#1 (#1)

(X,d) metric space, $\varphi: X \to X$. $\quad d'(x,y) = d(\varphi(x), \varphi(y))$.

distance d: 1) $d(x,y) \geq 0$, $d(x,y) = 0 \iff x = y$

2) $d(x,y) = d(y,x)$

3) $d(x,z) \leq d(x,y) + d(y,z)$

1) $d(x,y) \geq 0 \implies d'(x,y) \geq 0 \equiv d(\varphi(x), \varphi(y)) \geq 0$ always because $\varphi(x) \in X$

$d(x,y) = 0 \iff x = y \implies d(\varphi(x), \varphi(y)) = 0 \implies \varphi(x) = \varphi(y) \implies x = y$ only if φ is injective

2) $d(x,y) = d(y,x) \implies d(\varphi(x), \varphi(y)) = d(\varphi(y), \varphi(x)) \quad \forall x,y$

3) $d(x,z) \leq d(x,y) + d(y,z) \qquad d(\varphi(x), \varphi(z)) \leq d(\varphi(x), \varphi(y)) + d(\varphi(y), \varphi(z)) \quad \forall x,y,z$

$\implies \boxed{\varphi \text{ must be injective}}$

#2

(X,d) metric space, $\varphi: \mathbb{R}^+ \to \mathbb{R}$. $\quad d'(x,y) = \varphi(d(x,y))$

1) $d'(x,y) \geq 0 \implies \varphi(d(x,y)) \geq 0 \implies \varphi(\cdot) \geq 0$

1') $d'(x,y) = 0 \iff \varphi(d(x,y)) = 0 \implies \boxed{\varphi(0) = 0, \quad \varphi(x) > 0 \quad \forall x \in \mathbb{R}^+ \setminus \{0\}}$

2) $d'(x,y) = \varphi(d(x,y)) = \varphi(d(y,x)) = d'(y,x) \quad \forall x,y$

3) $d'(x,z) \leq d'(x,y) + d'(y,z)$

$\varphi(d(x,z)) \leq \varphi(d(x,y)) + \varphi(d(y,z))$

Let φ be increasing ($\varphi(t_1) < \varphi(t_2)$ if $t_1 < t_2$) and subadditive ($\varphi(t_1 + t_2) < \varphi(t_1) + \varphi(t_2)$).

Then: $s_1 := d(x,z)$, $\quad s_2 := d(x,y)$, $\quad s_3 := d(y,z)$.

Since d is a distance $\implies s_1 \leq s_2 + s_3$

$$\implies \varphi(s_1) \underbrace{\leq}_{\substack{\text{monotone}\\ \text{increasing}}} \varphi(s_2 + s_3) \underbrace{\leq}_{\substack{\text{sub-}\\ \text{additive}}} \varphi(s_2) + \varphi(s_3)$$

$\implies \boxed{\varphi \nearrow, \text{ sub-additive}}$

#3

d: $\quad d(x,y) = \sqrt[3]{|x-y|} \quad \forall x,y \in \mathbb{R}$

$d(x,y) = \varphi(|x-y|) = \sqrt[3]{|x-y|}$, where $\varphi(\cdot) = \sqrt[3]{\cdot}$

$\varphi(0) = 0$, $\varphi(x) > 0 \quad \forall x \in (0, \infty)$, $\varphi \nearrow$, φ is sub-additive

Since $|x-y|$ is a distance $\implies \varphi(|x-y|) = d(x,y)$ is a distance

#1 (#2) *

$\mu: \mathcal{M} \to [0, \infty)$: (i) $\mu(\emptyset) = 0$

(ii) μ finitely additive: $\{E_j\}_{j=1}^n$ disjoint : $\mu(\bigcup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j)$

(iii) μ σ-subadditive: $\{E_n\}_n$: $\mu(\bigcup_{n \in \mathbb{N}} E_n) \leq \sum_{n \in \mathbb{N}} \mu(E_n)$.

We need to prove that μ is σ-additive.

First we see that μ is monotone: $E \subset F \implies F = E \cup (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$.

Then we consider $\{E_n\}_n$ disjoint. Then $\forall n \in \mathbb{N}$: $\bigcup_{j=1}^n E_j \subseteq \bigcup_{j=1}^\infty E_j$ and so:

$\mu(\bigcup_{j=1}^\infty E_j) \geq \mu(\bigcup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j) \implies \mu(\bigcup_{j=1}^\infty E_j) \geq \sum_{j=1}^\infty \mu(E_j)$ ($\lim_{n \to \infty}$)

2 (# 2)

$\mu : \mathcal{M} \to [0,\infty)$
- $\mu(\emptyset) = 0$
- ((*)) μ finitely additive: $\{E_j\}_{j=1}^n$ disjoint: $\mu(\cup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j)$
- (*) continuous among increasing sequences: let $\{E_n\}_n \nearrow$ then: $\mu(\cup_{n=1}^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$

We need to prove that μ is σ-additive.

Let $\{E_n\}_{n\in\mathbb{N}}$ be disjoint and define $A_n := \cup_{j=1}^n E_j$, so that $\{A_n\}_n \nearrow$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} E_n$. $\Rightarrow \mu(\cup_{n=1}^\infty E_n) = \mu(\cup_{n=1}^\infty A_n) \overset{(*)}{=} \lim_{n\to\infty} \mu(A_n) \overset{\text{def.}}{=} \lim_{n\to\infty} \mu(\cup_{j=1}^n E_j)$

$\underset{((*))}{=} \lim_{n\to\infty} \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^\infty \mu(E_j)$

6

$A = \{x \in \mathbb{R} : 0 \le \operatorname{atan}(x) \le 1\} \Longrightarrow \operatorname{atan}(x)$ continuous $\Rightarrow$ measurable

$\Longrightarrow \{0 \le \operatorname{atan}(x)\}$ and $\{\operatorname{atan}(x) \le 1\}$ measurable $\Rightarrow \cap$ measurable

$B = \mathbb{Q} \cap [0,1]$ is countable $\Longrightarrow$ is measurable and of zero-measure

$C = \cap_{n=1}^\infty (1-\frac{1}{n}, 2+\frac{1}{3n}) \Longrightarrow (1-\frac{1}{n}, 2+\frac{1}{3n})$ measurable ~~th~~ since they're intervals

$\Longrightarrow$ countable intersection of measurable sets is measurable

$\Omega = A \cup B \cup C$ measurable as union of measurable sets.

