MARKOV PROCESS

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space
$(E, \mathcal{E})$ measurable space (E typically is finite / countable / $\mathbb{R}$ / $\mathbb{R}^n$ / $A \subseteq \mathbb{R}^n$)

Def. **STOCHASTIC PROCESS**: A stochastic process with values in E is a collection of random variables $X_t : \Omega \to E$ and it's denoted by $(X_t)_t$

Def. **MARKOV PROCESS**: $(X_t)_t$ is a Markov process if $(X_t)_t$ is a stochastic process with Markov property:

$$\mathbb{P}(X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m, \dots, X_{t_1} \in E_1) = \mathbb{P}(X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m)$$

$$\forall t_1 < t_2 < \dots < t_m < t_{m+1}, \quad \forall E_1, \dots, E_{m+1} \in \mathcal{E}, \quad \forall m \in \mathbb{N}$$

t_{m+1} = future
t_m = present
$t_1, \dots, t_{m-1}$ = past

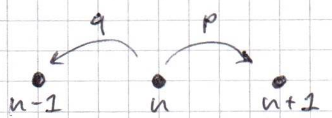
Example: Discrete Markov Chain

E finite / countable, $\mathcal{E} = \mathcal{P}(E)$, $t \in \mathbb{N} = \{0, 1, 2, \dots\}$

Markov property $\iff$ $\mathbb{P}(X_{t_{m+1}} = e_{m+1} \mid X_{t_m} = e_m, \dots, X_{t_1} = e_1) =$
$= \mathbb{P}(X_{t_{m+1}} = e_{m+1} \mid X_{t_m} = e_m)$

(with discrete Markov chains we can replace sets with elements of the set)

Example: Discrete time, $\mathbb{Z}$-valued Markov chain: Random walk on $\mathbb{Z}$



Starting from n, in a unit of time we jump to:
- $n+1$ with probability p
- $n-1$ with probability q

Defining the realization of $(Y_k)_{k \geq 1}$ independent random variables with values $\{-1, 1\}$, we can define the position at time n:

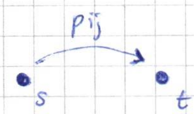
$$X_n = x_0 + \sum_{k=1}^{n} Y_k \qquad x_0 \in \mathbb{Z}$$

$(X_n)_{n \geq 1}$ is a Markov chain.

(Discrete time $t \in \mathbb{N} = \{0, 1, 2, \dots\}$ $\neq$ discrete values (state space))

Def. **TRANSITION PROBABILITIES**:

$$p_{ij}(t,s) := \mathbb{P}(X_t = j \mid X_s = i) \qquad s < t$$



Def. **TIME HOMOGENEOUS PROCESS**: The process $(X_t)_t$ is time homogeneous if $p_{ij}(t,s)$ depends only on $t-s$. In this case we can write $p_{ij}(t-s)$ or $p_{ij}^{(t-s)}$. For $t-s = 1$ we simply write p_{ij} or $p_{ij}^{(1)}$ and we call $(p_{ij})_{i,j \in E}$ the TRANSITION MATRIX.

Lemma: $\mathbb{P}(X_{t+2} = j \mid X_t = i) = p_{ij}^{(2)}$ power 2 of the step transition matrix

$$\phantom{\mathbb{P}(X_{t+2} = j \mid X_t = i)} = \sum_{k \in E} p_{ik} p_{kj}$$

proof.

$$\mathbb{P}(X_{t+2} = j \mid X_t = i) = \frac{\mathbb{P}(X_{t+2} = j, X_t = i)}{\mathbb{P}(X_t = i)}$$

$$\mathbb{P}(X_{t+2}=j \mid X_t=i) = \frac{\sum_{k\in E} \mathbb{P}(X_{t+2}=j, X_{t+1}=k, X_t=i)}{\mathbb{P}(X_t=i)}$$

Markov property

$$= \sum_{k\in E} \mathbb{P}(X_{t+2}=j \mid X_{t+1}=k, X_t=i)\, \frac{\mathbb{P}(X_{t+1}=k, X_t=i)}{\mathbb{P}(X_t=i)}$$

$$= \sum_{k\in E} \mathbb{P}(X_{t+2}=j \mid X_{t+1}=k)\, \mathbb{P}(X_{t+1}=k \mid X_t=i)$$

$$= \sum_{k\in E} p_{kj}\, p_{ik} \quad \leftarrow \text{elements of } (p_{ij}) \text{ power } 2 \qquad \blacksquare$$

More generally:

$$\mathbb{P}(X_{t_m}=j_m, X_{t_{m-1}}=j_{m-1}, \dots, X_0=j_0) = \mathbb{P}(X_0=j_0)\, p_{j_0 j_1}^{(t_1)}\, p_{j_1 j_2}^{(t_2-t_1)} \cdots p_{j_{m-1} j_m}^{(t_m-t_{m-1})}$$

with $(p_{ij})_{ij}$ transition matrix.

Properties: 1. $0 \le p_{ij} \le 1$

2. $\sum_{j\in E} p_{ij} = 1 \qquad \left(\sum_{j\in E} \mathbb{P}(X_1=j \mid X_0=i) = 1 \right)$

Suppose $(X_n)_{n\ge 0}$, $n\in\mathbb{N}$, E countable:

Def. ACCESSIBLE STATE: $i,j\in E$, j accessible from i if $\exists n \ge 0$ s.t.: $\mathbb{P}(X_n=j \mid X_0=i) = p_{ij}^{(n)} > 0$.

COMMUNICATIVE STATES: $i,j\in E$, j and i communicate if each one is accessible from the other one.

CLASS OF STATES: $C \subseteq E$ is a class of states if all states in C communicate and they do not communicate with states in E/C.

Example: Gambler's ruin problem

States $E=\{0,1,\dots,N\}$

$p+q=1$
$p,q>0$

Transition matrix:

$p_{ij}: i \to j$:

$$\begin{array}{c|c} & 0\ 1\ 2\ 3\ \cdots\ N\text{-}1\ N \quad \text{(going to)} \\ \hline \begin{matrix}0\\1\\2\\ \vdots \\ N-1\\N\end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1\ 0\ 0 \cdots & 0 \\ \operatorname{diag}(q,0,p) & \\ 0\ 0\ 0 \cdots & 1 \end{bmatrix}$$

(going from)

$p_{ij}=q$
$=\mathbb{P}(i\to j)=q$
$=\mathbb{P}(2\to 1)=q$

$\{0\}$ is a class of state since 0 communicates only with itself.

The classes are $\{0\}$, $\{N\}$, $\{1,2,3,\dots,N-1\}$

Example: Random queue at a counter:

- a counter processes 1 costumer per unit of time
- the number of costumers arriving in the system per unit of time is a random variable with density $(a_k)_{k\ge 0}$:

A_n = number of arrivals at time n : $A_1,\dots,A_n$ $\perp\!\!\!\perp$

$\mathbb{P}(A_n=k)=a_k$ = at the time n there comes A_n people with probability a_k

X_n = number of customers in the queue at time n

$$X_{n+1} = \begin{cases} \quad + A_{n+1} - 1 & \text{if } X_n > 0 \\ A_{n+1} & \text{if } X_n = 0 \end{cases}$$
$$= \underbrace{(X_n - 1)^+}_{\substack{\text{it appears} \\ \text{only if } X_n > 0}} + A_{n+1}$$

Transition matrix:

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & \cdots & & \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ \\ \end{matrix} & \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

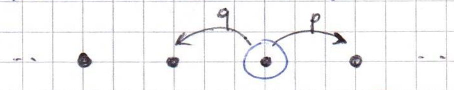
if a_k are sufficiently "generic", namely $a_0 > 0$, $a_1 > 0$, and $\exists\, \bar{n} \geq 2$ such that $a_{\bar{n}} > 0$, what are the classes of the states?

Starting from $i > 0$ one can reach 0 in i steps with probability $a_0^i > 0$.

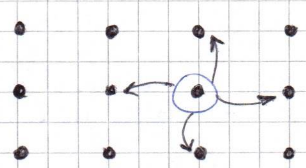
Starting from $i > 0$ one can reach any state $1 + (\bar{n} - 1)n$ and then any $j > i$ for $i < j < 1 + (\bar{n} - 1)n$

Def. IRREDUCIBLE MC: A Markov chain is irreducible if all states communicate.

(Examples:) – Random walk on $\mathbb{Z}$:



– Random walk on $\mathbb{Z}^2$:



in this case we need a 4 dimensional matrix for the representation

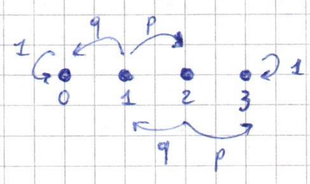
Def. RECURRENT STATE: $(X_n)_{n \geq 0}$ discrete time Markov chain.
A state is called recurrent if:

$$\mathbb{P}\left(\bigcup_{1 \leq n \leq \infty} \{X_n = i\} \mid X_0 = i \right) = 1 .$$

Namely, the probability of return to i in finite time (starting from i) is 1. We're sure to come back to i.
Otherwise the state i is called TRANSIENT.

Example: Gambler's ruin with $N = 3$:

Transition matrix: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad p, q > 0, \quad p + q = 1$



$0, 3$ are recurrent (absorbing states)
$1, 2$ are transient

Def. FIRST ENTRANCE TIME: Let i be a state.

$$T_i := \begin{cases} \min\{n \mid X_n = i\} & \text{if } \{n \geq 1 \mid X_n = i\} \neq \emptyset \\ +\infty & \text{if } \{n \geq 1 \mid X_n = i\} = \emptyset \end{cases}$$

T_i is called first entrance time (or first visit time) to i.

Notation: $f_{ji}^{(n)} = \mathbb{P}(T_i = n \mid X_0 = j)$

Prop. Renewal equations

$$\forall i,j \in E \quad : \quad \underbrace{p_{ij}^{(n)}}_{\text{probability of going from } i \text{ to } j \text{ in } n \text{ steps}} = \underbrace{\sum_{\nu=1}^{n} f_{ij}^{(\nu)} \, p_{jj}^{(n-\nu)}}_{\Sigma \text{ enter in } j \text{ after } \nu \text{ steps and then remain in } j \text{ (the } \Sigma \text{ represents all the possible cases)}}$$

proof.

