# KNAPSACK PROBLEM - 1D

$n$ objects : $i \in N = \{1, \dots, n\}$

$\forall i \in N$ : $p_i$ profit, $a_i$ weight

$b$ capacity of the knapsack

ILP:

$$\begin{aligned} \max \ & \textstyle\sum_i p_i x_i \\ & \textstyle\sum_i a_i x_i \le b \\ & x_i \in \{0,1\} \quad \forall i \in N \end{aligned}$$

## Cover inequality ?

A subset $C \subseteq N$ is a cover if it corresponds to a subset of indeces which items will not fit (all together) the knapsack.

If $C \subseteq N$ is a cover for $X$, the cover inequality $\sum_{j \in C} x_{ij} \le |C| - 1$ is a valid inequality for $X$.

A valid inequality is an inequality that is satisfied by all the feasible solutions. Since $C$ is a cover, it is impossible to put all the elements of $C$ in the knapsack, so at most $|C|-1$ of them can be put in together

$\Longrightarrow$ $\sum_{j \in C} x_{ij} \le |C| - 1$ is satisfied by all integer solutions of the problem (and so it's a valid inequality).

## Separation problem ?

Problem: given a fractional $\underline{x}^*$ with $0 \le x_j^* \le 1$ $\forall j \in N$, find a cover inequality that is violated by $\underline{x}^*$ or establish that none exists.

Since $\left[ \sum_{j \in C} x_j \le |C| - 1 \right] \Longleftrightarrow \left[ \sum_{j \in C} (1 - x_j) \ge 1 \right]$

The problem becomes:

$$\exists C \subseteq N \text{ s.t. } \underbrace{\textstyle\sum_{j \in C} a_j > b}_{C \text{ is a cover}} \quad \text{and} \quad \underbrace{\textstyle\sum_{j \in C} (1 - x_j^*) < 1}_{C \text{ violates the cover inequality}} \ ?$$

ILP formulation:

let $\underline{z} \in \{0,1\}^n$ characterizes $C$ $\quad (z_j = 1 \text{ if } j \in C)$

$\Longrightarrow$ $\xi := \min \left\{ \sum_{j \in N} (1 - x_j^*) z_j \ : \ \sum_j a_j z_j > b, \ \underline{z} \in \{0,1\}^n \right\}$

- if $\xi \ge 1 \Longrightarrow \underline{x}^*$ satisfies all cover inequalities
- if $\xi < 1 \Longrightarrow \sum_{j \in C} x_j \le |C| - 1$, $C = \{ j : z_j^* = 1, \ j \in N, \ \underline{z}^* \text{ optimal} \}$, is violated by $\underline{x}^*$ by a quantity $1 - \xi$

---

Consider $X = \{ \underline{x} \in \{0,1\}^6 : \ 12x_1 + 9x_2 + 7x_3 + 5x_4 + 5x_5 + 3x_6 \le 14 \}$

List all minimal inequalities:

| | | |
|---|---|---|
| $x_1 + x_2 \le 1$ | $x_2 + x_3 \le 1$ | $x_3 + x_4 + x_5 \le 2$ |
| $x_1 + x_3 \le 1$ | $x_2 + x_4 + x_5 \le 2$ | $x_3 + x_4 + x_6 \le 2$ |
| $x_1 + x_4 \le 1$ | $x_2 + x_4 + x_6 \le 2$ | $x_3 + x_5 + x_6 \le 2$ |
| $x_1 + x_5 \le 1$ | $x_2 + x_5 + x_6 \le 2$ | |
| $x_1 + x_6 \le 1$ | | |

Apply the lifting procedure from $x_3 + x_5 + x_6 \le 2$ :

$C = \{3, 5, 6\}$, $\quad N \setminus C = \{1, 2, 4\}$

- $\alpha_1 x_1 + x_3 + x_5 + x_6 \le 2$ :

  $x_1 = 0 \Longrightarrow \forall \alpha_1$

  $x_1 = 1 \Longrightarrow \alpha_1 = 2 - \max\{ x_3 + x_5 + x_6 : \ 7x_3 + 5x_5 + 3x_6 \le 14 - 12 = 2 \}$
  $= 2 - 0 = 2$

  $\Longrightarrow$ $2x_1 + x_3 + x_5 + x_6 \le 2$

- $\alpha_2 x_2 + 2x_1 + x_3 + x_5 + x_6 \le 2$ :

  $x_2 = 0 \Longrightarrow \forall \alpha_2$

  $x_2 = 1 \Longrightarrow \alpha_2 = 2 - \max\{ 2x_1 + x_3 + x_5 + x_6 : \ 12x_1 + 7x_3 + 5x_5 + 3x_6 \le 14 - 9 = 5 \}$
  $= 2 - 1 = 1$

  $\Longrightarrow$ $2x_1 + x_2 + x_3 + x_5 + x_6 \le 2$

- $\alpha_4 x_4 + 2x_1 + x_2 + x_3 + x_5 + x_6 \le 2$ :