Then $A = [0, \frac{\pi}{4})$, $C = [1, 2] \Longrightarrow \lambda(\Omega) = \frac{\pi}{4} + 1$

7

$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ x^6 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

f measurable $\Longleftrightarrow \{f > \alpha\}$ measurable $\forall \alpha \in \mathbb{R}$

$\Longleftrightarrow \begin{cases} \{x \in \mathbb{Q} : f(x) > \alpha\} \text{ meas.} \\ \{x \in \mathbb{R}\setminus\mathbb{Q} : f(x) > \alpha\} \text{ meas.} \end{cases}$ (and so their $\cup$ is meas.)

$\Longleftrightarrow$ $\{x \in \mathbb{Q} : f(x) > \alpha\} = \{x \in \mathbb{Q} : x^2 > \alpha\}$ measurable because $\mathbb{Q}$ is measurable and x^2 is continuous and so measurable; $\{x \in \mathbb{R}\setminus\mathbb{Q} : f(x) > \alpha\} = \{x \in \mathbb{R}\setminus\mathbb{Q} : x^6 > \alpha\}$ measurable because $\mathbb{R}\setminus\mathbb{Q}$ is measurable and x^6 is continuous and so measurable

f Lebesgue measurable $\Longleftarrow$ $f = g$ a.e. and g is continuous

<u>f Borel measurable</u> ! $\Longleftarrow$ $f(x) = \begin{cases} \varphi(x) & x \in A \\ \psi(x) & x \in B \end{cases}$ measurable if:

$[\{x \in A : \varphi(x) > \alpha\} \cup \{x \in B : \psi(x) > \alpha\}]$ meas.

A measurable, φ measurable; B measurable, ψ measurable

8

$f : \mathbb{R} \to \mathbb{R}$ C^∞, $g : \mathbb{R} \to \mathbb{R}$ non-measurable, s.t. $h = f \circ g$ is measurable

$f(x) = \cos(x)$ $\forall x \in \mathbb{R}$, $g(x) = 2\pi \mathbb{1}_E(x)$, E non-measurable $\Longrightarrow$ $x \in E^c$: $h(x) = \cos(0) = 1$; $x \in E$: $h(x) = \cos(2\pi) = 1$

$\Longrightarrow h$ measurable

#10 (#2)

$A = \{x \in [0, \frac{\pi}{2}] : 0 \le \sin(x) < \frac{\sqrt{2}}{2}\}$ measurable because $\sin(x)$ is measurable and so $\{0 \le \sin(x)\}$ and $\{\sin(x) < \frac{\sqrt{2}}{2}\}$ are measurable (and so their $\cap$)

$B = \{\text{algebraic numbers}\}$: let P be a polynomial with rational coefficients. Then P has a countable number of zeros. Moreover, the family of poly with rational coefficients is countable $\Longrightarrow$ B is countable, therefore meas and of zero measure

$C = \bigcap_{n=1}^{\infty} (-\frac{\pi}{4} - \frac{1}{n}, \frac{\pi}{8} + \frac{1}{2n})$ meas. because the countable $\cap$ of meas. sets is meas. (and each one is measurable since it's an interval)

$\Omega = A \cup B \cup C$ meas. because A, B, C are meas.

$A = [0, \frac{\pi}{4})$, $C = [-\frac{\pi}{4}, \frac{\pi}{8}]$ $\Longrightarrow$ $\mu(A \cup B \cup C) = \mu(A \cup C) = \mu([-\frac{\pi}{4}, \frac{\pi}{4})) = \frac{\pi}{2}$

#9

A non-measurable $\subset [0,1]$, $x_0 \in \mathbb{R}$. $A \times \{x_0\} \subset [0,1] \times \{x_0\} \Rightarrow$ $\lambda_2(A \times \{x_0\}) \le \lambda_1([0,1]) \lambda_1(\{x_0\}) = 0 \Longrightarrow$ since Lebesgue measure on $\mathbb{R}^2$ is a complete measure then $A \times \{x_0\}$ is measurable and of zero-measure.

#11

$g : [-1,1] \to \mathbb{R}$ measurable, $E := \{x \in [-1,1] : g(x) \ge 0\}$

$$f(x) = \begin{cases} x^4 & x \in E \\ x^5 & x \in [-1,1] \setminus E \end{cases}$$

f measurable $\Longleftrightarrow$ $\{x \in E : x^4 > \alpha\} \cup \{x \in [-1,1] \setminus E : x^5 > \alpha\}$ meas. $\forall \alpha \in \mathbb{R}$

since g is measurable $\Rightarrow$ E is measurable

$\Longrightarrow$ $[-1,1] \setminus E$ is measurable

x^4, x^5 continuous $\Rightarrow$ measurable

#12

$E_n = (\frac{1}{2^n} - \frac{1}{4^n}, \frac{1}{2^n})$ $\forall n \in \mathbb{N}^+$, $A = \bigcup_{n=1}^{\infty} E_n$

We notice that $\frac{1}{2^n} - \frac{1}{4^n} = \frac{1}{2^n}(1 - \frac{1}{2^n}) \ge \frac{1}{2^{n+1}} \Longleftrightarrow 1 - \frac{1}{2^n} \ge \frac{1}{2}$ which is true $\forall n \in \mathbb{N}^+$

$\Longrightarrow$

we have disjoint intervals and so:

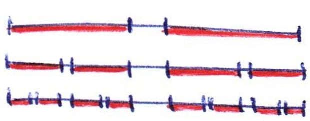
$\lambda(A) = \lambda(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n) = \sum_{n=1}^{\infty} (\frac{1}{4})^n = \frac{1}{3}$

#13

like 2.10. $A = (0, \frac{\pi}{4})$, $C = [0, 2]$ $\Rightarrow$ $\lambda(\Omega) = 2$

#27 *

Generalized Cantor set:



length( ) $= 1 - \frac{\alpha}{3}$

length( ) $= 1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2}$

length( ) $= 1 - \frac{\alpha}{3} - 2\frac{\alpha}{3^2} - 4\frac{\alpha}{3^3}$

$\Longrightarrow$ length (Generalized Cantor) $= 1 - \alpha\left(\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}\right) = 1 - \alpha\left(\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}\right) =$

$= 1 - \alpha \frac{1}{3}\left(\frac{1}{1 - 2/3}\right) = 1 - \alpha$

Then, the Generalized Cantor set has no interior points and its measure is $1 - \alpha$. It's enough to choose $\alpha \in (0,1)$ to have a positive (strictly) measure.