$p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i)$

$\{X_n = j\} = \bigcup_{\nu=1}^{n} \{X_n = j, T_j = \nu\} = \bigcup_{\nu=1}^{n} \{X_n = j, \dots, X_\nu = j, X_{\nu-1} \neq j, \dots\}$

disjoint events

$$\begin{aligned}
p_{ij}^{(n)} &= \textstyle\sum_{\nu=1}^{n} \mathbb{P}(X_n = j, \dots, X_\nu = j, X_{\nu-1} \neq j, \dots, X_0 = i) \\
&= \textstyle\sum_{\nu=1}^{n} \mathbb{P}(X_n = j, \dots \mid X_\nu = j, X_{\nu-1} \neq j, \dots)\, \mathbb{P}(X_\nu = j, \dots, X_0 = i) \\
&= \textstyle\sum_{\nu=1}^{n} \mathbb{P}(X_n = j, \dots \mid X_\nu = j)\, \mathbb{P}(X_\nu = j, \dots, X_0 = i) \\
&= \textstyle\sum_{\nu=1}^{n} p_{jj}^{(n-\nu)}\, \mathbb{P}(T_j = \nu \mid X_0 = i) \\
&= \textstyle\sum_{\nu=1}^{n} p_{jj}^{(n-\nu)}\, f_{ij}^{(\nu)}
\end{aligned}$$

■

Theorem: For any state i of a discrete Markov chain $(X_n)_n$ the following are equivalent:

(a) i is recurrent

(b) $\sum_{n\geq 0} p_{ii}^{(n)} = +\infty$ = the average time spent in i is $+\infty$

If the series in (b) is convergent ($\sum_{n\geq 0} p_{ii}^{(n)} < +\infty$) then:

$$\sum_{n\geq 0} p_{ii}^{(n)} = \frac{1}{1 - \mathbb{P}_i(T_i < \infty)}$$

$1 - \mathbb{P}_i(T_i < \infty)$ = probability of never returning in i

where $\mathbb{P}_i(\bullet) = \mathbb{P}(\bullet \mid X_0 = i)$

Meaning of **(b)**:

$$p_{ii}^{(n)} = \mathbb{P}(X_n = i \mid X_0 = i) = \underbrace{\mathbb{E}[\mathbb{1}_{\{X_n = i\}}]}_{\text{expectation with respect to } \mathbb{P}(\bullet \mid X_0 = i)}$$

$$\Longrightarrow \sum_{n\geq 0} p_{ii}^{(n)} = \sum_{n\geq 0} \mathbb{E}[\mathbb{1}_{\{X_n = i\}}] = \underbrace{\mathbb{E}\Big[\underbrace{\textstyle\sum_{n\geq 0} \mathbb{1}_{\{X_n = i\}}}_{\text{total time spent in } i}\Big]}_{\text{average time spent in } i}$$

$\Longrightarrow$ [i recurrent $\Longleftrightarrow$ (a): for sure we'll be back to i $\Longleftrightarrow$ (b): the (average) time spen in i is $+\infty$]

proof.

Recall: $f_{ii}^{(n)} = \mathbb{P}_i(T_i = n)$ = probability of first return to i after steps

We define:

$$\left.\begin{aligned} P_{ii}(s) &= \textstyle\sum_{n\geq 0} s^n p_{ii}^{(n)} \\ F_{ii}(s) &= \textstyle\sum_{n\geq 1} s^n f_{ii}^{(n)} \end{aligned}\right\} \text{power series, absolutely convergent if } |s| < 1$$

Renewal equations: $p_{ii}^{(n)} = \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}$ $\quad \cdot s^n$

$$\Longrightarrow s^n p_{ii}^{(n)} = \sum_{k=1}^{n} (s^k f_{ii}^{(k)})(s^{n-k} p_{ii}^{(n-k)})$$

$$\begin{aligned}\Longrightarrow \sum_{n=1}^{\infty} s^n p_{ii}^{(n)} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} (s^k f_{ii}^{(k)})(s^{n-k} p_{ii}^{(n-k)}) \\ &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} (s^k f_{ii}^{(k)})(s^{n-k} p_{ii}^{(n-k)}) \\ &= \sum_{k=1}^{\infty} (s^k f_{ii}^{(k)}) \sum_{n=k}^{\infty} (s^{n-k} p_{ii}^{(n-k)}) \\ &= \sum_{k=1}^{\infty} (s^k f_{ii}^{(k)}) \sum_{m=0}^{\infty} (s^m p_{ii}^{(m)}) \\ &= F_{ii}(s)\, P_{ii}(s)\end{aligned}$$

$$\Longrightarrow P_{ii}(s) - 1 = F_{ii}(s)\, P_{ii}(s)$$

$$\Longrightarrow \text{For } |s| < 1: \quad P_{ii}(s) = \frac{1}{1 - F_{ii}(s)} \qquad (*)$$

(a) $\Longrightarrow$ (b): i recurrent means $\mathbb{P}_i(T_i < \infty) = 1$, namely:

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1 \qquad (f_{ii}^{(n)} = \mathbb{P}_i(T_i = n))$$

$$\Longrightarrow \lim_{s \to 1^-} F_{ii}(s) = \lim_{s \to 1^-} \sum_{n=1}^{\infty} s^n f_{ii}^{(n)} = 1$$

$$\underbrace{\lim_{s \to 1^-} P_{ii}(s)}_{\sum_{n=0}^{\infty} p_{ii}^{(n)}} = \lim_{s \to 1^-} \frac{1}{1 - F_{ii}(s)} = +\infty$$

(b) $\Longrightarrow$ (a): (b) means that $\lim_{s \to 1^-} P_{ii}(s) = +\infty$

From $(*)$: $F_{ii}(s) = 1 - \frac{1}{P_{ii}(s)}$, then:

$$\lim_{s \to 1^-} F_{ii}(s) = 1 \quad \left(\text{i.e. } \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1\right) \iff \mathbb{P}_i(T_i < \infty) = 1$$

if the series is convergent: $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$:

$$P_{ii}(s) = \frac{1}{1 - F_{ii}(s)} \xrightarrow{s \to 1^-} \sum_{n=0}^{\infty} p_{ii}^{(n)} = \frac{1}{1 - \mathbb{P}_i(T_i < \infty)} \qquad \blacksquare$$

Remark: $\sum_{n \geq 0} p_{ii}^{(n)} = \mathbb{E}_i\Big[\underbrace{\sum_{n \geq 0} \mathbb{1}\{X_n = i\}}_{\text{number of visits in } i}\Big]$

If i is transient both of the terms are $< \infty$, and so:

$$i \text{ transient} \Longrightarrow \mathbb{P}\left(\sum_{n \geq 0} \mathbb{1}\{X_n = i\} = +\infty\right) = 0$$

Conclusion: a MC visits a transient state only a finite number of times with probability $= 1$

Corollary: Two communicating states i, j are both recurrent or both transient
(This implies that: let J be a class of states, then if $j \in J$ is recurrent (or transient) all the elements of J are recurrent (or transient))

proof.

Suppose that i is recurrent $\Longrightarrow \sum_{n \geq 0} p_{ii}^{(n)} = +\infty$

Since i and j communicate $\exists\, s, t \in \mathbb{N}: p_{ji}^{(s)} > 0,\ p_{ij}^{(t)} > 0$

$$\Longrightarrow p_{jj}^{(s+t+n)} = \sum_{k_1, k_2} p_{jk_1}^{(s)} p_{k_1k_2}^{(n)} p_{k_2j}^{(t)} \geq p_{ji}^{(s)} p_{ii}^{(n)} p_{ij}^{(t)}$$

$$\Rightarrow \sum_{n\geq 0} p_{jj}^{(s+t+n)} \geq \underbrace{p_{ji}^{(s)}}_{>0} \underbrace{\left(\sum_{n\geq 0} p_{ii}^{(n)}\right)}_{\substack{+\infty \\ \text{since } i \text{ is} \\ \text{recurrent}}} \underbrace{p_{ij}^{(t)}}_{>0} = +\infty$$

$$\Rightarrow \sum_{n\geq 0} p_{jj}^{(s+t+n)} = +\infty \Rightarrow j \text{ is recurrent.}$$

This is enough since every state is either recurrent or transient. ■

Corollary:

1. If j is transient $\Longrightarrow$ $\forall i \; \sum_{n\geq 0} p_{ij}^{(n)} < +\infty$
2. In particular (j transient) : $\lim_{n\to\infty} \mathbb{P}_i(X_n = j) = 0$
3. If the set of states E is finite then there exists at least a recurrent state (this is not necessary true for a transient state). (Notice: E is the set of states, not class of states)

if j is transient, starting from any i the Markov Chain goes through j almost certainly only a finite number of times

proof.

1. Renewal equation:

$$\begin{aligned}
\sum_{n\geq 1} p_{ij}^{(n)} &= \sum_{n\geq 1} \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_{ij}^{(k)} p_{jj}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{ij}^{(k)} \sum_{n=k}^{\infty} p_{jj}^{(n-k)} \\
&= \underbrace{\sum_{k=1}^{\infty} f_{ij}^{(k)}}_{\substack{\mathbb{P}_i(T_j<\infty) \\ \left(<1 \text{ since } j \text{ is transient}\right)}} \underbrace{\sum_{m=0}^{\infty} p_{jj}^{(m)}}_{\text{convergent } (<\infty)}
\end{aligned}$$

$$\Longrightarrow \sum_{n\geq 1} p_{ij}^{(n)} < \infty$$

2. $\sum_{n\geq 1} p_{ij}^{(n)} < \infty \Longrightarrow \lim_{n\to\infty} p_{ij}^{(n)} = 0$ $\equiv$ the probability of visiting j goes to 0 as $n \to \infty$

3. E finite. Suppose that all the states are transient $\Longrightarrow \lim_{n\to\infty} p_{ij}^{(n)} = 0 \;\; \forall i,j$

   but $\forall n: \sum_{j\in E} p_{ij}^{(n)} = 1$

   $\Longrightarrow$ (E finite) $n\to\infty$: $0 = \lim_{n\to\infty} \sum_{j\in E} p_{ij}^{(n)} = 1$, contraddiction

   $\Longrightarrow \exists$ at least 1 recurrent state. ■

Example: Random walk on $\mathbb{Z}$:

Application of the theorem (recurrent and transient states)

$p, q > 0, \quad p+q = 1$

$X_{n+1} = X_n + Y_{n+1}$ where $(Y_n)_n$ are iid: $\mathbb{P}(Y_m = 1) = p$, $\mathbb{P}(Y_m = -1) = q$

$X_n = x_0 + \sum_{k=1}^{n} Y_k \qquad \left(\frac{1}{n} \sum_{k=1}^{n} Y_k \xrightarrow{a.s.} \mathbb{E}[Y_1] = p - q \right)$

If we start from 0 we can find a moment (a time) for which after that moment we can't go back to 0 $\Longrightarrow$ 0 is transient.

LLN $\longrightarrow$ all states are transient if $p \neq q$

Law of large numbers

The MC is irreducible since all the states communicate.
($\Rightarrow$ all the states are either transient or recurrent)

Let's consider the state 0 and study the series:

$$\sum_{n\geq 0} p_{00}^{(n)} = \begin{cases} +\infty \\ < \infty \end{cases}$$

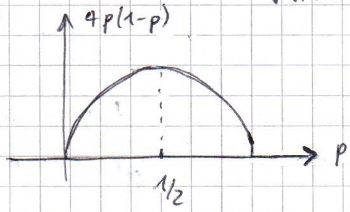
The probability of starting from 0 and came back to 0 after **n** steps:

$$p_{00}^{(n)} = \begin{cases} 0 & n = \text{odd} \\ \binom{2m}{m} p^m q^m & n = 2m \end{cases}$$

$$p_{00}^{(2m)} = \frac{(2m)!}{(m!)^2} (p(1-p))^m \sim \frac{(2m)^{2m} e^{-2m} \sqrt{4\pi m}}{m^{2m} e^{-2m} \, 2\pi m} (p(1-p))^m$$

Stirling: $\lim_{n\to\infty} \frac{n^n e^{-n} \sqrt{2\pi n}}{n!} = 1$

$$\Rightarrow p_{00}^{(2m)} \sim \frac{2^{2m}}{\sqrt{\pi m}} (p(1-p))^m = \frac{(4p(1-p))^m}{\sqrt{\pi m}}$$



$$p_{00}^{(2m)} = \begin{cases} \frac{1}{\sqrt{\pi m}} & p = \frac{1}{2} \quad (p = q) \\ \frac{(4p(1-p))^m}{\sqrt{\pi m}} & p \neq \frac{1}{2} \quad (p \neq q) \end{cases}$$

$$\Rightarrow \begin{cases} \bullet \; p = q = \frac{1}{2} : & \sum_{n\geq 0} p_{00}^{(n)} = +\infty & (\forall \text{ state is recurrent}) \\ \bullet \; p \neq q \; (\neq \frac{1}{2}) : & \sum_{n\geq} p_{00}^{(n)} < +\infty & (\forall \text{ state is transient}) \end{cases}$$

