KNAPSACK PROBLEM - 1D

n objects : $i \in N = \{1, .., n\}$
$\forall i \in N$: p_i profit, a_i weight
b capacity of the knapsack

ILP:

$$\max \sum_i p_i x_i$$
$$\sum_i a_i x_i \le b$$
$$x_i \in \{0,1\} \quad \forall i \in N$$

Cover inequality ?

A subset $C \subseteq N$ is a cover if it corresponds to a subset of indeces which items will not fit (all together) the knapsack.
if $C \subseteq N$ is a cover for X, the cover inequality $\sum_{j \in C} x_{ij} \le |C| - 1$ is a valid inequality for $\mathbf{X}$.

A valid inequality is an inequality that is satisfied by all the feasible solutions. Since C is a cover, it is impossible to put all the elements of C in the knapsack, so at most $|C|-1$ of them can be put in together
$\Longrightarrow$ $\sum_{j \in C} x_{ij} \le |C| - 1$ is satisfied by all integer solutions of the problem (and so it's a valid inequality).

Separation problem ?

Problem: given a fractional $\underline{x}^*$ with $0 \le x_j^* \le 1$ $\forall j \in N$, find a cover inequality that is violated by $\underline{x}^*$ or establish that none exists.

Since $\left[\sum_{j \in C} x_j \le |C| - 1 \right] \Longleftrightarrow \left[\sum_{j \in C} (1 - x_j) \ge 1 \right]$

The problem becomes:

$$\exists C \subseteq N \text{ s.t. } \underbrace{\sum_{j \in C} a_j > b}_{C \text{ is a cover}} \text{ and } \underbrace{\sum_{j \in C} (1 - x_j^*) < 1}_{C \text{ violates the cover inequality}} \text{ ?}$$

ILP formulation:
let $\underline{z} \in \{0,1\}^n$ charactenizes C ($z_j = 1$ if $j \in C$)

$$\Longrightarrow \quad \xi := \min \left\{ \sum_{j \in N} (1 - x_j^*) z_j \ : \ \sum_j a_j z_j > b, \ \underline{z} \in \{0,1\}^n \right\}$$

- if $\xi \le 1 \Longrightarrow$ $\underline{x}^*$ satisfies all cover inequalities
- if $\xi > 1 \Longrightarrow$ $\sum_{j \in C} x_j \le |C| - 1$, $C = \{ j : z_j^* = 1, \ j \in N, \ \underline{z}^* \text{ optimal} \}$, is violated by $\underline{x}^*$ by a quantity $1 - \xi$

Consider $X = \{ \underline{x} \in \{0,1\}^6 : 12x_1 + 9x_2 + 7x_3 + 5x_4 + 5x_5 + 3x_6 \leq 14 \}$

List all minimal inequalities:

$x_1 + x_2 \leq 1$
$x_1 + x_3 \leq 1$
$x_1 + x_4 \leq 1$
$x_1 + x_5 \leq 1$
$x_1 + x_6 \leq 1$

$x_2 + x_3 \leq 1$
$x_2 + x_4 + x_5 \leq 2$
$x_2 + x_4 + x_6 \leq 2$
$x_2 + x_5 + x_6 \leq 2$

$x_3 + x_4 + x_5 \leq 2$
$x_3 + x_4 + x_6 \leq 2$
$x_3 + x_5 + x_6 \leq 2$

Apply the lifting procedure from $x_3 + x_5 + x_6 \leq 2$:

$C = \{3,5,6\}$, $\quad N \setminus C = \{1,2,4\}$

- $\alpha_1 x_1 + x_3 + x_5 + x_6 \leq 2$;
  $x_1 = 0 \implies \forall \alpha_1$
  $x_1 = 1 \implies \alpha_1 = 2 - \max\{ x_3 + x_5 + x_6 : 7x_3 + 5x_5 + 3x_6 \leq 14 - 12 = 2 \}$
  $= 2 - 0 = 2$

  $\implies 2x_1 + x_3 + x_5 + x_6 \leq 2$

- $\alpha_2 x_2 + 2x_1 + x_3 + x_5 + x_6 \leq 2$;
  $x_2 = 0 \implies \forall \alpha_2$
  $x_2 = 1 \implies \alpha_2 = 2 - \max\{ 2x_1 + x_3 + x_5 + x_6 : 12x_1 + 7x_3 + 5x_5 + 3x_6 \leq 14 - 9 = 5 \}$
  $= 2 - 1 = 1$

