BANACH SPACES

NORMED SPACES

we always consider **real** vector spaces

Def. Let X be a vector space. A norm on X is a function $\|\cdot\| : X \to [0,+\infty)$ s.t.:

(i) $\forall x \in X: \|x\| \geq 0, \quad \|x\| = 0 \iff x = 0$

(ii) $\forall \alpha \in \mathbb{R}, x \in X: \quad \|\alpha x\| = |\alpha| \cdot \|x\|$

(iii) $\forall x, y \in X: \quad \|x+y\| \leq \|x\| + \|y\|$

Then, the pair $(X, \|\cdot\|)$ is called normed space.

Remark: $\big| \|x\| - \|y\| \big| \leq \|x-y\| \qquad \forall x, y \in X$

Remark: $d(x,y) := \|x-y\|$ is a distance on X.
Hence:

$$(X, \|\cdot\|) \text{ normed space} \implies (X, d) \text{ metric space}$$

notice that here we require that X is a vector space (it has a structure); when we defined metric spaces X could have been **any set**

Examples:

1. $\mathbb{R}^N$ with $\|x\|_p := \left(\sum_{i=1}^N |x_i|^p\right)^{1/p} \quad \forall p \in [1,\infty)$
   $\|x\|_\infty := \max_{i=1,\dots,N} |x_i|$
2. $C^0([a,b])$ with $\|f\|_\infty := \max_{x\in[a,b]} |f(x)|$
3. $L^1(X, \mathcal{A}, \mu)$ with $\|f\|_1 := \int_X |f(x)|\, dx \qquad (\cancel{\mathcal{L}^1})$
4. $L^\infty(X, \mathcal{A}, \mu)$ with $\|f\|_\infty := \operatorname{ess\,sup}_X |f| \qquad (\cancel{\mathcal{L}^\infty})$

   since if we take a function f inside of these spaces, if the norm of f is zero we can only say that the function is zero a.e., we cannot conclude that $f \equiv 0$
5. $\ell^p := \{ x = x^{(k)}$ sequence of real numbers s.t. $\sum_{k=1}^\infty |x^{(k)}|^p < \infty \}$
   with $\|x\|_p := \left(\sum_{k=1}^\infty |x^{(k)}|^p\right)^{1/p}$
6. $\ell^\infty := \{ x = x^{(k)}$ sequence of real numbers s.t. $\sup_{k\in\mathbb{N}} |x^{(k)}| < \infty \}$
   with $\|x\|_\infty := \sup_{k\in\mathbb{N}} |x^{(k)}|$
7. $C^k([a,b])$ with $\|f\|_{\infty,k} := \sum_{i=0}^k \|f^{(i)}\|_\infty \qquad (f^{(0)} = f, \ \|f^{(0)}\|_\infty = \|f\|_\infty)$
8. $BV([a,b])$ with $\|f\|_{BV} := \|f\|_1 + V_a^b(f)$
9. $AC([a,b])$ with $\|f\|_{AC} := \|f\|_1 + \|f'\|_1$

   instead of $\|f\|_1$ we can consider $f(a)$ and it would be (still) a norm (in both cases)

   we can write f' because if $f \in AC$ then f' exists a.e. and it is in L^1
10. $L^p(X, \mathcal{A}, \mu) \quad p \in [1,\infty]$

SEQUENCES AND SERIES (in **normed** spaces)

Def. Let $(X, \|\cdot\|)$ be a normed space, $\{x_n\}_n \subset X$, $x \in X$. We say that:

$$x_n \xrightarrow[n\to\infty]{} x \iff d(x_n, x) \xrightarrow[n\to\infty]{} 0 \iff \|x_n - x\| \xrightarrow[n\to\infty]{} 0$$

(x_n converges to x). convergence in the associated metric space. we can also use the notion of the norm

Remark: if $x_n \xrightarrow[n\to\infty]{} x \underset{(\not\Leftarrow)}{\implies} \|x_n\| \xrightarrow[n\to\infty]{} \|x\|$

In fact:

$$\underbrace{\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\|}_{\text{holds } \forall n \in \mathbb{N}} \xrightarrow[n\to\infty]{} 0 \implies \big| \|x_n\| - \|x\| \big| \xrightarrow[n\to\infty]{} 0$$

since $x_n \xrightarrow[n\to\infty]{} x$

Def. $\{x_n\}_n \subseteq X$ is a Cauchy sequence if $\forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N}$:

$$\underbrace{\|x_n - x_m\|}_{d(x_n, x_m)} < \varepsilon \qquad \forall n, m > \bar{n}$$

$d(x_n, x_m)$, hence the concept is the same as in metric spaces

Remark: $\{x_n\}_n$ convergent $\overset{\Rightarrow}{\not\Leftarrow}$ $\{x_n\}$ is Cauchy

Def. $\{x_n\} \subset X$ is bounded if $\exists M > 0$:

$$\|x_n\| < M \quad \forall n \in \mathbb{N}$$

Remark: $\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_n\}$ is bounded

Def. Let X be a normed space, $\{x_n\} \subset X$. Let:

$$S_n := x_0 + x_1 + \dots + x_n := \sum_{k=0}^{n} x_k$$

– we can do it because we have a structure and summing elements of the space gives an other element of the space

$\{S_n\}$ is called sequence of partial sums, or series.

The series $\{S_n\}$ is said to be convergent if there exists $x \in X$ s.t.:

$$S_n \xrightarrow[n \to \infty]{} x.$$

(convergent w.r.t. the norm of the normed space)

We say that $x = \sum_{k=0}^{\infty} x_k$ is the sum of the series. (sometimes we say that $x = \sum_{k=0}^{\infty} x_k$ is the series)

Remark: $\sum_{n=0}^{\infty} \|x_n\|$ convergent $\not\Rightarrow$ $\sum_{n=0}^{\infty} x_n$ convergent – this is true in $\mathbb{R}$, however not in general normed spaces

Def. $(X, \|\cdot\|)$ is complete $\Leftrightarrow$ (X, d) is complete

$\Leftrightarrow$ every Cauchy sequence in X is convergent

Def. A complete **normed** space is called Banach space.

BANACH SPACE

Examples: All the examples we have seen before.

SEPARABILITY

(Recall) (X, d) metric space (in general we don't need a norm) X is separable if $\exists A \subset X$ countable and dense in X. ($\bar{A} = X$).

Theorem: $C^0([a,b])$ is separable.

proof.

It is known that the set of polynomials is dense in $C^0([a,b])$. (Stone-Weierstrass theorem)

$\forall f \in C^0([a,b])$, $\forall \varepsilon > 0$ $\exists p =$ polynomial s.t.

$$\|f - p\|_\infty < \frac{\varepsilon}{2}$$

We can find a polynomial r with rational coefficients s.t.

$$\|p - r\|_\infty < \frac{\varepsilon}{2}$$

(rational numbers are dense in the set of real numbers, so it holds this inequality)

Hence:

$$\|f - r\|_\infty \le \|f - p\|_\infty + \|p - r\|_\infty < \varepsilon$$

$\Rightarrow$ $A = \{$polynomials with rational coefficients$\}$ dense in $C^0([a,b])$

Since A is countable $\Rightarrow$ $C^0([a,b])$ is separable. ∎

COMPLETNESS

Theorem: (Criterion for **completness**) (in normed spaces) – Characterization of **Banach** spaces

(i) Let X be a Banach space, $\{x_n\} \subset X$.

If $\sum_{n=1}^{\infty} \|x_n\|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ converges. (w.r.t. the norm of the normed space)

(ii) Let X be a normed space.

If for any $\{x_n\} \subset X$ s.t. $\sum_{n=1}^{\infty} \|x_n\|$ converges one has that also $\sum_{n=1}^{\infty} x_n$ converges $\Rightarrow$ X is a Banach space.

proof.

(i) $\sum_{n=1}^{+\infty} \|x_n\|$ converges $\Rightarrow$ Cauchy criterion for series of real numbers:

$\forall \varepsilon > 0 \;\; \exists \bar{n} \in \mathbb{N} : \;\; \forall m,n > \bar{n} \quad \sum_{i=m+1}^{n} \|x_i\| < \varepsilon$

$\sum_{i=0}^{+\infty} a_i$ is convergent $\Leftrightarrow$ $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall m,n > N : \sum_{i=m+1}^{n} a_i < \varepsilon$

In other terms:

$S_n - S_m = \sum_{i=m+1}^{n} x_i \qquad$ with $S_n = \sum_{i=1}^{n} x_i$

$\Longrightarrow \|S_n - S_m\| \le \sum_{i=m+1}^{n} \|x_i\| < \varepsilon$

This means that $\{S_n\}$ is a Cauchy sequence in X.

Since X is a Banach space:

$\Rightarrow \exists x \in X$ s.t. $S_n \xrightarrow[n \to \infty]{} x$

$\overset{\text{def.}}{\Longleftrightarrow} \sum_{i=1}^{+\infty} x_i = x$.

(ii) Let $\{y_n\}_n$ be a Cauchy sequence in X.

We can always assume that $\exists$ a subsequence $\{y_{n_k}\}_k \subset \{y_n\}_n$ s.t.:

$$\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{k^2} \qquad \forall k \in \mathbb{N}$$

$\longrightarrow \sum_{k=1}^{+\infty} \underbrace{\|y_{n_{k+1}} - y_{n_k}\|}_{:= x_k}$ converges (since $\le \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$)

By assumption we can deduce that $\sum_{k=1}^{+\infty} x_k$ converges.

we are assuming that if $\sum \|x_n\|$ converges then $\sum x_n$ converges too (we are starting from this to prove that we obtain a Banach space)

$\Longleftrightarrow \exists s \in X : \lim_{m \to \infty} \boxed{\sum_{k=1}^{m} x_k} = s$

$\Longleftrightarrow \lim_{m \to \infty} \boxed{(y_{n_{m+1}} - y_{n_1})} = s$

$\sum_{k=1}^{m} x_k = \sum_{k=1}^{m} (y_{n_{k+1}} - y_{n_k}) = y_{n_{m+1}} - y_{n_1}$

$\Longleftrightarrow \lim_{m \to \infty} y_{n_{m+1}} = s + y_{n_1}$

$\Longrightarrow y_n \xrightarrow[n \to \infty]{} s + y_{n_1} \in X$

General property:
- $\{y_n\}_n$ Cauchy
- $\{y_{n_m}\}_m$ converges

$\Longrightarrow \{y_n\}_n$ converges

$\Longrightarrow X$ is complete (since any Cauchy sequence converges)

$\Longleftrightarrow X$ is a Banach space. ■

COMPACTNESS

– very related with the dimension of the space

Let X be a normed space.

Let's recall the def. of ball:

$\forall x_0 \in X, \; r > 0 :$

$B_r(x_0) := \{x \in X : \|x - x_0\| < r\}$ open ball

$\bar{B}_r(x_0) := \{x \in X : \|x - x_0\| \le r\}$ closed ball

$\overline{B_r(x_0)}$ closure of $B_r(x_0)$

Remark: If X is Banach: $\bar{B}_r(x_0) \equiv \overline{B_r(x_0)}$.

Remark: (X, d) metric space: $\overline{B_r(x_0)} \subsetneq \bar{B}_r(x_0)$.

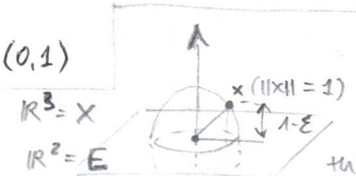
Lemma (**Riesz**).

Let X be a normed space, $E \subsetneq X$ closed subspace.
Then $\forall \varepsilon > 0 \;\; \exists x \in X$ s.t.

$$\|x\| = 1 \quad \text{and} \quad \underbrace{\operatorname{dist}(x, E)}_{:= \inf_{y \in E} \|x - y\|} \ge 1 - \varepsilon .$$

We take any closed subspace, then $\forall \varepsilon$ we can find an unitary element which is not in the subspace and which is s.t. its distance with the subspace is $\ge 1 - \varepsilon$

$\varepsilon \in (0,1)$



this case interpretation:
$\forall \varepsilon \in (0,1) \;\; \exists x \in B_1(0)$ s.t. $d(x, \mathbb{R}^2) \ge 1 - \varepsilon$

proof.

Let $y \in X \setminus E$.

Define: $d := \text{dist}(y, E) > 0$ (since E is closed)

Let $\varepsilon \in (0,1)$. We can choose $z \in E$ s.t.

$$d \le \|y - z\| \le \frac{d}{1-\varepsilon}$$

(z exists because $d = \text{dist}(y,E) := \inf_{\xi \in E} \|\xi - y\|$ and $\frac{d}{1-\varepsilon} > d$)

Let:

$$x := \frac{y-z}{\|y-z\|}$$

$$\forall \xi \in E: \quad \|x - \xi\| = \left\| \frac{y-z}{\|y-z\|} - \xi \right\| = \frac{1}{\|y-z\|} \|y - z - \xi\|y-z\|\| = \frac{1}{\|y-z\|} \underbrace{\|y - \underbrace{(z + \xi\|y-z\|)}_{\in E \text{ since } E \text{ is a subspace } (z,\xi \in E,\ \|y-z\| \in \mathbb{R})}\|}_{\ge d \text{ (since it's the distance of } y \text{ and an element of } E \text{ and } d := \inf_{\xi \in E} \|y-\xi\|)} \ge \frac{d}{\|y-z\|} \ge 1-\varepsilon$$

■

Theorem: (**Riesz**)

Let X be a normed space.
If the closed ball $\bar{B}_1(0)$ is compact $\Longrightarrow$ $\dim(X) < \infty$.

proof.

Let $x_1 \in \bar{B}_1(0)$ and $Y_1 := \text{span}\{x_1\} = \{y \in X : y = \alpha x_1,\ \alpha \in \mathbb{R}\}$ — vector space generated by the vector x_1

$\Longrightarrow$ Y_1 is closed (in a normed space, a vector subspace of finite dimension is closed)

If $X = Y_1 \Longrightarrow \dim(X) = 1$ and we have finished.

If $X \neq Y_1 \Longrightarrow$ we can use the Riesz lemma with $\varepsilon = \frac{1}{2}$ to find $x_2 \in \bar{B}_1(0)$ s.t.:

$$\|x_2 - x_1\| \ge \frac{1}{2}$$

since for $\varepsilon = \frac{1}{2}$ $\exists x_2$ s.t. $\text{dist}(x_2, Y_1) \ge 1-\varepsilon$
hence $\text{dist}(x_2, x_1) \ge 1 - \varepsilon = \frac{1}{2}$ since $x_1 \in Y_1$
(**Riesz lemma**)

Let $Y_2 := \text{span}\{x_1, x_2\} \Longrightarrow Y_2$ is closed.

If $X = Y_2 \Longrightarrow \dim(X) = 2$

If $X \neq Y_2 \Longrightarrow$ we can again use Riesz lemma to find $x_3 \in \bar{B}_1(0)$ s.t.:

$$\|x_3 - x_i\| \ge \frac{1}{2} \qquad i = 1, 2$$

We iterate this procedure.
If X is not finite dimensional this process can be iterated to yield a sequence $\{x_n\}_n \subseteq \bar{B}_1(0)$ s.t.:

$$\|x_i - x_j\| \ge \frac{1}{2} \qquad \forall i,j \in \mathbb{N},\ i \neq j$$

Hence $\{x_n\}_n$ is bounded (since $\|x_n\| \le 1\ \forall n \in \mathbb{N}$) but it has **no** convergent subsequence.

$\to$ $\bar{B}_1(0)$ is not (sequentially) compact.

we have a sequence of elements where the elements distance from each others at least $1/2$
$\Longrightarrow$ **not** a Cauchy sequence
$\Longrightarrow$ **not** convergent
$\Longrightarrow$ $\nexists$ converging subsequence

■

If $\exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $\{x_{n_k}\}$ converges then $\{x_{n_k}\}$ has to be Cauchy; however the whole sequence is s.t. every element distance at least $1/2$ from every other ($\nexists$ Cauchy subsequence)

Remark: $E \subseteq X$ closed and bounded is compact $\Longleftrightarrow$ $\dim(X) < \infty$

Remark: $F \subset C^0(X)$ bounded, closed, $\underbrace{\text{equicontinuous}}$ $\Longrightarrow$ F compact
(Ascoli–Arzelà)

this Hp. permits to deal with a space of infinite dimension ($C^0(X)$) as a space of """finite dimension""" (otherwise we couldn't a compact subspace (closed and bdd))

EQUIVALENT NORMS

Def. Let $(X, \|\cdot\|)$, $(X, \|\cdot\|_\#)$ be two normed spaces. (same vector space, different norms)
$\|\cdot\|, \|\cdot\|_\#$ are equivalent if $\exists m, M > 0$ s.t. :

$$m\|x\| \leq \|x\|_\# \leq M\|x\| \qquad \forall x \in X$$

Theorem: X normed space.
If $\dim(X) < \infty \Longrightarrow$ all norms on X are equivalent.

proof.

Consider $X = \mathbb{R}^n$, $\|\cdot\|_1$ and $\|\cdot\|$ = any norm.
In $\mathbb{R}^n$ we consider the standard basis:
$\{e_j\}_{j=1}^n$ (canonical basis)

$$\Longrightarrow x = \sum_{j=1}^n \alpha_j e_j \quad \Longrightarrow \quad \|x\|_1 = \sum_{j=1}^n |\alpha_j|$$

$$\Longrightarrow \|x\| \leq \sum_{j=1}^n |\alpha_j| \cdot \|e_j\| \leq M \cdot \sum_{j=1}^n |\alpha_j| = M \cdot \|x\|_1$$

$$M := \max_{j=1,\dots,n} \|e_j\|$$

Now we show: $\exists m > 0 : \|x\| \geq m \cdot \|x\|_1$.
We define $\phi : X \to \mathbb{R}_+$:

$$\phi(x) := \|x\|$$

$$\forall x, x_0 \in X : \quad |\phi(x) - \phi(x_0)| \leq \|x - x_0\| \leq M \cdot \|x - x_0\|_1$$

$(|\phi(x) - \phi(x_0)| = |\|x\| - \|x_0\|| \leq \|x - x_0\|)$

Reverse triangle inequality: $|\|x\| - \|y\|| \leq \|x - y\|$!

$\Longrightarrow \phi$ is Lipschitz in X
$\Longrightarrow \phi$ is continuous in X

Now we define:

$$K := \{x \in \mathbb{R}^n : \|x\|_1 = 1\} \quad \text{compact (since } \dim(X) = n < \infty)$$

$\Longrightarrow$ By Weierstrass theorem : $\exists m := \min_K \phi \geq 0$ (since $\phi : X \to \mathbb{R}_+$)

$$\frac{\|x\|}{\|x\|_1} = \left\| \frac{x}{\|x\|_1} \right\| = \phi\Big(\underbrace{\frac{x}{\|x\|_1}}_{\in K}\Big) \geq m \qquad \forall x \in X \setminus \{0\}$$

because $m = \min \phi$ on K

$\in K$ because $\left\|\frac{x}{\|x\|_1}\right\|_1 = 1$

If $m = 0 \Longrightarrow \exists x_{min} \neq 0$ s.t. $\|x_{min}\|_1 = 1$, $\|x_{min}\| = 0$
but this is a contraddiction : (since:)

$$\phi(x_{min}) = \|x_{min}\| = 0 \Longleftrightarrow x_{min} = 0$$

Thus $m > 0$ and :

$$\frac{\|x\|}{\|x\|_1} \geq m \Longleftrightarrow \|x\| \geq m \cdot \|x\|_1 . \qquad \blacksquare$$

Examples :

1. $C^1([a,b])$
$$\|f\|_{\infty,1} = \|f\|_\infty + \|f'\|_\infty$$
$$\|f\|_\# = |f(a)| + \|f'\|_\infty$$
} equivalent

2. $AC([a,b])$
$$\|f\|_{AC} = \|f\|_1 + \|f'\|_1$$
$$\|f\|_\# = |f(a)| + \|f'\|_1$$
} equivalent

3. $C^0([a,b])$: $\underbrace{\|\cdot\|_\infty}_{(C^0([a,b]), \|\cdot\|_\infty) \text{ is complete}}$ is not equivalent to $\underbrace{\|\cdot\|_1}_{(C^0([a,b]), \|\cdot\|_1) \text{ is not complete (not Banach)}}$

If two norms are equivalent also properties hold or not w.r.t. both norms. If two norms are **not** equivalent we can have different behavior in terms of properties.

