# DISCRETE TIME MARKOV CHAIN

## STATE CLASSIFICATION

- Stochastic process
  Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ a measure space.
  A stochastic process with values in $E$ is a collection of random variables $X_t : \Omega \to E$ and it's denoted by $(X_t)_t$.
- Markov Process
  A stochastic process $(X_t)_t$ is a Markov process if it holds the Markov property:
  $$\mathbb{P}(X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m, \dots, X_{t_1} \in E_1) = \mathbb{P}(X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m)$$
  $$\forall t_1 < t_2 < \dots < t_m < t_{m+1}, \quad \forall E_1, \dots, E_{m+1} \in \mathcal{E}, \quad \forall m \in \mathbb{N}$$
- Transition probabilities : $p_{ij}(t,s) := \mathbb{P}(X_t = j \mid X_s = i) \qquad s < t$
- Time homogeneous process
  The process $(X_t)_t$ is time homogeneous if $p_{ij}(t,s)$ depends only on $t-s$.
- Transition matrix : $(p_{ij})_{i,j \in E}$, where $p_{ij} = p_{ij}^{(1)}$ are the 1-step transition probabilities

  properties :
  - $0 \le p_{ij} \le 1 \qquad \forall i,j \in E$
  - $\sum_{j \in E} p_{ij} = 1 \qquad \forall i \in E \qquad \left(= \sum_{j \in E} \mathbb{P}(X_1 = j \mid X_0 = i) = 1\right)$
  - $\mathbb{P}(X_{t+2} = j \mid X_t = i) = \underbrace{p_{ij}^{(2)}}_{\text{power 2 of the step transition matrix}} = \sum_{k \in E} p_{ik} p_{kj}$
  - $\mathbb{P}(X_{t_m} = j_m, X_{t_{m-1}} = j_{m-1}, \dots, X_0 = j_0) = \mathbb{P}(X_0 = j_0)\, p_{j_0 j_1}^{(t_1)} \cdots p_{j_{m-1} j_m}^{(t_m - t_{m-1})}$
- Accessible state : $j$ accessible from $i$ if $\exists n \ge 0 : \mathbb{P}(X_n = j \mid X_0 = i) = p_{ij}^{(n)} > 0$
- Communicating states : $i$ and $j$ communicate if each one is accessible from the other one
- Class of states : $C \subseteq E$ is a class of state if all states in $C$ communicate and they do not communicate with states in $E \setminus C$
- Irreducible MC : a MC is irreducible if all the states communicate
- Recurrent state : $(X_n)_{n \ge 0}$ discrete MC :
  $$i \in E \text{ recurrent} \iff \mathbb{P}\left( \bigcup_{n=1}^{\infty} \{X_n = i\} \mid X_0 = i \right) = 1$$
  the probability of return to $i$ in finite time starting from $i$ is 1.
  If a state is not recurrent then is transient.
- First entrance time : $T_i = \begin{cases} \min\{n : X_n = i\} & \text{if } \{n \ge 1 : X_n = i\} \ne \emptyset \\ +\infty & \text{otherwise} \end{cases}$

  (first entrance/visit time in $i \in E$)
  $$f_{ji}^{(n)} = \mathbb{P}(T_i = n \mid X_0 = j) \quad = \text{probability of entering in } i \text{ only at the } n\text{-th step starting from } j$$
- Renewal equation :
  $$\underbrace{p_{ij}^{(n)}}_{\text{probability of going from } i \text{ to } j \text{ in } n \text{ steps}} = \underbrace{\sum_{\nu=1}^{n} f_{ij}^{(\nu)} p_{jj}^{(n-\nu)}}_{\text{$\Sigma$ probability of entering in $j$ after $\nu$ steps and remain in $j$ (the $\Sigma$ represents all possible cases)}}$$

- Thm. $(X_n)_{n\geq 0}$ discrete Markov Chain

$$\underbrace{i \text{ recurrent}}_{\text{we surely come back to } i} \iff \underbrace{\sum_{n\geq 0} p_{ii}^{(n)} = +\infty}_{\text{the average time spent in } i \text{ is infinite}}$$

$$\sum_{n\geq 0} p_{ii}^{(n)} = \sum_{n\geq 0} \mathbb{E}[\mathbb{1}\{X_n=i\}] = \mathbb{E}\left[\sum_{n\geq 0} \mathbb{1}\{X_n=i\}\right]$$

Otherwise (if $\sum_{n\geq 0} p_{ii}^{(n)} < \infty$) the state is transient and:

$$\sum_{n\geq 0} p_{ii}^{(n)} = \frac{1}{\underbrace{1-\mathbb{P}_i(T_i<\infty)}_{\text{probability of never returning in } i}}, \quad \mathbb{P}_i(\cdot) = \mathbb{P}(\cdot \mid X_0=i)$$

$\Longrightarrow$ a MC visits a transient state only a finite number of times

- Two communicating states are both recurrent or transient
  $\Longrightarrow$ the elements of a class of states are all recurrent or transient
- $j$ transient $\Longrightarrow$ $\begin{cases} \bullet\ \sum_{n\geq 0} p_{ij}^{(n)} < +\infty \quad \forall i \\ \bullet\ \lim_{n\to\infty} \mathbb{P}_i(X_n=j) = \mathbb{P}(X_n=j \mid X_0=i) = 0 \\ \quad (\text{the MC goes through } j \text{ only a finite number of times a.s.}) \end{cases}$
- $E$ finite $\Longrightarrow$ $\exists$ at least one recurrent state

## INVARIANT LAWS & ASYMPTOTIC BEHAVIOUR

- Invariant distribution
  $(X_n)_{n\geq 0}$ MC with transition matrix $(p_{ij})_{i,j\in E}$. Let $\pi = (\pi_i)_{i\in E}$ be a probability density on $E$ $(0\leq \pi_i \leq 1,\ \sum_{i\in E}\pi_i = 1)$.
  $\pi$ is an invariant distribution if $X_{n+1}\sim\pi$ whenever $X_n\sim\pi$ $\forall n$.

$$\pi \text{ invariant} \iff \pi_j = \sum_{i\in E} p_{ij}\pi_i \iff \pi P = \pi$$

the density is the left eigenvector of $P$ (eigenvalue $=1$)

- Properties:
  - $E$ finite $\Longrightarrow$ $\exists\pi$ (at least one)
  - $\exists\pi \not\Rightarrow \exists!\pi$
  - $E$ infinite $\not\Rightarrow$ $\exists/\nexists\ \pi$
- $(\pi_i)_{i\in E}$ probability density is reversable if: $\pi_i p_{ij} = \pi_j p_{ji}$ $\forall i,j\in E$
  If $(\pi_i)_{i\in E}$ is reversable $\Longrightarrow$ $(\pi_i)_{i\in E}$ is invariant
- Thm. Existence and unicity of invariant distributions:
  If limits $\lim_{n\to\infty} p_{ij}^{(n)}$
  - exists and are strictly positive
  - depend only on $j$

  $\Longrightarrow$ the MC has an unique invariant distribution $(\pi_i)_{i\in E}$
  moreover:

$$\lim_{n\to\infty}\mathbb{E}[f(X_n)] = \int f\,d\pi = \sum_{j\in E} f(j)\,\pi_j$$

  for all $f$ bounded.
  In this situation: $X_n \xrightarrow{n} \pi$ in distribution $(\mathbb{P}(X_n=j)\to\pi_j \ \forall j)$
- Period of a state: defined as MCD $\{n\geq 1 \mid p_{ii}^{(n)}>0\}$.
  If the period is 1 we call the state aperiodic.
  States of the same class have the same period (so in case of irreducible MC we can talk about periodic/aperiodic MC)

# STRONG MARKOV PROPERTY

- Stopping time
  The random variable $T : \Omega \to \mathbb{N}$ is a stopping time of the MC $(X_n)_{n\geq 0}$ if $\forall n$ the event $\{T \leq n\}$ belongs to the $\sigma$-algebra generated by $\{X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\}$. (with $i_n, \dots, i_0$ arbitrary states)
  The stopping time is $\perp\!\!\!\perp$ from the future.