  $\implies 2x_1 + x_2 + x_3 + x_5 + x_6 \leq 2$

- $\alpha_4 x_4 + 2x_1 + x_2 + x_3 + x_5 + x_6 \leq 2$:
  $x_4 = 0 \implies \forall \alpha_4$
  $x_4 = 1 \implies \alpha_4 = . - \max\{ 2x_1 + x_2 + x_3 + x_5 + x_6 : 12x_1 + 9x_2 + 7x_3 + 5x_5 + 3x_6 \leq 14 - 5 = 9 \}$
  $= 2 - 2 = 0$

  $\implies \boxed{2x_1 + x_2 + x_3 + x_5 + x_6 \leq 2}$

since we started from a minimal cover the lifting procedure ends with a facet inequality (facet of $conv(X)$)

KNAPSACK PROBLEM – multiple

m = richieste di calcolo
n = calcolatori
p_j = profitto del calcolo j
c_i = capacità dell'elaboratore i
w_{ij} = capacità necessaria per far girare la richiesta j sull'elaboratore i

$$\max \sum_i \sum_j x_{ij} p_j$$

s.t. $\sum_j x_{ij} w_{ij} \le c_i \quad \forall i \quad$ (1) capacity constraint

$\sum_i x_{ij} \le 1 \quad \forall j \quad$ (2) assignment constraint

$x_{ij} \in \{0,1\} \quad \forall i \, \forall j$

Two Lagrangian relaxations:

(write the sub-problems ~~and explain how they can be solved~~)

1. we relax the capacity constraint:

$$W_1 = \max \sum_i \sum_j x_{ij} p_j + \sum_i u_i \left(c_i - \sum_j x_{ij} w_{ij} \right)$$

$\sum_i x_{ij} \le 1 \quad \forall j$

$x_{ij} \in \{0,1\} \quad \forall i \forall j$

$u_i \ge 0 \quad \forall i$

2. we relax the assignment constraint:

$$W_2 = \max \sum_i \sum_j x_{ij} p_j + \sum_j v_j \left(1 - \sum_i x_{ij} \right)$$

$\sum_j x_{ij} w_{ij} \le c_i \quad \forall i$

$x_{ij} \in \{0,1\} \quad \forall i \forall j$

$v_j \ge 0 \quad \forall j$

How can they be solved?

1. $$\max \ldots = \max \sum_i \sum_j x_{ij} p_j + \sum_i u_i c_i - \sum_i \sum_j x_{ij} w_{ij} u_i$$
$$= \max \sum_i \sum_j x_{ij} (p_j - w_{ij} u_i) + \sum_i u_i c_i$$

In this way we have a new profit for each term: $\tilde{p}_j = p_j - w_{ij} u_i \ge 0$. According to this new profits $\tilde{p}$ we will insert the item j in i which maximizes $\tilde{p}_j$. Each item can be selected at most once.
(We can see it as assignment problem: assign j to the i for which $\tilde{p}_j$ is the highest)

2. $$\max \ldots = \max \sum_i \sum_j x_{ij} p_j + \sum_j v_j - \sum_j \sum_i v_j x_{ij}$$
$$= \max \sum_i \sum_j x_{ij} (p_j - v_j) + \sum_j v_j$$

The new profit for the item j is $\hat{p}_j = p_j - v_j \ge 0$. This time each item can be taken several times. We can decompose the original problem in m binary knapsack problems:

$$W^i = \max \sum_{j=1}^{n} (p_j - v_j) x_{ij}$$

$\sum_{j=1}^{n} w_j x_{ij} \le c_i \quad \forall i$

$x_{ij} \in \{0,1\} \quad \forall i \forall j$

(This is possible because we eliminated the linking constraints)

Which relaxation is stronger?

Given $\min\{\underline{c}^T\underline{x} : A\underline{x} \geq \underline{b},\ D\underline{x} \geq \underline{d},\ \underline{x} \in \mathbb{Z}^n\}$, $X = \{\underline{x} \in \mathbb{Z}^n,\ A\underline{x} \geq \underline{b}\}$,

let: $\begin{cases} w(\underline{u}) = \min\{\underline{c}^T\underline{x} + \underline{u}^T(\underline{d} - D\underline{x}) : A\underline{x} \geq \underline{b},\ \underline{x} \in \mathbb{Z}^n\} \\ w^* = \max\limits_{\underline{u} \geq \underline{0}} w(\underline{u}) \end{cases}$

$\longrightarrow \quad w^* = \min\{\underline{c}^T\underline{x} : D\underline{x} \geq \underline{d},\ \underline{x} \in \operatorname{conv}(X)\}$.

