POLITECNICO MILANO 1863

CRC

Markov Reliability and Availability Analysis

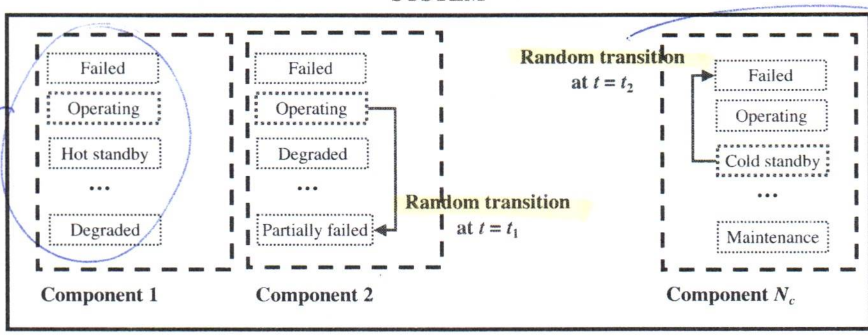
Enrico Zio

enrico.zio@polimi.it

General Framework

General Framework

SYSTEM



we're not restricting the component's states to just 2 states

there are random transitions, we cannot transform them into deterministic ones

We want to describe the stochastic process of the system evolution which occurs as a consequence of the stochastic process of the random transitions of the components in time

Under <u>specified</u> conditions:

Stochastic process of system evolution

=

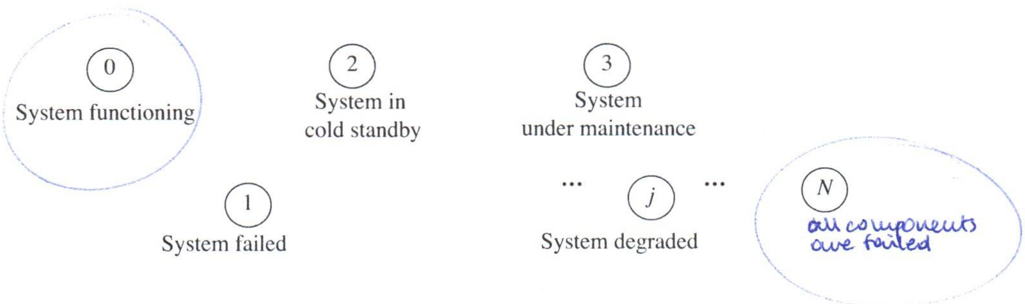
MARKOV PROCESS

Markov Processes: Basic Elements

Markov Processes: basic elements-the system

for our purposes

- The **system** can occupy a **finite** or **countably infinite** number N of states



Set of possible states $U = \{0, 1, 2, \ldots, N\}$
=
State-space of the random process

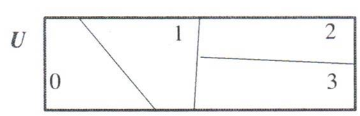
within 1 and N we have all the possible states of the system (all the possible system configurations)

Markov Processes: basic elements-the system states

- The **states** are:
  - **mutually exclusive** → the system must be **only** in **one** state at **each** time
  - **exhaustive** → the system must be in **one** state at **all** times

- **Example:**

Set of possible states $U = \{0, 1, 2, 3\}$



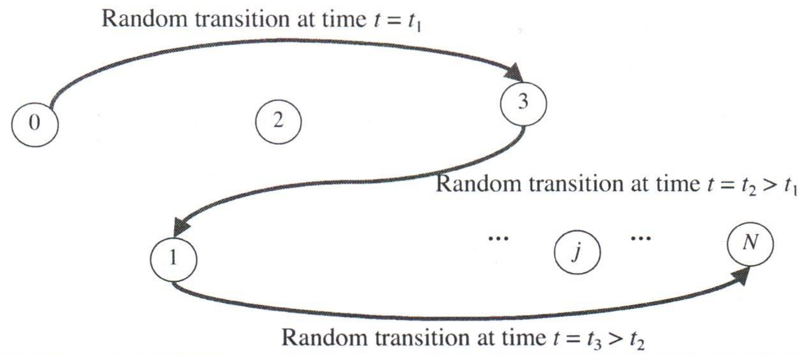
- **Mutually exclusive:** $P(\text{State} = i \cap \text{State} = j) = 0, \text{ if } i \neq j$
- **Exhaustive:** $P(U) = P(\text{State} = 0 \cup \text{State} = 1 \cup \text{State} = 2 \cup \text{State} = 3)$

$$= P(\text{State} = 0) + P(\text{State} = 1) + P(\text{State} = 2) + P(\text{State} = 3) = 1$$

since they're mutually exclusive

Markov Processes: basic elements-transition between states

- **Transitions** from one state to another occur **stochastically** (i.e., **randomly in time**)



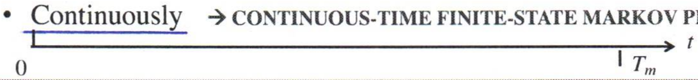
Markov Processes: basic elements-nathematical representation

- The **random** process of system transition in **time** can be described by an **integer random variable** $X(t)$

$X(t)=5$ → the system occupies **state number 5** at time t

- The **stochastic process** may be observed at:
  - Discrete times → **DISCRETE-TIME FINITE-STATE MARKOV CHAIN**

  - Continuously → **CONTINUOUS-TIME FINITE-STATE MARKOV PROCESS**



Discrete-Time Finite-State Markov Chain (DTFSMC)

The conceptual model: discrete times

- The stochastic process is **observed** at **discrete** times

they might be different, however we consider them all the same

$\Delta t(2) = t_2 - t_1$ $\Delta t(4) = t_4 - t_3$

$$t_n = t_{n-1} + \Delta t(n)$$

- **Hypotheses**:
  - The time interval $\Delta t(n)$ is **small** such that **only one** event (i.e., stochastic transition) can occur within it
  - For simplicity, $\Delta t(n) = \Delta t$ = constant

The conceptual model: finite state space

- The random process of system transition in time is described by an **integer random variable** $X(\cdot)$

- $X(n)$:= **system state** at time $t_n = n\Delta t$
  - $X(3) = 5$: the system occupies state 5 at time t_3

OBJECTIVE:
Compute the probability that the system is in a given state at a given time, for all possible states and times

$$P[X(n) = j], n = 1, 2, \ldots, N_{time}, j = 0, 1, \ldots, N$$

Objective:

$$P[X(n) = j], n = 1, 2, \ldots, N_{time}, j = 0, 1, \ldots, N$$

What do we need?

We need to describe the stochastic transitions (since we know where the initial state is)

The conceptual model: the Markov assumption

13

In general for stochastic processes:

- the **probability** of a **future** state of the system usually depends on its **entire life history**

$$P[X(n+1)=j]=P[X(n+1)=j\,|\,X(0)=x_0, X(1)=x_1, X(2)=x_2,\ldots, X(n)=x_n]$$

In Markov Processes:

- the **probability** of a **future** state of the system **only** depends on its **present state**

$$P[X(n+1)=j\,|\,\cancel{X(0)=x_0, X(1)=x_1, X(2)=x_2},\ldots, X(n)=x_n]$$
$$=$$
$$P[X(n+1)=j\,|\,X(n)=x_n]$$

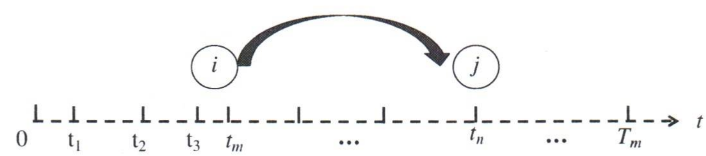
THE PROCESS HAS "NO MEMORY"

The conceptual model: the transition probabilities

14

- **Transition probability** that the system in state i at time t_m moves to state j at time t_n

$$p_{ij}(m,n)=P[X(n)=j\,|\,X(m)=i],\, n>m\geq 0$$
$$i=0,1,2,\ldots,N,\, j=0,1,2,\ldots,N$$



The conceptual model: properties of the transition probabilities (1

15

1. Transition probabilities $p_{ij}(m,n)$ are **larger than or equal to 0**

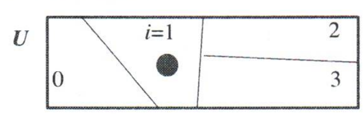
$$p_{ij}(m,n)\geq 0,\quad n>m\geq 0 \quad i=0,1,2,\ldots,N,\, j=0,1,2,\ldots,N$$

