SET THEORY

Let X be a set. We denote:

$$\mathcal{P}(X) := \{Y \mid Y \subseteq X\}$$ and it's called power set.

Let I be the index set. Then:

$\{E_i\}_{i \in I}$ is called family / collection indexed by I

$E_i \subseteq X \quad \forall i \in I.$

In the special case in which $I = \mathbb{N}$ we have $\{E_n\}_n$ ($\{E_n\}_{n\in\mathbb{N}}$) and we call it a sequence of sets.

Def. A sequence $\{E_n\}_n \subseteq \mathcal{P}(X)$ is said to be monotone increasing (or ascending) if:

$$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N} \qquad (E_n \nearrow)$$

The sequence is monotone decreasing (or descending) if:

$$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N} \qquad (E_n \searrow)$$

Def. Consider a family $\{E_i\}_{i\in I} \subseteq \mathcal{P}(X)$, we define:

$$\bigcup_{i\in I} E_i = \{x \in X : \exists i \in I, \ x \in E_i\}$$

$$\bigcap_{i\in I} E_i = \{x \in X : \forall i \in I, \ x \in E_i\}$$

A family $\{E_i\}_{i\in I}$ is said to be disjoint if: $E_i \cap E_j = \emptyset \quad \forall i,j \in I, \ i \neq j$

Ex. check the following identities:

$$[a,b] = \bigcap_{n=1}^{+\infty} \left(a - \tfrac{1}{n}, b + \tfrac{1}{n}\right) \qquad a,b \in \mathbb{R}, \quad a<b$$

$$(a,b) = \bigcup_{n=1}^{+\infty} \left[a + \tfrac{1}{n}, b - \tfrac{1}{n}\right]$$

$$(a,b] = \bigcap_{n=1}^{+\infty} \left(a, b + \tfrac{1}{n}\right)$$

$$(a,b) = \bigcup_{n=1}^{+\infty} \left(a, b - \tfrac{1}{n}\right]$$

Intersections/Unions can change sequences of open (/closed) intervals into closed (/open) ones.

Def. Consider a sequence $\{E_n\}_n \subseteq \mathcal{P}(X)$. We define:

they're both sets

$$\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{+\infty} \left[\bigcup_{n=k}^{+\infty} E_n\right] \subseteq X$$

$$\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{+\infty} \left[\bigcap_{n=k}^{+\infty} E_n\right] \subseteq X$$

If $\limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n \ :\Longrightarrow \ \lim_{n\to\infty} E_n = F$

Ex. Check that: $x \in \limsup_{n\to\infty} E_n \iff x \in E_n$ for infinitely many n

$$x \in \liminf_{n\to\infty} E_n \iff \exists k \in \mathbb{N} \text{ s.t. } x \in E_n \ \forall n \geq k$$

$$\left(\liminf_{n\to\infty} E_n\right)^c = \limsup_{n\to\infty} (E_n^c)$$

Ex. If $E_n \nearrow \ \Longrightarrow \ \lim_{n\to\infty} E_n = \bigcup_{n=1}^{+\infty} E_n$

If $E_n \searrow \ \Longrightarrow \ \lim_{n\to\infty} E_n = \bigcap_{n=1}^{+\infty} E_n$

Def. A family of sets $\{E_i\}_{i\in I}$ is called a cover (/covering) of X if:

$$X \subseteq \bigcup_{i\in I} E_i$$

A subfamily of a cover which itself forms a cover is called a subcover.
Instead of I we take only some $i \in I$

Def. Let $E \subseteq X$. The function: $\mathbb{1}_E : X \to \mathbb{R}$ is defined as:

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called characteristic / indicator function.

Ex. Let $E_1, E_2 \subseteq X$. $\Longrightarrow$

$$\mathbb{1}_{E_1 \cap E_2} = \mathbb{1}_{E_1} \cdot \mathbb{1}_{E_2}$$

$$\mathbb{1}_{E_1 \cup E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} - \mathbb{1}_{E_1}\mathbb{1}_{E_2}$$

Ex. Consider $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}(X)$ disjoint, $E := \bigcup_{k=1}^{+\infty} E_k$.

$$\Longrightarrow \mathbb{1}_E = \sum_{k=1}^{+\infty} \mathbb{1}_{E_k}$$

Ex. Consider $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$, $P := \limsup_{n\to\infty} E_n$, $Q := \liminf_{n\to\infty} E_n$

(here liminf / limsup are intended for sets)

$$\Longrightarrow \mathbb{1}_P = \limsup_{n\to\infty} \mathbb{1}_{E_n}$$

$$\mathbb{1}_Q = \liminf_{n\to\infty} \mathbb{1}_{E_n}$$

here liminf / lim sup are intended for real numbers ($\mathbb{1}_{E_n} \in \{0,1\}$)

$$E = \lim_{n\to\infty} E_n \Longleftrightarrow \mathbb{1}_E = \lim_{n\to\infty} \mathbb{1}_{E_n}$$

RELATIONS

Let X be a set. A relation in X is a subset $R \subseteq X \times X$.

Def. R is an equivalence relation if:

(i) $(x,x) \in R \quad \forall x \in X$ — reflexivity

(ii) $(x,y) \in R \Longrightarrow (y,x) \in R$ — symmetry

(iii) $\left.\begin{matrix}(x,y) \in R \\ (y,z) \in R\end{matrix}\right\} \Longrightarrow (x,z) \in R$ — transitivity

$(x,y) \in R \leftrightsquigarrow xRy$

E.g. $X = \{\text{straight lines}\}$

$xRy \Leftrightarrow x \parallel y \quad x,y \in X$

if x: ↗ $\Rightarrow E_x$:

if x: → $\Rightarrow E_x$

The quotient set is:

$\frac{X}{R} = \{ \ldots, \ldots, \ldots \}$

Def. Given $x \in X$ we define:

$$E_x := \{ y \in X : yRx \}, \quad R = \text{equivalence relation}$$

called equivalence class of X.

Remark: $X = \bigcup_{x\in X} E_x$ (disjoint union) $\longrightarrow$ The equivalence classes form a partition of the set X

Def. $\frac{X}{R} := \{ E_x : x \in X \}$ is called quotient set. (set of sets and the elements are all equivalence classes)

Def. A relation in a set X is an order relation if:

(i) $xRx \quad \forall x \in X$ — reflexivity

(ii) $xRy,\ yRx \Longrightarrow x = y$ — antisimmetry

(iii) $xRy,\ yRz \Longrightarrow xRz$ — transitivity

Example: In $\mathbb{R}$: $R = \leq$

because of this example, it is used to write whatever order relation with "$\leq$"
$R \leftrightsquigarrow \leq$

14/09

A set $P \subseteq X$ with an order relation is called a partially ordered set.
If $\forall x, y \in P$ either $\underset{\text{(general ordering)}}{x R y}$ $[x \leq y]$ or $y R x$ $[y \leq x]$ (general ordering) (R order relation):
$\Longrightarrow$ P is totally ordered or a chain.

Example:
- $(\mathbb{R}, \leq)$ is a chain ($\forall x,y \in \mathbb{R}$ it's either $x \leq y$ or $y \leq x$)
- $X = \mathbb{Z}^n$: $x \underset{}{\overset{\text{generic}}{\leq}} y \iff x$ divides y

  X is partially ordered, but it's not a chain (2 is in relation with 4 (2 divides 4), 3 is **not** in relation with 4
  $\Rightarrow$ we found two elements (3,4) s.t. neither xRy or yRx)

CARDINALITY

Def. X, Y are called equipotent if there exists a bijection $f: X \to Y$.

Def. The cardinality of X is the collection of all sets equipotent to X:
$$|X| := \{ Y \mid \exists f: X \to Y \text{ bijection} \}$$

f INJECTIVE if $\forall x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
f SURJECTIVE if $\mathrm{Im}(f) = Y$ $(f: X \to Y)$

Remark: X, Y are equipotent $\iff$ $|X| = |Y|$

Def. $|X| \leq |Y| \iff \exists f: X \to Y$ injective

If $|X| \leq |Y|$ and $|X| \neq |Y| \Longrightarrow |X| < |Y|$

they're sets, hence "$\leq$", "$<$" represents generic relations

Theorem (Schröder - Bernstein):
If $|X| \leq |Y|$ and $|Y| \leq |X| \longrightarrow |X| = |Y|$.

Theorem (Cantor): $|X| < |\mathcal{P}(X)|$

we can obtain a bigger cardinality through the operation of power set

Def. X is said to be infinite if there exists $E \subsetneq X$ s.t. $|X| = |E|$.
For example: $[0,1] \subset \mathbb{R}$ has the same cardinality of $\mathbb{R}$
$\mathbb{N}$ has the same cardinality of odd/even numbers

Def. X is finite if it is not infinite.

! Remark: X finite $\iff \exists!\, n \in \mathbb{N}$ s.t. $|X| = |\{1, 2, \dots, n\}|$
and we write $\boxed{|X| = n}$: number of elements.
abuse of notation

Def. X is said to be countable if $|X| \leq |\mathbb{N}|$.
X is said to be uncountable if it is not countable.

Remark:
(i) $\mathbb{Q}$ is countable
(ii) $\mathbb{R}$ is not countable (cardinality of continuum)
(iii) X, Y countable $\longrightarrow$ $X \times Y$ is countable
(iv) Countable union of countable sets is countable
(v) $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$

The **continuum hypothesis**:
There is no set whose cardinality is strictly between that of $\mathbb{N}$ and $\mathbb{R}$.

Axiom of choice:
Given any disjoint family of sets $\{E_i\}_{i \in I}$ ($\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$), there exists a set $V \subseteq X$ that contains exactly one element of every E_i, i.e.:
$$V \cap E_i = \{x_i\} \quad \forall i \in I.$$
$\Longrightarrow$ Given a collection of non-empty sets we can choose one element from each set

Better: given a collection of non-empty sets $\exists$ a criterium (function) to extract exactly 1 element per set ($\forall$ collection)

Now we see an equivalent formulation.

Consider $(P, \leq)$, a partially ordered set.

Def. $M \in P$ is called maximal element if:

$$M \leq x \text{ for some } x \in P \implies x = M.$$

general order relation
(that's why it's called maximal and not maximum)

Let $A \subseteq P$ and $u \in P$.

If $\forall a \in A$, $a \leq u \implies$ (Def.) u is called upper bound (we're **not** saying $u \in A$)

general order

Moreover: if $u \leq v$ $\forall$ upperbounds v of A $\implies$ $u := \sup A$ = minimum of the upperbounds

general order

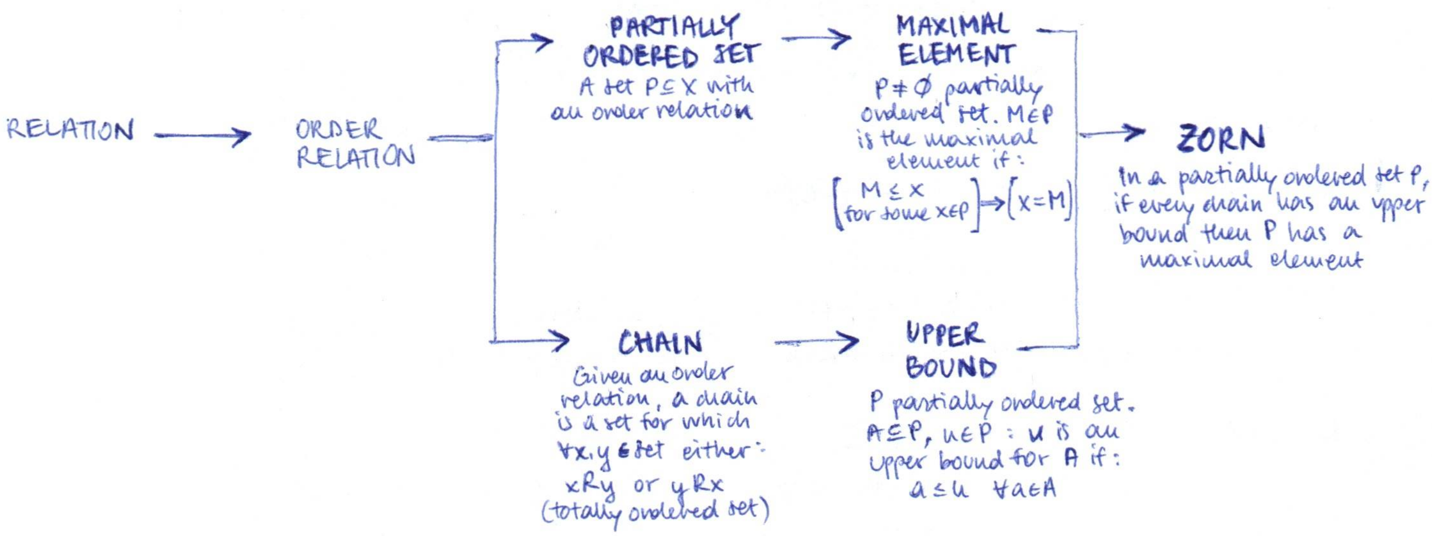
Zorn lemma:

In a partially ordered set P, if every chain has an upper bound then P has a maximal element. — equivalent to the Axiom choice

→ It tells us a condition which implies the existence of a maximal element w.r.t. any order relation

Ex. Prove that: $P = (0,1) \subset \mathbb{R}$ does not have a maximal element.

Remark: Axiom of choice $\iff$ Zorn's lemma



Ex. $P=(0,1)$ does not have a maximal element. $(\Rightarrow)$ The hypothesis of Zorn's lemma cannot be satisfied. We look for a chain in P which does not have an upper bound.
Consider $A=\{a_n\}$ where $a_n = 1-\frac{1}{n}$. (This is a chain since all elements are ordered $\longrightarrow$ w.r.t. $(\mathbb{R}, \leq)$)
A is a chain but A does not have an upper bound in P $\left(\sup_{n\in\mathbb{N}}\{a_n\} = 1 \notin P\right)$.
$\Longrightarrow$ the non-existence of a maximal element is not in contrast with Zorn

METRIC SPACES

Consider a metric space (X,d) and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$.

Def. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded if $\exists x_0 \in X, M>0$ s.t.

$$d(x_n, x_0) < M \qquad \forall n \in \mathbb{N}$$

Def. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if:

$$\forall \varepsilon>0 \quad \exists \bar{n}_\varepsilon \in \mathbb{N} : \quad d(x_m, x_n) < \varepsilon \quad \forall m,n > \bar{n}_\varepsilon .$$

Any convergent sequence is a Cauchy sequence, the contrary is not always true.

$\boxed{\text{+ Cauchy sequences are always bounded}}$!

Def. (X,d) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent

Examples:
- $(\mathbb{R}^N, d_p)$ is complete $\forall p \in [1,+\infty]$
- $\left(C^0([a,b]), d(x,y) = \max_{t\in[a,b]} |x(t)-y(t)|\right)$ is complete

  The same set $(C^0([a,b]))$ but with a different distance, for example the integral distance, is not complete. In order to speak about completeness we always have to refer at the set + distance.

SEPARABILITY (in metric spaces, X = metric space)

$\boxed{x_0 \text{ ACCUMULATION POINT of } A \text{ if } \forall r>0 \ (B_r(x_0)\cap A)\setminus\{x_0\} \neq \emptyset}$

Def. $A \subset X$ is dense in X if $\bar{A} = X$. $(\bar{A} = A \cup \{\text{accumulation points}\})$

Def. X is separable if $\exists A \subset X$ countable, dense in X.

A set is countable if its cardinality is the same of the cardinality of $\mathbb{N}$.

Ex. (Example) $\mathbb{R}$ is separable $(A = \mathbb{Q}, \ \bar{A} = \bar{\mathbb{Q}} = \mathbb{R} = X)$

Def. $E \subseteq X$ is said to be of first category (or meagre) in X, if E is the union of countably many nowhere dense sets in X.

$\boxed{\text{A set is of 1st category if it's a countable union of nowhere dense sets.}}$

Def. A set $E \subseteq X$ which is not of first category is said to be of second category (or not meagre) in X.

Def. $E \subseteq X$ is said to be nowhere dense if $\text{Int}(\bar{E}) = \emptyset$ = it does not contain any open ball

Example: $E = \mathbb{Z} \subset X = \mathbb{R} \Longrightarrow \bar{E} = E, \ \text{Int}(\bar{E}) = \text{Int}(E) = \emptyset \Longrightarrow$ nowhere dense (in $\mathbb{R}$)

Example: (same thing if $E = \mathbb{N}$) $E = \mathbb{Q} \subset X = \mathbb{R} \Longrightarrow$ ($\mathbb{Q}$ is countable) $\mathbb{Q} = \bigcup_{n=1}^{+\infty}\{q_n\} := \bigcup_{n=1}^{+\infty} A_n$ (singletons)

$\text{Int}(\bar{A_n}) = \text{Int}(A_n) = \emptyset$

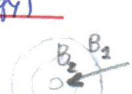
A_n is nowhere dense

$\mathbb{Q}$ is of first category in $\mathbb{R}$

$\mathbb{Q}$ is first category even if is not nowhere dense $(\bar{\mathbb{Q}} = \mathbb{R})$. However $\mathbb{Q}$ is a countable union of nowhere dense sets (singletons).
More in general: any countable set in $\mathbb{R}$ is 1st category (w.r.t. $\mathbb{R}$ topology) !

Def. A sequence of closed balls in a metric space X is called **nested** if:

$$B_1 \supset B_2 \supset \ldots \supset B_K \supset \ldots$$



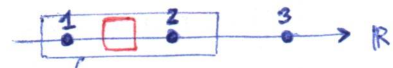
Let r_n be the radius of B_n; thus $\{r_n\}_n \searrow$. (decreasing)

Theorem (of nested balls, **Cantor** intersection theorem):
The metric space (X,d) is complete $\Longleftrightarrow$ any nested sequence of closed balls whose radius tends to 0 has non-empty intersection:

$$\bigcap_{n=1}^{+\infty} B_n \neq \emptyset$$

Lemma: X is a metric space and $A \subseteq X$ is nowhere dense.
Then every closed ball in X contains a closed ball which does not intersect A.

Example: $X = \mathbb{R}$, $A = \mathbb{N}$ $\quad$ $\forall$ Ball $\subset \mathbb{R}$ (closed): inside the ball we can find a closed interval (ball) that does not intersect any natural number :



given this "ball" (interval, closed) we can always find an other closed ball that does not intersect $\mathbb{N}$

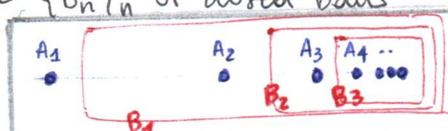
proof. (Lemma)

- Let $B \subset X$ be a closed ball. $\quad$ * B closed: $B = \bar{B}$
  Then $\operatorname{Int}(B) \overset{*}{=} \operatorname{Int}(\bar{B}) \not\subseteq \bar{A}$ since A is nowhere dense. $\Longrightarrow$ it does not contain any open ball ($\operatorname{Int}(B)$ is open)
  $\Longrightarrow \exists x_0 \in \boxed{\operatorname{Int}(B) \setminus \bar{A} := E.}$
  E is open since it's the difference between an open set ($\operatorname{int}(B)$) and a closed set ($\bar{A}$).
- By def of open sets $\exists B_r(x_0) \subset E$. $\quad$ (E is open and $x_0 \in E$)
- $\Longrightarrow \overline{B_{\frac{r}{2}}(x_0)} \subset E$
  If a ball centered in x_0 and of radius r is entirely contained in E then a ball centered in x_0 and of radius $\frac{r}{2}$ can also be closed and be contained, hence, the closure of the (open) ball centered in x_0 and with radius $\frac{r}{2}$ is in E.
- $\Longrightarrow \overline{B_{\frac{r}{2}}(x_0)} \cap A = \emptyset$. $\quad \Rightarrow$ It exists a closed ball inside B which does not intersect A.

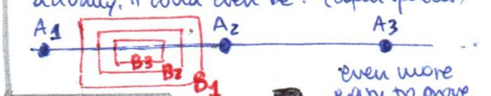
Theorem (**Baire** category theorem):
(X,d) is a complete metric space. $E = X \subseteq X$ is of second category in itself.
= A complete metric space is 2nd category in itself.

proof.

- Suppose by contraddiction that X is of first category, so that
  $$X = \bigcup_{n=1}^{+\infty} A_n, \quad A_n \subset X \text{ and nowhere dense } \quad (\operatorname{Int}(\bar{A}_n) = \emptyset)$$
  X 1st category, X complete
- By the previous lemma, there exists a closed ball B_1 of radius r_1 which does not intersect A_1. (since A_1 is, by assumption, nowhere dense) $(r_1 < 1)$
- By the lemma again, B_1 contains a closed ball B_2 of radius r_2 s.t. B_2 does not intersect A_2. $(r_2 < \frac{1}{2})$
- carrying on this process we obtain a nested sequence $\{B_n\}_n$ of closed balls such that: $\quad B_n \cap A_n = \emptyset \quad \forall n \in \mathbb{N}$
  and: $\quad r_n \xrightarrow[n \to \infty]{} 0$.



actually, it could even be: (equispaced)



even more easy to prove that $\bigcap_{n=1}^{+\infty} B_n = \emptyset$

- Thus, by the theorem of nested balls (since X is complete): $\boxed{\bigcap_{n=1}^{+\infty} B_n \neq \emptyset}$
- On the other hand:
  $$\left(\bigcap_{n=1}^{+\infty} B_n\right) \cap \left(\bigcup_{n=1}^{+\infty} A_n\right) = \emptyset$$
  but $\bigcup_{n=1}^{+\infty} A_n = X \longrightarrow \boxed{\bigcap_{n=1}^{+\infty} B_n = \emptyset} \longrightarrow$ contraddiction.
  hence, X is of 2nd category.

Corollary: Let (X,d) be a complete metric space. The intersection of a countable family of open sets dense in X is a set which is dense in X.
(X complete $\Longrightarrow$ any intersection of countable dense open sets is dense) = If X is complete and we take a countable family of open dense sets their intersection is still dense

proof.

- Let $\{A_n\}_{n \in \mathbb{N}} \subset X$ be a sequence of open dense sets. (each A_n: $\bar{A}_n = X$)
- By contraddiction, suppose that:
  $$E := \overline{\bigcap_{n \in \mathbb{N}} A_n} \subsetneq X$$
- E is closed so, by def., E^c is open $\Longrightarrow \exists$ a closed set $B \subset E^c$
- B is closed in X $\Longrightarrow \boxed{(B,d) \text{ is complete metric space.}}$
  closed set in a complete metric space

In general: (X,d) complete, $B \subseteq X$ closed $\Rightarrow (B,d)$ complete

- Therefore, by our construction:
  $$\left(\bigcap_{n \in \mathbb{N}} A_n\right) \cap B = \emptyset$$
  $$\Longrightarrow \bigcup_{n \in \mathbb{N}} \underbrace{(A_n^c \cap B)}_{:= G_n} = B \qquad (\bullet)$$

$(\bigcap A_n) \cap B = \emptyset$
$\bigcap A_n = (\bigcup A_n^c)^c$ De Morgan
$\Longrightarrow (\bigcup A_n^c) \cap B = B$

15/09

- $\Rightarrow \underbrace{\mathrm{Int}(\overline{A_n^c \cap B}) \subseteq \mathrm{Int}(\overline{A_n^c}) = \emptyset}_{(\bullet\bullet)}$ since $\bar{A}_n = X$, $\overline{\bar{A}_n}^c = \emptyset$ $\neq \overline{A_n^c}$

Notice: A_n open $\Rightarrow \mathrm{Int}(\overline{A_n^c}) = \mathrm{Int}(A_n^c) = \emptyset$ because $\bar{A}_n = X$

$B = \bigcup_{n\in\mathbb{N}} G_n$ and $\mathrm{Int}(\bar{G}_n) = \emptyset \rightarrow$ (G_n nowhere dense)

complete metric space

- $\Rightarrow$ $\boxed{B \text{ is of first category}}$ which is in contraddiction with the previous theorem

since it's the countable union of nowhere dense set

because B is complete !

- $\Rightarrow$ $E = X \Rightarrow$ the intersection is dense ■

(INTRODUCTION TO) COMPACTNESS

Def. A metric space X is said to be compact $\Longleftrightarrow$ any open cover has a finite subcover

Def. A metric space X is said to be sequentially compact $\Longleftrightarrow$ any sequence $\{x_n\}_n$ has a convergent subsequence

Def. A metric space X is said to be totally bounded

$\Longleftrightarrow \forall \varepsilon > 0 \;\; \exists A \subset X$, A finite s.t. $\forall x \in X \;\; \mathrm{dist}(x, A) < \varepsilon$

where $\mathrm{dist}(x,A) = \inf_{y\in A} d(x,y)$

we can always find ($\forall \varepsilon>0$) a set of points * that is close to all the points of **X**

*(a finite set of points)

Theorem : X metric space. The 3 conditions are equivalent :

Characterization of compact metric spaces

(i) X is compact

(ii) X is sequentially compact

(iii) X is complete and totally bounded

Note: X compact $\Rightarrow$ X complete (general relation)

Prop. $E \subseteq X$ is compact $\Longrightarrow$ E is closed and bounded (If $X = \mathbb{R}^N$ it holds $\Longleftrightarrow$)

(**HEINE-BOREL THEOREM**: $\forall E \subseteq \mathbb{R}$: E closed and bounded $\Leftrightarrow$ E compact)

Def. A subset S of a metric space (X,d) is **bounded** if $\exists r > 0 : \;\; \forall s,t \in S : \; d(s,t) < r$

16/09

We saw compactness in general metric spaces.
Now we see compactness when the metric space is $C^0(X)$.

COMPACTNESS IN $C^0(X)$

X is a compact metric space.

For instance : $X = [a,b] \subset \mathbb{R}$, $X = K \subseteq \mathbb{R}^N$ closed and bounded set.

We recall that:

$C^0(X) := \{ f : X \to \mathbb{R} \text{ continuous} \}$

We can define a distance :

$$d(f,g) = \sup_{x\in X} |f(x) - g(x)| \equiv \|f-g\|_\infty$$

for the moment this is only a notation

Def. Consider $A \subseteq C^0(X)$. A is equicontinuous if :

$$\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 : \; \forall f \in A, \; \forall x,y \in X \quad d(x,y) < \delta_\varepsilon \Longrightarrow |f(x) - f(y)| < \varepsilon$$

δ_ε does not depend on f, δ_ε is also independent of x

$\forall$ function in **A** we obtain for any ε the same δ ($\forall$ function!) $\Rightarrow$ equicontinuous

$\Longrightarrow$ uniform continuity

In "simple" continuity, if we take a set of continuous function, for every function f $\forall \varepsilon>0 \; \exists \delta_\varepsilon$ which depends on **f**. In the case of **equi**-continuity we have the ∄ from the function **f**. ($\forall \varepsilon \; \exists \delta_\varepsilon$ which is valid $\forall f \in A$)

Theorem (**Ascoli-Arzelà**):

sufficient condition for compactness (of a set in $C^0(X)$)

If $F \subset C^0(X)$ is : (**X** compact metric space)
- bounded
- closed
- equicontinuous

$\Longrightarrow$ **F** is compact

This theorem is like the Heine-Borel one with the difference that the dimension of $\mathbb{R}^N$ is **N**, while the dimension of $C^0(X)$ is ∞ (every element of $C^0(X)$ is a function). That's the reason why we need to add something (bounded and closed is not enough) $\rightarrow$ equicontinuity

CONTINUITY OF A FUNCTION
$f : X \to \mathbb{R}$ is continuous in $x_0 \in X$ means that : $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall x \in X$:
$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

proof.

If X is compact then $C^0(X)$ is complete

- $F \subset C^0(X)$, $C^0(X)$ complete $\Longrightarrow$ F closed $\Longrightarrow$ F complete (with the same distance of $C^0(X)$) $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$

Therefore if we show that F is totally bounded then the thesis follows.

- F equicontinuous $\Longleftrightarrow$ $\forall f \in F$, $\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$: $(\delta_\varepsilon := \delta)$
$\forall f \in F$, $\forall x, y \in X$: $d(x,y) < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$ $(*)$

- X compact $\Longleftrightarrow$ $X = \bigcup_{i=1}^{n} B_\delta(x_i)$ for some $\delta > 0$. (same δ as above) $(**)$

- F is bounded $\Longleftrightarrow$ $\exists k > 0$: $\forall f \in F$, $\forall x \in X$: $|f(x)| \le k$

F bounded if every one of its element is bounded ($\exists k>0$ that bounds all elems.)

- Let $E := [-k, k] \subseteq \mathbb{R}$. (same k as above)
- We define a mapping: $T : F \to E^n = [-k,k] \times [-k,k] \times \dots [-k,k]$ (n times)
$T(f) = (f(x_1), f(x_2), \dots, f(x_n))$ $\forall f \in F$
where x_i is the center of the ball $B_\delta(x_i)$ introduced before

- $E^n \subseteq \mathbb{R}^n$ is compact $\Longrightarrow$ E^n is totally bounded
cartesian product of compact intervals

Equivalent definition of a **TOTALLY BOUNDED** metric space. (X,d) totally bounded if $\forall r > 0$ $\exists x_1, \dots, x_n \in X$ s.t. $X \subseteq \bigcup_{i=1}^{n} B_r(x_i)$!