Since $\operatorname{conv}(X) \subseteq \{\underline{x} \in \mathbb{R}^n : A\underline{x} \geq \underline{b}\} \Longrightarrow z_{LP} \leq w^* \leq z^*$, so the lagrangian duality solution is at least as good as the LP relaxation.

But if $\operatorname{conv}(X) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \geq \underline{b}\} \Longrightarrow z_{LP} = w^*$

both lagrangian and LP are equal weak

Based on this we can say that 1. is as weak as the LP relaxation. Relaxing the constraints as in 2. is a better choice.

Write the lagrangian duals and explain how to solve them:

1. $w_1^* = \min\limits_{\underline{u} \geq \underline{0}} w_1(\underline{u})$, 2. $w_2^* = \min\limits_{\underline{u} \geq \underline{0}} w_2(\underline{u})$

We can solve them with the subgradient method.

(Recall: given $C \subseteq \mathbb{R}^n$, $f: C \to \mathbb{R}$ convex, $\underline{\gamma} \in \mathbb{R}^n$ is a subgradient of f at $\underline{x}^* \in C$ if: $f(\underline{x}) \geq f(\underline{x}^*) + \underline{\gamma}^T(\underline{x} - \underline{x}^*) \quad \forall \underline{x} \in C$.
We denote by $\partial f(\underline{x}^*)$ all the subgradients at $\underline{x}^*$)

Subgradient method:

- we select $\underline{u}_0$ and we set $k=0$
- at each iteration **k**:

  we solve $w(\underline{u}_k) = \min\limits_{x \in X}\{\underline{c}^T\underline{x} - \underline{u}_k^T(\underline{d} - D\underline{x})\} = \min\limits_{x \in X} L(\underline{x}, \underline{u}_k)$

  let $\underline{x}_k$ be the optimal solution $\Longrightarrow (\underline{d} - D\underline{x}_k) \in \partial w(\underline{u}_k)$
- $\underline{u}_{k+1} = \max\{\underline{0},\ \underline{u}_k + \alpha_k(\underline{d} - D\underline{x}_k)\}$

  $\alpha_k = \varepsilon_k \dfrac{\hat{w} - w(\underline{u}_k)}{\|\underline{d} - D\underline{x}_k\|}$
- $k := k+1$

Minimum cost flow

Directed graph $G = (V, A)$
$\forall (i,j) \in A$: k_{ij} capacity, c_{ij} unit cost
$\forall i \in V$: b_i demand (>0 source, <0 dest) : $\sum_i b_i = 0$

ILP formulation?

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\sum_{(h,i)\in\delta^-(i)} x_{hi} - \sum_{(i,j)\in\delta^+(i)} x_{ij} = b_i \qquad \forall i \in V \qquad (1)$$

$$x_{ij} \leq k_{ij} \qquad \forall (i,j) \in A \qquad (2)$$

$$x_{ij} \geq 0 \text{ integer} \qquad \forall (i,j) \in A$$

where $\delta^+(i) = \{(i,j) \in A : j \in V\}$
$\delta^-(i) = \{(h,i) \in A : h \in V\}$

Is it an ideal formulation?

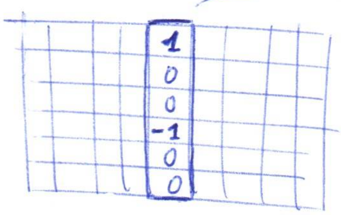
The matrix of constraints (1), (2) is TU.
If the matrix of constraints is TU, it sufficies to solve the LP relaxation instead of solving ILP. Since in some sense the converse is also true, we can say:

Let X be the feasible region for the original problem,
let P be the feasible region of the LP relaxation

since the constr. matrix is TU (and b_i, k_{ij} are integers) $\Longrightarrow$ $P = conv(X)$ (= def of ideal formulation)

Why is TU?

The constraints (1) create:



exactly 2 non-zero components for every column,

We can partition this as I_1 = all indeces, $I_2 = \emptyset$ and so this matrix is TU.
(sufficient cond. for being TU)
The constraint (2) creates $-I_{|V|}$ and so the whole matrix of constraints (1) and (2) is TU.

Shortest path problem?

