1 Set Theory

1.1. Write the definitions of: **sequence of sets $\{E_n\}_n$**, **increasing** and **decreasing sequence of sets $\{E_n\}_n$**, **$\limsup_{n\to\infty} E_n$**, **$\liminf_{n\to\infty} E_n$**, **$\lim_{n\to\infty} E$**.

Let X be a set. We denote $\mathcal{P}(X) := \{Y | Y \subseteq X\}$ the power set of X.
Let I be a (general) index set. Then $\{E_i\}_{i\in I}$ is called family or collection indexed by I and $E_i \subseteq X\ \forall i \in I$.
If $I = \mathbb{N}$ then $\{E_n\}_{n\in\mathbb{N}}$ is called sequence of sets.
A sequence $\{E_n\}_n \subseteq \mathcal{P}(X)$ is monotone increasing if $E_n \subseteq E_{n+1} \forall n \in \mathbb{N}$. $[E_n \nearrow]$
A sequence $\{E_n\}_n \subseteq \mathcal{P}(X)$ is monotone decreasing if $E_n \supseteq E_{n+1} \forall n \in \mathbb{N}$. $[E_n \searrow]$
Let $\{E_n\}_n \subseteq \mathcal{P}(X)$. We define:

$\limsup_n E_n := \bigcap_{k=1}^{\infty} \left[\bigcup_{n=k}^{\infty} E_n\right]$ $\qquad$ $\left(\bigcup_{i\in I} E_i = \{x \in X \colon \exists\, i \in I \ x \in E_i\}\right)$

$\liminf_n E_n := \bigcup_{k=1}^{\infty} \left[\bigcap_{n=k}^{\infty} E_n\right]$ $\qquad$ $\left(\bigcap_{i\in I} E_i = \{x \in X \colon \forall\, i \in I \ x \in E_i\}\right)$

If $\limsup_n E_n = \liminf_n E_n := F$ then we define $\lim_n E_n = F$.

1.2. Write the definitions of: **cover** (or covering) of a set, **subcover**, **open cover** of a set, **finite subcover**.

Let X be a set, I be a (general) index set. A family of sets $\{E_i\}_{i\in I}$ is called cover of X if $X \subseteq \bigcup_{i\in I} E_i$.
A subfamily of a cover which itself forms a cover is called a subcover.
A cover $\{E_i\}_{i\in I}$ is said to be open if every element E_i of the cover is an open set.
A subcover is said to be finite if it has a finite number of elements.

1.3. Write the definitions of: **equivalence relation**, **equivalence class**, **quotient set**.

Let X be a set. A relation in X is a subset $R \subseteq X \times X$ (denoted $(x, y) \in R$ or xRy).
The relation R in X is an equivalence relation if $\forall x, y, z \in X$ the following properties hold:

1. reflexivity: xRx
2. symmetry: $xRy \Rightarrow yRx$
3. transitivity: $xRy,\ yRz \Rightarrow xRz$

Given $x \in X$ and R relation, we define the equivalence class of x as: $E_x := \{y \in X : yRx\}$.
The equivalence classes form a partition of the set X.
The set $\frac{X}{R} := \{E_x : x \in X\}$ is called quotient set of X over the relation R.
The quotient set is a set of sets, the elements are equivalence classes.

1.4. Write the definitions of: **order relation**, **partially ordered set**, totally ordered set (or **chain**).
Provide examples of partially ordered sets and of totally ordered sets.

Let X be a set. A relation in X is a subset $R \subseteq X \times X$ (denoted $(x, y) \in R$ or xRy).
The relation R in X is an order relation if $\forall x, y, z \in X$ the following properties hold:

1. reflexivity: xRx
2. antisymmetry: $xRy,\ yRx \Rightarrow x = y$
3. transitivity: $xRy,\ yRz \Rightarrow xRz$

A set $P \subseteq X$ with an order relation R is called a partially ordered set.
If $\forall x, y \in P$ either xRy or yRx then P is a totally ordered set (or chain).
Examples:
- $(\mathbb{R}, \leq)$ is totally ordered, since $\forall x, y \in \mathbb{R}$ it holds either $x \leq y$ or $y \leq x$.
- $X = \mathbb{Z}^n$ with the relation $[x \leq y \Leftrightarrow x \text{ divides } y]$ is partially ordered but not totally ordered.
  For example, 2 is in relation with 4 (2 divides 4), but 3 is not in relation with 4.
  There exist two elements x, y (3,4) for which neither xRy nor yRx holds.

1.5. Write the definition of **equipotent sets**, **cardinality of a set**. State the **Schröder - Bernstein theorem** and the **Cantor theorem**.

Two sets X, Y are equipotent if there exists a bijection $f : X \to Y$.
The cardinality of X is defined as the collection of all sets equipotent to X, namely:
$|X| := \{Y \mid \exists f : X \to Y \text{ bijection }\}$.
Therefore we can say that X, Y are equipotent $\Leftrightarrow$ $|X| = |Y|$.

Theorem. **Schröder - Bernstein**
Let X, Y be two sets. If $|X| \leq |Y|$ and $|Y| \leq |X| \Rightarrow |X| = |Y|$.

Theorem. **Cantor**
Let X be any set. Then: $|X| \leq |\mathcal{P}(X)|$.
It is always possible to obtain a bigger cardinality through the operation of power set.

1.6. Write the definitions of: **infinite set**, **finite set**, **countable set**, **uncountable set**. Provide examples.

Let X be a set. Then X is said to be infinite if there exists $E \subsetneq X$ such that $|X| = |E|$.
The set X is said to be finite if it is not infinite.
The set X is said to be countable if $|X| \leq |\mathbb{N}|$.
The set X is said to be uncountable if it is not countable.
Examples:
- $\mathbb{Q}$ is countable
- $\mathbb{R}$ is uncountable
- if X, Y are countable, then $X \times Y$ is countable
- the countable union of countable sets is countable

1.7. Write the **continuum hypothesis** and the **axiom of choice**.

Axiom. Continuum hypothesis
There is no set whose cardinality is strictly between that of $\mathbb{N}$ and $\mathbb{R}$.

Axiom. Axiom of choice
Let X be a set, I a (general) index set. Given any disjoint family of sets $\{E_i\}_{i\in I} \subseteq \mathcal{P}(X)$, there exists a set $V \subseteq X$ that contains exactly one element of every E_i, namely: $V \cap E_i = \{x_i\}$ $\forall i \in I$.
Given a collection of non-empty sets, we can choose one element from each set.

1.8. Write the definitions of: **maximal element, upper bound, sup of a subset of a partially ordered set.** State the **Zorn's lemma**. Which is the relation between the axiom of choice and Zorn's lemma?

Let $(P, \leq)$ be a partially ordered set with the generic order relation $\leq$.
$M \in P$ is called maximal element of P if: $(M \leq x$ for some $x \in P) \Rightarrow (x = M)$.
Let $A \subseteq P$ and $u \in P$. We say that u is an upper bound if: $a \leq u$ $\forall a \in A$.
If $u \leq v$ $\forall v$ upper bound of A, then we say that u is the supremum of A and we denote it as: $u := \sup A$.
(Recall) In a partially ordered set $(P, \leq)$, if $\forall x, y \in P$ either $x \leq y$ or $y \leq x$ holds, then P is a chain.

Lemma. Zorn's lemma
Let $(P, \leq)$ be a partially ordered set.
If every chain has an upper bound, then P has a maximal element.

Zorn's lemma is an equivalent formulation of the axiom of choice, hence: AC $\Leftrightarrow$ Zorn's lemma.

2 Metric Spaces

Def. **Metric space**
Let $X \neq \varnothing$ be any set. A distance (metric) on X is a function $d : X \times X \to [0, \infty)$ s.t.:
1. $d(x, y) \geq 0$ $\forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$ $\forall x, y \in X$
3. $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality)

If d is a distance on X then (X, d) is a metric space.

$\mathbb{R}^N$	$d_p(x, y) = \left(\sum_{j=1}^N \lvert x_j - y_j\rvert^p\right)^{\frac{1}{p}}$ $\forall p \geq 1$
$\mathbb{R}^N$	$d_\infty(x, y) = \max_{j=1,..,N} \lvert x_j - y_j\rvert$
$C^0([a, b])$	$d(x, y) = \max_{t\in[a,b]} \lvert x(t) - y(t)\rvert$
$\ell^p = \{x = \{x^{(k)}\}_k \subseteq \mathbb{R} : \sum_{k=1}^\infty \lvert x^{(k)}\rvert^p < +\infty\}$	$d_p(x, y) = \left(\sum_{k=1}^\infty \lvert x^{(k)} - y^{(k)}\rvert^p\right)^{\frac{1}{p}}$
$\ell^\infty = \{x = \{x^{(k)}\}_k \subseteq \mathbb{R} : \sup_{k\in\mathbb{N}} \lvert x^{(k)}\rvert < +\infty\}$	$d_\infty(x, y) = \sup_{k\in\mathbb{N}} \lvert x^{(k)} - y^{(k)}\rvert$

Def. **Open ball**
Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. The open ball with center in x_0 and radius r is defined as: $B(x_0, r) = \{y \in Y : d(x, y) < r\}$

Def. **Open/Closed set**
(X, d) metric space. Let $A \subseteq X$.
A is open if: $\forall x \in A$ $\exists r = r_x > 0$ such that $B(x, r) \subseteq A$.
A is closed if $X \setminus A$ is open.

Def. **Basic notions**
(X, d) metric space. Let $A \subseteq X$, $x_0 \in X$:
- x_0 is an interior point of A if $\exists r > 0 : B(x_0, r) \subseteq A$
- x_0 is an exterior point of A if $\exists r > 0 : B(x_0, r) \subseteq X \setminus A$
- x_0 is a boundary point of A if $\forall r > 0 : B(x_0, r) \cap A \neq \varnothing$ and $B(x_0, r) \cap X \setminus A \neq \varnothing$
- x_0 is an accumulation point of A if $\forall r > 0 : \exists x_r \in (B(x_0, r) \cap A) \setminus \{x_0\}$
- interior of $A = int(A) := \{x \in X : x$ interior point of $A\}$
- exterior of $A = ext(A) := \{x \in X : x$ exterior point of $A\}$
- boundary of $A = \partial A := \{x \in X : x$ boundary point of $A\}$
- closure of $A = \overline{A} = cl(A) := A \cup \partial A$

Prop. A is open $\Leftrightarrow A = int(A)$ ($int(A)$ is the biggest open set which is contained in A)
A is closed $\Leftrightarrow \partial A \subseteq A = \overline{A}$ ($\overline{A}$ is the smallest closed set which contains A)

Def. **Sequence**
(X, d) metric space. A sequence in X is a map $x : \mathbb{N} \to X$ and we denote it as $x = \{x_n\}_n$.

Def. **Convergent sequence**
(X, d) metric space. A sequence $\{x_n\}_n \subseteq X$ converges to $x_0 \in X$, namely $x_n \xrightarrow[n\to\infty]{d} x_0$, if:
$\lim_{n\to\infty} d(x_n, x_0) = 0 \quad \Leftrightarrow \quad d(x_n, x_0) \xrightarrow[n\to\infty]{} 0$.
The sequence $\{d(x_n, x_0)\}_n$ is a sequence of real numbers. Let $\{a_n\}_n$ be a real sequence:
if $\forall \varepsilon > 0$ $\exists \bar{n} \in \mathbb{N}$ such that $\forall n \geq \bar{n}$ $|a_n - a| < \varepsilon \Rightarrow \lim_{n\to\infty} a_n = a$.

Prop. In $(\mathbb{R}^N, d_p)$, $\forall 1 \leq p \leq \infty$, it holds: $x_n \xrightarrow[n\to\infty]{d_p} x_0 \Leftrightarrow x_n^{(i)} \xrightarrow[n\to\infty]{} x_0^{(i)}$ $\forall i = 1, .., N$. $(x_n^{(i)}, x_0^{(i)} \in \mathbb{R})$
The convergence of the sequence is equivalent to the convergence of the components.

Prop. In $(C^0([a, b]), d)$ it holds: $x_n \xrightarrow[n\to\infty]{d} x_0 \Leftrightarrow \lim_{n\to\infty}(\max_{t\in[a,b]} |x_n(t) - x_0(t)|) = 0$.
This means that the sequence $\{x_n\}_n \subseteq C^0([a, b])$ has to uniformly converges to x_0.

Prop. **Closed set (with sequences)**
Let (X, d) be a metric space, $A \subseteq X$. The set A is closed if:
$\forall x_0 \in X$ for which $\exists \{x_n\}_n \subseteq A$ with $x_n \xrightarrow[n\to\infty]{} x_0$ (hence $x_0 \in \overline{A}$) $\Rightarrow x_0 \in A$.

Def. **Bounded sequence**
(X, d) metric space. A sequence $\{x_n\}_n \subseteq X$ is bounded if $\exists x_0 \in X$, $M > 0$ such that:
$d(x_n, x_0) < M$ $\forall n \in \mathbb{N}$

Thorem. **Uniqueness of the limit**
(X, d) metric space. If a sequence $\{x_n\}_n$ has a limit, then the limit is unique.

Def. **Cauchy sequence**
(X, d) metric space. A sequence $\{x_n\}_n$ is Cauchy if:
$\forall \varepsilon > 0 \ \exists \bar{n}_\varepsilon \in \mathbb{N}$: $d(x_m, x_n) < \varepsilon \ \forall n, m > \bar{n}_\varepsilon$.
A convergent sequence is Cauchy, not every Cauchy sequence converges.
Cauchy sequences are always bounded.

Def. **Complete metric space**
(X, d) metric space is said to be complete if every Cauchy sequence is convergent.
Examples:
- $(\mathbb{R}^N, d_p)$ is complete $\forall p \in [1, +\infty]$
- $(C^0([a, b]), d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|)$ is complete

Prop. (X, d) complete metric space, $B \subseteq X$ closed $\Rightarrow (B, d)$ complete.

Def. **Bounded subset of a metric space (X, d)**
A subset S of a metric space (X, d) is bounded if:
$\exists r > 0$ such that $\forall s, t \in S$: $d(s, t) < r$.

2.1 Separability

1.9. Write the definitions of: **dense set, separable metric space, nowhere dense set, set of first category, set of second category.** Provide an example of a nowehere dense and one of a set of first category.

Let (X, d) be a metric space.
A set $A \subset X$ is dense in X if the closure of A is equivalent to X, namely $\overline{A} = X$.
The metric space (X, d) is separable if $\exists A \subset X$ countable and dense in X.
A set $E \subseteq X$ is nowhere dense if $int(\overline{E}) = \varnothing$. A nowhere dense set does not contain any open ball.
A set $E \subseteq X$ is of first category (meagre) in X if E is the countable union of nowhere dense sets in X.
A set $E \subseteq X$ is of second category (not meagre) in X if it is not of first category.
Examples:
- $E = \mathbb{Z} \subset \mathbb{R} = X \Rightarrow \overline{E} = E \Rightarrow int(\overline{E}) = int(E) = \varnothing \Rightarrow \mathbb{Z}$ is nowehere dense (in $\mathbb{R}$)
- $E = \mathbb{Q} \subset \mathbb{R} = X \Rightarrow \mathbb{Q} = \bigcup_{n=1}^{\infty}\{q_n\} := \bigcup_{n=1}^{\infty} A_n$
  $\Rightarrow int(\overline{A_n}) = int(A_n) = \varnothing \Rightarrow A_n$ is nowhere dense
  $\Rightarrow \mathbb{Q}$ is of first category in $\mathbb{R}$
- any countable set in $\mathbb{R}$ is of first cateogry (in $\mathbb{R}$)

1.10. Write the definition of **sequence of nested balls.** State the **theorem of nested balls** (or the Cantor intersection theorem).

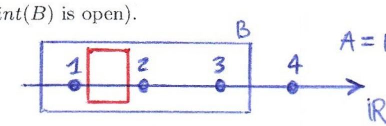
Let (X, d) be a metric space. A sequence $\{B_n\}_n$ of closed balls in a metric space X is called nested if:
$B_n \supset B_{n+1} \ \forall n \in \mathbb{N} \qquad (B_1 \supset B_2 \supset .. \supset B_k \supset ..)$.
Let r_n be the radius of B_n. Then $\{r_n\}_n$ is a decreasing sequence, namely $\{r_n\}_n \searrow$.

Theorem. Cantor intersection theorem (theorem of nested balls)
The metric space (X, d) is complete $\Leftrightarrow$ any nested sequence of closed balls whose radius tends to 0 has non-empty intersection, namely $\bigcap_{n=1}^{\infty} B_n \neq \varnothing$.

1.11. State and prove the **Baire category theorem** and its corollary. *(The proof includes also the preliminary lemma.)*

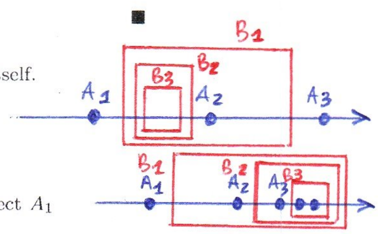
Lemma. Let (X, d) be a metric space and $A \subseteq X$ a nowhere dense set.
Then, every closed ball in X contains a closed ball which does not intersect A.

proof.
- Let $B \subset X$ be a closed ball, so that $B = \overline{B}$.
  Then $int(B) = int(\overline{B}) \not\subseteq \overline{A}$, since A is nowhere dense and cannot contain any open ball ($int(B)$ is open).
  $\Rightarrow \exists x_0 \in int(B) \setminus \overline{A} := E$.
  E is open since it's the difference between an open set ($int(B)$) and a closed set ($\overline{A}$).



- Since E is open and $x_0 \in E$, by definition of open sets: $\exists B_r(x_0) \subset E$.
- Since $B_r(x_0)$ is entirely contained in E:
  $\Rightarrow \overline{B_{\frac{r}{2}}(x_0)} \subset E$
- Since $E \cap A = \varnothing \Rightarrow \overline{B_{\frac{r}{2}}(x_0)} \cap A = \varnothing$.
  It exists a closed ball inside B which does not intersect A. ∎

Theorem. Baire category theorem
Let (X, d) be a complete metric space. Then $E = X \subseteq X$ is of second category in itself.
A complete metric space has to be of second category in itself.



proof.
- Suppose by contraddiction that X is of first category, so that:
  $X = \bigcup_{n=1}^{\infty} A_n$, $\quad A_n \subset X$ and nowhere dense $(int(\overline{A_n})) = \varnothing) \ \forall n \in \mathbb{N}$.
- By the previous lemma, there exists a closed ball B_1 of radius $r_1 < 1$ which does not intersect A_1 (since A_1 is, by assumption, nowhere dense).
- By the lemma again, B_1 contains a closed ball B_2 of radius $r_2 < \frac{1}{2}$ such that B_2 does not intersect A_2.
- Carrying on this process we obtain a nested sequence $\{B_n\}_n$ of closed balls such that:
  $B_n \cap A_n = \varnothing \quad \forall n \in \mathbb{N} \qquad$ and $\qquad r_n \xrightarrow[n\to\infty]{} 0$.
- Thus, since X is complete, by the theorem of nested balls: $\bigcap_{n=1}^{\infty} B_n \neq \varnothing$.
- On the other hand, since $B_n \cap A_n = \varnothing \ \forall n \in \mathbb{N}$: $(\bigcap_{n=1}^{\infty} B_n) \cap (\bigcup_{n=1}^{\infty} A_n) = \varnothing$.
  But: $\bigcup_{n=1}^{\infty} A_n = X \Rightarrow \bigcap_{n=1}^{\infty} B_n = \varnothing \Rightarrow$ contraddiction $\Rightarrow X$ must be of second category. ∎

Corollary. Let (X,d) be a complete metric space.
Then, the intersection of a countable family of open dense sets in X is a set which is dense in X.
Alternative from. Let (X,d) be a complete metric space.
Let $\{A_n\}_n \subset X$ be a countable family of open dense sets in X ($\overline{A_n} = X\ \forall n \in \mathbb{N}$).
Then, also $\bigcap_{n=1}^{\infty} A_n$ is dense in X, namely $\overline{\bigcap_{n=1}^{\infty} A_n} = X$.

The corollary says that density is preserved under countable intersection.

proof.

- Let $\{A_n\}_{n\in\mathbb{N}} \subset X$ be a sequence of open dense sets, so that $\overline{A_n} = X\ \forall n \in \mathbb{N}$.
  By contraddiction suppose that: $E := \overline{\bigcap_{n\in\mathbb{N}} A_n} \subsetneq X$.
- E is closed so, by definition, E^c is open $\Rightarrow \exists B \subset E^c$ closed.
  $\Rightarrow$ B is closed in X
  $\Rightarrow$ (B,d) is a complete metric space $\big((X,d)$ complete, $B \subseteq X$ closed $\Rightarrow$ (B,d) complete$\big)$
  $\Rightarrow$ B is of **second category** (by Baire category theorem).
- On the other hand, by our costruction:
  $\left(\bigcap_{n\in\mathbb{N}} A_n\right) \cap B = \varnothing \ \Rightarrow$ De Morgan law: $\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^c\right)^c$
  $\Rightarrow \left(\bigcup_{n\in\mathbb{N}} A_n^c\right) \cap B = B$
  $\Rightarrow \bigcup_{n\in\mathbb{N}} (A_n^c \cap B) = B$
  We define $G_n := A_n^c \cap B$ and so: $B = \bigcup_{n\in\mathbb{N}} G_n$.
- Since A_n is open and $\overline{A_n} = X \ \Rightarrow\ A_n^c$ is closed and $int(\overline{A_n^c}) = int(A_n^c) = \varnothing$.
  $\Rightarrow$ $int(\overline{G_n}) = int(\overline{A_n^c \cap B}) \subseteq int(\overline{A_n^c}) = int(A_n^c) = \varnothing$
  $\Rightarrow$ $B = \bigcup_{n\in\mathbb{N}} G_n$ and G_n is nowhere dense, since $int(\overline{G_n}) = \varnothing$
  $\Rightarrow$ B is of **first category** (countable union of nowhere dense sets)
  $\Rightarrow$ contraddiction $\Rightarrow$ $E = X$ ∎

2.2 Compactness

1.12. Write the definitions of: **compact** metric space, **sequentially compact** metric space, **totally bounded** metric space. Explain how these properties are related.

A metric space (X,d) is compact:
$\Leftrightarrow$ any open cover has a finite subcover.
$\Leftrightarrow$ $\forall \{x_i\}_{i=1}^n \subset X\ \exists \delta > 0$ such that: $X = \bigcup_{i=1}^n B_\delta(x_i)$.
A metric space (X,d) is sequentially compact:
$\Leftrightarrow$ any bounded sequence $\{x_n\}_n$ has a convergent subsequence.
A metric space (X,d) is totally bounded:
$\Leftrightarrow$ $\forall \varepsilon > 0\ \exists A \subset X$, A finite, such that $\forall x \in X$: $dist(x,A) := \inf_{y\in A} d(x,y) < \varepsilon$.
It is always possible to find a set of points that is close to all the points of X.
$\Leftrightarrow$ $\forall r > 0\ \exists \{x_i\}_{i=1}^n \subset X$ such that: $X \subseteq \bigcup_{i=1}^n B_r(x_i)$.
The above means that $\exists$ *a finite cover made of balls of radius that we choose.*

Theorem. **Characterization of compact metric spaces**
Let (X,d) be a metric space. The below conditions are equivalent:
(i) X is compact
(ii) X is sequentially compact
(iii) X is complete and totally bounded

Prop. Let (X,d) be a metric space.
- $E \subseteq X$ is compact $\Rightarrow$ E is closed and bounded.
- $E \subseteq \mathbb{R}^N$ is compact $\Leftrightarrow$ E is closed and bounded (Heine-Borel theorem).

2.3 Compactness in $C^0([a,b])$

Def. **Set of continuous functions $C^0(X)$**
Let (X,d) be a compact metric space. We define the space of continuous functions over X as:
$C^0(X) := \{f : X \to \mathbb{R} \text{ continuous}\}$.
On this set we define the distance: $d(f,g) = \sup_{x\in X} |f(x) - g(x)| = \|f - g\|_\infty$.

Prop. **Completeness of $C^0(X)$**
The metric space $(C^0(X), d)$ defined as above is complete.

1.13. Write the $\varepsilon - \delta$ definition of **equicontinuous** subset F of $C^0(X)$, where X is a compact metric space. Explain from which parameters δ depends. In particular, write the definition when $F = \{f_n\}_{n\in\mathbb{N}}$.

Let (X,d) be a compact metric space. The subset $F \subseteq C^0(X)$ is equicontinuous if:
$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0:\ \forall f \in F\ \ \forall x,y \in X\ \ \ d(x,y) < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$.
In this case, δ_ε does not depend on f. Moreover δ_ε is independent of x. In conclusion, δ_ε depends only on ε.
In particular, if $F = \{f_n\}_{n\in\mathbb{N}}$ then F is equicontinuous if:
$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0:\ \forall n \in \mathbb{N}\ \ \forall x,y \in X\ \ \ d(x,y) < \delta_\varepsilon \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$.

X compact, F equicontinuous, F bounded $\longrightarrow$ F totally bounded
F closed $\longrightarrow$ F complete
(F totally bounded, F complete) $\Longleftrightarrow$ F compact

1.14. State and prove the **Ascoli-Arzelà theorem**.

The Ascoli-Arzelà theorem is a sufficient condition for compactness (of a set in $C^0(X)$).

Theorem. **Ascoli-Arzelà**
Let (X, d) be a compact metric space.
If $F \subset C^0(X)$ is bounded, closed and equicontinuous, then F is compact.

proof.

- By the characterization of compact metric spaces: F **compact** $\Leftrightarrow$ F **complete and totally bounded**.
- $F \subset C^0(X)$, F closed, $C^0(X)$ complete $\Rightarrow$ F is **complete**. $\big((C^0(X), d)$ complete, $F \subset C^0(X)$ closed $\Rightarrow (F, d)$ complete$\big)$
- By hypothesis we have:
  - **(1)** F equicontinuous $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists \delta_\varepsilon := \delta > 0 : \ \forall f \in F\ \ \forall x, y \in X \quad d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
  - **(2)** X compact $\Leftrightarrow$ $X = \bigcup_{i=1}^{n} B_\delta(x_i)$ for some $\delta > 0$
  - **(3)** F bounded $\Leftrightarrow$ $\exists\, k > 0$: $\forall f \in F$, $\forall x \in X$: $|f(x)| \leq k$
- Let $E := [-k, k] \subset \mathbb{R}$.
  We define a mapping $T : F \to E^n$ such that:
  $T(f) = (f(x_1), f(x_2), .., f(x_n)) \quad \forall f \in F$
  where $\forall i = 1, .., n$, x_i is the center of the ball $B_\delta(x_i)$ introduced before.
  Since E^n is the cartesian product of compact intervals:
  $\Rightarrow$ $E^n \subset \mathbb{R}^n$ is compact
  $\Rightarrow$ E^n is totally bounded.
- Because of **(3)**, $\forall f \in F$ every component of $T(f)$ is such that $|\cdot| \leq k$:
  $\Rightarrow$ $T(F) \subseteq E^n$
  $\Rightarrow$ $T(F)$ is totally bounded. *(a subset of a totaly bounded set is totally bounded)*
- Let $\varepsilon > 0$. Since $T(F)$ is totally bounded, we can cover $T(F)$ by finitely many open balls of radius ε, say:
  $B_1, B_2, .., B_m$. $\left(T(F) \subseteq \bigcup_{j=1}^{m} B_j \right)$
  For each $j = 1, 2, .., m$ we choose a $f_j \in F$ such that: $T(f_j) \in B_j$.
- *Claim*: F is covered by a finite number of open balls of radius 4ε, which means that F is totally bounded.
  - Since $T(F)$ is covered by $\bigcup_{j=1}^{m} B_j$:
    $\Rightarrow$ $\forall f \in F$: $T(f) \in B_j$ for some $j \in \{1, 2, .., m\}$
    $\Rightarrow$ $d(T(f), T(f_j)) < 2\varepsilon$ since they're in the same ball B_j of radius ε
    $\Rightarrow$ $|f(x_i) - f_j(x_i)| \leq d(T(f), T(f_j)) < 2\varepsilon \quad \forall i = 1, 2, .., n \quad ({}^1)$
  - By **(2)**: $\forall x \in X\ \ \exists x_i \in X$: $d(x, x_i) < \delta$
    $\Rightarrow$ due to **(1)**: $\forall f \in F \quad |f(x) - f(x_i)| < \varepsilon \quad ({}^2)$
    Since it holds $\forall f \in F$, in particular it holds for f_j for each $j = 1, 2, .., m$:
    $\Rightarrow$ $|f_j(x) - f_j(x_i)| < \varepsilon \quad \forall i = 1, 2, .., n \quad ({}^3)$
  - Combining $({}^1)$, $({}^2)$ and $({}^3)$ we get:
    $$\begin{aligned} \forall x \in X: \quad |f(x) - f_j(x)| &= |f(x) \pm f(x_j) \pm f_j(x_i) - f_j(x)| \\ &\leq |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)| \\ &< \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon \end{aligned}$$
    $\Rightarrow$ $\forall f \in F\ \ \exists j \in \{1, .., m\}$: $d(f, f_j) < 4\varepsilon$
    $\Rightarrow$ if we consider f_j as the center of the ball $B_{4\varepsilon}(f_j)$ of radius 4ε, we conclude that:
    $\forall f \in F\ \ \exists j \in \{1, .., m\}$: $f \in B_{4\varepsilon}(f_j)$
    $\Rightarrow$ F is covered by m open balls of radius 4ε
    $\Rightarrow$ **F is totally bounded.** ■

$E \downarrow T(F) \downarrow T(F)$ totally bounded $\downarrow F$ totally bounded

1.15. Write the statement of the **Ascoli-Arzelà theorem** when the subset of $C^0(X)$ is a sequence $\{f_n\}_n$.

Theorem. **Ascoli-Arzelà** ***(for sequences)***
Let (X, d) be a compact metric space, $F = \{f_n\}_n \subset C^0(X)$.
If $\{f_n\}_n$ is bounded and equicontinuous then:
$\exists \{f_{n_k}\}_k \subset \{f_n\}_n$ such that: $f_{n_k} \xrightarrow[k\to\infty]{} f$ in $C^0(X)$ for some $f \in C^0(X)$.

