# Monte Carlo simulation of a QML estimation of an ARCH(1) model

Paulina Kurowska
Bingling Wang

Ladislaus von Bortkiewicz Chair of Statistics
Humboldt–Universität zu Berlin
https://github.com/paulinakurowska/SFM$_project.git$

# Outline

- ARCH(1) model
- Maximum Likelihood estimation
- Quasi Maximum Likelihood estimation
- Monte Carlo Set up
- Monte Carlo Simulation
- simulation Results
- Furthur information

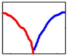
# Definition: ARCH(1) Model

- The process $\varepsilon_t$, $t \in \mathbb{Z}$, is ARCH(1), if $E[\varepsilon_t|\mathcal{F}_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2$$

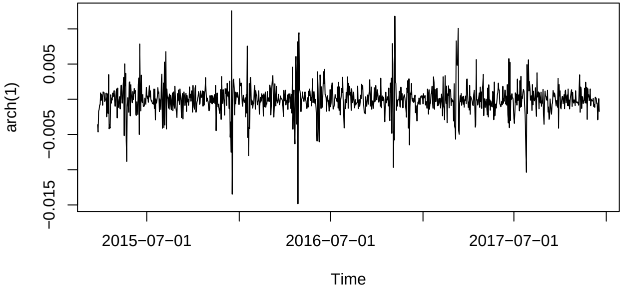
  with $\omega > 0, \alpha > 0$ and
- $Var(\varepsilon_t|\ F_{t-1}) = \sigma_t^2$ and $Z_t = \varepsilon_t/\sigma_t$ is i.i.d. (strong ARCH)
- $Var(\varepsilon_t|\ F_{t-1}) = \sigma_t^2$ (semi-strong ARCH)
- $\mathcal{P}(\varepsilon_t^2|1, \varepsilon_{t-1}, \varepsilon_{t-2}, ..., \varepsilon_{t-1}^2, \varepsilon_{t-2}^2, ...) = \sigma_t^2$(weak ARCH), where $\mathcal{P}$ is the best linear projection





# Properties of ARCH(1) process

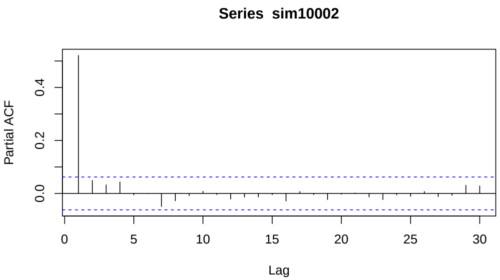
Necessary and sufficient condition for weak stationarity of a semi strong ARCH(1) process is$\alpha < 1$



# Estimation of ARCH(1) Models

- ARCH(1) process $\varepsilon_t$ has an AR(1) representation:

$$\varepsilon_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \eta_t$$
$$\eta_t = \sigma_t^2(Z_t^2 - 1)$$



# Maximum Likelihood Estimation (MLE)

- Assume pdf of $\varepsilon_t | \mathcal{F}_{t-1}$ normal

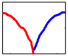
$$p(\varepsilon_t|\mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma_t} exp\left\{ -\frac{1}{2}\frac{\varepsilon_t^2}{\sigma_t^2} \right\}$$

- Log-likelihood function $l(\omega, \alpha)$ :

$$l(\omega,\alpha) = \sum_{t=2}^{n} l_t(\omega,\alpha) + logp_\varepsilon(\varepsilon_1)$$

where$l(\omega, \alpha)$=log $p(\varepsilon_t|\mathcal{F}_{t-1})$





# Maximum Likelihood Estimation (MLE)

- First order conditions

$$\sum_{t=2}^{n} \frac{\partial l_t^b}{\partial \omega} = 0 and \sum_{t=q+1}^{n} \frac{\partial l_t^b}{\partial \alpha} = 0$$

- Under the conditions
  1. $E[Z_t|\mathcal{F}_{t-1}] = 0$ and $E[Z_t^2|\mathcal{F}_{t-1}] = 1$
  2. $E[log(\alpha Z_t^2)|\mathcal{F}_{t-1}] < 0$(strong stationarity)

