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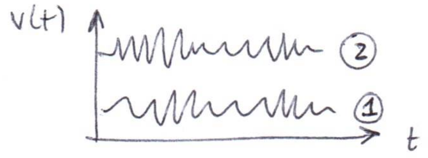
$\boxed{10/03}$

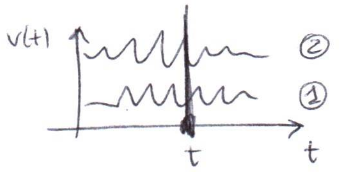
STOCHASTIC PROCESS = sequence of random variables in indexed order (indexed by time):



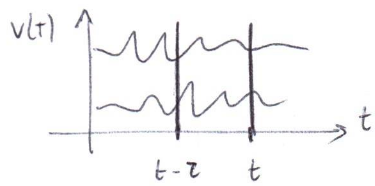
$v(t)$ = outcome of a random experiment

we can have 1 or 2 (:= realizations of the process)

- Expected value = $\mathbb{E}[v(t)]$ describes the average behaviour of the process ~~at a t~~ for each time t



- Covariance function = $\mathbb{E}\left[\left(v(t)-\mathbb{E}[v(t)]\right)\left(v(t-\tau)-\mathbb{E}[v(t-\tau)]\right)\right]$



If $\tau=0 \implies \mathbb{E}\left[\left(v(t)-\mathbb{E}[v(t)]\right)^2\right]$:= variance of the process (express how the process $v(t)$ is spread from the average value)

If $\tau \neq 0 \implies \gamma_v(t,\tau)$ express the correlation of the process for different time instants

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STATIONARY (STOCHASTIC) PROCESS

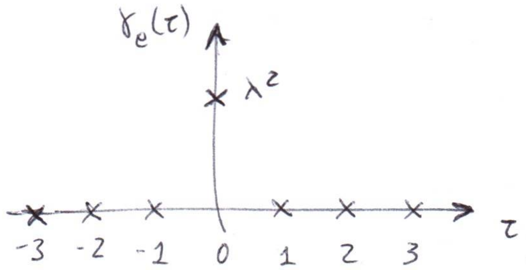
1. $\mathbb{E}[v(t)] = m \quad \forall t$
2. $\gamma_v(t,\tau) = \gamma_v(\tau) \quad (\perp\!\!\!\perp t)$

WHITE NOISE

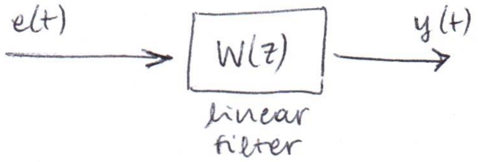
$e(t) \sim WN(\mu, \lambda^2)$:

1. $\mathbb{E}[e(t)] = \mu \quad \forall t$
2. $\gamma_e(\tau) = \begin{cases} \lambda^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases} \implies$ it's a signal completely uncorrelated

$\rightarrow$



Why is important? Because we use it for obtaining a stochastic process.



For example: $y(t) = e(t) + \frac{1}{2}e(t-1) - e(t-2)$ $e(t) \sim WN(0,1)$

These type of processes are called MOVING AVERAGE PROCESSES, in this case we have a MA(2).

We use on it the delay operator:

$$y(t) = e(t) + \frac{1}{2}z^{-1}e(t) - z^{-2}e(t) = \underbrace{\boxed{\left(1+\frac{1}{2}z^{-1} - z^{-2}\right)}}_{:= W(z) \text{ transfer function}} e(t)$$

- Expected value of the process:

$$\mathbb{E}[y(t)] = \mathbb{E}\left[e(t) + \frac{1}{2}e(t-1) - e(t-2)\right] = \cancel{\mathbb{E}[e(t)]} + \frac{1}{2}\cancel{\mathbb{E}[e(t-1)]} - \cancel{\mathbb{E}[e(t-2)]} = 0$$

- Covariance function:

$$\gamma(\tau=0) = \mathbb{E}[y^2(t)] = \mathbb{E}\left[\left(e(t)+\frac{1}{2}e(t-1) - e(t-2)\right)^2\right]$$

$$= \mathbb{E}\left[e^2(t) + \frac{1}{4}e^2(t-1) + e^2(t-2) + e(t)e(t-1) - 2e(t)e(t-2) - e(t-1)e(t-2)\right]$$

$$= \underbrace{\mathbb{E}[e^2(t)]}_{\lambda^2} + \frac{1}{4}\underbrace{\mathbb{E}[e^2(t-1)]}_{\lambda^2} + \underbrace{\mathbb{E}[e^2(t-2)]}_{\lambda^2} + \cancel{\mathbb{E}[e(t)e(t-1)]} - 2\cancel{\mathbb{E}[e(t)e(t-2)]} - \cancel{\mathbb{E}[e(t-1)e(t-2)]}$$

$$= 1 + \frac{1}{4}\cdot 1 + 1$$

$$= \frac{9}{4}$$

$$\gamma_y(\tau=1) = \mathbb{E}[y(t)\,y(t-1)] = \mathbb{E}\left[\left(e(t)+\frac{1}{2}e(t-1) - e(t-2)\right)y(t-1)\right]$$

$$= \mathbb{E}[e(t)y(t-1)] + \frac{1}{2}\mathbb{E}[e(t-1)y(t-1)] - \mathbb{E}[e(t-2)y(t-1)]$$

$$= \mathbb{E}\left[e(t)\left(e(t-1)+\frac{1}{2}e(t-2) - e(t-3)\right)\right] +$$

$$+ \frac{1}{2}\mathbb{E}\left[\underline{e(t-1)}\left(\underline{e(t-1)} + \frac{1}{2}e(t-2) - e(t-3)\right)\right] +$$

$$- \mathbb{E}\left[\underline{e(t-2)}\left(e(t-1) + \underline{\frac{1}{2}e(t-2)} - e(t-3)\right)\right]$$

only the e(·) with the same time istant have mean ≠ 0

$$= \frac{1}{2}\mathbb{E}[e^2(t-1)] - \frac{1}{2}\mathbb{E}[e^2(t-2)] = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 1 = 0$$

$$\gamma_y(\tau=2) = \mathbb{E}[y(t)\,y(t-2)] = \mathbb{E}\left[\left(e(t)+\tfrac{1}{2}e(t-1)-e(t-2)\right)\left(e(t-2)+\tfrac{1}{2}e(t-3)-e(t-4)\right)\right]$$

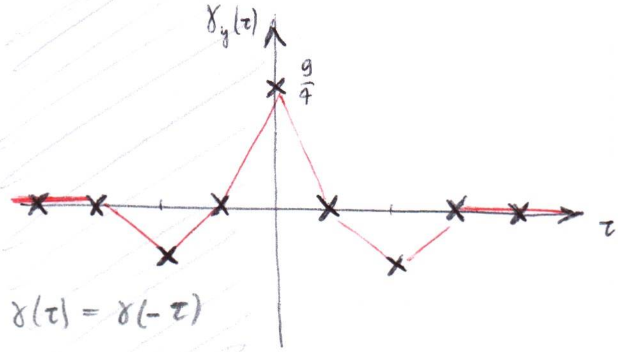
$$= \mathbb{E}[e^2(t-2)](-1) = -1$$

$$\gamma_y(\tau=3) = \mathbb{E}[y(t)\,y(t-3)] = \mathbb{E}\left[\left(e(t)+\tfrac{1}{2}e(t-1)-e(t-2)\right)\left(e(t-3)+\tfrac{1}{2}e(t-4)-e(t-5)\right)\right]$$

$= 0$ ($\nexists$ terms with the same time)

$$\gamma_y(\tau>2) = 0$$

$$\Longrightarrow \quad \gamma(\tau) = \begin{cases} 9/4 & \tau = 0 \\ 0 & \tau = 1 \\ -1 & \tau = 2 \\ 0 & \tau > 2 \end{cases} \quad :$$



$\gamma(\tau) = \gamma(-\tau)$

Remark: Given $MA(n)$ process: $y(t) = c_0 e(t) + c_1 e(t-1) + \ldots + c_n e(t-n)$ with $e(t) \sim WN(\mu, \lambda^2)$ we know that $y(t)$ is a stationary stochastic process $\forall \{c_0, c_1, \ldots, c_n\}$ with:

$$\gamma_y(\tau) = \begin{cases} (c_0^2 + c_1^2 + \ldots + c_n^2)\lambda^2 & \tau = 0 \\ (c_0c_1 + c_1c_2 + \ldots + c_{n-1}c_n)\lambda^2 & \tau = \pm 1 \\ \ldots & \\ c_0 c_n \lambda^2 & \tau = \pm n \\ 0 & |\tau| > n \end{cases}$$

Consider now the process:

$$y(t) = \tfrac{1}{2}y(t-1) - \tfrac{1}{4}y(t-2) + e(t) \qquad e(t) \sim WN(0,1)$$

$y(t)$ is an AUTOREGRESSIVE PROCESS: $AR(2)$

Using the delay operator:

$$y(t) = \tfrac{1}{2}z^{-1}y(t) - \tfrac{1}{4}z^{-2}y(t) + e(t)$$

$$= \left(\tfrac{1}{2}z^{-1} - \tfrac{1}{4}z^{-2}\right)y(t) + e(t)$$

$$\Longrightarrow \quad y(t) = \underbrace{\boxed{\frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}}}_{:=W(z)}\, e(t)$$

- Expected value:

$$\mathbb{E}[y(t)] = \mathbb{E}\left[\tfrac{1}{2}y(t-1) - \tfrac{1}{4}y(t-2) + e(t)\right] = \tfrac{1}{2}\mathbb{E}[y(t-1)] - \tfrac{1}{4}\mathbb{E}[y(t-2)] + \mathbb{E}[e(t)]$$

We assume that $y(t)$ is stationary $\Longrightarrow \mathbb{E}[y(t)] = \mathbb{E}[y(t-1)] = \mathbb{E}[y(t-2)] = m$

$$\Longrightarrow \mathbb{E}[y(t)] = \tfrac{1}{2}m - \tfrac{1}{4}m + 0 = m \quad \Longrightarrow \quad m = 0$$

THEOREM:
$$\left.\begin{array}{l} y(t) = W(z)\,\mu(t) \\ \mu(t) \text{ stationary process} \\ W(z) \text{ stable system} \end{array}\right\} \Longrightarrow \begin{array}{l} y(t) \text{ is a stationary} \\ \text{stochastic process} \end{array}$$

$\Longrightarrow$ we have to compute the poles of $W(z)$:

$$W(z) = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z^2}{z^2-\frac{1}{2}z+\frac{1}{4}}$$

For the poles: $z^2-\frac{1}{2}z+\frac{1}{4}=0 \Longrightarrow z_{1/2} = \frac{1}{4} \pm j\sqrt{\frac{3}{4}}$

$|z_{1,2}| = \sqrt{\frac{1}{16}+\frac{3}{16}} = \frac{1}{2} \Longrightarrow |z_{1,2}| < 1 \Longrightarrow W(z)$ is stable

$\Longrightarrow$ $y(t)$ is stationary $\Longrightarrow$ our Hp. of $y(t)$ stationary is good

$\Longrightarrow \mathbb{E}[y(t)] = 0$

- Covariance function:

$$\gamma_y(0) = \mathbb{E}[(y(t)-0)^2] = \mathbb{E}[y^2(t)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-1) - \tfrac{1}{4}y(t-2) + e(t)\right)^2\right]$$

$$= \mathbb{E}\left[\tfrac{1}{4}y^2(t-1) + \tfrac{1}{16}y^2(t-2) + e^2(t) - \tfrac{1}{4}y(t-1)y(t-2) + y(t-1)e(t) - \tfrac{1}{2}y(t-2)e(t)\right]$$

$$= \tfrac{1}{4}\mathbb{E}[y^2(t-1)] + \tfrac{1}{16}\mathbb{E}[y^2(t-2)] + \overset{\lambda^2=1}{\mathbb{E}[e^2(t)]} - \tfrac{1}{4}\mathbb{E}[y(t-1)y(t-2)] + \mathbb{E}[y(t-1)e(t)] - \tfrac{1}{2}\mathbb{E}[y(t-2)e(t)]$$

$y(t)$ stationary $\Rightarrow \mathbb{E}[(y(t-1))^2] = \mathbb{E}[(y(t-2))^2] = \gamma_y(0)$

$$= \tfrac{1}{4}\gamma_y(0) + \tfrac{1}{16}\gamma_y(0) + 1 + \cancel{\mathbb{E}[y(t-1)e(t)]} - \tfrac{1}{2}\overset{\text{analogamente}}{\cancel{\mathbb{E}[y(t-2)e(t)]}} - \tfrac{1}{4}\mathbb{E}[y(t-1)y(t-2)]$$

$y(t-1)$ is a process s.t.
is a function of $e(t-1), e(t-2), \ldots$
$\Rightarrow$ $y(t-1)$ and $e(t)$ are uncorrelated
$\Rightarrow \mathbb{E}[y(t-1)e(t)] = \mathbb{E}[y(t-1)]\,\underbrace{\cancel{\mathbb{E}[e(t)]}}_{(=0)} = 0$

$-\frac{1}{4}\gamma_y(1) =$

we have to compute $\gamma_y(1)$ to find $\gamma_y(0)$

$$\Longrightarrow \frac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1 \qquad (1)$$

$$\gamma_y(1) = \mathbb{E}[y(t)\,y(t-1)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-1) - \tfrac{1}{4}y(t-2) + e(t)\right)y(t-1)\right]$$

$$= \mathbb{E}\left[\tfrac{1}{2}y^2(t-1)\right] - \tfrac{1}{4}\mathbb{E}[y(t-2)y(t-1)] + \cancel{\mathbb{E}[e(t)\,y(t-1)]}$$

$$= \tfrac{1}{2}\gamma_y(0) - \tfrac{1}{4}\gamma_y(1) \qquad (2)$$

$$(1) + (2) \Longrightarrow \begin{cases} \gamma_y(0) = \dfrac{80}{63} \\[2mm] \gamma_y(1) = \dfrac{32}{63} \end{cases}$$

$$\gamma_y(2) = \mathbb{E}[y(t)\,y(t-2)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-1) - \tfrac{1}{4}y(t-2) + e(t)\right)y(t-2)\right]$$

$$= \tfrac{1}{2}\mathbb{E}[y(t-1)\,y(t-2)] - \tfrac{1}{4}\mathbb{E}[y^2(t-2)] + \cancel{\mathbb{E}[e(t)\,y(t-2)]}$$

$$= \tfrac{1}{2}\gamma_y(1) - \tfrac{1}{4}\gamma_y(0) = -\tfrac{4}{63}$$

$$\boxed{\gamma_y(\tau) = \tfrac{1}{2}\gamma_y(\tau-1) - \tfrac{1}{4}\gamma_y(\tau-2)} \qquad \tau \geq 2$$