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\sum_{(h,i)\in\delta^-(i)} x_{hi} - \sum_{(i,j)\in\delta^+(i)} x_{ij} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{else} \end{cases} \qquad \forall i \in V$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} \leq 1 \qquad \forall i \in V$$

$$x_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A$$

Maximum flow problem?

$$\max \quad \sum_{(i,j)\in A} x_{ij}$$

$$\sum_{(h,i)\in\delta^-(i)} x_{hi} = \sum_{(i,j)\in\delta^+(i)} x_{ij} \qquad \forall i \in V \setminus \{s,t\}$$

$$x_{ij} \leq k_{ij} \qquad \forall (i,j)\in A$$

$$x_{ij} \geq 0 \qquad \forall (i,j)\in A$$

UFL - Uncapacitated Facility location problem

$M = \{1, \dots, m\}$ clients, $i \in M$
$N = \{1, \dots, n\}$ candidates sites, $j \in N$
f_j = fixed costs for opening j
c_{ij} = transp. cost* if the whole demand of i is served by j * cost/profit
x_{ij} = fraction of demand of i satisfied by j
y_j = 1 if depot j is open

Two formulations ?

1. $$\max \sum_i \sum_j c_{ij} x_{ij} - \sum_j f_j y_j$$

$$\begin{array}{ll} \sum_{j \in N} x_{ij} = 1 & \forall i \in M \\ \boxed{\sum_{i \in M} x_{ij} \le m y_j \qquad \forall j \in N} & (*) \\ y_j \in \{0,1\} & \forall j \in N \\ 0 \le x_{ij} \le 1 & \forall j \in N, \forall i \in M \end{array}$$

2. as before, but we substitute (*) with: $\boxed{x_{ij} \le y_j \qquad \forall i \in M, \forall j \in N}$

The first formulation has **n** constraints in (*).
We substitute **n** constraints with **mxn** constraints in the second.

Which is stronger?

The second formulation is stronger than the first because the polyhedras of the LP relaxations (respectively P_2 for 2. and P_1 for 1.) are such that: $P_2 \subsetneq P_1$.

(obviously $P_2 \subset P_1$ because $\forall j$, summing $x_{ij} \le y_j$ over i leads to $\sum_i x_{ij} \le m y_j$, moreover $\exists (\underline{x}, \underline{y}) \in P_1 \setminus P_2$)

Lagrangian relaxation ?

Lagrangian subproblem:

$$w(\underline{u}) = \max \sum_i \sum_j c_{ij} x_{ij} - \sum_j f_j y_j + \sum_i u_i \left(1 - \sum_j x_{ij}\right)$$

$$= \max \sum_i \sum_j (c_{ij} - u_i) x_{ij} - \sum_j f_j y_j + \sum_j u_i$$

$$\begin{array}{ll} x_{ij} \le y_j & \forall i \in M, \forall j \in N \\ y_j \in \{0,1\} & \forall j \in N \\ 0 \le x_{ij} \le 1 & \forall i \in M, \forall j \in N \end{array}$$

The Lagrangian subproblem is equivalent to $|N|$ independent subproblems.
Each problem is:

$$w_j(\underline{u}) = \max \sum_i (c_{ij} - u_i) x_{ij} - f_j y_j$$

$$\begin{array}{ll} x_{ij} \le y_j & \forall i \\ y_j \in \{0,1\} & \\ 0 \le x_{ij} \le 1 & \forall i \end{array}$$

which can be solved by inspection, and so:

$$w_j(\underline{u}) = \max \left\{0, \sum_i \max\{c_{ij} - u_i, 0\} - f_j\right\}$$

$$\Longrightarrow w(\underline{u}) = \sum_j w_j(\underline{u}) + \sum_i u_i$$

The lagrangian dual is : $\min \ w(\underline{u})$
which can be solved with the subgradient method.

- select $\underline{u}_0$ and set $k=0$
- at the iteration **k** :

  solve $w(\underline{u}_k) = \min L(\underline{x}, \underline{u}_k)$

  let $\underline{x}_k$ be the optimal solution $\Longrightarrow$ $(\underline{d} - D\underline{x}_k) \in \partial w(\underline{u}_k)$
- $\underline{u}_{k+1} = \max \{ \underline{0}, \underline{u}_k + \alpha_k (\underline{d} - D\underline{x}_k) \}$ $\quad$ α_k opportunely chosen
- $k = k+1$

Apply the lagr. method to:

$m = 6$
$n = 5$

$(c_{ij}) =$

6	2	1	3	5
4	10	2	6	1
3	2	4	1	3
2	0	4	1	4
1	8	6	2	5
3	2	4	8	1

$\underline{u}_0 = [5, 6, 3, 2, 6, 4]^T$ $\qquad$ $\underline{f} = [2, 4, 5, 3, 3]^T$