(Recall) The sequence $\{f_n\}_n \subset C^0(X)$ is always closed (by definition).
(Recall) The sequence $\{f_n\}_n$ is bounded if $\exists M > 0$: $\forall n \in \mathbb{N}\ \ \sup_{x\in X} |f_n(x)| \leq M$.
(Recall) The thesis is the notion of sequentially compact.

Corollary. **Ascoli-Arzelà corollary**
Let (X, d) be a compact metric space, $C > 0$. Then, the set:
$E := \{f \in C^1([a, b]) : \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)| \leq C\}$
is compact in $C^0([a, b])$.

Corollary. **Ascoli-Arzelà corollary** ***(for sequences)***
Let (X, d) be a compact metric space, $\{f_n\}_n \subset C^1([a, b])$ and assume that $\exists C > 0$ such that:
$\sup_{x\in[a,b]} |f_n(x)| + \sup_{x\in[a,b]} |f_n'(x)| \leq C \quad \forall n \in \mathbb{N}$.
Then:
$\exists \{f_{n_k}\}_k \subset \{f_n\}_n$ such that: $f_{n_k} \xrightarrow[k\to\infty]{} f$ in $C^0(X)$ for some $f \in C^0(X)$.
Namely, $\{f_n\}_n$ is sequentially compact in $C^0([a, b])$.

3 Measure

3.1 σ-Algebra

2.1. Write the definitions of: **algebra, σ-algebra, measurable space, measurable set.** Show that if $\mathcal{A}$ is a σ-algebra and $\{E_k\}_k \subset \mathcal{A}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{A}$.

Let X be a set.
A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be an algebra if:
(i) $\varnothing \in \mathcal{A}$
(ii) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$
(iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
Property (i) and (ii) give that: $\varnothing^c = X \Rightarrow X \in \mathcal{A}$.
Property (ii) and (iii) gives that: $A, B \in \mathcal{A} \Rightarrow A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be an σ-algebra if:
(i) $\varnothing \in \mathcal{A}$
(ii) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$
(iii) $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{A}$

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. Then $(X, \mathcal{A})$ is called measurable space.
The elements of $\mathcal{A}$ are called measurable sets. Hence, a set is measurable if it belongs to a σ-algebra.
Prop. If $\mathcal{A}$ is a σ-algebra and $\{E_k\}_k \subset \mathcal{A} \Rightarrow \bigcap_{k=1}^{\infty} E_k \in \mathcal{A}$
proof.

$\{E_k\}_k \subset \mathcal{A} \Rightarrow E_k^c \in \mathcal{A} \quad \forall k \in \mathbb{N}$ by property (ii)
$\Rightarrow \bigcup_{k=1}^{\infty} E_k^c \in \mathcal{A}$ by property (iii)
$\Rightarrow \left(\bigcup_{k=1}^{\infty} E_k^c\right)^c \in \mathcal{A}$ by property (ii)
$\Rightarrow \left(\bigcup_{k=1}^{\infty} E_k^c\right)^c = \bigcap_{k=1}^{\infty} E_k \in \mathcal{A}$ by De Morgan's law. ■

2.2. State the theorem concerning the existence of the **σ-algebra generated by a given set**. Give an idea of the proof.

Theorem. **Generation of a σ-algebra**
Let X be a set, $S \subseteq \mathcal{P}(X)$. Then there exists a σ-algebra $\sigma_0(S)$ such that:
(i) $S \subseteq \sigma_0(S)$
(ii) $\forall\sigma$-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subset \mathcal{A}$ we have: $\sigma_0(S) \subseteq \mathcal{A}$

This theorem ensures that starting from $\forall S \subseteq \mathcal{P}(X)$ there exists a σ-algebra that contains S and that is contained in each σ-algebra that contains S. In some sense, any other σ-algebra that contains S is bigger (with respect to $\subseteq$) of $\sigma_0(S)$. We can conclude that $\sigma_0(S)$ is the smaller (with respect to $\subseteq$) σ-algebra containing S.

(idea) proof.

We define the set $\nu := \{\mathcal{A} \subseteq \mathcal{P}(X)$ such that $S \subseteq \mathcal{A}$, $\mathcal{A}$ σ-algebgra $)\}$ = all σ-algebras that contain S.
We define $\sigma_0(S) := \bigcap\{\mathcal{A} : \mathcal{A} \in \nu\}$ = intersection of all the σ-algebras containing S.
The intersection of all possible σ-algebras containing S is the smallest σ-algebra containing S.
The proof consists in proving that $\sigma_0(S)$ defined as above is actually a σ-algebra. (■)

In conclusion, $\sigma_0(S)$, which existence is guaranteed by the theorem, is called σ-algebra generated by S.

3.2 Borel Sets

2.3. Write the definition of the **Borel σ-algebra** in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\overline{\mathbb{R}})$ and $\mathbb{R}^N$.

Let (X, d) be a metric space.
In a metric space we have the notion of *open set*, hence we can define:
$\mathcal{G} := \{E \subseteq X : E \text{ is open}\}$.
This set $\mathcal{G}$, defined as the set containing all the open sets, is called topology.
It is possible to apply the theorem of generation of a σ-algebra starting from the set $\mathcal{G}$.
Then, the σ-algebra generated by open sets, namely $\sigma_0(\mathcal{G})$, is called Borel σ-algebra and it is denoted by $\mathcal{B}(X)$.
*In this way, $\sigma_0(\mathcal{G})$ is the intersection of all the σ-algebras which contain **all** the open sets.*
The elements of the Borel σ-algebra are called Borel measurable sets.

The following sets are Borel: (Borel is **not** only these sets)
- open sets
- closed sets
- countable intersections of open sets (G_δ)
- countable union of closed sets (F_σ)

To generate $\mathcal{B}(\mathbb{R})$ we consider all the open sets of $\mathbb{R}$. Then, we generate all the σ-algebras containing all the open sets of $\mathbb{R}$ and we intersect them. The intersection of these σ-algebras is a σ-algebra, in particular it is the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.
However, to generate the Borel σ-algebra it is enough to consider special families of sets:

(i)	$\mathbb{R}$	$\mathcal{B}(\mathbb{R}) = \sigma_0(I_1)$	$I_1 = \{(a,b) : a, b \in \mathbb{R}, a \leq b\}$
		$\mathcal{B}(\mathbb{R}) = \sigma_0(I_2)$	$I_2 = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$
		$\mathcal{B}(\mathbb{R}) = \sigma_0(I_3)$	$I_3 = \{(a,b] : a, b \in \mathbb{R}, a \leq b\}$
		$\mathcal{B}(\mathbb{R}) = \sigma_0(I_4)$	$I_4 = \{(a,b] : -\infty \leq a \leq b < \infty\} \cup \{(a,\infty) : a \in \mathbb{R}\}$
		$\mathcal{B}(\mathbb{R}) = \sigma_0(I_5)$	$I_5 = \{(a,\infty) : a \in \mathbb{R}\}$
(ii)	$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$	$\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\tilde{I}_1)$	$\tilde{I}_1 = \{(a,b) : a, b \in \mathbb{R}, a < b\} \cup \{[-\infty, b) : b \in \mathbb{R}\} \cup \{(a,\infty] : a \in \mathbb{R}\}$
		$\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\tilde{I}_2)$	$\tilde{I}_2 = \{(a,\infty] : a \in \mathbb{R}\}$
(iii)	$\mathbb{R}^N$	$\mathcal{B}(\mathbb{R}^N) = \sigma_0(K_1)$	$K_1 = \{N\text{-dimensional closed rectangles}\}$
		$\mathcal{B}(\mathbb{R}^N) = \sigma_0(K_2)$	$K_2 = \{N\text{-dimensional open rectangles}\}$

3.3 Measure

2.4. Write the definitions of: **measure, finite measure, σ-finite measure, measure space, probability space.** Provide some examples of measures.

Let X be a set, $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\varnothing \in \mathcal{C}$.
A function $\mu : \mathcal{C} \to \overline{\mathbb{R}}_+$ is a measure on $\mathcal{C}$ if:

(i) $\mu(\varnothing) = 0$

(ii) σ-additivity: $\forall \{E_k\}_k \subseteq \mathcal{C}$ disjoint such that $\bigcup_{k=1}^\infty E_k \in \mathcal{C}$, then:
$\mu(\bigcup_{k=1}^\infty E_k) = \sum_{k=1}^\infty \mu(E_k)$

In order to talk about the measure of $\bigcup_{k=1}^\infty E_k$, the requirement $\bigcup_{k=1}^\infty E_k \in \mathcal{C}$ is needed.
If $\mathcal{C}$ is a σ-algebra then $\forall \{E_k\}_k \subseteq \mathcal{C} \Rightarrow \bigcup_{k=1}^\infty E_k \in \mathcal{C}$, hence the requirement is always satisfied.

A measure μ is finite if $\mu(X) < \infty$.
A measure μ is σ-finite if there exists $\{E_k\}_k \subseteq \mathcal{C}$ such that: $X = \bigcup_{k=1}^\infty E_k$ and $\mu(E_k) < \infty \ \forall k \in \mathbb{N}$.

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra, $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure. Then $(X, \mathcal{A}, \mu)$ is called measure space.

- If μ is finite (/σ-finite) the measure space is called finite (/σ-finite).
- If $\mu(X) = 1$ then $(X, \mathcal{A}, \mu)$ is a probability space and μ is a probability measure.

Examples:
- Let X be a set, $\mathcal{C} \subseteq \mathcal{P}(X)$: $\varnothing \in \mathcal{C}$. The function μ: $\mu(\varnothing) := 0$, $\mu(E) := \infty \ \forall E \in \mathcal{C}$, $E \neq \varnothing$, is a measure.
- Let X be a set. The function μ: $\mathcal{P}(X) \to \overline{\mathbb{R}}_+$ such that: $\mu(E) := |E|$ if E is finite, $\mu(E) := \infty$ otherwise, with $|E|$ = number of elements of E, is a measure and it is called counting measure ($\mu^\#$).
- Let $X \neq \varnothing$ be a set, $x_0 \in X$. The function $\delta_{x_0} : \mathcal{P}(X) \to \mathbb{R}_+$ such that: $\delta_{x_0}(E) := 1$ if $x_0 \in E$, $\delta_{x_0}(E) = 0$ otherwise, is a measure and it is called Dirac measure concentrated at x_0.

2.5. State and prove the theorem regarding **properties of measures**.
For what concerns continuity w.r.t. a descending sequence $\{E_k\}_k$, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

Theorem. Properties of measures
Let $(X, \mathcal{A}, \mu)$ be a measure space. Then:

(i)	$\forall \{E_1, .., E_n\} \subseteq \mathcal{A}$ disjoint	:	$\mu(\bigcup_{k=1}^n E_k) = \sum_{k=1}^n \mu(E_k)$	(additivity)
(ii)	$\forall E, F \in \mathcal{A}$	:	$E \subseteq F \Rightarrow \mu(E) \leq \mu(F)$	(monotonicity)
(iii)	$\forall \{E_k\}_k \subseteq \mathcal{A}$	:	$\mu(\bigcup_{k=1}^\infty E_k) \leq \sum_{k=1}^\infty \mu(E_k)$	(σ-subadditivity)
(iv)	$\forall \{E_k\}_k \subseteq \mathcal{A} \nearrow$	:	$\mu(\bigcup_{k=1}^\infty E_k) = \lim_{k\to\infty} \mu(E_k)$	*(continuity of the measure)*
(v)	$\forall \{E_k\}_k \subseteq \mathcal{A} \searrow$, $\mu(E_1) < \infty$	:	$\mu(\bigcap_{k=1}^\infty E_k) = \lim_{k\to\infty} \mu(E_k)$	*(continuity of the measure)*

proof.

(i) Set $E_{n+1} = E_{n+2} = .. = \varnothing$.
(1) $\mu(\bigcup_{k=1}^\infty E_k) = \mu(\bigcup_{k=1}^n E_k \cup \bigcup_{k=n+1}^\infty E_k) = \mu(\bigcup_{k=1}^n \cup \bigcup_{k=n+1}^\infty \varnothing) = \mu(\bigcup_{k=1}^n \cup \varnothing) = \mu(\bigcup_{k=1}^n E_k)$
By property (i) of μ we have that $\mu(\varnothing) = 0$, hence:
(2) $\sum_{k=1}^\infty \mu(E_k) = \sum_{k=1}^n \mu(E_k) + \sum_{k=n+1}^\infty \mu(E_k) = \sum_{k=1}^n \mu(E_k) + \sum_{k=n+1}^\infty \mu(\varnothing) = \sum_{k=1}^n \mu(E_k)$
By property (ii) of the μ: (1) = (2), hence: $\mu(\bigcup_{k=1}^n E_k) = \sum_{k=1}^n \mu(E_k)$.

(ii) Since $E \subseteq F$ we can write: $F := E \cup (F \setminus E) \Rightarrow E \cap (F \setminus E) = \varnothing$. $\quad (*)$
Since in $(*)$ we have two disjoint sets we can apply property (i):
$\mu(F) = \mu(E) + \mu(F \setminus E) \geq (E)$.
The $\geq$ is guaranteed because μ is non-negative.

(iii) Let's define a new family of sets: $F_1 := E_1$, $F_n := E_n \setminus (\bigcup_{k=1}^{n-1} E_k)$ for $n = 2, 3, ...$
Then $\{F_n\}_n \subseteq \mathcal{A}$ is disjoint sequence of sets, $F_k \subseteq E_k \ \forall k \in \mathbb{N}$ and it holds: $\bigcup_{k=1}^\infty F_k = \bigcup_{k=1}^\infty E_k$.
Therefore, by σ-additivity ($\{F_k\}_k$ is disjoint) and by monotonicity ($F_k \subseteq E_k \ \forall k \in \mathbb{N}$):
$\mu(\bigcup_{k=1}^\infty E_k) = \mu(\bigcup_{k=1}^\infty F_k) = \sum_{k=1}^\infty \mu(F_k) \leq \sum_{k=1}^\infty \mu(E_k)$.

(iv) Let's define: $E_0 := \varnothing$, $F_k := E_k \setminus E_{k-1}$.
Then $\bigcup_{k=1}^n F_k = E_n$ and $\bigcup_{k=1}^\infty F_k = \bigcup_{k=1}^\infty E_k$.
Since $\{F_k\}_k$ are disjoint and by (i):
$\mu(\bigcup_{k=1}^\infty E_k) = \mu(\bigcup_{k=1}^\infty F_k) = \sum_{k=1}^\infty \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu(F_k) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^n F_k) = \lim_{n\to\infty} \mu(E_n)$.

(v) Let's define: $F_k := E_1 \setminus E_k$.
Then $\{F_k\}_k \nearrow$ and so, we can apply property (iv):
$\mu(\bigcup_{k=1}^\infty F_k) = \lim_{k\to\infty} \mu(F_k) = \mu(E_1) - \lim_{k\to\infty} \mu(E_k)$. $\quad$ (1)
The last equivalence is possible only because $\mu(E_1) < \infty$.
On the other hand:
$\bigcup_{k=1}^\infty F_k = \bigcup_{k=1}^\infty (E_1 \cap E_k^c) = E_1 \cap (\bigcup_{k=1}^\infty E_k^c) = E_1 \setminus (\bigcap_{k=1}^\infty E_k)$
$\Rightarrow \mu(\bigcup_{k=1}^\infty F_k) = \mu(E_1) - \mu(\bigcap_{k=1}^\infty E_k)$. $\quad$ (2)
By combining (1) and (2) we obtain:
$\lim_{k\to\infty} \mu(E_k) = \mu(\bigcap_{k=1}^\infty E_k)$. ∎

The hypothesis $\mu(E_1) < \infty$ is essential. In fact, if $\mu(E_1) = \infty$ then (v) fails.
Let's consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$, $\mu^\#$ counting measure, and:
$E_n := \{k \in \mathbb{N} : k \geq n\} \quad \forall n \in \mathbb{N}. \quad (\{E_n\}_n \searrow)$
Then:
(1) $\bigcap_{n=1}^\infty E_n = \varnothing \Rightarrow \mu^\#(\bigcap_{n=1}^\infty E_n) = \mu^\#(\varnothing) = 0$.
(2) $\mu^\#(E_n) = +\infty \ \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu^\#(E_n) = \infty$.
By combining (1) and (2) we obtain:
$\mu^\#(\bigcap_{n=1}^\infty E_n) \neq \lim_{n\to\infty} \mu^\#(E_n)$.
In particular, we underly that $\mu(E_1) = \infty$.

Lemma. **Borel-Cantelli**
Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{E_n\}_n \subseteq \mathcal{A}$.
If $\sum_{n=1}^{\infty} \mu(E_n) < \infty \Rightarrow \mu(\limsup_{n\to\infty} E_n) = \mu(\bigcap_{k=1}^{\infty} [\bigcup_{n=k}^{\infty} E_n]) = 0$.

proof.

Let's define: $F_k := \bigcup_{n=k}^{\infty} E_n \quad \forall k \in \mathbb{N} \Rightarrow \{F_k\}_k \searrow$.
Because of σ-subadditivity ($\{E_k\}_k \subseteq \mathcal{A}$: $\mu(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \mu(E_k)$)
$\mu(F_k) \leq \sum_{n=k}^{\infty} \mu(E_n)$. (1)
In particular:
$\mu(F_1) = \mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_k) < \infty$.
Because of (1) and because of the point (v) of the properties of measures:
$\mu(\limsup_n E_n) = \mu(\bigcap_{k=1}^{\infty} F_k) = \lim_k \mu(F_k) \leq \lim_k(\sum_{n=k}^{\infty} \mu(E_n)) = 0$ since $k \to \infty$. ■

3.4 Sets of Zero Measure

2.6. Write the definitions of: **sets of zero measure, negligible sets.**
What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

Let $(X, \mathcal{A}, \mu)$ be a measure space.
The set $N \subseteq X$ is said to be a set of zero measure if $N \in \mathcal{A}$ and $\mu(N) = 0$.
The set $E \subseteq X$ is said to be negligible if $\exists N \in \mathcal{A}$ such that $E \subseteq N$ and $\mu(N) = 0$.
We call $\mathcal{N}_\mu$ the collection of sets of zero measure.
We call $\mathcal{T}_\mu$ the collection of negligible sets.
A property P on X is said to be true almost everywhere (a.e.) if:
$\{x \in X : P(x)$ is false $\} \in \mathcal{N}_\mu$
namely, if the set where the propery is false is measurable and its measure is zero.
Examples:
- $f, g : X \to \overline{\mathbb{R}}$ are equal a.e. if: $\{x \in X : f(x) \neq g(x)\} \in \mathcal{N}_\mu$.
- $f : X \to \overline{\mathbb{R}}$ is finite a.e. if: $\{x \in X : f(x) = \pm\infty\} \in \mathcal{N}_\mu$.
- $D \in \mathcal{A}$, $f : D \to \overline{\mathbb{R}}$ is defined a.e. in X if: $D^c \in \mathcal{N}_\mu$.

2.7. Write the definition of **complete measure space**. State the theorem concerning the **existence of the completion of a measure space**. Give just an idea of the proof.

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if every negligible set is also measurable, namely $\mathcal{T}_\mu \subseteq \mathcal{A}$.
In such case, μ is said to be a complete measure and $\mathcal{A}$ is said to be a complete σ-algebra.
Furthermore, it holds that: $(X, \mathcal{A}, \mu)$ is complete $\Leftrightarrow \mathcal{N}_\mu = \mathcal{T}_\mu$.
Thus, a complete measure space is such that any negligible set is measurable with zero measure. — any subset of a zero-measure set is measurable with measure zero

Theorem. **Completion of a measure space**
Let $(X, \mathcal{A}, \mu)$ be a measure space. We define:
- $\overline{\mathcal{A}} := \{E \subseteq X : \exists F, G \in \mathcal{A}$ such that $F \subseteq E \subseteq G,\ \mu(G \setminus F) = 0\}$
- $\overline{\mu} : \overline{\mathcal{A}} \to \overline{\mathbb{R}}_+$ such that: $\overline{\mu}(E) := \mu(F)$

Then:
(i) $\overline{\mathcal{A}}$ is a σ-algebra which contains $\mathcal{A}$
(ii) $\overline{\mu}$ is a complete measure, $\overline{\mu}|_{\mathcal{A}} = \mu$, so $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space, more precisely it is the smallest (w.r.t. inclusion) complete measure space which contains $(X, \mathcal{A}, \mu)$.

(idea) proof.

It is straightforward to prove that $\mathcal{A} \subseteq \overline{\mathcal{A}}$. Indeed, if $E \in \mathcal{A}$ then we can choose $F = G = E$ and so $E \in \overline{\mathcal{A}}$.
Because of this, if $E \in \mathcal{A}$ it holds $\overline{\mu}(E) = \mu(E)$ and so we can conclude $\overline{\mu}|_{\mathcal{A}} = \mu$.

The overall idea is that we can construct a **complete** measure space out of **any** general measure space $(X, \mathcal{A}, \mu)$.
To do so, we need to enlarge the original σ-algebra $\mathcal{A}$ and to extend the original measure μ.
The sets that we need to add to the original σ-algebra $\mathcal{A}$ are the negligible sets and the extension of the original measure μ has to be such that the added negligible sets have zero-measure. Proceeding like this, we obtain $(X, \overline{\mathcal{A}}, \overline{\mu})$ which will be complete since all the negligible sets will belong to $\overline{\mathcal{A}}$ and will have zero measure through $\overline{\mu}$. (■)

3.5 Outer Measure

2.8. Write the definition of **outer measure**. State and prove the theorem concerning **generation of outer measure on a general set X**, starting from a set $K \in \mathcal{P}(X)$, containing $\varnothing$, and a function $\nu : K \to \overline{\mathbb{R}}_+$, $\nu(\varnothing) = 0$.
Intuitively, which is the meaning of (K, ν)?

Let X be a set.
A functin $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ is said to be an outer measure on X if:
(i) $\mu^*(\varnothing) = 0$
(ii) monotonicity: $\forall E_1, E_2 \in \mathcal{P}(X)$: $E_1 \subseteq E_2 \Rightarrow \mu^*(E_1) \leq \mu^*(E_2)$
(iii) σ-subadditivity: $\forall \{E_n\}_n \subseteq \mathcal{P}(X)$: $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

The outer measure is defined on $\mathcal{P}(X)$. In particular, since all the properties of the outer measure are valid for a measure, if μ is a measure on $\mathcal{P}(X)$ then μ is an outer measure.

Theorem. **Generation of outer measure**
Let X be a set, $K \in \mathcal{P}(X)$ be a set such that $\varnothing \in K$, $\nu : K \to \overline{\mathbb{R}}_+$ be a function such that $\nu(\varnothing) = 0$.
We define $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ such that:
- $\mu^*(E) := \inf_{\{I_n\}_n \subseteq K:\, E \subseteq \bigcup_{n=1}^{\infty} I_n} \{\sum_{n=1}^{\infty} \nu(I_n)\}$
  if $E \subseteq X$ can be covered by a countable union of sets $I_n \in K$
- $\mu^*(E) := +\infty$ otherwise

Then μ^* is an outer measure on X.

proof.

We have to prove that the 3 properties of the outer measure are fulfilled.

(i) As a general remark: if $I \in K$ then $\mu^*(I) \leq \nu(I)$, since μ^* is defined as the infimum.
Let's consider $\varnothing \in K$. Because of the above remark and since μ^* is non-negative:
$\Rightarrow \mu^*(\varnothing) \leq \nu(\varnothing) = 0 \Rightarrow \boldsymbol{\mu^*(\varnothing) = 0}$.

(ii) Consider $\boldsymbol{E_1 \subseteq E_2}$. If there exists a countable cover of E_2 then it is also a countable cover of E_1.
Since both E_1 and E_2 have a countable cover, both $\mu^*(E_1)$ and $\mu^*(E_2)$ are defined with the infimum.
but since $E_1 \subseteq E_2$ then, from the very definition of μ^* it follows:
$\boldsymbol{\mu^*(E_1) \leq \mu^*(E_2)}$.
If E_2 does not have a countable cover, then:
$\boldsymbol{\mu^*(E_1) \leq \mu^*(E_2)} = \infty$
hence, whatever $\mu^*(E_1)$ is, it holds $\mu^*(E_1) < \infty$.

(iii) Let $\{E_n\}_n \subseteq \mathcal{P}(X)$.
If $\sum_{n=1}^{\infty} \mu^*(E_n) = \infty$ then it is obious.
On the contrary, if: $\sum_{n=1}^{\infty} \mu^*(E_n) < \infty \Rightarrow \mu^*(E_n) < \infty \quad \forall n \in \mathbb{N}$.
By the definition of μ^*:
$\forall n \in \mathbb{N} \ \exists \{I_{n_k}\}_k \subseteq K$ such that: $E_n \subseteq \bigcup_{k=1}^{\infty} I_{n_k}$
By the definition of infimum, $\forall \varepsilon > 0$:
$\mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{\infty} \nu(I_{n_k})$
Since $\{I_{n_k}\}_k \subseteq K \ \forall n \in \mathbb{N}$:
$\Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n_k}$
$\Rightarrow \mu^*(\bigcup_{k=1}^{\infty} E_n) \leq \sum_{n,k=1}^{\infty} \nu(I_{n_k})$, since it is the infimum
$< \sum_{n=1}^{\infty} (\mu^*(E_n) + \frac{\varepsilon}{2^n})$
$= \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$.
Since $\varepsilon > 0$ is arbitrary:
$\boldsymbol{\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)}$. ■

Intuitively, we see the function ν as an *elementary measure* and $K \in \mathcal{P}(X)$ as a set made of *special sets* on which this elementary measure can be applied. Starting from these two objects we can define an outer measure on every set of $\mathcal{P}(X)$.

Consider $E \in \mathcal{P}(X)$. We try to cover E with a countable union of sets belonging to K, namely $I_n \in K \ \forall n \in \mathbb{N}$. If this is possible, we collect all the possible cover of E, we apply ν on every cover and we set the outer measure of E, namely $\mu^*(E)$, as the infimum of these values. If we cannot cover E with a countable union of sets belonging to K then we set $\mu^*(E) = \infty$. Proceeding like this, the resulting function μ^* is an outer measure defined on $\mathcal{P}(X)$.

2.9. What is the **Caratheodory condition**? How can it be stated in an equivalent way? Prove it.

Let X be a set, μ^* an outer measure on X.
A set $E \subseteq X$ satisfies the Caratheodory condition if: $\quad \mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \quad \forall Z \subseteq X$.
The Caratheodory condition is equivalent to say: $\quad \mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \quad \forall Z \subseteq X$.

proof.

It is enough to show that $\forall E \subseteq X$:
$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \quad \forall Z \subseteq X$. — if we show that this is always satisfied then the equality is equivalent to "≥"
Since $\forall E \subseteq X$ we can write $X = E \cup E^c$:
$Z = Z \cap X = Z \cap (E \cup E^c) = (Z \cap E) \cup (Z \cap E^c) \quad \forall Z \subseteq X$
By σ-subadditivity of the outer measure we get:
$\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \quad \forall Z \subseteq X$ ■

A set $E \subseteq X$ which satisfies the Caratheodory condition is said to be μ^*-measurable.

2.10. Can it exist a set of zero outer measure, which does not fulfill the Caratheodory condition? Prove it.

It cannot exist a set of zero outer measure which does not fulfill the Caratheodory condition.
Let X be a set, μ^* an outer measure on X, $E \subseteq X$.
Lemma. If $\mu^*(E) = 0$ then E is μ^*-measurable. ($\Leftrightarrow E$ satisfies the Caratheodory condition)
proof.

By monotonicity of μ^*:
$\mu^*(Z \cap E) + \mu^*(Z \cap E^c) \leq \mu^*(E) + \mu^*(Z) = \mu^*(Z) \quad \forall Z \subseteq X$
since $Z \cap E \subseteq E$, $Z \cap E^c \subseteq Z$ and $\mu^*(E) = 0$.
So, the Caratheodory condition is satisfied. ■

3.6 Generation of Measure

2.11. State the theorem concerning **generation of a measure as a restriction of an outer measure.**

Theorem. Let X be a set. Let μ^* be an outer measure on X. Then:
(i) the collection $\mathcal{L} := \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$ is a σ-algebra
equivalent formulation: $\mathcal{L} := \{E \subseteq X : E \textit{ satisfies Caratheodory condition}\}$
(ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$.

