A SHORT REVIEW OF METRIC SPACES

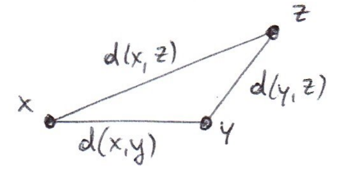
set + metric on the set (there is no structure like in vector spaces), it's just about sets)

(Bose, "Introduction to functional analysis")

any unstructured set

Def. Let $X \neq \emptyset$. A distance (metric) on X is a function $d: X \times X \to [0, +\infty)$ satisfying the following properties:

1. $d(x,y) \geq 0 \quad \forall x,y \in X$ and $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x) \quad \forall x,y \in X$ (symmetry)
3. Triangular inequality:
$$\forall x,y,z \in X : \quad d(x,z) \leq d(x,y) + d(y,z)$$



If d is a distance on X then (X, d) is a metric space.

They don't have linear or algebraic structure but, due to the presence of a distance function, they're a good environment to talk about closeness and approximation (hence: limits, sequences, converging sequences, ...)

a sequence x_n converges to x if the distance between x_n and x becomes smaller and smaller as $n \to \infty$

Example: In $\mathbb{R}^N$ $(N \geq 1)$ the function
$$d_E(x,y) = \left(\sum_{j=1}^{N} |x_j - y_j|^2\right)^{1/2}$$
is a distance (Euclidean distance).

To prove the triangular inequality we need the **CAUCHY-SCHWARTZ** ineq.:
$$\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

More generally, the functions:
$$d_p(x,y) = \left(\sum_{j=1}^{N} |x_j - y_j|^p\right)^{1/p} \qquad (p \geq 1)$$

MINKOWSKI inequality for triangular ineq.

$$d_\infty(x,y) = \max_{j=1,\dots,N} |x_j - y_j|$$
are all distances in $\mathbb{R}^N$.

Example: In the space $X = C^0([a,b])$ the function
$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$
is a distance.

continuous function from $[a,b]$ to $\mathbb{R}$

$|x(t) - y(t)|$ is continuous since 1. the sum of a finite number of continuous functions is continuous ($-y(\cdot)$ continuous), 2. the absolute value of a continuous function is continuous

$x, y : [a,b] \to \mathbb{R}$ continuous functions $\Rightarrow$ by the Weierstrass theorem, since $|x(t) - y(t)|$ is a continuous function, $|x(t) - y(t)|$ has a maximum (on $[a,b]$)

WEIERSTRASS: let $f \in C^0([a,b])$
$$\Rightarrow \begin{cases} \exists M = \max f(x) \\ \exists m = \min f(x) \end{cases} \quad x \in [a,b]$$

Example: For a fixed $p \geq 1$ we define:
$$\ell^p = \left\{ x = (x_n)_n \subseteq \mathbb{R} : \sum_{j=1}^{+\infty} |x_j|^p < +\infty \right\}$$

Also, we define:
$$\ell^\infty = \left\{ x = (x_n)_n \subseteq \mathbb{R} : \sup_{n \in \mathbb{N}} |x_n| < +\infty \right\}$$
= set of sequences in $\mathbb{R}$ that are bounded

For instance, the sequence:
$$x = (x_n)_n = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} \quad \text{(armonic series)}$$
since $\sum_{n=1}^{+\infty} \frac{1}{n^p} < +\infty \iff p > 1$
$$\Rightarrow (x_n)_n \in \ell^p \quad \forall\, 1 < p \leq +\infty \qquad \left((x_n)_n \notin \ell^1\right)$$

$\frac{1}{n} \leq 1 \quad \forall n$
$\Rightarrow \sup_{n \in \mathbb{N}} \left|\frac{1}{n}\right| < +\infty$

Question: for which $\alpha > 0$ the sequence:

$$\mathbf{x} = (x_n)_n = \frac{e^{1/n} - 1}{n^\alpha (\sin^2(n))} \in \ell^2 \quad ?$$

That is: $\sum_{j=1}^{+\infty} |x_j|^2 = \sum_{j=1}^{+\infty} \left| \frac{e^{1/j} - 1}{j^\alpha (\sin^2(j))} \right|^2 < +\infty$

The functions:

$$d_p(x,y) = d_p((x_n)_n, (y_n)_n) = \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{1/p}$$

$$d_\infty(x,y) = d_\infty((x_n)_n, (y_n)_n) = \sup \{ |x_n - y_n| \; n \in \mathbb{N} \}$$

are distances in (respectively) ℓ^p and ℓ^∞.
We can see these spaces as "$\mathbb{R}^N$ with $N = +\infty$".

Example: if $X \neq \emptyset$ is any set, the function:

$$d_D(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a distance. (discrete distance on X)

Since we have the notion of distance we can talk about convergence:

SEQUENCES IN METRIC SPACES

If (X,d) is a metric space, a sequence in X is a map:

$$x : \mathbb{N} \longrightarrow X$$

We can use the notation: $x_n = x(n)$ and $x \equiv (x_n)_n$.

We say that a sequence $(x_n)_n \subseteq X$ converges to $x_0 \in X$ if:

$$\underbrace{\lim_{n\to\infty} d(x_n, x_0) = 0}_{\text{limit of a sequence of real numbers}}$$

($d(x_n, x_0) \in \mathbb{R}$ sequence of real numbers)

we know the convergence of a sequence of real numbers: $\forall \varepsilon > 0 \; \exists \bar{n} \in \mathbb{N}$ s.t. $\forall n \geq \bar{n}$ $|a_n - a| < \varepsilon$ (*)
($:\Rightarrow \lim_{n\to\infty} a_n = a$)

In this case we write:

$$\lim_{n\to\infty} x_n \overset{d}{=} x_0 \quad \text{or} \quad x_n \xrightarrow[n\to\infty]{d} x_0 .$$

The role of the distance is crucial: on the same set we can define different distances which lead to different result (convergent / non-conv.)

Example: in $(\mathbb{R}^N, d_p)$ we have:

$$x_n \xrightarrow[n\to\infty]{d_p} x_0 \iff x_n^{(i)} \xrightarrow[n\to\infty]{} x_0^{(i)} \quad \forall i = 1, \dots, N$$

(where $x_n^{(i)}, x_0^{(i)} \in \mathbb{R}$)

Notice that this holds $\forall \; 1 \leq p \leq \infty$.
$\Longrightarrow$ In $\mathbb{R}^N$ we can choose whatever d_p: a sequence converges with all the p or a sequence does **not** converge $\forall p$. $\longleftrightarrow$ The convergence of the sequence is equivalent to the convergence of the components in $\mathbb{R}$ ($\forall p$) ! (*) !

Example: in $(C^0([a,b]), d)$ we have:

$$x_n \xrightarrow[n\to\infty]{d} x_0 \iff \text{the sequence } (x_n)_n \text{ uniformly converges to } x_0$$

$$\iff \lim_{n\to\infty} \left(\max_{t\in[a,b]} |x_n(t) - x_0(t)| \right) = 0$$

$\left(d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)| \right)$

Example: In (X, d_D) we have:

$$x_n \xrightarrow[n\to\infty]{d_D} x_0 \iff d_D(x_n, x_0) \xrightarrow[n\to\infty]{} 0$$

$$\iff \exists \bar{n} \in \mathbb{N} \text{ s.t. } x_n = x_0 \quad \forall n \geq \bar{n}$$

If we take the discrete distance, the only converging sequences are the ones that are constant from a point ($\bar{n}$).

Theorem: Let (X,d) be a metric space. If a sequence $(x_n)_n$ has a limit then this limit is unique.

(Idea: if $x_0, \tilde{x}_0$ are two limits of $(x_n)_n$, then:

$$\underbrace{d(x_0, \tilde{x}_0)}_{\geq 0} \overset{\text{triangular inequality}}{\leq} \underbrace{d(x_0, x_n)}_{\to 0} + \underbrace{d(x_n, \tilde{x}_0)}_{\to 0}$$

since as $n \to \infty$ the right side goes to 0 we have $d(x_0, \tilde{x}_0) = 0 \implies x_0 = \tilde{x}_0$ (1.))

A metric (distance) induces topology: with a distance we can define open/closed sets, compact sets, ..

TOPOLOGY IN METRIC SPACES

Let (X,d) be a metric space. If $x_0 \in X$ and $r > 0$, we define:

$$B(x_0, r) = \{ y \in X : d(x,y) < r \}$$

(open ball with center x_0 and radius r)

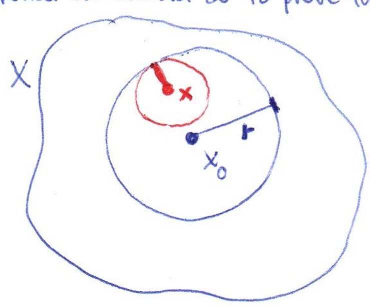
Def. Let $A \subseteq X$. We say that:

- A is open if $\forall x \in A \; \exists r = r_x > 0$ s.t. $B(x,r) \subseteq A$
- A is closed if $X \setminus A$ is open

Example: If (X,d) is a metric space:

- $B(x_0, r)$ is open ($\forall x_0 \in X$, $\forall r > 0$)
- $\{ y \in X : d(x_0, y) \leq r \}$ is closed
  (closed ball centered in x_0 with radius r)

What we should do to prove that the ball is open?



If we want to prove that the ball is open we have to prove that any point $x \in B(x_0, r)$ is such that $\exists r_x$ s.t. $B(x, r_x)$ is entirely contained in $B(x_0, r)$.

As a convenient r_x: $r_x = r - d(x, x_0)$

$\implies B(x, r - d(x,x_0)) \subseteq B(x_0, r)$

$\implies$ the ball is open
($\forall x$ we can find a radius s.t. we create a ball centered in x and with that radius entirely contained in $B(x_0, r)$)

Let $A \subseteq X$, $x_0 \in X$:

- x_0 is an interior point of A if $\exists r > 0 : B(x_0, r) \subseteq A$
  ($\implies$ a set is open if and only if all of its point are interior points)
- x_0 is an exterior point of A if $\exists r > 0 : B(x_0, r) \subseteq X \setminus A$
- x_0 is a boundary point of A if x_0 is not an interior nor exterior point, that is: $\forall r > 0 : B(x_0, r) \cap A \neq \emptyset$, $B(x_0, r) \cap (X \setminus A) \neq \emptyset$

We define:

- interior of A = $\text{int}(A) = \{ x \in X : x \text{ is an interior point of } A \}$
- exterior of A = $\text{ext}(A) = \{ x \in X : x \text{ is an exterior point of } A \}$
- boundary of A = $\partial A = \{ x \in X : x \text{ is a boundary point of } A \}$

We also define:

- closure of $A = \bar{A} = cl(A) = A \cup \partial A$

Remark: if $A \subseteq X$ is any set, then:

$$\mathrm{int}(A) \subseteq A \subseteq \bar{A}$$

Moreover: $\mathrm{int}(A)$ is open and $\bar{A}$ is closed

The $\mathrm{int}(A)$ is the biggest open set which is contained in A and $\bar{A}$ is the smallest closed set which contains A.

Properties:
- A is open $\iff A = \mathrm{int}(A)$
- A is closed $\iff A = \bar{A}$
  $\iff \partial A \subseteq A$

Def. Let (X, d) be a metric space.* X is separable if $\exists$ a countable set $E \subseteq X$ such that $\bar{E} = X$.

set which has the same cardinality as the set of natural numbers

Example: $(\mathbb{R}^N, d_p)$ is separable. $(E = \mathbb{Q}^N)$

* metric space $\rightarrow$ distance $\rightarrow$ open/closed balls $\rightarrow$ def. of closure
$\Longrightarrow$ it's necessary a metric space, it's not sufficient just a set

Theorem: Let (X,d) be a metric space, $A \subseteq X$ and $x_0 \in X$.
Then, the following assertions are equivalent:

(i) $x_0 \in \bar{A}$ $(= A \cup \partial A)$

(ii) $\exists (x_n)_n \subseteq A$ s.t. $x_n \xrightarrow[n\to\infty]{d} x_0$

} the points in the closure are the points which can be approximated by elements of the set A (we can get as close as we want to points of $\bar{A}$ through points of A)

Remark: Let (X,d) be a separable metric space and let E be a countable set s.t. $E \subseteq X$, $\bar{E} = X$. Then, $\forall x \in X$ $(= \bar{E})$ there exists a sequence $(x_n)_n \subseteq E$ s.t.

$$x_n \xrightarrow[n\to\infty]{d} x.$$

(by the previous theorem)

Hence: we can approximate any point in X with a sequence of points in E.
("approximation" is intended in the sense of limit).

CONTINUOUS FUNCTIONS BETWEEN METRIC SPACES

Let (X,d) and (Y,ρ) be metric spaces, let $f: X \to Y$.

Def. We say that f is continuous if $\forall (x_n)_n \subseteq X$ with $x_n \xrightarrow[n\to\infty]{d} x_0$ then:

$$f(x_n) \xrightarrow[n\to\infty]{\rho} f(x_0)$$

In other words, f is continuous if:

$$\left(\lim_{n\to\infty} x_n \overset{d}{=} x_0\right) \Longrightarrow \left(\lim_{n\to\infty} f(x_n) \overset{\rho}{=} f(x_0) \left(= f\left(\lim_{n\to\infty} x_n\right)\right)\right)$$

– In some sense it exchanges the evaluation of the limit and $f(\cdot)$

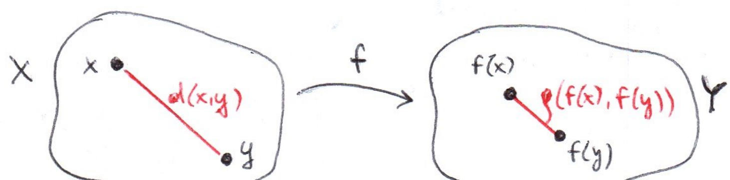
As a particular class of continuous functions:

Def. Let (X,d) and (Y,ρ) be metric spaces, let $f: X \to Y$. We say that f is Lipschitz (-continuous) if $\exists L > 0$ s.t.:

$$\rho(f(x), f(y)) \leq L \cdot d(x,y) \qquad \forall x,y \in X$$

– we can control the distance between $f(x)$ and $f(y)$ in the metric space (Y,ρ) with a constant times the distance between x and y in (X,d).

In particular, if $L < 1$ we say that f is a contraction.



notice that $f(x)$ and $f(y)$ are closer than x and y by a factor that is always the same $(\forall x,y \in X)$

Remark: If $f: (X,d) \to (Y,\rho)$ is Lipschitz then f is continuous. (!)
In fact, if $(x_n)_n \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d} x_0$, then:

$$(0 \leq)\ \underbrace{\rho(f(x_n), f(x_0))}_{\downarrow\, 0} \leq L \cdot \underbrace{d(x_n, x_0)}_{\downarrow\, 0} \qquad \forall n \in \mathbb{N}$$

$f(x_n) \xrightarrow{\rho} f(x_0) \Longleftarrow 0 \Longleftarrow 0$ because $x_n \xrightarrow{d} x_0$

Ex. 1 Let (X,d) be a metric space and let $z_0 \in X$ be fixed. Show that the function $f: X \to \mathbb{R}$ defined by: $(f: (X,d) \to (\mathbb{R}, d_{euclidean}))$

$$f(x) = d(x, z_0)$$

is continuous.

Solution: Let $(x_n)_n \subseteq X$ be a sequence s.t. $x_n \xrightarrow[n\to\infty]{d} x_0 \in X$.
Then, by triangular's inequality we have:

$$d(x_n, z_0) \leq d(x_n, x_0) + d(x_0, z_0)$$

$$\Longrightarrow \boxed{d(x_n, z_0) - d(x_0, z_0) \leq d(x_n, x_0)}$$

Also: $$d(x_0, z_0) \leq d(x_0, x_n) + d(x_n, z_0)$$

$$\Longrightarrow \boxed{d(x_0, z_0) - d(x_n, z_0) \leq d(x_0, x_n)}$$

Hence : $|\underbrace{d(x_n,z_0)}_{f(x_n)} - \underbrace{d(x_0,z_0)}_{f(x_0)}| \leq d(x_n,x_0)$

$\Longrightarrow \quad |f(x_n) - f(x_0)| \leq d(x_n,x_0) \quad \forall n \in \mathbb{N}.$

By letting $n \to \infty$ we get : (the right-hand side goes to zero and so the left-hand side)

$$\underbrace{\lim_{n\to\infty} |f(x_n) - f(x_0)| = 0}_{\lim_{n\to\infty} d_e(f(x_n),f(x_0))=0} \quad \underset{\text{"which means"}}{\Longleftrightarrow} \quad \lim_{n\to\infty} f(x_n) \overset{d_e}{=} f(x_0)$$

Remark: By re-run the same computation with $x_n \rightsquigarrow x$ and $x_0 \rightsquigarrow y$ we get :

$$|f(x) - f(y)| \leq d(x,y) \quad \forall x,y \in X$$

Hence, f is lipschitz with $\mathbf{L} = \mathbf{1}$.

Notice that here we used $|f(x)-f(y)|$, but generally is $\rho(f(x),f(y))$. This time is like this because $(Y,\rho) = (\mathbb{R}, d_e)$.

Another approach we could use to solve **Ex. 1** : show that the function is lipschitz and so, in particular, is continuous

Ex. 2 Let $\mathbf{V}$ be a (real) vector space and let $\mathbf{d}$ be a distance on $\mathbf{V}$. (so that (V,d) is a metric space).