MA(∞) REPRESENTATION

1. SUBSTITUTION :

$$y(t) = \tfrac{1}{2}y(t-1) - \tfrac{1}{4}y(t-2) + e(t)$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}y(t-2) - \tfrac{1}{4}y(t-3) + e(t-1)\right] - \tfrac{1}{4}y(t-2) + e(t)$$

$$= e(t) + \tfrac{1}{2}e(t-1) + \cancel{\tfrac{1}{4}y(t-2)} - \tfrac{1}{8}y(t-3) - \cancel{\tfrac{1}{4}y(t-2)}$$

$$= e(t) + \tfrac{1}{2}e(t-1) - \tfrac{1}{8}\left(\tfrac{1}{2}y(t-4) - \tfrac{1}{4}y(t-5) + e(t-3)\right)$$

$$= e(t) + \tfrac{1}{2}e(t-1) - \tfrac{1}{8}e(t-3) - \tfrac{1}{16}y(t-4) + \tfrac{1}{32}y(t-5)$$

$$= f(e(t), e(t-1), e(t-2), \ldots)$$

2. LONG DIVISION

$$\begin{array}{l|l}
\dfrac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} & 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} \\
\hline
/ \quad \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} & 1 + \frac{1}{2}z^{-1} - \frac{1}{8}z^{-3} - \frac{1}{16}z^{-4} \ldots \\
\quad \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} & \\
\hline
/ \quad / \quad -\frac{1}{8}z^{-3} & \\
\qquad -\frac{1}{8}z^{-3} + \frac{1}{16}z^{-4} - \frac{1}{32}z^{-5} & \\
\qquad \ldots &
\end{array}$$

these are the same coefficients that we found in the previous representation.
These are the **c**oefficients of MA(∞) representation of $y(t)$

(EX. 1)

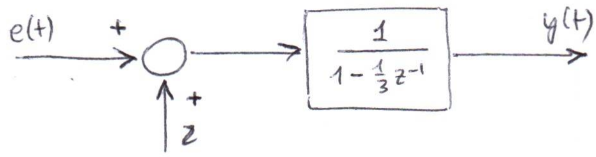
11/03

AR(1): $y(t) = \frac{1}{3} y(t-1) + e(t) + 2$ $\qquad e(t) \sim WN(1, 1)$

constant signal

$$y(t) = \frac{1}{3} z^{-1} y(t) + e(t) + 2$$

$$\left(1 - \frac{1}{3} z^{-1}\right) y(t) = e(t) + 2 \quad :$$



$$W(z) = \frac{1}{1 - \frac{1}{3} z^{-1}} = \frac{z}{z - \frac{1}{3}} \quad : \quad \left.\begin{array}{l} \text{zero}: \ z = 0 \\ \text{pole}: \ z = \frac{1}{3} \end{array}\right\} \Longrightarrow \begin{array}{l} \text{(for the theorem)} \\ y(t) \text{ is stationary} \end{array}$$

- Expected value:

$$\mathbb{E}[y(t)] = \mathbb{E}\left[\frac{1}{3} y(t-1) + e(t) + 2\right] = \frac{1}{3} \underbrace{\mathbb{E}[y(t-1)]}_{= \mathbb{E}[y(t)]} + \mathbb{E}[e(t)] + 2$$

$= \mathbb{E}[y(t)]$ since it's a stationary process

$$= \frac{1}{3} \mathbb{E}[y(t)] + 1 + 2$$

$$\left(1 - \frac{1}{3}\right) \mathbb{E}[y(t)] = 3 \Longrightarrow \mathbb{E}[y(t)] = \frac{9}{2}$$

- Covariance function:

Note: $\mathbb{E}[e(t)\, e(t-1)] \neq 0$

What is null in this case?

$$\mathbb{E}[(e(t) - 1)(e(t-1) - 1)] = 0$$

! DETREND or DEBIAS PROCESS

$$\tilde{e}(t) = e(t) - 1 \Longrightarrow \mathbb{E}[\tilde{e}(t)] = 0 \qquad \tilde{e}(t) \sim WN(0, 1)$$

$$\tilde{y}(t) = y(t) - \frac{9}{2} \Longrightarrow \mathbb{E}[\tilde{y}(t)] = 0$$

$$\gamma_y(\tau) = \mathbb{E}\Big[\underbrace{\boxed{\left(y(t) - \tfrac{9}{2}\right)}}_{\tilde{y}(t)} \underbrace{\boxed{\left(y(t-\tau) - \tfrac{9}{2}\right)}}_{\tilde{y}(t-\tau)}\Big] = \mathbb{E}[\tilde{y}(t)\, \tilde{y}(t-\tau)] = \gamma_{\tilde{y}}(\tau) \qquad (*)$$

$$\begin{cases} y(t) = \tilde{y}(t) + \frac{9}{2} \\ e(t) = \tilde{e}(t) + 1 \end{cases}$$

$$\longrightarrow \tilde{y}(t) + \frac{9}{2} = \frac{1}{3}\left[\tilde{y}(t-1) + \frac{9}{2}\right] + \tilde{e}(t) + 1 + 2$$

$$\longrightarrow \boxed{\tilde{y}(t) = \frac{1}{3} \tilde{y}(t-1) + \tilde{e}(t)}$$

$$\gamma_{\tilde{y}}(0) = \mathbb{E}[\tilde{y}^2(t)] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)^2\right]$$

$$= \frac{1}{9} \underbrace{\mathbb{E}[\tilde{y}^2(t-1)]}_{\text{since it's stationary}} + \mathbb{E}[\tilde{e}^2(t)] + \cancel{\frac{2}{3} \mathbb{E}[\tilde{y}(t-1)\, \tilde{e}(t)]}$$

$f(\tilde{e}(t-1), \tilde{e}(t-2), \ldots) \perp\!\!\!\perp \tilde{e}(t)$

$$= \frac{1}{9} \gamma_{\tilde{y}}(0) + 1$$

$$\Longrightarrow \gamma_{\tilde{y}}(0) = \frac{9}{8}$$

$$\gamma_{\tilde{y}}(1) = \mathbb{E}[\tilde{y}(t)\,\tilde{y}(t-1)] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)\tilde{y}(t-1)\right]$$

$$= \frac{1}{3}\mathbb{E}[\tilde{y}^2(t-1)] + \cancel{\mathbb{E}[\tilde{e}(t)\,\tilde{y}(t-1)]}$$

$$= \frac{1}{3}\gamma_{\tilde{y}}(0) = \frac{3}{8}$$

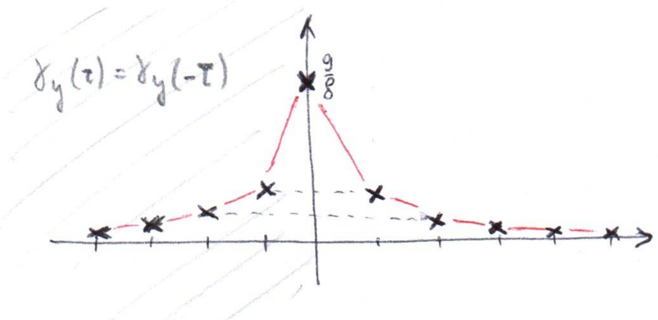
$$\gamma_{\tilde{y}}(2) = \mathbb{E}[\tilde{y}(t)\,\tilde{y}(t-2)] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)\tilde{y}(t-2)\right]$$

$$= \frac{1}{3}\mathbb{E}[\tilde{y}(t-1)\,\tilde{y}(t-2)] + \cancel{\mathbb{E}[\tilde{e}(t)\,\tilde{y}(t-2)]}$$

$$= \frac{1}{3}\gamma_{\tilde{y}}(1) = \frac{1}{8}$$

$$\gamma_{\tilde{y}}(\tau) = \frac{1}{3}\gamma_{\tilde{y}}(\tau-1) \qquad \Longrightarrow \qquad \gamma_{\tilde{y}}(\tau) = \begin{cases} \frac{9}{8} & \tau = 0 \\ \frac{1}{3}\gamma_{\tilde{y}}(\tau-1) & \tau > 0 \end{cases}$$

(*) $\gamma_y(\tau) = \gamma_{\tilde{y}}(\tau)$:



Remark: EULER-WALKER FORMULA
(for the covariance function)

$$y(t) = a\,y(t-1) + e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

$$\Longrightarrow \gamma_y(\tau) = \begin{cases} \frac{1}{1-a^2}\lambda^2 & \tau = 0 \\ a\,\gamma_y(\tau-1) & \tau \geq 1 \end{cases}$$

Moreover: $\gamma(\tau) = \gamma(-\tau)$

(EX. 2)

$$y(t) = \underbrace{\frac{1}{2}y(t-1) + \eta(t)}_{AR(1)} \quad \underbrace{\eta(t-1)}_{MA(1)} \qquad \eta(t) \sim WN(1, 9)$$

$\Longrightarrow$ $y(t)$ is an ARMA (1,1)

$$y(t) = \frac{1}{2}z^{-1}y(t) + \eta(t) \quad z^{-1}\eta(t)$$

$$\Longrightarrow y(t) = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}\eta(t) = \frac{z-1}{z-\frac{1}{2}}\eta(t) \Longrightarrow \begin{array}{l} \text{zero}: z = 0 \\ \text{pole}: z = \frac{1}{2} \end{array}$$

$\Longrightarrow$ (for the theorem) : $y(t)$ is stationary

- Expected value:

$$\mathbb{E}[y(t)] = \mathbb{E}\left[\tfrac{1}{2}y(t-1) + \eta(t) - \eta(t-1)\right] = \tfrac{1}{2}\mathbb{E}[y(t)] + \mathbb{E}[\eta(t)] - \mathbb{E}[\eta(t-1)]$$

$$= \tfrac{1}{2}\mathbb{E}[y(t)] + 1 - 1$$

$$\Longrightarrow \mathbb{E}[y(t)] = 0$$

- Covariance function:

$$e(t) = \eta(t) - 1$$

$$\mathbb{E}[e(t)] = \mathbb{E}[\eta(t)] - 1 = 0$$

$$\mathbb{E}[e^2(t)] = \mathbb{E}[(\eta(t)-1)^2] = \mathbb{E}[\eta^2(t)] + 1 - \mathbb{E}[2\eta(t)] = 9$$

$$\Longrightarrow e(t) \sim WN(0,9)$$

$$\Longrightarrow y(t) = \tfrac{1}{2}y(t-1) + e(t) \cancel{+1} - e(t-1) \cancel{-1}$$

$$\boxed{y(t) = \tfrac{1}{2}y(t-1) + e(t) - e(t-1)}$$

$$\gamma_y(0) = \mathbb{E}[y^2(t)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-1) + e(t) - e(t-1)\right)^2\right]$$

$$= \tfrac{1}{4}\underbrace{\mathbb{E}[y^2(t-1)]}_{\substack{\gamma_y(0) \\ \text{since the process} \\ \text{is stationary}}} + \underbrace{\mathbb{E}[e^2(t)]}_{9 = Var(e) = 9} + \underbrace{\mathbb{E}[e^2(t-1)]}_{} + \underbrace{\cancel{\mathbb{E}[y(t-1)e(t)]}}_{\substack{y(t-1) \perp\!\!\!\perp e(t) \\ \text{and } \mathbb{E}[e(t)]=0, \\ \mathbb{E}[y(t)]=0}} - \underbrace{\mathbb{E}[y(t-1)e(t-1)]}_{?} - 2\underbrace{\cancel{\mathbb{E}[e(t)e(t-1)]}}_{\substack{e(t) \perp\!\!\!\perp e(t-1) \\ \text{and} \\ \mathbb{E}[e(t)] = \mathbb{E}[e(t-1)] = 0}}$$

$$\mathbb{E}[y(t-1)e(t-1)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-2) + e(t-1) - e(t-2)\right)e(t-1)\right]$$

$$= \tfrac{1}{2}\cancel{\mathbb{E}[y(t-2)e(t-1)]} + \mathbb{E}[e^2(t-1)] - \cancel{\mathbb{E}[e(t-1)e(t-2)]}$$

$$= Var(e(t)) = 9$$

$$= \tfrac{1}{4}\gamma_y(0) + 9 + \cancel{9 - 9}$$

$$\Longrightarrow \gamma_y(0) = 12$$

$$\gamma_y(1) = \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-1) + e(t) - e(t-1)\right)y(t-1)\right]$$

$$= \tfrac{1}{2}\underbrace{\mathbb{E}[y^2(t-1)]}_{\substack{\gamma_y(0) \\ \text{(stationary)}}} + \underbrace{\cancel{\mathbb{E}[e(t)y(t-1)]}}_{\substack{e(t) \perp\!\!\!\perp y(t-1) \\ \mathbb{E}[e(t)]=0}} - \underbrace{\mathbb{E}[e(t-1)y(t-1)]}_{9 \text{ (previously)}}$$

$$= \tfrac{1}{2}\gamma_y(0) - 9 = -3$$

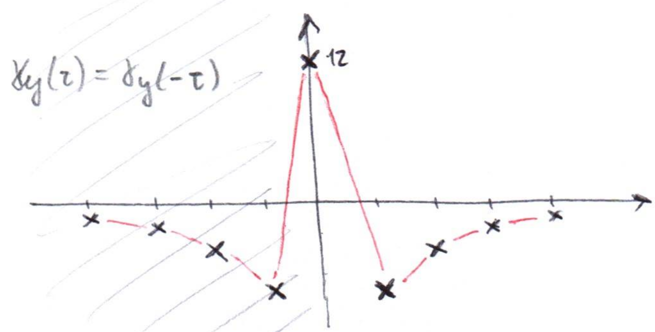
$$\gamma_y(2) = \mathbb{E}[y(t)y(t-2)] = \mathbb{E}\left[\left(\tfrac{1}{2}y(t-1) + e(t) - e(t-1)\right)y(t-2)\right]$$

$$= \tfrac{1}{2}\mathbb{E}[y(t-1)y(t-2)] + \cancel{\mathbb{E}[e(t)y(t-2)]} - \cancel{\mathbb{E}[e(t-1)y(t-2)]}$$

$$= \tfrac{1}{2}\gamma_y(1) = -\tfrac{3}{2}$$

$$\Longrightarrow \quad \gamma_y(\tau) = \begin{cases} 12 & \tau = 0 \\ -3 & \tau = 1 \\ \frac{1}{2}\gamma_y(\tau-1) & \tau \geq 2 \end{cases}$$

:



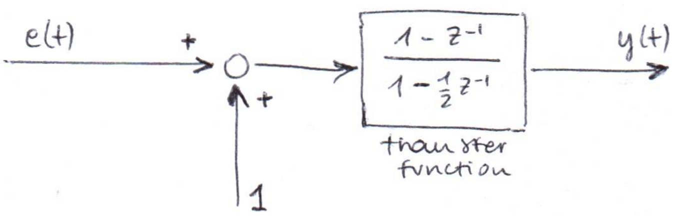
Remark:
property of covariance function

1. $\gamma_y(0) \geq 0$
2. $|\gamma_y(\tau)| \leq \gamma(0)$
3. $\gamma_y(\tau) = \gamma_y(-\tau)$

!

