$p_{ij}(t,s) = \mathbb{P}(X_t = j \mid X_s = i)$ $\longrightarrow$
- $0 \le p_{ij} \le 1$
- $\sum_{j\in E} p_{ij} = 1$

$\sum_{j\in E} \mathbb{P}(X_1 = j \mid X_0 = i) = 1$

$p_{ij}^{(2)} = \sum_{k\in E} p_{ik}\, p_{kj} = \mathbb{P}(X_{t+2} = j \mid X_t = i)$

$\mathbb{P}(X_{t_m} = j_m, X_{t_{m-1}} = j_{m-1}, \dots, X_0 = j_0) = \mathbb{P}(X_0 = j_0)\, p_{j_0 j_1}^{(t_1)}\, p_{j_1 j_2}^{(t_2 - t_1)} \cdots p_{j_{m-1} j_m}^{(t_m - t_{m-1})}$

$P = (p_{ij})$ transition matrix

STATE CLASSIFICATION

Accessible state: j accessible from i if $\exists\, n \ge 0 : \mathbb{P}(X_n = j \mid X_0 = i) = p_{ij}^{(n)} > 0$

Communicating states: j and i communicate if one is accessible from the other

Class of states: $C \subseteq E$ c.o.s. if all states in C communicate and they don't communicate with states in E/C

Irreducibility: $(X_n)_{n\ge 0}$ irreducible $\iff$ all the states communicate

Recurrent state: i recurrent $\iff \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{X_n = i\} \mid X_0 = i\right) = 1$

Transient state: i transient $\iff$ i <u>not</u> recurrent

First entrance: $T_i = \begin{cases} \min\{n : X_n = i\} & \text{if } \{n \ge 1 : X_n = i\} \ne \emptyset \\ +\infty & \text{otherwise} \end{cases}$

probability of entering in i only at the n-th step starting from j $= f_{ji}^{(n)} = \mathbb{P}(T_i = n \mid X_0 = j)$

NB:
$(T_j < \infty) = \left(\bigcup_{n=1}^{\infty} (X_n = j)\right)$
$(T_j = 1) = (X_1 = j)$
$(T_j = n) = (X_n = j) \cap_{m=1}^{n-1} (X_m \ne j)$

Renewal equation: $p_{ij}^{(n)} = \sum_{\nu=1}^{n} f_{ij}^{(\nu)}\, p_{jj}^{(\nu)} \quad \forall n \ge 1$

i recurrent $\iff \sum_{n=0}^{+\infty} p_{ii}^{(n)} = +\infty$

$\sum_{n=0}^{+\infty} p_{ii}^{(n)} = \underbrace{\mathbb{E}_i\left[\sum_{n=0}^{+\infty} \mathbb{1}\{X_n = i\}\right]}_{\text{mean time of being in } i}$

i transient $\iff \sum_{n=0}^{+\infty} p_{ii}^{(n)} = \dfrac{1}{1 - \boxed{\mathbb{P}_i(T_i < \infty)}} < \infty$

$\mathbb{P}_i(T_j < \infty) = \sum_{n=1}^{\infty} f_{ij}^{(n)} = \sum_{n=1}^{\infty} \mathbb{P}_i(T_j = n)$

i, j communicating $\Longrightarrow$ both recurrent / transient

j transient $\Longrightarrow \begin{cases} \lim_{n\to\infty} \mathbb{P}_i(X_n = j) = \mathbb{P}(X_n = j \mid X_0 = i) = 0 \\ \forall i \ \sum_{n\ge 0} p_{ij}^{(n)} < +\infty \end{cases}$

E finite $\Longrightarrow \exists$ at least one recurrent state

$p_{ii}^{(n)} = \mathbb{E}_i[\mathbb{1}\{X_n = i\}]$

INVARIANT LAWS, ASYMPTOTIC BEHAVIOUR

Invariant distribution $(X_n)_{n\ge 0}$ MC, $(p_{ij})_{i,j\in E}$ transition matrix

How much do we stay in a state?

$\pi = (\pi_i)_{i\in E}$ probability density $(0 \le \pi_i \le 1,\ \sum_{i\in E} \pi_i = 1)$

π invariant distribution if $X_{n+1} \sim \pi$ whenever $X_n \sim \pi$ $\forall n$.

π invariant $\iff \pi_j = \sum_{i\in E} p_{ij}\, \pi_i$

$\iff \pi P = \pi$

E finite $\Longrightarrow \exists\, \pi$ (at least one)

$\exists\, \pi \not\Rightarrow \exists!\, \pi$

E infinite $\not\Rightarrow \exists / \nexists\, \pi$

- Reversable probability density $(\pi_i)_{i\in E}$:

  $(\pi_i)_{i\in E}$ reversable $\iff$ $\pi_i \, p_{ij} = \pi_j \, p_{ji}$

  $(\pi_i)_{i\in E}$ reversable $\Longrightarrow$ $(\pi_i)_{i\in E}$ invariant

- $$\left.\begin{array}{l} p_{ij}^{(n)} \xrightarrow{n} \mu_j \qquad \perp\!\!\!\perp i \\ \sum_{j\in E} \mu_j = 1 \end{array}\right\} \Longrightarrow \quad \mu_j = \sum_{i\in E} \mu_i \, p_{ij}$$

- $\exists \lim_{n\to\infty} p_{ij}^{(n)} > 0 \quad \perp\!\!\!\perp i \quad \Longrightarrow \quad \exists! \, \pi$ invariant distr., and, $\forall f$ bounded $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \int f \, d\pi = \sum_{j\in E} f(j)\pi_j$

Period of a state $= MCD\{n \geq 1 : p_{ii}^{(n)} > 0\}$

- if period $= 1 \rightarrow$ "aperiodic"
- states of the same class have the same period

STRONG MARKOV PROPERTY

$T : \Omega \to \mathbb{N}$ stopping time $\iff$ $\forall n$ $\{T \leq n\}$ belongs to the σ-algebra generated by $\{X_n = i_n, \dots, X_0 = i_0\}$