$c_{ij} - u_i$ (only if > 0) $\Longrightarrow$

$$\begin{array}{c|ccccc} & & & j & & \\ \hline & 1 & - & - & - & - \\ & - & 4 & - & - & - \\ i & - & - & 1 & - & - \\ & - & - & 2 & - & 2 \\ & - & 2 & - & - & - \\ & - & - & - & 4 & - \end{array}$$

$\Longrightarrow$ $x_{11} = x_{22} = x_{33} =$
$= x_{43} = x_{45} = x_{52} = x_{64} = 1$

$w(\underline{u}_0) = \sum_j w_j(\underline{u}_0) + \sum_i (\underline{u}_0)_i$

$w_j(\underline{u}_0) = \max \{ 0, \sum_{\text{elementi}} \boxed{j} - f_j \}$

$\Longrightarrow$ $w_1(\underline{u}_0) = \max\{0, 1-2\} = 0$
$w_2(\underline{u}_0) = \max\{0, 6-4\} = 2$
$w_3(\underline{u}_0) = \max\{0, 3-5\} = 0$
$w_4(\underline{u}_0) = \max\{0, 4-3\} = 1$
$w_5(\underline{u}_0) = \max\{0, 2-3\} = 0$

$\Longrightarrow$ $w(\underline{u}_0) = 3 + (5+6+3+2+6+4) = 3 + 26 = 29$

$\gamma_i^0 = 1 - \Sigma_j x_{ij} = 1 - \left(\text{somma tutte le } \mathbf{x} \text{ che iniziano con } i \right)$

$\underline{\gamma}^0 = [0, 0, 0, -1, 0, 0] \Rightarrow \underline{u}_1 = \underline{u}_0 + \alpha_0 \underline{\gamma}^0$

STSP - Symmetric Traveling Salesman Problem

undirected graph $G = (V, E)$
c_e = cost for the edge $e \in E$

Two ILP formulations ?

1. $$\min \sum_{e \in E} c_e x_e$$
$$\sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\boxed{\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V,\ S \neq \emptyset,\ 1 \leq |S| \leq n}$$
$$x_e \in \{0,1\} \qquad \forall e \in E$$

CUT SET INEQUALITIES (CUT)

2. $$\min \sum_{e \in E} c_e x_e$$
$$\sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\boxed{\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subset V,\ S \neq \emptyset,\ |S| \geq 2}$$
$$x_e \in \{0,1\}$$

SUBTOUR ELIMINATION INEQUALITIES (SEC)

$\delta(S) = \{ \{i,j\} \in E : i \in S,\ j \in V \setminus S \}$
$E(S) = \{ \{i,j\} \in E : i \in S,\ j \in S \}$

For both formulations we have exponential number of constraints.

What is the relation between the two?

The two formulations are equally strong:

from $\sum_{e \in \delta(i)} x_e = 2 \implies \sum_{i \in S} \sum_{e \in \delta(i)} x_e = 2|S|$
$\implies \sum_{e \in \delta(S)} x_e + 2 \sum_{e \in E(S)} x_e = 2|S|$

(SEC) → (CUT): From (SEC) we have: $2 \sum_{e \in E(S)} x_e \leq 2|S| - 2$
$\implies 2 \leq 2|S| - 2 \sum_{e \in E(S)} x_e = \sum_{e \in \delta(S)} x_e$ (CUT)

(CUT) ⇒ (SEC): from (CUT) we have: $\sum_{e \in E(S)} x_e \geq 2$
$\implies 2 \leq \sum_{e \in E(S)} x_e = 2|S| - 2 \sum_{e \in E(S)} x_e$
$\implies \sum_{e \in E(S)} x_e \leq |S| - 1$ (SEC)

Lagrangian relaxation based on 1-trees?

We start from the 2. formulation.
We can rewrite the formulation as:

$$\min \sum_{e \in E} c_e x_e$$
$$\sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\sum_{e \in E(S)} x_e = |S| - 1 \qquad \forall S \subset V,\ S \neq \emptyset,\ |S| \geq 2,\ 1 \notin S$$
$$\sum_{e \in E} x_e = n$$
$$x_e \in \{0,1\}$$

we can add it w.l.o.g. because if we specify the subcycle condition for S then it's respected also for $V \setminus S$ (so it's enough to have half of the (SEC))

Starting from this formulation we relax (lagrangianize) the constraint: $\sum_{e \in \delta(i)} x_e = 2$ for all the $i \in V \setminus \{1\}$. In this way we obtain the lagrangian relaxation for the STSP based on 1-tree.

