Stochastic Dynamical Models

June 21^{st}, 2018

EXERCISES

Exercice 1. In a far country with a big public debt live two clumsy politicians Donald and Sam, say. Each time Donald speaks the interest on the public debt grows by 1 percentage point, if Sam speaks it raises by 2 points (regardless of whether Donald speaks or is silent), if both are silent the interest decreases by one point. Suppose that: (i) at any time Sam speaks with probability $3/7$ and Donald speaks with probability $1/2$; (ii) they speak or stay silent independently of each other; (ii) the interest on the public debt can not be negative, (iii) if it reaches 8 points the government will fall.

(a) Construct a discrete time Markov chain model describing the random behaviour of the interest rate.

(b) Classify the states of the Markov chain.

(c) If the initial value of the interest is 1 point, what is the probability that the government will eventually fall?

(d) If the initial value of the interest is 1 point, compute the mean time before the next goverment fall. (*Find the solution up to three constants* c_1, c_2, c_3 *depending on three boundary conditions, do not try to compute* c_1, c_2, c_3).

tempi di soggiorno

(e) If the initial value of the interest is 1 point write down the linear system allowing one to compute the mean time, before the government fall, in which the interest will be strictly smaller than 3 points. (*Do not try to work out computations and explicitly solve the system*).

Exercice 2. A car washing system does the job in two stages. When a car enters in the system, it first goes through stage 1, then stage 2, and finally it goes away. Service times in both stages are independent (and independent of the car) with exponential distribution with parameters a and b respectively. Potential customers arrive according to a Poisson process with parameter $\lambda > 0$ (independent of all other exponential random variables) but new cars enter in the system only if there are no other cars (at stage 1 and at stage 2).

(1) Write the transition rate matrix Q of a three-state $0, 1, 2$ continuous time Markov chain model.

(2) Find the frequency of visit in each state in the stationary regime.

Suppose now that $\lambda = 1/24$ (i.e. the interarrival time of two cars is an exponential random variable with average 24 minutes, to fix the ideas), $a = 1/4$ and $b = 1/2$ and denote by $(X_t)_{t\geq 0}$ the three-state Markov chain.

(3) Let $f : \{0, 1, 2\} \to \mathbb{R}$ be the function $f(0) = 24$, $f(1) = 0$, $f(2) = 16$. Show that the stochastic process $(M_t)_{t\geq 0}$ defined by

$$M_t = f(X_t) - \int_0^t \left(4 \cdot \mathbb{1}_{\{X_s>0\}} - \mathbb{1}_{\{X_s=0\}}\right) ds$$

is a martingale with respect to the natural filtration of the process $(X_t)_{t\geq 0}$.

(4) Find the average service time (stage 1 + stage 2) applying the stopping theorem. How can we interpret the result?

(5) Fill the matrix P_t of transition probabilities at time t of the Markov chain model

$$P_t = \begin{bmatrix} \frac{4}{5} + \mathrm{e}^{-\frac{3}{8}t} - \frac{4}{5}\,\mathrm{e}^{-\frac{5}{12}t} & \frac{2}{15} - \frac{1}{3}\,\mathrm{e}^{-\frac{3}{8}t} + \frac{1}{5}\,\mathrm{e}^{-\frac{5}{12}t} & \frac{1}{15} - \frac{2}{3}\,\mathrm{e}^{-\frac{3}{8}t} + \frac{3}{5}\,\mathrm{e}^{-\frac{5}{12}t} \\ \frac{4}{5} - 8\mathrm{e}^{-\frac{3}{8}t} + \frac{36}{5}\,\mathrm{e}^{-\frac{5}{12}t} & \frac{2}{15} + \frac{8}{3}\,\mathrm{e}^{-\frac{3}{8}t} - \frac{9}{5}\,\mathrm{e}^{-\frac{5}{12}t} & \frac{1}{15} + \frac{16}{3}\,\mathrm{e}^{-\frac{3}{8}t} - \frac{27}{5}\,\mathrm{e}^{-\frac{5}{12}t} \\ * & * & * \end{bmatrix}$$

Stochastic Dynamical Models

July 13^{th}, 2018

EXERCISES

Exercice 1. Un tale scommette tante volte sullo stesso evento. In ogni scommessa vince 1 moneta con probabilità p e perde 1 moneta con probabilità $1 - p$. Supponiamo che i risultati nelle varie scommesse siano indipendenti. Il capitale del giocatore al tempo 0 è di i monete con $1 \leq i < N$ e il gioco continua finché perde tutti oppure il suo capitale raggiunge N ($N > 2$) monete.

(a) Costruire una catena di Markov che descriva il capitale del giocatore.