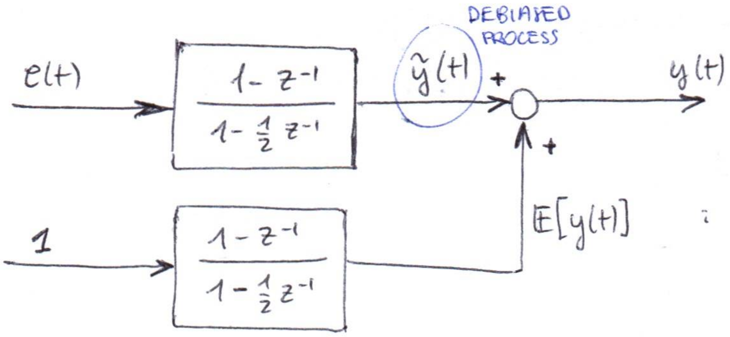
Usefull to recognize if a plot is a covariance function or not

(another way to do the DETREND or DEBIAS PROCESS)



$e(t) \sim WN(0,9)$

Since the transfer function is linear:



:

$$\mathbb{E}[y(t)] = \left.\frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=1} \cdot 1 = 0$$

(Spectrum = Covariance in the frequency domain) 17/03

Remarks:

- THEOREM OF THE GAIN

$\Longrightarrow$ FINAL VALUE THEOREM

$$y(t) = W(z)\, u(t)$$

$$y(t \to \infty) = \lim_{z \to 1} W(z)\, U(z) \cdot (z-1)$$

We considered $u(t)$ constant, so: $u(t) = \bar{u}$

$$\Longrightarrow U(z) = \frac{z}{z-1}\,\bar{u}$$

$$\Longrightarrow y(t) = \lim_{z\to 1} W(z)\, \frac{z}{\cancel{z-1}}\,\bar{u}\,\cancel{(z-1)} = W(z\ \ 1)\cdot\bar{u}$$

- OPERATOR DELAY z^{-1}

$$y(t-k) \longrightarrow z^{-k}\, Y(z) = z^{-k}\, y(t)$$

- Linearity: $Z\left[\alpha f(t) + \beta g(t)\right] = \alpha F(z) + \beta G(z)$
- Convolution: $\boxed{F(z)\, G(z) \xRightarrow[z^{-1}]{} f(t) * g(t)} = \sum_k f(k)\, g(t-k) = \sum_k f(t-k)\, g(k)$

$\Rightarrow$ When we considered:

$$u(t) \longrightarrow \boxed{W(z)} \longrightarrow y(t) \quad ; \quad Y(z) = W(z)\, U(z)$$

$$\Longrightarrow y(t) = \sum_{\tau=-\infty}^{+\infty} h(t-\tau)\, u(\tau) \qquad \left(h(t) \xrightarrow{z^{-1}} W(z) \right)$$

impulsive response of the system

---

time domain $\longrightarrow$ frequency domain
(covariance function) (spectrum = discrete fourier's transform of the cov. fun)

SPECTRUM of the stochastic process:

$$\Gamma_y(\omega) := \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau)\, e^{-j\omega\tau}$$

:= discrete fourier's transform of the covariance function

Properties:

1. $\Gamma_y(\omega)$ is real
2. $\Gamma_y(\omega) \geq 0$
3. $\Gamma_y(\omega) = \Gamma_y(-\omega)$
4. $\Gamma_y(\omega) = \Gamma_y(\omega + 2k\pi)$

$\underset{\text{period}}{T} = 2\pi \Longrightarrow$ we represent the spectrum in $[-\pi, \pi]$

5. $y(t) = W(z)\, u(t) \implies \boxed{\Gamma_y(\omega) = \left| W(z)\big|_{z=e^{j\omega}} \right|^2 \Gamma_u(\omega)}$ Fundamental thm.

$u(t) \sim WN(0, \lambda^2) \implies \Gamma_y(\omega) = \left| W(z)\big|_{z=e^{j\omega}} \right|^2 \lambda^2$

$(\Gamma_u(\omega) = \lambda^2 \quad \forall \omega)$

6. $\gamma_y(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_y(\omega)\, d\omega$ (this can be a check for $\Gamma_y(\omega)$)

(EX. 1)

$y(t) = \frac{1}{2} y(t-1) + e(t) - e(t-1)$ $\qquad e(t) \sim WN(0,9)$

$$\gamma(\tau) = \begin{cases} 12 & \tau = 0 \\ -3 & |\tau| = 1 \\ \frac{1}{2}\gamma_y(|\tau|-1) & |\tau| \geq 2 \end{cases}$$

Compute the spectrum.

(1) By def. $\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau)\, e^{-j\omega\tau}$

DEF.

$$= 12\, e^{j\omega 0} - 3e^{-j\omega} - 3e^{j\omega} - \frac{3}{2}e^{-j\omega 2} - \frac{3}{2}e^{j\omega 2} - \frac{3}{4}e^{-j\omega 3} - \frac{3}{4}e^{-j\omega 3} - \dots$$

$$= 12 - 6\left(\frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega 2} + \frac{1}{8}e^{-j\omega 3} + \dots + \frac{1}{2}e^{+j\omega} + \frac{1}{4}e^{+j\omega 2} + \frac{1}{8}e^{+j\omega 3} + \dots\right)$$

$$= 12 - 6\left(-1 + \boxed{1 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega 2} + \frac{1}{8}e^{-j\omega 3} + \dots} + -1 + \boxed{1 + \frac{1}{2}e^{j\omega} + \frac{1}{4}e^{j\omega 2} + \frac{1}{8}e^{j\omega 3} + \dots}\right)$$

$\sum_{i=0}^{+\infty} a^i = \frac{1}{1-a}, \quad |a|<1$

$$= 12 - 6\left(-1 + \boxed{\sum_{i=0}^{+\infty}\left(\frac{1}{2}e^{-j\omega}\right)^i} - 1 + \boxed{\sum_{i=0}^{+\infty}\left(\frac{1}{2}e^{j\omega}\right)^i}\right)$$

$$= 24 - 6\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{2}e^{j\omega}}\right)$$

$$= 24 - 6\left(\frac{2 - \frac{1}{2}(e^{j\omega} + e^{-j\omega})}{1 + \frac{1}{4} + \frac{1}{2}e^{-j\omega} + \frac{1}{2}e^{j\omega}}\right)$$

$\rightarrow = \cos(\omega)$

$\rightarrow = (\omega)$

$$= \frac{18 - 18\cos(\omega)}{\frac{5}{4} - \cos(\omega)}$$

(2) THM.

$$W(z) = \frac{z-1}{z-\frac{1}{2}}, \quad \Gamma_e(\omega) = 9 \quad (e \sim WN(0,9))$$

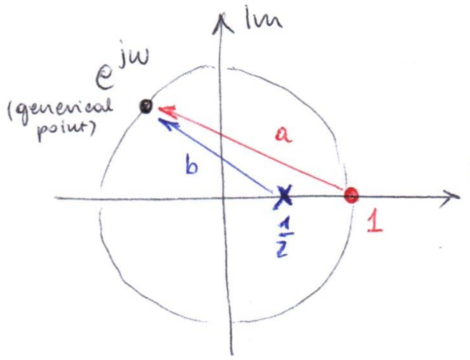
The system is stable $(z = \frac{1}{2}) \Longrightarrow$ we use the theorem:

$$\Gamma_y(\omega) = \left| \frac{e^{j\omega}-1}{e^{j\omega}-\frac{1}{2}} \right|^2 \cdot 9 = \frac{(e^{j\omega}-1)(e^{-j\omega}-1)}{(e^{j\omega}-\frac{1}{2})(e^{-j\omega}-\frac{1}{2})} \cdot 9$$

$$= \frac{(1-e^{j\omega}-e^{-j\omega}+1)}{(1-\frac{1}{2}e^{j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{4})} \cdot 9$$

$$= \frac{2-\frac{2}{2}(e^{j\omega}+e^{-j\omega})}{\frac{5}{4}-\frac{1}{2}(e^{j\omega}+e^{-j\omega})} \cdot 9 = \frac{18-18\cos(\omega)}{\frac{5}{4}-\cos(\omega)}$$

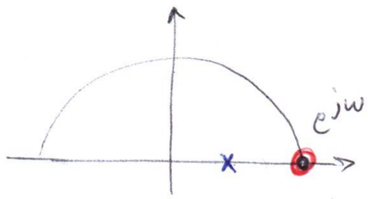
(3) GRAPHICAL (qualitative)



$$\Gamma_y(\omega) = \frac{\|a\|^2}{\|b\|^2} \cdot 9$$

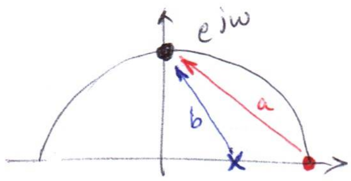
We need to check a phew $\underline{\omega}$:

$\omega = 0$:



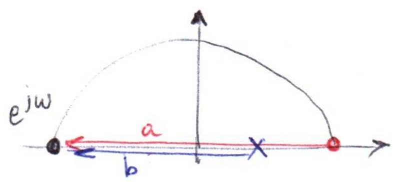
$\|a\| = 0 \Longrightarrow \Gamma_y(0) = 0$

$\omega = \frac{\pi}{2}$:



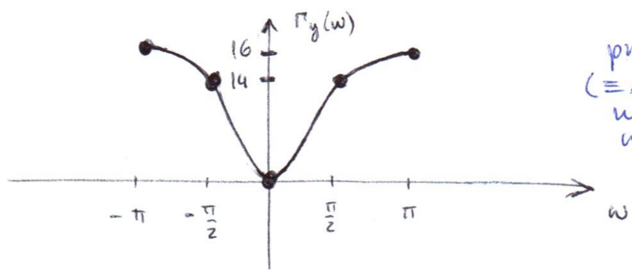
$$\left.\begin{aligned} \|a\| &= \sqrt{2} \\ \|b\| &= \sqrt{1+\frac{1}{4}} \end{aligned}\right\} \Rightarrow \Gamma_y\left(\frac{\pi}{2}\right) = \frac{72}{5} \simeq 14$$

$\omega = \pi$:



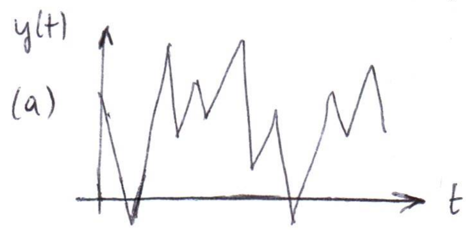
$$\left.\begin{aligned} \|a\| &= 2 \\ \|b\| &= \frac{3}{2} \end{aligned}\right\} \Rightarrow \Gamma_y(\pi) = 16$$

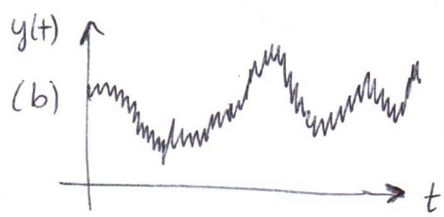
$\Longrightarrow$



predominanza di ALTE frequenze (≡ segnale che varia molto rapidamente nel dominio del tempo)

Which realization of this process is more likely?





Lo spettro favorisce alte frequenze. Il grafico (a) contiene variazioni molto veloci come modulo y(t), quindi cambiamenti "ad alta frequenza" (RICORDIAMO che i segnali sono a tempo discreto)

The high frequency are predominant. In (a) the signal is very fast, (b) is slow (has low frequences) $\Longrightarrow$ (a) is more likely.

(EX. 2)

$$y(t) = \left[\left(1 - z^{-1} + z^{-2}\right)\left(1 + \frac{3}{2} z^{-1}\right)\right] e(t), \qquad e(t) \sim WN(0, 1)$$

Compute the spectrum.

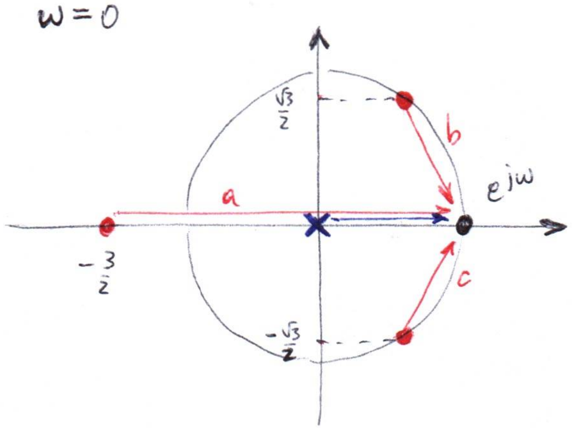
$$y(t) = \left(\frac{z^2 - z + 1}{z^2}\right)\left(\frac{z+3}{2z}\right) e(t) = \frac{(z^2 - z + 1)(z+}{z^3} \, e(t) :$$

- poles: $z = 0$ (3)
- zeros: $z = -\frac{3}{2}$, $z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$\Big\} \Longrightarrow$ $y(t)$ is stationary

$$\Longrightarrow \Gamma_y(w) = \frac{\left|e^{jw} + \frac{3}{2}\right|^2 \left|e^{jw} - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right|^2 \left|e^{jw} - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right|^2}{\left(\left|e^{jw}\right|^2\right)^3} \cdot 1$$

$w = 0$



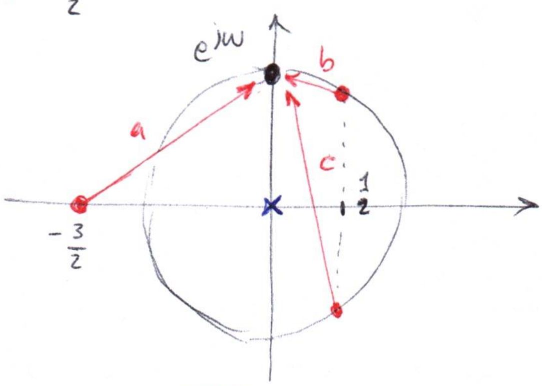
$\|a\| = 1 + \frac{3}{2}$

$\|b\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$\|c\| = 1$

$\Rightarrow \Gamma_y(0) = \frac{\|a\|^2 \cdot \|b\|^2 \cdot \|c\|^2}{1} \cdot 1 = \frac{25}{4}$

$w = \frac{\pi}{2}$



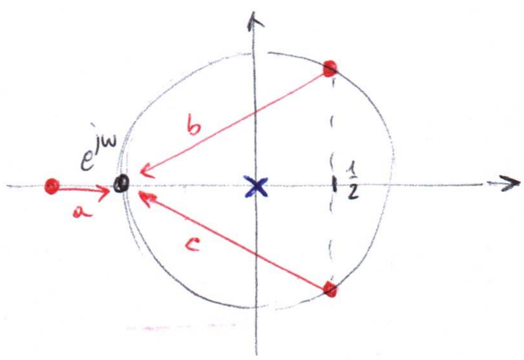
$\|a\| = \sqrt{\left(\frac{3}{2}\right)^2 + 1}$

$\|b\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(1 - \frac{\sqrt{3}}{2}\right)^2}$

$\|c\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(1 + \frac{\sqrt{3}}{2}\right)^2}$

$\Rightarrow \Gamma_y\left(\frac{\pi}{2}\right) = [\,..\,] = \frac{13}{4}$

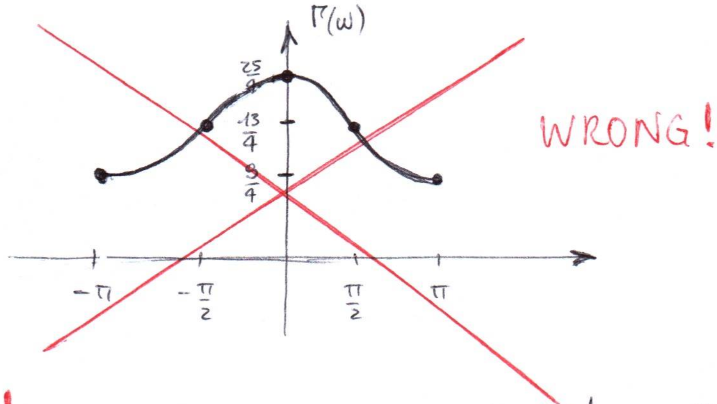
$w = \pi$



$\|a\| = \frac{1}{2}$

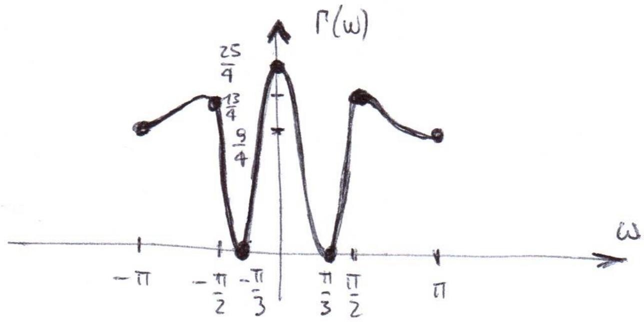
$\|b\| = \|c\| = \sqrt{\left(1 + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}$

$\Rightarrow \Gamma_y(\pi) = [\,..\,] = \frac{9}{4}$

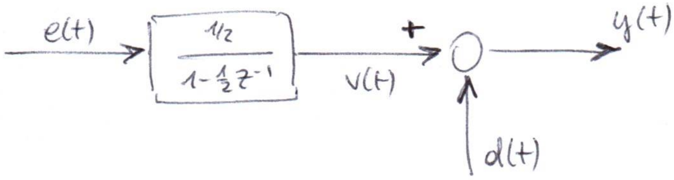


! We lost the zeros on the unit disk!
(If we had a pole the system wouldn't be asintotically stable, with a zero we have that the spectrum is 0 in that point)

$\Gamma_y\left(\omega = \frac{\pi}{3}\right) = 0$



(EX 1)



$e(t) \sim WN(0,1)$

1. $d(t) \sim WN(0,1)$
   $e(t) \perp d(t)$ = completely uncorrelated
2. $d(t) = -\frac{1}{3} e(t)$

$\Gamma_y(\omega)$?

1. $y(t) = v(t) + d(t)$

$\mathbb{E}[y(t)] = \mathbb{E}[v(t)] + \mathbb{E}[d(t)]$

$v(t) = \frac{1/2}{1-\frac{1}{2}z^{-1}} e(t) \Rightarrow v(t) = \frac{1}{2} v(t-1) + \frac{1}{2} e(t)$

$\mathbb{E}[v(t)] = \frac{1}{2}\mathbb{E}[v(t-1)] + \frac{1}{2}\mathbb{E}[e(t)]$

$W(z) = \frac{\frac{1}{2}z}{z-\frac{1}{2}}$

$\Longrightarrow$ stable $\Longrightarrow$ stationary

$\Longrightarrow \mathbb{E}[v(t)] = \frac{1}{2}\mathbb{E}[v(t)] + 0 \Longrightarrow \mathbb{E}[v(t)] = 0$

$= 0$

$$\gamma_y(\tau) = \mathbb{E}[y(t)y(t-\tau)] = \mathbb{E}[(v(t)+d(t))(v(t-\tau) \quad d(t-\tau))]$$

$$= \mathbb{E}[v(t)v(t-\tau) + d(t)v(t-\tau) + v(t)d(t-\tau) + d(t)d(t-\tau)]$$

$$= \underbrace{\mathbb{E}[v(t)v(t-\tau)]}_{\gamma_v(\tau)} + \underbrace{\cancel{\mathbb{E}[d(t)v(t-\tau)]}}_{\text{because they're uncorrelated}} + \mathbb{E}[v(t)d(t \quad \tau)] + \underbrace{\mathbb{E}[d(t)d(t-\tau)]}_{\gamma_d(\tau)}$$

$$= \gamma_v(\tau) + \gamma_d(\tau)$$

$$\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau} = \sum_{\tau=-\infty}^{+\infty} \gamma_v(\tau) e^{-j\omega\tau} + \sum_{\tau=-\infty}^{+\infty} \gamma_d(\tau) e^{-j\omega\tau}$$

$$\Gamma_v(\omega) = \left| \frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}} \right|^2_{z=e^{j\omega}} \Gamma_e(\omega) = \frac{|1/2|^2}{|1-\frac{1}{2}e^{-j\omega}|^2}$$

$$= \frac{1/4}{(1-\frac{1}{2}e^{j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= \frac{1/4}{1-\frac{1}{2}(e^{j\omega}+e^{-j\omega})+\frac{1}{4}}$$

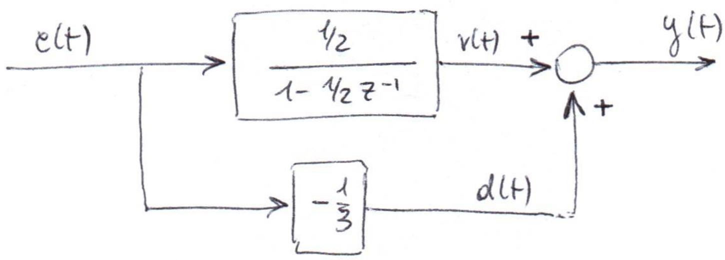
$$= \frac{1/4}{\frac{5}{4} - \cos(\omega)}$$

$\Gamma_d(\omega) = 1 \quad \forall \omega$

$$\Rightarrow \Gamma_y(\omega) = \frac{1/4}{\frac{5}{4}-\cos(\omega)} + 1$$

2. $d(t) = -\frac{1}{3} e(t)$:

(FOR US)



$$y(t) = \left(\frac{1/2}{1-\frac{1}{2}z^{-1}} - \frac{1}{3} \right) e(t) = \frac{1}{6} \frac{z+1}{z-\frac{1}{2}} e(t) \longrightarrow \Gamma_y(\omega) \text{ with}$$

1. Fundamental theorem
2. graphical method

ESTIMATION OF PROCESS CHARACTERISTICS

$$\hat{\gamma}_y(\tau) := \frac{1}{N-\tau} \sum_{t=1}^{N-\tau} y(t)\, y(t-\tau)$$

unbiased and consistent

N s.t. $\tau < \frac{N}{4}$, $N \geq 50$

$$\left(\hat{\gamma}_y(\tau) = \frac{1}{N-|\tau|} \sum_{t=1}^{N-|\tau|} y(t)\; y(t+|\tau|) \right)$$

$$\hat{\gamma}'_y(\tau) = \frac{1}{N} \sum_{t=1}^{N-|\tau|} y(t)\, y(t+|\tau|) \qquad |\tau| \leq N-1$$

not correct, but for $N \to \infty$ is asymptotically correct, it also work for N large and τ small

---

$$\hat{\Gamma}_y(\omega) := \sum_{\tau=-N+1}^{N-1} \hat{\gamma}_y(\tau)\, e^{-j\omega\tau}$$

$$\hat{\Gamma}'_y(\omega) := \sum_{\tau=-N+1}^{N-1} \hat{\gamma}'_y(\tau)\, e^{-j\omega\tau}$$

$\rightarrow$ we prefer this because $\hat{\Gamma}_y(\omega)$ can be negative

$$\hat{\Gamma}'_y(\omega) = \frac{1}{N} \Bigg(\underbrace{\sum_{t=1}^{N} y(t)\, e^{-j\omega t}}_{\text{Fourier's discrete transform of } y(t)} \Bigg)^2$$

- not unbiased
- not consistent

$$\bar{\hat{\Gamma}}(\omega) = \frac{1}{r} \sum_{i=1}^{r} \hat{\Gamma}_i(\omega)$$

PREDICTION

$y(t),\ y(t-1),\ y(t-2)$ process data $\Big] \Longrightarrow \hat{y}(t+k|t)$ (k = prediction horizon)

(EX.)

$$y(t) = \frac{1}{2}y(t-2) + \eta(t) + 4\eta(t-1) \qquad \eta(t) \sim WN(0,1)$$

$\boxed{\hat{y}(t+1|t)\ ?}$

$$y(t) - \frac{1}{2}z^{-2}y(t) = \eta(t) + 4z^{-1}\eta(t) \Longrightarrow y(t) = \frac{1+4z^{-1}}{1-\frac{1}{2}z^{-2}}\eta(t)$$

$\Longrightarrow y(t) = \dfrac{z(z+4)}{z^2-\frac{1}{2}}\eta(t)$:
- zeros: $z=0,\ z=-4$
- poles: $z = \frac{1}{\sqrt{2}},\ z = -\frac{1}{\sqrt{2}}$

$\Longrightarrow y(t)$ is stationary

1. Check the canonical representation:

$y(t) = W(z)\,u(t)$

$$W(z) = \frac{N(z)}{D(z)} = \frac{(1+4z^{-1})}{(1-\frac{1}{2}z^{-2})} = \frac{z(z+4)}{z^2-\frac{1}{2}}$$

CANONICAL REPRESENTATION

1. $N(z)$ and $D(z)$ must have the same degree
2. $N(z)$ and $D(z)$ must be monic (the coefficient of the heigest power of z must be $= 1$)
3. $N(z)$ and $D(z)$ must be coprime (no common factor to semplify)
4. zeros and poles must be inside the unit circle

$\Longrightarrow$ 4. is not verified (zero: $z =$

How can we remove this zero?

$$y(t) = \frac{1+4z^{-1}}{1-\frac{1}{2}z^{-2}} \cdot \frac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{4}z^{-1}}\eta(t) = \frac{1+\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-2}} \cdot \boxed{\frac{1+4z^{-1}}{1+\frac{1}{4}z^{-1}}\eta(t)} := e(t)$$

GAIN THEOREM

$e(t) = \dfrac{1+4z^{-1}}{1+\frac{1}{4}z^{-1}}\eta(t)$:

- $\mathbb{E}[e(t)] = \dfrac{1+4}{1+\frac{1}{4}}\mathbb{E}[\eta(t)] = 0$

- $$\Gamma_e(\omega) = \frac{|1+4e^{-j\omega}|^2}{|1+\frac{1}{4}e^{-j\omega}|^2}\cdot 1 = \frac{(1+4e^{-j\omega})(1+4e^{+j\omega})}{(1+\frac{1}{4}e^{-j\omega})(1+\frac{1}{4}e^{+j\omega})} = \frac{1+4e^{j\omega}+4e^{-j\omega}+16}{1+\frac{1}{4}e^{j\omega}+\frac{1}{4}e^{-j\omega}+\frac{1}{16}} = 16 \quad \forall \omega$$

(1.) $\Longrightarrow$ $e(t) \sim WN(0, 16)$

$\Longrightarrow$ CANONICAL REPRESENTATION: $y(t) = \dfrac{(1+\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-2})} e(t) \qquad e(t) \sim WN(0,16)$

$\Longrightarrow$ ARMA (2,1)

2. Prediction computation

2.1. LONG DIVISION:

$$\begin{array}{l|l} 1 + \frac{1}{4}z^{-1} & 1-\frac{1}{2}z^{-2} \\ \hline 1 \qquad\quad - \frac{1}{2}z^{-2} & \underbrace{1}_{E(z)} \\ \hline / \quad \underbrace{\frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}}_{F(z)} & \end{array}$$

$$y(t) = \frac{C(z)}{A(z)} e(t) = \left(E(z) + \frac{F(z)}{A(z)}\right) e(t) = 1 + \left(\frac{\frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}}{1-\frac{1}{2}z^{-2}}\right) e(t)$$

$$= e(t) + \frac{\frac{1}{4} + \frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-2}} e(t-1)$$

unpredictable at time t-1

predictable at time t-1: $f(e(t-1), e(t-2), \dots)$ MA(∞)

$$\boxed{\hat{y}(t|t-1) = \frac{\frac{1}{4} + \frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-2}} e(t-1)}$$

predictable part of the process

PREDICTION FROM THE **NOISE**

but we have $y(\cdot)$, not $e(\cdot)$

2.2.

$$y(t) = \frac{C(z)}{A(z)} e(t) \xrightarrow[\text{repr.}]{\text{thanks to the canonical}} e(t) = \frac{A(z)}{C(z)} y(t)$$

$$\hat{y}(t|t-1) = \frac{(\frac{1}{4}z^{-1} + \frac{1}{2}z^{-2})}{1-\frac{1}{2}z^{-2}} e(t)$$

$$= \frac{(\frac{1}{4}z^{-1} + \frac{1}{2}z^{-2})}{\cancel{(1-\frac{1}{2}z^{-2})}} \frac{\cancel{(1-\frac{1}{2}z^{-2})}}{(1+\frac{1}{4}z^{-1})} y(t)$$

$$= \boxed{\frac{\frac{1}{4} + \frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}} y(t-1)}$$

PREDICTOR FROM THE **DATA**

$$\Longrightarrow \hat{y}(t+1|t) = \frac{\frac{1}{4} + \frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}} y(t)$$

$$\Longrightarrow (1+\tfrac{1}{4}z^{-1})\hat{y}(t+1|t) = (\tfrac{1}{4} + \tfrac{1}{2}z^{-1}) y(t)$$

$$\Longrightarrow \boxed{\hat{y}(t+1|t) = -\frac{1}{4}\hat{y}(t|t-1) + \frac{1}{4}y(t) + \frac{1}{2}y(t-1)}$$

PREDICTOR FROM THE **DATA** IN **TIME** DOMAIN

t	$y(t)$	$\hat{y}(t+1\|t)$
≤ 0	0	$\hat{y}(1\|0)$
1	1	$\hat{y}(2\|1)$
2	0	$\hat{y}(3\|2)$
3	$-\frac{1}{2}$	$\hat{y}(4\|2)$

W use the predictor from the data in time domain:

$$\hat{y}(t+1|t) = -\frac{1}{4}\hat{y}(t|t-1) + \frac{1}{4}y(t) + \frac{1}{2}y(t-1)$$

- $\hat{y}(1|0) = 0$
- $\hat{y}(2|1) = -\frac{1}{4}\hat{y}(1|0) + \frac{1}{4}y(1) + \frac{1}{2}y(0) = \frac{1}{4}$
- $\hat{y}(3|2) = -\frac{1}{4}\hat{y}(2|1) + \frac{1}{4}y(2) + \frac{1}{2}y(1) = \frac{7}{16}$
- $\hat{y}(4|3) = -\frac{1}{4}\hat{y}(3|2) + \frac{1}{4}y(3) + \frac{1}{2}y(2) = -\frac{15}{64}$

Another way to predict:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{A(z)} \cdot e(t) \qquad \text{PREDICTOR FROM } \textbf{NOISE}$$

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} \, y(t) \qquad \text{PREDICTOR FROM } \textbf{DATA}$$

$$\Longrightarrow \quad \hat{y}(t|t-1) = \frac{(1+\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-2})}{(1-\frac{1}{2}z^{-2})} e(t) = \frac{(\cancel{1}+\frac{1}{4}z^{-1} \cancel{-1} + \frac{1}{2}z^{-2})}{1-\frac{1}{2}z^{-2}} e(t)$$

$$= \frac{\frac{1}{4}+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-2}} e(t-1) \qquad (\text{from NOISE})$$

For us: check the "from DATA" formula

---

3. Predictor error

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1)$$

$$y(t) = e(t) + \frac{\frac{1}{4}-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-2}} e(t-1)$$

$$\hat{y}(t|t-1) = \frac{\frac{1}{4}-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-2}} e(t-1)$$

$$\longrightarrow \quad \varepsilon(t|t-1) = e(t)$$

Unpredictable part

$\Longrightarrow$ OPTIMAL PREDICTION

$$\mathbb{E}[\varepsilon(t|t-1)^2] = \mathbb{E}[e^2(t)] = 16$$

variance of the prediction error

$$\boxed{\hat{y}(t|t-2)\ ?}$$

$$y(t) = \frac{1+\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-2}}\, e(t) \quad , \qquad e(t) \sim WN(0,16)$$

2.1.