#1 (#3)

$\lim_{n\to\infty} \int_1^{+\infty} \frac{\sin(nx)}{x^3 e^{n\sqrt{x}}} dx$

since $f_n(x) = \frac{\sin(nx)}{x^3 e^{n\sqrt{x}}} \longrightarrow 0 \quad \forall x \in [1,\infty)$ and $|f_n(x)| \leq \frac{1}{x^3} \quad \forall n \in \mathbb{N}$

where $\frac{1}{x^3} \in L^1([1,\infty)) \Rightarrow$ we can apply **DCT** and conclude: $\int_1^\infty f_n dx \to \int_1^\infty 0\, dx$

$\Longrightarrow \lim_{n\to\infty} \int_1^\infty f_n(x) dx = 0$

#2

$$\int_0^\pi \sum_{n=1}^\infty \frac{|\sin(nx)|}{2^n} dx = \int_0^\pi \sum_{n=1}^\infty \varphi_n(x)\, dx$$

Every $\varphi_n(x)$ is continuous and positive and therefore measurable (and ≥ 0)
We can apply the series integration **corollary** to conclude:

$$\int_0^\pi \sum_{n=1}^\infty \frac{|\sin(nx)|}{2^n} dx = \sum_{n=1}^\infty \int_0^\pi \frac{|\sin(nx)|}{2^n} dx = \sum_{n=1}^\infty \frac{1}{2^n} \frac{1}{n} \int_0^{n\pi} |\sin(y)| dy$$
$$= \sum_{n=1}^\infty \frac{1}{2^n} \cdot \cancel{\frac{1}{n}} \cancel{n} \int_0^\pi \sin(y) dy$$
$$= \sum_{n=1}^\infty \frac{1}{2^n} \cdot 2 = \sum_{n=0}^\infty \frac{1}{2^n} = 2$$

#3

$f_n(x) = \frac{\sin(nx)}{n^2 x^{3/2}} \qquad \forall x \in (0,\infty)$

$\forall x \in (0,\infty) \quad \lim_{n\to\infty} f_n(x) = 0$, moreover $\begin{cases} |f_n(x)| \leq \frac{1}{x^{3/2}} & \text{in } (1,\infty) \\ |f_n(x)| \sim \frac{nx}{n^2 x^{3/2}} = \frac{1}{n x^{1/2}} \leq \frac{1}{x^{1/2}} & \text{in } (0,1) \end{cases}$

Thus $f_n \in L^1((0,\infty))$ and:
Therefore we can apply **DCT** and conclude: (since $1/x^{3/2} \in L^1((1,\infty))$, $1/x^{1/2} \in L^1((0,1))$)

$$\int_0^\infty |f_n(x) - f(x)|\, dx \longrightarrow 0 \iff f_n \longrightarrow 0 \text{ in } L^1.$$

#4

$$\int_0^\pi \sin^2(nx)dx = \frac{1}{n}\int_0^{\pi n} \sin^2(y)dy = \int_0^\pi \sin^2(y)dy = \cancel{\left[-\cos(x)\sin(x)\right]_0^\pi} + \int_0^\pi \cos^2(x)\, dx$$
$$= \int_0^\pi (1 - \sin^2(x))\, dx \Longrightarrow 2\int_0^\pi \sin^2(x)\, dx = \int_0^\pi 1\, dx = \pi$$

$\Longrightarrow \int_0^\pi \sin^2(x) dx = \frac{\pi^2}{2} \Longrightarrow \liminf_{n\to\infty} \int_0^\pi \sin^2(nx)\, dx = \frac{\pi^2}{2}$

$\liminf_{n\to\infty} \sin^2(nx) = 0 \quad \forall x \in [0,\pi] \quad \Longrightarrow \int_0^\pi \liminf \sin^2(nx)\, dx = 0$

$\Longrightarrow$ This is an example of strict inequality of Fatou's.

#5

$f: [a,b] \to [0,+\infty)$ measurable, bounded and non-negative. $\forall E \subseteq [a,b]$
measurable: $\mu(E) = \int_E f(x)\, dx$.

$\mathbb{1}_{E\cup F}(x) = \mathbb{1}_E(x) + \mathbb{1}_F(x) - \mathbb{1}_{E\cap F}(x) \leq \mathbb{1}_E(x) + \mathbb{1}_F(x)$

$\Longrightarrow \mu(E\cup F) = \int_{E\cup F} f\, dx \leq \int_E f\, dx + \int_F dx = \mu(E) + \mu(F)$

let $\{E_n\} \subseteq [a,b]$. Define $A_n = \bigcup_{j=1}^n E_j$, $A = \bigcup_{j=1}^\infty E_j$.

$\mu(A) = \int_A f\, dx = \int_a^b f\mathbb{1}_A\, dx = \int_a^b \lim_{n\to\infty} f\mathbb{1}_{A_n} dx = \lim_{n\to\infty} \int_a^b f\mathbb{1}_{A_n} d\mu = \lim_{n\to\infty} \mu(A_n)$

$= \lim_{n\to\infty} \mu(\bigcup_{j=1}^n E_j) \leq \lim_{n\to\infty} \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^\infty \mu(E_j)$

$\{\mathbb{1}_{A_n}\}\nearrow$ and $\mathbb{1}_{A_n} \to \mathbb{1}_A$
(we can apply **MCT**)

#6 (#3)

$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}\,dx$

$\frac{x}{1+x^{2n}} \geq 0$ and $\lim_{n\to\infty} \frac{x}{1+x^{2n}} = x \quad \forall x \in (0,1)$.

Moreover $\frac{x}{1+x^{2n}} \leq \frac{x}{1+x^{2(n+1)}} \Rightarrow$ MCT : $\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}\,dx = \int_0^1 x\,dx = 1$

#7

$E_n := (q_n - \frac{1}{2^{n+1}}, q_n + \frac{1}{2^{n+1}}) \quad \forall n \in \mathbb{N}^+$, $f(x) = \sum_{n=1}^{\infty} \mathbb{1}_{E_n}(x) \quad \forall x \in \mathbb{R}$, $\int_{\mathbb{R}} f(x)\,dx$

We define $f_n(x) = \sum_{j=1}^{n} \mathbb{1}_{E_j}(x)$ so that $\{f_n\}_n$ is measurable, $\geq 0 \ \forall n$ and $\nearrow$.