$\Longrightarrow T(F) \subseteq E^n \Longrightarrow T(F)$ is totally bounded *
every element of $T(f)$ $\forall f \in F$ is s.t. each component is $|\cdot| \le k$
(since it is a subset of a totally bounded set)
(the cover of E^n is also a cover for its subsets)

- Let $\varepsilon > 0$. We can cover $T(F)$ by finitely many open balls of radius ε. (by ↑) say $B_1, B_2, \dots, B_m$. (We can do it because $T(F)$ is totally bounded)
- For each $j = 1, 2, \dots, m$ we choose $f_j \in F$ s.t.
$T(f_j) \in B_j$

* A subset of a totally bounded set is totally bdd

- Claim: F is covered by a finite number of open balls of radius 4ε (which means that F is totally bounded)

totally bounded $\Longleftrightarrow$ $\exists$ finite cover made of balls of radius that we choos

- In fact, let $f \in F$. For some $j \in \{1, 2, \dots, m\}$:
$T(f) \in B_j$ since all $T(F)$ is covered by $\bigcup_{i=1}^{m} B_i$ then $\forall f$: $T(f)$ must belong to some of the B_j
- Then : $d(T(f), T(f_j)) < 2\varepsilon$ (because they're in the same ball B_j of radius ε)
Hence : $|f(x_i) - f_j(x_i)| \le d(T(f), T(f_j)) < 2\varepsilon$ $\forall i = 1, 2, \dots, n$ $(\bullet)$
- By $(**)$, $\forall x \in X$ $\exists x_i \in X$: $d(x, x_i) < \delta$.
Therefore, due to $(*)$, $\forall f \in F$ $|f(x) - f(x_i)| < \varepsilon$, so that:
$|f_j(x) - f_j(x_i)| < \varepsilon$ $\forall i = 1, 2, \dots, n$ $(\bullet\bullet)$
- We combine $(\bullet)$ and $(\bullet\bullet)$ and we get:

$$\begin{aligned} |f(x) - f_j(x)| &= |f(x) - f(x_i) + f(x_i) - f_j(x_i) + f_j(x_i) - f_j(x)| \\ &\le |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)| \\ &< \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon \quad (\forall x \in X) \end{aligned}$$

$\Longrightarrow d(f, f_j) < 4\varepsilon$ and hence we consider f_j as center of the ball $B_{4\varepsilon}(f_j)$ and 4ε the radius

- Thus $f \in B_{4\varepsilon}(f_j) \subseteq F$. $(\forall f \in F)$
- $\Longrightarrow$ F is covered by m open balls of radius 4ε.
The claim has been proved. By the claim the thesis follows. ■

Remark: Consider a special subset of $C^0(X)$: $F = \{f_n\}_n \subset C^0(X) \Longrightarrow F$ is closed.
a sequence is closed ($f_n \in C^0(X)$ like $x_n \in \mathbb{R}$ (e.g.) and $\{x_n\}_n$ is closed in $\mathbb{R}$)
So, the hypothesis of the theorem become :
$\{f_n\}_n \subset C^0(X)$ bounded and equicontinuous
$\exists M > 0 : \forall n \in \mathbb{N}$ $\sup_{x \in X} |f_n(x)| \le M$
and the thesis is :
$\exists \{f_{n_k}\} \subset \{f_n\}$ s.t. $f_{n_k} \xrightarrow{k \to \infty} f$ in $C^0(X)$ } we can use the notion of sequentially compactness
for some $f \in C^0(X)$.

Corollary: Consider $C>0$. Consider the set:

$$E:=\{f\in C^1([a,b]) : \sup_{x\in[a,b]}|f(x)| + \sup_{x\in[a,b]}|f'(x)| \le C\}$$

since boundness is given by $\sup|f(x)|<..$ then equicontinuity will be given by $\sup|f'(x)|$

E is compact in $C^0([a,b])$.
This condition for sequences is:
consider $\{f_n\}_n \subset C^1([a,b])$ and assume that $\exists C>0$:

$$\sup_{x\in[a,b]}|f_n(x)| + \sup_{x\in[a,b]}|f_n'(x)| \le C \qquad \forall n\in\mathbb{N}$$

$$\Longrightarrow \exists \{f_{n_k}\}_k \subset \{f_n\}_n \text{ s.t.: } f_{n_k} \xrightarrow[k\to+\infty]{} f \text{ in } C^0([a,b]).$$

20/09

MEASURE (Chapter)

σ-ALGEBRA

Let X be a set.

Def. A family $\mathcal{A}\subseteq\mathcal{P}(X)$ is said to be an algebra if:
(i) $\emptyset\in\mathcal{A}$
(ii) $E\in\mathcal{A} \Longrightarrow E^c\in\mathcal{A}$
(iii) $A,B\in\mathcal{A} \Longrightarrow A\cup B\in\mathcal{A}$

Remark: (i) $X\in\mathcal{A} \Longleftarrow \emptyset^c = X$
(ii) $A,B\in\mathcal{A} \Longrightarrow A\cap B\in\mathcal{A}$
$(A\cap B = (A^c\cup B^c)^c)$ $A,B\in\mathcal{A} \Rightarrow A^c,B^c\in\mathcal{A} \Rightarrow A^c\cup B^c\in\mathcal{A}$ (iii) $\Rightarrow (A^c\cup B^c)^c\in\mathcal{A}$ (ii)

Def. A family $\mathcal{A}\subseteq\mathcal{P}(X)$ is said to be a σ-algebra if:
(i) $\emptyset\in\mathcal{A}$
(ii) $E\in\mathcal{A} \Longrightarrow E^c\in\mathcal{A}$
(iii) $\{E_k\}_{k\in\mathbb{N}}\subseteq\mathcal{A} \Longrightarrow \bigcup_{k=1}^{+\infty}E_k\in\mathcal{A}$ → difference with the algebra: in an algebra the union holds for 2 sets (∀), here it holds as countable union of sets (sequence of sets)

$(X,\mathcal{A})$ is called measurable space, the elements of $\mathcal{A}$ are called measurable sets. (A set is **measurable** if it belongs to a σ-algebra)

Remark: $\{E_k\}_k\subseteq\mathcal{A} \Longrightarrow \bigcap_{k=1}^{+\infty}E_k = \left[\bigcup_{k=1}^{+\infty}E_k^c\right]^c \subset\mathcal{A}$ (by (ii) and (iii))

How can we construct a σ-algebra?

Theorem (Generation of σ-algebra): Let $S\subseteq\mathcal{P}(X)$. Then there exists a σ-algebra $\sigma_0(S)$ s.t.:
(i) $S\subseteq\sigma_0(S)$
(ii) ∀ σ-algebra $\mathcal{A}\subseteq\mathcal{P}(X)$, s.t. $\mathcal{A}\supseteq S$, we have: $\mathcal{A}\supseteq\sigma_0(S)$
($\mathcal{A}$ contains both S and $\sigma_0(S)$)

This theorem ensures that, starting from $\forall S\subseteq\mathcal{P}(X)$ there exist a σ-algebra that contains S (i) and that is contained in ∀ σ-algebra that contains S (ii).
In some sense: any other σ-algebra that contains S is bigger (w.r.t. ⊆) of $\sigma_0(S)$ ⟶ $\sigma_0(S)$ is the smaller σ-algebra containing S

"proof." (idea)

We define the set $\mathcal{U}:=\{\mathcal{A}\subseteq\mathcal{P}(X) \text{ s.t. } \mathcal{A}\supseteq S,\ \mathcal{A}\ \sigma\text{-algebra}\}$ = all σ-algebras that contain S
We define $\sigma_0(S):=\bigcap\{\mathcal{A} : \mathcal{A}\in\mathcal{U}\}$ = intersection of all the σ-algebras containing S
(the intersection of all possible σ-algebras containing S is the smallest σ-algebra containing S)
The proof consists in proving that $\sigma_0(S)$ so defined is actually a σ-algebra.

Def. $\underbrace{\sigma_0(S)}_{\text{which existence is guaranteed by the theorem}}$ is called σ-algebra generated by S.

BOREL SETS

Consider a metric space (X,d).
In a metric space we have the notion of open set, hence we can define:

$$\mathcal{G} := \{ E \subseteq X : E \text{ is open} \}.$$

This set $\mathcal{G}$ is called topology. $\Longrightarrow$ A **topology** is a set, in a metric space, containing all open sets. (**Note**: it is possible to define a topology even without the concept of metric spaces, for us it is enough to say that $\mathcal{G}$ is constructed by means of open sets of a metric space)

We can do what we did before substituting S with $\mathcal{G}$.
We want to construct a σ-algebra not in general, but starting from a set X, which is a metric space.
To construct a σ-algebra we need some structure, otherwise is very difficult. That's why we start with a metric space.

Def. The σ-algebra $\sigma_0(\mathcal{G})$, generated by open sets, is called Borel σ-algebra and its elements are called Borel measurable sets.
The Borel σ-algebra will be denoted by $\mathcal{B}(X)$.

$\sigma_0(\mathcal{G})$ = intersection of all the σ-algebras which contain all the open sets
The elements of $\sigma_0(\mathcal{G})$ are sets (Borel measurable sets)

Remark: The following sets are Borel: (Borel is **not** only these sets)
(i) open sets
(ii) closed sets
(iii) countable intersections of open sets (G_δ)
(iv) countable union of closed sets (F_σ)

Now we need a description of Borel σ-algebra constructed on $X = \mathbb{R} / \mathbb{R}^N$

Prop. How to construct Borel σ-algebra in $\mathbb{R}/\mathbb{R}^N$

(i) $\mathcal{B}(\mathbb{R}) = \sigma_0(I)$ $\quad I = \{(a,b) : a,b \in \mathbb{R},\ a \leq b\}$
$= \sigma_0(I_1)$ $\quad I_1 = \{[a,b] : a,b \in \mathbb{R},\ a \leq b\}$
$= \sigma_0(I_2)$ $\quad I_2 = \{(a,b] : a,b \in \mathbb{R},\ a \leq b\}$
$= \sigma_0(I_0)$ $\quad I_0 = \{(a,b] : -\infty \leq a \leq b < \infty\} \cup \{(a,\infty) : a \in \mathbb{R}\}$
$= \sigma_0(\hat{I})$ $\quad \hat{I} = \{(a,\infty) : a \in \mathbb{R}\}$

This is an important property: we don't need all the open sets, it's enough a special family of sets. !

(ii) $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ = extended $\mathbb{R}$
$\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\tilde{I})$ $\quad \tilde{I} = \{(a,b) : a,b \in \mathbb{R},\ a<b\} \cup \{[-\infty,b) : b \in \mathbb{R}\} \cup \{(a,+\infty] : a \in \mathbb{R}\}$
$= \sigma_0(\tilde{I}_1)$ $\quad \tilde{I}_1 = \{(a,+\infty] : a \in \mathbb{R}\}$

(iii) $\mathcal{B}(\mathbb{R}^N) = \sigma_0(K_1)$ $\quad K_1 = \{N\text{-dim closed rectangles}\}$
$= \sigma_0(K_2)$ $\quad K_2 = \{N\text{-dim opened rectangles}\}$

How is $\mathcal{B}(\mathbb{R})$ generated (by def)? We consider all the open sets of $\mathbb{R}$. Then we consider all the σ-algebras containing all the open sets of $\mathbb{R}$ and we intersect them. The intersection of these σ-algebras is itself a σ-algebra, more specificly: $\mathcal{B}(\mathbb{R})$.
Alternatively we can do the same thing but not with all the open sets of $\mathbb{R}$, but with all (a,b) s.t. ..., or $[a,b]$ s.t. ... These properties allow us to work on smaller families of sets.
(e.g. $I = \{(a,b) : a,b \in \mathbb{R}, a<b\} \subsetneq$ all open sets of $\mathbb{R}$ $\}$)

MEASURE

Let X be a set, $\mathcal{C} \subseteq \mathcal{P}(X)$: $\phi \in \mathcal{C}$.

Def. A function $\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$ is a measure on $\mathcal{C}$ if:
(i) $\mu(\emptyset) = 0$
(ii) $\forall \{E_k\}_k \subseteq \mathcal{C}$ disjoint s.t. $\cup_{k=1}^{+\infty} E_k \in \mathcal{C}$ then:
$$\mu\left(\cup_{k=1}^{+\infty} E_k\right) = \sum_{k=1}^{+\infty} \mu(E_k)$$
σ-additivity

(we're requiring that $\cup_k E_k \in \mathcal{C}$ so that it makes sense to say "measure of $\cup_k E_k$")

Remark: If $\mathcal{C}$ is a σ-algebra, then $\cup_{k=1}^{+\infty} E_k \in \mathcal{C}$
(the requirement is clearly satisfied
$\Longrightarrow$ it'll be natural to define a measure on a σ-algebra)

20/09

Def. μ is finite if $\mu(X) < +\infty$.

Def. μ is σ-finite if there exists $\{E_k\}_k \subseteq \mathcal{C}$ s.t.:
$$\begin{cases} X = \bigcup_{k=1}^{+\infty} E_k \\ \mu(E_k) < +\infty \quad \forall k \in \mathbb{N} \end{cases}$$
we're saying that the measure of a set X can be infinite but X can be written as union of a sequence of sets E_k and the measure of each E_k is finite

Def. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra, $\mu: \mathcal{A} \to \overline{\mathbb{R}}_+$ be a measure.
$(X, \mathcal{A}, \mu)$ is called measure space. (set, σ-algebra on that set, measure defined on the σ-alg) (the measure is evaluated on the elements of the σ-algebra)

- If μ is finite (σ-finite) the measure space is called finite (σ-finite)
- If $\mu(X) = 1$ then $(X, \mathcal{A}, \mu)$ is a probability space and μ is a probability measure

in fact the elements of the σ-algebra are called measurable sets (measurable in the sense that we can compute their measure, so we can evaluate $\mu(\cdot)$ of that set)

Examples:

1. X general set, $\mathcal{C} \subseteq \mathcal{P}(X)$ s.t. $\emptyset \in \mathcal{C}$.
   Then if we define $\mu(\cdot)$ as:
   $\mu(\emptyset) := 0$
   $\mu(E) := \infty \quad \forall E \in \mathcal{C},\ E \neq \emptyset$
   $\Longrightarrow \mu$ is a measure
2. Define: $\mu: \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ s.t.:
   $$\mu(E) := \begin{cases} |E| & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is not finite} \end{cases}$$
   $|E|$ = number of elements of E
   $\Longrightarrow \mu$ is a measure, called counting measure (denoted: $\mu^{\#}$)
   counting measure allows us to connect integrals and series (we'll see it)
   - $\mu^{\#}$ is finite if X is finite
   - $\mu^{\#}$ is σ-finite if X is countable (the counting measure considers the number of elements of a set, it is σ-finite if the whole set X can be written as the union of elements of a sequence ($\Rightarrow$ countable family of sets) and the measure of every element of the sequence has to be of finite measure)
3. $X \neq \emptyset$, $x_0 \in X$. We define $\delta_{x_0}: \mathcal{P}(X) \to \mathbb{R}_+$ as:
   $$\delta_{x_0}(E) := \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{otherwise} \end{cases}$$
   $\Longrightarrow \delta_{x_0}$ is a measure, called Dirac measure concentrated at x_0

Theorem: (properties of measures)
Let $(X, \mathcal{A}, \mu)$ be a measure space.
Then:

for every finite family of subsets of the σ-algebra

(additivity) (i) $\forall \{E_1, .., E_n\} \subseteq \mathcal{A}$ disjoint : $\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$ — we're going from sequence (*) to finite family

(monotonicity) (ii) $\forall E, F \in \mathcal{A}$, $E \subseteq F \Longrightarrow \mu(E) \leq \mu(F)$

(σ-subadditivity) (iii) $\forall \{E_k\}_k \subseteq \mathcal{A}$: $\mu\left(\bigcup_{k=1}^{+\infty} E_k\right) \leq \sum_{k=1}^{+\infty} \mu(E_k)$

(iv) $\forall \{E_k\}_k \subseteq \mathcal{A} \nearrow$: $\mu\left(\bigcup_{k=1}^{+\infty} E_k\right) = \lim_{k \to +\infty} \mu(E_k)$

(v) $\forall \{E_k\}_k \subseteq \mathcal{A} \searrow$ and $\underline{\mu(E_1) < \infty}$!! : $\mu\left(\bigcap_{k=1}^{+\infty} E_k\right) = \lim_{k \to +\infty} \mu(E_k)$

these two properties sometimes are called continuity of the measure

(*) defined in the measure

proof.

(i) Set $E_{n+1} = E_{n+2} = \ldots = \emptyset$. By property (ii) (and (i)) of μ the thesis follows:
$$\mu\left(\bigcup_{k=1}^{+\infty} E_k\right) = \mu\Big(\underbrace{\bigcup_{k=1}^{n} E_k \cup \bigcup_{k=n+1}^{+\infty} \emptyset}_{= \bigcup_{k=1}^{n} E_n}\Big) = \mu\left(\bigcup_{k=1}^{n} E_k\right)$$
$$\overset{(ii)}{=} \sum_{k=1}^{+\infty} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k) + \underbrace{\sum_{k=n+1}^{+\infty} \mu(\emptyset)}_{=0}$$
measure zero because they're empty (property (i))
$$\Longrightarrow \mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$$

(iii) We define a new family of sets:

$$\begin{cases} F_1 := E_1 \\ F_n := E_n \setminus \left(\bigcup_{k=1}^{n-1} E_k \right) & n = 2,3,\dots \end{cases}$$

$\{F_n\}_n \subseteq \mathcal{A}$ disjoint, $F_k \subseteq E_k \ \forall k \in \mathbb{N}$ and it holds: $\bigcup_{k=1}^{+\infty} F_k = \bigcup_{k=1}^{+\infty} E_k$.

By monotonicity (since $F_k \subseteq E_k \ \forall k \in \mathbb{N}$):

$$\mu\left(\bigcup_{k=1}^{+\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{+\infty} F_k\right) = \sum_{k=1}^{+\infty} \mu(F_k) \leq \sum_{k=1}^{+\infty} \mu(E_k)$$

$$\Longrightarrow \mu\left(\bigcup_{k=1}^{+\infty} E_k\right) \leq \sum_{k=1}^{+\infty} \mu(E_k)$$

we're using the σ-additivity of the measure since $\{F_k\}_k$ are disjoint

$F_k \subseteq E_k \ \forall k \in \mathbb{N}$ means (implies): $\mu(F_k) \leq \mu(E_k)$ and since it holds $\forall k$ the same inequality holds for $\sum_{k=1}^{+\infty}$

$(E \subseteq F)$

(ii) Let's define: $F := E \cup (F \setminus E) \Longrightarrow E \cap (F \setminus E) = \emptyset \quad (*)$

(more than "let's define" is a "let's write F as")

because they're disjoint $(*)$: $\mu(F) \overset{(i)}{=} \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(the measure is non-negative)

(iv) We define: $\begin{cases} E_0 := \emptyset \\ F_k := E_k \setminus E_{k-1} \end{cases}$

$$\Longrightarrow \boxed{\bigcup_{k=1}^{n} F_k = E_n} \quad \text{and} \quad \bigcup_{k=1}^{+\infty} F_k = \bigcup_{k=1}^{+\infty} E_k$$

but $\{F_k\}_k$ is now disjoint, so:

$$\mu\left(\bigcup_{k=1}^{+\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{+\infty} F_k\right) = \sum_{k=1}^{+\infty} \mu(F_k) = \lim_{n\to+\infty} \boxed{\sum_{k=1}^{n} \mu(F_k)} = \lim_{n\to+\infty} \boxed{\mu(E_n)}$$

since $\{F_k\}_k$ are disjoint we can write $\sum_{k=1}^{n} \mu(F_k) = \mu\left(\bigcup_{k=1}^{n} F_k\right)$

(v) Let's define: $F_k := E_1 \setminus E_k$.

$$\{F_k\}_k \nearrow \overset{(iv)}{\Longrightarrow} \mu\left(\bigcup_{k=1}^{+\infty} F_k\right) = \lim_{k\to+\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to+\infty} \mu(E_k) \qquad (1)$$

We can do this because $\mu(E_1) < +\infty$

Moreover: $\bigcup_{k=1}^{+\infty} F_k = \bigcup_{k=1}^{+\infty} (E_1 \cap E_k^c) = E_1 \cap \left(\bigcup_{k=1}^{+\infty} E_k^c\right) = E_1 \setminus \left(\bigcap_{k=1}^{+\infty} E_k\right)$

$$\Longrightarrow \mu\left(\bigcup_{k=1}^{+\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{+\infty} E_k\right) \qquad (2)$$

$$(1) \text{ and } (2) \Longrightarrow \lim_{k\to+\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{+\infty} E_k\right) \quad \blacksquare$$

Remark: (v) fails if $\mu(E_1) = +\infty$.

EXERCISES

21/09

Ex. 1 Consider ℓ^∞. Check that ℓ^∞ is <u>not</u> separable.

Solution: Let's denote $\{x^{(k)}\}_k$ instead of $\{x_k\}_k$ for the sequences in ℓ^∞.

1. Find an uncountable set in ℓ^∞
2. Prove that, because of 1, there cannot be a countable dense set

$$\{x^{(k)}\}_k \in \ell^\infty \iff \{x^{(k)}\}_k \text{ is bounded}$$

We define:

$$E := \{\{x^{(k)}\} \in \ell^\infty : x^{(k)} \in \{0,1\}\}$$

We can see $\{x^{(k)}\}$ as a function:

$$\{x^{(k)}\} : \mathbb{N} \to \{0,1\}$$

$\boxed{E \text{ is not countable.}}$ *

* Suppose that it is. Since it's countable we can list all the elements:

$$\begin{bmatrix} \{x_1^{(k)}\} \\ \{x_2^{(k)}\} \\ \vdots \\ \{x_j^{(k)}\} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & \cdots \\ x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \cdots \\ \vdots & \vdots & & \\ x_j^{(1)} & x_j^{(2)} & \cdots & x_j^{(j)} \end{bmatrix}$$

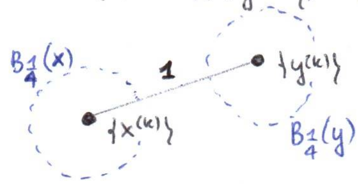
We define now $\{y^{(k)}\}$ s.t.:

$$y^{(k)} = \begin{cases} 1 & \text{if } x_k^{(k)} = 0 \\ 0 & \text{if } x_k^{(k)} = 1 \end{cases}$$

then $\{y^{(k)}\} \in E$ but $\{y^{(k)}\} \neq \{x_j^{(k)}\} \ \forall j \in \mathbb{N}$

$\to$ E cannot be countable

Moreover, $\{x^{(k)}\}, \{y^{(k)}\} \in E \longrightarrow d(\{x^{(k)}\}, \{y^{(k)}\}) = 1$



Consider $\forall x = \{x^{(k)}\}$ and consider $B_{\frac14}(x)$:

$$\Longrightarrow B_{\frac14}(x) \cap B_{\frac14}(y) = \emptyset \qquad x \neq y \quad (x, y \in E)$$

The family $\{B_{\frac14}(x)\}_{x \in E}$ is uncountable and disjoint.

Let $A \subset \ell^\infty$ be a dense set in ℓ^∞, i.e. $\bar{A} = \ell^\infty$:

$\Longrightarrow \forall x \in E: \quad B_{\frac14}(x) \cap A \neq \emptyset$

$\Longrightarrow$ A is uncountable

$\Longrightarrow$ ℓ^∞ cannot have a countable dense set

$\Longrightarrow$ ℓ^∞ is not separable

We created an uncountable family of disjoint balls. If A is dense in ℓ^∞ then it has at least one element in each one of the balls.

$\to$ A must be uncountable

Ex. 2 Let $\{f_n\} \subset C^1([a,b])$. Suppose that $\exists M > 0$:

$$(*) \quad |f_n(x)| \le M \qquad \forall x \in [a,b], \ n \in \mathbb{N}$$

$$(**) \quad |f_n'(x)| \le M \qquad \forall x \in [a,b], \ n \in \mathbb{N}$$

Show that $\exists \{f_{n_k}\}_k \subset \{f_n\}_n$ s.t. $f_{n_k} \xrightarrow[k\to\infty]{} f$ in $C^0([a,b])$

Here the key assumption is boundness of derivative. Compactness in $C^0(\cdot)$ is given by boundness in $C^1(\cdot)$!

Solution:

The idea is to use the Ascoli-Arzelà theorem (we just have to show that the hypothesis are satisfied)

$(*) \Longrightarrow \{f_n\}$ is <u>bounded</u> in $C^0([a,b])$

$\forall n \in \mathbb{N}, \ \forall x, y \in [a,b]: \ \exists \xi \in (a,b):$

$$f_n(y) - f_n(x) = f_n'(\xi)(y-x)$$

$$|f_n(y) - f_n(x)| = |f_n'(\xi)| \cdot |y-x|$$

$$\le M \cdot |y-x|$$

Since the Lipschitz constant does not depend on **n** we have "equi"-Lips.

$\Longrightarrow \{f_n\}$ is <u>equicontinuous</u> ($\Longleftarrow \{f_n\}$ is equi-Lipschitz)

(stronger than equi-continuous)

Consider: $F := \overline{\{f_n\}} \subset C^0([a,b]) \to \overline{\{f_n\}}$ is the set of the points of the sequence and the accumulation points of the sequence (limit points of the sequence)

F is closed, bounded and equicontinuous**

$\xrightarrow{\text{Ascoli Arzelà}} \overline{\{f_n\}}$ is compact $\iff \exists f \in C^0([a,b]), \ \{f_{n_k}\} \subset \{f_n\}$:

"which means"

$$f_{n_k} \xrightarrow[k\to\infty]{} f \text{ in } C^0([a,b])$$

ASCOLI-ARZELÀ FOR SEQUENCES

If $\{f_n\} \subset C^0(X)$, X compact, and $\{f_n\}$ is bounded and equicontinuous

$\Longrightarrow \exists \{f_{n_k}\} \subset \{f_n\}$ s.t. $f_{n_k} \xrightarrow[k]{} f$ in $C^0(X)$

** When we take the closure boundness and equicontinuity are preserved

Ex. 3 Let $\mathcal{A}$ be an algebra.
Show that $\mathcal{A}$ is a σ-algebra $\Longleftrightarrow \left[\begin{array}{l} \{E_k\} \subseteq \mathcal{A} \\ \{E_k\} \nearrow \end{array} \longrightarrow \bigcup_{k=1}^{+\infty} E_k \in \mathcal{A} \right]$ (*)

Solution: If $\mathcal{A}$ is a σ-algebra, it is obvious. (since (*) is satisfied for every sequence, so in particular for increasing sequences)
Viceversa, suppose (*).
Given any $\{E_n\} \subseteq \mathcal{A}$ we define:

$$F_k := \bigcup_{n=1}^{k} E_n$$

$\mathcal{A}$ is an algebra $\Longrightarrow F_k \in \mathcal{A}$. (since F_k is the finite union of sets)
Moreover, $\{F_k\} \nearrow$ and $\bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k$

$\{F_k\} \nearrow \overset{(*)}{\Downarrow} \left[\bigcup_{k=1}^{+\infty} F_k\right] \in \mathcal{A}$ * we're proving "

$\underbrace{\bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k}$ $\Downarrow$ since ($\bigcup_{k=1}^{+\infty} F_k \in \mathcal{A}$)* and since $\bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k$:

$$\left[\bigcup_{k=1}^{+\infty} E_k\right] \in \mathcal{A}$$

Remark: (v) fails if $\mu(E_1) = +\infty$.
In fact, consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$ and: $\mu^\#$ = counting measure

$$E_n := \{k \in \mathbb{N} : k \geq n\} \searrow \quad \forall n \in \mathbb{N}$$

$$\bigcap_{n=1}^{+\infty} E_n = \emptyset \Longrightarrow \mu^\#\left(\bigcap_{n=1}^{+\infty} E_n\right) = \mu^\#(\emptyset) = 0$$

$$\left.\begin{array}{l} \mu^\#(E_n) = +\infty \quad \forall n \in \mathbb{N} \\ \text{in particular: } \mu^\#(E_1) = +\infty \end{array}\right\} \longrightarrow \lim_{n\to\infty} \mu^\#(E_n) = +\infty \neq \mu^\#\left(\bigcap_{n=1}^{+\infty} E_n\right)$$

In conclusion: the assumption: $\mu^\#(E_1) < +\infty$ cannot be removed

Lemma: (**Borel - Cantelli**)
$(X, \mathcal{A}, \mu)$ measure space. $\{E_n\}_n \subseteq \mathcal{A}$.
If $\sum_{n=1}^{+\infty} \mu(E_n) < +\infty \longrightarrow \mu\left(\underbrace{\limsup_{n\to\infty} E_n}_{\bigcap_{k=1}^{+\infty}\left[\bigcup_{n=k}^{+\infty} E_n\right]}\right) = 0$

proof.
Let's define: $F_k := \bigcup_{n=k}^{+\infty} E_n \quad k \in \mathbb{N}$.
$\Longrightarrow \{F_k\} \searrow$
Because of σ-subadditivity: $\left(\{E_k\} \subseteq \mathcal{A} : \mu\left(\bigcup_{k=1}^{+\infty} E_k\right) \leq \sum_{k=1}^{+\infty} \mu(E_k)\right)$

$$\mu(F_k) \leq \sum_{n=k}^{+\infty} \mu(E_n) \qquad (*)$$

in particular: $\mu(F_1) < +\infty$. $\left(\mu(F_1) = \mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \leq \sum_{n=1}^{+\infty} \mu(E_n) < +\infty\right)$

$$\mu\left(\limsup_{n\to\infty} E_n\right) = \mu\left(\bigcap_{k=1}^{+\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k) \overset{(*)}{\leq} \lim_{k\to\infty}\left(\sum_{n=k}^{+\infty} \mu(E_n)\right) = 0$$

since $k \to \infty$

previous proposition (v.):
$\forall \{E_k\} \subseteq \mathcal{A}$, $\{E_k\} \searrow$ and $\mu(E_1) < \infty$
$\Longrightarrow \mu\left(\bigcap_{k=1}^{+\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$

■

SETS OF ZERO MEASURE

22/09

$(X, \mathcal{A}, \mu)$ measure space.

Def. $N \subseteq X$ is said to be a set of zero measure if $N \in \mathcal{A}$ and $\mu(N) = 0$.

