Introduction to Nonparametric Statistics (cheatsheet)

1 Compare Two Independent Samples

1.1 Wilcoxon Rank Sum Test

Given two treatments, we are interested in testing:

H_0: two treatments are equivalent (in location)
H_1: two treatments differ (in location)

It can be one or two-sided.

Given samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$. Let $N = n + m$. Find the ranks S_i for each $Y_i,\ i \in \{1, .., n\}$.

The statistic

$$W_s = S_1 + \ldots + S_n$$

has mean and variance

$$E[W_s] = \frac{1}{2} n(N+1)$$
$$V[W_s] = \frac{1}{12} mn(N+1)$$

under H_0.

The null hypothesis is rejected if W_s is either too big or too small (depending on H_1).

Limiting Distribution

When m and n are sufficiently large, $\frac{W_s - E[W_s]}{\sqrt{V[W_s]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, set the mid-ranks to where there are ties. The statistic W_s^* will have the same mean and limiting distribution, but different variance, i.e.:

$$V[W_s^*] = \frac{mn(N+1)}{12} - \frac{mn}{12N(N-1)} \sum_{i=1}^{l} (d_i^3 - d_i)$$

and f is the desity of F.

Power

Wilcoxon Rank Sum statistic tests for shilf. Given a distribution F such that $X_i \sim F$ and $Y_i + \Delta \sim F$, we can find the minimum sample size $m = n$ that achive a power of at least β to reject H_0 at level α.

$$m = n \geq n_W(\alpha, \beta) = \frac{(z_\alpha + z_\beta)^2}{6\Delta^2 \{f^*(0)\}^2}$$

where

$$f^*(0) = \int_{-\infty}^{\infty} \{f(t)\}^2 dt$$

Pitman's asymptotic relative efficiency

This test can be compared with Student's t-test

$$e_{W_s,T}(F) = 12\sigma^2\{f^*(0)\}^2$$

where σ^2 is the variance for associated with F.

Notes

It has been proven by Hodges and Lehmann in 1956 that $e_{W_s} \geq 0.864$ always.

[Mann-Whitney-Wilcoxon Test] ["Extended Wilcoxon's" Test] [Normal Score Test]

1.2 Mann-Whitney-Wilcoxon Test

Given two treatments, we are interested in testing:

H_0: two treatments are equivalent (in location)

H_1: two treatments differ (in location)

It can be one or two-sided.

Given samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$. Let $N = n + m$.

The statistic

$$W_{XY} = W_s - \frac{1}{2}n(n+1) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{1}(X_i < Y_i)$$

has mean and variance

$$E[W_s] = \frac{1}{2}mn$$
$$V[W_s] = \frac{1}{12}mn(N+1)$$

under H_0.

The null hypothesis is rejected if W_s is either too big or too small (depending on H_1).

Distribution

A table for this statistic is available for small n and m.

Limiting Distribution

When m and n are sufficiently large, $\frac{W_{XY}-E[W_{XY}]}{\sqrt{V[W_{XY}]}} \approx \mathcal{N}(0,1)$.

Notes

This is design to make it easier to tabulate the distribution for Wilcoxon rank sum statistic.

[Wilcoxon Rank Sum Test] ["Extended Wilcoxon's" Test] [Normal Score Test]

1.3 Kolmogorov-Smirnov Test

Given two treatments, we are interested in testing:

H_0: two treatments are equivalent

H_1: two treatments differ

It can be one or two-sided.

Given samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$. Compare their empirical CDF.

$$F_m(x) = \frac{1}{m}\sum_{i=1}^{m}(X_i \leq x) \qquad G_n(y) = \frac{1}{n}\sum_{i=1}^{n}(Y_i \leq y)$$

The test statistic is defined as

$$D_{m,n} = \max_{t\in\mathbb{R}}(|F_m(t) - G_n(t)|)$$

The null hypothesis is rejected if $D_{m,n}$ is either too big.