- First entrance time : $T_j = \inf\{n \geq 1 : X_n = j\}$
  (If $\{n \geq 1 : X_n = j\} = \emptyset \implies T_j = +\infty$)

- First exit time : $U_j = \inf\{n \geq 1 : X_n \neq j\}$
  (If $\{n \geq 1 : X_n \neq j\} = \emptyset \implies U_j = +\infty$)
  law of the first exit time:
  If $(X_n)_{n\geq 0}$ is a MC with $(p_{ij})_{i,j\in E}$ s.t. $p_{jj} \in (0,1)$ $\forall j \in E$
  - the stopping time $U_j$ : $U_j \sim \mathcal{G}(1 - p_{jj})$ (geometric distribution)
  - $(X_{U_j})(\omega) = \sum_{n\geq 1} X_n(\omega) \mathbb{1}_{\{U_j = n\}}(\omega)$ satisfies:
  $$\mathbb{P}_j(X_{U_j} = k) = \frac{p_{jk}}{1 - p_{jj}} \qquad \forall k \neq j$$

- Thm. Restarted MC :
  - $(X_n)_{n\geq 0}$ MC
  - $T$ stopping time
  - $(Y_n)_{n\geq 0}$ restarted MC :
  $$Y_n(\omega) = \begin{cases} X_{T(\omega)+n}(\omega) & \text{if } T(\omega) < \infty \\ \text{arbitrary} & \text{else} \end{cases}$$

  $\implies$ $(Y_n)_{n\geq 0}$ is a **MC** with the same $(p_{ij})_{i,j\in E}$ of $(X_n)_{n\geq 0}$

- Thm. Strong Markov Property :
  - $(X_n)_{n\geq 0}$ MC
  - $T$ stopping time
  - $(p_{ij})_{i,j\in E}$ transition matrix

  $\implies$ $\underbrace{(X_{T\wedge n})_{n\geq 0}}_{\text{stopped MC}} \perp\!\!\!\perp \underbrace{(X_{T+n})_{n\geq 0}}_{\text{restarted MC}}$

  w.r.t. $\mathbb{P}(\bullet \mid X_T = i, T < \infty)$

  where $(T \wedge n)(\omega) = \min\{T(\omega), n\}$

# TRANSIENCY, RECURRENCY, ABSORPTION

- Thm. The number of visits of a recurrent state is infinite almost surely.
  proof.
  $N_i := \sum_{n\geq 1} \mathbb{1}_{\{X_n = i\}}$ = number of returns in the state $i$.
  let $(T_i^{(n)})_{n\geq 1}$ be s.t. :
  $$\begin{cases} T_i^{(1)} = \inf\{n \geq 1 : X_n = i\} & \text{time of the } 1^{st} \text{ return} \\ T_i^{(k+1)} = \inf\{n > T_i^{(k)} : X_n = i\} & \text{time of the } (k+1)^{th} \text{ return} \end{cases}$$
  $i$ recurrent $\iff \mathbb{P}_i(T_i^{(1)} < +\infty) = 1 \iff \mathbb{P}_i(N_i \geq 1) = 1$.
  $T_i^{(2)}$ is the first visit time in $i$ for the MC restarted from time $T_i^{(1)}$, but the transition matrix is the same $\implies \mathbb{P}_i(T_i^{(2)} < +\infty) = 1$
  By induction:
  $$\mathbb{P}_i(T_i^{(k)} < +\infty) = 1 \ \forall k \iff \mathbb{P}_i(N_i \geq k) = 1 \ \forall k$$
  $$\implies \mathbb{P}_i(N_i = +\infty) = 1$$
  ∎

- Thm. Probability of staying forever in transient states

$T$ = set of transient states

$U_i = \mathbb{P}_i(\cap_{n\geq 1} \{X_n \in T\}) \qquad i \in T$

= probability of remaining in $T$ forever

$\Longrightarrow (U_i)_{i\in T}$ is the biggest solution s.t. $0 \leq U_i \leq 1 \quad \forall i \in T$ that satisfies the system of equations:

$$U_i = \sum_{j\in E} p_{ij} U_j$$

(Remark: generally there is no unique solution, but if $T$ is finite $\Rightarrow \exists !$ sol.: $U_i = 0 \quad \forall i \in T$)

proof.

We define $U_i^{(n)} = \mathbb{P}_i(X_n \in T, .., X_1 \in T)$

= probability of staying in transient states from $1$ to $n$

$(U_i^{(n)})_{n\geq 1}$ is a non increasing sequence (the prob. of staying longer is lower)

$U_i = \lim_{n\to\infty} U_i^{(n)} = \lim_{n\to\infty} \mathbb{P}_i(X_n \in T, .., X_1 \in T)$

$= \mathbb{P}_i(\cap_{n\geq 1} \{X_n \in T, .., X_1 \in T\})$

$U_i^{(n+1)} = \mathbb{P}_i(X_{n+1} \in T, .., X_2 \in T, X_1 \in T)$

$= \mathbb{P}_i(X_{n+1} \in T, .., X_2 \in T \mid X_1 \in T)\ \mathbb{P}_i(X_1 \in T)$

$= \sum_{j\in T} \underbrace{\mathbb{P}_i(X_{n+1} \in T, .., X_2 \in T \mid X_1 = j)}_{\text{probability of remaining in } T \text{ } n \text{ steps starting from } X_1 = j}\ \mathbb{P}_i(X_1 = j)$

$= \sum_{j\in T} U_j^{(n)} p_{ij}$

Applying the limits on both sides: $U_i = \sum_{j\in T} p_{ij} U_j \qquad \forall i \in T$

Now we want to prove that $(U_i)_{i\in T}$ is the biggest $[0,1]$-valued solution.

Suppose we have a bigger solution $V_i = \sum_{j\in T} p_{ij} V_j \qquad \forall i \in T, \quad 0 \leq V_i \leq 1$

$\Longrightarrow U_i^{(1)} = \mathbb{P}_i(X_1 \in T) = \sum_{j\in T} p_{ij} \geq \sum_{j\in T} p_{ij} V_j = V_i$

$\Longrightarrow U_i^{(1)} \geq V_i \qquad \forall i \in T$

By induction: $U_i^{(n)} \geq V_i \qquad \forall i \in T$

$\Longrightarrow U_i^{(n+1)} = \sum_{j\in T} p_{ij} U_j^{(n)} \geq \sum_{j\in T} p_{ij} V_j = V_i$

by considering the limit: $U_i^{(n)} \to U_i \quad \Longrightarrow \quad U_i \geq V_i$, contraddiction.