$T_j = \inf\{n \geq 1 : X_n = j\}$: $\quad \{T_j \leq n\} = \bigcup_{m=1}^{n} \{X_m = j\}$

first entrance time $\quad \{T_j = n\} = \{X_n = j, X_{n-1} \neq j, \dots, X_1 \neq j\}$

$U_j = \inf\{n \geq 1 : X_n \neq j\}$: $\quad \{U_j \leq n\} = \{U_j > n\}^c = \{X_n = j, \dots, X_1 = j\}^c$

first exit time

If $(X_n)_{n\geq 0}$ MC with $(p_{ij})_{i,j\in E}$ s.t. $p_{jj} \in (0,1)$

$$\Longrightarrow \begin{cases} U_j \sim \mathcal{G}(1 - p_{jj}) & (Y \sim \mathcal{G}(\theta) : \; \mathbb{P}(Y = k) = (1-\theta)^{k-1}\theta) \\ (X_{U_j})(\omega) = \sum_{n\geq 1} X_n(\omega) \, \mathbb{1}_{\{U_j = n\}}(\omega) & : \quad \mathbb{P}_j(X_{U_j} = k) = \dfrac{p_{jk}}{1 - p_{jj}} \quad \forall k \neq j \end{cases}$$

$(X_n)_{n\geq 0}$ MC, T stopping time :

$$Y_n(\omega) = \begin{cases} X_{T(\omega)+n}(\omega) & \text{if } T(\omega) < +\infty \\ \text{arbitrary} & T(\omega) = +\infty \end{cases} \quad \longrightarrow \quad (Y_n)_{n\geq 0} \text{ is a MC with the same } (p_{ij})_{i,j\in E} \text{ of } (X_n)_{n\geq 0}$$

restarted MC

Strong Markov property :

$$\left.\begin{array}{l} (X_n)_{n\geq 0} \text{ MC} \\ (p_{ij})_{i,j\in E} \text{ transition matrix} \\ T \text{ stopping time} \end{array}\right\} \longrightarrow \underbrace{(X_{T\wedge n})_{n\geq 0}}_{\text{stopped MC}} \; \perp\!\!\!\perp \; \underbrace{(X_{T+n})_{n\geq 0}}_{\text{restarted MC}}$$

min $\{T(\omega), n\}$

w.r.t. $\mathbb{P}(\cdot \mid X_T = i, T < \infty)$

TRANSIENCY, RECURRENCY, ABSORPTION

The number of visits of a recurrent state is (a.s.) infinite.

Probability of staying forever in a transient state

T = set of transient states

$U_i = \mathbb{P}_i(\bigcap_{n=1}^{\infty}\{X_n \in T\})$ $\quad i \in T$ $\quad$ = prob. of staying for ever in T

$\Longrightarrow$ U_i is the biggest $[0,1]$-solution of: $\boxed{U_i = \sum_{j\in T} p_{ij} U_j}$ $\quad i \in T$

(T finite $\Longrightarrow$ $U_i = 0$)

Absorption probability in a recurrent class

V_i = prob. of absorption in a recurrent class C starting from $i \in T$

$\Longrightarrow$ V_i is the smallest $[0,1]$-solution of: $\boxed{V_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} V_j}$ $\quad i\in T$

(T finite $\Longrightarrow \exists!\, (V_i)_{i\in T}$)

Mean absorption time in a recurrent class

E finite, C = unique recurrent class

w_i = mean absorption time in C, $i \in T$

$\Longrightarrow$ w_i is finite and satisfies: $\boxed{w_i = 1 + \sum_{j\in T} p_{ij} w_j}$ $\quad i \in T$

(V is a $\mathbb{N}$-valued r.v. $\longrightarrow$ $\mathbb{E}[V] = \sum_{n=0}^{\infty} \mathbb{P}(V > n)$)

Transience criterium

$(X_n)_{n\geq 0}$ irreducible MC, E countable

$\Longrightarrow$ $(X_n)_{n\geq 0}$ transient $\Longleftrightarrow$ $\exists$ bounded, non-constant solution of:

$\sum_{k\in E} p_{jk} y_k = y_j \quad \forall j \in E$ (but one at most)

Recurrence criterium

$(X_n)_{n\geq 0}$ irreducible MC.

If $\exists (y_j)_{j\in E}$ s.t. :
- $\sum_{k\in E} p_{jk} y_k \leq y_j \quad \forall j \in E$ but one at most
- $\lim_{k\to\infty} y_k = +\infty$ $\equiv$ $\forall M>0\ \exists F \subseteq E$ finite s.t. $y_k > M \quad \forall k \notin F$

$\Longrightarrow$ $(X_n)_{n\geq 0}$ is recurrent.

INVARIANT LAWS, ASYMPTOTIC BEHAVIOUR (pt. 2)

Sufficient cond. for $\exists \pi$ = invariant distribution

$(X_n)_{n\geq 0}$ irreducible MC

If $\exists (y_j)_j, (x_j)_j$ unbounded ($\lim_{j\to\infty} x_j = \lim_j y_j = +\infty$) s.t.