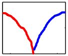
  ML estimators are consistent

# Quasi Maximum likelihood Estimation (QMLE)

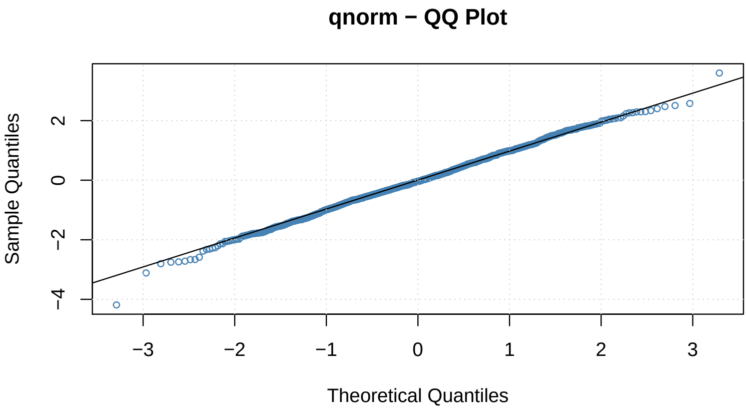
- If $Z_t$ is not normally distributed, $\hat{\theta}$ is consistent,and asymptotically normally distrubuted, but not efficient. In this case the method is interpreted as quasi ML (QML).
- The robust variance-covariance matrix of the estimates equals the (Eicker-White) sandwich estimator, i.e.

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} N(0, \mathcal{H}^{-1} I \mathcal{H}^{-1}) \tag{1}$$





# QQ Plot of standardized residuals



# Requirements

- Control over the random variables and parameters (selection of pdf for random numbers)
- Reproducibility
- Efficience
- Documented steps
  Source: Elements of Computational Statistics, Gentle 2002

# Monte Carlo Set up

R-Package: fGarch
Testing for different RNG: Marsenne Twister,Knuth-TAOCP-2002, Wichmann-Hill with seed=123.
We define the function called 'simulation' which takes as input size of the dataset and returns the string of:

- averaged value of estimated $\hat{\alpha}$
- standard deviation of estimated $\hat{\alpha}$ from true parameter $\alpha = 0.9$
- number of $\hat{\alpha}$ which are larger or equal to 1 (evidence of the stationarity violation)

over k=1000 replications for datasets of size n=100,250,500,1000.

# Monte Carlo Set up continued

For building that simulation we use the function fitGarch() from fGarch package with specified conditional distribution to be calculated with the Quasi Maximum Likelihood Estimation, the simulated ARCH(1) models are derived with the help of the function garchSim().

# Monte Carlo Set up continued

QMLE assumes normal distribution and uses robust standard errors for inference. Bollerslev and Wooldridge (1992) proved that if the mean and the volatility equations are correctly specified, the QML estimates are consistent and asymptotically normally distributed. However, the estimates are not efficient.

# Results of Monte Carlo Simulation

RNG: Marsenne Twister

| n | $k^{-1}\sum_{j=1}^{k}\hat{\alpha}_j$ | $\sqrt{k^{-1}\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$ | $\#(\alpha_j \geq 1)$ |
|---|---|---|---|
| 100 | 0.824 | 0.209 | 34.5% |
| 250 | 0.868 | 0.129 | 23.2% |
| 500 | 0.880 | 0.095 | 17.3% |
| 1000 | 0.890 | 0.073 | 7.4% |

# Results with other RNG

RNG: Knuth-TAOCP-2002

| n | $k^{-}1\sum_{j=1}^{k}\hat{\alpha}_j$ | $\sqrt{k^{-}1\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$ | $\#(\alpha_j \geq 1)$ |
|---|---|---|---|
| 100 | 0.806 | 0.219 | 27.6% |
| 250 | 0.852 | 0.133 | 19.5% |
| 500 | 0.886 | 0.095 | 17.5% |
| 1000 | 0.893 | 0.073 | 9.2% |

RNG: Wichmann-Hill

| n | $k^{-}1\sum_{j=1}^{k}\hat{\alpha}_j$ | $\sqrt{k^{-}1\sum_{j=1}^{k}(\hat{\alpha}_j-\alpha)^2}$ | $\#(\alpha_j \geq 1)$ |
|---|---|---|---|
| 100 | 0.809 | 0.216 | 29.5% |
| 250 | 0.861 | 0.134 | 24.4% |
| 500 | 0.879 | 0.099 | 16.2% |
| 1000 | 0.891 | 0.073 | 9% |

# For Further Reading


Gentle, James E.
*Elements of Computational Statistics*

R-Package
*fGarch, Nov 2017*
available on

Bollersev, T. , Wooldridge, M.
*Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances*