BANACH SPACES (L^p/ℓ^p spaces)

Ex. 1 Let $X = \ell^p$ $(1 \le p < +\infty)$ and let: $S := \{ x \in \ell^p : \|x\| = 1 \}$
(remember that $x \in \ell^p \iff x = (x_k)_k$ and $\sum_{k=1}^{+\infty} |x_k|^p < +\infty$).
Prove that:
(i) S is **closed** (in ℓ^p)
(ii) S is **not compact** (by hand)

Solution: (i) We recall that S is closed if and only if $S = \bar{S}$, and this is equivalent to say that:

$\forall x_0 \in \ell^p$ for which $\exists (x_n)_n \subseteq S$ with $x_n \xrightarrow[n\to\infty]{} x_0$ $\left(\begin{array}{c}\text{hence}\\ x_0 \in \bar{S}\end{array}\right)$
$\Longrightarrow x_0 \in S$

Let then $x_0 \in \bar{S}$ and let $(x_n)_n \subseteq S$ s.t. $x_n \xrightarrow[n\to\infty]{} x_0$ $(\|x_n - x_0\| \xrightarrow[n\to\infty]{} 0)$.
Since $x_n \in S$ $\forall n \in \mathbb{N}$, we have $\|x_n\| = 1$ $\forall n \in \mathbb{N}$, hence:

$$\|x_0\| = \lim_{n\to\infty} \underbrace{\|x_n\|}_{=1\ \forall n} = 1 \Longrightarrow x_0 \in S$$

If $\|x_n - x_0\| \to 0$ then $\|x_n\| \to \|x_0\|$

Summing up we have proved that $\bar{S} \subseteq S \Longrightarrow S = \bar{S}$
(it is always true that $S \subseteq \bar{S}$).

(ii) To prove that $S \subseteq \ell^p$ is **not** compact, it sufficies to show (to find) a sequence in S which does not possess a converging subsequence.
To this end, we consider the sequence $(x_n)_n$, where

$$x_n^{(k)} = x_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

($(x_n)_n$ is a sequence of S, so every x_n is an element of S: hence, x_n is a sequence and $(x_n)_n$ is a sequence of sequences)

In other words, we have:
- $x_1 = (1, 0, \dots, 0, \dots)$
- $x_2 = (0, 1, \dots, 0, \dots)$
- $x_n = (0, 0, \dots, 1, \dots)$

↑ n-th element

! actually we can see $(x_n)_n$ as a sequence of functions where x_n is a function defined on $\mathbb{N}$ $(x_n(k) : \mathbb{N} \to \mathbb{R})$

It is easy to see that $(x_n)_n \subseteq S$: in fact, $\forall n \in \mathbb{N}$ we have:

$$\|x_n\| = \Big(\sum_{k=1}^{+\infty} \underbrace{|x_n^{(k)}|^p}_{=0 \text{ if } k\neq n} \Big)^{1/p} = \big(\underbrace{|x_n^{(n)}|^p}_{=1} \big)^{1/p} = 1$$

On the other hand $(x_n)_n$ cannot admit converging sub-sequences: indeed $\forall n \in \mathbb{N}$, $n \neq m$ we have:

$$\|x_n - x_m\|^p = d(x_n, x_m)^p = \sum_{k=1}^{+\infty} |x_n^{(k)} - x_m^{(k)}|^p = |x_n^{(n)}|^p + |x_m^{(m)}|^p = 2$$

$x_n^{(k)} = 1 \Leftrightarrow k = n$, $x_m^{(k)} = 1 \Leftrightarrow m = k$

Hence, the mutual distance between two distinct elements of the sequence $(x_n)_n$ is constant, and it does not shrink to 0.

Remark: The very same argument can be used also in the case $p = +\infty$, the unique difference is that:

$$\|x\| = \sup_{k \in \mathbb{N}} |x_k| \qquad \forall x \in \underbrace{\ell^\infty}_{\{x=(x_k)_k:\ \sup_k |x_k| < \infty\}}$$

Ex. 2 Let $X = \ell^p$ $(1 \le p \le +\infty)$ and let $(x_n)_n \subseteq \ell^p$. Moreover let $x_0 \in \ell^p$ be such that $x_n \xrightarrow[n\to\infty]{} x_0$.

Prove that:

$$\lim_{n\to\infty} x_n^{(k)} = x_0^{(k)} \qquad \forall k \in \mathbb{N}$$

In other words: convergence in $\ell^p \Longrightarrow$ pointwise convergence.

$(x_n)_n =$ sequence of sequences which converges to $x_0 =$ sequence in ℓ^p.

Solution: $(1 \le p < +\infty)$.

Let $(x_n)_n \subseteq \ell^p$ and let $x_0 \in \ell^p$ be such that $x_n \xrightarrow[n\to\infty]{} x_0$, that is: $\|x_n - x_0\| \xrightarrow[n\to\infty]{} 0$.

Then $\forall k \in \mathbb{N}$ we have:

$$0 \le |x_n^{(k)} - x_0^{(k)}| \le \left(\sum_{j=1}^{+\infty} |x_n^{(j)} - x_0^{(j)}|^p\right)^{1/p} = \underbrace{\|x_n - x_0\|}_{\longrightarrow 0}$$

Hence, by the comparison theorem:

$$|x_n^{(k)} - x_0^{(k)}| \xrightarrow[n\to\infty]{} 0$$

For the case $p = +\infty$ the proof is analogous (just replace $\sum_j |x_n^{(j)} - x_0^{(j)}|$ with $\sup_j |x_n^{(j)} - x_0^{(j)}|$)

Remark: In general, the pointwise convergence does **not** imply the ℓ^p-convergence. For example if $(x_n)_n \subseteq \ell^p$ is given by

$$x_n^{(k)} = \begin{cases} 1 & k = n \\ 0 & k \ne n \end{cases}$$

then $x_n^{(k)} \xrightarrow[n\to\infty]{} 0$ (since $x_n^{(k)} = 0$ $\forall n > k$), but:

$$\|x_n - 0\| = \left(\sum_{k=1}^{+\infty} |x_n^{(k)}|^p\right)^{1/p} = 1 \qquad \forall n \in \mathbb{N}$$

($\Longrightarrow x_n \not\to 0$ in ℓ^p)

Ex. 3 In the space ℓ^2, consider the sequence $(x_n)_n$ given by:

$$x_n(k) = x_n^{(k)} = \frac{1}{n+k} \qquad \forall k, n \in \mathbb{N}.$$

a sequence $(x_n)_n \subseteq \ell^2$ means that $\forall n$ we are giving a sequence $(x_n^{(k)})_k$ in $\mathbb{R}$

In other words, the sequence $(x_n)_n$ is defined as:

$$x_1 = \left(\frac{1}{1+k}\right)_{k\in\mathbb{N}} = \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k+1}, \dots\right)$$

$$x_2 = \left(\frac{1}{2+k}\right)_{k\in\mathbb{N}} = \left(\frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{k+2}, \dots\right)$$

$$\vdots$$

Prove that: $x_n \xrightarrow[n\to\infty]{} 0$ in ℓ^2.

Solution: We first check that $x_n \in \ell^2 \quad \forall n \in \mathbb{N}$.
To this end it suffices to notice that: $\forall n \in \mathbb{N}$ fixed:

$$\|x_n\|^2 = \sum_{k=1}^{+\infty} |x_n^{(k)}|^2 = \sum_{k=1}^{+\infty} \frac{1}{(n+k)^2} \sim \sum_{k=1}^{+\infty} \frac{1}{k^2} < \infty$$

← not required by the exercise

Hence: $x_n \in \ell^2 \ \forall n \in \mathbb{N}$.
As for the convergence of x_n to 0, we have:

$$\|x_n - 0\|^2 = d(x_n, 0)^2 = \sum_{k=1}^{\infty} |x_n^{(k)} - 0|^2 = \sum_{k=1}^{\infty} \frac{1}{(n+k)^2} = \sum_{h=n+1}^{\infty} \frac{1}{h^2} \quad \forall n \in \mathbb{N}$$

$n+k=h$
$k=h-n$

As a consequence, since $\sum_{h=1}^{+\infty} \frac{1}{h^2} < \infty$ then:

$$\lim_{n\to\infty} \|x_n\|^2 = \lim_{n\to\infty} \sum_{h=n+1}^{+\infty} \frac{1}{h^2} = 0 = \lim_{n\to\infty} \Big(\underbrace{\sum_{h=1}^{\infty} \frac{1}{h^2}}_{:=\lambda} - \underbrace{\sum_{h=1}^{n} \frac{1}{h^2}}_{\xrightarrow[n\to\infty]{} \lambda} \Big)$$

and this prove that: $x_n \xrightarrow[n\to\infty]{} 0$ in ℓ^p.

Ex. 4 Given $\alpha > 0$, consider the sequence $(x_n)_n$ defined by

$$x_n(k) = x_n^{(k)} = \begin{cases} \frac{1}{k^\alpha} & 1 \le k \le n \\ 0 & k > n \end{cases}$$

In other words, $(x_n)_n$ is given by:

- $x_1 = \left(\frac{1}{1^\alpha}, 0, 0, ..\right) = (1, 0, 0, ..)$
- $x_2 = \left(\frac{1}{1^\alpha}, \frac{1}{2^\alpha}, ..\right) = (1, \frac{1}{2^\alpha}, 0, 0, ..)$.

- find pointwise convergent element x_0
- check when $x_0 \in \ell^p$
- prove that $x_n \to x_0$ in ℓ^p

Determine the values of $1 \le p \le \infty$ for which $(x_n)_n \subseteq \ell^p$.
For these values of p study the convergence of $(x_n)_n$ in ℓ^p.

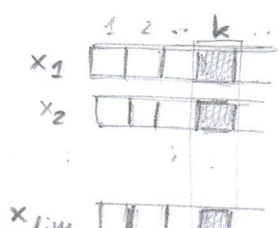
Solution: We claim that $(x_n)_n \subseteq \ell^p \quad \forall 1 \le p \le \infty$.
Indeed, since $x_n^{(k)} \neq 0$ only for a finite number of indeces (for $1 \le k \le n$), we have:

- $\sum_{k=1}^{+\infty} |x_n^{(k)}| = \sum_{k=1}^{n} |x_n^{(k)}| = \sum_{k=1}^{n} \left(\frac{1}{k}\right)^\alpha < \infty \qquad (1 \le p < \infty)$
- $\sup_{k\in\mathbb{N}} |x_n^{(k)}| = \sup_{k=1,..,n} |x_n^{(k)}| = \max_{k=1,..,n} |x_n^{(k)}| = \max_{k=1,..,n} \left(\frac{1}{k}\right)^\alpha < \infty \qquad (= 1)$

Hence $(x_n)_n \subseteq \ell^p \quad \forall 1 \le p \le \infty$.

(second part) → we use **Ex. 2**
To study the convergence of $(x_n)_n$ in ℓ^p we first observe that $(x_n)_n$ is pointwise convergent: in fact

$$(*) \qquad \exists \lim_{n\to\infty} x_n^{(k)} = \left(\frac{1}{k}\right)^\alpha \qquad \forall k \in \mathbb{N}$$



by Ex. 2, if x_n converges to something in ℓ^p then the limit is the pointwise limit
⟹ we already have the candidate limit for the convergence in ℓ^p

we find the candidate limit through pointwise convergence

- To prove (*) it sufficies to prove that, if $k \in \mathbb{N}$ is fixed, then for every $n \geq k$, we have:

somehow k plays the role of x in the case of functions (we can see the series as: $x_n : \mathbb{N} \to \mathbb{R}$)

$$x_n(k) = x_n^{(k)} = \left(\frac{1}{k}\right)^\alpha$$

$$\Longrightarrow \exists \lim_{n\to\infty} x_n^{(k)} = \left(\frac{1}{k}\right)^\alpha$$

Summing up, $(x_n)_n$ is pointwise convergent to:

$$x_0 = \left(\frac{1}{k^\alpha}\right)_{k\in\mathbb{N}}$$

(more precisely: $\begin{cases} x_0 : \mathbb{N} \to \mathbb{R} \\ x_0(k) = x_0^{(k)} = \left(\frac{1}{k}\right)^\alpha \end{cases}$)

Hence, if $(x_n)_n$ is convergent in ℓ^p $(1 \leq p \leq \infty)$, then its limit must be x_0 (since ℓ^p convergence implies pointwise convergence).

we check when the candidate limit (x_0) is in ℓ^p

- We then start by finding the values of $\alpha > 0$ for which

$$x_0 = \left(\frac{1}{k^\alpha}\right)_{k\in\mathbb{N}} \in \ell^p.$$

If $1 \leq p < \infty$, we have:

$$\sum_{k=1}^\infty |x_0^{(k)}|^p = \sum_{k=1}^\infty \frac{1}{k^{\alpha p}} < \infty \iff \alpha p > 1 \iff \alpha > \frac{1}{p}$$

(generalized armonic series)

If, instead, $p = +\infty$, then:

$$\sup_{k\in\mathbb{N}} |x_0^{(k)}| = \sup_{k\in\mathbb{N}} \left(\frac{1}{k^\alpha}\right) = 1 < \infty$$

Summing up, we have:

- $x_0 \in \ell^p$ $(1 \leq p < \infty) \iff \alpha > \frac{1}{p}$ (1)
- $x_0 \in \ell^\infty \iff \forall \alpha > 0$ (2)

we check the ℓ^p convergence (only for cases in which $x_0 \in \ell^p$)

- In view of (1) and (2) let us check that:

(1)' $x_n \xrightarrow[n\to\infty]{} x_0$ in ℓ^p $(1 \leq p < \infty)$ if $\alpha > \frac{1}{p}$

(2)' $x_n \xrightarrow[n\to\infty]{} x_0$ in ℓ^∞ $\forall \alpha > 0$

Proof of (1)':

By def., $1 \leq p < \infty$ and $\alpha > \frac{1}{p}$, we have:

$$\|x_n - x_0\|^p = \sum_{k=1}^\infty |\underbrace{x_n^{(k)}}_{\frac{1}{k^\alpha} \text{ if } k \leq n} - \underbrace{x_0^{(k)}}_{\frac{1}{k^\alpha}\ \forall k}|^p$$

$$= \sum_{k=n+1}^\infty \frac{1}{k^{\alpha p}} \xrightarrow[n\to\infty]{} 0 \quad \text{since } \sum_{k=1}^\infty \frac{1}{k^{\alpha p}} < \infty$$

Hence: $x_n \xrightarrow[n\to\infty]{} x_0$ in ℓ^p

Proof of (2)':

By def., for every $\alpha > 0$ we have:

$$\|x_n - x_0\|_\infty = \sup_{k\in\mathbb{N}} \underbrace{|x_n^{(k)} - x_0^{(k)}|}_{= 0 \text{ if } 1 \leq k \leq n}$$

$$= \sup_{k \geq n+1} \left(\frac{1}{k}\right)^\alpha$$

$$= \frac{1}{(n+1)^\alpha} \xrightarrow[n\to\infty]{} 0 \quad \text{since } \alpha > 0 \text{ (and } \left(\tfrac{1}{k^\alpha}\right)_{k\in\mathbb{N}} \text{ is decreasing)}$$

In the case we cannot compute the sup explicitly: 18/11

[Notice that: $\lim_{n\to\infty}\left(\sup_{k\geq n+1}\frac{1}{k^\alpha}\right) = \limsup_{n\to\infty}\frac{1}{n^\alpha} = 0$

$= \lim_{n\to\infty}\frac{1}{n^\alpha} = 0$]

Hence: $x_n \xrightarrow[n\to\infty]{} x_0$ in ℓ^∞ $\forall \alpha > 0$

Remark: We stress that the convergence of x_n to x_0 in ℓ^p $(1\leq p\leq\infty)$ is not a consequence of the fact that:

$$x_n \xrightarrow[n\to\infty]{} x_0 \quad \text{pointwise and} \quad x_0 \in \ell^p.$$

This is only a necessary condition*, but the ℓ^p-convergence must be proved by showing that $\|x_n - x_0\|_{\ell^p} \xrightarrow[n\to\infty]{} 0$

It can happen that we have pointwise convergence to x_0, $x_0\in\ell^p$ but there is no ℓ^p convergence.

* since the ℓ^p convergence implies the pointwise convergence

LP SPACES

Ex. 1 Let $(X, \mathcal{A}, \mu)$ be a measure space and let:

$$f \in L^p(X) \cap L^q(X)$$

for some $1\leq p < q < \infty$.

Prove that $f\in L^r(X)$ for every $p<r<q$.

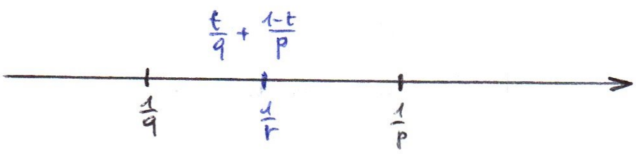
Solution: The conclusion is obvious if $\mu(X)<\infty$, since in this case we know that:

$$L^q(X) \subseteq L^r \qquad (\text{as } q>r)$$

If, instead, $\mu(X)=\infty$, we procede as follows.

First of all, since $p<r<q$ we have:

$$\frac{1}{q} < \frac{1}{r} < \frac{1}{p} \implies \exists t\in(0,1): \frac{1}{r} = \frac{t}{q} + \frac{1-t}{p}$$



Hence, we can write:

$$\int_X |f|^r\,d\mu = \int_X |f|^{rt}|f|^{r(1-t)}\,d\mu = \int_X |f|^{q\cdot\frac{rt}{q}}|f|^{p\frac{r(1-t)}{p}}\,d\mu = \int_X |f|^{p\alpha}|f|^{q\beta}\,d\mu$$

25/11

where $\alpha = \frac{r(1-t)}{p}$, $\beta = \frac{rt}{q}$.

We now aim to apply Holder's inequality with exponents:

$\frac{1}{\alpha}$ and $\frac{1}{\beta}$. (as p and q)

To this end we observe that: (we need to prove $p,q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$)

(1) $\frac{1}{\beta} > 1$, since $q>r>rt$

(2) $\frac{1}{\alpha} = \frac{p}{r(1-t)} > 1$ since $\frac{1}{r} = \frac{t}{q}+\frac{1-t}{p} \implies \frac{1}{r} > \frac{1-t}{p}$

(3) $\frac{1}{\left(\frac{1}{\alpha}\right)} + \frac{1}{\left(\frac{1}{\beta}\right)} = \alpha+\beta = \frac{r(1-t)}{p} + \frac{rt}{q} = r\underbrace{\left(\frac{1-t}{p}+\frac{t}{q}\right)}_{1/r} = 1$

We can then apply Holder's inequality obtaining:

$$\int_X |f|^r d\mu = \int_X |f|^{p\alpha} |f|^{q\beta} d\mu$$
$$\leq \left(\int_X |f|^p d\mu\right)^{\alpha} \left(\int_X |f|^q d\mu\right)^{\beta}$$
$$= \left(\int_X |f|^p d\mu\right)^{\frac{r(1-t)}{p}} \left(\int_X |f|^q d\mu\right)^{\frac{rt}{q}}$$
$$= \|f\|_{L^p}^{r(1-t)} \|f\|_{L^q}^{rt} < \infty \implies f \in L^r(X)$$

$\int_X \left||f|^{p\alpha}|f|^{q\beta}\right| d\mu \leq \left(\int_X |f|^{p\alpha \frac{1}{\alpha}} d\mu\right)^{\alpha} \left(\int_X |f|^{q\beta\frac{1}{\beta}} d\mu\right)^{\beta}$

$(\|fg\|_1 \leq \|f\|_p \|g\|_q)$

From this we also obtain the interpolation inequality:

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{1-t} \|f\|_{L^p}^{t} \quad \text{where} \quad \frac{1}{r} = \frac{t}{q} + \frac{1-t}{p}$$

t interpolation parameter

Remark: If $(X, \mathcal{A}, \mu)$ is a measure space and $f \in L^p(X) \cap L^\infty(X)$ (we are choosing $q = \infty$), then one can prove that:

$$f \in L^q(X) \qquad \forall q > p.$$

Indeed if $q > p$ is fixed, we have:

$$\int_X |f|^q d\mu = \int_X |f|^p |f|^{q-p} d\mu \qquad (q-p > 0)$$
$$\leq \|f\|_{L^\infty}^{q-p} \int_X |f|^p d\mu$$
$$= \|f\|_{L^\infty}^{q-p} \|f\|_{L^p}^{p} < \infty \implies f \in L^q(X)$$

From this we obtain the interpolation inequality:

$$\|f\|_{L^q} \leq \|f\|_{L^\infty}^{1-\frac{p}{q}} \|f\|_{L^p}^{\frac{p}{q}}$$

Ex. 2 Consider the sequence of functions: $\{f_n\}_n$ where:

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = \sqrt{n}\, x^n$$

Study the convergence of $\{f_n\}_n$ in L^p spaces. $(1 \leq p \leq \infty)$ *

Solution: We first observe that, since $f_n \in C([0,1])$ and $I = [0,1]$ is a compact interval (hence $\lambda(I) < \infty$), we have:

$$f_n \in L^\infty(I) \subseteq L^p(I) \quad \forall\, 1 \leq p \leq \infty$$

(since I has finite measure we have the inclusion of all L^p)

recall to 18/10

Moreover, since the L^p-convergence implies the pointwise convergence a.e.**, to find the "candidate L^p-limit" of $\{f_n\}_n$ we study the pointwise limit of the sequence.

We have:

- if $x = 0$ then $f_n(x) = f_n(0) = 0 \quad \forall n \in \mathbb{N}$

  $\implies f_n(0) \xrightarrow[n\to\infty]{} 0$

- if $0 < x < 1$ then $f_n(x) = \sqrt{n}\, x^n = n^{\frac{1}{2}} x^n$, thus, since $x^n \xrightarrow[n\to\infty]{} 0$ faster than any power of n, we get:

  $\implies f_n(x) \xrightarrow[n\to\infty]{} 0$

- if $x = 1$, then $f_n(x) = f_n(1) = \sqrt{n}$

  $\implies f_n(1) \xrightarrow[n\to\infty]{} +\infty$

Summing up, we can say that for every $0 \leq x \leq 1$ we have:

$$f_n(x) \xrightarrow[n\to\infty]{} f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ +\infty & x = 1 \end{cases}$$

! *The problem makes sense if we recognize that the sequence belongs to all L^p spaces (and the limit too), so we need to prove it

** up to a subsequence

In particular, $f_n \xrightarrow[n\to\infty]{} 0$ pointwise for a.a. $x \in [0,1]$. $(\forall x \neq 1)$ 25/11

We can conclude that, if $\{f_n\}_n$ has a limit in L^p, then this limit must be 0:

- Hence, we are left to check if:

$$f_n \xrightarrow[n\to\infty]{} 0 \text{ in } L^p(I) \underset{p<\infty}{\iff} \lim_{n\to\infty} \int_0^1 f_n^p\,dx = 0 \qquad (f_n \geq 0)$$

To study the above limit one can try to use the dominated convergence theorem (by finding $g \in L^1(I)$ such that $f_n^p \leq g$), but it is simpler to compute explicitly the integral: $\forall p<\infty$:

$$\int_0^1 f_n^p\,dx = n^{p/2}\int_0^1 x^{np}\,dx = n^{p/2}\left[\frac{x^{np+1}}{np+1}\right]_0^1 = \frac{n^{p/2}}{np+1}$$

As a consequence we obtain:

$$\lim_{n\to\infty}\int_0^1 f_n^p\,dx = \lim_{n\to\infty}\frac{n^{p/2}}{np+1} = \begin{cases} 0 & \text{if } p/2 < 1 \\ +\infty & \text{if } p/2 > 1 \\ \frac{1}{p} & \text{if } p/2 = 1 \end{cases} = \begin{cases} 0 & \text{if } p < 2 \\ +\infty & \text{if } p > 2. \\ \frac{1}{2} & \text{if } p = 2 \end{cases}$$

Summing up, if $1 \leq p < \infty$ we have:

$$f_n \xrightarrow[n\to\infty]{} 0 \text{ in } L^p \iff 1 \leq p < 2.$$

- Finally, we study the convergence in $L^\infty(I)$.