LEBESGUE SPACES

Let $(X, \mathcal{A}, \mu)$ be a measure space (μ general measure). Let $p \in [1, +\infty]$.

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \{ f : X \to \overline{\mathbb{R}} \text{ measurable}, \int_X |f|^p \, d\mu < \infty \}.$$

Two functions $f, g \in \mathcal{L}^p$ are in relation:

$$f \sim g \iff f = g \text{ a.e. in } X \qquad \text{(equivalence relation)}$$

Moreover, we define:

$$L^p(X, \mathcal{A}, \mu) := \frac{\mathcal{L}^p(X, \mathcal{A}, \mu)}{\sim}$$

The elements of L^p are equivalence classes ($[f] \in L^p$).

Lemma. Let $p \in [1, \infty)$, $a \geq 0$, $b \geq 0$. Then:

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \qquad (1)$$

proof.

$$(1) \iff \left(\frac{a+b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{2}. \qquad (2)$$

The function $x \mapsto x^p$ is convex $(p \geq 1)$.
By definition of convexity $\Rightarrow$ (2). ∎

$\varphi(\cdot)$ convex if:
$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$ $\forall x, y$, $\forall t \in [0,1]$
here we're choosing $t = \frac{1}{2}$

Lemma. L^p is a vector space.

proof.

If $f, g \in L^p \Rightarrow f, g$ are finite a.e. in X ($\int_X |f|^p d\mu < \infty \iff f$ finite a.e.)
Let $\lambda \in \mathbb{R} \Rightarrow f + \lambda g$ * is measurable **

* well defined only because f, g finite a.e. (otherwise with $\lambda = -1$ we could have $\infty - \infty$)
** since both are measurable

$$\int_X |f + \lambda g|^p d\mu \leq 2^{p-1}\left(\int_X |f|^p d\mu + \lambda \int_X |g|^p d\mu\right) \quad \text{by the previous lemma}$$

$< \infty$ since $f, g \in L^p$ ($\int_X |f|^p d\mu < \infty$, $\int_X |g|^p d\mu < \infty$) ∎

Def. Consider $p, q \in [1, \infty]$. They are conjugate if $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, or $p = 1, q = \infty$, or $q = 1, p = \infty$.

$$\begin{cases} p, q \in (1, \infty) & \frac{1}{p} + \frac{1}{q} = 1 \\ p = 1, q = \infty \\ p = \infty, q = 1 \end{cases}$$

Lemma. (Young inequality)
Let $p \in (1, \infty)$, $a > 0$, $b > 0$. Then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{with } p, q \text{ conjugate } \left(q = \frac{p}{p-1}\right).$$

proof.

We define: $\varphi(x) = e^x$, which is convex.
We write: $a \cdot b = e^{\log(a)} e^{\log(b)} = e^{\frac{1}{p}\log(a^p)} \cdot {}^{\frac{1}{q}\log(b^q)}$.
Since φ is convex:

$$\varphi(tx + (1-t)y) \leq t\,\varphi(x) + (1-t)\,\varphi(y) \qquad \forall x, y \in \mathbb{R}_+,\ t \in (0,1)$$

Now we consider

$$t = \frac{1}{p} \in (0,1) \text{ and } 1 - t = 1 - \frac{1}{p} = \frac{p-1}{p} = \frac{1}{p/_{p-1}} = \frac{1}{q}$$

Moreover we take $x = \log(a^p)$, $y = \log(b^q)$.

$$\Rightarrow ab = e^{\frac{1}{p}\log(a^p)} e^{\frac{1}{q}\log(b^q)} \leq \frac{1}{p} e^{\log(a^p)} + \frac{1}{q} e^{\log(b^q)} = \frac{a^p}{p} + \frac{b^q}{q}$$

■

Theorem (Young's consequence) : (**Hölder inequality**)

Let $f, g \in \mathcal{M}(X, \mathcal{A})$, $p, q \in [1, \infty]$ conjugate.

Then:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad (*)$$

where: $f \in L^p$
$$\|f\|_p := \left(\int_X |f|^p dx\right)^{1/p}$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_X |f|$$

and so:

$$\int_X |fg| \, d\mu \leq \left[\int_X |f|^p d\mu\right]^{1/p} \left[\int_X |g|^q d\mu\right]^{1/q}$$

proof.

(i) $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$.

- $(*)$ is obvious if $\|f\|_p \cdot \|g\|_q = +\infty$. ($\|f \cdot g\|_1$ can be anything)
- If $\|f\|_p \cdot \|g\|_q = 0$ then $f = 0$ a.e. or $g = 0$ a.e.
  $\Rightarrow fg = 0$ a.e. $\Rightarrow \|fg\|_1 = 0$
- Now, suppose that $\|f\|_p$ and $\|g\|_q$ are finite and $\neq 0$.
  Let $x \in X$. Set:
  $$a := \frac{|f(x)|^p}{\|f\|_p^p}, \quad b := \frac{|g(x)|^q}{\|g\|_q^q}$$
  We apply Young inequality:
  $$a^{1/p} b^{1/q} = \underbrace{\frac{|f(x)|}{\|f\|_p}}_{A} \cdot \underbrace{\frac{|g(x)|}{\|g\|_q}}_{B} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$
  $AB \leq \frac{1}{p} A^p + \frac{1}{q} B^q$

$$\Rightarrow \frac{1}{\|f\|_p \|g\|_q} \int_X |fg| \, d\mu \leq \frac{1}{p} \underbrace{\frac{\int_X |f|^p d\mu}{\|f\|_p^p}}_{=1} + \frac{1}{q} \underbrace{\frac{\int_X |g|^q d\mu}{\|g\|_q^q}}_{=1} = \frac{1}{p} + \frac{1}{q} = 1$$

Therefore:

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q \quad \left(\Leftrightarrow \quad \|fg\|_1 = \|f\|_p \|g\|_q\right)$$

(ii) $p = 1$, $q = \infty$.

$|g| \leq \|g\|_\infty$ a.e. in X

$\Rightarrow |fg| \leq |f| \, \|g\|_\infty$ a.e. in X

$$\Rightarrow \|fg\|_1 = \int_X |fg| \, d\mu \leq \|g\|_\infty \int_X |f| \, d\mu = \|f\|_1 \|g\|_\infty$$

■

Theorem. (**Minkowski inequality**).

(Hölder's consequence)

Let $f, g \in L^p(X, \mathcal{A}, \mu)$, $p \in [1, \infty]$. Then:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

proof.

(i) $p \in (1, \infty)$.

$$\|f+g\|_p^p = \int_X |f+g|^p d\mu = \int_X |f+g| \cdot |f+g|^{p-1} d\mu$$
$$\leq \int_X |f||f+g|^{p-1} d\mu + \int_X |g| |f+g|^{p-1} d\mu$$

In view of Hölder inequality: $\left(q = \frac{p}{p-1}\right)$

$$\int_X |f| \cdot |f+g|^{p-1} d\mu \leq \|f\|_p \| |f+g|^{p-1} \|_q$$
$$\int_X |g||f+g|^{p-1} d\mu \leq \|g\|_p \| |f+g|^{p-1} \|_q$$

Now:

$$\| |f+g|^{p-1} \|_q = \left(\int_X |f+g|^{(p-1)q} d\mu\right)^{1/q} \quad \text{where } q = \frac{p}{p-1}$$
$$= \left(\int_X |f+g|^p d\mu\right)^{\frac{1}{q}}$$
$$= \left(\|f+g\|_p^p\right)^{1/q} = \|f+g\|_p^{p/q}$$

It follows that:

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f+g\|_p^{p/q}$$

$$\Longrightarrow \|f+g\|_p^{p-p/q} \leq \|f\|_p + \|g\|_p$$

$$\Longrightarrow p - \frac{p}{q} = p - \frac{p}{p/_{p-1}} = p - p + 1 = 1$$

$$\Longrightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

(ii) Let $p = 1$:

$$\|f+g\|_1 = \int_X |f+g| d\mu \leq \int_X (|f| + |g|) d\mu = \|f\|_1 + \|g\|_1$$

(iii) Let $p = \infty$:

$$\|f+g\|_\infty = \operatorname*{ess\,sup}_X |f+g| \leq \operatorname*{ess\,sup}_X (|f| + |g|)$$
$$\leq \operatorname*{ess\,sup}_X |f| + \operatorname*{ess\,sup}_X |g|$$
$$= \|f\|_\infty + \|g\|_\infty \qquad \blacksquare$$

Corollary: $L^p(X, \mathcal{A}, \mu)$ is a normed space with:

(Minkowski's consequence)

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}, \quad p \in [1, \infty)$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_X |f|$$

proof.

$p = 1$, $q = \infty$ is already done.

We consider the other cases.

The function $\|\cdot\|_p : L^p \longrightarrow [0, +\infty)$ is well defined.

$\|f\|_p \geq 0 \quad \forall f \in L^p, \quad \|f\|_p = 0 \iff f = 0$ in L^p
($\iff f = 0$ a.e. in X)

$$\|\alpha f\|_p = \left(\int_X |\alpha f|^p \, d\mu\right)^{1/p} = |\alpha| \left(\int_X |f|^p \, d\mu\right)^{1/p} = |\alpha| \cdot \|f\|_p$$

Minkowski inequality implies that:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \qquad \blacksquare$$

Theorem: (Inclusion of L^p spaces)
Suppose that $\mu(X) < \infty$. Then:

$$1 \leq p \leq q \leq \infty \longrightarrow L^q(X, \mathcal{A}, \mu) \subseteq L^p(X, \mathcal{A}, \mu).$$

proof.
The thesis follows if we show that there is a constant $C = C(X, p, q) > 0$:

$$\|f\|_p \leq C \cdot \|f\|_q \qquad \forall f \in L^q \quad (\Rightarrow f \in L^p)$$

- Suppose $q = \infty$.

$$\|f\|_p^p = \int_X |f|^p \, d\mu \leq \|f\|_\infty^p \, \mu(X).$$

$$\longrightarrow \|f\|_p \leq \underbrace{(\mu(X))^{1/p}}_{= C} \|f\|_\infty$$

($\mu(X)$ has to be finite)

- Now let $q < \infty$.

$$\|f\|_p^p = \int_X |f|^p \, d\mu = \int_X 1 \cdot |f|^p \, d\mu \leq \left(\int_X 1^r \, d\mu\right)^{\frac{1}{r}} \left(\int_X |f|^{p \cdot s} \, d\mu\right)^{\frac{1}{s}}$$

Take $ps = q \Rightarrow \frac{1}{s} = \frac{p}{q} \longrightarrow \frac{1}{r} = 1 - \frac{1}{s} = \frac{q-p}{q}$

Hence:

$$\|f\|_p^p \leq (\mu(X))^{\frac{q-p}{q}} \left(\int_X |f|^q \, d\mu\right)^{p/q}$$

$$\|f\|_p \leq \underbrace{(\mu(X))^{\frac{q-p}{qp}}}_{C} \left(\int_X |f|^q \, d\mu\right)^{1/q} = C \cdot \|f\|_q \qquad \blacksquare$$

Remark: $\mu(X)$ is essential. If $\mu(X) = \infty$, in general, the previous inclusion is false.
Counterexample:

$$f(x) = \frac{1}{x} \qquad x \in (1, \infty) \qquad (\lambda((1,\infty)) = \infty)$$

$f \notin L^1((1,\infty))$ but $f \in L^2((1,\infty))$ since $\int_1^\infty \frac{1}{x^2} \, dx < \infty$.

COMPLETENESS OF LP SPACES

Theorem: $L^p(X, \mathcal{A}, \mu)$ is a Banach space $\forall p \in [1, \infty]$.

(Recall) $(X, \|\cdot\|_X)$ normed space, $\{f_n\} \subset X$.
If we have the implication:

$$\sum_{n=1}^\infty \|f_n\|_X \text{ converges} \Longrightarrow \underbrace{\sum_{n=1}^\infty f_n \text{ converges in } X}_{\text{in the sense of the normed space}}$$

for every sequence

then X is complete.

proof.

- let $p \in [1, \infty)$. → characterization of completeness in terms of series (+ MCT, DCT)

Claim: let $\{f_n\} \subset L^p$. If $\sum_{n=1}^{\infty} \|f_n\|_p$ converges (series of non-negative real numbers) then $\sum_{n=1}^{\infty} f_n$ converges (series of elements of the normed space) in L^p.

From the claim the thesis follows by a previous result.

Now we prove the claim.

We define:

$$g_k := \sum_{n=1}^{k} |f_n| \qquad \forall k \in \mathbb{N}.$$

$$\Longrightarrow \|g_k\|_p \underset{\substack{\text{Minkowski}\\ \text{inequality}}}{\leq} \|f_1\|_p + \dots + \|f_k\|_p \leq \sum_{n=1}^{+\infty} \|f_n\|_p =: \underbrace{M \in \mathbb{R}_+}_{\text{the series converges by assumption}}$$

let:

$$g(x) := \sum_{n=1}^{\infty} |f_n(x)|$$

We observe that $\{g_k\} \nearrow$, g_k is measurable $\forall k \in \mathbb{N}$

$\Longrightarrow$ $\{|g_k|^p\} \nearrow$, $|g_k|^p$ is measurable

$$\Longrightarrow \underbrace{\lim_{k\to\infty} \int_X |g_k|^p \, d\mu}_{\leq M^p} \overset{MCT}{=} \int_X \underbrace{\lim_{k\to\infty} |g_k|^p}_{\lim_{k\to\infty} |g_k| = |g|} d\mu = \int_X |g|^p \, d\mu \leq M^p$$

since $\int_X |g_k|^p d\mu \leq M^p$

$\Longrightarrow$ $g \in L^p$ $\Longrightarrow$ g is finite a.e. in X (hence the definition of g makes sense)

$\Longrightarrow$ $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely a.e. in X.

let:

$$s(x) := \sum_{n=1}^{\infty} f_n(x)$$

$$s_k(x) := \sum_{n=1}^{k} f_n(x)$$

$$\Longrightarrow s_k \xrightarrow[k\to\infty]{} s \quad \text{a.e. in } X \qquad (1)$$

$$|s_k - s|^p \leq \left|\sum_{n=k+1}^{\infty} f_n\right|^p \leq \Big(\underbrace{\sum_{n=k+1}^{\infty} |f_n|}_{\leq g}\Big)^p \leq \underbrace{g^p \in L^1}_{\text{(since:)} \iff g \in L^p} \qquad (2)$$

By DCT: ((1)+(2))

$$\lim_{k\to\infty} \int_X |s_k - s|^p \, d\mu = 0 \iff \sum_{n=1}^{\infty} f_n \text{ converges in } L^p.$$

- Now, let $p = \infty$. → Definition of complete space: [we take a Cauchy sequence and we prove that it converges to an element of the same space in the norm of the space]

let $\{f_n\}_n \subset L^\infty$ be a Cauchy sequence. → we check that $\forall$ Cauchy converges

set:

$$A_n := \{x \in X : |f_n(x)| > \|f_n\|_\infty\}$$

$$B_{nm} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$$

$\forall n, m \in \mathbb{N}$ $\quad \mu(A_n) = \mu(B_{nm}) = 0$ (since $\{f_n\}_n \subset L^\infty$) ⌒ $f_n - f_m \in L^\infty$ hence $\mu(B_{nm}) = 0$ (L^∞ is a vector space)

$$E := \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n,m\in\mathbb{N}} B_{nm}\right)$$

$\Longrightarrow$ $\mu(E) = 0$

$\{f_n\}_n$ Cauchy in L^∞ $\Longrightarrow$ $\{f_n\}_n$ bounded in L^∞

$(\iff \exists K > 0 : \|f_n\|_\infty \leq K \quad \forall n \in \mathbb{N})$

Moreover $\exists f$ measurable, bounded s.t.
$$f_n \xrightarrow[n\to\infty]{} f \quad \text{in } E^c.$$

pointwise convergence because it's a sequence of real numbers $\forall x$ and in E^c it holds Cauchy condition for $\mathbb{R}^n \longrightarrow$ the seq. converges in E^c: $|f_n(x)-f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon$

Cauchy condition: (in L^∞)

$\forall \varepsilon > 0 \quad \exists \bar{n} \in \mathbb{N}$ s.t. $\forall x \in E^c \quad \forall m,n > \bar{n}$:
$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon$$

Letting $m \to \infty$: (we exploit the pointwise convergence)
$$|f_n(x) - f(x)| < \varepsilon \qquad \forall n > \bar{n},\ \forall x \in E^c$$

↳ uniform convergence in E^c

$\Longrightarrow f_n \xrightarrow[n\to\infty]{} f$ uniformly in E^c

$\underset{\mu(E)=0}{\Longrightarrow} f_n \xrightarrow[n\to\infty]{} f$ in $L^\infty(X)$.

→ since E^c is such that $\mu(E)=0$ we have uniform convergence a.e., which is the L^∞ convergence ■

CONVERGENCE IN L^p, A.E. AND IN MEASURE

$$f_n \longrightarrow f \text{ in } L^p \iff \|f_n - f\|_p \xrightarrow[n\to\infty]{} 0$$
$$\iff \int_X |f_n - f|^p d\mu \xrightarrow[n\to\infty]{} 0$$

Prop. $f_n \xrightarrow[n\to\infty]{} f$ in $L^p \Longrightarrow f_n \xrightarrow[n\to\infty]{} f$ in measure (i.e. $\forall \varepsilon > 0 \ \ \mu(\{|f_n - f| \geq \varepsilon\}) \xrightarrow[n\to\infty]{} 0$)

proof.

It is the same as in the case $p=1$.
Suppose by contraddiction that $f_n \not\xrightarrow[n\to\infty]{} f$ in measure.
Then $\exists \varepsilon > 0, \sigma > 0$ s.t. :
$$\mu(\{|f_n - f| \geq \varepsilon\}) \geq \sigma \quad \text{for infinitely many } n \in \mathbb{N}.$$
$$\|f_n - f\|_p^p = \int_X |f_n - f|^p d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f|^p d\mu$$
$$\geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon^p d\mu$$
$$= \varepsilon^p \mu(\{|f_n - f| \geq \varepsilon\}) \geq \varepsilon^p \sigma > 0$$
for infinitely many $n \in \mathbb{N}$
$\longrightarrow f_n \not\xrightarrow[n\to\infty]{} f$ in L^p ■

Corollary. $f_n \xrightarrow[n\to\infty]{} f$ in L^p, $p \in [1,\infty] \Longrightarrow \exists \{f_{n_k}\}_k \subset \{f_n\}_n$ s.t.
$$f_{n_k} \xrightarrow[k\to\infty]{} f \quad \text{a.e. in } X$$

proof.

$p = \infty$ obvious (conv. in $L^\infty \longrightarrow$ conv. a.e. for $\{f_n\}$)

convergence in L^∞ is uniform convergence a.e. (hence it implies a.e. pointwise conv.) !

$p \in [1,\infty)$: $f_n \xrightarrow[n]{} f$ in measure $\Longrightarrow$ thesis ■

previous theorem (convergence in measure implies a convergent a.e. subsequence)

SEPARABILITY IN L^p

Def. Let $g : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$
$$\operatorname{supp} g := \overline{\{x \in \Omega : g(x) \neq 0\}}$$

Def. $C_c^0(\mathbb{R}) := \{ f \in C^0(\mathbb{R}) : \operatorname{supp} f \text{ is compact} \}$.

continuous with compact support

Theorem: (Lusin)

measurable set

Let $\Omega \in \mathcal{L}(\mathbb{R})$, $\lambda(\Omega) < \infty$.] e.g. $[a,b]$

$L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$

Let $f: \mathbb{R} \to \mathbb{R}$ measurable, $f \equiv 0$ in Ω^c.