$\sum_{k\geq 0} p_{jk} y_k \leq y_j - x_j \quad \forall j$ $\longrightarrow$ $\exists! \, (\pi_j)_j$ invariant for the MC

Sojourn time in $S \subseteq T$: $\quad T_S = \sum_{n\geq 0} \mathbb{1}\{X_n \in S\}$

Moment generating function: $\quad m_i = \mathbb{E}_i[z^T] = \mathbb{E}_i[e^{tT}] \quad \forall i \in E$

properties of the moment generating f. of $T = T_S$: (if S is finite $\rightarrow$ m_i is well defined)

- $m_i(z) = \sum_{k\geq 0} z^k \, \mathbb{P}_i(T = k)$
- $m_i'(z) = \mathbb{E}_i\left[\frac{d}{dz}\left(e^{T\log(z)}\right)\right] = \mathbb{E}_i\left[z^{T-1}\right]$
- $m_i(z) = \begin{cases} 1 & i \text{ recurrent} \\ z\sum_j p_{ij} m_j(z) & i \in S \\ \sum_j p_{ij} m_j(z) & i \in T\setminus S \end{cases}$

Transition probabilities & Transition rates

$(X_t)_{t\geq 0}$ r.v. with values in E (countable / finite)

Time homogeneous : $\mathbb{P}(X_{t+s}=j \mid X_s=i) = \mathbb{P}(X_t=j \mid X_0=i) := p_{ij}(t)$

Transition matrix at time t : $P_t = (p_{ij}(t))_{i,j\in E}$:

- $0 \leq p_{ij}(t) \leq 1$
- $\sum_{j\in E} p_{ij}(t) = 1$
- $P_{t+s} = P_t P_s$
- $(X_t)_{t\geq 0}$ continuous in probability* $\Rightarrow t \mapsto p_{ij}(t)$ continuous
- $P_0 = I$

$* \lim_{s\to t} \mathbb{P}(X_s \neq X_t) = 0$

Chapman-Kolmogorov : $p_{ij}(t+s) = \sum_{k\in E} p_{ik}(t)\, p_{kj}(s)$

Transition rate matrix : $Q = (q_{ij})_{i,j\in E}$:

- $q_{ij} = \lim_{t\to 0^+} \frac{p_{ij}(t)}{t} \geq 0$
- $q_{ii} = \lim_{t\to 0^+} \frac{p_{ii}(t)-1}{t} \leq 0$
- $\sum_{j\in E} q_{ij} = 0$

- $q_{ii} = 0 \Rightarrow i$ absorbing $(p_{ii}(t)=1 \ \forall t)$
- $q_{ii} = -\infty \Rightarrow i$ istantaneous state
- $\exists q_{ij}$ if $p_{ij}(t)$ are continuous

FKE : $p_{ij}'(t) = \sum_{k\in E} p_{ik}(t)\, q_{kj}$

BKE : $p_{ij}'(t) = \sum_{k\in E} q_{ik}\, p_{kj}(t)$

$+$ initial cond : $p_{ij}(0) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

FKE $\Rightarrow P_t' = P_t Q$

~~$x' = ax \rightarrow x(t) = e^{ta}x_0$~~

Solution of Kolmogorov equations

- If $\sup |q_{ii}| < +\infty \Rightarrow$ BKE and FKE have the same solution :

$$\boxed{P_t = e^{tQ}}$$

Exit time from a state & discrete skeleton

$T_i = \inf\{t>0 : X_t \neq i\}$ $\quad i\in E$ $\quad$:= Exit time (from i)

- If $-\infty < q_{ii} < 0 \Rightarrow T_i \sim \mathcal{E}(-q_{ii})$
- If $-\infty < q_{ii} < 0 \Rightarrow \forall j\neq i$: $\boxed{\mathbb{P}_i(X_{T_i}=j)} = \frac{q_{ij}}{-q_{ii}}$

state visited when leaving i

probability of going to j after leaving i

It's a distribution :

$$\frac{\sum_{j\neq i} q_{ij}}{-q_{ii}} = \frac{-q_{ii}}{-q_{ii}} = 1$$

$(X_{T_i}$ takes values in $E\setminus\{i\})$

This motivate the introduction to the discrete skeleton

$(X_n)_{n\geq 0} \Rightarrow (T_n)_{n\geq 0}$: $\begin{cases} T_0 = 0 \\ T_n = \inf\{t \geq T_{n-1} : X_t \neq X_{T_{n-1}}\} \end{cases}$

(p. 8)

$(Y_n)_{n\geq 0}$ ...

Discrete MC $\overset{?}{\longleftrightarrow}$ Continuous MC :

$\exists$ a discrete MC associated to a continuous MC : DISCRETE SKELETON

$$\boxed{q_{ii} \in (-\infty, 0]}$$

$(\hat{p}_{ij})_{i,j \in E}$ discrete skeleton :
$$\hat{p}_{ij} = \begin{cases} \dfrac{q_{ij}}{-q_{ii}} & q_{ii} \neq 0 \\ 1 & q_{ii} = 0, \ i = j \\ 0 & (q_{ii} = 0, \ i \neq j) \vee (q_{ii} \neq 0, \ i = j) \end{cases}$$

$(X_t)_{t \geq 0}, \ Q = (q_{ij})_{i,j \in E} \longrightarrow (Y_n)_{n \geq 0}, \ \hat{P} = (\hat{p}_{ij})_{i,j \in E}$

we can define a stochastic matrix $\hat{P}$ starting from Q and associate it to a discrete time MC $(Y_n)_{n \geq 0}$

POISSON PROCESS

Poisson process:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ & \cdots & & & & \end{bmatrix} \quad \lambda > 0$$

$$i \xrightarrow{\sim \mathcal{E}(\lambda)} i+1$$

$$\hat{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & & \end{bmatrix} \quad \text{discrete skeleton}$$

- $(N_t)_{t \geq 0}$ MC with $Q = (q_{ij})_{i,j \in E}$;
$$\mathbb{P}_i(N_t = n \mid N_0 = 0) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \qquad \left(N_t \mid N_0 = 0 \sim \mathcal{P}(\lambda t) \right)$$

- $N_{t_1}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}$ are $\perp\!\!\!\perp$ w.r.t. $\mathbb{P}_0$ $\quad \Big| \ \forall n, \ \forall t_1 < t_2 < \dots < t_n$
$$\Longrightarrow N_{t_k} - N_{t_{k-1}} \sim \mathcal{P}(\lambda(t_k - t_{k-1}))$$