Distribution

The cumulative distribution $P(D_{m,n} \geq \frac{a}{n})$ when $m = n = 1, .., 20$ and $a = 0, \ldots, n$ is tabulated. Also, if $m = n$ and $d = \frac{a}{n}$, then

$$P(D_{m,n} \geq d) = \frac{2}{\binom{2n}{n}} \left\{ \binom{2n}{n-a} - \binom{2n}{n-2a} + \binom{2n}{n-3a} \pm \ldots \right\}$$

Limiting Distribution

When $min(m, n) \to \infty$, $P\left(\sqrt{\frac{mn}{m+n}} D_{m,n} \geq t\right) = K(t)$, where

$$K(t) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2t^2}$$

[Wilcoxon Rank Sum Test] [Mann-Whitney-Wilcoxon Test] [Siegel-Tukey Test] [Ansari-Bradley Test] ["Extended Wilcoxon's" Test] [Normal Score Test]

1.4 Normal Score Test

Given two treatments, we are interested in testing:

H_0: two treatments are equivalent

H_1: two treatments differ

It can be one or two-sided.

Given samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$. Find the ranks S_i for each $Y_i,\ i \in \{1, .., n\}$. Let $a_N(S_i)$ be a function such that $a_N(s) = E[X_{(s)}]$ where $X_{(s)}$ is the sth order statisitic of $X_1, \ldots, X_N \sim F$ for some Gaussian distribution F.

The test statistic is defined as

$$N_s = \sum_{i=1}^{n} a_N(S_i)$$

Notes

Compared to Student's t-test, this statistic satistisfies $e_{N_s,T} \geq 1$.

[Wilcoxon Rank Sum Test] [Mann-Whitney-Wilcoxon Test] ["Extended Wilcoxon's" Test]

2 Deal with Paired Data

2.1 Sign Test

Given paired data (X_i, Y_i), we are interested in testing:

H_0: two treatments are equivalent ($E(X) = E(Y)$)

H_1: two treatments differ ($E(X) \neq E(Y)$)

It can be one or two-sided.

Given paired data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$. Let $D_i = Y_i - X_i$ for $i = 1, \ldots, N$. Observe that under H_0, $P(D_i > 0) = P(D_i < 0) = 1/2$. Find the ranks S_i for each $Y_i,\ i \in \{1, .., n\}$.

The statistic

$$S_N = \#\{i | D_i > 0\}$$

has mean and variance

$$E[S_N] = \frac{N}{2}$$
$$V[S_N] = \frac{N}{4}$$

under H_0.

The null hypothesis is rejected if S_N is too big or too small (depending on H_1).

Distribution

As S_N can be expressed as a sum of Bernoulli random variables, $S_N \sim \mathrm{Bin}(N, 1/2)$.

Limiting Distribution

When N is sufficiently large, $\frac{S_N - E[S_N]}{\sqrt{V[S_N]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, we ignore the pairs of data which a difference of zero, i.e.:if there are exactly N_0 zeros,

$$S_{N+} \sim \mathrm{Bin}(N - N_0, 1/2)$$

.

Power

This statistic tests for shilf. Given a distribution F such that $X_i \sim F$, $Y_i + \Delta \sim F$ and $L(z - \Delta) = P(Y_i - \Delta - X_i < z - \Delta)$, we can find the minimum sample size N that achive a power of at least β to reject H_0 at level α.

$$\sqrt{N} \geq \frac{\frac{z_\alpha}{2} + \sqrt{p(1-p)}z_\beta}{p - \frac{1}{2}}$$

where $p = L(\Delta)$.