In order to say that $\mu(N)=0$ N must be measurable ($N \in \mathcal{A}$)

Def. $E \subseteq X$ is said to be negligible if $\exists N \in \mathcal{A}$ s.t.
$E \subseteq N$ and $\mu(N) = 0$

Here we don't require that E is measurable

Def. We call: $\mathcal{N}_\mu :=$ collection of sets of zero measure
$\mathcal{T}_\mu :=$ collection of negligible sets

Ex. $X = \{a, b, c\}$; $\mathcal{A} = \{\emptyset, \{a\}, \{b,c\}, X\}$ is a σ-algebra.
(hence we have a measurable space)
To have a measure space:

$$\left.\begin{aligned} &\mu(X) = \mu(\{a\}) = 1 \\ &\mu(\emptyset) = \mu(\{b,c\}) = 0 \end{aligned}\right\} \Longrightarrow \mu \text{ is a measure.}$$

Exercise: check that:
- $\mathcal{A}$ is a σ-algebra
- μ is a measure

The set $\{b,c\}$ is a set of zero measure. ($N = \{b,c\} \in \mathcal{A}$, $\mu(N) = 0$)
The sets $\{b\}, \{c\}$ are negligible sets. ($\{b\}, \{c\} \notin \mathcal{A}$ $\Rightarrow$ they're not measurable) (as singletons)

$N = \{b, c\} \in \mathcal{N}_\mu$, $\{b\}, \{c\} \in \mathcal{T}_\mu$

In fact:

$$\left.\begin{aligned} &\{b\}, \{c\} \notin \mathcal{A} \\ &\{b\} \subseteq \{b,c\} \\ &\{c\} \subseteq \{b,c\} \end{aligned}\right\} \rightarrow \{b\}, \{c\} \text{ are negligible but they are **not** sets of zero measure!}$$

Def. A property P on X is said to be true almost everywhere (a.e.) if
$\{x \in X : P(x) \text{ is false}\} \in \mathcal{N}_\mu$ = a property is true almost everywhere if the set where it fails is a "small set", which means that the set is measurable and its measure is zero

Examples:

1. $f, g : X \to \overline{\mathbb{R}}$ are equal almost everywhere if:
$\{x \in X : f(x) \neq g(x)\} \in \mathcal{N}_\mu$
2. $f : X \to \overline{\mathbb{R}}$ is finite a.e. if:
$\{x \in X : f(x) = \pm\infty\} \in \mathcal{N}_\mu$
3. $f : D \to \overline{\mathbb{R}}$ ($D \in \mathcal{A}$) is defined a.e. in X if: $D^c \in \mathcal{N}_\mu$
If D is measurable ($D \in \mathcal{A}$) then also D^c is measurable. If $\mu(D^c) = 0$ we say that the function f is defined in X almost everywhere.

Remark: Equality a.e. is an equivalence relation in
$\overline{\mathbb{R}}^X := \{f : X \to \overline{\mathbb{R}}\}$

Def. A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if $\mathcal{T}_\mu \subseteq \mathcal{A}$.
(= if every negligible set is also measurable)
In such case, μ is a complete measure and $\mathcal{A}$ is a complete σ-algebra. *

Remark: If $\mathcal{N}_\mu = \mathcal{T}_\mu \Longleftrightarrow (X, \mathcal{A}, \mu)$ is complete
any negligible set is measurable with zero measure

* We can say $(X, \mathcal{A}, \mu)$ is a complete measure space.

COMPLETNESS: Any subset of a zero-measure set is measurable with measure zero

Given a measure space (∀), we can construct the smallest measure space (which contains the previous one) which is **complete**. To do so we have to construct a new σ-algebra (enlarge the original one) and a new measure defined in the new σ-algebra. The new measure is an extension of the old one so, the new measure on the old $\mathcal{A}$ is the same as the old measure. We have to add all the negligible sets to the σ-algebra and we have to extend the measure in such way that the added negligible sets have zero-measure.

Let $\bar{\mathcal{A}} := \{ E \subseteq X : \underbrace{\exists F, G \in \mathcal{A}}_{\text{they're measurable}} \text{ s.t. } F \subseteq E \subseteq G \,,\ \mu(G \setminus F) = 0 \}$.

We define a new measure:

$$\bar{\mu} : \bar{\mathcal{A}} \to \bar{\mathbb{R}}_+ \,, \quad \bar{\mu}(E) := \mu(F) .$$

Notice that $\boxed{\mathcal{A} \subseteq \bar{\mathcal{A}}}$ if $E \in \mathcal{A}$ then (we can choose:) $F = G = E$, so $E \in \bar{\mathcal{A}}$.

Because of this, if $E \in \mathcal{A}$:

$$\bar{\mu}(E) = \mu(E) \Longrightarrow \boxed{\bar{\mu}|_{\mathcal{A}} = \mu} .$$

Theorem: (Completion of a measure space) let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) $\bar{\mathcal{A}}$ is a σ-algebra which contains $\mathcal{A}$ $(\mathcal{A} \subseteq \bar{\mathcal{A}})$

(ii) $\bar{\mu}$ is a complete measure, $\bar{\mu}|_{\mathcal{A}} = \mu$, so $(X, \bar{\mathcal{A}}, \bar{\mu})$ is a complete measure space, more precisely it is the smallest (w.r.t. inclusion) measure space which contains $(X, \mathcal{A}, \mu)$ and which is complete

Given a measure space there exists a new measure space which is complete and it is the smallest complete measure space which contains the previous one.

OUTER MEASURE

(useful to construct a measure)

Let X be a set.

Def. A function $\mu^* : P(X) \to \bar{\mathbb{R}}_+$ is said to be an outer measure on X if:

(the outer measure is defined on the power set, not on a σ-algebra)

(i) $\mu^*(\emptyset) = 0$

monotone (ii) $E_1 \subseteq E_2 \Longrightarrow \mu^*(E_1) \leq \mu^*(E_2)$ — $\forall E_1, E_2$: they don't have to belong to a σ-algebra because there is none! The outer measure is defined on $P(X)$.

σ-subadditive $\boxed{\text{(iii) } \mu^*(\bigcup_{n=1}^{+\infty} E_n) \leq \sum_{n=1}^{+\infty} \mu^*(E_n)}$ ≠ measure because measures have **σ-additivity**, here we just have **σ-subadditivity** (in measures we start from σ-additivity and we obtain also σ-subadditivity, here the requirement is only σ-subadditivity) (the requirements on the measure are stronger than the ones on outer measure ⟹ we can imagine that if we have an outer measure we need something more to construct a measure)

Remark: If μ is a measure on $P(X)$ then μ is a outer measure.

How can we construct an **outer measure?**

Let $K \subseteq P(X)$ with $\emptyset \in K$.

Let $\vartheta : K \to \bar{\mathbb{R}}_+$ be a function s.t. $\vartheta(\emptyset) = 0$.

Define $\mu^* : P(X) \to \bar{\mathbb{R}}_+$ in such way:

- $\mu^*(E) := \inf \{ \sum_{n=1}^{\infty} \vartheta(I_n) : E \subseteq \bigcup_{n=1}^{+\infty} I_n ,\ \{I_n\} \subseteq K \}$

if $E \subseteq X$ can be covered by a countable union of sets $I_n \in K$;

- $\mu^*(E) := +\infty$ otherwise.

ϑ is an "elementary measure" and K is made of special sets where this elementary measure is applied.

Then we introduce whatever set **E**: how can we measure it? By means of the elementary measure applied to special sets. We try to cover the general set **E** with an union of these special sets I_n (on these special sets we can compute the elementary measure then we take the infimum)

Remark: If $I \in K$ then $\underbrace{\mu^*(I) \leq \vartheta(I)}_{\text{since } \mu^* \text{ is an infimum}}$

Theorem: (generation of outer measure)

Let $K \subseteq P(X)$ s.t. $\emptyset \in K$, $\vartheta : K \to \bar{\mathbb{R}}_+$ s.t. $\vartheta(\emptyset) = 0$.

Then μ^* (defined above) is an outer measure on X. (we say on X but it is def. on $P(X)$)

proof.

We have to prove that the **3** properties are fullfilled.

(i) Consider $\emptyset \in K$. $\boxed{\mu^*(\emptyset) \leq \vartheta(\emptyset)}_{\text{Remark}} = 0 \Longrightarrow \mu^*(\emptyset) = 0$ $(\mu^* \geq 0)$

(ii) Consider $E_1 \subseteq E_2$. If there exists a countable cover of E_2, it is also a countable cover of E_1. From the very definition of μ^* it follows: $\mu^*(E_1) \leq \mu^*(E_2)$. → Both E_1 and E_2 have a countable cover so both $\mu^*(E_1)$ and $\mu^*(E_2)$ are defined with the infimum but since $E_1 \subseteq E_2$ then $\mu^*(E_1) \leq \mu^*(E_2)$ as a consequence

If E_2 does **not** have a countable cover, then:

$$\mu^*(E_1) \leq \underbrace{\mu^*(E_2) = +\infty}_{\text{by def.}}$$

(whatever $\mu^*(E_1)$ it holds $\mu^*(E_1) \leq +\infty$)

(iii) If $\sum_{n=1}^{+\infty} \mu^*(E_n) = +\infty$ then it is obvious. 22/09

On the contrary, suppose that: $\sum_{n=1}^{+\infty} \mu^*(E_n) < +\infty \Rightarrow \mu^*(E_n) < +\infty \quad \forall n \in \mathbb{N}$

By the definition of μ^* and that of inf:

$\forall \varepsilon > 0 \quad \forall n \in \mathbb{N} \quad \exists \{I_{n_k}\}_k \subseteq K$ s.t.

$$E_n \subseteq \bigcup_{k=1}^{+\infty} I_{n_k}$$

before $\{I_n\}$ was the covering of a general set E. Now we're considering the covering of each E_n, so the index of the covering is k. (We consider a cover of any set E_n: then we have a covering and the index of the covering is k)

By definition of infimum:

$$\mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{+\infty} \nu(I_{n_k})$$

Since:

$$\bigcup_{n=1}^{+\infty} E_n \subseteq \bigcup_{n,k=1}^{+\infty} I_{n_k} \qquad \left(\{I_{n_k}\}_k \subseteq K \quad \forall n \in \mathbb{N}\right)$$

it follows that:

$$\mu^*\left(\bigcup_{k=1}^{+\infty} E_n\right) \leq \sum_{n,k=1}^{+\infty} \nu(I_{n_k}) \quad \text{(since it's the infimum)}$$

$$< \sum_{n=1}^{+\infty}\left(\mu^*(E_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{+\infty} \mu^*(E_n) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, then:

$$\mu^*\left(\bigcup_{n=1}^{+\infty} E_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(E_n)$$

■

General strategy of the definition of the outer measure μ^*. We have a special set K and a special function ν. The subsets of K are special subsets and ν is the elementary measure. By using special subsets and the elementary measure we can construct an outer measure, which is defined by means of a countable union of special sets. On special sets we know the elementary measure (since ν can be computed on these sets).

Def. μ^* defined as above is called outer measure generated by (K, ν).

GENERATION OF MEASURE

By combining ν and K we define an outer measure and it is defined on $\mathcal{P}(X)$. The important thing is that if we start from ν and K we can arrive to $\mathcal{P}(X)$ and to do so we use countable unions. We defined an outer measure, now we want to obtain a measure.

Let μ^* be an outer measure on X. (even if, actually, μ^* is defined on $\mathcal{P}(X)$)

Def. $E \subseteq X$ is said to be μ^*-measurable if:

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \qquad \forall Z \subseteq X$$

Caratheodory condition

Note: we're saying E μ^*-meas., **not** E measurable (which means that E belongs to the σ-algebra)

actually, this is the real one:

Lemma: $E \subseteq X$ is μ^*-measurable $\Longleftrightarrow$ $\boxed{\mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \quad \forall Z \subseteq X}$

proof.

It is enough to show that $\forall E \subseteq X$:

$$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \qquad \forall Z \subseteq X \qquad (1)$$

we want to show that this is always satisfied, so that, to obtain the equality, we need the right-hand side of the lemma

Now we write:

$$Z = Z \cap X = Z \cap (E \cup E^c) = (Z \cap E) \cup (Z \cap E^c) \qquad (X = E \cup E^c)$$

By sub-additivity we get: ((iii) of outer measure)

$$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^c)$$

■

Lemma: If $\mu^*(E) = 0$ then E is μ^*-measurable.

proof.

$\forall Z \subseteq X$, by monotonicity of μ^* we have:

$$\mu^*(\underbrace{Z \cap E}_{\subseteq E}) + \mu^*(\underbrace{Z \cap E^c}_{\subseteq Z}) \leq \underbrace{\cancel{\mu^*(E)}}_{=0} + \mu^*(Z) = \mu^*(Z)$$

So, by the previous lemma, the Caratheodory condition is satisfied and the thesis follows.

■

Theorem: Let μ^* be an outer measure on X. Then:

(i) the collection
$$\mathcal{L} := \{ E \subseteq X : E \text{ is } \mu^*\text{-measurable} \}$$
is a σ-algebra.

← $\{E$ for which **Caratheodory** condition is satisfied$\}$

(ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$

$\Longrightarrow$ The outer measure combined with Caratheodory condition allows us to construct a **measure**.

If we have an outer measure we can consider the set containing all sets satisfying caratheodory condition. This set is a σ-algebra. If we restrict the outer measure (μ^*) to this σ-algebra it becomes a complete measure.

Summary:

X set $\longrightarrow$ (K, ν) $\longrightarrow$ μ^* outer measure

K sets containing special subsets (e.g. rectangles)

ν elementary measure (e.g. area of rect.)

μ^* outer measure + **Caratheodory** condition $\Longrightarrow$ $\left\{ \begin{array}{l} \mathcal{L} \ \sigma\text{-algebra} \\ \mu = \mu^*|_{\mathcal{L}} \text{ measure} \end{array} \right.$

$\Downarrow$

$(X, \mathcal{L}, \mu)$ **complete** measure space

LEBESGUE MEASURE

28/09

Remark: $(X, \mathcal{L}, \mu)$ with $\mu = \mu^*|_{\mathcal{L}}$ is a complete measure space.

proof.

Consider $N \in \mathcal{L}$ s.t. $\mu(N) = 0$ (so, by def. of measure μ: $\mu^*(N) = 0$).
Consider $E \subseteq N$:

μ^* monotone $\Rightarrow 0 \leq \mu^*(E) \leq \mu^*(N) = 0$
$\Rightarrow \mu^*(E) = 0$
$\Rightarrow$ E is μ^*-measurable, i.e. E satisfies Caratheodory condition $\Longleftrightarrow$ $E \in \mathcal{L}$ and so we can write $\mu(E)$, which by def. is: $\mu(E) = \mu^*(E) = 0$

by a previous lemma

we showed that every subset of a zero-measure set is measurable and of zero-measure ■

Summary (construction of a measure):
Let X be a set. We defined (K, ν) from which we can construct an outer measure μ^*. We defined the Caratheodory condition and we created $\mathcal{L}$ σ-algebra on which we restricted μ^*, obtaining a complete measure $\mu = \mu^*|_{\mathcal{L}}$.
In conclusion: $(X, \mathcal{L}, \mu)$ is a complete measure space.

($\mathcal{L}$: all subsets of X which satisfy the Caratheodory condition)

Consider an interval $I \subseteq \mathbb{R}$: $I = (a,b)$ $a, b \in \mathbb{R}$ s.t. $a \leq b$.
We know that the length of the interval is:

$$\ell(I) = b - a$$

$$\ell(I) = +\infty \quad \text{if } I \text{ is unbounded}$$

Starting from this we want to define the measure of a general set.

The length of an interval is the **measure** of the interval. How can we measure a general set?

We construct a σ-algebra whose elements are the Lebesgue-measurable sets, and a measure λ, called Lebesgue measure.

λ possesses two properties: (we want λ to satisfy two properties:)

(i) $\lambda(I) = \ell(I)$ $\quad I \subseteq \mathbb{R}$ interval] we are extending the notion of length of an interval, so what we're constructing must coincide with the length, in case of intervals

(ii) $\lambda(E + x_0) = \lambda(E)$ $\quad x_0 \in \mathbb{R}$, $E \subseteq \mathbb{R}$, $\underbrace{E + x_0 := \{x + x_0 : x \in E\}}_{\text{translation}}$

Precise definition of Lebesgue measure:
Consider $\mathcal{I} = \{(a,b] : a, b \in \mathbb{R}, a \leq b\}$, $\emptyset \in \mathcal{I}$ $(b = a)$.
We define a function $\ell : \mathcal{I} \longrightarrow \mathbb{R}_+$:

$$\begin{cases} \ell(\emptyset) := 0 \\ \ell((a,b]) := b - a \end{cases}$$

We choose $X = \mathbb{R}$, $(K, \nu) = (\mathcal{I}, \ell)$ and we get the Lebesgue measure (by means of the previous construction).
More precisely: $X = \mathbb{R}$, $(K, \nu) = (\mathcal{I}, \ell) \rightsquigarrow \lambda^*$ outer measure on $\mathbb{R}$: λ^* is def. on $\mathcal{P}(\mathbb{R})$, this is just a notation

$$\lambda^*(E) := \inf \left\{ \sum_{n=1}^{+\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{+\infty} I_n, \ \{I_n\} \subseteq \mathcal{I} \right\}$$

if $E \subseteq \mathbb{R}$ can be covered by a countable union of elements of $\mathcal{I}$

$$\lambda^*(E) := +\infty \quad \text{otherwise.}$$

Remark: $I \subseteq \mathbb{R}$ interval $\Longrightarrow \lambda^*(I) = \ell(I)$

Def. λ^* generated by $(\mathcal{I}, \ell)$ is called Lebesgue outer measure on $\mathbb{R}$.
The λ^*-measurable sets are called Lebesgue measurable sets (sets which satisfy the Caratheodory condition).
The corresponding σ-algebra $\mathcal{L}$ is called Lebesgue σ-algebra and it is denoted by $\mathcal{L}(\mathbb{R})$. The measure $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$ is called Lebesgue measure on $\mathbb{R}$.

Remark: $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space.

IN $\mathbb{R}^N$

Let's consider $\mathcal{I}^N := \{ \prod_{k=1}^N (a_k, b_k] : a_k, b_k \in \mathbb{R}, \ a_k \le b_k \ \forall k = 1, .., N \}$.
We define:

$$\begin{cases} \ell^N(\emptyset) = 0 \\ \ell^N(\prod_{k=1}^N (a_k, b_k]) = \prod_{k=1}^N (b_k - a_k) \end{cases}$$

We now consider $X = \mathbb{R}^N$, $(K, \nu) = (\mathcal{I}^N, \ell^N)$ and we repeat the same construction as before:

$$X = \mathbb{R}^N, \ (K, \nu) = (\mathcal{I}^N, \ell^N) \rightsquigarrow \lambda^{*,N} + \text{Caratheodory condition} \Longrightarrow (\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), \lambda^N)$$

complete measure space
$\lambda^N = \lambda^{*,N}|_{\mathcal{L}(\mathbb{R}^N)}$
N-dimensional Lebesgue measure

(again) IN $\mathbb{R}$

Prop. Any countable subset $E \subseteq \mathbb{R}$ is Lebesgue measurable and its measure is zero. $(\lambda(E) = 0)$

proof.

- Let $a \in \mathbb{R}$.
  $\{a\} \subseteq (a - \varepsilon, a] \quad \forall \varepsilon > 0 \xrightarrow{\text{def. } \lambda^*} \lambda^*((a-\varepsilon, a]) = \varepsilon \quad \forall \varepsilon > 0$
  $\longrightarrow \lambda^*(\{a\}) \le \varepsilon$ since λ^* is monotone $(\forall \varepsilon > 0)$
  $\Longrightarrow \lambda^*(\{a\}) = 0$
  $\Longrightarrow \{a\} \in \mathcal{L}$ and $\lambda(\{a\}) = 0$
- λ^* is σ-subadditive, hence:

$$\lambda^*(E) = \lambda^*(\cup_{n=1}^{+\infty} \{a_n\}) \le \sum_{n=1}^{+\infty} \underbrace{\lambda^*(\{a_n\})}_{=0} = 0$$

since E is countable: $E = \cup_{n=1}^{+\infty} \{a_n\}$

$\longrightarrow \lambda^*(E) = 0$ (since $\lambda^*(\cdot) \ge 0$) $\Longrightarrow E \in \mathcal{L}, \ \lambda(E) = 0$. ■

(Lemma) $\mu^*(E) = 0 \rightarrow E$ μ^*-meas.

Prop. $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \quad (\subsetneq) \Longrightarrow$ Borel sets are Lebesgue measurable

proof.

$\mathcal{B}(\mathbb{R}) = \sigma_0((a, +\infty)). \quad (a \in \mathbb{R})$
Therefore, it is enough to show that $(a, +\infty) \in \mathcal{L}(\mathbb{R})$.

- Let $A \subseteq \mathbb{R}$ be any set.
- We assume that $a \notin A$, otherwise we replace A by $A \setminus \{a\}$.
  (since this leaves the outer measure unchanged)
- We have to show that:

$$\lambda^*(A_1) + \lambda^*(A_2) \le \lambda^*(A) \qquad (1)$$

  where $A_1 := A \cap (-\infty, a]$, $A_2 := A \cap (a, +\infty)$.
  (i.e. the Caratheodory condition with $E = (a, +\infty)$, $Z = A$)
- Since $\lambda^*(A)$ is defined as an infimum, to verify (1) it is necessary and sufficient to show that for any countable collection $\{I_k\}$ of open bounded intervals that cover A:

$$\lambda^*(A_1) + \lambda^*(A_2) \le \sum_{k=1}^{+\infty} \ell(I_k)$$

If we have the inequality for all series (any countable collection $\{I_k\}$) then we have also the inequality for the infimum

- For each $k \in \mathbb{N}$ we define:

$$I_k' := I_k \cap (-\infty, a]$$
$$I_k'' := I_k \cap (a, +\infty)$$

- Then I_k' and I_k'' are disjoint intervals and:

$$\ell(I_k) = \ell(I_k') + \ell(I_k'')$$

- $\{I_k'\}$ is a countable cover of A_1.
- $\{I_k''\}$ is a countable cover of A_2.
- Hence: (by def. of outer measure:)

$$\lambda^*(A_1) \le \sum_{k=1}^{+\infty} \ell(I_k')$$
$$\lambda^*(A_2) \le \sum_{k=1}^{+\infty} \ell(I_k'')$$

- Therefore:

$$\lambda^*(A_1) + \lambda^*(A_2) \le \sum_{k=1}^{+\infty} \ell(I_k') + \sum_{k=1}^{+\infty} \ell(I_k'') = \sum_{k=1}^{+\infty} [\ell(I_k') + \ell(I_k'')] = \sum_{k=1}^{+\infty} \ell(I_k)$$

$\Longrightarrow$ the proof is complete. ■

Prop. The translate of a $\mathcal{L}$-measurable set is $\mathcal{L}$-measurable.

proof.

Let E be $\mathcal{L}$-measurable set.

Clearly, λ^* is translation invariant. That's because λ^* is defined by means of length of interval and length of interval is translation invariant

Hence $\forall A \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}$:

$$\lambda^*(A) = \lambda^*(A - x_0) = \lambda^*((A-x_0)\cap E) + \lambda^*((A-x_0)\cap E^c)$$
(Caratheodory condition for $E \in \mathcal{L}(\mathbb{R})$)
$$= \lambda^*(A \cap (E+x_0)) + \lambda^*(A \cap (E+x_0)^c)$$

$\Longrightarrow (E + x_0) \in \mathcal{L}(\mathbb{R})$. ■

Theorem: $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$. ! *

This is an alternative definition. We defined the Lebesgue measure space as a **restriction** of another space. In this view, instead, the Lebesgue measure is an **extension**.

REGULARITY OF LEBESGUE MEASURE

Important: in points (ii)-(v) the thesis is that $E \in \mathcal{L}(\mathbb{R})$ (it's not an hypothesis) hence we cannot write λ, we have to write λ^*

Theorem: Let $E \subseteq \mathbb{R}$. The following statements are equivalent:

Approximation of sets which are **Lebesgue-measurable** (through **Borel** sets)

(i) $E \in \mathcal{L}(\mathbb{R})$

(ii) $\forall \varepsilon > 0\ \exists A \subseteq \mathbb{R}$ open s.t. $E \subseteq A$: $\lambda^*(A \setminus E) < \varepsilon$

(iii) $\exists G \subseteq \mathbb{R}$ of class G_δ s.t. $E \subseteq G$: $\lambda^*(G \setminus E) = 0$ $\qquad G_\delta = \bigcap_\infty$ open

outer approximation by open sets and G_δ sets

(iv) $\forall \varepsilon > 0\ \exists C \subseteq \mathbb{R}$ closed s.t. $C \subseteq E$: $\lambda^*(E \setminus C) < \varepsilon$

(v) $\exists F \subseteq \mathbb{R}$ of class F_σ s.t. $F \subseteq E$: $\lambda^*(E \setminus F) = 0$ $\qquad F_\sigma = \bigcup_\infty$ closed

inner approximation by closed sets and F_σ sets

* The sets that are missing in the Borel σ-algebra are the zero-measure sets. There are zero-measure sets, but too few.

APPROXIMATING (in terms of measure) **A LEBESGUE SET WITH BOREL SETS**

$\mathcal{L}(\mathbb{R})$ is bigger than $\mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ has all closed/open sets. Hence we cannot hope that only open/closed sets are Lebesgue measurable. But, can we approximate Lebesgue measurable sets by means of open/closed sets? "Approximate" means that: can we find an open/closed set which measure is almost equal to the measure of a Lebesgue measurable set ($\forall$ Lebesgue measurable set)? Yes, we can approximate a Lebesgue measurable set through an open set (ii), a closed set (iv), a G_δ set (iii), a F_σ set (v). (Actually it's even more, it is a **definition**: a set is Lebesgue only if it can be approximated through these sets)

From the property (v) we deduce that

$\forall E \in \mathcal{L}(\mathbb{R}) \quad \exists F \subseteq \mathbb{R}$, F_σ set s.t.

$F \subseteq E$ and $\lambda^*(E \setminus F) = 0$

$\Longrightarrow \quad E = F \cup (E \setminus F)$

$\Longrightarrow$ every Lebesgue set can be written as the union of a Borel set and a zero λ-measure set.

On account of this decomposition it's not surprising to discover that $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of the Borel measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$

Lemma: (Excision property). If $A \in \mathcal{L}(\mathbb{R})$ and $\lambda^*(A) < +\infty$, $A \subseteq B$ then: $\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$

29/09

Notice that if we assume that a set is Lebesgue measurable then writing λ or λ^* is the same (here we write λ^* because we'll use Caratheodory condition which is defined on λ^*)

(**Notice II**: we can write $\lambda(A)$ but we have to write $\lambda^*(B)$ because we **don't** say that $B \in \mathcal{L}(\mathbb{R})$)

proof.

Since $A \in \mathcal{L}(\mathbb{R})$: $\lambda^*(B) = \lambda^*(\underbrace{B \cap A}_{=A}) + \lambda^*(\underbrace{B \cap A^c}_{=B\setminus A})$

$= \lambda^*(A) + \lambda^*(B \setminus A)$

$\Rightarrow \lambda^*(B) - \lambda^*(A) = \lambda^*(B \setminus A)$ ■

proof. (Regularity theorem) (partial proof: (i) $\overset{1.}{\Rightarrow}$ (ii) $\overset{2.}{\Rightarrow}$ (iii) $\overset{3.}{\Rightarrow}$ (i))

$E \in \mathcal{L}(\mathbb{R})$, $\lambda(E) < +\infty$. (We make these assumptions to simplify the proof)

1.

By the definition of outer measure: $\exists \{I_k\}$ which covers E for which: ($\{I_k\}$ **open** bounded intervals)

$$\sum_{k=1}^{+\infty} \ell(I_k) < \lambda^*(E) + \varepsilon$$ arbitrary positive number (fixed)

We define $O := \bigcup_{k=1}^{+\infty} I_k$. Then O is open and $E \subseteq O$.

$$\lambda^*(O) \overset{*}{\leq} \sum_{k=1}^{+\infty} \ell(I_k) < \lambda^*(E) + \varepsilon$$ * σ-subadditivity

$$\Rightarrow \lambda^*(O) - \lambda^*(E) < \varepsilon$$

By assumption $E \in \mathcal{L}(\mathbb{R})$ and $\lambda^*(E) < +\infty$, so by the excision property:

$$\Rightarrow \lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$$

2.

$\forall k \in \mathbb{N}$ we can find $O_k \supseteq E$ open for which:

$$\lambda^*(O_k \setminus E) < \frac{1}{k}$$

because of property (ii) we can find an open set that contains E and s.t. the measure of this open set minus E is $< \varepsilon$. We choose $\varepsilon = \frac{1}{k}$ and we call the set: O_k

Define $G = \bigcap_{k=1}^{+\infty} O_k$. Then G is a G_δ set and $\underbrace{G \supseteq E}_{E \subseteq O_k \, \forall k}$.

Moreover: $\forall k \in \mathbb{N}$

$$G \setminus E \subseteq O_k \setminus E$$ (since G is smaller than any O_k)

$\lambda^* \nearrow$ (monotone) $\Rightarrow \lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < \frac{1}{k}$

$\underset{k \to \infty}{\Rightarrow} \lambda^*(G \setminus E) = 0$

3.

$\lambda^*(G \setminus E) = 0 \Rightarrow G \setminus E \in \mathcal{L}(\mathbb{R})$ (**Lemma**: if $\lambda^*(A) = 0 \rightarrow A$ is λ^*-measurable)

Moreover $G \in \mathcal{L}(\mathbb{R})$ since $G \in G_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.

$\Rightarrow E = G \cap (G \setminus E)^c \in \mathcal{L}(\mathbb{R})$

$G \in \mathcal{L}(\mathbb{R})$
$G \setminus E \in \mathcal{L}(\mathbb{R}) \Rightarrow (G \setminus E)^c \in \mathcal{L}(\mathbb{R})$
$G \cap (G \setminus E)^c \in \mathcal{L}(\mathbb{R})$ ■

The Lebesgue measure cannot be extended on the whole $\mathcal{P}(\mathbb{R})$
$\Rightarrow$ there exist sets which are **not** Lebesgue-measurable:

NON MEASURABLE SETS

Lemma: Let $E \subseteq \mathbb{R}$ be measurable and bounded. Suppose that there is a bounded countably infinite set $\Sigma \subseteq \mathbb{R}$ for which $\{\underbrace{\sigma + E}_{\text{translation}}\}_{\sigma \in \Sigma}$ is a disjoint family. Then $\lambda(E) = 0$.

proof.