Then $\forall \varepsilon > 0 \ \exists g \in C_c^0(\mathbb{R})$ s.t.:

$$\lambda(\{x \in \mathbb{R} : f(x) \neq g(x)\}) < \varepsilon$$

Moreover we can choose g: $\left[\sup_{\mathbb{R}} |g| \leq \sup_{\mathbb{R}} |f|\right]$

we can find a continuous function (g) which is close to f in a suitable way:
- the set where the two functions are different has small measure (as we want) (g is "close" to f)
- the supremum of $g \leq$ supremum of f

Def. Let $\tilde{\mathcal{S}}(\mathbb{R}) := \{ s: \mathbb{R} \to \mathbb{R} \text{ simple}, \ \lambda(\text{supp } s) < \infty \}$

Remark: $s \in \tilde{\mathcal{S}}(\mathbb{R}) \iff$ s is simple and $s \neq 0$ on a set of finite measure

$\overset{*}{\iff}$ s is simple and belongs to L^p ($s \in L^p(\mathbb{R})$, $p \in [1, \infty)$)

Theorem: $\tilde{\mathcal{S}}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ $\forall p \in [1, \infty)$.

proof.

Let $f \in L^p(\mathbb{R})$. — we have to prove that $\exists$ a sequence of elements of $\tilde{\mathcal{S}}(\mathbb{R})$ converging to f in L^p. — 1st way of proving density

In addition we suppose that $f \geq 0$ a.e. in $\mathbb{R}$.

$\xrightarrow{\text{simple approx. theorem}}$ $\exists \underbrace{\{s_n\}_n}_{\in \mathcal{S}(\mathbb{R})}$ s.t. $0 \leq s_n \leq f$ $\quad (\{s_n\} \subset \mathcal{S}(\mathbb{R}))$

$\{s_n\} \nearrow$

$s_n \xrightarrow[n\to\infty]{} f$ a.e. in $\mathbb{R}$

$\Longrightarrow \{s_n\} \subset L^p$ (since $0 \leq s_n \leq f$ and $f \in L^p$)

$\overset{*}{\Longrightarrow} \{s_n\} \subset \tilde{\mathcal{S}}(\mathbb{R})$

$\overset{DCT}{\Longrightarrow} s_n \xrightarrow[n\to\infty]{} f$ in $L^p(\mathbb{R})$. ($\leftarrow$ meaning of dense) — we found a sequence in $\tilde{\mathcal{S}}$ that approximate $f \in L^p$

If f is sign-changing, we argue as above for f_+ and f_-. The thesis follows since $f = f_+ - f_-$. ∎

Theorem: $C_c^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ $\forall p \in [1, \infty)$.

proof.

Let $\varepsilon > 0$, $f \in L^p(\mathbb{R})$. — we have to prove that $\forall \varepsilon > 0 \ \exists$ a function of $C_c^0(\mathbb{R})$ such that the distance of this function and f is $< \varepsilon$. — 2nd way of proving density

By the previous theorem there exists $s \in \tilde{\mathcal{S}}(\mathbb{R})$ such that:

$$\|f - s\|_p < \varepsilon.$$

By Lusin theorem $\exists g \in C_c^0(\mathbb{R})$ s.t. $\lambda(\{g \neq s\}) < \varepsilon$, $\|g\|_\infty \leq \|s\|_\infty$.

The aim is to prove that $\|f - g\|_p$ is small as we want:

If $\sup|g| \leq \sup|s|$ then for sure: $\text{ess sup}|g| \leq \text{ess sup}|s|$, $\|g\|_\infty \leq \|s\|_\infty$

$$\begin{aligned}
\|f-g\|_p &\leq \|f-s\|_p + \|s-g\|_p = \|f-s\|_p + \|g-s\|_p \\
&< \varepsilon + \left(\int_{\mathbb{R}} |g-s|^p \, d\lambda\right)^{1/p} \\
&= \varepsilon + \left(\int_{\{g \neq s\}} |g-s|^p \, d\lambda\right)^{1/p} \\
&\leq \varepsilon + \left(\int_{\{g \neq s\}} (|g| + |s|)^p \, d\lambda\right)^{1/p} \\
&\leq \varepsilon + \left[(\|g\|_\infty + \|s\|_\infty)^p \int_{\{g \neq s\}} d\lambda\right]^{1/p} \\
&= \varepsilon + (\|g\|_\infty + \|s\|_\infty)\left[\lambda(\{g \neq s\})\right]^{1/p} \\
&< \varepsilon + 2\|s\|_\infty \varepsilon^{1/p}
\end{aligned}$$

in the integral: $|g| + |s| \leq \|g\| + \|s\|$

∎

Theorem: Let $p \in [1, \infty)$. Then, $L^p(\mathbb{R})$ is separable.

proof.

Let $\varepsilon > 0$, $f \in L^p(\mathbb{R})$.
By the previous theorem $\exists g \in C_c^0(\mathbb{R})$ s.t.

$$\|f - g\|_p < \frac{\varepsilon}{2}.$$

$\exists \bar{n} \in \mathbb{N}$ s.t. $\operatorname{supp} g \subset [\bar{n}, \bar{n}]$.
Since $C^0([-\bar{n},\bar{n}])$ is separable, there exist a polynomial Q with rational coefficients such that:
$\Rightarrow$ countable

$$\|g - Q\mathbb{1}_{[-\bar{n},\bar{n}]}\|_{L^\infty(\mathbb{R})} = \|g - Q\|_{L^\infty([-\bar{n},\bar{n}])} < \frac{\varepsilon}{2(2\bar{n})^{1/p}}$$

$C^0(\cdot)$ separable

by previous results we assume that this is small as we want, we decide:

Therefore:

$$\underbrace{\|f - Q\mathbb{1}_{[-\bar{n},\bar{n}]}\|_p}_{\text{countable } Q \text{ polynomial}, \ \bar{n} \in \mathbb{N}} \leq \|f - g\|_p + \|g - Q\mathbb{1}_{[-\bar{n},\bar{n}]}\|_p$$
$$< \frac{\varepsilon}{2} + \left(\int_{[-\bar{n},\bar{n}]} |g - Q|^p \, d\lambda\right)^{1/p}$$
$$< \frac{\varepsilon}{2} + \left(\lambda([-\bar{n},\bar{n}]) \, \|g - Q\|^p_{L^\infty([-\bar{n},\bar{n}])}\right)^{1/p}$$
$$= \frac{\varepsilon}{2} + \left(2\bar{n} \, \|g - Q\|^p_{L^\infty([-\bar{n},\bar{n}])}\right)^{1/p}$$
$$< \frac{\varepsilon}{2} + (2\bar{n})^{1/p} \frac{\varepsilon}{2(2\bar{n})^{1/p}} = \varepsilon \qquad \blacksquare$$

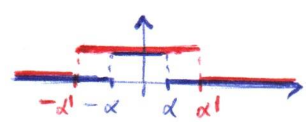
$\forall f \ \exists \ Q\mathbb{1}_{[-\bar{n},\bar{n}]}$ such that the distance between these two functions is small as we want and this implies separability since the set of functions of this type ($Q\mathbb{1}_{[-\bar{n},\bar{n}]}$) is countable.

Theorem: $L^\infty(\mathbb{R})$ is **not** separable.
$(L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda))$

proof.

$\{\mathbb{1}_{[-\alpha,\alpha]}\}_{\alpha>0} \subset L^\infty(\mathbb{R})$.
Since $\alpha \in \mathbb{R} \Rightarrow \{\mathbb{1}_{[-\alpha,\alpha]}\}_{\alpha>0}$ is not countable.
Consider now $\alpha, \alpha' \in \mathbb{R}$: $(\alpha' \geq \alpha)$:

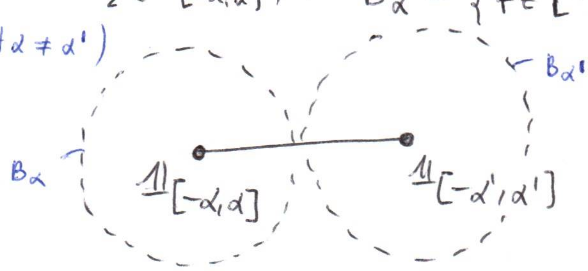
$$\|\mathbb{1}_{[-\alpha,\alpha]} - \mathbb{1}_{[-\alpha',\alpha']}\|_\infty = 1 \quad \text{if} \quad \alpha \neq \alpha'.$$



Inside $L^\infty(\mathbb{R})$ we consider a ball of radius $\frac{1}{2}$ and centered in $\mathbb{1}_{[-\alpha,\alpha]}$:

$$B_{\frac{1}{2}}(\mathbb{1}_{[-\alpha,\alpha]}) := B_\alpha = \{f \in L^\infty(\mathbb{R}) : \|\mathbb{1}_{[-\alpha,\alpha]} - f\|_\infty < \tfrac{1}{2}\}$$

$(\forall \alpha \neq \alpha')$



$$B_\alpha \cap B_{\alpha'} = \emptyset \qquad \forall \alpha \neq \alpha'$$

Let $Z \subset L^\infty(\mathbb{R})$, Z dense in $L^\infty(\mathbb{R})$:
$\Rightarrow \forall \alpha > 0 \quad B_\alpha \cap Z \neq \emptyset$ (Z dense)
however, if we change ball:
$\alpha' \neq \alpha \quad B_\alpha \cap B_{\alpha'} = \emptyset$
and:
$\{B_\alpha\}_{\alpha>0}$ is not countable

$\Rightarrow Z$ is **not** countable

Hence, if there is a dense set in L^∞ then the set is uncountable.

$\longrightarrow$ L^∞ cannot be separable since $\nexists$ countable dense sets.

$\longrightarrow$ $L^\infty(\mathbb{R})$ is **not** separable. ■

The same results (separability) hold if we consider open subsets of $\mathbb{R}^n$ instead of $\mathbb{R}^n$:

If $\Omega \subseteq \mathbb{R}^n$ open: $L^p(\Omega)$ is separable $\forall p \in [1,\infty)$

$L^\infty(\Omega)$ is **not** separable.

ℓ^p SPACES

$p \in [1,\infty]$: $\quad \ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$

Consider now a sequence of real numbers: $\{a_n\}_n \subset \mathbb{R}$.

If we define $f: \mathbb{N} \to \mathbb{R}$: $f \equiv \{a_n\}_n$ then we see f as a sequence.

$$\|f\|_p = \left(\int_{\mathbb{N}} |f|^p d\mu^\#\right)^{1/p} = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \qquad p \in [1,\infty)$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{n\in\mathbb{N}} |f(n)| = \sup_{n\in\mathbb{N}} a_n$$

Exactly as in L^p, it holds: (however the proofs are different)

$\longrightarrow$ ℓ^p $p \in [1,\infty)$ is separable.

In fact, consider:

$$c_0 := \{\{a_n\}_n \subset \mathbb{Q} : a_n = 0 \quad \forall n > k, \ k \in \mathbb{N}\}.$$

This set is countable and dense in ℓ^p.

In c_0 there are:

$k=1$	a_1	0	0	0	..
$k=2$	a_1	a_2	0	0	..
$k=3$	a_1	a_2	a_3	0	..

$\longrightarrow$ ℓ^∞ is **not** separable.

Inequality for L^1 functions:

Theorem: (Jensen inequality)

Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$.

Let $f \in L^1$, $\varphi: \mathbb{R} \to \mathbb{R}$ be <u>convex</u>. (φ convex, **not** f)

Then:

$$\varphi\Big(\underbrace{\int_X f \, d\mu}_{\in\mathbb{R}}\Big) \leq \int_X (\varphi \circ f) \, d\mu.$$

Example: $\varphi(x) = e^x \implies e^{\int_X f \, d\mu} \leq \int e^f \, d\mu$.

LINEAR OPERATORS

Let X, Y be two vector spaces.

Def. A mapping $T: X \to Y$ is a linear operator if:

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) \qquad \forall x_1, x_2 \in X, \ \alpha, \beta \in \mathbb{R}.$$

In particular, if $Y = \mathbb{R}$ then T is called functional.
(= a functional is a linear operator $T: X \to \mathbb{R}$)

Remark: $\left.\begin{aligned} T(0) &= T(0 \cdot x) = 0 \cdot T(x) = 0 \\ T(\alpha \cdot x) &= \alpha \cdot T(x) \end{aligned}\right\}$ the image of 0 by means of a linear operator is always zero

!!! (From here, let X, Y be **normed spaces**).

Def. Let X, Y be normed spaces. We say that T is bounded if $\exists M > 0$ s.t.:

$$\|T(x)\|_Y \leq M \|x\|_X \qquad \forall x \in X.$$

Def. $T: X \to Y$ is continuous in $x_0 \in X$: $\qquad$ (T linear operator)

$$\iff \left[\forall \{x_n\}_n \subset X \ \text{ s.t. } x_n \xrightarrow[n \to \infty]{} x_0 \in X \implies T(x_n) \xrightarrow[n \to \infty]{} T(x_0) \right].$$

another form:

$$\left(\|x_n - x_0\|_X \xrightarrow[n \to \infty]{} 0 \implies \|T(x_n) - T(x_0)\|_Y \xrightarrow[n \to \infty]{} 0 \right)$$

Theorem: Let $T: X \to Y$ be a linear operator.
Then, the following statements are equivalent:

(i) T is bounded
(ii) T is Lipschitz $\left(\iff \exists L > 0 : \|T(x) - T(y)\|_Y \leq L \|x - y\|_X \ \ \forall x, y \in X \right)$
(iii) T is continuous at $x_0 = 0 \in X$ (0 always belong to X because 0 always belongs to a vector space)

proof.

(i) $\implies$ (ii)
Since T is bounded:

$$\|T(x) - T(y)\|_Y \underset{\substack{\uparrow \\ \text{linearity}}}{=} \|T(x-y)\|_Y \underset{\substack{\uparrow \\ \text{boundness}}}{\leq} M \|x - y\|_X \qquad \forall x, y \in X$$

(ii) $\implies$ (iii)
Consider $\{x_n\}_n \subset X$: $x_n \xrightarrow[n \to \infty]{} 0$.
Since T is Lipschitz:

$$\|T(x_n) - T(0)\|_Y = \|T(x_n)\|_Y \underset{(ii)}{\leq} M \underbrace{\|x_n\|_X}_{\xrightarrow[n \to \infty]{} 0}$$

($T(0) = 0$ from the above remark)

More precisely:
$$\begin{aligned} \|T(x_n)\|_Y &= \|T(x_n) - T(0)\|_Y && \text{(Lip.)} \\ &\leq M \|x_n - 0\|_X && \text{((ii))} \\ &= M \|x_n\|_X \xrightarrow[n \to \infty]{} 0 \end{aligned}$$

$$\implies \|T(x_n)\|_Y \xrightarrow[n \to \infty]{} 0$$
$$\implies T(x_n) \xrightarrow[n \to \infty]{} 0 = T(0)$$

(iii) $\implies$ (i)
Suppose by contradiction that T is not bounded.
Then, there exists $\{x_n\} \subset X$, $x_n \neq 0$ s.t.

$$\|T(x_n)\|_Y \geq n \|x_n\|_X \quad *$$

We have that:

$$z_n := \frac{x_n}{n \|x_n\|_X} \xrightarrow[n \to \infty]{} 0 \quad \left(\Leftarrow \underbrace{\|z_n\|_X = \frac{1}{n} \xrightarrow[n \to \infty]{} 0}_{\text{we do the reasoning on this}} \right)$$

since we have that:

$$\implies T(z_n) = \frac{1}{n \|x_n\|_X} T(x_n)$$

$$\implies \|T(z_n)\|_Y = \frac{1}{n \|x_n\|_X} \|T(x_n)\|_Y \overset{*}{\geq} \frac{1}{n \|x_n\|_X} n \|x_n\|_X = 1$$

$\Longrightarrow T(z_n) \underset{n\to\infty}{\not\longrightarrow} T(0) = 0$ (while $z_n \underset{n\to\infty}{\longrightarrow} 0$)

($T(z_n) \not\to T(0)$ because if $T(z_n)$ converges to $T(0)=0$ then also the norms must converge) but the norm is $\equiv 1$, cannot conv.

Therefore, T is not continuous at $x_0 = 0$, contradiction. $\blacksquare$

We say that:

T is continuous at $X \overset{\text{def.}}{\Longleftrightarrow} T$ is continuous at **any** $x_0 \in X$.

T is lipschitz $\Longrightarrow$ T is continuous at X

In fact: $\forall x_0 \in X$, $\{x_n\}_n \subset X$, $x_n \longrightarrow x_0$ $\Rightarrow$

$$\| T(x_n) - T(x_0) \|_Y \le L \| x_n - x_0 \|_X \underset{n\to\infty}{\longrightarrow} 0$$

$$\Rightarrow T(x_n) \underset{n\to\infty}{\longrightarrow} T(x_0).$$

In view of the above considerations:

- Remark: T is continuous in $X \Longleftrightarrow T$ is continuous at $x_0 = 0$ in X.

continuity in the whole X implies continuity in x_0, continuity in x_0 implies lipschitz and lipschitz implies continuity in the whole X

Let X, Y be two normed spaces (fixed).

We define:

$$\mathcal{L}(X,Y) := \{ T : X \to Y \ : \ T \text{ is linear and continuous} \}$$
$$= \{ T : X \to Y \ : \ T \text{ is linear and bounded} \}$$

bounded is equivalent to continuous at $x_0 = 0 \in X$, which is equivalent to continuous in X

If $X = Y \longrightarrow \mathcal{L}(X,Y) \equiv \mathcal{L}(X)$.

Now we want to study the properties of this new set $\mathcal{L}(X,Y)$. (set of operators between X and Y)

Remark: $\mathcal{L}(X,Y)$ is a vector space.

In fact, $\underbrace{\forall T, S \in \mathcal{L}(X,Y)}_{T,S \text{ are two operators}}$, $\lambda \in \mathbb{R}$:

$$(T + \lambda S)(x) = T(x) + \lambda S(x) \in \mathcal{L}(X,Y)$$

If both are linear and bounded then also their sum is linear and bdd $\Longrightarrow \mathcal{L}(X,Y)$ is closed w.r.t. linear combination of its elements $\longrightarrow \mathcal{L}(X,Y)$ is a vector space

We now want to define a norm on this new space.

If $T \in \mathcal{L}(X)$, then:

$$\exists M > 0 : \ \| T(x) \|_Y \le M \qquad \forall x \in X, \ \|x\|_X \le 1$$

$$(\| T(x) \|_Y \underset{\text{bounded}}{\le} M \|x\|_X \le M \text{ since } \|x\|_X \le 1)$$

$$\sup_{x\in X, \|x\|_X \le 1} \| T(x) \|_Y \quad (\le M)$$

thanks to this, we know that $\forall x : \|x\|_X \le 1$ the sup $\|T(x)\|_Y$ cannot be $+\infty$

is a norm on $\mathcal{L}(X,Y)$. $\Longrightarrow$ not only $\mathcal{L}(X,Y)$ has the structure of a vector space, we can also define a norm on it

$$\Longrightarrow \| T \|_{\mathcal{L}(X,Y)} := \sup_{\substack{\|x\|_X \le 1 \\ x \in X}} \| T(x) \|_Y$$

$\longrightarrow$ $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is a normed space.