Moreover: (for $p = q = \frac{1}{2}$)

$$p_{00}^{(2m)} = \binom{2m}{m} (pq)^m \Rightarrow P_{00}(s) = \sum_{n\geq 0} s^n p_{00}^{(n)}$$

$$\Rightarrow P_{00}(s) = \sum_{m\geq 0} s^{2m} p_{00}^{(2m)} = \sum_{m\geq 0} s^{2m} \binom{2m}{m} (pq)^m = \frac{1}{\sqrt{1 - 4pqs^2}} \qquad |s| < 1$$

$$\Rightarrow F_{00}(s) = 1 - \frac{1}{P_{00}(s)} = 1 - \sqrt{1 - 4pqs^2} = 1 - \sqrt{1 - s^2}$$

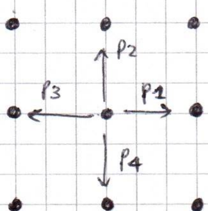
$$\Rightarrow F_{00}(s) = \sum_{n\geq 1} s^n \, \mathbb{P}_0(T_0 = n) = 1 - \sqrt{1 - s^2}$$

$$\frac{\partial}{\partial s} \Rightarrow \sum_{n\geq 1} n s^{n-1} \, \mathbb{P}_0(T_0 = n) = \frac{s}{(1 - s^2)^{3/2}}$$

$$\lim_{s\to 1^-} \Rightarrow \sum_{n\geq 1} n \, \mathbb{P}_0(T_0 = n) = \mathbb{E}_0[T_0] = +\infty$$

so even if the return to 0 is almost certain, the return time is ∞

Example: Random walk on $\mathbb{Z}^2$:



$p_1 + p_2 + p_3 + p_4 = 1$

- p_i all equal $\Rightarrow$ $p_i = \frac{1}{4}$ $\forall i$, all states are recurrent
- p_i all different $\rightarrow$ all states are transient

$$p_{00}^{(2m)} = \sum_{h+k=m} \frac{(2m)!}{(h!)^2 (k!)^2} \left(\frac{1}{4}\right)^{2m}$$

$\longleftrightarrow$ h

$\updownarrow$ k

$$p_{00}^{(2m)} = \sum_{k=0}^{m} \frac{(2m)!}{(k!)^2((m-k)!)^2}\left(\frac{1}{4}\right)^{2m}$$

$$= \binom{2m}{m}\left(\frac{1}{4}\right)^{2m}\sum_{k=0}^{m}\binom{m}{k}\binom{m}{m-k}$$

$$= \binom{2m}{m}\left(\frac{1}{4}\right)^{2m}\binom{2m}{m}$$

$$= \left(\binom{2m}{m}\left(\frac{1}{4}\right)^{m}\right)^2 \sim \left(\frac{1}{\sqrt{\pi m}}\right)^2 = \frac{1}{\pi m}$$

$\Longrightarrow \sum_{n\geq 0} p_{00}^{(n)} = +\infty \Longrightarrow$ $(0,0)$ is recurrent (and so all the states are)

Example: Random walk on $\mathbb{Z}^d$:

$$\mathbb{P}\left(X_{n+1} = (j_1,\dots,j_d) \mid X_n = (i_1,\dots,i_d)\right) = \begin{cases} \frac{1}{2d} & \text{if } \sum_{k=1}^{d} |i_k - j_k| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Considering the MC irreducible: are the states transient or recurrent?

$$\mathbb{P}(X_{2n} = \underline{0} \mid X_0 = \underline{0}) = p_{00}^{(2n)}$$

$$= \sum_{k_1+\dots+k_d=n} \frac{(2n)!}{(k_1!)^2\cdots(k_d!)^2}\left(\frac{1}{2d}\right)^{2n}$$

$$= \binom{2n}{n}\left(\frac{1}{2d}\right)^{2n}\sum_{k_1+\dots+k_d=n}\frac{(n!)^2}{(k_1!)^2\cdots(k_d!)^2}$$

$$\sum_{k_1+\dots+k_d=n}\frac{(n!)^2}{(k_1!)^2\cdots(k_d!)^2} \leq \max_{k_1+\dots+k_d=n}\left\{\frac{n!}{k_1!\cdots k_d!}\right\}\cdot \boxed{\sum_{k_1+\dots+k_d=n}\frac{n!}{k_1!\cdots k_d!}}$$

$$= \max_{k_1+\dots+k_d=n}\left\{\frac{n!}{k_1!\cdots k_d!}\right\}\cdot d^n$$

$$(a_1+\dots+a_d)^n = \sum_{\sum k_j = n}\frac{n!}{(k_1!)\cdots(k!)}\left(a_1^{k_1}\cdots a_d^{k_d}\right)$$

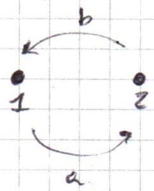
$$\Longrightarrow p_{00}^{(2n)} \leq d^n \max_{k_1+\dots+k_d=n}\left\{\frac{n!}{k_1!\cdots k_d!}\right\}\cdot\binom{2n}{n}\left(\frac{1}{2d}\right)^{2n} =$$

$$= d^n\frac{n!}{\left(\left(\frac{n}{d}\right)!\right)^d}\binom{2n}{n}\left(\frac{1}{2d}\right)^{2m}$$

$$\sim \frac{d^{d/2}}{\pi^{d/2}\, 2^{(d-1)/2}\, n^{d/2}} = \frac{\text{constant}(d)}{n^{d/2}}$$

$\Longrightarrow$ if $d \geq 3$ $\sum_{n\geq 0} p_{00}^{(n)} < +\infty \longrightarrow$ all the states are transient

We want now to focus on the invariant distribution.
for example:



$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

We would like to know how much we stay in a state. The invariant distribution answers to this problem:

$$\mathbb{P}(X_n = 1) \overset{n \text{ big}}{\approx} \pi_1 \quad \text{of an invariant distribution}$$

$\uparrow$
asymptotical
distribution

Def. INVARIANT DISTRIBUTION: $(X_n)_{n\geq 0}$ MC with transition matrix $(p_{ij})_{i,j}$ and $\pi = (\pi_i)_{i\in E}$ probability density of E $(0 \leq \pi_i \leq 1,\ \sum_i \pi_i = 1)$.
~~π is an invariant density if $X_n \sim \pi$ $\forall n \geq 0$.~~

We can write it as:
π is an invariant density if $X_{n+1} \sim \pi$ whenever $X_n \sim \pi$ $\forall n$.

Prop. $\mathbb{P}(X_{n+1} = j) = \sum_{i\in E} \mathbb{P}(X_{n+1} = j \mid X_n = i)\, \mathbb{P}(X_n = i)$

If $\mathbb{P}(X_n = i) = \pi_i \Longrightarrow \mathbb{P}(X_{n+1} = j) = \sum_{i\in E} p_{ij}\pi_i$

$\Longrightarrow$ If $X_n \sim \pi$, $X_{n+1} \sim \pi \iff \pi_j = \sum_{i\in E} p_{ij}\pi_i$ = the density is a left eigenve of P with eigenvalue $= 1$

Example: $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

$$\pi P = \pi \iff \begin{cases} \pi_1(1-a) + \pi_2 b = \pi_1 \\ \pi_1 a + (1-b)\pi_2 = \pi_2 \end{cases} \iff \pi_1 a = \pi_2 b$$

$$\Longrightarrow \begin{cases} \pi_1 + \pi_2 = 1 \\ \pi_1 a = \pi_2 b \end{cases} \Longrightarrow \pi = \left[\frac{b}{a+b}, \frac{a}{a+b}\right]$$

$$\Longrightarrow \lim_{n\to\infty} \mathbb{P}(X_n = 1) = \pi_1 = \frac{b}{a+b}$$

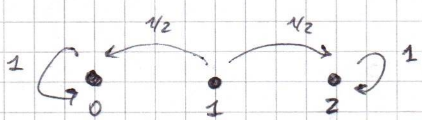
if we know the invariant distribution we can quantify the frequence of arriving

Theorem: If E is finite then there exists at least one invariant density.

Remark: The invariant is not necessarily unique:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$ (Gambler's ruin with $p = q = \frac{1}{2}$, $N = 2$)

$E = \{0,1,2\}$: there are two absorbing states: $\{0\}, \{2\}$



$$\Longrightarrow \begin{cases} \pi = [1, 0, 0] \\ \pi = [0, 0, 1] \end{cases}$$

because if we are in 0 (or 2) we remain in 0 (or 2)

$\mathbb{P}(X_n = 0) = 1 \Longrightarrow \pi P = [1, 0, 0]P$

$\mathbb{P}(X_n = 2) = 1 \Longrightarrow \pi P = [0, 0, 1]P$

Convex combination:

$$\lambda[1,0,0] + (1-\lambda)[0,0,1] = \underbrace{[\lambda, 0, 1-\lambda]}_{\text{they're all invariant distributions}} \quad \lambda \in [0,1]$$

$$\begin{cases} \mu = \mu P \\ \nu = \nu P \end{cases} \Longrightarrow \lambda\mu + (1-\lambda)\nu = (\lambda\mu + (1-\lambda)\nu)P$$

Remark: Infinite MC may not have invariant distributions

Random walk on $\mathbb{Z}$: $\mathbb{P}(X_n = 0) = \frac{1}{\sqrt{\pi n}} \longrightarrow 0$

Rigorous computation:

$$\pi P = \pi \quad : \quad p_{ij} = \begin{cases} 1/2 & j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$\Longrightarrow \pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1}$

$\Longrightarrow \pi_{i+1} - \pi_i = \pi_i - \pi_{i-1}$

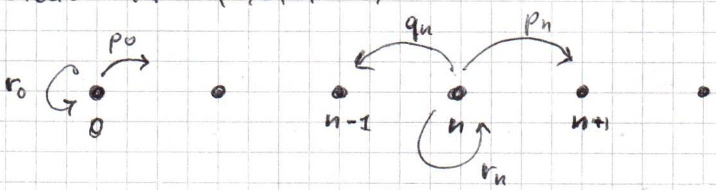
$\Longrightarrow \pi_i = \pi_0 + \sum_{k=}^{i} (\pi_i - \pi_{i-1}) = \pi_0 + (\pi_1 - \pi_0)\, i$

$\longrightarrow$
- $\pi_1 \neq \pi_0 \Longrightarrow \pi_i$ can be negative for $i \to -\infty \Longrightarrow$ no solutions
- $\pi_1 = \pi_0 \Longrightarrow \pi_i = \pi_0\ \forall i \Longrightarrow \sum_{i\in\mathbb{Z}} \pi_i = +\infty \Longrightarrow$ no solutions

Example: Discrete birth and death process

(only the name of the model, it actually has a lot of applications)

Status $\mathbb{N} = \{0, 1, 2, \dots\}$



$$P = \begin{bmatrix} r_0 & p_0 & \dots \\ q_1 & r_1 & p_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & q_2 & r_2 & p_2 & 0 & 0 & 0 & \dots \\ 0 & 0 & q_3 & r_3 & p_3 & 0 & 0 & \dots \\ & & \vdots \end{bmatrix}$$

$r_i, q_i, p_i \geq 0$
$r_0 + p_0 = 1$
$r_i + p_i + q_i = 1 \quad \forall i \neq$

r_i = no change
$p_i = \mathbb{P}(\text{birth})$
$q_i = \mathbb{P}(\text{death})$

This MC is irreducible $\Rightarrow p_i > 0, \; q_i > 0$.