$\forall \sigma \in \Sigma$: $\sigma + E$ is measurable (the translation of an $\mathcal{L}$-meas. set is still $\mathcal{L}$-meas.)

$$\lambda\left(\bigcup_{\sigma \in \Sigma} (\sigma + E)\right) \underset{\substack{\sigma\text{-additivity of } \lambda \\ \text{(since the sets are disjoint)}}}{=} \sum_{\sigma \in \Sigma} \lambda(\sigma + E) = \sum_{\sigma \in \Sigma} \lambda(E)$$

E and Σ are bounded $\Rightarrow$ and $\bigcup_{\sigma \in \Sigma} (\sigma + E)$ is bounded. $\rightarrow$ $\bigcup_{\sigma \in \Sigma} (\sigma + \Sigma)$ is contained in a bounded interval and its measure is less than the measure of the interval (which is a finite measure)

$\Rightarrow \bigcup_{\sigma \in \Sigma} (\sigma + E)$ has finite measure

If $\lambda(E) > 0$ then $\lambda\left(\bigcup_{\sigma \in \Sigma} (\sigma + E)\right) = +\infty$. because Σ is infinite

Thus $\lambda(E) = 0$. ■

Consider $\emptyset \neq E \subseteq \mathbb{R}$ and $x, y \in E$.
We say that x and y are in relation if:

$$x \sim y \iff x - y \in \mathbb{Q}$$ $\neq$ from saying $x, y \in \mathbb{Q}$!

This is an equivalence relation. (reflexivity, simmetry, transitivity)
Hence:
There is a disjoint decomposition of E into the collection of equivalence classes.
From the axiom of choice we infer that we can construct a set **V** containing exactly one member of each equivalence class.

the equivalence classes form a partition

How do we construct the equivalence classes? We take a point x and its equivalence class is made of all y s.t. x ~ y, then we take another point, etc.

* Properties of **V**:

(i) the difference of two points of V is **not** rational
$(\iff \forall \Sigma \subseteq \mathbb{Q} : \{\sigma + V\}_{\sigma \in \Sigma}$ is a disjoint family)

if the difference is rational they belong to the same class, but we're taking elements from different classes!

(ii) $\forall x \in E \ \exists c \in V$ for which: $x = c + q$ with $q \in \mathbb{Q}$

Theorem: (Vitali) **

Any measurable bounded set $E \subseteq \mathbb{R}$ with $\lambda(E) > 0$ contains a subset that is not $\mathcal{L}$-measurable.
(ideally **bounded** can be removed, however we prove the theorem under this assumption)

!! proof.

claim: **V** is not measurable. (V constructed above)

- Let $\Sigma \subset \mathbb{Q}$ be any bounded countably (infinite) set.
- Suppose by contraddiction that V is measurable. (Lebesgue measurable)
- Lemma + (i)(↑) $\Longrightarrow \lambda(V) = 0$
- Moreover:

$$\lambda\left(\bigcup_{\sigma \in \Sigma} (\sigma + V)\right) = \sum_{\sigma \in \Sigma} \lambda(\sigma + V) = \sum_{\sigma \in \Sigma} \lambda(V) = 0$$

- We now make a special choice of Σ.
$\exists b > 0 : \ E \subset [-b, b]$. (since E is bounded)
We choose: $\Sigma := [-2b, 2b] \cap \mathbb{Q}$.
$\longrightarrow \Sigma$ is bounded and countably infinite.
- We claim:

$$E \subseteq \bigcup_{\sigma \in \Sigma} (\sigma + V) \qquad (*)$$

- In fact, by (ii)(↑), if $x \in E$ then there is $c \in V$ for which $x = c + q$ with $q \in \mathbb{Q}$. ($c \in V$)
- But $x, c \in [-b, b]$ so that $q \in [-2b, 2b]$. (so $x - c \in [-b, b]$)
- Thus $(*)$ holds.
- This gives a contraddiction. In fact:

$$\lambda(E) > 0, \ E \subseteq \bigcup_{\sigma \in \Sigma} (\sigma + V), \ \lambda\left(\bigcup_{\sigma \in \Sigma} (\sigma + V)\right) = 0$$

this cannot be since $\lambda \nearrow$ (monotone increasing) ∎

$\longrightarrow$ **V** cannot be measurable.

* property (i) is a consequence of the fact that equivalence classes are disjoint
property (ii) is a consequence of the fact that we have a partition (hence the union is the whole set **E**)

** **Thm. VITALI**:
$\forall E \in \mathcal{L}(\mathbb{R})$ bounded s.t. $\lambda(E) > 0$
$\exists F \subset E$ s.t. $F \notin \mathcal{L}(\mathbb{R})$
(hence: $\mathcal{L}(\mathbb{R}) \subsetneq \mathcal{P}(\mathbb{R})$)

(29/09)

- construction of the **Lebesgue measure** as measure induced by the outer measure (the outer measure is defined by the means of intervals and lengths of intervals in $\mathbb{R}$ (similarly in $\mathbb{R}^N$))

- **Properties** of Lebesgue measure
    1. every countable set is measurable and its measure is zero
    2. translation invariant
    3. $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$
    4. regularity

MEASURABLE FUNCTIONS

Def. Consider two measurable spaces: $(X, \mathcal{A}), (X', \mathcal{A}')$. A function $f: X \to X'$ is said to be measurable if:

$$f^{-1}(E) \in \mathcal{A} \qquad \forall E \in \mathcal{A}'$$

$\forall E$ measurable (w.r.t. $\mathcal{A}'$) $f^{-1}(E)$ measurable (w.r.t. $\mathcal{A}$)

Prop. Consider $(X, \mathcal{A}), (X', \mathcal{A}'), (X'', \mathcal{A}'')$ measurable spaces.
Let $f: X \to X'$ and $g: X' \to X''$ be measurable.
Then $g \circ f: X \to X''$ is measurable.

proof.

$\forall E \in \mathcal{A}': f^{-1}(E) \in \mathcal{A}$

$\forall F \in \mathcal{A}'': g^{-1}(F) \in \mathcal{A}'$

$\forall F \in \mathcal{A}'': (g \circ f)^{-1}(F) = f^{-1}(\underbrace{g^{-1}(F)}_{:= E \in \mathcal{A}'}) = f^{-1}(E) \in \mathcal{A}$ ∎

we call it E and by the previous property it belongs to $\mathcal{A}'$

Theorem: Let $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ be measurable spaces.
Let $\mathcal{E}' \subseteq \mathcal{P}(X')$ s.t. $\sigma_0(\mathcal{E}') = \mathcal{A}'$. the σ-algebra generated by $\mathcal{E}'$ is $\mathcal{A}'$
Then: $f: X \to X'$ is measurable $\iff$ $f^{-1}(E) \in \mathcal{A}$ $\quad \forall E \in \mathcal{E}'$ (1)

Equivalent notion of measurable functions

in def. of measurable we require $\forall E \in \mathcal{A}'$, this requires only that $E \in \mathcal{E}'$ (where $\mathcal{E}'$ generates the σ-algebra $\mathcal{A}'$)

proof.

- Suppose f measurable.
  $\Rightarrow$ Since $\mathcal{E}' \subseteq \mathcal{A} \implies f^{-1}(E) \in \mathcal{A} \;\; \forall E \in \mathcal{A} \supseteq \mathcal{E}'$
- $\Leftarrow$ Now suppose that (1) holds.
  Let: $\Sigma := \{E \subseteq X' : f^{-1}(E) \in \mathcal{A}\}$.
  Σ is a σ-algebra. (we omit the details)
  (1) $\implies \mathcal{E}' \subseteq \Sigma$. ($\forall E \in \mathcal{E}': f^{-1}(E) \in \mathcal{A}$, Σ is bigger since $f^{-1}(E) \in \mathcal{A}$ $\forall E \subseteq X'$)
  Thus: $\mathcal{A}' = \sigma_0(\mathcal{E}') \subseteq \Sigma$
  $\implies f$ is measurable (since we're saying $\forall E \in \mathcal{A}'$: $\underbrace{f^{-1}(E) \in \mathcal{A}}_{\text{condition of } \Sigma}$) ∎

Def. Let $(X, \mathcal{L})$ be a measurable space. Let (X', d') be a metric space and on this metric space we consider the Borel σ-algebra constructed by means of open sets w.r.t. the metric d': $(X', \mathcal{B}')$ measurable space.
Moreover, let $f: X \to X'$ be measurable.
Then, we say that f is Lebesgue measurable.
($\mathcal{L}$ is the σ-algebra constructed as the collection of sets that satisfy Caratheodory condition)

Def. Let $(X, d), (X', d')$ be two metric spaces and let $(X, \mathcal{B}), (X', \mathcal{B}')$ be the measurable spaces s.t. the σ-algebra is the Borel σ-algebra constructed by means of open sets w.r.t. the metrics d and d' (respectively).
If $f: X \to X'$ is measurable then we say that f is Borel measurable.

$\Longrightarrow$ **Measurability strongly depends on the σ-algebra**
(the starting σ-algebra since in X' we consider always the Borel σ-algebra)

Prop. $f: X \to X'$ continuous $\implies f$ Borel measurable = any continuous function is measurable according to Borel

proof.

f continuous $\overset{\text{def.}}{\iff} \forall E \in \mathcal{E}' \;\; f^{-1}(E) \in \mathcal{E} \subseteq \mathcal{B}$ with $\mathcal{E}'$ open sets in X', $\mathcal{E}$ open sets in X
$\mathcal{E}'$ family of all open sets in X'
$\mathcal{E}$ family of all open sets in X

f continuous $\overset{\text{def.}}{\iff}$ (i.e.) the inverse image of an open set is an open set.

but $\mathcal{B}' = \underbrace{\sigma_0(\mathcal{E}')}_{\sigma\text{-algebra generated by } \mathcal{E}'} \implies f$ is Borel measurable. (we're using the previous theorem) ∎

29/09

Prop. $f: X \to X'$ continuous $\Longrightarrow$ f Lebesgue measurable

proof.

f continuous $\Longleftrightarrow$ $\forall E \in \mathcal{E}'$ $f^{-1}(E) \in \mathcal{E}$
but $\mathcal{B}' = \sigma_0(\mathcal{E}')$ and: $\mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{L}$
$\Longrightarrow$ $f^{-1}(E) \in \mathcal{E} \subseteq \mathcal{L}$ $\forall E \in \mathcal{E}'$, $\mathcal{E}'$ s.t. $\mathcal{B}' = \sigma_0(\mathcal{E}')$
$\Longrightarrow$ f is Lebesgue measurable ■
(because of the previous theorem with $\mathcal{A}' = \mathcal{B}'$, $\mathcal{A} = \mathcal{L}$)

Remark: $f: X \to X'$ Lebesgue measurable $\Longleftrightarrow$ $\forall E \in \mathcal{E}'$ $f^{-1}(E) \in \mathcal{L}$, $\mathcal{E}' = \{$open sets $\subseteq X'\}$

[**i.e.** $\forall$ open set in X', its inverse image is Lebesgue measurable]

That's basically the previous theorem with X' equipped with the Borel measurable sets (so, instead of considering all Borel sets we can consider open sets).

Notice: sometimes this remark is given as a definition of Lebesgue measurable function

$(X, \mathcal{L})$
$(X', \mathcal{B})$

5/10

Remark on the chapter "**Measure**" (a property of the outer measure)

Remark: There are disjoint sets $A, B \subset \mathbb{R}$ s.t.

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$$

proof.

Assume by contraddiction that $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ $\forall A, B \subset \mathbb{R}$, $A \cap B = \emptyset$
Then, every set satisfies the caratheodory condition:
In fact $\forall E \subset \mathbb{R}$, $\forall Z \subseteq \mathbb{R}$:

assuming that the equality holds

$$\lambda^*(\underbrace{E \cap Z}_{:=A}) + \lambda^*(\underbrace{E^c \cap Z}_{:=B}) = \lambda^*(A \cup B) = \lambda^*(Z)$$

"which is exactly"

$\Longrightarrow$ E is measurable.
This is a contraddiction (e.g. Vitali set) — $\exists$ non-measurable sets ■

REAL VALUED FUNCTIONS

Def. $(X, \mathcal{A})$ measurable space, $f: X \to \overline{\mathbb{R}}$. Consider the set of meas. functions:

$$\mathcal{M}(X, \mathcal{A}) := \{f: X \to \overline{\mathbb{R}} \text{ measurable}^*\} = \text{set of measurable functions}$$

$$\mathcal{M}_+(X, \mathcal{A}) := \{f: X \to \overline{\mathbb{R}} \text{ measurable}^*,\ f \geq 0 \text{ in } X\}$$

$\forall \alpha \in \mathbb{R}$ let:

$$\{f > \alpha\} = \{x \in X : f(x) > \alpha\} := f^{-1}((\alpha, +\infty])$$

similarly $\rightarrow$ $\{f \geq \alpha\}$, $\{f > \alpha\}$, $\{f \leq \alpha\}$

generally speaking measurable = Lebesgue measurable
$f: (X, \mathcal{L}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$
L-measurable if $\forall E \in \mathcal{B}(\overline{\mathbb{R}})$: $f^{-1}(E) \in \mathcal{L}(X)$ Lebesgue σ-algebra
here we consider: $f: (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$
measurable if $\forall E \in \mathcal{B}(\overline{\mathbb{R}})$: $f^{-1}(E) \in \mathcal{A}$

Theorem: The following statements are equivalent:

useful to understand whether f is measurable or not

(i) f is measurable
(ii) $\{f > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$ $(\{f > \alpha\} \in \mathcal{A})$
(iii) $\{f \geq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$ $(\{f \geq \alpha\} \in \mathcal{A})$
(iv) $\{f < \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$ $(\{f < \alpha\} \in \mathcal{A})$
(v) $\{f \leq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$ $(\{f \leq \alpha\} \in \mathcal{A})$

* the inverse image of every subset of the Borel σ-algebra belongs to $\mathcal{A}$

proof. ((i) $\overset{1.}{\Longleftrightarrow}$ (ii), (ii) $\overset{2.}{\Rightarrow}$ (iii), (iii) $\overset{3.}{\Rightarrow}$ (iv), (iv) $\overset{4.}{\Rightarrow}$ (v), (v) $\overset{5.}{\Rightarrow}$ (ii))

1. $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\{(\alpha, +\infty] : \alpha \in \mathbb{R}\}) = \sigma_0(\mathcal{E})$ ($\mathcal{E}$ to use the previous lesson notation)

By a previous theorem*, f is measurable $\Longleftrightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{E}$
$\Longleftrightarrow f^{-1}((\alpha, +\infty]) \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$
$\Longleftrightarrow \{f > \alpha\} \in \mathcal{A}$

thanks to the theorem* we can work only with $(\alpha, +\infty]$ instead of considering every set of $\mathcal{B}(\overline{\mathbb{R}})$ (that's a great thing)

2. $\{f \geq \alpha\} = \bigcap_{n=1}^{+\infty} \underbrace{\{f > \alpha - \frac{1}{n}\}}_{\text{measurable (by property (ii))}} \in \mathcal{A}$ countable $\cap$ of measurable sets $\Rightarrow$ measurable

3. $\{f < \alpha\} = \underbrace{\{f \geq \alpha\}^c}_{\text{measurable (by property (iii))}} \in \mathcal{A}$ by def. of σ-algebra

4. $\{f \leq \alpha\} = \bigcap_{n=1}^{+\infty} \underbrace{\{f < \alpha + \frac{1}{n}\}}_{\text{measurable (by property (iv))}} \in \mathcal{A}$

5. $\{f > \alpha\} = \underbrace{\{f \leq \alpha\}^c}_{\text{measurable (by prop. (v))}} \in \mathcal{A}$ ■

Let $f, g : X \to \overline{\mathbb{R}}$. We define:

$\{f < g\} = \{x \in X : f(x) < g(x)\}$

$\{f \leq g\} = \{x \in X : f(x) \leq g(x)\}$, $\{f = g\}$, ..

Theorem: Let $\boxed{f, g \in \mathcal{M}(X, \mathcal{A})}$ The following statements hold:

f,g measurable

(i) $\{f < g\} \in \mathcal{A}$ (measurable)
(ii) $\{f \leq g\} \in \mathcal{A}$
(iii) $\{f = g\} \in \mathcal{A}$

proof.

(i) $\{f < g\} = \bigcup_{r \in \mathbb{Q}} \underbrace{[\underbrace{\{f < r\}}_{\in \mathcal{A}} \cap \underbrace{\{r < g\}}_{\in \mathcal{A}}]}_{\in \mathcal{A}} \in \mathcal{A}$ the countable union of measurable sets is measurable ($\mathbb{Q}$ because we needed "countable" union)

because f,g are measurable and $r \in \mathbb{Q} \subset \mathbb{R}$ (hence, by the previous theorem $\in \mathcal{A}$)

$\in \mathcal{A}$ because it's the intersection of 2 measurable sets

(ii) By (i): $\{f > g\} \in \mathcal{A} \Longrightarrow \{f > g\}^c \in \mathcal{A}$ where $\{f > g\}^c = \{f \leq g\}$

(iii) $\{f = g\} = \{f \geq g\} \cap \{f \leq g\} \in \mathcal{A}$ intersection of measurable sets ■

Theorem: Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$. Then: $\underbrace{\sup_{n \in \mathbb{N}} f_n, \ \inf_{n \in \mathbb{N}} f_n}_{\text{these are 2 functions in } x \,!} \in \mathcal{M}(X, \mathcal{A})$

proof.

$\{\sup_n f_n > \alpha\} = \bigcup_{n=1}^{+\infty} \underbrace{\{f_n > \alpha\}}_{\in \mathcal{A}} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R} \Longleftrightarrow \sup_n f_n$ is measurable

union of measurable sets

by the previous theorem ("equivalent statements")

$\inf_n f_n = -\sup_n (-f_n) \in \mathcal{M}(X, \mathcal{A})$ ■

f_n measurable $\Rightarrow -f_n$ measurable
sup measurable $\Rightarrow$ - sup measurable

Corollary: $f, g \in \mathcal{M}(X, \mathcal{A}) \Longrightarrow \max\{f,g\}, \min\{f,g\}, f^+, f^- \in \mathcal{M}(X,\mathcal{A})$

positive/negative parts

max/min come from the measurability of inf/sup, f^+/f^- come from the measurability of max/min

Theorem: $\{f_n\} \subseteq \mathcal{M}(X,\mathcal{A})$. Then: $\limsup_n f_n, \liminf_n f_n \in \mathcal{M}(X,\mathcal{A})$

proof.

$$\limsup_n f_n = \inf_{k \geq 1}\left(\sup_{n \geq k} f_n\right) \in \mathcal{M}$$

because if f_n is measurable then $\sup f_n$ is measurable, then the inf of something measurable is measurable

$$\liminf_n f_n = -\limsup_n (-f_n) \in \mathcal{M}$$

■

Theorem: $f, g: X \to \mathbb{R}$, $f, g \in \mathcal{M}(X,\mathcal{A})$. Then $f+g, f \cdot g \in \mathcal{M}$

not $\overline{\mathbb{R}}$ (because we consider $f+g$ and $\infty - \infty$ wouldn't be defined

proof.

- We introduce: $\varphi: X \to \mathbb{R}^2$:

$$\varphi(x) = (f(x), g(x)) \qquad x \in X$$

- Moreover we consider:

$$\left.\begin{array}{ll} \psi: \mathbb{R}^2 \to \mathbb{R} & \psi(s,t) = s+t \\ \chi: \mathbb{R}^2 \to \mathbb{R} & \chi(s,t) = s \cdot t \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \psi \circ \varphi = f+g \\ \chi \circ \varphi = f \cdot g \end{array}\right.$$

compositions:

- $\psi, \chi \in C^0(\mathbb{R}^2) \Longrightarrow \psi, \chi$ are measurable (continuous functions are measurable)
- We claim that φ is measurable.
- In fact, φ is measurable $\iff \forall E \subseteq \mathbb{R}^2 \quad \varphi^{-1}(E) \in \mathcal{A}$
- We consider: $E = R := (a,b) \times (c,d)$ (open rectangle)
- Then:

in principle we should consider every Borel set of $\mathbb{R}^2$, however we know (because of the theorem*) that it's sufficient to consider open sets of $\mathbb{R}^2$

* of the previous lesson

(that's why we consider $(a,b) \times (c,d)$)

$$\begin{aligned} \varphi^{-1}(R) &= \{x \in X : (f(x), g(x)) \in R\} \\ &= \{x \in X : f(x) \in (a,b),\ g(x) \in (c,d)\} \\ &= \underbrace{\underbrace{f^{-1}((a,b))}_{\in \mathcal{A}} \cap \underbrace{g^{-1}((c,d))}_{\in \mathcal{A}}}_{\in \mathcal{A}} \end{aligned}$$

because f, g are measurable (hence the inverse image of every open interval is measurable)

intersection of measurable sets

- We showed it not for every open sets, only for rectangles, but it's enough since:

$\forall E \subseteq \mathbb{R}^2$ open:

$$E = \bigcup_{k=1}^{+\infty} R_k \qquad R_k = (a_k, b_k) \times (c_k, d_k)$$

in $\mathbb{R}^n$: every open set can be written as a countable union of rectangles (open)

- $$\Longrightarrow \varphi^{-1}(E) = \varphi^{-1}\left(\bigcup_{k=1}^{+\infty} R_k\right) = \underbrace{\bigcup_{k=1}^{+\infty} \underbrace{\varphi^{-1}(R_k)}_{\in \mathcal{A}}}_{\in \mathcal{A}}$$

(we have proved it above)

(countable union of measurable sets)

- The claim has been proved. $\Longrightarrow$ $\psi \circ \varphi, \chi \circ \varphi$ are measurable $\Longrightarrow$ thesis

the composition of 2 measurable functions is measurable

■

Corollary: let $f: X \to \mathbb{R}$:

(i) $f \in \mathcal{M}(X,\mathcal{A}) \iff f^{\pm} \in \mathcal{M}_+(X,\mathcal{A})$ (its positive and negative parts are measurable)

(ii) $f \in \mathcal{M}(X,\mathcal{A}) \Longrightarrow |f| \in \mathcal{M}(X,\mathcal{A})$ (the opposite **isn't** true in general)

proof.

$f = f^+ - f^-$, $\quad f^+ = \max\{f, 0\}$, $\quad f^- = \max\{-f, 0\}$

f measurable $\Rightarrow$ $-f$ measurable $\Rightarrow$ $\max\{f,0\}, \max\{-f,0\}$ measurable $\Rightarrow$ f^+/f^- measurable

f^+ and f^- measurable $\Rightarrow$ $f^+ - f^-$ is measurable (by previous theorem)

$|f| = f^+ + f^-$: $\quad f \in \mathcal{M} \xrightarrow{(i)} f^+, f^- \in \mathcal{M} \xrightarrow{\text{by previous theorem}} f^+ + f^- \in \mathcal{M}$

■

Lemma: $A \subseteq X$: $\mathbb{1}_A \in \mathcal{M} \Longleftrightarrow A \in \mathcal{A}$ $\quad \left(\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \right)$

proof.

$$\{\mathbb{1}_A > \alpha\} = \begin{cases} X & \text{if } \alpha < 0 \\ A & \text{if } \alpha \in [0,1) \\ \emptyset & \text{if } \alpha \geq 1 \end{cases} \qquad \forall \alpha \in \mathbb{R} \qquad \blacksquare$$

(by def.) $X, \emptyset$ are always measurable $\rightarrow [\{\mathbb{1}_A > \alpha\}$ meas. $\Longleftrightarrow A \in \mathcal{A}$ (measurable)$]$

Remark: $|f| \in \mathcal{M} \not\Rightarrow f \in \mathcal{M}$

proof.

consider: $E \subseteq X$, $E \notin \mathcal{A}$: (not measurable) $\quad f(x) := \mathbb{1}_E(x) - \mathbb{1}_{E^c}(x) = \begin{cases} 1 & x \in E \\ -1 & x \in E^c \end{cases}$

$E \notin \mathcal{A} \Longrightarrow f \notin \mathcal{M}$ (by the def.)

But $|f| \equiv 1 \in \mathcal{M}$ $\blacksquare$

SIMPLE FUNCTIONS

Def. Let X be a set. A function $s: X \to \mathbb{R}$ is said to be a simple function if $s(X)$ is finite. (**simple functions**: functions that takes only a finite number of values)

$$s(X) = \{c_1, \dots, c_n\} \qquad c_k \neq c_\ell \quad k \neq \ell, \quad c_i \in \mathbb{R} \quad \forall i = 1, \dots, n$$

$$s = \sum_{k=1}^n c_k \mathbb{1}_{E_k} \qquad \textbf{canonical form}$$

$$E_k = \{x \in X : s(x) = c_k\} \qquad k = 1, \dots, n$$

$$X = \bigcup_{k=1}^n E_k, \qquad E_k \cap E_\ell = \emptyset \qquad k \neq \ell \quad (\text{since } c_k \neq c_\ell \ k \neq \ell)$$

Remark: The canonical form is unique.

Remark: $s \in \mathcal{M}(X, \mathcal{A}) \Longleftrightarrow E_k \in \mathcal{A} \quad \forall k = 1, \dots, n$ (by the previous lemma)

Moreover we introduce:

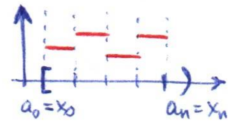
$$\mathcal{S}(X, \mathcal{A}) = \{s : X \to \mathbb{R} \text{ measurable simple functions}\}$$

$$\mathcal{S}_+(X, \mathcal{A}) = \{s : X \to \mathbb{R} \text{ measurable simple functions}, \ s \geq 0\}$$

Remark: Consider $I = [a_0, a_1)$ and a partition: $P := \{a_0 := x_0 < x_1 < \dots < x_n = a_1\}$.
Consider the function $f : I \to \mathbb{R}$:

$$f(x) = \sum_{\ell=0}^{n-1} c_\ell \mathbb{1}_{[x_\ell, x_{\ell+1})}(x) \qquad c_i \in \mathbb{R} \qquad i = 0, \dots, n-1$$

This function is the step function. $\rightarrow$ • finite number of values • constant on every interval



Theorem (the **Simple Approximation** theorem)

Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$. Then there exists a sequence $\{s_n\}_n$ of simple functions s.t.:

$$s_n \xrightarrow[n \to \infty]{} f \quad (\text{pointwise}) \text{ in } X. \qquad (\text{pointwise} \Longrightarrow s_n \to f \ \forall x \in X)$$

Moreover:

(i) if f is measurable then $\{s_n\}_n$ is measurable
$f \in \mathcal{M}(X, \mathcal{A}) \Longrightarrow \{s_n\} \subseteq \mathcal{S}(X, \mathcal{A})$

(ii) $f \geq 0 \Longrightarrow \{s_n\} \nearrow, \quad 0 \leq s_n \leq f$

(iii) f bounded $\Longrightarrow s_n \xrightarrow[n \to \infty]{} f$ uniformly in X

proof. (sketch)

We assume $f \geq 0$, f bounded. (for simplicity)

$\Longrightarrow \exists M > 0 : \quad 0 \leq f(x) \leq M \quad \forall x \in X$

Suppose $M = 1$: (w.l.o.g., otherwise we consider $\frac{f}{M}$)

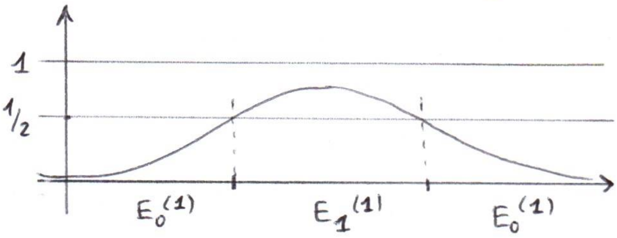
$\Longrightarrow \quad f : X \to [0,1]$.

Divide $[0,1]$ in 2^n intervals of length 2^{-n} for each $n \in \mathbb{N}$.