Prop.
$$\boxed{\| T \|_{\mathcal{L}} = \sup_{\substack{x \in X, \\ \|x\|_X = 1}} \| T(x) \|_Y = \sup_{\substack{x \in X \\ x \ne 0}} \frac{\| T(x) \|_Y}{\|x\|_X}}$$

useful for exercises where we need to compute the norm of an operator

proof.

$$\sup_{x\in X, \|x\|_X \le 1} \| T(x) \|_Y \ge \sup_{x \in X, \|x\|_X = 1} \| T(x) \|_Y \quad \text{(sup on a bigger set)}$$

If $\|x\|_X \le 1$, $x \ne 0$, then:

$$\| T(x) \|_Y = \|x\|_X \cdot \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y$$
$$\le \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y \qquad \frac{x}{\|x\|_X} \text{ has norm } 1$$

$\sup_{\|x\| \le 1} \|T(x)\|_Y \le \sup_{\|z\| = 1} \|T(z)\|$ since $z := \frac{x}{\|x\|_X}$ is such that $\|z\| = 1$

$$\Longrightarrow \sup_{\|x\|_X \le 1, x \in X} \| T(x) \|_Y \le \sup_{x \in X, \|x\|_X = 1} \| T(x) \|_Y$$

because we saw that $\forall x : \|x\|_X \le 1$, $\|T(x)\|$ is less than $\|T(\cdot)\|$ evaluated on a vector whose norm is 1

$\Rightarrow \|T\|_{\mathcal{L}} = \sup_{x\in X,\ \|x\|_X = 1} \|T(x)\|_Y$ 22/11

Furthermore; $\forall x \in X \setminus \{0\}$

$$\frac{\|T(x)\|_Y}{\|x\|_X} = \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y := \|T(z)\|_Y \text{ (with } \|z\|_X = 1) \Longrightarrow \underbrace{\sup_{x\in X\setminus\{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{\|z\|=1} \|T(z)\|_Y}_{*}$$

$$\Longrightarrow \sup_{x\in X,\ x\neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} = \underbrace{\sup_{x\in X,\ \|x\|_X \le 1} \|T(x)\|_Y}_{*} = \|T\|_{\mathcal{L}} \quad \blacksquare$$

Theorem: Let X be a normed space and Y be a Banach space. Then $\mathcal{L}(X,Y)$ is a Banach space. $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}})$ Banach space

Def. Let $T \in \mathcal{L}(X,Y)$. We define:

$$Ker(T) := \{x \in X : T(x) = 0\} \quad \text{kernel of } T.$$

$Ker(T)$ is a vector space.

Remark: (i) $Ker(T)$ is closed
(ii) T is injective $\Longleftrightarrow$ $Ker(T) = \{0\}$

Def. X, Y normed spaces. X and Y are isomorphic if $\exists T \in \mathcal{L}(X,Y)$ invertible with: $T^{-1} \in \mathcal{L}(X,Y)$.

Def. $T \in \mathcal{L}(X,Y)$ is called isometry if it preserves the norm:

$$\|T(x)\|_Y = \|x\|_X \quad \forall x \in X.$$

Def.* $X \subseteq Y$ (X subspace of Y). We define: $J: X \to Y$: (mapping)

$$J(x) = x \quad \forall x \in X.$$

J is an embedding if $J \in \mathcal{L}(X,Y)$.
We write: $X \hookrightarrow Y$.

$\boxed{* \ (X, \|\cdot\|_X),\ (Y, \|\cdot\|_Y) \text{ normed spaces}}$

Remark: $J \in \mathcal{L}(X,Y)$ embedding $\longrightarrow$ $\exists M > 0$: $\underbrace{\underbrace{\|J(x)\|_Y}_{\|x\|_Y} \le M\|x\|_X}$

$\Longrightarrow$ we have a relation between the X-norm of x and the Y-norm of x

we have already shown that $\mathcal{L}(X,Y)$ is a vector space and that we can make it a norm space, so, we just need to prove completeness (and we do it by definition) ($= \forall$ Cauchy sequence converges)

proof. (of the above theorem) 23/11

- Let $\{T_n\}_n \subset \mathcal{L}(X,Y)$ be a Cauchy sequence.
This means that ($\Longleftrightarrow$):

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \text{ s.t.}: \quad \|T_m - T_n\|_{\mathcal{L}} < \varepsilon \quad \forall n, m > n_\varepsilon$$

$\Longrightarrow \forall x \in X$:

$$\left.\begin{aligned} \|T_m(x) - T_n(x)\|_Y &\le \|T_m - T_n\|_{\mathcal{L}} \|x\|_X \\ &< \varepsilon \|x\|_X \quad \{T_n\}_n \text{ Cauchy} \quad \forall n, m > n_\varepsilon \end{aligned}\right\} (*)$$

We showed that $\forall x \in X$: $\{T_n(x)\}_n \subset Y$ is a Cauchy sequence. ($\forall x$ fixed)
Hence, since Y is a Banach space:

$$\exists y \in Y \text{ s.t. } T_n(x) \xrightarrow[n\to\infty]{} y \text{ in } Y. \qquad (y = y(x) \text{ dependency})$$

- We define another operator $T: X \to Y$:

$$\forall x \in X, \quad T(x) := y$$

we do this to underly the dependence of y on x

- In particular, T is linear.
In fact, $\forall x_1, x_2 \in X$ with $T(x_1) = y_1$, $T(x_2) = y_2$ and $\alpha, \beta \in \mathbb{R}$, we have:

$$T_n(\alpha x_1 + \beta x_2) = \alpha T_n(x_1) + \beta T_n(x_2)$$

By definition $\alpha T_n(x_1) \longrightarrow \alpha T(x_1)$, $\beta T_n(x_2) \longrightarrow \beta T(x_2)$.

since $\alpha T_n(x_1) \to \alpha y_1$, $\beta T_n(x_2) \to \beta y_2$ and we defined the operator T as: $T(x) = y$ (we're not linking T to T_n yet) (we're just using T as a way to refer to y)

$$\Longrightarrow T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$$

since $T_n(\alpha x_1 + \beta x_2) \xrightarrow[n\to\infty]{} T(\alpha x_1 + \beta x_2)$.

but not because $T_n \to T$, it's just because $T_n(\alpha x_1 + \beta x_2) \xrightarrow[n]{} y := T(\cdot\cdot)$

- We claim that T is bounded.

$\forall x \in X$, $\forall \varepsilon > 0$, $\forall m > n > n_\varepsilon$:

$$\begin{aligned}\|T(x) - T_n(x)\|_Y &= \|T(x) - T_m(x) + T_m(x) - T_n(x)\|_Y \\ &\leq \|T(x) - T_m(x)\|_Y + \|T_m(x) - T_n(x)\|_Y \\ &\leq \|T(x) - T_m(x)\|_Y + \varepsilon \|x\|_X \end{aligned}$$

$(*)$

Letting $m \to \infty$, by def. of limit:

$$\|T(x) - T_m(x)\|_Y \longrightarrow 0$$

$T_m(x) \to y := T(x)$

$$\Longrightarrow \|T(x) - T_n(x)\|_Y \leq \varepsilon \|x\|_X \qquad ((*))$$

so:

$$\begin{aligned}\|T(x)\|_Y &= \|T(x) - T_m(x) + T_m(x)\|_Y \\ &\leq \|T(x) - T_m(x)\|_Y + \|T_m(x)\|_Y \\ &\leq \varepsilon \|x\|_X + \underbrace{\|T_m\|_{\mathcal{L}}}_{\leq M} \|x\|_X \end{aligned}$$

$\leq M$ since $\{T_m\}_m$ is Cauchy and so, bounded — $M \perp\!\!\!\perp n$

$$\leq (\varepsilon + M)\|x\|_X$$

Hence, T is bounded.

$\Longrightarrow T \in \mathcal{L}(X,Y)$

until now we just used the object T as something to refer to y, now we show that this T is the limit of the T_n ($\downarrow$)

- Now we show that $T_n \xrightarrow[n\to\infty]{} T$ in $\mathcal{L}(X,Y)$.

$$\|T(x) - T_n(x)\|_Y \leq \varepsilon \|x\|_X \qquad ((*))$$

$$\Longrightarrow \|T - T_n\|_{\mathcal{L}} \leq \varepsilon \quad \forall n > n_\varepsilon.$$

$$\Longleftrightarrow T_n \xrightarrow[n\to\infty]{} T \text{ in } \mathcal{L}$$

$\Longrightarrow \mathcal{L}(X,Y)$ is complete and so, it's Banach. ∎

UNIFORM BOUNDED PRINCIPLE (UBP)

(/ Banach - Steinhaus)

If we have pointwise boundness then we get automatically uniform boundness

Theorem: (UBP theorem / Banach-Steinhaus)

Let X, Y be Banach spaces, $\mathcal{F} \subset \mathcal{L}(X,Y)$ (family of linear bdd operators).

Assume that:

x is fixed, we take the supremum of $\|T(x)\|_Y$ over all the $T \in \mathcal{F}$

(PB condition) $\forall x \in X \ \exists M_x > 0 : \sup_{T \in \mathcal{F}} \|T(x)\|_Y \leq M_x$

(in this case we say that $\mathcal{F}$ is pointwise bounded).

Then:

(UB condition) $\exists K > 0 : \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}} \leq K$

(in this case we say that $\mathcal{F}$ is uniformly bounded).

$$\left(\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}} = \sup_{T\in\mathcal{F}} \sup_{\|x\|\leq 1} \|T(x)\|_Y \right)$$

proof.

- $\forall n \in \mathbb{N}$ let $C_n := \{x \in X : \|T(x)\|_Y \le n \quad \forall T \in \mathcal{F}\}$.

Then, C_n is closed.

Indeed, let $\{x_m\}_m \subset C_n$, $x_m \xrightarrow[m\to\infty]{} x_0 \in X$. In order to prove that C_n is closed we have to prove that $x_0 \in C_n$

Therefore:

$$T(x_m) \xrightarrow[m\to\infty]{} T(x_0)$$

(by the continuity of T $(\mathcal{F} \subset \mathcal{L}(X,Y))$)

$$\Longrightarrow \underbrace{\|T(x_m)\|_Y}_{\le n} \xrightarrow[m\to\infty]{} \|T(x_0)\|_Y$$

(in any Banach space if a sequence converges to a point then we have also convergence of the norms)

$$\Longrightarrow \|T(x_0)\|_Y \le n$$
$$\Longrightarrow x_0 \in C_n$$
$$\Longrightarrow C_n \text{ is closed.}$$

- (PB) $\Longrightarrow X = \bigcup_{n=1}^{\infty} C_n$.

(PB condition): $\forall x \in X \; \exists M_x > 0$ s.t. $\sup_{T\in\mathcal{F}} \|T(x)\|_Y \le M_x$, hence $\forall x \; \exists n$ large enough for which $x \in C_n$ $\Longrightarrow X = \bigcup_{n=1}^{\infty} C_n$

- Due to Baire theorem there exists $n_0 \in \mathbb{N}$ s.t. $\operatorname{int}(C_{n_0}) \ne \emptyset$.

$$\Longrightarrow \overline{B(x_0,\varepsilon)} \subset C_{n_0}$$

If the interior of C_{n_0} is not empty then we can find a closed ball, we call it $\overline{B(x_0,\varepsilon)}$ entirely contained in C_{n_0}

By Baire's theorem X is of second category in itself (since X is a complete normed space). Hence, since X is a countable union of sets (C_n) then these sets have to be: $\operatorname{int}(C_n) \ne \emptyset$ for at least one n

otherwise, if $\operatorname{int}(C_n) = \emptyset \;\forall n$ then X is the countable union of nowhere dense set, i.e. of first category (↯)

- if $\|z\|_X \le \varepsilon \longrightarrow z + x_0 \in \overline{B(x_0,\varepsilon)} \subset C_{n_0}$

$$\begin{aligned}
\longrightarrow \|T(z)\|_Y &= \|T(z) - T(x_0) + T(x_0)\|_Y \\
&\le \|T(z) + T(x_0)\|_Y + \|T(x_0)\|_Y \\
&= \|\underbrace{T(z+x_0)}_{\in C_{n_0}}\|_Y + \underbrace{\|T(x_0)\|_Y}_{\in C_{n_0}} \\
&\le 2n_0 \qquad \forall T \in \mathcal{F}
\end{aligned}$$

if $x_0 \in C_{n_0}$ then $\|T(x_0)\|_Y \le n_0$ (by def. of C_n)

from here we can take whatever x_0, it doesn't matter, it just matters that $\exists x_0, \varepsilon$: $B_\varepsilon(x_0) \subset C_{n_0}$ (it does **not** have to be the x_0 of $x_m \to x_0$)

- Then, $\forall x \in X \setminus \{0\}$:

$$\begin{aligned}
\|T(x)\|_Y &= \frac{\|x\|_X}{\varepsilon} \left\| T\left(\frac{\varepsilon x}{\|x\|_X}\right)\right\|_Y \\
&\le \frac{\|x\|_X}{\varepsilon} 2n_0
\end{aligned}$$

$\left\|\frac{\varepsilon x}{\|x\|_X}\right\|_X = \varepsilon$, $z := \frac{\varepsilon x}{\|x\|_X}$

above we proved $\|T(z)\|_Y \le 2n_0 \;\; \forall z : \|z\|_X \le \varepsilon$ (and here we have $z := \frac{\varepsilon x}{\|x\|_X}$ s.t. $\|z\|_X = \varepsilon$)

$$\Longrightarrow \|T\|_{\mathcal{L}} \le K := \frac{2n_0}{\varepsilon} \qquad \forall T \in \mathcal{F}$$

$$\Longrightarrow \sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}} \le K.$$

∎

Corollary (useful for exercises): X, Y Banach spaces, $\{T_n\}_n \subset \mathcal{L}(X,Y)$. Assume that $\forall x \in X \; \exists \lim_{n\to\infty} T_n(x)$.

Let:

$$T(x) := \lim_{n\to\infty} T_n(x).$$

Then, $T \in \mathcal{L}(X,Y)$.

(Note: a "sequence" is "a family of", we can apply UBP)

If a sequence of bounded operators converges pointwise, then the "limit" operator is bounded.

proof.

- $T : X \to Y$ is linear. (same proof as the one for the "$(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}})$ is Banach if Y is Banach")
- $\{T_n(x)\}_n$ is bounded $\forall x \in X$. (is bounded since it's converging $(\{T_n(x)\}_n)$, we now need to prove that also $\{T_n\}_n$ is bounded)

by hypothesis: $\exists \lim T_n(x) := T(x)$

$$\Longleftrightarrow \forall x \in X \quad \exists M_x > 0 \text{ s.t. } \|T_n(x)\|_Y \le M_x \quad \forall n \in \mathbb{N}.$$

$$\Longrightarrow \sup_{n\in\mathbb{N}} \|T_n(x)\|_Y \le M_x$$

By UBP there exists $M > 0$:

$$\sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{L}} \le M$$

$\Rightarrow \|T_n(x)\|_Y \le M\|x\|_X \quad \forall x \in X,\ n \in \mathbb{N}$

Now: Since we have Banach spaces, if $T_n(x) \to T(x)$ then $\|T_n(x)\|_Y \to \|T(x)\|_Y$

$$\|T(x)\|_Y = \lim_{n\to\infty} \|T_n(x)\|_Y \le M\|x\|_X \qquad \forall x \in X.$$

$\Rightarrow$ T is bounded (since $\|T(x)\|_Y \le M\|x\|_X \quad \forall x \in X$)

$\Rightarrow T \in \mathcal{L}(X,Y)$. ■

OPEN MAPPING THEOREM

We recall that: $T \in \mathcal{L}(X,Y) \iff T$ linear bounded operator (= mapping)

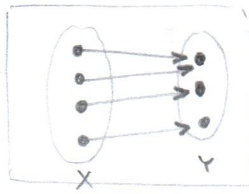
Def. X, Y normed spaces, $T: X \to Y$ is said to be an open mapping if for any open set $G \subset X$: $T(G) \subset Y$ is open.
An open mapping transports open sets into open sets.

(*)
$f: \mathbb{R} \to \mathbb{R}$ is continuous $\iff \forall E \subseteq \mathbb{R}$ open $f^{-1}(E) \subseteq \mathbb{R}$ is open
$\overset{\Longrightarrow}{\not\Longleftarrow}$ f is an open mapping

Here $\mathbb{R}$ is a space of dimension 1, but we're working in a setting where also dimension ∞ is considered. Can we say that a continuous mapping is an open mapping also in our new settings? The following theorem answers the question:

Theorem: (open mapping theorem) (OMT)

Let X, Y be Banach spaces, $T \in \mathcal{L}(X,Y)$ surjective.
Then T is an open mapping.



$Im(X) = Y$

Corollary: (Inverse Bounded Mapping (IBM))

(consequence of OMT)

Let $T \in \mathcal{L}(X,Y)$, T bijective. (X, Y Banach spaces!)
Then $T^{-1} \in \mathcal{L}(X,Y)$.

proof.

$\exists T^{-1}: Y \to X$ and it is linear. (since T is bijective)

claim: T^{-1} is continuous.

T^{-1} continuous $\iff (T^{-1})^{-1}(E) \subset Y$ is open $\forall E \subseteq X$ open. (*)

We have that:

$(T^{-1})^{-1}(E) = T(E)$ is open thanks to OMT. ■

$\Rightarrow \forall E \subseteq X$ open $(T^{-1})^{-1}(E) \subset Y$ is open

$\iff T^{-1}$ is continuous

$\Rightarrow T^{-1} \in \mathcal{L}(X,Y)$ since it's also linear

We have generalized a property of vector space of finite dimension ((*)) to the infinite dimension case by means of the open mapping theorem.

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Corollary: Consider two Banach spaces $(X, \|\cdot\|), (X, \|\cdot\|_*)$.
(consequence of IBM)
(Same vector space, different norms).
Suppose that there exists $M > 0$ s.t.:

$$\|x\|_* \le M\|x\| \qquad \forall x \in X$$

Then $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, that is:

$\exists m > 0$ s.t.:

$$m\|x\| \le \|x\|_* \le M\|x\| \qquad \forall x \in X.$$

proof.

Consider the identity operator $\underset{(\|\cdot\|)}{I : X} \to \underset{(\|\cdot\|_*)}{X}$: $I(x) = x$

Then:

$$\| \underbrace{I(x)}_{x} \|_* \le M \|x\| \quad \forall x \in X$$

(this holds by assumption, since $\|I(x)\|_* = \|x\|_*$ (by def. of I) and by Hp. $\|x\|_* \le M\|x\| \ \forall x \in X$)

I is bijective and bounded. (bijective because it is the identity, bdd because $\|I(x)\|_* \le M\|x\| \ \forall x$)

By the inverse bounded mapping theorem:

$$I^{-1} : \underset{(\|\cdot\|_*)}{X} \to \underset{(\|\cdot\|)}{X} \quad : \quad I^{-1}(x) = x$$

I^{-1} is linear and bounded:

$$\Longrightarrow \| \underbrace{I^{-1}(x)}_{x} \| \le m' \|x\|_*$$

$\left(\|x\|_* \ge \frac{1}{m'} \|x\| := m \|x\| \right)$ ∎

CLOSED GRAPH THEOREM

Def. X, Y normed spaces, $T : X \to Y$ linear operator is called closed if:

$\forall \{x_n\}_n \subseteq X, \ \forall \{T(x_n)\}_n \subseteq Y$:

$$\left. \begin{array}{l} x_n \longrightarrow x \text{ in } X \\ T(x_n) \longrightarrow y \text{ in } Y \end{array} \right\} \Longrightarrow T(x) = y.$$

(if x_n converges to x in X and the image of x_n through the operator T converges to some y in Y then we must have that $y = T(x)$)

Remark: $T \in \mathcal{L}(X,Y) \Longrightarrow T$ is closed

In fact, by definition of continuity:

$$\left. \begin{array}{l} x_n \to x \\ T(x_n) \to T(x) := y \end{array} \right\} \quad \text{so} \quad y = T(x)$$

linear, continuous $\underset{\not\Longleftarrow}{\Longrightarrow}$ closed

(without further assumptions, if X, Y are Banach spaces then "$\Longleftrightarrow$")

Def. $T : X \to Y$. We define:

$$\text{graph}(T) := \{ (x, T(x)) : x \in X \} \subseteq X \times Y.$$

Remark: X, Y Banach spaces, then: $(X \times Y, \underbrace{\|x\|_X + \|y\|_Y}_{\text{usually called "graph norm"}})$ is a Banach space

Prop. $T : X \to Y$ linear and closed $\Longleftrightarrow$ graph(T) is closed

T is an operator $\overset{\neq}{\longleftrightarrow}$ graph(T) is a subset of a Banach space

proof. $(\Rightarrow)$

let $\{(x_n, T(x_n))\} \subset \text{graph}(T)$ be such that:

$$(x_n, T(x_n)) \xrightarrow[n\to\infty]{} (x, y) \in X \times Y.$$

(in general we don't know if $y \in \text{graph}(T)$)

This implies that: (in view of the definition of the graph norm:)

$$x_n \to x, \quad T(x_n) \to y \quad \longleftarrow \left(\Longleftarrow \left[(x_n, T(x_n)) \xrightarrow[n]{} (x,y) \right] \right)$$

$$T \text{ closed} \overset{\text{def.}}{\Longrightarrow} y = T(x) \Longrightarrow y \in T(X) \Longleftrightarrow (x, \underbrace{y}_{=T(x)}) \in \text{graph}(T)$$

$\Rightarrow$ graph(T) is closed.