Pitman's asymptotic relative efficiency

This test can be compared with Student's t-test

$$e_{S_N,T} = 4\tau^2\{l(0)\}^2$$

where l is the density of L and τ^2 is the variance associated with L; and with Wilcoxon signed-rank test

$$e_{S_N,V_s} = \frac{\{l(0)\}^2}{3\{l^*(0)\}^2}$$

where

$$l^*(0) = \int_{-\infty}^{\infty} \{l(z)\}^2 dz$$

[Wilcoxon Signed-Rank Test]

2.2 Wilcoxon Signed-Rank Test

Given paired data (X_i, Y_i), we are interested in testing:

H_0: two treatments are equivalent ($E(X) = E(Y)$)

H_1: two treatments differ ($E(X) \neq E(Y)$)

It can be one or two-sided.

Given paired data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$. Let $D_i = Y_i - X_i$ and $Z_i = |D_i|$ for $i = 1, \ldots, N$. Rank the Z_i's. With $|D_{(1)}| < \ldots < |D_{(N)}|$ compute

$$I_i = \begin{cases} 1 & \text{if } D_{(i)} > 0 \\ 0 & \text{if } D_{(i)} < 0 \end{cases}$$

The statistic

$$V_s = \sum_{i=1}^{N} iI_i$$

has mean and variance

$$E[V_s] = \frac{N(N+1)}{4}$$
$$V[V_s] = \frac{N(N+1)(2N+1)}{24}$$

under H_0.

Dual Statisitc

The dual statistic is

$$V_r = \sum_{i=1}^{N} i(1-I_i)$$

such that \equation{equation*}V_s+V_r = \frac{1}{2}N(N+1)\end{equation*}

The null hypothesis is rejected if V_s is too big or too small (depending on H_1).

Distribution

The cumulative distribution of V_s is tabulated for small N.

Limiting Distribution

When N is sufficiently large, $\frac{V_s - E[V_s]}{\sqrt{V[V_s]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, we score the Z_i's with the mid-ranks and discard the ranks for the Z_i's that are equal to zero. The mean and the variance are both affected.

$$E[V_s^*] = \frac{N(N+1)}{4} - \frac{d_0(d_0+1)}{4}$$
$$V[V_s^*] = \frac{N(N+1)(2N+1)}{24} - \frac{d_0(d_0+1)(2d_0+1)}{24} - \frac{\sum_{i=1}^{l} d_i(d_i+1)(d_i-1)}{48}$$

Furtheremore,

$$V_s + V_r = \frac{1}{2}N(N+1) - \frac{1}{2}d_0(d_0+1)$$

Power

This statistic tests for shilf. Given a distribution F such that $X_i \sim F$, $Y_i + \Delta \sim F$ and $L(z-\Delta) = P(Y_i - \Delta - X_i < z - \Delta)$, we can find the minimum sample size N that achive a power of at least β to reject H_0 at level α.

$$N_{\min} \approx \left(\frac{z_\alpha + z_\beta}{\sqrt{12}\Delta l^*(0)} \right)^2$$

where

$$l^*(0) = \int_{-\infty}^{\infty} \{l(z)\}^2 dz$$

and l is the density of L.

Pitman's asymptotic relative efficiency

This test can be compared with Student's t-test

$$e_{S_N,T} = 12\tau^2\{l^*(0)\}^2$$

τ^2 is the variance associated with L; and with Sign test

$$e_{S_N,V_s} = \frac{\{l(0)\}^2}{3\{l^*(0)\}^2}$$

[Sign Test] [Wilcoxon Rank Sum Test]

2.3 "Wilcoxon Combo"

Given blocked data, we are interested in testing:

H_0: no treatment effect

H_1: treatment effect does in the same direction for all block$

It can be one or two-sided.

Given b blocks with m_k controls and n_k treatments such that $m_k + n_k = N_k$ for $k = 1, \ldots, b$, compute the Wilcoxon rank sum statistic for each block.

$$W_{s_k} = \sum_{i=1}^{n_k} S_{ki}$$

where S_{ki} is the rank of the ith data in block k. Let $c_k = \frac{1}{N_k+1}$ for $k = 1, \ldots, b$.