Show that : $\overline{B(0,r)} \subseteq \hat{B}(0,r) := \{x \in V : d(x,0) \leq r\} \quad \forall r \geq 0$

Solution: Let $x_0 \in \overline{B(0,r)}$.

Then $\exists (x_n)_n \subseteq B(0,r)$ s.t. $x_n \xrightarrow[n\to\infty]{d} x_0$. (Theorem)

Thus, since $x_n \in B(0,r) \ \forall n \in \mathbb{N}$:

$$d(0,x_n) < r \quad \forall n \in \mathbb{N}$$

from this, by letting $n \to \infty$ and using Ex. 1 (with $z_0 = 0$) :

$$d(0,x_0) = \lim_{n\to\infty} d(0,x_n) \leq r \quad \Longrightarrow \quad x_0 \in \hat{B}(0,r).$$

we can do this because in Ex. 1 we proved that the distance function with one point fixed is continuous

By the property of limit : (strict inequalities change to weak inequalities over limits)

$d(0,x_n) < r \ \forall n \Rightarrow \lim_{n\to\infty} d(0,x_n) \leq r$

We defined :

- $B(x_0,r)$ open
- $\hat{B}(x_0,r) = \{y \in X : d(x_0,y) \leq r\}$ closed.

Is it (always) true that :

$\overline{B(x_0,r)} = \hat{B}(x_0,r)$?

No, it depends on the distance. However, we can give a condition s.t. $\overline{B(x_0,r)} = \hat{B}(x_0,r)$

Generally : the closure of the open ball is $\neq$ from the closed ball

Ex. 2 (continuation) Show that in general : $\overline{B(0,r)} \subsetneq \hat{B}(0,r)$

Solution : We consider in $V = \mathbb{R}$ the discrete distance :

$$d_D(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

In general when we look for patological cases (counterproofs etc.) we go with discrete distance

Then, choosing $r = 1$, we have:

- $B(0,r) = \{x : d(x,0) < 1\} = \{0\} \Longrightarrow \overline{B(r,0)} = \{0\}$
- $\hat{B}(0,r) = \{x : d(x,0) \leq 1\} = \mathbb{R}$

Hence : $\overline{B(0,1)} = \{0\} \subsetneq \mathbb{R} = \hat{B}(0,1)$

Ex. 2 (continuation) Assume that:

$\exists$ some conditions s.t. $\overline{B(0,r)} = \hat{B}(0,r)$

(H1) $\forall$ fixed $x \in V$, the map:

$$m_x : \mathbb{R} \to V \quad , \quad m_x(\lambda) = \lambda x \in V$$

that's because V is a vector space (an element of V multiplied by a scalar is still in V)

is continuous, that is:

$$((\lambda_n)_n \subseteq \mathbb{R}, \lambda_n \to \lambda) \Longrightarrow \underbrace{\lambda_n x \xrightarrow[n\to\infty]{d} \lambda_0 x}_{m_x(\lambda_n) \xrightarrow[n\to\infty]{d} m_x(\lambda_0)}$$

(H2) $\forall$ fixed $x \in V \setminus \{0\}$, the map:

$$f_x : \mathbb{R} \to \mathbb{R} , \quad f_x(\lambda) := d(\lambda x, 0)$$

is injective (1-1) when restricted to $[0, +\infty)$

this is because if we take, for instance, $(\mathbb{R}, d_E)$ then $d(\lambda x, 0) = d(-\lambda x, 0)$

Show that, in this case, : $\overline{B(0,r)} = \hat{B}(0,r)$

! Solution: Since we know that $\overline{B(0,r)} \subseteq \hat{B}(0,r)$, it suffices to prove that if $x_0 \in \hat{B}(0,r)$ then $x_0 \in \overline{B(0,r)}$. Actually the only interesting case is when $\underline{d(x_0,0) = r}$.

if $d(x_0,0) < r$ then of course $x_0 \in \overline{B(0,r)}$ since it's even $x_0 \in B(0,r)$

we approximate x_0 with a sequence of points $(x_n)_n$ (for this we need the continuity assumed in (H1))

(1) We define:

$$x_n := \frac{n}{n+1} \cdot x_0 \in V$$

$x_0 \in V$, $\lambda_n = \frac{n}{n+1} \in \mathbb{R}$ $\Longrightarrow x_n \in V$ (again, since V is a vector space)

Since $\lambda_n \xrightarrow[n\to\infty]{} 1$, by (H1) we have :

$$x_n = \lambda_n x_0 = m_{x_0}(\lambda_n) \xrightarrow[n\to\infty]{d} m_{x_0}(1) = x_0$$

so we have a sequence of points which converges to x_0 in the distance d. If we're able to prove that the sequence of points is all contained in the ball then we can say that x_0 is in the closure (since it's the limit of a sequence of points in the ball)

Hence, if we prove that $(x_n)_n \subseteq B(0,r)$ then we can conclude that :

$$x_0 = \lim_{n\to\infty} x_n \in \overline{B(0,r)}$$

we prove that the whole sequence $(x_n)_n$ is in the ball (and so we get for free that x_0 is in the closure) (for this we need (H2))

(2) To prove that $(x_n)_n \subseteq B(0,r)$ we use (H2).
Since f_{x_0} is 1-1 and continuous (since m_{x_0} and $d(\cdot,0)$ are continuous) then f_{x_0} is monotone(*) in particular, since:

$$\underbrace{f_{x_0}(0) = d(0 \cdot x_0, 0)}_{f_{x_0}(\lambda) = d(\lambda x_0, 0)} = 0 < \underbrace{r = d(x_0,0)}_{\text{that's the case we're considering}} = f_{x_0}(1)$$

characterization of closure that we gave in the previous lesson (Theorem)

then f_{x_0} is increasing. As a consequence we have:

$$d(x_n,0) \overset{\text{def.}}{=} d\left(\frac{n}{n+1} x_0, 0\right) \overset{\text{def.}}{=} f_{x_0}\left(\frac{n}{n+1}\right) < f_{x_0}(1) \overset{\text{def.}}{=} d(x_0,0) = r$$

and thus $x_n \in B(0,r)$ $\forall n \in \mathbb{N}$. (since $d(x_n,0) < r$)

and somehow we can say that f_{x_0} is the composition of m_{x_0} and $d(\cdot,0)$

(*) $\left. \begin{array}{l} f : I \to \mathbb{R} \\ \text{continuous} \\ \text{injective} \end{array} \right\} \Longrightarrow$ monotone (increasing/decreasing)

Remark: Both assumptions (H1)-(H2) cannot be dropped. In fact, if we consider $(\mathbb{R}, d_0)$ where:

$$d_0(x,y) = \min\{|x-y|, 1\}$$

we can check that (H1) is satisfied while (H2) is **not**. this is because the distance at some point becomes constant (becomes 1 if $|x-y|$ is too large) $\longrightarrow$ $f_x(\lambda)$ is **not** injective

In this case we have:

- $\overline{B(0,1)} = \{x \in \mathbb{R} : |x| \leq 1\}$
- $\hat{B}(0,1) = \{x \in \mathbb{R} : d_0(x,0) \leq 1\} = \mathbb{R}$

$\Longrightarrow \overline{B(0,1)} \subsetneq \hat{B}(0,1)$

CAUCHY SEQUENCES

Def. Let (X,d) be a metric space and let $(x_n)_n \subseteq X$. We say that $(x_n)_n$ is a cauchy sequence if:

$$\forall \varepsilon > 0 \quad \exists \bar{n} = \bar{n}_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_m) < \varepsilon \qquad \forall n, m \in \mathbb{N},\ n, m \geq \bar{n}_\varepsilon$$

Remark: If $(x_n)_n \subseteq X$ converges to some $x_0 \in X$ then $(x_n)_n$ is a Cauchy sequence.
The converse, in general, is **not** true.
For example, consider:

$$x_n = \left(1 + \frac{1}{n}\right)^n \in \mathbb{Q} \qquad (\mathbb{Q}, d_e)$$

Then, $(x_n)_n$ is a Cauchy sequence in $\mathbb{Q}$ but it's **not** convergent in $\mathbb{Q}$ (it does not have limit in $\mathbb{Q}$) $(e = \lim_{n\to\infty} x_n \notin \mathbb{Q})$

Def. We say that (X,d) is complete if every Cauchy sequence in X has a limit in X.
(**E.g.** $\mathbb{Q}$ is not complete because it has a Cauchy sequence that does **not** have a limit in $\mathbb{Q}$)

Examples: The spaces $(\mathbb{R}^N, d_e)$ are complete.
The space $(\mathbb{Q}, d_e)$ is **not** complete.

If we have a non-complete metric space we can always fill the holes (in some sense): we can always construct a bigger metric space which is **complete** and s.t. the old metric space is **dense** in the new one.

Theorem: Let (X,d) be a **non**-complete metric space. Then, there exists a complete metric space $(\tilde{X}, \tilde{d})$ such that:

- $X \subseteq \tilde{X}$
- $\tilde{d}(x,y) = d(x,y) \quad \forall x,y \in X$ (the distances coincide on X)
- $\boxed{\overline{X} = \tilde{X} \quad (X \text{ is dense in } \tilde{X})}$ $\longrightarrow$ this condition leads to the fact that $(\tilde{X}, \tilde{d})$ is the **smallest** complete metric space whic contains X (this condition says "we are just adding the points that are missing in **X**")

Example: If we start from $(\mathbb{Q}, d_e)$ then $(\tilde{X}, \tilde{d}) = (\mathbb{R}, d_e)$.

MEASURES AND σ-ALGEBRAS

Ex. 1 In $\mathbb{R}$, prove that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{I})$ where $\mathcal{I} = \{(a,b) \subseteq \mathbb{R} : a < b\}$.

Solution: We recall that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O})$ where:

$$\mathcal{O} = \{A \subseteq \mathbb{R} : A \text{ is open}\}.$$

Since $\mathcal{I} \subseteq \mathcal{O}$ we clearly have: $\sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathbb{R})$.
To prove the reverse inclusion, let $A \subseteq \mathbb{R}$ be open.
Then, $\forall x \in A \quad \exists (p_x, q_x) \subseteq A$ such that:

$$x \in (p_x, q_x) \quad \text{and} \quad \boxed{p_x, q_x \in \mathbb{Q}}$$

$\longrightarrow$ σ-algebras are closed under countable unions, not arbitrary ones. Since we can choose intervals whoose extremes are rational number then we can write **A** as union of intervals with extremes of rational number (which is a **countable** family of intervals)

As a consequence, A is the union of a countable family of open intervals. Thus $A \in \sigma(\mathcal{I})$ and hence

$$\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{I})$$

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Ex. 2 Prove that, in $\mathbb{R}$, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{I})$ where $\mathcal{I} = \{[a,b) \subseteq \mathbb{R} : a < b\}$.

Solution: We first prove that $\sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathbb{R})$.

We can write:

$$[a,b) = \bigcap_{n=1}^{+\infty} \left(a - \frac{1}{n}, b\right) \qquad (*)$$

proof of $(*)$:

if $x \in [a,b)$ then: $a - \frac{1}{n} < a \leq x < b \quad \forall n \in \mathbb{N}$

$\Longrightarrow x \in \left(a - \frac{1}{n}, b\right) \ \forall n \in \mathbb{N} \Longrightarrow x \in \bigcap_{n=1}^{+\infty} \left(a - \frac{1}{n}, b\right)$

Conversely, if $x \in \bigcap_{n=1}^{+\infty} \left(a - \frac{1}{n}, b\right)$ then:

$x > a - \frac{1}{n} \ \forall n \in \mathbb{N} \longrightarrow x \geq a \Longrightarrow x \in [a,b)$

Since $[a,b)$ is a countable intersection of open sets then

$$[a,b) \in \mathcal{B}(\mathbb{R}) \Longrightarrow \sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathbb{R})$$

$\Longrightarrow$ every element of the set $\mathcal{I}$ is a Borel set because it is the intersection of open sets (and open sets are the generators of $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$, being a σ-algebra, is closed under countable intersection)

To prove the opposite inclusion we use Ex. 1 and we show that $(a,b) \subseteq \sigma(\mathcal{I}) \ \forall a < b$.

Since we have:

$$(a,b) = \bigcup_{n=n_0}^{+\infty} \left[a + \frac{1}{n}, b\right)$$

where $n_0 \in \mathbb{N}$ is such that $a + \frac{1}{n} < b \ \forall n \geq n_0$, we see that (a,b) is a countable union of elements in $\mathcal{I}$, hence:

$$(a,b) \in \sigma(\mathcal{I}) \Longrightarrow \sigma(\mathcal{J}) \underset{\text{Ex. 1}}{=} \mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{I})$$

Ex. 1 Let $c \in \mathbb{R}$ and let:

$$r_c = \{(x,y) \in \mathbb{R}^2 : y = c\}$$

Show that $\lambda(r_c) = 0$.

Show that a line has zero Lebesgue measure

Solution: First of all: $r_c \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$ (r_c is closed). $\Rightarrow$ r_c is measurable

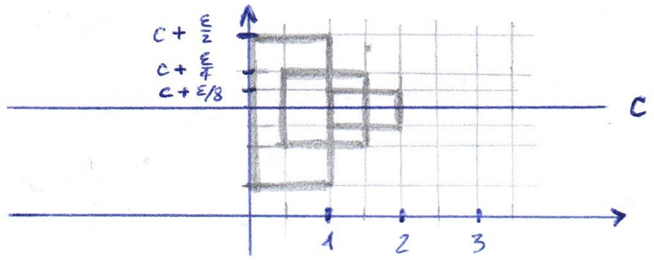
Moreover:

$$\lambda(r_c) = \inf \left\{ \sum_{n=1}^{+\infty} \lambda_0(I_n) : r_c \subseteq \bigcup_{n=1}^{+\infty} I_n, \ I_n \in \mathcal{I} \right\}$$

$\mathcal{I}$: class of all the open rectangles $(a,b) \times (c,d)$ (hence if $I_n = (a,b) \times (c,d)$ then $\lambda_0(I_n) = (b-a)\cdot(d-c)$)

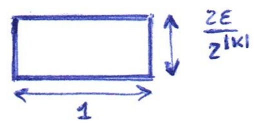
Let now $\varepsilon > 0$ be fixed and let:

$$I_k(\varepsilon) := \left(\frac{k-1}{2}, \frac{k+1}{2}\right) \times \left(c - \frac{\varepsilon}{2^{|k|}}, c + \frac{\varepsilon}{2^{|k|}}\right) \qquad k \in \mathbb{Z}$$



$k=1$: $(0,1) \times (c-\varepsilon/2, c+\varepsilon/2)$
$k=2$: $(1/2, 3/2) \times (c-\varepsilon/4, c+\varepsilon/4)$
$k=3$: $(1,2) \times (c-\varepsilon/8, c+\varepsilon/8)$
$\vdots$

Same thing for negative k ($k \in \mathbb{Z}$)
These rectangles have all the same length ($=1$) but the height $\downarrow$:



Using this open cover, we get:

$$\sum_{k\in\mathbb{Z}} \lambda_0(I_k(\varepsilon)) = \sum_{k\in\mathbb{Z}} 1 \cdot \frac{2\varepsilon}{2^{|k|}} = 2\varepsilon \sum_{k\in\mathbb{Z}} \frac{1}{2^{|k|}}$$

$$= 2\varepsilon \Big(\underbrace{1}_{k=0} + 2 \underbrace{\sum_{n=1}^{+\infty} \frac{1}{2^n}}_{=1}\Big)$$

$$= 6\varepsilon$$

Hence:

$$\lambda(r_c) = \inf \left\{ \sum_{n=1}^{+\infty} \lambda_0(I_n) : r_c \subseteq \bigcup_{n=1}^{+\infty} I_n \right\}$$

$$\leq \sum_{k\in\mathbb{Z}} \lambda_0(I_k(\varepsilon)) = 6\varepsilon$$

Due to the arbitrariness of $\varepsilon > 0$ we have that:

$$0 \leq \lambda(r_c) \leq 6\varepsilon \quad \forall \varepsilon > 0 \ \Rightarrow \ \lambda(r_c) = 0$$

Remark: In the same way we can prove that:

$$r'_c = \{(x,y) \in \mathbb{R}^2 : x = c\}$$

has zero Lebesgue measure $\forall c \in \mathbb{R}$.

In this case we have to choose a base which becomes smaller and smaller and a constant height

More generally, this is true of any line $y = mx + q$ $(m, q \in \mathbb{R})$

Note: the Lebesgue measure is invariant by isometry in the plane so we can always reconduct the situation to the case of horizontal line if we rotate the general $y = mx + q$

Ex. 2 Characterization of the measure

Let $(X, \mathcal{A})$ be a measure space and let $\mu : \mathcal{A} \to [0, +\infty]$ s.t.: (σ-algebra)

(1) $\mu(\emptyset) = 0$ $\quad \emptyset \in \sigma$-algebra $\forall \sigma$-algebra

(2) μ is finite additive, that is:

$$\forall E_1, \dots, E_n \subseteq \mathcal{A} \text{ s.t. } E_i \cap E_j = \emptyset \quad i \neq j :$$

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$$

(3) μ is continuous along increasing sequences:

$$\forall (E_n)_n \subseteq \mathcal{A} \text{ s.t. } E_i \subseteq E_{i+1} :$$

$$\mu\left(\bigcup_{n=1}^{+\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n)$$

Prove that μ is a measure on $(X, \mathcal{A})$.