We also prove the remark ($T$ finite $\Rightarrow U_i = 0 \quad \forall i$)

$U_i = \sum_{j\in T} p_{ij} U_j = \sum_{j\in T} p_{ij} \left(\sum_{k\in T} p_{jk} U_k\right) = \sum_{j\in T} \sum_{k\in T} p_{ij} p_{jk} U_k$

$\leq \sum_{k\in T} \left(\sum_{j\in T} p_{ij} p_{jk}\right) U_k = \sum_{k\in T} p_{ik}^{(2)} U_k$

By iterating $n$ times:

$U_i = \sum_{k_1,..,k_n\in T} (p_{ik_1} \cdots p_{k_{n-1}k_n}) U_{k_n} \leq \sum_{\substack{k_1,..,k_{n-1}\in E \\ k_n \in T}} (\cdots) U_{k_n} =$

$= \sum_{k_n\in T} p_{ik_n}^{(n)} U_{k_n}$

$\Longrightarrow \forall n: \quad U_i \leq \sum_{j\in T} p_{ij}^{(n)} U_j \leq \sum_{j\in T} p_{ij}^{(n)} \leq 1$

And since $\forall j \in T \quad p_{ij}^{(n)} \longrightarrow 0$ (transient) $\Longrightarrow U_i = 0 \qquad \forall i \in T$. ■

- Example: Gambler's win vs. a bank

$(X_n)_{n\geq 0}$ : $X_n$ = capital of the player at time $n$, $E = \mathbb{N}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ & \cdots & & & & \end{bmatrix} \qquad p+q=1, \quad p,q>0$$

Classes:
- $\{0\}$ recurrent $\quad p_{00}^{(n)} = 1$, $\sum_{n\geq 1} p_{00}^{(n)} = +\infty$
- $\{1,2,3,\dots\}$ transient $\quad \mathbb{P}_i(\cap_{i\geq 1}(X_n = i)) < 1-q^i < 1$

We want to evaluate the probability that the Gambler is never ruined, i.e. the probability of remain in $T = \{1,2,3,\dots\}$

$$U_i = \sum_{j\geq 1} p_{ij} U_j \quad \Longrightarrow \quad \begin{cases} U_1 = p\,U_2 & \text{(since } U_0 = 0) \\ U_i = p\,U_{i+1} + q\,U_{i-1} & i > 1 \end{cases}$$

Characteristic equation: $p\lambda^2 - \lambda + q = 0 \implies \lambda_1 = 1, \ \lambda_2 = q/p$

- $p \neq q$ : $U_i = A(1)^i + B\left(\frac{q}{p}\right)^i$

  since $U_0 = 0 \implies A = -B \implies U_i = A\left(1 - \left(\frac{q}{p}\right)^i\right)$

  if $p<q$ $\quad A = 0$ and so $U_i = 0 \ \forall i$ $\quad$ (we need $0 \leq U_i \leq 1$)

  if $p>q$ then $\lim_{i\to\infty} U_i = A$ and in order to get the bigger solution $A = 1 \implies U_i = 1 - (q/p)^i$

- $p = q = \frac{1}{2}$ : $U_i = A + Bi \implies \begin{matrix} U_0 = 0 \implies A = 0 \\ U_1 = pU_2 \implies B = 0 \end{matrix} \implies U_i = 0$

- Thm. Absorption probability in a recurrent class

  $T$ = set of transient states

  $V_i$ = probability of absorption in a recurrent class $C$ starting from $i \in T$

  $\implies (V_i)_{i\in T}$ is the smallest $[0,1]$-valued solution of:

$$V_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} V_j$$

(Remark: $T$ finite $\implies \exists!\,(V_i)_{i\in T}$)

proof.

We define: $V_i^{(n)} = \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C)$

$\qquad\qquad\qquad$ = probability of absorption in $C$ at time $n$

$$V_i^{(1)} = \sum_{j\in C} p_{ij}$$

$$\begin{aligned} V_i^{(n)} &= \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C) \qquad\qquad n \geq 2 \\ &= \sum_{j\in T} \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, \dots, X_2 \notin C, X_1 = j) \\ &= \sum_{j\in T} \mathbb{P}_i(X_n \in C, X_{n-1} \notin C, \dots, X_2 \notin C \mid X_1 = j)\, \mathbb{P}_i(X_1 = j) \\ &= \sum_{j\in T} V_j^{(n-1)} p_{ij} \end{aligned}$$

The event "absorption in $C$" : $\cup_{n\geq 1}\{X_n \in C\} = \underbrace{\cup_{n\geq 1}\{X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C\}}_{\text{disjoint union}}$

$$\begin{aligned} V_i &= \mathbb{P}_i(\cup_{n\geq 1}\{X_n \in C\}) = \mathbb{P}_i(\cup_{n\geq 1}\{X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C\}) \\ &= \sum_{n\geq 1} \mathbb{P}_i(\{X_n \in C, X_{n-1} \notin C, \dots, X_1 \notin C\}) \\ &= \sum_{n\geq 1} V_i^{(n)} \\ &= V_i^{(1)} + \sum_{n\geq 2} V_i^{(n)} \\ &= \sum_{j\in C} p_{ij} + \sum_{n\geq 2}\left(\sum_{j\in T} p_{ij} V_j^{(n-1)}\right) \\ &= \sum_{j\in C} p_{ij} + \sum_{j\in T}\left(\sum_{k\geq 1} V_j^{(k)}\right) \\ &= \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} V_j \end{aligned}$$

Moreover $(V_i)_i$ is the smaller $[0,1]$-valued solution. Let $(X_i)_i$ be another $[0,1]$-valued solution of: $X_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} X_j$

$$\Longrightarrow X_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} X_j = V_i^{(1)} + \sum_{j\in T} p_{ij} X_j > V_i^{(1)} \quad \forall i \in T$$

Suppose $X_i \geq \sum_{k=1}^{n} V_i^{(k)}$ :

$$X_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} X_j \geq \underbrace{\sum_{j\in C} p_{ij}}_{V_i^{(1)}} + \underbrace{\sum_{j\in T} p_{ij} \sum_{k=1}^{n} V_i^{(k)}}_{\sum_{k=1}^{n}\left(\sum_{j\in T} p_{ij} V_i^{(k)}\right)} = \sum_{k=1}^{n+1} V_i^{(k)}$$

$$V_i^{(1)} + \sum_{k=1}^{n}\left(\sum_{j\in T} p_{ij} V_i^{(k)}\right)$$

$$V_i^{(1)} + \sum_{k=1}^{n} V_i^{(k+1)}$$

By induction we showed $X_i \geq \sum_{k=1}^{n} V_i^{(k)}$ $\forall n$, and so by taking the limit with $n \to \infty$ we obtain $X_i \geq V_i$ $\forall i \in T$, contraddiction. ■

- Example: Gambler's win vs. a bank

We want to determine the win probability (absorption to $\{0\}$)

$$V_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} V_j \Longrightarrow \begin{cases} V_1 = pV_2 + q & (V_0 = 1) \\ V_i = pV_{i+1} + qV_{i-1} & i \geq 2 \end{cases}$$

General solution:

$$V_i = A + B(q/p)^i \Longrightarrow V_0 = 1 : \quad A + B = 1$$

$$V_i = A + (1-A)(q/p)^i \Longrightarrow V_i \xrightarrow{i\to\infty} A \text{ , we want the smallest solution}$$

$$\Longrightarrow A = 0$$

$$V_i = \left(\tfrac{q}{p}\right)^i$$

- lemma: $V$ $\mathbb{N}$-valued r.v. $\Longrightarrow \mathbb{E}[V] = \sum_{n=0}^{\infty} \mathbb{P}(V > n)$

proof.

$$\mathbb{E}[V] = \sum_{m\geq 1} m \cdot \mathbb{P}(V=m) = \sum_{m\geq 1} \sum_{k=1}^{m} 1 \cdot \mathbb{P}(V=m)$$

$$= \sum_{k\geq 1} \sum_{m=k}^{\infty} \mathbb{P}(V=m)$$

$$= \sum_{k\geq 1} \mathbb{P}(V \geq k)$$

$$= \sum_{n\geq 0} \mathbb{P}(V > n)$$ ■

- Thm. Mean absorption time in a recurrent class

$E$ = set of states, $E$ finite

$C$ = unique recurrent class

$w_i$ = mean absorption time in $C$

If $E$ is finite and $\exists!$ recurrent class $C \Longrightarrow w_i$ is finite and satisfies:

$$w_i = 1 + \sum_{j\in T} p_{ij} w_j \qquad \forall i \in T$$

proof.