The statistic

$$W_s^{\text{combo}} = \sum_{k=1}^{b} c_k W_{s_k}$$

has mean and variance

$$E[W_s^{\text{combo}}] = \frac{1}{2}\sum_{k=1}^{b} n_k$$

$$V[W_s^{\text{combo}}] = \sum_{k=1}^{b} \frac{n_k m_k}{12(N_k+1)}$$

under H_0.

The null hypothesis is rejected if W_s^{combo} is too big or too small (depending on H_1).

Limiting Distribution

If $b \to \infty$ and $\max(N_k)$ is bounded, or if b is fixed and $\min(m_k, m_k) \to \infty$, then $\frac{W_s^{\text{combo}} - E[W_s^{\text{combo}}]}{\sqrt{V[W_s^{\text{combo}}]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, assign with mid-ranks and we must find the mean and variance by ourselves.

$$E[W_s^{\text{combo}}] = \sum_{k=1}^{b} \frac{1}{N_k+1} E[W_{s_k}]$$

$$V[W_s^{\text{combo}}] = \sum_{k=1}^{b} \left(\frac{1}{N_k+1}\right)^2 V[W_{s_k}]$$

[Sign Test] [Wilcoxon Signed-Rank Test] [Wilcoxon Rank Sum Test]

3 Compare Several Treatments

3.1 Kruskal-Wallis Test

Given many treatments, we are interested in testing:

H_0: the treatments are equivalent

H_1: at leat one treatment differ

This is only two-sided.

Given s treatments and $N = \sum_{i=1}^{s} n_i$ data points in total, rank all the data and sum all the rank for each treatment. Let R_{ij} denote the rank of treatment i for the jth data point.

$$\begin{aligned}
\bar{R}_{i\cdot} &= \frac{1}{n_i} \sum_{k=1}^{n_i} R_{ik} \\
\bar{R}_{\cdot\cdot} &= \frac{1}{N} \sum_{i=1}^{s} \sum_{k=1}^{n_i} R_{ik} = \frac{N+1}{2} \\
W_i &= n_i \bar{R}_{i\cdot}
\end{aligned}$$

The test statistic is defined as

$$\begin{aligned}
K &= \frac{12}{N(N+1)} \sum_{i=1}^{s} n_i \left(\bar{R}_{i\cdot} - \frac{N+1}{2} \right)^2 \\
&= \frac{12}{N(N+1)} \left(\sum_{i=1}^{s} \frac{W_i^2}{n_i} \right) - 3(N+1)
\end{aligned}$$

The null hypothesis is rejected if K is too small.

Distribution

The cumulative distribution of K is tabulated for 3 groups of sample size at most 5.

Limiting Distribution

As in the case of $s = 2$, it can be shown that K is a quadratic function of Wilcoxon's statistic, W_2 has normal limiting distribution. It follows by Central Limit Theorem that $K \approx \chi^2(s-1)$ when the size $n_1, \dots, n_s$ are large.

Encountering Tries

When there are ties, we score the data with the mid-ranks.

$$K^* = \frac{\frac{12}{N(N+1)} \sum_{i=1}^{s} \frac{W_i^2}{n_i} - 3(N+1)}{1 - \sum_{j=l}^{l} \frac{d_j^3 - d_j}{N^3 - N}}$$

If $s = 2$ and there are a lot of ties, we have

$$\bar{K}^* = \frac{n_{\cdot\cdot} - 1}{n_{\cdot\cdot}} \sum_{k=1}^{2} \sum_{i=1}^{t} \frac{(n_{ki} - n_{k\cdot} n_{\cdot i} / n_{\cdot\cdot})^2}{n_{k\cdot} n_{\cdot i} / n_{\cdot\cdot}}$$

Furthermore, letting A_i and B_i denote the number of subjects giving a higher score for the first and second treatment respectively,

$$\begin{aligned}
\bar{R}_i^* &= A_i \left(\frac{m+1}{2} \right) + B_i \left(m + \frac{n+1}{2} \right) \\
\bar{K}^* &= \frac{N(N-1)}{mn} \left(\sum_{i=1}^{s} \frac{A_i^2}{n_i} - \frac{m^2}{N} \right)
\end{aligned}$$

K^* and $\bar{K}^*$ all have limiting chi-squared distribution.