To this end we observe that, since $f_n \in C(I)$ we have: I compact !! 29/11

$$\|f_n\|_{L^\infty} = \operatorname*{ess\,sup}_I |f_n| = \sup_I |f_n| = \sup_I f_n \geq f_n(1) = \sqrt{n}$$

the continuity plays an essential role: without continuity ess sup $\neq$ sup and we cannot say ess sup $|f_n| \geq f_n(1)$

From this we get:

$$\|f_n\|_{L^\infty} \xrightarrow[n\to\infty]{} \infty$$

so that $\{f_n\}_n$ does not converges to 0 in $L^\infty(I)$.

! **Remark:** (Trick) useful if we cannot do $\|f_n\|_{L^\infty} \geq (\dots) \to \infty$ (as above)

The fact that $\{f_n\}_n$ does not converge to 0 in $L^\infty(I)$ can be deduced from the following argument: since $\lambda(I)$ is finite $(\lambda(I)<\infty)$ we know that $L^\infty(I) \subseteq L^p(I) \subseteq L^q(I)$ $\forall 1 \leq q \leq p \leq \infty$ and the inclusions are continuous, that is:

$$(f_n \longrightarrow f_0 \text{ in } L^\infty(I)) \implies (f_n \longrightarrow f_0 \text{ in } L^p(I) \quad \forall 1\leq p<\infty)$$

Since $f_n \longrightarrow 0$ in $L^p(I)$ only if $1 \leq p < 2$, we deduce that $\{f_n\}_n$ cannot converge to 0 in $L^\infty(I)$.

Ex. 3 Prove that the sequence $f_n : [0,1] \to \mathbb{R}$:

$$f_n(x) := \frac{\sin(nx)\,e^{-nx}}{\sqrt[3]{x}}$$

converges to $f \equiv 0$ in $L^p([0,1])$ $\forall 1 \leq p < 3$.

Solution: • We first observe that $\{f_n\}_n \subseteq L^p(I)$ for all $1 \le p \le \infty$.
In fact $I = [0,1]$ is compact and so $\lambda(I) < \infty$.
On the other hand, $f_n \in C(I)$, since:

$$\exists \lim_{x \to 0^+} f_n(x) = 0 \qquad \forall n \in \mathbb{N}$$

$$\left(f_n \underset{x \to 0^+}{\sim} \frac{nx}{\sqrt[3]{x}} = n\sqrt[3]{x^2} \right)$$

Hence, as before: $\{f_n\}_n \subseteq L^\infty(I) \subseteq L^p(I) \quad \forall 1 \le p < \infty$.

• To prove that $f_n \xrightarrow[n\to\infty]{} 0$ in $L^p(I)$ (for $1 \le p < 3$) we try to apply the **D**ominated **C**onvergence **T**heorem, since $\|f_n\|_{L^p}$ cannot be explicitly computed.
To apply the DCT, we need to check that:

(1) $f_n(x) \xrightarrow[n\to\infty]{} 0$ for a.e. $x \in I = [0,1]$

(2) $\exists g \in L^1(I)$ s.t. $|f_n|^p \le g$ a.e. in I, $\forall n \in \mathbb{N}$.

It is like applying DCT to $|f|^p$:
$|f_n|^p \xrightarrow[n\to\infty]{} 0$ pointwise (by point (1)) and if we think to the sequence $|f_n|^p$ as $|h_n|$ then $h_n \in L^1$, because $f_n \in L^p$, and it is dominated by a function g $\perp\!\!\!\perp n$, $g \in L^1$

As for (1) we see that:

• if $x = 0$, then: $f_n(x) = f_n(0) = 0 \quad \forall n \in \mathbb{N}$
$\Longrightarrow f_n(0) \xrightarrow[n\to\infty]{} 0$

• if $0 < x \le 1$, then:

$$|f_n(x)| \le \frac{e^{-nx}}{\sqrt[3]{x}} \xrightarrow[n\to\infty]{} 0$$

Hence: $f_n \xrightarrow[n\to\infty]{} 0$ pointwise in $I = [0,1]$.
As for (2) we notice that, if $1 \le p < 3$, we have the estimate:

$$|f_n(x)|^p = \frac{|\sin(nx)|^p e^{-npx}}{\sqrt[3]{x^p}} \le \frac{e^{-npx}}{x^{p/3}} \le \frac{1}{x^{p/3}} =: g(x) \qquad (*)$$

and $g(x) = \frac{1}{x^{p/3}} \in L^1([0,1])$ since $\frac{p}{3} < 1 \iff p < 3$.
In view of (1) and (2) we can then apply the DCT obtaining:

$$\lim_{n\to\infty} \|f_n\|_{L^p}^p = \underbrace{\lim_{n\to\infty} \int_0^1 |f_n|^p \, dx}_{= \int_0^1 \left(\lim_{n\to\infty} |f_n|^p \right) dx} = 0$$

By definition, this shows that $f_n \xrightarrow[n\to\infty]{} 0$ in $L^p(I)$ if $1 \le p < 3$.

Remark: It can be proved that $f_n \not\to 0$ in $L^p(I)$ if $p \ge 3$.
In other words, there do not exist a dominating function $g' \in L^1(I)$ for $|f_n|^p$ when $p \ge 3$.

(*) still holds even if $p \ge 3$, however if $p \ge 3$ then $g(x) = \frac{1}{x^{p/3}} \notin L^1([0,1])$.
One may wonder: if it possible to find another g, say g', such that $|f_n(x)|^p \le g'(x) \ \forall n$ and such that $g' \in L^1([0,1])$ for $p \ge 3$? **No.**
The point is not that we cannot find it, the point is that it cannot exist.
(since $f_n \not\to 0$ in $L^p(I)$, $p \ge 3$)

OPERATORS BETWEEN BANACH SPACES

Ex. 1 Consider the map $T : \ell^2 \to \ell^2$ defined as:

$$T(x) := \left\{ \frac{n}{1+n^2} x_n \right\}_{n \in \mathbb{N}} \subseteq \ell^2$$

Prove that T is linear and continuous and that $\|T\|_{\mathcal{L}} = \frac{1}{2}$.

Solution: • We first prove that T is well-posed in the sense that:

$$T(x) \in \ell^2 \quad \forall x = \{x_n\}_n \in \ell^2.$$

To this end we observe that, for every $n \in \mathbb{N}$, one has:

$$(a_n :=) \; \frac{n}{1+n^2} \leq a_1 = \frac{1}{2}.$$

As a consequence, for every $x = \{x_n\}_n \in \ell^2$ we have:

$$\begin{aligned} \sum_{n=1}^{\infty} |T(x)_n|^2 &= \sum_{n=1}^{\infty} |a_n x_n|^2 \\ &= \sum_{n=1}^{\infty} a_n^2 x_n^2 \\ &\leq \sum_{n=1}^{\infty} \frac{1}{4} x_n^2 \\ &= \frac{1}{4} \|x_n\|_{\ell^2}^2 < \infty \end{aligned}$$

and this proves that $T(x) \in \ell^2$.
We notice that the above estimate can be re-written as:

$$\|T(x)\|_{\ell^2}^2 \leq \frac{1}{4} \|x\|_{\ell^2}^2 \quad \forall x \in \ell^2 \qquad (*)$$

• We now turn to prove that T is linear and continuous.
As for the linearity of T we have: $\forall x, y \in \ell^2$, $\alpha, \beta \in \mathbb{R}$:

$x = \{x_n\}_n$
$(x)_n = x_n$

$$\begin{aligned} T(\alpha x + \beta y) &= \{a_n (\alpha x + \beta y)_n\}_{n \in \mathbb{N}} \\ &= \{a_n (\alpha x_n + \beta y_n)\}_{n \in \mathbb{N}} \\ &= \{a_n (\alpha x_n) + a_n (\beta y_n)\}_{n \in \mathbb{N}} \\ &= \{\alpha (a_n x_n) + \beta (a_n y_n)\}_{n \in \mathbb{N}} \\ &= \{\alpha T(x)_n + \beta T(y)_n\}_{n \in \mathbb{N}} \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

n-th element of $T(x)$ / $T(y)$

• Now we have proved the linearity of T, the continuity of this map follows immediately from its boundness, which is proved in $(*)$.
In particular, we have:

$$(*) \implies \|T(x)\|_{\ell^2} \leq \frac{1}{2} \|x\|_{\ell^2}$$

$$\implies \|T\|_{\mathcal{L}} = \sup_{x \neq 0} \frac{\|T(x)\|_{\ell^2}}{\|x\|_{\ell^2}} \leq \frac{1}{2}$$

• To complete the solution, we need to prove that

$$\|T\|_{\mathcal{L}} = \frac{1}{2}$$

to this end it suffices to show that $\exists$ an element $x \neq 0$ s.t.
$$\frac{\|T(x)\|_{\ell^2}}{\|x\|_{\ell^2}} \geq \frac{1}{2}$$

To this end we consider the sequence:

$$e_1 = (1, 0, 0, \dots, 0, \dots) \in \ell^2$$

Clearly: $\|e_1\|_{\ell^2} = 1$.

Moreover, we have:

$$\|T(e_1)\|_{\ell^2} = \left(\sum_{n=1}^{\infty} |a_n (e_1)_n|^2\right)^{1/2} = (|a_1|^2)^{1/2} = a_1 = \frac{1}{2}$$

As a consequence, we obtain:

$$\frac{1}{2} \geq \|T\|_{\mathcal{L}} = \sup_{x \neq 0} \frac{\|T(x)\|_{\ell^2}}{\|x\|_{\ell^2}} \geq \frac{\|T(e_1)\|_{\ell^2}}{\|e_1\|_{\ell^2}} = \frac{1}{2}$$

since we have "sup"

and this proves that $\|T\|_{\mathcal{L}} = \frac{1}{2}$.

Ex. 2 Consider the map $T : C([-1,1]) \longrightarrow \mathbb{R}$:

$$T(f) := \int_{-1}^{1} g(x)\, f(x)\, dx$$

$\left(C([-1,1]), \max_{[-1,1]} |\cdot|\right)$

we consider this norm since I is compact and the elements are continuous (hence sup $\rightarrow$ max)

where g is the piece-wise constant map:

$$g(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ -1 & \text{if } -1 \leq x \leq 0 \end{cases}$$

Prove that T is linear and continuous and that $\|T\|_{\mathcal{L}} = 2$.