This theorem states that an outer measure combined with the Caratheodory condition allows us to construct a measure. Starting from an outer measure μ^*, we consider the family of sets that satisfy the Caratheodory condition. This family, $\mathcal{L}$, is a σ-algebra. The outer measure restricted to this σ-algebra, $\mu = \mu^*|_{\mathcal{L}}$, is a complete measure.
In conclusion, $(X, \mathcal{L}, \mu)$ is a complete measure space.

X set $\rightarrow (K, \nu) \rightarrow$ μ^* outer measure, Caratheodory condition $\rightarrow (X, \mathcal{L}, \mu)$ complete meas. space
$\mathcal{L} = \{E \subseteq X : E \text{ sat. caratheodory}\}$
$\mu = \mu^*|_{\mathcal{L}}$

3.1. Show that the measure induced by an outer measure on the σ-algebra of all sets fulfilling the Caratheodory condition is complete.

Let X be a set and let μ^* be an outer measure on X.
We define $\mathcal{L} := \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\} = \{E \subseteq X : E \text{ satisfies Caratheodory condition}\}$, which is a σ-algebra.
Then, the measure induced by μ^* on $\mathcal{L}$, namely $\mu = \mu^*|_{\mathcal{L}}$, is complete.

proof.

Consider $N \in \mathcal{L}$ such that $\mu^*(N) = \mu(N) = 0$. (since $N \in \mathcal{L}$ and $\mu = \mu^*|_{\mathcal{L}}$)
Consider $E \subset N$:
μ^* monotone $\Rightarrow 0 \leq \mu^*(E) \leq \mu^*(N) = 0$
$\Rightarrow \mu^*(E) = 0$
$\Rightarrow$ by the lemma which states that if $\mu^*(E) = 0$ then E is μ^*-measurable:
$\Rightarrow$ E is μ^*-measurable, namely E satisfies Caratheodory condition
$\Leftrightarrow$ $E \in \mathcal{L}$, hence we can write $\mu(E)$, which by definition is $\mu(E) = \mu^*(E) = 0$.
We showed that every subset of a zero-measure set is measurable and of zero-measure:
$\Rightarrow \mu = \mu^*|_{\mathcal{L}}$ is complete. ■

3.7 Lebesgue Measure

3.2. Describe the **construction of the Lebesgue measure** in $\mathbb{R}$ and in $\mathbb{R}^n$.

Let X be a set.

Construction of an outer measure μ^* starting from (K, ν).
We define (K, ν) from which we can construct an outer measure μ^* on every set of $\mathcal{P}(X)$.
We see the function ν as an *elementary measure* and $K \subseteq \mathcal{P}(X)$ as a set made of *special sets* on which this elementary measure can be applied. Consider $E \in \mathcal{P}(X)$. We try to cover E with a countable union of sets belonging to K, namely $I_n \in K$ $\forall n \in \mathbb{N}$. If this is possible, we collect all the possible cover of E, we apply ν on every cover and we set the outer measure of E, namely $\mu^*(E)$, as the infimum of these values. If we cannot cover E with a countable union of sets belonging to K then we set $\mu^*(E) = \infty$. Proceeding like this, the resulting function μ^* is an outer measure defined on $\mathcal{P}(X)$.

Construction of a measure μ starting from an outer measure μ^*.
We define $\mathcal{L}$ as the collection of sets that satisfy the Caratheodory condition. $\mathcal{L}$ is a σ-algebra. We restrict the outer measure μ^* on the σ-algebra $\mathcal{L}$, obtaining a complete measure $\mu = \mu^*|_{\mathcal{L}}$. Thus, $(X, \mathcal{L}, \mu)$ is a complete measure space.

Construction of the Lebesgue measure on $\mathbb{R}$.

1. **Context.**
   Consider an interval $I \subseteq \mathbb{R}$: $I = (a, b)$ with $a, b \in \mathbb{R}, a \leq b$.
   The length of such interval is: $\ell(I) = b - a$, if the interval is bounded, $\ell(I) = +\infty$ otherwise.
   Starting from this, we want to define the measure of a general set (not only for intervals).
   The goal is to construct a specific σ-algebra, whose elements will be called Lebesgue-measurable sets, and a specific measure λ, called Lebesgue measure. In particular, we want λ to satisfy two properties:
   (i) $\lambda(I) = \ell(I)$ $\qquad I \subseteq \mathbb{R}$ interval
   (ii) $\lambda(E + x_0) = \lambda(E)$ $\qquad x_0 \in \mathbb{R},\ E \subseteq \mathbb{R},\ E + x_0 := \{x + x_0 : x \in E\}$
   If so, the resulting measure λ will be the natural extension of the length of an interval.
2. **Construction.**
   Consider $\mathcal{I} = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$, $\varnothing \in \mathcal{I}$.
   We define a function $\ell : \mathcal{I} \to \mathbb{R}_+$:
   - $\ell(\varnothing) := 0$
   - $\ell((a, b]) := b - a$

   We choose $X = \mathbb{R}$, $(K, \nu) = (\mathcal{I}, \ell)$ and, by means of the previous construction, we get the Lebesgue measure.
   More precisely, with $(\mathcal{I}, \ell)$ we obtain an outer measure λ^* on $\mathbb{R}$ such that:
   - $\lambda^*(E) := \inf_{\{I_n\}_n \subseteq \mathcal{I}:\ E \subseteq \bigcup_{n=1}^{\infty} I_n} \left\{ \sum_{n=1}^{\infty} \ell(I_n) \right\}$
     if $E \subseteq X$ can be covered by a countable union of sets $I_n \in \mathcal{I}$
   - $\lambda^*(E) := +\infty$ otherwise

   Notice that, by definition, if $I \subseteq \mathbb{R}$ is an interval then $\lambda^*(I) = \ell(I)$.
   Then, combining λ^* and the Caratheodory condition, we get the σ-algebra $\mathcal{L}$.
   The outer measure λ^* restricted on this σ-algebra $\mathcal{L}$ defines the Lebesgue measure on $\mathbb{R}$.
3. **Definition.**
   The outer measure λ^* generated by $(\mathcal{I}, \ell)$ is called Lebesgue outer measure on $\mathbb{R}$.
   The λ^*-measurable sets are called Lebesgue measurable sets.
   The corresponding σ-algebra $\mathcal{L}$ is called Lebesgue σ-algebra and it is denoted by $\mathcal{L}(\mathbb{R})$.
   The measure $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$ is called Lebesgue measure on $\mathbb{R}$.
   $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space.

Construction of the Lebesgue measure on $\mathbb{R}^N$.
Consider $\mathcal{I}^N := \{\prod_{k=1}^{N}(a_k, b_k] : a_k, b_k \in \mathbb{R}, a_k \leq b_k\ \forall k = 1, .., N\}$.
We define a function $\ell^N : \mathcal{I}^N \to \mathbb{R}_+$:

- $\ell^N(\varnothing) := 0$
- $\ell^N(\prod_{k=1}^{N}(a_k, b_k]) := \prod_{k=1}^{N}(b_k - a_k)$

We choose $X = \mathbb{R}^N$, $(K, \nu) = (\mathcal{I}^N, \ell^N)$ and we repeat the same construction as before.
With $(\mathcal{I}^N, \ell^N)$ we obtain an outer measure $\lambda^{*,N}$ on $\mathbb{R}^N$.
Combining the outer measure $\lambda^{*,N}$ with the Caratheodory condition, we get the σ-algebra $\mathcal{L}(\mathbb{R}^N)$.
The outer measure $\lambda^{*,N}$ restricted on $\mathcal{L}(\mathbb{R}^N)$ define the Lebesgue measure on $\mathbb{R}^N$, namely $\lambda^N := \lambda^{*,N}|_{\mathcal{L}(\mathbb{R})}$.
Thus, $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), \lambda^N)$ is a complete measure space, with λ^N which is the N-dimensional Lebesgue measure.

4.1. Are there disjoint subsets $A, B \subset \mathbb{R}$ such that $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$? Justify the answer.

Consider the Lebesgue outer measure λ^* on $\mathbb{R}$. There are disjoint sets $A, B \subset \mathbb{R}$ such that:
$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$.

proof.

Assume by contraddiction that:
$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B) \quad \forall A, B \subset \mathbb{R}$ disjoint $\quad (*)$
Then, every set satisfies the Caratheodory condition.
In fact $\forall E \subset \mathbb{R}$, $\forall Z \subseteq \mathbb{R}$ we can choose $A = E \cap Z$, $B = E^c \cap Z$, so that $A \cup B = Z$:
$\Rightarrow \lambda^*(E \cap Z) + \lambda^*(E^c \cap Z) = \lambda^*(A) + \lambda^*(B) = \lambda^*(A \cup B) = \lambda^*(Z)$
$\Rightarrow \forall E \subset \mathbb{R}$ is measurable.
This is a contraddiction because there exist non-measurable sets, for example the Vitali set. (3.7.)
Hence, there exist A, B such that in $(*)$ holds the strict inequality. ■

3.3. Prove that any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

Consider $(X, \mathcal{L}(\mathbb{R}), \lambda)$. Any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

proof.

- Let $a \in \mathbb{R}$.
  $\Rightarrow \{a\} \subseteq (a - \varepsilon, a] \quad \forall \varepsilon > 0$
  $\Rightarrow \lambda^*((a - \varepsilon, a]) = \varepsilon \quad \forall \varepsilon > 0$, by definition of λ^*
  $\Rightarrow \lambda^*(\{a\}) \leq \varepsilon \quad \forall \varepsilon > 0$, since λ^* is monotone
  $\Rightarrow \lambda^*(\{a\}) = 0$
  $\Rightarrow \{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$.
- Since E is countable $E = \bigcup_{n=1}^{\infty} \{a_n\}$, and since λ^* is σ-subadditive:
  $\Rightarrow \lambda^*(E) = \lambda^*(\bigcup_{n=1}^{\infty} \{a_n\}) \leq \sum_{n=1}^{\infty} \lambda^*(\{a_n\}) = 0$, since $\lambda^*(\{a_n\}) = 0$
  $\Rightarrow \lambda^*(E) = 0$, since λ^* is non-negative
  $\Rightarrow E \in \mathcal{L}(\mathbb{R})$ and $\lambda(E) = 0$. ■

3.4. Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.

proof.

Since $\mathcal{B}(\mathbb{R}) = \sigma_0((a, +\infty))$, with $a \in \mathbb{R}$, it is enough to show that $(a, +\infty) \in \mathcal{L}(\mathbb{R})$.

- Let $A \subseteq \mathbb{R}$ be any set.
  We assume that $a \notin A$, otherwise we replace A by $A \setminus \{a\}$, since $\lambda^*(A)$ would remain unchanged.
- We have to show that the Caratheodory condition holds for $E = (a, +\infty)$, namely:
  $\lambda^*(A \cap E^c) + \lambda^*(A \cap E) := \lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A) \quad (1)$
  where $A_1 := A \cap (-\infty, a]$, $A_2 := A \cap (a, +\infty)$.
- Since $\lambda^*(A)$ is defined as an infimum, to verify (1) it is necessary and sufficient to show that for any countable collection $\{I_k\}_k$ of open bounded intervals that cover A:
  $\lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k)$.
  In fact, if the inequality holds for any countable collection $\{I_k\}_k$, then it will also hold for the infimum.
- For each $k \in \mathbb{N}$ we define:
  $I'_k := I_k \cap (-\infty, a]$
  $I''_k := I_k \cap (a, +\infty)$.
  Then, I'_k and I''_k are disjoint intervals and:
  $\ell(I_k) = \ell(I'_k) + \ell(I''_k)$.
- Furthermore, $\{I'_k\}_k$ is a countable cover of A_1 and $\{I''_k\}_k$ is a countable cover of A_2.
  Hence, by definition of outer measure:
  $\lambda^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I'_k)$
  $\lambda^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I''_k)$.
  Therefore:
  $$\begin{aligned}\lambda^*(A_1) + \lambda^*(A_2) &\leq \sum_{k=1}^{\infty} \ell(I'_K) + \sum_{k=1}^{\infty} \ell(I''_k) \\ &= \sum_{k=1}^{\infty} [\ell(I'_k) + \ell(I''_k)] \\ &= \sum_{k=1}^{\infty} \ell(I_k)\end{aligned}$$ ■

Theorem. $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$.

3.5. Prove that the translate of a measurable set is measurable.

Consider $(X, \mathcal{L}(\mathbb{R}), \lambda)$. The translate of a measurable set is measurable.

proof.

Let E be a measurable set.
Since λ^* is defined by means of length of inverval and the length of interval is translation invariant:
$\Rightarrow \lambda^*$ is translation invariant.
Hence, $\forall A \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}$:
$$\begin{aligned}\lambda^*(A) &= \lambda^*(A - x_0) \\ &= \lambda^*((A - x_0) \cap E) + \lambda^*((A - x_0) \cap E^c), \text{ Caratheodory condition for } E \in \mathcal{L}(\mathbb{R}) \\ &= \lambda^*(A \cap (E + x_0)) + \lambda^*(A \cap (E + x_0)^c)\end{aligned}$$
$\Rightarrow (E + x_0) \in \mathcal{L}(\mathbb{R})$. ■

3.6. Write the **excision property** and prove it.

Lemma. **Excission property**
Consider $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and the Lebesgue outer measure λ^*.
If $A \in \mathcal{L}(\mathbb{R})$ is such that $\lambda^*(A) < \infty$ and $A \subseteq B$, then:
$\lambda^*(B \setminus A) = \lambda^*(B) - \lambda^*(A)$.

proof.

Since $A \in \mathcal{L}(\mathbb{R})$ and since $A \subseteq B$:
$\Rightarrow \lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \cap A^c) = \lambda^*(A) + \lambda^*(B \setminus A)$
$\Rightarrow \lambda^*(B) - \lambda^*(A) = \lambda^*(B \setminus A)$. ■

Theorem. **Regularity of Lebesgue measure** *(approximation of a Lebesgue set with Borel sets)*
Let $E \subseteq \mathbb{R}$. The following statements are equivalent:

(i)	$E \in \mathcal{L}(\mathbb{R})$				
(ii)	$\forall \varepsilon > 0\ \exists A \subseteq \mathbb{R}$ open	s.t.	$E \subseteq A$:	$\lambda^*(A \setminus E) < \varepsilon$	(outer approx.)
(iii)	$\exists G \subseteq \mathbb{R}$ of class $G_\delta (= \bigcap_\infty$ open)	s.t.	$E \subseteq G$:	$\lambda^*(G \setminus E) = 0$	(outer approx.)
(iv)	$\forall \varepsilon > 0\ \exists C \subseteq \mathbb{R}$ closed	s.t.	$C \subseteq E$:	$\lambda^*(E \setminus C) < \varepsilon$	(inner approx.)
(v)	$\exists F \subseteq \mathbb{R}$ of class $F_\sigma (= \bigcup_\infty$ closed)	s.t.	$F \subseteq E$:	$\lambda^*(E \setminus F) = 0$	(inner approx.)

proof.

from property (v) we conclude that
$\forall E \in \mathcal{L}(\mathbb{R})\ \exists F \subseteq \mathbb{R}$
F F_σ set, such that
$F \subseteq E$ and $\lambda^*(E \setminus F) = 0$
$\Longrightarrow E = F \cup (E \setminus F)$
$\Longrightarrow$ every Lebesgue set can be written as the union of a Borel set and a zero λ-measure set. On account of this decomposition, it is not surprising that $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$

We give a simplified partial proof: (i) $\overset{1.}{\Rightarrow}$ (ii), (ii) $\overset{2.}{\Rightarrow}$ (iii), (iii) $\overset{3.}{\Rightarrow}$ (i), with $E \in \mathcal{L}(\mathbb{R})$, $\lambda(E) < \infty$.

1. By the definition of outer measure, $\exists \{I_k\}_k$ which covers E for which: ($\{I_k\}_k$ are of the form $(a,b]$ and $\bigcup_n (a_n, b_n]$ is open)
   $\sum_{k=1}^{\infty} \ell(I_k) < \lambda^*(E) + \varepsilon$.
   Let $O := \bigcup_{k=1}^{\infty} I_k$. Then O is open and $E \subseteq O$.
   By σ-subadditivity:
   $\Rightarrow \lambda^*(O) \leq \sum_{k=1}^{\infty} \ell(I_k) < \lambda^*(E) + \varepsilon$
   $\Rightarrow \lambda^*(O) - \lambda^*(E) < \varepsilon$.
   By assumption $E \in \mathcal{L}(\mathbb{R})$ and $\lambda^*(E) < \infty$, so, by excision property:
   $\lambda^*(O \setminus E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$.
2. $\forall k \in \mathbb{N}$ we can find $O_k \supseteq E$ open for which:
   $\lambda^*(O_k \setminus E) < \frac{1}{k}$.
   Define $G := \bigcap_{k=1}^{\infty} O_k$. Then G is a G_δ set and $G \supseteq E$.
   Moreover, $\forall k \in \mathbb{N}$: $G \setminus E \subseteq O_k \setminus E$.
   Since λ^* is monotone:
   $\Rightarrow \lambda^*(G \setminus E) \leq \lambda^*(O_k \setminus E) < \frac{1}{k}$
   $\Rightarrow \lambda^*(G \setminus E) = 0$.
3. Because of a property of the outer measure, $\lambda^*(G \setminus E) = 0 \Rightarrow G \setminus E \in \mathcal{L}(\mathbb{R})$.
   $\Rightarrow (G \setminus E)^c \in \mathcal{L}(\mathbb{R})$
   Moreover, $G \in \mathcal{L}(\mathbb{R})$ since $G \in G_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.
   $\Rightarrow E = G \cap (G \setminus E)^c \in \mathcal{L}(\mathbb{R})$. ■

3.8 Non Measurable Sets

3.7. Define the **Vitali set**. Prove that any measurable bounded set $E \subset \mathbb{R}$ with $\lambda(E) > 0$ contains a subset which is not Lebesgue measurable.

Consider $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$.

Lemma. Let $E \subseteq \mathbb{R}$ be measurable and bounded. Suppose that there is a bounded and countably infinite set $\Sigma \subseteq \mathbb{R}$ for which $\{\sigma + E\}_{\sigma \in \Sigma}$ is a disjoint family. Then $\lambda(E) = 0$.

proof.

Since the translation of a measurable set is meausrable, then $\sigma + E$ is measurable $\forall \sigma \in \Sigma$.
Since $\{\sigma + E\}_{\sigma \in \Sigma}$ is a disjoint family, by the σ-additivity of λ:
$\Rightarrow \lambda(\bigcup_{\sigma \in \Sigma} (\sigma + E)) = \sum_{\sigma \in \Sigma} \lambda(\sigma + E) = \sum_{\sigma \in \Sigma} \lambda(E)$ (1)
Since E and Σ are bounded:
$\Rightarrow \bigcup_{\sigma \in \Sigma} (\sigma + E)$ is bounded
$\Rightarrow \lambda(\bigcup_{\sigma \in \Sigma} (\sigma + E)) < \infty$ (2)
Combining (1) and (2):
$\Rightarrow \lambda(\bigcup_{\sigma \in \Sigma} (\sigma + E)) = \sum_{\sigma \in \Sigma} \lambda(E) < \infty$
$\Rightarrow \sum_{\sigma \in \Sigma} \lambda(E) < \infty$
But, since Σ is countably infinite, if $\lambda(E) > 0$ then $\sum_{\sigma \in \Sigma} \lambda(E) = \infty$
$\Rightarrow \lambda(E) = 0$. ■

Construction of the Vitali set.
Consider $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and $x, y \in E$. We define the relation $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$, which is an equivalence relation. Given an equivalence relation, the equivalence classes form a partition of the set on which the relation is defined. Thus, there is a disjoint decomposition of E into the collection of equivalence classes. By the axiom of choice, we can construct a set $V \subseteq E$, called Vitali set, containing exactly one member of each equivalence class. The Vitali set V is such that:
(i) the difference of two points of V is **not** rational, otherwise the two points would be in the same equivalence class ($\Leftrightarrow \forall \Sigma \subseteq \mathbb{Q} : \{\sigma + V\}_{\sigma \in \Sigma}$ is a disjoint family)
(ii) $\forall x \in E\ \ \exists c \in V$ for which: $x = c + q$ with $q \in \mathbb{Q}$

Theorem. **Vitali**
Any measurable bounded set $E \subset \mathbb{R}$ with $\lambda(E) > 0$ contains a subset which is not Lebesgue measurable.
Namely, $\forall E \in \mathcal{L}(\mathbb{R})$ bounded s.t. $\lambda(E) > 0\ \exists F \subset E$ s.t. $F \notin \mathcal{L}(\mathbb{R})$. Thus, $\mathcal{L}(\mathbb{R}) \subsetneq \mathcal{P}(\mathbb{R})$.

proof.

Consider the Vitali set V.
Since we can construct V starting from $\forall E \subset \mathbb{R}$ with $\lambda(E) > 0$, the thesis follows if we prove that $V \notin \mathcal{L}(\mathbb{R})$.

- Let $\Sigma \subset \mathbb{Q}$ be any bounded countably infinite set.
- Suppose by contraddiction that V is Lebesgue measurable.
  By property (i) of V and by the previous lemma:
  $\Rightarrow \lambda(V) = 0$.
  Moreover:
  $\lambda(\bigcup_{\sigma \in \Sigma}(\sigma + V)) = \sum_{\sigma \in \Sigma} \lambda(\sigma + V) = \sum_{\sigma \in \Sigma} \lambda(V) = 0$.
- Since E is bounded:
  $\Rightarrow \exists b > 0$ such that $E \subset [-b, b]$.
  Thus, we choose $\Sigma := [-2b, 2b] \cap \mathbb{Q}$. In this way, Σ is bounded and countably infinite.
- *Claim*:
  $E \subseteq \bigcup_{\sigma \in \Sigma}(\sigma + V) \quad (*)$
  In fact, by the property (ii) of V:
  $\Rightarrow$ if $x \in E$ then $\exists c \in V$ for which: $x = c + q$ with $q \in \mathbb{Q}$
  $\Rightarrow x, c \in [-b, b]$
  $\Rightarrow q = x - c \in [-2b, 2b]$
  $\Rightarrow \forall x \in E \;\; \exists c \in V \;\; \exists q \in \Sigma$ such that: $x = q + c$
  $\Rightarrow$ every element of E is written as the sum of an element of Σ and an element of V
  $\Leftrightarrow (*)$ holds.
  However, this is a contraddiction, in fact:
  $\lambda(E) > 0, \;\; E \subseteq \bigcup_{\sigma \in \Sigma}(\sigma + V), \;\; \lambda$ is monotone
  $\Rightarrow \mathbf{0 < \lambda(E)} \leq \lambda(\bigcup_{\sigma \in \Sigma}(\sigma + V)) = \mathbf{0}$.
  Hence, V **cannot** be Lebesgue measurable. ■

contraddiction: $V \in \mathcal{L}(\mathbb{R})$
$\downarrow$
$\lambda(V) = 0$
$\downarrow$
$\lambda(\bigcup_{\sigma \in \Sigma}(\sigma + V)) = 0$
$\downarrow$
$E \subseteq \bigcup_{\sigma \in \Sigma}(\sigma + V)$
$\downarrow$
$\lambda(E) = 0$ ↯

4 Measurable Functions

3.8. Write the definition of **measurable function**. Show the measurability of the composite function.

Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
A function $f : X \to X'$ is said to be measurable if: $f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{A}'$.

Prop. Consider $(X, \mathcal{A})$, $(X', \mathcal{A}')$, $(X'', \mathcal{A}'')$ measurable spaces.
Let $f : X \to X'$ and $g : X' \to X''$ be measurable.
Then $g \circ f : X \to X''$ is measurable.

proof.

By the measurability of f and g we have:
$\forall E \in \mathcal{A}' : f^{-1}(E) \in \mathcal{A}$
$\forall F \in \mathcal{A}'' : g^{-1}(F) \in \mathcal{A}'$
Combining the two results we obtain:
$\forall F \in \mathcal{A}'' : (g \circ f)^{-1}(F) = f^{-1}(\underbrace{g^{-1}(F)}_{=:E \in \mathcal{A}'}) = f^{-1}(E) \in \mathcal{A}$ ■

3.9. Characterize measurability of functions and prove it.

Theorem. **Characterization of measurability of functions**
Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
Let $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$. ($\mathcal{C}'$ generates $\mathcal{A}'$)
Then: $f : X \to X'$ is measurable $\Leftrightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \boldsymbol{\mathcal{C}'}$.

proof.

($\Rightarrow$) Suppose f measurable. Since $\mathcal{C}' \subseteq \mathcal{A} \Rightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{A}' \supseteq \mathcal{C}'$.
($\Leftarrow$) Let's define the set $\Sigma := \{E \subseteq X' : f^{-1}(E) \in \mathcal{A}\}$, which is a σ-algebra.
Suppose it holds the right-hand-side of the thesis:
$\Rightarrow \forall E \in \mathcal{C}' \Rightarrow f^{-1}(E) \in \mathcal{A} \Rightarrow E \in \Sigma$
$\Rightarrow \mathcal{C}' \subseteq \Sigma$
$\Rightarrow \mathcal{A}' = \sigma_0(\mathcal{C}') \subseteq \Sigma$
$\Leftrightarrow [\forall E \in \mathcal{A}' \Rightarrow E \in \Sigma]$
$\Leftrightarrow [\forall E \in \mathcal{A}' \Rightarrow f^{-1}(E) \in \mathcal{A}]$
$\Leftrightarrow f$ is measurable. ■

3.10. Write the definitions of: **Borel measurable functions, Lebesgue measurable functions**.

Let $(X, \mathcal{L})$ be a measurable space. Let (X', d') be a metric space and let $(X', \mathcal{B}')$ be the measurable space such that the σ-algebra is the Borel σ-algebra constructed by means of open sets w.r.t. the metric d'.
If $f : X \to X'$ is measurable then we say that f is Lebesgue measurable.
$f : (X, \mathcal{L}) \to (X', \mathcal{B}')$, *where $\mathcal{L}$ is the σ-algebra constructed as the collection of sets that satisfy Caratheodory condition.*

Let (X, d), (X', d') be two metric spaces and let $(X, \mathcal{B})$, $(X', \mathcal{B}')$ be the measurable spaces such that the σ-algebras are the Borel σ-algebras constructed by means of open sets w.r.t. the metrics d and d' (respectively).
If $f : X \to X'$ is measurable then we say that f is Borel measurable.
$f : (X, \mathcal{B}) \to (X', \mathcal{B}')$.

3.11. Prove that continuous functions are both Borel and Lebesgue measurable.

(Recall) Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces. Let $\mathcal{C} := \{E \subseteq X : E \text{ open}\}$, $\mathcal{C}' := \{E \subseteq X' : E \text{ open}\}$.
A function $f : X \to X'$ is continuous if: $f^{-1}(E) \in \mathcal{C} \quad \forall E \in \mathcal{C}'$. (the inverse image of an open set is open)

(Recall) **Theorem. Characterization of measurability of functions**
Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
Let $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$.
Then: $f : X \to X'$ is measurable $\Leftrightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \boldsymbol{\mathcal{C}'}$.

Prop. Let $f : X \to X'$ be continuous. Then f is both Borel measurable and Lebesgue measurable.

proof.

Let $\mathcal{C} := \{E \subseteq X : E \text{ open}\}$, $\mathcal{C}' := \{E \subseteq X' : E \text{ open}\}$.

(Borel) The function f is continuous:
$\Leftrightarrow f^{-1}(E) \in \mathcal{C} \quad \forall E \in \mathcal{C}'$
$\Rightarrow$ since $\mathcal{B}' = \sigma_0(\mathcal{C}')$, we apply the characterization of measurability of functions:
$\Rightarrow f^{-1}(E) \in \mathcal{C} \subseteq \mathcal{B} \quad \forall E \in \mathcal{B}'$
$\Rightarrow f$ is Borel measurable.

(Lebesgue) The function f is continuous:
$\Leftrightarrow f^{-1}(E) \in \mathcal{C} \quad \forall E \in \mathcal{C}'$
$\Rightarrow$ since $\mathcal{B}' = \sigma_0(\mathcal{C}')$, we apply the characterization of measurability of functions:
$\Rightarrow f^{-1}(E) \in \mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{L} \quad \forall E \in \mathcal{B}'$
$\Rightarrow f$ is Lebesgue measurable. ∎

3.12. Characterize Lebesgue measurability of functions and prove it.

(Recall) Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
A function $f : X \to X'$ is measurable if: $f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{A}'$.

(Recall) Let $(X, \mathcal{L})$ be a measurable space. Let (X', d') be a metric space and let $(X', \mathcal{B}')$ be the measurable space such that the σ-algebra is the Borel σ-algebra constructed by means of open sets w.r.t. the metric d'.
If $f : X \to X'$ is measurable then we say that f is Lebesgue measurable.