[Jonckheere Test] [Chacko-Shorack Test] [Friedman's Test]

3.2 Friedman's Test

Given balanced block design with many treatments, we are interested in testing:

H_0: no difference between the treatments
H_1: there is a difference between the level of the response variable
Note that the alternative is not assumed to be specific, but the difference between the treatments should not be in terms of dispersion, say. This is only two-sided.

Given balanced design with n blocks and s treatments, rank the data in each block individually. For R_ij denoting the rank of treatment i in block j,

$$\bar{R}_{i\cdot} = \frac{1}{n}\sum_{k=1}^{n_i} R_{ik}$$

$$\bar{R}_{\cdot\cdot} = \frac{1}{ns}\sum_{i=1}^{s}\sum_{k=1}^{n_i} R_{ik} = \frac{s+1}{2}$$

The test statistic is defined as

$$\begin{aligned} Q &= \frac{12n}{s(s+1)}\sum_{i=1}^{s}\left(\bar{R}_{i\cdot} - \frac{s+1}{2}\right)^2 \\ &= \frac{12}{ns(s+1)}\sum_{i=1}^{s}\bar{R}_{i\cdot}^2 - 3n(s+1) \end{aligned}$$

The null hypothesis is rejected if Q is too small.

Distribution

The cumulative distribution of Q is tabulated for $s \leq 4$ and $n \leq 8$.

Limiting Distribution

Assymptotically, $Q \sim \chi^2_{(s-1)}$.

Encountering Tries

When there are ties, we score the data with the mid-ranks.

$$Q = \frac{\frac{12n}{s(s+1)}\sum_{i=1}^{s}\left(\bar{R}^*_{i\cdot} - \frac{s+1}{2}\right)^2}{1 - \frac{1}{ns(s^2-1)}\sum_{j=1}^{n}\sum_{i=1}^{l_j}(d_{ij}^3 - d_{ij})}$$

with limiting distribution $\chi^2_{(s-1)}$.

In the case when $s = 2$,

$$Q = \frac{4}{N}\left(A - \frac{N}{2}\right)$$

, where A is the number of block where treatment 1 rank first. In this case, the limiting distribution is $\chi^2_{(1)}$, as A follows binomial distribution.

[Kruskal-Wallis Test] [Jonckheere Test] [Chacko-Shorack Test] [Cochran's Test] [Aligned Rank Test]

3.3 Cochran's Test

Given balanced block design with many treatments, we are interested in testing:
H_0: no difference between the treatments
H_1: there is a difference between the level of the response variable
Note that the alternative is not assumed to be specific, but the difference between the treatments should not be in terms of dispersion, say. This is only two-sided.

This is a special case of Friedman's test where there are s treatments and the response variable is dichotomous (response 0 and 1). Observe that there are a lot of tries. Let L_j denote the number of response 1's in block j, and B_i, the total number of response 1 for treatment i.

The test statistic is defined as

$$Q = \frac{(s-1)\{s\sum_{i=1}^{s} B_i^2 - (\sum_{i=1}^{s} B_i)^2\}}{s\sum_{j=1}^{s} L_j - \sum_{j=1}^{s} L_j^2}$$

The null hypothesis is rejected if Q is too small.

Limiting Distribution

Assymptotically, $Q \sim \chi^2_{(s-1)}$.

[Friedman's Test] [Aligned Rank Test]

3.4 McNemar's Test

Given balanced block design with many treatments, we are interested in testing:

H_0: no difference between the treatments

H_1: there is a difference between the level of the response variable

Note that the alternative is not assumed to be specific, but the difference between the treatments should not be in terms of dispersion, say. This is only two-sided.