Recall: 1-tree is a spanning tree on $V \setminus \{1\}$ plus two edges incident in node 1.
We obtain the lagrangian subproblem:

$$w(\underline{u}) = \min \sum_{e \in E} c_e x_e + \sum_{i \in V} u_i \left(2 - \sum_{e \in \delta(i)} x_e\right)$$

$$= \min \sum_{e \in E} (c_e - u_i - u_j) x_e + 2 \sum_{i \in V} u_i$$

$$\sum_{e \in \delta(1)} x_e = 2$$

$$\sum_{e \in E(S)} x_e \le |S| - 1 \qquad \forall S \subset V: S \neq \emptyset,\ |S| \ge 2,\ 1 \notin S$$

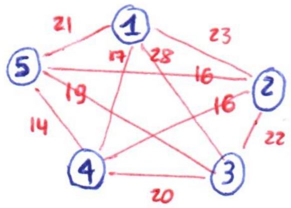
$$\sum_{e \in E} x_e = n$$

$$x_e \in \{0,1\} \qquad \forall e \in E$$

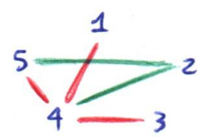
This lagrangian subproblem can be solved looking for 1-trees with a greedy algorithm (minimum cost 1-tree with costs $\tilde{c}_e = c_e - u_i - u_j$).
The lagrangian dual problem ($:= \max \{ w(\underline{u}) : u_1 = 0,\ \underline{u} \in \mathbb{R}^{|V|} \}$) can be solved with the $\boxed{\text{subgradient method.}}$ *

Apply the 1st iteration to:

$\underline{u}_0 = [-1, 0, -1, 3, -1]^T$ considering the node 2 the special node.



$$C = \begin{array}{|c|c|c|c|c|} \hline - & 23 & 28 & 17 & 21 \\ \hline & - & 22 & 16 & 16 \\ \hline & & - & 20 & 19 \\ \hline & & & - & 14 \\ \hline & & & & - \\ \hline \end{array} \xrightarrow{c_{ij}^0 = c_{ij} - u_i^0 - u_j^0} C^0 = \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & \\ - & 24 & 30 & 15 & 23 & 1 \\ & - & 23 & 13 & 17 & 2 \\ & & - & 18 & 21 & 3 \\ & & & - & 12 & 4 \\ & & & & - & 5 \end{array}$$



$$x_{14} = x_{24} = x_{25} = x_{34} = x_{45} = 1$$

$$w(\underline{u}^0) = (\text{cost 1-tree}) + \left(2 \sum_i u_i^0\right)$$
$$= (12 + 15 + 18 + 13 + 17) - 0 = 75$$

$$\gamma_i^0 = 2 - \begin{pmatrix} \text{number of edges} \\ \text{going out of } i \end{pmatrix}$$

$$\underline{\gamma}^0 = [1, 0, 1, -2, 0] \implies \underline{u}_1 = \underline{u}_0 + \alpha_0 \underline{\gamma}^0$$

*
- we select $\underline{u}_0$ and we set $k = 0$
- at each k: we solve $w(\underline{u}_k) = \min \boxed{\{ \underline{c}^T \underline{x} + \underline{u}_k^T (\underline{d} - D\underline{x}) \}}$ lagrangian function $L(\underline{x}, \underline{u})$

  let $\underline{x}_k$ be optimal sol. $\implies (\underline{d} - D\underline{x}_k) \in \partial w(\underline{u}_k)$
- $\underline{u}_{k+1} = \max \{ \underline{0}, \underline{u}_k + \alpha_k (\underline{d} - D\underline{x}_k) \}$, $\quad \alpha_k = \varepsilon_k \dfrac{\hat{w} - w(\underline{u}_k)}{\|\underline{d} - D\underline{x}_k\|}$
- $k = k + 1$

- m production plants $\quad 1 \le i \le m$
- n clients $\quad 1 \le j \le n$
- p_i = production capacity of i
- d_j = demand of client j
- f_{ij} = fixed cost $i \to j$
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+ each client served from at most k plants