Then we define :

$$E_k^{(n)} := \left\{ x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\} \qquad k = 0, \dots, 2^n - 1$$

$n = 1$
(we divide the interval $[0,1]$ in 2)



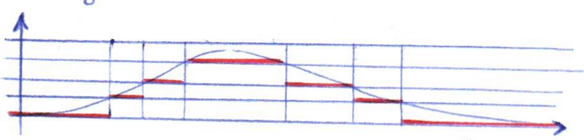
the range is divided in 2 subsets, they're **not** intervals
($E_0^{(1)}$ is not an interval, it's the union of 2 disjoint intervals)

Define:

$$S_n(x) = \sum_{k=0}^{2^n - 1} \frac{k}{2^n} \mathbb{1}_{E_k^{(n)}}(x) \qquad x \in X,\ n \in \mathbb{N}$$

$\longrightarrow \{S_n\}$ has all the desired properties ■

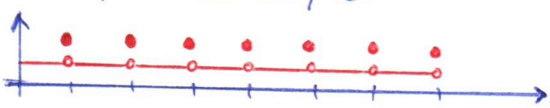
What is this doing?



as $n \to \infty$ the approximation gets better and better

Remark: simple function $\not\Rightarrow$ / $\Leftarrow$ step function

a general simple function may be:



(or even the characteristic function of the Cantor set is simple, but not step)

on the other hand, a step function is a linear combination of characteristic functions of **intervals**

6/10

ESSENTIALLY BOUNDED FUNCTIONS

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Def. $\forall N \in \mathcal{N}_\mu$ we can define the supremum

$$\alpha_N := \sup_{x \in N^c} f(x) \qquad f : X \to \overline{\mathbb{R}}$$

$\mathcal{N}_\mu =$ all subsets of X that are measurable and of zero-measure

$N_1, N_2 \in \mathcal{N}_\mu$, $N_2 \subseteq N_1 \Longrightarrow \alpha_{N_1} \leq \alpha_{N_2}$ (since the sup is defined by means of complementary sets)

Def. The essential supremum of a function is:

$$\operatorname*{ess\,sup}_X f := \inf \left\{ \sup_{x \in N^c} f(x) : N \in \mathcal{N}_\mu \right\} = \inf_{N \in \mathcal{N}_\mu} \left\{ \sup_{x \in N^c} f(x) \right\} = \inf_{N \in \mathcal{N}_\mu} \alpha_N$$

Similarly:

$$\operatorname*{ess\,inf}_X f := \sup \left\{ \inf_{x \in N^c} f(x) : N \in \mathcal{N}_\mu \right\} = \sup_{N \in \mathcal{N}_\mu} \left\{ \inf_{x \in N^c} f(x) \right\}$$

Prop. Let $f \in \mathcal{M}(X, \mathcal{A})$. Then $\exists N \in \mathcal{N}_\mu$ s.t. :

$$\operatorname*{ess\,sup}_X f(x) = \sup_{x \in N^c} f(x)$$

$$\inf_{N \in \mathcal{N}_\mu} \alpha_N = \alpha_N$$

$\exists N \in \mathcal{N}_\mu$ for which the infimum is "realized" (so the infimum becomes a minimum)

Moreover: $f(x) \le \operatorname{ess\,sup}_X f(x)$ a.e. in X $\quad (f \in \mathcal{M}(X,\mathcal{A}))$

proof. (both properties)

- If $\operatorname{ess\,sup}_X f(x) = \infty \implies$ it is obvious

$$\boxed{\begin{array}{l} 1 \quad \inf_{N \in \mathcal{N}_\mu} \alpha_N \le \alpha_N \le \alpha_{N_k} < \left[\inf_{N \in \mathcal{N}_\mu} \alpha_N\right] + \frac{1}{k} \\ 2 \quad \inf_{N \in \mathcal{N}_\mu} \alpha_N = \alpha_N \end{array}}$$

- Suppose that $\operatorname{ess\,sup}_X f(x) < \infty$.
  $\forall k \in \mathbb{N} \ \exists N_k \in \mathcal{N}_\mu$ s.t.

$$\sup_{x \in N_k^c} f < \operatorname{ess\,sup}_X f + \frac{1}{k} \quad \text{(by def of infimum)} \quad \left[\alpha_{N_k} < \inf_{N \in \mathcal{N}_\mu} \alpha_N + \frac{1}{k}\right]$$

- Define: $N := \bigcup_{k=1}^{+\infty} N_k$
  $\implies N \in \mathcal{N}_\mu$ since the N_k is measurable and of zero-measure
- Moreover:

$$N^c = \bigcap_{k=1}^{+\infty} N_k^c \subseteq N_k^c \quad \forall k \in \mathbb{N}$$

(the intersection of all subsets is contained in each subset)

- $\implies \underline{\forall k \in \mathbb{N}}$ $\boxed{\operatorname{ess\,sup}_X f \le \sup_{N^c} f \le \sup_{N_k^c} f < \operatorname{ess\,sup}_X f + \frac{1}{k}}$ 1
- $\implies$ letting $k \to +\infty$:

$$\boxed{\operatorname{ess\,sup}_X f = \sup_{N^c} f} \quad 2$$

since $N^c \subseteq X$ and the ess sup is defined as infimum

similarly: $N^c \subseteq N_k^c$

- We define a special set:

since f is measurable $\Rightarrow \{f > \alpha\}$ with $\alpha = \operatorname{ess\,sup}_X f$ is measurable

$$\bar{N} := \{f > \operatorname{ess\,sup}_X f\} \in \mathcal{A}$$

$\leftarrow \bar{N}$ is of the form of the previous lesson theorem (used for measurability) with $\alpha = \operatorname{ess\,sup}_X f$

- $\bar{N} \subseteq N \implies \bar{N} \in \mathcal{N}_\mu \implies f \le \operatorname{ess\,sup}_X f$ a.e. in X

■

Prop. $f \in \mathcal{M}(X,\mathcal{A}) \implies \operatorname{ess\,sup}_X f = -\operatorname{ess\,inf}_X (-f)$

Prop. $f \in \mathcal{M}(X,\mathcal{A}) \implies \operatorname{ess\,sup}_X (kf) = k \operatorname{ess\,sup}_X (f) \quad \forall k \ge 0$

if $k<0$ we use both proposition to obtain the conclusion

Prop. $f, g \in \mathcal{M}(X,\mathcal{A})$ then:

(i) $f \le g$ <u>a.e.</u> in $X \implies \operatorname{ess\,sup}_X f \le \operatorname{ess\,sup}_X g$

(ii) $\operatorname{ess\,sup}_X (f+g) \le \operatorname{ess\,sup}_X f + \operatorname{ess\,sup}_X g$

(iii) $f = g$ <u>a.e.</u> in $X \implies \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X g$ (/ess inf)

(iv) $g \ge 0$ <u>a.e.</u> in $X \implies f \cdot g \le (\operatorname{ess\,sup}_X f) \cdot g$ a.e. in X

The key point of everything is **a.e.** Only if we consider things a.e. we can make conclusions on ess sup/ess inf

We're saying that what happens on sets of zero measures is **not** important (the important things happen out of sets of zero measures)

Remark: $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ continuous $\implies \operatorname{ess\,sup}_A f = \operatorname{ess\,inf}_A f$

Def. We say that $f \in \mathcal{M}(X,\mathcal{A})$ is essentially bounded in X if:

$$\operatorname{ess\,sup}_X |f| < \infty$$

$\exists M \ge 0$ s.t. $\mu(\{x \in X : |f(x)| \ge M\}) = 0$

We define:

$$\mathcal{L}^\infty(X,\mathcal{A},\mu) := \{f : X \to \bar{\mathbb{R}} \text{ s.t. } f \in \mathcal{M}(X,\mathcal{A}) \text{ essentially bounded}\} = \text{set of all essentially bounded functions}$$

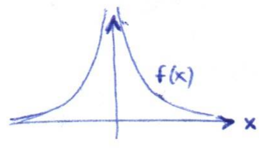
Remark: (i) $f \in \mathcal{L}^\infty \implies \boxed{f \text{ finite a.e. in } X}$, because: $|f| \le \operatorname{ess\,sup}_X |f| < \infty$ a.e. in X

If $f = \pm\infty$ then only on a set of zero measure

(ii) f finite a.e. in $X \not\Rightarrow f \in \mathcal{L}^\infty$

essentially bounded is stronger than finite a.e.

$$f(x) = \begin{cases} \frac{1}{|x|} & x \ne 0 \\ +\infty & x = 0 \end{cases} \quad (f : \mathbb{R} \to \bar{\mathbb{R}})$$



f is finite in $E := \mathbb{R} \setminus \{0\} \implies f$ is finite a.e. in $\mathbb{R}$ since $\underline{\mu(\{0\}) = 0}$

In this context $\mu(\cdot)$ is the Lebesgue measure

but:

$$\operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X |f| = +\infty$$

this is because as $x \to 0$ the function goes $+\infty$ (the limit of the function is $+\infty$ and hence the supremum is $+\infty$)

$\implies f$ is <u>not</u> essentially bounded

- $\forall x \in N^c$ is s.t. $f(x) \overset{\text{def. of sup}}{\leq} \sup\limits_{y \in N^c} f(y) \overset{\text{first part of the prop.}}{=} \operatorname{ess\,sup}\limits_{X} f$

- $\bar{N} := \{x \in X : f(x) > \operatorname{ess\,sup}\limits_{X} f\} \implies \begin{cases} \forall x \in \bar{N} : x \notin N^c \\ \forall x \in N^c : x \notin \bar{N} \end{cases}$

$\implies \bar{N} \cap N^c = \emptyset$

$\implies \bar{N} \subseteq N$

→ general measures
→ Lebesgue measure ($\mathbb{R}$ / $\mathbb{R}^N$)
→ properties of Lebesgue measure
→ measurable functions
→ properties of measurable functions
→ among all measurable functions:
  → simple functions
    → step functions
  → essentially bounded functions
→ Lebesgue integral:
1. non-negative simple functions
2. non-negative measurable functions
3. general measurable functions (including sign-changing functions)

THE LEBESGUE INTEGRAL

it's a name, we are **not** considering only the lebesgue measure

INTEGRAL OF NONNEGATIVE SIMPLE FUNCTIONS

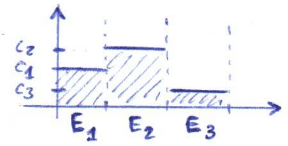
set of all non-negative simple functions

Let $(X, \mathcal{A}, \mu)$ be a measure space, $s \in \mathcal{S}_+(X, \mathcal{A})$ ($s = \sum_{k=1}^n c_k \mathbb{1}_{E_k}$ canonical form, with $c_1, c_2, \dots, c_n \in \mathbb{R}_+$, $E_k \in \mathcal{A}$ $k = 1, \dots, n$ pairly disjoint, $X = \bigcup_{k=1}^n E_k$)

$\mathcal{S}_+(X,\mathcal{A})$ (since s is measurable

Def. The Lebesgue integral is defined as:

$$\int_X s \, d\mu := \sum_{k=1}^n c_k \cdot \mu(E_k)$$



If $E \in \mathcal{A}$, we set: (E has to be measurable, otherwise we cannot consider its measure)

$$\int_E s \, d\mu := \int_X s \cdot \mathbb{1}_E \, d\mu$$

Remark: (i) $s \cdot \mathbb{1}_E = \sum_{k=1}^n c_k \cdot \mathbb{1}_{E_k \cap E}$ and so:

$$\int_E s \, d\mu = \sum_{k=1}^n c_k \cdot \mu(E_k \cap E)$$

(ii) $\int_X \mathbb{1}_E \, d\mu = \mu(E) \quad \forall E \in \mathcal{A}$

proof.

$$\underbrace{\mathbb{1}_E(x) = \sum_{k=1}^2 c_k \mathbb{1}_{E_k}(x)}_{\text{simple function}} \qquad E_1 = E, \; c_1 = 1, \; E_2 = E^c, \; c_2 = 0$$

$$\Rightarrow \int_X \mathbb{1}_E \, d\mu = c_1 \mu(E_1) + c_2 \mu(E_2) = \mu(E) \qquad \blacksquare$$

(iii) $\int_N s \, d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$

proof.

$$\int_N s \, d\mu = \sum_{k=1}^n c_k \mu(\underbrace{E_k \cap N}_{\in \mathcal{A}}) = 0$$

$\in \mathcal{A}$, (intersection of measurable)

$E_k \cap N \subseteq N$, $\mu(N) = 0$

and $\mu(E_k \cap N) \le \lambda(N) = 0 \quad \forall k$ $\blacksquare$

Prop. Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $c \geq 0$, then: $\int_X c \cdot s \, d\mu = c \cdot \int_X s \, d\mu$ (i)

Prop. Let $s, t \in \mathcal{S}_+(X, \mathcal{A})$, then: $\int_X (s+t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$ (ii)

Prop. Let $s, t \in \mathcal{S}_+(X, \mathcal{A})$, $s \le t \Rightarrow \int_X s \, d\mu \le \int_X t \, d\mu$ (iii)

Prop. Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $E, F \in \mathcal{A}$, $E \subseteq F \Rightarrow \int_E s \, d\mu \le \int_F s \, d\mu$ (iv)

Prop. Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $\forall E \in \mathcal{A}$:

$$\varphi(E) := \int_E s \, d\mu, \quad \varphi: \mathcal{A} \to \overline{\mathbb{R}}_+ \text{ is a measure}$$

way of defining new measures (by means of the integral)

! starting from a measure space with a measure μ it's possible to define another measure by using μ.

the variable of this function is a measurable set E. We take a simple function s and we fix it. This function, which takes a measurable set and gives a number, is not only a function but a measure

proof.

- $\varphi(\emptyset) = 0$ because $\mu(\emptyset) = 0$. and $\int_N s \, d\mu = 0 \; \forall N \in \mathcal{N}_\mu$??

  $\boxed{\text{(iv) of the previous prop.} \Rightarrow \text{monotonicity}}$

- Now σ-additivity:

  Let $\{E_k\} \subseteq \mathcal{A}$ disjoint, $E := \bigcup_{k=1}^\infty E_k$

  $\boxed{s := \sum_{\ell=1}^m d_\ell \mathbb{1}_{F_\ell}}$ canonical form with new letters (we introduce F_ℓ because we already used E_k)

$$\varphi(E) = \sum_{\ell=1}^m d_\ell \mu(F_\ell \cap E) = \sum_{\ell=1}^m \sum_{k=1}^\infty d_\ell \mu(F_\ell \cap E_k) = \sum_{k=1}^\infty \boxed{\sum_{\ell=1}^m d_\ell \mu(F_\ell \cap E_k)}$$

μ is a measure and $\forall$ fixed: $\{F_\ell \cap E_k\}$ is a disjoint family

since everything is ≥ 0 we can exchange the series

$\overset{(\text{def.})}{=} \int_{E_k} s \, d\mu$

$$= \sum_{k=1}^\infty \varphi(E_k)$$

by def. $\int_{E_k} s \, d\mu = \varphi(E_k)$ $\blacksquare$

INTEGRAL OF NONNEGATIVE MEASURABLE FUNCTIONS

Consider $f: X \to \overline{\mathbb{R}}_+$ and $(X, \mathcal{A}, \mu)$ measure space.

Def. $f \in \mathcal{M}_+(X, \overline{\mathbb{R}}_+)$. We define the integral as:

$$\int_X f \, d\mu := \sup_{s \in \mathcal{S}_f} \int_X s \, d\mu \qquad \mathcal{S}_f := \{ s \in \mathcal{S}_+(X, \mathcal{A}) : s \leq f \text{ in } X \}$$

this has been already defined

If $E \in \mathcal{A}$, we set:

$$\int_E f \, d\mu := \int_X f \, \mathbb{1}_E \, d\mu$$

Remark: $\mathcal{S}_f \neq \emptyset$

because of: (simple approx. theorem $:\Rightarrow \exists \{s_n\} \subseteq \mathcal{S}_f : \{s_n\} \nearrow, \; s_n \xrightarrow[n\to\infty]{} f$ in X) $\Rightarrow \mathcal{S}_f \neq \emptyset$

It is also possible to define the integral as:

$$\int_X f \, d\mu := \lim_{n \to \infty} \int_X s_n \, d\mu \quad \longrightarrow \text{ we exploit the simple approx. theorem}$$

$\in \mathbb{R}$!

Properties: as before ($s \rightsquigarrow f$)

Remark: $\int_N f \, d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$ because $\int_N s \, d\mu = 0 \quad \forall s \in \mathcal{S}_+(X, \mathcal{A})$

since the integral of f is defined by means of the integral of s, if the integral of s is zero we take the supremum and it is zero

Prop. (Chebychev inequality)

Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Then $\forall c > 0$:

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

proof.

- $E_c := \{ f \geq c \} \in \mathcal{A}$ (because $f \in \mathcal{M}_+(X, \mathcal{A})$)
- $c \, \mathbb{1}_{E_c} \leq f \, \mathbb{1}_{E_c}$ $\quad \mathbb{1}_{E_c} = 0$ when we're out E_c, in E_c ($\mathbb{1}_{E_c} = 1$) we have $f \geq c$
- $\Rightarrow \int_X f \, d\mu \underset{E_c \subseteq X}{\geq} \int_{E_c} f \, d\mu = \int_X f \, \mathbb{1}_{E_c} \, d\mu \geq \int_X c \cdot \mathbb{1}_{E_c} \, d\mu = c \cdot \int_X \mathbb{1}_{E_c} \, d\mu = c \cdot \mu(E_c)$

$\Rightarrow$ dividing by c we obtain the inequality ■

12/10

Two consequences of Chebychev inequality:

(1) **Prop.** Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be s.t. $\int_X f \, d\mu < +\infty$.
Then, f is finite a.e. in X.

$\mathcal{M}_+(X, \mathcal{A})$ non-negative measurable functions

That's **intuitive**: if f is infinite on a set which has positive (>0) measure then we obtain $\int_X f \, d\mu = +\infty$.

proof.

Thesis $\Leftrightarrow \mu(\{f = \infty\}) = 0$.

$\{f = \infty\} = \bigcap_{n=1}^{\infty} \{ f > n \}$.

Let $E_n := \{ f > n \}$. Then:

(a) $\{E_n\}_n \searrow$

(b) $\mu(E_n) \leq \frac{1}{n} \int_X f \, d\mu \quad \forall n \in \mathbb{N}$ (Chebyshev inequality)

$$\mu(\{f = \infty\}) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n) = 0$$ ■

the sequence is decreasing and $\mu(E_1) < \infty$

$\rightarrow$ since $\mu(E_n) \leq \frac{1}{n} \int_X f \, d\mu$, as $n \to \infty$ it goes to 0 (also: $\int_X f \, d\mu < \infty$)

(2) **Lemma** (**Vanishing lemma** for non-negative functions)

Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be s.t. $\int_X f \, d\mu = 0$.
Then $f = 0$ a.e. in X.

proof.

Thesis $\iff \mu(\{f>0\})=0$ $(f\in\mathcal{M}_+(X,\mathcal{A}))$

$\{f>0\} = \bigcup_{n=1}^{\infty}\{f\geq \frac{1}{n}\} := \bigcup_{n=1}^{\infty} F_n$

$\{F_n\}_n \nearrow$, $\frac{1}{n}\mathbb{1}_{F_n} \leq f\,\mathbb{1}_{F_n}$

$\mu(F_n)\leq n\int_X f\,d\mu = 0 \quad \forall n\in\mathbb{N}$

$\Rightarrow \mu(\{f>0\}) = \mu(\bigcup_{n=1}^{\infty}F_n) = \lim_{n\to\infty}\underbrace{\mu(F_n)}_{=0} = 0$

again we use the continuity of the measure (prop.)

■

CONVERGENCE THEOREMS

Theorem (Beppo Levi theorem, **Monotone Convergence** theorem)

Let $\{f_n\}\subseteq\mathcal{M}_+(X,\mathcal{A})$, $f: X\to\overline{\mathbb{R}}_+$ be s.t.:

(i) $f_n\leq f_{n+1}$ in X $\forall n\in\mathbb{N}$ ← crucial assumption: the sequence is **monotone** increasing

(ii) $f_n \xrightarrow{n\to\infty} f$ pointwise in X

Then:

$$\int_X f_n\,d\mu \xrightarrow{n\to\infty}\int_X f\,d\mu \qquad \left(\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X \overbrace{\lim_{n\to\infty} f_n}^{:=f}\,d\mu\right)$$

→ the limit of the integrals is the integral of the limit (we have passed the limit under the integral sign)

proof.

$\boxed{f\in\mathcal{M}_+(X,\mathcal{A})}$ is non-negative because it's the pointwise limit of non-negative functions

However, why is it measurable? We have proved that if we have a sequence of measurable functions then liminf and limsup are measurable. The limit, if it exists, is equal to both liminf and limsup. As a consequence the limit is measurable.

$\int_X f_n\,d\mu \leq \int_X f_{n+1}\,d\mu \leq \int_X f\,d\mu \quad (*)$

$\Rightarrow \boxed{\exists\ \alpha := \lim_{n\to\infty}\int_X f_n\,d\mu, \quad \alpha\leq\int_X f\,d\mu}$

This is because, thanks to $(*)$, we can deduce that the sequence $\{\int_X f_n\,d\mu\}_n$ is monotone $\Rightarrow$ a monotone sequence of real numbers has a limit (we call this limit α) and this limit, by $(*)$, is $\leq \int_X f\,d\mu$

Claim: $\alpha\geq\int_X f\,d\mu$.

In fact: $\forall\varepsilon\in(0,1)$, $\forall s\in\mathcal{S}_f$ (all simple functions smaller than f) let:

$E_n := \{(1-\varepsilon)s\leq f_n\} \quad \forall n\in\mathbb{N}$.

We have that:

(a) $\{E_n\}\subseteq\mathcal{A}$ (i.e. each E_n is measurable) — that's true because s and f_n are measurable and we know that if $f,g\in\mathcal{M}(X,\mathcal{A}) \Rightarrow \{f\leq g\}\in\mathcal{A}$

(b) $\{E_n\}\nearrow$ because $\{f_n\}\nearrow$

(c) $\boxed{X=\bigcup_{n=1}^{\infty}E_n}$

About (c):

clearly $\boxed{\bigcup_{n=1}^{\infty}E_n\subseteq X}$ (since by definition $\forall E_n: E_n\subseteq X$), so we have to check that $X\subseteq\bigcup_{n=1}^{\infty}E_n$.

Let $x\in X$. If $f(x)=+\infty$, then $\exists\bar{n}\in\mathbb{N}$ s.t. $\forall n>\bar{n}$:

$(1-\varepsilon)s(x) < f_n(x)$ (since $f_n(x)\to+\infty$)

$\Rightarrow x\in E_n \quad \forall n>\bar{n}$

If $f(x)<\infty$ then $\exists\bar{n}\in\mathbb{N}$ s.t. $\forall n>\bar{n}$:

$(1-\varepsilon)s(x) \underset{s\in\mathcal{S}_f}{\leq} (1-\varepsilon)f(x) < f_n(x)$

$\Rightarrow x\in E_n \quad \forall n>\bar{n}$

Therefore $X=\bigcup_{n=1}^{\infty}E_n$.

Observe that:

$(1-\varepsilon)\int_{E_n}s\,d\mu \leq \int_{E_n}f_n\,d\mu \leq \int_X f_n\,d\mu$ (since $E_n\subseteq X$ $\forall n\in\mathbb{N}$ and f is non-negative)

Let $n\to\infty$:

$(1-\varepsilon)\int_X s\,d\mu \leq \lim_{n\to\infty}\int_X f_n\,d\mu = \alpha$

$\varepsilon>0$ is arbitrary, hence: (also s is arbitrary)

$\int_X s\,d\mu\leq\alpha \Rightarrow \underbrace{\sup_{s\in\mathcal{S}_f}\int_X s\,d\mu}_{\text{this is the definition of } \int_X f\,d\mu} \leq\alpha \Rightarrow \int_X f\,d\mu\leq\alpha$ ■

In the previous proof we used:

Remark: $\lim_{n\to\infty} \int_{E_n} s \, d\mu = \int_X s \, d\mu$

proof.

$\varphi(E_n) := \int_{E_n} s \, d\mu \implies \varphi$ is a measure

$\{E_n\} \nearrow$ and so:

$$\lim_{n\to\infty} \varphi(E_n) \overset{\text{since } \varphi \text{ is a measure}}{=} \varphi\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{\text{point (c) of the previous proof}}{=} \varphi(X) = \int_X s \, d\mu.$$

■

Lemma (Fatou's lemma)

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then:

$$\int_X \underbrace{\left[\liminf_{n\to\infty} f_n\right]}_{\substack{\in \mathcal{M}_+(X,\mathcal{A}) \\ \text{(so the integral is well defined)}}} d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$$

proof.

$\liminf_{n\to\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$ since $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$

We define $g_k : X \to \overline{\mathbb{R}}$: $g_k := \inf_{n \ge k} f_n$

(Why? We recall: $\liminf_n f_n = \sup_{k\ge 1} g_k \overset{\text{since } g_k \text{ is increasing } (\{g_k\}\nearrow)}{=} \lim_{k\to\infty} g_k$)

Then:

(a) $\{g_k\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, $\{g_k\} \nearrow$ ($\{g_k\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ because the class of measurable functions is closed w.r.t. sup/inf)

(b) $g_k \le f_k \quad \forall k \in \mathbb{N}$ (by def.)

(c) $\lim_{k\to\infty} g_k = \sup_{k\ge 1} g_k = \liminf_{n\to\infty} f_n$

(b) $\implies \int_X g_k \, d\mu \le \int_X f_k \, d\mu \quad \forall k \in \mathbb{N}$ ($\int$ is monotone)

$$\underbrace{\liminf_{k\to\infty} \int_X g_k \, d\mu}_{\underbrace{\lim_{k\to\infty} \int_X g_k \, d\mu}_{\substack{(MCT): \text{ (since } \{g_k\}\nearrow \text{ (monotone))} \\ \int_X [\lim_{k\to\infty} g_k] \, d\mu \overset{(c)}{=} \int_X [\liminf_{n\to\infty} f_n] \, d\mu}}} \le \liminf_{k\to\infty} \int_X f_k \, d\mu$$

$\{g_k\} \nearrow \implies \{\int_X g_k \, d\mu\} \nearrow$ increasing sequence of real numbers $\implies$ the lim inf of a monotone sequence is lim

■

Remark: $(X, \mathcal{A}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$, $f_n := \mathbb{1}_{\{n\}}$:

There are cases in which Fatou's lemma has "<" and not "≤"

$$\lim_{n\to\infty} f_n = 0 \implies \int_{\mathbb{N}} \underbrace{\left[\liminf_{n\to\infty} f_n\right]}_{\lim_{n\to\infty} f_n = 0 \text{ (since the limit } \exists)} d\mu^\# = 0$$

On the other hand:

$$\int_{\mathbb{N}} f_n \, d\mu^\# = 1 \quad \forall n \in \mathbb{N}$$

($\int_{\mathbb{N}} f_n \, d\mu^\# = \sum_{k\in\mathbb{N}} f_n(k)$) the integral w.r.t. the counting measure is just a Σ

$$\implies \lim_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu^\# = 1$$

Hence:

$$0 = \int_{\mathbb{N}} \left[\liminf_{n\to\infty} f_n\right] d\mu < 1 = \lim_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu^\#$$

Theorem: (Integration of series)

Consider $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then:

$$\int_X \left(\sum_{n=1}^{+\infty} f_n\right) d\mu = \sum_{n=1}^{+\infty} \int_X f_n \, d\mu$$

proof.

$\sum_{n=1}^{+\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$ (the series is a limit ($\lim_{k\to\infty} \sum_{n=1}^{k}$) and measurable functions are closed w.r.t. the passage to the limit (moreover meas. functions are also closed for finite sums))

If $\sum_{n=1}^{+\infty} f_n \notin \mathcal{M}_+(X, \mathcal{A})$ the integral would **not** be defined.
(for the moment we defined integrals for simple functions and $\mathcal{M}_+(X, \mathcal{A})$ functions)

We define:

$$\sigma_k := \sum_{n=1}^{k} f_n \quad , \quad \lim_{k\to+\infty} \sigma_k =: \sum_{n=1}^{+\infty} f_n$$

$$\sigma_k \in \mathcal{M}_+ \, , \; \{\sigma_k\} \nearrow \quad (\Longleftarrow f_k \geq 0)$$

$$\int_X \sigma_k \, d\mu = \sum_{n=1}^{k} \int_X f_n \, d\mu$$

> since it's a **finite** sum, by property of the integral we can write the equality

$$\Longrightarrow \underbrace{\lim_{k\to\infty} \int_X \sigma_k \, d\mu}_{\substack{(\text{MCT}:) \\ \underbrace{\int_X \left(\lim_{k\to\infty} \sigma_k\right) d\mu}_{\int_X \left(\sum_{k=1}^{+\infty} f_n\right) d\mu}}} = \underbrace{\lim_{k\to+\infty} \sum_{n=1}^{k} \int_X f_n \, d\mu}_{\sum_{n=1}^{+\infty} \int_X f_n \, d\mu}$$

both limits exist since they're monotone quantities

∎

Theorem: let $f \in \mathcal{M}_+(X, \mathcal{A})$.

(i) Consider $\nu : \mathcal{A} \to \overline{\mathbb{R}}_+$ (last time we had **s** instead of **f**, a simple function)

$$\boxed{\nu(E) := \int_E f \, d\mu} \quad (E \in \mathcal{A})$$

Lebesgue integral + (generic) **measure** + measurable function generate a new measure (here μ is a generic measure, it's not, for example, Lebesgue)

is a measure. (defined by the means of **f**)

(ii) let $g \in \mathcal{M}_+(X, \mathcal{A})$. Then:

$$\int_X g \, d\nu = \int_X g f \, d\mu$$

since ν is a measure we can define the integral w.r.t. this measure

(iii) $\forall E \in \mathcal{A}$, $\mu(E) = 0 \Longrightarrow \nu(E) = 0$

proof.

(i) $\nu(\emptyset) = 0$ since $\mu(\emptyset) = 0$. the integral computed on a set of zero measure is zero

Let $E := \bigcup_{k=1}^{\infty} E_k$, $\{E_k\} \subseteq \mathcal{A}$ disjoint.

Then:

$$\nu(E) = \int_X f \mathbb{1}_E \, d\mu = \int_X f \sum_{k=1}^{\infty} \mathbb{1}_{E_k} \, d\mu = \sum_{k=1}^{+\infty} \underbrace{\boxed{\int_X f \mathbb{1}_{E_k} \, d\mu}}_{\nu(E_k)} = \sum_{k=1}^{+\infty} \nu(E_k)$$

theorem (integration for series)

(ii) let $g \equiv s \in \mathcal{S}_+(X, \mathcal{A})$: $\quad s = \sum_{k=1}^{n} c_k \mathbb{1}_{F_k}$, $\{F_k\} \subseteq \mathcal{A}$ disjoint, $X = \bigcup_{k=1}^{n} F_k$ (canonical form)

Then:

$$\int_X s \, d\nu \overset{\text{def. of integral of simple function}}{=} \sum_{k=1}^{n} c_k \nu(F_k) = \sum_{k=1}^{n} c_k \int_{F_k} f \, d\mu = \int_X \left(\sum_{k=1}^{n} c_k f \mathbb{1}_{F_k}\right) d\mu = \int_X f \boxed{\sum_{k=1}^{n} c_k \mathbb{1}_{F_k}} \, d\mu = \int_X f s \, d\mu$$

def. of $\nu(F_k)$

$\Rightarrow$ we have proved the thesis for the special case for which **g** is a simple function

If $g \in \mathcal{M}_+(X, \mathcal{A})$ then, by approximation we get the thesis.