Moreover $f_n \longrightarrow f$ pointwise in X. We can thus apply MCT:

$$\int_{\mathbb{R}} f(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} \sum_{j=1}^{n} \mathbb{1}_{E_j}(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} \int_{\mathbb{R}} \mathbb{1}_{E_j}(x)\,dx$$

$$= \sum_{j=1}^{\infty} 2\left(\frac{1}{2^{j+1}}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}\,\frac{1}{1-1/2} = 1$$

#8

$f_n(x) = \sin(x)\,\mathbb{1}_{[\frac{1}{n}, 2\pi]}(x) \quad \forall x \in [0, 2\pi]$: $\lim_{n\to\infty} \int_0^{2\pi} f_n(x)\,dx$

$\sin(x) \in \mathcal{M}$, $\mathbb{1}_{[\frac{1}{n}, 2\pi]} \in \mathcal{M}_+ \Longrightarrow f_n \in \mathcal{M} \ \forall n \in \mathbb{N}$

Moreover $f_n \longrightarrow f$ in $[0, 2\pi)$, $f = \sin(x)\,\mathbb{1}_{[0,2\pi]}(x)$

and $|f_n| \leq \mathbb{1}_{[0,2\pi]}(x) \in L^1([0,2\pi])$

$\Longrightarrow$ DCT

$\Longrightarrow \lim_{n\to\infty} \int_0^{2\pi} f_n(x)\,dx = \int_0^{2\pi} \lim_{n\to\infty} f_n\,dx = \int_0^{2\pi} \sin(x)\,\mathbb{1}_{[0,2\pi]}(x)\,dx = [-\cos(x)]_0^{2\pi} = 0$

#9

$f_n(x) = \frac{1}{1+x^2}\,\mathbb{1}_{[n,n+1]}(x) \quad \forall x \geq 0$, $g(x) = \sum_{n=0}^{\infty} f_n(x)$, $\int_0^{\infty} g(x)\,dx$

Since $f_n \in \mathcal{M}_+ \ \forall n \Rightarrow \int_0^{\infty} \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_0^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{1+x^2}\,\mathbb{1}_{[n,n+1]}(x)\,dx$

$= \sum_{n=0}^{\infty} \int_n^{n+1} \frac{1}{1+x^2}\,dx = \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$

#10

$f_n(x) = \frac{\tanh(nx)}{\sqrt{\sin(x)}} \cdot \frac{1}{2+nx^3} \quad \forall x \in (0,2)$, $\lim_{n\to\infty} \int_0^2 f_n(x)\,dx$

$f_n(x) = \frac{e^{2nx}-1}{(e^{2nx}+1)\sqrt{\sin(x)}\,(2+nx^3)} \longrightarrow 0$ a.e. in $(0,2)$

Moreover : $|f_n(x)| \leq \frac{1}{2\sqrt{\sin(x)}}$ a.e. in $(0,2)$ and $\frac{1}{2\sqrt{\sin(x)}} \in L^1((0,2))$

$\Longrightarrow$ DCT : $\lim_{n\to\infty} \int_0^2 f_n(x)\,dx = \int_0^2 \lim_{n\to\infty} f_n(x)\,dx = \int_0^2 0\,dx = 0$

#11

$f_n(x) = (1-\frac{x}{n})^n\,\mathbb{1}_{[0,\log(n)]}(x) \quad x \in (0,\infty)$, $f(x) = e^{-x}$, $f_n \to f$ in L^1 ?

$f_n(x) \longrightarrow f(x)$ a.e. in $(0,\infty)$

Moreover $f_n(x) < f(x) \ \forall n$ $\left(\boxed{1-y \leq e^{-y} \ \forall y \in [0,1]} : (1-\frac{x}{n})^n \leq (e^{-\frac{x}{n}})^n = e^{-x} \ \forall n\right)$

Therefore $|f_n| \leq f = e^{-x} \in L^1((0,\infty)) \Longrightarrow$ DCT theorem

$\Rightarrow f_n \to f$ in L^1

$\boxed{\#\,12}$

$f_n(x) = \frac{1}{n^\alpha} \mathbb{1}_{[0,n]}(x)\, dx \qquad \alpha>0: \ f_n \to 0$ in L^1? $\qquad \alpha>0: f_n \to 0$ in meas.?

$$\int_{\mathbb{R}} |f_n(x) - f(x)|\,dx = \int_{\mathbb{R}} |f_n(x) - 0|\,dx = \int_{\mathbb{R}} \frac{1}{n^\alpha}\mathbb{1}_{[0,n]}\,dx = \int_0^n \frac{1}{n^\alpha}\,dx = \frac{1}{n^{\alpha-1}} \to 0$$

only if $\alpha - 1 > 0 \iff \boxed{\alpha > 1}$

For $\alpha>1$ we have also convergence in measure, however we can have more.

$f_n \to 0$ in measure $\iff \mu(\{|f_n - 0| \geq \varepsilon\}) \longrightarrow 0$

$|f_n - 0| = \frac{1}{n^\alpha}\mathbb{1}_{[0,n]}$: $\quad$ if $\varepsilon \leq \frac{1}{n^\alpha} \implies \mu(\{|f_n - f| \geq \varepsilon\}) = n$

$\qquad\qquad\qquad\qquad\quad$ if $\varepsilon > \frac{1}{n^\alpha} \implies \mu(\{|f_n - f| \geq \varepsilon\}) = 0$

Then, $\forall n > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha}}$: $\mu(\{|f_n - f| \geq \varepsilon\}) = 0$

Therefore, for $n \to \infty$ we conclude that $f_n \to 0$ in measure $\forall \alpha > 0$.

13

$f_n(x) = \frac{\sin(n^2 x)}{x^3} e^{-n\sqrt{x}} \qquad \forall x > 2, \qquad \lim_{n\to\infty} \int_2^\infty f_n(x)\,dx$

$f_n(x) \to 0$ a.e. in $(2,\infty)$, moreover $|f_n| \leq \frac{1}{x^3} \in L^1((2,\infty))$

so we can apply DCT: $\lim_{n\to\infty} \int_2^\infty f_n(x)\,dx = \int_2^\infty \lim_{n\to\infty} f_n(x)\,dx = 0$

14

$$f(x) = \begin{cases} x^2 & x \in [0,1] \\ 0 & x \in (1,3) \\ 1 & x \in [3,4] \end{cases} \qquad \mu(E) = \int_E f(x)\,dx$$

- μ well defined $\forall E \subseteq [0,4]$ measurable?

$f(x) = x^2 \mathbb{1}_{[0,1]} + 0 \cdot \mathbb{1}_{(1,3)} + \mathbb{1}_{[3,4]} \implies f$ measurable $\forall x \in [0,4]$

Moreover $0 \leq f(x) \leq 1 \quad \forall x \in [0,4]$ and since $[0,4]$ is bounded then f is L^1 integrable on every measurable set $E \subseteq [0,4]$. Thus, μ is well-def.