? MILP for: min total transport cost while satisfying demands and capacities

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x_{ij}^1 = amount $i \to j$ if ≤ 100

x_{ij}^2 = amount $i \to j$ if > 100

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$$z_{ij} = \begin{cases} 1 & \text{if } x_{ij}^2 > 0 \\ 0 & \text{else} \end{cases}$$

$$\longrightarrow \quad \min \ \sum_i \sum_j \left[f_{ij} y_{ij} + x_{ij}^1 t_{ij}^1 + x_{ij}^2 t_{ij}^2 \right]$$

$$\text{s.t.} \quad x_{ij}^1 + x_{ij}^2 \le q_{ij} y_{ij} \qquad \forall i \ \forall j$$

$$\sum_i x_{ij}^1 + x_{ij}^2 \ge d_j \qquad \forall j$$

$$\sum_j x_{ij}^1 + x_{ij}^2 \le p_i \qquad \forall i$$

$$\sum_i y_{ij} \le k \qquad \forall j$$

$$0 \le x_{ij}^1 \le 100(1 - z_{ij})$$

$$100 z_{ij} \le x_{ij}^2 \le q_{ij} z_{ij}$$

$$x_{ij}^1, x_{ij}^2 \ge 0, \quad y_{ij} \in \{0,1\}, \quad z_{ij} \in \{0,1\} \qquad \forall i \ \forall j$$

- k ambulances to locate
- $S = \{1, \dots, m\}$ candidate sites $\quad i \in S$
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z = maximum time needed to arrive, $\in \mathbb{R}_+$

$\Longrightarrow$ min z

s.t. $\sum_i y_i = k$		posiziono **k** ambulanze
$x_{ij} \leq y_i$	$\forall i \forall j$	da **i** arriva in **j** se in **i** c'è
$\sum_i x_{ij} = 1$	$\forall j$	un'ambulanza arriva sempre
$t_{ij} x_{ij} \leq z$	$\forall i \forall j$	z = max time needed
$x_{ij} \in \{0,1\}$, $y_i \in \{0,1\}$, $z \geq 0$		$\forall i \forall j$

+ two ambulances at every location: the first must be there ≤ 8 min, the second will be sent if the first is occupied

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$y_i \in \{0,1\}$, $x_{ij}^1 \in \{0,1\}$, $x_{ij}^2 \in \{0,1\}$, $W \geq 0$	$\forall i \forall j$	

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\+ $(\sum \text{earnings})_{\forall \text{ client}} \leq 1.1 \, (\sum \text{earnings})_{\text{lowest earnings}}$

\+ the flow of each canal must be at least α times its capacity $(\alpha \in (0,1))$

\+ " if both canals $(v_1, v_2) \in A$ and $(v_3, v_4) \in A$ are built then only one of the two: $(w_1, w_2) \in A$, $(w_3, w_4) \in A$ can be build "

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x_{ij} = quantity from i to j

z_i = quantity that remains in i

$\Longrightarrow$

$$\min \sum_i \sum_j y_{ij} c_{ij} - \sum_i g_i z_i$$

s.t.

$$x_{ij} \leq \alpha u_{ij} y_{ij} \qquad \forall i \; \forall j$$

$$b_i \leq z_i \leq B_i \qquad \forall i$$

$$z_i = \sum_{(h,i) \in \delta^-(i)} x_{hi} - \sum_{(i,j) \in \delta^+(i)} x_{ij} \qquad \forall i$$

$$g_i z_i \leq 1.1 \, g_j z_j \qquad \forall i \; \forall j$$

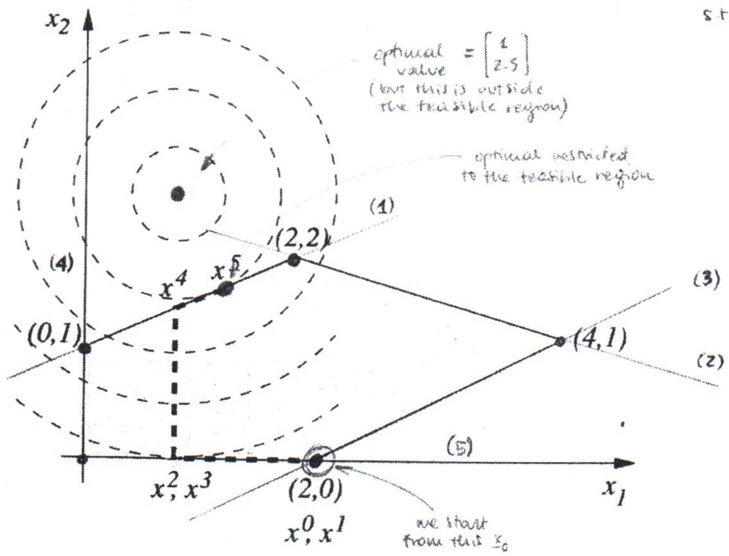
$$y_{w_1 w_2} + y_{w_3 w_4} \leq 3 - (y_{v_1 v_2} + y_{v_3 v_4})$$

$$x_{ij} \geq 0, \; y_{ij} \in \{0,1\}, \; z_i \geq 0 \qquad \forall i \; \forall j$$

$$\min\ q(x_1,x_2) = (x_1-1)^2 + (x_2-2.5)^2$$

s.t.