This is a special case of Friedman's and Cochran's tests where there are 2 treatments and the response variable is dichotomous (response pass and fail). Observe that there are a lot of tries. Let L_j denote the number of response 1's in block j, and B_i, the total number of response 1 for treatment i. The data can be illustrated in a little table.

		Evaluation 1	
		Pass	Fail
Evaluation 2	Pass	a	b
	Fail	c	d

The test statistic is defined as

$$Q^* = \frac{(B_1 - B_2)^2}{\sum_{j=1}^{N} L_j(2 - L_j)} = \frac{(C-B)^2}{B+C}$$

The null hypothesis is rejected if Q^* is too small.

Limiting Distribution

Assymptotically, $Q \sim \chi^2_{(s-1)}$.

Notes

As $B \sim \text{Bin}(k, 1/2)$ where $k = B + C$, the equivalent test would be $P\left(\left|B - \frac{k}{2}\right| \geq \left|b - \frac{k}{2}\right|\right)$. The one-sided test would thus be $P(B \geq b)$, which is simply the plain old sign test.

[Sign Test] [Friedman's Test] [Aligned Rank Test]

3.5 Aligned Rank Test

Given balanced block design with many treatments, we are interested in testing:

H_0: no difference between the treatments

H_1: there is a difference between the level of the response variable

Note that the alternative is not assumed to be specific, but the difference between the treatments should not be in terms of dispersion, say. This is only two-sided.

This test is designed to compare withing blocks. Given balanced design with s treatments and n blocks, substract each response with the mean of the responses in its corresponding block. Rank all the data. Let $\hat{R}_{ij}$ be the rank of the treatments i at block j.

$$\hat{R}_{i.} = \frac{1}{n}\sum_{k=1}^{n} \hat{R}_{ik}$$

Let

$$\Lambda = \frac{s-1}{\sum_{i=1}^{s}\sum_{j=1}^{n}\hat{R}_{ij}^{2} - \frac{1}{s}\sum_{j=1}^{n}(s\hat{R}_{.j})^2}$$

The test statistic is defined as

$$\hat{Q} = \Lambda\left(\frac{(sn)^3(sn+1)(2sn+1)}{6} - \frac{1}{4}sn^2(sn+1)^2\right)$$

The null hypothesis is rejected if Q is too small.

Limiting Distribution

Assymptotically, $Q \sim \chi^2_{(s-1)}$.

[Friedman's Test] [Chacko-Shorack Test] [Cochran's Test]

4 Test Independence and Trend

4.1 Spearman' Test (trends)

Given data collected at different points in time, we are interested in testing:

H_0: no difference between variables at different time

H_1: there is a trend

This is only one-sided.

Given samples $X_1, \ldots, X_N$ collected at time $t_1, \ldots, t_N$ respectively. Rank the X_i and place them in order of time. Let R_i be the rank of the data point collected at time t_i.

The statistic

$$D = \sum_{i=1}^{N}(R_i - i)^2 = \frac{N(N+1)(2N+1)}{3} - 2\sum_{i=1}^{N} iR_i$$

has mean and variance

$$E[D] = \frac{N^3 - N}{6}$$
$$V[D] = \frac{N^2(N+1)^2(N-1)}{36}$$

under H_0.

Dual Statisitc

The dual statistic is

$$D' = \sum_{i=1}^{N} iR_i - \frac{1}{6}N(N+1)(N+2)$$

The null hypothesis is rejected if D is too small or if D' is too large.

Distribution

A table for this statistic is available for small $N \leq 11$.

Limiting Distribution

When N is sufficiently large, $\frac{D-E[D]}{\sqrt{V[D]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, assign the data with mid-ranks. Both mean and variance are affected.

$$E[D^*] = \frac{N^3 - N}{6} - \frac{1}{12}\sum_{i=1}^{l}(d_i^3 - d_i)$$
$$V[D^*] = \frac{N^2(N+1)^2(N-1)}{36}\left(1 - \sum_{i=1}^{l}\frac{d_i^3 - d_i}{N^3 - N}\right)$$

[Mann' Test] [D+ Test] [Spearman's Test (independence)]

4.2 Mann' Test

Given data collected at different points in time, we are interested in testing:

H_0: no difference between variables at different time

H_1: there is a trend

This is only one-sided.