- $\mu([\frac{1}{2}, \frac{7}{2}]) = \int_0^{\frac{1}{2}} f(x)\,dx + \int_3^{7/2} f(x)\,dx = \int_0^{1/2} x^2\,dx + \int_3^{7/2} 1\,dx = \frac{13}{24}$

- $\xi = \eta \quad \mu$-a.e. $\iff \xi = \eta$ a.e. in $[0,1] \cup [3,4]$ since

since $\mu(E) = 0 \quad \forall E \subseteq [1,3)$
$\mu(E) = 0 \quad \forall E \subseteq [0,1] \cup [3,4] \iff \lambda(E) = 0$

15

$f_n(x) = \frac{\cos(nx)}{x^2(x+n^2)} \qquad x \in (1,\infty)$

$|f_n| \leq \frac{1}{x^3} \in L^1((1,\infty)), \quad f_n \to 0 \quad \forall x \in (1,\infty) \xrightarrow{DCT} f_n \to 0$ in L^1

16

$\{a_k\}_{k=1}^\infty = \{1,2,4,1,2,4,1,2,4,\dots\}$: $\int_0^1 \sum_{k=1}^\infty a_k \mathbb{1}_{(\frac{1}{2^k}, \frac{1}{2^{k-1}})}(x)\,dx$

Let $f_k(x) = a_k \mathbb{1}_{(\frac{1}{2^k}, \frac{1}{2^{k-1}})}$, then $f_n \in \mathcal{M}^+ \ \forall n \implies$ Corollary MCT:

$$\int_0^1 \sum_{k=1}^\infty f_k(x)\,dx = \sum_{k=1}^\infty \int_0^1 f_k(x)\,dx = \sum_{k=1}^\infty \int_0^1 a_k \mathbb{1}_{(\frac{1}{2^k}, \frac{1}{2^{k-1}})}(x)\,dx$$
$$= \sum_{k=1}^\infty a_k \frac{1}{2^k} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 4 \cdot \frac{1}{2^3} + 1 \cdot \frac{1}{2^4} + 2 \cdot \frac{1}{2^5} + 4 \cdot \frac{1}{2^6} + \dots$$
$$= \left(1 \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}\right) + \left(\frac{1}{2^7} + \dots\right) = 3\frac{1}{2} + 3\left(\frac{1}{2^4}\right) + 3\left(\frac{1}{2^7}\right) + \dots$$
$$= 3 \sum_{k=0}^{+\infty} \frac{1}{2^{1+3k}} = \frac{3}{2} \sum_{k=0}^{\infty} \left(\frac{1}{8}\right)^k = \frac{12}{7}$$

$\{b_n\}$ analogous, $= \dots = \frac{27}{13}$

#17 (#3)

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \int_0^1 (1-x)\, x^{k+1}\, dx \qquad \boxed{\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\log(1-x)}$$

$\sum_{k=0}^{\infty} \int_0^1 \frac{(1-x)\,x^{k+1}}{k+1}\, dx$ since $\frac{(1-x)\,x^{k+1}}{k+1}$ is measurable we can apply MCT corollary to conclude: $\sum_{k=0}^{\infty} \int_0^1 \frac{(1-x)\,x^{k+1}}{k+1}\, dx = \int_0^1 (1-x) \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}\, dx$

$= \int_0^1 (1-x)(-\log(1-x))\, dx = \int_1^0 y \log(y)\, dy = -\left(-\frac{1}{2^2}\right) = \frac{1}{4}$

#18

$$\sum_{n=1}^{\infty} \int_0^1 x^n (1-x)\, e^x\, dx$$

since $x^n(1-x)e^x$ is measurable $\overset{(\geq 0)}{\Longrightarrow}$ $\int_0^1 \sum_{n=1}^{\infty} x^n (1-x)\, e^x\, dx = \int_0^1 (1-x)\, e^x \sum_{n=1}^{\infty} x^n\, dx$

$= \int_0^1 \cancel{(1-x)}\, e^x \frac{x}{\cancel{1-x}}\, dx = \int_0^1 e^x x\, dx = 1$

#19

$f(x) = \sum_{k=0}^{\infty} 2^k \mathbb{1}_{\left[\frac{1}{4^{k+2}}, \frac{1}{4^k}\right)}(x) \qquad \forall x \in [0,1)$

$\int_0^1 |f(x)|\, dx = \int_0^1 \left| \sum_{k=0}^{\infty} 2^k \mathbb{1}_{[\cdot\cdot,\cdot\cdot)}(x) \right| dx = \int_0^1 \sum_{k=0}^{\infty} \underbrace{2^k \mathbb{1}_{\left[\frac{1}{4^{k+2}}, \frac{1}{4^k}\right)}(x)}_{\text{measurable, } \geq 0\ \forall k}\, dx$

$= \sum_{k=0}^{\infty} \int_0^1 2^k \mathbb{1}_{\left[\frac{1}{4^{k+2}}, \frac{1}{4^k}\right)}(x)\, dx$

$= \sum_{k=0}^{\infty} 2^k \left(\frac{1}{4^k} - \frac{1}{4^{k+2}} \right) = \left(1 - \frac{1}{2^4}\right) \sum_{k=0}^{\infty} \frac{1}{2^k} = \left(\frac{2^4 - 1}{2^4}\right) \cdot \frac{1}{1 - 1/2} = [\ldots] = \frac{15}{8}$

$\Longrightarrow f \in L^1([0,1])$

analogously with $f(x) = \sum_{n=0}^{\infty} \frac{1}{5^n} \mathbb{1}_{[3^n, 3^{n+2})}(x)$ .. $\int_1^{\infty} |f(x)|\, dx = 20 \Rightarrow f \in L^1((1,\infty))$

#20

$f_n(x) = \left(\frac{1}{n^2} - \frac{e^{-x^2}}{n^3} \right) \mathbb{1}_{[1,n]}(x) \qquad x \geq 1.$

$f_n(x)$ is such that: $f_n \geq 0$ because $\frac{1}{n^2} > \frac{1}{n^3 e^{x^2}}$ $\left(1 > \frac{1}{n e^{x^2}}\ (x \geq 1)\right)$

and so: $|f_n(x)| = \left(\frac{1}{n^2} - \frac{e^{-x^2}}{n^3} \right) \mathbb{1}_{[1,n]}(x) \leq \boxed{\frac{1}{n^2} \mathbb{1}_{[1,n]}(x) \leq \frac{1}{x^2} \mathbb{1}_{[1,n)}(x) \leq \frac{1}{x^2}}$

$\Longrightarrow \frac{1}{x^2} \in L^1((1,\infty)) \Longrightarrow f_n \in L^1$

We can apply DCT: $\lim_n \int_1^{\infty} f_n(x)\, dx = \int_1^{\infty} \lim_n f_n(x)\, dx = \int_1^{\infty} 0\, dx = 0.$

$f_n \leq f_{n+1}$? $\quad \frac{n - e^{-x^2}}{n^3} \overset{?}{\leq} \frac{(n+1) - e^{-x^2}}{(n+1)^3} \Longrightarrow [\ldots] \Rightarrow \frac{n(3n^2+3n)}{3n^2+3n+1} \overset{?}{\leq} e^{-x^2}$

If yes then $\frac{n(3n^2+3n)}{3n^2+3n+1} \leq 1$ but $\longrightarrow [\ldots] \Longrightarrow \frac{1}{n^2} + \frac{1}{3n^2} \geq 1$ th

which is false since with $n=2$ (e.g.) $\frac{1}{4} + \frac{1}{24} \not\geq 1$

$\Longrightarrow$ We cannot apply **MCT**.