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Solution: In view of (1) we only need to prove that μ is σ-additive on $(X, \mathcal{A})$.

Let then $(A_n)_n \subseteq \mathcal{A}$ be s.t. $A_i \cap A_j = \emptyset \quad \forall i,j \in \mathbb{N} \ (i \neq j)$.

For every $n \in \mathbb{N}$ we define:

$$E_n := \bigcup_{i=1}^{n} A_i$$

Clearly $(E_n)_n \subseteq \mathcal{A}$ and $E_n \subseteq E_{n+1} \ \forall n \in \mathbb{N}$.

Then, by (2) and (3) we have:

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \\
&\overset{(3)}{=} \lim_{n \to \infty} \mu(E_n) \\
&\overset{\text{def. of } E_n}{=} \lim_{n \to \infty} \mu\left(\bigcup_{k=1}^{n} A_k\right) \\
&\overset{(2)}{=} \lim_{n \to \infty} \sum_{k=1}^{n} A_k \\
&\overset{\text{def. of } \sum_{n=1}^{+\infty}}{=} \sum_{k=1}^{+\infty} A_k
\end{aligned}$$

Hence μ is σ-additive $\Longrightarrow$ μ is a measure

σ-additivity $\Longleftrightarrow$ $\begin{cases} \text{finite additivity} \\ \text{continuity along increasing sequences} \end{cases}$

THE CANTOR SET

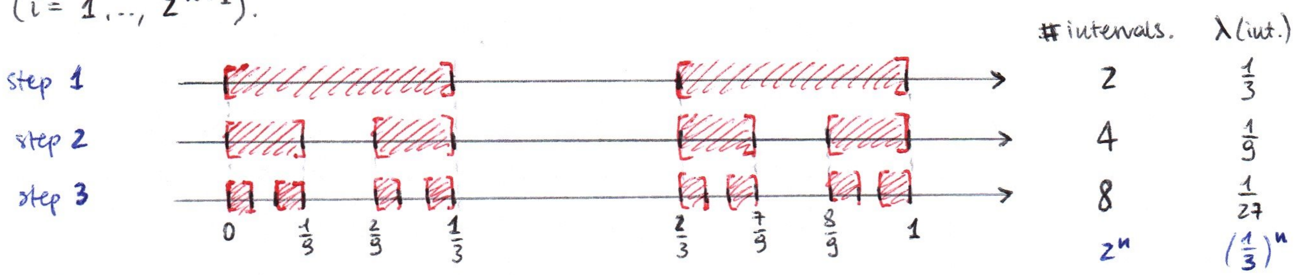
The cantor set is a subset $C \subset [0,1] \subseteq \mathbb{R}$ which can be inductively constructed as follows:

1. we remove from $I_0 = [0,1]$ the open interval
$$I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$$
and we define:
$$J_{1,1} = \left[0, \frac{1}{3}\right], \quad J_{1,2} = \left[\frac{2}{3}, 1\right]$$

2. we remove from $J_{1,i}$ $(i = 1,2)$ the interval (open) $I_{2,i}$ with the same center as $J_{1,i}$ and length $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. Explicitly:
   - from $J_{1,1}$ we remove $I_{2,1} = \left(\frac{1}{9}, \frac{2}{9}\right)$
   - from $J_{1,2}$ we remove $I_{2,2} = \left(\frac{7}{9}, \frac{8}{9}\right)$

   Then we define:
$$J_{2,1} = \left[0, \frac{1}{9}\right], \quad J_{2,2} = \left[\frac{2}{9}, \frac{1}{3}\right], \quad J_{2,3} = \left[\frac{2}{3}, \frac{7}{9}\right], \quad J_{2,4} = \left[\frac{8}{9}, 1\right]$$

By proceeding in this way, after n steps we have removed from $[0,1]$ a number of open intervals which is: $1 + 2 + 4 + \dots + 2^{n-1}$.
in fact, at each step $k = 1, \dots, n$ we remove 2^{k-1} intervals $I_{k,i}$ $(i = 1, \dots, 2^{k-1})$ of length: $\lambda(I_{k,i}) = \left(\frac{1}{3}\right)^k$.
The "remaining" set is the disjoint union of 2^n closed intervals $J_{n,i}$ $(i = 1, \dots, 2^n)$ with length: $\lambda(J_{n,i}) = \left(\frac{1}{3}\right)^n$.
We then proceed by removing from $J_{n,i}$ $(i = 1, \dots, 2^n)$ the open interval $I_{n+1,i}$ with the same center as $J_{n,i}$ and with length: $\lambda(I_{n+1,i}) = \left(\frac{1}{3}\right)^{n+1}$.
The remaining set is the disjoint union of 2^{n+1} closed intervals denoted by $J_{n+1,i}$ $(i = 1, \dots, 2^{n+1})$.



Then, we define:
$$C_n = \bigcup_{k=1}^{2^n} J_{n,k} \qquad \left(\text{notice that: } C_n \supseteq C_{n+1}\right)$$

and the **Cantor set** is:
$$C := \bigcap_{n=1}^{+\infty} C_n$$

Some **properties** of C:

1. C is closed. In fact, since $J_{n,k}$ is closed for every $n \in \mathbb{N}$ and $k = 1, \dots, 2^n$ the set:
$$C_n = \bigcup_{k=1}^{2^n} J_{n,k} \quad \text{is closed}$$
As a consequence
$$C = \bigcap_{n=1}^{+\infty} C_n \quad \text{is closed}$$

2. $C \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. This follows immediately from the fact that C is closed. (closed sets $\in \mathcal{B}(\mathbb{R})$)

3. $\lambda(C) = 0$. In fact:

$$\lambda(C) \leq \lambda(C_n) = \lambda\left(\bigcup_{k=1}^{2^n} J_{n,k}\right) \overset{\text{since } J_{n,k} \text{ are disjoint}}{=} \sum_{k=1}^{2^n} \lambda(J_{n,k}) = \underbrace{\sum_{k=1}^{2^n} \left(\tfrac{1}{3}\right)^n}_{2^n \left(\frac{1}{3}\right)^n} = \left(\tfrac{2}{3}\right)^n$$

From this, letting $n \to \infty$ we get $\lambda(C) = 0$.

4. $\mathrm{int}(C) = \emptyset$. In fact, since $\mathrm{int}(C) \subseteq C$ and $\lambda(C) = 0$ then $\lambda(\mathrm{int}(C)) = 0$. (since $\lambda \uparrow$)
Since $\mathrm{int}(C)$ is open, then we necessarily have: $\mathrm{int}(C) = \emptyset$.

the complement of C is dense in $[0,1]$

5. $\overline{[0,1] \setminus C} = [0,1]$. In fact if $x_0 \in [0,1]$ and $r > 0$ is fixed then
$$(x_0 - r, x_0 + r) \cap ([0,1] \setminus C) \neq \emptyset$$
otherwise $(x_0 - r, x_0 + r)$ would be entirely contained in C, but $\mathrm{int}(C) = \emptyset$
since $\mathrm{int}(C) = \emptyset$ (hence $(x_0 - r, x_0 + r) \not\subseteq C$) → "cannot be entirely contained in C"

6. C is uncountable and $\mathrm{card}(C) = |C| = \mathrm{card}(\mathbb{R}) = |\mathbb{R}|$. even if the interior is empty and its leb-measure is zero

more like curiosities than necessary properties

7. $C = \left\{ x \in [0,1] : x = \sum_{n=1}^{+\infty} \frac{x_n}{3^n},\ x_n \in \{0, 2\} \right\}$

→ The cantor set is the set of points s.t. in base 3 they don't have "1" in the expansion
(notice that that's how we write in base 3, in base 10 it would have been: $124 = \frac{1}{10^{-2}} + \frac{2}{10^{-1}} + \frac{4}{10^0}$)

$(\underbrace{0.02}_{\frac{0}{3} + \frac{2}{3^2} = \frac{2}{9}} \in C,\quad \underbrace{0.12}_{\frac{1}{3} + \frac{2}{3^2} = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}} \notin C)$

THE VITALI FUNCTION

The Vitali function is a map $f : [0,1] \to [0,1]$ which can be inductively defined as follows:

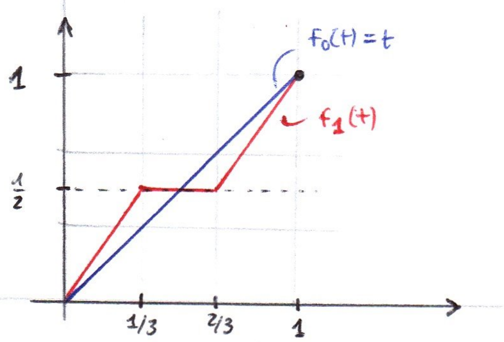
1. we start from the identity map $f_0 : [0,1] \to [0,1]$, $f_0(t) = t$.
We split the interval $[0,1]$ into 3 intervals with the same length $\ell = \frac{1}{3}$, that is:

$$J_{1,1} = \left[0, \tfrac{1}{3}\right], \quad I_{1,1} = \left(\tfrac{1}{3}, \tfrac{2}{3}\right), \quad J_{1,2} = \left[\tfrac{2}{3}, 1\right]$$

and we define $f_1 : [0,1] \to [0,1]$ as follows:

- $f_1 \in C([0,1])$, $f_1(0) = 0$, $f_1(1) = 1$
- f_1 is constant* equal to $f_0\left(\frac{1}{2}\right) = \frac{1}{2}$ — center of $I_{1,1}$
  * on $I_{1,1}$
- f_1 is affine on $C_1 = J_{1,1} \cup J_{1,2}$

"affine" = its graph is a line



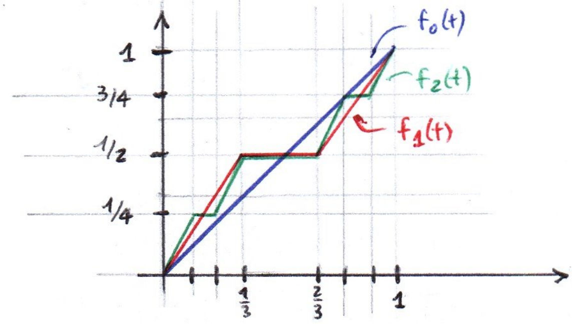
The function f_1 can be written explicitly:

$$f_1(t) = \begin{cases} \frac{3}{2} t & t \in J_{1,1} = \left[0, \frac{1}{3}\right] \\ \frac{1}{2} & t \in I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right) \\ \frac{3}{2} t - \frac{1}{2} & t \in J_{1,2} = \left[\frac{2}{3}, 1\right] \end{cases}$$

It can be easily proved that:

$$\max_{[0,1]} |f_1 - f_0| = \frac{1}{2}\cdot\frac{1}{3} = \left(\begin{matrix}\text{half of the}\\ \text{slope of } f_0\end{matrix}\right)\cdot\left(\begin{matrix}\text{length of}\\ [0,1]/3\end{matrix}\right)$$

2. We repeat the same procedure starting from f_1 on the intervals $J_{1,i}$



More precisely we split $J_{1,i}$ into 3 intervals with the same length $\ell = \left(\frac{1}{3}\right)^2$, that is:

$$J_{2,1} = \left[0, \tfrac{1}{9}\right] \qquad I_{2,1} = \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \qquad J_{2,2} = \left[\tfrac{2}{9}, \tfrac{1}{3}\right]$$

$$J_{2,3} = \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \qquad I_{2,2} = \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \qquad J_{2,4} = \left[\tfrac{8}{9}, 1\right]$$

We then define f_2 as follows:

- $f_2 \in C([0,1])$ and $f_2(0) = 0$, $f_2(1) = 1$
- on each $J_{1,i}$, f_2 is constructed in such a way that
  - (i) f_2 is constant $I_{2,i}$, equal to the value of f_1 on the center of $I_{2,i}$
  - (ii) f_2 is affine on $J_{1,i} \setminus I_{2,i}$

As before, f_2 can be written explicitly:

$$f_2(t) = \begin{cases} 9/4\, t & 0 \le t \le 1/9 & J_{2,1} \\ 1/4 & 1/9 < t < 2/9 & I_{2,1} \\ 9/4\, t - 1/4 & 2/9 \le t \le 1/3 & J_{2,2} \\ 1/2 & 1/3 < t < 2/3 & I_{1,1} \\ 9/4\, t - 1 & 2/3 \le t \le 7/9 & J_{2,3} \\ 3/4 & 7/9 < t < 8/9 & I_{2,2} \\ 9/4\, t - 5/4 & 8/9 \le t \le 1 & J_{2,4} \end{cases}$$

$J_{2,1}, I_{2,1}, J_{2,2}$: $J_{1,1}$; $J_{2,3}, I_{2,2}, J_{2,4}$: $J_{1,2}$

Moreover, one has:

$\frac{1}{2}\cdot$ slope(f_1) $\qquad$ $\frac{1}{3}$ length of the interval

- $$\max_{J_{1,1}} |f_2 - f_1| = \frac{1}{2}\cdot\left(\frac{3}{2}\right)\cdot\frac{1}{3}\cdot\left(\frac{1}{3}\right) = \frac{3}{4}\cdot\frac{1}{9} = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2$$

- $$\max_{I_{1,1}} |f_2 - f_1| = 0$$

- $$\max_{J_{1,2}} |f_2 - f_1| = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2$$ on $J_{1,2}$ we have the same situation as $J_{1,1}$

Summing up: $\max_{[0,1]} |f_2 - f_1| = \frac{1}{3}\left(\frac{1}{2}\right)^2$ – we see that f_2 and f_1 are closer than f_1 and f_0

By iterating this argument we obtain a sequence $(f_n)_{n\in\mathbb{N}}$ of functions s.t.:

(i) $f_n \in C([0,1])$, $0 \le f_n \le 1$, $f_n(0) = 0$, $f_n(1) = 1$

(ii) $f_n \nearrow$ in $[0,1]$ (not strictly increasing but increasing)

(iii) f_n is constant in $\Omega_n = \bigcup_{j=1}^{n} \bigcup_{k=1}^{2^{j-1}} I_{j,k}$, and it is affine on $[0,1] \setminus \Omega_n = C_n = \bigcup_{k=1}^{2^n} J_{n,k}$. ($\Omega_n$ gets larger as $n \to +\infty$)

More precisely, the slope of f_n on C_n is $\left(\frac{3}{2}\right)^n$.

(iv) For all $n, p \in \mathbb{N}$ we have:

$$\underbrace{f_{n+p}}_{\text{is constant on: } \Omega_{n+p} \supseteq \Omega_n} = f_n = \text{const} \quad \text{on } \Omega_n$$

(f_{n+p} is constant on a bigger set (Ω_{n+p}), but on Ω_n is constant and coincide with f_n)

(v) For every n: $\max_{[0,1]} |f_{n+1} - f_n| = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^{n+1}$

Using (v) we see that $(f_n)_n$ is a **Cauchy** sequence in $C([0,1])$. Indeed, if $m, n \in \mathbb{N}$ and $m > n$:

$$\begin{aligned} d(f_n, f_m) = \max_{[0,1]} |f_m - f_n| &\le \sum_{k=n}^{m-1} \max_{[0,1]} |f_{k+1} - f_k| \\ &= \frac{1}{3} \sum_{k=n}^{m-1} \left(\frac{1}{2}\right)^{k+1} \\ &= \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}\left[1 + \frac{1}{2} + \dots + \left(\frac{1}{2}\right)^{m-(n+1)}\right] \\ &\le \frac{1}{3}\left(\frac{1}{2}\right)^{n+1} \underbrace{\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k}_{2} = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^n \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

Hence $d(f_n, f_m) \to 0$ as $n, m \to \infty$. (Both n, m since we can exchange n with m and we obtain the same)

Since $C([0,1])$ is complete (every Cauchy sequence is convergent), $\exists V \in C([0,1])$ s.t.:

$$\lim_{n\to\infty} \max_{[0,1]} |V - f_n| = 0 \qquad \left(f_n \xrightarrow[n\to\infty]{\text{(uniformly)}} V \text{ on } [0,1] \right)$$

This V is the Vitali function. (⊥ Vitali set)

Some **properties** of V:

1. $0 \le V \le 1$, $V(0) = 0$, $V(1) = 1$
   (since this is true of f_n $\forall n \in \mathbb{N}$)

2. $V \nearrow$ in $[0,1]$
   (since this is true of f_n $\forall n \in \mathbb{N}$)

and V is the (pointwise) limit

3. $V([0,1]) = [0,1]$ since $V \in C([0,1])$ and it satisfies 1.
   ($\forall y \in [0,1]$ $\exists t : V(t) = y$)

Bolzano theorem: if f is a continuous function whose domain contains the interval $[a,b]$ then f takes on any given value between $f(a)$ and $f(b)$ at some point within the interval

4. $\exists V' = 0$ a.e. in $[0,1]$. In fact, since:
$$f_{n+p} = f_n = \text{constant} \quad \text{in } \Omega_n \qquad \forall n, p \in \mathbb{N}$$
then: $V = f_n = \text{constant}$ in Ω_n $\forall n \in \mathbb{N}$.
Hence, V is constant on the open set:
$$\Omega = \bigcup_{n=1}^{+\infty} \Omega_n = [0,1] \setminus C$$
In particular $\exists V'(t) = 0$ $\forall t \in \Omega$.
From this, since $\lambda(C) = 0$, we get: $\exists V' = 0$ a.e. in $[0,1]$. ($\forall t \in [0,1] \setminus C$)