Solution: • We first observe that T is well-posed in the sense that:

$$\boxed{T(x) \in \mathbb{R} \iff gf \in L^1([-1,1]) \quad \forall f \in C([-1,1])}$$

In fact, if $f \in C([-1,1])$, then $f \in L^p([-1,1])$ $\forall 1 \leq p \leq \infty$,] continuous function on a compact interval
as a consequence, since $|g(x)| = 1$ $\forall -1 \leq x \leq 1$, we have:

$$|fg| = |f| \Longrightarrow fg \in L^p([-1,1]) \quad \forall 1 \leq p \leq \infty$$

• We now turn to prove that T is linear and continuous.
As for the linearity, for every $f_1, f_2 \in C([-1,1])$ and every $\alpha, \beta \in \mathbb{R}$ we have:

$$T(\alpha f_1 + \beta f_2) = \int_{-1}^{1} g(x)\left(\alpha f_1(x) + \beta f_2(x)\right) dx = \alpha \int_{-1}^{1} g(x) f_1(x)\, dx + \beta \int_{-1}^{1} g(x) f_2(x)\, dx = \alpha T(f_1) + \beta T(f_2)$$

• Since we have proved that T is linear, to prove the continuity of this map we can prove that T is bounded.
In fact, for every $f \in C([-1,1])$ we have:

$$\|T(f)\|_{\mathbb{R}} = |T(f)| \leq \int_{-1}^{1} |g(x)| \cdot |f(x)|\, dx \leq \max_{[-1,1]} |f| \cdot \int_{-1}^{1} |g(x)|\, dx = \|f\|_{C([-1,1])} \cdot 2$$

This proves that T is bounded from $C([-1,1])$ to $\mathbb{R}$ (hence T is continuous) and:

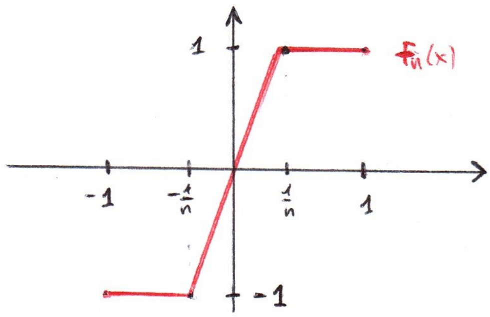
$$\|T\|_{\mathcal{L}} = \sup_{f \neq 0} \frac{|T(f)|}{\|f\|_{C([-1,1])}} \leq 2.$$

- To complete the solution we prove that $\|T\|_{X} \geq 2$ (which will imply that $\|T\|_X = 2$).

To this end we consider the sequence $\{f_n\}_n \subseteq C([-1,1])$ def. as:

$$f_n(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq -\frac{1}{n} \\ nx & \text{if } -\frac{1}{n} < x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

the goal is to find a sequence $\{f_n\}_n$ s.t.: $\dfrac{|T(f_n)|}{\|f_n\|_{C([-1,1])}} \longrightarrow 2$



(Not important for the solution, it's just an idea on why we should choose these functions:
Note: it is easy to prove that $f_n \longrightarrow g$ pointwise on $[-1,1]$.)

we are approximating by continuous functions the function g, which defines (essentially) the operator T

Now, for every $n \in \mathbb{N}$ we have:

(1) $\|f_n\|_{C([-1,1])} = \max\limits_{[-1,1]} |f_n| = 1$

(2)
$$\begin{aligned} T(f_n) &= \int_{-1}^{1} g(x) f_n(x)\,dx \\ &= 2\int_0^1 g(x) f_n(x)\,dx \\ &= 2\int_0^1 f_n(x)\,dx \\ &= 2\left(\int_0^{1/n} nx\,dx + \int_{1/n}^{1} 1\,dx\right) \\ &= 2\left(\left[\frac{nx^2}{2}\right]_0^{1/n} + 1 - \frac{1}{n}\right) = 2 - \frac{1}{n} > 0 \end{aligned}$$

As a consequence we obtain:

$$\|T(f_n)\|_{\mathbb{R}} = |T(f_n)| = 2 - \frac{1}{n} \qquad \forall n \in \mathbb{N}.$$

By combining (1) and (2) we then get:

$$\|T\| = \sup_{f \neq 0} \frac{|T(f)|}{\|f\|_{C([-1,1])}} \geq \frac{|T(f_n)|}{\|f_n\|_{C([-1,1])}} = 2 - \frac{1}{n} \qquad \forall n \in \mathbb{N}$$

From which we derive that:

$$\|T\| \geq \lim_{n \to \infty} \left(2 - \frac{1}{n}\right) = 2$$

Remark: Notice that, in this case there does **not** exist a function $f_0 \in C([-1,1])$ for which we have:

$$\|T\| = \frac{|T(f_0)|}{\|f\|_{C([-1,1])}}$$

Hence, $\|T\|$ is a "real" sup, and not a maximum.

! **Ex. 3** Consider the map $T : L^\infty([0,1]) \longrightarrow L^\infty([0,1])$:

$$T(f) = e^{-x} \int_0^x e^y f(y)\, dy$$
$(T(f)(x))$

prove that T is linear and continuous and compute $\|T\|_{\mathcal{X}}$.
Moreover, verify if T is injective and or surjective.

Solution: • As in the previous exercises, we first prove that $T(f)$ is well posed in the sense that :

$$T(f) \in L^\infty([0,1]) \qquad \forall f \in L^\infty([0,1])$$

Indeed, for every $f \in L^\infty([0,1])$ we have: for a.a. $x \in [0,1]$:

$$|T(f)| \leq e^x \int_0^x e^y |f(y)|\, dy$$
$(|T(f)(x)|)$
$$\leq \|f\|_{L^\infty} e^{-x} \int_0^x e^y dy$$
$$= \|f\|_{L^\infty} e^{-x} \left[e^y\right]_0^x$$
$$= \|f\|_{L^\infty} e^{-x} (e^x - 1) = \|f\|_{L^\infty} (1-e^{-x}) \leq \|f\|_{L^\infty}$$

As a consequence we derive that : $T(f) \in L^\infty([0,1])$ and:

$$\|T(f)\|_{L^\infty} \leq \|f\|_{L^\infty} \qquad \forall f \in L^\infty([0,1]) \qquad (*)$$

Notice that: once the linearity of T has been proved, from $(*)$ we immediately infer that T is continuous. and

$$\|T\|_{\mathcal{X}} = \sup_{f \neq 0} \frac{\|T(f)\|_{L^\infty}}{\|f\|_{L^\infty}} \leq 1$$

• We then turn to prove the linearity of T.
To this end we choose $f, g \in L^\infty([0,1])$, and $\alpha, \beta \in \mathbb{R}$. Then:

$$T(\alpha f + \beta g) = e^{-x} \int_0^x e^y (\alpha f + \beta g)(y)\, dy$$
$$= e^{-x} \int_0^x e^y (\alpha f(y) + \beta g(y))\, dy$$
$$= \alpha \left[e^{-x} \int_0^x e^y f(y) dy\right] + \beta \left[e^{-x} \int_0^x e^y g(y)\, dy\right]$$
$$= \alpha T(f) + \beta T(g)$$

• Hence T is linear and, by $(*)$, T is also continuous with :

$$\|T\|_{\mathcal{X}} = \sup_{f \neq 0} \frac{\|T(f)\|_{L^\infty}}{\|f\|_{L^\infty}} \leq 1$$

• We now turn to compute the exact value of $\|T\|_{\mathcal{X}}$.
To this end we first observe that $(*)$ is **not** sharp and it can be refined. In fact, since $e^{-1} \leq e^{-x} \leq 1$ $\forall x \in [0,1]$, we have:

$$|T(f)| \leq \|f\|_{L^\infty} (1-e^{-x}) \leq \|f\|_{L^\infty} (1-e^{-1}) \qquad \forall f \in L^\infty([0,1])$$

As a consequence we obtain the sharper bound :

$$\|T\|_{\mathcal{X}} = \sup_{f \neq 0} \frac{\|T(f)\|_{L^\infty}}{\|f\|_{L^\infty}} \leq 1 - e^{-1} \qquad (< 1)$$

On the other hand, choosing $f \equiv 1 \in L^\infty([0,1])$ one has:

$$\|T\|_{\mathcal{X}} \geq \frac{\|T(f)\|_{L^\infty}}{\|f\|_{L^\infty}} = \frac{1-e^{-x}}{1} \leftarrow \text{the } \sup(\cdot) = 1-e^{-1}$$

$$\Longrightarrow \|T\|_{\mathcal{X}} = 1-e^{-1}$$

One has:

$$T(f) = T(1) = e^{-x}\int_0^x e^y \cdot 1\, dy = e^{-x}\int_0^x e^y\, dy = e^{-x}\left[e^y\right]_{y=0}^{y=x} = e^{-x}(e^x - 1) = 1 - e^{-x}$$

and hence: $\|T(1)\|_{L^\infty} = 1 - e^{-1}$. $= \max_{x\in[0,1]} 1-e^{-x}$ (max since it's continuous on a compact interval)

As a consequence we get:

$$\|T\| = \sup_{f\neq 0} \frac{\|T(f)\|_{L^\infty}}{\|f\|_{L^\infty}} \geq \frac{\|T(1)\|_{L^\infty}}{\|1\|_{L^\infty}} = 1 - e^{-1}$$

Summing up we can conclude that:

$$\|T\| \overset{*}{=} \max_{f\neq 0} \frac{\|T(f)\|_{L^\infty}}{\|f\|_{L^\infty}} = 1 - e^{-1}$$

* we can write max because we found a function that realizes the sup

!!! • To finish the solution we now study the injectivity and the surjectivity of T.

$T: L^\infty \to L^\infty$ is injective if $\forall f, g \in L^\infty$ $f \neq g$ $\Rightarrow$ $T(g), T(f) \in L^\infty$ are such that: $T(f) \neq T(g)$

As for the injectivity, let $f, g \in L^\infty([0,1])$ be such that: (for a.a. $x \in [0,1]$)

$$e^{-x}\int_0^x e^y f(y)\,dy = T(f) = T(g) = e^{-x}\int_0^x e^y g(y)\,dy$$

Then, since $e^{-x} > 0$ $\forall x \in \mathbb{R}$, we get:

$$\int_0^x e^y f(y)\,dy = \int_0^x e^y g(y)\,dy \qquad \text{for a.e. } x \in [0,1] \qquad (*)$$

Now, since $f, g \in L^\infty([0,1]) \overset{**}{\subseteq} L^p([0,1])$ $\forall p \in [1,\infty]$ and the exponential map is continuous on $[0,1]$ (hence it's bounded), we deduce that the functions:

** $[0,1]$ is of finite measure

$$F(y) := e^y f(y) \qquad \text{and} \qquad G(y) := e^y g(y)$$

are in $L^\infty([0,1])$.

As a consequence, $F, G \in L^1([0,1])$. From this we infer that the two integral functions:

$$\mathcal{F} := \int_0^x F(y)\,dy \qquad \text{and} \qquad \mathcal{G}(x) := \int_0^x G(y)\,dy$$

satisfy the following property:

1. $\mathcal{F}, \mathcal{G} \in AC([0,1])$ (as integral functions of L^1 functions)
2. $\exists\, \mathcal{F}'(x) = F(x) = e^x f(x)$ for a.e. $x \in [0,1]$
3. $\exists\, \mathcal{G}'(x) = G(x) = e^x g(x)$ for a.e. $x \in [0,1]$

we need $F, G \in L^1([0,1])$ to obtain these properties (because we need integrals of L^1 functions)

From this, since $\mathcal{F}(x) = \mathcal{G}(x)$ $\forall x \in [0,1]$ (from $(*)$), we obtain for a.e. $x \in [0,1]$ that:

$$e^x f(x) = \mathcal{F}'(x) = \mathcal{G}'(x) = e^x g(x).$$

As a consequence, we conclude that $f = g$ a.e. in $[0,1]$.

Summing up, we have proved that:

$$\left[\, f, g \in L^\infty([0,1]),\ T(f) = T(g)\,\right] \Longrightarrow \underbrace{f = g \text{ a.e. in } [0,1]}_{\text{therefore } f=g \text{ in } L^\infty([0,1])}.$$

and this shows that T is injective.

• Finally, as for the surjectivity of T, we observe that:

$$T(f) = e^{-x}\int_0^x e^y f(y)\,dy = e^{-x}\, \mathcal{Y}(x) \in C([0,1])$$

$T(f)$ is the product of a smooth function e^{-x} and an $AC([0,1])$ function, therefore their product is (at least) continuous, but we know that $L^\infty([0,1])$ has also some discontinuity functions (which cannot be mapped through T)

From this we conclude that:

$$T(L^\infty([0,1])) \subseteq C([0,1]) \subsetneq L^\infty([0,1])$$

and thus, T is **not** surjective.