$(\Leftarrow)$

let $\{(x_n, T(x_n))\} \subset \text{graph}(T)$ be such that: $(x_n, T(x_n)) \xrightarrow[n\to\infty]{} (x,y) \in X \times Y$.

Again, this implies: $x_n \to x$, $T(x_n) \to y$.

graph(T) is closed $\Longrightarrow (x,y) \in \text{graph}(T)$

$\Longrightarrow y = T(x)$, otherwise it cannot belong to graph(T)

$\Longrightarrow T$ is closed ∎

Theorem: (Closed graph theorem)

Let $T: X \to Y$ be a linear closed operator. (X, Y Banach spaces)!
Then $T \in \mathcal{L}(X,Y)$.

Note: " $T: X \to Y$ linear closed operator " $\Longleftrightarrow$ " graph(T) is closed "

proof.

We consider the graph norm: $\|x\|_* = \|x\|_X + \|T(x)\|_Y$.
This is a norm on X.
$\Longrightarrow (X, \|\cdot\|_*)$ is a Banach space. $\leftarrow$ the assumption that T is closed is needed to prove this ($(X, \|\cdot\|_*)$ is Banach)

Clearly:

$$\|x\|_X \leq \|x\|_* \qquad \forall x \in X \qquad (\|T(x)\|_Y \geq 0) - \text{since it is a norm}$$

By the previous corollary: $\exists M > 0$ s.t.:

$$\|x\|_* = \|x\|_X + \|T(x)\|_Y \leq M\|x\|_X \qquad \forall x \in X$$

$\Longrightarrow \|T(x)\|_Y \leq (M-1)\|x\|_X \qquad \forall x \in X$
$\Longrightarrow T$ is bounded
$\Longrightarrow T \in \mathcal{L}(X,Y)$ (linear by Hp.) ∎

DUALITY AND REFLEXIVITY

DUAL SPACE

Let X be a normed space. We define the dual space of X as:

$$X^* := \mathcal{L}(X, \mathbb{R}) = \{T: X \to \mathbb{R} \text{ operator, linear and continuous}\}.$$

If X is a Banach space $\Longrightarrow X^*$ is Banach, with the norm:

$$\forall L \in X^*: \quad \|L\|_* := \sup_{\|x\|_X = 1} |L(x)|$$

$$(\|L\|_* := \|L\|_{\mathcal{L}(X,\mathbb{R})}, \quad |L(x)| = \|L(x)\|_Y \text{ if } Y = \mathbb{R})$$

since the norm in $\mathbb{R}$ is $|\cdot|$

Prop. Let $L: X \to \mathbb{R}$ be linear, $L \neq 0$.
The following statements are equivalent:
(i) $L \in X^* = \mathcal{L}(X, \mathbb{R})$ ($\Rightarrow L$ is also bounded)
(ii) $Ker(L)$ is closed
(iii) $Ker(L)$ is **not** dense in X

characterization of $L \in X^*$

We want now to study the dual space of an example of Banach space: $L^p(X, \mathcal{A}, \mu)$.

Example: Consider $X = L^p(X, \mathcal{A}, \mu)$. ($(X, \mathcal{A}, \mu)$ measure space)
Consider two conjugate exponents p, q $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.
Let's consider $g \in L^q$.
We define: the operator:

$$T_g : L^p \longrightarrow \mathbb{R} \quad \text{s.t.}$$

(T_g because the operator depends on g)

$$\forall f \in L^p: \quad T_g(f) := \int_X fg \, d\mu$$

T_g is obviously linear.

are we sure that if we take $g \in L^q$ and $f \in L^p$ then we can compute $\int_X fg\,d\mu$ and this integral is a real number?

Is T_g well defined? Yes; because of Holder inequality:

$$|T_g(f)| = \left|\int_X fg \, d\mu\right| \leq \|f\|_p \|g\|_q \qquad \forall f \in L^p$$

$\hookrightarrow \left|\int_X fg\,d\mu\right| \leq \left|\int_X |fg|\,d\mu\right| \leq \|f\|_p \|g\|_q$

$$\Longrightarrow \|T_g\|_{X^*} \leq \|g\|_q$$

(the operator is bounded and)

! The norm $\|T_g\|_*$ is a supremum and we found: $\|T_g\|_* \leq \|g\|_q$. Hence, if we find a function s.t. the modulus of $T_g(f)$ is exactly $\|g\|_q$ then the supremum is a maximum and the norm is: $\|T_g\|_* = \|g\|_q$

Consider: $\tilde{f} = \dfrac{|g|^{q-2} g}{\|g\|_q^{q-1}}$ $(g \not\equiv 0)$

Then:

$$T_g(\tilde{f}) = \int_X \frac{|g|^{q-2} g}{\|g\|_q^{q-1}} \, g \, d\mu = \frac{\int_X |g|^q d\mu}{\|g\|_q^{q-1}} = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q$$

we have found a function that realizes the supremum

$$\Longrightarrow \boxed{\|T_g\|_{X^*} = \|g\|_{L^q}}$$

T_g is a linear, bounded operator from L^p to $\mathbb{R}$

$\Longleftrightarrow T \in \mathcal{L}(L^p, \mathbb{R})$

$\Longleftrightarrow T \in (L^p)^*$ = dual space of L^p

! we have found an operator belonging to the dual space of L^p. In general, if we want to have an element of the dual space of L^p it is enough to consider a function $g \in L^q$ (q, p conjugate). What can we say about the contrary implication? If we start from an element of the dual space of L^p, what can we say about it? If we consider any element of the dual space of L^p, is this element (functional) related to the conjugate L^p space? (i.e. L^q)

Now we study $(L^p)^*$.

We know how to construct an element of $(L^p)^*$, but what can we say about all elements of $(L^p)^*$?

Theorem: Characterization of $(L^p)^*$ $p \in (1, \infty)$

Let $(X, \mathcal{A}, \mu)$ be a complete measure space, $p \in (1, \infty)$.
For any $T \in (L^p)^*$ $\exists!\, g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that:

$$T(f) = \int_X f g \, d\mu \qquad \forall f \in L^p$$

Moreover:

$$\|T\|_{(L^p)^*} = \|g\|_{L^q}.$$

because of this theorem we conclude that every element of $(L^p)^*$ ($\forall$ linear bdd operator from L^p to $\mathbb{R}$) can be written using a function $g \in L^q$ ($\forall$ element $\in (L^p)^*$ $\exists!\, g$) such that: $T(\cdot) = \int_X \cdot \, g \, d\mu$

any element of $(L^p)^*$ is an integral of a function times g (g is unique for each elem.)

Remark: The same holds when $p = 1$, $q = \infty$, provided that μ is σ-finite.

Remark: Consider $\mathbb{R}^n$ / or $\Omega \subseteq \mathbb{R}^n$ measurable:

$$(L^p)^* = L^q \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < \infty$$

$$(L^1)^* = L^\infty$$

However:

$$(L^\infty)^* \neq L^1.$$

HAHN - BANACH THEOREMS

Def. Let X be a vector space. A mapping (/function) $p: X \to \mathbb{R}$ is called sublinear functional if:

$$p(x+y) \leq p(x) + p(y) \qquad \forall x, y \in X$$

$$p(\alpha x) = \alpha p(x) \qquad \forall x \in X, \ \alpha \in \mathbb{R}_+$$

Theorem (Hahn-Banach) - dominated extension form / general form

Let X be a vector space and p be a sublinear functional on X.
Let Y be a subspace of X. ($Y \subsetneq X$)
Let $f: Y \to \mathbb{R}$ be a linear continuous functional s.t.

$$f(y) \leq p(y) \quad \forall y \in Y \qquad \text{(i.e. } f \text{ is dominated by } p\text{)}$$

Then, there exists an extension F of f to the whole X s.t.

- $F \in X^*$
- $F(x) \leq p(x) \quad \forall x \in X.$

(Extension: $F(y) = f(y) \quad \forall y \in Y$)

where f is defined: $f = F$

The key word is **extension**: the Hahn-Banach theorem is a thm. concerning the extension of linear operators defined on $Y \subsetneq X$ (the extension is from Y to X). In order to do this we need to control the functional in terms of something. The object which controls the functional is a sublinear functional p.

proof.

- Let $\mathbb{F}$ be the set of all pairs (Y_α, f_α), where:
  (i) Y_α subspace of X, $Y_\alpha \supseteq Y$
  (ii) f_α is an extension of f to Y_α
  (iii) $f_\alpha(x) \leq p(x) \quad \forall x \in Y_\alpha$

  We observe that $\mathbb{F} \neq \emptyset$ because by hypothesis $(Y, f) \in \mathbb{F}$.
- We can define an order relation: ($\leq$ = "is in relation with")

$$(Y_\alpha, f_\alpha) \leq (Y_\beta, f_\beta) \underset{\text{(we def. as:)}}{\overset{\text{def.}}{\Longleftrightarrow}} \begin{cases} Y_\alpha \subseteq Y_\beta \\ f_\beta \text{ is an extension of } f_\alpha \text{ to } Y_\beta \end{cases}$$

$\Longrightarrow$ $\leq$ order relation.

$\mathbb{F}$ is a partially ordered set.

RECALL
- A set P with an order relation is called partially ordered set
- If $\forall x, y \in P$ it's either xRy or yRx then P is totally ordered / chain
- $M \in P$ is a maximal element if: $M \leq x$ for some $x \in P \Rightarrow x = M$
- Let $A \subseteq P$, $u \in P$. If $\forall a \in A: a \leq u \Rightarrow u$ is called upper bound
- **ZORN**'s LEMMA: in a partially ordered set P, if every chain has an upper bound then P has a maximal element.

- In $(\mathbb{F}, \leq)$ every chain has an upper bound.
  Indeed, let:

$$\mathcal{C} := \{(Y_\alpha, f_\alpha) : \alpha \in A\} \subset \mathbb{F} \text{ be a chain.}$$

Let:

$$\tilde{Y} := \cup_{\alpha \in A} Y_\alpha \qquad \text{(subspace of } X)$$

Define:

$$\tilde{f} : \tilde{Y} \to \mathbb{R}.$$

$\forall y \in \tilde{Y} \Longrightarrow y \in Y_\alpha$ for some $\alpha \in A$ and we define:

$$\tilde{f}(y) := f_\alpha(y)$$

Then:
(i) $\tilde{f}$ is linear
~~(ii) $(\tilde{Y}, \tilde{f}) \in \mathcal{C}$~~
(iii) $(\tilde{Y}, \tilde{f})$ is an upper bound for $\mathcal{C}$

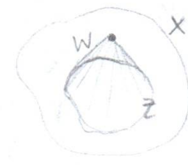
this is because, for all elements of $\mathcal{C}$ namely (Y_α, f_α), it holds:
$\begin{cases} Y_\alpha \subseteq \cup_{\alpha \in A} Y_\alpha := \tilde{Y} \\ \tilde{f} \text{ is an extension of } f_\alpha \text{ to } \tilde{Y} \end{cases}$ therefore: $(Y_\alpha, f_\alpha) \leq (\tilde{Y}, \tilde{f})$

- By Zorn's lemma there exists a maximal element (Z, g) of $\mathbb{F}$.
  This means that if $(U, h) \in \mathbb{F}$ and $(Z, g) \leq (U, h)$ then:

$$\begin{cases} Z = U \\ g = h \end{cases}$$

- To get the thesis, it is sufficient to prove that $Z = X$, that is: g is the extension of f to X.
- So, let us assume by contraddiction that $Z \subsetneq X$.
- Let $x_0 \in X \setminus Z$. We define:

$$W := \{z + t x_0 : z \in Z, t \in \mathbb{R}\}$$



(**not** proper since X is not a set but a space (same for Z, W))

- On this set W we define a function $\phi : W \to \mathbb{R}$:

$$\phi(w) = g(z) + t \cdot c$$

where $c \in \mathbb{R}$ is fixed.

Observe that ϕ is an extension of g to W. if we have an element of Z then it is also an element of W with $t = 0$ and so $\phi(z) = g(z) \; \forall z \in Z \in W$

- If it is possible to select $c \in \mathbb{R}$ so that:

$$\phi(w) \leq p(w) \qquad \forall w \in W$$

then:

$$(W, \phi) \in \mathbb{F} \quad \text{and} \quad (Z, g) \leq (W, \phi)$$

$\begin{cases} W \supseteq Z \supseteq Y \\ \phi \text{ extension of } g \text{ on } W \\ \text{(where } g \text{ is an extension of } f) \end{cases} \Longrightarrow \begin{cases} (W, \phi) \in \mathbb{F} \\ (Z, g) \leq (W, \phi) \end{cases}$

in contraddiction with the maximality of (Z, g).

Claim: such $c \in \mathbb{R}$ exists.

- in fact, for $z_1, z_2 \in Z$:

$$g(z_1) - g(z_2) \underset{g \text{ is linear}}{=} g(z_1 - z_2) \le p(z_1 - z_2) = p(z_1 + x_0 - x_0 - z_2) \le p(z_1 + x_0) + p(-x_0 - z_2)$$

sublinear

$$\Longrightarrow \underbrace{-p(-x_0 - z_2) - g(z_2)}_{\text{independent of } z_1} \le \underbrace{p(z_1 + x_0) - g(z_1)}_{\text{independent of } z_2}$$

$\Rightarrow \exists c \in \mathbb{R} : \forall z \in Z:$

$$-p(-x_0 - z) - g(z) \le c \le p(z + x_0) - g(z) \qquad (*)$$

- Now, let $w := z + t x_0 \in W$.

If $t = 0 \Longrightarrow w = z$ and $\phi(w) = g(z) \le p(z) = p(w)$

since ϕ is an extension but on $Z: \phi = g$ (and $z \in Z$)

Thus: $\phi(w) \le p(w)$.

- let $t > 0$. So $\frac{z}{t} \in Z$. (Z subspace)

$$c \le p\left(\frac{z}{t} + x_0\right) - g\left(\frac{z}{t}\right)$$ (from the right part of $(*)$)

$$\Longrightarrow tc \le p(z + t x_0) - g(z)$$

$$\Longrightarrow \phi(w) = g(z) + tc \le p(z + t x_0) = p(w).$$

Therefore:

$$\phi(w) \le p(w).$$

- when $t < 0$ the same holds. (from the left part of $(*)$)
- Thus, the claim has been shown. $\rightarrow$ contraddiction with maximality of (Z, g) $\Rightarrow$ it must be $Z = X$ ∎

Theorem (Hahn - Banach) - continuous extension

Let X be a normed space, Y subspace of X, $f \in Y^*$ (linear continuous functional). Then, there exists $F \in X^*$ s.t.:

"in terms of"

- $\|F\|_{X^*} = \|f\|_{Y^*}$
- $F(y) = f(y) \quad \forall y \in Y$

Stronger **Hp.** $\rightarrow$ stronger **th.**
The difference is that we require a normed space (not only a vector space) and we obtain that we still get an extension, but this time we get an equality with the sublinear function (in the sense that: we use the norm as a sublinear function and we get $\|F\|_{X^*} = \|f\|_{Y^*}$)

proof.

- Define $p: X \to \mathbb{R}$ s.t.:

$$p(x) := \|f\|_{Y^*} \|x\|_X \qquad \forall x \in X.$$

- p is a sublinear functional on X, moreover:

$$f(y) \le p(y) \quad \forall y \in Y$$ (since it always holds: $|f(y)| \le \|f\|_{Y^*} \|y\|_X$) (by property of operators)

- By the H-B theorem (general form) there exists $F \in X^*$: F is an extension of f and:

$$F(x) \le p(x) \quad \forall x \in X$$

- Hence:

$$\|F\|_{X^*} \le \|f\|_{Y^*} \qquad \left(|F(x)| \le p(x) = \|f\|_{Y^*} \|x\|_X \Longrightarrow \|F\|_{X^*} \le \|f\|_{Y^*}\right)$$

- Since F is an extension of f:

$$\|f\|_{Y^*} \le \|F\|_{X^*} \qquad \left(\|f\|_{Y^*} = \sup_{\substack{x \ne 0 \\ x \in Y}} \frac{|f(x)|}{\|x\|_X} \le \sup_{\substack{x \ne 0 \\ x \in X}} \frac{|f(x)|}{\|x\|_X} = \|F\|_{X^*}\right)$$

since we do the supremum on a bigger set ($Y \subsetneq X$)

- $\Longrightarrow \|f\|_{Y^*} = \|F\|_{X^*}$. ∎

CONSEQUENCES OF H-B THEOREM

We consider the **continuous extension**
X normed space, $Y \subsetneq X$, $f \in X^*$
$\Rightarrow \exists F \in X^*: \|F\|_{X^*} = \|f\|_{Y^*}$, $F(y) = f(y)\ \forall y \in Y$

1. **Corollary**: Let $x_0 \in X \setminus \{0\}$, where X is a normed space.
Then $\exists L_{x_0} \in X^*$ such that:
- $\|L_{x_0}\|_{X^*} = 1$
- $L_{x_0}(x_0) = \|x_0\|$

If $x_0 \neq 0$ we can always construct an operator of norm 1 such that at x_0 the value is exactly the norm of x_0
(operator $\rightarrow$ functional)

proof.

Consider $Y := \{\lambda x_0 : \lambda \in \mathbb{R}\} \equiv \text{span}\{x_0\}$ and define $L_0 : Y \to \mathbb{R}$ s.t.:
$$L_0(\lambda x_0) := \lambda \|x_0\|$$
which is linear and bounded.
By H-B theorem (continuous extension form) $\exists \tilde{L}_0 : X \to \mathbb{R}$ s.t.
- $\tilde{L}_0 \in X^*$
- $\|\tilde{L}_0\|_{X^*} = \|L_0\|_{Y^*} = \sup\limits_{\substack{\lambda x_0 \in Y \\ \|\lambda x_0\| = 1}} |\lambda \|x_0\|| = 1$
- $\tilde{L}_0(x_0) = L_0(x_0) = \|x_0\|$ $(\lambda = 1)$

The proof is complete setting $L_{x_0} := \tilde{L}_0$. ∎

The point is: we want a functional with some specific properties on a special point x_0. Therefore, we can construct a functional on the span of that special point and then we can extend this functional through Hahn-Banach theorem.

2. **Corollary**: Let $y, z \in X$, $L(y) = L(z)$ $\forall L \in X^*$.
Then $y = z$.

the elements of X^* separate the elements of X

proof.

Suppose by contradiction that $\exists y, z \in X$, $y \neq z$ s.t.
$$L(y) = L(z) \quad \forall L \in X^*.$$
Define now: $x := y - z \neq 0$.
$$L(x) = L(y - z) = L(y) - L(z) = 0 \quad \forall L \in X^*$$
$\Longrightarrow$ by the previous corollary: (since $x \neq 0 \Longrightarrow \exists L_x \in X^* : \|L_x\|_{X^*} = 1,\ L_x(x) = \|x\|_X (\neq 0)$)
$$\exists L_x \in X^* \text{ s.t. } L_x(x) = \|x\| \neq 0$$
since $x \neq 0$
$\Longrightarrow$ contradiction. ∎

3. **Corollary**: Let X be a normed space, Y a closed subspace of X and $x_0 \in X \setminus Y$. Then $\exists f \in X^*$ with $\|f\|_{X^*} = 1$ such that:
- $f(y) = 0 \quad \forall y \in Y$
- $f(x_0) = \text{dist}(x_0, Y) := \inf\limits_{\xi \in Y} \|x_0 - \xi\|$

We can always construct an operator which is not zero, but it is zero on a closed subspace.