Invariant distribution π:

$$\pi = \pi P \iff \begin{cases} \pi_0 = r_0 \pi_0 + q_1 \pi_1 \\ \pi_n = p_{n-1} \pi_{n-1} + r_n \pi_n + q_{n+1} \pi_{n+1} \end{cases}$$

$$\Rightarrow \pi_1 = \frac{1 - r_0}{q_1} \pi_0 = \frac{p_0}{q_1} \pi_0$$

$$\pi_2 = \pi_1 \frac{1 - r_1}{q_2} - \frac{p_0}{q_2} \pi_0 = \frac{p_0 p_1}{q_1 q_2} \pi_0$$

$$\vdots$$

$$\pi_n = \frac{p_0 p_1 \cdots p_{n-1}}{q_1 q_2 \cdots q_n} \pi_0$$

$(\pi_n)_{n \geq 0}$ are positive numbers if $\pi_0 > 0$.

$$\sum_{n \geq 0} \pi_n = 1 \iff \pi_0 + \pi_0 \sum_{n \geq 1} \frac{p_0 p_1 \cdots p_{n-1}}{q_1 \cdots q_n} = 1$$

If the series converges we can define: $Z := 1 + \sum_{n \geq 1} \frac{p_0 p_1 \cdots p_{n-1}}{q_1 \cdots q_n}$

$$\Rightarrow \pi_n = \frac{1}{Z} \cdot \frac{p_0 p_1 \cdots p_{n-1}}{q_1 q_2 \cdots q_n}$$

(unique) invariant distribution

If the series diverges there exists no invariant distribution.

Remark:

$$\left.\begin{aligned} \pi_n P_{n(n+1)} &= \frac{1}{Z} \frac{p_0 \cdots p_{n-1}}{q_1 \cdots q_n} p_n \\ \pi_{n+1} P_{(n+1)n} &= \frac{1}{Z} \frac{p_0 \cdots p_n}{q_1 \cdots \cancel{q_{n+1}}} \cancel{q_{n+1}} \end{aligned}\right\} \text{they're } =$$

Given π_n, if all the variables have the same invariant distribution, going on or going back is the same

Def. **REVERSABLE PROBABILITY DENSITY**: A probability density $(\pi_i)_{i \in E}$ is reversable with refer to a stochastic matrix $(p_{ij})_{i,j \in E}$ if:

$$\pi_i p_{ij} = \pi_j p_{ji} \qquad \forall i, j \in E$$

Remark: Not every invariant density is a reversable density, but if a density is reversable then it's also invariant

$$\text{reversable } (\pi_i)_i \; \underset{\not\Leftarrow}{\Rightarrow} \; (\pi_i)_i \text{ is invariant}$$

proof.

$$\underbrace{\sum_{i\in E} \pi_i p_{ij}}_{(\pi P)_j} \underset{\text{by reversability}}{=} \sum_{j\in E} \pi_i p_{ij} = \pi_i \sum_{j\in E} p_{ij} = \pi_i \qquad \blacksquare$$

Remark: If $p_{ij}^{(n)} \xrightarrow{n} \mu_j$ (converges) and $\lim_{n\to\infty} p_{ij}^{(n)} = \mu_j \ \forall i$ (so it depends only on j) and $\sum_{j\in E}\mu_j$ (which is ≤ 1) satisfies $\sum_{j\in E}\mu_j = 1 \Longrightarrow \mu_j = \sum_i \mu_i p_{ij}$ is an invariant density

(because $p_{ij}^{(n+1)} \longrightarrow \mu_j \Rightarrow \sum_{k\in E} p_{ik}^{(n)} p_{kj} \xrightarrow{n\to\infty} \sum_{k\in E}\mu_k p_{kj}$)

Theorem: If limits $\lim_{n\to\infty} p_{ij}^{(n)}$ exist and are strictly positive and depend only on j $\Longrightarrow$ MC has a unique invariant distribution π:

$$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \int f \, d\pi = \sum_{j\in E} f(j)\,\pi_j$$

for all f bounded.

In this situation: $X_n \to \pi$ in distribution $(\mathbb{P}(X_n=j) \longrightarrow \pi_j \ \forall j)$

Def. **PERIOD OF A STATE**: The period of the state $i\in E$ is defined as:

$$\underset{\text{maximum common divisor}}{\mathrm{MCD}} \{ n \geq 1 \mid p_{ii}^{(n)} > 0 \}$$

Ex. random walk: $p_{ii}^{(n)} = \begin{cases} 0 & n \text{ odd} \\ \neq 0 & n \text{ even} \end{cases} \Longrightarrow$ period $= 2$

Def. **APERIODIC STATE**: If the period is 1 the state is called aperiodic.

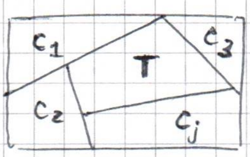
Prop. States of the same class have the same period.
(for irreducible MC all states have the same period and we can talk about periodic / aperiodic Markov chains)

Theorem: Limits of subsequences drawn from $\frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} \longrightarrow L$ determine invariant densities (also for periodic MC)

$$\underbrace{\frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)}}_{\text{mean frequency of visits in } j} = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}_i[\mathbb{1}_{\{X_k=j\}}] = \underbrace{\mathbb{E}_i\Big[\underbrace{\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_k=j\}}}_{\text{frequency of visit in } j}\Big]}_{\text{mean frequency of visit in } j}$$

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_k=j\}} \xrightarrow{\text{a.s.}} \pi_j$$

Reminder (Absorption probabilities / Hitting probabilities)



C_i = recurrent classes
T = transient class

$$h_i := \mathbb{P}_i\Big(\bigcup_{n\geq 1} \{X_n \in C\}\Big)$$

= probability of absorption in C_1 starting from a transient state

$$h_i = \sum_{j\in C_1} p_{ij} + \sum_{j\in T} p_{ij} h_j \qquad (*)$$

the probabilities $(h_i)_{i\in T}$ are the smallest solution with values in $[0,1]$ of $(*)$. If T is finite $\Longrightarrow$ $(h_i)_{i\in T}$ are the only solutions with values in $[0,1]$ of $(*)$.

(Recall) Markov property for discrete MC:

$$\mathbb{P}(X_{n+1}=j \mid X_n=i_n, X_{n-1}=i_{n-1}, ..) = \mathbb{P}(X_{n+1}=j \mid X_n=i_n) \qquad \forall n, j, i_1, .., i_n$$

Notice that here we have fixed times $(n+1, n, n-1, ..)$.

Def. STOPPING TIME: The random variable $T: \Omega \to \mathbb{N}$ is a stopping time of the Markov chain $(X_n)_{n\geq 0}$ if $\forall n$ the event $\{T \leq n\}$ belongs to the σ-algebra generated by $\{X_n=i_n, X_{n-1}=i_{n-1}, .., X_0=i_0\}$. (with $i_n, .., i_0$ arbitrary states in the countable set of states)

The stopping time T is $\perp\!\!\!\perp$ from the future.

If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces and $f: E \to F$ is a function, then the σ-algebra generated by f (denoted by $f^{-1}(\mathcal{F})$) is the set of $f^{-1}(B)$ $B \in \mathcal{F}$.

The set of the events $\{X_n=i_n, X_{n-1}=i_{n-1}, .., X_0=i_0\}$ represent the history of observations till time n.

Example: First entrance time T_j:

$$T_j = \inf_{n \geq 1} \{n \mid X_n = j\} \qquad (\text{if } \{n \mid X_n=j\} = \emptyset \Rightarrow T_j = +\infty)$$

The time of the first visit in the state j is a stopping time, because:

$$\underbrace{\{T_j \leq n\}}_{\left[\begin{array}{c}\text{the first entrance in } j \\ \text{is at the step } n \text{ at most}\end{array}\right]} = \underbrace{\cup_{m=1}^{n} \{X_m = j\}}_{\left[\begin{array}{c}\exists\, m \leq n \text{ for which} \\ X_m = j\end{array}\right]}$$

$$\{T_j \leq n\} = \bigcup_{m=1}^{n} \{X_m = j\} = \bigcup_{m=1}^{n} \bigcup_{i_1, .., i_n} \{X_n=i_n, .., X_{m+1}=i_{m+1}, X_m=i_m, ..\}$$

$$\{T_j = n\} = \{X_n=j, X_{n-1}\neq j, X_{n-2}\neq j, .., X_1 \neq j\}$$

Example: First exit time U_j:

$$U_j = \inf_{n\geq 1} \{n \mid X_n \neq j\} \qquad (\text{if } \{n \mid X_n \neq j\} = \emptyset \Rightarrow U_j = +\infty)$$

$$\{U_j \leq n\} = \{U_j > n\}^c = \{X_n = j, .., X_1 = j\}^c$$

Prop.
Law of the first exit time
$(X_n)_{n\geq 0}$ MC with transition matrix $(p_{ij})_{i,j}$. If $p_{jj} \in (0,1)$ then the stopping time U_j has a geometric* density with parameter $1-p_{jj}$ and the random variable X_{U_j}:

$$(X_{U_j})(\omega) = \sum_{n\geq 1} X_n(\omega)\, \mathbb{1}_{\{U_j = n\}}(\omega)$$

satisfies: $\mathbb{P}_j(X_{U_j} = k) = \dfrac{p_{jk}}{1-p_{jj}} \qquad \forall k \neq j$

* the Geometric distribution is the only one without memory, that's why it's correlated with Markov's property.

proof.

Geometric distribution: $\mathbb{P}(U_j = n) = ?$

$$\mathbb{P}(U_j > n) = \mathbb{P}(X_n = j, .., X_1 = j) = \sum_k p_{jj}\, p_{jj} \cdots p_{jk}\, \mathbb{P}(X_0 = k)$$

If we consider $\mathbb{P}(\cdot) = \mathbb{P}_j(\cdot) = \mathbb{P}(\cdot \mid X_0 = j) \Rightarrow \mathbb{P}_j(U_j > n) = (p_{jj})^n$

$$\rightarrow (n>0): \quad \mathbb{P}_j(U_j = n) \overset{!}{=} \mathbb{P}_j(U_j > n-1) - \mathbb{P}_j(U_j > n) \overset{!}{=} (p_{jj})^{n-1} - (p_{jj})^n = (p_{jj})^{n-1}(1-p_{jj})$$

So the parameter is $1-p_{jj}$.