Birth-Death processes

$(X_t)_{t \geq 0}$: X_t = # individuals at time t

- PURE BIRTH : $n \xrightarrow{\sim \mathcal{E}(\lambda_n)} n+1 \quad \lambda_n > 0$

  special cases :
  1. $\lambda_n = \lambda > 0 \Longrightarrow$ Poisson process
  2. $\lambda_n = (n+1)\lambda \Longrightarrow$ waiting time for a new arrival $\sim \mathcal{E}(\lambda(n+1))$
  3. $\lambda_n = n\lambda \Longrightarrow$ Yule process
  $$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & -2\lambda & 2\lambda & 0 & \cdots \end{bmatrix}$$
  $$\mathbb{P}_i(X_t = k) = p_{ik}(t) = \binom{k-1}{k-i} e^{-\lambda i t} (1 - e^{-\lambda t})^{k-i}$$

- PURE DEATH : $n-1 \xleftarrow{\sim \mathcal{E}(} n$
$$Q = \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ \vdots \end{bmatrix}$$

POPULATION DYNAMICS

states classification

j accessible from i if $\exists t : p_{ij}(t) > 0$

i,j communicate if each one is accessible from the other

(+) class of states, irreducible MC (p. 14)

j accessible from i $\Longleftrightarrow$ $\exists\, n : \underbrace{i_1, \dots, i_n}_{\neq i,j}$ s.t. $\boxed{q_{ii_1} \cdots q_{i_n j} > 0}$

i recurrent : $\mathbb{P}_i\left(\omega \mid \underbrace{\{t \geq 0 \mid X_t(\omega) = i\}}\ \text{is unbounded}\right) = 1$

i transient : $\mathbb{P}_i\left(\omega \mid \underbrace{\{t \geq 0 \mid X_t(\omega) = i\}}_{\text{we visit } i \text{ for times } t \geq 0 \text{ in an unbounded subset of } [0,+\infty)}\ \text{is unbounded}\right) = 0$

Theorem:
- i recurrent (/transient) for the continuous MC $(X_t)_{t\geq 0}$ $\Longleftrightarrow$ i recurrent (/transient) for the discrete skeleton
- i is either recurrent or transient and all the states of a class of states are (all) recurrent / transient

(+) p. 15

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invariant density

Invariant densities

$(X_t)_{t\geq 0}$ continuous time MC
$(P_t)_{t\geq 0}$ transition semigroup
$X_0 \sim \mu$ $(\mathbb{P}(X_0 = i) = \mu_i \ \forall i)$

$\longrightarrow$ $\mathbb{P}(X_t = j) = \sum_i \mathbb{P}(X_t = j \mid X_0 = i)\, \mathbb{P}(X_0 = i)$
$= \sum_i p_{ij}(t)\, \mu_i$

$(\mu_i)_{i\in E}$ invariant density $\Longleftrightarrow$ $\boxed{\mu_j = \sum_{i \in E} \mu_i p_{ij}(t) \quad \forall j\, \forall t}$

- $\left(\frac{d}{dt}\mu_j = \sum_{i\in E} \mu_i \frac{d}{dt} p_{ij}(t)\right)$ at 0 : $\boxed{0 = \sum_{i\in E} \mu_i q_{ij}}$

- $q_{ii} > -\infty$
  $(p_{ij}(t))_{i,j}$ unique sol. of FKE, BKE $\longrightarrow$ $(\mu_i)_{i\in E}$ s.t. $\begin{cases} \sum_{i\in E} \mu_i = 1 \\ \mu_i \geq 0 \quad \forall i \end{cases}$ is an invariant density
  $\Longleftrightarrow$ $\boxed{0 = \sum_{i\in E} \mu_i q_{ij} \quad \forall j \in E}$
  $(\pi Q = 0)$

- E finite $\Rightarrow$ $\exists$ at least 1 invariant
- $(\pi_i)_{i\in E}$ inv. density for $(X_t)_{t\geq 0}$
  $\Longleftrightarrow$ $(\pi_j(-q_{jj}))_{j\in E}$ * is an invariant density for the discrete skeleton $(Y_n)_{n\geq 0}$

  * (to be normalized)

- Ergodic theorem:
  Discrete positive recurrent irreducible MC:
  $$\frac{\sum_{k=1}^{n} \mathbb{1}\{X_k = i\}}{n} \longrightarrow \pi_i \quad \text{invariant density}$$
  Continuous positive recurrent MC:
  $$\frac{\int_0^t \mathbb{1}\{X_s = i\}\, ds}{t} \longrightarrow \pi_i \quad \text{inv. dens.}$$
  $$\pi_i = \frac{1}{-q_{ii}\, \mathbb{E}_i[T_i]}$$
  T_i exit time

POPULATION DYNAMICS

Population dynamics: $X = \#$ individuals at time t

$$n-1 \xleftarrow{\mathcal{E}(\mu_n)} n \xrightarrow{\mathcal{E}(\lambda_n)} n+1$$

- **PURE BIRTH** (not irreducible)

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ & & \cdots & & & \end{bmatrix}$$

1. $\lambda_n = \lambda \quad \forall n \Rightarrow$ Poisson process
2. $\lambda_n = (n+1)\lambda$
3. $\lambda_n = n\lambda \longrightarrow$ Yule process

Case:

2. $\mathbb{P}_i(X_t = k) = p_{ik}(t) = \binom{k}{i} e^{-\lambda(i+1)t}\left(1 - e^{-\lambda t}\right)^{k-i}$

3. $\mathbb{P}_i(X_t = k) = p_{ik}(t) = \binom{k-1}{k-i} e^{-\lambda i t}\left(1 - e^{-\lambda t}\right)^{k-i}$

- **PURE DEATH** (not irreducible)

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -\mu_1 & 0 & 0 & 0 & \cdots \\ 0 & \mu_2 & -\mu_2 & 0 & 0 & \cdots \\ 0 & 0 & \mu_3 & -\mu_3 & 0 & \cdots \\ & & \cdots & & & \end{bmatrix}$$

1. $\mu_n = \mu$
2. $\mu_n = n\mu$

Case:

2. $\mathbb{P}_i(X_t = k) = p_{ik}(t) = \binom{i}{k} e^{-\mu i t}\left(e^{\mu t} - 1\right)^{k-i}$

$\qquad \binom{i}{k} e^{-\mu i t}\left(1 - e^{-\mu t}\right)^{i-k}$

- **BIRTH-DEATH PROCESSES** (irreducible iff $\lambda_n > 0 \ \forall n \geq 0$, $\mu_n > 0 \ \forall n > 0$)

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & -\lambda_2 & 0 & \cdots \\ & & \cdots & & & \end{bmatrix}$$

$n \to n+1 \quad \sim \mathcal{E}(\lambda_n)$

$n \to n-1 \quad \sim \mathcal{E}(\mu_n)$

$\Longrightarrow$ $B_n \sim \mathcal{E}(\lambda_n)$, $D_n \sim \mathcal{E}(\mu_n)$

we leave the state n after a random time, which is the minimum of B_n, D_n

$\Longrightarrow$ we're leaving $n \sim \mathcal{E}(\lambda_n + \mu_n)$

$$\mathbb{P}(B_n < D_n) = \frac{\lambda_n}{\mu_n + \lambda_n} \quad \left(= \frac{q_{ii}}{-q_{ii}} \right)$$

invariant:

$$\pi_n = \left(\frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right) \cdot \left(\frac{1}{1 + \sum_{n\geq 1} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}}\right)$$

Explosion (in finite time):

pure birth process with trans. rates λ_n

If λ_n are s.t. $\sum_{n\geq 0} \frac{1}{\lambda_n} < +\infty \Longrightarrow$ In a finite time X_t diverges (explode)

Conditional expectation

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space.
$\mathcal{G} \subseteq \mathcal{F}$ sub σ-algebra.
X random variable s.t. $\mathbb{E}[|X|] < \infty$

$\Longrightarrow$ we call conditional expectation of X w.r.t. $\mathcal{G}$ any $\mathcal{G}$-measurable random variable Y s.t. : $\int_G X \, d\mathbb{P} = \int_G Y \, d\mathbb{P} \quad \forall G \in \mathcal{G}$

(Thm.) X $\mathcal{F}$-meas., integrable
$\mathbb{E}[|X|] < \infty$
$\mathcal{G} \subseteq \mathcal{F}$

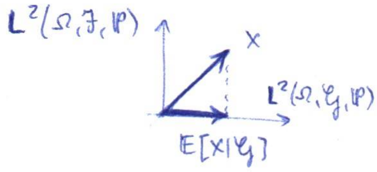
$\longrightarrow$ $\exists Y$ $\mathcal{G}$-meas. s.t.
$\int_G X \, d\mathbb{P} = \int_G Y \, d\mathbb{P} \quad \forall G \in \mathcal{G}$
(it's unique, up to sets $G \in \mathcal{G} : \mathbb{P}(G) = 0$)

- $\mathbb{E}[aX_1 + bX_2 | \mathcal{G}] = a\mathbb{E}[X_1|\mathcal{G}] + b\mathbb{E}[X_2|\mathcal{G}] \qquad (\mathbb{E}[|X_1|] < \infty, \mathbb{E}[|X_2|] < \infty)$
- $X_1 \geq 0 \Longrightarrow \mathbb{E}[X_1|\mathcal{G}] \geq 0 \qquad (\mathbb{P}(X_1 \geq X_2) = 1 \Rightarrow \mathbb{E}[X_1|\mathcal{G}] \geq \mathbb{E}[X_2|\mathcal{G}])$
- X_1 constant $\Longrightarrow \mathbb{E}[X_1|\mathcal{G}] = X_1 = \text{constant}$
- $\mathbb{E}[\mathbb{E}[X|\mathcal{G}] \,|\, \mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ if $\mathcal{H} \subseteq \mathcal{G}$
- MARKOV CHAINS
  - Discrete: $(X_n)_{n \geq 0}$, P transition matrix
    $f: E \to \mathbb{R}$, $\mathbb{E}[|f(X_n)|] < +\infty \quad \forall n$
    $$\boxed{\mathbb{E}[f(X_{n+m}) | \sigma(X_n, \dots, X_0)] = P^m f(X_n)}$$
    $\mathbb{E}[f(X_{n+1}) | \sigma(X_n, \dots, X_0)] = Pf(X_n)$, $(Pf)(j) = \sum_{k \in E} p_{jk} f(X_n)$
  - Continuous: $(X_t)_{t \geq 0}$, $(P_t)_{t \geq 0}$ transition semi group
    $f: E \to \mathbb{R}$, $\mathbb{E}[|f(X_t)|] < +\infty \quad \forall t$
    $$\boxed{\mathbb{E}[f(X_t) | \sigma(X_r \,|\, r \leq s)] = (P_{t-s} f)(X_s)}$$
- X r.v. $\mathcal{F}$-meas., $\mathbb{E}[|X|^p] < \infty$
  Y r.v. $\mathcal{G}$-meas., $\mathbb{E}[|Y|^q] < \infty$
  $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{G} \subseteq \mathcal{F}$
  $\longrightarrow \mathbb{E}[XY|\mathcal{G}] = Y\,\mathbb{E}[X|\mathcal{G}]$
- X $\mathcal{F}$-measurable, $\mathcal{G} \subseteq \mathcal{F}$
  $X \perp\!\!\!\perp \mathcal{G}$ (i.e. $\sigma(X) \perp\!\!\!\perp \mathcal{G}$
  i.e. $\mathbb{P}(\{X \in A\}, G) = \mathbb{P}(X \in A)\mathbb{P}(G)$
  $\forall G \in \mathcal{G}$)
  $\longrightarrow \mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$
- X, $\mathbb{E}[|X|] < +\infty \longrightarrow \mathbb{E}[|\mathbb{E}[X|\mathcal{G}]|] < +\infty$
  $\mathbb{E}[|X|^p] < +\infty$, $p \in [1, \infty) \longrightarrow \mathbb{E}[|\mathbb{E}[X|\mathcal{G}]^p|] \leq \mathbb{E}[|X|^p]$

Conditional expectation

- (Jensen): $\left.\begin{array}{l} X : \mathbb{E}[|X|] < +\infty \\ \varphi : \mathbb{R} \to \mathbb{R} \text{ convex} \\ \mathbb{E}[|\varphi(X)|] < +\infty \end{array}\right\} \longrightarrow \varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}]$

- $\left.\begin{array}{l} X : \mathbb{E}[|X|^2] < +\infty \\ \mathcal{G} \subseteq \mathcal{F} \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \mathbb{E}\left[|\mathbb{E}[X|\mathcal{G}]|^2\right] < +\infty \\ \min\limits_{\substack{Y \ \mathcal{G}\text{-meas.} \\ \mathbb{E}[|Y|^2] < +\infty}} \mathbb{E}[|X-Y|^2] = \mathbb{E}\left[|X - \mathbb{E}[X|\mathcal{G}]|^2\right] \end{array}\right.$



X is $\mathcal{F}$-meas. The min squared error we do by approximating X with a $\mathcal{G}$-meas. function is $(\uparrow)$.
$\Longrightarrow$ the conditional expectation is the MSE approx. of X with a $\mathcal{G}$-meas. function

$$\mathbb{E}[|X-Y|^2] = \mathbb{E}\left[|X - \mathbb{E}[X|\mathcal{G}]|^2\right] + \mathbb{E}\left[|\mathbb{E}[X|\mathcal{G}] - Y|^2\right]$$

- $\left.\begin{array}{l} X \text{ r.v. with density } f \\ \mathcal{G} \perp\!\!\!\perp X \quad (\mathcal{G} \perp\!\!\!\perp \sigma(X)) \\ Y_1, \dots, Y_n \ \mathcal{G} \text{ meas.} \end{array}\right| \longrightarrow \mathbb{E}\left[h(X, Y_1, \dots, Y_n) | \mathcal{G}\right] = \int_{\mathbb{R}} h(x, Y_1, \dots, Y_n) f(x)\, dx$

Markov Processes

Markov process: $(X_t)_{t \geq 0}$ $X_t : \to$: $\forall t_1 < \dots < t_m$, $\forall E_1, \dots, E_{m+1} \in \mathcal{E}$:

$$\mathbb{P}(X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m, \dots, X_{t_1} \in E_1) = \mathbb{P}(X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m)$$

Homogeneous if: $\mathbb{P}(X_{t+s} \in E_2 | X_s \in E_1) \perp\!\!\!\perp s$

Transition kernels: $\boxed{P_t(x, A) = \mathbb{P}(X_t \in A | X_0 = x)}$

(formal): $P_t : E \times \mathcal{E} \to [0,1]$:

1. $P_t(x, \bullet)$ is a prob. meas. on $\mathcal{E}$ $\forall x \in E$
2. $P_t(\bullet, A)$ is $\mathcal{E}$-meas. $\forall A \in \mathcal{E}$
3. $\underbrace{\mathbb{P}(X_{t+s} \in A | X_s = x)}_{\mathbb{E}[\mathbb{1}_A(X_{t+s}) | X_s = x]} = \int_A P_t(x, dy)$

the trans. kernel is the collection $(P_t)_t$

- Continuous MC: (with discrete state-space): $P_t(i, \{j\}) = p_{ij}(t)$
- $\mathbb{E}[X_{t+s} | \sigma(X_s)] = \int_E y \, P_t(X_s, dy)$
- $\mathbb{E}[f(X_{t+s}) | \sigma(X_s)] = \int_E f(y)\, P_t(X_s, dy)$ $\quad$ ($f : E \to \mathbb{R}$ meas. s.t. $\mathbb{E}[f(X_{t+s})] < \infty$)
- CHAPMAN-KOLMOGOROV:

$$\boxed{\mathbb{P}(X_t \in A | X_r = x) = P_{t-r}(x, A) = \int_A P_{t-s}(y, A)\, P_{s-r}(x, dy)}$$

discrete case: $P_{n+m}(x, A) = \int_A P_n(y, A)\, P_m(x, dy)$

$$\mathbb{P}(X_t = j | X_r = k) = p_{kj}^{(t-r)} = \sum_h p_{kh}^{(s-r)} p_{hj}^{(t-s)}$$

MP: stationarity

- $X_s \sim \mu_s$: $\mathbb{E}[f(X_{t+s})g(X_s)] = \int_E g(x)\mu_s(dx) \int_E f(y) P_t(x,dy)$

Def. $(X_t)_{t\geq 0}$ MC with values in $(E,\mathcal{E})$.
μ measure on $\mathcal{E}$.
$(X_t)_{t\geq 0}$ invariant / stationary when $X_0 \sim \mu \Rightarrow X_t \sim \mu \quad \forall t$
μ invariant $\iff \boxed{\mu(A) = \int_E P_t(x,A)\mu(dx)} \quad \forall A \in \mathcal{E}$

- $X_s \sim \mu_s$: $\mathbb{P}(X_{t+s} \in A) = \mu_{t+s}(A) = \int_E \mu_s(dx) P_t(x,A)$

MP: irreducibility, recurrency

Def. $(X_t)_{t\geq 0}$ MC with values in $(E,\mathcal{E})$, $(P_t)_{t\geq 0}$ transition kernels
$(X_t)_{t\geq 0}$ irreducible w.r.t. a reference measure φ on $\mathcal{E}$ $\iff$ $\forall x \in E \quad \forall A \in \mathcal{E}: \varphi(A) > 0$
$\exists t > 0 : \quad P_t(x,A) > 0$