For every $n \geq 1$: $\mathbb{P}_i(V > n) = \mathbb{P}_i(X_n \in T, .., X_1 \in T) = U_i^{(n)}$

= probability of staying in transient states from the $1^{st}$ step to the $n^{th}$

We know that:

$$U_i^{(n+1)} = \sum_{j\in T} p_{ij} U_j^{(n)} \Longrightarrow U_i^{(n+2)} = \sum_{j\in T}\sum_{k\in T} p_{ij} p_{jk} U_k^{(n)}$$

$$\leq \sum_{k\in T}\left(\sum_{j\in T} p_{ij} p_{jk}\right) U_k^{(n)} =$$

$$= \sum_{k\in T} p_{ik}^{(2)} U_k^{(n)}$$

Iterating we obtain: $U_i^{(n+m)} \leq \sum_{j\in T} p_{ij}^{(m)} U_j^{(n)}$

If $i,j$ are transient $\Longrightarrow \lim_{m\to\infty} p_{ij}^{(m)} = 0$

and so, if $T$ is finite, we can chose $m$ s.t. $\sum_{j\in T} p_{ij}^{(m)} < 1$ $\forall i$

and we set $M = \max_{i\in T} \left\{ \sum_{j\in T} p_{ij}^{(m)} \right\} < 1$

$$\Longrightarrow U_i^{(n)} \le \sum_{j\in T} p_{ij}^{(m)} U_j^{(n-m)}$$
$$\le \max_{j\in T} U_j^{(n-m)} \sum_{j\in T} p_{ij}^{(m)}$$
$$\le \max_{j\in T} U_j^{(n-m)} M$$

$\Longrightarrow$ the probability of remaining forever in transient states is exponentially decreasing

Iterating:

$$\max_{i\in T} U_i^{(n)} \le M^2 \max_{i\in T} U_i^{(n-m-m)} \le \dots \le M^{\frac{n}{m}} \underbrace{\max_{i\in T} U_i^{(n-(\frac{n}{m})m)}}_{\le 1 \text{ since it's a probability}}$$

$$\Longrightarrow \mathbb{P}_i(V>n) \le \max U_i^{(n)} \le M^{\frac{n}{m}} \le M^{\frac{n}{m}-1} \le M^{-1}\left(M^{\frac{1}{m}}\right)^n$$
$$\Longrightarrow \mathbb{E}_i[V] = \sum_{n\ge 0} \mathbb{P}_i(V>n) \le M^{-1}\sum_{n\ge 0}\left(M^{\frac{1}{m}}\right)^n = \left(M\left(1-M^{1/m}\right)\right)^{-1} < +\infty$$

$\Longrightarrow$ $w_i$ are finite.

Now we want to prove the formula.

$$w_i = \mathbb{E}_i[V] = \sum_{n\ge 1} n\cdot \mathbb{P}_i(V=n)$$
$$= \underbrace{\sum_{n\ge 1} n\cdot \mathbb{P}_i(V=n, X_1\in T)}_{} + \underbrace{\sum_{n\ge 1} n\cdot \mathbb{P}_i(V=n, X_1\in C)}_{}$$

if $X_1 \in T$ then $V$ is at least 2 since $X_0\in T$, $X_1\in T$ (we can be in $C$ only from $X_2$)

$$\{V=n, X_1\in C\} = \begin{cases} \{X_1\in C\} & n=1 \\ \emptyset & n>1 \end{cases}$$

since $X_1\in C$ then it cannot be, for instance, $V=2$ (since we're already in $C$)

$$= \left[\sum_{n\ge 2} n\cdot \sum_{j\in T} \mathbb{P}_i(V=n, X_1=j)\right] + \sum_{j\in C} p_{ij}$$
$$= \left[\sum_{n\ge 2} n\cdot \sum_{j\in T} \underbrace{\mathbb{P}_i(V=n\mid X_1=j)}_{} p_{ij}\right] + \sum_{j\in C} p_{ij}$$

$$\mathbb{P}(V=n\mid X_1=j, X_0=i) = \mathbb{P}(V=n\mid X_1=j)$$
$$= \mathbb{P}(V=n-1\mid X_0=j)$$
$$= \mathbb{P}_j(V=n-1)$$

$$= \left[\sum_{n\ge 1} (n+1) \sum_{j\in T} \mathbb{P}_j(V=n)\, p_{ij}\right] + \sum_{j\in C} p_{ij}$$
$$= \left[\sum_{j\in T} \underbrace{\sum_{k\ge 1} n\, \mathbb{P}_j(V=n)}_{\mathbb{E}_j[V]=w_j} p_{ij}\right] + \underbrace{\left[\sum_{j\in T} p_{ij} \underbrace{\sum_{n\ge 1} \mathbb{P}_j(V=n)}_{1}\right] + \left[\sum_{j\in C} p_{ij}\right]}_{1}$$
$$= \sum_{j\in T} w_j p_{ij} + 1 \qquad \blacksquare$$

- Example: coupons collector.
The collection is made of $N$ pictures. At time $n$ we buy an envelope containing a random picture. How many pictures do we have to buy to complete the collection?
$(X_n)_{n\ge 0}$ MC: $X_n = \#$ different pictures collected at time $n$ $\quad (X_0 = 0)$

Transition probabilities:

$$\mathbb{P}(X_{n+1}=k+1\mid X_n=k) = \frac{N-k}{N}$$
$$\mathbb{P}(X_{n+1}=k\mid X_n=k) = \frac{k}{N}$$
$$\mathbb{P}(X_{n+1}=j\mid X_n=k) = 0 \qquad j\ne k, k+1$$

Transition matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & & & \\ 0 & 1/N & \frac{N-1}{N} & 0 & 0 & \cdots & & & \\ 0 & 0 & 2/N & \frac{N-2}{N} & 0 & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \cdots & \cdots & \cdots & & & & 0 & \frac{N-1}{N} & \frac{1}{N} \\ \cdots & \cdots & \cdots & & & & 0 & 0 & 1 \end{bmatrix}$$

$T$ = transient states = $\{0, 1, 2, .., N-1\}$
$C$ = recurrent class = $\{N\}$

$w_i$ = mean time for reaching $N$ starting from $i$

$$\begin{cases} w_i = 1 + p_{ii} w_i + p_{ii+} w_{i+1} & i = 0, .., N-1 \\ w_N = 0 \end{cases}$$

$$\Longrightarrow \quad w_i = 1 + \frac{i}{N} w_i + \frac{N-i}{N} w_{i+1}$$

$$\Longrightarrow \quad w_i = \frac{N}{N-i} + \frac{N}{N-i-1} + .. + \frac{N}{N-(N-1)}$$

$$\Longrightarrow \quad w_0 = \sum_{k=0}^{N-1} \frac{N}{N-k} = N \sum_{h=1}^{N} \frac{1}{h} \qquad \approx N \log(N)$$

- Example: Gambler's ruin.
We consider the fair case $p = q = \frac{1}{2}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & .. & & & \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & .. & & & \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & .. & & & \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & .. & & & \\ \vdots & & \vdots & & & & & & & \\ 0 & 0 & .. & & & & .. & 1/2 & 0 & 1/2 \\ 0 & 0 & .. & & & & .. & 0 & 0 & 1 \end{bmatrix}$$

states = $\{0, 1, 2, .., N\}$
Absorbing classes: $\{0\}, \{N\}$

We want to compute the mean ruin time knowing that the ruin will occurs eventually,
we want the mean w.r.t. $\mathbb{P}_i(\bullet \mid \bigcap_{n \geq 1} \{X_n \neq N\})$
(we have to exclude the other absorbing class ($\{N\}$))

$$\begin{cases} w_0 = 0 \\ w_i = 1 + \frac{1}{2} w_{i-1} + \frac{1}{2} w_{i+1} & i = 1, .., N-1 \\ w_{N-1} = 1 + w_{N-2} \end{cases}$$

$\longleftarrow$ we exclude the possibility of going to $N$

Homogeneous $\Longrightarrow$ $w_i = + i$ $\quad (\lambda_{1/2} = 1)$
Particular $\Longrightarrow$ $w_i = ai^2 + bi + c \Longrightarrow [..] \Longrightarrow a = -1, b = 0, c = 0$
Complete $\Longrightarrow$ $w_i = A + Bi - i^2$
Boundaries $\Longrightarrow$

$$\left.\begin{array}{l} w_0 = 0 \\ w_1 = 1 + \frac{1}{2} w_2 \\ w_{N-1} = 1 + w_{N-2} \end{array}\right\} \Longrightarrow w_i = -i^2 + 2(N-1)i$$

mean duration of the game if we'll be ruined

- Thm. Transience criterium
$(X_n)_{n \geq 0}$ irreducible MC (with countable space $E$)
$(X_n)_{n \geq 0}$ transient $\Longleftrightarrow$ $\exists$ bounded non-constant solution of:

$$(*) \quad \sum_{k \in E} p_{jk} y_k = y_j \qquad \forall j \in E \text{ but one}$$

proof.
($\Longrightarrow$) Suppose that $(X_n)_{n \geq 0}$ is transient.
We denote $e$ the unique state for which $(*)$ doesn't hold.
We consider a transformed MC in which $e$ is absorbing:

$$\tilde{p}_{ij} = \begin{cases} p_{ij} & j \neq e \\ \delta_{ij} & j = e \end{cases}$$

Since the original MC is transient (the transformed is not), there exist $i \in E$ such that:

$$\tilde{V}_i = \mathbb{P}_i(T_e < +\infty) < 1$$

$T_e$ = first entrance time in $e$

It must be $< 1$, otherwise, if $\mathbb{P}_i(T_e < +\infty) = 1 \quad \forall i$ :

$$\mathbb{P}_e(T_e<+\infty) = p_{ee} + \sum_{k\neq e} p_{ek}\, \mathbb{P}_k(T_e<+\infty) = p_{ee} + \sum_{k\neq e} p_{ek} = 1$$

which contraddicts the transiency of $(X_n)_n$.

$\Longrightarrow \tilde{V}_i < 1$ for some $i$ and $\tilde{V}_e = 1$

$\Longrightarrow (\tilde{V}_i)_{i\in E}$ is $[0,1]$- valued (bounded) and non constant

Moreover, $(\tilde{V}_i)_{i\in E}$ are the absorption probabilities in $e$ for the transformed MC, and so:

$$\tilde{V}_i = \tilde{p}_{ie} + \sum_{k\neq e} \tilde{p}_{ik}\tilde{V}_k \quad \xrightarrow[\substack{\text{since } \tilde{p}_{ik}=p_{ik} \\ \text{(for } k\neq e\text{) and} \\ \tilde{V}_e = 1}]{} \quad \tilde{V}_i = \sum_{k\in E} p_{ik}\tilde{V}_k$$

and so $(*)$ holds.

$(\Longleftarrow)$ We suppose $\sum_{k\in E} p_{ik}y_k = y_i \quad \forall i\in E\setminus\{e\}$ $((*))$, $(y_i)_{i\in E}$ bounded and non constant.

If we consider $(\tilde{p}_{ij})_{i,j\in E}$ as before we get: $\sum_{k\in E}\tilde{p}_{ik}y_k = y_i \quad \forall i\in E$

Iterating: $\sum_{k\in E}\sum_{h\in E}\tilde{p}_{ik}\tilde{p}_{kh}y_h = \sum_{h\in E}\left(\sum_{k\in E}\tilde{p}_{ik}\tilde{p}_{kh}\right)y_h = \sum_h \tilde{p}_{ih}^{(2)}y_h = y_i$

Iterating $n$ times: $\sum_{k\in E}\tilde{p}_{ik}^{(n)}y_k = y_i$

If the original MC was recurrent, the transformed MC would converge to the state $e$, namely:

$$\lim_{n\to\infty}\tilde{p}_{ie}^{(n)} = 1 \qquad \forall i\in E$$

(since $\lim_{n\to\infty}\tilde{p}_{ie}^{(n)} = \lim_{n\to\infty}\mathbb{P}_i(\tilde{X}_n = e) = \lim_{n\to\infty}\mathbb{P}_i(T_e\le n) = 1 \Leftrightarrow$ recurrence)

$$\begin{aligned}
\Longrightarrow \forall j\neq e:\quad |y_j - y_e| &= \lim_n |y_j - p_{je}^{(n)}y_e| \\
&= \lim_n |y_j - (y_j - \textstyle\sum_{k\neq e} p_{jk}^{(n)}y_k)| \\
&= \lim_n |\textstyle\sum_{k\neq e} p_{jk}^{(n)}y_k| \\
&\le \sup_k |y_k|\cdot \lim_n |\textstyle\sum_{k\neq e} p_{jk}^{(n)}| \\
&\le \sup_k |y_k|\cdot \lim_n (1-\tilde{p}_{je}^{(n)}) = 0
\end{aligned}$$

$\Longrightarrow y_j = y_e \;\; \forall j\neq e \Longrightarrow (y_i)_i$ constant $\Longrightarrow$ contraddiction. ■

- **Thm.** Recurrence criterium

  $(X_n)_{n\ge 0}$ irreducible MC.

  If $\exists (y_i)_{i\in E}$ s.t. :
  - $\sum_{k\in E} p_{jk}y_k \le y_j \quad \forall j\in E$ but one
  - $\lim_{k\to\infty} y_k = +\infty$

  $\Longrightarrow$ the MC $(X_n)_{n\ge0}$ is recurrent.

- **Example**: Queue model
  - one customer per unit of time
  - number of customers arriving at each time : $A_n$

    $\mathbb{P}(A_n = k) = a_k, \quad a_k\in[0,1], \quad \sum_k a_k = 1$
  - number of customers arriving at different times are $\perp\!\!\!\perp$

Transition matrix:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & & & & \end{bmatrix}$$

$a_0 > 0$

$0 < a_0 + a_1 < 1$

The MC is irreducible (there is always the possibility of coming back to 0)

Is the MC transient or recurrent? (transient → the queue explodes)

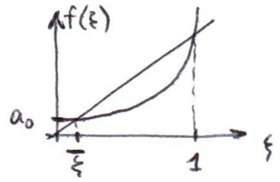
Let $\lambda = \sum_{k\geq 1} k \cdot a_k$ = average number of arriving customers

- $\lambda > 1$ : (intuitively transient)

  We use the first criteria: we look for sol. of $\sum_{k\geq 0} p_{jk}\, y_k = y_j \quad \forall j>0$ of the form $y_k = \xi^k$ : $(0<\xi<1)$

  $$\longrightarrow \sum_{k\geq j-1} a_{k-(j-1)}\, \xi^k = \xi^j$$
  $$\sum_{k\geq j-1} a_{k-(j-1)}\, \xi^{k-(j-1)} = \xi$$
  $$\sum_{h\geq 0} a_h\, \xi^h = \xi$$

  Consider $f(\xi) = \sum_{h\geq 0} a_h \xi^h$ :

  $a_k \geq 0 \Rightarrow \xi \mapsto f(\xi)$ increasing

  We know that $f(1) = \sum_{h\geq} h a_h = \lambda$

  Moreover, by Abel: $f'(1) = \sum_{h\geq 0} h a_h = \lambda \Longrightarrow$ For $\lambda > 1$ $\exists \bar{\xi} \in (0,1)$ s.t. $f(\bar{\xi}) = \bar{\xi}$ and we have $(y_i)_i$ with $y_i = \bar{\xi}^i$ bounded and non constant

- $\lambda \leq 1$ : (intuitively recurrent)

  We look for an unbounded solution of $\sum_{k\geq 0} p_{jk}\, y_k \leq y_j \quad \forall j>0$ and we chose $y_k = k$ :

  $$\begin{aligned}\sum_{k\geq 0} p_{jk}\, y_k &= \sum_{k\geq j-1} a_{k-(j-1)}\, k \\ &= \sum_{h\geq 0} a_h (h + (j-1)) \\ &= \sum_{h\geq 0} a_h h + \sum_{h\geq 0} a_h (j-1) \\ &= \lambda + (j-1) = (\lambda - 1) + y_j \leq y_j \quad \text{for } \lambda \leq 1\end{aligned}$$

  Thus the MC is recurrent.