For which $\alpha \in \mathbb{R}$: $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} f_n(x) \in L^1([0,\infty))$?

$\int_0^{\infty} \left| \sum_{n=1}^{\infty} \frac{f_n(x)}{n^\alpha} \right| dx = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{f_n(x)}{n^\alpha}\, dx \overset{\text{corollary}}{=} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{f_n(x)}{n^\alpha}\, dx = \sum_{n=1}^{\infty} \int_1^n \left(\frac{1}{n^{2+\alpha}} - \frac{e^{-x^2}}{n^{3+\alpha}} \right) dx$

$\leq \sum_{n=1}^{\infty} \int_1^n \frac{1}{n^{2+\alpha}}\, dx = \sum_{n=1}^{\infty} \frac{n-1}{n^{2+\alpha}} \leq \sum_{n=1}^{\infty} \frac{n}{n^{2+\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} < \infty$ if $\boxed{\alpha > 0}$

#21

$$\boxed{\int_0^{\infty} x^n e^{-x}\, dx = \Gamma(n+1) = n!}$$

$f(x) = \frac{x}{e^x - 1} \qquad x > 0$

$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{x}{e^{nx}}\, dx \overset{\text{corollary}}{=} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{x}{e^{nx}}\, dx = \int_0^{\infty} x \sum_{n=1}^{\infty} \left(\frac{1}{e^x}\right)^n dx = \int_0^{\infty} x \frac{1}{e^x} \frac{e^x}{1-e^x}\, dx = \int_0^{\infty} \frac{x}{1-e^x}\, dx$

Moreover:

$\int_0^{\infty} f(x)\, dx = \sum_{n=1}^{\infty} \int_0^{\infty} x e^{-nx}\, dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{y}{n^2} e^{-y}\, dy = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}$

(then, since $f \geq 0 \Longrightarrow |f| = f \Longrightarrow \int |f| = \int f < \infty \Longrightarrow f \in L^1$)

#22

$$f(x) = \frac{1}{x^\alpha + x^\beta} \qquad x>0 \qquad \beta > \alpha > 0$$

$f \in L^1$? $\quad f(x) \sim \frac{1}{x^\alpha} \quad x \to 0+$
$\quad f(x) \sim \frac{1}{x^\beta} \quad x \to \infty$ $\Bigg\} \implies \alpha \in (0,1), \ \beta > 1$

$$f_n(x) = \frac{n \, \mathrm{atan}\left(\frac{x}{n^2}\right)}{x^\alpha + x^n} \qquad x>0$$

$$f_n(x) \sim \frac{n \frac{1}{n^2} x}{x^\alpha + x^n} = \frac{\frac{1}{n}}{x^{\alpha-1} + x^{n-1}} \le \frac{1}{x^{\alpha-1} + x^{n-1}} \in L^1((0,\infty)) \text{ if } \alpha - 1 < 1 \quad \alpha < 2$$

for such values: $f_n(x) \longrightarrow 0$ and $|f_n| \le \frac{1}{x^{\alpha-1}} + \frac{1}{x^2} \in L^1((0,\infty))$

so we can apply DCT: $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = 0$

30

$f : [0,2] \to \mathbb{R}$ bounded, increasing, continuous at $x_0 = 0$. let $f_n(x) = f\left(\frac{x}{n}\right)$

$\implies$ f_n measurable $\forall n$, moreover $f_n(x) = f\left(\frac{x}{n}\right) \le c$ since f is bounded

$\implies$ $f_n \longrightarrow f(0)$ a.e. since it's continuous, then: $\lim_{n\to\infty} \int f_n(x) = \int_0^2 f(0)\,dx$

$= 2f(0)$

54

$f(x,y) = x e^{-x^2(1+y^2)} \qquad \forall x,y : (x,y) \in [0,\infty) \times [0,\infty) \ : \qquad \int_0^\infty e^{-x^2} dx$

$f(x,y)$ measurable and ≥ 0, therefore, by Fubini-Tonelli:

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \int_0^\infty \left(\int_0^\infty x e^{-x^2(1+y^2)}\,dx \right) dy$$

$$\int_0^\infty x e^{-x^2(1+y^2)}\,dx \to \left[\begin{array}{l} x^2(1+y^2) = \xi \\ 2x(1+y^2)\,dx = d\xi \end{array} \right] \to \int_0^\infty \frac{1}{2(1+y^2)} e^{-\xi}\,d\xi = \frac{1}{2(1+y^2)} \int_0^\infty e^{-\xi}\,d\xi$$

$$= \frac{1}{2(1+y^2)}$$

$$\longrightarrow \int_0^\infty \frac{1}{2(1+y^2)}\,dy = \frac{1}{2} \left[\mathrm{atan}(y)\right]_0^\infty = \frac{\pi}{4}$$

$$\implies \int_0^\infty \int_0^\infty f(x,y)\,dy\,dx = \int_0^\infty e^{-x^2} \int_0^\infty x e^{-x^2y^2}\,dy\,dx \qquad \begin{array}{l} xy = z \\ x\,dy = dz \end{array}$$

$$= \int_0^\infty e^{-x^2} \int_0^\infty e^{-z^2}\,dz\,dx$$

$$= \left(\int_0^\infty e^{-x^2}\,dx \right)^2$$

$$\longrightarrow \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

#56

#2 (#4)

$$f(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 0 & x=0 \end{cases} \quad \notin BV: \qquad 0 = x_0 < x_1 = \varepsilon < x_2 = 1$$

$$\sum_{k=1}^{2} |f(x_k) - f(x_{k-1})| = |f(x_2) - f(x_1)| + |f(x_1) - f(x_0)| = \left|1 - \frac{1}{\varepsilon}\right| + \left|\frac{1}{\varepsilon}\right| = \frac{2}{\varepsilon} - 1 \xrightarrow[\varepsilon \to 0]{} \infty$$

$\Longrightarrow V_0^1(f) = \infty, \quad f \notin BV$

#3

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \in [0,1] \setminus \mathbb{Q} \end{cases} \quad \notin BV: \qquad 0 = x_0 < x_1 < \ldots < x_n = 1$$

such that

$x_i \in \mathbb{Q}$ i even, $\quad x_i \in [0,1] \setminus \mathbb{Q}$ i odd

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = |-1-1| \cdot n = 2n \xrightarrow[n \to \infty]{} \infty$$