(**definition** of **probability**)

2. Transition probabilities **must sum to 1**

$$\sum_{all\, j} p_{ij}(m,n)=\sum_{j=0}^{N} p_{ij}(m,n)=1,\, n>m\geq 0 \quad i=0,1,2,\ldots,N$$

(the **set of states** is **exhaustive**)



$$\sum_{j=0}^{3} p_{1j}(m,n)=1,\, n>m\geq 0$$

Starting from $i = 1$, the system either **remains in $i = 1$** or it goes **somewhere else, i.e.,** to **$j = 0$ or 2 or 3**

The conceptual model: properties of the transition probabilities (2

16

3. $p_{ij}(m,n)=\sum_k p_{ik}(m,r)p_{kj}(r,n) \quad i=0,1,2,\dots,N, j=0,1,2,\dots,N$

proof.

$$p[X(n)=j, X(m)=i]=\sum_k p[X(n)=j, X(r)=k, X(m)=i] \quad \text{(theorem of total probability)}$$

↓ **conditional probability**

$$=\sum_k p[X(n)=j \mid X(r)=k, X(m)=i]P[X(r)=k, X(m)=i]$$

↓ **Markov assumption**

$$=\sum_k p[X(n)=j \mid X(r)=k]P[X(r)=k, X(m)=i]$$

$$p_{ij}(m,n)=P[X(n)=j \mid X(m)=i]=\frac{P[X(n)=j, X(m)=i]}{P[X(m)=i]} \quad \text{(conditional probability)}$$

↓ **formula above**

$$=\sum_k p[X(n)=j \mid X(r)=k]\frac{P[X(r)=k, X(m)=i]}{P[X(m)=i]}$$

↓ **conditional probability**

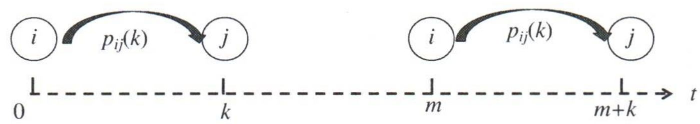
$$=\sum_k P[X(n)=j \mid X(r)=k]P[X(r)=k \mid X(m)=i]=\sum_k p_{kj}(r,n)p_{ik}(m,r) \quad \blacksquare$$

The conceptual model: stationary transition probabilities

17

- If the **transition probability** $p_{ij}(m,n)$ depends on the **interval** $(t_n - t_m)$ and **not** on the **individual times** t_m and t_n, then
  - the **transition probabilities** are **stationary**
  - the **Markov process** is **homogeneous in time**

$$p_{ij}(m,n)=p_{ij}(m,m+\overbrace{(n-m)}^{k \text{ time steps}})=p_{ij}(m,m+k)=P[X(m+k)=j \mid X(m)=i]$$
$$=P[X(k)=j \mid X(0)=i]$$
$$=p_{ij}(k), \quad k \geq 0 \quad i=0,1,2,\dots,N, j=0,1,2,\dots,N$$



This means that if the system is again in state "i" it doesn't remember that it was already in "i" some time before, and the probability of going to "j" in k steps is the same as it was when the system was in "i" the first time

The conceptual model: one-step stationary transition probabilities

18

We need to determine the **stationary** transition probabilities at the k-th time step

$$p_{ij}(k), k \geq 0$$

Markov assumption (see back-up slides…)

We need to know **only** the **stationary** **one-step** transition probabilities

$$p_{ij}(1)=p_{ij}$$

$(i=0,1,2,\dots,N, j=0,1,2,\dots,N)$

= probability that a system is in state i and at the next time is in state j
$i \rightarrow j$ in 1 step

The conceptual model: the transition probability matrix

states in which the system arrives (in 1 step)