$Y \sim \mathcal{G}(\vartheta) \iff \mathbb{P}(Y=n) = (1-\vartheta)^{n-1}\vartheta$

Formula's proof:

$$\mathbb{P}_j(X_{U_j}=k) = \sum_{n=1}^{\infty} \mathbb{P}_j(X_n=k, U_j=n)$$
$$= \sum_{n=1}^{\infty} \mathbb{P}_j(X_n=k, X_n \neq j, X_{n-1}=j, \dots, X_1=j)$$

If $k=j \implies \mathbb{P}_j(X_n=k, X_n\neq j, X_{n-1}=j,\dots,X_1=j)=0$

If $k\neq j \implies \sum_{n=1}^{\infty} \mathbb{P}_j(X_n=k, X_n\neq j, X_{n-1}=j,\dots,X_1=j) =$

$$= \sum_{n=1}^{\infty} \underbrace{p_{jj}\, p_{jj} \cdots p_{jj}}_{n-1 \text{ factors}}\, p_{jk}$$

$$= p_{jk} \sum_{n\geq} p_{jj}^{\,n-1} = p_{jk} \frac{1}{1-p_{jj}} \quad \blacksquare$$

Theorem (Restarted Markov chain): $(X_n)_{n\geq 0}$ MC, T stopping time (a.s. finite)

$$Y_n(\omega) := \begin{cases} X_{T(\omega)+n}(\omega) & \text{if } T(\omega) < +\infty \\ \text{arbitrary} & \text{if } T(\omega) = +\infty \end{cases}$$

$(Y_n)_{n\geq 0}$ is a Markov chain with the same transition matrix of the MC $(X_n)_{n\geq 0}$.

restart the process after the stopping time

proof.

We must check that Y_n is a random variable $\forall n$ and that $\forall n\ \forall j, i_1, \dots, i_n \in E$:

$$\mathbb{P}(Y_{n+1}=j \mid Y_n=i_n, \dots, Y_1=i_1) = \mathbb{P}(Y_{n+1}=j \mid Y_n=i_n) = p_{i_n j}$$

(Markov property)

(trans. matrix P)

namely:

$$\mathbb{P}(X_{T+(n+1)}=j \mid X_{T+n}=i_n, \dots, X_{T+1}=i_1) \overset{*}{=} \mathbb{P}(X_{T_{n+1}}=i_{n+1} \mid X_{T_n}=i_n) = p_{i_n j}$$

$$\overset{*}{=} \frac{\mathbb{P}(X_{T+n+1}=j, X_{T+n}=i_n, \dots)}{\mathbb{P}(X_{T+n}=i_n, \dots)}$$

$$= \frac{\sum_{m=1}^{\infty} \mathbb{P}(X_{m+n+1}=j, X_{m+n}=i_n, \dots, T=m)}{\mathbb{P}(X_{T+n}=i_n, \dots)}$$

$$= \frac{\sum_{m=1}^{\infty} \mathbb{P}(X_{m+n+1}=j \mid X_{m+n}=i_n, \dots, T=m)\, \mathbb{P}(X_{m+n}=i_n, \dots, T=m)}{\mathbb{P}(X_{T+n}=i_n, \dots)}$$

$$= \frac{\sum_{m=1}^{\infty} p_{i_n j}\, \mathbb{P}(X_{m+n}=i_n, \dots, T=m)}{\mathbb{P}(X_{T+n}=i_n, \dots, X_{T+1}=i_1)}$$

$$= p_{i_n j}$$

Same computation for $\mathbb{P}(X_{T+n+1}=j \mid X_{T+n}=i_n) = p_{i_n j}$ $\blacksquare$

Theorem (Strong Markov property): $(X_n)_{n\geq 0}$ MC, $(p_{ij})_{i,j\in}$ transition matrix, T stopping time

$(X_{T\wedge n})_{n\geq 0}$ $\quad (T\wedge n)(\omega) = \min\{T(\omega), n\}$ $\quad$ MC stopped at T

$(Y_n)_{n\geq 0} = (X_{T+n})_{n\geq 0}$ $\quad$ MC restarted at T

The stopped MC and restarted MC are $\perp\!\!\!\perp$ w.r.t. $\mathbb{P}(\cdot \mid X_T=i, T<\infty)$

proof. (Hint)

We have to check that all pairs of events:

$$\{Y_m = j_m, .., Y_0 = j_0\}, \quad \{X_{T\wedge n} = i_n, .., X_T = i_0\} \quad \forall m,n \ \forall j_k, i_k$$

are independent. " ■ "

Theorem: The number of visits of a recurrent state is infinite almost surely.

transiency, recurrency and absorption

proof.

We define the number of visits in $i := N_i = \sum_{n\geq 1} \mathbb{1}_{\{X_n = i\}}$.

Times of visit:

$$\begin{cases} T_i^{(1)} = \inf_{n\geq 1} \{n \mid X_n = i\} & \text{time of the } 1^{st} \text{ visit} \\ \vdots & \\ T_i^{(k+1)} = \inf_{n > T_i^{(k)}} \{n \mid X_n = i\} & \text{time of the } (k+1)^{th} \text{ visit} \end{cases}$$

i recurrent $\underset{\text{(def.)}}{\Longleftrightarrow} P_i(T_i^{(1)} < +\infty) = 1 \iff \mathbb{P}_i(N_i \geq 1) = 1$

$T_i^{(2)}$ is the first visit time for the MC restarted from time $T^{(1)}$.
This MC has the same transition matrix

$$\Longrightarrow \mathbb{P}_i(T_i^{(2)} < \infty) = 1$$

By induction: $\mathbb{P}_i(T_i^{(k)} < +\infty) = 1 \quad \forall k$

$\iff \mathbb{P}_i(N_i \geq k) = 1 \quad \forall k$

$\iff \mathbb{P}_i(N_i = +\infty) = 1$ ■

Theorem (Probability of staying forever in transient states):
Let T be the set of transient states.
The probability of remain for ever in transient states is defined as:

$$U_i = \mathbb{P}_i\left(\bigcap_{n=1}^{+\infty} \{X_n \in T\}\right) \quad \text{if } i \in T$$

Then, $(U_i)_{i\in T}$ is the biggest solution with $0 \leq U_i \leq 1 \quad \forall i \in T$ of the system of equations:

$$U_i = \sum_{j\in T} p_{ij} U_j$$

Remark: Generally there is no unique solution, but if T is finite then the only solution is $U_i = 0$.

proof. (Thm.)

We define $U_i^{(n)} = \mathbb{P}_i(X_n \in T, .., X_1 \in T)$ = probability of staying in transient states from 1 to n (starting from 0)

$(U_i^{(n)})_{n\geq 1}$ is a non increasing sequence (since the probability of staying longer in transient states is lower than the probability of staying shorter)

$$U_i = \mathbb{P}_i\left(\bigcap_{n\geq 1} \{X_n \in T, .., X_1 \in T\}\right)$$

$$= \lim_{n\to\infty} U_i^{(n)} = \lim_{n\to\infty} \mathbb{P}_i(X_n \in T, .., X_1 \in T)$$

$$U_i^{(n+1)} = \mathbb{P}_i(X_{n+1} \in T, .., X_2 \in T, X_1 \in T)$$

$$= \underbrace{\mathbb{P}_i(X_{n+1} \in T, .., X_2 \in T \mid X_1 \in T)}_{\mathbb{P}(\text{remain } n \text{ steps in } T \text{ starting from } X_1 \in T} \mathbb{P}_i(X_1 \in T)$$

More precisely, we have:

$$U_i^{(n+1)} = \sum_{j\in T} \mathbb{P}_i(X_{n+1}\in T, \dots, X_2 \in T, X_1 = j)$$

$$= \sum_{j\in T} \underbrace{\mathbb{P}_i(X_{n+1}\in T, \dots, X_2\in T \mid X_1 = j)}_{\substack{\text{probability of remaining}\\ n \text{ steps in } T \text{ starting from}\\ X_1 = j \text{ (and not } X_0 = i)}} \underbrace{\mathbb{P}_i(X_1 = j)}_{p_{ij}}$$

$$= \sum_{j\in T} U_j^{(n)} p_{ij}$$

Applying the limits on both sides we get: $U_i = \sum_{j\in T} p_{ij} U_j \quad \forall i\in T.$

Now we have to prove that is the biggest $[0,1)$-valued solution.

Suppose we have a bigger solution $V_i = \sum_{j\in T} p_{ij} V_j \quad \forall i \in T, \; 0\le V_i \le 1$

$$\Longrightarrow U_i^{(1)} = \mathbb{P}_i(X_1\in T) = \sum_{j\in T} p_{ij} \underset{\substack{\uparrow\\ \text{since } 0\le V_j\le 1}}{\geq} \sum_{j\in T} p_{ij} V_j = V_i \Longrightarrow U_i^{(1)} \geq V_i \quad \forall i \in T$$

By induction: $U_i^{(n)} \geq V_i \quad \forall i\in T$, then:

$$U_i^{(n+1)} = \sum_{j\in T} p_{ij} U_j^{(n)} \geq \sum_{j\in T} p_{ij} V_j = V_i$$

And so, by considering the limit for $n\to\infty$ we have that $U_i^{(n)}$ increases to U_i and so $U_i \geq V_i \quad \forall i \in T$, which is a contraddiction.

We also prove the remark: T finite $\Longrightarrow U_i = 0 \quad \forall i \in T$

$$U_i = \sum_{j\in T} p_{ij} U_j = \sum_{j\in T} p_{ij} \sum_{k\in T} p_{jk} U_k$$

$$= \sum_{j\in T}\sum_{k\in T} p_{ij} p_{jk} U_k$$

$$\le \sum_{k\in T}\left(\sum_{j\in T} p_{ij}p_{jk}\right) U_k = \sum_{k\in T} p_{ik}^{(2)} U_k$$

By iterating n times we get:

$$U_i = \sum_{k_1,\dots,k_n\in T} p_{ik_1}\cdots p_{k_{n-1}k_n} U_{k_n}$$

$$\le \sum_{\substack{k_1,\dots,k_{n-1}\in E\\ k_n\in T}} (\dots)\, U_{k_n} = \sum_{k_n\in T} p_{ik_n}^{(n)} U_{k_n}$$

$$\Longrightarrow \forall n: \; U_i \le \sum_{j\in T} p_{ij}^{(n)} U_j \underset{U_j\in[0,1]}{\le} \sum_{j\in T} p_{ij}^{(n)} \le 1$$

Since $\forall j \in T \;\; p_{ij}^{(n)} \xrightarrow{n\to\infty} 0$ (transient) $\Longrightarrow U_i = 0 \quad \forall i\in T$ ■

Example: Gambler's win problem (against a bank)

$(X_n)_{n\geq 0}$ MC, X_n = player's capital at time n, $E \;\; \mathbb{N}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ & & \cdots & & & \end{bmatrix} \qquad \begin{matrix} p+q = 1 \\ p,q > 0 \end{matrix}$$

Classes: $\{0\}$ recurrent $\qquad p_{00}^{(n)} = 1, \; \sum_{n\geq 1} p_{00}^{(n)} = +\infty$

$\{1,2,3,\dots\}$ transient $\qquad \forall i \geq 1 \;\; \mathbb{P}_i\left(\bigcap_{i\geq 1}(X_n = i)\right) < 1 - q^i < 1$

We want to calculate the probability that the gambler is never ruined.
(This is equivalent to calculate the probability of remain forever in the transient states $T = \{1, 2, 3, ..\} = \mathbb{N} \setminus \{0\}$)

$$U_i = \sum_{j \geq 1} p_{ij} U_j \quad : \quad \begin{cases} U_1 = p U_2 & (U_0 = 0) \\ U_i = p U_{i+1} + q U_{i-1} & i > 1 \end{cases}$$

General difference equation:

$$\left. \begin{array}{l} a y_{i+1} + b y_i + c y_{i-1} = 0 \\ a, b, c \in \mathbb{R}, \ a \neq 0 \end{array} \right\} \Longrightarrow \begin{cases} y_i = k_1 \lambda_1^i + k_2 \lambda_2^i \\ \lambda_1, \lambda_2 : \quad a\lambda^2 + b\lambda + c = 0 \\ (\text{if } \lambda_1 \neq \lambda_2) \\ (\text{if } \lambda_1 = \lambda_2 : \quad y_i = (k_1 + i k_2) \lambda_1^i) \end{cases}$$