  $x_4 = 0 \Longrightarrow \forall \alpha_4$

  $x_4 = 1 \Longrightarrow \alpha_4 = . - \max\{ 2x_1 + x_2 + x_3 + x_5 + x_6 : \ 12x_1 + 9x_2 + 7x_3 + 5x_5 + 3x_6 \le 14 - 5 = 9 \}$
  $= 2 - 2 = 0$

$\longrightarrow$ $\boxed{2x_1 + x_2 + x_3 + x_5 + x_6 \le 2}$

Since we started from a minimal cover the lifting procedure ends with a facet inequality (facet of $conv(X)$)

# KNAPSACK PROBLEM-multiple

$m$ = richieste di calcolo
$n$ = calcolatori
$p_j$ = profitto del calcolo $j$
$c_i$ = capacita' dell'elaboratore $i$
$w_{ij}$ = capacita' necessaria per far girare la richiesta $j$ sull'elaboratore $i$

$$\max \sum_i \sum_j x_{ij} p_j$$
s.t.
$\sum_j x_{ij} w_{ij} \le c_i \quad \forall i \qquad$ (1) capacity constraint
$\sum_i x_{ij} \le 1 \quad \forall j \qquad$ (2) assignment constraint
$x_{ij} \in \{0,1\} \quad \forall i \forall j$

## Two lagrangian relaxations:

(write the sub-problems ~~and explain how they can be solved~~)

**1.** we relax the capacity constraint:

$$W_1 = \max \sum_i \sum_j x_{ij} p_j + \sum_i u_i \left(c_i - \sum_j x_{ij} w_{ij}\right)$$
$\sum_i x_{ij} \le 1 \quad \forall j$
$x_{ij} \in \{0,1\} \quad \forall i \forall j$
$u_i \ge 0 \quad \forall i$

**2.** we relax the assignment constraint:

$$W_2 = \max \sum_i \sum_j x_{ij} p_j + \sum_j v_j \left(1 - \sum_i x_{ij}\right)$$
$\sum_j x_{ij} w_{ij} \le c_i \quad \forall i$
$x_{ij} \in \{0,1\} \quad \forall i \forall j$
$v_j \ge 0 \quad \forall j$

## How can they be solved?

**1.** $\max \ldots = \max \sum_i \sum_j x_{ij} p_j + \sum_i u_i c_i - \sum_i \sum_j x_{ij} w_{ij} u_i$

$= \max \sum_i \sum_j x_{ij} (p_j - w_{ij} u_i) + \sum_i u_i c_i$

In this way we have a new profit for each term: $\tilde{p}_j = p_j - w_{ij} u_i \ge 0$. According to this new profits $\tilde{p}$ we will insert the item $j$ in $i$ which maximizes $\tilde{p}_j$. Each item can be selected at most once.
(We can see it as assignment problem: assign $j$ to the $i$ for which $\tilde{p}_j$ is the highest)

**2.** $\max \ldots = \max \sum_i \sum_j x_{ij} p_j + \sum_j v_j - \sum_j \sum_i v_j x_{ij}$

$= \max \sum_i \sum_j x_{ij} (p_j - v_j) + \sum_j v_j$

The new profit for the item $j$ is $\hat{p}_j = p_j - v_j \ge 0$. This time each item can be taken several times. We can decompose the original problem in **m** binary knapsack problems:

$$W^i = \max \sum_{j=1}^{n} (p_j - v_j) x_{ij}$$
$\sum_{j=1}^{n} w_j x_{ij} \le c_i \quad \forall i$
$x_{ij} \in \{0,1\} \quad \forall i \ \forall j$

(This is possible because we eliminated the linking constraints)

## Which relaxation is stronger?

Given $\min \{\underline{c}^T \underline{x} : A\underline{x} \ge \underline{b},\ D\underline{x} \ge \underline{d},\ \underline{x} \in \mathbb{Z}^n\}$, $\quad X = \{\underline{x} \in \mathbb{Z}^n,\ A\underline{x} \ge \underline{b}\}$,

let: $\begin{cases} w(\underline{u}) = \min \{\underline{c}^T \underline{x} + \underline{u}^T(\underline{d} - D\underline{x}) : A\underline{x} \ge \underline{b},\ \underline{x} \in \mathbb{Z}^n\} \\ W^* = \max_{\underline{u} \ge \underline{0}} w(\underline{u}) \end{cases}$

$\longrightarrow W^* = \min \{\underline{c}^T \underline{x} : D\underline{x} \ge \underline{d},\ \underline{x} \in \text{conv}(X)\}$.

Since $\text{conv}(X) \subseteq \{\underline{x} \in \mathbb{R}^n : A\underline{x} \ge \underline{b}\} \implies z_{LP} \le W^* \le z^*$, so the lagrangian duality solution is at least as good as the LP relaxation.

But if $\text{conv}(X) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \ge \underline{b}\} \implies z_{LP} = W^*$
both lagrangian and LP are equal weak

Based on this we can say that **1.** is as weak as the LP relaxation. Relaxing the constraints as in **2.** is a better choice.