Properties:

$$\underline{\underline{A}} = \begin{array}{c|cccc} i/j & 0 & 1 & \dots & N \\ \hline 0 & p_{00} & p_{01} & \cdots & p_{0N} \\ 1 & p_{10} & p_{11} & \cdots & p_{1N} \\ \dots & \dots & \dots & \dots & \dots \\ N & p_{N0} & p_{N1} & \cdots & p_{NN} \end{array} \quad \begin{array}{c} \\ =1 \\ =1 \\ \vdots \\ =1 \end{array}$$

states in which the system is

- $\dim\left(\underline{\underline{A}}\right) = (N+1)\times(N+1)$
- $0 \le p_{ij} \le 1, \forall i, j \in \{0,1,2,\dots,N\}$

(all elements are **probabilities**)

only (N+1)xN elements need to be known (since each row sums to 1)

- $\sum_{j=0}^{N} p_{ij} = 1, i = 0,1,2,\dots,N$

(the **set of states** is **exhaustive**)

$\underline{\underline{A}}$ **is a Stochastic Matrix**

- Given the **stationary one-step** transition probabilities p_{ij} $(i = 0,1,2,\dots,N, j = 0,1,2,\dots,N)$

Go back to the **OBJECTIVE:**

Compute $P[X(n)=j], n=1,2,\dots,N_{time}, j=0,1,\dots,N$

Unconditional state probabilities (1)

- Compute the probability that the system is in a given state at a given time, for all possible states and times

$$P[X(n)=j] = P_j(n), n=1,2,\dots,N_{time}, j=0,1,\dots,N$$

- Introduce the row vector:

this vector is our objective at time n

$$\underline{P}(n) = \left[P_0(n) P_1(n) \dots P_j(n) \dots P_N(n)\right]$$ = probabilities of the system being in state 0, 1, 2, …, N at the n-th time step

- Initialize the vector $\underline{P}(n)$ at time step $n = 0$:

$$\underline{P}(0) = \underline{C} = \left[C_0\, C_1 \dots C_j \dots C_N\right]$$

in our case we can assume that the initial state of the system is the perfect functioning (state 0) and so: $\underline{P}(0) = [1, 0, 0, \dots, 0]$

Unconditional state probabilities (2)

22

$$P_j(1) = P[X(1) = j]$$

↓ theorem of total probability + Markov assumption

$$= \sum_{i=0}^{N} P[X(1) = j | X(0) = i] \cdot P[X(0) = i]$$

↓ homogeneous process

$$= \sum_{i=0}^{N} p_{ij} C_i = p_{0j} \cdot C_0 + p_{1j} \cdot C_1 + p_{2j} \cdot C_2 + \ldots + p_{Nj} \cdot C_N,$$

$$with\ j = 0,1,2,\ldots,N$$

Using Matrix Notation:

$$\underline{P}(1) = \underline{C} \cdot \underline{\underline{A}}$$

Unconditional state probabilities (3)

23

- At the second time step $n = 2$:

$$P_j(2) = P[X(2) = j]$$

$$= \sum_{k=0}^{N} P[X(2) = j | X(1) = k] \cdot P[X(1) = k]$$

$$= \sum_{k=0}^{N} p_{kj} \cdot P_k(1)$$

$$= P_0(1) \cdot p_{0j} + P_1(1) \cdot p_{1j} + P_2(1) \cdot p_{2j} + \ldots + P_N(1) \cdot p_{Nj},$$

$$with\ j = 0,1,2,\ldots,N$$

$$\underline{P}(2) = \underline{P}(1) \cdot \underline{\underline{A}} = (\underline{C}\,\underline{\underline{A}})\underline{\underline{A}} = \underline{C}\,\underline{\underline{A}}^2$$

Proceeding in the same recursive way…

FUNDAMENTAL EQUATION OF THE HOMOGENEOUS DISCRETE-TIME DISCRETE-STATE MARKOV PROCESS

$$\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$$

this is the equation that we need to solve to get the objective (full description of the occupancies of states of the system at any time n)

Multi-step transition probabilities (1)

24

FUNDAMENTAL EQUATION $\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$

Define:

$$\underline{\underline{A}}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & \ldots & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & \ldots & p_{1N}(n) \\ \ldots & \ldots & \ldots & \ldots \\ p_{N0}(n) & p_{N1}(n) & \ldots & p_{NN}(n) \end{pmatrix}$$

n-th step transition probability matrix

$$p_{ij}(n) = P[X(n) = j \mid X(0) = i]$$

probability of arriving in state j after n steps given that the initial state was i

