207	30.1%
250	0.869	0.128	24.1%
500	0.883	0.097	17.2%
1000	0.890	0.072	9.3%

Table 2: RNG: Knuth-TAOCP-2002

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Results with other RNG

n	$k^{-1}\sum_{j=1}^{k}\hat{\alpha}_j$	$\sqrt{k^{-1}\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$	$\#(\alpha_j \geq 1)$
100	0.822	0.207	31.1%
250	0.868	0.132	24.5%
500	0.884	0.096	17.9%
1000	0.896	0.070	9.7%

Table 3: RNG: Wichmann-Hill

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For Further Reading

James E Gentle.
Elements of computational statistics.
QA276, 4:G455.

D Wuertz, Y Chalabi, and M Miklovic.
fgarch: Rmetrics-autoregressive conditional heteroskedastic modelling, r package version 290.76, 2008

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Statistics of financial markets
Forth Edition, Springer-Verlag Berlin Heidelberg, 2015.