Long division:

$$\begin{array}{l|l}
1+\frac{1}{4}z^{-1} & 1-\frac{1}{2}z^{-2} \\
1 \qquad\qquad -\frac{1}{2}z^{-2} & \overline{1+\frac{1}{4}z^{-1}} \\
\hline
/ \quad \frac{1}{4}z^{-1}+\frac{1}{2}z^{-2} & \\
\quad \frac{1}{4}z^{-1} \qquad\qquad -\frac{1}{8}z^{-3} & \\
\hline
\qquad / \quad \frac{1}{2}z^{-2}+\frac{1}{8}z^{-3} &
\end{array}$$

$$\Longrightarrow \quad y(t) = \left(E(z)+\frac{F(z)}{A(z)}\right)e(t) = \left(1+\frac{1}{4}z^{-1}+\frac{\frac{1}{2}z^{-2}+\frac{1}{8}z^{-3}}{1-\frac{1}{2}z^{-2}}\right)e(t)$$

$$= e(t) + \frac{1}{4}e(t-1) + \frac{\frac{1}{2}+\frac{1}{8}z^{-1}}{1-\frac{1}{2}z^{-2}}\, e(t-2)$$

unpredictable at time t 2

predictable at time t-2

$$\boxed{\hat{y}(t|t-2) = \frac{\frac{1}{2}+\frac{1}{8}z^{-1}}{1-\frac{1}{2}z^{-2}}\, e(t-2)}$$

the only predictable part of y(t) ↑

PREDICTOR FROM THE **NOISE**

2.2.

$\Longrightarrow$ thanks to the canonical repr.

$$\hat{y}(t|t-2) = \frac{(\frac{1}{2}z^{-2}+\frac{1}{8}z^{-3})}{\cancel{(1-\frac{1}{2}z^{-2})}} \cdot \frac{\cancel{(1-\frac{1}{2}z^{-2})}}{(1+\frac{1}{4}z^{-1})}\, y(t)$$

$$= \boxed{\frac{\frac{1}{2}+\frac{1}{8}z^{-1}}{1+\frac{1}{4}z^{-1}}\, y(t-2)}$$

PREDICTION FROM THE **DATA**

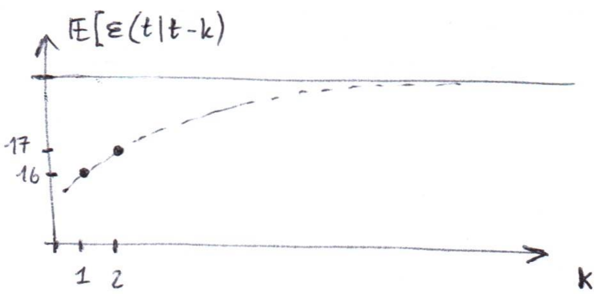
3.

Prediction error:

$$\varepsilon(t|t-2) = y(t) - \hat{y}(t|t-2) = E(z)\, e(t) \quad \Longrightarrow \quad MA(2)$$

$$\mathbb{E}[\varepsilon(t|t-2)^2] = \mathbb{E}\left[\left(e(t)+\frac{1}{4}e(t-1)\right)^2\right] = \mathbb{E}\left[e(t)^2\right] + \frac{1}{16}\,\mathbb{E}\left[e(t-1)^2\right] = 16+1 = 17$$

In general:



$\mathbb{E}[y(t)^2]$ = variance of the process ($\mathbb{E}[y(t)]=0$)

$$\mathbb{E}[\varepsilon(t|t-k)^2] \underset{k\to\infty}{=} \underbrace{\mathbb{E}\left[\left(E(z)\, e(t)\right)^2\right]}_{MA(\infty)} = \mathbb{E}[y(t)^2] \qquad \Rightarrow \quad \hat{y}(t|t-\infty) = 0$$

$$\mathbb{E}\left[(y(t) - \hat{y}(t|t-\infty))^2\right] = \mathbb{E}\left[(y(t) - 0)^2\right] = \gamma_y(0)$$

$$\Rightarrow \hat{y}(t|t-\infty) = 0$$

In general:

$$\mathbb{E}[y(t)] \neq 0 \Longrightarrow \mathbb{E}\left[(y(t) - \mathbb{E}[y(t)])^2\right] = \gamma_y(0)$$

$$\Longrightarrow \boxed{\hat{y}(t|t-\infty) = \mathbb{E}[y(t)]}$$

with an infinity prediction horizon the only thing we can say about the process is its expected value

PREDICTION

$y(t) = W(z)\, e(t)$ $\qquad e(t) \sim WN(0, \lambda^2)$

$= \dfrac{C(z)}{A(z)} e(t)$

long division with **k** steps

$= \left(E(z) + \dfrac{F(z)}{A(z)} \right) e(t)$

$= \underbrace{\boxed{E(z) e(t)}}_{\text{unpredictable (at time) } t-k} + \underbrace{\boxed{\dfrac{F(z)}{A(z)} e(t)}}_{\text{predictable } t-k}$

$$\Longrightarrow \boxed{\hat{y}(t|t-k) = \frac{F(z)}{A(z)} e(t)}$$ PREDICTOR FROM THE **NOISE**

since we used the canonical representation:

$e(t) = \dfrac{A(z)}{C(z)} y(t) \quad \Longrightarrow \quad \hat{y}(t|t-k) = \dfrac{F(z)}{A(z)} \cdot \dfrac{A(z)}{C(z)} y(t)$

$$\Longrightarrow \boxed{\hat{y}(t|t-k) = \frac{F(z)}{C(z)} y(t)}$$ PREDICTOR FROM THE **DATA**

We always have the prediction error:

$\varepsilon(t|t-k) = y(t) - \hat{y}(t|t-k)$

$= E(z) e(t) + \cancel{\dfrac{F(z)}{A(z)} e(t)} - \cancel{\dfrac{F(z)}{A(z)} e(t)}$

$$\Longrightarrow \boxed{\varepsilon(t|t-k) = \underbrace{E(z)\, e(t)}_{\text{unpredictable part of the process}}}$$ **ERROR**

Variance of the predictor error:

$\mathbb{E}\left[(E(z) e(t))^2 \right] \Longrightarrow$ variance of $MA(k)$

- $k = 1$; $\quad \mathbb{E}\left[(E(z) e(t))^2 \right] = \mathbb{E}\left[e(t)^2 \right] = Var(e(t))$
- $k \to \infty$: $\quad \varepsilon(t|t-k) = E(z) e(t) = MA(\infty) = y(t)$

  $\mathbb{E}\left[(\varepsilon(t|t-k))^2 \right] = \gamma_y(0)$ (variance of the process)

  $\gamma_y(0) = \mathbb{E}\left[(y(t) - \mathbb{E}[y(t)])^2 \right]$

$\gamma_y(0) = \mathbb{E}\left[(y(t) - \mathbb{E}[y(t)])^2 \right] = \mathbb{E}\left[(y(t) - \hat{y}(t|t-k))^2 \right]$

$\Longrightarrow \hat{y}(t|t-k) = \mathbb{E}[y(t)]$ as $k \to \infty$

the only thing we can say with an ∞ horizon is the expected value of the process

$$\hat{y}(t|t-2) = \frac{\frac{1}{2}+\frac{1}{8}z^{-1}}{1+\frac{1}{4}z^{-1}}\, y(t-2) = \frac{1}{2}\,\frac{\cancel{1+\frac{1}{4}z^{-1}}}{\cancel{1+\frac{1}{4}z^{-1}}}\, y(t-2) = \frac{1}{2}y(t-2)$$

(yesterday's exercise)

$$y(t) = \underbrace{\frac{1}{2}y(t-2)}_{\text{predictable, in fact this is the predictor } \hat{y}(t|t-2)} + \underbrace{e(t) + \frac{1}{4}e(t-1)}_{\text{unpredictable at time } t-2}$$

$\boxed{\hat{y}(t|t-3)\,?}$

$$\cancel{y(t) = \underbrace{\frac{1}{2}y(t-2) + e(t) + \frac{1}{4}e(t-1)}_{\text{fully unpredictable at time } t-3} \implies \hat{y}(t|t-3) = 0}$$

WRONG

$y(t-2) \not\perp y(k) \quad k = t-3, t-4, \dots \implies$ $y(t-2)$ is predictable, only $e(t)$ and $e(t-1)$ are unpredictable from $t-3$

$$y(t-2) = \frac{1}{2}y(t-4) + e(t-2) + \frac{1}{4}e(t-3)$$

$$\implies y(t) = \frac{1}{4}y(t-4) + \frac{1}{2}e(t-2) + \frac{1}{8}e(t-3) + e(t) + \frac{1}{4}e(t-1)$$

$$= \underbrace{\frac{1}{4}y(t-4) + \frac{1}{8}e(t-3)}_{\text{predictable}} + \underbrace{\frac{1}{2}e(t-2) + \frac{1}{4}e(t-1) + e(t)}_{\text{unpredictable}}$$

$$\implies \hat{y}(t|t-3) = \frac{1}{4}y(t-4) + \frac{1}{8}e(t-3)$$

depend also on the NOISE
$\Rightarrow$ to get rid of it we do the LONG DIVISION (ONLY WAY)

(Ex.)

$y(t) = e(t) + 5e(t-1)$, $e(t) \sim WN(\underline{1}, 1)$ $\rightarrow$ we cannot procede as before

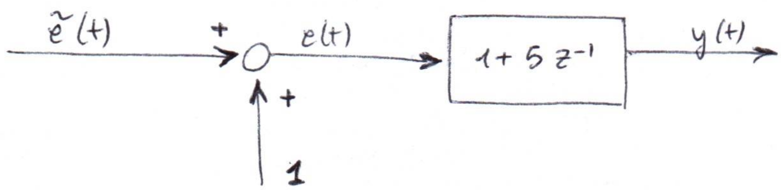
because:

$\hat{y}(t|t-1) = \mathbb{E}[e(t)] + 5e(t-1)$
$= 1 + 5e(t-1)$

$\boxed{\hat{y}(t|t-1)\,?}$

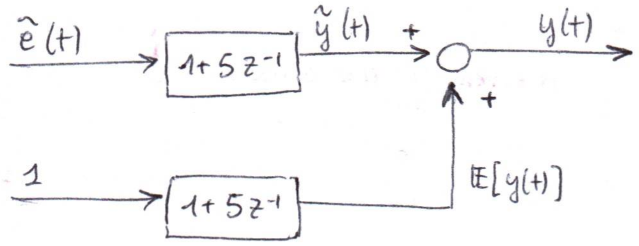
In order to avoid problems: **DEBIAS PROCESS**

$$\boxed{\begin{aligned} \tilde{e}(t) &:= e(t) - \mathbb{E}[e(t)] = e(t) - 1 \\ \tilde{y}(t) &:= y(t) - \mathbb{E}[y(t)] \end{aligned}}$$



$y(t) = f(\eta(t))$ $\quad \eta \sim WN(a,b)$, $a \neq 0$

1. DEBIAS $\rightarrow \tilde{y}$
2. $\hat{\tilde{y}}$ from noise
3. $\hat{\tilde{y}}$ from data
4. $\hat{\tilde{y}} \longrightarrow \hat{y}$



$$\mathbb{E}[y(t)] = 1 \cdot (1+5) = 6$$

$$\tilde{y}(t) = \tilde{e}(t) + 5\,\tilde{e}(t-1) \qquad \text{with:} \quad \begin{cases} \tilde{y}(t) = y(t) - 6 \\ \tilde{e}(t) = e(t) - 1 \end{cases}$$

$$\boxed{\hat{\tilde{y}}(t|t-1) \ ?}$$

$$\tilde{y}(t) = (1+5z^{-1})\,\tilde{e}(t) = \frac{z+5}{z}\,\tilde{e}(t)$$

canonical representation?

1. N, D same degree? Y
2. N, D monic? Y
3. N, D coprime? Y
4. zeros and poles in the unit disc? **NO.**

zero: $z = -5$

pole: $z = 0$

We have to remove this zero:

$$\tilde{y}(t) = (1+5z^{-1})\,\frac{(1+\frac{1}{5}z^{-1})}{(1+\frac{1}{5}z^{-1})}\,\tilde{e}(t)$$

$$= \left(1+\tfrac{1}{5}z^{-1}\right)\boxed{\left(\frac{1+5z^{-1}}{1+\frac{1}{5}z^{-1}}\right)\tilde{e}(t)} := \eta(t)$$

$$= \left(1+\tfrac{1}{5}z^{-1}\right)\eta(t), \qquad \eta(t) \sim WN(0, 5^2) = WN(0, 25)$$

$$\Longrightarrow \quad \tilde{y}(t) = \underbrace{\eta(t)}_{\text{unpred. at time } t-1} + \underbrace{\tfrac{1}{5}\,\eta(t-1)}_{\text{predictable at time } t-1}$$

$$\Longrightarrow \quad \boxed{\hat{\tilde{y}}(t|t-1) = \tfrac{1}{5}\,\eta(t-1)}$$

PREDICTOR FROM THE **NOISE**

$$\eta(t) = \frac{1}{1+\frac{1}{5}z^{-1}}\,\tilde{y}(t) \quad \Longrightarrow \quad \boxed{\hat{\tilde{y}}(t|t-1) = \frac{1}{5}\cdot\frac{z^{-1}}{1+\frac{1}{5}z^{-1}}\,\tilde{y}(t)}$$

PREDICTOR FROM THE **DATA**

$$\boxed{\Longrightarrow \quad \hat{y}(t|t-1)\ ?}$$

$$\hat{y}(t|t-1) = \widehat{\tilde{y}(t)+6} = \hat{\tilde{y}}(t|t-1) + 6$$

$$\Longrightarrow \quad \hat{y}(t|t-1) = \frac{1/5}{1+\frac{1}{5}z^{-1}}\,\tilde{y}(t-1) + 6$$

$$= \frac{1/5}{1+\frac{1}{5}z^{-1}}\,\big(y(t-1)-6\big) + 6$$

$$\Longrightarrow \hat{y}(t|t-1) = \frac{1/5}{1+\frac{1}{5}z^{-1}}\, y(t-1) - \frac{6/5}{1+1/5} + 6$$

theorem of THE GAIN : !

$$\frac{1/5}{1+\frac{1}{5}z^{-1}} \cdot 6 = \frac{6/5}{1+1/5}$$

we consider $z=1$ because we have as an input a constant signal

$$\Longrightarrow \boxed{\hat{y}(t|t-1) = \frac{1/5}{1+\frac{1}{5}z^{-1}}\, y(t-1) + 5}$$

FINAL PREDICTOR

(EX.)

$$\frac{1}{2}y(t) = -\frac{1}{3}y(t-1) - \frac{1}{18}y(t-2) + 3e(t-2) - 8e(t-3) - 3e(t-4), \quad \boxed{\hat{y}(t|t-1)\,?}$$

$$\frac{1}{2}y(t) + \frac{1}{3}z^{-1}y(t) + \frac{1}{18}z^{-2}y(t) = 3z^{-2}e(t) - 8z^{-3}e(t) - 3z^{-4}e(t)$$

$$\Longrightarrow y(t) = \frac{3z^{-2} - 8z^{-3} - 3z^{-4}}{\underbrace{\frac{1}{2} + \frac{1}{3}z^{-1} + \frac{1}{18}z^{-2}}}\, e(t)$$

canonical ? NO.