Def. $(X_n)_{n\geq 0}$ MC, $E \subseteq \mathbb{R}^d$, $\mathcal{E} = \mathcal{B}(E)$ $\Rightarrow$ $A \in \mathcal{E}$: $N_A = \sum_{n\geq 0} \mathbb{1}_{\{X_n \in A\}} = \sum_{n\geq 0} \mathbb{1}_A(X_n)$

number of visits in the set A

A Harris recurrent if $\forall x$: $\mathbb{P}_x(N_A = +\infty) = 1$ $(\iff \mathbb{E}_x[N_A] = +\infty)$
$(X_n)_n$ Harris recurrent if $\exists \varphi$: $\forall A \in E$: $\varphi(A) > 0$ is Harris recurrent

LLN : $(X_n)_{n\geq 0}$ Harris recurrent
φ invariant measure
$f: E \to \mathbb{R}$ $\mathcal{E}$-meas. s.t.
$\int_E |f(x)| \varphi(dx) < +\infty$
$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=0}^{n} f(X_k)}{n} = \int_E f(x)\varphi(dx)$

(almost surely)

(Discrete space analogy: $\lim_{n\to\infty} \frac{\sum_{k=0}^{n} f(X_k)}{n} = \sum_{j\in E} f(j)\pi_j$, $(\pi_j)_j$ invariant)

Martingale

$(\Omega, \mathcal{F}, \mathbb{P})$ prob. space
$(M_t)_{t\geq 0}$ real random var.
$(\mathcal{F}_t)_{t\geq 0}$ increasing family of sub σ-algebra of $\mathcal{F}$
(increasing : $s<t$: $\mathcal{F}_s \subseteq \mathcal{F}_t$)
$\Rightarrow$ $(M_t)_{t\geq 0}$ is a martingale if:
1. $\mathbb{E}[|M_t|] < \infty \quad \forall t$
2. M_t is $\mathcal{F}$-measurable $\forall t$
3. $\forall s<t$: $\boxed{\mathbb{E}[M_t | \mathcal{F}_s] = M_s} \quad \forall t$

- $(X_n)_{n\geq 0}$ <u>discrete</u> MC
f measurable : $\mathbb{E}[|f(X_n)|] < +\infty \ \forall n$
$\mathcal{F}_n = \sigma(X_0, \dots, X_n)$
$(f: E \to \mathbb{R})$
$\Rightarrow$
1. $\boxed{\mathbb{E}[f(X_n) | \mathcal{F}_n] = (P^{n-m} f)(X_n)} \ \forall m<n$
2. $\boxed{M_n := f(X_n) - \sum_{k=0}^{n-1} Pf(X_k) - f(X_k)}$
$(M_n)_{n\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\geq 0}$

- $(X_t)_{t\geq 0}$ <u>continuous</u> MC (discr. space)
$f: E \to \mathbb{R}$ meas. $\mathbb{E}[|f(X_t)|] < +\infty \quad \forall t$
$\mathbb{E}[|Qf(X_t)|] < +\infty \quad \forall t$
$\Rightarrow \boxed{M_t = f(X_t) - \int_0^t (Qf)(X_s)\,ds} \quad \forall t \geq 0$
$(M_t)_{t\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$

Martingale

$(\Omega, \mathcal{F}, \mathbb{P})$ prob. space
$(\mathcal{F}_t)_{t\geq 0}$ filtration $\Rightarrow$ $T : \Omega \to [0, +\infty]$ stopping time for the filtration $(\mathcal{F}_t)_{t\geq 0}$ if :

$$\{T \leq t\} \in \mathcal{F}_t \quad \forall t$$

eq. if $\{T > t\} \in \mathcal{F}_t \quad \forall t$

- $(M_t)_{t\geq 0}$ martingale, $t \in$ discrete set
  $(\mathcal{F}_t)_{t\geq 0}$ filtration
  T stopping time for $(\mathcal{F}_t)_{t\geq 0}$

$$\Longrightarrow \underbrace{(M_{T\wedge t})_t}_{\text{stopped process}} \text{ martingale s.t. } \mathbb{E}[M_{T\wedge t}] = \mathbb{E}[M_0] = \mathbb{E}[M_t] \quad \forall t$$

martingale have constant expectation

By MCT : $\mathbb{E}_1[t \wedge T] \xrightarrow{t \to \infty} \mathbb{E}[T]$

Fatou : $\mathbb{E}_1[M_{t\wedge T}] \xrightarrow{t \to \infty} \mathbb{E}[M_T]$

Mean permanence time in a continuous-time queue+service MC $(Q_t)_{t\geq 0}$

→ $W_t = S_t + Y_t \mathbb{1}_{\{Q_t \geq s\}}$ where:
- W_t: permanence time in the system
- S_t: service time
- Y_t: waiting time in the queue
- s: maximum customers served by the system at once.

→ Under stationarity: $Q_t \sim \pi$

→ if $S_t \sim \mathcal{E}(\lambda)$ then:
- $\mathbb{E}[S_t] = \frac{1}{\lambda}$
- if $Q_t = K \geq s$: $Y_t | Q_t = K \sim \sum_{j=1}^{K-s+1} \mathcal{E}(s\cdot\lambda) \equiv \Gamma(K-s+1, s\lambda)$

$\Rightarrow \mathbb{E}[Y_t | Q_t = K] = (K-s+1)/s\lambda$

- $\mathbb{E}_\pi[Y_t \mathbb{1}_{\{Q_t\geq s\}}] = \sum_{\substack{K\geq s\\ K\in E}} \mathbb{E}_\pi[Y_t|Q_t=K]\,\mathbb{P}_\pi\{Q_t=K\} = \sum_{\substack{K\geq s\\ K\in E}} \mathbb{E}_\pi[Y_t|Q_t=K]\,\pi_K$

→ Check Little's Law:

if arrivals are distributed as a Poisson process with parameter θ:

$$\mathbb{E}_\pi[Q_t] = \sum_{j\in E} j\,\mathbb{P}_\pi\{Q_t=j\} = \sum_{j\in E} j\pi_j \overset{\downarrow}{\underset{?}{=}} \theta\,\mathbb{E}_\pi[W_t]$$

Markov processes

→ Check if $(X_n)_{n\geq 0}$ (discrete time) or $(X_t)_{t\geq 0}$ (continuous time) are Markov processes

- if $X_{n+1} = F(X_n, V_{n+1})$ with $(V_n)_{n\geq 1}$ i.i.d., just use the definition;
- if $X_n = g(B_n)$ or $X_t = g(B_t)$, just use the definition and the fact that (B) is a MP.