## Write the lagrangian duals and explain how to solve them:

**1.** $W_1^* = \min_{\underline{u} \ge \underline{0}} W_1(\underline{u})$, $\qquad$ **2.** $W_2^* = \min_{\underline{u} \ge \underline{0}} W_2(\underline{u})$

We can solve them with the subgradient method.
(Recall: given $C \subseteq \mathbb{R}^n$, $f: C \to \mathbb{R}$ convex, $\underline{\gamma} \in \mathbb{R}^n$ is a subgradient of $f$ at $\underline{x}^* \in C$ if: $f(\underline{x}) \ge f(\underline{x}^*) + \underline{\gamma}^T(\underline{x} - \underline{x}^*) \quad \forall \underline{x} \in C$.
We denote by $\partial f(\underline{x}^*)$ all the subgradients at $\underline{x}^*$)

Subgradient method:

- we select $\underline{u}_0$ and we set $k=0$
- at each iteration $k$:
  we solve $w(\underline{u}_k) = \min_{x \in X} \{\underline{c}^T \underline{x} - \underline{u}_k^T(\underline{d} - D\underline{x})\} = \min_{x \in X} L(\underline{x}, \underline{u}_k)$

  Let $\underline{x}_k$ be the optimal solution $\implies (\underline{d} - D\underline{x}_k) \in \partial w(\underline{u}_k)$
- $\underline{u}_{k+1} = \max \{\underline{0},\ \underline{u}_k + \alpha_k(\underline{d} - D\underline{x}_k)\}$

  $\alpha_k = \varepsilon_k \dfrac{\hat{w} - w(\underline{u}_k)}{\|\underline{d} - D\underline{x}_k\|}$
- $k := k+1$

# Minimum cost flow

Directed graph $G=(V,A)$
$\forall (i,j)\in A$ : $k_{ij}$ capacity, $c_{ij}$ unit cost
$\forall i \in V$ : $b_i$ demand ($>0$ source, $<0$ dest) : $\sum_i b_i = 0$

## ILP formulation?

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$
$$\sum_{(h,i)\in\delta^-(i)} x_{hi} - \sum_{(i,j)\in\delta^+(i)} x_{ij} = b_i \quad \forall i\in V \quad (1)$$
$$x_{ij} \le k_{ij} \quad \forall (i,j)\in A \quad (2)$$
$$x_{ij} \ge 0 \text{ integer} \quad \forall (i,j)\in A$$

where $\delta^+(i) = \{(i,j)\in A : j\in V\}$
$\delta^-(i) = \{(h,i)\in A : h\in V\}$

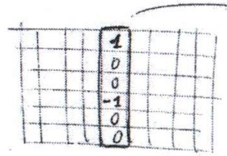
## Is it an ideal formulation?

The matrix of constraints (1), (2) is TU.
If the matrix of constraints is TU, it sufficies to solve the LP relexation instead of solving ILP. Since in some sense the converse is also true, we can say:
Let $X$ be the feasible region for the original problem,
Let $P$ be the feasible region of the LP relaxation

since the constr. matrix is TU (and $b_i$, $k_{ij}$ are integers) $\Longrightarrow$ $P = \text{conv}(X)$ (= def of ideal formulation)

## Why is TU?

The constraints (1) create:




exactly 2 non-zero components for every column,

We can partition this as $I_1$ = all indeces, $I_2=\emptyset$ and so this matrix is TU. (sufficient cond. for being TU)
The constraint (2) creates $-I_{|V|}$ and so the whole matrix of constraints (1) and (2) is TU.

## Shortest path problem?

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$
$$\sum_{(h,i)\in\delta^-(i)} x_{hi} - \sum_{(i,j)\in\delta^+(i)} x_{ij} = \begin{cases} 1 & \text{if } i=s \\ -1 & \text{if } i=t \\ 0 & \text{else} \end{cases} \quad \forall i\in V$$
$$\sum_{(i,j)\in\delta^+(i)} x_{ij} \le 1 \quad \forall i\in V$$
$$x_{ij}\in\{0,1\} \quad \forall (i,j)\in A$$

## Maximum flow problem?

$$\max \sum_{(i,j)\in A} x_{ij}$$
$$\sum_{(h,i)\in\delta^-(i)} x_{hi} = \sum_{(i,j)\in\delta^+(i)} x_{ij} \quad \forall i\in V\setminus\{s,t\}$$
$$x_{ij}\le k_{ij} \quad \forall (i,j)\in A$$
$$x_{ij}\ge 0 \quad \forall (i,j)\in A$$

③ The idea of cutting plane methods is to improve an initial formulation by adding valid inequalities, because the ideal formulation is often difficult to find.
So we don't want to find exactly $conv(X)$, but we want to bring out $\underline{x}^*_{LP}$ as a vertex of $P$.
If we solve the LP relaxation of an ILP and find $\underline{x}^*_{LP}$ fractional, we try to solve the separation problem that is find a cutting plane for $\underline{x}^*_{LP}$ and add that valid inequality (if $\exists$).
If we solve LP, at the end, and we find $\underline{x}^*_{LP}$ fractional, at least we have a stronger formulation.
A valid inequality is necessary to describe $P$, if $P$ is full dimensional, $\iff$ it defines a facet of $P$.

An example can be the binary knapsack problem:
$X = \{ \underline{x} \in \{0,1\}^m : \sum_{j=1}^{m} a_j x_j \le b \}$

$C \subseteq N$, cover $\longrightarrow \sum_{j \in C} x_j \le |C| - 1$ cover inequalities are valid for $X$
They can be generated looking for a cover $C \subseteq N$ s.t. $\sum_{j \in C} a_j > b$ and, if we start from a minimal cover, we can strengthen them by using the lifting procedure and so generate a facet defining inequality for $conv(X)$.
Slow convergence of cutting plane methods can be helped integrating them with B&B in Branch and Cut methods.

④ Lagrange relaxation, the idea is to delete the "complicating constraints" and to add to the objective function a term, for each deleted constraint, with a multiplier s.t. it penalizes the violation of the constraint.

$\min \{ \underline{c}^T \underline{x} : A\underline{x} \ge \underline{b},\ D\underline{x} \ge \underline{d},\ \underline{x} \in \mathbb{Z}^n \} \rightarrow$ Delete $D\underline{x} \ge \underline{d}$

$w(\underline{u}) = \min \{ \underbrace{\underline{c}^T \underline{x} + \underline{u}^T (\underline{d} - D\underline{x})}_{\text{Lagrangian function}} : A\underline{x} \ge \underline{b},\ \underline{x} \in \mathbb{Z}^m \}$ Lagrangian = subproblem =

↑ dual function $\quad \forall \underline{u} \ge \underline{0}$

Lagrangian dual : $\max_{\underline{u} \ge \underline{0}} w(\underline{u}) = w^*$

An example is STSP (with SEC). Its Lagrangian relaxation is:

$$\min \sum_{e \in E} (c_e - u_i - u_j) x_e + 2 \sum_{i \in V} u_i \quad = w(\underline{u})$$
$$\text{s.t.} \quad \sum_{e \in \delta(1)} x_e = 2$$
$$\sum_{e \in E(S)} x_e \le |S| - 1, \quad S \subset V, |S| \ge 2, 1 \notin S$$
$$\sum_{e \in E} x_e = m$$
$$x_e \in \{0,1\} \quad \forall e \in E$$

(We relaxed all (DEG) constraints except for node 1, assumption $u_1 = 0$)

- $\forall \underline{u} \ge \underline{0}, \forall \underline{x} \in X, \quad w(\underline{u}) \le \underline{c}^T \underline{x}$ (weak duality)
- If $\exists \tilde{\underline{u}} \ge \underline{0}, \tilde{\underline{x}} \in X$ s.t. $w(\tilde{\underline{u}}) = \underline{c}^T \tilde{\underline{x}} \Rightarrow \tilde{\underline{x}}$ and $\tilde{\underline{u}}$ are optimal solutions for the primal and the dual
- In general, $z_{LP} \le w^* \le z^*$. Sometimes Lagrangian relaxation is as weak as LP ($w^* = z_{LP}$) $\Rightarrow$ better to solve LP.

The Lagrangian dual can be solved with the subgradient method, because $w(\underline{u})$ is a concave piecewise $\mathcal{C}^1$ function.
$\forall\, \underline{x}(\tilde{\underline{u}}) \in X(\tilde{\underline{u}})$, the vector $(\underline{d} - D\underline{x}(\tilde{\underline{u}})) \in \partial w(\tilde{\underline{u}}) \rightarrow$ characterization of the subgradients of $w(\underline{u})$ at $\tilde{\underline{u}}$

Column generation method:

ILMP ⟶ LPM ⟶ LPRM

ILMP: integer linear master problem
LPM: LP relaxation of the master problem
LPRM: restricted problem (we consider only a subset $J_K$ of all the variables $J$)

⟶ Pricing subproblem: we look for a new improving non basic variable

If $\exists$ a non basic variable which improve: $J_{K+1} := J_K \cup \{new\}$ ⟶ LPRM

If $\nexists$ ⟶ the optimal solution of LPRM is also optimal for LPM

general idea:
do not consider all variables explicitly, new variables are generated when needed

- single source, $n$ clients
- $G = (N \cup \{0\}, A)$ $N = \{1,..,n\}$, $0$ = source, $(i,j) \in A$ arcs
- $c_{ij}$ = cost for $(i,j)$
- $u_{ij}$ = max flow capacity
- $b_i \le$ need of client $i \le B_i$ $\forall i$
- $g_i$ = dollars for gallon

+ $(\Sigma \text{ earnings})_{\forall \text{client}} \le 1.1 \; (\Sigma \text{earnings})_{\text{lowest earnings}}$
+ the flow of each canal must be at least $\alpha$ times its capacity $(\alpha \in (0,1))$
+ " if both canals $(v_1,v_2) \in A$ and $(v_3,v_4) \in A$ are built then only one of the two : $(w_1,w_2) \in A$ , $(w_3,w_4) \in A$ can be build "

? MILP for: minimizing the total building cost while providing the service in a fair way

---

$$y_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is build} \\ 0 & \text{else} \end{cases}$$

$x_{ij}$ = quantity from $i$ to $j$

$z_i$ = quantity that remains in $i$

$\longrightarrow$ min $\sum_i \sum_j y_{ij} c_{ij} - \sum_i g_i z_i$

| | | |
|---|---|---|
| s.t. | $x_{ij} \le \alpha u_{ij} y_{ij}$ | $\forall i \; \forall j$ |
| | $b_i \le z_i \le B_i$ | $\forall i$ |
| | $z_i = \sum_{(h,i) \in \delta^-(i)} x_{hi} - \sum_{(i,j) \in \delta^+(i)} x_{ij}$ | $\forall i$ |
| | $g_i z_i \le 1.1 \; g_j z_j$ | $\forall i \; \forall j$ |
| | $y_{w_1 w_2} + y_{w_3 w_4} \le 3 - (y_{v_1 v_2} + y_{v_3 v_4})$ | |
| | $x_{ij} \ge 0, \; y_{ij} \in \{0,1\}, \; z_i \ge 0$ | $\forall i \; \forall j$ |

- k ambulances to locate
- $S = \{1,..,m\}$ candidate sites $i \in S$
- $C = \{1,..,n\}$ emergency locations $j \in S$
- $t_{ij}$ time from $i$ to $j$

+ at every call an ambulance goes
+ $\forall$ candidate sites has at most 1 ambulance

? LP for: where to locate ambulances, which ambulance send to which call s.t.: minimize the max time needed to arrive to the call location

---

$$y_i = \begin{cases} 1 & \text{if an ambulance is located at } i \\ 0 & \text{else} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if ambulance } i \text{ goes to } j \\ 0 & \text{else} \end{cases}$$

$z$ = maximum time needed to arrive, $\in \mathbb{R}_+$

$\Longrightarrow$ min $z$

| | | | |
|---|---|---|---|
| s.t. | $\sum_i y_i = k$ | | posiziono k ambulanze |
| | $x_{ij} \le y_i$ | $\forall i \; \forall j$ | da $i$ arriva in $j$ se in $i$ c'è |
| | $\sum_i x_{ij} = 1$ | $\forall j$ | un'ambulanza arriva sempre |
| | $t_{ij} \, x_{ij} \le z$ | $\forall i \; \forall j$ | $z$ = max time needed |
| | $x_{ij} \in \{0,1\}, \; y_i \in \{0,1\}, \; z \ge 0$ | | $\forall i \; \forall j$ |

---

+ two ambulances at every location : the first must be there $\le 8$ min, the second will be sent if the first is occupied

? minimize the max time for the second ambulance to arrive

$$x_{ij}^1 = \begin{cases} 1 & \text{if from } i \text{ goes to } j \text{ as first} \\ 0 & \text{else} \end{cases}$$

$$x_{ij}^2 = \begin{cases} 1 & \text{if from } i \text{ goes to } j \text{ as second} \\ 0 & \text{else} \end{cases}$$

$W$ = max time for the second to arrive

$\longrightarrow$ min $W$

| | | | |
|---|---|---|---|
| s.t. | $\sum_i y_i = k$ | | posiziono k ambulanze |
| | $x_{ij}^1 \le y_i$ , $x_{ij}^2 \le y_i$ | $\forall i \forall j$ | da $i$ va a $j$ se in $i$ c'è |
| | $x_{ij}^1 + x_{ij}^2 \le 1$ | $\forall i \forall j$ | va o la prima o la seconda |
| | $x_{ij}^1 \, t_{ij} \le 8$ | $\forall i \forall j$ | la prima ci mette $\le 8$ min |
| | $x_{ij}^2 \, t_{ij} \le W$ | $\forall i \; \forall j$ | W = max time needed |
| | $\sum_i x_{ij}^1 + x_{ij}^2 = 1$ | $\forall j$ | una ci deve comunque arrivare |
| | $y_i \in \{0,1\}, \; x_{ij}^1 \in \{0,1\}, \; x_{ij}^2 \in \{0,1\}, \; W \ge 0$ | $\forall i \; \forall j$ | |

- $m$ production plants $\quad 1 \le i \le m$
- $n$ clients $\quad 1 \le j \le n$
- $p_i$ = production capacity of $i$
- $d_j$ = demand of client $j$
- $f_{ij}$ = fixed cost $i \to j$
- $t_{ij}^1$ = unit transport $i \to j$ up to 100 units
  $t_{ij}^2$ = unit transport $i \to j$ for more than 100 units $\quad (t_{ij}^1 > t_{ij}^2)$
- $q_{ij}$ = max amount that can be shipped $i \to j$

+ each client served from at most $k$ plants

? MILP for: min total transport cost while satisfying demands and capacities

---

$x_{ij}^1$ = amount $i \to j$ if $\le 100$

$x_{ij}^2$ = amount $i \to j$ if $> 100$

$$y_{ij} = \begin{cases} 1 & \text{if } x_{ij}^1 + x_{ij}^2 > 0 \\ 0 & \text{else} \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if } x_{ij}^2 > 0 \\ 0 & \text{else} \end{cases}$$

$$\longrightarrow \quad \min \sum_i \sum_j \left[ f_{ij} y_{ij} + x_{ij}^1 t_{ij}^1 + x_{ij}^2 t_{ij}^2 \right]$$

$$\begin{aligned}
\text{s.t.} \quad & x_{ij}^1 + x_{ij}^2 \le q_{ij} y_{ij} && \forall i \ \forall j \\
& \textstyle\sum_i x_{ij}^1 + x_{ij}^2 \ge d_j && \forall j \\
& \textstyle\sum_j x_{ij}^1 + x_{ij}^2 \le p_i && \forall i \\
& \textstyle\sum_i y_{ij} \le k && \forall j \\
& 0 \le x_{ij}^1 \le 100(1 - z_{ij}) \\
& 100 z_{ij} \le x_{ij}^2 \le q_{ij} z_{ij} \\
& x_{ij}^1, x_{ij}^2 \ge 0, \quad y_{ij} \in \{0,1\}, \quad z_{ij} \in \{0,1\} && \forall i \ \forall j
\end{aligned}$$

---

- $m$ impianti $\quad i$
- $n$ clienti $\quad j$
- $p_i$ = capacità $i$
- $d_j$ = domanda $j$
- $f_{ij}$ = costo fisso $i \to j$
- $t_{ij}$ = costo unitario $i \to j$

+ 1 client → max $k$ plants

---

$x_{ij}$ = quantità $i \to j$

$$y_{ij} = \begin{cases} 1 & x_{ij} > 0 \\ 0 & x_{ij} = 0 \end{cases}$$

---

$$\begin{aligned}
\min \quad & \textstyle\sum_i \sum_j f_{ij} y_{ij} + \sum_i \sum_j x_{ij} t_{ij} \\
\text{s.t.} \quad & \textstyle\sum_j x_{ij} \le p_i && \forall i \\
& \textstyle\sum_i x_{ij} \ge d_j && \forall j \\
& \textstyle\sum_i y_{ij} \le k && \forall j \\
& x_{ij} \ge 0 \\
& y_{ij} \in \{0,1\} \\
& x_{ij} \le M y_{ij} && \forall i \ \forall j
\end{aligned}$$

# UFL - Uncapacitated facility location

$M = \{1, \dots, m\}$ clients, $i \in M$
$N = \{1, \dots, n\}$ candidates sites, $j \in N$
$f_j$ = fixed costs for opening $j$
$c_{ij}$ = transp. cost* if the whole demand of $i$ is served by $j$     * cost/profit
$x_{ij}$ = fraction of demand of $i$ satisfied by $j$
$y_j$ = 1 if depot $j$ is open

## Two formulations ?

1. $\max \sum_i \sum_j c_{ij} x_{ij} - \sum_j f_j y_j$

$$\sum_{j \in N} x_{ij} = 1 \quad \forall i \in M$$
$$\boxed{\sum_{i \in M} x_{ij} \le m y_j \quad \forall j \in N} \quad (*)$$
$$y_j \in \{0,1\} \quad \forall j \in N$$
$$0 \le x_{ij} \le 1 \quad \forall j \in N, \forall i \in M$$

2. as before, but we substitute (*) with: $\boxed{x_{ij} \le y_j \quad \forall i \in M, \forall j \in N}$

The first formulation has **n** constraints in (*).
We substitute **n** constraints with **m×n** constraints in the second.

## Which is stronger ?

The second formulation is stronger than the first because the polyhedras of the LP relaxations (respectively $P_2$ for 2. and $P_1$ for 1.) are such that:
$P_2 \subsetneq P_1$.

(obviously $P_2 \subset P_1$ because $\forall j$, summing $x_{ij} \le y_j$ over $i$ leads to $\sum_i x_{ij} \le m y_j$, moreover $\exists (\underline{x}, \underline{y}) \in P_1 \setminus P_2$)

## Lagrangian relaxation ?

Lagrangian subproblem:

$$w(\underline{u}) = \max \sum_i \sum_j c_{ij} x_{ij} - \sum_j f_j y_j + \sum_i u_i \left(1 - \sum_j x_{ij}\right)$$

$$= \max \sum_i \sum_j (c_{ij} - u_i) x_{ij} - \sum_j f_j y_j + \sum_i u_i$$
$$x_{ij} \le y_j \quad \forall i \in M, \forall j \in N$$
$$y_j \in \{0,1\} \quad \forall j \in N$$
$$0 \le x_{ij} \le 1 \quad \forall i \in M, \forall j \in N$$

The lagrangian subproblem is equivalent to $|N|$ independent subproblems.
Each problem is:

$$w_j(\underline{u}) = \max \sum_i (c_{ij} - u_i) x_{ij} - f_j y_j$$
$$x_{ij} \le y_j \quad \forall i$$
$$y_j \in \{0,1\}$$
$$0 \le x_{ij} \le 1 \quad \forall i$$

which can be solved by inspection, and so:

$$w_j(\underline{u}) = \max \left\{0, \sum_i \max\{c_{ij} - u_i, 0\} - f_j\right\}$$

$$\longrightarrow w(\underline{u}) = \sum_j w_j(\underline{u}) + \sum_i u_i$$

The lagrangian dual is: $\min w(\underline{u})$
which can be solved with the subgradient method.