(iii) $\mu(E) = 0 \Rightarrow \int_E f \, d\mu = 0 \overset{\text{by def.}}{\Longleftrightarrow} \nu(E) = 0$ ∎

In some sense, in this theorem we're saying " $d\nu = f \cdot d\mu$ ".
In some sense, this means that : " $\frac{d\nu}{d\mu} = f$ "
(f is the derivative of the measure ν w.r.t. the measure μ)

13/10

SET OF ZERO MEASURE AND INTEGRALS

Theorem: $f, g \in \mathcal{M}_+(X, \mathcal{A})$: $f = g$ a.e. in X. Then:

$$\int_X f \, d\mu = \int_X g \, d\mu$$

proof.

We define $N := \{x \in X : f \neq g\}$.

$N \in \mathcal{A}$, $\mu(N) = 0$ ($f, g \in \mathcal{M}_+$ and $f = g$ a.e.) saying $f=g$ a.e. already means that $f \neq g$ on a set of zero measure (which automatically implies that this set is meas.)

$\int_N f \, d\mu = \int_N g \, d\mu = 0$ (integral on a set of zero measure)

$$\int_X f \, d\mu = \underbrace{\int_N f \, d\mu}_{=0} + \int_{N^c} f \, d\mu = \int_{N^c} f \, d\mu = \int_{N^c} g \, d\mu = \int_{N^c} g \, d\mu + \underbrace{\int_N g \, d\mu}_{\text{we can add it since it's zero}} = \int_X g \, d\mu$$ ∎

on N^c we have $f=g$ since N is where $f \neq g$

Corollary : $f \in \mathcal{M}_+(X, \mathcal{A})$:

(i) $\int_X f \, d\mu = 0 \Longleftrightarrow$ (ii) $f = 0$ a.e. in X

proof.

(i) $\Rightarrow$ (ii) proved already (Vanishing lemma)

(ii) $\Rightarrow$ (i) previous theorem with $g = 0$, since $\int_X 0 \, d\mu = 0$ ∎

Hence, in the definition of $\int_X f \, d\mu$, $f \in \mathcal{M}_+(X, \mathcal{A})$, the sets of zero measure are not essential.

if we take 2 functions that are equal a.e. then we can consider the integral of the first or of the second, it's =.

Theorem (Beppo Levi theorem, or MCT, refined version)

$f_n, f \in \mathcal{M}_+(X, \mathcal{A})$ defined a.e. in X s.t. :

(i) $f_n \leq f_{n+1}$ a.e. in X $\forall n \in \mathbb{N}$

(ii) $f_n \xrightarrow[n \to \infty]{} f$ a.e. in X

Then :

$$\int_X f_n \, d\mu \xrightarrow[n \to \infty]{} \int_X f \, d\mu$$

The proof is more or less the same: the idea is that it is important only the function on sets of measure $\neq 0$.

INTEGRABLE FUNCTIONS

$(X, \mathcal{A}, \mu)$ measure space. $f : X \to \overline{\mathbb{R}}$.

Def. f is said to be integrable on X if $f \in \mathcal{M}(X, \mathcal{A})$ and :

$$\int_X f_+ \, d\mu < \infty, \quad \int_X f_- \, d\mu < \infty$$

this 2 objects are well defined since $f \in \mathcal{M} \Rightarrow f_\pm \in \mathcal{M}$, moreover both f_+ and f_- are non-negative
$\Rightarrow$ for them we know the notion of integral

We denote by :

$$\mathcal{L}^1(X, \mathcal{A}, \mu) := \{ f : X \to \overline{\mathbb{R}} \text{ integrable in } X \}$$

Hence, the integral of a (general) measurable function comes from the integral of non-negative measurable function (which comes from the integral of simple function)

simple $\to$ non-negative $\to$ measurable

Def. Let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. We define the integral as:

$$\int_X f\,d\mu := \int_X f_+\,d\mu - \int_X f_-\,d\mu \qquad (f = f_+ - f_-)$$

this is always **well defined** because $f \in \mathcal{L}^1$ hence both $\int_X f_+\,d\mu$ and $\int_X f_-\,d\mu$ are $< \infty$

This object is the **Lebesgue integral** of f in X.
If $E \in \mathcal{A}$, we set:

$$\int_E f\,d\mu := \int_X f\,\mathbb{1}_E\,d\mu = \int_E f_+\,d\mu - \int_E f_-\,d\mu$$

Prop. $f: X \to \overline{\mathbb{R}}$. Then:

(i) $f \in \mathcal{L}^1 \iff f_\pm \in \mathcal{L}^1$

$\boxed{\text{(ii) } f \in \mathcal{L}^1 \iff f \in \mathcal{M},\ |f| \in \mathcal{L}^1}$! alternative definition of $\mathcal{L}^1$ functions

(iii) $f \in \mathcal{L}^1 \implies \left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu$

proof.

(i) $(f_+)_+ = f_+$, $(f_+)_- = 0$, $(f_-)_+ = f_-$, $(f_-)_- = 0$ (first we have to say: $f \in \mathcal{M} \iff f_\pm \in \mathcal{M}$ otherwise it makes no sense talking about the integral)

$f_+ \in \mathcal{L}^1 \iff \int_X f_+\,d\mu < \infty$

$f_- \in \mathcal{L}^1 \iff \int_X f_-\,d\mu < \infty$

$\implies f \in \mathcal{L}^1 \iff f_\pm \in \mathcal{L}^1$ *

$f_+ \in \mathcal{L}^1 \iff \int (f_+)_\pm\,d\mu < \infty$ but we know that $(f_+)_+ = f_+$ and $(f_+)_- = 0$, hence it's enough to say: $f_+ \in \mathcal{L}^1 \iff \int f_+\,d\mu < \infty$ (it's not necessary to control $(f_+)_-$)

* $f \in \mathcal{L}^1 \iff \int f_+ < \infty, \int f_- < \infty$
$f_\pm \in \mathcal{L}^1 \iff \int f_+ < \infty, \int f_- < \infty$
$\implies f \in \mathcal{L}^1 \iff f_\pm \in \mathcal{L}^1$

(ii) $(\Leftarrow)$ $|f| \in \mathcal{L}^1 \overset{\text{def.}}{\iff} |f| \in \mathcal{M},\ \int_X |f|\,d\mu < \infty$

this is enough since: $|f|_+ = |f|$, $|f|_- = 0$

but:

$$\int_X |f|\,d\mu = \int_X (f_+ + f_-)\,d\mu = \int_X f_+\,d\mu + \int_X f_-\,d\mu$$

hence the assumpt. $\int |f| < \infty \iff \int_X f_+\,d\mu < \infty,\ \int_X f_-\,d\mu < \infty \overset{f \in \mathcal{M}}{\implies} f \in \mathcal{L}^1$

$(\Rightarrow)$ Let $f \in \mathcal{L}^1$.

$f \in \mathcal{M} \implies |f| \in \mathcal{M}$:

$$\int_X |f|\,d\mu = \int_X f_+\,d\mu + \int_X f_-\,d\mu < \infty$$

since $f \in \mathcal{L}^1$ both the integrals are finite

$\implies |f| \in \mathcal{L}^1$

(iii) $$\left|\int_X f\,d\mu\right| \overset{\text{def.}}{=} \left|\int_X f_+\,d\mu - \int_X f_-\,d\mu\right| \overset{\text{triang. ineq.}}{\le} \underbrace{\underbrace{\int_X f_+\,d\mu}_{\ge 0} + \underbrace{\int_X f_-\,d\mu}_{\ge 0}}_{\text{we omit the modulus}} = \int_X |f|\,d\mu$$ ∎

! The (ii) property $\iff f \in \mathcal{M},\ \int_X |f|\,d\mu < \infty$. (since $|f| \in \mathcal{L}^1 \iff \int_X |f|\,d\mu < \infty$) (because $|f|_+ = |f|$, $|f|_- = |f|$)

Prop. $\mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space.

proof.

Let $f, g \in \mathcal{L}^1$, $\lambda \in \mathbb{R}$

$\implies f_\pm, g_\pm$ finite a.e. in X (since $f, g \in \mathcal{L}^1 \implies \int f_\pm < \infty, \int g_\pm < \infty \implies f_\pm, g_\pm$ finite a.e.)

$\implies f, g$ finite a.e. in X ($f_\pm, g_\pm$ finite a.e. $\implies f = f_+ - f_-$, $g = g_+ - g_-$ finite a.e.)

$h := f + \lambda g$ defined a.e. in X (h measurable) (since both f, g are finite a.e. and measurable)

$\implies \int_X |h|\,d\mu \le \int_X |f|\,d\mu + |\lambda| \int_X |g|\,d\mu < \infty$

$\implies h$ measurable, $\int_X |h|\,d\mu < \infty \implies h \in \mathcal{L}^1$ (by the previous prop. (ii))

$\implies \mathcal{L}^1$ is a vector space

since we considered two elements of the space and we proved that the space is closed w.r.t. product by a real number and w.r.t. the sum ∎

Remark: $f, g \in \mathcal{L}^1$, $\lambda \in \mathbb{R}$. Then:

(i) $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$

(ii) $\int_X \lambda f\,d\mu = \lambda \int_X f\,d\mu$

Theorem: let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ be s.t.:

$$\int_E f\,d\mu = 0 \qquad \forall E \in \mathcal{A}.$$

Then $f = 0$ a.e. in X.

proof.

Consider: $E_+ := \{x \in X : f(x) \geq 0\}$, $E_- := \{x \in X : f(x) < 0\}$

Both $E_+, E_- \in \mathcal{A}$. ($f \in \mathcal{L}^1 \Rightarrow f \in \mathcal{M} \Rightarrow \{f \geq 0\}, \{f < 0\} \in \mathcal{A}$)

$\left.\begin{array}{l}\int_{E_+} f\,d\mu = 0 \longrightarrow f = 0 \text{ a.e. in } E_+ \\ \int_{E_-} f\,d\mu = 0 \longrightarrow f = 0 \text{ a.e. in } E_-\end{array}\right\}$ because of vanishing lemma (one with a positive funct. one with a negative one, but we can do it)

$\longrightarrow f = 0$ a.e. in $E_+ \cup E_- = X$ $\blacksquare$

$E_+ = \{f \geq 0\}$
$E_- = \{f < 0\}$

* $\int_E f\,d\mu = 0 \ \forall E \in \mathcal{A}$ ($E_+ \in \mathcal{A}$, $E_- \in \mathcal{A}$) hence both integrals $= 0$ (this holds only because of the arbitrariety of E)

Theorem: let $f \in \mathcal{L}^1$, $g \in \mathcal{M}$, $f = g$ a.e. in X. Then $g \in \mathcal{L}^1$ and:

$$\int_X g\,d\mu = \int_X f\,d\mu$$

proof.

$f = g$ a.e. $\Longrightarrow f_+ = g_+$, $f_- = g_-$ a.e. in X — if two functions are $\mathcal{M}_+$ and equal a.e. then their integral is the same (theorem from 13/10 - begin)

$\Longrightarrow \int_X f_+\,d\mu = \int_X g_+\,d\mu$, $\int_X f_-\,d\mu = \int_X g_-\,d\mu$ — and this implies the thesis since the integral is the difference of these two objects $\blacksquare$

LEBESGUE THEOREM

(Dominated Convergence Theorem)

Theorem: Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$, $f \in \mathcal{M}(X, \mathcal{A})$ be s.t. $f_n \xrightarrow[n\to\infty]{} f$ a.e..
Suppose that $\exists g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ s.t.:

$$|f_n| \leq g \quad \text{a.e. in } X \quad \forall n \in \mathbb{N}$$

Then, $\forall n \in \mathbb{N}$, $f_n, f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ and:

$$\int_X |f_n - f|\,d\mu \xrightarrow[n\to\infty]{} 0$$

In particular:

$$\int_X f_n\,d\mu \xrightarrow[n\to\infty]{} \int_X f\,d\mu \qquad \left(\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X \underbrace{\left(\lim_{n\to\infty} f_n\right)}_{:= f} d\mu\right)$$

proof.

From the assumptions $|f_n| \leq g$ a.e. in X $\forall n \in \mathbb{N}$.

$\Longrightarrow |f| \leq g$ a.e. in X since $f_n \xrightarrow[n]{} f$ a.e. in X

$\Longrightarrow \int_X |f|\,d\mu \leq \int_X g\,d\mu$, $\int_X |f_n|\,d\mu \leq \int_X g\,d\mu$ (since $|f| \leq g$, $|f_n| \leq g$ $\forall n$)

$\Longrightarrow g \in \mathcal{L}^1$, $\int_X g\,d\mu < \infty$ (since $g \in \mathcal{L}^1 \Longrightarrow \int_X g\,d\mu < \infty$, and hence:)

$\Longrightarrow f_n, f \in \mathcal{L}^1$ (since $f_n, f \in \mathcal{M}$, $|f|, |f_n|$ are s.t. $\int |f| < \infty$, $\int |f_n| < \infty$) (II charact. of $\mathcal{L}^1$)

$\Longrightarrow f_n, f$ are finite a.e. ($f, f_n \in \mathcal{L}^1 \Rightarrow \int f_+ < \infty, \int f_- < \infty, \ldots \Rightarrow \int f = \int f_+ - \int f_- < \infty, \ldots$)

Define: $\boxed{g_n := 2g - |f_n - f|}$ $n \in \mathbb{N}$ (we can def. g_n because all g, f, f_n are finite a.e.)

$|f_n - f| \leq |f_n| + |f| \leq 2g$ a.e. in X $\forall n \in \mathbb{N}$

$\Longrightarrow g_n \geq 0$ a.e. in X $\forall n \in \mathbb{N}$

$\Longrightarrow \{g_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$ since it's defined as difference of 2 measurable functions ($g \in \mathcal{M}$ since $g \in \mathcal{L}^1$, $f, f_n \in \mathcal{M} \Rightarrow f - f_n \in \mathcal{M} \Rightarrow |f - f_n| \in \mathcal{M} \Rightarrow g - |f - f_n| \in \mathcal{M}$)

$$2\int_X g\,d\mu = \int_X \left(\lim_{n\to\infty} g_n\right) d\mu$$

since $f_n \to f$, the pointwise limit of g_n is $2g$

$$\leq \liminf_{n\to\infty} \int_X g_n\,d\mu$$

Fatou's lemma

$$\overset{\text{def. of } g_n}{=} \liminf_{n\to\infty} \int_X [2g - |f_n - f|]\,d\mu$$

$$\overset{g \perp n}{=} \int_X 2g\,d\mu + \liminf_{n\to\infty}\left(-\int_X |f_n - f|\,d\mu\right)$$

$$= \int_X 2g\,d\mu - \limsup_{n\to\infty}\left(\int_X |f_n - f|\,d\mu\right)$$

$$\Rightarrow \limsup_{n\to\infty} \int_X |f_n - f|\,d\mu \leq 0$$

$$\Rightarrow \limsup_{n\to\infty} \int_X |f_n - f|\,d\mu = 0 \xrightarrow[\limsup = \lim]{\text{for positive sequences}} \lim_{n\to\infty}\int_X |f_n - f|\,d\mu = 0$$

$$\left|\int_X f_n\,d\mu - \int_X f\,d\mu\right| = \left|\int_X (f_n - f)\,d\mu\right| \leq \int_X |f_n - f|\,d\mu \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \int_X f_n\,d\mu \xrightarrow[n\to\infty]{} \int_X f\,d\mu \qquad \blacksquare$$

Remark: If : (i) $\mu(X) < \infty$

(ii) $\exists M > 0 : |f_n| \leq M$ a.e. in X $\forall n \in \mathbb{N}$

then we can take $g := M$.

Indeed:

$$\int_X g\,d\mu = M \cdot \mu(X) < \infty$$

RIEMANN AND LEBESGUE INTEGRAL

Theorem: $I = [a,b] \subset \mathbb{R}$, $f : I \to \mathbb{R}$ bounded. Then:

(i) $f \in \mathcal{R}(I)$ $\Longleftrightarrow$ (ii) f is continuous λ-a.e. in I

Riemann-integral

$=$ f is continuous a.e. with Lebesgue measure

Integrability according to Riemann can be characterized by means of the Lebesgue measure

Theorem: $I = [a,b]$, $f \in \mathcal{R}(I)$. Then: (I has to be **bounded**!)

(i) $f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$

(ii) $\int_I f\,d\lambda = \int_a^b f(x)\,dx$

The Lebesgue integral is exactly the Riemann integral

The situation is different when we consider improper integrals.

We define:

$$\mathcal{R}^g(I) := \{ f : I \to \mathbb{R} \text{ integrable in the generalized sense} \}$$

we are considering improper integrals

Here I can be: $I = (a,b)$, $\alpha, \beta \in \overline{\mathbb{R}}$.

Theorem: (i) $f \in \mathcal{R}^g(I) \Rightarrow f \in \mathcal{M}(I, \mathcal{L}(I))$

(ii) $|f| \in \mathcal{R}^g(I) \Rightarrow f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and: $\int_I f\,d\lambda = \int_\alpha^\beta f(x)\,dx$

the Lebesgue integral is exactly the improper integral in the sense of Riemann

Remark: Consider $f : I = [0, +\infty) \to \mathbb{R}$ s.t. $f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ (**Dirichlet** function)

$$\Rightarrow f \in \mathcal{R}^g(I) : \int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

but : $\int_{\mathbb{R}^+} \left|\frac{\sin(x)}{x}\right| d\lambda = +\infty \longrightarrow f \notin \mathcal{L}^1$

Lebesgue integral is a generalization of Riemann integral when we are in a bdd interval. But, if we consider improper integrals, Lebesgue integral is concerned with the modulus of f!

Remark: Consider $I = [0,1]$, $f = \mathbb{1}_{I \cap \mathbb{Q}}$.
$\Longrightarrow f \notin \mathcal{R}(I)$ but $f \in \mathcal{L}^1$ and $\int_I f \, d\lambda = 0$

$\Longrightarrow$ when we compare Riemann and Lebesgue we have to consider the interval where the function is defined. If the interval is bold then Lebesgue integral is better than Riemann. If we consider improper integrals, the situation is $\neq$ since Lebesgue integral is concerned with $|f|$. ($\exists$ functions Riemann-integrable in the improper sense but s.t. $\notin \mathcal{L}^1$).

16/10

LEBESGUE INTEGRATION

- Consequence of DCT (we also did it for MCT)

Theorem: (Integration for series)
Let $\{f_n\} \subseteq \mathcal{L}^1(X, \mathcal{A}, \mu)$ be s.t.

$$\sum_{n=1}^{+\infty} \int_X |f_n| \, d\mu < \infty.$$

Then, the series $\sum_{n=1}^{+\infty} f_n$ converges a.e. in X and:

$$\int_X \left(\sum_{n=1}^{+\infty} f_n \right) d\mu = \sum_{n=1}^{+\infty} \int_X f_n \, d\mu$$

proof.

$$\int_X \left(\sum_{n=1}^{+\infty} |f_n| \right) d\mu \overset{MCT}{=} \sum_{n=1}^{+\infty} \int_X |f_n| \, d\mu \overset{Hp.}{<} \infty$$

$\Longrightarrow \sum_{n=1}^{+\infty} |f_n| \in \mathcal{L}^1$

$\Longrightarrow \sum_{n=1}^{+\infty} |f_n|$ is finite a.e. in X (since $\in \mathcal{L}^1$)

$\Longrightarrow \sum_{n=1}^{+\infty} f_n$ absolutely converges a.e. in X (another way to say the same thing as above)

We define:

$$\sigma_k := \sum_{n=1}^{k} f_n$$

$\Longrightarrow |\sigma_k| = \left| \sum_{n=1}^{k} f_n \right| \leq \sum_{n=1}^{k} |f_n| \leq \sum_{n=1}^{+\infty} |f_n| := g$

$\Longrightarrow g \in \mathcal{L}^1$ (we have proved it)

Moreover:

$$\sigma_k \xrightarrow[k \to \infty]{\text{by def.}} \sum_{n=1}^{+\infty} f_n$$

this σ_k respects all the Hp. of the DCT: $\sigma_k \to \sum_{n=1}^{\infty} f_n$ (which converges because g conv.) and $\exists g \in \mathcal{L}^1$ s.t. : $|\sigma_k| \leq g \;\; \forall k$

$$\Rightarrow \sum_{n=1}^{+\infty} \int_X f_n \, d\mu = \lim_{k \to \infty} \sum_{n=1}^{k} \int_X f_n \, d\mu \overset{*}{=} \lim_{k \to \infty} \int_X \sum_{n=1}^{k} f_n \, d\mu$$

$$= \lim_{k \to \infty} \int_X \sigma_k \, d\mu \overset{DCT}{=} \int_X \lim_{k \to \infty} \sigma_k \, d\mu = \int_X \left(\sum_{n=1}^{+\infty} f_n \right) d\mu \quad \blacksquare$$

* finite sum

DERIVATIVE OF MEASURES

Def. $(X, \mathcal{A})$ measurable space, μ, ν measures. A function $\phi \in \mathcal{M}_+(X, \mathcal{A})$ is said to be the **Radon-Nikodym derivative** of ν w.r.t. μ if:

$$\nu(E) = \int_E \phi \, d\mu \qquad \forall E \in \mathcal{A}$$

We write: $\phi = \dfrac{d\nu}{d\mu}$.

(Only to have an **idea**:) (with a little abuse of notation)

$$d\nu(E) = \cancel{d}\!\!\int_E \phi \, d\mu \qquad \forall E \in \mathcal{A}$$

$$\Rightarrow \frac{d\nu(E)}{d\mu} = \frac{\phi \, \cancel{d\mu}}{\cancel{d\mu}} \quad \Longrightarrow \quad \frac{d\nu}{d\mu} = \phi$$

Def. We say that ν is absolutely continuous w.r.t. μ if:

$$\mu(E)=0 \implies \nu(E)=0$$

We write: $\nu \ll \mu$.

ν and μ are two general measures on the same measure space (we are not saying that one is the derivative of the other)

Theorem: (Radon-Nikodym)

$(X, \mathcal{A})$ measurable space; μ, ν measures.
Assume that $\nu \ll \mu$ and that μ is σ-finite.
Then: $\frac{d\nu}{d\mu}$ exists.

essential

We know that if $f \in \mathcal{M}_+$ then $\nu(E)=\int_E f\,d\mu \quad \forall E \in \mathcal{A}$ is a measure. However, in general, is it true that if we have two generic measures ν and μ there exists ϕ s.t. $\phi = d\nu/d\mu$?
(i.e. a ϕ s.t. we can represent one measure in terms of the other)
No, only under some assumptions.

Theorem: (uniqueness of the R-N derivative)

$\frac{d\nu}{d\mu}$ is unique. (a.e.)

proof.

By contraddiction suppose that $\exists\, \phi_1, \phi_2 \in \mathcal{M}_+(X,\mathcal{A})$ and both are R-N derivative of ν w.r.t. μ.

$$\implies \nu(E) = \int_E \phi_1\,d\mu = \int_E \phi_2\,d\mu \quad \forall E \in \mathcal{A}$$

$$\implies \int_E (\phi_1 - \phi_2)\,d\mu = 0 \quad \forall E \in \mathcal{A}$$

$$\implies \phi_1 - \phi_2 = 0 \text{ a.e. in } X \iff \phi_1 = \phi_2 \text{ a.e. in } X \quad \blacksquare$$

uniqueness is meant w.r.t. equality almost everywhere

THE SPACES L^1 AND L^∞

Consider a measure space $(X, \mathcal{A}, \mu)$.
We say that two functions are in relation if:

$$f \sim g \iff f = g \text{ a.e. in } X$$

$\sim$ is an equivalence relation. (reflexivity, simmetry, transitivity) $\rightarrow$ we can construct equivalence classes
We consider $\mathcal{L}^1(X, \mathcal{A}, \mu)$ and we define:

$$L^1(X,\mathcal{A},\mu) := \frac{\mathcal{L}^1(X,\mathcal{A},\mu)}{\sim}$$

in $\mathcal{L}^1$ we have f, in L^1 we have $[f]$.
However, to simplify we say $f \in L^1$

Two functions $f, g \in \mathcal{L}^1$ and s.t. $f=g$ a.e. are the same element in L^1.
If $f, g \in L^1$ and f is not $g \Rightarrow f \neq g$ a.e.

Lemma: L^1 is a metric space with

$$d(f,g) := \int_X |f-g|\,d\mu \quad \forall f, g \in L^1$$

proof.

$d: L^1 \times L^1 \to \mathbb{R}_+$

$$d(f,g) \leq \underbrace{\int_X |f|\,d\mu}_{<\infty} + \underbrace{\int_X |g|\,d\mu}_{<\infty} \in \mathbb{R}_+ \quad \text{since both } f, g \in L^1$$

well defined + non-negative

$d(f,g) = d(g,f)$ (simmetry)

$$d(f,g) = \int_X |f-g|\,d\mu \leq \int_X (|f-h| + |h-g|)\,d\mu \leq \underbrace{\int_X |f-h|\,d\mu + \int_X |h-g|\,d\mu}_{d(f,h)+d(h,g)}$$

(triangular inequality)

$d(f,g) = 0 \iff f = g$:

If $f=g \Rightarrow d(f,g) = \underbrace{\int_X |f-g|\,d\mu}_{=0 \text{ since } f=g} = 0$

$d(f,g) = 0 \iff \int_X |f-g| \, d\mu = 0 \implies |f-g| = 0$ a.e. in X (since $|f-g| \geq 0$)

$\iff f = g$ a.e. in X

$\iff f = g$ (in L^1) where functions equal a.e. are the same object $\blacksquare$

Remark: $\mathcal{L}^1$ is **not** a metric space. (the conclusion that $f=g$ a.e. does **not** imply that $f=g$ in $\mathcal{L}^1$)

Remark: L^1 is also a vector space.

Analogously we define:

$$L^\infty(X,\mathcal{A},\mu) := \frac{\mathcal{L}^\infty(X,\mathcal{A},\mu)}{\sim}$$

Lemma: L^∞ is a metric space with:

$$d(f,g) := \operatorname{ess\,sup}_X |f-g| \qquad \forall f,g \in L^\infty$$

Remark: L^∞ is a vector space.

TYPES OF CONVERGENCE (chapter)

Let $\{f_n\} \subseteq \mathcal{M}(X,\mathcal{A})$, $f_n : X \to \mathbb{R}$, $f : X \to \overline{\mathbb{R}}$.

- Pointwise convergence: $f_n \xrightarrow[n]{} f$ pointwise in X

  $\iff \forall x \in X : f_n(x) \xrightarrow[n\to\infty]{} f(x)$

- Uniform convergence: $f_n \xrightarrow[n\to\infty]{} f$ uniformly in X

  $\iff \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$

- Convergence a.e.: $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X

  $\iff \{x \in X : f_n(x) \xrightarrow[n\to\infty]{} f(x)\}^c$ is measurable and has zero-measure.

  = pointwise convergence for almost any x

- Convergence in L^1:

  $\{f_n\} \subseteq L^1$, $f \in L^1$, $f_n \xrightarrow[n\to\infty]{} f$ in L^1

  $\iff \underbrace{\int_X |f_n(x) - f(x)| \, d\mu}_{d(f_n,f) \text{ in } L^1} \xrightarrow[n\to\infty]{} 0$

Remark: in **MCT** and **DCT** the thesis is: $f_n \xrightarrow[n\to\infty]{} f$ in L^1. we can reformulate the thesis of MCT, DCT

- Convergence in L^∞:

  $\{f_n\} \subseteq L^\infty$, $f \in L^\infty$, : $f_n \xrightarrow[n\to\infty]{} f$ in L^∞

  $\iff \underbrace{\operatorname{ess\,sup}_X |f_n - f|}_{d(f_n,f) \text{ in } L^\infty} \xrightarrow[n\to\infty]{} 0$

- Convergence in measure

  $f_n, f \in \mathcal{M}(X,\mathcal{A})$ finite a.e. in X, $f_n \xrightarrow[n\to\infty]{} f$ in measure

  $\iff \forall \varepsilon > 0$:

  $$\mu(\underbrace{\{|f_n - f| \geq \varepsilon\}}_{\text{set } \in \mathcal{A} \text{ (since } f_n \text{ and } f \text{ are measurable)}}) \xrightarrow[n\to\infty]{} 0.$$

  notice that $\{|f_n - f| \geq \varepsilon\}$ is a sequence of sets, however $\mu(\{|f_n - f| \geq \varepsilon\})$ is a sequence of real numbers ($\mu(\{|f_n - f| \geq \varepsilon\}) \in \mathbb{R}_+$)

another formulation of conv. in measure:

16/10

Remark: $\forall \varepsilon > 0 \quad \forall \sigma > 0 \quad \exists \bar{n} \in \mathbb{N}$ s.t.:

$$\mu(\{|f_n - f| \geq \varepsilon\}) < \sigma.$$

Now we want to study **conv. relationships**. We know: uniform $\overset{\Rightarrow}{\not\Leftarrow}$ pointwise, pointwise $\overset{\Rightarrow}{\not\Leftarrow}$ pointwise a.e.

Theorem: let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ finite a.e.. $\forall \varepsilon > 0$ define:

$$B_n^\varepsilon := \{|f_n - f| \geq \varepsilon\} \qquad (n \in \mathbb{N})$$

2 equivalent formulations of convergence a.e.

The following statements are equivalent:

(i) $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X

(ii) $\mu(\{\limsup\limits_{n\to\infty} B_n^\varepsilon\}) = 0 \qquad \forall \varepsilon > 0$

B_n^ε = sequence of sets, hence we can define lim sup

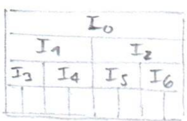
Theorem: let $\underline{\mu(X) < \infty}$. Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ finite a.e. in X.
The following statements are equivalent:

(i) $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X

(ii) $\lim\limits_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} B_n^\varepsilon\right) = 0 \qquad \forall \varepsilon > 0$

Remark: convergence in measure $\not\Rightarrow$ convergence a.e.