(b) Supponiamo che il giocatore cambi la strategia e decida, se il suo capitale raggiunge N, di puntare tutto in modo tale che il suo capitale raggiunga $2N$ (e si ritira dal gioco) con probabilità p oppure perda tutto e quindi sia rovinato con probabilità $1-p$. Costruire una catena di Markov che descriva il capitale del giocatore con questa nuova strategia.

(c) Calcolare la probabilità di vincere $2N$ partendo da un capitale iniziale i con $1 \leq i < N$

(d) Calcolare la durata media del gioco nel caso in cui $p = 1/2$.

Exercice 2. Un bambino sfoglia un album di fotografie di N pagine ($N \geq 8$) guardando una pagina per unità di tempo. Ad ogni unità di tempo sceglie una pagina a caso tra quelle non ancora viste. Quando ha visto tutte le pagine ricomincia dall'inizio. Sia X_n il numero delle pagine già viste al tempo n ($n \in \mathbb{N}$, $n \geq 1$). Il processo $(X_n)_{n\geq 1}$ è una catena di Markov a tempo discreto? (La risposta deve essere ben motivata).

Exercice 3. Un porto ha due moli per lo scarico di minerali. Una flotta di 4 navi trasporta i minerali a uno dei due moli. Quando una nave entra in porto, si dirige al molo 1 se libero, al molo 2 se è libero e 1 è occupato e aspetta fino a quando se ne liberi uno se sono entrambi occupati. Lo scarico al molo 1 dura in media $1/a$ mentre al molo 2 dura in media $1/b$. Il tempo di scarico a entrambi i moli è una variabile aleatoria con densità esponenziale. Il tempo trascorso tra quando una nave lascia il porto vuota e ritorna carica è una variabile aleatoria esponenziale con media $1/\lambda$. Tutte le variabili esponenziali sono indipendenti.

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(1) Costruire un modello con una catena di Markov $(X_t)_{t\geq 0}$ a tempo continuo, spazio degli stati $\{0,1,2,3,4\}$ (numero di navi in porto) e classificare gli stati.
(2) Calcolare le densità invarianti.
(3) Se C è il costo per unità di tempo per nave in attesa di un molo libero, qual è il costo medio in condizioni stazionarie?
(4) Sia $f : \{0,1,2,3,4\} \to \mathbb{R}$ la funzione $f(0) = f(1) = f(2) = 1, f(3) = \theta, f(4) = 0$. Trovare il valore del parametro θ in modo tale che $(Qf)(3) = (Qf)(4)$.
(5) Scrivere la formula esplicita per Qf e verificare che il processo $(M_t)_{t\geq 0}$ definito da
$$M_t = f(X_t) - \int_0^t (Qf)(X_s)ds$$
è una martingala.
(6) Se ci sono quattro navi in porto, calcolare il tempo medio di esaurimento della coda (ovvero il tempo medio in cui si ci saranno per la prima volta due navi ai moli e nessuna in attesa) applicando il teorema d'arresto.

Stochastic Dynamical Models

February 14, 2018

Exercises

Exercice 1. Matteo moves along an infinite stairway whose steps are numbered by natural numbers $i \in \{0, 1, 2, \dots\}$. If, at time n, he is on the step $i > 0$ at time $n+1$ he is on the step $i+1$ (resp. $i-1$) with probability $1/3$ (resp. $2/3$). Anytime he reaches the step 0 he tosses a fair coin and he is stays on the step 0 or jumps to the step 1 according to the result. Let X_n be the number of the step at time n.

(1) Write the transition matrix of a discrete time Markov chain model.
(2) Is the Markov chain irreducible? Are the states periodic?
(3) Are states recurrent or transient? Does the Markov chain admit invariant densities?
(4) Compute the mean time to reach the step 0 starting from any state $i > 0$.

Suppose now that step number 5 is slippery so that whoever descends quickly (not Matteo!) and puts his foot on that step slips and instantly falls to step 0. A friend of Matteo, Luca, say, who starts at step $j > 5$, trying to quickly descend along the stairway at each unit of time goes down one step with probability $1/2$ and two steps with probability $1/2$. Compute the probability that the friend Luca falls on step number 5.