Multi-step transition probabilities (2)

EXAMPLE WITH N = 2 STATES AND n = 2 time steps

$$\underline{\underline{A}} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad (i = 0,1, j = 0, 1)$$

"PHYSICAL" MEANING: the probability of being in 0 after 2 steps is the probability of remaining in 0 both steps + probability of moving to 1 at the first step and to return to 0 at the second

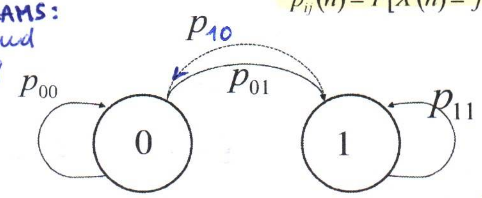
$$\underline{\underline{A}}^2 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \cdot \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \boxed{p_{00} \cdot p_{00} + p_{01} \cdot p_{10}} & \boxed{p_{00} \cdot p_{01} + p_{01} \cdot p_{11}} \\ p_{10} \cdot p_{00} + p_{11} \cdot p_{10} & p_{10} \cdot p_{01} + p_{11} \cdot p_{11} \end{pmatrix}$$

WHAT IS THE "PHYSICAL" MEANING?

Multi-step transition probabilities (3)

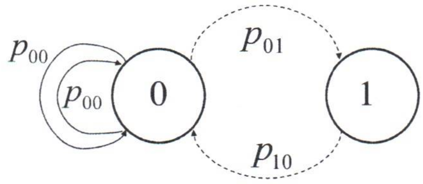
$$p_{ij}(n) = P[X(n) = j \mid X(0) = i]$$

MARKOV DIAGRAMS: represent states and transitions among the states



$$p_{01}(2) = p_{00} \cdot p_{01} + p_{01} \cdot p_{11} = \mathbb{P}(\text{either we go to 1 at the first step and we remain in 1 at the second step, either we stay in 0 at the first step and we go to 1 at the second})$$

probability that the system goes from 0 to 1 in 2 steps



$$p_{00}(2) = p_{00} \cdot p_{00} + p_{01} \cdot p_{10}$$

$p_{ij}(n)$ is the sum of the probabilities of all trajectories with length n which originate in state i and end in state j

Example 1: wet and dry days in a town (1)

- Stochastic process of raining in a town (transitions between wet and dry days)

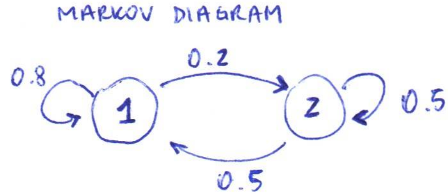
DISCRETE STATES
State 1: dry day
State 2: wet day

DISCRETE TIME
Time step = 1 day

TRANSITION MATRIX:

$$\underline{\underline{A}} = \begin{array}{cc} \text{dry} & \text{wet} \\ \left[\begin{matrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{matrix}\right. & \left.\right] \end{array} \begin{array}{l} \text{dry} \\ \text{wet} \end{array}$$

MARKOV DIAGRAM



Question: If today the weather is dry, what is the probability that it will be **dry two days from now**?

$$p_{12}(2) = p_{11}p_{11} + p_{12}p_{21} = 0.8 \cdot 0.8 + 0.2 \cdot 0.5 = 0.74$$

Otherwise:

$$A^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.74 & 0.26 \\ 0.65 & 0.35 \end{bmatrix}$$

Then: $\underline{P}(2) = \underline{P}(0)\,\underline{\underline{A}}^2$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.74 & 0.26 \\ 0.65 & 0.35 \end{bmatrix} = \begin{bmatrix} 0.74 & 0.26 \end{bmatrix}$$

Solution to the fundamental equation

Solution to the fundamental equation (1)

$$\begin{cases} \underline{P}(n) = \underline{P}(0)\underline{\underline{A}}^n \\ \underline{P}(0) = \underline{C} \end{cases}$$