WEAK CONVERGENCE IN L^p

Ex. 1 Let $I = (0,+\infty)$ and let $\{f_n\}_n$ be the sequence of functions on I defined as follows:

$$f_n : I \to \mathbb{R}$$

$$f_n(x) = \mathbb{1}_{[n,n+1]}(x).$$

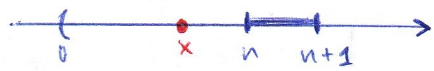
Prove the following facts:

1. $f_n \xrightarrow[n\to\infty]{} 0$ pointwise on I
2. $\{f_n\}_n \subseteq L^p(I)$ $\forall\, 1 \le p \le \infty$ but $f_n \not\xrightarrow[n\to\infty]{} 0$ in $L^p(I)$
3. $f_n \xrightarrow[n\to\infty]{} 0$ in $L^p(I)$ $\forall p > 1$ $(p \neq \infty)$
4. $f_n \not\xrightarrow[n\to\infty]{} 0$ in $L^1(I)$

Solution: ! 1. Let $x \in I = (0,+\infty)$ be fixed.
Then, for every $n > x$ we have that $f_n(x) = \mathbb{1}_{[n,n+1]}(x) = 0$.
As a consequence:

$$\exists \lim_{n\to\infty} f_n(x) = 0$$

If n is large enough then $f_n(x) \equiv 0$ $\forall x \in I$
$\Rightarrow$ pointwise convergence to 0
$\forall x \in I$ $\exists n$:



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2. We first consider the case $p < +\infty$.
For every $n \in \mathbb{N}$ we have:

$$\int_0^{+\infty} |f_n(x)|^p\,dx = \int_n^{n+1} 1\,dx = 1 \quad (< +\infty)$$

As a consequence we derive that:

• $f_n \in L^p(I)$ $\forall n \in \mathbb{N}$ (since $\int_0^{+\infty} |f_n|^p\,dx < \infty$)

• $f_n \not\to 0$ in $L^p(I)$ since:

$$\|f_n\|_{L^p} = \left(\int_0^{+\infty} |f_n|^p\,dx\right)^{1/p} = 1 \quad \forall n \in \mathbb{N}$$

(in particular $\|f_n\|_{L^p} = \|f_n - 0\|_{L^p} \not\to 0$ as $n \to \infty$)

We now consider the case $p = \infty$.
We have:

$$\operatorname*{ess\,sup}_{(0,+\infty)} |f_n| = 1 \quad \forall n \in \mathbb{N} \quad (< +\infty)$$

As a consequence we derive that:

• $f_n \in L^\infty(I)$ $\forall n \in \mathbb{N}$ (since $\operatorname*{ess\,sup}_I |f_n| < \infty$)

- $f_n \not\to 0$ in L^∞ since:

$$\|f_n\|_{L^\infty} = \operatorname{ess\,sup}_I |f_n| = 1 \quad \forall n \in \mathbb{N}$$

(in particular: $\|f_n\|_{L^\infty} = \|f_n - 0\|_{L^\infty} \not\to 0$)

Remark: since $f_n \not\to 0$ in $L^p(I)$ but since $f_n \to 0$ pointwise on I, we can say that $\{f_n\}_n$ cannot have a limit in L^p (the L^p convergence $\forall p$ implies pointwise convergence a.e. (up to a subsequence) therefore, if it exist, the L^p limit must coincide with the pointwise a.e. limit

3. Let $1<p<\infty$ be fixed.
By definition, the sequence $\{f_n\}_n$ weakly converges to 0 in $L^p(I)$ if: (and only if)

$$\lim_{n\to\infty} T(f_n) = 0 \qquad \forall T \in (L^p(I))^*$$

On the other hand, since $1<p<\infty$, by the Riesz theorem (repr. theorem) we know that $(L^p(I))^* = L^q(I)$ where $q = p^* = \frac{p}{p-1} \in (1,\infty)$
As a consequence;

$$f_n \xrightarrow[n\to\infty]{} 0 \text{ in } L^p(I) \iff \boxed{\lim_{n\to\infty} \underbrace{\int_0^{+\infty} f_n(x)g(x)}_{T(f_n)} = 0 \quad \forall g \in L^q(I).} \qquad (*)$$

We then prove that $(*)$ holds.
To this end, for every fixed $g \in L^q(I)$ we write:

$$\left|\int_0^{+\infty} f_n g\, dx\right| \le \int_0^{+\infty} |f_n|\cdot|g|\, dx = \int_0^{+\infty} \left(|f_n|^{1-\frac{1}{p}} |g|\right) |f_n|^{\frac{1}{p}}\, dx$$

From this, by Holder's inequality with exponents p and $q=p^*$ we obtain:

$(*)$

$$\left|\int_0^{+\infty} f_n\cdot g\, dx\right| \le \underbrace{\left(\int_0^{+\infty} |f_n|\, dx\right)^{1/p}}_{=1 \ \forall n\in\mathbb{N} \text{ (we proved it)}} \left(\int_0^{+\infty} |f_n|^{\overbrace{q(1-\frac{1}{p})}^{1}} |g|^q\, dx\right)^{1/q} = \left(\int_0^{+\infty} f_n\cdot|g|^q\, dx\right)^{1/q}$$

To complete the proof of $(*)$ we observe that, setting:

$$h_n := f_n \cdot |g|^q \qquad (n\in\mathbb{N})$$

the following properties hold:

(1) $h_n \xrightarrow[n\to\infty]{} 0$ pointwise on I (since this is true of $\{f_n\}_n$)

(2) $0 \le h_n \le |g|^q \in L^1(I)$ (since $|f_n| \le 1$ and $g \in L^q(I)$)

We can then apply DCT obtaining:

$$\lim_{n\to\infty} \int_0^{+\infty} f_n\cdot|g|^q\, dx = \int_0^{+\infty} \left(\lim_{n\to\infty} f_n\cdot|g|^q\right) dx = 0$$

As a consequence of this, we conclude that

$$\exists \lim_{n\to\infty} \underbrace{\int_0^{+\infty} f_n(x)g(x)\, dx}_{0<|\cdots| \le \left(\int_0^{+\infty} f_n |g|^q dx\right)^{1/q} \ (*)} = 0$$

4. Again, by the Riesz theorem we know that $(L^1(I))^* = L^\infty(I)$.
As a consequence:

$$f_n \longrightarrow 0 \text{ in } L^1(I) \iff \underbrace{\lim_{n\to\infty} \int_0^{+\infty} f_n \cdot g \, dx}_{T(f_n) \text{ with } T\in (L^1(I))^*} = 0 \quad \forall g \in L^\infty(I)$$

On the other hand, choosing $g \equiv 1 \in L^\infty(I)$, we have:

$$\int_0^{+\infty} f_n g \, dx = \int_0^{+\infty} f_n \, dx = 1 \qquad \forall n \in \mathbb{N}$$

and hence: $\int_0^{+\infty} f_n \cdot 1 \, dx \not\to 0$ as $n \to \infty$.

This is enough to conclude that $f_n \underset{n\to\infty}{\not\rightharpoonup} 0$ in $L^1(I)$.

Remark: The weak convergence of $\{f_n\}_n$ in $L^\infty(I)$ is delicate since $(L^\infty(I))^* \neq L^1(I)$. Hence, it is **not** true that

$$f_n \rightharpoonup 0 \text{ in } L^\infty(I) \iff \boxed{\lim_{n\to\infty} \int_0^{+\infty} f_n g \, dx = 0 \quad \forall g \in L^1(I)} \quad (\bullet)$$

However, since $L^\infty(I) = (L^1(I))^*$ we see that the validity of $(\bullet)$ characterize the weak $*$ convergence in $L^\infty(I)$, if we think of an L^∞-function as a bounded linear functional operator on $L^1(I)$.

in some sense we are not considering f_n as a function but as a continuous linear operator. We identify f_n with the linear continuous operator from L^1 to $\mathbb{R}$ associated with f_n. That's why we can talk about weak $*$ convergence (that is a convergence for operators)

In our case, when $f_n = \mathbb{1}_{[n,n+1]}$, a direct application of the DCT shows that:

$$\underbrace{\lim_{n\to\infty} \int_0^{+\infty} f_n \cdot g \, dx}_{\lim_{n\to\infty} \int_n^{n+1} g \, dx} = 0 \qquad \forall g \in L^1(I)$$

$$\lim_{n\to\infty} \int_n^{n+1} g \, dx = 0$$

$g \in L^1(I)$ and so at some point it must become zero, otherwise $\int_0^{+\infty} g \, dx = \infty$

and hence $f_n \xrightarrow{*} 0$ in $L^\infty(I)$

‼ **Theorem**: Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $1 < p < \infty$.
Moreover, let $f_0 \in L^p(\Omega)$ and let $\{f_n\}_n \subseteq L^p(\Omega)$ be bounded. ($\|f_n\|_{L^p} \leq c \;\; \forall n \in \mathbb{N}$)
Then, the following statements are equivalent:

(1) $f_n \underset{n\to\infty}{\rightharpoonup} f_0$ in $L^p(\Omega)$, that is:

$$\lim_{n\to\infty} \int_\Omega f_n \cdot g \, dx = \int_\Omega f_0 \cdot g \, dx \quad \forall g \in L^q(\Omega) \qquad \left(q = p^* = \frac{p}{p-1}\right)$$

(2) For every $\phi \in C_c^\infty(\Omega)$ we have:

$$\lim_{n\to\infty} \int_\Omega f_n \cdot \phi \, dx = \int_\Omega f_0 \cdot \phi \, dx$$

Remark: The proof of this fact relies on the fact that $C_c^\infty(\Omega)$ is dense in $L^q(\Omega)$ for every $1 < q < \infty$.

Ex. 2 Let $I = (0, 2\pi) \subseteq \mathbb{R}$ and let $\{f_n\}_n$ be the seq. of functions defined as:

$$f_n : I = (0, 2\pi) \to \mathbb{R}, \quad f_n(x) = \sin(nx)$$

Prove the following facts:

(1) $\{f_n\}_n \subseteq L^p(I) \qquad \forall 1 \leq p \leq \infty$

(2) $f_n \rightharpoonup 0$ in $L^p(I) \qquad 1 < p < \infty$

Solution: 1. For every $n \in \mathbb{N}$ we clearly have that:

$$|f_n(x)| = |\sin(nx)| \leq 1$$

$\Rightarrow f_n \in L^\infty(I)$.

As a consequence, since $\lambda(I) < \infty$ $(\lambda(I) = 2\pi)$, we also have:

$$f_n \in L^p(I) \quad \forall p \in [1,\infty) \qquad (L^p(I) \supseteq L^\infty(I) \quad \forall p \in [1,\infty))$$

2. To prove that $f_n \xrightarrow[n\to\infty]{} 0$ in $L^p(I)$ $(1<p<\infty)$ we use the previous theorem.

* Hence, we show that:

$$\lim_{n\to\infty} \int_0^{2\pi} \sin(nx)\,\phi\,dx = 0 \qquad \forall \phi \in C_c^\infty(I)$$

Let then $\phi \in C_c^\infty(I)$ be fixed.

Integrating by parts (since both $f_n, \phi \in C^\infty(I)$), we get:

possible only because the functions are smooth (with a generic L^q function we could have not been able to do that)

$$\int_0^{2\pi} f_n \cdot \phi\,dx = \int_0^{2\pi} \underbrace{\sin(nx)}_{f'} \cdot \underbrace{\phi}_{g}\,dx$$

$$= \left[-\frac{\cos(nx)}{n}\cdot\phi\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\cos(nx)\cdot\phi'\,dx$$

$$= \left(-\frac{\cos(2\pi n)}{n}\cdot\phi(2\pi) + \frac{\cos(0)}{n}\cdot\phi(0)\right) + \frac{1}{n}\int_0^{2\pi}\cos(nx)\cdot\phi'\,dx$$

$(\phi(0) = \phi(2\pi) = 0$ since $\operatorname{supp}(\phi) = [a,b] \subseteq (0,2\pi))$

$$= \frac{1}{n}\int_0^{2\pi}\cos(nx)\cdot\phi'\,dx$$

As a consequence we obtain:

$$\left|\int_0^{2\pi} f_n\cdot\phi\,dx\right| = \frac{1}{n}\left|\int_0^{2\pi}\cos(nx)\,\phi'\,dx\right|$$

$$\leq \frac{1}{n}\int_0^{2\pi}\underbrace{|\cos(nx)|}_{\leq 1}\cdot\underbrace{|\phi'|}_{\leq \max|\phi'|}\,dx$$

true because ϕ' is smooth and compactly supported, so it is bounded

$$\leq \frac{1}{n}\left(\max_I|\phi'|\cdot\int_0^{2\pi}1\,dx\right)$$

$$= \max_I|\phi'|\cdot\frac{2\pi}{n} \xrightarrow[n\to\infty]{} 0$$

and so we can conclude (by the previous theorem) that $f_n \to 0$ in $L^p(I)$ (for $1<p<\infty$) as $n\to\infty$.