- Thm. Sufficient condition for $\exists \pi$ (invariant distribution)

  $(X_n)_{n\geq 0}$ irreducible MC.

  If $\exists (y_j)_j, (x_j)_j$ both unbounded $(y_j, x_j \xrightarrow{j} +\infty)$ such that:

  $$\sum_{k\geq 0} p_{jk}\, y_k \leq y_j - x_j \quad \forall j$$

  $\Longrightarrow$ the MC admits a unique invariant density

- Example: Queue model

  We assume $\sum_{k\geq 0} k^2 a_k < \infty$

  We consider $y_j = j^2$ :
  $$\begin{aligned}\sum_{k\geq 0} p_{jk}\, y_k &= \sum_{k\geq j-1} a_{k-(j-1)}\, k^2 \\ &= \sum_{h\geq 0} a_h (h + (j-1))^2 \\ &= \sum_h h^2 a_h + 2(j-1)\sum_h h a_h + (j-1)^2 \sum_h a_h \\ &= m_2 + 2(j-1)\lambda + (j-1)^2 \\ &= j^2 - (2(1-\lambda)j - (1-2\lambda+m_2)) \\ &= y_j - x_j\end{aligned}$$

  Since both $y_j$ and $x_j$ goes to $\infty$ (if $\lambda < 1$) $\longrightarrow \exists!$ invariant distribution (if $\lambda < 1$)

- Sojourn time

  We call $T$ = set of transient states. Let $S \subseteq T$. We define the sojourn time in $S$ the total time spent in $S$: $T_S = \sum_{n\geq 0} \mathbb{1}_{\{X_n \in S\}}$

- Moment generating function of a random var. T : $m_i = \mathbb{E}_i[z^T] \quad \forall i \in E$

Note: if S is finite $\Rightarrow$ $m_i = \mathbb{E}_i[z^{T_S}]$ is well defined

Properties:

1. $m_i(z) = \sum_{k \geq 0} z^k \mathbb{P}_i(T=k)$
2. $m_i'(z) = \mathbb{E}_i[z^{T-1}]$
3. $\begin{cases} m_i(z) = 1 & i \text{ recurrent} \\ m_i(z) = z \sum_j p_{ij} m_j(z) & i \in S \\ m_i(z) = \sum_j p_{ij} m_j(z) & i \in T \setminus S \end{cases}$

proof.

- $i$ recurrent $\Rightarrow T = 0 \Rightarrow \mathbb{E}_i[z^T] = 1$
- $i \in T$ : $T_S = \mathbb{1}\{X_0 \in S\} + \sum_{n \geq 1} \mathbb{1}\{X_n \in S\} = \mathbb{1}\{X_0 \in S\} + \tilde{T}$

$$\begin{aligned} \mathbb{E}_i[z^T] &= \sum_j \mathbb{E}_i[z^T \mid X_1 = j] \, \mathbb{P}_i(X_1 = j) \\ &= \sum_j \mathbb{E}_i[z^{\mathbb{1}\{X_0 \in S\}} z^{\tilde{T}} \mid X_1 = j] \, \mathbb{P}_i(X_1 = j) \\ &= z^{\mathbb{1}\{X_0 \in S\}} \sum_j \mathbb{E}[z^{\tilde{T}} \mid X_1 = j, X_0 = i] \, p_{ij} \\ &= z^{\mathbb{1}\{X_0 \in S\}} \sum_j \mathbb{E}_j[z^T] \, p_{ij} \end{aligned}$$

$$\Rightarrow \begin{cases} i \in S : & m_i(z) = z \sum_j \mathbb{E}_j[z^T] \, p_{ij} \\ i \in T \setminus S : & m_i(z) = \sum_j \mathbb{E}_j[z^T] \, p_{ij} \end{cases}$$

$\blacksquare$

- Example: Gambler's win

Symmetric case $p = q = 1/2$

Finite state space := $\{0,1,2,3\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Moment generator function of the total time spent in $S = \{1\}$ starting from $\{1\}$:

$$\begin{cases} m_0(z) = m_3(z) = 1 \\ m_1(z) = z \left( \frac{1}{2} m_2(z) + \frac{1}{2} m_0(z) \right) \\ m_2(z) = \frac{1}{2} m_1(z) + \frac{1}{2} \end{cases}$$

$$\Rightarrow \quad m_1(z) = \frac{3/4 \, z}{1 - z/4}, \qquad m_2(z) = \frac{1 + z/2}{2(1 - z/4)}$$

# CONTINUOUS TIME MARKOV CHAIN

## TRANSITION PROBABILITIES & RATES

- Markov property : $\mathbb{P}(X_{t_{n+1}}=j \mid X_{t_n}=i, \dots, X_{t_0}=i_0) = \mathbb{P}(X_{t_{n+1}}=j \mid X_{t_n}=i)$
- Time homogeneity : $\mathbb{P}(X_{t+s}=j \mid X \; =i) = \mathbb{P}(X_t=j \mid X_0=i) = p_{ij}(t)$
- Transition semigroup : $P_t = (p_{ij}(t))_{i,j\in E}$

  properties :
  - $0 \le p_{ij}(t) \le 1 \qquad \forall i,j \in E, \;\; \forall t > 0$
  - $\sum_{j\in E} p_{ij}(t) = 1$
  - $P_{t+s} = P_t P_s$
  - $P_0 = I$

- Chapman-Kolmogorov equation : $p_{ij}(t+s) = \sum_{k\in E} p_{ik}(t)\, p_{kj}(s)$
- In regular situations we know that the following limits exist :

$$q_{ij} = \lim_{t\to 0^+} \frac{p_{ij}(t)}{t} \quad i\neq j \quad , \qquad q_{ii} = \lim_{t\to 0^+} \frac{-1+p_{ii}(t)}{t}$$

They are the derivatives of transition probabilities in 0 (since $p_{ij}(0)=\delta_{ij}$) and they're called transition rates. Moreover : $q_{ii} \le 0, \;\; q_{ij} \ge 0$

- Kolmogorov equations

  Forward KE : $\quad p_{ij}'(t) = \sum_{k\in E} p_{ik}(t)\, q_{kj}$

  Backward KE : $\quad p_{ij}'(t) = \sum_{k\in E} q_{ik}\, p_{kj}(t)$

- Transition rate matrix : $Q = (q_{ij})_{i,j\in E}$

  properties :
  - $\sum_{j\in E} q_{ij} = 0$
  - $q_{ij} \ge 0 , \quad q_{ii} \le 0$
  - (from FKE) : $P_t' = P_t Q$

- Thm. Solution of Kolmogorov equations

  If $\sup |q_{ii}| < +\infty \implies$ FKE and BKE have the same solution : $P_t = e^{tQ}$

## EXIT TIME & DISCRETE SKELETON

- $T_i = \inf\{t>0 : X_t \neq i\}$ = exit time from the state $i \qquad (i\in E)$