In our case (defining $U_0 = 0$):

$$p U_{i+1} - U_i + q U_{i-1} = 0 \Longrightarrow U_i = p U_{i+1} + q U_{i-1} \qquad \forall i > 1$$

characteristic equation: $p\lambda^2 - \lambda + q = 0 \Longrightarrow \lambda_1 = 1, \ \lambda_2 = \frac{q}{p}$

- $p \neq q$: Generic solution: $U_i = k_1 1^i + k_2 \left(\frac{q}{p}\right)^i$
  $U_0 = 0 \Longleftrightarrow k_2 = -k_1$ and so:
  $U_i = k_1 \left(1 - \left(\frac{q}{p}\right)^i\right)$:
  - $q > p$: $U_i \in [0,1] \Longleftrightarrow k_1 = 0$, and so:
    $U_i = 0 \ \forall i \Longleftrightarrow$ Ruin, i.e. absorption to 0
  - $q < p$: $\lim_{i \to \infty} U_i = k_1$, in order to get the bigger solution we get $k_1 = 1$, and so:
    $U_i = 1 - \left(\frac{q}{p}\right)^i$ = probability that the gambler becomes ∞-rich
    $\left(\frac{q}{p}\right)^i$ = probability of the ruin (= probability of absorption in 0)
- $p = q = \frac{1}{2}$: $U_i = k_1 + k_2 i$: $U_i = 0 \Longrightarrow i k_2 = -k_1$
  $U_0 = 0 \Longrightarrow k_1 = 0 \Longrightarrow k_2 = 0$

Theorem (Absorption probability in a recurrent class):
The probability of absorption in a recurrent class C is defined as $(V_i)_{i \in T}$.
V_i is the smallest $[0,1]$-valued solution of:

$$V_i = \sum_{j \in C} p_{ij} + \sum_{j \in T} p_{ij} V_j$$

Moreover, if T is finite, the solution $((V_i)_i)$ is unique.

proof.

We define $V_i^{(n)} = \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, .., X_1 \notin C)$ = probability of absorption in C at time n

$$V_i^{(1)} = \sum_{j \in C} p_{ij}$$

$$\begin{aligned} V_i^{(n)} &= \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, .., X_1 \notin C) \qquad (n > 1) \\ &= \sum_{j \in T} \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, .., X_2 \notin C, X_1 = j) \\ &= \sum_{j \in T} \mathbb{P}_i(X_n \in C, .., X_2 \notin C \mid X_1 = j) \ \mathbb{P}(X_1 = j) \\ &= \sum_{j \in T} V_i^{(n-1)} \ p_{ij} \end{aligned}$$

The event "absorption in C" can be written as: $\bigcup_{n=1}^{\infty} \{X_n \in C\}$

Moreover: $\bigcup_{n\geq 1} \{X_n \in C\} = \bigcup_{n\geq 1} \{X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C\}$

disjoint union

$$V_i^{(n)} = \sum_{j\in T} p_{ij} V_j^{(n-1)}$$

$$\begin{aligned}
\Longrightarrow \quad V_i &= \mathbb{P}_i\left(\bigcup_{n\geq 1} \{X_n \in C\}\right) \\
&= \mathbb{P}_i\left(\bigcup_{n\geq 1} \{X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C\}\right) \\
&= \sum_{n\geq 1} \mathbb{P}_i\left(\{X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C\}\right) \\
&= \sum_{n\geq 1} V_i^{(n)} \\
&= V_i^{(1)} + \sum_{n>1} V_i^{(n)} \\
&= \sum_{j\in C} p_{ij} + \sum_{n\geq 2} \sum_{j\in T} p_{ij} V_j^{(n-1)} \\
&= \sum_{j\in C} p_{ij} + \sum_{j\in T} \sum_{m\geq 1} p_{ij} V_j^{(m)} \\
&= \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} V_j
\end{aligned}$$

Moreover, it's the smallest $[0,1]$-valued solution:

let $(X_i)_{i\in T}$ be another $[0,1]$-valued solution of $X_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} X_j$

Since $V_i^{(1)} = \sum_{j\in C} p_{ij}$ we have:

$$X_i = V_i^{(1)} + \sum_{j\in T} p_{ij} X_j \geq V_i^{(1)} \qquad \forall i \in T.$$

Suppose $X_i \geq \sum_{k=1}^{n} V_i^{(k)}$

$$X_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} X_j \geq \underbrace{\sum_{j\in C} p_{ij}}_{V_j^{(1)}} + \underbrace{\sum_{j\in T} p_{ij} \sum_{k=1}^{n} V_j^{(k)}}_{\sum_{k=1}^{n}\left(\sum_{j\in T} p_{ij} V_j^{(k)}\right)} = \sum_{k=1}^{n+1} V_i^{(k)}$$

$$V_j^{(1)} + \sum_{k=1}^{n} V_j^{(k+1)}$$

So, by induction, we showed that $X_i \geq \sum_{k=1}^{n} V_i^{(k)} \quad \forall n$

$\Longrightarrow$ taking the lim $n\to\infty$: $X_i \geq V_i \quad \forall i \in T$ ■

Example: Gambler's win problem

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ 0 & q & 0 & p & \cdots \\ 0 & 0 & q & 0 & \cdots \\ & \vdots & & & \end{bmatrix}$$

states $= \{0, 1, 2, 3, \dots\}$

$p > q$ (since if $q \geq p$ the probability of win is 1)

$\{0\}$ absorbing, $\{1, 2, 3, \dots\}$ transient

Ruin $\Longleftrightarrow$ absorption in 0 : $C = \{0\}$, $T = \{1, 2, 3, \dots\}$

$$\begin{cases} V_1 = pV_2 + q \\ V_i = pV_{i+1} + qV_{i-1} & i > 1 \end{cases}$$

Or, with the convention $V_0 = 1$:

$$\begin{cases} V_0 = 1 \\ V_i = p V_{i+1} + q V_{i-1} & i \geq 1 \end{cases}$$

General solution:

$$V_i = k_1 + k_2 \left(\frac{q}{p}\right)^i$$

and since $V_0 = 1 \Rightarrow k_1 + k_2 = 1 \Rightarrow V_i = k_1 + (1-k_1)\left(\frac{q}{p}\right)^i$

We want the smallest $[0-1]$-valued solution:

$i \to \infty \quad V_i \to k_1 \in [0,1] \Rightarrow k_1 = 0$ is the smallest

$$\Rightarrow V_i = \left(\frac{q}{p}\right)^i$$

Theorem (Mean absorption time in recurrent classes): Suppose that the set of states E is finite and there is an unique recurrent class C. Then, the mean absorption time w_i in C is finite and satisfies:

$$w_i = 1 + \sum_{j \in T} p_{ij} w_j \qquad i \in T$$

(w_i is the unique solution)

Interpretation: the mean time for absorption in C (w_i) is the time for the 1-step absorption ($=1$) + mean time for absorption in C after moving to a different transient state j

Lemma: Let V be a $\mathbb{N}$-valued random variable $\Rightarrow \mathbb{E}[V] = \sum_{n=0}^{\infty} \mathbb{P}(V > n)$

proof. (Lemma)

$$\mathbb{E}[V] = \sum_{m=1}^{\infty} m \, \mathbb{P}(V=m) = \sum_{m=1}^{\infty} \sum_{k=1} 1 \cdot \mathbb{P}(V=m)$$
$$= \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \mathbb{P}(V=m)$$
$$= \sum_{k=1}^{\infty} \mathbb{P}(V \geq k)$$
$$= \sum_{n=0}^{\infty} \mathbb{P}(V > n) \qquad \blacksquare$$

proof. (Theorem)

Step 1. Check that w_i are finite by: $\exists c \in (0,1)$: $\mathbb{P}_i(V>n) \leq c^n$, where V is the absorption time in C.

$\mathbb{P}_i(V>n) = \mathbb{P}(X_n \in T, \dots, X_1 \in T) := U_i^{(n)}$ = probability of staying in transient states from 1 to n steps

$$U_i^{(n+1)} = \sum_{j \in T} p_{ij} U_j^{(n)}$$

$$U_i^{(n+2)} = \sum_{\substack{j \in T \\ k \in T}} p_{ij} p_{jk} U_k^{(n)} \leq \sum_{k \in T} \left(\sum_{j \in E} p_{ij} p_{jk} \right) U_k^{(n)} =$$
$$= \sum_{k \in T} p_{ik}^{(2)} U_k^{(n)}$$

Iterating: $U_i^{(n+m)} \leq \sum_{j \in T} p_{ij}^{(m)} U_j^{(n)}$

If i,j are transient $\left(\sum_{n \geq 1} p_{ij}^{(n)} < \infty\right) \Rightarrow \lim_{m \to \infty} p_{ij}^{(m)} = 0$

$\Rightarrow$ If T is finite we can chose m s.t. $\sum_{j \in T} p_{ij}^{(m)} < 1 \quad \forall i$ and

we set $M = \max_{i \in T} \left\{ \sum_{j \in T} p_{ij}^{(m)} \right\}$, that will be < 1.

Then: $U_i^{(m)} \le \sum_{j\in T} p_{ij}^{(m)} U_j^{(n-m)}$

$$\le \max_{j\in T} U_j^{(n-m)} \sum_{j\in T} p_{ij}^{(m)}$$

$$\le \max_{j\in T} U_j^{(n-m)} M \longrightarrow$$ the probability of remaining forever in a transient state is exponentially decreasing

Iterating: $\max_{i\in T} U_i^{(n)} \le M^2 \max_{i\in T} U_i^{(n-2m)} \le \dots$

$$\le M^{\frac{n}{m}} \underbrace{\max_{i\in T} U_i^{(n-(\frac{n}{m})m)}}_{\le 1 \text{ since they're probabilities}}$$

$$\Longrightarrow \mathbb{P}_i(V>n) \le \max_{i\in T} U_i^{(n)} \le M^{\frac{n}{m}} \le M^{\frac{n}{m}-1}$$

$$\le M^{-1}(M^{\frac{1}{m}})^n$$

$$\Longrightarrow \mathbb{E}_i[V] = \sum_{n=0}^{\infty} \mathbb{P}_i(V>n) \le M^{-1}\sum_{n=0}^{\infty} (M^{\frac{1}{m}})^n =$$

$$\le (M(1-M^{1/m}))^{-1} < \infty$$

Step 2. Check the formula: $w_i = 1 + \sum_{j\in T} p_{ij} w_j$

$$\mathbb{E}_i[V] = \sum_{n=1}^{\infty} n\,\mathbb{P}_i(V=n)$$

$$= \underbrace{\sum_{n=1}^{\infty} n\,\mathbb{P}_i(V=n, X_1\in T)}_{\text{if } X_1\in T \text{ then } V\ge 2 \text{ since we're starting from } X_0\in T,\ X_1\in T \Rightarrow \text{ we can be in } C \text{ not before than } X_2.} + \underbrace{\sum_{n=1}^{\infty} n\,\mathbb{P}_i(V=n, X_1\in C)}$$

$\{V=n, X_1\in C\} = \begin{cases} \emptyset & n>1 \\ \{X_1\in C\} & n=1 \end{cases}$