Remark: The function V has this "strange" property:

- $V(1) - V(0) = 1$
- $\int_0^1 V'(t)\,dt = 0$

$$\Longrightarrow \quad V(1) - V(0) \neq \int_0^1 V'(t)\,dt$$

the Vitali function does **not** satisfy the fundamental theorem of calculus (for the integrals), but that's okay because the function is **not** C^1 (fundamental theorem holds for C^1 functions)

MEASURABLE FUNCTIONS

Ex. 1 For each of the functions:

(i) $f_1(t) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(ii) $f_2(t) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases}$

(iii) $f_3(x) = \mathbb{1}_A(x)$ where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

discuss the $\mathcal{L}$-measurability (Lebesgue-meas.) and the $\mathcal{B}$-measurability (Borel-meas.).

f_1 continuous $\rightarrow$ easiest situation
f_2 obtained by modifying a continuous function in a finite number of points (it could also have been countable), hence on a set of zero lebesgue-measure
f_3 indicator function of a $\mathcal{B}$-set

Solution: (i) Since $\lim_{x \to 0} f_1(x) = 0$, we can say that f_1 is continuous on $\mathbb{R}$ (out of 0 is surely continuous).
Hence, f_1 is Borel-measurable, that is:
$$f_1 : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ is measurable},$$
in particular, since $\mathcal{L}(\mathbb{R}) \supseteq \mathcal{B}(\mathbb{R})$, we have:
$$f_1 : (\mathbb{R}, \mathcal{L}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ is measurable},$$
that is: f_1 is $\mathcal{L}$-measurable

! (*) If a function f coincides a.e. with a continuous function g then f is both $\mathcal{B}$-meas. and $\mathcal{L}$-meas, since g is. What happens on sets of zero measure does **not** affect the measure of a function

(ii) since $f_2(0) = 1 \neq 0 \Longrightarrow f_2$ is not continuous on $\mathbb{R}$.
However, since $f_2(x) = f_1(x)$ $\forall x \neq 0$ we have that $f_2 = f_1$ a.e. on $\mathbb{R}$.
Thus, by (i)* we conclude that f_2 is both Borel-measurable and Lebesgue-measurable.
($\mathcal{B}$-measurable, $\mathcal{L}$-measurable)

The point is that we modify f_1 only on one point and of course this does not affect the measurability both w.r.t. Borel and Lebesgue. (*)

* and since $\{0\} \in \mathcal{B}(\mathbb{R})$ (we have to say it because we have to add that the measure of $\{0\}$ is zero (it must be **measurable**))

(iii) The measurability of $f_3 = \mathbb{1}_A$ is equivalent to the measurability of the set A.
On the other hand, since $A \in \mathcal{B}(\mathbb{R})$ (as countable union of closed sets), we have that $f_3 = \mathbb{1}_A$ is $\mathcal{B}$-measurable.
From this, since $\mathcal{L}(\mathbb{R}) \supseteq \mathcal{B}(\mathbb{R})$, we have that f_3 is also $\mathcal{L}$-measurable.

Ex. 2 Prove that $f: \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} 2^x & x \in C \text{ (Cantor set)} \\ e^x - x^2 & x \notin C \end{cases}$$

is $\mathcal{B}$-measurable and $\mathcal{L}$-measurable.

Solution: Since $\lambda(C) = 0$, we can say that $f = g$ a.e. on $\mathbb{R}$, where g is the continuous function:

$$g(x) = e^x - x^2$$

The set where $g \neq f$ is the Cantor set → uncountable but of zero Lebesgue measure.

As a consequence, since g is both $\mathcal{B}$-measurable and $\mathcal{L}$-measurable, (since C is closed and $C \in \mathcal{B}(\mathbb{R})$) we can conclude that f is $\mathcal{B}$-measurable and $\mathcal{L}$-measurable.

More precisely: g is continuous, hence is $\mathcal{L}$-measurable, and so f is $\mathcal{L}$-measurable.**
Moreover, since g is continuous it is also $\mathcal{B}$-measurable and since $C \in \mathcal{B}(\mathbb{R})$ (is closed) $\Rightarrow$ f is $\mathcal{B}$-measurable.
(we need $C \in \mathcal{B}(\mathbb{R})$ to say that its measure is zero also in $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$)
** since $f \neq g$ on a set of zero λ-measure.

Ex. 3 Prove that the function $f: \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} 2^x & x \in [0,1] \\ e^x - x^2 & x \notin [0,1] \end{cases}$$

is $\mathcal{B}$-measurable and $\mathcal{L}$-measurable.

Solution: If we consider the maps:

$$f_1(x) = 2^x, \qquad f_2(x) = e^x - x^2$$

we can write:

$$f(x) = f_1(x)\,\mathbb{1}_{[0,1]}(x) + f_2(x)\,\mathbb{1}_{\mathbb{R}\setminus[0,1]}(x)$$

Since $\underbrace{f_1 \in C(\mathbb{R})}_{\text{hence } f_1 \text{ is } \mathcal{B}\text{-meas. and } \mathcal{L}\text{-meas.}}$ and $\underbrace{[0,1] \in \mathcal{B}(\mathbb{R})}_{\text{hence } \mathbb{1}_{[0,1]} \text{ is } \mathcal{B}\text{-meas. and } \mathcal{L}\text{-meas.}} \longrightarrow f_1\,\mathbb{1}_{[0,1]}$ is both $\mathcal{B}$-measurable and $\mathcal{L}$-measurable.

Similarly, since $f_2 \in C(\mathbb{R})$ and $\mathbb{R}\setminus[0,1] \in \mathcal{B}(\mathbb{R})$

$\Rightarrow f_2 \cdot \mathbb{1}_{\mathbb{R}\setminus[0,1]}$ is both $\mathcal{B}$-measurable and $\mathcal{L}$-measurable

$\Rightarrow f = (f_1 \cdot \mathbb{1}_{[0,1]}) + (f_2 \cdot \mathbb{1}_{\mathbb{R}\setminus[0,1]})$ is both $\mathcal{B}$-measurable and $\mathcal{L}$-measurable.

$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda)$ is not complete.

We show that $\exists F \subseteq \mathbb{R}^n$ $(n=1)$ s.t. $\lambda(F)=0$ but $F \notin \mathcal{B}(\mathbb{R}^n)$.
The fact that $\lambda(F)=0$ means that $F \in \mathcal{L}(\mathbb{R}^n)$.
In part, this proves that $\mathcal{B}(\mathbb{R}^n) \subsetneq \mathcal{L}(\mathbb{R}^n)$.

proof.
Let $V: [0,1] \to [0,1]$ be the Vitali function and let $h: [0,1] \to [0,1]$ be s.t.:

$$h(t) = \frac{1}{2}(t + V(t)).$$

Using the properties of V we see that:

- $h \in C([0,1])$ and $h([0,1]) = [0,1]$ — continuous and maps $[0,1]$ into $[0,1]$ (in particular it's bijective)
- h is strictly increasing, hence:

$$f := h^{-1} \in C([0,1])$$

$h([0,1]) = [0,1] \Rightarrow$ surjective
strictly increasing $\Rightarrow$ injective
$\Rightarrow$ bijective $\Rightarrow$ h^{-1} is continuous

Moreover, if $C \subseteq [0,1]$ is the Cantor set,:

$$h([0,1]\setminus C) = h\left(\bigcup_{n\geq 1} \bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)$$

(where $I_{n,k}$ are the intervals removed at each step) is a set in $[0,1]$ and $h([0,1]\setminus C)$ has positive Lebesgue measure:

$$\lambda(h([0,1]\setminus C)) = \frac{1}{2}.$$

(Notice that $\bigcup_{n\geq 1} \bigcup_{k=1}^{2^{n-1}} I_{n,k}$ is an open set (union of open sets))

As a consequence: $\lambda(h(C)) = \frac{1}{2} > 0$ $\rightarrow$ $h(\cdot)$ maps the Cantor set into a set of positive λ-meas.

(since we have: $\underbrace{h(C) \cup h([0,1]\setminus C)}_{\lambda(\cdot)=\frac{1}{2}} = h([0,1]) = \underbrace{[0,1]}_{\lambda(\cdot)=1}$)

and hence: $\exists E \subseteq h(C)$ s.t. $E \notin \mathcal{L}(\mathbb{R})$.
(E is the Vitali set).

(**Vitali theorem**: any measurable bdd set $\subseteq \mathbb{R}$ with positive Lebesgue-measure, like in this case the set $h(C)$, contains a subset which is not measurable)

We defined:

$$F := h^{-1}(E)$$

Since $E \subseteq h(C)$, then $F = h^{-1}(E) \subseteq h^{-1}(h(C)) = C$ — because h is bijective
$\Rightarrow$ since we know that $\lambda(C)=0$ we have $\lambda(F)=0$
$\Rightarrow$ $F \in \mathcal{L}(\mathbb{R})$ even if E is not

We are looking at the image of the Cantor set by $h(\cdot)$, we are taking a subset of this image $(:= E)$ and we consider the pre-image of this subset $(h^{-1}(E))$. Of course $h^{-1}(E) \subseteq C$ because $h(\cdot)$ is bijective. Moreover $\lambda(h^{-1}(E)) \leq \lambda(C) = 0$.
$\Rightarrow$ $h^{-1}(E)$ has 0 λ-measure and so $h^{-1}(E) \in \mathcal{L}(\mathbb{R})$ because of the completness of $\mathcal{L}(\mathbb{R})$ (if a set has zero λ-meas. then it's $\in \mathcal{L}(\mathbb{R})$)

We claim that $F \notin \mathcal{B}(\mathbb{R})$.
In fact, if by contraddiction $F \in \mathcal{B}(\mathbb{R})$ then

$$f^{-1}(F) \in \mathcal{B}(\mathbb{R}) \quad \text{since } f = h^{-1} \text{ is continuous.}$$

continuous functions are Borel measurable so, if $F \in \mathcal{B}(\mathbb{R})$ then $f^{-1}(F) \in \mathcal{B}(\mathbb{R})$.
we can't do the same reasoning also for the Lebesgue-measurability because:

- f continuous and Borel-meas $\Rightarrow f: (X, \mathcal{B}) \to (X', \mathcal{B}')$
- f continuous and Leb.-meas: $\Rightarrow f: (X, \mathcal{L}) \to (X', \mathcal{B}')$

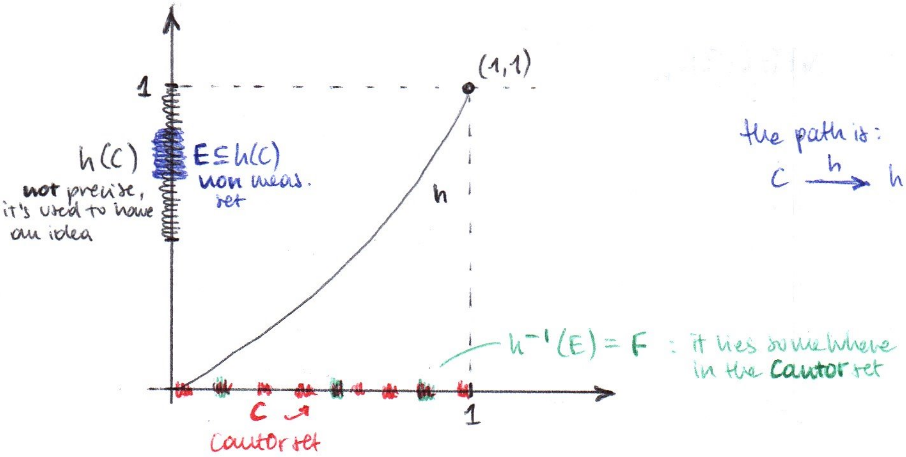
On the other hand:

$$f^{-1}(F) = h(F) = h(h^{-1}(E)) = E \notin \mathcal{L}(\mathbb{R}).$$

Hence we cannot have $F \in \mathcal{B}(\mathbb{R})$.

Summing up, we have proved that:

- $F \in \mathcal{L}(\mathbb{R})$ and $\lambda(F) = 0$
- $F \notin \mathcal{B}(\mathbb{R})$



the path is:

$$C \xrightarrow{h} h(C) \longrightarrow E \xrightarrow{h^{-1}} h^{-1}(E) \in \mathcal{L}(\mathbb{R}) \;\&\; \notin \mathcal{B}(\mathbb{R})$$

The above construction can be used to prove that the composition of $\mathcal{L}$-measurable functions is **not** (in general) $\mathcal{L}$-measurable.

we know that the composition of measurable functions is always measurable, but the point is that we have to care about the σ-algebras involved.

We remind that, if:

- $f : (X, \mathcal{A}) \to (X', \underline{\mathcal{A}'})$ is $(\mathcal{A}, \mathcal{A}')$-measurable
- $g : (X', \underline{\mathcal{A}'}) \to (X'', \mathcal{A}'')$ is $(\mathcal{A}', \mathcal{A}'')$-measurable

then: $g \circ f : (X, \mathcal{A}) \to (X'', \mathcal{A}'')$ is $(\mathcal{A}, \mathcal{A}'')$-measurable.

However, if $f, g : \mathbb{R} \to \mathbb{R}$ are $\mathcal{L}$-measurable we have that:

- $f : (\mathbb{R}, \mathcal{L}(\mathbb{R})) \longrightarrow (\mathbb{R}, \underline{\mathcal{B}(\mathbb{R})})$ is $(\mathcal{L}, \mathcal{B})$-measurable
- $g : (\mathbb{R}, \underline{\mathcal{L}(\mathbb{R})}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $(\mathcal{L}, \mathcal{B})$-measurable

hence, $\mathcal{A}' = \mathcal{B}(\mathbb{R}) \neq \mathcal{L}(\mathbb{R})$ and, in general, the composition $(g \circ f)$ is not $\mathcal{L}$-measurable. Indeed, if $B \in \mathcal{B}(\mathbb{R})$ then:

$$(g \circ f)^{-1}(B) = f^{-1}\big(\underbrace{g^{-1}(B)}_{g^{-1}(B) \in \mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})}\big)$$

For $g \circ f$ to be $\mathcal{L}$-measurable, we need to know that:

$$g^{-1}(B) \in \mathcal{B}(\mathbb{R}) \qquad \forall B \in \mathcal{B}(\mathbb{R})$$

that is, g must be $\mathcal{B}$-measurable (e.g. continuous).

As a concrete example, if we take:

$$f = h^{-1} \qquad \text{and} \qquad g = \mathbb{1}_F \quad (F \in \mathcal{L}(\mathbb{R}),\ F \notin \mathcal{B}(\mathbb{R}))$$

$\left(h(t) = \frac{1}{2}(t + V(t)),\ F \subseteq C, \text{ as before} \right)$

then we can see that $g \circ f$ is not $\mathcal{L}$-measurable.

(even if both f and g are $\underline{\mathcal{L}\text{-measurable}}$).

since f is continuous and g is the indicator function of a set with zero-Lebesgue-measure (so it is measurable) and so the $\mathbb{1}$ is measurable

$$\left(f \in C([0,1]) \implies f \in \mathcal{L}(\mathbb{R}) \atop F \in \mathcal{L}(\mathbb{R}) \implies \mathbb{1}_F \in \mathcal{L}(\mathbb{R}) \right)$$

LIMITS FOR LEBESGUE'S INTEGRAL

Ex. 1 Compute, if it $\exists$, :

$$\lim_{n\to\infty} \int_0^1 nxe^{-nx}\,dx$$

Solution: In this case, we can solve the exercise by two different approaches:

(1) Compute explicitly the integral

$$\int_0^1 xe^{-nx}\,dx = \left[\frac{e^{-nx}}{-n}\cdot x\right]_0^1 - \int_0^1 \frac{e^{-nx}}{-n}\,dx$$

$$= [\,..\,] = \frac{e^n-(1+n)}{e^n n^2}$$

$$\lim_{n\to\infty} n\int_0^1 xe^{-nx}\,dx = \lim_{n\to\infty} \frac{e^n-(1+n)}{e^n\cdot n} = 0$$

(2) Use some theorems for the Lebesgue integral.
We first observe that, setting:

$$f_n(x) := nxe^{-nx}\,dx \qquad x\in[0,1]$$

we have:

- $f_n \in C([0,1]) \longrightarrow f_n \in L^1([0,1])$ since $[0,1]$ is compact $\Big]$ $*$
- $\forall x\in[0,1]$:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[x\,\frac{n}{e^{nx}}\right] = 0$$

fixed x this is a sequence of real numbers

(notice that $f_n(0)=0 \quad \forall n\in\mathbb{N}$)

If we can apply some theorem allowing us to pass to the limit under the integral, we get:

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \underbrace{\lim_{n\to\infty} f_n(x)}_{0\ \forall x\in[0,1]}\,dx = 0 \qquad (*)$$

$*$

BOUNDNESS THM.
If f is continuous on a closed and bounded interval then it is bounded on the interval.