  If $-\infty < q_{ii} < 0 \implies T_i \sim \mathcal{E}(-q_{ii})$

  $\implies \mathbb{P}_i(X_{T_i}=j) = \dfrac{q_{ij}}{-q_{ii}} \qquad \forall j\neq i$

  where $X_{T_i}$ = state we visit after leaving $i$

  If $q_{ii} = +\infty \implies i$ is an istantaneous state

  If $q_{ii} = 0 \implies i$ is absorbing : $p_{ii}(t) = 1 \quad \forall t$

- We can associate a discrete MC to a continuous one through the Discrete Skeleton : $(\hat{p}_{ij})_{i,j\in E}$

$$\hat{p}_{ij} = \begin{cases} \dfrac{q_{ij}}{-q_{ii}} & q_{ii}\neq 0, \;\; i\neq j \\ 1 & q_{ii}=0, \;\; i=j \\ 0 & (q_{ii}=0, \; i\neq j) \vee (q_{ii}\neq 0, \; i=j) \end{cases}$$

# POISSON PROCESS

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ \cdots \end{bmatrix} \Longrightarrow \underset{\text{discrete skeleton}}{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots \end{bmatrix}$$

- The states are $\{0,1,2,\dots\}$ and the situation is:
  $(i) \rightsquigarrow (i+1)$ we go out of the state $i$ after an exponential time $(\mathcal{E}(\lambda))$
  We define the MC $(N_t)_{t\geq 0}$ and we set $\lambda > 0$.
- $(N_t)_{t\geq 0}$ MC with $Q = (q_{ij})_{i,j\in E}$ is such that:
  $$\mathbb{P}_i(N_t = n \mid N_0 = 0) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad N_t \mid N_0 = 0 \sim \mathcal{P}(\lambda t)$$
- $(N_t)_{t\geq 0}$ MC with $Q = (q_{ij})_{i,j\in E}$ is such that:
  $N_{t_1}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}$ are $\perp\!\!\!\perp$ with respect to $\mathbb{P}_0$ $(\forall n, \forall t_1,\dots,t_n)$
  Moreover: $N_{t_k} - N_{t_{k-1}} \sim \mathcal{P}(\lambda(t_k - t_{k-1}))$

# POPULATION DYNAMICS

$(X_t)_{t\geq 0}$ MC : $X_t = \#$ individuals at time $t$
Generic case of BIRTH-DEATH PROCESS :

$(n-1) \leftsquigarrow (n) \rightsquigarrow (n+1)$

In every moment we can :
- $n \to n+1 \quad \sim \mathcal{E}(\lambda_n)$
- $n \to n-1 \quad \sim \mathcal{E}(\mu_n)$

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\mu_3+\lambda_3) & \lambda_3 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

$B_n \sim \mathcal{E}(\lambda_n)$, $D_n \sim \mathcal{E}(\mu_n)$
The MC is irreducible iff $\lambda_n > 0 \ \forall n$, $\mu_n > 0 \ \forall n$.
We leave the state $n$ after a random time, which is the minimum of $B_n, D_n$:
$\Rightarrow$ leaving time $\sim \mathcal{E}(\lambda_n + \mu_n)$
$$\mathbb{P}(B_n < D_n) = \frac{\lambda_n}{\mu_n + \lambda_n} \qquad \left(= \frac{q_{i\,i+1}}{-q_{ii}}\right)$$

- Invariant density
  $$\pi_n = \left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)\left(\frac{1}{1+\sum_{n\geq 1}\frac{\lambda_0\cdots\lambda_{n-1}}{\mu_1\cdots\mu_n}}\right)$$

- Explosion in finite time: (pure birth process with $\lambda_n$)
  If $\lambda_n$ are such that: $\sum_{n\geq 0}\frac{1}{\lambda_n} < +\infty \Longrightarrow$ In finite time $(X_n)_n$ diverges

# STATES CLASSIFICATION

- $j$ is accessible from $i$ if $\exists t : p_{ij}(t) > 0$
- $i$ and $j$ communicate if one is accessible from the other

(def. 2) $j$ accessible from $i \iff \exists n : i_1,\dots,i_n \ : \ q_{ii_1}\cdots q_{i_nj} > 0$

- $i$ recurrent : $\mathbb{P}_i(\omega \mid \{t\geq 0 : X_t(\omega) = i\}$ is unbounded$) = 1$
  $i$ transient : $\mathbb{P}_i(\omega \mid \{t\geq 0 : X_t(\omega) = i\}$ is unbounded$) = 0$
- Thm. $i$ recurrent (/transient) for the continuous MC $\iff$ $i$ recurrent (/transient) for the discrete skeleton (associated discrete MC)

# INVARIANT DENSITIES

- $(X_t)_{t\geq 0}$ continuous time MC. $(P_t)_{t\geq 0}$ transition semigroup.
  $(\mu_i)_{i\in E}$ invariant density $\iff$ $\mu_j = \sum_{i\in E} \mu_i p_{ij}(t)$ $\quad \forall j \; \forall t$

  properties:
  - $\sum_{i\in E} \mu_i = 1$
  - $\mu_i \geq 0 \quad \forall i \in E$
  - If $q_{ii} > -\infty$ and $(p_{ij}(t))_{ij}$ is the unique sol. of FKE, BKE
    $\Longrightarrow$ $(\mu_i)_i$ is an invariant density $\iff$ $0 = \sum_{i\in E} \mu_i q_{ij} \quad \forall j \in E$
  - E finite $\Longrightarrow$ $\exists$ at least 1 invariant density
  - Invariant densities of the continuous MC are invariant densities of the discrete skeleton
    $(\pi_i)_i$ for continuous, $(\pi_i(-q_{ii}))_i$ for discrete

- Ergodic theorem
  - discrete positive recurrent irreducible MC: $\quad \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}\{X_k = i\} \xrightarrow{n} \pi_i$
  - continuous positive recurrent irreducible MC: $\quad \frac{1}{t}\int_0^t \mathbb{1}\{X_s = i\}\, ds \xrightarrow{n} \pi_i$

# CONDITIONAL EXPECTATION

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space. $\mathcal{G} \subseteq \mathcal{F}$ sub-$\sigma$-algebra.

- X random variable s.t. $\mathbb{E}[|X|] < +\infty$. We call conditional expectation of X w.r.t. $\mathcal{G}$ any $\mathcal{G}$-measurable random variable **Y** s.t.:
$$\int_G X \, d\mathbb{P} = \int_G Y \, d\mathbb{P} \qquad \forall G \in \mathcal{G}$$

- Thm. Existence of conditional expectation
  X $\mathcal{F}$-measurable, integrable, $\mathbb{E}[|X|] < +\infty$, $\mathcal{G} \subseteq \mathcal{F}$
  $\Longrightarrow$ $\exists Y$ $\mathcal{G}$-measurable : $\int_G X\, d\mathbb{P} = \int_G Y \, d\mathbb{P} \qquad \forall G \in \mathcal{G}$
  (unique up to sets $G \in \mathcal{G}$ : $\mathbb{P}(G) = 0$)

- Properties:
  - $\mathbb{E}[aX_1 + bX_2 | \mathcal{G}] = a\mathbb{E}[X_1|\mathcal{G}] + b\mathbb{E}[X_2|\mathcal{G}]$
  - $X_1 \geq 0 \Longrightarrow \mathbb{E}[X_1|\mathcal{G}] \geq 0$
  - $\mathbb{P}(X_1 > X_2) = 1 \Longrightarrow \mathbb{E}[X_1|\mathcal{G}] \geq \mathbb{E}[X_2|\mathcal{G}]$
  - $X_1$ constant $\Longrightarrow \mathbb{E}[X_1|\mathcal{G}] = X_1 =$ constant
  - $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ if $\mathcal{H} \subseteq \mathcal{G}$

- $(X_n)_{n\geq 0}$ discrete MC, P transition matrix, $f: E \to \mathbb{R}$, $\mathbb{E}[|f(X_n)|] < \infty \quad \forall n$
  $\Longrightarrow$ $\mathbb{E}[f(X_{n+m}) | \sigma(X_0, \dots, X_n)] = P^m f(X_n)$