* To this end we first observe that $\{f_n\}_n$ is bounded in $L^p(I)$ $(\forall\, 1<p<+\infty)$. Indeed:

$$\|f_n\|_{L^p} = \left(\int_0^{2\pi}|f_n|^p\,dx\right)^{1/p} \leq (2\pi)^{1/p} \qquad \forall n\in\mathbb{N}.$$

$f_n \xrightarrow{*} 0$ in $L^\infty(I)$:

we consider $\{f_n\}_n \subseteq L^\infty(I) = (L^1(I))^*$
(we are considering $\{f_n\}_n$ as a sequence of functionals from $L^1(I)$ to $\mathbb{R}$)

The correlation functional - function is at the base of the representation theorem:

$$f_n \xrightarrow{*} 0 \text{ in } L^\infty(I)$$

$$\Longleftrightarrow \int_I f_n g \, dx \longrightarrow \int_I f g \, dx \quad \forall g \in L^1(\Omega)$$

COMPACT OPERATORS AND HILBERT SPACES

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Ex. 1 Consider the operator $T \in \mathcal{L}(\ell^2, \ell^2)$ defined as:

$$T(x) = \left\{ \frac{n}{1+n^2} x_n \right\}_{n \in \mathbb{N}}$$

(for all $x = \{x_n\}_n \in \ell^2$).

Prove the following facts:

(1) T is compact ($T \in K(\ell^2, \ell^2)$)

(2) T is symmetric

Moreover find the eigenvalues and the eigenfunctions of T.

Solution: We remind that this operator T has been studied previously, and we proved that:

$$T \in \mathcal{L}(\ell^2, \ell^2) \quad \text{and} \quad \|T\|_{\mathcal{L}} = \frac{1}{2}.$$

23/11 Ex. 1

We now turn to prove that T is compact.
To this end we could try to use the very definition of compactness, hence, to prove that:

$$\overline{T(E)} \subseteq \ell^2 \text{ is compact for all bounded set } E \subseteq \ell^2.$$

- However, since we do not have a good characterization of the compact subsets of ℓ^2, we follow a different approach, namely we prove that $\exists \{T_N\}_N \subseteq K(\ell^2, \ell^2)$ such that:

$$\lim_{N \to \infty} T_N = T \quad \text{in } \mathcal{L}(\ell^2, \ell^2)$$

- Let then $N \in \mathbb{N}$ be fixed and let $T_N : \ell^2 \to \ell^2$ be the operator defined as follows:

$$T_N(x) := \{y_n\}_{n \in \mathbb{N}} \quad \text{where} \quad y_n = \begin{cases} \frac{n}{1+n^2} x_n & n \le N \\ 0 & n > N \end{cases}$$

Notice that T_N is a "truncated version" of T.
In fact:

$$T_N(x) = \left(\frac{1}{2} x_1, \frac{2}{5} x_2, \dots, \frac{N}{1+N^2} x_N, 0, \dots, 0, \dots \right).$$

It is very easy to see that $T_N \in \mathcal{L}(\ell^2, \ell^2)$.

- Moreover T_N is compact since it is a finite-rank operator.
Indeed we have:

$$\mathrm{Im}(T_N) \subseteq \{ x \in \ell^2 : x_k = 0 \ \forall k \ge N+1 \} \cong \mathbb{R}^N$$

$$(x = (x_1, \dots, x_N, 0, \dots, 0, \dots))$$

and thus;

$$\dim(\mathrm{Im}(T_N)) \le N < \infty.$$ (it's actually $\dim(\mathrm{Im}(T_N)) = N$, but it's not imp.)

- We now prove that:

$$\lim_{N \to \infty} T_N = T \text{ in } \mathcal{L}(\ell^2, \ell^2) \iff \|T_N - T\|_{\mathcal{L}} \xrightarrow[N \to \infty]{} 0$$

To this end, for every $x \in \ell^2$ we observe that:

$$\|T_N(x) - T(x)\|_{\ell^2}^2 = \sum_{k=1}^{\infty} |\underbrace{T_N(x)^{(k)}}_{= T(x)^{(k)} \text{ if } k \le N} - T(x)^{(k)}|^2$$

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$$\|T_N(x) - T(x)\|_{\ell^2}^2 = \sum_{k=N+1}^{\infty} |\underbrace{T_N(x)^{(k)}}_{=0 \text{ if } k \geq N+1} - T(x)^{(k)}|^2$$

$$= \sum_{k=N+1}^{\infty} |T(x)^{(k)}|^2$$

$$= \sum_{k=N+1}^{\infty} \left(\frac{k}{1+k^2}|x_k|\right)^2$$

$$\leq \left(\sup_{k \geq N+1}\left(\frac{k}{1+k^2}\right)\right)^2 \sum_{k=N+1}^{\infty} |x_k|^2$$

$$\leq \left(\sup_{k \geq N+1}\left(\frac{k}{1+k^2}\right)\right)^2 \underbrace{\sum_{k=1}^{\infty} |x_k|^2}_{\|x\|_{\ell^2}^2}$$

As a consequence we can conclude that:

$$\|T_N - T\|_{\mathcal{L}} = \sup_{x \neq 0} \frac{\|T_N(x) - T(x)\|_{\ell^2}}{\|x\|_{\ell^2}}$$

$$\leq \sup_{k \geq N+1}\left(\frac{k}{1+k^2}\right)$$

From which, since $\frac{n}{1+n^2} \longrightarrow 0$, we have:

$$\lim_{N\to\infty}\left(\sup_{k\geq N+1}\left(\frac{k}{1+k^2}\right)\right) \overset{\text{def.}}{=} \limsup_{N\to\infty}\left(\frac{N}{N^2+1}\right)$$

$$= \lim_{N\to\infty} \frac{N}{1+N^2} = 0$$

!

and hence we conclude that: $\|T_N - T\|_{\mathcal{L}} \xrightarrow[N\to\infty]{} 0$.

- Summing up, we have that $T \in \mathcal{K}(\ell^2, \ell^2)$ as the limit of a sequence of finite rank operators (which are compact).
- Remark: Using the same argument we can prove that the operator $T: \ell^2 \to \ell^2$ defined by:

$$T(x) := \{a_n x_n\}_{n\in\mathbb{N}} \qquad \text{with } \{a_n\}_n \in \ell^\infty$$

is compact, provided that: $a_n \xrightarrow[n\to\infty]{} 0$.

(2) Since ℓ^2 is a Hilbert space we can study the symmetry of T, which means that: 22/12

$$\langle T(x), y\rangle_{\ell^2} = \langle x, T(y)\rangle_{\ell^2} \qquad \forall x, y \in \ell^2$$

In our case we have:

$$\langle T(x), y\rangle_{\ell^2} = \sum_{n=1}^{\infty} (T(x)_n y_n)$$

$$= \sum_{n=1}^{\infty}\left[\left(\frac{n}{n^2+1}x_n\right)y_n\right]$$

$$= \sum_{n=1}^{\infty}\left[x_n\left(\frac{n}{n^2+1}y_n\right)\right]$$

$$= \sum_{n=1}^{\infty}\left[x_n T(y)_n\right]$$

$$= \langle x, T(y)\rangle_{\ell^2} \qquad \forall x, y \in \ell^2$$

and thus T is symmetric.

(3) We remind that $\lambda \in \mathbb{R}$ is an eigenvalue of T if there exists $x \in \ell^2$, $x \neq 0$ such that:

$$T(x) = \lambda x.$$

In this case x is an eigenvector of T.

- In our case, using the explicit definition of T, we have:

$$T(x) = \lambda x \text{ in } \ell^2 \iff T(x)_n = \lambda x_n \quad \forall n \in \mathbb{N}$$
$$\iff \frac{n}{1+n^2} x_n = \lambda x_n \quad \forall n \in \mathbb{N}$$

Hence, since we are looking for non-zero solutions of the equation $T(x) = \lambda x$, we can assume that:

$$A := \{n \in \mathbb{N} : x_n \neq 0\} \neq \emptyset,$$

and thus, we obtain $\frac{n}{1+n^2} = \lambda \quad \forall n \in A \neq \emptyset$.

On the other hand, since $\underbrace{\lambda \text{ cannot depend on } n}$ and since the sequence:

in the sense that every λ can depend on n, but every λ is fixed. if $\lambda_1 = \frac{1}{1+1} = \frac{1}{2}$ then we have $T(x) = \lambda x = \frac{1}{2}x \iff (\frac{1}{2}x_1, \frac{2}{1+4}x_2, \dots) = \frac{1}{2}(x_1, x_2, x_3, \dots)$ (fixed)

$$a_n = \frac{n}{1+n^2}$$

is strictly decreasing (hence $a_n \neq a_m$ if $n \neq m$), we conclude that the equation $T(x) = \lambda x$ has non-zero solutions if and only if $\lambda = \frac{n}{1+n^2}$ and $x = (x_k)_{k\in\mathbb{N}}$ is such that:

$$x_k = 0 \quad \forall k \neq n.$$

Summing up, the eigenvalues of T are exactly the elements of the sequence $\left(a_n = \frac{n}{1+n^2}\right)_{n\in\mathbb{N}}$, that is, T possesses a sequence of eigenvalues, $\forall n \in \mathbb{N}$ the eigenvectors associated with $\lambda = \lambda_n = \frac{n}{n^2+1}$ are given by $x = (x_k)_{k\in\mathbb{N}}$ s.t.

$$x_k = 0 \quad \forall k \neq n.$$

In other words, the eigenspace associated with λ_n is:

$$E(\lambda_n) = \{x = (x_k)_k \in \ell^2 : x_k = 0 \quad \forall k \neq n\}$$
$$= \operatorname{span}(\{e_n = (0, \dots, 0, \underbrace{1}_{n\text{-th component}}, 0, \dots, 0, \dots)\})$$

Theorem: Let X be an infinite dimensional Banach space and let $T \in K(X,X)$. ($T: X \to X$ is compact). Then, T is **not** bijective.

Ex. 2 Let $X = C([0,1])$ and let $T \in \mathcal{L}(X,X)$ be the operator defined as:

$$T(f) := u_0 f \quad \text{where} \quad u_0(x) = \begin{cases} \frac{e^x - 1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Prove that:

(1) T is bijective

(2) T is **not** compact

Solution: (1) We start by proving that T is injective. To this end we observe that, by definition, $u_0 \in C([0,1])$ and $u_0(x) \neq 0 \ \forall x \in [0,1]$. Thus, if $f, g \in X = C([0,1])$ s.t. $T(f) = T(g)$ in X, we get:

$$T(f) = T(g) \text{ in } X \Rightarrow T(f)(x) = T(g)(x) \quad \forall x \in [0,1]$$

$$T(f) = T(g) \text{ in } X \Rightarrow \underbrace{u_0(x)}_{\neq 0} f(x) = \underbrace{u_0(x)}_{\neq 0} g(x) \quad \forall x \in [0,1]$$
$$\Rightarrow f(x) = g(x) \quad \forall x \in [0,1]$$
$$\Rightarrow f = g$$

Therefore T is injective.
As for surjectivity of T we procede in the same way:
if $g \in X$ we have:

$$T(f) = g \text{ in } X \iff T(f)(x) = g(x) \quad \forall x \in [0,1]$$
$$\iff \underbrace{u_0(x)}_{\neq 0} f(x) = g(x) \quad \forall x \in [0,1]$$

Thus, since $u_0(x) \neq 0$ for every $0 \leq x \leq 1$ we can define:
$$f := \frac{g}{u_0} \in X = C([0,1])$$
which clearly satisfies $T(f) = g$ in X.
Hence, T is surjective.