$$-x_1 + 2x_2 - 2 \le 0 \quad (1)$$
$$x_1 + 2x_2 - 6 \le 0 \quad (2)$$
$$x_1 - 2x_2 - 2 \le 0 \quad (3)$$
$$-x_1 \le 0 \quad (4)$$
$$-x_2 \le 0 \quad (5)$$



0 : We start from $\underline{x}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $W_0 = \{3,5\}$

$\underline{x}_0$ is optimal in W_0 $\Longrightarrow$ $\underline{d}_0 = \underline{0}$ $\Longrightarrow$ $\underline{x}_1 = \underline{x}_0 + \alpha_0 \underline{d}_0 = \underline{x}_0$

KKT conditions $\left(Q\underline{x}_0 + \underline{c} + \sum_{i \in W_0} u_i^0 \underline{a}_i = \underline{0}\right)$ $\Longrightarrow$ $(u_3, u_5) = (-2,-1)$

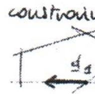
$\Longrightarrow$ we delete the constraint (3) from W_0 : $W_1 = \{5\}$

1 : $\underline{x}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $W_1 = \{5\}$

$\min\{q(\underline{x}_1 + \underline{d}) : \underline{a}_i^T \underline{d} = 0 \quad i \in W_1\}$ $\Longrightarrow$ $\underline{d}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (which does not violate constraints $\Rightarrow \alpha_1 = 1$)

$\underline{x}_2 = \underline{x}_1 + \underline{d}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

we don't have constraints (added) in $\underline{x}_2$ $\Longrightarrow$ $W_2 = \{5\}$



using "all" $\underline{d}_1$ we are still in the feasible region, so we keep all $\underline{d}_1$

2 : $\underline{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $W_2 = \{5\}$

$\underline{x}_2$ is optimal in W_2 $\rightarrow$ $\underline{d}_2 = \underline{0}$ $\Longrightarrow$ $\underline{x}_3 = \underline{x}_2$

KKT conditions $\Longrightarrow$ $u_5 = -5$ $\Longrightarrow$ $W_3 = W_2 \setminus \{5\} = \{\} = \emptyset$

3 : $\underline{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $W_3 = \emptyset$

we are determining $\underline{d}_3$ in a subspace which imposes no constraints with equality (it's an unconstrained case $\Longrightarrow$ for sure α_3 has to be analized (it'll probably be $\neq 1$))

it cannot be like that: α_3 has to be < 1

$\underline{d}_3 = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$, $\alpha_3 = 0.6$ $\Longrightarrow$ $\underline{x}_4 = \underline{x}_3 + \alpha_3 \underline{d}_3 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$

At $\underline{x}_4$ we have a new constraint : $W_4 = \{1\}$

4 : $\underline{x}_4 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$, $W_4 = \{1\}$

$\underline{d}_4 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$ ($\underline{x}_5 = \underline{x}_4 + \underline{d}_4$ does not violate constraints) $\Longrightarrow$ $\underline{x}_5 = \underline{x}_4 + \underline{d}_4 = \begin{bmatrix} 1.4 \\ 1.7 \end{bmatrix}$

Moreover we don't have additional constraints : $W_5 = W_4 = \{1\}$

5 : $\underline{x}_5 = \begin{bmatrix} 1.4 \\ 1.7 \end{bmatrix}$, $W_5 = \{1\}$

$\underline{x}_5$ is optimal for W_5 $\Rightarrow$ $\underline{d}_5 = \underline{0}$

KKT conditions $\longrightarrow$ $u_1 = 1.25 \ge 0$ $\longrightarrow$ $\underline{x}_5$ is optimal for the original

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- m production plants $\quad 1 \leq i \leq m$
- n clients $\quad 1 \leq j \leq n$
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s.t. $x_{ij}^1 + x_{ij}^2 \leq q_{ij} \, y_{ij}$	$\forall i \; \forall j$
$\sum_i x_{ij}^1 + x_{ij}^2 \geq d_j$	$\forall j$
$\sum_j x_{ij}^1 + x_{ij}^2 \leq p_i$	$\forall i$
$\sum_i y_{ij} \leq k$	$\forall j$
$0 \leq x_{ij}^1 \leq 100(1 - z_{ij})$ $100 z_{ij} \leq x_{ij}^2 \leq q_{ij} z_{ij}$	
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$x_{ij} \ge 0, \ y_{ij} \in \{0,1\}, \ z_i \ge 0$	$\forall i \ \forall j$