(Recall) **Theorem. Characterization of measurability of functions**
Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
Let $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$.
Then: $f : X \to X'$ is measurable $\Leftrightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \boldsymbol{\mathcal{C}'}$.

A function $f : X \to X'$ is Lebesgue measurable $\Leftrightarrow f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{C}' = \{E \subseteq X' : E \text{ open}\}$.

proof.

$(\Rightarrow)$ Suppose f Lebesgue measurable:
$\Leftrightarrow f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{B}'$
$\Rightarrow f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{C} \subseteq \mathcal{B}'$, since $\mathcal{C}' \subseteq \mathcal{B}'$.

$(\Leftarrow)$ Sppose it holds the right-hand-side of the thesis:
$\Rightarrow$ since $\mathcal{B}' = \sigma_0(\mathcal{C}')$, we apply the characterization of measurability of functions:
$\Rightarrow f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{B}'$
$\Leftrightarrow f$ is Lebesgue measurable. ∎

4.1 Real Valued Functions

4.2. Establish and show all equivalent statements to the fact that $f : X \to \overline{\mathbb{R}}$ is measurable.

(Recall) Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
A function $f : X \to X'$ is measurable if: $f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{A}'$.

(Recall) **Theorem. Characterization of measurability of functions**
Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be two measurable spaces.
Let $\mathcal{C}' \subseteq \mathcal{P}(X')$ such that $\sigma_0(\mathcal{C}') = \mathcal{A}'$.
Then: $f : X \to X'$ is measurable $\Leftrightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \boldsymbol{\mathcal{C}'}$.

Let $(X, \mathcal{A})$ be a measurable space.
Let $f : X \to \overline{\mathbb{R}}$, more precisely $f : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$.
Consider the set of measurable functions:
$\mathcal{M}(X, \mathcal{A}) := \{f : X \to \overline{\mathbb{R}} \text{ measurable}\}$
$\mathcal{M}_+(X, \mathcal{A}) := \{f : X \to \overline{\mathbb{R}} \text{ measurable}, f \geq 0 \text{ in } X\}$.
Moreover, $\forall \alpha > 0$ let:
$\{f > \alpha\} := \{x \in X : f(x) > \alpha\} = f^{-1}((\alpha, \infty])$
$\{f \geq \alpha\} := \{x \in X : f(x) \geq \alpha\}$
$\{f < \alpha\} := \{x \in X : f(x) < \alpha\}$
$\{f \leq \alpha\} := \{x \in X : f(x) \leq \alpha\}$.

Theorem. The following statements are equivalent:
(i) f is measurable, namely $f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{B}(\overline{\mathbb{R}})$
(ii) $\{f > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$, namely $\{f > \alpha\} \in \mathcal{A}$
(iii) $\{f \geq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$, namely $\{f \geq \alpha\} \in \mathcal{A}$
(iv) $\{f < \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$, namely $\{f < \alpha\} \in \mathcal{A}$
(v) $\{f \leq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$, namely $\{f \leq \alpha\} \in \mathcal{A}$

proof.

Steps: (i) $\overset{1.}{\Leftrightarrow}$ (ii), (ii) $\overset{2.}{\Rightarrow}$ (iii), (iii) $\overset{3.}{\Rightarrow}$ (iv), (iv) $\overset{4.}{\Rightarrow}$ (v), (v) $\overset{5.}{\Rightarrow}$ (ii).

1. It holds $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\{(\alpha, \infty] : \alpha \in \mathbb{R}\}) := \sigma_0(\mathcal{C})$.
$\Rightarrow$ we apply the characterization of measurability of functions:
$\Rightarrow f$ is measurable $\Leftrightarrow f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{C}$
$\Leftrightarrow f^{-1}((\alpha, \infty]) \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$
$\Leftrightarrow \{f > \alpha\} \in \mathcal{A}$.

2. It is possible to write: $\{f \geq \alpha\} = \bigcap_{n=1}^{\infty}\{f > \alpha - \frac{1}{n}\}$
$\Rightarrow$ $\{f > \alpha - \frac{1}{n}\} \in \mathcal{A}$, by point (ii)
$\Rightarrow$ $\bigcap_{n=1}^{\infty}\{f > \alpha - \frac{1}{n}\} \in \mathcal{A}$, since the countable intersection of measurable sets is measurable.
3. It is possible to write: $\{f < \alpha\} = \{f \geq \alpha\}^c$
$\Rightarrow$ $\{f \geq \alpha\} \in \mathcal{A}$, by point (iii)
$\Rightarrow$ $\{f \geq \alpha\}^c \in \mathcal{A}$, by definition of σ-algebra.
4. It is possible to write: $\{f \leq \alpha\} = \bigcap_{n=1}^{\infty}\{f < \alpha + \frac{1}{n}\}$
$\Rightarrow$ $\{f < \alpha + \frac{1}{n}\} \in \mathcal{A}$, by point (iv)
$\Rightarrow$ $\bigcap_{n=1}^{\infty}\{f < \alpha + \frac{1}{n}\} \in \mathcal{A}$, since the countable intersection of measurable sets is measurable.
5. It is possible to write: $\{f > \alpha\} = \{f \leq \alpha\}^c$
$\Rightarrow$ $\{f \leq \alpha\} \in \mathcal{A}$, by point (v)
$\Rightarrow$ $\{f \leq \alpha\}^c \in \mathcal{A}$, by definition of σ-algebra. ■

4.3. Let $f, g \in \mathcal{M}(X, \mathcal{A})$. What can we say about measurability of $\{f < g\}$, $\{f \leq g\}$, $\{f = g\}$? Justify the answer.

Let $(X, \mathcal{A})$ be a measurable space.
Let $f, g : X \to \overline{\mathbb{R}}$.
Consider the sets:
$\{f < g\} := \{x \in X : f(x) < g(x)\}$
$\{f \leq g\} := \{x \in X : f(x) \leq g(x)\}$
$\{f = g\} := \{x \in X : f(x) = g(x)\}$

Theorem. Let $f, g \in \mathcal{M}(X, \mathcal{A})$. The following statements hold:
(i) $\{f < g\}$ is measurable, namely $\{f < g\} \in \mathcal{A}$
(ii) $\{f \leq g\}$ is measurable, namely $\{f \leq g\} \in \mathcal{A}$
(iii) $\{f = g\}$ is measurable, namely $\{f = g\} \in \mathcal{A}$

proof.
(Recall) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow \{f > \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R} \Leftrightarrow \{f < \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$.
(i) It is possible to write: $\{f < g\} = \bigcup_{r \in \mathbb{Q}} [\{f < r\} \cap \{r < g\}]$
$\Rightarrow$ $\{f < r\}, \{r < g\} \in \mathcal{A}$, because $f, g \in \mathcal{M}(X, \mathcal{A})$ and $r \in \mathbb{Q} \subset \mathbb{R}$
$\Rightarrow$ $\{f < r\} \cap \{r < g\} \in \mathcal{A}$, since the intersection of two measurable sets is measurable
$\Rightarrow$ $\bigcup_{r \in \mathbb{Q}} [\{f < r\} \cap \{r < g\}] \in \mathcal{A}$, since the countable union of measurable sets is measurable.
(ii) It is possible to write: $\{f \leq g\} = \{f > g\}^c$
$\Rightarrow$ $\{f > g\} \in \mathcal{A}$, by property (i)
$\Rightarrow$ $\{f > g\}^c \in \mathcal{A}$, by definition of σ-algebra.
(iii) It is possible to write: $\{f = g\} = \{f \geq g\} \cap \{f \leq g\}$
$\Rightarrow$ $\{f \geq g\}, \{f \leq g\} \in \mathcal{A}$, by property (ii)
$\Rightarrow$ $\{f \geq g\} \cap \{f \leq g\} \in \mathcal{A}$, by definition of σ-algebra. ■

4.4. Let $\{f_n\}_n \subset \mathcal{M}(X, \mathcal{A})$. Show that $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$.
Can there exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$? Why?

Theorem. Let $(X, \mathcal{A})$ be a measurable space, $\{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$. Then $\sup_n f_n$, $\inf_n f_n \in \mathcal{M}(X, \mathcal{A})$.
proof.
(Recall) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow \{f > \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$.
($\sup_n$) It is possible to write: $\{\sup_n f_n > \alpha\} = \bigcup_{n=1}^{\infty}\{f_n > \alpha\} \quad \forall \alpha \in \mathbb{R}$
$\Rightarrow$ $\{f_n > \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}, \forall n \in \mathbb{N}$, because $f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$
$\Rightarrow$ $\bigcup_{n=1}^{\infty}\{f_n > \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$, since the countable union of measurable sets is measurable
$\Rightarrow$ $\{\sup_n f_n > \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$
$\Leftrightarrow$ $\sup_n f_n$ is measurable.
($\inf_n$) It is possible to write: $\inf_n f_n = -\sup_n(-f_n)$
$\Rightarrow$ $-f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$, since $f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$
$\Rightarrow$ $\sup_n(-f_n) \in \mathcal{M}(X, \mathcal{A})$
$\Rightarrow$ $-\sup_n(-f_n) \in \mathcal{M}(X, \mathcal{A})$. ■

Theorem. Let $(X, \mathcal{A})$ be a measurable space, $\{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$. Then $\limsup_n f_n$, $\liminf_n f_n \in \mathcal{M}(X, \mathcal{A})$.
proof.
($\limsup_n$) By definition: $\limsup_n f_n = \inf_{k \geq 1} (\sup_{n \geq k} f_n)$
$\Rightarrow$ $\sup_n f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$, since $f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$
$\Rightarrow$ $\inf_k (\sup_n f_n) \in \mathcal{M}(X, \mathcal{A})$.
($\liminf_n$) It is possible to write: $\liminf_n f_n = -\limsup_n(-f_n)$
$\Rightarrow$ $-f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$, since $f_n \in \mathcal{M}(X, \mathcal{A}) \quad \forall n \in \mathbb{N}$
$\Rightarrow$ $\limsup_n(-f_n) \in \mathcal{M}(X, \mathcal{A})$
$\Rightarrow$ $-\limsup_n(-f_n) \in \mathcal{M}(X, \mathcal{A})$. ■

There **cannot** exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$.
This is because, if $f, g \in \mathcal{M}(X, \mathcal{A})$ then $\max\{f, g\}$, $\min\{f, g\}$, $f_{\pm} \in \mathcal{M}(X, \mathcal{A})$.
In particular, if $\exists \max\{f, g\}$, *then* $\max\{f, g\} = \sup\{f, g\}$, *which we proved is measurable.*

4.5. Let $f, g \in \mathcal{M}(X, \mathcal{A})$. Show that $f + g$, $f \cdot g \in \mathcal{M}(X, \mathcal{A})$.

Theorem. Let $(X, \mathcal{A})$ be a measurable space. Let $f, g : X \to \mathbb{R}$ such that $f, g \in \mathcal{M}(X, \mathcal{A})$.
Then $f + g$, $f \cdot g \in \mathcal{M}(X, \mathcal{A})$.

proof.

- We introduce the functions:
  $\varphi : X \to \mathbb{R}^2 \quad \varphi(x) \ := (f(x), g(x)) \quad x \in X$
  $\psi : \mathbb{R}^2 \to \mathbb{R} \quad \psi(s,t) := s + t$
  $\chi : \mathbb{R}^2 \to \mathbb{R} \quad \chi(s,t) := s \cdot t.$
  So that, in particular: $\boldsymbol{\psi \circ \varphi = f + g,\ \chi \circ \varphi = f \cdot g}$.
- Since $\psi, \chi \in C^0(\mathbb{R}^2) \Rightarrow$ $\boldsymbol{\psi, \chi}$ **are measurable** (continuous functions are measurable).
- *Claim*: φ is measurable.
  1. We consider $E = R = (a,b) \times (c,d)$, namely open rectangles:
     $$\begin{aligned}\varphi^{-1}(R) &= \{x \in X : (f(x), g(x)) \in R\}\\ &= \{x \in X : f(x) \in (a,b),\ g(x) \in (c,d)\}\\ &= f^{-1}((a,b)) \cap g^{-1}((c,d))\end{aligned}$$
     Since $f, g \in \mathcal{M}(X, \mathcal{A})$:
     $\Leftrightarrow\ f^{-1}(E),\ g^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{B}(\mathbb{R})$
     $\Rightarrow\ f^{-1}((a,b)),\ g^{-1}((c,d)) \in \mathcal{A}$, since $(a,b),\ (c,d) \in \mathcal{B}(\mathbb{R})$
     $\Rightarrow\ f^{-1}((a,b)) \cap g^{-1}((c,d)) \in \mathcal{A}$, by definition of σ-algebra
     $\Rightarrow\ \boldsymbol{\varphi^{-1}(R) \in \mathcal{A} \quad \forall R}$ **open rectangle.**
  2. Every open set in $\mathbb{R}^2$ can be written as a countable union of open rectangles, namely:
     $\forall E \in \mathbb{R}^2$ open: $\quad E = \bigcup_{k=1}^{\infty} R_k \quad$ where $R_k = (a_k, b_k) \times (c_k, d_k)$.
     Thus, $\forall E \in \mathbb{R}^2$ open:
     $\varphi^{-1}(E) = \varphi^{-1}(\bigcup_{k=1}^{\infty} R_k) = \bigcup_{k=1}^{\infty} \varphi^{-1}(R_k)$.
     We proved that $\varphi^{-1}(R) \in \mathcal{A}\ \ \forall R$ open rectangle and the countable union of measurable sets is measurable:
     $\Rightarrow\ \boldsymbol{\varphi^{-1}(E) \in \mathcal{A} \quad \forall E \subseteq \mathbb{R}^2}$ **open.**
  3. By definition:
     φ is measurable $\Leftrightarrow\ \varphi^{-1}(E) \in \mathcal{A} \quad \forall E \subseteq \mathbb{R}^2$.
     To prove that φ is measurable, we should consider every Borel set of $\mathbb{R}^2$, however, by the characterization of measurability of functions, it is enough to consider only open sets in $\mathbb{R}^2$ (since the σ-algebra generated by open sets of $\mathbb{R}^2$ is equivalent to the Borel σ-algebra in $\mathbb{R}^2$.
     Since we proved that $\varphi^{-1}(E) \in \mathcal{A} \quad \forall E \subseteq \mathbb{R}^2$ open:
     $\Rightarrow\ \boldsymbol{\varphi}$ **is measurable.**
- The composition of two measurable functions is measurable:
  $\Rightarrow \boldsymbol{\psi \circ \varphi = f + g,\ \chi \circ \varphi = f \cdot g}$ **are measurable.** ■

4.6. Prove that A is measurable if and only if $\mathbb{1}_A$ is a measurable function.

Lemma. Let $(X, \mathcal{A})$ be a measurable space.
Let $A \subseteq X$ and let $\mathbb{1}_A : X \to \{0,1\}$ be such that $\mathbb{1}_A(x) \Leftrightarrow x \in A$.
Then $A \in \mathcal{A} \Leftrightarrow \mathbb{1}_A \in \mathcal{M}(X, \mathcal{A})$.

proof.

It is possible to write:
$$\{\mathbb{1}_A > \alpha\} = \{x \in X \colon \mathbb{1}_A(x) > \alpha\} = \begin{cases} X & \text{if } \alpha < 0 \\ A & \text{if } \alpha \in [0,1) \\ \varnothing & \text{if } \alpha \geq 1 \end{cases} \qquad \forall \alpha \in \mathbb{R}.$$
Since X and $\varnothing$ are always measurable $\Rightarrow \big[A \in \mathcal{A} \Leftrightarrow \mathbb{1}_A \in \mathcal{M}(X, \mathcal{A})\big]$. ■

4.7. Prove or disprove the following statements:
(a) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow f_\pm \in \mathcal{M}_+(X, \mathcal{A})$
(b) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow |f| \in \mathcal{M}(X, \mathcal{A})$

Let $(X, \mathcal{A})$ be a measurable space. Let $f : X \to \mathbb{R}$.
The following relations holds:
(i) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow f_\pm \in \mathcal{M}_+(X, \mathcal{A})$
(ii) $f \in \mathcal{M}(X, \mathcal{A}) \Rightarrow |f| \in \mathcal{M}(X, \mathcal{A})$
(iii) $f \in \mathcal{M}(X, \mathcal{A}) \not\Leftarrow |f| \in \mathcal{M}(X, \mathcal{A})$

proof.

(i) By definition: $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$, $f = f_+ - f_-$.
  ($\Rightarrow$) Since f is measurable:
    $\Rightarrow\ -f$ measurable
    $\Rightarrow\ \max\{f,0\}, \max\{-f,0\}$ measurable (since also $f \equiv 0$ is measurable)
    $\Rightarrow\ f_\pm$ measurable.
  ($\Leftarrow$) Since $f_\pm$ are measurable:
    $\Rightarrow\ -f_-$ measurable
    $\Rightarrow\ f_+ + (-f_-)$ measurable, since the sum of two measurable functions is measurable
    $\Rightarrow\ f$ measurable.
(ii) By definition: $|f| = f_+ + f_-$.
  Since f is measurable:
    $\Rightarrow\ f_\pm$ measurable, by property (i)
    $\Rightarrow\ f_+ + f_-$ measurable, since the sum of two measurable functions is measurable
    $\Rightarrow\ |f|$ measurable.
(iii) Let $E \subseteq X$ be a non measurable set, namely $E \notin \mathcal{A}$.
  Let $f(x) := \mathbb{1}_E(x) - \mathbb{1}_{E^c}(x) = \begin{cases} 1 & x \in E \\ -1 & x \in E^c \end{cases}$
  Since, by definition, $E \notin \mathcal{A} \Rightarrow f \notin \mathcal{M}(X, \mathcal{A})$.
  However, $|f| \equiv 1 \in \mathcal{M}(X, \mathcal{A})$. ■

4.2 Simple Functions

4.8. Write the definition of **simple function**. What is its canonical form? How can we characterize measurability of a simple function? Write the definition of **step function**.

Let X be a set.
A function $s : X \to \mathbb{R}$ is said to be a simple function if $S(X)$ is finite.
Thus, simple functions are functions that take only a finite number of values.
For every simple function s we can define:

$$\begin{aligned} s(X) &= \{c_1, .., c_n\} && c_i \in \mathbb{R} \quad \forall i = 1, .., n \quad \text{such that: } c_i \neq c_j \quad \forall i \neq j \\ E_k &= \{x \in X : s(x) = c_k\} && \forall k = 1, .., n \\ X &= \textstyle\bigcup_{k=1}^{n} E_k && E_i \cap E_j = \varnothing \quad \forall i \neq j \end{aligned}$$

Then, we can write the simple function s in a unique canonical form:
$s = \sum_{k=1}^{n} c_k \, \mathbb{1}_{E_k}$.
Consider $(X, \mathcal{A})$ measurable space. It is true that $A \subseteq X$ is measurable $\Leftrightarrow$ $\mathbb{1}_A$ is a measurable function.
Thus, relying on the canonical form, we can characterize the measurability of simple functions:
$s \in \mathcal{M}(X, \mathcal{A}) \;\Leftrightarrow\; E_k \in \mathcal{A} \quad \forall k = 1, .., n.$
Moreover, we introduce the following sets:
$\mathcal{S}(X, \mathcal{A}) \;= \{s : X \to \mathbb{R}$ measurable simple function$\}$
$\mathcal{S}_+(X, \mathcal{A}) = \{s : X \to \mathbb{R}$ measurable simple function, $s \geq 0\}$.
Consider an interval $I = [a_0, a_1)$ and a partition $\mathcal{P} := \{a_0 = x_0 < x_1 < .. < x_n = a_1\}$.
Then, function $f : I \to \mathbb{R}$ is said to be a step function if:
$f(x) = \sum_{k=0}^{n-1} c_k \, \mathbb{1}_{[x_k, x_{k+1})}(x) \qquad c_k \in \mathbb{R} \quad \forall k \in 0, .., n-1.$
Step functions have finite number of values and they are constant on every interval.
Therefore, step functions are special cases of simple functions.

simple function $\Longleftarrow$ / $\not\Longrightarrow$ step function
e.g. $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ is a simple function but not a step function

4.9. State and give a sketch of the proof of the **Simple Approximation Theorem**.

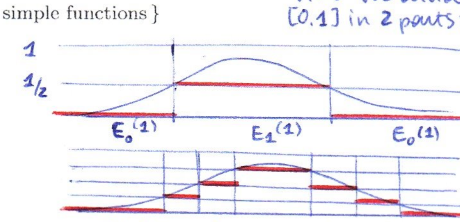
Theorem. **Simple Approximation Theorem**
Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$.
Then, there exists a sequence $\{s_n\}_n$ of simple functions such that:
$s_n \xrightarrow[n\to\infty]{} f$ pointwise in X, namely $s_n \xrightarrow[n\to\infty]{} f \;\; \forall x \in X$.
Moreover:
(i) $f \in \mathcal{M}(X, \mathcal{A}) \Rightarrow \{s_n\}_n \subseteq \mathcal{S}(X, \mathcal{A}) = \{s : X \to \mathbb{R}$ measurable simple functions$\}$
(ii) $f \geq 0 \Rightarrow \{s_n\}_n \nearrow, \; 0 \leq s_n \leq f$
(iii) f bounded $\Rightarrow s_n \xrightarrow[n\to\infty]{} f$ uniformly in X

(sketch) proof.
Let assume that $f \geq 0$ and f bounded:
$\Rightarrow \exists\, M > 0$ such that $0 \leq f(x) \leq M \quad \forall x \in X$.
Without loss of generality we suppose $M = 1$ (otherwise we can consider $\frac{f}{M}$):
$\Rightarrow f : X \to [0, 1]$.
We divide the interval $[0, 1]$ in 2^n intervals of length 2^{-n} for each $n \in \mathbb{N}$.
Then, for each $n \in \mathbb{N}$ we define:

$$\begin{aligned} E_k^{(n)} &:= \{x \in X : \tfrac{k}{2^n} \leq f(x) < \tfrac{k+1}{2^n}\} && \forall k = 0, .., 2^n - 1 \\ s_n(x) &:= \textstyle\sum_{k=0}^{2^n-1} \tfrac{k}{2^n} \, \mathbb{1}_{E_k^{(n)}}(x) && x \in X, \forall n \in \mathbb{N} \end{aligned}$$

The sequence $\{s_n\}_n$ has the desired properties. (■)

n=0 we divide [0,1] in 2 parts:



as $n \to \infty$ the approximation gets better

4.3 Essentially Bounded Functions

4.10. Write the definitions of $\operatorname{ess\,sup}_X f$, $\operatorname{ess\,inf}_X f$. State their properties and prove some of them.

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$.
Given a measure space, it is always possible to define:
$\mathcal{N}_\mu := \{E \subseteq X : E$ measurable, $\mu(E) = 0\}$.
Then, the essential supremum and the essential infimum of f are defined as:
$\operatorname{ess\,sup}_X f := \inf_{N \in \mathcal{N}_\mu} \{\sup_{x \in N^c} f(x)\}$
$\operatorname{ess\,inf}_X f := \sup_{N \in \mathcal{N}_\mu} \{\inf_{x \in N^c} f(x)\}$

Prop. Let $f \in \mathcal{M}(X, \mathcal{A})$. Then, $\exists\, N \in \mathcal{N}_\mu$ such that:
$\operatorname{ess\,sup}_X f(x) = \inf_{N \in \mathcal{N}_\mu} \{\sup_{x \in N^c} f(x)\} = \sup_{x \in N^c} f(x)$.
Moreover:
$f(x) \leq \operatorname{ess\,sup}_X f(x) \quad$ a.e. in X.

($\exists\, N \in \mathcal{N}_\mu$ for which the inf is realized)

proof.

1.1. If $\operatorname{ess\,sup}_X f = \infty$ then it is obvious.
Indeed, $\operatorname{ess\,sup}_X f$ is defined as the infimum over $N \in \mathcal{N}_\mu$ of $\sup_{x \in N^c} f(x)$.
Thus, if $\operatorname{ess\,sup}_X f = \infty$ it must be that $\exists\, N \in \mathcal{N}_\mu$ such that $\sup_{x \in N^c} f(x) = \infty$.

1.2. Suppose that $\operatorname{ess\,sup}_X f < \infty$:

$\Rightarrow \forall k \in \mathbb{N} \;\; \exists\, N_k \in \mathcal{N}_\mu$: $\sup_{x \in N_k^c} f < \operatorname{ess\,sup}_X f + \frac{1}{k}$ — by definition of infimum
$\Rightarrow N := \bigcup_{k=1}^{\infty} N_k \in \mathcal{N}_\mu$ — since $\forall k \in \mathbb{N}$: $N_k \in \mathcal{N}_\mu$ (measurable and of zero-measure)
$\Rightarrow N^c = \bigcap_{k=1}^{\infty} N_k^c \subseteq N_k^c \quad \forall k \in \mathbb{N}$ — since the intersection of subsets is contained in each subset
$\Rightarrow \forall k \in \mathbb{N}$: $\operatorname{ess\,sup}_X f \leq \sup_{N^c} f$ — by definition of infimum
$\leq \sup_{N_k^c} f$ — since $N^c \subseteq N_k^c$
$< \operatorname{ess\,sup}_X f + \frac{1}{k}$

Letting $k \to \infty \Rightarrow \operatorname{ess\,sup}_X f = \sup_{N^c} f$.

2. Let $N \in \mathcal{N}_\mu$ be the set of the first part of the proposition.
Then, by the definition of supremum and by the first result of the proposition:
$\Rightarrow \forall x \in N^c$: $f(x) \le \sup_{x\in N^c} f(x) = \operatorname{ess\,sup}_X f(x)$. $(*)$
Let $\overline{N} := \{x \in X : f(x) > \operatorname{ess\,sup}_X f(x)\}$.
Because of $(*)$:
$\Rightarrow \forall x \in \overline{N}$: $x \notin N^c$ and $\forall x \in N^c$: $x \notin \overline{N}$
$\Rightarrow \overline{N} \cap N^c = \varnothing$
$\Rightarrow \overline{N} \subseteq N \in \mathcal{N}_\mu$.
$\Rightarrow \overline{N} \in \mathcal{N}_\mu$
$\Rightarrow f \le \operatorname{ess\,sup}_X f$ a.e. in X. ■

Prop. Let $f \in \mathcal{M}(X, \mathcal{A})$. Then:
(i) $\operatorname{ess\,sup}_X f = -\operatorname{ess\,inf}_X(-f)$
(ii) $\operatorname{ess\,sup}_X(kf) = k \operatorname{ess\,sup}_X f \quad \forall k \ge 0$

Prop. Let $f, g \in \mathcal{M}(X, \mathcal{A})$. Then:
(i) $f \le g$ a.e. in $X \Rightarrow \operatorname{ess\,sup}_X f \le \operatorname{ess\,sup}_X g$
(ii) $\operatorname{ess\,sup}_X(f+g) \le \operatorname{ess\,sup}_X f + \operatorname{ess\,sup}_X g$
(iii) $f = g$ a.e. in $X \Rightarrow \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X g$
$f = g$ a.e. in $X \Rightarrow \operatorname{ess\,inf}_X f = \operatorname{ess\,inf}_X g$
(iv) $g \ge 0$ a.e. in $X \Rightarrow f \cdot g \le (\operatorname{ess\,sup}_X f) \cdot g$ a.e. in X

Prop. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous. Then $\operatorname{ess\,sup}_A f = \operatorname{ess\,inf}_A f$.

Lastly, we say that $f \in \mathcal{M}(X, \mathcal{A})$ is essentially bounded in X if:
$\operatorname{ess\,sup}_X |f| < \infty$. $\Longleftrightarrow \exists M \geq 0 : \mu(\{x\in X : |f(x)| > M\}) = 0 \Longleftrightarrow f = g$ a.e. where g is a bdd function

4.11. What is $\mathcal{L}^\infty$? Which is the relation between functions finite a.e. and essentially bounded functions? Justify the answer.

(Recall) Starting from a measurable space $(X, \mathcal{A}, \mu)$, it is always possible to define:
$\mathcal{N}_\mu := \{E \subseteq X : E \text{ measurable}, \mu(E) = 0\}$.
Then, the essential supremum of $f : X \to \overline{\mathbb{R}}$ is defined as:
$\operatorname{ess\,sup}_X f := \inf_{N\in\mathcal{N}_\mu} \{\sup_{x\in N^c} f(x)\}$.
We say that $f \in \mathcal{M}(X, \mathcal{A})$ is essentially bounded in X if:
$\operatorname{ess\,sup}_X |f| < \infty$.

Starting from a measurable space $(X, \mathcal{A}, \mu)$, we define:
$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}) \text{ essentially bounded}\}$.
Then, the following statements hold:
(i) $f \in \mathcal{L}^\infty \Rightarrow f$ finite a.e. in X
(ii) $f \in \mathcal{L}^\infty \not\Leftarrow f$ finite a.e. in X

proof.