In fact: consider $(I, \mathcal{L}(I), \lambda)$, $I = [0,1]$. and define $\{I_n\}$ as:



$I_0 = I$

$I_1 = [0, \frac{1}{2}], \quad I_2 = [\frac{1}{2}, 1]$

$I_3 = [0, \frac{1}{4}], \quad I_4 = [\frac{1}{4}, \frac{1}{2}], \quad I_5 = [\frac{1}{2}, \frac{3}{4}], \quad I_6 = [\frac{3}{4}, 1]$

For all $p \in \mathbb{N} \Rightarrow 2^p$ intervals, each of length 2^{-p}

Moreover, consider:

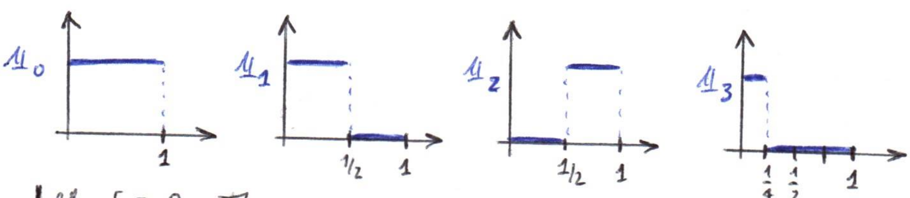
$1\!\!1_n := 1\!\!1_{I_n}$ (**Rademacher** sequence) - or: typewriter sequence

$$\Longrightarrow \begin{cases} 1\!\!1_n \not\xrightarrow[n\to\infty]{} 0 & \text{a.e. in } [0,1] \\ 1\!\!1_n \xrightarrow[n\to\infty]{} 0 & \text{in measure} \end{cases}$$

However, if we consider $\{[0, \frac{1}{2^k}]\} \subseteq I_n$

$\Longrightarrow \{1\!\!1_{n_k}\} \subseteq \{1\!\!1_n\}$. $1\!\!1_{n_k} \xrightarrow[k\to\infty]{} 0$ a.e. in $[0,1]$

19/10



- this sequence does **not** converge almost everywhere

Let $f \equiv 0$. Then:

$$B_n^\varepsilon := \{1\!\!1_n \geq \varepsilon\} \qquad \forall \varepsilon \in (0,1).$$

$B_0^\varepsilon = [0,1]$
$B_1^\varepsilon = [0, \frac{1}{2}]$
$B_2^\varepsilon = [\frac{1}{2}, 1]$
$\vdots$

$\forall k \in \mathbb{N}$:

$$\lambda\left(\bigcup_{n=k}^{\infty} B_n^\varepsilon\right) = \lambda([0,1]) = 1$$

By a previous lemma:

$$1\!\!1_n \not\rightarrow f \quad \text{a.e. in } [0,1]$$

(above **Theorem**:
f_n conv. a.e. $\Longleftrightarrow \lim\limits_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} B_n^\varepsilon\right) = 0$ but here it's $= 1$)

On the other hand:

$$\lim_{n\to\infty} \mu(B_n^\varepsilon) = \lim_{n\to\infty} \frac{1}{2^n} = 0 \iff \mathbb{1}_n \xrightarrow[n\to\infty]{} 0 \text{ in measure}$$

We consider now the subsequence:

$\{[0, \frac{1}{2^k}]\} \subseteq I$ (we consider always the first interval $\forall$ step)

Then: $\mathbb{1}_{n_k} \xrightarrow[k\to\infty]{} 0$ a.e. since $\forall x$ $\exists k$ big enoug s.t. the interval on which $\mathbb{1}_{n_k} = 1$ is on the left of x

Theorem: let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ finite a.e. in X.
If $f_n \xrightarrow[n\to\infty]{} f$ in measure, then there exists a subsequence $\{f_{n_k}\}_k$ s.t.

$$f_{n_k} \xrightarrow[n\to\infty]{} f \text{ a.e. in } X$$

proof.

$f_n \xrightarrow[n\to\infty]{} f$ in measure $\overset{\text{def.}}{\iff}$ with $\varepsilon = \sigma = \frac{1}{2^p}$ $p \in \mathbb{N}$: — thanks to this, instead of saying "$\forall \varepsilon, \forall \sigma$" we can say only "$\forall p$"

$\forall p \in \mathbb{N}: \exists n_p \in \mathbb{N} : \forall n \geq n_p$:

$$\mu(\{|f_n - f| \geq \underbrace{\tfrac{1}{2^p}}_{\varepsilon}\}) < \underbrace{\tfrac{1}{2^p}}_{\sigma}$$

set $n = n_p$:

$$B_p := \{|f_{n_p} - f| \geq \frac{1}{2^p}\}$$

$$\implies \mu(B_p) < \frac{1}{2^p} \quad \forall p \in \mathbb{N}$$

Now we define:

$$A_k := \bigcup_{p=k}^\infty B_p, \quad A := \bigcap_{k=1}^\infty A_k = \limsup_{p\to\infty} B_p$$

$$\implies \sum_{p=1}^\infty \mu(B_p) < \sum_{p=1}^\infty \frac{1}{2^p} = 1$$

$\overset{\text{Borel-cantelli}}{\implies} \mu(A) = 0$ **Borel-cantelli:** $\left(\sum_n \mu(E_n) < \infty \implies \mu(\limsup E_n) = 0\right)$

Consider $x \in A^c \overset{*}{\implies} \exists k \in \mathbb{N} : x \in \bigcap_{p=k}^\infty B_p^c$

$\implies \exists k \in \mathbb{N} : \forall p \geq k : |f_{n_p}(x) - f(x)| < \frac{1}{2^p}$

$\overset{\text{(def.)}}{\implies} f_{n_p}(x) \xrightarrow[p\to\infty]{} f(x)$

* $x \in A^c$
$x \in (\bigcap A_k)^c$
$x \in \bigcup (A_k)^c$
$x \in \bigcup (\bigcup B_p)^c$
$x \in \bigcup (\bigcap B_p^c)$

we are saying that the convergence happens $\forall x \in A^c$ and the complementary set ($A^{c^c} = A$) (set on which we **don't** have conv.) has measure zero → this is the def. of convergence **a.e.**

$\mu(A) = 0 \implies f_{n_p} \xrightarrow[p\to\infty]{} f$ a.e. in X → convergence is satisfied out of a set of zero measure $\blacksquare$

Theorem: let $\mu(X) < \infty$. let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X.
If $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X, then $f_n \xrightarrow[n\to\infty]{} f$ in measure.

proof.

By a previous lemma: (thm.)

$$\lim_{n\to\infty} \mu(B_n^\varepsilon) = \lim_{n\to\infty} \mu(\{|f_n - f| > \varepsilon\}) = 0 \quad \forall \varepsilon > 0$$

The assumption $\mu(X) < \infty$ is needed for the char. of convergence a.e. that we're using

(since $\lim_{k\to\infty} \mu(\bigcup_{n=k}^\infty B_n^\varepsilon) = 0$, $B_n^\varepsilon \subseteq \bigcup_{n=k}^\infty B_n^\varepsilon$ and μ is monotone) $\blacksquare$

def. conv. a.e.

however the Hp. is needed **not only** for the proof.

Remark: $\mu(X) = \infty \implies$ convergence a.e. in $X \not\Rightarrow$ convergence in measure

Consider $f_n := \mathbb{1}_{[n, \infty)}$:

- $f_n \xrightarrow[n\to\infty]{} 0$ in $\mathbb{R}$ (pointwise $\Rightarrow$ pointwise **a.e.**)
- $\mu(\{f_n \geq \frac{1}{2}\}) = +\infty$ $\quad (\{f_n \geq \frac{1}{2}\} = [n, +\infty))$ $\forall n \in \mathbb{N}$

Theorem: let $f_n, f \in L^1(X, \mathcal{A}, \mu)$.
If $f_n \longrightarrow f$ in L^1 then $f_n \longrightarrow f$ in measure.

proof.

Suppose, by contraddiction that $f_n \not\to f$ in measure.
Then $\exists\, \varepsilon > 0, \sigma > 0$ s.t.

$$\mu(\{|f_n - f| \geq \varepsilon\}) \geq \sigma$$

for infinitely many n in $\mathbb{N}$.
Thus:

since $|f_n - f| \geq 0$

$$d_{L^1}(f_n, f) = \int_X |f_n - f|\, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f|\, d\mu$$

$$\geq \varepsilon \int_{\{|f_n - f| \geq \varepsilon\}} d\mu$$

$$= \varepsilon\, \mu(\{|f_n - f| \geq \varepsilon\}) \geq \varepsilon \sigma > 0$$

for infinitely many $n \in \mathbb{N}$
$\Longrightarrow f_n \not\to f$ in L^1. ■

Corollary: If $f_n \xrightarrow[n\to\infty]{} f$ in L^1 then there exists a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ s.t.
$f_{n_k} \xrightarrow[k\to\infty]{} f$ a.e. in X

proof.

It follows by combining the two previous results. ■

Remark: $f_n \xrightarrow[n\to\infty]{} f$ in measure$_{(\lambda)}$ $\not\Longrightarrow$ $f_n \xrightarrow[n\to\infty]{} f$ in L^1

Consider: $f_n(x) := n\, \mathbb{1}_{[0, \frac{1}{n}]}(x) \qquad x \in [0,1]$:

- $f_n \xrightarrow[n\to\infty]{} 0$ a.e. in $[0,1] \Longrightarrow f_n \xrightarrow[n\to\infty]{} 0$ in measure
- $\int_0^1 f_n(x)\, dx = n \int_0^1 \mathbb{1}_{[0, \frac{1}{n}]}(x)\, dx = n \cdot \frac{1}{n} = 1 \quad \forall n \in \mathbb{N}$.

$\Longrightarrow f_n \not\to 0$ in $L^1([0,1])$

in general looking for convergence in measure is **difficult**, if possible we prefer to go through convergence **a.e.** ($\lambda([0,1]) = 1 < \infty$)

Here we've saying also another thing: $f_n \to f$ a.e. $\not\Longrightarrow$ $f_n \to f$ in L^1 (we can use the same example)

PRODUCT MEASURE

Let $(X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2)$ be measurable spaces.
Let $R \subseteq \mathcal{P}(X_1 \times X_2)$:

$$R := \{\underbrace{E_1 \times E_2}_{\text{measurable rectangle}} : E_1 \in \mathcal{A}, E_2 \in \mathcal{A}\}$$

– it is **not** generally a rectangle since both E_1 and E_2 are just measurable sets (not intervals necessarily), we say rectangle because we consider the cartesian product of sets

We define: $\boxed{\sigma_0(R) = \mathcal{A}_1 \times \mathcal{A}_2}$, the product σ-algebra.
Then $(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is the product measurable space.

σ-algebra generated by all the above sets (of R) (we call it $\mathcal{A}_1 \times \mathcal{A}_2$)

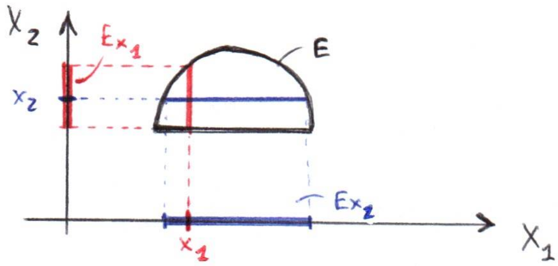
Def. Let $E \subseteq X_1 \times X_2$. $\forall x_1 \in X_1$:

$$E_{x_1} := \{x_2 \in X_2 : (x_1, x_2) \in E\} \subseteq X_2$$

is the x_1-section of E.
Similarly $\forall x_2 \in X_2$:

$$E_{x_2} := \{x_1 \in X_1 : (x_1, x_2) \in E\} \subseteq X_1$$

is the x_2-section of E.



If we have a measurable set in the product σ-algebra also all sections are measurable in the σ-algebra they belong.

Prop. Let $E \in \mathcal{A}_1 \times \mathcal{A}_2$. Then $E_{x_1} \in \mathcal{A}_2$ and $E_{x_2} \in \mathcal{A}_1$ $\forall x_1 \in X_1$, $\forall x_2 \in X_2$.

Let $(X_1, \mathcal{A}_1, \mu_1), (X_2, \mathcal{A}_2, \mu_2)$ be two measure spaces. Let $E \in \mathcal{A}_1 \times \mathcal{A}_2$. Then:

$E_{x_1} \in \mathcal{A}_2$ $\forall x_1 \in X_1$: $\varphi_1 : X_1 \to \overline{\mathbb{R}}_+$ s.t.

$$\varphi_1(x_1) = \mu_2(E_{x_1}) \qquad x_1 \in X_1$$

$E_{x_2} \in \mathcal{A}_1$ $\forall x_2 \in X_2$: $\varphi_2 : X_2 \to \overline{\mathbb{R}}_+$ s.t.

$$\varphi_2(x_2) = \mu_1(E_{x_2}) \qquad x_2 \in X_2$$

Theorem: Let $(X_1, \mathcal{A}_1, \mu_1), (X_2, \mathcal{A}_2, \mu_2)$ be two <u>σ-finite</u> measure spaces.
Let $E \in \mathcal{A}_1 \times \mathcal{A}_2$. Then:

(i) $\varphi_i \in \mathcal{M}_+(X_i, \mathcal{A}_i)$ $\qquad i = 1,2$

(ii) $\displaystyle\int_{X_1} \underbrace{\varphi_1(x_1)}_{\mu_2(E_{x_1})} d\mu_1 = \int_{X_2} \underbrace{\varphi_2(x_2)}_{\mu_1(E_{x_2})} d\mu_2$

φ_i is measurable and non-negative (since it is a measure) hence we can define its integral. (it can be $+\infty$ but is well defined).
This basically says that a set in $\mathcal{A}_1 \times \mathcal{A}_2$ can be measured by its sections (and obviously the measures correspond)

Prop. The function $\mu_1 \times \mu_2 : \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow \overline{\mathbb{R}}_+$:

$$(\mu_1 \times \mu_2)(E) := \int_{X_1} \mu_2(E_{x_1})\, d\mu_1 = \int_{X_2} \mu_1(E_{x_2})\, d\mu_2$$

is a σ-finite measure.

whatever subset, if we cover the subset with horizontal lines we find a measure, the same measure can be obtained if we cover the same set with vertical lines.

20/10

Example: Consider $(X_1, \mathcal{A}_1, \mu_1) = (X_2, \mathcal{A}_2, \mu_2) = (\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$.
Consider the Vitali set in $[0,1]$: $V \subseteq [0,1]$.
(V **not** measurable according to Lebesgue: $V \notin \mathcal{L}(\mathbb{R})$)
Define now: $E := \{x_0\} \times V$ $\quad (x_0 \in \mathbb{R})$.

$\Longrightarrow$ $E_{x_0} = V \notin \mathcal{L}(\mathbb{R})$

$\Longrightarrow$ $E \notin \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$ by def of the σ-alg (we know that: $E \in \mathcal{A}_1 \times \mathcal{A}_2 \Rightarrow E_{x_2} \in \mathcal{A}_1$, $E_{x_1} \in \mathcal{A}_2$, since we negate the right-hand side also the left-hand side does **not** hold)

Observe that:

$$E \subseteq F := \{x_0\} \times [0,1]$$

with $F \in \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$ s.t.

$$(\lambda \times \lambda)(F) = \int_{\mathbb{R}} \lambda(\{x_0\})\, d\lambda = 0$$ since $\lambda(\{x_0\}) = 0$ (Lebesgue measure of a point)

$\Longrightarrow$ $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}), \lambda \times \lambda)$ is <u>**not** complete</u>.

since there is a set E not measurable but contained in a set which is measurable and whose measure is zero (F) — E is negligible but not measurable

Remark: Consider $(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), \lambda_m)$, $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$.
Then, consider $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$, where λ_{m+n} is the Lebesgue measure constructed by means of the $(m+n)$-dimensional rectangles.

$\Longrightarrow$ $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ is complete.

$\mathcal{L}(\mathbb{R}^{m+n})$ and λ_{m+n} are $\perp\!\!\!\perp$ from $\mathcal{L}(\mathbb{R}^m), \mathcal{L}(\mathbb{R}^n)$ and λ_m, λ_n

We can consider also another measure on this space:

$$(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$$

here we're considering the product measure, which is $\not\!\perp\!\!\!\perp$ from $\mathcal{L}(\mathbb{R}^m), \mathcal{L}(\mathbb{R}^n)$, in fact it totally depends on them

which is **not** complete.

$$\Longrightarrow (\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n}) \neq (\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$$

Theorem: The first measure space is the completion of the second one.

Consider $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ measure spaces.
Consider $f \in \mathcal{M}_+(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$. Then:

$$\forall x_1 \in X_1 : \quad f(x_1, \bullet) \in \mathcal{M}_+(X_2, \mathcal{A}_2)$$
$$\forall x_2 \in X_2 : \quad f(\bullet, x_2) \in \mathcal{M}_+(X_1, \mathcal{A}_1)$$

Moreover, consider:

$$\varphi_1 : X_1 \to \overline{\mathbb{R}}_+ : \quad \varphi_1(x_1) = \int_{X_2} f(x_1, x_2)\, d\mu_2 \qquad x_1 \in X_1$$
$$\varphi_2 : X_2 \to \overline{\mathbb{R}}_+ : \quad \varphi_2(x_2) = \int_{X_1} f(x_1, x_2)\, d\mu_1 \qquad x_2 \in X_2$$

fixed

Theorem: **(Tonelli)**

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two σ-finite measure spaces.
Let $f \in \mathcal{M}_+(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$. (**only Hp**: f non-negative and measurable (the integrals can be $+\infty$ but it's not a problem, for non-negative functions the integral can be $+\infty$))
Then:

(i) $\varphi_i \in \mathcal{M}_+(X_i, \mathcal{A}_i) \qquad i = 1, 2$

(*) (ii) $$\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2) = \int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, d\mu_2 \right] d\mu_1 = \int_{X_1} \varphi_1(x_1)\, d\mu_1$$
$$= \int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, d\mu_1 \right] d\mu_2 = \int_{X_2} \varphi_2(x_2)\, d\mu_2$$

(applications:)
(reduction formula,
change of order of integration)

Theorem: **(Fubini)**

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two σ-finite measure spaces.
Let $f \in L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$.
Then:

(i) $f(x_1, \bullet) \in L^1(X_2, \mathcal{A}_2, \mu_2)$ for a.e. $x_1 \in X_1$,
$f(\bullet, x_2) \in L^1(X_1, \mathcal{A}_1, \mu_1)$ for a.e. $x_2 \in X_2$

we need to add "a.e." because a function in L^1 is properly defined only almost everywhere

(ii) $\varphi_1 \in L^1(X_1, \mathcal{A}_1, \mu_1)$,
$\varphi_2 \in L^1(X_2, \mathcal{A}_2, \mu_2)$

in (i) we said that $f(x_1, \bullet), f(\bullet, x_2) \in L^1$. Since they are in L^1 their integral is well defined and so we can consider φ_1 and φ_2.

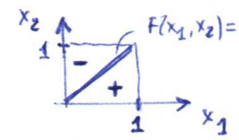
(iii) (*) holds

Remark: the hypothesis of $f \in L^1(X_1 \times X_2)$ is essential.

Consider: $f(x_1,x_2) := \dfrac{x_1^2 - x_2^2}{(x_1^2+x_2^2)^2}$ $\quad x_1, x_2 \in [0,1]$

Here: $(X_i, \mathcal{A}_i, \mu_i) = ([0,1], \mathcal{L}([0,1]), \lambda)$ $\quad i = 1,2$

- Statement: $f \notin L^1(X_1 \times X_2)$.



(proof.) $f_+ = \begin{cases} 0 & x_1 \le x_2 \\ f & x_1 > x_2 \end{cases}$ $\in \mathcal{M}_+(X_1 \times X_2)$ (since it's continuous and non-negative)

$$\int_{X_1 \times X_2} f_+(x_1,x_2)\, d(\lambda \times \lambda) \underset{\text{Tonelli theorem}}{=} \int_{X_1}\left[\int_{X_2} f_+(x_1,x_2)\, d\lambda\right] d\lambda = [..] = +\infty$$

$\Longrightarrow f \notin L^1(X_1 \times X_2)$

- Statement:

$$\left.\begin{array}{l} \int_{X_1}\left[\int_{X_2} f(x_1,x_2)\, dx_2\right] dx_1 = \frac{\pi}{4} \\ \int_{X_2}\left[\int_{X_1} f(x_1,x_2)\, dx_1\right] dx_2 = -\frac{\pi}{4} \end{array}\right\} \neq \quad \text{because } f \notin L^1$$

so, to use the formula we have to be sure that the Hp. hold

BV AND AC FUNCTIONS

Consider $f \in L^1([a,b])$ and define: $\quad F(x) := \int_a^x f(t)\, dt \quad \forall x \in [a,b]$.

we should write $d\lambda$ (we'll always consider the Lebesgue measure), but for simplicity: dt

Def. $x_0 \in [a,b]$ is a lebesgue point for f if:

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)|\, dt = 0$$

x_0 is a **Lebesgue point** if we can approximate $f(x_0)$ with an integral average (e.g. jumps are not lebesgue points)

Theorem: If $f \in L^1([a,b])$ then almost all $x_0 \in [a,b]$ are lebesgue points for f.

Theorem: (1^{st} Fundamental Theorem of Calculus)

If $f \in L^1([a,b])$, then F is differentiable a.e. in $[a,b]$ and

$$F' = f \quad \text{a.e. in } [a,b].$$

proof.

Let $x \in [a,b]$ be a lebesgue point for f and $h \neq 0$ s.t. $x+h \in [a,b]$.

h must be small enough, otherwise we cannot compute $F(x+h)$

$$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\left[\int_x^{x+h} (f(t) - f(x))\, dt\right]$$

$\dfrac{F(x+h)-F(x)}{h} = \dfrac{\int_x^{x+h} f(t)dt}{h}$ by the above definition then we add $f(x)$ which is $\perp\!\!\perp$ from dt

$$\Longrightarrow \left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \le \underbrace{\frac{1}{|h|}\left|\int_x^{x+h} |f(t) - f(x)|\, dt\right|}_{\to 0 \text{ as } h \to 0 \text{ since } x \text{ is a lebesgue point}}$$

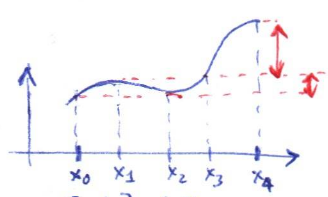
$$\Longrightarrow \underbrace{\lim_{h \to 0} \frac{F(x+h)-F(x)}{h}}_{:= F'(x)} = f(x)$$

by preceding theorem: $F' = f$ a.e. in $[a,b]$. (since a.e. x is a lebesgue point) ■

Let $I = [a,b]$, $f: I \to \mathbb{R}$. Let

$$P = \{a = x_0 < x_1 < \dots < x_n = b\} \quad n \in \mathbb{N}$$

be a partition of the interval I.



Define:

$$v_a^b(f;P) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

:= variation on $[a,b]$ of the function f w.r.t. the partition P

If a function is constant, the variation is zero. If a function variates a lot then v_a^b is very big.

Let $\mathcal{P}$ be the collection of all partitions of $[a,b]$.

the partition is important: changing the partition we change v_a^b

Def. $V_a^b(f) := \sup_{P \in \mathcal{P}} v_a^b(f;P)$:= total variation. 20/10

Def. If $V_a^b(f) < \infty$ then we say that f is a function of bounded variation.
The set of all functions of bounded variations in $[a,b]$ is denoted by $BV([a,b])$.

Remark: (i) $f:[a,b] \to \mathbb{R}$ monotone $\implies V_a^b(f) = |f(b) - f(a)| < \infty$ since $f(b)$ and $f(a)$ are numbers ($< \infty$)
$\implies f \in BV([a,b])$

(ii) $f \in BV([a,b]) \implies f$ is bounded
Indeed:
$$\sup_{x \in [a,b]} |f(x)| \leq |f(a)| + V_a^b(f) < \infty \longrightarrow f \text{ bounded}$$
f unbounded $\implies f \notin BV([a,b])$

(iii) Consider:
$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad x \in [0, \tfrac{2}{\pi}]$$
$$P_n = \{0, x_{n-1}, x_{n-2}, \dots, x_1, x_0 = \tfrac{2}{\pi}\}, \quad x_\ell = \frac{2}{(2\ell+1)\pi}$$
(a partition is just an ordered set of points) (ℓ indexes name)
$$\implies \sum_{\ell=0}^{n-1} |f(x_{\ell+1}) - f(x_\ell)| \xrightarrow[n\to\infty]{} \infty$$
$$\implies f \notin BV([0, \tfrac{2}{\pi}])$$

26/10

BV([a,b]) has a structure: it is a vector space*. Indeed (↓)

Prop. (i) Let $f \in BV([a,b])$, $\lambda \in \mathbb{R}$. Then $\lambda f \in BV([a,b])$ and:
$$V_a^b(\lambda f) = |\lambda| V_a^b(f)$$
(ii) Let $f, g \in BV([a,b])$. Then $f+g \in BV([a,b])$ and
$$V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$$

Remark: (i)* $BV([a,b])$ is a vector space
(ii) $V_a^b(f) = 0 \implies f = \text{constant}$

Remark: Consider $BV([a,b])$ and $V_a^b(f)$. In general, $V_a^b(f)$ is **not** a norm, otherwise $V_a^b(f) = 0$ would imply that $f = 0$ (which it false).
Consider:
$$\|f\| := |f(a)| + V_a^b(f).$$
This is a norm on $BV([a,b])$.

Theorem: Let $f \in BV([a,b])$.
(i) $c \in [a,b]$. Then $V_a^b(f) = V_a^c(f) + V_c^b(f)$
(ii) The function $x \mapsto V_a^x(f)$ is increasing, $x \in [a,b]$

Theorem: Consider $f:[a,b] \to \mathbb{R}$. The following statements are equivalent:
(i) $f \in BV([a,b])$
(ii) $\exists \varphi, \psi : [a,b] \to \mathbb{R}$ increasing s.t.
$$f = \varphi - \psi \qquad := \text{Jordan decomposition}$$
a BV function can always be written as the difference of two increasing functions
(also, if a function can be written as the difference of two increasing functions $\implies$ the function is BV)

Theorem: let $f: I \to \mathbb{R}$ monotone. Then f is differentiable a.e. in $\mathbb{R}$.

Theorem: let $f: I = [a,b] \to \mathbb{R}$ be increasing. Then $f' \in L^1(I)$, furthermore:

$$\int_I f' \, d\lambda \leq f(b) - f(a)$$

since f is increasing we know that f', where it exists, is ≥ 0. Now we also have an upper bound (for $\int f'$) (so: $0 \leq \int_I f' d\lambda \leq f(b) - f(a)$)

this results with the previous give us that a BV function is differentiable a.e.
($f \in BV \Rightarrow f$ is the difference of two increasing functions. Each of these increasing functs. is differentiable a.e. $\Rightarrow$ the difference of these two is differentiable and so, also f is differentiable a.e.)
$\Downarrow$
If $f \in BV$ then f admits a derivative a.e.

proof.

f' exists a.e. in $[a,b]$. (by previous theorem, since f is increasing)

$f \nearrow \implies f \in \mathcal{R}([a,b]) \implies f \in \mathcal{M}(I, \mathcal{L}(I))$

an increasing function is always integrable according to Riemann, moreover we saw that if a function is Riemann-integrable then it is measurable

Set $f(x) := f(b) \quad \forall x > b$.

Define:

$$g_n(x) := \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}, \quad x \in I$$

$\implies g_n \in \mathcal{M}_+(I, \mathcal{L}(I))$ ($\mathcal{M}$ because f is measurable, non-negative because f is increasing)

Moreover:

$$\lim_{n\to\infty} g_n(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \quad \text{for a.a. } x \in I$$

$\implies f' \in \mathcal{M}_+(I, \mathcal{L}(I))$ ($g_n \in \mathcal{M}_+ \Rightarrow \limsup, \liminf \in \mathcal{M}_+ \implies$ if it exists, $\lim \in \mathcal{M}_+$)

By Fatou's lemma:

$$\int_a^b \underbrace{f'(x)}_{\liminf = \lim \text{ and it has a name } (f')} d\lambda \leq \liminf_{n\to\infty} \int_a^b g_n(x) \, d\lambda$$

$$= \liminf_{n\to\infty} \left[n\int_a^b f(x+\tfrac{1}{n}) \, dx - n\int_a^b f(x) \, dx \right]$$

$$= \liminf_{n\to\infty} \left[n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(y) \, dy - n\int_a^b f(x) \, dx \right]$$

$y = x + \frac{1}{n}$
$dy = dx$
(this is the reason why we extended f after b, so it makes sense the $\int$)

$$= \liminf_{n\to\infty} \left[n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x) \, dx - n\int_a^b f(x) \, dx \right]$$

$$= \liminf_{n\to\infty} \left[n\int_b^{b+\frac{1}{n}} f(x) \, dx - n\int_a^{a+\frac{1}{n}} f(x) \, dx \right]$$

since f is increasing:
$\int_b^{b+\frac{1}{n}} f \leq \frac{1}{n} \cdot \underbrace{f(b+\tfrac{1}{n})}_{= f(b)}$
$\int_a^{a+\frac{1}{n}} f \leq \frac{1}{n} \cdot f(a)$

$$\leq \liminf_{n\to\infty} \left[\cancel{n} \cdot \tfrac{1}{\cancel{n}} f(b) - \cancel{n} \cdot \tfrac{1}{\cancel{n}} f(a) \right]$$

$$= f(b) - f(a) \qquad \blacksquare$$

very useful to conclude that a function is not BV (if its $f' \notin L^1 \Rightarrow f \notin BV$)

Corollary: If $f \in BV([a,b])$ then f' exists a.e. in $[a,b]$ and $f' \in L^1([a,b])$.

(By Jordan decomposition and by previous results (notice that if φ and ψ are s.t. $\varphi', \psi' \in L^1$, then also $\varphi' - \psi'$ is in L^1 since L^1 is a vector space))

ABSOLUTELY CONTINUOUS FUNCTIONS

stronger than BV

Let $J \subseteq \mathbb{R}$ be an interval, $f: J \to \mathbb{R}$.