- no same degree
- no monic

$$y(t) = \frac{3z^{-2}}{\frac{1}{2}}\, \frac{(1-\frac{8}{3}z^{-1} - z^{-2})}{(1-\frac{2}{3}z^{-1} + \frac{1}{9}z^{-2})}\, e(t)$$

$:= \eta(t) = 6\,e(t-1), \quad \eta \sim WN(0,36)$

$$\Longrightarrow y(t) = \frac{1-\frac{8}{3}z^{-1} - z^{-2}}{1-\frac{2}{3}z^{-1}+\frac{1}{9}z^{-2}}\,\eta(t)$$

$$= \frac{z^2 - \frac{8}{3}z - 1}{z^2 - \frac{2}{3}z + \frac{1}{9}}\,\eta(t) :$$

- zeros : $z = 3, \quad z = -\frac{1}{3}$
- poles : $z = -\frac{1}{3}, \quad z = -\frac{1}{3}$

$$= \frac{(z-3)\cancel{(z+\frac{1}{3})}}{(z+\frac{1}{3})^{\cancel{2}}}\,\eta(t)$$

$$= \frac{z-3}{z+\frac{1}{3}}\,\eta(t) = \frac{1-3z^{-1}}{1+\frac{1}{3}z^{-1}}\,\eta(t)$$

$$= \frac{(1-\frac{1}{3}z^{-1})}{(1+\frac{1}{3}z^{-1})}\,\boxed{\frac{1-3z^{-1}}{1-\frac{1}{3}z^{-1}}\,\eta(t)}$$

$:= w(t), \quad w(t) \sim WN(0, 9\cdot 36)$

$$\Longrightarrow y(t) = \frac{(1-\frac{1}{3}z^{-1})}{(1+\frac{1}{3}z^{-1})}\, w(t)$$

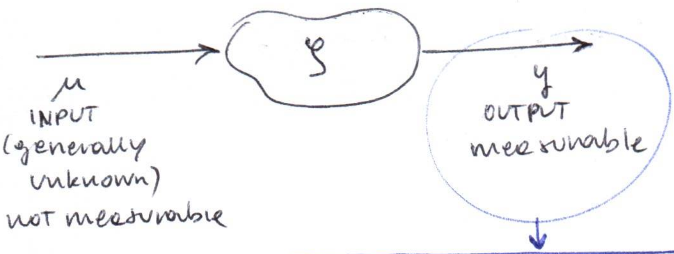
$$\Longrightarrow \hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)}\, y(t)$$

$$= \frac{(\cancel{1}-\frac{1}{3}z^{-1}) - (\cancel{1}+\frac{1}{3}z^{-1})}{1-\frac{1}{3}z^{-1}}\, y(t)$$

$$= \boxed{\frac{-\frac{2}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}\, y(t)}$$

PREDICTOR FROM THE DATA

IDENTIFICATION



- Date
- suppose a family model
- suppose input $\sim WN$
- $\hat{y}$
- ε
- $\hat{\theta}_n$, $\hat{\sigma}_n^2$

Starting from $\{y(1), y(2), \dots, y(N)\}$ we want to deduce the system date generator:

$$y(t) = \frac{C(z)}{A(z)} e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

We can only suppose a family models to approximate our system

We focus on a family model: (we suppose a model family)

$$\mathcal{M}: \; y(t) = \boxed{\frac{C_m(z,\theta)}{A_m(z,\theta)}} \xi(t) \qquad \xi(t) \sim WN(0, \sigma^2) \quad \text{(we suppose as an input a WN)}$$

CANONICAL REPRESENTATION

$\Longrightarrow$ AIM: find θ and σ^2 s.t. $y(t)$ (model) describes well the date

We use the PEM method:

PREDICTION ERROR METHOD (P.E.M.)

$$\mathcal{M} \longrightarrow \hat{\mathcal{M}}: \; \hat{y}(t|t-1,\theta) = \hat{W}(z) y(t)$$

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t|t-1,\theta)$$

prediction error

$$J_N(\theta) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon(t,\theta)^2 \longrightarrow \begin{cases} \hat{\theta}_N = \arg\min_\theta J_N(\theta) \\ \hat{\sigma}_N^2 = J_N(\hat{\theta}_N) \end{cases}$$

sample variance

What can we expect?

If $\xi \in \mathcal{M} \Longrightarrow \varepsilon(t, \hat{\theta}_N) = e(t) \quad (e(t) \sim WN) \; : \; \mathbb{E}[(\varepsilon(t,\hat{\theta}_N))^2] = \lambda^2$

This method depends on $N \Longrightarrow$ is not so good.

We can resolve this having $N \longrightarrow \infty$.

What happen if we have $\{y(1), \dots, y(\infty)\}$?

$$\underbrace{\begin{aligned} J_N(\theta) &= \frac{1}{N}\sum_{t=1}^{N} \varepsilon(t,\theta)^2 \\ \hat{\theta}_N &= \arg\min_\theta J_N(\theta) \\ \hat{\sigma}_N^2 &= J_N(\hat{\theta}_N) \end{aligned}}_{\text{performance indexes}} \quad \begin{aligned} &\xrightarrow{N\to\infty} \\ &\xrightarrow{N\to\infty} \\ &\xrightarrow{N\to\infty} \end{aligned} \quad \begin{aligned} \bar{J}(\theta) &= \mathbb{E}[\varepsilon(t,\theta)^2] \\ \theta^* &= \arg\min_\theta \bar{J}(\theta) \\ \sigma^{2*} &= \bar{J}(\theta^*) \end{aligned}$$

PROBABILISTIC INDEXES

(ASYMPTOTIC ANALYSIS)

$\mathcal{M}: \quad y(t) = \dfrac{C_m(z,\vartheta)}{A_m(z,\theta)} \xi(t) \qquad \xi(t) \sim WN(0, \sigma^2)$

1. $\hat{y}(t|t-1) = \left[\dfrac{C_m(z,\theta) - A_m(z,\theta)}{C_m(z,\theta)}\right] y(t)$ (simplified formula)

one step predictor

2. $\varepsilon(t,\theta) = y(t) - \hat{y}(t|t-1) = \left[1 - \dfrac{C_m(z,\theta) - A_m(z,\theta)}{C_m(z,\theta)}\right] y(t)$

predictor error

$= \dfrac{A_m(z,\vartheta)}{C_m(z,\vartheta)} y(t)$

$= \dfrac{A_m(z,\theta)}{C_m(z,\vartheta)} \dfrac{C(z)}{A(z)} e(t)$ $\qquad y(t) = \dfrac{C(z)}{A(z)} e(t)$

3. $\bar{J}(\theta) = \mathbb{E}\left[(\varepsilon(t,\theta))^2\right]$ variance of predictor error

$\hat{\theta}^* = \underset{\theta}{\arg\min}\ \bar{J}(\theta)$

$\sigma^{2*} = \bar{J}(\hat{\theta}^*)$

(EX.)

$S: \quad y(t) = e(t) + \frac{1}{2} e(t-1) \qquad e(t) \sim WN(0,1) \qquad MA(1)$

$\mathcal{M}: \quad y(t) = a\,y(t-1) + \xi(t) \qquad \xi(t) \sim WN(0,\sigma^2) \qquad AR(1)$

θ^*, σ^{2*} ?

1. $y(t) = \dfrac{1}{1-az^{-1}} \xi(t) \qquad |a| < 1$ (Hp.)

$\hat{y}(t|t-1) = a\,y(t-1)$

2. $\varepsilon(t,a) = y(t) - \hat{y}(t|t-1) = (1-az^{-1})\,y(t)$

$= (1-az^{-1})(1+\frac{1}{2}z^{-1})\,e(t)$ $\qquad y(t) = (1+\frac{1}{2}z^{-1})\,e(t)$

$= (1+\frac{1}{2}z^{-1} - az^{-1} - \frac{1}{2}az^{-2})\,e(t)$

$= e(t) + (\frac{1}{2}-a)\,e(t-1) - \frac{1}{2}a\,e(t-2)$

3. $\bar{J}(a) = \mathbb{E}\left[\varepsilon(t,a)^2\right]$

$= \mathbb{E}[e(t)^2] + (\frac{1}{2}-a)^2\,\mathbb{E}[e(t-1)^2] + \frac{1}{4}a^2\,\mathbb{E}[e(t-2)^2]$ $\qquad$ the double products semplificates

$= 1 + \frac{1}{4} + a^2 - a + \frac{1}{4}a^2$

$= \frac{5}{4} - a + \frac{5}{4}a^2$

$(\bar{J}(a))' = -1 + \frac{5}{4}\cdot 2a = 0 \implies a^* = \frac{2}{5}$ $\quad (< 1 \ (\text{Hp. } |a|<1))$

$\implies \sigma^{2*} = \bar{J}(a^*) = \frac{5}{4} - \frac{2}{5} + \frac{5}{4}\left(\frac{2}{5}\right)^2 = \frac{21}{20}$

$$\Longrightarrow \underbrace{\varepsilon(t, a^*)} = e(t) + \frac{1}{10} e(t-1) - \frac{1}{5} e(t-2)$$

is not a WN as the best result that we could obtain (this is obvious because the system is an MA(1) and we chose AR(1), so $S \notin \mathcal{M}$)

$\longrightarrow$ the prediction error is a colored noise, but it's not so bad (1. $\sigma^{2*} \simeq Var(e(t)) = 1$) 2. the coefficients that make the input not a WN ($\frac{1}{10}$ and $-\frac{1}{5}$) are "small")

(**EX.**)

$$S: \; y(t) = e(t) + \frac{1}{2} e(t-1) \quad ; \quad e(t) \sim WN(0,1) \qquad MA(1)$$

$$\mathcal{M}: \; y(t) = \eta(t) + b\,\eta(t-1) \quad ; \quad \eta(t) \sim WN(0, \lambda^2) \qquad MA(1)$$

$\vartheta^* = b^*?$ $\quad \lambda^{2*} = ?$

$\Longrightarrow$ since they're both MA(1): $b^* = \frac{1}{2}$, $\lambda^* = 1$

How to procede if we don't notice that?

Hp. $|b| < 1$

1. $$\hat{y}(t|t-1) = \frac{C_m(z) - A_m(z)}{C_m(z)} y(t) = \left[\frac{(1+bz^{-1}) - 1}{1+bz^{-1}}\right] y(t) = \frac{bz^{-1}}{1+bz^{-1}} y(t)$$

2. $$\varepsilon(t,b) = y(t) - \hat{y}(t|t-1) = \left(1 - \frac{bz^{-1}}{1+bz^{-1}}\right) y(t) = \frac{1}{1+bz^{-1}} y(t) = \frac{1}{1+bz^{-1}} \left(1 + \frac{1}{2} z^{-1}\right) e(t) = e(t) \quad \text{if } b = \frac{1}{2} \; ; \quad \mathbb{E}[\varepsilon(t,b)^2] = 1 = \mathbb{E}[e(t)^2]$$

If we don't notice that is optimal for $b = \frac{1}{2}$?

3. $$\mathbb{E}[\varepsilon(t,b)^2] = \mathbb{E}\left[\left(-b\,\varepsilon(t-1) + e(t) + \frac{1}{2} e(t-1)\right)^2\right] \qquad e(t) \sim WN(0,1)$$

$$= \mathbb{E}[b^2 \varepsilon(t-1)^2] + \mathbb{E}[e(t)^2] + \frac{1}{4}\mathbb{E}[e(t-1)^2] - \cancel{2b\,\mathbb{E}[\varepsilon(t-1)e(t)]} - b\,\mathbb{E}[\varepsilon(t-1)e(t-1)] + \cancel{\mathbb{E}[e(t)e(t-1)]}$$

$\varepsilon(t-1) = f(e(t-1), e(t-2), \dots)$ so they're $\perp\!\!\!\perp$

$$= b^2 \underbrace{\mathbb{E}[\varepsilon(t,b)^2]}_{\text{stochastic stationary process } (|b|<1)} + 1 + \frac{1}{4} - b\,\underbrace{\mathbb{E}[\varepsilon(t-1)e(t-1)]}$$

$$\mathbb{E}[\varepsilon(t-1)e(t-1)] = \mathbb{E}\left[\left(-b\,\varepsilon(t-2) + e(t-1) + \frac{1}{2} e(t-2)\right) e(t-1)\right] = -\cancel{b\,\mathbb{E}[\varepsilon(t-2)e(t-1)]} + \mathbb{E}[e(t-1)^2] + \cancel{\frac{1}{2}\mathbb{E}[e(t-1)e(t-2)]} = 1$$

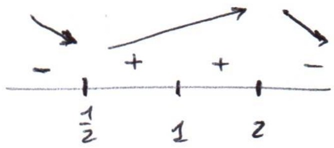
$$= b^2\,\mathbb{E}[\varepsilon(t,b)^2] + \frac{5}{4} + b$$

(3.) $\Longrightarrow \quad \bar{J}(b) = \dfrac{\frac{5}{4} - b}{1 - b^2} \qquad (|b| < 1)$

$$(\bar{J}(b))' = \frac{-(1-b^2) + (\frac{5}{4} - b)(2b)}{(1-b^2)^2} = \frac{-b^2 + \frac{5}{2}b - 1}{(1-b^2)^2}$$

$D > 0 \quad \forall b$

$N \geq 0 : \; -b^2 + \frac{5}{2}b - 1 \geq 0 \quad : \quad N \geq 0 \quad \frac{1}{2} < b < 2 \qquad (|b| < 1)$



$b^* = \frac{1}{2}$

$$\bar{J}(b^*) = \frac{\frac{5}{4} - \frac{1}{2}}{1 - (\frac{1}{2})^2} = 1 \qquad (= Var(e(t)))$$

$$\varepsilon(t) = -\tfrac{1}{2}\varepsilon(t-1) + e(t) + \tfrac{1}{2} e(t-1)$$

$$\varepsilon(t) + \tfrac{1}{2} z^{-1} \varepsilon(t) = e(t) + \tfrac{1}{2} z^{-1} e(t)$$

$$\Longrightarrow \quad \varepsilon(t) = \frac{\cancel{1 + \frac{1}{2} z^{-1}}}{\cancel{1 + \frac{1}{2} z^{-1}}} e(t)$$

(EX.)

$S : y(t) = 3e(t) + 9e(t-1) \quad , \quad e(t) \sim WN(0,1) \qquad MA(1)$

$\mathcal{M} : y(t) = \eta(t) + b\eta(t-1) \quad , \quad (t) \sim WN(0, \lambda^2) \qquad MA(1)$

$\vartheta^* = b^*$? $\quad \lambda^{2*}$?

Hp. $|b| < 1$

$S \in \mathcal{M}$ but S is not written in the canonical representation!

$$y(t) = (3 + 9z^{-1}) e(t) = 3(1 + 3z^{-1}) e(t) =$$

$$3 \, \frac{1 + 3z^{-1}}{1 + \frac{1}{3} z^{-1}} \left(1 + \tfrac{1}{3} z^{-1}\right) e(t) \quad := \xi(t)$$

$$\xi(t) = \underbrace{3 \, \frac{(1 + 3z^{-1})}{(1 + \frac{1}{3} z^{-1})}}_{\text{all pass filter}} (t) \longrightarrow \xi(t) \sim WN(0, 9 \cdot 3^2) = WN(0, 81)$$

$$\Longrightarrow \quad y(t) = \left(1 + \tfrac{1}{3} z^{-1}\right) \xi(t)$$

$$\Longrightarrow \quad b^* = \frac{1}{3} \quad , \quad \lambda^{2*} = 81$$

(Ex.)