$\Longrightarrow V_0^1(f) = \infty, \quad f \notin BV$

$\sin(x)$ is 1-Lipschitz: $\quad |\sin(t) - \sin(s)| \leq |t-s| \quad \forall t,s \in \mathbb{R}$

#8

$$f(x) = \begin{cases} \dfrac{\sin(\log(x))}{\log^2(x)} & x \in (0, \frac{1}{2}] \\ 0 & x = 0 \end{cases}$$

0. $f \in C([0, \frac{1}{2}])$ since $\lim_{x \to 0^+} f(x) = 0$

1. f differentiable a.e., in fact:

$$f'(x) = \frac{N'D - ND'}{D^2} = \frac{\cos(\log(x))}{x \log^2(x)} - \frac{2 \sin(\log(x))}{x \log^3(x)} \qquad \forall x \in (0, \tfrac{1}{2}]$$

(a.e. in $[0, \frac{1}{2}]$)

2. $f' \in L^1([0, \frac{1}{2}])$, in fact:

$$|f'(x)| \leq \left|\frac{\cos(\log(x))}{x \log^2(x)}\right| + \left|\frac{2 \sin(\log(x))}{x \log^3(x)}\right|$$

$$\leq \frac{1}{x \log^2(x)} + \frac{2}{x \log^3(x)}$$

$$\int_0^{\frac{1}{2}} \frac{1}{x \log^2(x)} dx = \int_0^{\frac{1}{2}} \frac{1}{x} \frac{1}{(\log(x))^2} dx = \int_0^{\frac{1}{2}} f'(x) \frac{1}{(f(x))^2} dx = \left[-\frac{1}{f(x)}\right]_0^{\frac{1}{2}} = \left[-\frac{1}{\log(x)}\right]_0^{\frac{1}{2}} = \frac{1}{\log(2)}$$

$$\int_0^{\frac{1}{2}} \frac{2}{x \log^3(x)} dx = \int_0^{\frac{1}{2}} 2 \cdot \frac{1}{x} \cdot \frac{1}{(\log(x))^3} dx = 2 \left[\frac{(f(x))^{-3+1}}{-3+1}\right]_0^{\frac{1}{2}} = \left[-\frac{1}{(\log(x))^2}\right]_0^{\frac{1}{2}} = \frac{1}{\log^2(2)}$$

$\Longrightarrow |f'(x)| \leq g(x) \in L^1([0, \frac{1}{2}])$

$\Longrightarrow f' \in L^1([0, \frac{1}{2}])$

3. Since $f \in C^1([\xi, \frac{1}{2}]) \Longrightarrow f(x) = f(\xi) + \int_\xi^x f'(t)\, dt \quad \forall 0 < \xi \leq x \leq \frac{1}{2}$

Moreover, since $f' \in L^1([0, \frac{1}{2}])$, as $\xi \to 0$:

$$f(x) = f(0) + \int_0^x f'(t)\, dt$$

$\Longrightarrow \left[1,2,3 \Longrightarrow f \in AC([0, \tfrac{1}{2}])\right]$

10

$$f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases} \qquad \alpha, \beta : f \in AC ?$$

- $f \in C^0([0,1])$
- $f'(x) = \alpha x^{\alpha-1} \sin\left(\frac{1}{x^\beta}\right) - \beta x^{\alpha-\beta-1} \cos\left(\frac{1}{x^\beta}\right) \qquad \forall x \in (0,1]$
- $|f'(x)| \leq \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} \qquad \forall x \in (0,1]$
    - $x^{\alpha-1} \in L^1([0,1])$ if $\alpha > 0$
    - $x^{\alpha-\beta-1} \in L^1([0,1])$ if $\frac{\alpha}{\beta} > 1$

  $\Big\}\ f' \in L^1([0,1])$ if $0 < \alpha$, $\frac{\alpha}{\beta} > 1$
- $f \in C^1([\xi,1]) \implies f(x) = f(\xi) + \int_\xi^x f'(t)\,dt \qquad \forall\ 0 < \xi \leq x \leq 1$

  and since $f' \in L^1([0,1])$, as $\xi \to 0$: $f(x) = f(0) + \int_0^x f'(t)\,dt \qquad \forall x \in [0,1]$

$\implies f \in AC([0,1]) \implies f \in BV([0,1])$

if $\frac{\alpha}{\beta} \geq 1$ $f \notin BV([0,1]) \longrightarrow$ counter example

5.2

$\|f\|_1 = \int_a^b |f(x)|\,dx, \quad \|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$

Since : $\int_a^b |f(x)|\,dx \le \int_a^b \sup_{x\in[a,b]} |f(x)|\,dx \le \|f\|_\infty (b-a)$

we need to prove that $\nexists M : \; \|f\|_\infty \le M \|f\|_1$ for at least one $f \in C^0([a,b])$.

Consider, for example:

$$f_n(x) = \begin{cases} 1 - n(x-a) & x \in [a, a+\frac{1}{n}] \\ 0 & x \in (a+\frac{1}{n}, b] \end{cases}$$

$$\Longrightarrow \int_a^b |f(x)|\,dx = \int_a^{a+\frac{1}{n}} 1 - n(x-a)\,dx = \Big[(1+na)x - n\frac{x^2}{2}\Big]_a^{a+\frac{1}{n}} = [\,..\,] = \frac{1}{2n} \xrightarrow[n\to\infty]{} 0$$

$\Longrightarrow \nexists M > 0$ s.t. $\forall n \in \mathbb{N}$:

$$1 = \|f\|_\infty \le M \cdot \|f\|_1 = \frac{M}{n}$$

5.3

$T : L^2((0,1)) \to L^2((0,1))$: $\quad (Tf)(x) = x \int_0^1 f(t)\,dt \quad \forall x \in (0,1)$

- linear :

$$\begin{aligned} T(\alpha f + \beta g)(x) &= x \int_0^1 (\alpha f(t) + \beta g(t))\,dt \\ &= \alpha\, x \int_0^1 f(t)\,dt + \beta\, x \int_0^1 g(t)\,dt \\ &= \alpha T(f)(x) + \beta T(g)(x) \end{aligned}$$

- Bounded :

$$\|Tf\|_2^2 = \int_0^1 |T(f)(x)|^2\,dx = \int_0^1 \Big(x \int_0^1 f(t)\,dt\Big)^2 dt$$

$$\begin{aligned} \Big|\int_0^1 f(t)\,dt\Big| \le \Big|\int_0^1 |f(t)|\,dt\Big| &= \big|\,\|f\|_1\big| \\ &\le \big|\,\|f\|_2 \cdot \|1\|_2\big| \\ &= \big|\,\|f\|_2\big| \end{aligned}$$

$$\Longrightarrow \Big(\int_0^1 f(t)\,dt\Big)^2 = \Big|\int_0^1 f(t)\,dt\Big|^2 \le \|f\|_2^2$$

$$\le \int_0^1 x^2 \|f\|_2^2\,dx = \|f\|_2^2\,\frac{1}{3}$$

$\Longrightarrow$ linear and bounded $\equiv$ linear and continuous

$\boxed{\#5.4}$ *

$T : L^1((0,1)) \to L^\infty((0,1))$: $(Tf)(x) = \int_0^x e^t f(t)\, dt \quad \forall x \in (0,1)$