By means of this corollary we can state the following theorem (hence also this theorem comes from Hahn-Banach)

Theorem: Let X be a normed space. If X^* separable $\Longrightarrow$ X is separable.

THE DUAL OF L^∞

Let $\Omega \subseteq \mathbb{R}^n$ be measurable. Consider $L^1(\Omega, \mathcal{L}(\mathbb{R}^n), \lambda)$.
Let $g \in L^1$.
Define $L_g : L^\infty \longrightarrow \mathbb{R}$ as:
$$L_g(f) := \int_\Omega f g \, d\lambda \qquad \forall f \in L^\infty.$$
Then L_g is linear. Moreover:
$$|L_g(f)| \leq \|f\|_\infty \|g\|_1$$
$$\Longrightarrow \|L_g\|_{L^\infty} \leq \|g\|_1.$$

consider now: $\tilde{f} := \text{sign}(g)$, then: $(\text{sign}(g) \in L^\infty)$

$$|L_g(f)| = \|g\|_1 \qquad \left(L_g(\tilde{f}) = \int_\Omega g \cdot \text{sign}(g)\, d\lambda = \int_\Omega |g|\, d\lambda = \|g\|_1 \right)$$

$$\Longrightarrow \|L_g\|_{(L^\infty)^*} = \|g\|_1 .$$

! Remark: $\exists L \in (L^\infty)^*$ such that L is **not** of the form of L_g with $g \in L^1$.

("proof" of the Remark):

In fact, consider $L_0 \in [C_c^0(\mathbb{R}^n)]^*$, $(C_c^0(\mathbb{R}^n), \|\cdot\|_\infty)$, defined as:

$$L_0(f) := f(0) \qquad \forall f \in C_c^0(\mathbb{R}^n)$$

We know that $C_c^0(\mathbb{R}^n)$ is a subspace of L^∞.

By H-B theorem, $\exists L \in [L^\infty(\mathbb{R}^n)]^*$ which is an extension of L_0.

We claim that $\nexists g \in L^1(\mathbb{R}^n)$ s.t.

$$L(f) = \int_{\mathbb{R}^n} g f \, d\lambda \qquad \forall f \in L^\infty(\mathbb{R}^n)$$

Suppose by contraddiction that such g exists. Then:

$$L(f) = L_0(f) = \int_{\mathbb{R}^n} g f\, d\lambda = f(0) \qquad \forall f \in C_c^0(\mathbb{R}^n)$$

Among all $f \in C_c^0(\mathbb{R})$ there are f s.t. $f(0) = 0$. For those f, the above would be:

$$L(f) = L_0(f) = \int_{\mathbb{R}^n} g f \, d\lambda = f(0) = 0$$

and the only way is that $gf = 0$ a.e. in $\mathbb{R}^n$, but, for the arbitrariety of f it must be $g = 0$ a.e. in $\mathbb{R}^n$.

If $g = 0$ a.e. in $\mathbb{R}^n \Longrightarrow L \equiv 0$, which is a contraddiction. $\left(\int_{\mathbb{R}^n} fg\, d\lambda = 0 \quad \forall f \in L^\infty(\mathbb{R}^n) \right)$

(even just because $\exists f \in C_c^0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ such that $f(0) \neq 0$, and hence:

$$0 \neq f(0) = L_0(f) = L(f) = \int_{\mathbb{R}^n} g f\, d\lambda = 0)$$

REFLEXIVE SPACES

Let X be a normed space and let X^* be its dual. $(X^* = \mathcal{L}(X, \mathbb{R}))$

The dual of X^*, i.e. $(X^*)^* \equiv X^{**}$ is called the bidual or second dual of X.

For each $x \in X$ we define:

$$\Lambda_x : X^* \longrightarrow \mathbb{R}$$

$$\Lambda_x(L) = L(x) \qquad \forall L \in X^* \qquad \text{(the definition makes sense because } L(x) \in \mathbb{R})$$

$$\Longrightarrow |\Lambda_x(L)| = |L(x)| \leq \|L\|_{X^*} \|x\|_X \qquad (1)$$

Moreover, Λ_x is linear.

Thus, Λ_x is linear and bounded (1) on X^* $\overset{\text{which means}}{\Longleftrightarrow}$ $\Lambda_x \in X^{**}$.

Moreover: (from (1):)

$$\|\Lambda_x\|_{X^{**}} \leq \|x\|_X \qquad (2)$$

it follows that there is a mapping $\tau : X \to X^{**}$ s.t.:

$$\forall x \in X \ : \quad \tau(x) = \Lambda_x$$

Def. τ is called canonical map.

Notice that (2): $\quad \|\Lambda_x\|_{X^{**}} = \|\tau(x)\|_{X^{**}} \leq \|x\|_X$

Theorem: τ is linear and $\|\tau(x)\|_{X^{**}} = \|x\|_X \quad \forall x \in X$.

proof.

linearity is obvious.

From (2) we get that: $\|x\|_X \geq \|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} \quad \forall x \in X$.

It remains to show: $\|x\|_X \leq \|\tau(x)\|_{X^{**}} \quad \forall x \in X$.

By a corollary of H-B theorem, for each $x \in X$, $x \neq 0$:

$$\exists L \in X^* \text{ with } \|L\|_{X^*} = 1, \quad L(x) = \|x\|_X$$

Therefore:

$$\|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} = \sup_{\|L\|_{X^*}=1} |\Lambda_x(L)| = \sup_{\|L\|_{X^*}=1} |L(x)| \geq \|x\|_X$$

(why $\geq$?) since maybe there are others $L \in X^*$ s.t. $\|L\|_{X^*} = 1$ and $L(x) \geq \|x\|_X$, but for sure there is at least one L s.t. $\|L\|_{X^*}=1$ and $L(x) = \|x\|_X$.

Thus, the thesis follows. ■

Corollary: The canonical map $\tau : X \to \tau(X)$ is an isometric isomorphism. — but we need to considerate $\tau(X)$ instead of X^{**}

Corollary: $\tau(X)$ is closed in X^{**}. (X Banach space)*

why? $\left.\begin{array}{l} X \text{ complete}^* \\ \tau \text{ isometry} \end{array}\right\} \Longrightarrow \tau(X)$ complete $\Longrightarrow \tau(X)$ closed

the image of a complete space through an isometry is complete (isometries preserve the norms)

(in general a complete space is closed)

Def. If $\tau(X) = X^{**}$ ($\Longleftrightarrow \tau$ is surjective) then X is said to be reflexive.

PROPERTIES OF REFLEXIVE SPACES

Theorem: let X be a reflexive Banach space. Then every closed subspace of X is reflexive.

Theorem: let X be a Banach space. Then:

$$X \text{ reflexive} \Longleftrightarrow X^* \text{ is reflexive}$$

Theorem: let X be a reflexive Banach space. Then:

$$X \text{ separable} \Longrightarrow X^* \text{ separable}.$$

In general: X^* separable $\overset{\Longrightarrow}{\not\Longleftarrow}$ X separable. However, if X is reflexive then: $\left.\begin{array}{l} X \text{ separable} \\ X \text{ reflexive} \end{array}\right\} \Longrightarrow X^*$ separable

Theorem: let $p \in (1,\infty)$. Then L^p is reflexive.

proof.

consider p, q: $\frac{1}{p} + \frac{1}{q} = 1$.

then: $(L^p)^* = L^q$ (since $p \neq \infty$) (if $p=1$ then $q=\infty$ to this point, from here we consider $p \neq 1, \infty$)

$$(L^p)^{**} = (L^q)^* = L^p$$

$\Longrightarrow L^p$ is reflexive. ■

Def. X Banach space. X is uniformly convex if $\forall \varepsilon > 0 \ \exists \delta > 0$:

$$\forall x, y \in X, \ \|x\| \leq 1, \ \|y\| \leq 1, \ \|x-y\| > \varepsilon$$
$$\Longrightarrow \left\| \frac{x+y}{2} \right\| < 1 - \delta$$

Remark: L^p, $p \in (1,\infty)$ is uniformly convex.

This follows from Clarkson inequalities: $p \in [1,2)$, $p \geq 2$

Clarkson inequalities

- $1 < p < 2$: $\left\| \frac{f+g}{2} \right\|_p^q + \left\| \frac{f-g}{2} \right\|_p^q \leq \left(\frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right) \right)^{q/p}$ $\quad f, g \in L^p$, p, q conj.
- $p \geq 2$: $\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right)$ $\quad f, g \in L^p$

Theorem : (Millmann-Pettis)

Let X be a Banach space and uniformly convex. Then X is reflexive.

In the previous theorem we obtained the reflexivity of L^p spaces by using the representation of its dual (we exploited the characterization). However, through Millmann-Pettis theorem we can obtain reflexivity without going through the characterization. (L^p uniformly convex, Banach $\Rightarrow$ reflexive)

WEAK CONVERGENCE

Def. Let X be a Banach space, $\{x_n\}_n \subset X$, $x \in X$.

We say that $x_n \xrightharpoonup[n\to\infty]{} x$ if : (namely x_n weakly converges to x)

$$L(x_n) \xrightarrow[n\to\infty]{} L(x) \quad \forall L \in X^*$$

(notice that the convergence $L(x_n) \to L(x)$ is a convergence of real numbers since $L \in X^*$ and so $L(x) \in \mathbb{R}$ $\forall x \in X$)

- Remark : $\overset{\text{in } X}{x_n \xrightarrow[n\to\infty]{} x} \implies x_n \xrightharpoonup[n\to\infty]{} x$

In fact : $\forall L \in X^*$:

$$|L(x_n) - L(x)| = |L(x_n - x)| \leq \|L\|_{X^*} \underbrace{\|x_n - x\|_X}_{\xrightarrow[n\to\infty]{} 0 \text{ if } x_n \xrightarrow[n\to\infty]{} x}$$

by def. of norm of L in X^*

$$\implies L(x_n) \xrightarrow[n\to\infty]{} L(x) \overset{\text{def.}}{\iff} x_n \xrightharpoonup[n\to\infty]{} x$$

- Remark : $x_n \xrightharpoonup[n\to\infty]{} x \;\not\Longrightarrow\; x_n \xrightarrow[n\to\infty]{} x$.

In fact, consider : $X = \ell^2$, $\{e_n\}_n \subset \ell^2$: $e_n^{(k)} = \delta_{kn}$.

$$\begin{array}{l} \quad\;\; 1\;\,2\;\,3 \qquad\quad n \\ e_1 = (1, 0, 0, \dots, 0, \dots) \\ e_2 = (0, 1, 0, \dots, 0, \dots) \\ e_3 = (0, 0, 1, \dots, 0, \dots) \\ e_n = (0, 0, 0, \dots, 1, \dots) \end{array}$$

(we're considering a sequence $\{e_n\}_n$ where every element e_n is in ℓ^2 and so it's a sequence)

We recall that $(\ell^2)^* = \ell^2$.

Instead of writing :

$$L(e_n) \longrightarrow L(x) \quad \forall L \in (\ell^2)^*$$

we can write :

$$\sum_{j=1}^{\infty} e_n^{(j)} x^{(j)} = x^{(n)} \xrightarrow[n\to\infty]{} 0 \quad \forall x \in \ell^2$$

since :

$$L(e_n) = \sum_{j=1}^{\infty} e_n^{(j)} x^{(j)}$$

representation theorem : (called Riesz repr. theorem)

CHARACTERIZATION OF $(L^p)^*$ (special case of $(\ell^2)^*$)
$\forall L \in (\ell^2)^*$ $\exists!\, x \in \ell^2$ such that :
$L(y) = \sum_{j=1}^{\infty} y^{(j)} x^{(j)}$, $\boxed{\forall y \in \ell^2}$

Then : $L(e_n) = \sum_{j=1}^{\infty} e_n^{(j)} x^{(j)} = x^{(n)}$ which must converge to zero because $x \in \ell^2$

However it holds :

$$\|e_n\|_{\ell^2} = 1 \quad \forall n \in \mathbb{N} \implies e_n \not\xrightarrow[n\to\infty]{} 0 \text{ in } \ell^2.$$

Consider from now on $p \in [1, \infty)$.

- **!** Remark : Let $\{f_n\}_n \subset L^p$, $f \in L^p$:

$$\boxed{f_n \xrightharpoonup[n\to\infty]{} f}, \text{ i.e. } L(f_n) \xrightarrow[n\to\infty]{} L(f) \quad \forall L \in (L^p)^* = L^{p'} = L^q$$

$\overset{\text{RIESZ THEOREM}}{\iff}$ (representation theorem)

$$\boxed{\int_\Omega f_n g \, d\lambda \xrightarrow[n\to\infty]{} \int_\Omega f g \, d\lambda \quad \forall g \in L^{p'} = L^q}$$

- **!** Remark : Let $\{x_n\}_n \subset \ell^p$, $x \in \ell^p$:

$$\boxed{x_n \xrightharpoonup[n\to\infty]{} x} \text{ in } \ell^p, \text{ i.e. } L(x_n) \xrightarrow[n\to\infty]{} L(x) \quad \forall L \in (\ell^p)^* = \ell^{p'} = \ell^q$$

$\iff$ (representation theorem for ℓ^p)

$$\boxed{\sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \xrightarrow[n\to\infty]{} \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \quad \forall y \in \ell^{p'} = \ell^q}$$

Prop. $\{x_n\}_n$ weakly converges $\Longrightarrow$ the weak limit is unique.

proof.

Suppose by contraddiction that $x_n \xrightharpoonup[n]{} x_1$, $x_n \xrightharpoonup[n]{} x_2$, $x_1 \neq x_2$.

$\Longrightarrow |L(x_n) - L(x_1)| \xrightarrow[n\to\infty]{} 0$, $|L(x_n) - L(x_2)| \xrightarrow[n\to\infty]{} 0$ $\quad \forall L \in X^*$] these are convergences in the set of real numbers and convergences of sequences of real numbers imply that the limit is unique $\Rightarrow L(x_1) = L(x_2)$

$\Longrightarrow L(x_1) = L(x_2) \quad \forall L \in X^*$

as a consequence of H-B (2nd corollary)

$\Longrightarrow x_1 = x_2$. ■

Prop. If $x_n \xrightharpoonup[n\to\infty]{} x$ then $\{x_n\}_n$ is bounded.

(consequence of UBP)

Prop. If $x_n \xrightharpoonup[n\to\infty]{} x$ then $\liminf\limits_{n\to\infty} \|x_n\|_X \geq \|x\|_X$

(equivalently: $x \longmapsto \|x\|_X$ is lower semicontinuous w.r.t. weak convergence)

proof.

Let $x \in X \setminus \{0\}$, $L \in X^*$: $\|L\|_{X^*} = 1$, $L(x) = \|x\|$. (1st corollary of H-B)

Then:

by def. of weak convergence (which implies the equivalence $\forall L$)

$$\|x\| = L(x) = \lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} |L(x_n)| \quad \text{(since } \|x\| \geq 0)$$

On the other hand:

$$|L(x_n)| \leq \|L\|_{X^*} \|x_n\|_X = \|x_n\|_X \quad \text{(by def. of norm in } X^*)$$

$$\Longrightarrow \|x\|_X = \lim_{n\to\infty} |L(x_n)| \leq \underbrace{\liminf_{n\to\infty} \|x_n\|_X}_{} . \quad ■$$

based on the previous expression ($|L(x_n)| \leq \|x_n\|_X$) we would like to take the limit, however we don't know if the limit exists $\Rightarrow$ **liminf** (always exists)

Prop. $\left.\begin{array}{l} x_n \xrightharpoonup[n]{} x \text{ in } X \\ L_n \xrightarrow[n]{} L \text{ in } X^* \end{array}\right\} \Longrightarrow \underbrace{L_n(x_n) \xrightarrow[n\to\infty]{} L(x)}_{L_n(x_n) \in \mathbb{R},\ L(x) \in \mathbb{R} \quad \text{(converges in } \mathbb{R})}$

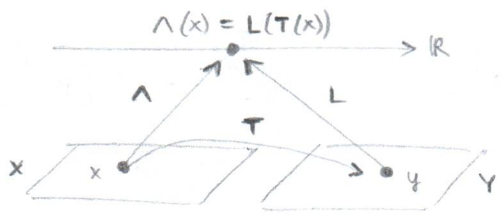
proof.

$$\begin{aligned} |L_n(x_n) - L(x)| &= |L_n(x_n) - L(x_n) + L(x_n) - L(x)| \\ &\leq |L_n(x_n) - L(x_n)| + |L(x_n) - L(x)| \\ &\leq \underbrace{\|L_n - L\|_{X^*}}_{\substack{\xrightarrow[n\to\infty]{} 0 \\ \text{(by Hp.)} \\ L_n \to L \text{ in } X^*}} \underbrace{\|x_n\|_X}_{\substack{\leq M \\ (\exists M>0:\ \|x_n\|_X \leq M\ \forall n\in\mathbb{N})}} + \underbrace{|L(x_n) - L(x)|}_{\substack{\xrightarrow[n\to\infty]{} 0 \text{ (by Hp.) } x_n \rightharpoonup x \text{ in } X}} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

■

Prop. X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$. Then:

$$x_n \xrightharpoonup[n]{} x \Longrightarrow T(x_n) \xrightharpoonup[n]{} T(x).$$



proof.

Let $L \in Y^*$.

Define $\Lambda : X \to \mathbb{R}$ s.t. : $\Lambda(x) := L[T(x)] \quad \forall x \in X$.

Therefore, since $\Lambda \in X^*$;

$\Longrightarrow \Lambda(x_n) \xrightarrow[n]{} \Lambda(x)$ (by def. of $x_n \rightharpoonup x$)

$\Longrightarrow L(T(x_n)) \xrightarrow[n]{} L(T(x)) \iff T(x_n) \xrightharpoonup[n]{} T(x)$. ■

Prop. Let X be reflexive. If $\{L(x_n)\}_n$ converges $\forall L \in X^*$ then $\exists!\, x \in X$ such that:

$$x_n \xrightharpoonup[n\to\infty]{} x \ .$$

WEAK* CONVERGENCE

Def. We say that $\{L_n\}_n \subset X^*$ (dual of a Banach space X) converges weakly * to L if:

$$L_n(x) \longrightarrow L(x) \qquad \forall x \in X$$

notice that $L_n(x), L(x) \in \mathbb{R}$ (therefore it is a convergence in $\mathbb{R}$)

We write:

$$L_n \xrightarrow[n\to\infty]{*} L.$$

Prop. $L_n \xrightarrow[n\to\infty]{} L$ in $X^* \implies L_n \xrightarrow[n\to\infty]{*} L$

$\Longleftarrow$ true if X is reflexive

we know that if X is a Banach space then also X^* is a Banach space and so we can define the notion of weak conv. in X^*. What is the relation between weak convergence in X^* and weak * convergence?

(Recall: X reflexive $\iff \tau(X) = X^{**}$
$\iff \forall \phi \in X^{**} \ \exists x \in X : \ \phi(f) = f(x) \quad \forall f \in X^*$)

we can identify through the canonical map the Banach space X and its second dual $\Rightarrow$ every element of the second dual can be therefore represented in terms of an element of the space X

proof.

$(\Rightarrow)$ $L_n \longrightarrow L$ in X^* $\overset{\text{def.}}{\iff} \Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L) \quad \forall \Lambda \in X^{**}$

$\implies \Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L) \quad \forall \Lambda \in \tau(X) \subseteq X^{**}$

$\iff L_n(x) \xrightarrow[n\to\infty]{} L(x) \quad \forall x \in X$

$\overset{\text{def.}}{\iff} L_n \xrightarrow[n\to\infty]{*} L$ in X^*

$(\Leftarrow)$ $\tau(X) = X^{**}$: $L_n \xrightarrow{*} L$ in $X^* \iff L_n(x) \longrightarrow L(x) \ \forall x \in X$

$\iff \Lambda(L_n) \longrightarrow \Lambda(L) \ \forall \Lambda \in \tau(X) = X^{**}$

$\iff L_n \longrightarrow L$ in X^* ■

Prop. Let X be a Banach space.