- select $\underline{u}_0$ and set $k=0$
- at the iteration $k$:
  solve $w(\underline{u}_k) = \min L(\underline{x}, \underline{u}_k)$
  let $\underline{x}_k$ be the optimal solution $\Longrightarrow (\underline{d} - D\underline{x}_k) \in \partial w(\underline{u}_k)$
- $\underline{u}_{k+1} = \max\{\underline{0}, \underline{u}_k + \alpha_k(\underline{d} - D\underline{x}_k)\}$    $\alpha_k$ opportunely chosen
- $k = k+1$

## Apply the lagr. method to:

$m = 6$
$n = 5$

$$(c_{ij}) = \begin{bmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{bmatrix} \qquad \underline{u}_0 = \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 6 \\ 4 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 3 \end{bmatrix}$$

$c_{ij} - u_i$ (only if $>0$) $\Longrightarrow$

$$\begin{array}{c|ccccc} & & & j & & \\ \hline & 1 & - & - & - & - \\ & - & 4 & - & - & - \\ i & - & - & 1 & - & - \\ & - & - & 2 & - & 2 \\ & - & 2 & - & - & - \\ & - & - & - & 4 & - \end{array}$$

$\longrightarrow x_{11} = x_{22} = x_{33} = x_{43} = x_{45} = x_{52} = x_{64} = 1$

$$w(\underline{u}_0) = \sum_j w_j(\underline{u}_0) + \sum_i (\underline{u}_0)_i$$
$$w_j(\underline{u}_0) = \max\left\{0, \sum_{\text{elementi}} - f_j\right\}$$

$\longrightarrow$
$w_1(\underline{u}_0) = \max\{0, 1-2\} = 0$
$w_2(\underline{u}_0) = \max\{0, 6-4\} = 2$
$w_3(\underline{u}_0) = \max\{0, 3-5\} = 0$
$w_4(\underline{u}_0) = \max\{0, 4-3\} = 1$
$w_5(\underline{u}_0) = \max\{0, 2-3\} = 0$

$\longrightarrow w(\underline{u}_0) = 3 + (5+6+3+2+6+4) = 3 + 26 = 29$

$\gamma_i^0 = 1 - \sum_j x_{ij} = 1 - $ (somma tutte le x che uniscono con $i$)

$\underline{\gamma}^0 = [0, 0, 0, -1, 0, 0] \Longrightarrow \underline{u}_2 = \underline{u}_0 + \alpha_0 \underline{\gamma}^0$

**SUBGRADIENT METHOD**

- we select $\underline{u}_0$ and $k := 0$
- at each iteration $k$:
  $w(\underline{u}_k) = \min_{\underline{x} \in X} L(\underline{x}, \underline{u}_k)$
  $L(\underline{x}, \underline{u}_k) = \underline{c}^T\underline{x} - \underline{u}_k^T(\underline{d} - D\underline{x})$
  $\underline{x}_k$ optimal
  $\Longrightarrow (\underline{d} - D\underline{x}_k) \in \partial w(\underline{u}_k)$
- $\underline{u}_{k+1} = \max\{\underline{0}, \underline{u}_k + \alpha_k(\underline{d} - D\underline{x}_k)\}$
  $\alpha_k$ choosen opport.
- $k := k+1$

# STSP – Symmetric Travelling Salesman Problem

undirected graph $G = (V, E)$
$c_e$ = cost for the edge $e \in E$

## Two ILP formulations ?

**1.** min $\sum_{e \in E} c_e x_e$

$$\sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\boxed{\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V,\ S \neq \emptyset,\ 1 \leq |S| \leq n}$$
$$x_e \in \{0,1\} \qquad \forall e \in E$$

CUT SET INEQUALITIES (CUT)

**2.** min $\sum_{e \in E} c_e x_e$

$$\sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\boxed{\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subset V,\ S \neq \emptyset,\ |S| \geq 2}$$
$$x_e \in \{0,1\}$$

SUBTOUR ELIMINATION INEQUALITIES (SEC)

$\delta(S) = \{ \{i,j\} \in E : i \in S, j \in V \setminus S \}$
$E(S) = \{ \{i,j\} \in E : i \in S, j \in S \}$

For both formulations we have exponential number of constraints.