$f\in L^1 \Leftrightarrow \begin{cases} f\in\mathcal{M} \\ \int_X |f|\,d\mu<\infty\end{cases}$

$f\in L^\infty \Leftrightarrow \begin{cases} f\in\mathcal{M} \\ \text{ess sup}\,|f|<\infty\end{cases}$

f bounded $\Rightarrow f\le M$
$\Rightarrow \int_I |f|\,d\mu \le M\mu(I)$
$\Rightarrow \text{ess sup}_I\,|f| \le M$

Hence:
$f\in C([a,b])$
$\Downarrow$
$f\in L^1,\ f\in L^\infty$

In fact, we can apply **DCT**.
Since the function:

$$h(t) = te^{-t}$$

is continuous and: $\lim_{t\to\infty} te^{-t} = 0$

$\Rightarrow \exists M>0$ s.t. :

$$0\le te^{-t}\le M \qquad \forall t\ge 0$$

Hence: $0\le f_n(x) = (nx)e^{-nx} = h(nx) \le M \quad \forall x\in[0,1],\ \forall n\in\mathbb{N}$

Then, choosing $g(x)=M$ $(x\in[0,1])$, we have:

(i) $g\in L^1([0,1])$ $\quad \left(\int_0^1 g(x)\,dx = M(1-0) = M\right)$

(ii) $0\le f_n(x)\le g(x) \quad \forall x\in[0,1] \quad \forall n\in\mathbb{N}$

We can apply DCT and obtain: $(*)$.

Ex. 2 Compute: $\lim_{n\to\infty} \int_0^1 e^{\frac{x^2}{n}}\,dx$

Solution: This time the integrals cannot be easily computed, hence we try to apply some theorem allowing to conclude that:

$$\lim_{n\to\infty} \int_0^1 e^{\frac{x^2}{n}}\,dx = \int_0^1 \lim e^{\frac{x^2}{n}}\,dx$$

First of all, setting:

$$f_n(x) = e^{x^2/n}$$

we see that:

- $f_n \in C([0,1]) \implies f_n \in L^1([0,1])$ since $[0,1]$ is compact
- $\forall x \in [0,1]$:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{\frac{x^2}{n}} = 1 \qquad \left(\text{since } \tfrac{x^2}{n} \to 0 \text{ and } e^{\frac{x^2}{n}} \text{ is continuous}\right)$$

(notice that: $f_n(0) = 1 \quad \forall n \in \mathbb{N}$)

We then observe that $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$:

$$n < n+1 \implies \frac{x^2}{n+1} \leq \frac{x^2}{n} \implies f_{n+1}(x) = \exp\left(\frac{x^2}{n+1}\right) \leq \exp\left(\frac{x^2}{n}\right) = f_n(x)$$

$\implies$ we cannot apply **MCT**, but: we still can exploit the monotonicity:

$$0 \leq f_n(x) \leq f_1(x) = \underbrace{e^{x^2}}_{g(x)\ (\geq 1\ \forall x)} \qquad \forall x \in [0,1],\ \forall n \in \mathbb{N}$$

notice that we cannot consider $-f_n(x)$ to force the application of **MCT**, because $-f_n(x)$ are not positive functions

Hence, choosing $g(x) = e^{x^2}$ we have:

- $g \in C([0,1]) \implies g \in L^1([0,1])$
- $0 \leq f_n(x) \leq g(x) \quad \forall x \in [0,1] \quad \forall n \in \mathbb{N}$

we can apply **DCT**, obtaining:

$$\lim_{n\to\infty} \int_0^1 e^{\frac{x^2}{n}}\,dx = \int_0^1 \underbrace{\lim_{n\to\infty} e^{\frac{x^2}{n}}}_{=1\ \forall x \in [0,1]}\,dx = 1$$

Ex. 3 Compute: $\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}\,dx$

Solution: The integrals cannot be explicitly computed (at least, for $n \geq 2$), hence we try to apply some theorems allowing to pass to the lim under the integral.
First of all, we set:

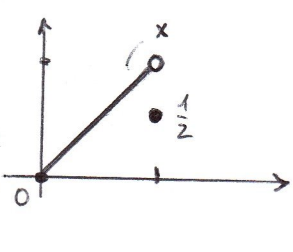
$$f_n(x) = \frac{x}{1+x^{2n}} \qquad x \in [0,1]$$

Clearly $f_n \in L^1([0,1])$ (since f_n is continuous on $[0,1]$). ratio of 2 polynomials and the denominator $\neq 0$
moreover, reminding that:

$$\lim_{n\to\infty} x^{2n} = \lim_{n\to\infty} (x^2)^n = \begin{cases} 0 & x^2 < 1 \\ +\infty & x^2 > 1 \\ 1 & x^2 = 1 \end{cases}$$

For every $x \in [0,1]$ we have:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \end{cases}$$



Hence, $f_n(x) \xrightarrow[n\to\infty]{} x$ for a.e. $x \in [0,1]$ $\quad (\forall x \neq 1)$

Finally, for every $x \in [0,1]$ and $n \in \mathbb{N}$:

$$0 \leq f_n(x) \leq \frac{x}{1} = x = g(x)$$

Thus, setting $g(x) = x$ we have:

- $g \in C([0,1]) \implies g \in L^1([0,1])$
- $0 \leq f_n(x) \leq g(x) \quad \forall x \in [0,1] \quad \forall n \in \mathbb{N}$

We can apply **DCT**, obtaining:

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}\,dx = \int_0^1 \underbrace{\lim_{n\to\infty} \frac{x}{1+x^{2n}}}_{= x \text{ a.e. on } [0,1]}\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

Ex. 4 Compute: $\displaystyle\lim_{n\to\infty} \int_I \frac{x}{1+x^{2n}}\,dx$ where $I = [0,+\infty)$

Solution: As above, the integrals cannot be computed, hence we try to pass to the limit under the integral. We set:

$$f_n(x) = \frac{x}{1+x^{2n}} \qquad x \geq 0$$

Since $f_n \in C(I)$ and: $f_n \sim \dfrac{x}{x^{2n}} = \dfrac{1}{x^{2n-1}}$ as $x \to \infty$

we see that $f_n \in L^1([0,\infty))$ $\forall n \in \mathbb{N}$ s.t.

$$2n-1 > 1 \iff 2n > 2 \iff n > 1 \iff n \geq 2$$

since I is not compact we can't conclude that f_n is L^1 only by the fact that f_n is C (continuous)

Moreover: (from ex. 3)

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+x^{2n}} = \begin{cases} x & 0 \leq x < 1 \\ 1/2 & x = 1 \\ 0 & x > 1 \end{cases}$$

Hence: $f_n(x) \xrightarrow[n\to\infty]{} x\,\mathbb{1}_{[0,1]}$ for a.a. $x \geq 0$ $\quad (\forall x \neq 1)$

To prove that DCT can be applied, we cannot choose $g(x) = x$ (as before). Indeed:

$$0 \leq f_n(x) = \frac{x}{1+x^{2n}} \leq x \quad \forall x \geq 0 \qquad \text{but} \qquad g(x) \notin L^1(I).$$

We then write:

$$\int_I f_n(x)\,dx = \int_0^1 \frac{x}{1+x^{2n}}\,dx + \int_1^\infty \frac{x}{1+x^{2n}}\,dx := (1) + (2)$$

and we apply DCT for the two integrals separately.

(1) by Ex. 3: $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^{2n}}\,dx = \frac{1}{2}$

(2) We observe that:

$$0 \leq f_n(x) \leq \frac{x}{x^{2n}} = \frac{1}{x^{2n-1}} \qquad \forall x \geq 1$$

moreover, if $n \geq 2$, $\forall x \geq 1$ we have:

$$x^2 \leq x^{2n-1} \implies \frac{1}{x^{2n-1}} \leq \frac{1}{x^2} \qquad \forall x \geq 1$$

Choosing $g(x) = \dfrac{1}{x^2}$:

- $g \in L^1([1,+\infty))$
- $0 \leq f_n(x) \leq g(x) \quad \forall x \geq 1 \quad \forall n \in \mathbb{N}$

Then, we can apply **DCT**, obtaining: 18/10

$$\lim_{n\to\infty} \int_1^{+\infty} \frac{x}{1+x^{2n}}\,dx = \int_1^{+\infty} \underbrace{\lim_{n\to\infty} \frac{x}{1+x^{2n}}}_{\substack{=0 \text{ if } x>1 \\ (\text{a.e. on } [1,+\infty))}}\,dx = 0$$

Summing up:

$$\lim_{n\to\infty} \int_I \frac{x}{1+x^{2n}}\,dx = \lim_{n\to\infty}\int_0^1 \frac{x}{1+x^{2n}}\,dx + \lim_{n\to\infty}\int_1^{+\infty} \frac{x}{1+x^{2n}}\,dx = \frac{1}{2}+0=\frac{1}{2}$$

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Ex. 5 Compute: $\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-\pi x}\,dx$

Solution: First, we write:

$$\int_0^n \left(1+\frac{x}{n}\right)^n e^{-\pi x}\,dx = \int_0^\infty \underbrace{\left(1+\frac{x}{n}\right)^n \mathbb{1}_{[0,n]}\, e^{-\pi x}}_{:= f_n(x)}\,dx$$

Then, since the sequence:

$$a_n = \left(1+\frac{x}{n}\right)^n$$

is increasing ($\forall$ fixed $x\in\mathbb{R}$) and:

$$\lim_{n\to\infty} a_n = e^x$$

we deduce that:

(1) the sequence f_n is increasing (as a sequence (w.r.t. n)), that is: $\forall x\in[0,+\infty)$:

$$f_n(x) = \left(1+\frac{x}{n}\right)^n \mathbb{1}_{[0,n]}\, e^{-\pi x} \le \left(1+\frac{x}{n+1}\right)^{n+1} \mathbb{1}_{[0,n+1]}\, e^{-\pi x} = f_{n+1}(x)$$

(2) $\forall x\in[0,+\infty)$ we have:

$$f_n(x) = \left(1+\frac{x}{n}\right)^n \mathbb{1}_{[0,n]}\, e^{-\pi x} = \left(1+\frac{x}{n}\right)^n e^{-\pi x} \quad \forall n\in\mathbb{N}: n\ge x$$

$$f_n(x) \xrightarrow[n\to\infty]{} e^x e^{-\pi x} = e^{-(\pi-1)x}$$

(3) $f_n(x)\ge 0 \quad \forall x\in[0,+\infty) \quad \forall n\in\mathbb{N}$

Using (1), (2), (3) together with the fact that f_n is $\mathcal{X}$-measurable $\forall n\in\mathbb{N}$ (f_n is the product of a continuous function times $\mathbb{1}_{[0,n]}$), we can apply the Beppo Levi theorem (**MCT**), obtaining:

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-\pi x}\,dx = \lim_{n\to\infty}\int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty}\lim_{n\to\infty} f_n(x)\,dx = \int_0^{+\infty} e^{-(\pi-1)x}\,dx = \left[-\frac{e^{-(\pi-1)}}{\pi-1}\right]_0^{+\infty} = \frac{1}{\pi-1}$$

Remark: since $(f_n(x))_n$ is increasing $\forall x\ge 0$ we also have that:

$$0\le f_n(x) \le \lim_{n\to\infty} f_n(x) = e^{-(\pi-1)x} =: g(x)$$

Hence, one can also apply **DCT** (since $g\in L^1([0,+\infty))$)

Ex. 6 Show that the sequence $f_n : [0,\pi] \longrightarrow \mathbb{R}$:

$$f_n(x) = \begin{cases} 2^n \sin(2^n x) & 0 \le x \le \pi/2^n \\ 0 & \pi/2^n < x \le \pi \end{cases}$$

converges to $f(x) \equiv 0$ as $n \to \infty$ but $f_n \not\to f$ in $L^1([0,\pi])$ (pointwise convergence but **not** in L^1)

Solution: We first notice that $f_n \in C([0,\pi])$, so that $f_n \in L^1([0,\pi])$ $\forall n \in \mathbb{N}$. $\left(f_n\left(\frac{\pi}{2^n}\right) = 0\right)$

To prove that: $\lim_{n\to\infty} f_n(x) = 0$ $\forall x \in [0,\pi]$ we distinguish 2 cases:

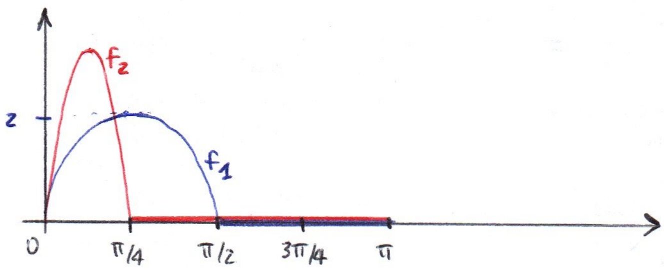
- if $x = 0$ then $f_n(x) = f_n(0) = 0$ $\forall n \in \mathbb{N}$ $\Rightarrow$ $f_n(0) \xrightarrow[n\to\infty]{} 0$
- if $0 < x \le \pi$, $\exists \bar{n} = \bar{n}_x \in \mathbb{N}$: $\frac{\pi}{2^n} < x$ $\forall n \ge \bar{n}_x$

  (since $\frac{\pi}{2^n} \longrightarrow 0$ as $n \to \infty$)

  hence: $f_n(x) = 0$ $\forall n \in \mathbb{N}$, $n \ge \bar{n}_x$

  (since $f_n \equiv 0$ out of $[0, \frac{\pi}{2^n}]$)

  and thus: $f_n(x) \xrightarrow[n\to\infty]{} 0$



On the other hand, since $f_n \ge 0$ $\forall n \in \mathbb{N}$, we have:

$$\underbrace{\int_0^\pi |f_n(x) - 0|\, dx}_{d(f_n,0) \text{ in } L^1} = \int_0^\pi f_n(x)\, dx = \int_0^{\frac{\pi}{2^n}} 2^n \sin(2^n x)\, dx$$

$$= \left[-\cos(2^n x)\right]_0^{\frac{\pi}{2^n}} = 2 \qquad \forall n$$

As a consequence:

$$\lim_{n\to\infty} d(f_n, 0) = 2 > 0 \implies f_n \not\to 0 \text{ in } L^1([0,1])$$

Remark: try to prove the same facts for:

$$f_n(x) = \begin{cases} 2^n \cos(2^n x) & 0 \le x \le \frac{\pi}{2^n} \\ 0 & \frac{\pi}{2^n} < x \le \pi \end{cases} \qquad \notin C([0,\pi])$$

(this function has a jump in $x = \frac{\pi}{2^n}$)

Notice that, in this case:

$$\underbrace{\lim_{n\to\infty} \int_0^\pi f_n(x)\, dx = 0}_{\ne\, d(f_n,0) \text{ in } L^1([0,\pi])}$$

since the function change sign ($|f_n| \ne f_n$)

Remark: since $f_n \xrightarrow[n\to\infty]{} 0$ pointwise in $[0,\pi]$ but f_n does not converge to $f \equiv 0$ in $L^1([0,\pi])$, we deduce that there cannot exist $g \in L^1([0,\pi])$ s.t.

$$0 \le f_n(x) \le g(x) \quad \forall n \in \mathbb{N}, \ \forall x \in [0,\pi]$$

(otherwise we could apply **DCT** and conclude that, since f_n goes to zero, we can pass the limit under the integral and obtain that f_n goes to zero also in L^1)

CONVERGENCE FOR SEQUENCES OF FUNCTIONS

Ex. 1 Study the convergence (pointwise, L^1, in measure) of the sequence:

$$f_n(x) := \cos(x^n) \qquad 0 \leq x \leq 1$$

Solution: We start with the pointwise convergence.
To this end, we observe that

$$\exists \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = \end{cases}$$

Hence, by continuity of $t \mapsto \cos(t)$ we have:

$$\exists \lim_{n\to\infty} f_n(x) = f_0(x) \qquad \forall x \in [0,1]$$

$\cos(\cdot)$ is a continuous function so the point is to find the limit of its argument
(if f is continuous: $\lim f(g(x)) = f(\lim g(x))$)

where: $f_0 : [0,1] \longrightarrow \mathbb{R}$ is given by:

$$f_0(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ \cos(1) & \text{if } x = 1 \end{cases}$$

Thus, we can say that $(f_n)_n$ converges pointwisely to f_0 on $[0,1]$, and in particular:

$$f_n(x) \xrightarrow[n\to\infty]{} 1 \qquad \text{a.e. on } [0,1]$$

Note: since $f_0 \notin C([0,1])$ we can say that $(f_n)_n$ does **not** converge to f_0 uniformly on $[0,1]$.

As for the L^1-convergence, we have to prove that:

$$\lim_{n\to\infty} \underbrace{\int_0^1 |f_n(x) - f_0(x)|\, dx}_{d(f_n, f_0) \text{ in } L^1([0,1])} = 0$$

We observe that, since $f_0 = 1$ a.e. on $[0,1]$ and $f_n(x) = \cos(x^n) \leq 1$, we can write:

$$d(f_n, f_0) = \int_0^1 |\cos(x^n) - 1|\, dx = \int_0^1 \underbrace{(1 - \cos(x^n))}_{\geq 0}\, dx$$

On the other hand, we have:

$$|1 - \cos(x^n)| = (1 - \cos(x^n)) \leq \underbrace{2}_{:= g(x)} \qquad \forall x \in [0,1],\ \forall n \in \mathbb{N}$$

Hence, setting $g(x) = 2$ ($\forall x \in [0,1]$) we have:

- $g \in L^1([0,1])$ $\quad \left(\int_0^1 g\, dx = 2\right)$
- $0 \leq 1 - \cos(x^n) \leq g(x) \quad \forall x \in [0,1] \quad \forall n \in \mathbb{N}$

We can then apply the DCT, obtaining:

$$\lim_{n\to\infty} \int_0^1 |1 - \cos(x^n)|\, dx = \int_0^1 \underbrace{\lim_{n\to\infty} |1 - \cos(x^n)|}_{\substack{=0 \\ \text{since } f_n(x) \xrightarrow[n\to\infty]{} 1}}\, dx = 0$$

Finally, since $f_n \xrightarrow[n]{} 1$ in $L^1([0,1])$, we can conclude that $f_n \xrightarrow[n\to\infty]{} 1$ in measure on $[0,1]$.