(2) By point (1), T is bijective. Thus, since $X = C([0,1])$ is an infinite-dimensional Banach space, we derive from the previous theorem that T cannot be compact.

Ex. 3 Let $\Omega \subseteq \mathbb{R}^n$ be an open set with $\lambda(\Omega) < \infty$ and let:
$$\mathcal{C} = \{ f \in L^2(\Omega) : f = \text{const a.e. in } \Omega \}$$
Prove the following facts:

(1) $\mathcal{C}$ is a closed subspace of $L^2(\Omega)$

(2) $\forall g \in L^2(\Omega)$ one has $P_{\mathcal{C}}(g) = \frac{1}{\lambda(\Omega)} \int_\Omega g \, dx$

(3) determine $\mathcal{C}^\perp$ (in $L^2(\Omega)$).

Solution: ! (1) We first observe that, since $\lambda(\Omega) < \infty$, then $\mathcal{C} \subseteq L^2(\Omega)$.
Moreover, any function in $\mathcal{C}$ is proportional (in $L^2(\Omega)$) to:
$$f_0(x) \equiv 1 \in \mathcal{C},$$
that is: if $f \in \mathcal{C}$ then $f = c \cdot f_0$ a.e. in Ω.
Hence we have:
$$\mathcal{C} = \operatorname{span}(\{f_0 \equiv 1\})$$
and thus $\mathcal{C}$ is a subspace of $L^2(\Omega)$.
In particular, since $\mathcal{C}$ is finite dimensional ($\dim(\mathcal{C}) = 1$), then $\mathcal{C}$ is also closed.

final-dimensional vector spaces are closed

! (2) Since $\mathcal{C}$ is a closed subspace of $L^2(\Omega)$, the orthogonal projection of an element $g \in L^2(\Omega)$ on $\mathcal{C}$ is well defined; moreover, $P_{\mathcal{C}}(g)$ is characterized by the following properties:

(a) $P_{\mathcal{C}}(g) \in \mathcal{C}$

(b) $\langle g - P_{\mathcal{C}}(g), f \rangle_{L^2} = 0 \quad \forall f \in \mathcal{C}$

Now, since $P_{\mathcal{C}}(g) \in \mathcal{C}$ then $P_{\mathcal{C}}(g) \in \mathcal{C}$ is constant a.e. in Ω. As a consequence, using property (b) with $f = f_0 \equiv 1 \in \mathcal{C}$, we get:

$$
\begin{aligned}
0 = \langle g - P_{\mathcal{C}}(g), f_0 \rangle_{L^2} &= \int_\Omega (g - P_{\mathcal{C}}(g)) \cdot \underbrace{f_0}_{=1} \, dx \\
&= \int_\Omega (g - P_{\mathcal{C}}(g)) \, dx \\
&= \int_\Omega g \, dx - \int_\Omega \underbrace{P_{\mathcal{C}}(g)}_{\in \mathcal{C}} \, dx \\
&= \int_\Omega g \, dx - \lambda(\Omega) P_{\mathcal{C}}(g)
\end{aligned}
$$

From this we obtain that $P_{\mathcal{C}}(g) = \frac{1}{\lambda(\Omega)} \int_\Omega g \, dx \in \mathcal{C}$.

(3) By definition, $\mathcal{C}^\perp$ is the subspace of $L^2(\Omega)$ given by:

$$\mathcal{C}^\perp = \{ g \in L^2(\Omega) : \langle g, f \rangle = 0 \quad \forall f \in \mathcal{C} \}$$

On the other hand, since $\mathcal{C} = \mathrm{span}(\{f_0 \equiv 1\})$, we have:

$$\mathcal{C}^\perp = \{ g \in L^2(\Omega) ; \langle g, f_0 \rangle = 0 \}$$

As a consequence we obtain:

$$
\begin{aligned}
g \in \mathcal{C}^\perp &\iff \langle g, f_0 \rangle_{L^2} = 0 \\
&\iff \int_\Omega g f_0 \, dx = 0 \\
&\iff \int_\Omega g \, dx = 0 .
\end{aligned}
$$

Summing up, we conclude that:

$$\mathcal{C}^\perp = \{ g \in L^2(\Omega) : \int_\Omega g \, dx = 0 \}.$$

Remark: Since $\mathcal{C} \subseteq L^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, we know that $L^2(\Omega) = \mathcal{C} \oplus \mathcal{C}^\perp$, that is: for every $g \in L^2(\Omega)$:

$$
\begin{aligned}
g &= \underbrace{P_{\mathcal{C}}(g)}_{\in \mathcal{C}} + \underbrace{(g - P_{\mathcal{C}}(g))}_{\in \mathcal{C}^\perp} \\
&= \underbrace{\frac{1}{\lambda(\Omega)} \int_\Omega g \, dx}_{P_{\mathcal{C}}(g)} + \underbrace{\left(g - \frac{1}{\lambda(\Omega)} \int_\Omega g \, dx \right)}_{h \in \mathcal{C}^\perp}
\end{aligned}
$$

Notice that since $h \in \mathcal{C}^\perp$ we have:

$\int_\Omega h \, dx = 0.$

$$x = (x_1, x_2, x_3, \dots, x_n, \dots)$$

$$T(x) = \left(\frac{1}{1+1} x_1, \quad \frac{2}{1+4} x_2, \quad \frac{3}{1+9} x_3, \dots, \quad \frac{n}{1+n^2} x_n, \dots \right)$$

non c'è ($\nexists$) un λ tale che ciascun x_k sia uguale a λx_k
siccome dentro $T(x)$ ciascun x_k ha un coefficiente (λ) diverso,
però:

$\forall n \in \mathbb{N}$:

$$x = (0, \dots, 0, x_n, 0, \dots, 0)$$

$$T(x) = \left(0, \dots, 0, \frac{n}{1+n^2} x_n, 0, \dots, 0\right)$$

$\Longrightarrow \quad T(x) = \lambda x \quad$ con $\quad x = (0, \dots, 0, x_n, 0, \dots)$

$$\lambda = \frac{n}{1+n^2}$$

Ex. 4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set with $\lambda(\Omega) < \infty$.
Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$
Moreover let :

$$\underset{\text{function}}{K} = K(x,y) \in L^q(\Omega \times \Omega) \iff \int_{\Omega\times\Omega} |K(x,y)|^q \, dx\, dy < \infty$$

We then consider the map:

$$T : L^p(\Omega) \to L^q(\Omega),$$

$$T(f)(x) := \int_\Omega K(x,y) f(y)\, dy$$

Prove the following facts:

(1) $T \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$

(2) T is compact — Banach-Alaoglu

Ex. 1 $X = \ell^p \quad p \in (1, \infty)$.

$T : X \to X$ such that: $\quad (T(x))^{(k)} = \text{atan}\left(\frac{1}{k} + 1\right) x^{(k)}$

1. T is linear and continuous
2. Determine $\|T\|_X$
3. Is T injective? Is T surjective?
4. $\xi_n = \{\xi_n^{(k)}\}_k \quad : \quad \xi_n^{(k)} = \delta_{nk}$.
   Show that:
   $$T(\xi_n) \xrightarrow[n\to\infty]{} 0 \quad \text{in } \ell^p$$
5. $p = 2$: T is symmetric

Solution: **1.**
$$\left[\begin{array}{ll} T(x+y) = T(x) + T(y) & \forall x, y \in \ell^p \\ T(\alpha x) = \alpha T(x) & \forall x \in \ell^p, \alpha \in \mathbb{R} \end{array}\right] : \quad \begin{array}{l}(1.1)\\(1.2)\end{array}$$

$$\begin{aligned}(T(x+y))^{(k)} &= \text{atan}\left(\tfrac{1}{k}+1\right)(x+y)^{(k)} \\ &= \text{atan}\left(\tfrac{1}{k}+1\right) x^{(k)} + \text{atan}\left(\tfrac{1}{k}+1\right) y^{(k)} \\ &= (T(x))^{(k)} + (T(y))^{(k)} \quad \Longleftrightarrow \quad (1.1)\end{aligned}$$

$$\begin{aligned}(T(\alpha x))^{(k)} &= \text{atan}\left(1+\tfrac{1}{k}\right)(\alpha x)^{(k)} \\ &= \alpha \, \text{atan}\left(1+\tfrac{1}{k}\right) x^{(k)} \\ &= \alpha (T(x))^{(k)} \quad \Longleftrightarrow \quad (1.2)\end{aligned}$$

$\Longrightarrow$ T is linear

Moreover, $\forall x \in \ell^p$:

$$\begin{aligned}\|T(x)\|_{\ell^p} &= \left(\textstyle\sum_{k=1}^{\infty} \left((T(x))^{(k)}\right)^p\right)^{1/p} \\ &= \left(\textstyle\sum_{k=1}^{\infty} \left(\text{atan}\left(1+\tfrac{1}{k}\right) x^{(k)}\right)^p\right)^{1/p} \\ &\le \beta \left(\textstyle\sum_{k=1}^{\infty} (x^{(k)})^p\right)^{1/p} \\ &= \beta \|x\|_{\ell^p}\end{aligned}$$

$$\beta := \sup_{k \ge 1} \text{atan}\left(1+\tfrac{1}{k}\right) = \text{atan}(2)$$

$\Longrightarrow$ T is bounded

$\Longrightarrow$ T is continuous

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Ex. 1 **2.** $\|T\|_{\chi} = \sup_{\|x\|_{\ell^p} \leq 1} \|T(x)\|_{\ell^p} \leq \beta$

Consider $e_1 = (1, 0, 0, \dots, 0, \dots)$:

$$\|T(e_1)\|_{\ell^p} = \left(\sum_{k=1}^{\infty} \left((T(e_1))^{(k)} \right)^p \right)^{1/p} = \left(\sum_{k=1}^{\infty} \left(\operatorname{atan}\left(\tfrac{1}{k}+1\right) e_1^{(k)} \right)^p \right)^{1/p} = \operatorname{atan}\left(\tfrac{1}{1}+1\right) = \operatorname{atan}(2) = \beta$$

$\Longrightarrow \|T\|_{\chi} = \beta$.

3. Injectivity:

$$T(x) = T(y) \iff \cancel{\operatorname{atan}\left(1+\tfrac{1}{k}\right)} x^{(k)} = \cancel{\operatorname{atan}\left(1+\tfrac{1}{k}\right)} y^{(k)}$$
$$\iff x^{(k)} = y^{(k)}$$
$$\iff x = y \quad \forall\, x, y \in \ell^p$$

So T is injective.

Surjectivity:

$\forall y \in \ell^p$ there exist $x \in \ell^p$ such that $T(x) = y$

!!! How to procede? Solve $T(x)=y$ for x and check that the x is in ℓ^p

$\forall y \in \ell^p \quad T(x) = y: \quad (x \in \ell^p) \qquad (*)$

$$(T(x))^{(k)} = y^{(k)} \iff \underbrace{\operatorname{atan}\left(1+\tfrac{1}{k}\right)}_{>0} x^{(k)} = y^{(k)}$$
$$\iff x^{(k)} = \frac{1}{\operatorname{atan}\left(1+\frac{1}{k}\right)} y^{(k)}$$

candidate solution to $(*)$
(candidate because it must be in ℓ^p)

$x = \{x^{(k)}\}_k \in \ell^p$? $\|x\|_{\ell^p} < \infty$?

$$\|x\|_{\ell^p} = \left(\sum_{k=1}^{\infty} \left(\frac{y^{(k)}}{\underbrace{\operatorname{atan}\left(\frac{1}{k}+1\right)}_{\geq \frac{\pi}{4}}} \right)^p \right)^{1/p} \leq \frac{4}{\pi} \left(\sum_{k=1}^{\infty} (y^{(k)})^p \right)^{1/p} = \frac{4}{\pi} \|y\|_{\ell^p} < \infty$$

$\left(\operatorname{atan}\left(1+\tfrac{1}{k}\right) \geq \operatorname{atan}(1) = \tfrac{\pi}{4}\right)$

$\longrightarrow x \in \ell^p$

$\Longrightarrow$ T is surjective.