## What is the relation between the two?

The two formulations are equally strong:

from $\sum_{e \in \delta(i)} x_e = 2 \implies \sum_{i \in S} \sum_{e \in \delta(i)} x_e = 2|S|$

$\implies \sum_{e \in \delta(S)} x_e + 2 \sum_{e \in E(S)} x_e = 2|S|$

(SEC) → (CUT): From (SEC) we have: $2 \sum_{e \in E(S)} x_e \leq 2|S| - 2$

$\implies 2 \leq 2|S| - 2 \sum_{e \in E(S)} x_e = \sum_{e \in \delta(S)} x_e$ (CUT)

(CUT) ⇒ (SEC): from (CUT) we have: $\sum_{e \in E(S)} x_e \geq 2$

$\implies 2 \leq \sum_{e \in E(S)} x_e = 2|S| - 2 \sum_{e \in E(S)} x_e$

$\implies \sum_{e \in E(S)} x_e \leq |S| - 1$ (SEC)

## Lagrangian relaxation based on 1-trees?

We start from the 2. formulation.
We can rewrite the formulation as:

min $\sum_{e \in E} c_e x_e$

$$\sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\sum_{e \in E(S)} x_e = |S| - 1 \qquad \forall S \subset V,\ S \neq \emptyset,\ |S| \geq 2,\ 1 \notin S$$
$$\sum_{e \in E} x_e = n$$
$$x_e \in \{0, 1\}$$

we can add it w.l.o.g. because if we specify the subcycle condition for $S$ then it's respected also for $V \setminus S$ (so it's enough to have half of the (SEC))

Starting from this formulation we relax (lagrangianize) the constraint: $\sum_{e \in \delta(i)} x_e = 2$ for all the $i \in V \setminus \{1\}$. In this way we obtain the lagrangian relaxation for the STSP based on 1-tree.

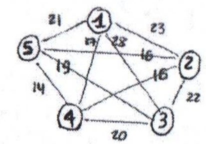
Recall: 1-tree is a spanning tree on $V \setminus \{1\}$ plus two edges incident in node 1.
We obtain the lagrangian subproblem:

$$W(\underline{u}) = \min \sum_{e \in E} c_e x_e + \sum_{i \in V} u_i \Big(2 - \sum_{e \in \delta(i)} x_e\Big)$$
$$= \min \sum_{e \in E} (c_e - u_i - u_j) x_e + 2 \sum_{i \in V} u_i$$
$$\sum_{e \in \delta(1)} x_e = 2$$
$$\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subset V : S \neq \emptyset,\ |S| \geq 2,\ 1 \notin S$$
$$\sum_{e \in E} x_e = n$$
$$x_e \in \{0,1\} \qquad \forall e \in E$$

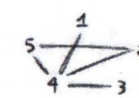
This lagrangian subproblem can be solved looking for 1-trees with a greedy algorithm (minimum cost 1-tree with costs $\tilde{c}_e = c_e - u_i - u_j$).
The Lagrangian dual problem ($:= \max \{ W(\underline{u}) : u_1 = 0,\ \underline{u} \in \mathbb{R}^{|V|} \}$) can be solved with the $\boxed{\text{subgradient method.}}$ ✱

## Apply the 1st iteration to:

$\underline{u}_0 = [-1, 0, -1, 3, -1]^T$ considering the node 2 the special node.




$$C = \begin{bmatrix} - & 23 & 28 & 17 & 21 \\ & - & 22 & 16 & 16 \\ & & - & 20 & 19 \\ & & & - & 14 \\ & & & & - \end{bmatrix} \xrightarrow{c_{ij}^0 = c_{ij} - u_i^0 - u_j^0} C^0 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & - & 24 & 30 & 15 & 23 \\ 2 & & - & 23 & 13 & 17 \\ 3 & & & - & 18 & 21 \\ 4 & & & & - & 12 \\ 5 & & & & & - \end{array}$$




$x_{14} = x_{24} = x_{25} = x_{34} = x_{45} = 1$

$W(\underline{u}^0) = (\text{cost 1-tree}) + (2 \sum_i u_i^0)$
$= (12 + 15 + 18 + 13 + 17) - 0 = 75$

$\gamma_i^0 = 2 - \binom{\text{number of edges}}{\text{going out of } i}$

$\underline{\gamma}^0 = [1, 0, 1, -2, 0] \implies \underline{u}_1 = \underline{u}_0 + \alpha_0 \underline{\gamma}^0$

✱
- we select $\underline{u}_0$ and we set $k = 0$
- at each $k$: we solve $W(\underline{u}_k) = \min \{ \boxed{\underline{c}^T \underline{x} + \underline{u}_k^T (\underline{d} - D\underline{x})} \}$ — Lagrangian function $L(\underline{x}, \underline{u})$
  let $\underline{x}_k$ be optimal sol. $\longrightarrow (\underline{d} - D\underline{x}_k) \in \partial W(\underline{u}_k)$
- $\underline{u}_{k+1} = \max \{ \underline{0},\ \underline{u}_k + \alpha_k (\underline{d} - D\underline{x}_k) \}$, $\alpha_k = \varepsilon_k \dfrac{\hat{W} - W(\underline{u}_k)}{\|\underline{d} - D\underline{x}_k\|}$
- $k = k + 1$