→ Find its transition kernel

- if $X_{n+1} = F(X_n, V_{n+1})$: $\mathbb{P}\{X_{n+1}\in A | X_n = x\} = \mathbb{P}\{F(x,V_{n+1})\in A | X_n = x\} = \mathbb{P}\{F(x,V_{n+1})\in A\}$
call $Y = F(x, V_{n+1})$ and compute the cumulative distribution function of Y using the one of V_{n+1}: take derivatives and find the probability density function of Y.

- if $X_n = g(B_n)$ or $X_t = g(B_t)$, just use the fact that $(B)\sim\mathcal{N} \Rightarrow (X)\sim\mathcal{N}$ if $g \equiv$ linear combination and the theorem:

$$\begin{pmatrix} X\\ Y\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} m_X\\ m_Y\end{pmatrix}, \begin{pmatrix} C_X & C_{XY}\\ C_{XY} & C_Y\end{pmatrix}\right) \Rightarrow X|Y=y \sim \mathcal{N}\left(m_X + C_{XY}C_Y^{-1}(y-m_Y),\, C_X - C_{XY}C_Y^{-1}C_{YX}\right)$$

→ Is it irreducible?

By definition: $(X_n)_{n\geq 0}$ is irreducible if and only if $\forall x\in E, \forall A\in\mathcal{E}$ s.t. $\varphi(A)>0$ $\exists t>0$ s.t. $P_t(x,A)>0$

When $E=\mathbb{R}$, just consider the Lebesgue measure φ: if $P_t(x,A)>0\ \forall x \Leftrightarrow \varphi(A)>0$ then $(X_n)_{n\geq 0}$ is irreducible.

→ Is it homogeneous?

By definition: $(X_n)_{t\geq 0}$ is a homogeneous MP if and only if $P_t(y,A) = \mathbb{P}\{X_{t+s}\in A | X_s = x\}$ depends only on t, doesn't depend on s.

- <u>Classify the states in an irreducible MC with discrete time and infinite E</u>

  → Consider $T_i = \inf\{n \geq 1 \mid X_n = i\}$ first return time to state i (i.e. 0)

  → Compute $v_i = \mathbb{P}_i\{T_i < +\infty\}$ as the hitting probability of set $\{i\}$ starting from i, i.e. the probability of returning to i starting from i.

  → $v_i = \begin{cases} 1 & \Rightarrow i \text{ is recurrent} \Rightarrow \text{the MC is recurrent} \\ <1 & \Rightarrow i \text{ is transient} \Rightarrow \text{the MC is transient} \end{cases}$

  Computation: $v_j = p_{ji} + \sum_{k \in I \setminus \{i\}} p_{jk} v_k$

  Constraints: $0 \leq v_j \leq 1 \quad \forall j$
  v_j is minimal
  boundary condition(s)

- <u>Determine the law of $T_A = \inf\{n \geq 1 \mid X_n \in A\}$</u>

  → $\mathbb{P}_i\{T_A = n\} = \mathbb{P}\{X_n \in A, X_{n-1} \notin A, \dots, X_1 \notin A \mid X_0 = i\} = \begin{pmatrix}\text{Markov} \\ \text{Property}\end{pmatrix} = \dfrac{\mathbb{P}\{X_n \in A \mid X_{n-1} \notin A\} \cdot \ldots \cdot \mathbb{P}\{X_1 \notin A \mid X_0 = i\} \cdot \cancel{\mathbb{P}\{X_0 = i\}}}{\cancel{\mathbb{P}\{X_0 = i\}}} =$

  $= \sum_{\substack{i_n \in A \\ i_{n-1}, \dots, i_1 \notin A}} p_{i_{n-1}, i_n} \cdot \ldots \cdot p_{i, i_1}$

  → evaluate the 1-step probabilities according to the particular MC in analysis, with the caution not to reach A before n steps neither to reach another absorbing class $C \neq A$, forcing $T_A = +\infty$.

  <u>Remark</u>: • if the MC is irreducible and we find $f_{00} = \mathbb{P}_0\{T_0 < +\infty\} = 1$ using the above probability $\sum_{n=1}^{+\infty} \mathbb{P}_0\{T_0 = n\}$, then the MC is recurrent (in this particular case, $A = \{0\}$).

  • having the law of T_0, we can easily compute $\mathbb{E}_0[T_0]$; if this quantity is finite and 0 is recurrent, then 0 is positive recurrent. Moreover if the MC is irreducible then the whole MC is positive recurrent, thus it exists a unique invariant distribution.

- <u>Classify the states in an irreducible MC with continuous time and countable E</u>

  → state i is recurrent/transient for a continuous time MC $(X_t)_{t \geq 0}$ if and only if i is recurrent/transient for its discrete skeleton $(Y_n)_{n \geq 0}$

  → we write the recurrent equation using the transition probabilities of the discrete skeleton:
  $y_j = \sum_{k=0}^{+\infty} \hat{p}_{jk} y_k \quad$ where $\hat{p}_{jk} = \left\{ \frac{q_{jk}}{-q_{jj}} \text{ if } j \neq k, q_{jj} \neq 0;\ 0 \text{ if } (j = k, q_{jj} \neq 0) \vee (j \neq k, q_{jj} = 0);\ 1 \text{ if } j = k, q_{jj} = 0 \right\}$

  → if it has a bounded non-constant solution $(y_j)_j$ for all $j \in E$ except for one $\bar{j}$ at most, then the MC is transient; if this doesn't exist then the MC is recurrent.