! **4.** $(T(\xi_n))^{(k)} = \operatorname{atan}\left(1+\tfrac{1}{k}\right) \delta_{kn} := \eta_n^{(k)}$

We want to show that $\eta_n = \{\eta_n^{(k)}\}$ weakly converges in ℓ^p:

$$\eta_n \xrightarrow[n\to\infty]{} 0 \text{ in } \ell^p$$

$$\iff \underbrace{L(\eta_n) \xrightarrow[n\to\infty]{} L(0)}_{\text{convergence in } \mathbb{R} \text{ with } L(0)=0} \quad \forall L \in (\ell^p)^*$$

$$\iff \sum_{k=1}^{\infty} \eta_n^{(k)} y^{(k)} \xrightarrow[n\to\infty]{} 0 \qquad \forall y \in \ell^q$$

representation theorem

$$\sum_{k=1}^{\infty} \eta_n^{(k)} y^{(k)} = \underbrace{\operatorname{atan}\left(1+\frac{1}{n}\right)} y^{(n)}$$

$$\xrightarrow[n\to\infty]{} \operatorname{atan}(1) = \frac{\pi}{4}$$

! $y = \{y^{(k)}\}_k \in \ell^p \implies y^{(k)} \xrightarrow[k\to\infty]{} 0$

$$\implies \sum_{k=1}^{\infty} \eta_n^{(k)} y^{(k)} = \underbrace{\operatorname{atan}\left(1+\frac{1}{n}\right)}_{\xrightarrow[n\to\infty]{} \frac{\pi}{4}} \underbrace{y^{(n)}}_{\xrightarrow[n\to\infty]{} 0} \xrightarrow[n\to\infty]{} 0$$

$\implies \{\eta_n\}_n$ weakly converges

5. Is the operator symmetric?

$\forall x, y \in \ell^2$ (only ℓ^2 is Hilbert, symmetry is studied only on Hilbert)

$$\langle T(x), y\rangle = \langle x, T(y)\rangle \quad \forall x, y \in \ell^2 \quad ?$$

$$\begin{aligned}
\langle T(x), y\rangle &= \sum_{k=1}^{\infty} (T(x))^{(k)} y^{(k)} \\
&= \sum_{k=1}^{\infty} \operatorname{atan}\left(1+\frac{1}{k}\right) x^{(k)} y^{(k)} \\
&= \sum_{k=1}^{\infty} x^{(k)} \left(\operatorname{atan}\left(1+\frac{1}{k}\right) y^{(k)}\right) \\
&= \sum_{k=1}^{\infty} x^{(k)} (T(y))^{(k)} \\
&= \langle x, T(y)\rangle
\end{aligned}$$

Ex. 2 $X = C^0([0,1]), \|\cdot\|_\infty$.

$T: X \to X$ such that : $(T(f))(x) = \int_0^x \cosh(t) f(t)\, dt \quad \forall x \in [0,1]$!

I part

1. Prove that T is linear and continuous
2. Prove that T is compact
3. Determine the eigenvalues of T, if they exist — for Banach spaces we use the same def. that we use for Hilbert spaces

II part

4. $\varphi_n(x) = n^{2/p} e^{-n^2 x} \quad x \in [0,1]$, study strong and weak convergence of $\{\varphi_n\}_n$ in $L^p([0,1])$ $p \in (1,\infty)$
5. // in $L^1([0,1])$.

Solution: **1.** T is obviously linear.

$\cosh(t) \geq 0 \implies \int_0^x \cosh(t)\,dt \leq \int_0^1 \cosh(t)\,dt \quad \forall x \in [0,1]$

$$\begin{aligned}
|(T(f))(x)| &\leq \|f\|_\infty \int_0^x \cosh(t)\, dt \\
&\leq \|f\|_\infty \sinh(1) \quad \forall x \in [0,1]
\end{aligned}$$

$\sinh(1)$ — $\int_0^1 \cosh(t)\,dt$

! it is not important to write exactly the integral, but it is imp. to write something ⫫ of x

$\implies \|T(f)\|_\infty \leq \sinh(1) \|f\|_\infty \quad \forall f \in X$

$\implies T$ is bounded

$\implies T$ is continuous

2. T is compact $\overset{\text{def.}}{\iff} \forall E \subset X \quad \overline{T(E)}$ compact

$K \subset X$ compact $\xleftarrow{\text{Ascoli Arzelà thm.}}$ K bounded, closed, equicontinuous we need to prove that:

$T(E)$ bounded$_1$ and equicontinuous$_2$, then:

$\Longrightarrow \overline{T(E)}$ is compact. ($\overline{T(E)}$ is closed by def.)

1

E is bounded $\Longleftrightarrow \exists C_0 > 0 : \|f\|_\infty \le C_0 \quad \forall f \in E$.

$T(E)$ is bounded $\Longleftrightarrow \exists C_1 > 0 : \|\varphi\|_\infty \le C_1 \quad \forall \varphi \in T(E)$

$\Longleftrightarrow \exists C_1 > 0 : \|T(f)\|_\infty \le C_1 \quad \forall f \in E$

$\varphi \in T(E)$ only if it is the image through T of some function $f \in E$

We have:

$$\|T(f)\|_\infty \le \sinh(1) \|f\|_\infty \le \sinh(1)\, C_0 := C_1 \quad \forall f \in E$$

$\Longrightarrow T(E)$ is bounded

2

$T(E)$ equicontinuous $\Longleftrightarrow \forall \varepsilon > 0 \;\exists \delta_\varepsilon > 0 :$

(in blue an alternative formulation for this specific case)

$\forall \varphi \in T(E)$, $\forall f \in E$, $\forall x, y \in [0,1] \quad |x-y| < \delta_\varepsilon$

$\Longrightarrow |\varphi(x) - \varphi(y)| < \varepsilon$, $|(T(f))(x) - (T(f))(y)| < \varepsilon$

$$\begin{aligned}
|(T(f))(x) - (T(f))(y)| &= \left| \int_0^x \cosh(t) f(t)\, dt - \int_0^y \cosh(t) f(t)\, dt \right| \\
&\le \left| \int_y^x \cosh(t) |f(t)|\, dt \right| \\
&\le \|f\|_\infty \left| \int_y^x \cosh(t)\, dt \right| \\
&\le C_0 |x-y| \cosh(1) \\
&\le C_0 \cosh(1)\, \delta_\varepsilon < \varepsilon
\end{aligned}$$

$$\left(0 < \delta_\varepsilon < \frac{\varepsilon}{C_0 \cosh(1)} \right)$$

here we can simply write $\sup_{t \in [x,y]} |\cosh(t)|$

$\Longrightarrow T(E)$ is equicontinuous.

$\Longrightarrow \overline{T(E)}$ compact $\quad \forall E \subset X$ bounded

$\Longrightarrow T$ compact

3. $T(f) = \lambda f \qquad \lambda \in \mathbb{R}, \quad f \in X \setminus \{0\}$

$$\int_0^x \cosh(t) f(t)\, dt = \lambda f(x) \qquad \forall x \in [0,1]$$

f is continuous and so the lhs is differentiable (since the lhs is the integral of a continuous function $\Rightarrow$ it is differentiable). Since lhs = rhs then also rhs is differentiable. Therefore we can differentiate the equality.

by differentiating both sides:

$$\begin{cases} \cosh(x) f(x) = \lambda f'(x) & \forall x \in [0,1] \\ f(0) = 0 \end{cases}$$

$y = f(x)$

$$\Longleftrightarrow \begin{cases} y' - \frac{1}{\lambda} \cosh(x) y = 0 \\ y(0) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} y(x) = C e^{1/\lambda \sinh(x)} & \forall C \in \mathbb{R} \\ y(0) = C = 0 \Longleftrightarrow C = 0 \end{cases}$$

$\Longrightarrow y(x) = 0 \Longleftrightarrow f(x) = 0$

$\Longrightarrow T$ does not admit eigenvalues

4. $\varphi_n \xrightarrow[n\uparrow\infty]{} 0 \qquad \forall x \in [0,1]$

$$\|\varphi_n\|_{L^p}^p = \int_0^1 n^2 e^{-n^2 p x}\, dx = \frac{1}{p}\left(1 - e^{-n^2 p}\right) \xrightarrow[n\uparrow\infty]{} \frac{1}{p}$$

since the sequence $\{\varphi_n\}_n$ conv. pointwise to 0 but $\{\|\varphi_n\|\}_n$ does not conv. to zero:

$\Longrightarrow$ φ_n does not converge strongly in ℓ^p $\quad p \in [1,\infty)$.

We observe that $\{\varphi_n\}_n$ is bounded in L^p $\quad (1<p<\infty)$. (since the sequence of the norms converges to $1/p$)

Weak convergence follows if:

$$\forall g \in C_c^\infty([0,1]) : \quad \int_0^1 \varphi_n(x)\, g(x)\, dx \xrightarrow[n\to\infty]{} 0$$

integrating by parts:

$$\int_0^1 \varphi_n(x)\, g(x)\, dx = \cancel{\left[-n^{\frac{2}{p}-2}\, e^{-n^2 x}\, g(x)\right]_0^1} + \int_0^1 n^{\frac{2}{p}-2}\, e^{-n^2 x}\, g'(x)\, dx$$

$$= \int_0^1 \frac{\varphi_n}{n^2}\, g'(x)\, dx \qquad ((*))$$

$= 0$ because $g(0) = g(1) = 0$

$$\left|\frac{1}{n}\int_0^1 \varphi_n\, g'(x)\, dx\right| \le \frac{1}{n^2}\|\varphi_n g'\|_{L^1}$$

$$\le \underbrace{\frac{1}{n^2}\underbrace{\|\varphi_n\|_{L^p}}_{\substack{\le C_0 \\ (\forall n \in \mathbb{N}) \\ (\{\varphi_n\}_n \text{ bdd in } L^p)}} \underbrace{\|g'\|_{L^q}}_{\substack{\le C_1 \\ (g \in C_c^\infty)}}}_{\xrightarrow[n\to\infty]{} 0} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

Then: $\quad ((*)) \xrightarrow[n\to\infty]{} 0$

$\Longrightarrow \varphi_n \xrightarrow[n\to\infty]{} 0$ in $L^p([0,1]) \quad \forall p \in (1,\infty)$.

5. $\varphi_n \xrightarrow[n\to\infty]{} 0$ in $L^1([0,1])$

$\underset{\substack{\text{repr. thm.} \\ (L^1)^* = L^\infty}}{\Longleftrightarrow} \quad \int_0^1 \varphi_n(x)\, g(x)\, dx \xrightarrow[n\to\infty]{} 0 \quad \forall g \in L^\infty([0,1])$

with $g \equiv 1$: $\quad \int_0^1 \underbrace{\varphi_n(x)}_{\ge 0}\, dx = \|\varphi_n\|_{L^1} \xrightarrow[n\to\infty]{} 1 \quad \left(\frac{1}{p} \text{ with } p=1\right)$

$\Longrightarrow \varphi_n \not\xrightarrow[n\to\infty]{} 0$ in $L^1([0,1])$.

Compact operator?
If $X = C^0([a,b])$
$\Longrightarrow$ ASCOLI ARZELA

Weak convergence in L^p
with $1<p<\infty$
$\Longrightarrow$ theorem for $g \in C_c^\infty$
with integration by parts