Ex. 2 Study the convergence of: $f_n(x) = n \cdot \mathbb{1}_{[0,\frac{1}{n}]}(x) \qquad x \in \mathbb{R}$.

Solution: We start with the pointwise convergence of f_n.
To this end we distinguish 3 cases:

(1) If $x<0$: $f_n(x) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \exists \lim_{n\to\infty} f_n(x) = 0$

(2) If $x=0$: $f_n(x) = n \quad \forall n \in \mathbb{N}$

$\Rightarrow \exists \lim_{n\to\infty} f_n(x) = +\infty$

(3) If $x>0$: $\exists \bar{n} = \bar{n}_x \in \mathbb{N}$ s.t. $\frac{1}{n} < x \quad \forall n \geq \bar{n}_x$

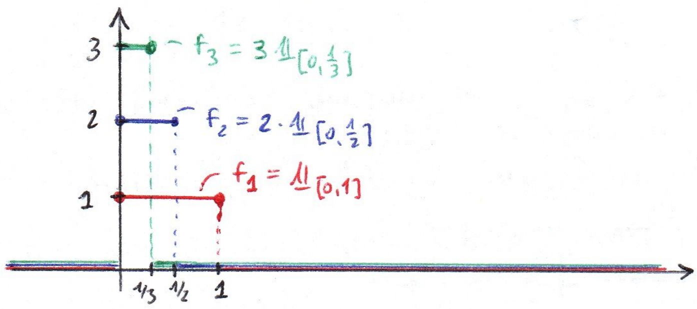
$\Rightarrow f_n(x) = 0 \quad \forall n \geq \bar{n}_x$

$\Rightarrow \exists \lim_{n\to\infty} f_n(x) = 0$

· Summing up, we have that:

$$\lim_{n\to\infty} f_n(x) = f_0(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

In particular: $f_n(x) \xrightarrow[n\to\infty]{} 0$ a.e. $\mathbb{R}$



As for the L^1-convergence we have:

$$d(f_n, f_0) = d(f_n, 0) = \int_{\mathbb{R}} |f_n(x) - 0|\, dx = \int_{\mathbb{R}} f_n(x)\, dx$$

$$= \int_{\mathbb{R}} n \cdot \mathbb{1}_{[0,\frac{1}{n}]}(x)\, dx = n \int_0^{\frac{1}{n}} 1\, dx = 1 \qquad \forall n \in \mathbb{N}$$

Hence: $d(f_n, f_0) \not\xrightarrow[n\to\infty]{} 0$, so that:

$$f_n \not\xrightarrow[n\to\infty]{} f_0 \quad \text{in } L^1(\mathbb{R})$$

Note: In particular, there cannot exist a function $g \in L^1(\mathbb{R})$ s.t.
$0 \leq f_n(x) \leq g(x) \quad \forall n \in \mathbb{N}$ and $\forall x \in \mathbb{R}$. — otherwise, if $\exists g \in L^1$ s.t. .. we could apply DCT and we would conclude that $f_n \to f_0$ in L^1 but this is **not** the case

Note: $$\int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\, dx = \int_{\mathbb{R}} f_0(x)\, dx = 0$$

$$< \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = 1$$

It holds Fatou's lemma with strict "<".

Finally, let us study the convergence in measure of $(f_n)_n$.
Let then $\varepsilon > 0$ be fixed.
Since $f_n \equiv 0$ on $\mathbb{R} \setminus [0, \frac{1}{n}]$ we necessarily have that:

$$E_n := \{x \in \mathbb{R} : |f_n(x) - f_0(x)| \geq \varepsilon\}$$

$$= \{x \in \mathbb{R} : f_n(x) \geq \varepsilon\} \subseteq [0, \tfrac{1}{n}] \qquad \forall n \in \mathbb{N}$$

Hence: $\lambda(E_n) = \lambda(\{f_n \geq \varepsilon\}) \leq \lambda([0, \frac{1}{n}]) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$ 25/10

As a consequence: $f_n \xrightarrow[n\to\infty]{} 0$ in measure, on $\mathbb{R}$.

Ex. 3 For every $\alpha > 0$ study the convergence of:

$$f_n(x) := n^\alpha \mathbb{1}_{[n^2-n, n^2+n]} \qquad x \in \mathbb{R}$$

Solution: We start with the pointwise convergence.

To this end, we observe that if $x \in \mathbb{R}$ is fixed then $\exists \bar{n} = \bar{n}_x \in \mathbb{N}$ s.t.

$$x < n^2 - n \qquad \forall n \geq \bar{n}_x$$

$$(n^2 - n \xrightarrow[n\to\infty]{} \infty)$$

Hence: $f_n(x) = 0 \quad \forall n \geq \bar{n}_x$

$$\Longrightarrow \exists \lim_{n\to\infty} f_n(x) = 0$$

Summing up: $f_n \xrightarrow[n\to\infty]{} 0$ pointwisely on $\mathbb{R}$.

As regards the L^1 convergence, we have:

$$\underbrace{d(f_n, 0)}_{L^1 \text{ distance}} = \int_{\mathbb{R}} |f_n(x) - 0| \, dx = \int_{\mathbb{R}} f_n(x) \, dx$$

$$= n^\alpha \int_{\mathbb{R}} \mathbb{1}_{[n^2-n, n^2+n]}(x) \, dx$$

$$= n^\alpha (\cancel{n^2} + n - (\cancel{n^2} - n)) = 2n^{1+\alpha}$$

Hence, $d(f_n, 0) \xrightarrow[n\to\infty]{} \infty$ and thus:

$$f_n \not\xrightarrow[n\to\infty]{} 0 \quad \text{in } L^1(\mathbb{R})$$

Finally, let us study the convergence in measure of $(f_n)_n$.

We then fix $\varepsilon > 0$ and we study the set:

$$E_n := \{x \in \mathbb{R} : f_n(x) \geq \varepsilon\} = \{x \in \mathbb{R} : |f_n(x) - 0| \geq \varepsilon\}$$

It is not restrictive to assume $\varepsilon < 1$.

We then observe that: $\forall n \in \mathbb{N}$

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin [n^2-n, n^2+n] \\ n^\alpha \underbrace{\geq 1}_{\forall \alpha > 0, \forall n \in \mathbb{N}} > \varepsilon & \text{if } x \in [n^2-n, n^2+n] \end{cases}$$

As a consequence:

$$E_n = \{f_n \geq \varepsilon\} = [n^2-n, n^2+n]$$

$$\Longrightarrow \lambda(E_n) = \lambda([n^2-n, n^2+n]) = 2n \xrightarrow[n\to\infty]{} \infty$$

Hence: $f_n \not\xrightarrow[n\to\infty]{} 0$ in measure.

Remark: Homework: Convergence of:

$$f_n(x) := n^\alpha \mathbb{1}_{[n^2-n, n^2+n]} \qquad x \in \mathbb{R}, \ \alpha < 0$$

(hint: $\sup_{\mathbb{R}} f_n = n^\alpha \xrightarrow[n\to\infty]{} 0$)

- we always have the pointwise conv. to zero
- we have uniform convergence to zero in this case ($\sup f_n \to 0$)
- for some α we have conv. in L^1, for others no
- we always have convergence in measure (since we have uniform conv.)

Ex. 4 Show that: $f(x,y) := \dfrac{\sin(x)}{x^2+y^2} \in L^1([0,1]\times\mathbb{R})$

Solution: Since $f \geq 0$ on $A := [0,1]\times\mathbb{R}$, we need to prove that:

$$\int_A f(x,y)\,dx\,dy < +\infty$$

(integral of a non-negative measurable function)

the integral for sure exists, either finite or $+\infty$, because the function is non-negative and measurable (since it's the quotient of $\sin(x)$, which is continuous, and of x^2+y^2, which is a polynomial) (measurable both w.r.t. Borel and Lebesgue)

Notice that: f is **not** continuous at $(0,0)$ and:

$$\lim_{x\to 0^+} f(x,0) = \lim_{x\to 0^+} \frac{\sin(x)}{x^2} = +\infty$$

this is just to underly that it is **not** trivial to prove that $\int_A f(x,y)\,dx\,dy < \infty$

Since, again, $f \geq 0$, we can use Fubini's theorem:

$$\begin{aligned}
\int_A f(x,y)\,dx\,dy &= \int_0^1 \left(\int_{\mathbb{R}} \frac{\sin(x)}{x^2+y^2}\,dy \right) dx \\
&= \int_0^1 \sin(x) \left(\int_{\mathbb{R}} \frac{1}{x^2+y^2}\,dy \right) dx \\
&= \int_0^1 \sin(x) \left(\int_{\mathbb{R}} \frac{1}{x^2} \frac{1}{1+\left(\frac{y}{x}\right)^2}\,dy \right) dx \\
&= \int_0^1 \sin(x) \left[\frac{1}{x} \operatorname{atan}\left(\frac{y}{x}\right) \right]_{-\infty}^{+\infty} dx \\
&= \int_0^1 \sin(x) \cdot \frac{\pi}{x}\,dx \\
&= \pi \int_0^1 \frac{\sin(x)}{x}\,dx
\end{aligned}$$

From this, since $x \mapsto \dfrac{\sin(x)}{x}$ is bounded near 0, we conclude that:

$$\int_A f(x,y)\,dx\,dy = \pi \int_0^1 \underbrace{\frac{\sin(x)}{x}}\,dx < \infty$$

since this function is bounded in $[0,1]$: $0 \leq \dfrac{\sin(x)}{x} \leq M \quad \forall x \in [0,1]$

$$\Longrightarrow f \in L^1(A)$$

BV AND AC FUNCTIONS

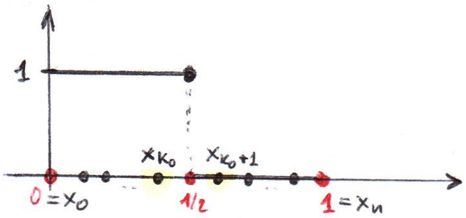
Ex. 1 Consider the function: $f:[0,1]\to\mathbb{R}$, $f(x) = \mathbb{1}_{[0,\frac{1}{2}]}(x)$.
Discuss if $f\in BV([0,1])$, $f\in AC([0,1])$.

Solution: Since $f\notin C([0,1])$ we can immediately say that $f\notin AC([0,1])$.
(since $AC([0,1])\subseteq C([0,1])$)

Let us prove that $f\in BV([0,1])$ with 2 different approaches.

1. Definition.

Let $\sigma = \{0=x_0<x_1<x_2<\dots<x_n=1\}$ be any fixed partition of $[0,1]$.



We define $k_0 := \max\{k=0,\dots,n \text{ s.t. } x_k\le \frac{1}{2}\}$. $(\ne\emptyset)$

x_{k_0} is the last point of the partition for which $f(\cdot)=1$

Then:

$$\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)| = \sum_{k=0}^{k_0}|f(x_{k+1})-f(x_k)| + \underbrace{\sum_{k=k_0+1}^{n-1}|f(x_{k+1})-f(x_k)|}_{(=0 \text{ if } k_0=n-1)}$$

(in the sense that we don't consider it)

$$= \underbrace{\sum_{k=0}^{k_0-1}|f(x_{k+1})-f(x_k)|}_{(1)} + \underbrace{|f(x_{k_0+1})-f(x_{k_0})|}_{(2)} + \underbrace{\sum_{k=k_0+1}^{n-1}|f(x_{k+1})-f(x_k)|}_{(3)}$$

(1) • Now, for $k\le k_0$: $x_k\in[0,\frac{1}{2}]$ and thus:

$$|f(x_{k+1})-f(x_k)| = |\mathbb{1}_{[0,\frac{1}{2}]}(x_{k+1}) - \mathbb{1}_{[0,\frac{1}{2}]}(x_k)| = |1-1| = 0 \qquad \forall k=0,\dots,k_0-1$$

(3) • Similarly, if $k\ge k_0+1$, $x_k>\frac{1}{2}$ and thus:

$$|f(x_{k+1})-f(x_k)| = |\mathbb{1}_{[0,\frac{1}{2}]}(x_{k+1}) - \mathbb{1}_{[0,\frac{1}{2}]}(x_k)| = |0-0| = 0 \qquad \forall k\ge k_0+1$$

• Then:

$$\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)| = |f(x_{k_0+1})-f(x_{k_0})| = (2)$$
$$= |\underbrace{\mathbb{1}_{[0,\frac{1}{2}]}(x_{k_0+1})}_{\substack{=0\\ \text{since } x_{k_0+1}>\frac{1}{2}}} - \underbrace{\mathbb{1}_{[0,\frac{1}{2}]}(x_{k_0})}_{\substack{=1\\ \text{since } x_{k_0}\le\frac{1}{2}}}|$$
$$= 1$$

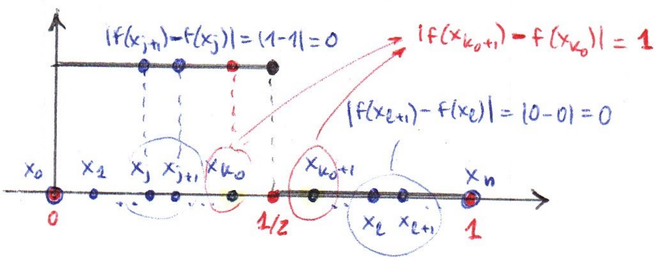
Summing up: for every partition $\{x_0,\dots,x_n\}$ of $[0,1]$ we have:

$$\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)| = 1$$

$\Longrightarrow$ $V_0^1(f) = 1 < +\infty$

Hence, $f\in BV([0,1])$.

Graphically:



2. The function $f = \mathbb{1}_{[0,\frac{1}{2}]}$ is non-increasing in $[0,1]$.
Indeed: if $0 \le x \le y \le 1$ we have:

- if $x \le y \le \frac{1}{2}$: $f(x) = f(y) = 1 \implies f(x) \geq f(y)$
- if $\frac{1}{2} < x \le y$: $f(x) = f(y) = 0 \implies f(x) \geq f(y)$
- if $x \le \frac{1}{2} < y$: $f(x) = 1,\ f(y) = 0 \implies f(x) \geq f(y)$

Hence, $f \in BV([0,1])$ and:

$$V_0^1(f) = f(0) - f(1) = 1$$

Remark: Since $f = \mathbb{1}_{[0,\frac{1}{2}]} \in BV([0,1])$ then $\exists f'(x)$ a.e. in $[0,1]$ and $f' \in L^1([0,1])$.
In fact, $\forall x \neq \frac{1}{2}$ $\exists f'(x) = 0$ and so: $\exists f'(x) = 0$ a.e. in $[0,1]$, $f' \in L^1([0,1])$
However, $f \notin AC([0,1])$ and in fact:

$$f(x) = \mathbb{1}_{[0,\frac{1}{2}]}(x) \neq \underbrace{1}_{f(0)} + \int_0^x \underbrace{f'(t)}_{=0 \text{ a.e.}}\,dt = 1 \quad \underline{\text{if } x > \tfrac{1}{2}}$$

for example: $-1 = f(1) - f(0) \neq \int_0^1 f'(t)\,dt = 0$

Ex. 2 Prove that if $\alpha > \beta > 0$ then $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} x^\alpha \cos\left(\frac{1}{x^\beta}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases}$
is in $AC([0,1])$.

Solution: We first observe that $f \in C([0,1])$ since:

$$\lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} x^\alpha \underbrace{\cos\left(\frac{1}{x^\beta}\right)}_{\text{bounded } (\in [-1,1])} = 0 = f(0)$$

(remember that $\alpha > 0$).

To prove it we use the characterization of AC:
- $\exists$ derivative a.e.
- the derivative is L^1
- it holds the fundamental formula of calculus

Moreover, f is differentiable on $(0,1]$, where we have:

$$\exists f'(x) = \alpha x^{\alpha-1}\cos\left(\frac{1}{x^\beta}\right) + x^\alpha\left(-\sin\left(\frac{1}{x^\beta}\right)\right)\left(-\frac{\beta}{x^{\beta+1}}\right)$$
$$= \alpha x^{\alpha-1}\cos\left(\frac{1}{x^\beta}\right) + \beta x^{\alpha-\beta-1}\sin\left(\frac{1}{x^\beta}\right)$$

!!