Given samples $X_1, \ldots, X_N$ collected at time $t_1, \ldots, t_N$ respectively. Rank the X_i and place them in order of time. Let R_i be the rank of the data point collected at time t_i, and let

$$U_{ij} = \begin{cases} 1 & \text{if } R_i < R_j \\ 0 & \text{if } R_i > R_j \end{cases}$$

The statistic

$$B = \sum_{i<j} U_{ij}$$

has mean and variance

$$E[B] = \frac{1}{4}N(N+1)$$
$$V[B] = \frac{1}{72}(2N^3 + 3N^2 - 5N)$$

under H_0.

The null hypothesis is rejected if B is too large.

Limiting Distribution

When N is sufficiently large, $\frac{B - E[B]}{\sqrt{V[B]}} \approx \mathcal{N}(0,1)$.

[Spearman' Test (trends)] [B+ Test]

4.3 Spearman's Test (independence)

Given two series of data collected at differnt points in time, we are interested in testing:

H_0: the two variables are stochastically independent

H_1: the two variables are stochastically ependent

This is only one-sided.

Given two series of data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ collected at different time. Get ranks R_i and S_i for each each series, $i = 1, \ldots, N$.

The statistic

$$D = \sum_{i=1}^{N}(R_i - S_i)^2$$

has mean and variance

$$E[D] = \frac{N^3 - N}{6}$$
$$V[D] = \frac{N^2(N+1)^2(N-1)}{36}$$

under H_0.

The null hypothesis is rejected if D is too small.

Distribution

A table for this statistic is available for small $N \leq 11$.

Limiting Distribution

When N is sufficiently large, $\frac{D-E[D]}{\sqrt{V[D]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, assign the data with mid-ranks. Both mean and variance are affected.

$$
\begin{aligned}
E[D^*] &= \frac{N^3 - N}{6} - \frac{1}{12}\sum_{i=1}^{l}(d_i^3 - d_i) \\
V[D^*] &= \frac{N^2(N+1)^2(N-1)}{36}\left(1 - \sum_{i=1}^{l}\frac{d_i^3 - d_i}{N^3 - N}\right)
\end{aligned}
$$

[D+ Test] [Spearman' Test (trends)] [Kendall's Test] [Spearman's Rho]

4.4 Kendall's Test

Given two series of data collected at differnt points in time, we are interested in testing:

H_0: the two variables are stochastically independent
H_1: the two variables are stochastically ependent

This is only one-sided.

Given two series of data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ collected at different time. Get ranks R_i and S_i for each each series, $i = 1, \ldots, N$. Order the R_i's' according to the order of the S_i's. Let

$$
U_{ij} = \begin{cases} 1 & \text{if } R_i < R_j \\ 0 & \text{if } R_i > R_j \end{cases}
$$

The statistic

$$
B = \sum_{i<j} U_{ij}
$$

has mean and variance

$$
\begin{aligned}
E[B] &= \frac{1}{4}N(N+1) \\
V[B] &= \frac{1}{72}(2N^3 + 3N^2 - 5N)
\end{aligned}
$$

under H_0.

The null hypothesis is rejected if B is too large.

Limiting Distribution

When N is sufficiently large, $\frac{B-E[B]}{\sqrt{V[B]}} \approx \mathcal{N}(0,1)$.

Notes

B can be interpreted as the number of concordant pairs.

[Mann' Test] [B+ Test] [Spearman's Test (independence)] [Kendall's Tau]

4.5 Spearman's Rho

Given two series of data collected at differnt points in time, we are interested in testing:

H_0: the two variables are stochastically independent
H_1: the two variables are stochastically ependent

This is only one-sided.

Given two series of data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ collected at different time. Get Spearman's statistic D.