Exercice 2. Let P_t $(t \geq 0)$ be the 4×4 matrix

$$P_t = \begin{bmatrix} \frac{1}{6} + \frac{1}{3}\,\mathrm{e}^{-3t} + \frac{1-\alpha}{2}\,\mathrm{e}^{-4t} & \frac{1}{3} - \frac{1}{3}\,\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\,\mathrm{e}^{-3t} & \frac{1}{6} + \frac{1}{3}\,\mathrm{e}^{-3t} - \frac{1-\alpha}{2}\,\mathrm{e}^{-4t} \\ \frac{1}{6} + \frac{1}{3}\,\mathrm{e}^{-3t} - \frac{1}{2}\,\mathrm{e}^{-4t} & \frac{1}{3} + \frac{2}{3}\,\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\,\mathrm{e}^{-3t} & \frac{1}{6} - \frac{2}{3}\,\mathrm{e}^{-3t} + \frac{1}{2}\,\mathrm{e}^{-4t} \\ \frac{1}{6} - \frac{2}{3}\,\mathrm{e}^{-3t} + \frac{1}{2}\,\mathrm{e}^{-4t} & \frac{1}{3} - \frac{1}{3}\,\mathrm{e}^{-3t} & \frac{1}{3} + \frac{2}{3}\,\mathrm{e}^{-3t} & \frac{1}{6} + \frac{1}{3}\,\mathrm{e}^{-3t} - \frac{1}{2}\,\mathrm{e}^{-4t} \\ \frac{1}{6} + \frac{1}{3}\,\mathrm{e}^{-3t} - \frac{1}{2}\,\mathrm{e}^{-4t} & \frac{1}{3} - \frac{1}{3}\,\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\,\mathrm{e}^{-3t} & \frac{1}{6} + \frac{1}{3}\,\mathrm{e}^{-3t} + \frac{1}{2}\,\mathrm{e}^{-4t} \end{bmatrix}$$

(1) Determine the only value of the parameter $\alpha \in \mathbb{R}$ for which $(P_t)_{t \geq 0}$ is the transition semigroup of a continuous time Markov chain $(X_t)_{t \geq 0}$ with states $\{1, 2, 3, 4\}$.
(2) For the above α determine the matrix Q of transition rates.
(3) Is the Markov chain recurrent or transient? In the first case determine all the invariant densities.
(4) Let $f : \{1, 2, 3, 4\} \to \{0, 1\}$ be the function

$$f(1) = f(4) = 0, \;\; f(1) = -1, \;\; f(2) = 1$$

Show that the family of random variables $(M_t)_{t \geq 0}$ defined by

$$M_t = \mathrm{e}^{3t} f(X_t)$$

is a martingale with respect to the natural filtration of $(X_t)_{t \geq 0}$.
(5) Study $\lim_{t \to \infty} M_t$ (in distribution, in probability ...).

Stochastic Dynamical Models

January 17, 2018

Exercises

Exercice 1. Let $(X_n)_{n\geq 0}$ be a discrete time Markov chain with state space $I = \{1,2,3,4,5\}$ and transition matrix

$$P = \begin{bmatrix} \frac{2}{3} & * & 0 & 0 & 0 \\ * & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & * & 0 \\ 0 & 0 & * & \frac{1}{2} & \frac{1}{4} \\ 0 & * & 0 & 0 & 1 \end{bmatrix}$$

(1) Fill the transition matrix with the appropriate values.

(2) Draw the graph of the Markov chain. Is it irreducible?

(3) Determine the period of each state and say if the the Markov chain aperiodic?

(4) Does there exist a unique invariant distribution? (The answer must be motivated!) If the answer is in the affirmative find the invariant distribution. If not determine all invariant distributions.

(5) Compute the probability that the Markov chain reaches the state 5 starting from the state 3.

(6) Compute (if it exists) $\lim_{n\to\infty} P^n$

(7) What is the mean sojourn time in the set $\{3,4\}$?

(8) Let $f : \{1,2,3,4,5\} \to \mathbb{R}$ be the function $f(1) = f(2) = 1, f(3) = f(4) = 4, f(5) = 0$. Show that the process $(M_n)_{n\geq 0}$ defined by $M_0 = f(X_0)$ and

$$M_n = f(X_n) + \sum_{k=0}^{n-1} \mathbb{1}_{\{3,4\}}(X_k)$$

for $n \geq 1$ is a martingale.

(9) How about the limit as n goes to infinity of M_n?

(10) Find (again) the mean sojourn time in the set $\{3,4\}$ applying the stopping theorem.

Exercice 2. Consider a labour market in which there is only one job with a monthly salary s. Moreover, the workforce can be divided into two groups: "committed workers" and "lazy workers". An unemployed worker will be hired in a random time exponentially distributed with parameter λ and an employed worked will be fired (for force majeure clauses) in a random time exponentially distributed with parameter μ.

However, a lazy worker who has been surprised slacking is immediately fired. Suppose that the next supervisor's inspection is also exponentially distributed with parameter θ and, as a result of the inspection, any lazy worker will be fired.

A committed worker always gets his salary s, but a lazy worker can use his saved energy to do a second job (always immediately available) with a monthly salary m.