(i) If $f \in \mathcal{L}^\infty$ then $|f| \le \operatorname{ess\,sup}_X |f| < \infty$ a.e. in X and so $|f| < \infty$ a.e. in X.
(ii) Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ such that:
$$f(x) := \begin{cases} \frac{1}{|x|} & x \neq 0 \\ +\infty & x = 0 \end{cases}$$
Since f is finite in $\mathbb{R} \setminus \{0\} \Rightarrow f$ is finite a.e. in $\mathbb{R}$, since $\mu(\{0\}) = 0$.
However, since as $x \to 0$ the function goes $+\infty$:
$\Rightarrow \nexists M \ge 0$: $\mu(\{x \in X : |f(x)| > M\}) = 0$
$\Leftrightarrow \operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_X |f| = \infty$
$\Rightarrow f$ is **not** essentially bounded. ■

5 The Lebesgue Integral

5.1 Integral of Non-Negative Simple Functions

4.12. Write the definitions of the **Lebesgue integral of a non-negative measurable simple function** over X and over a measurable subset $E \subseteq X$. Write the main properties of the integral and prove some of them.

Let $(X, \mathcal{A}, \mu)$ be a measure space.
Let $s \in \mathcal{S}_+(X, \mathcal{A})$, namely a non-negative simple function, and let s be written in the canonical form:
$s = \sum_{k=1}^n c_k \mathbb{1}_{E_k}$ $\quad c_1, .., c_n \in \mathbb{R}_+$, $E_k \in \mathcal{A}$ $\forall k = 1, .., n$, pairly disjoint and such that $X = \bigcup_{k=1}^n E_K$.
Then, the Lebesgue integral of s over X is defined as:
$\int_X s\, d\mu := \sum_{k=1}^n c_k\, \mu(E_k)$.
If $E \in \mathcal{A}$, the Lebesgue integral of s over E is defined as:
$\int_E s\, d\mu := \int_X s \mathbb{1}_E\, d\mu$.

Prop. Let $s \in \mathcal{S}_+(X, \mathcal{A})$ and let $E \in \mathcal{A}$. Then:
(i) $\int_E s\, d\mu = \sum_{k=1}^\infty c_k\, \mu(E_k \cap E)$
(ii) $\int_X \mathbb{1}_E\, d\mu = \mu(E)$
(iii) $\int_N s\, d\mu = 0 \quad \forall N \in \mathcal{N}_\mu$

proof.

(i) The thesis follows from the fact that: $s \mathbb{1}_E = \sum_{k=1}^n c_k \mathbb{1}_{E_k \cap E}$.
(ii) It is possible to write: $\mathbb{1}_E(x) = \sum_{k=1}^2 c_k \mathbb{1}_{E_k}(x)$, with $E_1 = E$, $E_2 = E^c$, $c_1 = 1$, $c_2 = 0$
$\Rightarrow \mathbb{1}_E$ is a simple function
$\Rightarrow \int_X \mathbb{1}_E\, d\mu = \sum_{k=1}^2 c_k\, \mu(E_k) = c_1\, \mu(E_1) + c_2\, \mu(E_2) = \mu(E)$.

(iii) By property (i):

$\Rightarrow \int_N s\,d\mu = \sum_{k=1}^{n} c_k\,\mu(E_k \cap N)$

$\Rightarrow \forall k \in \mathbb{N}$: $E_k \cap N \in \mathcal{A}$, since it is the intersection of measurable sets

$\Rightarrow \forall k \in \mathbb{N}$: $0 \leq \mu(E_k \cap N) \leq \mu(N) = 0$, since $E_k \cap N \subseteq N$

$\Rightarrow \int_N s\,d\mu = 0.$ ■

Prop. Let $s, t \in \mathcal{S}_+(X, \mathcal{A})$. Then:

(i) $c \geq 0 \Rightarrow \int_X c \cdot s\,d\mu = c \cdot \int_X s\,d\mu$

(ii) $\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$

(iii) $s \leq t \Rightarrow \int_X s\,d\mu \leq \int_X t\,d\mu$

(iv) $E, F \in \mathcal{A}$, $E \subseteq F \Rightarrow \int_E s\,d\mu \leq \int_F s\,d\mu$

4.13. Let $s \in \mathcal{S}_+(X, \mathcal{A})$. For any $E \in \mathcal{A}$, let $\varphi(E) := \int_E s\,d\mu$. Prove that φ is a measure.

Prop. Let $(X, \mathcal{A}, \mu)$ be a measure space, $s \in \mathcal{S}_+(X, \mathcal{A})$.
Then, the function $\varphi : \mathcal{A} \to \overline{\mathbb{R}}_+$ defined as:

$\varphi(E) := \int_E s\,d\mu \quad \forall E \in \mathcal{A}$

is a measure.

Starting from a measure space $(X, \mathcal{A}, \mu)$ we can define another measure by means of the measure μ, a simple function s and the Lebesgue integral. The variable of the function φ is E: given any $E \in \mathcal{A}$, φ provides a new measure for E.

proof.

(i) Since $\mu(\varnothing) = 0 \Rightarrow \varnothing \in \mathcal{N}_\mu$.
Since $\int_N s\,d\mu = 0 \quad \forall N \in \mathcal{N}_\mu \Rightarrow \varphi(\varnothing) = \int_\varnothing s\,d\mu = 0.$

(ii) Let $\{E_k\}_k \subseteq \mathcal{A}$ be a disjoint collection of sets and let $E := \bigcup_{k=1}^{\infty} E_k$.
It is possible to write the simple function s using the canonical form: $s := \sum_{j=1}^{m} d_j\,\mathbb{1}_{F_j}$.

$\Rightarrow \varphi(E) = \sum_{j=1}^{m} d_j\,\mu(F_j \cap E)$

$= \sum_{j=1}^{m} \sum_{k=1}^{\infty} d_j\,\mu(F_j \cap E_k)$ since μ is a measure and $\forall j$ fixed $\{F_j \cap E_k\}_k$ is a disjoint family

$= \sum_{k=1}^{\infty} \sum_{j=1}^{m} d_j\,\mu(F_j \cap E_k)$ since everything is ≥ 0 we can exchange the series

$= \sum_{k=1}^{\infty} \varphi(E_k)$ since $\sum_{j=1}^{m} d_j\,\mu(F_j \cap E_k) = \int_{E_k} s\,d\mu = \varphi(E_k)$

$\Rightarrow$ σ-additivity holds. ■

5.2 Integral of Non-Negative Measurable Functions

4.14. Write two possible equivalent definitions of **Lebesgue integral of a measurable non-negative function.**

(Recall) **Theorem. Simple Approximation Theorem**

Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$.
Then, there exists a sequence $\{s_n\}_n$ of simple functions such that:

$s_n \xrightarrow[n\to\infty]{} f$ pointwise in X, namely $s_n \xrightarrow[n\to\infty]{} f \ \forall x \in X$.

Moreover:

(i) $f \in \mathcal{M}(X, \mathcal{A}) \Rightarrow \{s_n\}_n \subseteq \mathcal{S}(X, \mathcal{A}) = \{s : X \to \mathbb{R}$ measurable simple functions$\}$

(ii) $f \geq 0 \Rightarrow \{s_n\}_n \nearrow, 0 \leq s_n \leq f$

(iii) f bounded $\Rightarrow s_n \xrightarrow[n\to\infty]{} f$ uniformly in X

Let $(X, \mathcal{A}, \mu)$ be a measure space.
Let $f : X \to \overline{\mathbb{R}}_+$ such that $f \in \mathcal{M}_+(X, \mathcal{A})$.
Then, the Lebesgue integral of f over X is defined as:

$\int_X f\,d\mu := \sup_{s \in \mathcal{S}_f} \int_X s\,d\mu \quad \mathcal{S}_f := \{s \in \mathcal{S}_+(X, \mathcal{A}) : s \leq f \text{ in } X\}.$

If $E \in \mathcal{A}$, the Lebesgue integral of f over E is defined as:

$\int_E f\,d\mu := \int_X f\,\mathbb{1}_E\,d\mu.$

Because of the simple approximation theorem, we know that if $f \in \mathcal{M}_+(X, \mathcal{A})$ then there exists $\{s_n\}_n \subseteq \mathcal{S}(X, \mathcal{A})$ such that $\{s_n\}_n \nearrow$, $0 \leq s_n \leq f$ and such that $s_n \xrightarrow[n\to\infty]{} f$ in X. Hence, $\mathcal{S}_f \neq \varnothing$.

Exploiting the simple approximation theorem, it is possible to define the integral of f over X also as:

$\int_X f\,d\mu := \lim_{n\to\infty} \int_X s_n\,d\mu. \quad$ (where: $\{\int_X s_n\,d\mu\}_n \subseteq \mathbb{R}_+$)

Prop. Let $f, g : X \to \overline{\mathbb{R}}_+$ such that $f, g \in \mathcal{M}_+(X, \mathcal{A})$. Then:

(i) $c \geq 0 \Rightarrow \int_X c \cdot f\,d\mu = c \cdot \int_X f\,d\mu$

(ii) $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$

(iii) $f \leq g \Rightarrow \int_X f\,d\mu \leq \int_X g\,d\mu$

(iv) $E, F \in \mathcal{A}$, $E \subseteq F \Rightarrow \int_E f\,d\mu \leq \int_F f\,d\mu$

(v) $\forall N \in \mathcal{N}_\mu \ \int_N f\,d\mu = 0$

4.15. State and prove the **Chebychev inequality**.

Theorem. Chebychev inequality

Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in \mathcal{M}_+(X, \mathcal{A})$. Then, $\forall c > 0$:

$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f\,d\mu \leq \frac{1}{c} \int_X f\,d\mu.$

proof.

Since $f \in \mathcal{M}_+(X, \mathcal{A})$, we define $E_c := \{f \geq c\} \subseteq X$ and we conclude $E_c \in \mathcal{A}$:

$\Rightarrow c\,\mathbb{1}_{E_c} \leq f\,\mathbb{1}_{E_c}$, because $\mathbb{1}_{E_c} = 0$ when we're outside E_c, inside E_c it holds $c \leq f$

$\Rightarrow \int_X f d\mu \geq \int_{E_c} f\,d\mu = \int_X f\,\mathbb{1}_{E_c}\,d\mu \geq \int_X c\,\mathbb{1}_{E_c}\,d\mu = c \int_X \mathbb{1}_{E_c}\,d\mu = c\,\mu(E_c)$

$\Rightarrow$ dividing by c we obtain the thesis. ■

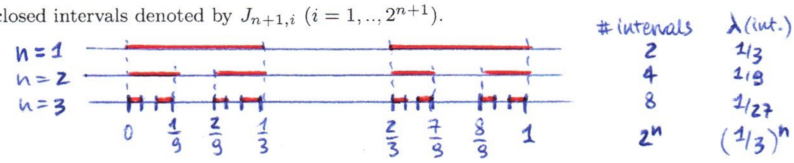
4.16. Define the **Cantor set**. State its main properties and prove some of them.

The Cantor set is a subset $C \subset [0,1] \subseteq \mathbb{R}$ which can be inductively constructed as follow.

1. We remove from $I_0 = [0,1]$ the open interval:
   $I_{1,1} = (\frac{1}{3}, \frac{2}{3})$
   and we define:
   $J_{1,1} = [0, \frac{1}{3}], \quad J_{1,2} = [\frac{2}{3}, 1].$
2. We remove from $J_{1,i}$ $(i = 1, 2)$ the open interval $I_{2,i}$ with the same center as $J_{1,i}$ and length $(\frac{1}{3})^2 = \frac{1}{9}$.
   Explicitly:
   - from $J_{1,1}$ we remove $I_{2,1} = (\frac{1}{9}, \frac{2}{9})$
   - from $J_{1,2}$ we remove $I_{2,2} = (\frac{7}{9}, \frac{8}{9})$

   Then, we define:
   $J_{2,1} = [0, \frac{1}{9}], \quad J_{2,2} = [\frac{2}{9}, \frac{1}{3}], \quad J_{2,3} = [\frac{2}{3}, \frac{7}{9}], \quad J_{2,4} = [\frac{8}{9}, 1].$

By proceding in this way, after n steps we have removed from $[0,1]$ a number of open intervals which is: $1+2+4+..+2^{n-1}$.
In fact, at each step $k = 1, .., n$ we remove 2^{k-1} intervals $I_{k,i}$ $(i = 1, .., 2^{k-1})$ with length: $\lambda(I_{k,i}) = (\frac{1}{3})^k$.
The remaining set is the disjoint union of 2^n closed intervals $J_{n,i}$ $(i = 1, .., 2^n)$ with length: $\lambda(J_{n,i}) = (\frac{1}{3})^n$.
We then proceed by removing from $J_{n,i}$ $(i = 1, .., 2^n)$ the open interval $I_{n+1,i}$ with the same center as $J_{n,i}$ and with length: $\lambda(I_{n+1,i}) = (\frac{1}{3})^{n+1}$.
The remaining set is the disjoint union of 2^{n+1} closed intervals denoted by $J_{n+1,i}$ $(i = 1, .., 2^{n+1})$.
Then, we define:
$C_n := \bigcup_{k=1}^{2^n} J_{n,k}$.
Lastly, we define the Cantor set as:
$C := \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} J_{n,k}$.



Properties of the Cantor set C:

1. C is closed.
   proof.
   Since $\forall n \in \mathbb{N}$, $\forall k = 1, .., 2^n$ $J_{n,k}$ is closed $\Rightarrow$ $C_n = \bigcup_{k=1}^{2^n} J_{n,k}$ is closed $\Rightarrow$ $C = \bigcap_{n=1}^{\infty} C_n$ is closed. ■
2. $C \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.
   proof.
   This follows immediately from the fact that C is closed (since closed sets $\in \mathcal{B}(\mathbb{R})$). ■
3. $\lambda(C) = 0$.
   proof.
   Since $C \subseteq C_n$ $\forall n \in \mathbb{N}$:
   $\Rightarrow \lambda(C) \leq \lambda(C_n) = \lambda(\bigcup_{k=1}^{2^n} J_{n,k})$
   $= \sum_{k=1}^{2^n} \lambda(J_{n,k})$, since $J_{n,k}$ are disjoint
   $= \sum_{k=1}^{2^n} (\frac{1}{3})^n = 2^n (\frac{1}{3})^n = (\frac{2}{3})^n$
   $\Rightarrow$ letting $n \to \infty$ we get $\lambda(C) = 0$. ■
4. $int(C) = \varnothing$.
   proof.
   Since $int(C) \subseteq C$ and $\lambda(C) = 0 \Rightarrow \lambda(int(C)) = 0$, since λ is monotone.
   Since $int(C)$ is open $\Rightarrow$ $int(C) = \varnothing$ necessarily. ■
5. $\overline{[0,1] \setminus C} = [0,1]$.
   proof.
   The property says that the complement of C is dense in $[0,1]$.
   Let $x_0 \in [0,1]$ and $r > 0$ be fixed:
   $\Rightarrow (x_0 - r, x_0 + r) \cap ([0,1] \setminus C) \neq \varnothing$
   otherwise $(x_0 - r, x_0 + r)$ would be entirely contained in C, but $int(C) = \varnothing$.
   $\Rightarrow$ $[0,1] \setminus C$ is dense in $[0,1]$ and so, its closure is equal to $[0,1]$. ■
6. C is uncountable and $|C| = |\mathbb{R}|$, namely C has the same cardinality as $\mathbb{R}$.
7. $C = \{x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{x_n}{3^n},\ x_n \in \{0, 2\}\}$.

5.1. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu < \infty$. Show that f is finite a.e. in X.

Prop. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu < \infty$.
Then f is finite a.e. in X.

proof.
The thesis follows $\Leftrightarrow$ $\mu(\{f = \infty\}) = 0$.
It is possible to write $\{f = \infty\} = \bigcap_{n=1}^{\infty} \{f > n\}$.
Let $E_n := \{f > n\}$:
(a) $\{E_n\}_n \searrow$
(b) $\mu(E_n) \leq \frac{1}{n} \int_X f\, d\mu$ $\forall n \in \mathbb{N}$ (Chebychev inequality)

$\Rightarrow \mu(\{f = \infty\}) = \mu(\bigcap_{n=1}^{\infty} E_n)$
$= \lim_{n\to\infty} \mu(E_n)$ since $\{E_n\}_n \searrow$ and $\mu(E_1) < \infty$, so we apply the continuity of measure
$\leq \lim_{n\to\infty} \frac{1}{n} \int_X f\, d\mu$ because of (b)
$= 0$ since $\int_X f\, d\mu < \infty$.

$\left(\mu(E_1) \leq \int_X f\, d\mu < \infty \right.$ (b) ■

5.2. Prove the **vanishing lemma** for functions $f \in \mathcal{M}_+(X, \mathcal{A})$.

Lemma. **Vanishing lemma** for non-negative functions
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu = 0$.
Then $f = 0$ a.e. in X.

proof.

Since $f \in \mathcal{M}_+(X, \mathcal{A})$, the thesis follows $\Leftrightarrow$ $\mu(\{f > 0\}) = 0$.
It is possible to write $\{f > 0\} = \bigcup_{n=1}^{\infty} \{f \geq \frac{1}{n}\}$.
Let $F_n := \{f \geq \frac{1}{n}\}$:

(a) $\{F_n\}_n \nearrow$

(b) $\mu(F_n) \leq n \int_X f\, d\mu = 0 \quad \forall n \in \mathbb{N}$ (Chebychev inequality and hypothesis $\left(\int_X f\, d\mu = 0\right)$)

$\Rightarrow$ $\boldsymbol{\mu(\{f > 0\})} = \mu(\bigcup_{n=1}^{\infty} F_n)$
$= \lim_{n\to\infty} \mu(F_n)$ since $\{F_n\}_n \nearrow$, so we apply the continuity of measure
$\leq \lim_{n\to\infty} n \int_X f\, d\mu = \mathbf{0}$ because of (c). ■

5.3 Convergence Theorems - part 1

5.3. State and prove the **Monotone Convergence Theorem**.

Theorem. **Monotone Convergence Theorem (Beppo Levi) - MCT**
Let $(X, \mathcal{A}, \mu)$ be a measure space.
Let $\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$, $f : X \to \overline{\mathbb{R}}_+$ be such that:

(i) $f_n \leq f_{n+1}$ in X $\forall n \in \mathbb{N}$

(ii) $f_n \xrightarrow[n\to\infty]{} f$ pointwise in X

Then:
$\int_X f_n\, d\mu \xrightarrow[n\to\infty]{} \int_X f\, d\mu$. $\quad \left(\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X \underbrace{\lim_{n\to\infty} f_n}_{:=f}\, d\mu\right)$

proof.

Since $\{f_n\}_n \subset \mathcal{M}_+(X, \mathcal{A})$ $\Rightarrow$ $\limsup_n f_n, \liminf_n f_n \in \mathcal{M}_+(X, \mathcal{A})$.
Moreover, since $\{f_n\}_n$ is pointwise converging $\Rightarrow$ $\limsup_n f_n = \liminf_n f_n = \lim_n f_n = f \in \mathcal{M}_+(X, \mathcal{A})$.
Since the sequence $\{f_n\}_n$ is monotone increasing (by (i)):
$\Rightarrow$ $\int_X f_n\, d\mu \leq \int_X f_{n+1}\, d\mu \leq \int_X f\, d\mu \quad (*)$
$\Rightarrow$ $\{\int_X f_n\, d\mu\}_n$ is a monotone sequence of real numbers, so, it admits a limit
$\Rightarrow$ $\exists\, \boldsymbol{\alpha := \lim_{n\to\infty} \int_X f_n\, d\mu}$
$\Rightarrow$ $\boldsymbol{\alpha \leq \int_X f\, d\mu}$, because of $(*)$.

Claim: $\alpha \geq \int_X f\, d\mu$.

- In fact $\forall \varepsilon > 0$, $\forall s \in \mathcal{S}_f = \{s : X \to \mathbb{R}$ measurable simple function, $0 \leq s \leq f$ in $X\}$ let:
  $E_n := \{(1-\varepsilon)\, s \leq f_n\} \quad \forall n \in \mathbb{N}$
  so that:
  (a) $\{E_n\}_n \subseteq \mathcal{A}$
  namely E_n is measurable $\forall n \in \mathbb{N}$, which follows from the fact that s and f_n ($\forall n \in \mathbb{N}$) are measurable (if f, g are measurable then also $\{f \leq g\}$ is measurable)
  (b) $\{E_n\}_n \nearrow$ because $\{f_n\}_n \nearrow$
  (c) $X = \bigcup_{n=1}^{\infty} E_n$
  proof.
  Clearly $\bigcup_{n=1}^{\infty} E_n \subseteq X$, since by definition $E_n \subseteq X$ $\forall n \in \mathbb{N}$. $\quad E_n = \{x \in X : (1-\varepsilon)s(x) \leq f_n(x)\}$
  Let $x \in X$. (we need to prove that $x \in \bigcup_{n=1}^{\infty} E_n$)
  If $f(x) = \infty$:
  $\Rightarrow$ $\exists \bar{n} \in \mathbb{N}$: $(1-\varepsilon)\, s(x) < f_n(x)$ $\forall n > \bar{n}$, since $f_n(x) \xrightarrow[n\to\infty]{} f(x) = \infty$ $\Rightarrow$ $x \in E_n$ $\forall n > \bar{n}$.
  If $f(x) < \infty$:
  $\Rightarrow$ $\exists \bar{n} \in \mathbb{N}$: $(1-\varepsilon)\, \underbrace{s(x) \leq (1-\varepsilon)\, f(x)}_{s \in \mathcal{S}_f} \leq f_n(x)$ $\Rightarrow$ $x \in E_n$ $\forall n > \bar{n}$.
  Therefore, $X \subseteq \bigcup_{n=1}^{\infty} E_n$, and so, $X = \bigcup_{n=1}^{\infty} E_n$. ■

- We now observe that:
  $(1-\varepsilon) \int_{E_n} s\, d\mu \leq \int_{E_n} f_n\, d\mu \underbrace{\leq \int_X f_n\, d\mu}_{f_n \geq 0,\ E_n \subseteq X\ \forall n \in \mathbb{N}} \quad (**)$

- Because of (c), it holds:
  $\lim_{n\to\infty} \int_{E_n} s\, d\mu = \int_X s\, d\mu$.
  In fact, $\forall E \in \mathcal{A}$ define $\varphi(E) := \int_E s\, d\mu$, so that φ is a measure.
  $\Rightarrow$ $\lim_{n\to\infty} \varphi(E_n) \underbrace{=}_{\varphi \text{ measure},\ \{E_n\}_n \nearrow} \varphi(\bigcup_{n=1}^{\infty} E_n) \underbrace{=}_{(c)} \varphi(X) = \int_X s\, d\mu$

- Considering $(**)$ and letting $n \to \infty$:
  $(1-\varepsilon) \int_X s\, d\mu \leq \lim_{n\to\infty} \int_X f_n\, d\mu = \alpha$
  $\Rightarrow$ $\int_X s\, d\mu \leq \alpha$, since ε is arbitrary
  $\Rightarrow$ $\boldsymbol{\int_X f\, d\mu} = \sup_{s \in \mathcal{S}_f} \int_X s\, d\mu \boldsymbol{\leq \alpha}$. ■

5.4. State and prove the theorem concerning **integration of series** with general terms $f_n \in \mathcal{M}_+(X, \mathcal{A})$.

Theorem. **Series integration ($f_n \in \mathcal{M}_+(X, \mathcal{A})$)**
Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then:
$\int_X \left(\sum_{n=1}^{\infty} f_n \, d\mu\right) = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$.

proof.

Since f_n measurable and positive $\forall n \in \mathbb{N}$:
$\Rightarrow \sum_{n=1}^{k} f_n \in \mathcal{M}_+(X, \mathcal{A})$
$\Rightarrow \sum_{n=1}^{\infty} f_n := \lim_{k\to\infty} \sum_{n=1}^{k} f_n \in \mathcal{M}_+(X, \mathcal{A})$.
We define:
$\sigma_k := \sum_{n=1}^{k} f_n$.
In this way:
(a) $\lim_{k\to\infty} \sigma_k = \sum_{n=1}^{\infty} f_n$
(b) $\{\sigma_k\}_k \subseteq \mathcal{M}_+(X, \mathcal{A})$
(c) $\{\sigma_k\}_k \nearrow$, since $f_k \geq 0 \quad \forall k \in \mathbb{N}$
Then, since σ_k is a finite sum, it is possible to apply the property of the integral to obtain:
$\Rightarrow \int_X \sigma_k \, d\mu = \sum_{n=1}^{k} \int_X f_n \, d\mu \quad \forall k \in \mathbb{N}$

$$\Rightarrow \quad \underbrace{\lim_{k\to\infty} \int_X \sigma_k \, d\mu}_{\substack{\text{by MCT:} \\ \underbrace{\int_X \left(\lim_{k\to\infty} \sigma_k\right) d\mu}_{\substack{\text{equivalent to:} \\ \int_X \left(\sum_{k=1}^{\infty} f_n\right) d\mu}}}} \quad = \quad \underbrace{\lim_{k\to\infty} \sum_{n=1}^{k} \int_X f_n \, d\mu}_{\substack{\text{equivalent to:} \\ \sum_{n=1}^{\infty} \int_X f_n \, d\mu}}$$

■

5.5. State and prove the **Fatou's Lemma**.

Lemma. **Fatou's Lemma**
Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then:
$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$.

proof.

Since f_n are measurable and positive $\forall n \in \mathbb{N} \Rightarrow \liminf_{n\to\infty} f_n \in \mathcal{M}_+(X, \mathcal{A})$.
We define $g_k : X \to \overline{\mathbb{R}}$ as:
$g_k := \inf_{n \geq k} f_n$.
In this way:
(a) $\{g_k\}_k \subseteq \mathcal{M}_+(X, \mathcal{A})$
(b) $\{g_k\}_k \nearrow$
(c) $g_k \leq f_k \quad \forall k \in \mathbb{N}$
(d) $\liminf_{n\to\infty} f_n = \sup_{k\geq 1} \inf_{n\geq k} f_n = \sup_{k \geq 1} g_k = \lim_{k\to\infty} g_k$
Then, starting from (c) it is possible to obtain:
$\Rightarrow \int_X g_k \, d\mu \leq \int_X f_k d\mu \quad \forall k \in \mathbb{N}$

$$\Rightarrow \quad \underbrace{\liminf_{k\to\infty} \int_X g_k \, d\mu}_{\substack{\{g_k\}_k \nearrow \, \Rightarrow \, \{\int_X g_k \, d\mu\}_k \nearrow, \\ \{\int_X g_k \, d\mu\}_k \subseteq \mathbb{R} \text{ and a monotone} \\ \text{sequence of real numbers} \\ \text{always admits a limit:} \\ \underbrace{\lim_{k\to\infty} \int_X g_k \, d\mu}_{\substack{\text{by MCT:} \\ \underbrace{\int_X \left(\lim_{k\to\infty} g_k\right) d\mu}_{\substack{\text{by (d):} \\ \int_X \left(\liminf_{n\to\infty} f_n\right) d\mu}}}}}} \quad \leq \quad \liminf_{k\to\infty} \int_X f_k \, d\mu$$

■

Prop. There are cases in which Fatou's lemma holds with strict inequality.
Let $(X, \mathcal{A}, \mathbb{N}) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^{\#})$, $f_n = \mathbb{1}_{\{n\}}$.
Then:
- $\lim_{n\to\infty} f_n = 0 \Rightarrow \int_{\mathbb{N}} \left(\liminf_{n\to\infty} f_n\right) d\mu^{\#} = \int_{\mathbb{N}} \left(\lim_{n\to\infty} f_n\right) d\mu^{\#} = 0$
- $\int_{\mathbb{N}} f_n \, d\mu^{\#} = \sum_{k \in \mathbb{N}} f_n(k) = 1 \quad \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu^{\#} = 1$

5.6. Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Show that $\nu(E) := \int_E f \, d\mu$ is a measure, state and prove its properties.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in \mathcal{M}_+(X, \mathcal{A})$.
Then, the function $\nu : \mathcal{A} \to \overline{\mathbb{R}}_+$ defined as:
$\nu(E) := \int_E f \, d\mu \quad \forall E \in \mathcal{A}$
is a measure.
Moreover, it holds:
(i) $g \in \mathcal{M}_+(X, \mathcal{A}) \Rightarrow \int_X g \, d\nu = \int_X g f \, d\mu$
(ii) $\forall E \in \mathcal{A}$: $\mu(E) = 0 \Rightarrow \nu(E) = 0$

proof.

- Since μ is a measure $\Rightarrow \mu(\varnothing) = 0 \Rightarrow \int_\varnothing f\,d\mu = 0 \Leftrightarrow \nu(\varnothing) = 0$.
  Let $\{E_n\}_n \subseteq \mathcal{A}$ be a disjoint collection of sets and let $E := \bigcup_{k=1}^\infty E_k$.
  $\Rightarrow \nu(E) = \int_X f\,\mathbb{1}_E\,d\mu$
  $= \int_X f \sum_{k=1}^\infty \mathbb{1}_{E_k}\,d\mu$
  $= \sum_{k=1}^\infty \int_X f\,\mathbb{1}_{E_k}\,d\mu$ because of the integration for series' theorem
  $= \sum_{k=1}^\infty \nu(E_k)$ since $\int_X f\,\mathbb{1}_{E_k}\,d\mu = \nu(E_k)$
  $\Rightarrow$ σ-additivity holds.