- linear:

$$T(\alpha f + \beta g)(x) = \int_0^x e^t (\alpha f(t) + \beta g(t))\, dt$$
$$= \alpha \int_0^x e^t f(t)\, dt + \beta \int_0^x e^t g(t)\, dt$$
$$= \alpha\, T(f)(x) + \beta\, T(g)(x)$$

- Bounded:

$f \in L^1((0,1)) \implies \|Tf\|_\infty = \sup_{x \in (0,1)} |(Tf)(x)| = \sup_{x\in(0,1)} \left| \int_0^x e^t f(t)\, dt \right|$

$$\leq \sup_{x \in (0,1)} \int_0^x e^t |f(t)|\, dt$$
$$= \int_0^1 e^t |f(t)|\, dt = \|gf\|_1 \quad (g(t) = e^t)$$
$$\leq \|f\|_1 \cdot \|g\|_\infty$$
$$\|g\|_\infty = \sup_{t \in (0,1)} |g(t)| = \sup_{t\in(0,1)} e^t = e$$
$$= e\|f\|_1$$

$\|Tf\|_\infty \leq e\|f\|_1$

$\|T\|_{\mathcal{L}} \leq e$

let's prove: $\|T\|_{\mathcal{L}} = e$ $(\downarrow)$

- Norm of T:

let's consider: $f_n(x) := \begin{cases} n & 1 - \frac{1}{n} \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\implies \|f_n\|_1 = \int_0^1 |f_n|\, dx = \int_{1-\frac{1}{n}}^1 n\, dx = [nx]_{1-\frac{1}{n}}^1 = 1$$

$$\|T(f_n)\|_\infty = \sup_{x\in(0,1)} T(f_n) = \sup_{x\in(0,1)} \int_0^x e^t n \mathbb{1}_{[1-\frac{1}{n},1)}(t)\, dt$$
$$= \int_0^1 e^t n \mathbb{1}_{[1-\frac{1}{n},1)}(t)\, dt$$
$$= \int_{1-\frac{1}{n}}^1 n e^t\, dt = ne^1 - ne^{1-\frac{1}{n}} = n\left(e - e^{1-\frac{1}{n}}\right)$$
$$= e\left(\frac{1 - e^{-1/n}}{1/n}\right)$$
$$\geq e$$

$$\implies \|T\|_{\mathcal{L}} = e$$

\# 5.9

$\{x_n\}_n \subset \ell^2 \quad : \quad x_n^{(k)} := \dfrac{1}{n+k} \qquad \forall\, k,n \in \mathbb{N}$

$\{x_n\} \longrightarrow 0$ in ℓ^2 ?

First we check that $x_n \in \ell^2$:

$$x_n \in \ell^2 \iff \sum_{k=1}^{\infty} |x_n^{(k)}|^2 = \sum_{k=1}^{\infty} \left|\frac{1}{n+k}\right|^2 < \infty$$

$$= \sum_{h=n+1}^{\infty} \left|\frac{1}{h}\right|^2 < \sum_{h=1}^{\infty} \left|\frac{1}{h}\right|^2 < \infty \quad \checkmark$$

Convergence:

$$\| x_n - 0 \|_2^2 = \| x_n \|_2^2 = \sum_{k=1}^{\infty} |x_n^{(k)}|^2 = \sum_{h=n+1}^{\infty} \left|\frac{1}{h}\right|^2 < \sum_{h=1}^{\infty} \left|\frac{1}{h}\right|^2 < \infty$$

$\Longrightarrow x_n \xrightarrow[n\to\infty]{} 0$ in ℓ^2.

\# 5.12

$\{x_n\}_n : \quad x_n^{(k)} = \begin{cases} \frac{1}{k} & 1 \le k \le n \\ 0 & \text{otherwise} \end{cases}$

$n=1$	1	0	0	0	0	0	0
$n=2$	1	$\frac{1}{2}$	0	0	0	0	0 .
$n=3$	1	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	..

$\{x_n\}_n$ converges pointwise to $x = \{\frac{1}{k}\}_k$.

($\forall k$ fixed: $x_n^{(k)} \to \frac{1}{k}$)

- ℓ^2:
$$\| x_n - x \|_2^2 = \sum_{k=1}^{\infty} |x_n^{(k)} - x^{(k)}|^2$$
$$= \sum_{k=1}^{n} \left| \frac{1}{k} - \frac{1}{k} \right|^2 + \sum_{k=n+1}^{\infty} \left| 0 - \frac{1}{k} \right|^2$$
$$= \sum_{k=n+1}^{\infty} \left| \frac{1}{k} \right|^2 < \sum_{k=1}^{\infty} \left| \frac{1}{k} \right|^2 < \infty$$
$\Longrightarrow x_n \longrightarrow x$ in ℓ^2

- ℓ^1: If x_n converges to something then this something is x. However, $x \in \ell^1$ $\left(\sum_{k=1}^{\infty} \left|\frac{1}{k}\right| = \infty \right)$ hence $x_n \underset{n\to\infty}{\not\to} x$.

#5.5

$|T(f)| \le \int_0^1 |f(x)|\,dx \le \|f\|_\infty \cdot 1 \implies \|T\|_{\Sigma} \le 1$

Moreover, let $f_n(x) = x^{1/n}$; $T(f_n) = \int_0^1 x^{\frac{1}{n}}\,dx = \left[\frac{x^{1+1/n}}{1+1/n} \right]_0^1 = \frac{1}{1+\frac{1}{n}} \xrightarrow[n\to\infty]{} 1$

$\implies \|T\|_{\Sigma} = 1.$

(Moreover T is linear and bounded $\Rightarrow$ well defined : E^*)

If it $\exists f$: $\|f\|_\infty = 1$ and $T(f) = 1$ then:

$T(f) = \int_0^1 f(x)\,dx = \|f\|_\infty = \int_0^1 \|f\|_\infty\,dx \implies \int_0^1 (f(x) - \|f\|_\infty)\,dx =$

and so : $f(x) = \|f\|_\infty$ a.e. $\implies$ $f = 1$ a.e. $\lightning$ because to be $f = 1$ a.e. then $f = 1$ $\forall x \in [0,1]$ since it's continuous however $f(0) = 1 \neq 0$.