Now, since $\boxed{f \in C^1((0,1])}$, we have:

$$f(x) = f(\xi) + \int_\xi^x f'(t)\,dt \quad \text{for } 0 < \xi \le x \le 1.$$

here is more correct to write: $f \in C^1([\xi,1])$ $\forall \xi \in (0,1]$

If $f \in C([a,b])$ then: $F(x) := \int_a^x f(t)\,dt + F(a)$
- is uniformly cont. on $[a,b]$
- is differentiable on (a,b)
- is s.t.: $F'(x) = f(x)$ $\forall x \in (a,b)$

If $f' \in L^1([0,1])$ then, by letting $\xi \to 0$ we get:

$$f(x) = f(0) + \int_0^x f'(t)\,dt \quad \forall x \in [0,1].$$

given by:
- integrability of f' ($f' \in L^1$)
- continuity of f: ($\lim_{x\to 0} f(x) = f(0)$)

So, we only need to prove that $f' \in L^1([0,1])$.
(Notice that is fundamental to prove that $\lim_{x\to 0} f(x) = f(0)$) !
To prove that $f' \in L^1([0,1])$:
we notice that: $|\sin(\frac{1}{x^\beta})|, |\cos(\frac{1}{x^\beta})| \le 1$ hence:

$$\int_0^1 |f'(x)|\,dx \le \alpha\int_0^1 x^{\alpha-1}\,dx + \int_0^1 x^{\alpha-\beta-1}\,dx$$
$$= \alpha\int_0^1 \frac{1}{x^{1-\alpha}}\,dx + \int_0^1 \frac{1}{x^{1+\beta-\alpha}}\,dx$$

$$\left.\begin{array}{l} \text{since } \alpha > 0 \implies 1-\alpha < 1 \\ \text{since } \beta < \alpha \implies 1+\beta-\alpha < 1 \end{array}\right\} \implies f' \in L^1([0,1])$$

we recall that:

$$\int_0^1 \frac{1}{x^\gamma}\,dx < \infty \iff \gamma < 1$$

Ex. 3 Consider $\forall \alpha \in \mathbb{R}$ the function: $f_\alpha : [0,1] \to \mathbb{R}$ s.t.:

$$f_\alpha(x) = \begin{cases} e^{x^\alpha} & 0 < x \le 1 \\ 1 & x = 0 \end{cases}$$

(1) Prove that $f_\alpha \in AC([0,1])$ if $\alpha > 0$

(2) For $\alpha = -1$, prove that $f_\alpha \notin BV([0,1])$ using the definition

Solution: (1) We fix $\alpha > 0$ and we first observe that $f_\alpha \in C([0,1])$. In fact, since $\alpha > 0$, we have:

$$\lim_{x \to 0^+} x^\alpha = 0 \implies \lim_{x \to 0^+} f_\alpha(x) = e^0 = 1 = f_\alpha(0).$$

To prove that $f_\alpha \in AC([0,1])$ we show:

(i) $\exists f_\alpha'(x)$ for a.e. $x \in [0,1]$

(ii) $f_\alpha' \in L^1([0,1])$

(iii) f_α satisfies the fundamental thm. of calculus.

Proof of (i): $\forall x > 0$ we have:

$$f_\alpha'(x) = (e^{x^\alpha})' = e^{x^\alpha} \alpha x^{\alpha - 1} = \alpha x^{\alpha-1} e^{x^\alpha}$$

Hence, f_α is differentiable for all $x \in (0,1]$
$\implies f_\alpha$ is differentiable a.e. in $[0,1]$.

Proof of (ii): We first observe that, since $f_\alpha \in C([0,1])$, there exists some $M > 0$ such that:

$$0 < f_\alpha(x) = e^{x^\alpha} \le M \quad \forall x \in (0,1].$$

As a consequence, since $\alpha > 0$:

$$0 < f_\alpha'(x) = \alpha x^{\alpha-1} e^{x^\alpha} \le \alpha M x^{\alpha - 1} = \frac{\alpha M}{x^{1-\alpha}} \quad \forall x \in (0,1]$$

Finally, since $1 - \alpha < 1$ (as $\alpha > 0$):

$$\frac{1}{x^{1-\alpha}} \in L^1([0,1]) \implies \underbrace{f_\alpha'}_{\ge 0} \left(\le \frac{\alpha M}{x^{1-\alpha}}\right) \in L^1([0,1])$$

(important, otherwise we need to estimate $|f_\alpha'|$)

Proof of (iii): We first observe that, since $f_\alpha' = \alpha x^{\alpha-1} e^{x^\alpha} \in C([0,1])$, the function f_α is $C^1([0,1])$, hence, by fundamental thm. of calculus for C^1-functions, we have:

$$f_\alpha(x) - f_\alpha(\xi) = \int_\xi^x f_\alpha'(t)\,dt \quad \forall 0 < \xi \le x \le 1$$

From this, letting $\xi \to 0^+$ we obtain:

- $f_\alpha(\xi) \xrightarrow[\xi \to 0^+]{} f_\alpha(0) = 1$ since $f_\alpha \in C([0,1])$
- $\displaystyle\lim_{\xi \to 0^+} \int_\xi^x f_\alpha'(t)\,dt = \int_0^x f_\alpha'(t)\,dt$ since $f_\alpha' \in L^1([0,1])$

As a consequence we infer that:

$$\underbrace{f_\alpha(x) - f_\alpha(0)}_{(=1)} = \lim_{\xi \to 0^+} (f_\alpha(x) - f_\alpha(\xi))$$

$$f_\alpha(x) - f_\alpha(0) = \lim_{\xi \to 0^+} \int_\xi^x f'_\alpha(t)\,dt$$
$$= \int_0^x f'_\alpha(t)\,dt \qquad \forall\, 0 \le x \le 1$$

And this proves that f_α satisfies the fundamental theorem of calculus.

Summing up, since (i), (ii), (iii) hold, we can conclude that $f_\alpha \in AC([0,1])$ if $\alpha > 0$.

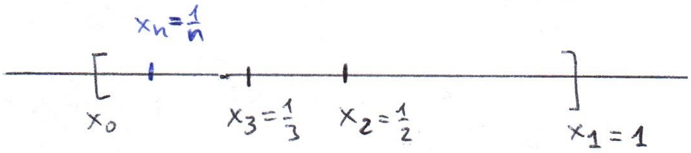
(2) We first notice that when $\alpha < 0$, the function f_α is differentiable a.e. on $[0,1]$, (actually $\forall x \neq 0$), but:
$$f'_\alpha \notin L^1([0,1]).$$
Hence, $f_\alpha \notin BV([0,1])$.

- We verify this fact for $\alpha = -1$ using the explicit def. of BV functions. To this end we observe that, when $\alpha = -1$, we have:

$$f_\alpha(x) = f_{-1}(x) = \begin{cases} e^{1/x} & \text{if } 0 < x \le 1 \\ 1 & \text{if } x = 0 \end{cases}$$

(in particular $f_{-1} \notin C([0,1])$.

Then, we fix $n \geq 2$ and we consider the partition of $[0,1]$ defined as:
$$\sigma_n = \{x_0, \dots, x_n\} \text{ where } x_0 = 0, \quad x_k = \frac{1}{k} \quad (k = 1, \dots, n)$$



Computing the variation of f_{-1} w.r.t. σ_n, we get:

$x_k = \frac{1}{k}$, $k = 1, \dots, n$

$$\sum_{k=0}^{n-1} |f_{-1}(x_{k+1}) - f_{-1}(x_k)| \geq \sum_{k=1}^{n-1} |f_{-1}(x_{k+1}) - f_{-1}(x_k)|$$
$$= \sum_{k=1}^{n-1} \left| \exp\left\{1/\tfrac{1}{k+1}\right\} - \exp\left\{1/\tfrac{1}{k}\right\}\right|$$
$$= \sum_{k=1}^{n-1} |e^{k+1} - e^k|$$
$$= \sum_{k=1}^{n-1} (e^{k+1} - e^k) \qquad e^x \text{ increasing}$$
$$= e^n - e$$

As a consequence we obtain:
$$V_0^1(f_{-1}) \geq \sum_{k=0}^{n-1} |f_{-1}(x_{k+1}) - f_{-1}(x_k)| \geq e^n - e \qquad \forall n \in \mathbb{N}$$

Thus, since $e^n - e \xrightarrow[n \to \infty]{} \infty$, we conclude that:

$$V_0^1(f_{-1}) = +\infty \implies f_{-1} \notin BV([0,1])$$

EXERCISES ON REAL ANALYSIS

Ex. (28/1/21)

Consider the sequence: $f_n(x) = \dfrac{n+5}{1+n^2x^2}$ $\quad x \in [-1,1]$

1. measurability, integrability of f_n
2. convergence a.e., pointwise
3. $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\,dx$
4. convergence in measure, in L^1, in L^∞

Solution:

1. $\forall n \in \mathbb{N}$ $\quad f_n \in C^0([-1,1]) \Longrightarrow f_n$ measurable

when we write measurable without specifying if it is Lebesgue/Borel $\Rightarrow$ **Lebesgue** measurable

Moreover: $f_n \in C^0([-1,1])$ on $[-1,1]$ which is compact
$\Longrightarrow f_n \in L^1([-1,1])$

continuous function on a compact interval $\Rightarrow L^1$ (since it's Riemann, in particular L^1)

2. $\forall x \in [-1,1] \setminus \{0\}$: $f_n(x) \xrightarrow[n\to\infty]{} 0$

If $x=0$: $f_n(0) \xrightarrow[n\to\infty]{} +\infty$

$\Longrightarrow f_n \xrightarrow[n\to\infty]{} 0$ a.e. in $[-1,1]$

4. $\left.\begin{array}{l} f_n \xrightarrow[n\to\infty]{} 0 \text{ a.e. in } [-1,1] \\ \lambda([-1,1]) < \infty \end{array}\right\} \Longrightarrow f_n \xrightarrow[n\to\infty]{} 0$ in measure in $[-1,1]$

3. $$\int_{-1}^{1} f_n(x)\,dx = \int_{-1}^{1} \frac{n+5}{1+n^2x^2}\,dx \qquad y = nx,\ dy = n\,dx$$
$$= \frac{1}{n}\int_{-n}^{n} \frac{n+5}{1+y^2}\,dy$$
$$= \frac{1}{\cancel{n}}\cancel{n}\left(1+\frac{5}{n}\right)\int_{-n}^{n} \frac{1}{1+y^2}\,dy$$
$$= \left(1+\frac{5}{n}\right)\left[\operatorname{atan}(y)\right]_{-n}^{n} \xrightarrow[n\to\infty]{} \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

! we cannot apply convergence theorems because we don't have convergence in L^1 (we'll check it in (4.))

4. Since $f_n \geq 0$:

$$\|f_n\|_{L^1} = \int_{-1}^{1} f_n(x)\,dx \xrightarrow[n\to\infty]{} \pi$$

$\Longrightarrow \{f_n\}_n$ does **not** converge in L^1.

this is because convergence in L^1 implies convergence a.e. of a subsequence, but a.e. the sequence converges to zero (so if the limit exists it must be zero), but if we have convergence to zero in L^1 we have to have: $\|f_n\|_{L^1} \to 0$

Since $f_n \in C^0([-1,1])$:

$$\|f_n\|_{L^\infty} \overset{*}{=} \max_{x\in[-1,1]} |f_n(x)| = f_n(0) \xrightarrow[n\to\infty]{} \infty$$

* since the function is continuous: sup $\to$ max

$\Longrightarrow \{f_n\}_n$ does **not** converge in L^∞

Ex. (18/2/21)

Let $\alpha > 0$. Consider: $\sum_{k=1}^{\infty} \dfrac{1}{x^\alpha + k^\alpha}$ $\quad x \geq 0$ $\qquad$ * pointwise in $[0,+\infty)$

1. find $\alpha > 0$ for which the series converges* $\left(F_\alpha(x) : \sum_{k=1}^{\infty} \dfrac{1}{x^\alpha + k^\alpha},\ x \geq 0 \right)$ for such α we def:
2. find $\alpha > 0$ for which:
$$\int_0^{+\infty} F_\alpha(x)\,dx = \int_0^{+\infty} \sum_{k=1}^{\infty} \frac{1}{x^\alpha + k^\alpha}\,dx = \sum_{k=1}^{\infty} \int_0^{+\infty} \frac{1}{x^\alpha + k^\alpha}\,dx$$

notice that $\exists F_\alpha$ only for α at 1.

3. find $\alpha > 0$ for which: $F_\alpha \in L^1([0,+\infty))$
4. find $\alpha > 0$ for which: $F_\alpha \in BV([0,1])$, compute $V_0^1(F_\alpha)$

hint [this means (almost always) $\rightarrow$ monotone functions!]

Solution: 1. convergence $\Longleftrightarrow$ $\alpha > 1$ $\quad \forall x \in [0,+\infty)$

2. $\forall \alpha > 1$ we define:

$$f_k^\alpha(x) := \frac{1}{x^\alpha + k^\alpha} > 0 \qquad x \in [0,+\infty)$$

$f_k^\alpha(x)$ is positive and measurable $\Longrightarrow$ Corollary of MCT:

$$\int_0^{+\infty} \sum_{k=1}^{\infty} f_k^\alpha(x)\,dx = \sum_{k=1}^{\infty} \int_0^{+\infty} f_k^\alpha(x)\,dx$$

$\Longrightarrow \alpha > 1$

3. $$\int_0^{+\infty} f_k^\alpha(x)\,dx = \frac{1}{k^\alpha}\int_0^{\infty} \frac{1}{(\frac{x}{k})^\alpha + 1}\,dx \qquad y = x/k,\ dy = \tfrac{1}{k}dx$$

$$= \frac{k}{k^\alpha} \underbrace{\int_0^{+\infty} \frac{1}{1+y^\alpha}\,dy}_{:= c_\alpha}$$

(for every $\alpha > 1$ we know that the integral converges, we don't care to what) $\Longrightarrow c_\alpha \in \mathbb{R}$

(α because it depends on α, but we don't care)

$$= \frac{c_\alpha}{k^{\alpha-1}}$$

for the integrability of F_α we use the corollary of the MCT: to study $\int_0^{+\infty} F_\alpha\,dx$ we use that $\int_0^{+\infty}\sum^{\infty} = \sum^{\infty}\int_0^{+\infty}$ and we study only $\int_0^{+\infty}$

$$\Longrightarrow \int_0^{+\infty} F_\alpha(x)\,dx = \sum_{k=1}^{\infty}\int_0^{+\infty} f_k^\alpha(x)\,dx$$

$$= \sum_{k=1}^{\infty} \frac{c_\alpha}{k^{\alpha-1}} < \infty \Longleftrightarrow \alpha - 1 > 1 \Longleftrightarrow \alpha > 2$$

4. $F_\alpha \searrow$ in $[0,1]$ (monotone decreasing) $\forall \alpha$ $\Longrightarrow$ $F_\alpha \in BV([0,1])$ and $V_0^1(F_\alpha) = F_\alpha(0) - F_\alpha(1)$

Ex. (13/7/21)

Consider the sequence: $$f_n(x) = \frac{\sin(nx)}{nx + x^2} + \frac{n}{nx^{3/2} + 1} \qquad x \in [1,+\infty),\ n \in \mathbb{N}$$

1. convergence a.e. in $[1,+\infty)$, $f_n \in L^1$
2. convergence in L^1, in measure
3. Is $\lim_{n\to\infty} \int_0^{+\infty} f_n(x)\,dx < \infty$?

hint [if it's asked L^1 and then $\lim_{n\to\infty}\int_0^\infty$ then we probably going to use some convergence results. In Ex. 1 of today there was the $\lim_{n\to\infty}\int_0^\infty$ first and then L^1]

Solution: 1. $$\boxed{f_n(x) \xrightarrow[n\to\infty]{} \frac{1}{x^{3/2}} := f(x)} \qquad \forall x \in [1,+\infty)$$

since $\frac{|\sin(nx)|}{nx+x^2} \le \frac{1}{nx+x^2} \xrightarrow{n\to\infty} 0$ and $\frac{n}{nx^{3/2}+1} \sim \frac{1}{x^{3/2}}$ as $n \to \infty$

$\Longrightarrow$ pointwise convergence in $[1,+\infty)$

$\Longrightarrow$ convergence a.e. in $[1,+\infty)$.

!! to check if a function is L^1 we use the characteriz.:
- measurable (f)
- $\int |f_n(x)|\,dx < \infty$

$\Updownarrow$

$f \in L^1$

f_n continuous $\Longrightarrow$ f_n measurable in $[1,+\infty)$

$$|f_n(x)| \le \frac{|\sin(nx)|}{nx+x^2} + \frac{n}{nx^{3/2}+1} \le \frac{1}{x^2} + \frac{1}{x^{3/2}} := h(x) \qquad \forall n \in \mathbb{N}$$

it's continuous and so measurable

$$\Longrightarrow \int_1^{\infty} |f_n(x)|\,dx \le \int_1^{\infty} h(x)\,dx < \infty$$

$\Longrightarrow f_n \in L^1([1,\infty))$ $\quad \forall n \in \mathbb{N}$

2. $|f_n| \leq h$ a.e. in $[1,\infty)$ $\forall n \in \mathbb{N}$, $h \in L^1([1,\infty))$

$\overset{DCT}{\Longrightarrow}$ $f_n \xrightarrow[n\to\infty]{} f$ in $L^1([1,\infty))$

$\Longrightarrow$ $f_n \xrightarrow[n\to\infty]{} f$ in measure

3. $\lim_{n\to\infty} \int_1^{+\infty} f_n(x)\,dx \underset{(2.)}{=} \int_1^{+\infty} \lim_{n\to\infty} f_n(x)\,dx$

$\overset{!}{=} \int_1^{\infty} f(x)\,dx < \infty$ and also $0 < \int_1^{\infty} f(x)\,dx$

Ex. (2.10)

Consider: $A := \{x \in [0, \frac{\pi}{2}] : 0 \leq \sin(x) \leq \frac{\sqrt{2}}{2}\}$

$B := \{$algebraic real numbers$\}$ – real numbers that solve polynomial equation with rational coefficients (algebraic real numbers = zeros of a polynomial with rational coefficients)

$C := \bigcap_{n=1}^{\infty} \left(-\frac{\pi}{4} - \frac{1}{n}, \frac{\pi}{8} + \frac{1}{2^n}\right)$

$\Omega := A \cup B \cup C$

$\Omega \in \mathcal{L}(\mathbb{R})$? $\lambda(\Omega) = ?$

Solution: A: $x \mapsto \sin(x)$ is continuous $\Longrightarrow$ measurable

$\Longrightarrow$ A measurable $(A \in \mathcal{L}(\mathbb{R}))$ (because of the characterization of measurable sets)

B: $P(x)$ polynomial with rational coefficients.