(i) Let $g \equiv s \in \mathcal{S}_+(X,\mathcal{A})$. It is possible to write the simple function s using the canonical form:
$s := \sum_{k=1}^n c_k\,\mathbb{1}_{F_k}$, with $\{F_k\}_k \subseteq \mathcal{A}$ disjoint, $X = \bigcup_{k=1}^n F_k$.
Then:
$\Rightarrow \int_X s\,d\nu = \sum_{k=1}^n c_k\,\nu(F_k)$ by definition of simple function
$= \sum_{k=1}^n c_k \int_{F_k} f\,d\mu$ by definition of $\nu(F_k)$
$= \sum_{k=1}^n c_k \int_X f\,\mathbb{1}_{F_k}\,d\mu$
$= \int_X \left(\sum_{k=1}^n c_k\,f\,\mathbb{1}_{F_k}\right) d\mu$
$= \int_X f \sum_{k=1}^n c_k\,\mathbb{1}_{F_k}\,d\mu$
$= \int_X f\,s\,d\mu$ by definition of s
Thus, we proved the thesis for the special case in which g is a simple function.
If $g \in \mathcal{M}_+(X,\mathcal{A})$ then, by the simple approximation theorem, we get the thesis.

(ii) If $\mu(E) = 0 \Rightarrow \int_E f\,d\mu = 0 \Leftrightarrow \nu(E) = 0$. ■

5.4 Set of Zero Measure and Integrals

5.7. Let $f, g \in \mathcal{M}_+(X,\mathcal{A})$ with $f = g$ a.e. in X. Show that $\int_X f\,d\mu = \int_X g\,d\mu$.

Theorem. Let $(X,\mathcal{A},\mu)$ be a measure space, $f, g \in \mathcal{M}_+(X,\mathcal{A})$ such that $f = g$ a.e. in X.
Then $\int_X f\,d\mu = \int_X g\,d\mu$.

proof.

Let $N := \{x \in X : f \neq g\}$.
Since $f, g \in \mathcal{M}_+(X,\mathcal{A})$ and $f = g$ a.e. in X:
$\Rightarrow N \in \mathcal{A}$, $\mu(N) = 0$
$\Rightarrow \int_N f\,d\mu = \int_N g\,d\mu = 0$, as they are integral on a set of zero measure
$\Rightarrow \int_X f\,d\mu = \underbrace{\int_N f\,d\mu}_{=0} + \int_{N^c} f\,d\mu = \underbrace{\int_{N^c} f\,d\mu = \int_{N^c} g\,d\mu}_{\text{on } N^c:\ f=g} = \int_{N^c} g\,d\mu + \underbrace{\int_N g\,d\mu}_{=0} = \int_X g\,d\mu$. ■

Corollary. Let $(X,\mathcal{A},\mu)$ be a measure space, $f \in \mathcal{M}_+(X,\mathcal{A})$. Then:
(i) $\int_X f\,d\mu = 0 \Leftrightarrow$ (ii) $f = 0$ a.e. in X.

proof.

(i) $\Rightarrow$ (ii): proved by the Vanishing lemma.
(i) $\Leftarrow$ (ii): previous theorem with $g = 0$, since $\int_X 0\,d\mu = 0$. ■

Therefore, in the definition of $\int_X f\,d\mu$, $f \in \mathcal{M}_+(X,\mathcal{A})$, the sets of zero measure are not essential.

Theorem. **Monotone convergence theorem (Beppo Levi) - refined version**
Let $(X,\mathcal{A},\mu)$ be a measure space.
Let $f_n, f \in \mathcal{M}_+(X,\mathcal{A})$ defined a.e. in X be such that:
(i) $f_n \le f_{n+1}$ a.e. in X $\forall n \in \mathbb{N}$
(ii) $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X
Then:
$\int_X f_n\,d\mu \xrightarrow[n\to\infty]{} \int_X f\,d\mu$.

5.5 Integrable Functions

5.8. Write the definition of: **integrable functions, Lebesgue integral, $\mathcal{L}^1(X,\mathcal{A},\mu)$.**

Let $(X,\mathcal{A},\mu)$ be a measure space.
The function $f : X \to \overline{\mathbb{R}}$ is said to be integrable on X if $f \in \mathcal{M}(X,\mathcal{A})$ and:
$\int_X f_+\,d\mu < \infty$, $\int_X f_-\,d\mu < \infty$.
We denote by:
$\mathcal{L}^1(X,\mathcal{A},\mu) := \{f : X \to \overline{\mathbb{R}} : f \text{ is integrable in } X\}$.
Then, the Lebesgue integral of a function $f \in \mathcal{L}^1(X,\mathcal{A},\mu)$ is defined as:
$\int_X f\,d\mu := \int_X f_+\,d\mu - \int_X f_-\,d\mu$. $(f = f_+ - f_-)$
This is always well defined because $f \in \mathcal{L}^1(X,\mathcal{A},\mu)$ and so, both $\int_X f_+\,d\mu$ and $\int_X f_-\,d\mu$ are finite.
If $E \in \mathcal{A}$, the Lebesgue integral of f over E is defined as:
$\int_E f\,d\mu = \int_X f\,\mathbb{1}_E\,d\mu = \int_E f_+\,d\mu - \int_E f_-\,d\mu$.

$f \in \mathcal{L}^1(X,\mathcal{A},\mu)$
$\Longrightarrow f$ finite a.e. in X

5.9. Let $f : X \to \overline{\mathbb{R}}$. How the integrability of f is related to that of $f_\pm$ and of $|f|$? Justify the answer. Show that if $f \in \mathcal{L}^1$, then $|\int_X f\,d\mu| \le \int_X |f|\,d\mu$.

(Recall) Let $(X, \mathcal{A}, \mu)$ be a measure space.
The function $f : X \to \overline{\mathbb{R}}$ is said to be integrable on X if $f \in \mathcal{M}(X, \mathcal{A})$ and: $\int_X f_+\,d\mu < \infty$, $\int_X f_-\,d\mu < \infty$.
We denote by: $\mathcal{L}^1(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : \text{integrable in } X\}$.

Prop. Let $(X, \mathcal{A}, \mu)$ be a measure space, $f : X \to \overline{\mathbb{R}}$. Then:
(i) $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow f_\pm \in \mathcal{L}^1(X, \mathcal{A}, \mu)$
(ii) $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow f \in \mathcal{M}(X, \mathcal{A}),\ |f| \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow f \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f|\,d\mu < \infty$
(iii) $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Rightarrow |\int_X f\,d\mu| \le \int_X |f|\,d\mu$

Property (ii) is an alternative characterization of $\mathcal{L}^1(X, \mathcal{A}, \mu)$ functions.

proof.

(i) By previous results we known that $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow f_\pm \in \mathcal{M}(X, \mathcal{A})$.
Moreover, it holds: $(f_+)_+ = f_+$, $(f_+)_- = 0$, $(f_-)_+ = f_-$, $(f_-)_- = 0$.

$$\Rightarrow f_+ \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow \int_X (f_+)_+\,d\mu < \infty,\ \int_X (f_+)_-\,d\mu < \infty$$
$$\Leftrightarrow \int_X f_+\,d\mu < \infty, \text{ since } (f_+)_+ = f_+ \text{ and } (f_+)_- = 0$$
$$\Rightarrow f_- \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow \int_X (f_-)_+\,d\mu < \infty,\ \int_X (f_-)_-\,d\mu < \infty$$
$$\Leftrightarrow \int_X f_-\,d\mu < \infty, \text{ since } (f_-)_+ = f_- \text{ and } (f_-)_- = 0$$
$$\Rightarrow f_\pm \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow \int_X f_\pm\,d\mu < \infty \Leftrightarrow f \in \mathcal{L}^1(X, \mathcal{A}, \mu).$$

(ii) ($\Leftarrow$) By definition:
$$|f| \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Leftrightarrow |f| \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f|_\pm\,d\mu < \infty$$
$$\Leftrightarrow |f| \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f|\,d\mu < \infty, \text{ since } |f|_+ = |f| \text{ and } |f|_- = 0$$
Moreover, since $|f| = f_+ + f_-$:
$$\Rightarrow \int_X |f|\,d\mu = \int_X (f_+ + f_-)\,d\mu = \int_X f_+\,d\mu + \int_X f_-\,d\mu$$
Therefore, if $f \in \mathcal{M}(X, \mathcal{A})$ and $|f| \in \mathcal{L}^1(X, \mathcal{A}, \mu)$:
$$\Rightarrow \int_X |f|\,d\mu < \infty \Leftrightarrow \int_X f_\pm\,d\mu < \infty \Leftrightarrow f \in \mathcal{L}^1(X, \mathcal{A}, \mu).$$
($\Rightarrow$) Since $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Rightarrow f \in \mathcal{M}(X, \mathcal{A}) \Rightarrow |f| \in \mathcal{M}(X, \mathcal{A})$.
Moreover, since $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$:
$$\Rightarrow \int_X f_\pm\,d\mu < \infty$$
$$\Rightarrow \int_X |f|\,d\mu = \int_X f_+\,d\mu + \int_X f_-\,d\mu < \infty$$
$$\Rightarrow |f| \in \mathcal{L}^1(X, \mathcal{A}, \mu).$$

(iii)
$$\begin{aligned} |\textstyle\int_X f\,d\mu| &= |\textstyle\int_X f_+\,d\mu - \int_X f_-\,d\mu| && \text{since } f = f_+ - f_- \\ &\le |\textstyle\int_X f_+\,d\mu| + |\int_X f_-\,d\mu| && \text{triangular inequality} \\ &= \textstyle\int_X f_+\,d\mu + \int_X f_-\,d\mu && \text{since } f_\pm \ge 0 \text{ and so } \textstyle\int_X f_\pm\,d\mu \ge 0 \\ &= \textstyle\int_X |f|\,d\mu. \end{aligned}$$
■

5.10. Prove that $\mathcal{L}^1$ is a vector space.

(Recall) Let $(X, \mathcal{A}, \mu)$ be a measure space.
The function $f : X \to \overline{\mathbb{R}}$ is said to be integrable on X if $f \in \mathcal{M}(X, \mathcal{A})$ and: $\int_X f_+\,d\mu < \infty$, $\int_X f_-\,d\mu < \infty$.
We denote by: $\mathcal{L}^1(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : \text{integrable in } X\}$.
Equivalently, $\mathcal{L}^1(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f|\,d\mu < \infty\}$.

Prop. $\mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space.

proof.

Let $f, g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$, $\lambda \in \mathbb{R}$.
$\Rightarrow \int_X f_\pm\,d\mu < \infty$, $\int_X g_\pm\,d\mu < \infty$
$\Rightarrow f_\pm$, $g_\pm$ are finite a.e. in X
$\Rightarrow f = f_+ - f_-$, $g = g_+ - g_-$ are finite a.e. in X.
Since f, g are both measurable and finite a.e., the function $h := f + \lambda g$ is defined a.e. and measurable:
$\Rightarrow \int_X |h|\,d\mu \le \int_X |f|\,d\mu + |\lambda| \int_X |g|\,d\mu < \infty$
$\Rightarrow h$ measurable, $\int_X |h|\,d\mu < \infty$
$\Rightarrow h \in \mathcal{L}^1(X, \mathcal{A}, \mu)$
$\Rightarrow \mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space. *(it is closed w.r.t. sum and w.r.t. product by a real number)* ■

5.11. Let $f \in \mathcal{L}^1$ be such that $\int_E f\,d\mu = 0\ \forall E \in \mathcal{A}$. Show that $f = 0$ a.e. in X.

(Recall) **Lemma.** **Vanishing lemma** for non-negative functions
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\,d\mu = 0$.
Then $f = 0$ a.e. in X.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ such that $\int_E f\,d\mu = 0\ \forall E \in \mathcal{A}$.
Then $f = 0$ a.e. in X.

proof.

Let $E_+ := \{x \in X : f(x) \ge 0\}$, $E_- := \{x \in X : f(x) < 0\}$.
Since $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Rightarrow f \in \mathcal{M}(X, \mathcal{A}) \Rightarrow E_+ = \{f \ge 0\}$, $E_- = \{f < 0\} \in \mathcal{A}$.
Thus, the hypothesis $\int_E f\,d\mu = 0$ holds also for $E = E_+$, $E = E_-$.
In E_+: $f \ge 0 \Rightarrow$ by vanishing lemma: $\int_{E_+} f\,d\mu = 0 \Rightarrow f = 0$ a.e. in E_+.
In E_-: $f < 0 \Rightarrow$ by vanishing lemma: $\int_{E_-} (-f)\,d\mu = -\int_{E_-} f\,d\mu = 0 \Rightarrow -f = 0$ a.e. in $E_- \Rightarrow f = 0$ a.e. in E_-.
Then, $f = 0$ a.e. in $E_+ \cup E_- = X$. ■

5.12. Let $f \in \mathcal{L}^1$, $g \in \mathcal{M}$, $f = g$ a.e. in X. Show that $g \in \mathcal{L}^1$ and $\int_X g\,d\mu = \int_X f\,d\mu$.

(Recall) **Theorem.** Let $(X, \mathcal{A}, \mu)$ be a measure space, $f, g \in \mathcal{M}_+(X, \mathcal{A})$ such that $f = g$ a.e. in X. Then $\int_X f\,d\mu = \int_X g\,d\mu$.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$, $g \in \mathcal{M}(X, \mathcal{A})$ such that $f = g$ a.e. in X. Then $g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ and $\int_X f\,d\mu = \int_X g\,d\mu$.

proof.

Since $f \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Rightarrow f \in \mathcal{M}(X, \mathcal{A})$.
Since $f, g \in \mathcal{M}(X, \mathcal{A})$ and $f = g$ a.e. in $X \Rightarrow f_\pm, g_\pm \in \mathcal{M}_+(X, \mathcal{A})$ and $f_+ = g_+$, $f_- = g_-$ a.e. in X.
By a previous result, if $f, g \in \mathcal{M}_+(X, \mathcal{A})$ such that $f = g$ a.e. in X then $\int_X f\,d\mu = \int_X g\,d\mu$:
$\Rightarrow \int_X f_+\,d\mu = \int_X g_+\,d\mu$, $\int_X f_-\,d\mu = \int_X g_-\,d\mu$.
Since $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$:
$\Rightarrow \int_X g_+\,d\mu = \int_X f_+\,d\mu < \infty$, $\int_X g_-\,d\mu = \int_X f_-\,d\mu < \infty$
$\Rightarrow g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ and:
$$\int_X g\,d\mu = \int_X g_+\,d\mu - \int_X g_-\,d\mu = \int_X f_+\,d\mu - \int_X f_-\,d\mu = \int_X f\,d\mu.$$
■

5.6 Convergence Theorems - part 2

5.13. State and prove the **Lebesgue theorem**.

Theorem. **Dominated Convergence Theorem (Lebesgue) - DCT**
Let $(X, \mathcal{A}, \mu)$ be a measure space.
Let $\{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$, $f \in \mathcal{M}(X, \mathcal{A})$ be such that $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X.
Suppose that $\exists\, g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ such that:
$|f_n| \le g$ a.e. in X $\forall n \in \mathbb{N}$.
Then:
(i) $f_n, f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ $\forall n \in \mathbb{N}$
(ii) $\int_X |f_n - f|\,d\mu \xrightarrow[n\to\infty]{} 0$
(iii) $\int_X f_n\,d\mu \xrightarrow[n\to\infty]{} \int_X f\,d\mu$ $\quad \left(\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X \underbrace{\lim_{n\to\infty} f_n}_{:=f}\,d\mu\right)$

proof.

(i) We check that $f_n, f \in \mathcal{L}^1(X, \mathcal{A}, \mu) = \{f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}),\ \int_X |f|\,d\mu < \infty\}$.
Since $f_n, f \in \mathcal{M}(X, \mathcal{A})$ $\forall n \in \mathbb{N} \Rightarrow |f_n|, |f| \in \mathcal{M}(X, \mathcal{A})$ $\forall n \in \mathbb{N}$.
By assumptions $|f_n| \le g$ a.e. in X $\forall n \in \mathbb{N}$, and so, since $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X:
$\Rightarrow |f| \le g$ a.e. in X.
By assumptions $g \in \mathcal{L}^1(X, \mathcal{A}, \mu)$, which implies $\int_X g\,d\mu < \infty$.
Therefore, since $|f| \le g$, $|f_n| \le g$ $\forall n \in \mathbb{N}$:
$$\Rightarrow \begin{array}{l} \int_X |f|\,d\mu \le \int_X g\,d\mu < \infty \Rightarrow \boldsymbol{f \in \mathcal{L}^1(X, \mathcal{A}, \mu)} \\ \int_X |f_n|\,d\mu \le \int_X g\,d\mu < \infty \Rightarrow \boldsymbol{f_n \in \mathcal{L}^1(X, \mathcal{A}, \mu)} \quad \forall n \in \mathbb{N} \end{array}$$
(ii) Since g, f, f_n $\forall n \in \mathbb{N}$ are finite a.e. we can define $g_n := 2g - |f_n - f|$ $\forall n \in \mathbb{N}$.
Then, $\{g_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$, since every g_n is defined as the difference of two measurable functions.
$(g \in \mathcal{L}^1(X, \mathcal{A}, \mu) \Rightarrow g \in \mathcal{M}(X, \mathcal{A})$ and $f, f_n \in \mathcal{M}(X, \mathcal{A}) \Rightarrow |f - f_n| \in \mathcal{M}(X, \mathcal{A}))$
Moreover, since $|f_n - f| \le |f_n| + |f| \le 2g$ a.e. in X $\forall n \in \mathbb{N}$:
$\Rightarrow g_n \ge 0$ a.e. in X $\forall n \in \mathbb{N}$
$\Rightarrow \{g_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$
Then:
$$\begin{aligned} 2\int_X g\,d\mu &= \int_X \left(\lim_{n\to\infty} g_n\right) d\mu && \text{since } f_n \xrightarrow[n\to\infty]{} f\text{, the pointwise limit of } g_n \text{ is } 2g \\ &\le \liminf_{n\to\infty} \int_X g_n\,d\mu && \text{Fatou's lemma} \\ &= \liminf_{n\to\infty} \int_X (2g - |f_n - f|)\,d\mu && \text{by definition of } g_n \\ &= \int_X 2g\,d\mu + \liminf_{n\to\infty}\left(-\int_X |f_n - f|\,d\mu\right) && \text{since } g \text{ does not depend on } n \\ &= \int_X 2g\,d\mu - \limsup_{n\to\infty}\left(\int_X |f_n - f|\,d\mu\right) \end{aligned}$$
$\Rightarrow \limsup_{n\to\infty} \int_X |f_n - f|\,d\mu \le 0$
$\Rightarrow \limsup_{n\to\infty} \int_X |f_n - f|\,d\mu = 0$
But, for positive sequences $lim\ sup = lim$:
$\Rightarrow \boldsymbol{\lim_{n\to\infty} \int_X |f_n - f|\,d\mu = 0}$.
(iii) By result of point (ii):
$\left|\int_X f_n\,d\mu - \int_X f\,d\mu\right| = \left|\int_X (f_n - f)\,d\mu\right| \le \int_X |f_n - f|\,d\mu \xrightarrow[n\to\infty]{} 0$
$\Rightarrow \boldsymbol{\int_X f_n\,d\mu \xrightarrow[n\to\infty]{} \int_X f\,d\mu}$. ■

Prop. In the context of the Lebesgue theorem (DCT), if the following conditions hold:
(i) $\mu(X) < \infty$
(ii) $\exists\, M > 0$: $|f_n| \le M$ a.e. in X $\forall n \in \mathbb{N}$
then we can take $g := M$.
Indeed: $\int_X g\,d\mu = M\,\mu(X) < \infty$.

5.15. State and prove the theorem concerning **integration for series** with general terms $f_n \in \mathcal{L}^1$.

Theorem. **Series integration ($f_n \in \mathcal{L}^1(X, \mathcal{A}, \mu)$)**
Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{f_n\}_n \subseteq \mathcal{L}^1(X, \mathcal{A}, \mu)$ be such that:
$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$.
Then, the series $\sum_{n=1}^{\infty} f_n$ converges a.e. in X and:
$\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$.

proof.

Since $\{f_n\}_n \subseteq \mathcal{L}^1(X, \mathcal{A}, \mu) \Rightarrow \{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$
$\Rightarrow \{|f_n|\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$
$\Rightarrow \sum_{n=1}^{k} |f_n| \in \mathcal{M}_+(X, \mathcal{A})$
$\Rightarrow \sum_{n=1}^{\infty} |f_n| := \lim_{k\to\infty} \sum_{n=1}^{k} |f_n| \in \mathcal{M}_+(X, \mathcal{A})$.

Then, since $\sum_{n=1}^{k} |f_n|$ is positive, measurable, monotone and converging, we apply MCT:
$\Rightarrow \int_X \left(\sum_{n=1}^{\infty} |f_n|\right) d\mu = \sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$, by assumptions
$\Rightarrow \sum_{n=1}^{\infty} |f_n| \in \mathcal{L}^1(X, \mathcal{A}, \mu)$
$\Rightarrow \sum_{n=1}^{\infty} |f_n|$ is finite a.e. in X
$\Rightarrow \sum_{n=1}^{\infty} f_n$ absolutely converges a.e. in X ($\Rightarrow \sum_{n=1}^{\infty} f_n$ converges a.e. in X)

We define:
$\sigma_k := \sum_{n=1}^{k} f_n$.

In this way:
(i) $\{\sigma_k\}_k \subseteq \mathcal{M}(X, \mathcal{A})$
(ii) $\sigma_k \xrightarrow[k\to\infty]{} \sum_{n=1}^{\infty} f_n$
(iii) $|\sigma_k| = |\sum_{n=1}^{k} f_n| \leq \sum_{n=1}^{k} |f_n| \leq \sum_{n=1}^{\infty} |f_n| := g \in \mathcal{L}^1(X, \mathcal{A}, \mu) \quad \forall k \in \mathbb{N}$

Hence, $\{\sigma_k\}_k$ satisfies al the assumptions of DCT:

$\Rightarrow \underbrace{\lim_{k\to\infty} \int_X \sigma_k \, d\mu}_{\text{equivalent to:}} = \int_X \lim_{k\to\infty} \sigma_k \, d\mu = \int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu$

$\underbrace{\lim_{k\to\infty} \int_X \sum_{n=1}^{k} f_n \, d\mu}_{\text{since it is a finite sum:}}$

$\underbrace{\lim_{k\to\infty} \sum_{n=1}^{k} \int_X f_n \, d\mu}_{\text{equivalent to:}}$

$\sum_{n=1}^{\infty} \int_X f_n \, d\mu$ ∎

5.7 Riemann and Lebesgue Integrals

5.14. Describe the relations between Riemann and Lebesgue integrals.

Consider $I = [a, b]$, with $a, b \in \mathbb{R}$.
Let $\mathcal{R}(I) := \{f : I = [a, b] \to \mathbb{R} : f \text{ Riemann-integrable}\}$.

Theorem. Let $I = [a, b] \subset \mathbb{R}$, $f : I \to \mathbb{R}$ bounded. Then:
$f \in \mathcal{R}(I) \Leftrightarrow f$ is continuous λ-a.e. in I. *(the set of discontinuity points of f has zero λ-measure)*

Theorem. Let $I = [a, b] \subset \mathbb{R}$. Then:
$f \in \mathcal{R}(I) \Rightarrow f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and $\int_I f \, d\lambda = \int_a^b f(x) \, dx$

The situation is different when we consider improper integrals.
Consider $I = (\alpha, \beta)$ with $\alpha, \beta \in \overline{\mathbb{R}}$.
Let $\mathcal{R}^g(I) := \{f : I \to \mathbb{R} : \text{integrable in the generalized sense}\}$.

Theorem. Let $I = (\alpha, \beta)$, with $\alpha, \beta \in \overline{\mathbb{R}}$. Then:
(i) $f \in \mathcal{R}^g(I) \Rightarrow f \in \mathcal{M}(I, \mathcal{L}(I))$
(ii) $|f| \in \mathcal{R}^g(I) \Rightarrow f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and $\int_I f \, d\lambda = \int_\alpha^\beta f(x) \, dx$

In the comparison of Lebesgue and Riemann we have to consider the interval wehere the function is defined. If the interval is bounded, the Lebesgue integral is a generalization of the Riemann integral. If, instead, we consider improper integrals, the Lebesgue integral is concerned with $|f|$.

Examples:
- Let $I = [0, 1]$, $f = \mathbb{1}_{I\cap\mathbb{Q}}$. Then $f \notin \mathcal{R}(I)$, however $f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and $\int_I f \, d\lambda = 0$.
- Let $f : I = [0, \infty) \to \mathbb{R}$ be the Dirichlet function, namely:
$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
Then $f \in \mathcal{R}^g(I)$ since $\int_0^\infty \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}$, however $\int_{\mathbb{R}_+} |\frac{\sin(x)}{x}| \, d\lambda = \infty \Rightarrow f \notin \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$.

5.8 Derivative of Measures

5.16. Write the definition of **Randon-Nikodym derivative** of a measure ν w.r.t. a measure μ.
When ν is said to be absolutely continuous w.r.t. μ?

Let $(X, \mathcal{A})$ be a measurable space and let μ, ν be two measures.
A function $\Phi \in \mathcal{M}_+(X, \mathcal{A})$ is said to be the Radon-Nikodym derivative of ν w.r.t. μ if:
$\nu(E) = \int_E \Phi \, d\mu \quad \forall E \in \mathcal{A}$.
We denote it: $\Phi = \frac{d\nu}{d\mu}$.
Let μ, ν be two general measures *(not necessarily one the Radon-Nikodym derivative of the other)*.
We say that ν is absolutely continuous w.r.t. μ if:
$\forall E \in \mathcal{A}$ such that $\mu(E) = 0 \Rightarrow \nu(E) = 0$.
In this case we write $\nu \ll \mu$.

5.17. State the **Radon-Nikodym theorem**. Prove the uniqueness of the derivative of a measure.

(Recall) Let X be a set, $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\varnothing \in \mathcal{C}$, $\mu : \mathcal{C} \to \overline{R}_+$ a measure on $\mathcal{C}$.
Then μ is σ-finite if there exists $\{E_k\}_k \subseteq \mathcal{C}$ such that: $X = \bigcup_{k=1}^{\infty} E_k$ and $\mu(E_k) < \infty \ \forall k \in \mathbb{N}$.

Theorem. **Radon-Nikodym**
Let $(X, \mathcal{A})$ be a measurable space and let μ, ν be two measures.
Assume that $\nu \ll \mu$ and that μ is σ-finite.
Then $\frac{d\nu}{d\mu}$ exists.

Theorem. **Uniqueness of the Radon-Nikodym derivative**
Let $(X, \mathcal{A})$ be a measurable space and let μ, ν be two measures.
If it exists, the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is unique a.e..

proof.

By contraddiction, suppose that $\exists\, \Phi_1, \Phi_2 \in \mathcal{M}_+(X, \mathcal{A})$ and both are R-N derivative of ν w.r.t. μ.
$\Rightarrow \nu(E) = \int_E \Phi_1 \, d\mu = \int_E \Phi_2 \, d\mu \quad \forall E \in \mathcal{A}$
$\Rightarrow \int_E (\Phi_1 - \Phi_2) \, d\mu = 0 \quad \forall E \in \mathcal{A}$
$\Rightarrow \Phi_1 - \Phi_2 = 0$ a.e. in X $\Leftrightarrow$ $\Phi_1 = \Phi_2$ a.e. in X. ■

5.9 L^1 and L^∞ Spaces

5.18. Write the definitions of L^1 and of L^∞. Show that they are metric spaces. Are $\mathcal{L}^1$ and $\mathcal{L}^\infty$ metric spaces?

(Recall) Let $(X, \mathcal{A}, \mu)$ be a measurable space.
A function $f : X \to \overline{\mathbb{R}}$ is integrable on X if $f \in \mathcal{M}(X, \mathcal{A})$ and $\int_X f_+ \, d\mu < \infty$, $\int_X f_- \, d\mu < \infty$.
Then we define:
$\mathcal{L}^1(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : f \text{ is integrable in } X\}$.
Let $\mathcal{N}_\mu := \{E \subseteq X : E \text{ measurable}, \mu(E) = 0\}$.
The essential supremum of $f : X \to \overline{\mathbb{R}}$ is defined as $\operatorname{ess\,sup}_X f := \inf_{N \in \mathcal{N}_\mu} \{\sup_{x \in N^c} f(x)\}$.
A function $f \in \mathcal{M}(X, \mathcal{A})$ is essentially bounded in X if $\operatorname{ess\,sup}_X |f| < \infty$.
Then we define:
$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}) \text{ essentially bounded}\}$.

Let $(X, \mathcal{A}, \mu)$ be a measure space.
We say that two functionfs f, g are in relation $f \sim g \Leftrightarrow f = g$ a.e. in X, which is an equivalence relation.
We consider $\mathcal{L}^1(X, \mathcal{A}, \mu)$ and $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$ and we define:

$L^1(X, \mathcal{A}, \mu) := \frac{\mathcal{L}^1(X, \mathcal{A}, \mu)}{\sim}$, $\quad L^\infty(X, \mathcal{A}, \mu) := \frac{\mathcal{L}^\infty(X, \mathcal{A}, \mu)}{\sim}$.