Denote by $\mathbb{F}(J)$ the set of finite collections of closed subintervals of J, without interior points in common.

Def. A function $f: J \to \mathbb{R}$ is said to be absolutely continuous in J, if

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall \{[a_k, b_k]\}_k \in \mathbb{F}(J)$ $k = 1, \dots, n$ for which

$$\sum_{k=1}^n (b_k - a_k) < \delta$$

one has:

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon.$$

(stronger of **both** continuity and uniform continuity)

$(f: X \to \mathbb{R})$ f uniformly continuous $\iff \left[\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x, y \in X \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \right]$

Remark: (i) If we take $\mathcal{F}(J)$ families made of only one interval, we easily get that:

$$f \text{ absolutely continuous in } [a,b] \quad \underset{\not\Leftarrow}{\Longrightarrow} \quad f \text{ uniformly continuous in } [a,b]$$

absolutely continuous coincides with the def. of uniform continuous if we consider $[x,y]$ instead of $\{[a_k,b_k]\}_k$ (with $x,y \in [a,b]$, $x \leq y$)

we denote:

$$AC([a,b]) := \{ f : [a,b] \to \mathbb{R} \text{ absolutely continuous} \}$$

Counterexample ($\not\Leftarrow$)

$$f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & x \in [-1,1] \setminus \{0\} \\ 0 & x = 0 \end{cases} \quad \notin AC$$

but f is uniformly continuous in $[-1,1]$.

Lipschitz is stronger than uniformly continuous, indeed it is strong enough to imply AC

(ii) f lipschitz in $[a,b] \Longrightarrow f \in AC([a,b])$

proof:

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \overset{f \text{ Lip}}{\leq} \sum_{k=1}^n L(b_k - a_k) \quad \forall \{[a_k,b_k]\}_k$$

$$= L \sum_{k=1}^n (b_k - a_k)$$

we get rid of $|\cdot|$ since $b_k \geq a_k$

$$< L \cdot \delta = \varepsilon \quad \text{if } \delta = \frac{\varepsilon}{L} \qquad \blacksquare$$

The contrary does not hold ($\not\Leftarrow$).

Counterexample:

$$f(x) = \sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}}\, dt \quad \text{is AC, but } \textbf{not} \text{ Lip.}$$

The following theorem holds in very general cases. That's why we consider a generic measure μ (and **not** λ) and we consider $f : X \to \overline{\mathbb{R}}$. — Both things that we do **not** assume in this chapter

Theorem: Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be s.t. $\int_X f\, d\mu < \infty$. (If $f \in \mathcal{M}_+(X,\mathcal{A})$ then $\int_X f\, d$ can be $+\infty$, that's why we need to specify)

Then, $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\underbrace{\forall E \in \mathcal{A},\ \mu(E) < \delta}_{\text{for all } E \in \mathcal{A} \text{ s.t. } \mu(E) < \delta}$:

$$\int_E f\, d\mu < \varepsilon$$

proof.

Consider $F_n := \{f < n\}$ $(n \in \mathbb{N})$. $\{F_n\}_n$ sequence of sets

$F_n \in \mathcal{A}$, $\{F_n\} \nearrow$. (f measurable $\Rightarrow$ F_n is measurable, since $n \nearrow \Rightarrow \{F_n\}_n \nearrow$)

Then:

$$X = \{f = \infty\} \cup \{f < \infty\} = \{f = \infty\} \cup \left[\bigcup_{n=1}^\infty F_n\right] \qquad (f : X \to \overline{\mathbb{R}})$$

Since:

$$\int_X f\, d\mu < \infty \Longrightarrow \left[f \text{ finite a.e. in } X \Longleftrightarrow \mu(\{f = \infty\}) = 0\right]$$

Moreover:

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_{F_n} f\, d\mu \Longleftrightarrow \forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N} \text{ s.t. } \forall n > \bar{n}:$$

previous property (a few chapters ago). We can say it because X is the union of all the F_n plus a set whose measure is zero

$$\underbrace{\left| \int_{F_n} f\, d\mu - \int_X f\, d\mu \right|}_{= \int_{F_n^c} f\, d\mu} < \frac{\varepsilon}{2}$$

Therefore, for any fixed $n > \bar{n}$ we obtain that:

$$\int_E f\, d\mu = \int_{E \cap F_n} f\, d\mu + \int_{E \cap F_n^c} f\, d\mu < \int_{E \cap F_n} n\, d\mu + \underbrace{\int_{F_n^c} f\, d\mu}$$

$$< n\mu(E) + \frac{\varepsilon}{2} = \varepsilon$$

since $f < n$ on F_n (then we'll say: $\int_{E \cap F_n} n\, d\mu = n\mu(E \cap F_n) < n \cdot \mu(E)$)

$> \int_{F_n^c \cap E} f\, d\mu$, and now we use the above result: $\int_{F_n^c} f\, d\mu < \frac{\varepsilon}{2}$

provided that: $\delta := \dfrac{\varepsilon}{2n}$. $\blacksquare$

Corollary: let $f \in L^1([a,b]) = L^1(I)$. Then:

$$F(x) := \int_{[a,x]} f \, d\lambda \qquad x \in [a,b]$$

is absolutely continuous in I.

If we have an L^1 function, its integral function is absolutely continuous.

proof.

Let $E := \bigcup_{k=1}^n [a_k, b_k]$ with $\{[a_k, b_k]\} \in \mathcal{F}(J)$

$\lambda(E) = \sum_{k=1}^n (b_k - a_k)$. (since the intervals don't have interior points in common)

Now we consider: the concept of AC function for F

$$\sum_{k=1}^n |F(b_k) - F(a_k)| = \sum_{k=1}^n \left| \int_{[a_k,b_k]} f \, d\lambda \right| \quad \text{(by def.) (we have a cancellation)}$$

$$\leq \sum_{k=1}^n \int_{[a_k,b_k]} |f| \, d\lambda$$

$$= \int_E |f| \, d\lambda \quad \text{since } E = \bigcup_{k=1}^n [a_k, b_k]$$

Now we use the previous theorem applied on $|f|$ and E: $\left(\int_E |f| d\lambda < \infty \Rightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall E \text{ s.t. } \mu(E) < \delta : \int_E |f| d\mu < \varepsilon\right)$

$$\left.\begin{array}{l} \forall \varepsilon > 0 \;\; \exists \delta > 0 : \underbrace{\sum_{k=1}^n (b_k - a_k) < \delta}_{\mu(E) < \delta} \\ \Rightarrow \underbrace{\sum_{k=1}^n |F(b_k) - F(a_k)| < \varepsilon}_{\sum_{k=1}^n |F(b_k) - F(a_k)| \leq \int_E |f| d\lambda < \varepsilon} \end{array}\right\} \iff F \text{ is AC.}$$

Theorem: $f \in AC([a,b]) \Longrightarrow f \in BV([a,b])$

Moreover: $x \longmapsto V_a^x(f)$ is AC.

Remark: $f \in BV([a,b]) \not\Rightarrow f \in AC([a,b])$

indeed, consider:

$$f(x) = \begin{cases} -1 & x \in [0,1] \\ 1 & x \in [-1,0) \end{cases}$$

$f \in BV([-1,1])$, $f \notin AC([-1,1])$ (since it's **not** even continuous)

Theorem: let $f : [a,b] \to \mathbb{R}$. The following statements are equivalent:

(i) $f \in AC([a,b])$

(ii) $\exists \varphi, \psi \in AC([a,b]) \nearrow$ s.t. $f = \varphi - \psi$

Lemma: $f \in AC$, $f \nearrow$ in $[a,b]$.

If $f' = 0$ a.e. in $[a,b]$ $\Rightarrow$ $f \equiv c$ in $[a,b]$ for some $c \in \mathbb{R}$

(f is constant)

Theorem: (2nd Fundamental Theorem of Calculus)

let $F : [a,b] \to \mathbb{R}$. The following statements are equivalent:

(i) $F \in AC([a,b])$

(ii) F is differentiable a.e. in $[a,b]$ with $F' \in L^1([a,b])$ and:

$$F(x) = F(a) + \int_{[a,x]} F' \, d\lambda \qquad \forall x \in [a,b]$$

Very useful for exercices ("prove that a function is **AC**")

- calculate the first derivative
- prove that the first derivative is in L^1
- try to show the fundamental formula (expecially for x for which we can't compute the derivative (exploiting the limit))

(ii) $\Rightarrow$ (i) is a consequence of a previous result, since we already proved that $F(x) = \int_a^x f \, d\lambda$ is AC (where f is an L^1 function (and here we have the same since we're saying that F' is an L^1 function)) and the constant factor ($F(a)$) is **not** relevant.

We have to prove (i) $\Rightarrow$ (ii).

proof.

$(i) \Longrightarrow (ii)$:

$F \in AC([a,b]) \Longrightarrow F \in BV([a,b])$

$\Longrightarrow F$ differentiable a.e. in $[a,b]$ and $F' \in L^1([a,b])$

Suppose, in addition, that F is increasing.

Set :

$$G(x) := \int_{[a,x]} F' \, d\lambda \qquad x \in [a,b]$$

*[G is the integral function of an L^1 function $\Rightarrow G \in AC([a,b])$
$\Rightarrow G \in BV([a,b])$
$\Rightarrow G$ differentiable a.e.]

Hence, G is differentiable a.e. in $[a,b]$ and :

$$(F-G)' = 0 \quad \text{a.e. in } [a,b]$$

(actually G is differentiable a.e. even only because it is the integral of an L^1 function (1^{st} FTC). moreover, because of 1^{st} FTC : $G' = F'$ a.e.)

moreover, $G \in AC([a,b])$. ← This, instead, comes from *

$\Longrightarrow F - G \in AC([a,b])$ (AC is a vector space (and both $F, G \in AC$))

Now we consider $a \le x_1 \le x_2 \le b$:

$$[F(x_2) - G(x_2)] - [F(x_1) - G(x_1)] \underset{\text{by def. of } G}{=} F(x_2) - F(x_1) - \int_{[x_1,x_2]} F' \, d\lambda \underset{\substack{\text{theorem for} \\ \text{increasing functions}}}{\geq} 0$$

$\Longrightarrow F - G$ is increasing

$\Longrightarrow$ by yesterday's lemma : ($f \in AC$, $f \nearrow$, $f' = 0 \Rightarrow f$ constant)

$$F(x) - G(x) = \text{constant} \overset{\text{e.g.}}{=} F(a) - G(a) \qquad (G(a) = 0 \text{ by def.})$$

$$= F(a) \qquad \forall x \in [a,b]$$

$\Longrightarrow F(x) - G(x) = F(a) \quad \forall x \in [a,b]$

is the thesis, when $F \nearrow$.

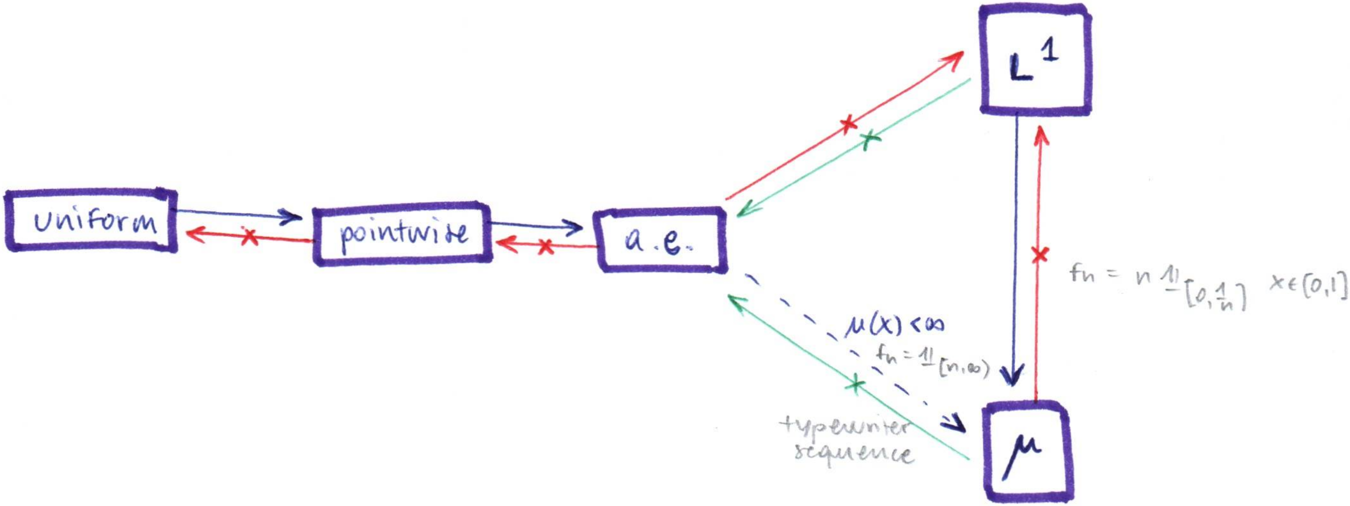
For a general F, it is sufficient to observe that

$$F = \Psi - \varphi \qquad \text{with } \Psi, \varphi \nearrow, \ AC.$$

So, we apply the above result to Ψ and φ, then the thesis follows for F, too.

$(ii) \Longrightarrow (i)$:

Theorem already seen. ■



$\longrightarrow$ always implies

$\dashrightarrow$ implies under condition

$\not\longrightarrow$ not implies

$\not\longrightarrow$ not implies, but $\exists$ a subsequence s.t. converges

14/09/21 - Exercices (1)

$[a,b] = \bigcap_{n=1}^{+\infty} (a-\frac{1}{n}, b+\frac{1}{n})$

1. $x \in [a,b] \Rightarrow a \leq x \leq b \Rightarrow [a,b] \subseteq (a-\frac{1}{n}, b+\frac{1}{n}) \ \forall n \geq 1 : a-\frac{1}{n} < x < b+\frac{1}{n}$
$\Rightarrow x \in (a-\frac{1}{n}, b+\frac{1}{n}) \ \forall n \geq 1 \Rightarrow x \in \bigcap_{n=1}^{+\infty}(a-\frac{1}{n}, b+\frac{1}{n})$

2. $\boxed{x \in \bigcap_{n=1}^{+\infty}(a-\frac{1}{n}, b+\frac{1}{n}) \Rightarrow x \in [a,b]} \equiv \boxed{x \notin [a,b] \Rightarrow x \notin \bigcap_{n=1}^{+\infty}(a-\frac{1}{n}, b+\frac{1}{n})}$

$x \notin [a,b] \Rightarrow (x<a) \vee (x>b) \Rightarrow$ without loss of generality $x<a$
$\Rightarrow \forall x<a \ \exists n_1, n_2 \in \mathbb{N}$ s.t. $a-\frac{1}{n_1} < x < a-\frac{1}{n_2} < a$
$\Rightarrow \exists n$ (in particular: n_2) s.t. $x \notin (a-\frac{1}{n}, b+\frac{1}{n})$
$\Rightarrow x \notin \bigcap_{n=1}^{+\infty}(a-\frac{1}{n}, b+\frac{1}{n})$ because x needs to be in $(a-\frac{1}{n}, b+\frac{1}{n}) \ \forall n \geq 1$

$(a,b) = \bigcup_{n=1}^{+\infty} [a+\frac{1}{n}, b-\frac{1}{n}]$

1. $x \in \bigcup_{n=1}^{+\infty} [a+\frac{1}{n}, b-\frac{1}{n}] \Rightarrow \exists n \in \mathbb{N}: x \in [a+\frac{1}{n}, b-\frac{1}{n}] \Rightarrow \exists n \in \mathbb{N}: a+\frac{1}{n} \leq x \leq b-\frac{1}{n}$
$\Rightarrow [a+\frac{1}{n}, b-\frac{1}{n}] \subset (a,b) \ \forall n \geq 1 : a<x<b \Rightarrow x \in (a,b)$

2. $x \notin \bigcup_{n=1}^{+\infty} [a+\frac{1}{n}, b-\frac{1}{n}] \Rightarrow \nexists n: a+\frac{1}{n} \leq x \leq b-\frac{1}{n} \Rightarrow (x<a+\frac{1}{n}) \vee (x>b-\frac{1}{n}) \ \forall n \geq 1$
$\Rightarrow$ without loss of generality: $x<a+\frac{1}{n}$
$\Rightarrow$ suppose that $x>a$ $(a<x<a+\frac{1}{n})$
there always $\exists n_1, n_2$ s.t.:
$a < a+\frac{1}{n_1} < x < a+\frac{1}{n_2}$
but it's a contraddiction since $x<a+\frac{1}{n} \ \forall n \geq 1$
$\Rightarrow x$ cannot be $>a \Rightarrow x \leq a \Rightarrow x \notin (a,b)$

$(a,b] = \bigcap_{n=1}^{+\infty} (a, b+\frac{1}{n})$

1. $x \in (a,b] \Rightarrow a<x\leq b \Rightarrow (a,b] \subset (a, b+\frac{1}{n}) \ \forall n \geq 1 : a<x<b+\frac{1}{n} \ \forall n \geq 1$
$\Rightarrow x \in (a, b+\frac{1}{n}) \ \forall n \geq 1 \Rightarrow x \in \bigcap_{n=1}^{+\infty}(a, b+\frac{1}{n})$

2. $x \notin (a,b] \Rightarrow (x \leq a) \vee (x>b)$
If $x \leq a \Rightarrow x \notin (a, b+\frac{1}{n}) \ \forall n \geq 1 \Rightarrow x \notin \bigcap_{n=1}^{+\infty}(a, b+\frac{1}{n})$
If $x>b \Rightarrow \forall x>b \ \exists n_1, n_2 \in \mathbb{N}: b<b+\frac{1}{n_1}<x<b+\frac{1}{n_2}$
$\Rightarrow \exists n \ (=n_1)$ s.t. $x \notin (a, b+\frac{1}{n})$
$\Rightarrow x \notin \bigcap_{n=1}^{+\infty}(a, b+\frac{1}{n})$

$(a,b) = \bigcup_{n=1}^{+\infty} (a, b-\frac{1}{n}]$

1. $x \in \bigcup_{n=1}^{+\infty} (a, b-\frac{1}{n}] \Rightarrow \exists n \in \mathbb{N}$ s.t. $a<x\leq b-\frac{1}{n} \Rightarrow (a, b-\frac{1}{n}] \subset (a,b) \ \forall n \geq 1 : x \in (a,b)$

2. $x \notin \bigcup_{n=1}^{+\infty} (a, b-\frac{1}{n}] \Rightarrow \nexists n$ s.t. $a<x\leq b-\frac{1}{n} \Rightarrow (x \leq a) \vee (x>b-\frac{1}{n}) \ \forall n \geq 1$
$\Rightarrow$ If $x \leq a \Rightarrow x \notin (a,b)$
If $x>b-\frac{1}{n} \Rightarrow$ suppose that $x<b$ $(b-\frac{1}{n}<x<b)$:
we can always find $n_1, n_2 \in \mathbb{N}$ s.t.:
$b-\frac{1}{n_1} < x < b-\frac{1}{n_2} < b$
but it's a contraddiction because:
$x > b-\frac{1}{n} \ \forall n \geq 1$
$\Rightarrow x$ cannot be $<b \Rightarrow x \geq b$
$\Rightarrow x \notin (a,b)$

$x \in \limsup E_n \iff x \in E_n$ for infinitely many n

$$x \in \limsup E_n \iff x \in \bigcap_{k=1}^{+\infty}\left(\bigcup_{n=k}^{+\infty} E_n\right) \iff x \in \bigcup_{n=k}^{+\infty} E_n \quad \forall k \geq 1$$
$$\iff \forall k \geq 1 \ \exists n \geq k : x \in E_n$$
$$\iff \exists n_j : x \in E_{n_j} \quad \forall j \geq 1$$
$$\iff x \in E_n \text{ for infinitely many } n$$

$x \in \liminf E_n \iff \exists k \in \mathbb{N} : x \in E_n \quad \forall n \geq k$

$$x \in \liminf E_n \iff x \in \bigcup_{k=1}^{+\infty}\left(\bigcap_{n=k}^{+\infty} E_n\right) \iff \exists k \geq 1 : x \in \bigcap_{n=k}^{+\infty} E_n$$
$$\iff \exists k \geq 1 : x \in E_n \quad \forall n \geq k$$

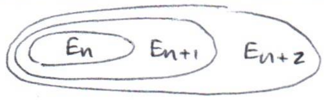
$(\liminf E_n)^c = \limsup (E_n^c)$

$$(\liminf E_n)^c = \left(\bigcup_{k=1}^{+\infty}\left[\bigcap_{n=k}^{+\infty} E_n\right]\right)^c = \bigcap_{k=1}^{+\infty}\left[\bigcap_{n=k}^{+\infty} E_n\right]^c = \bigcap_{k=1}^{+\infty}\left[\bigcup_{n=k}^{+\infty} E_n^c\right] = \limsup(E_n^c)$$

De Morgan (I) : $\left(\bigcup_k A_k\right)^c = \bigcap_k (A_k^c)$

De Morgan (II) : $\left(\bigcap_k A_k\right)^c = \bigcup_k (A_k^c)$

$E_n \nearrow \implies \lim E_n = \bigcup_{n=1}^{+\infty} E_n := E$



1. $\boxed{E \subseteq \liminf E_n}$

$x \in E \Rightarrow x \in \bigcup_{n=1}^{+\infty} E_n \rightarrow \exists n : x \in E_n \rightarrow$ since $E_n \nearrow$ $\exists \bar{n} \geq 1 : x \in E_n \ \forall n \geq \bar{n}$

$\Rightarrow x \in E_n \ \forall n \in \mathbb{N} \setminus$ a finite number of elements

$\Rightarrow x \in \liminf E_n$

2. $\boxed{\limsup E_n \subseteq E}$

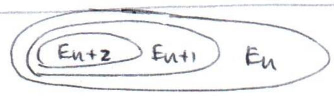
$x \in \limsup E_n \Rightarrow x \in \bigcap_{k=1}^{+\infty}\left[\bigcup_{n=k}^{+\infty} E_n\right] \Rightarrow \forall k \geq 1 \ x \in \bigcup_{n=k}^{+\infty} E_n$

$\Rightarrow \forall k \geq 1 \ \exists n \geq k : x \in E_n \Rightarrow$ considering all E_n $(E = \bigcup_{n=1}^{+\infty} E_n)$ for sure we have $x \in E$

3. We know $\liminf E_n \subseteq \limsup E_n$ + 1. + 2. $\rightarrow \liminf E_n = \limsup E_n = E$

$(\limsup E_n \subseteq E \subseteq \liminf E_n \Rightarrow \limsup E_n \subseteq \liminf E_n)$

$E_n \searrow \implies \lim E_n = \bigcap_{n=1}^{+\infty} E_n := E$



1. $\boxed{E \subseteq \liminf E_n}$

$x \in E \Rightarrow x \in \bigcap_{n=1}^{+\infty} E_n \rightarrow \forall n \geq 1 \ x \in E_n \Rightarrow \exists k \ (=1) \ \forall n \geq k \ x \in E_n$

$\Rightarrow x \in \bigcup_{k=1}^{+\infty} \bigcap_{n=k}^{+\infty} E_n \rightarrow x \in \liminf E_n$

2. $\boxed{\limsup E_n \subseteq E}$

$x \in \limsup E_n \rightarrow x \in \bigcap_{k=1}^{+\infty} \bigcup_{n=k}^{+\infty} E_n \rightarrow x \in \bigcap_{k=1}^{+\infty} E_k \rightarrow x \in E$

$\bigcup_{n=k}^{+\infty} E_n = E_k \quad (E_n \searrow)$

3. $\limsup E_n \subseteq E \subseteq \liminf E_n \implies \limsup E_n = \liminf E_n = E$

14/09/21 - Exercices (2)

$\mathbb{1}_{E_1 \cap E_2} = \mathbb{1}_{E_1} \cdot \mathbb{1}_{E_2}$

$\mathbb{1}_{E_1 \cap E_2}(x) = 1 \iff x \in E_1 \cap E_2 \iff x \in E_j \quad j = 1,2 \iff \mathbb{1}_{E_j}(x) = 1 \quad j = 1,2$
$\iff \prod_{j=1}^{2} \mathbb{1}_{E_j}(x) = 1$

$\mathbb{1}_{E_1 \cup E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} - \mathbb{1}_{E_1 \cap E_2}$

1. $\boxed{\mathbb{1}_{E_1 \cup E_2}(x) = 0 \implies \mathbb{1}_{E_1}(x) + \mathbb{1}_{E_2}(x) - \mathbb{1}_{E_1 \cap E_2}(x) = 0}$

$\left.\begin{array}{l} E_1, E_2 \subseteq E_1 \cup E_2 \\ E_1 \cap E_2 \subseteq E_1 \cup E_2 \end{array}\right\} \implies \left[\mathbb{1}_{E_1 \cup E_2}(x) = 0 \implies \mathbb{1}_{E_1}(x) = \mathbb{1}_{E_2}(x) = \mathbb{1}_{E_1 \cap E_2}(x) = 0\right]$

because if $A \subseteq B \subseteq C \longrightarrow \forall x \in C : \mathbb{1}_A(x) \leq \mathbb{1}_B(x)$

2. $\boxed{\mathbb{1}_{E_1}(x) + \mathbb{1}_{E_2}(x) - \mathbb{1}_{E_1 \cap E_2}(x) = 0 \implies \mathbb{1}_{E_1 \cup E_2}(x) = 0}$

$\mathbb{1}_{E_1}(x) + \mathbb{1}_{E_2}(x) - \mathbb{1}_{E_1 \cap E_2}(x) = 0 \implies \mathbb{1}_{E_1}(x) + \mathbb{1}_{E_2}(x) = \mathbb{1}_{E_1 \cap E_2}(x) \quad (*)$

$\implies$ if $x \in E_1 \cap E_2$ then $x \in E_1$, $x \in E_2$ and $(*)$: $1 + 1 = 2$ ↯

$\implies x \notin E_1 \cap E_2 \implies \mathbb{1}_{E_1 \cap E_2}(x) = 0 \implies \mathbb{1}_{E_1}(x) = \mathbb{1}_{E_2}(x) = 0$

$\implies x \notin E_1,\ x \notin E_2 \implies \mathbb{1}_{E_1 \cup E_2}(x) = 0$

$\{E_k\}_{k \in \mathbb{N}}$ disjoint, $E := \bigcup_{k=1}^{+\infty} E_k \implies \mathbb{1}_E = \sum_{k=1}^{+\infty} \mathbb{1}_{E_k}$

Knowing: $\mathbb{1}_{E_1 \cup E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} - \mathbb{1}_{E_1 \cap E_2}$:

$\mathbb{1}_E = \mathbb{1}_{E_1 \cup \left(\bigcup_{k=2}^{+\infty} E_k\right)} = \mathbb{1}_{E_1} + \mathbb{1}_{\bigcup_{k=2}^{\infty} E_k} - \cancel{\mathbb{1}_{E_1 \cap \left[\bigcup_{k=2}^{+\infty} E_k\right]}}$ disjoint

iteratively: $\mathbb{1}_E = \sum_{k=1}^{+\infty} \mathbb{1}_{E_k}$

Alternatively: $\mathbb{1}_E = 0 \iff \sum_{k=1}^{+\infty} \mathbb{1}_{E_k} = 0$:

$\mathbb{1}_E(x) = 0 \iff x \notin E \iff x \notin \bigcup_{k=1}^{+\infty} E_k \iff \nexists k : x \in E_k$
$\iff \mathbb{1}_{E_k}(x) = 0 \quad \forall k \geq 1 \iff \sum_{k=1}^{+\infty} \mathbb{1}_{E_k}(x) = 0$

$\mathbb{1}_{\limsup E_n} = \limsup \mathbb{1}_{E_n}$

$\limsup \mathbb{1}_{E_n}(x) = 1 \iff \inf_k \sup_{n \geq k} \mathbb{1}_{E_n}(x) = 1 \iff \inf_{k \geq 1} \sup \{\mathbb{1}_{E_k}(x), \mathbb{1}_{E_{k+1}}(x), \dots\} = 1$
$\iff x \in E_n$ for infinitely many n
$\iff x \in \limsup E_n$
$\iff \mathbb{1}_{\limsup E_n}(x) = 1$

$\mathbb{1}_{\liminf E_n} = \liminf \mathbb{1}_{E_n}$

$\liminf \mathbb{1}_{E_n}(x) = 1 \iff \sup_{k \geq 1} \inf_{n \geq k} \mathbb{1}_{E_n}(x) = 1 \iff \sup_{k \geq 1} \inf \{\mathbb{1}_{E_k}(x), \mathbb{1}_{E_{k+1}}(x), \dots\} = 1$
$\iff x \in E_n \quad \forall n \in \mathbb{N} \setminus$ a finite number of elements
$\iff x \in \liminf E_n$
$\iff \mathbb{1}_{\liminf E_n}(x)$

$E = \lim E_n \iff \mathbb{1}_E = \lim \mathbb{1}_{E_n}$

$\lim E_n = E \iff \limsup E_n = \liminf E_n = \lim E_n = E$

$\iff \forall x \;\; \mathbb{1}_{\limsup E_n}(x) = \mathbb{1}_{\liminf E_n}(x) = \mathbb{1}_{\lim E_n}(x) = \mathbb{1}_E(x)$

$(0,1)$ does not have a maximal element

Assume that there is a maximal element and that it is $M \in (0,1)$.

$M < 1$ (since $M \in (0,1)$) $\Longrightarrow \exists M' : \;\; M' = \frac{M+1}{2}$ = middle point between **M** and **1**

$\Longrightarrow M < M' < 1 \Longrightarrow M$ is not maximal

In upper bound terms:

$(0,1)$ is a chain, so let $A = (0,1) \subseteq P = (0,1)$.

$\nexists u \in P = (0,1)$ s.t. $u \geq a \;\; \forall a \in (0,1)$ (because of the above reasoning)

$\Longrightarrow \exists$ a chain with no upper bound

$\Longrightarrow$ Zorn's lemma doesn't apply