- If $\{L_n\}_n \subset X^*$ weakly * converges $\Rightarrow$ the limit is unique
- $L_n \xrightarrow{*} L \implies \{L_n\}_n$ is bounded
- $L_n \xrightarrow{*} L \implies \liminf\limits_{n\to\infty} \|L_n\|_{X^*} \geq \|L\|_{X^*}$

  ($L \longmapsto \|L\|_{X^*}$ is lower semicontinuous w.r.t. weak * convergence)
- $\left.\begin{array}{l} L_n \xrightarrow{*} L \\ x_n \longrightarrow x \end{array}\right\} \implies L_n(x_n) \xrightarrow[n\to\infty]{} L(x)$

Theorem: Let X be a separable Banach space. Then any bounded sequence $\{L_n\} \subset X^*$ admits a subsequence that weakly* converges to some $L \in X^*$.

(Banach-Alaoglu theorem)

Remark: Consider a sequence $\{f_n\} \subset L^\infty(\Omega)$. Let $\Omega \subseteq \mathbb{R}^n$ be measurable and consider $(\Omega, \mathcal{X}(\Omega), \lambda)$.

Let $L_n : L^1(\Omega) \to \mathbb{R}$ defined as:

$$L_n(g) := \int_\Omega f_n g \, d\lambda \qquad \forall g \in L^1(\Omega)$$

We assume that $\{f_n\}_n$ is bounded: (notice: it's an assumption, it is not guaranteed)

$\{f_n\}_n$ bounded in $L^\infty \iff \exists C > 0 : \ \|f_n\|_\infty \leq C \quad \forall$

$\Rightarrow \{L_n\}_n$ is bounded in $(L^1)^*$

In fact:

$$|L_n(g)| \leq \|f_n\|_\infty \|g\|_1 \leq C \|g\|_1 \qquad \forall n \in \mathbb{N}$$

Every element of L^∞ is bounded, however a sequence $\{f_n\}_n \subset L^\infty$ is not necessarily bdd. Take for example any sequence $\{f_n\}_n$ s.t. $\text{ess sup} |f_n| \leq n$. In this case the seq. would not be bounded.

$\Rightarrow \|L_n\|_{(L^1)^*} \leq C$

$\Rightarrow$ We apply Banach-Alaoglu:

L^1 is a separable Banach space. $\{L_n\}_n \subseteq (L^1)^*$ is bounded $\Rightarrow \exists \{L_{n_h}\}_h \subset \{L_n\}_n : L_{n_h} \xrightarrow{*} L \in (L^1)^*$

$\exists \{L_{n_h}\}_h \subset \{L_n\}_n : \quad L_{n_h} \xrightarrow{*} L \quad$ for some $L \in (L^1)^*$

$\overset{\text{def.}}{\Leftrightarrow} \quad L_{n_h}(g) \xrightarrow[h\to\infty]{} L(g) \quad \forall g \in L^1$

Then, in view of the representation theorem (for L^p):

$$\exists! f \in L^\infty \text{ s.t. } L(g) = \int_\Omega fg\,d\lambda \qquad \forall g \in L^1(\Omega)$$

$$\Rightarrow \underbrace{\int_\Omega f_{n_h} g\,d\lambda}_{\text{definition of } L_{n_h}(g) \text{ (we have done it at the beginning)}} \xrightarrow[h\to\infty]{} \int_\Omega fg\,d\lambda \qquad \forall g \in L^1(\Omega)$$

important for exercises ! [Therefore we say that $\{f_n\}_n$ bounded in L^∞ admits a subsequence $\{f_{n_h}\}_h$ which weakly * converges to some $f \in L^\infty(\Omega)$.

Corollary: Let X be a separable and reflexive Banach space. Then any bounded sequence $\{x_n\} \subset X$ possesses a subsequence which weakly converges.

proof.

- X^* separable ($\Leftrightarrow$ X separable) since X is a reflexive Banach space;
- $\{\tau(x_n)\}_n \subset X^{**}$ is bounded $= \{\Lambda_{x_n}\}_n \subset X^{**}$ *

$\Rightarrow$ we apply Banach-Alaoglu to X^*: $\exists \{\tau(x_{n_h})\}_h$ such that:

$$\Lambda_{x_{n_h}} = \tau(x_{n_h}) \xrightarrow{*} \Lambda \quad \text{for some } \Lambda \in X^{**}$$

This means that:

$$\boxed{\Lambda_{x_{n_h}}(L) = [\tau(x_{n_h})](L) \longrightarrow \Lambda(L) \qquad \forall L \in X^*}$$

by def. of $\tau(\cdot)$ and since X is reflexive:

$$\Lambda_{x_{n_h}}(L) = [\tau(x_{n_h})](L) = L(x_{n_h})$$

$$\Lambda(L) = L(x) \qquad x := \tau^{-1}(\Lambda)$$

$$\Rightarrow \boxed{L(x_{n_h}) \longrightarrow L(x) \qquad \forall L \in X^*}$$

$$\Leftrightarrow \quad x_{n_h} \rightharpoonup x. \qquad \blacksquare$$

Theorem: (**Eberlein - Smulyan**)
Let X be a Banach space.
If any bounded sequence contains a weakly convergent subsequence then X is reflexive.

*
$\tau(x_n) = \Lambda_{x_n}$
$\Lambda_{x_n}(L) = L(x_n)$
$|\Lambda_{x_n}(L)| = |L(x_n)| \leq \|L\|_{X^*} \|x_n\|_X$
since $\{x_n\}_n$ is bounded $\Rightarrow \exists C>0 : \|x_n\|_X \leq C \quad \forall n \in \mathbb{N}$
$\Rightarrow |\Lambda_{x_n}(L)| \leq C \cdot \|L\|_{X^*}$
$\Rightarrow \{\Lambda_{x_n}\}_n = \{\tau(x_n)\}_n$ is bounded

COMPACT OPERATORS

Let X, Y be Banach spaces.

Def. $K : X \to Y$ linear operator is said to be compact if:

$$\forall E \subseteq X \text{ bounded} : \overline{K(E)} \text{ is compact.}$$

notice that we are **not** assuming that it's bounded

A linear mapping is compact if it transforms bounded subsets into subsets whose closure is compact.

Remark: $\{x_n\} \subset X$ bounded $\Rightarrow$ $\{K(x_n)\}$ has a subsequence which converges in Y strongly

Prop. $K : X \to Y$ linear and compact $\Rightarrow$ $K \in \mathcal{L}(X, Y)$

this is why we don't need to assume boundness/continuity in the definition

Def. $T \in \mathcal{L}(X,Y)$ has finite rank if $\dim(\mathrm{Im}(T)) < \infty$.

Remark: $T \in \mathcal{L}(X,Y)$, rank(T) finite $\underset{\not\Leftarrow}{\Rightarrow}$ T compact

Def. Let $Y \subset X$ be a subspace. Consider the identity $I : Y \to X$. If I is compact then we say that Y is compactly embedded in X and we write: $Y \Subset X$.

Def. $K(X,Y) := \{K \in \mathcal{L}(X,Y) : K \text{ compact}\}$

If $X = Y$ we write: $K(X,X) = K(X)$.

Theorem: (i) If $K \in K(X,Y)$ then:

$$(1) \quad x_n \xrightarrow[n\to\infty]{} x \Rightarrow K(x_n) \xrightarrow[n\to\infty]{} K(x)$$

a compact operator is weak-strong continuous

(ii) If X is reflexive and $K \in \mathcal{L}(X,Y)$ satisfies (1), then $K \in K(X,Y)$.

Corollary: $K(X,Y)$ is a Banach space.

HILBERT SPACES

Def. Let H be a vector space. A simmetric, positive definite bilinear form on $H \times H$ is called scalar product (or inner product).

$p : H \times H \to \mathbb{R}$: (simmetric, positive definite bilinear form)

(i) $p(x,x) \geq 0 \quad \forall x \in H, \quad p(x,x) = 0 \iff x = 0$

(ii) $p(x,y) = p(y,x)$

(iii) $p(\alpha x + \beta y, z) = \alpha p(x,z) + \beta p(y,z) \quad \forall x,y,z \in H, \ \alpha, \beta \in \mathbb{R}$

We set: $\langle x, y \rangle := p(x,y) \quad (= (x,y) = x \cdot y)$

Def. H vector space with a scaler product is called pre-Hilbert space (or inner (scaler) product space).

Prop. (i) (Cauchy-Schwartz):

$$|\langle x,y \rangle| \leq \sqrt{\langle x,x \rangle} \sqrt{\langle y,y \rangle}$$

(ii) $\|x\| := \sqrt{\langle x,x \rangle} \quad \forall x \in H$ is a norm on H.

A norm induces a distance, a scalar product induces a norm which induces a distance. Therefore, a **pre-Hilbert** space is a normed space and a metric space.

Def. $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space when $(H, \|\cdot\|)$ (where the norm $\|\cdot\|$ is the one induced by $\langle\cdot,\cdot\rangle$) is a Banach space
$\iff$ (H, d) (d induced by $\|\cdot\|$, $\|\cdot\|$ induced by $\langle\cdot,\cdot\rangle$) is a complete metric space.

Examples:

1. $(C([a,b]), \langle\cdot,\cdot\rangle)$, $\langle f,g\rangle = \int_a^b fg\,dx$ is a pre-Hilbert space
2. $(L^2(X,\mathcal{A},\mu), \langle\cdot,\cdot\rangle)$, $\langle f,g\rangle = \int_X fg\,d\mu$ is a Hilbert space.
3. $(\ell^2, \langle\cdot,\cdot\rangle)$, $\langle x,y\rangle = \sum_{n=0}^{\infty} x(n)\,y(n)$ is a Hilbert space.

Def. $x,y \in H$ are orthogonal $\iff$ $\langle x,y\rangle = 0$

Theorem: H pre-Hilbert space. Then $\forall x,y \in H$:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(parallelogram identity / law)

proof.

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$\overset{\text{bilinearity and simmetry}}{=} \langle x,x\rangle + \cancel{2\langle x,y\rangle} + \langle y,y\rangle + \langle x,x\rangle - \cancel{2\langle x,y\rangle} + \langle y,y\rangle$$

$$= 2\|x\|^2 + 2\|y\|^2 \quad \blacksquare$$

Remark: $L^p(X,\mathcal{A},\mu)$ $p \neq 2$ is **not** a pre-Hilbert space.
The fact is that parallelogram identity characterizes Hilbert spaces (so pre-Hilbert spaces somehow are involved) and on L^p with $p \neq 2$ parallelogram identity is **not** true. Therefore on L^p $p \neq 2$ we cannot consider an inner product.

ORTHOGONAL PROJECTIONS

Theorem: Let H be a Hilbert space and let $\emptyset \neq S \subseteq H$ be a $\underbrace{\text{convex}}$ closed subset.
Then $\exists!\, h \in S$ such that:

$$\min_{s\in S} \|s\| = \|h\|.$$

for every pair of points of S the segment joining these points is contained in S.

Theorem (Minimal distance)
Let H be a Hilbert space and let $\emptyset \neq S \subseteq H$ be a convex closed subset.
Then $\forall f \in H$ $\exists!\, h \in S$ such that:

$$\underbrace{\|f - h\|}_{dist(f,h)} = \min_{s\in S} \|f - s\| =: dist(f, S)$$

We have an Hilbert space and a convex closed subset. We are saying that for every element in the bigger space there exists a unique element in the smaller convex subset such that the distance between f and h realizes the distance between f and S.

proof.

Let $\tilde{S} := \{v - f : v \in S\}$.
By the previous theorem $\exists!\, \xi \in \tilde{S}$ such that:

$$\|\xi\| = \min_{\eta\in\tilde{S}} \|\eta\|$$

Since $\xi \in \tilde{S}$:* $\xi = h - f \implies h = \xi + f$

* $\exists h \in S$ such that

$$\longrightarrow \|\xi\| = \|h - f\| = \min_{s\in S} \|s - f\| =: dist(f,S) \quad \blacksquare$$

Remark: S closed subspace is a special case.
(of the minimal distance theorem)

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proof. (first thm. of the "Orthogonal Projections")

1. Let $d := \inf_{s \in S} \|s\|$.

we use convexity of S

Then $\exists \{s_n\}_n \subset S$ such that: ($\{s_n\}_n$ is called minimizing sequence)

- $\|s_n\| \geq d$
- $\|s_n\| \longrightarrow d$

Claim: the sequence $\{s_n\}_n$ is Cauchy.
In fact, by the parallelogram law:

$$\|s_m - s_n\|^2 = 2\{\|s_m\|^2 + \|s_n\|^2\} - \|s_m + s_n\|^2$$
$$\leq 2\{\|s_m\|^2 + \|s_n\|^2\} - 4d^2$$

this is because:

$$\frac{s_m + s_n}{2} \in S \quad \text{(since S is convex)}$$

$\Longrightarrow \left\|\frac{s_m+s_n}{2}\right\| = \frac{1}{2}\|s_m + s_n\| \geq d$ (since $d = \inf_{s\in S}\|s\|$ and $\frac{s_n + s_m}{2} \in S$)

$\Longrightarrow \|s_m + s_n\|^2 \geq 4d^2$

$(-\|s_m + s_n\|^2 \leq -4d^2)$

Moreover:

$\|s_n\|^2 \longrightarrow d^2 \quad \Longrightarrow \quad \forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N}$ s.t. $\forall m, n > \bar{n}$:

$$\|s_m - s_n\|^2 \leq 2(d^2 + \varepsilon) + 2(d^2 + \varepsilon) - 4d^2$$

starting from:
$\|s_m - s_n\|^2 \leq 2(\|s_m\|^2 + \|s_n\|^2) - 4d^2$
and considering that
$\|s_n\|^2 \longrightarrow d^2$

$$\leq 4\varepsilon$$

$\Longleftrightarrow \quad \forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N}$ s.t. $\forall n, m > \bar{n}$:
$\|s_m - s_n\|^2 < 4\varepsilon$

$\Longleftrightarrow \quad \{s_n\}_n$ is a Cauchy sequence.

2. H is a Hilbert space $\Longrightarrow$ $(H, \|\cdot\|)$, where $\|\cdot\|$ is the norm induced by the scalar product $\langle\cdot,\cdot\rangle$, is a Banach space

we use closedness of S

$\Longrightarrow$ (H, d), where d is the metric induced by the norm $\|\cdot\|$, is a complete metric space

$\Longrightarrow$ $\{s_n\}_n$ is a Cauchy sequence in a complete metric space

$\Longrightarrow$ $\exists h \in H$ such that:

$$\|s_n - h\| \xrightarrow[n \to \infty]{} 0$$

We have used completeness of H to deduce that such element h belongs to H, but we also know that $\{s_n\}_n$ not only belongs to H, but it belongs to S, which is closed.

$\Longrightarrow$ we have a convergent sequence contained in S so, since S is closed:

$\Longrightarrow h \in S$.

Claim: $\|h\| = d = \min_{s \in S} \|s\|$.

In fact: $\big| \|s_n\| - \|h\| \big| \leq \|s_n - h\| \xrightarrow[n\to\infty]{} 0$

$\Longrightarrow \|s_n\| \to \|h\|$ (since $s_n \to h$)

Since $\|s_n\| \geq d \Longrightarrow \|h\| \geq d$.

Therefore it remains to prove $\|h\| \leq d$.

By definition of limit:

$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n > n_\varepsilon$:

- $\|s_n\| < d + \varepsilon$
- $\|s_n - h\| < \varepsilon$

Hence:

$$\|h\| = \|s_n - (s_n - h)\| \leq \|s_n\| + \|s_n - h\|$$
$$< d + \varepsilon + \varepsilon = d + 2\varepsilon$$

but $\varepsilon > 0$ is arbitrary, thus:

$\|h\| \leq d$.

3. $h \in S$ is unique.

In fact, by contraddiction suppose that $\exists \tilde{h} \in S$:

- $\|\tilde{h}\| = \|h\| = d$
- $\|\tilde{h} - h\| \geq \varepsilon > 0$ (another way to say $\tilde{h} \neq h$)

By the parallelogram law:

$$\|\tilde{h} + h\|^2 = 2(\|\tilde{h}\|^2 + \|h\|^2) - \|\tilde{h} - h\|^2$$
$$\leq 4d^2 - \varepsilon^2 \quad \text{(by the previous two points)}$$
$$< 4d^2 \quad (*)$$

Moreover, since S is convex:

$$\frac{\tilde{h}+h}{2} \in S \Longrightarrow \left\| \frac{\tilde{h}+h}{2} \right\| = \frac{1}{2}\|\tilde{h}+h\| \geq d$$
$$\Longrightarrow \|\tilde{h} + h\| \geq 4d^2 \quad (**)$$

But $(*)$ and $(**)$ are a contraddiction.

Therefore, it cannot exist $\tilde{h} \in S$ satisfying the same properties as h but such that $\tilde{h} \neq h$.

$\Longrightarrow h \in S$ is unique. ∎

Prop. Let H be a pre-Hilbert space and consider a subset $S \subseteq H$.

We have:

(i) $S^\perp := \{ f \in H : \langle f, g \rangle = 0 \ \forall g \in S \}$ is a closed subset

(ii) if S is a subspace then $S \cap S^\perp = \{0\}$.

Theorem: (Projection theorem)

Let H be a Hilbert space and consider $V \subseteq H$ closed subspace.

Then $\forall f \in H \ \exists! f_1, f_2$ satisfying:

(i) $f_1 \in V$, $f_2 \in V^\perp$

(ii) $f = f_1 + f_2$

} we have an orthogonal decomposition of f w.r.t. the subspace V.

proof.

Let $f \in H$ be fixed, and let f_1 be the unique element of V such that:

$$\| f - f_1 \| = \min_{s \in V} \| f - s \|. \qquad \text{(follows from the minimal distance thm.)}$$

Set:

$$f_2 = f - f_1.$$

Then:

$$f = f_1 + f_2.$$

Furthermore we claim that $f_2 \in V^{\perp}$, that is:

$$\langle f_2 , s \rangle = 0 \qquad \forall s \in V.$$

Indeed, V is a subspace:

$$\Longrightarrow f_1 + \lambda s \in V \qquad \forall \lambda \in \mathbb{R} \qquad (f_1, s \in V, \ \lambda \in \mathbb{R})$$

$$\Longrightarrow \| f - f_1 \|^2 \le \| f - (f_1 + \lambda s) \|^2$$

(since f_1 is s.t. $\| f - \bullet \|^2$ is minimum so, any other element in V is s.t. $\| f - (\cdot\cdot) \|^2 \geq \| f - f_1 \|^2$)

$$= \| f - f_1 \|^2 + \lambda^2 \| s \|^2 - 2\lambda \langle f - f_1 , s \rangle$$

$$\| f - f_1 - \lambda s \|^2 = \langle f - f_1 - \lambda s , f - f_1 - \lambda s \rangle$$
$$= \langle f - f_1 , f - f_1 - \lambda s \rangle - \langle \lambda s , f - f_1 - \lambda s \rangle$$
$$= \langle f - f_1 , f - f_1 \rangle - \langle f - f_1 , \lambda s \rangle - \cdots$$

$$\Longrightarrow \lambda^2 \| s \|^2 - 2 \lambda \langle f - f_1 , s \rangle \geq 0$$

Let $s \neq 0$.

For: $\lambda = \dfrac{\langle f - f_1 , s \rangle}{\| s \|^2}$ we get:

$$\| s \|^2 \frac{\langle f - f_1 , s \rangle^2}{\| s \|^4} - 2 \cdot \frac{\langle f - f_1 , s \rangle}{\| s \|^2} \langle f - f_1 , s \rangle \geq 0$$

$$\Longrightarrow \frac{\langle f - f_1 , s \rangle^2}{\| s \|^2} \le 0 \iff \frac{\langle f_2 , s \rangle^2}{\| s \|^2} \le 0$$

$\Longrightarrow$ it must be $= 0$ since it's a non-negative quantity.