$\boxed{02/04}$

$S: \; y(t) = e(t) + \frac{1}{2} e(t-1)$ $\qquad e(t) \sim WN(0, 1)$ $\qquad MA(1)$

$M: \; y(t) = \frac{1}{1 + az^{-1} + bz^{-2}} \eta(t)$ $\qquad \eta(t) \sim WN(0, \lambda^2)$ $\qquad AR(2)$

$\theta^* = \begin{bmatrix} a \\ b \end{bmatrix} ? \; , \; \lambda^{2*} = ?$

Family model $\neq$ system

Hp: $\; z^2 + az + b = 0 \qquad |z_{1,2}| < 1$

the solutions must be in the unit disc

1. $y(t) = -a y(t-1) - b y(t-2) + \eta(t)$

only unpredictable term at time $t-1$

$\hat{y}(t|t-1) = -a y(t-1) - b y(t-2)$

2. $\varepsilon(t) = y(t) - \hat{y}(t|t-1) = y(t) + a y(t-1) + b y(t-2)$

$$= (1 + az^{-1} + bz^{-2}) y(t)$$

$$= (1 + az^{-1} + bz^{-2})(1 + \tfrac{1}{2} z^{-1}) e(t)$$

$$= (1 + \tfrac{1}{2} z^{-1} + az^{-1} + \tfrac{1}{2} a z^{-2} + bz^{-2} + \tfrac{1}{2} bz^{-3}) e(t)$$

$$= \left(1 + (a + \tfrac{1}{2}) z^{-1} + (\tfrac{1}{2} a + b) z^{-2} + \tfrac{1}{2} bz^{-3}\right) e(t)$$

$$= e(t) + (a + \tfrac{1}{2}) e(t-1) + (\tfrac{1}{2} a + b) e(t-2) + \tfrac{1}{2} b e(t-3)$$

3. $\mathbb{E}[\varepsilon(t)^2] = \bar{J}(\theta) = \mathbb{E}[e(t)^2] + (a + \frac{1}{2})^2 \mathbb{E}[e(t-1)^2] + (\frac{1}{2} a + b)^2 \mathbb{E}[e(t-2)^2] + \frac{1}{4} \mathbb{E}[e(t-3)^2]$

$$= 1 + (a + \tfrac{1}{2})^2 + (\tfrac{1}{2} a + b)^2 + \tfrac{1}{4}$$

$$= \tfrac{5}{4} + a + ab + \tfrac{5}{4} a^2 + \tfrac{5}{4} b^2$$

$$\nabla \mathbb{E}[\varepsilon(t)^2] = 0$$

$$\begin{cases} \frac{\partial J}{\partial a} = 1 + b + \frac{5}{4} 2a = 0 \\ \frac{\partial J}{\partial b} = a + \frac{5}{4} 2b = 0 \end{cases} \Longrightarrow \begin{cases} a^* = -\frac{10}{21} \\ b^* = \frac{4}{21} \end{cases}$$

We check the Hp. : $\; z^2 - \frac{10}{21} z - \frac{4}{21} = 0 \Longrightarrow z_{1/2} \;\; \begin{matrix} \nearrow 0.735 \\ \searrow -0.259 \end{matrix}$

$$\underbrace{\mathbb{E}[\varepsilon(t)^2]}_{= \lambda^{2*}} = \frac{5}{4} - \frac{10}{21} + \left(-\frac{10}{21}\right)\left(\frac{4}{21}\right) + \frac{5}{4}\left(-\frac{10}{21}\right)^2 + \frac{5}{4}\left(\frac{4}{21}\right) = 1,011$$

not to different from the real variance of the process $\Rightarrow$ the mode is quite good

$$\varepsilon(t) = e(t) + \underbrace{\left(\frac{1}{2} - \frac{10}{21}\right)} e(t-1) + \underbrace{\left(\frac{4}{21} - \frac{5}{21}\right)} e(t-2) + \underbrace{\frac{2}{21}} e(t-3)$$

these coeffs. are less than 1 $\Rightarrow$ the error is just sligthly colored

Check if the model is good:

- λ^{2*} must be similar to $Var(e(t))$
- $\varepsilon(t)$ must be as similar as possible to a WN

(EX.)

$S: \ y(t) = e(t) + \frac{1}{3} e(t-1)$ $\quad e(t) \sim WN(0,1)$ $\quad$ MA (1)

$\mathcal{M}: \ y(t) = -a y(t-1) + \eta(t)$ $\quad \eta(t) \sim WN(0, \lambda^2)$ $\quad$ AR (1) (Hp. $|a| < 1$)

a^* that minimize $\bar{J}(a)$, λ^{2*} ?

1. $$\hat{y}(t|t-1) = \frac{C_m(z) - A_m(z)}{C_m(z)} y(t) = \frac{1-(1+az^{-1})}{1} y(t) = -a y(t-1)$$

2. $$\varepsilon(t) = y(t) - \hat{y}(t|t-1) = y(t) + a y(t-1) = (1+az^{-1}) y(t)$$

$$y(t) = \frac{C(z)}{A(z)} e(t) = \frac{1+\frac{1}{3}z^{-1}}{1} e(t)$$

$$= (1+az^{-1})\left(1+\frac{1}{3}z^{-1}\right) e(t) = \left(1+\left(\frac{1}{3}+a\right)z^{-1} + \frac{1}{3}az^{-2}\right) e(t)$$

3. $$\bar{J}(a) = \mathbb{E}[\varepsilon(t)^2] = 1 + \frac{1}{9} + a^2 + \frac{2}{3}a + \frac{1}{9}a^2$$

$$\frac{d\bar{J}}{da} = \frac{2}{3} + \frac{20}{9}a = 0 \implies a^* = -\frac{3}{10}$$

$$\lambda^{2*} = \bar{J}(a^*) = 1 + \left(\frac{1}{3} - \frac{3}{10}\right)^2 + \left(\frac{1}{9} \cdot \frac{9}{100}\right) = \frac{91}{90} \simeq 1.01$$

(similar to the variance of the WN that generates the data)

Alternative way: (YULE-WALKER PROCEDURE)

$$\varepsilon(t) = y(t) - \hat{y}(t|t-1) = y(t) + a y(t-1)$$

$$\mathbb{E}[\varepsilon(t)^2] = \mathbb{E}[(y(t) + a y(t-1))^2] = \underbrace{\mathbb{E}[y(t)^2]}_{\gamma_y(0)} + a^2 \underbrace{\mathbb{E}[y(t-1)^2]}_{\gamma_y(0)} + 2a \underbrace{\mathbb{E}[y(t)y(t-1)]}_{\gamma_y(1)}$$

$$= (1+a^2)\gamma_y(0) + 2a\gamma_y(1)$$

$$\frac{d\,\mathbb{E}[\varepsilon(t)^2]}{da} = 2a\gamma_y(0) + 2\gamma_y(1) = 0 \longrightarrow a^* = -\frac{\gamma_y(1)}{\gamma_y(0)}$$

$$\gamma_y(0) = \mathbb{E}[(e(t) + \tfrac{1}{3}e(t-1))^2] = \mathbb{E}[e(t)^2] + \tfrac{1}{9}\mathbb{E}[e(t-1)^2] = 1 + \tfrac{1}{9} = \tfrac{10}{9}$$

$$\gamma_y(1) = \mathbb{E}[y(t)\,y(t-1)] = \mathbb{E}[(e(t) + \tfrac{1}{3}e(t-1))(e(t-1) + \tfrac{1}{3}e(t-2))]$$

$$= \mathbb{E}[\tfrac{1}{3}e(t-1)^2] = \tfrac{1}{3}$$

all the other semplifies $(=0)$

$$\Longrightarrow \quad a^* = -\frac{1/3}{10/9} = -\frac{3}{10}$$

Remark: Identify a system $MA(1)$:

- family model $AR(1)$: $\lambda^{2*} = 1.05$
- family model $AR(2)$: $\lambda^{2*} = 1.011$ smaller : $AR(2)$ is better than $AR(1)$ (obvious since $AR(1) \subset AR(2)$)

If we increase the order of the family the result is better.

(EX.)

$$S: y(t) = \tfrac{1}{3}y(t-1) + u(t-1) + \eta(t) - \tfrac{1}{2}\eta(t-1)$$

exogenous input

$$\left.\begin{array}{l} \eta \sim WN(0,1) \\ u \sim WN(0,1) \end{array}\right\} \perp$$

$\Longrightarrow$ ARMAX $(1,1,1)$

$$M: y(t) = -a\,y(t-1) + b\,u(t-1) + e(t)$$

$e \sim WN(0, \lambda^2)$

$\Longrightarrow$ AR $(1,1)$

$$J(\theta) = \mathbb{E}[\varepsilon(t,\theta)^2], \quad \theta^* = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \lambda^{2*} ?$$

1. $$\boxed{\hat{y}(t+k|t) = \frac{\tilde{F}(z)}{C(z)}y(t) + \frac{B(z)E(z)}{C(z)}u(t)} :$$

$C(z) = A(z)E(z) + z^{-k}\tilde{F}(z)$

$E(z)$ = result of the long division for k-steps

$z^{-k}\tilde{F}(z)$ = remainder of the long division

So we compute the long division (1-step, ARX model):

$$\begin{array}{l|l} 1 & 1 + az^{-1} \\ \cline{2-2} 1 + az^{-1} & 1 \\ \hline / \quad -az^{-1} & \end{array}$$

$\Longrightarrow$

$C(z) = 1$

$B(z) = b$

$\tilde{F}(z) = -a$

$E(z) = 1$

$$\Longrightarrow \quad \hat{y}(t+1|t) = -a\,y(t) + b\,u(t)$$

2. $$\varepsilon(t,\theta) = y(t) - \hat{y}(t|t-1) = y(t) + a\,y(t-1) - b\,u(t-1)$$

3. $$\mathbb{E}[\varepsilon(t,\theta)^2] = \mathbb{E}[(y(t) + a\,y(t-1) - b\,u(t-1))^2]$$

$$= \mathbb{E}[y(t)^2] + a^2\mathbb{E}[y(t-1)^2] + b^2\mathbb{E}[u(t-1)^2] + 2a\,\mathbb{E}[y(t)\,y(t-1)] +$$

$$- 2b\,\mathbb{E}[y(t)\,u(t-1)] - 2ab\,\mathbb{E}[y(t-1)\,u(t-1)]$$

$$\mathbb{E}[\varepsilon(t)^2] = \gamma_y(0) + a^2\gamma_y(0) + b^2 + 2a\,\gamma_y(1) - 2b\,\mathbb{E}[y(t)\,u(t-1)] - 2ab\,\underbrace{\mathbb{E}[y(t-1)\,u(t-1)]}$$

$$\mathbb{E}[y(t-1)\,u(t-1)] =$$
$$= \mathbb{E}[(-a\,y(t-2) + b\,u(t-2) + e(t-1))\,u(t-1)] =$$
$$= -a\,\mathbb{E}[y(t-2)u(t-1)] + b\,\mathbb{E}[u(t-2)u(t-1)] + \mathbb{E}[e(t-1)u(t-1)]$$
$$y(t-2) = f(e(t-2), \dots) \qquad \perp \text{ and } \mathbb{E} = 0$$

$$\mathbb{E}[y(t)\,u(t-1)] =$$
$$= \mathbb{E}[(-a\,y(t-1) + b\,u(t-1) + e(t))\,u(t-1)] =$$
$$= -a\,\underbrace{\mathbb{E}[y(t-1)\,u(t-1)]}_{\text{checked on the right}} + \mathbb{E}[u(t-1)^2] + \underbrace{\mathbb{E}[e(t)\,u(t-1)]}_{\perp} =$$
$$= b$$

WE HAVE TO TAKE THE **REAL $y(t)$**, NOT THE MODEL:

- $\mathbb{E}[y(t-1)\,u(t-1)] = 0$
- $\mathbb{E}[y(t)\,u(t-1)] = 1$

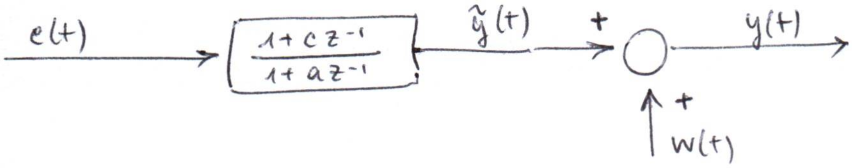
$$\mathbb{E}[\varepsilon(t)^2] = (1+a^2)\gamma_y(0) + 2a\,\gamma_y(1) + b^2 - 2b$$
$$\overset{!}{=} (1+a^2)\gamma_y(0) + 2a\,\gamma_y(1) + b^2 - 2b$$

$$\begin{cases} \dfrac{\partial \mathbb{E}[\varepsilon(t)^2]}{\partial a} = 2a\,\gamma_y(0) + 2\gamma_y(1) = 0 \\[2ex] \dfrac{\partial \mathbb{E}[\varepsilon(t)^2]}{\partial b} = 2b - 2\mathbb{E}[y(t)\,u(t-1)] = 0 \end{cases} \Longrightarrow \begin{cases} a^* = -\dfrac{\gamma_y(1)}{\gamma_y(0)} \\[2ex] b^* = \mathbb{E}[y(t)\,u(t-1)] \overset{!}{=} 1 \end{cases}$$

$$\mathbb{E}[y(t)\,y(t)] = \gamma_y(0) = \frac{69}{32}$$
$$\mathbb{E}[y(t)\,y(t-1)] = \gamma(1) = \frac{7}{32}$$
$$\mathbb{E}[y(t)\,u(t-1)] = \mathbb{E}\left[\left(\tfrac{1}{3}y(t-1) + u(t-1) + \eta(t) - \tfrac{1}{2}\eta(t-1)\right)u(t-1)\right]$$
$$\overset{!}{=} \tfrac{1}{3}\mathbb{E}[y(t-1)\,u(t-1)] + \underbrace{\mathbb{E}[u(t-1)^2]}_{1} + \underbrace{\cancel{\mathbb{E}[\eta(t)u(t-1)]}}_{\perp} - \tfrac{1}{2}\underbrace{\cancel{\mathbb{E}[\eta(t-1)u(t-1)]}}_{\perp}$$
$$\overset{!}{=} \tfrac{1}{3}\mathbb{E}\left[\left(\tfrac{1}{3}y(t-2) + u(t-2) - \eta(t-1) - \tfrac{1}{2}\eta(t-2)\right)u(t-1)\right]$$
$$\overset{!}{=} 1$$

$$\Longrightarrow \begin{cases} a^* = -\dfrac{\gamma_y(1)}{\gamma_y(0)} = -\dfrac{7}{69} \\[2ex] b^* = \underbrace{\mathbb{E}[y(t)\,u(t-1)]}_{\text{to find with the information about the \underline{real} system}} = 1 \end{cases}$$

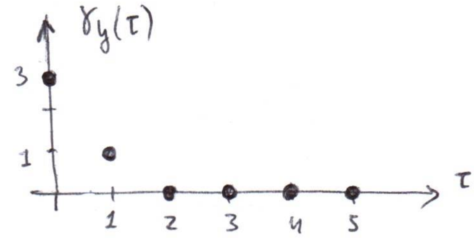
(Ex.)