- $(X_t)_{t\geq 0}$ continuous MC, $P_t$ trans. semigroup, $f: E \to \mathbb{R}$, $\mathbb{E}[|f(X_n)|] < \infty \quad \forall n$
  $\Longrightarrow$ $\mathbb{E}[f(X_t) | \sigma(X_r : r \leq s)] = (P_{t-s} f)(X_n)$

- Thm. Min square error approximation of a random variable
  X r.v. with $\mathbb{E}[|X|^2] < \infty$, $\mathcal{G} \subseteq \mathcal{F}$
  $\Longrightarrow$ $\mathbb{E}[|\mathbb{E}[X|\mathcal{G}]|^2] < \infty$ and $\displaystyle\min_{\substack{Y\ \mathcal{G}\text{-measurable} \\ \mathbb{E}[|Y|^2]<\infty}} \mathbb{E}[|X-Y|^2] = \mathbb{E}[|X - \mathbb{E}[X|\mathcal{G}]|^2]$

- Thm. Freezing lemma
  X r.v. with density f, $\mathcal{G} \perp\!\!\!\perp X$ $(\sigma(X) \perp\!\!\!\perp \mathcal{G})$, $Y_1, \dots, Y_n$ $\mathcal{G}$-meas.
  $\Longrightarrow$ $\mathbb{E}[h(X, Y_1, \dots, Y_n) | \mathcal{G}] = \int_{\mathbb{R}} h(x, Y_1, \dots, Y_n) f(x)\, dx$

# MARKOV PROCESS

- $(X_t)_{t\geq 0}$ collection of random variables $X_t : \Omega \to E$ is a Markov Process if
  $\mathbb{P}(X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m, \dots, X_{t_1} \in E_1) = \mathbb{P}(X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m)$
  $\forall n, \ \forall t_1 < \dots < t_{m+1}, \ \forall E_1, \dots, E_{m+1} \in \mathcal{E}$
- $(X_t)_{t\geq 0}$ MP homogeneous if: $\mathbb{P}(X_{t+s} \in E_2 \mid X_t \in E_1) \perp\!\!\!\perp s$
- Transition kernels : $P_t(x, A) = \mathbb{P}(X_t \in A \mid X_0 = x)$
  (def. 2) collection of $P_t : E \times \mathcal{E} \longrightarrow [0,1]$ such that:
  1. $P_t(x, \bullet)$ is a probability measure on $\mathcal{E}$ $\forall x \in E$
  2. $P_t(\bullet, A)$ is $\mathcal{E}$-measurable $\forall A \in \mathcal{E}$
  3. $\mathbb{P}(X_{t+s} \in A \mid X_s = x) = \int_A P_t(x, dy)$
- Thm. MP and conditional expectation
  $(X_t)_{t\geq 0}$ MP time homogeneous, $P_t$ transition kernel
  $f : E \to \mathbb{R}$ measurable : $\mathbb{E}[f(X_{t+s})] < \infty$
  $\Longrightarrow \mathbb{E}[f(X_{t+s}) \mid \sigma(X_s)] = \int_E f(y)\, P_t(X_s, dy)$
- STATIONARITY
  Let $(X_t)_{t\geq 0}$ be an MP and $\mu$ a measure on $\mathcal{E}$.
  $\mu$ is invariant if when $X_0 \sim \mu$ then $X_t \sim \mu$ $\forall t > 0$.
  $\mu$ is invariant $\Longleftrightarrow \mu(A) = \int_E P_t(x, A)\, \mu(dx)$ $\forall A \in \mathcal{E}$
- IRREDUCIBILITY
  let $(X_t)_{t\geq 0}$ be an MP with values in $(E, \mathcal{E})$ and transition kernels $(P_t)_t$.
  $(X_t)_{t\geq 0}$ is irreducible w.r.t. a reference measure $\varphi$ on $\mathcal{E}$ if
  $\forall x \in E, \ \forall A \in \mathcal{E}$ s.t. $\varphi(A) > 0$ : $\exists t > 0$ s.t. $P_t(x, A) > 0$.
- HARRIS RECURRENCE
  $(X_n)_{n\geq 0}$ discrete MC, $E \subseteq \mathbb{R}^d$, $\mathcal{E} = \mathcal{B}(E)$.
  We set $N_A = \sum_{n\geq 0} \mathbb{1}_{\{X_n \in A\}}$ = number of visits in the set $A$ $(\forall A \in \mathcal{E})$.
  Then:
  - $A$ is Harris recurrent if $\forall x$ : $\mathbb{P}_x(N_A = +\infty) = 1$ $(\forall x \in A)$
  - $(X_n)_n$ Harris recurrent if $\exists \varphi$ ; $\forall A \in \mathcal{E}$ with $\varphi(A) > 0$ are H.R.
- LLN
  $(X_n)_{n\geq 0}$ Harris recurrent, $\varphi$ invariant measure, $f : E \to \mathbb{R}$ $\mathcal{E}$-meas. : $\int_E |f(x)|\varphi(dx) < \infty$
  $\Longrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} f(X_k) = \int_E f(x)\, \varphi(dx)$

# MARTINGALES

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space, $(M_t)_{t\geq 0}$ real random variables,
  $(\mathcal{F}_t)_{t\geq 0}$ increasing family of sub $\sigma$-algebra of $\mathcal{F}$ $(s < t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t)$.
  $(M_t)_{t\geq 0}$ is a martingale if:
  1. $\mathbb{E}[|M_t|] < +\infty$ $\forall t$
  2. $M_t$ is $\mathcal{F}$-measurable $\forall t$
  3. $\forall s < t$ : $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ (Martingale property)

- $(X_n)_{n\geq 0}$ discrete MC
  $f : E \to \mathbb{R}$ measurable
  $\mathbb{E}[|f(X_n)|] < +\infty \quad \forall n$
  $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$

$\Longrightarrow$

- $\mathbb{E}[f(X_n) | \mathcal{F}_m] = (P^{n-m} f)(X_m) \qquad (m<n)$
- $M_n := f(X_n) - \sum_{k=0}^{n-1} Pf(X_k) - f(X_k)$
  $(M_n)_{n\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\geq 0}$

- $(X_t)_{t\geq 0}$ continuous MC
  $f : E \to \mathbb{R}$ measurable
  $\mathbb{E}[|f(X_n)|] < +\infty$
  $\mathbb{E}[|Qf(X_t)|] < +\infty \quad \forall t$
  $\mathcal{F}_t = \sigma(X_r : r \leq s)$

$\Longrightarrow$ $M_t = f(X_t) - \int_0^t (Qf)(X_s)\, ds \qquad \forall t \geq 0$
$(M_t)_{t\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$

- $T : \Omega \to [0, +\infty]$ stopping time for the filtration $(\mathcal{F}_t)_{t\geq 0}$ if $\{T \leq t\} \in \mathcal{F}_t \ \forall t$.
- Stopping theorem
  $(M_t)_{t\geq 0}$ martingale (where $t$ belongs to a discrete set)
  $(\mathcal{F}_t)_{t\geq 0}$ filtration
  $T$ stopping time for the filtration
  $\Longrightarrow$ The stopped process $(M_{t\wedge T})_t$ is also a martingale
  moreover:
  $$\mathbb{E}[M_{T\wedge t}] = \mathbb{E}[M_0] = \mathbb{E}[M_t] \qquad \forall t$$
  (martingales have constant expectation)

Note: by MCT $\mathbb{E}_i[t \wedge T] \xrightarrow{t} \mathbb{E}[T]$
by fatou $\mathbb{E}_i[M_{t\wedge T}] \xrightarrow{t} \mathbb{E}[M_T]$