(a) Write a Markov chain model and determine the fraction of time of his life that a committed worker will spend with an employment. In the same way determine the fraction of time in which a lazy worker will be unemployed (without keeping into account the second job).
(b) Determine a condition on s for making more profitable to be a committed worker.
(c) If θ goes to $+\infty$, how can be interpreted the above condition?

Esercizio 2. Consideriamo un mercato del lavoro dove c'è un solo tipo di impiego con salario mensile s. Inoltre la forza lavoro si divide in due gruppi: "lavoratori operosi" e "lavoratori pigri". Un disoccupato trova lavoro in un tempo casuale distribuito esponenzialmente con parametro λ e un lavoratore, operoso o pigro, perderà il posto per cause di forza maggiore in un tempo distribuito esponenzialmente con parametro μ.

Ogni lavoratore pigro colto sul fatto viene licenziato immediatamente. Supponiamo che la prossima visita di un ispettore avverrà in un tempo distribuito esponenzialmente con parametro θ e, di conseguenza, ogni lavoratore pigro sarà licenziato in tronco.

Un lavoratore operoso riceve sempre il suo salario mensile s, ma un lavoratore pigro può utilizzare la sua energia risparmiata per un secondo lavoro (sempre disponibile e magari in nero)in cui riceverà un salario mensile m.

(a) Scrivere un modello con catene di Markov a tempo continuo e determinare la frazione di tempo della vita lavorativa in cui un lavoratore solerte sarà disoccupato. Analogamente determinare la frazione di tempo della vita lavorativa in cui un lavoratore pigro sarà disoccupato (senza tener conto del secondo lavoro).
(b) Determinare una condizione su s affinché convegna a un lavoratore essere solerte.
(c) Se θ tende a $+\infty$, come si può interpretare questa condizione?

$s \cdot \pi^{\text{hard working}}_{\text{employed}} \geq m$ $\equiv$ is more profitable to be committed worker if you earn more money with your main job (in the fraction of time that you are employed) than with your second job

Stochastic dynamical models

July 21, 2017

Exercises

Esercizio 1. Two players, A e B, decide to play a game of head (H) or tail (T) with the following rule: the coin is tossed several times and

- A wins if occurs (T) in two consecutive tosses;
- B wins if occurs (H) in two consecutive tosses.

The coin is not fair and the probability that head occurs in each toss is equal to 1/3.

Let $(X_n)_{n\geq 0}$ be the process defined by

$$X_n = \begin{cases} TT & \text{if at the } n+1\text{-th toss occured } T \text{ and at the } n+2\text{-th toss occurs } T \\ CT & \text{if at the } n+1\text{-th toss occurred } C \text{ and at the } n+2\text{-th toss occurs } T \\ TC & \text{if at the } n+1\text{-th toss occurred } T \text{ and at the } n+2\text{-th toss occurs } C \\ CC & \text{if at the } n+1\text{-th toss occurred } C \text{ and at the } n+2\text{-th toss occurs } C \end{cases}$$

For instance, if occur the sequence $CCTTT\ldots$, then $X_0 = CC$, $X_1 = CT$, $X_2 = TT$, $X_3 = TT$ and so on. The process $(X_n)_{n\geq 0}$ is a discrete time Markov chain with states space $I = \{TT, CT, TC, CC\}$.

(1) Determine the transition matrix and the initial distribution of the Markov chain $(X_n)_{n\geq 0}$.
(2) What is the probability that the first winner is A?
(3) On average, how many tosses are needed for the first win of one of two players?
(4) Determine (if it exists) an invariant distribution for the process.What is the law of X_n for each $n \in \mathbb{N}$?

Esercizio 2. In a small call center there are 3 telephone operators. Suppose that the number of the incoming calls is described by a Poisson process with parameter $1/5$ (one call each 5 minutes), and the duration of each call is independent of the others and has an exponential distribution with mean 3 minutes. The calls arriving when the 3 lines are busy are put on hold, and then redirected to the first free line, according to the order of arrival. Suppose moreover that it is possible to put on hold at most 2 calls and that the following ones are lost.
Denote with $(Q_t)_t$ the number of calls at the time t (calls on hold+calls with the operators), with $Q_0 = 0$.

(1) Determine the transition rates matrix of the chain $(Q_t)_t$.
(2) What is the probability that in the first 15 minutes arrive at most 2 calls?
(3) What is the probability that the second call arrives after 6 minutes? and the probability that the first call arrives after 1 minute and the second one before of 3 minutes?
(4) Does exist an invariant distribution π? If yes, determine it.
(5) Compute in stationary conditions the mean permanence time in the system of a call arriving at the time t.
(6) In stationary conditions what is the probability that the calls are lost?