(since $X_1 \in C$ then it cannot be (for instance) $V=2$, since we're already in C)

$$= \left[\sum_{n=2}^{+\infty} n \sum_{j\in T} \mathbb{P}_i(V=n, X_1=j)\right] + \sum_{j\in C} p_{ij}$$

$$= \left[\sum_{n=2}^{\infty} n \sum_{j\in T} \underbrace{\mathbb{P}_i(V=n \mid X_1=j)}\, p_{ij}\right] + \sum_{j\in C} p_{ij}$$

$$= \mathbb{P}(V=n \mid X_1=j, X_0=i) \overset{M}{=} \mathbb{P}(V=n \mid X_1=j)$$

$$= \mathbb{P}(V=n-1 \mid X_0=j) = \mathbb{P}_j(V=n-1)$$

$$= \left[\sum_{n=1}^{\infty} (n+1) \sum_{j\in T} \mathbb{P}_j(V=n)\, p_{ij}\right] + \sum_{j\in C} p_{ij}$$

$$= \left[\sum_{j\in T} \underbrace{\sum_{n=1}^{\infty} n\,\mathbb{P}_j(V=n)}_{\mathbb{E}_j[V] = w_j} p_{ij}\right] + \underbrace{\left[\sum_{j\in T} p_{ij} \underbrace{\sum_{n=1}^{\infty} \mathbb{P}_j(V=n)}_{1}\right] + \left[\sum_{j\in C} p_{ij}\right]}_{1}$$

$$\Longrightarrow w_i = \sum_{j\in T} w_j p_{ij} + 1 \qquad \blacksquare$$

Remark: If there is a situation with two (or more) recurrent classes C_1, C_2 and there is a strictly positive probability of ending in C_1 and a strictly positive prob. of ending in C_2, then the mean absorption time of ending in C_1 is ∞ (since we can end in C_2).
To evaluate the "true mean" for, for instance, C_1 we have to neglect all C_i $i \neq 1$.

Example: Coupon collector problem.
The collection is made of N pictures. At time n we buy an envelope containing a random picture. How many pictures do we have to buy to complete the collection?

$(X_n)_{n \geq 0}$ MC : X_n = # different pictures collected at time n
$(X_0 = 0)$

Transitions probabilities:

$$\mathbb{P}(X_{n+1} = k+1 \mid X_n = k) = \frac{N-k}{N}$$

$$\mathbb{P}(X_{n+1} = k \mid X_n = k) = \frac{k}{N}$$

$$\mathbb{P}(X_{n+1} = j \mid X_n = k) = 0 \quad \forall j \neq k, k+1$$

Transition matrix:

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & & & \\ 0 & 1/N & \frac{N-1}{N} & 0 & 0 & \cdots & & & \\ 0 & 0 & 2/N & \frac{N-2}{N} & 0 & \cdots & & & \\ & & \cdots & & & & & & \\ & & \cdots & & & & 0 & \frac{N-1}{N} & \frac{1}{N} \\ & & \cdots & & & & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} T = \{0, 1, \dots, N-1\} \\ C = \{N\} \end{array}$$

w_i = mean time for reaching N starting from i : $\begin{cases} w_i = 1 + p_{ii} w_i + p_{i,i+1} w_{i+1} & i = 0, \dots, N-1 \\ w_N = 0 \end{cases}$

$$\Longrightarrow \quad w_i = 1 + \frac{i}{N} w_i + \frac{N-i}{N} w_{i+1}$$

$$\Longrightarrow \quad [\dots] \quad \Longrightarrow \quad w_i = \frac{N}{N-i} + \frac{N}{N-i-1} + \dots + \frac{N}{N-(N-1)} \qquad (i = 0, \dots, N-1)$$

$$\Longrightarrow \quad w_0 = \sum_{k=0}^{N-1} \frac{N}{N-k} = N \sum_{h=1}^{N} \frac{1}{h} \underset{N \text{ large}}{\approx} N \log(N)$$

If a collector has $N \log(N)$ pictures (generally):

- probability that there is no second copy of a given copy:

$$\left(\frac{N-1}{N}\right)^{N \log(N)} = \left(\left(\frac{N-1}{N}\right)^{N}\right)^{\log(N)} \approx e^{-\log(N)} = \frac{1}{N}$$

- probability that k fixed copies have no second copy:

$$\left(\frac{N-k}{N}\right)^{N \log(N)} = \left(1 - \frac{k}{N}\right)^{N \log(N)} \approx e^{-k \log(N)} = \left(\frac{1}{N}\right)^{k}$$

Example: Gambler's win against a bank
We consider the fair case of the game, namely $p = q = 1/2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots & & & \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & & & & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & & & & \cdots & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} \text{states} = \{0, 1, 2, \dots, N\} \\ \text{Absorbing classes: } \{0\}, \{N\} \end{array}$$

$\forall i \in \{1, \dots, N-1\}$ we are interested in computing the mean win time w.r.t. $\mathbb{P}_i(\cdot \mid \underbrace{\cap_{n=1}^{\infty} (X_n \neq N)}_{})$

win occurs with probability $= 1$

We have to exclude the possibility of going from N-1 to N because $\{N\}$ is a recurrent class and if we consider it too then the mean of absorption in $C_1 = \{0\}$ will be ∞.
(since we would consider the case in which we end up in $C_2 = \{N\}$)

$$\begin{cases} w_0 = 0 \\ w_i = 1 + \frac{1}{2} w_{i-1} + \frac{1}{2} w_{i+1} & 1 \le i < N-1 \\ w_{N-1} = 1 + w_{N-2} \end{cases}$$

This is a non-homogeneous finite diff. equation.
The associated homogeneous eq. and charact. eq.:

$$\left.\begin{array}{l} y_i = \frac{1}{2} y_{i+1} + \frac{1}{2} y_{i-1} \\ \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda = 1) \end{array}\right\} \Longrightarrow \quad y_i = c_1 + i c_2$$

(general solution)

We try with: $w_i = ai^2 + bi + c \quad (*)$

$$\Longrightarrow \quad 1 + \frac{1}{2} w_{i-1} + \frac{1}{2} w_{i+1} = 1 + \frac{1}{2} a(i-1)^2 + \frac{1}{2} b(i-1) + c\frac{1}{2} + $$
$$+ \frac{1}{2} a(i+1)^2 + \frac{1}{2} b(i-1) + c\frac{1}{2}$$
$$= 1 + ai^2 + a + bi + c$$

We compare $1 + ai^2 + a + bi + c$ with $(*)$ and we obtain:

$a + c + 1 = 0 \quad \longrightarrow$ we consider the simpler: $a = -1$, $c = 0$

$\Longrightarrow$ special solution: $w_i = -i^2$

general solution: $y_i = c_1 + i c_2$

$\Longrightarrow$ boundary conditions $(w_i = -i^2 + c_1 + i c_2)$:

$$\left.\begin{array}{l} w_0 = 0 \\ w_1 = 1 + \frac{1}{2} w_2 \end{array}\right\} \longrightarrow \quad c_1 = 0$$

$$w_{N-1} = 1 + w_{N-2} \longrightarrow \quad c_2 = 1 - (N-2)^2 + (N-1)^2 = 2N-2$$

$\Longrightarrow$ Ultimate solution:

$$\mathbf{w_i} = -i^2 + 2(N-1)i$$

= mean duration of the game if you're sure you'll be ruined

Theorem (Transience criterion):
$(X_n)_{n\ge0}$ irreducible MC (with countable space $\mathbf{E}$) is transient if and only if there exists a bounded non-constant solution of:

$$\underbrace{\sum_{k\in E} p_{jk}\, y_k = y_j}_{((*))} \qquad \text{for all } j \in E \text{ but one.}$$

for all $j \in E$ except for (at most) one.

proof. ()

1. This condition is necessary:

Suppose $(X_n)_{n\ge0}$ transient. We denote by $\mathbf{e}$ the (unique) state for which $((*))$ does not hold.
We consider the transformed MC in which $\mathbf{e}$ is absorbing. Its transition matrix is:

$$\tilde{p}_{ij} = \begin{cases} \delta_{ij} & j = e \\ p_{ij} & j \ e \end{cases}$$

Since the given MC is transient (the transformed is not), there exists $i \in \mathbf{E}$ such that:

$$\tilde{V}_i := \mathbb{P}_i(T_e < +\infty) < 1 \qquad (T_e = \text{first entrance time in } \mathbf{e})$$

< 1 because if $\tilde{V}_i = \mathbb{P}_i(T_e < \infty) = 1 \quad \forall i$ then:

$$\mathbb{P}_e(T_e < +\infty) = p_{ee} + \sum_{k \neq e} p_{ek} \mathbb{P}_k(T_e < +\infty)$$
$$= p_{ee} + \sum_{k \neq e} p_{ek} = 1$$

which contraddicts the transiency of $(X_n)_{n \geq 0}$.

$\Longrightarrow \tilde{V}_i < 1$ for some i and $\tilde{V}_e = 1$.

$\Longrightarrow (V_i)_{i \in E}$ is $[0,1]$-valued (bounded) and non-constant.

Moreover, $(V_i)_{i \in E}$ are the absorption probabilities in e for the MC with $(\tilde{p}_{ij})_{i,j \in E}$, therefore:

$$\tilde{V}_i = \tilde{p}_{ie} + \sum_{k \neq e} \tilde{p}_{ik} \tilde{V}_k \qquad (\bigstar)$$

but $\tilde{p}_{ik} = p_{ik} \;\; \forall k \neq e$ (if $i \neq e$), $\tilde{V}_e = 1 \Longrightarrow \tilde{V}_i = \sum_{k \in E} p_{ik} \tilde{V}_k$

$\Longrightarrow (\bigstar)$ and $((*))$ are equivalent and satisfied

2. This condition is sufficient:

Suppose: $\sum_{k \in E} p_{ik} y_k = y_i \quad \forall i \in E \setminus \{e\} \qquad (\blacktriangle)$

where y_i is bounded and non-constant. ($(y_i)_{i \in E}$).

Then, if we consider $(\tilde{p}_{ij})_{i,j \in E}$ as before, from $(\blacktriangle)$ we get:

$$\sum_{k \in E} \tilde{p}_{ik} y_k = y_i \qquad \forall i \in E$$

Iterating: $\underbrace{\sum_{k \in E, h \in E} \tilde{p}_{ik} \tilde{p}_{kh} y_h}_{\sum_{h \in E} \left(\sum_{k \in E} \tilde{p}_{ik} \tilde{p}_{kh} \right) y_h \; = \; \sum_{h \in E} \tilde{p}_{ih}^{(2)} y_h} = y_i$

Iterating n times:

$$\sum_{k \in E} \tilde{p}_{ik}^{(n)} y_k = y_i$$

If the MC $(X_n)_{n \geq 0}$ was recurrent then the transformed MC $(\tilde{p}_{ij})_{i,j}$ would converge to the state e, namely:

$$\lim_{n \to \infty} \tilde{p}_{ie}^{(n)} = 1 \qquad \forall i \in E$$

$(\lim_{n \to \infty} \tilde{p}_{je}^{(n)} = \lim_{n \to \infty} \mathbb{P}_j(\tilde{X}_n = e) = \lim_{n \to \infty} \mathbb{P}_j(T_e \leq n) = 1 \Longleftrightarrow$ recurrence$)$

Then $\forall j \neq e$:
$$|y_j - y_e| = \lim_n |y_j - \tilde{p}_{je}^{(n)} y_e|$$
$$= \lim_n |y_j - (y_j - \textstyle\sum_{k \neq e} \tilde{p}_{jk}^{(n)} y_k)| \quad \circledast$$
$$= \lim_n |\textstyle\sum_{k \neq e} \tilde{p}_{jk}^{(n)} y_k|$$
$$\leq \sup_k |y_k| \lim_n \textstyle\sum_{k \neq e} \tilde{p}_{jk}^{(n)}$$
$$\leq \sup_k |y_k| \lim_n (1 - \tilde{p}_{je}^{(n)}) = 0$$

$\circledast \; \sum_{k \in E} \tilde{p}_{jk}^{(n)} y_k = \sum_{k \neq e} \tilde{p}_{jk}^{(n)} y_k + \tilde{p}_{je}^{(n)} y_e$
$= y_j \quad \forall j \in E$

$\Longrightarrow y_j = y_e \quad \forall j \neq e$

$\Longrightarrow (y_i)_{i \in E}$ constant $\Longrightarrow$ contraddiction $\blacksquare$

Theorem (Recurrence criterion): $(X_n)_{n\geq 0}$ irreducible MC. If there exists a family $(y_j)_{j\in E}$ such that:

- $\sum_{k\in E} p_{jk}\, y_k \leq y_j \quad \forall j\in E$ but (at most) one
- $\lim_{k\to\infty} y_k = +\infty \quad (\forall M>0\ \exists F\subseteq E$ finite s.t. $y_k > M\ \forall k\notin F)$

then the Markov chain $(X_n)_{n\geq 0}$ is recurrent.