The statistic

$$\rho_N = 1 - \frac{6D}{N^3 - N} \qquad 0 \leq \rho_N \leq 1$$

has mean and variance

$$E[\rho_N] = 1 - \frac{6}{N^3 - N} E[D] = 0$$
$$V[\rho_N] = \frac{1}{N-1}$$

under H_0.

The null hypothesis is rejected if ρ_N is far from 0.

Limiting Distribution

When N is sufficiently large, $\frac{\rho_N - E[\rho_N]}{\sqrt{V[\rho_N]}} \approx \mathcal{N}(0,1)$.

Encountering Tries

When there are ties, Get Spearman's statistic D^*.

[Spearman's Test (independence)]

4.6 Kendall's Tau

Given two series of data collected at differnt points in time, we are interested in testing:

H_0: the two variables are stochastically independent
H_1: the two variables are stochastically ependent

This is only one-sided.

Given two series of data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ collected at different time. Get Kendall's statistic B. Let C be the number of discordant pairs.

The statistic

$$\tau_N = 2\frac{B}{\binom{N}{2}} - 1 = \frac{B - C}{\binom{N}{2}} \qquad 0 \leq \tau_N \leq 1$$

has mean and variance

$$E[\tau_N] = \frac{4}{N(N-1)} E[B] - 1 = 0$$
$$V[\tau_N] = \frac{2(2N+5)}{9N(N-1)}$$

under H_0.

The null hypothesis is rejected if τ_N is far from 0.

Limiting Distribution

When N is sufficiently large, $\frac{\tau_N - E[\tau_N]}{\sqrt{V[\tau_N]}} \approx \mathcal{N}(0,1)$.

[Kendall's Test]

4.7 Copula-Based Test of Independence

Given two series of data collected at differnt points in time, we are interested in testing:

H_0: the two variables are stochastically independent
H_1: the two variables are stochastically ependent

This is only one-sided.

Given two series of data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ collected at different time. Get ranks R_i and S_i for each each series, $i = 1, \ldots, N$. Construct a function $J(r, s)$ such that

- $\int_0^1 J(u, v)du \text{ (and } dv) = 0 \leq \int_0^s \int_0^t J(u, v)dvdu$
- J can be expressed as a finite sum of square integrable functions that are monotime in each of their arguments
- $J(uv) = -J(1-u, v) = -J(u, 1-v)$

The test statistic is defined as

$$T_N^J = \frac{1}{N}\sum_{i=1}^{N} J\left(\frac{R_i}{N+1}, \frac{S_i}{N+1}\right)$$

[Locally Most Powerful Test] [Waerden's Test] [Spearman's Rho] [Kendall's Tau]

4.8 Locally Most Powerful Test

Given two series of data collected at differnt points in time, we are interested in testing:

H_0: the two variables are stochastically independent
H_1: the two variables are stochastically ependent

This is only one-sided.

Given two series of data $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ collected at different time and their copula $C_\theta(u, v)$. Get ranks R_i and S_i for each each series, $i = 1, \ldots, N$. Construct a function $T(r, s)$ such that

$$T(r, s) = E\left[\frac{\dot{c}_{\theta_0}}{c_{\theta_0}}(U_{(r)}, V_{(s)})\right]$$

where

$$c_{\theta_0}(u, v) = \frac{\delta}{\delta u \delta v} C_\theta(u, v)$$
$$\dot{c}_{\theta_0}(u, v) = \frac{\delta}{\delta \theta} c_{\theta_0}(u, v)$$

and $U_{(r)}$ and $V_{(s)}$ are the rth and sth order statistics of $U_1, \ldots, U_N \sim \mathcal{U}(0, 1)$ and $V_1, \ldots, V_N \sim \mathcal{U}(0, 1)$ respectively.

The test statistic is defined as

$$T_N^J = \frac{1}{N}\sum_{i=1}^{N} T(R_i, S_i)$$

[Copula-Based Test of Independence] [Waerden's Test]