$P(x) = 0$ has a finite number of solutions. (degree $\in \mathbb{N}$)

The family of such $P(x)$ is countable.

$\Longrightarrow$ B is countable $(B \subset \mathbb{R})$

$\forall$ degree we have infinite but countable number of polynomials. $\forall$ polynomial we have finite number of solution.
$\Longrightarrow$ combining: B is countable (countable + countable + finite)

$\Longrightarrow$ $\boxed{B \in \mathcal{L}(\mathbb{R}),\ B \text{ is s.t. } \lambda(B) = 0}$

any countable subset of $\mathbb{R}$ is measurable and its measure is zero

C: C is a countable intersection of intervals

$\Longrightarrow$ $C \in \mathcal{L}(\mathbb{R})$

$\Longrightarrow$ $\Omega \in \mathcal{L}(\mathbb{R})$

In particular:

$A = [0, \frac{\pi}{4}]$, $C = [-\frac{\pi}{4}, \frac{\pi}{8}]$

$\Longrightarrow$ $\Omega = [-\frac{\pi}{4}, \frac{\pi}{4}] \cup B$, $B \in \mathcal{N}_\lambda$

$\Longrightarrow$ $\lambda(\Omega) = \lambda([-\frac{\pi}{4}, \frac{\pi}{4}]) = \frac{\pi}{2}$

Ex. (3.36)

Consider a measurable set $E \in \mathcal{L}(\mathbb{R})$, $f, g \in L^1(E)$.

Define:
$$\left.\begin{aligned} h(t) &:= \int_{E_t} f(x)\,dx \\ E_t &:= \{x \in E : g(x) > t\} \end{aligned}\right\} \quad t \in \mathbb{R}$$

Compare:

$$\lim_{t\to+\infty} h(t), \quad \lim_{t\to-\infty} h(t)$$

Solution: $g \in L^1(E) \implies \lambda(\{x \in E : g(x) = \pm\infty\}) = 0$ since if a function is in L^1 then it's finite a.e.

$$\implies \begin{cases} \mathbb{1}_{E_t}(x) \xrightarrow[t\to\infty]{} 0 & \text{a.e. in } E \\ \mathbb{1}_{E_t}(x) \xrightarrow[t\to-\infty]{} 1 & \text{a.e. in } E \end{cases}$$

$h(t) = \int_E f \mathbb{1}_{E_t}\, dx \quad , \quad f\mathbb{1}_{E_t} := \xi_t$ ← this is the **key point**

$$\implies \left.\begin{cases} \xi_t \xrightarrow[t\to\infty]{} 0 & \text{a.e. in } E \\ \xi_t \xrightarrow[t\to-\infty]{} f & \text{a.e. in } E \\ |\xi_t| \le |f| & \text{a.e. in } E,\ \forall t \in \mathbb{R} \end{cases}\right\} \longrightarrow \text{DCT}$$

(convergence a.e + dominating function)

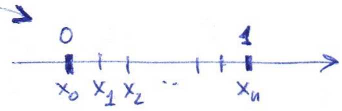
$$\implies \begin{cases} \lim\limits_{t\to\infty} h(t) = \lim\limits_{t\to\infty} \int_E \xi_t(x)\,dx \overset{\text{DCT}}{=} \int_E \underbrace{\lim\limits_{t\to\infty} \xi_t(x)}_{=0}\,dx = 0 \\ \lim\limits_{t\to-\infty} h(t) \overset{\text{DCT}}{=} \int_E \lim\limits_{t\to-\infty} \xi_t(x)\,dx = \int_E f(x)\,dx \end{cases}$$

Ex. (4.3)

let $f(x) := \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$.

$f \in BV([0,1])$?

Solution: Consider a partition P_n: $\begin{cases} x_i \in \mathbb{Q} & i \text{ even} \\ x_i \notin \mathbb{Q} & i \text{ odd} \end{cases}$



$$V_0^1(f; P_n) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = 2n \qquad (\forall i : |f(x_i) - f(x_{i-1})| = 2)$$

$$\implies V_0^1(f) = \sup_{P \in \mathcal{P}} V_0^1(f; P) \ge \sup_{P_n} V_0^1(f; P_n) = \lim_{n\to\infty} 2n = +\infty$$

$$\implies V_0^1(f) = +\infty$$

$$\implies f \notin BV([0,1])$$

Ex. (4.8)

let $f(x) := \begin{cases} \dfrac{\sin(\log(x))}{\log^2(x)} & x \in (0, \frac{1}{2}] \\ 0 & x = 0 \end{cases}$.

$f \in AC([0, \frac{1}{2}])$?

Solution: $\lim\limits_{x\to0^+} f(x) = 0 \implies f \in C^0([0, \frac{1}{2}])$.

From here we can procede and we need to prove:
- $f' \in L^1$ ($\exists f'$ a.e. and $\in L^1$)
- the fundamental formula holds

$f \in C^1((0, \frac{1}{2}]) \implies \forall\, 0 < \xi < x \le \frac{1}{2}: \quad f(x) - f(\xi) = \int_\xi^x f'(t)\,dt \qquad (f' \in C^0)$

If $f' \in L^1([0, \frac{1}{2}])$ then: (as $\xi \to 0^+$)

$$f(x) - f(0) = \int_0^x f'(t)\,dt \qquad \forall x \in [0, \tfrac{1}{2}]$$

$$\longrightarrow f \in AC([0, \tfrac{1}{2}])$$

$f' \in L^1$: $\forall x \in (0, \frac{1}{2})$: $$f'(x) = [\ldots] = \underbrace{-\frac{2}{x}\frac{1}{\log^3(x)}\sin(\log(x))}_{:=\eta_1} + \underbrace{\frac{1}{x}\frac{1}{\log^2(x)}\cos(\log(x))}_{:=\eta_2}$$

Then:

$$\int_0^{1/2} |\eta_1(x)|\,dx \overset{\substack{|\sin(\cdot)| < 1 \\ \text{and } \log^3(x) < 0}}{\le} -\int_0^{1/2} \frac{2}{x}\frac{1}{\log^3(x)}\,dx = \frac{1}{\log^2(2)} < \infty$$

$$\int_0^{1/2} |\eta_2(x)|\,dx \le \int_0^{1/2} \frac{1}{x\log^2(x)}\,dx = \frac{1}{\log(2)} < \infty$$

→ not important the values

f' continuous in $(0, 1/2] \implies f'$ is measurable

$$\int_0^{1/2} |f'(x)|\,dx < \infty \implies f' \in L^1$$

Why, as $\xi \to 0^+$, we get $\int_0^{+\infty}$?

The question is :

$$\lim_{\xi \to 0} \int_\xi^b f'(t)\, dt \ ?$$

with b fixed and $f' \in L^1([0,b])$.

$$\int_\xi^b f'(t)\, dt = \int_0^b \underbrace{f'(t)\, \mathbb{1}_{[\xi,b]}(t)}\, dt$$

we call this
$:= \sigma_\xi(t)$
(where ξ replace the usual n in a sequence and t is the variable)

$$\begin{cases} \sigma_\xi(t) \xrightarrow[\xi \to 0]{} f'(t) & \text{a.e. in } [0,b] \\ |\sigma_\xi(t)| \leq f'(t) & \text{a.e. in } [0,b] \ \forall \xi \end{cases}$$

$$\overset{DCT}{\Longrightarrow} \lim_{\xi \to 0} \int_\xi^b f'(t)\, dt = \lim_{\xi \to 0} \int_0^b \sigma_\xi(t)\, dt$$

$$= \int_0^b \lim_{\xi \to 0} \sigma_\xi(t)\, dt$$

$$= \int_0^b f'(t)\, dt$$

note: if we don't know that $f' \in L^1$ we cannot apply DCT and so the conclusion is not true in general (the conclusion that $\int_\xi^b \xrightarrow{\xi \to 0} \int_0^b$)

Note: when we solve exercises we can take this for granted

Ex. **1** (08/09/21) 04/11

Consider the functions $f, g : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

hint [if we are forced to use the definition then, probably, we have to prove that $\notin BV$]

1. Determine if f is $BV([0,1])$ by using the def. of BV function.
2. Is g differentiable a.e. on $[0,1]$? If so, determine the a.e. derivative of g and say if it belongs to $L^1([0,1])$.
3. Is $g \in AC([0,1])$? Is $g \in BV([0,1])$?

Solution:

- The answers 2., 3. are contained in the solution of Ex. **2** of 28/10/2021 where we have proved that:

$$h(x) = \begin{cases} x^\alpha \cos\left(\frac{1}{x^\beta}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

is an $AC([0,1])$ $\forall \alpha > \beta > 0$.
In particular we have proved that:

(i) $h \in C([0,1])$

(ii) $\exists h'(x)$ $\forall x \in (0,1]$ and $h' \in L^1([0,1])$

(iii) h satisfies the fundamental theorem of calculus since $h \in C^1([\xi, 1])$ $\forall 0 < \xi \leq 1$

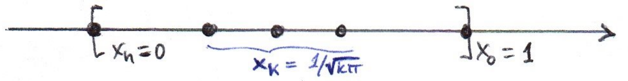
- We are left to discuss the 1st question.
f is a particular case of h, corresponding to $\alpha = \beta = 2$.
Moreover, $f \in C([0,1])$, since:

$$\exists \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 \overbrace{\cos\left(\frac{1}{x^2}\right)}^{\left|\cos\left(\frac{1}{x^2}\right)\right| \leq 1} = 0$$

however the continuity does not say anything about the total variation. In fact

We now prove that $f \notin BV([0,1])$.
To this end we fix $n \geq 4$ and we consider the partition:

$$x_n = 0, \qquad x_k = \frac{1}{\sqrt{k\pi}} \quad (k = 1, .., n-1) \qquad x_0 = 1$$



Using this partition, we have:

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \geq \sum_{k=2}^{n-1} |f(x_k) - f(x_{k-1})|$$

$$= \sum_{k=2}^{n-1} \Big| \underbrace{\frac{1}{k\pi} \cos(k\pi)}_{f\left(\frac{1}{\sqrt{k\pi}}\right)} - \underbrace{\frac{1}{(k-1)\pi} \cos((k-1)\pi)}_{f\left(\frac{1}{\sqrt{(k-1)\pi}}\right)} \Big|$$

$$= \sum_{k=2}^{n-1} \left| \frac{1}{k\pi} (-1)^k - \frac{1}{(k-1)\pi} (-1)^{k-1} \right|$$

$$= \sum_{k=2}^{n-1} \left| \frac{1}{k\pi} + \frac{1}{(k-1)\pi} \right|$$

$$\geq \frac{1}{\pi} \sum_{k=2}^{n-1} \frac{1}{k-1}$$

From this, reminding that:

$$\lim_{n \to \infty} \sum_{k=2}^{n-1} \frac{1}{k-1} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

we conclude that:

$$V_0^1(f) = \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = +\infty$$

$$\Big(\geq \underbrace{\frac{1}{\pi} \sum_{k=2}^{n-1} \frac{1}{k-1}}_{\xrightarrow[n\to\infty]{} \infty} \quad \forall n \geq 4 \Big)$$

Remark: By using the same argument try to prove that $h \notin BV([0,1])$ whenever $\alpha, \beta > 0$ but $\beta \leq \alpha$.
This argument corresponds to the particular choice $\alpha = \beta = 2$.

Ex. 2 (21/06/2021 - Ex. 1)

Let $g : \mathbb{R} \to \mathbb{R}$: $g(x) = \dfrac{x}{1+x^2}$.

For every n we define:

$$f_n(x) := g(nx)\, \mathbb{1}_{[-n,n]}(x) = \frac{nx}{1+(nx)^2}\, \mathbb{1}_{[-n,n]}$$

1. Study the a.e. convergence and the uniform convergence of f_n
2. Study the convergence of f_n in L^1 and in measure

Solution: 1. It's easy to show that $f_n(x) \xrightarrow[n\to\infty]{} 0 \quad \forall x \in \mathbb{R}$.
Indeed, if x is fixed then $\forall n \in \mathbb{N}$ with $n \geq |x|$ (>0) we have:

$$f_n(x) = \frac{nx}{1+(nx)^2} \cdot \underbrace{\mathbb{1}_{[-n,n]}(x)}_{1 \text{ since } |x| \leq n}$$

$$= \frac{nx}{1+(nx)^2} \sim \frac{1}{nx} \xrightarrow[n\to\infty]{} 0$$

If $x=0$ then $f_n(0) = 0 \quad \forall n \in \mathbb{N} \implies f_n(0) \xrightarrow[n\to\infty]{} 0$

Summing up, the sequence $(f_n)_n$ converges to the fun.
$g(x) \equiv 0$
pointwise in $\mathbb{R}$. (in particular a.e. in $\mathbb{R}$)
As regards the uniform convergence, we first observe that if $(f_n)_n$ is uniformly convergent to some function, then this function must be equal to $g(x) \equiv 0$.
(since uniform convergence $\Rightarrow$ pointwise convergence).
On the other hand:

$$f_n\left(\tfrac{1}{n}\right) = \underbrace{\frac{n \cdot \frac{1}{n}}{1+n^2\left(\frac{1}{n}\right)^2}}_{\frac{1}{2}} \; \underbrace{\mathbb{1}_{[-n,n]}\left(\tfrac{1}{n}\right)}_{1}$$

$$= \frac{1}{2} \quad \forall n \in \mathbb{N}$$

and so, we can conclude that:

convergent function

$$\sup_{\mathbb{R}} |f_n - 0| = \sup_{\mathbb{R}} |f_n| \geq \frac{1}{2} \quad \forall n \in \mathbb{N}$$

and hence:

$$\sup_{\mathbb{R}} |f_n| \underset{n\to\infty}{\not\longrightarrow} 0 \quad \Longrightarrow \quad f_n \text{ is not uniformly convergent to } g(x) \equiv 0$$

2. Since the L^1 convergence implies the convergence in measure we start to study the L^1 convergence of $(f_n)_n$.
To this end we observe that, since the L^1-convergence implies the a.e. pointwise convergence (up to a subsequence), the only possible L^1 limit is again $g(x) \equiv 0$.
Hence, let us see if $f_n \xrightarrow[n\to\infty]{} 0$ in $L^1(\mathbb{R})$.
We have:

$$d(f_n, 0) = \|f_n - 0\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f_n - 0| \, dx$$

$$= \int_{\mathbb{R}} \left| \frac{nx}{1+n^2x^2} \right| \mathbb{1}_{[-n,n]}(x) \, dx$$

$$= \int_{-n}^{n} \left| \frac{nx}{1+n^2x^2} \right| dx$$

symmetric interval and $|\cdot|$

$$= 2\int_0^n \frac{nx}{1+n^2x^2} \, dx$$

$$= 2\int_0^{+\infty} \frac{nx}{1+n^2x^2} \mathbb{1}_{[0,n]}(x) \, dx$$

Since $\mathbb{1}_{[0,n]} \xrightarrow[n\to\infty]{} 1$ and:

$$\frac{nx}{1+n^2x^2} \sim \frac{1}{nx} \leq \frac{1}{x}$$

$\left(\frac{1}{x} \notin L^1([1,+\infty))\right)$ we cannot find a dominating function $g \in L^1([1,+\infty))$ allowing to apply the DCT.
We then compute explicitly the integral:

$$2\int_0^n \frac{nx}{1+n^2x^2} \, dx = 2\int_0^{n^2} \frac{y}{1+y^2} \cdot \frac{1}{n} \, dy$$

$y = nx, \quad x = \frac{1}{n}y$
$dx = \frac{1}{n}dy$

$$= \frac{1}{n}\int_0^{n^2} \frac{2y}{1+y^2} \, dy$$

$$= \frac{1}{n}\left[\log(1+y^2)\right]_0^{n^2}$$

$$= \frac{1}{n}\log(1+n^4)$$

As a consequence we get:

$$\lim_{n\to\infty} \|f_n - 0\|_{L^1(\mathbb{R})} = \lim \frac{\log(1+n^4)}{n} = 0$$

and this proves that $f_n \xrightarrow[n\to\infty]{} 0$ in $L^1(\mathbb{R})$.
In particular, we can also conclude "for free" that $f_n \xrightarrow[n\to\infty]{} 0$ in measure.

Remark/Ex : Even if the DCT cannot be applied, by explicitly computing the integral we have proved that :

$$f_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } L^1(\mathbb{R}).$$

Try to prove that $\forall p > 1$ we also have that:

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n|^p \, dx = 0 \qquad \left(\Leftrightarrow f_n \xrightarrow[n\to\infty]{} 0 \text{ in } L^p(\mathbb{R})\right)$$

Hint: this integral cannot be computed, but this time $(p>1)$ we have that :

$$|f_n(x)|^p = \left|\frac{nx}{1+n^2x^2}\right|^p \underbrace{\mathbb{1}_{[-n,n]}(x)}_{\leq 1}$$

$$\leq \frac{1}{|x|^p} \in L^1(\mathbb{R}) \quad (\text{" at } \infty \text{"}) \implies (\text{DCT})$$