The elements of $L^1(X, \mathcal{A}, \mu)$ and $L^\infty(X, \mathcal{A}, \mu)$ are equivalence classes defined by the equivalence relation.
However, to simplify we'll say $f \in L^1(X, \mathcal{A}, \mu)$ or $f \in L^\infty(X, \mathcal{A}, \mu)$.

Lemma.
1. $L^1(X, \mathcal{A}, \mu)$ is a metric space with $d(f, g) := \int_X |f - g| \, d\mu \quad \forall f, g \in L^1(X, \mathcal{A}, \mu)$.
2. $L^\infty(X, \mathcal{A}, \mu)$ is a metric space with $d(f, g) := \operatorname{ess\,sup}_X |f - g| \quad \forall f, g \in L^\infty(X, \mathcal{A}, \mu)$.

proof.

1. Let f, g, $h \in L^1(X, \mathcal{A}, \mu)$.
   The function $d : L^1(X, \mathcal{A}, \mu) \times L^1(X, \mathcal{A}, \mu) \to \mathbb{R}_+$ is such that:
   - $f, g \in L^1(X, \mathcal{A}, \mu)$ (which implies $\int_X |f| \, d\mu < \infty$, $\int_X |g| \, d\mu < \infty$)
     $\Rightarrow d(f, g) = \int_X |f - g| \, d\mu \le \int_X |f| \, d\mu + \int_X |g| \, d\mu \in \mathbb{R}_+$
   - $d(f, g) = \int_X |f - g| \, d\mu = \int_X |g - f| \, d\mu = d(g, f)$
   - $d(f, g) = \int_X |f - g| \, d\mu \le \int_X (|f - h| + |h - g|) \, d\mu = \int_X |f - h| \, d\mu + \int_X |h - g| \, d\mu = d(f, h) + d(h, g)$
   - $f = g \Rightarrow d(f, g) = \int_X |f - g| \, d\mu = 0$
     $d(f, g) = \int_X |f - g| \, d\mu = 0 \Rightarrow |f - g| = 0$ a.e. in X, since $|f - g| \ge 0$
     $\Rightarrow f = g$ a.e. in X
     $\Rightarrow f = g$ in $L^1(X, \mathcal{A}, \mu)$

   Therefore, d is a distance and $(L^1(X, \mathcal{A}, \mu), d)$ a metric space.
2. Let f, g, $h \in L^\infty(X, \mathcal{A}, \mu)$.
   The function $d : L^\infty(X, \mathcal{A}, \mu) \times L^\infty(X, \mathcal{A}, \mu) \to \mathbb{R}_+$ is such that:
   - $f, g \in L^\infty(X, \mathcal{A}, \mu)$ (which implies $\operatorname{ess\,sup}_X |f| < \infty$, $\operatorname{ess\,sup}_X |g| < \infty$)
     $\Rightarrow d(f, g) = \operatorname{ess\,sup}_X |f - g| \le \operatorname{ess\,sup}_X |f| + \operatorname{ess\,sup}_X |g| \in \mathbb{R}_+$
   - $d(f, g) = \operatorname{ess\,sup}_X |f - g| = \operatorname{ess\,sup}_X |g - f| = d(g, f)$
   - $d(f, g) = \operatorname{ess\,sup}_X |f - g| \le \operatorname{ess\,sup}_X (|f - h| + |h - g|) \le \operatorname{ess\,sup}_X |f - h| + \operatorname{ess\,sup}_X |h - g| = d(f, h) + d(h, g)$
   - $f = g \Rightarrow d(f, g) = \operatorname{ess\,sup}_X |f - g| \, d\mu = 0$
     $d(f, g) = \operatorname{ess\,sup}_X |f - g| = 0 \Rightarrow |f - g| = 0$ a.e. in X, since $|f - g| \ge 0$
     $\Rightarrow f = g$ a.e. in X
     $\Rightarrow f = g$ in $L^\infty(X, \mathcal{A}, \mu)$

   Therefore, d is a distance and $(L^\infty(X, \mathcal{A}, \mu), d)$ a metric space. ■

However, $\mathcal{L}^1(X, \mathcal{A}, \mu)$ and $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$ are **not** metric spaces.
In fact, $d(f, g) = 0 \Rightarrow f = g$ only a.e. in X, which is not sufficient to say that $f = g$ in $\mathcal{L}^1(X, \mathcal{A}, \mu)$ or $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$.
Conversely, if $f = g$ a.e. in X then they belong to the same equivalence class and so they are equivalent in $L^1(X, \mathcal{A}, \mu)$ and $L^\infty(X, \mathcal{A}, \mu)$.

6 Types of Convergence

5.19. For a sequence of functions $\{f_n\}_n \subset \mathcal{M}$, write the definitions of: **pointwise** convergence, **uniform** convergence, **almost everywhere** convergence, convergence in $\boldsymbol{L^1}$, convergence in $\boldsymbol{L^\infty}$, convergence **in measure**.

Let $(X, \mathcal{A})$ be a measurable space.
Let $\{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$, $f_n : X \to \mathbb{R}$, $f : X \to \overline{\mathbb{R}}$.

Pointwise convergence	$f_n \xrightarrow[n\to\infty]{} f$ pointwise in X	$\Leftrightarrow$	$\forall x \in X : f_n(x) \xrightarrow[n\to\infty]{} f(x)$
Uniform convergence	$f_n \xrightarrow[n\to\infty]{} f$ uniformly in X	$\Leftrightarrow$	$\sup_{x\in X} \lvert f_n(x) - f(x)\rvert \xrightarrow[n\to\infty]{} 0$
Convergence a.e.	$f_n \xrightarrow[n\to\infty]{} f$ a.e. in X	$\Leftrightarrow$	$\{x \in X : f_n(x) \xrightarrow[n\to\infty]{} f(x)\}^c$ is measurable and of zero measure *pointwise convergence for almost any $x \in X$*
Convergence in L^1 $\{f_n\}_n \subseteq L^1$, $f \in L^1$	$f_n \xrightarrow[n\to\infty]{} f$ in L^1	$\Leftrightarrow$	$\int_X \lvert f_n(x) - f(x)\rvert \, d\mu =: d(f_n, f) \xrightarrow[n\to\infty]{} 0$
Convergence in L^∞ $\{f_n\}_n \subseteq L^\infty$, $f \in L^\infty$	$f_n \xrightarrow[n\to\infty]{} f$ in L^∞	$\Leftrightarrow$	$\text{ess sup}_X \lvert f_n - f\rvert =: d(f_n, f) \xrightarrow[n\to\infty]{} 0$
Convergence in measure $f \in \mathcal{M}(X, \mathcal{A})$ f, f_n finite a.e. in X	$f_n \xrightarrow[n\to\infty]{} f$ in measure	$\Leftrightarrow$ $\Leftrightarrow$	$\forall \varepsilon > 0 : \mu(\{\lvert f_n - f\rvert \geq \varepsilon\}) \xrightarrow[n\to\infty]{} 0$; $\forall \varepsilon > 0 \ \forall \sigma > 0 \ \exists \bar{n} \in \mathbb{N} :$ $\mu(\{\lvert f_n - f\rvert \geq \varepsilon\}) < \sigma \ \ \forall n \geq \bar{n}$

The above expression is well-defined because $\forall n \in \mathbb{N}$: $\{|f_n - f| \geq \varepsilon\} \in \mathcal{A}$ (since f_n and f are measurable functions). Furthermore, $\{|f_n - f| \geq \varepsilon\}$ is a sequence of sets, instead, $\mu(\{|f_n - f| \geq \varepsilon\})$ is a sqeuence of real numbers ($\in \mathbb{R}_+$).

5.20. Give two equivalent formulations of a.e. convergence.

Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $\mathcal{A}$
Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X $\forall n \in \mathbb{N}$ and let $B_n^\varepsilon := \{|f_n - f| \geq \varepsilon\}$ $\forall \varepsilon > 0$ $\forall n \in \mathbb{N}$.

Prop. The following statements hold:

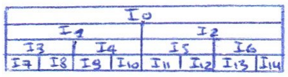
(i) $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X $\Leftrightarrow$ $\{x \in X : f_n(x) \xrightarrow[n\to\infty]{} f(x)\}^c$ is measurable and of zero measure

(ii) $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X $\Leftrightarrow$ $\mu(\ \limsup_{n\to\infty} B_n^\varepsilon\ ) = 0 \quad \forall \varepsilon > 0$

Moreover, if $\mu(X) < \infty$, it holds:

(iii) $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X $\Leftrightarrow$ $\lim_{k\to\infty} \mu(\bigcup_{n=k}^\infty B_n^\varepsilon) = 0 \quad \forall \varepsilon > 0$

5.21. Explain how the **Rademacher sequence** (also called the typewriter sequence) is defined.
In particular, write explicitely $I_1, I_2, .., I_{14}$.



Let $I = [0, 1]$, consider $(I, \mathcal{L}(I), \lambda)$ and define $\{I_n\}_n$ as follow:

$p = 0$: $\quad I_0 = I$

$p = 1$: $\quad I_1 = [0, \frac{1}{2}]$, $\ I_2 = [\frac{1}{2}, 1]$

$p = 2$: $\quad I_3 = [0, \frac{1}{4}]$, $\ I_4 = [\frac{1}{4}, \frac{1}{2}]$, $\ I_5 = [\frac{1}{2}, \frac{3}{4}]$, $\ I_6 = [\frac{3}{4}, 1]$

$p = 3$: $\quad I_7 = [0, \frac{1}{8}]$, $\ I_8 = [\frac{1}{8}, \frac{1}{4}]$, $\ I_9 = [\frac{1}{4}, \frac{3}{8}]$, $\ I_{10} = [\frac{3}{8}, \frac{1}{2}]$, $\ I_{11} = [\frac{1}{2}, \frac{5}{8}]$, $\ I_{12} = [\frac{5}{8}, \frac{3}{4}]$, $\ I_{13} = [\frac{3}{4}, \frac{7}{8}]$, $\ I_{14} = [\frac{7}{8}, 1]$

...

In this way, $\forall p = 0, 1, ..$ we obtain 2^p intervals, each one of length 2^{-p}.
An analytical version of this sequence is given by: $I_{n-1} = [\frac{n}{2^p} - 1, \frac{n+1}{2^p} - 1]$ $\ \forall p \geq 0,\ n \in [2^p, 2^{p+1})$.
Then, the Rademacher sequence (typewriter sequence) $\{f_n\}_n$ is defined as $f_n = \mathbb{1}_{I_n}$

6.1. By means of a counterexample, show that in general $f_n \to f$ in measure does not imply that $f_n \to f$ a.e..

in measure
$\not\Downarrow$
a.e.

Let $I = [0, 1]$ and consider $(I, \mathcal{L}(I), \lambda)$. Let $\{\mathbb{1}_{I_n}\}_n = \{\mathbb{1}_n\}_n$ be the Rademacher sequence.
Then:

(i) $\mathbb{1}_n \not\xrightarrow[n\to\infty]{} 0$ a.e. in $[0, 1]$

(ii) $\mathbb{1}_n \xrightarrow[n\to\infty]{} 0$ in measure

(iii) $\{I_{n_k}\}_k := \{[0, \frac{1}{2^k}]\}_k \subseteq \{I_n\}_n \Rightarrow \{\mathbb{1}_{n_k}\}_k \subseteq \{\mathbb{1}_n\}_n$ is such that: $\mathbb{1}_{n_k} \xrightarrow[k\to\infty]{} 0$ a.e. in $[0, 1]$

proof.

(i) We'll exploit an equivalent formulation of convergence a.e., namely:
Let $(X, \mathcal{A})$ be a measurable space, μ a measure on $\mathcal{A}$. Let $\mu(X) < \infty$.
Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. $\forall n \in \mathbb{N}$ and let $B_n^\varepsilon := \{|f_n - f| \geq \varepsilon\}$. Then:
$f_n \xrightarrow[n\to\infty]{} f$ a.e. in X $\Leftrightarrow$ $\lim_{k\to\infty} \mu(\bigcup_{n=k}^\infty B_n^\varepsilon) = 0 \quad \forall \varepsilon > 0$.
Let $f \equiv 0$.
Then $B_n^\varepsilon = \{|f_n - f| \geq \varepsilon\} = \{\mathbb{1}_n \geq \varepsilon\}$ $\forall \varepsilon \in (0, 1)$ $\forall n \in \mathbb{N}$. $\quad (B_0^\varepsilon = [0, 1],\ B_1^\varepsilon = [0, \frac{1}{2}],\ B_2^\varepsilon = [\frac{1}{2}, 1], ..)$
$\Rightarrow \lim_{k\to\infty} \lambda(\bigcup_{n=k}^\infty B_n^\varepsilon) = \lim_{k\to\infty} \lambda([0, 1]) = \lim_{k\to\infty} 1 = 1 \neq 0$
$\Rightarrow f_n \not\xrightarrow[n\to\infty]{} 0$ a.e. in X.

(ii) From (i) we have:
$\forall \varepsilon \in (0, 1)$: $\lim_{n\to\infty} \lambda(B_n^\varepsilon) = \lim_{n\to\infty} \frac{1}{2^n} = 0 \Leftrightarrow \mathbb{1}_n \xrightarrow[n\to\infty]{} 0$ in measure.

(iii) We consider now the subsequence $\{[0, \frac{1}{2^k}]\}_k \subseteq I_n$.
Then, $\forall x \in [0, 1]$ $\exists k$ big enough such that the interval on which $\mathbb{1}_{n_k} = 1$ is on the left of x.
$\Rightarrow \mathbb{1}_{n_k} \xrightarrow[k\to\infty]{} 0$ a.e. in X. ■

6.2. Prove that convergence in measure implies convergence a.e. up to subsequences.

(Recall) **Lemma.** **Borel-Cantelli**
Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{E_n\}_n \subseteq \mathcal{A}$.
If $\sum_{n=1}^{\infty} \mu(E_n) < \infty \Rightarrow \mu(\limsup_{n\to\infty} E_n) = \mu(\bigcap_{k=1}^{\infty} [\bigcup_{n=k}^{\infty} E_n]) = 0$.

Theorem. Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $\mathcal{A}$.
Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X $\forall n \in \mathbb{N}$.
If $f_n \xrightarrow[n\to\infty]{} f$ in measure $\Rightarrow \exists \{f_{n_k}\}_k \subseteq \{f_n\}_n$ such that: $f_{n_k} \xrightarrow[k\to\infty]{} f$ a.e. in X.

proof.

By definition, $f_n \xrightarrow[n\to\infty]{} f$ in measure $\Leftrightarrow \forall \varepsilon > 0\ \forall \sigma > 0\ \exists \bar{n} \in \mathbb{N} : \mu(\{|f_n - f| \geq \varepsilon\}) < \sigma\ \forall n \geq \bar{n}$.
We can choose $\varepsilon = \sigma = \frac{1}{2^p}$, with $p \in \mathbb{N}$. In this way we obtain the characterization:
$f_n \xrightarrow[n\to\infty]{} f$ in measure $\Leftrightarrow \forall p \in \mathbb{N}\ \exists n_p \in \mathbb{N}\ \forall n \geq n_p : \mu(\{|f_n - f| \geq \frac{1}{2^p}\}) < \frac{1}{2^p}$.
Setting $n = n_p$ we obtain $B_p = \{|f_{n_p} - f| \geq \frac{1}{2^p}\}$.
$\Rightarrow \mu(B_p) < \frac{1}{2^p}\ \forall p \in \mathbb{N}$.
Now we define:
$A_k := \bigcup_{p=k}^{\infty} B_p$
$A := \bigcap_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty} \bigcup_{p=k}^{\infty} B_p = \limsup_{p\to\infty} B_p$
$\Rightarrow \sum_{p=1}^{\infty} \mu(B_p) < \sum_{p=1}^{\infty} \frac{1}{2^p} = 1$
$\Rightarrow \mu(A) = 0$, by Borel-Cantelli lemma
Consider then $x \in A^c$:
$\Rightarrow x \in A^c = (\bigcap_{k=1}^{\infty} A_k)^c = \bigcup_{k=1}^{\infty} A_k^c = \bigcup_{k=1}^{\infty} (\bigcup_{p=k}^{\infty} B_p)^c = \bigcup_{k=1}^{\infty} (\bigcap_{p=k}^{\infty} B_p^c)$
$\Rightarrow \exists k \in \mathbb{N} : x \in \bigcap_{p=k}^{\infty} B_p^c$
$\Rightarrow \exists k \in \mathbb{N}\ \forall p \geq k : |f_{n_p}(x) - f(x)| < \frac{1}{2^p}\ (\xrightarrow[p\to\infty]{} 0)$
$\Rightarrow f_{n_p}(x) \xrightarrow[p\to\infty]{} f(x)$.
Thus, the convergence is guaranteed $\forall x \in A^c$.
The complementary of where we have convergence, namely $A^{c^c} = A$, is such that $\mu(A) = 0$.
Therefore, the convergence is satisfied out of a set of zero measure.
$\Rightarrow f_{n_p} \xrightarrow[p\to\infty]{} f$ a.e. in X. ∎

in measure
$\Downarrow$
$\exists \{f_{n_k}\}_k \to f$ a.e.

6.3. Under which hypothesis on X does convergence a.e. imply convergence in measure? Show this property. What happens if one omits the key assumption on X?

Theorem. Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $\mathcal{A}$. Let $\mu(X) < \infty$.
Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X $\forall n \in \mathbb{N}$.
If $f_n \xrightarrow[n\to\infty]{} f$ a.e. $\Rightarrow f_n \xrightarrow[n\to\infty]{} f$ in measure.

proof.

The aim is to prove that $\forall \varepsilon > 0 : \mu(\{|f_n - f| \geq \varepsilon\}) \xrightarrow[n\to\infty]{} 0 \Leftrightarrow f_n \xrightarrow[n\to\infty]{} f$ in measure.
For the sake of later convenience, since this is only a matter of notation we change the index:
$\forall \varepsilon > 0 : \mu(\{|f_k - f| \geq \varepsilon\}) \xrightarrow[k\to\infty]{} 0 \Leftrightarrow f_k \xrightarrow[k\to\infty]{} f$ in measure.
We'll exploit an equivalent formulation of convergence a.e., namely:
Let $(X, \mathcal{A})$ be a measurable space, μ a measure on $\mathcal{A}$. Let $\mu(X) < \infty$.
Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. $\forall n \in \mathbb{N}$ and let $B_n^\varepsilon := \{|f_n - f| \geq \varepsilon\}$. Then:
$f_n \xrightarrow[n\to\infty]{} f$ a.e. in $X \Leftrightarrow \lim_{k\to\infty} \mu(\bigcup_{n=k}^{\infty} B_n^\varepsilon) = 0\quad \forall \varepsilon > 0$.
Starting from convergence a.e., since $B_k^\varepsilon \subseteq \bigcup_{n=k}^{\infty} B_n^\varepsilon$ and μ is monotone:
$\Rightarrow \lim_{k\to\infty} \mu(\{|f_k - f| \geq \varepsilon\}) = \lim_{k\to\infty} \mu(B_k^\varepsilon) \leq \lim_{k\to\infty} \mu(\bigcup_{n=k}^{\infty} B_n^\varepsilon) = 0\quad \forall \varepsilon > 0$
$\Rightarrow f_k \xrightarrow[k\to\infty]{} f$ in measure. ∎

a.e.
$\mu(X) < \infty$
$\Downarrow$
in measure

The assumption that $\mu(X) < \infty$ is needed in the proof to introduce the equivalent formulation of convergence a.e..
However, the assumption that $\mu(X) < \infty$ is essential not only for the proof, but also for the validity of the theorem.
In fact, if $\mu(X) = \infty$ then convergence a.e. in X does **not** imply convergence in measure.
Consider $X = \mathbb{R}$, $f_n := \mathbb{1}_{[n,\infty)}$. Then:

- $f_n \xrightarrow[n\to\infty]{} 0$ pointwise in $\mathbb{R} \Rightarrow f_n \xrightarrow[n\to\infty]{} 0$ a.e. in $\mathbb{R}$
- $\forall n \in \mathbb{N}$: $\{f_n \geq \frac{1}{2}\} = [n, \infty) \Rightarrow \mu(\{f_n \geq \frac{1}{2}\}) = \infty \Rightarrow f_n \not\xrightarrow[n\to\infty]{} 0$ in measure

6.4. Show that convergence in L^1 implies convergence in measure.

Theorem. Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $\mathcal{A}$.
Let $f_n, f \in L^1(X, \mathcal{A}, \mu)$ $\forall n \in \mathbb{N}$.
If $f_n \xrightarrow[n\to\infty]{} f$ in $L^1 \Rightarrow f_n \xrightarrow[n\to\infty]{} f$ in measure.

proof.

We recall that $f_n \xrightarrow[n\to\infty]{} f$ in measure $\Leftrightarrow \forall \varepsilon > 0\ \forall \sigma > 0\ \exists \bar{n} \in \mathbb{N} : \mu(\{|f_n - f| \geq \varepsilon\}) < \sigma\ \forall n \geq \bar{n}$.
Suppose by contraddiction that $f_n \not\xrightarrow[n\to\infty]{} f$ in measure.
Then $\exists \varepsilon > 0$, $\sigma > 0$ such that:
$\mu(\{|f_n - f| \geq \varepsilon\}) \geq \sigma$
for infinitely many n in $\mathbb{N}$.
Thus:
$d_{L^1}(f_n, f) = \int_X |f_n - f|\, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f|\, d\mu \geq \varepsilon \int_{\{|f_n - f| \geq \varepsilon\}} d\mu = \varepsilon\, \mu(\{|f_n - f| \geq \varepsilon\}) \geq \varepsilon\sigma > 0$
for infinitely many n in $\mathbb{N}$.
$\Rightarrow f_n \not\xrightarrow[n\to\infty]{} f$ in L^1, which is a contraddiction. ∎

L^1
$\Downarrow$
in measure

6.5. Show that convergence in L^1 implies convergence a.e. up to subsequences.

L^1
$\Downarrow$
$\exists \{f_{n_k}\}_k \to f$ a.e.

Theorem. Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $\mathcal{A}$.
Let $f_n,\ f \in L^1(X, \mathcal{A}, \mu)\ \forall n \in \mathbb{N}$.
If $f_n \xrightarrow[n\to\infty]{} f$ in $L^1 \Rightarrow \exists \{f_{n_k}\}_k \subseteq \{f_n\}_n$ such that: $f_{n_k} \xrightarrow[k\to\infty]{} f$ a.e. in X.

proof.

If $f_n \xrightarrow[n\to\infty]{} f$ in L^1 then $f_n \xrightarrow[n\to\infty]{} f$ in measure.
If $f_n \xrightarrow[n\to\infty]{} f$ in measure then $\exists \{f_{n_k}\}_k \subseteq \{f_n\}_n$ such that $f_{n_k} \xrightarrow[k\to\infty]{} f$ a.e. in X. ■

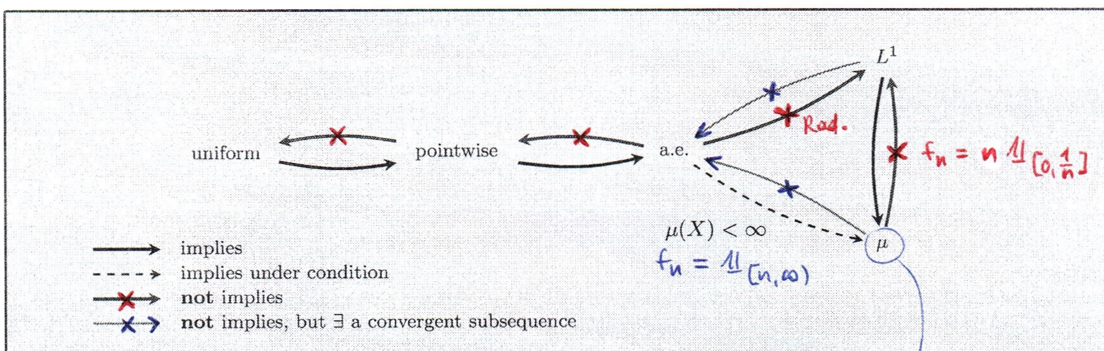
6.6. Does convergence in measure imply convergence in L^1? Does convergence a.e. imply convergence in L^1? Justify the answer.

in measure
$\not\Downarrow$
L^1

a.e.
$\not\Downarrow$
L^1

Let $(X, \mathcal{A}, \mu)$ be a measure space, $f_n,\ f \in L^1(X, \mathcal{A}, \mu)\ \forall n \in \mathbb{N}$.
Then, neither convergence in measure nor convergence a.e. implies convergence in L^1.
Consider $X = [0, 1]$, the Lebesgue measure λ and the sequence $f_n(x) := n\mathbb{1}_{[0,\frac{1}{n}]}(x)$. Then:

- $f_n \xrightarrow[n\to\infty]{} 0$ a.e. in $[0, 1] \Rightarrow f_n \xrightarrow[n\to\infty]{} 0$ in measure, since $\lambda(X) = \lambda([0, 1]) = 1 < \infty$
- $\int_0^1 |f_n(x) - 0|\, dx = \int_0^1 f_n(x)\, dx = n \int_0^1 \mathbb{1}_{[0,\frac{1}{n}]}(x)\, dx = n\frac{1}{n} = 1\ \forall n \in \mathbb{N} \Rightarrow f_n \not\xrightarrow[n\to\infty]{} 0$ in $L^1([0, 1])$



in general looking for convergence in μ is difficult ($\Rightarrow$ a.e. if $\mu(X) < \infty$, L^1 otherwise)

7 Product Measure

6.7. Write the definitions of: **product measurable space**, **section of measurable set**. What is the **product measure**? Why is the definition well-posed?

Let $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ be two measurable spaces.
Let $R := \{E_1 \times E_2 : E_1 \in \mathcal{A}_1,\ E_2 \in \mathcal{A}_2\} \subseteq \mathcal{P}(X_1 \times X_2)$.
We define the product σ-algebra as the σ-algebra generated by R and we denote it as $\sigma_0(R) = \mathcal{A}_1 \times \mathcal{A}_2$.
Then $(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is the product measurable space.
Let $E \subseteq X_1 \times X_2$.

- $\forall x_1 \in X_1$ we define the x_1-section of E as: $E_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in E\} \subseteq X_2$
- $\forall x_2 \in X_2$ we define the x_2-section of E as: $E_{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in E\} \subseteq X_1$

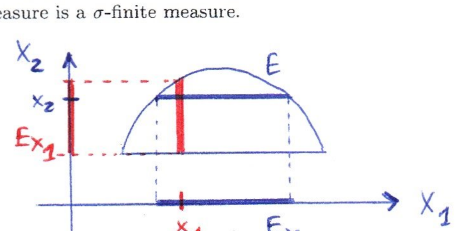
Let $E \in \mathcal{A}_1 \times \mathcal{A}_2$. Then $E_{x_1} \in \mathcal{A}_2\ \forall x_1 \in X_1$ and $E_{x_2} \in \mathcal{A}_1\ \forall x_2 \in X_2$.
*If we have a measurable set in the product σ-algebra then also **all** sections are measurable in the σ-algebra they belong.*

Theorem. Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two σ-finite measure spaces. Let $E \in \mathcal{A}_1 \times \mathcal{A}_2$.
We define:

- $\varphi_1 : X_1 \to \overline{\mathbb{R}}_+$ such that: $\varphi_1(x_1) := \mu_2(E_{x_1})\quad x_1 \in X_1$
- $\varphi_2 : X_2 \to \overline{\mathbb{R}}_+$ such that: $\varphi_2(x_2) := \mu_1(E_{x_2})\quad x_2 \in X_2$

Then:
(i) $\varphi_i \in \mathcal{M}_+(X_i, \mathcal{A}_i)\quad i = 1, 2$
(ii) $\int_{X_1} \varphi_1(x_1)\, d\mu_1 = \int_{X_2} \varphi_2(x_2)\, d\mu_2 \quad\Leftrightarrow\quad \int_{X_1} \mu_2(E_{x_1})\, d\mu_1 = \int_{X_2} \mu_1(E_{x_2})\, d\mu_2$

φ_i is measurable and non-negative (since it's a measure), hence we can define its integral (it can be $+\infty$, but still it would be well defined). This theorem says that a set in $\mathcal{A}_1 \times \mathcal{A}_2$ can be measured by its sections (and, obviously, if we measure horizontally or vertically we obtain two equal measures).
Therefore, we define the product measure as the function $\mu_1 \times \mu_2 : \mathcal{A}_1 \times \mathcal{A}_2 \to \overline{\mathbb{R}}_+$:
$(\mu_1 \times \mu_2)(E) := \int_{X_1} \mu_2(E_{x_1})\, d\mu_1 = \int_{X_2} \mu_1(E_{x_2})\, d\mu_2$
The product measure is a σ-finite measure.



$\varphi(x_1) = \mu_2(E_{x_1})$
$\varphi(x_2) = \mu_1(E_{x_2})$

6.8. Is the product measure space complete? Justify the answer. Which is the relation between $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ and $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$?

The product measure space is **not** complete.
To prove it, let $(X_1, \mathcal{A}_1, \mu_1) = (X_2, \mathcal{A}_2, \mu_2) = (\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and consider the Vitali set in $[0,1] : V \subseteq [0,1]$.
V is **not** measurable according to Lebesgue: $V \notin \mathcal{L}(\mathbb{R})$.
Define now: $E := \{x_0\} \times V$, with $x_0 \in \mathbb{R}$.