$e(t) \sim WN(0, 1)$
$w(t) \sim WN(0, 1)$
$|a| < 1, \quad c \in \mathbb{R}$
$e(t) \perp w(t)$

1. Identify a, c s.t. $\gamma_y(\tau)$ is:

$$\gamma_y(\tau) = \begin{cases} 3 & \tau = 0 \\ 1 & |\tau| = 1 \\ 0 & |\tau| \geq 2 \end{cases}$$



The process is MA(1) (looking at the covariance function)

$\Longrightarrow a = 0$

$\Longrightarrow y(t) = (1 + cz^{-1}) e(t) + w(t)$

$$\mathbb{E}[(y(t))^2] = \mathbb{E}[(e(t) + c\, e(t-1) + w(t))^2]$$
$$= \mathbb{E}[e(t)^2] + c^2 \mathbb{E}[e(t-1)^2] + \mathbb{E}[w(t)^2] \quad \text{(the others are } = 0)$$
$$= 1 + c^2 + 1$$
$$= \gamma_y(0) = 3 \quad \Longrightarrow \quad c^2 = 1 \quad \Longrightarrow \quad c = \pm 1$$

$$\mathbb{E}[y(t)\, y(t-1)] = \gamma_y(1) = 1$$
$$= \mathbb{E}[(e(t) + c\, e(t-1) + w(t))(e(t-1) + c\, e(t-2) + w(t-1))]$$
$$= \mathbb{E}[c\, e(t-1)^2]$$
$$= c \quad \Longrightarrow \quad c = 1$$

2. Identify a, c if $\gamma_y(0) = 5, \quad \gamma_y(\tau) = 0 \quad \tau \neq 0$.

$\Longrightarrow y(t) \sim WN(0, 5)$

$\Longrightarrow y(t) = w(t) + \boxed{\frac{1 + cz^{-1}}{1 + az^{-1}}} e(t)$

this must be an ALL PASS filter

$\Longrightarrow c = \frac{1}{a}$

$\Longrightarrow \frac{1 + cz^{-1}}{1 + az^{-1}} e(t) \sim WN(0, c^2) \quad := \quad \eta \sim WN(0, c^2)$

$\Longrightarrow \mathbb{E}[y(t)^2] \overset{[\dots]}{=} \mathbb{E}[w(t)^2] + \mathbb{E}[\eta(t)^2] = 1 + c^2 = 5 \quad \Longrightarrow \quad c = \pm 2$

$a = \pm \frac{1}{2}$

(EX.) (Prediction)

$$y(t) = \frac{(1-2z^{-1})(1+\frac{1}{4}z^{-1})(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})}\, e(t) \qquad e(t) \sim WN(0,1)$$

Canonical? Predictors?

$$y(t) = \frac{\left(\frac{z-2}{z}\right)\left(\frac{z+\frac{1}{4}}{z}\right)\left(\frac{z-\frac{1}{4}z}{z}\right)}{\left(\frac{z-\frac{1}{2}}{z}\right)} = \frac{(z-2)}{z}\,\frac{(z+\frac{1}{4})}{z}\,\frac{(z-\frac{1}{4})}{\cancel{z}}\,\frac{\cancel{z}}{(z-\frac{1}{2})}\, e(t)$$

- monic
- same degree
- no semplifications
- zeros & poles?

We have a bad zero to remove (ALL-PASS filter)

$$y(t) = \frac{(1-2z^{-1})(1+\frac{1}{4}z^{-1})(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})}\, e(t) \quad := \eta(t) \sim WN(0, 2^2)$$

$$\Longrightarrow \quad y(t) = \left(1-\frac{1}{16}z^{-2}\right)\eta(t) \qquad \eta(t) \sim WN(0,4)$$

We use the long division:

$$\begin{array}{ccc|c}
1 & az^{-1} & -\frac{1}{16}z^{-2} & 1 \\
1 & & & \hline 1 + az^{-1} - \frac{1}{16}z^{-2} \\
\hline
/ & az^{-1} & -\frac{1}{16}z^{-2} & \\
 & az^{-1} & & \\
\hline
 & / & -\frac{1}{16}z^{-2} & \\
 & & -\frac{1}{16}z^{-2} & \\
\hline
 & & / &
\end{array}$$

$k=3$ steps
$k=2$ steps
$k=1$ steps

$\longrightarrow$ the predictor for $k=3$ is 0

$k=1$:
$$y(t) = \left(1 + \frac{az^{-1} - \frac{1}{16}z^{-2}}{1}\right)\eta(t) = \left(1 + az^{-1} - \frac{1}{16}z^{-2}\right)\eta(t)$$

$$\hat{y}(t|t-1) = \left(az^{-1} - \frac{1}{16}z^{-2}\right)\eta(t) \qquad \text{(from noise)}$$

$$= \left(az^{-1} - \frac{1}{16}z^{-2}\right)\cdot\left(\frac{1}{1-\frac{1}{16}z^{-2}}\right) y(t) \qquad \text{(from data)}$$

$$\hat{y}(t|t-1) = \frac{-\frac{1}{16}z^{-2}}{1-\frac{1}{16}z^{-2}}\, y(t) \qquad \text{(since } a=1\text{)}$$

$$\hat{y}(t|t-1) = \frac{1}{16}\hat{y}(t-2|t-3) - \frac{1}{16}y(t-2) \qquad \text{(time domain)}$$

$k = 2$: $y(t) = \left[(1 + a z^{-1}) + \frac{(-\frac{1}{16} z^{-2})}{1} \right] \cdot \eta(t)$

$$\hat{y}(t|t-2) = \frac{-\frac{1}{16} z^{-2}}{1} \eta(t) \qquad \text{(from noise)}$$

$$= \frac{-\frac{1}{16} z^{-2}}{1 - \frac{1}{16} z^{-2}} y(t) \qquad \text{(from data)}$$

$$\hat{y}(t|t-2) = \frac{1}{16} \hat{y}(t-2|t-4) - \frac{1}{16} y(t-2) \qquad \text{(time domain)}$$

$k = 3$: $y(t) = \left[(1 + a z^{-1} - \frac{1}{16} z^{-2}) + 0 \right] \eta(t)$

$$\hat{y}(t|t-3) = 0$$

Analysis of the prediction error?

$$\mathbb{E}[\varepsilon(t|t-k)^2] = \mathbb{E}[(E(z)\, \eta(t))^2]$$

$$= \begin{cases} \bullet\ k=1 : & \mathbb{E}[\varepsilon(t|t-1)^2] = \mathbb{E}[\eta(t)^2] = 4 \\ \bullet\ k=2 : & \mathbb{E}[\varepsilon(t|t-2)^2] = \mathbb{E}[((1 + \underbrace{a z^{-1}}_{a=0})\, \eta(t))^2] = 4 \\ \bullet\ k=3 : & \mathbb{E}[\varepsilon(t|t-3)^2] = \mathbb{E}[y(t)^2] = (1 + \frac{1}{16^2})\, 4 = 4.016 \end{cases}$$

(EX.) (Identification)

S : $y(t) = -0.5\, y(t-1) + \xi(t)$ $\qquad \xi(t) \sim WN(0, 1)$

$\mathcal{M}$: $y(t) = a\, y(t-2) + e(t)$ $\qquad e(t) \sim WN(0, \lambda^2)$

a in order to minimize the 1-step ahead predictor error?

$$\hat{y}(t|t-1) = a\, y(t-2)$$

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) - a\, y(t-2)$$

$$\mathbb{E}[\varepsilon(t|t-1)^2] = \mathbb{E}[(y(t) - a\, y(t-2))^2]$$

$$= \mathbb{E}[y(t)^2] + a^2\, \mathbb{E}[y(t-2)^2] - 2a\, \mathbb{E}[y(t)\, y(t-2)]$$

$$= \gamma_y(0) + a^2 \gamma_y(0) - 2a\, \gamma_y(2)$$

$$\frac{dJ}{da} = 2a\, \gamma_y(0) - 2\gamma_y(2) = 0 \longrightarrow a^* = \frac{\gamma_y(2)}{\gamma_y(0)}$$

$$\gamma_y(0) = \frac{1}{1 - a^2} = \frac{1}{1 - (0.5)^2} = \frac{4}{3}$$

EULER-WALKER FORMULA

a in $y(t)$ of S, not $y(t)$ of $\mathcal{M}$

$\gamma_y(1) = -0.5\, \gamma_y(0)$

$\gamma_y(2) = -0.5\,(-0.5\, \gamma_y(0)) = 0.25\, \gamma_y(0)$

$\Longrightarrow \quad a^* = 0.25$

(EX.)

$y(t) = 3 + v(t)$

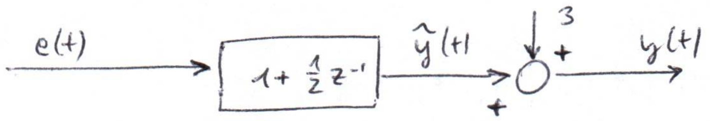
Find the optimal predictor if: $\quad (\hat{y}(t|t-k) = ?)$

1. $v(t) \sim WN(0,1)$:

$\Longrightarrow y(t) \sim WN(3,4)$

$\Longrightarrow \hat{y}(t|t-k) = \mathbb{E}[y(t)] = 3$

2. $v(t) = e(t) + \frac{1}{2} e(t-1) \qquad e(t) \sim WN(0,1)$:



We first consider the DEBIAS PROCESS. $\quad ; \quad y(t) = \tilde{y}(t) + 3$

$k = 1: \quad \hat{\tilde{y}}(t|t-1) = \dfrac{1/2}{1 + \frac{1}{2} z^{-1}}\, \tilde{y}(t-1)$

$$\hat{y}(t|t-1) = \frac{1/2}{1+\frac{1}{2}z^{-1}}\left(y(t-1) - 3\right) + 3$$

$$= \frac{1/2}{1+\frac{1}{2}z^{-1}}\, y(t-1) - \frac{3/2}{3/2} + 3$$

by theorem of the GAIN

$$= -\frac{1}{2}\hat{y}(t-1|t-2) + \frac{1}{2} y(t-1) + 2$$

ATTENTION when we don't have e will mean

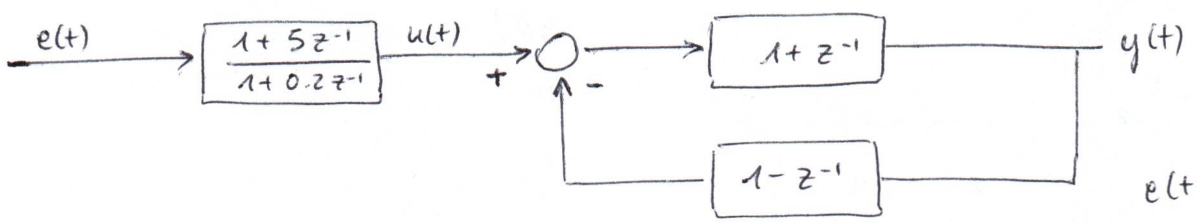
$k = 2: \quad \tilde{y}(t) = e(t) + \frac{1}{2} e(t-1)$

$(k \geq 2)$

$\Longrightarrow \hat{\tilde{y}}(t|t-2) = \mathbb{E}[\tilde{y}(t)] = 0$

$\Longrightarrow \hat{y}(t|t-2) = \mathbb{E}[y(t)] = 3$

(EX.) (Rias colto)



$e(t) \sim WN(0,1)$

$$u(t) = \frac{1+5z^{-1}}{1+\frac{1}{5}z^{-1}}\, e(t)$$

$u(t) \sim WN(0, 25)$
since it's an
All PASS filter

$$\Longrightarrow \Gamma_u(\omega) = 25 \quad \forall \omega$$

$$u(t) = \frac{(z+5)}{z} \, \frac{z \cdot 5}{(5z+1)}\, e(t) = \frac{(z+5)}{(z+\frac{1}{5})}\, e(t)$$

$$y(t) = (1-z^{-1})\left(u(t) - (1-z^{-1})\, y(t)\right)$$

$$= \frac{1-z^{-1}}{1+(1-z^{-1})^2}\, u(t) = \frac{z(z-1)}{2z^2-2z+1}\, u(t)$$

$$z_{1/2} = \frac{1}{2} \pm j\frac{1}{2}$$

$$= \frac{z(z-1)}{z\left(z-\frac{1}{2}-j\frac{1}{2}\right)\left(z-\frac{1}{2}+j\frac{1}{2}\right)}\, u(t)$$

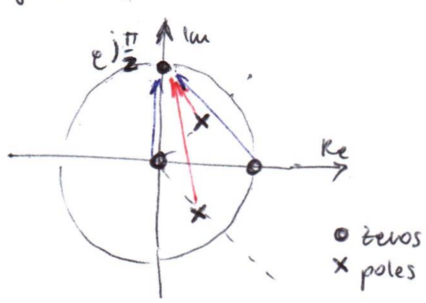
$$\Gamma_y(\omega) = |W(e^{j\omega})|^2\, \Gamma_u(\omega)$$

$$= \frac{|e^{j\omega}(e^{j\omega}-1)|^2}{|(e^{j\omega}-\frac{1}{2}-\frac{j}{2})|^2\, |e^{j\omega}-\frac{1}{2}+\frac{j}{2}|^2} \cdot \frac{25}{4}$$

We use the graphical approach:

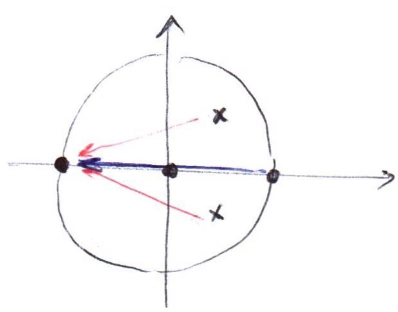
$\omega = 0$: $\Gamma_y(0) = 0$

$\omega = \frac{\pi}{2}$:



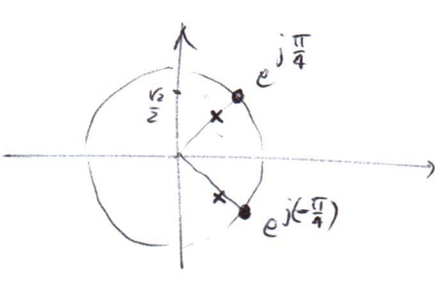
$$\Gamma_y\left(\frac{\pi}{2}\right) = \frac{25}{4} \cdot \frac{1 \cdot (\sqrt{2})^2}{\left(\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2}\right)^2 \left(\sqrt{(\frac{1}{2})^2+(\frac{3}{2})^2}\right)^2} = 10$$

$\omega = \pi$:



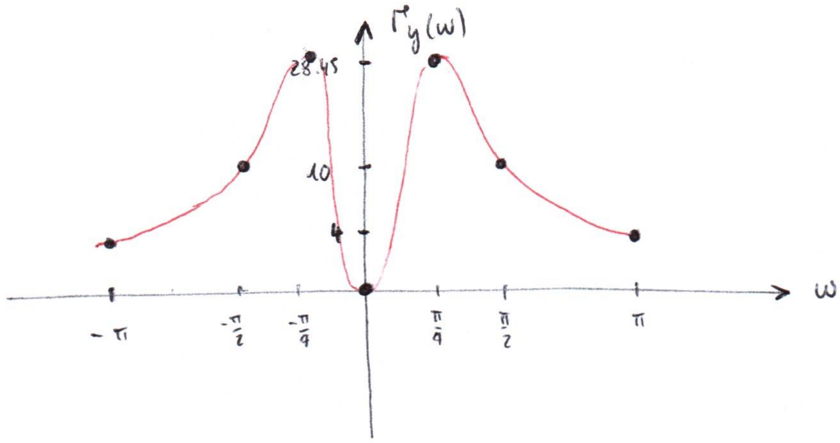
$$\Gamma_y(\pi) = \frac{25}{4} \cdot \frac{1^2 \cdot 2^2}{\left(\sqrt{(\frac{1}{2})^2+(\frac{3}{2})^2}\right)^{2\cdot 2}} = 4$$

Since we have poles in the unit disc we evaluate the closest points to these poles:



$$\Gamma_y\left(\frac{\pi}{4}\right) = \frac{25}{4} \cdot \frac{1^2 \cdot \left(\sqrt{(1-\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}\right)^2}{(1-\frac{\sqrt{2}}{2})^2 \left(\sqrt{(\frac{\sqrt{2}}{2}-\frac{1}{2})^2+(\frac{1}{2}+\frac{\sqrt{2}}{2})^2}\right)^2}$$

$$= \frac{25}{4} \, \frac{(2-\sqrt{2})}{(3-2\sqrt{2})} \approx 28{,}45$$



(last exercise on Beep)