$\Longrightarrow \langle f_2 , s \rangle = 0 \quad \forall s \in V \quad s \neq 0$

$\Longrightarrow f_2 \in V^{\perp}$. ∎

By means of this theorem we can define an operator.

Let H be a Hilbert space and let $V \subseteq H$ be a closed subspace.
Then we can define two mappings:

$$P : H \to V \qquad P(f) = f_1$$
$$Q : H \to V^{\perp} \qquad Q(f) = f_2 \qquad \forall f \in H$$

P is the orthogonal projector on V and Q is the orthogonal projector on $V^{\perp}$.

From the preceding theorem we have that:

(i) $P(f) + Q(f) = f \qquad \forall f \in H$

(ii) $f \in V \Longrightarrow P(f) = f, \ Q(f) = 0$
$\quad\ f \in V^{\perp} \Longrightarrow P(f) = 0, \ Q(f) = f$

(iii) $\| f - P(f) \| = \operatorname{dist}(f, V)$ (consequence of the previous theorem) = the projection is exactly the element realizing the distance

(iv) $\| f \|^2 = \| P(f) \|^2 + \| Q(f) \|^2$

(v) P, Q are linear.
Furthermore (from the previous point):
$$\|P(f)\| \leq \|f\|, \qquad \|Q(f)\| \leq \|f\|$$
$\Longrightarrow$ P, Q are bounded
$\Longrightarrow$ $P, Q \in \mathcal{L}(H)$ (linear and bounded operator on H)

DUALITY

Theorem: (Riesz)
Let H be a Hilbert space and let H^* be its dual.
(set of all linear and continuous operators from H to $\mathbb{R}$ (and it is a Banach space)) (since H is Banach)

Then $\forall F \in H^*$ $\exists!\, g \in H$ such that:
$$F(f) = \langle f, g \rangle \qquad \forall f \in H.$$
We are saying that we can represent every element of the dual space by means of an unique element of the space H. In some sense, H and H^* are "the same". The action of every element of the dual can be expressed totally by means of the scalar product with elements of H.

Furthermore: $\|F\|_{H^*} = \|g\|_H$.

proof.

1. If $F \equiv 0$ ($F(f) = 0 \ \forall f \in H$) then we take $g = 0$ and we've done.
If $F \neq 0$, we put:
$$N := \{f \in H : F(f) = 0\}$$ = elements of H that are mapped into 0 by F
N is a closed subset of H. (N is the kernel of F, so it's a closed subset)
If we have a closed subset of a Hilbert space then:
$$\exists f_2 \in N^\perp, \quad f_2 \neq 0$$ (this is due to the fact that $F \not\equiv 0$ and so $N \subsetneq H$)
Set:
$$h := \frac{f_2}{F(f_2)} \in N^\perp$$ (since $N^\perp$ is a subspace)
where $F(f_2) \neq 0$ since $f_2 \in N^\perp$ and $F(\cdot) = 0$ if $\cdot \in N$.
Therefore $F(f_2) \neq 0$ in view of the definition of N and because:
$$N \cap N^\perp = \{0\}.$$
Let:
$$m := f - F(f)\,h \qquad \forall f \in H$$
Then:
$$F(m) = F(f) - F(f)\underbrace{F(h)}_{F(h) = \frac{1}{F(f_2)} \cdot F(f_2) = 1}$$ ($F(f) \in \mathbb{R}$ and $F(\cdot)$ is linear)
$$= F(f) - F(f) = 0 \longrightarrow m \in N$$
$\Longrightarrow$ since $m = f - F(f)h$ and since $m \in N$:
$$F(f)\,h \in N^\perp$$
$$\Longrightarrow f = \underbrace{m}_{\in N} + \underbrace{F(f)\,h}_{\in N^\perp}$$ orthogonal decomposition of f

2. Let:
$$g := \frac{h}{\|h\|^2} \in N^\perp$$ (since $h \in N^\perp$)

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$\Longrightarrow \langle m, g \rangle = 0 \qquad (m \in N,\ g \in N^{\perp}) \qquad (1)$

Moreover:

$$F(g) = \frac{\overbrace{F(h)}^{=1}}{\|h\|^2} = \frac{1}{\|h\|^2} = \|g\|^2 = \langle g, g \rangle \qquad (2)$$

$$\Longrightarrow f = m + F(f)\,h = m + F(f)\cdot \frac{g}{\|g\|^2} \qquad (3)$$

$$h = g\,\|h\|^2 = \frac{g}{\|g\|^2}$$

We combine together (1), (2), (3):

$$F(f) = \langle m + F(f)\frac{g}{\|g\|^2}, g \rangle = \langle f, g \rangle \qquad \forall f \in H$$

$$= \cancel{\langle m, g \rangle} + \langle F(f)\frac{g}{\|g\|^2}, g \rangle$$

$$= \langle f - m, g \rangle$$

$$= \langle f, g \rangle - \cancel{\langle m, g \rangle}$$

$$|F(f)| = |\langle f, g \rangle| \le \|f\| \cdot \|g\| \qquad \forall f \in H$$

$$\Longrightarrow \|F\|_{H^*} \le \|g\|_H$$

But from (2):

$$|F(g)| = \|g\|^2$$

$$\Longrightarrow \|F\|_{H^*} = \|g\|_H.$$

3. (uniqueness)

Suppose by contraddiction that there exists $\tilde{g} \in H$, $\tilde{g} \neq g$ such that:

$$F(f) = \langle f, \tilde{g} \rangle = \langle f, g \rangle \qquad \forall f \in H.$$

Then:

$$0 = F(f) - F(f) = \langle f, g \rangle - \langle f, \tilde{g} \rangle = \langle f, g - \tilde{g} \rangle \qquad \forall f \in H.$$

For $f = g - \tilde{g}$ we get:

$$\langle g - \tilde{g}, g - \tilde{g} \rangle = 0 \iff g = \tilde{g}. \qquad \blacksquare$$

(contraddiction $\Longrightarrow g = \tilde{g}$)

We underly that in this proof we have explicitly constructed the element g and we have done it by means of consequences of orthogonal projections. (a crucial point is the orthogonal decomposition of f)

ORTHONORMAL BASES

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Let H be a Hilbert space.

Def. A set $S \subseteq H$ is called orthonormal if:

(i) $f \perp g \qquad \forall f, g \in S$

(ii) $\|f\| = 1 \qquad \forall f \in S.$

Def. An orthonormal set $S \subseteq H$ is called complete or an orthonormal basis if $S^{\perp} = \{0\}$.

Theorem: Let H be a Hilbert space. If H possesses at least two different elements then H has an orthonormal basis.

- H separable $\Longrightarrow$ Gram – Schmidt method
- H non separable $\Longrightarrow$ Zorn's lemma

Notice that: H separable $\iff$ H has an orthonormal countable basis

Example: $H = \ell^2$: $\{ e_n^{(k)} = \delta_{nk} \}_n$ is an orthonormal basis

Example: $H = L^2([-\pi, \pi])$: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \right\}_{n \geq 1}$ is an orthonormal basis

Theorem: (Bessel inequality)
Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset H$ be an orthonormal set. we don't know if it is an orthonormal basis
Then $\forall f \in H$:

$$\sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \leq \|f\|^2 .$$

Theorem: Let H be a separable Hilbert space, $\{\varphi_n\}_n$ an orthonormal basis.
Then $\forall f \in H$:

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n .$$

Moreover, $\forall f, g \in H$:

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \langle g, \varphi_n \rangle$$

in particular:

$$\|f\|^2 = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle^2 \qquad \text{(Parseval identity)}$$

The coefficients $\langle f, \varphi_n \rangle$ are called Fourier coefficients.

We recall that: (a definition)
Remark: $x_n \xrightarrow[n\to\infty]{} x$ in H $\overset{\text{def.}}{\iff}$ $L(x_n) \xrightarrow[n\to\infty]{} L(x)$ $\forall L \in H^*$
$\overset{\text{Riesz}}{\iff}$ $\langle x_n, y \rangle \xrightarrow[n\to\infty]{} \langle x, y \rangle$ $\forall y \in H$

Corollary: Let H be a separable space and let $\{\varphi_n\}_n$ be an orthonormal basis. Then:

- $\varphi_n \xrightharpoonup[n\to\infty]{} 0$
- $\varphi_n \not\xrightarrow[n\to\infty]{} 0$

proof.
$\forall f \in H$, Parseval identity implies that:

$\|f\|^2 = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle^2$ → $\|f\|^2 \in \mathbb{R}$ and so $\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle^2$ converges (as a sequence of real numbers) in particular:

$\Longrightarrow \langle f, \varphi_n \rangle \xrightarrow[n\to\infty]{} 0 \quad \forall f \in H$

$\overset{\text{def.}}{\iff} \varphi_n \rightharpoonup 0$

On the other hand: $\|\varphi_n\| = 1 \quad \forall n \in \mathbb{N}$.
Thus $\varphi_n \not\to 0$. ■

Theorem: (Riesz - Fisher)
Let H be a separable Hilbert space. Then H is isomorphic and isometric to ℓ^2.

in fact, there exist a mapping:

$$\mathcal{F} : H \to \ell^2$$
$$\mathcal{F}(f) = \langle f, \varphi_n \rangle = x^{(n)}$$

where $\{\varphi_n\}_n$ is an orthonormal basis.

Prop. H Hilbert space $\overset{\text{parallelogram law}}{\Longrightarrow}$ H is uniformly convex
$\Longrightarrow$ H is reflexive (Millman-Pettis)

(we just repeat other reflexive spaces examples (the following are **not** Hilbert spaces))
Other reflexive spaces are $L^p(\Omega)$ $1 < p < \infty$ ($L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive).

LINEAR AND BOUNDED OPERATORS ON HILBERT SPACES

Let H be a Hilbert space.

Prop. If $T \in \mathcal{L}(H)$. Then: $\|T\|_{\mathcal{L}} = \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle|$.

On H we have also another formula for the operator norm

$$\|T\|_{\mathcal{L}} = \sup_{\|x\|\le 1} \|T(x)\|_Y = \sup_{\|x\|=1} \|T(x)\|_Y = \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}$$

proof.

By Cauchy-Schwartz:

$$|\langle T(x), y\rangle| \le \|T(x)\| \cdot \|y\|$$

$$\Longrightarrow \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle| \le \|T(x)\|$$

$$\Longrightarrow \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle| \le \sup_{\|x\|=1} \|T(x)\|$$

$$\Longrightarrow \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle| \le \|T\|_{\mathcal{L}} \qquad (*)$$

For $y = \frac{T(x)}{\|T(x)\|}$ (such that $T(x) \neq 0$):

$$\langle T(x), y\rangle = \langle T(x), \frac{T(x)}{\|T(x)\|}\rangle = \|T(x)\|$$

therefore in general we have the inequality $(*)$, but on a special element the equality holds $\Longrightarrow$ the supremum is realized

$$\Longrightarrow \sup_{\|x\|=\|y\|=1} |\langle T(x), y\rangle| = \|T\|_{\mathcal{L}}. \qquad \blacksquare$$

SYMMETRIC OPERATORS

Def. We say that $T \in \mathcal{L}(H)$ is symmetric if:

$$\langle T(x), y\rangle = \langle x, T(y)\rangle \qquad \forall x, y \in H.$$

Prop. Let $T \in \mathcal{L}(H)$ be a symmetric operator. Then:

$$\|T\|_{\mathcal{L}} = \sup_{\|x\|=1} |\langle T(x), x\rangle|.$$

proof.

- Let $\alpha := \sup_{\|x\|=1} |\langle T(x), x\rangle|$.

$|\langle T(x), x\rangle| \le \|T(x)\| \cdot \|x\|$

$\alpha =: \sup_{\|x\|=1} |\langle T(x), x\rangle| \le \sup_{\|x\|=1} \|T(x)\| := \|T\|_{\mathcal{L}}$

$$\|T\|_{\mathcal{L}} = \sup_{\|x\|=1} \|T(x)\| \ge \sup_{\|x\|=1} \underbrace{|\langle T(x), x\rangle|}_{\le \|T(x)\| \cdot \|x\|} = \alpha$$

- Observe that:

$$\begin{aligned} 4\langle T(x), y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\ &\le \alpha(\|x+y\|^2 + \|x-y\|^2) \\ &\quad (\text{since } \langle T(\xi), \xi\rangle \le \alpha\|\xi\|^2 \quad \forall \xi \in H) \\ &= 2\alpha(\|x\|^2 + \|y\|^2) \quad \text{by parallelogram law.} \end{aligned}$$

$\alpha = \sup_{\|x\|=1} |\langle T(x), x\rangle|$

$\alpha \ge \langle T(\underbrace{\tfrac{\xi}{\|\xi\|}}), \underbrace{\tfrac{\xi}{\|\xi\|}}\rangle$ element of $\|\cdot\|=1$

$\frac{1}{\|\xi\|^2}\langle T(\xi), \xi\rangle$

Take $y = \frac{\|x\|}{\|T(x)\|} T(x)$ $(T(x) \neq 0)$

$$4\underbrace{\langle T(x), T(x)\rangle}_{\|T(x)\|^2} \frac{\|x\|}{\|T(x)\|} \le 2\alpha\left(\|x\|^2 + \frac{\|x\|^2}{\|T(x)\|^2}\|T(x)\|^2\right)$$

$$\Longrightarrow 4\|T(x)\| \cdot \|x\| \le 4\alpha\|x\|^2$$

$$\Longrightarrow \|T(x)\| \le \alpha\|x\|$$

$\Longrightarrow \underbrace{\sup_{\|x\|=1} \|T(x)\|}_{\|T\|_{\mathcal{L}}} \leq \alpha$ ■

EIGENVALUES AND EIGENVECTORS

Let H be a separable Hilbert space.

Def. $\lambda \in \mathbb{R}$ is an eigenvalue for $T \in \mathcal{L}(H)$ if there exists $v \in H \setminus \{0\}$ such that:

$$T(v) = \lambda v.$$

v is called eigenvector associated with λ:

$$V_\lambda := \{v \in H : T(v) = \lambda v\}$$

is called eigenspace associated with λ.

Remark: (i) V_λ is closed

(ii) $|\lambda| \leq \|T\|_{\mathcal{L}}$ (λ any eigenvalue of T)

Prop. If $T \in \mathcal{L}(H)$ is symmetric, then eigenvectors associated with distinct eigenvalues are orthogonal. The eigenvalues are at most countable.

Prop. Let H be a separable Hilbert space. If $K \in K(X,Y)$ (compact operator) is symmetric, then: $\|K\|_{\mathcal{L}}$ or $-\|K\|_{\mathcal{L}}$ is an eigenvalue of K.

Prop. Let H be a separable Hilbert space. Let $K \in K(X,Y)$. Suppose that $\lambda \neq 0$ is an eigenvalue of K. Then: $\dim(V_\lambda) < \infty$.

proof.

Suppose by contradiction that $\dim(V_\lambda) = \infty$.
Let $\{v_n\}_n$ be an orthonormal basis of V_λ.
Hence:

$$v_n \rightharpoonup 0, \quad \|v_n\| = 1 \quad (\text{so } v_n \not\to 0).$$

On the other hand:

$$K(v_n) = \lambda v_n \quad \forall n \in \mathbb{N}.$$

K compact $\Rightarrow K(v_n) \longrightarrow 0$ (a compact operator is weak-strong continuous and $v_n \rightharpoonup 0$)
$\Rightarrow v_n \longrightarrow 0$, which is a contradiction. ■

SPECTRAL THEOREM

Let H be a separable Hilbert space.

Def. Let $T \in \mathcal{L}(H)$ be a symmetric operator.

$$\sigma(T) := \{\lambda \in \mathbb{R} : \lambda \text{ eigenvalue of } T\}$$

is called spectrum of T.

$$R(T) = \sigma(T)^c$$

is called resolvent.

Theorem: (Spectral theorem)
Let $K \in K(H)$ be a symmetric operator. Then $\sigma(T)$ is either finite or it is a sequence $\{\lambda_n\}_n : \lambda_n \longrightarrow 0$. (convergence in $\mathbb{R}$, $\lambda_n \in \mathbb{R}$ $\forall n \in \mathbb{N}$)
Moreover, the eigenvectors can be chosen in such way that they are an orthonormal basis of H.

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Theorem: (Fredholm alternative)

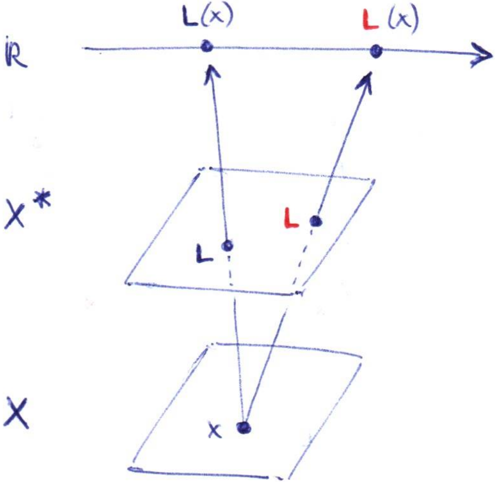
Let H be a separable Hilbert space, $T \in K(H)$ symmetric, $\mu \neq 0$. $(\mu \in \mathbb{R})$

Then either:

(a) $\forall y \in H \quad \exists ! \, x \in H$ such that:

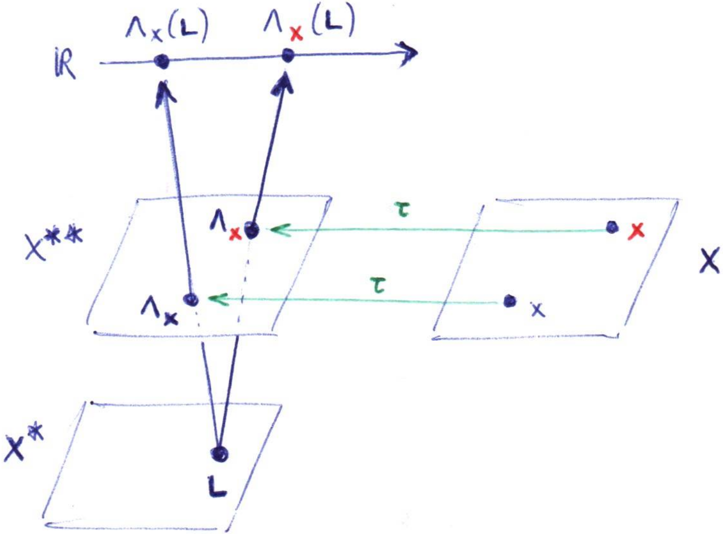
$$\mu x - T(x) = y$$

(b) $\lambda = \mu$ is an eigenvalue of T



$L : X \to \mathbb{R}$

$L \in X^*$



$\Lambda_x : X^* \to \mathbb{R}$

$\Lambda_x \in X^{**}$

$\tau : X \to X^{**}$

What is the path?
If we start from the set where functions are defined ($\mathbb{R}$) we need **5** levels:

- $\rightarrow$ $\mathbb{R}$ set of real numbers
- $\rightarrow$ $f : \mathbb{R} \to \mathbb{R}$ function between real numbers
- $\rightarrow$ $L^p(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : \dots \}$ set of all functions between real numbers (s.t. ..)
- $\rightarrow$ $T : L^p \to L^p$ operator between L^p spaces
- $\rightarrow$ $\mathcal{L}(L^p) = \{ T : L^p \to L^p : \dots \}$ set of all operators between L^p spaces

A (generic) transformation **L** from X to Y is an **ISOMETRY** if :

$$\forall x \in X \quad : \quad \|x\|_X = \|L(x)\|_Y$$

(the "length" of the element is preserved under the transformation)

Two normed spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are called **ISOMORPHIC** if there exist a linear bijection between the two which is continuous, i.e. if $\exists T \in \mathcal{L}(X,Y)$ s.t. $T^{-1} \in \mathcal{L}(X,Y)$.