Example: Application of this criteria to a queue model.

- the counter serves one customer per unit of time
- the number of customers arriving in the system at each unit of time is a random variable A_n = arrivals at n s.t.

$$\mathbb{P}(A_n = k) = a_k \qquad 0\leq a_k \leq 1\ ,\quad \sum_k a_k = 1$$

- the number of customers arriving at different times are $\perp\!\!\!\perp$

Transition matrix:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ & \cdots & & & \end{bmatrix} \qquad \left.\begin{array}{l} a_0 > 0 \\ 0 < a_0 + a_1 < 1 \end{array}\right\} \text{there is always the possibility of 0 arrivals}$$

$\Longrightarrow$ the Markov chain is irreducible

We want to find out if this MC is transient or recurrent
(transient = the queue explodes (at finite time)
recurrent = the queue can be very long but it'll go back to 0).

The average number of customers arriving at each time:

$$\lambda = \sum_{k\geq 1} a_k k \qquad \text{intuitively: } \begin{array}{l}\lambda > 1 \Longrightarrow \text{transient} \\ \lambda \leq 1 \Longrightarrow \text{recurrent}\end{array}$$

- $\boxed{\lambda > 1}$
We want to prove that is transient $\longrightarrow$ first criteria.
We look for solutions of:

$$\sum_{k\geq 0} p_{jk}\, y_k = y_j \qquad \forall j > 0 \qquad (j\neq 0)$$

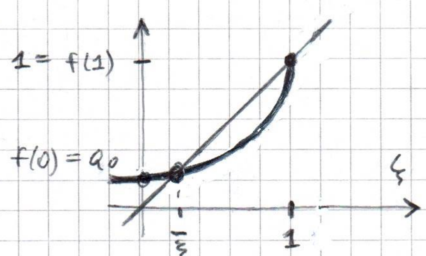
of the form $y_k = \xi^k$ with $0<\xi<1$

$$\Longrightarrow \sum_{k\geq j-1} a_{k-j+1}\, \xi^k = \xi^j$$

$$\Longleftrightarrow \sum_{k\geq j-1} a_{k-j+1}\, \xi^{k-j+1} = \xi$$

$$\Longleftrightarrow \sum_{h\geq 0} a_h\, \xi^h = \xi$$

$$f(\ ) := \sum_{h\geq 0} a_h \xi^h \qquad |\xi|\leq 1\ :$$



$a_h \geq 0 \Longrightarrow \xi \mapsto f(\xi)$ is increasing in $[0,1]$

$$f(1) = \sum_{h\geq 1} h\, a_h = \lambda$$

$$f'(1) = \lambda > 1$$

$\Longrightarrow \exists\, \xi\in(0,1)$ such that $f(\xi)=\xi$

- $\boxed{\lambda \leq 1}$

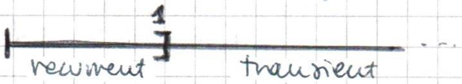
  we want to prove that is recurrent $\Longrightarrow$ second criteria.
  We look for the unbounded solution of:

$$\sum_{k \geq 0} p_{jk}\, y_k = y_j \quad \forall j > 0$$

Chose $y_k = k$, for $j > 0$:

$$\begin{aligned}\sum_{k\geq 0} p_{jk}\, y_k &= \sum_{k \geq j-1} a_{k-j+1}\, k \\ &= \sum_{h\geq 0} a_h\,(h + (j-1)) \\ &= \sum_{h\geq 0} a_h h + (j-1)\sum_{h\geq 0} a_h \\ &= \lambda + j - 1 = (\lambda - 1) + j = (\lambda-1) + y_j \leq 1 \quad (\lambda \leq 1)\end{aligned}$$

So:



the smallest is λ $\Rightarrow$ the smallest the number of arrivals

Indeed for $0 < \lambda < 1$ it is fast recurrent $\Longleftrightarrow$ $\exists$ invariant distribution
(the system has a stationary state)

Theorem (Sufficient condition for the existence of invariant distributions)
$(X_n)_{n\geq 0}$ irreducible MC.
If we find $(y_j)_j$, $(x_j)_j$ both unbounded $\left(\lim_{j\to+\infty} y_j = \lim_{j\to+\infty} x_j = +\infty\right)$ s.t.:

$$\sum_{k\geq 0} p_{jk}\, y_k \leq y_j - x_j \quad \forall j$$

$\Longrightarrow$ the MC admits a unique invariant density

Example: Application at the queue model.
We assume $\sum_{k\geq 0} k^2 a_k < +\infty$ (= the arrivals have a finite second order moment, we call it $\boldsymbol{m_2}$)

Consider $y_j = j^2$

$$\begin{aligned} j>0: \sum_{k\geq 0} p_{jk}\, y_k &= \sum_{k\geq j-1} a_{k-j+1}\, k^2 \\ &= \sum_{h\geq 0} a_h\,(h + (j-1))^2 \\ &= \sum_{h\geq 0} h^2 a_h + 2(j-1)\sum_{h\geq 0} h a_h + (j-1)^2 \sum_{h\geq 0} a_h \\ &= m_2 + 2(j-1)\lambda + (j-1)^2 \\ &= j^2 - 2j + 1 + 2j\lambda - 2\lambda + m_2 \\ &= j^2 - 2(1-\lambda)j + (1 - 2\lambda + m_2) \\ &= y_j - (2(1-\lambda)j - (1-2\lambda+m_2)) \\ &:= y_j - x_j \qquad (x_j \longrightarrow \infty \text{ as } j \to \infty) \quad \text{if } \lambda < 1)\end{aligned}$$

$\Rightarrow$ The MC admits a unique invariant density if $\lambda < 1$.

Def. **SOJOURN TIME**: We call T = set of transient states. Consider the subset $S \subseteq T$.
We define the sojourn time in S the total time spent in S:

$$T_S = \sum_{n\geq 0} \mathbb{1}\{X_n \in S\}$$

Def. **MOMENT GENERATING FUNCTION**: the moment gen. fun. of a random var. T is:

$$m_i = \mathbb{E}_i[\, z^T\,] \qquad \forall i \in E$$

Notice that: if S ($\subseteq T$ = transient states) is finite, then the moment generator function:

$$m_i = \mathbb{E}_i[z^{T_S}] \quad \text{is well defined}$$

Alternative notation: $\mathbb{E}[z^T] = \mathbb{E}[e^{Tt}]$ $(= \phi(t), \ t: e^t = z)$

Properties:
(of the moment generator function of $T = T_S$)

- $m_i(z) = \sum_{k \geq 0} z^k \, \mathbb{P}_i(T=k)$

  it can be seen as Taylor series, where $\mathbb{P}_i(T=0)$ is the constant, $\mathbb{P}_i(T=1)$ is the coeff. multiplying the first order term, etc...

- $m_i'(z) = \mathbb{E}_i\left[\frac{d}{dz}\left(e^{T\log(z)}\right)\right] = \mathbb{E}_i\left[\frac{1}{z} z^T\right] = \mathbb{E}_i[z^{T-1}]$

(1) • $m_i(z) = 1$ if i is recurrent

(2) • $m_i(z) = z \sum_j p_{ij} m_j(z)$ if $i \in S$

(3) • $m_i(z) = \sum_j p_{ij} m_j(z)$ if $i \in T \setminus S$ = (transient states) $\setminus S$

proof.

(1) i recurrent $\Longrightarrow T=0 \Longrightarrow \mathbb{E}_i[z^T] = 1$

(2)
(3) $i \in T$ = transient states

$$T_S = \mathbb{1}\{X_0 \in S\} + \sum_{n \geq 1} \mathbb{1}\{X_n \in S\} := \mathbb{1}\{X_0 \in S\} + \tilde{T}$$

$$\mathbb{E}_i[z^T] = \sum_{j \in E} \mathbb{E}_i[z^T \mid X_1 = j] \, \mathbb{P}_i(X_1 = j)$$

$$= \sum_{j \in E} \mathbb{E}_i[z^{\mathbb{1}\{X_0 \in S\}} z^{\tilde{T}} \mid X_1 = j] \, \mathbb{P}_i(X_1 = j)$$

$z^{\mathbb{1}\{X_0 \in S\}} = \begin{cases} z^0 = 1 & i \in S \\ z^1 = z & i \notin S \end{cases}$

$$= z^{\mathbb{1}\{i \in S\}} \sum_{j \in E} \mathbb{E}_i[z^{\tilde{T}} \mid X_1 = j] \, p_{ij}$$

$$= z^{\mathbb{1}\{i \in S\}} \sum_{j \in E} \mathbb{E}[z^{\tilde{T}} \mid X_1 = j, X_0 = i] \, p_{ij}$$

Markov property

$$= z^{\mathbb{1}\{i \in S\}} \sum_{j \in E} \mathbb{E}[z^{\tilde{T}} \mid X_1 = j] \, p_{ij}$$

?

$$= z^{\mathbb{1}\{i \in S\}} \sum_{j \in E} \mathbb{E}_j[z^T] \, p_{ij}$$

$\Longrightarrow$ • $i \in S$: $\quad m_i = z \sum_{j \in E} \mathbb{E}_j[z^T] \, p_{ij}$

• $i \notin S$: $\quad m_i = \sum_{j \in E} \mathbb{E}_i[z^T] \, p_{ij}$ ∎

Example: Gambler's ruin
symmetric case ($p = q = \frac{1}{2}$)
finite state space $:= \{0, 1, 2, 3\}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We want to compute the moment generator function of the total time spent in $S = \{1\}$ starting from the state 1.

$$\begin{cases} m_0(z) = m_3(z) = 1 \\ m_1(z) = z\left(\frac{1}{2} m_2(z) + \frac{1}{2} m_0(z)\right) = z\left(\frac{1}{2} m_2(z) + \frac{1}{2}\right) \\ m_2(z) = \frac{1}{2} m_1(z) + \frac{1}{2} \end{cases}$$

$$\Longrightarrow m_1(z) = \frac{3/4 \, z}{1 - z/4}, \quad m_2(z) = \frac{1 + z/2}{2(1 - z/4)}$$