$\Rightarrow E_{x_0} = V \notin \mathcal{L}(\mathbb{R})$

$\Rightarrow E \notin \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$

This is due to a property of the product σ-algebra: if $E \in \mathcal{A}_1 \times \mathcal{A}_2 \Rightarrow E_{x_2} \in \mathcal{A}_1$ and $E_{x_1} \in \mathcal{A}_2$.
Since we negate the right-hand side, also the left-hand side doesn't hold.
Now, observe that:

$E \subseteq F := \{x_0\} \times [0,1]$

with $F \in \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$ such that:

$(\lambda \times \lambda)(F) = \int_{\mathbb{R}} \lambda(\{x_0\})\, d\lambda = 0$

since the Lebesgue measure of a point is zero ($\lambda(\{x_0\}) = 0$).
We conclude that there is a set, E, which is **not** measurable but it is contained in a set which is measurable and of zero measure, F. Therefore $(\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}), \lambda \times \lambda)$ is **not** complete.

Consider $(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), \lambda_m)$, $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$.

- Consider $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$, where λ_{m+n} is the Lebesgue measure constructed by means of the $(m+n)$-dimensional rectangles. Because of its construction, $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ is **complete**.
- Consider $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$. This measure space is constructed by means of the product measure and hence (differently from before) is not independent from $(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), \lambda_m)$ and $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$.
  The product measure space is not complete, therefore $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$ is **not complete**.

We can conclude that $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n}) \neq (\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$.
Theorem. $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ is the competion of $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$.

6.9. State the **Tonelli theorem**.

Theorem. **Tonelli**
Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two σ-finite measure spaces.
Let $f \in \mathcal{M}_+(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$.
As a consequence, it holds:

- $\forall x_1 \in X_1 : \quad f(x_1, \cdot) \in \mathcal{M}_+(X_2, \mathcal{A}_2)$
- $\forall x_2 \in X_2 : \quad f(\cdot, x_2) \in \mathcal{M}_+(X_1, \mathcal{A}_1)$

Therefore, we define:

- $\psi_1 : X_1 \to \overline{\mathbb{R}}_+ : \quad \psi_1(x_1) := \int_{X_2} f(x_1, x_2)\, d\mu_2 \quad x_1 \in X_1$
- $\psi_2 : X_2 \to \overline{\mathbb{R}}_+ : \quad \psi_2(x_2) := \int_{X_1} f(x_1, x_2)\, d\mu_1 \quad x_2 \in X_2$

fixed

Then:

(i) $\psi_i \in \mathcal{M}_+(X_i, \mathcal{A}_i) \quad i = 1, 2$

(ii) $$\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2) = \int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, d\mu_2\right] d\mu_1 = \int_{X_1} \psi_1(x_1)\, d\mu_1$$
$$= \int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, d\mu_1\right] d\mu_2 = \int_{X_2} \psi_2(x_2)\, d\mu_2$$

6.10. State the **Fubini theorem**.
By means of a counterexample, show that it is not possible to omit the hypothesis $f \in L^1$.

Theorem. **Fubini**
Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be two σ-finite measure spaces.
Let $f \in L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$.
We define:

- $\psi_1 : X_1 \to \overline{\mathbb{R}}_+ : \quad \psi_1(x_1) := \int_{X_2} f(x_1, x_2)\, d\mu_2 \quad x_1 \in X_1$
- $\psi_2 : X_2 \to \overline{\mathbb{R}}_+ : \quad \psi_2(x_2) := \int_{X_1} f(x_1, x_2)\, d\mu_1 \quad x_2 \in X_2$

Then:

(i) $f(x_1, \cdot) \in L^1(X_2, \mathcal{A}_2, \mu_2) \quad$ for a.e. $x_1 \in X_1$
$\quad\ f(\cdot, x_2) \in L^1(X_1, \mathcal{A}_1, \mu_1) \quad$ for a.e. $x_2 \in X_2$

(ii) $\psi_1 \in L^1(X_1, \mathcal{A}_1, \mu_1)$
$\quad\ \psi_2 \in L^1(X_2, \mathcal{A}_2, \mu_2)$

(iii) $$\int_{X_1 \times X_2} f(x_1, x_2)\, d(\mu_1 \times \mu_2) = \int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, d\mu_2\right] d\mu_1 = \int_{X_1} \psi_1(x_1)\, d\mu_1$$
$$= \int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, d\mu_1\right] d\mu_2 = \int_{X_2} \psi_2(x_2)\, d\mu_2$$

The hypothesis of $f \in L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$ is essential.
Consider $(X_i, \mathcal{A}_i, \mu_i) = ([0,1], \mathcal{L}([0,1]), \lambda)$ $i = 1, 2$ and $f(x_1, x_2) = \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}$ with $x_1, x_2 \in [0,1]$.
Then $f \notin L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$.
This is because:

$$f_+ = \begin{cases} 0 & \text{if } x_1 \leq x_2 \\ f & \text{if } x_1 > x_2 \end{cases} \quad \in \mathcal{M}_+(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$$

Therefore, we can apply Tonelli theorem:

$\int_{X_1 \times X_2} f_+(x_1, x_2)\, d(\lambda \times \lambda) = \int_{X_1} \left[\int_{X_2} f_+(x_1, x_2)\, d\lambda\right] d\lambda = \infty$

$\Rightarrow f \notin L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$

Moreover:

$$\int_{X_1} \left[\int_{X_2} f(x_1, x_2)\, dx_2\right] dx_1 = \frac{\pi}{4} \quad \neq \quad \int_{X_2} \left[\int_{X_1} f(x_1, x_2)\, dx_1\right] dx_2 = -\frac{\pi}{4}$$

and this is because $f \notin L^1(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$.

8 BV and AC Functions

8.1 BV Functions

6.11. Write the definition of **Lebesgue point**. What is about the measure of the set of points that are not Lebesgue points for a function $f \in L^1$?

Let $f : [a,b] \in L^1([a,b])$. We say that $x_0 \in [a,b]$ is a Lebesgue point for f if:

$\lim_{h\to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)|\, dt = 0$.

Namely, x_0 is a Lebesgue point if we can approximate $f(x_0)$ with an integral average (e.g. jumps are not Lebesgue points).

Theorem. If $f \in L^1([a,b])$ then almost all $x_0 \in [a,b]$ are Lebesgue points for f.

Therefore, if $f \in L^1([a,b])$ then the measure of the set of points that are not Lebesgue points is zero.

6.12. State and prove the **First Fundamental Theorem of Calculus** for $f \in L^1$.

Let $f \in L^1([a,b])$ and define $F(x) := \int_a^x f(t)\, dt \quad \forall x \in [a,b]$.

We consider the Lebesgue measure λ, however to underly the variable of integration we write dt or dx.

Theorem. **First Fundamental Theorem of Calculus (1^{st} FTC)**

If $f \in L^1([a,b])$ then F is differentiable a.e. in $[a,b]$ and $F' = f$ a.e. in $[a,b]$.

proof.

Let $x \in [a,b]$ be a Lebesgue point for f and let $h \neq 0$ be such that $x + h \in [a,b]$.

Then:

$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{\int_x^{x+h} f(t)\, dt}{h} - f(x) = \frac{1}{h}\left(\int_x^{x+h} f(t) - f(x)\, dt\right)$

$\Rightarrow \left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \leq \underbrace{\frac{1}{|h|}\left|\int_x^{x+h} |f(t) - f(x)|\, dt\right|}_{\xrightarrow[h\to 0]{} 0 \text{ since } x \text{ is a Lebesgue point}}$

$\Rightarrow F'(x) =: \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$.

We recall that if $f \in L^1([a,b])$ then almost all $x \in [a,b]$ are Lebesgue points for f.

Therefore, $F' = f$ a.e. in $[a,b]$. ∎

6.13. Let $f : [a,b] \to \mathbb{R}$. Write the definition of: **variation of f** relative to a partition of $[a,b]$, **total variation of f** over $[a,b]$, **function of bounded variation**.

Let $I = [a,b]$ and let $f : I \to \mathbb{R}$.

Let $P = \{a = x_0 < x_1 < .. < x_n = b\}$, with $n \in \mathbb{N}$, be a partition of the interval I.

We define the variation of f relative to the partition P of $[a,b]$ as:

$v_a^b(f; P) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$.

Let $\mathcal{P}$ be the collection of all partitions of $[a,b]$.

Then, we define the total variation of f over $[a,b]$ as:

$V_a^b(f) = \sup_{P\in\mathcal{P}} v_a^b(f; P)$.

If $V_a^b(f) < \infty$ then we say that f is a function of bounded variation.

The set of all functions of bounded vriations in $[a,b]$ is denoted by $BV([a,b])$.

6.14. Let $f : [a,b] \to \mathbb{R}$ be monotone. Why $f \in BV([a,b])$? Show that if $f \in BV([a,b])$, then f is bounded.

If $f : [a,b] \to \mathbb{R}$ is monotone then $V_a^b(f) = |f(b) - f(a)| < \infty$ and so $f \in BV([a,b])$.

If $f \in BV([a,b])$ then f is bounded.

monotone $\Rightarrow$ BV

bounded $\underset{\not\Rightarrow}{\Leftarrow}$ BV

In fact, it always holds:

$\sup_{x\in[a,b]} |f(x)| \leq |f(a)| + V_a^b(f)$

Therefore, if f is unbounded $\Rightarrow \sup_{x\in[a,b]} |f(x)| = \infty \Rightarrow f \notin BV([a,b])$.

However, the contrary does not hold, namely if f is bounded $\not\Rightarrow f \in BV([a,b])$.

Consider a bounded function, for example f defined as:

$f = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \in (0, \frac{2}{\pi}] \\ 0 & \text{if } x = 0 \end{cases}$

Moreover, consider the partition $P_n = \{0, x_{n-1}, x_{n-2}, .., x_1, x_0 = \frac{2}{\pi}\}$ with $x_i = \frac{2}{(2\,i+1)\pi}$.

$\Rightarrow \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \xrightarrow[n\to\infty]{} \infty$

$\Rightarrow f \notin BV([0, \frac{2}{\pi}])$.

Prop. $BV([a,b])$ is a vector space.

In fact, let $f, g \in BV([a,b])$, $\lambda \in \mathbb{R}$. Then:

(i) $\lambda f \in BV([a,b])$ and $V_a^b(\lambda f) = |\lambda|\, V_a^b(f)$

(ii) $f + g \in BV([a,b])$ and $V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$

Prop. Let $f \in BV([a,b])$. Then:

(i) $V_a^b(f) = V_a^c(f) + V_c^b(f) \quad \forall c \in [a,b]$

(ii) $x \mapsto V_a^x(f)$, with $x \in [a,b]$, is increasing

Prop. In general $V_a^b(f)$ is not a norm on $BV([a,b])$, otherwise $V_a^b(f) = 0$ would imply that $f = 0$, which is not true since $V_a^b(f) = 0 \Rightarrow f = $ constant. Instead, $||f|| := |f(a)| + V_a^b(f)$ is a norm on $BV([a,b])$.

7.1. What is the **Jordan decomposition** of a BV function? Why a function of bounded variation is differentiable a.e.?

Theorem. Let $f : [a,b] \to \mathbb{R}$. Then $f \in BV([a,b]) \Leftrightarrow \exists\, \varphi,\, \psi : [a,b] \to \mathbb{R}$ increasing such that $f = \varphi - \psi$.
The decomposition $f = \varphi - \psi$ is called Jordan decomposition.

Theorem. Let $f : [a,b] \to \mathbb{R}$ be monotone. Then f is differentiable a.e. in $[a,b]$.

Because of the two previous theorems we can conclude that if $f \in BV([a,b])$ then f is differentiable a.e. in $[a,b]$.
In fact, if $f \in BV([a,b])$ then $\exists\, \varphi,\, \psi : [a,b] \to \mathbb{R}$ increasing such that $f = \varphi - \psi$ (Jordan decomposition).
Since both φ and ψ are monotone then both φ and ψ are differentiable a.e. in $[a,b]$.
Therefore, since f is the difference of two functions differentiable a.e. in $[a,b]$ then f is differentiable a.e. in $[a,b]$.

7.2. Let $f : [a,b] \to \mathbb{R}$ be increasing. What can we say about f' and $\int_{[a,b]} f'\, d\lambda$? Justify the answer.

Theorem. Let $f : [a,b] \to \mathbb{R}$ be increasing. Then $f' \in L^1([a,b])$ and $\int_{[a,b]} f'\, d\lambda \leq f(b) - f(a)$.

proof.

Since f is increasing then f is differentiable a.e. in $[a,b]$, which means that f' exists a.e. in $[a,b]$.
Moreover, since an increasing function is always integrable according to Riemann and since Riemann-integrability implies the measurability of a funcition:

$f \nearrow \Rightarrow f \in \mathcal{R}([a,b]) \Rightarrow f \in \mathcal{M}([a,b], \mathcal{L}([a,b]))$.

We set $f(x) := f(b)\ \forall x > b$.
We define:

$$g_n(x) := \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} \quad \forall x \in [a,b]$$

Then, since f is measurable and since f is increasing:

$g_n \in \mathcal{M}_+([a,b], \mathcal{L}([a,b]))$.

Moreover:

$\lim_{n\to\infty} g_n(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ for a.a. $x \in [a,b]$.

Then, since $g_n \in \mathcal{M}_+([a,b], \mathcal{L}([a,b]))$, also $\limsup_n g_n$, $\liminf_n g_n \in \mathcal{M}_+([a,b], \mathcal{L}([a,b]))$, threfore:

$f' \in \mathcal{M}_+([a,b], \mathcal{L}([a,b]))$

By Fatou's lemma:

$$\begin{aligned}
\int_a^b f'(x)\, dx =: \int_a^b \lim_{n\to\infty} g_n(x)\, dx = \int_a^b \liminf_{n\to\infty} g_n(x)\, dx &\leq \liminf_{n\to\infty} \int_a^b g_n(x)\, dx \\
&= \liminf_{n\to\infty} \left[n \int_a^b f(x+\tfrac{1}{n})\, dx - n \int_a^b f(x)\, dx\right] \\
&= \liminf_{n\to\infty} \left[n \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)\, dx - n \int_a^b f(x)\, dx\right] \\
&= \liminf_{n\to\infty} \left[n \int_b^{b+\frac{1}{n}} f(x)\, dx - n \int_a^{a+\frac{1}{n}} f(x)\, dx\right] \\
&\overset{(*)}{\leq} \liminf_{n\to\infty} \left[n \tfrac{1}{n} f(b) - n \tfrac{1}{n} f(a)\right] \\
&= f(b) - f(a).
\end{aligned}$$

$(*)$ f is increasing: $\int_a^{a+\frac{1}{n}} f\, dx \geq \frac{1}{n} f(a)$, by definition of f for $x > b$: $\int_b^{b+\frac{1}{n}} f\, dx \leq \frac{1}{n} f(b)$ ■

7.3. Can there exists a function $f \in BV([a,b])$ with $f' \notin L^1([a,b])$? Justify the answer.

If $f \in BV([a,b])$ then f' exists a.e. in $[a,b]$ and $f' \in L^1([a,b])$.

proof.

If $f \in BV([a,b])$, by Jordan decomposition we know that $\exists\, \varphi,\, \psi : [a,b] \to \mathbb{R}$ increasing such that $f = \varphi - \psi$.
Since φ and ψ are increasing then $\varphi',\, \psi' \in L^1([a,b])$.
Since $L^1([a,b])$ is a vector space then $f' = \varphi' - \psi' \in L^1([a,b])$. ■

Therefore, it cannot exist a function $f \in BV([a,b])$ such that $f' \notin L^1([a,b])$.

8.2 AC Functions

7.4. Write the definition of **absolutely continuous function**. Show that an absolutely continuous function is also uniformly continuous, but the viceversa is not true. Furthermore, a Lipschitz function is absolutely continuous, but the viceversa is not true.

Let $J \subseteq \mathbb{R}$ be an interval.
Let $\mathcal{F}(J)$ be the set of finite collections of closed subintervals of J without interior points in common.
Then, a function $f : J \to \mathbb{R}$ is said to be absolutely continuous in J if:

$\forall \varepsilon > 0\ \exists \delta > 0$ such that:

$\forall \{[a_k, b_k]\}_{k=1,..,n} \in \mathcal{F}(J)$ for which $\sum_{k=1}^n (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$.

We denote $AC([a,b]) := \{f : [a,b] \to \mathbb{R}$ absolutely continuous $\}$.

Prop. The following propositions hold:

(i) $f \in AC([a,b]) \Rightarrow f$ uniformly continuous in $[a,b]$
(ii) $f \in AC([a,b]) \not\Leftarrow f$ uniformly continuous in $[a,b]$
(iii) f Lipschitz in $[a,b] \Rightarrow f \in AC([a,b])$
(iv) f Lipschitz in $[a,b] \not\Leftarrow f \in AC([a,b])$

proof.

(i) A function $f : [a,b] \to \mathbb{R}$ is uniformly continuous if:
$\forall \varepsilon > 0\ \exists \delta > 0$ such that: $\forall x, y \in [a,b]$ for which $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Thus, if we consider $[x,y]$ as special case of $\{[a_k, b_k]\}_{k=1,..,n}$, with $x, y \in [a,b]$, $x \leq y$, then from the definition of absolutely continuity we get uniform continuity.

(ii) The function f defined as:

$$f = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \in [-1,1] \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

is not absolutely continuous in $[-1,1]$, however f is uniformly continuous in $[-1,1]$.

(iii) Consider $\{[a_k, b_k]\}_{k=1,\dots,n} \in \mathcal{F}$ for which $\sum_{k=1}^n (b_k - a_k) < \delta$.

Then, by definition of Lipschitz function in $[a,b]$, $\exists\, L > 0$ such that:

$$\begin{aligned}\sum_{k=1}^n |f(b_k) - f(a_k)| &\le \sum_{k=1}^n L\,(b_k - a_k) \\ &= L \sum_{k=1}^n (b_k - a_k) \\ &< L\,\delta = \varepsilon\end{aligned}$$

choosing $\delta := \frac{\varepsilon}{L}$ we get $f \in AC([a,b])$.

(iv) The function f defined as:

$$f = \sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}}\, dt$$

is absolutely continuous, however f is not Lipschitz. ■

Absolutely continuity is stronger than both continuity and uniform continuity, however is not stronger than Lipschitz.

7.5. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu < \infty$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \in \mathcal{A}$ with $\mu(E) < \delta$ there holds $\int_E f\, d\mu < \varepsilon$.

Theorem. Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f\, d\mu < \infty$. Then $\forall \varepsilon > 0\ \exists \delta > 0$ such that: $\forall E \in \mathcal{A}$ for which $\mu(E) < \delta \Rightarrow \int_E f\, d\mu < \varepsilon$.

proof.

Consider the sequence of sets $\{F_n\}_n$ with $F_n := \{f < n\}$ $\forall n \in \mathbb{N}$.

Since f is measurable then F_n is measurable for all $n \in \mathbb{N}$. Moreover, since $n \nearrow$ then also $\{F_n\}_n \nearrow$.

Then, we can write:

$$X = \{f = \infty\} \cup \{f < \infty\} = \{f = \infty\} \cup \left[\, \bigcup_{n=1}^{\infty} F_n \,\right] \quad (*)$$

Since $\int_X f\, d\mu < \infty$ we can conlcude that f is finite a.e. in X, which means:

$\mu(\{f = \infty\}) = 0$.

Thus, combining this result with $(*)$, we obtain:

$$\int_X f\, d\mu = \lim_{n \to \infty} \int_{F_n} f\, d\mu$$

which is equivalent to say that $\forall \varepsilon > 0\ \exists \bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$:

$$\underbrace{\left| \int_{F_n} f\, d\mu - \int_X f\, d\mu \right|}_{\text{equivalent to: } \int_{F_n^c} f\, d\mu} < \frac{\varepsilon}{2}$$

Therefore, for any fixed $n > \bar{n}$ we obtain that:

$$\int_E f\, d\mu = \int_{E \cap F_n} f\, d\mu + \int_{E \cap F_n^c} f\, d\mu < \underbrace{\int_{E \cap F_n} n\, d\mu}_{f < n \text{ on } F_n} + \underbrace{\int_{F_n^c} f\, d\mu}_{E \cap F_n^c \subseteq F_n^c} < n\, \mu(E \cap F_n) + \frac{\varepsilon}{2} \le n\, \underbrace{\mu(E)}_{<\delta} + \frac{\varepsilon}{2} = \varepsilon$$

choosing $\delta = \frac{\varepsilon}{2n}$. ■

7.6. Show that if $f \in L^1([a,b])$, then $F(x) := \int_{[a,x]} f\, d\lambda$ is absolutely continuous in $[a,b]$.

Let $f \in L^1([a,b])$. Then $F(x) := \int_{[a,x]} f\, d\lambda$ is $AC([a,b])$ for all $x \in [a,b]$.

The integral function of an L^1 function is absolutely continuous.

proof.

Let $\mathcal{F}([a,b])$ be the set of finite collections of closed subintervals of $[a,b]$ without interior points in common.

Let $E := \bigcup_{k=1}^n [a_k, b_k]$, with $\{[a_k, b_k]\}_{k=1,\dots,n} \in \mathcal{F}([a,b])$.

Since the intervals do not have interior points in common, we have that:

$\lambda(E) = \sum_{k=1}^n (b_k - a_k)$.

Now we consider:

$$\sum_{k=1}^n |F(b_k) - F(a_k)| = \sum_{k=1}^n \left| \int_{[a_k, b_k]} f\, d\lambda \right| \le \sum_{k=1}^n \int_{[a_k, b_k]} |f|\, d\lambda = \int_E |f|\, d\lambda.$$

We recall a theorem stating that:

If $f \in \mathcal{M}_+(X, \mathcal{A})$ is such that $\int_X f\, d\mu < \infty$ then $\forall \varepsilon > 0\ \exists \delta > 0$ such that: $\forall E \in \mathcal{A}$ for which $\mu(E) < \delta \Rightarrow \int_E f\, d\lambda < \varepsilon$.

We can apply this theorem on $|f|$ (since $|f| \in \mathcal{M}_+([a,b], \mathcal{L}([a,b]))$ and we obtain that:

$\forall \varepsilon > 0\ \exists \delta > 0$ such that: $\lambda(E) = \sum_{k=1}^n (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^n |F(b_k) - F(a_k)| \le \int_E |f|\, d\lambda < \varepsilon$

which is the definition of F being absolutely continuous. ■

7.7. Which is the relation between the spaces $BV([a,b])$ and $AC([a,b])$?

Theorem. If $f \in AC([a,b]) \Rightarrow f \in BV([a,b])$.

Moreover: $x \mapsto V_a^x(f) \in AC([a,b])$.

However, the reverse does not hold, namely: $f \in BV([a,b]) \nRightarrow f \in AC([a,b])$.

In fact, consider the function f defined as:

$$f = \begin{cases} 1 & x \in [0,1] \\ -1 & x \in [-1,0) \end{cases}$$

Then $f \in BV([-1,1])$, however $f \notin AC([-1,1])$ (since it is not even continuous).

Theorem. Let $f : [a,b] \to \mathbb{R}$. Then $f \in AC([a,b]) \Leftrightarrow \exists \varphi, \psi \in AC([a,b])$ increasing such that $f = \varphi - \psi$.

Lemma. Let $f \in AC([a,b])$, $f \nearrow$ in $[a,b]$. If $f' = 0$ a.e. in $[a,b]$ then $f \equiv c$ in $[a,b]$ for some $c \in \mathbb{R}$.

7.8. State and prove the **Second Fundamental Theorem of Calculus.**

Theorem. **Second Fundamental Theorem of Calculus (2^{nd} FTC)**

Let $F : [a, b] \to \mathbb{R}$. The following statementes are equivalent:

(i) $F \in AC([a, b])$

(ii) F is differentiable a.e. in $[a, b]$ with $F' \in L^1([a, b])$ and:

$F(x) = F(a) + \int_{[a,x]} F'\, d\lambda \quad \forall x \in [a, b]$

proof.

(ii) $\Rightarrow$ (i) It is known that if $f \in L^1([a, b]) \Rightarrow F(x) := \int_{[a,x]} f\, d\lambda$ is $AC([a, b])$ for all $x \in [a, b]$.

proof.

Let $\mathcal{F}([a, b])$ be the set of finite collections of closed subintervals of $[a, b]$ without interior points in common and let $E := \bigcup_{k=1}^n [a_k, b_k]$, with $\{[a_k, b_k]\}_{k=1,..,n} \in \mathcal{F}([a, b])$.

Since the intervals do not have interior points in common, we have that:

$\lambda(E) = \sum_{k=1}^n (b_k - a_k)$.

Now we consider:

$\sum_{k=1}^n |F(b_k) - F(a_k)| = \sum_{k=1}^n \left| \int_{[a_k,b_k]} f\, d\lambda \right| \leq \sum_{k=1}^n \int_{[a_k,b_k]} |f|\, d\lambda = \int_E |f|\, d\lambda$.

We recall a theorem stating that:

If $f \in \mathcal{M}_+(X, \mathcal{A})$ is such that $\int_X f\, d\mu < \infty$ then $\forall \varepsilon > 0\ \exists \delta > 0$ such that:

$\forall E \in \mathcal{A}$ for which $\mu(E) < \delta \Rightarrow \int_E f\, d\lambda < \varepsilon$.

We can apply this theorem on $|f|$ (since $|f| \in \mathcal{M}_+([a, b], \mathcal{L}([a, b]))$ and obtain that:

$\forall \varepsilon > 0\ \exists \delta > 0$ such that: $\lambda(E) = \sum_{k=1}^n (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^n |F(b_k) - F(a_k)| \leq \int_E |f|\, d\lambda < \varepsilon$

which is the definition of F being absolutely continuous. ■

Thus, if we set $F' = f$ we get (i), since a constant factor ($F(a)$) does not affect F being in $AC([a, b])$.

(i) $\Rightarrow$ (ii) For the second part of the proof we recall the following results.

1. If $f \in L^1([a, b]) \Rightarrow F(x) := \int_{[a,x]} f\, d\lambda$ is differentiable a.e. in $[a, b]$ and $F' = f$ a.e. in $[a, b]$. (1 FTC)
2. If $f \in L^1([a, b]) \Rightarrow F(x) := \int_{[a,x]} f\, d\lambda$ is $AC([a, b])$ for all $x \in [a, b]$.
3. If $f : [a, b] \to \mathbb{R}$ is increasing $\Rightarrow f' \in L^1([a, b])$ and $\int_{[a,b]} f'\, d\lambda \leq f(b) - f(a)$.
4. If $f \in AC([a, b])$, $f \nearrow$, $f' = 0$ a.e. in $[a, b] \Rightarrow f \equiv c$ in $[a, b]$, for some $c \in \mathbb{R}$.

Since $F \in AC([a, b]) \Rightarrow F \in BV([a, b]) \Rightarrow F$ is differentiable a.e. in $[a, b]$ and $F' \in L^1([a, b])$.

- Suppose, in addition, that $\boldsymbol{F}$ **is increasing**.

We set:

$G(x) := \int_{[a,x]} F'\, d\lambda \quad \forall x \in [a, b]$.

Since $F'(x) \in L^1([a, b])$, by the 1^{st} FTC (1):

$\Rightarrow G(x)$ is differentiable a.e. in $[a, b]$ and $G' = F'$ a.e. in $[a, b]$

$\Rightarrow \boxed{(F - G)' = 0 \text{ a.e. in } [a, b].}$

Moreover, since G is the integral function of an $L^1([a, b])$ function, by (2) we get:

$\Rightarrow G \in AC([a, b])$.

Thus, since both $F, G \in AC([a, b])$ and since $AC([a, b])$ is a vector space:

$\Rightarrow \boxed{F - G \in AC([a, b]).}$

Now we consider: $a \leq x_1 \leq x_2 \leq b$:

$$(F(x_2) - G(x_2)) - (F(x_1) - G(x_1)) = \underbrace{F(x_2) - F(x_1) - \int_{[x_1,x_2]} F'\, d\lambda}_{\text{by definition of } G} \underbrace{\geq 0}_{(*)}$$

where $(*)$ is due to the fact that F is increasing and thus (3) holds.

Therefore, $\boxed{F - G \text{ is increasing.}}$

Moreover, since $F - G \in AC([a, b])$, $F - G \nearrow$, $(F - G)' = 0$ a.e. in $[a, b]$, by (4) we get:

$\Rightarrow F(x) - G(x) = \text{constant} \quad \forall x \in [a, b]$

$\Rightarrow F(x) - G(x) = F(a) - G(a) \quad \forall x \in [a, b]$

$\Rightarrow F(x) - G(x) = F(a) \quad \forall x \in [a, b]$

which is the thesis (for F increasing).

- For a **general** $\boldsymbol{F}$ we recall that, if $F \in AC([a, b]) \Rightarrow \exists \varphi, \psi \in AC([a, b])$ increasing such that $F = \varphi - \psi$.

So, we can apply the above reasoning to φ and ψ and then the thesis follows for F too. ■