SOLVE THE ASSOCIATED EIGENVALUE PROBLEM

i) Set the **eigenvalue problem** $\underline{V} \cdot \underline{\underline{A}} = \omega \cdot \underline{V}$ $\underline{V}$ eigenvectors, ω eigenvalues

ii) Write the **homogeneous form** $\underline{V} \cdot \left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}}\right) = 0$

iii) Find **non-trivial solutions** by setting $\det\left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}}\right) = 0$

iv) From $\det\left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}}\right) = 0$ compute the **eigenvalues** $\omega_j, j = 0,1,...,N$

v) Set the *N* **eigenvalue problems** $\underline{V}_j \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V}_j \quad j = 0,1,...,N$

vi) From $\underline{V}_j \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V}_j$ compute the **eigenvectors** $\underline{V}_j, j = 0,1,...,N$

Solution to the fundamental equation (2)

The **eigenvectors** $\underline{V}_j$ span the $(N + 1)$-dimensional space and can be used as a **basis** to write **any** $(N + 1)$-dimensional vector as a **linear combination** of them

$\Longrightarrow$ we can write $\underline{P}(n)$ and $\underline{C}$ with this new basis

$$\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j \quad \text{AND} \quad \underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$$

WE NEED TO FIND THE COEFFICIENTS α_j AND $c_j, j = 0,1,...,N$

FIND THE COEFFICIENTS $c_j, j = 0,1,\ldots,N$ FOR $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$

SOLVE THE ASSOCIATED ADJOINT EIGENVALUE PROBLEM

i) Set the **adjoint eigenvalue problem**

$$\underline{V}^+ \cdot \underline{\underline{A}}^+ = \omega^+ \cdot \underline{V}^+$$

ii) Since for **real valued** matrices $\underline{\underline{A}}^+ = \underline{\underline{A}}^T$ then:

$$\underline{V}^+ \cdot \underline{\underline{A}}^+ = \omega^+ \cdot \underline{V}^+ \Rightarrow \underline{V}^+ \cdot \underline{\underline{A}}^T = \omega^+ \cdot \underline{V}^+$$

iii) Since the eigenvalues $\omega_j^+, j = 0,1,\ldots,N$ depend **only** on $\det\left(\underline{\underline{A}}^T\right) = \det\left(\underline{\underline{A}}\right)$

$$\Rightarrow \omega_j^+ = \omega_j, j = 0,1,\ldots,N$$

the eigenvalues do not change, however the eigenvectors yes because A is now transpose

iv) From $\underline{V}_j^+ \cdot \underline{\underline{A}}^+ = \omega_j \cdot \underline{V}_j^+, j = 0,1,\ldots,N$ compute the adjoint eigenvectors $\underline{V}_j^+, j = 0,1,\ldots,N$

v) By **definition** of the adjoint problem and since $\underline{V}_j^+$ and $\underline{V}_j$ are **orthonormal**

$$\Rightarrow <\underline{V}_j^+, \underline{V}_i> \equiv \underline{V}_j^+ \cdot \underline{V}_i^T = \begin{cases} 0 & if\ i \neq j \\ k & otherwise \end{cases}$$

vi) Multiply the left- and right-hand sides of $\underline{C} = \sum_{i=0}^{N} c_i \underline{V}_i$ by $\underline{V}_j^+$

$$\left\langle \underline{V}_j^+, \underline{C} \right\rangle = \sum_{i=0}^{N} c_i \left\langle \underline{V}_j^+, \underline{V}_i \right\rangle = c_j \left\langle \underline{V}_j^+, \underline{V}_j \right\rangle \rightarrow \boxed{c_j = \frac{\left\langle \underline{V}_j^+, \underline{C} \right\rangle}{\left\langle \underline{V}_j^+, \underline{V}_j \right\rangle}}$$

(orthonormality)

- **Find the coefficients** $\alpha_j, j = 0,1,\ldots,N$ **for** $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j$
- Use $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j$, $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$ and $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$

a) Substitute $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$ into $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$ to obtain $\underline{P}(n) = \left(\sum_{j=0}^{N} c_j \underline{V}_j \right) \cdot \underline{\underline{A}}^n$

b) Set $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j = \underline{C} \cdot \underline{\underline{A}}^n = \left(\sum_{j=0}^{N} c_j \underline{V}_j \right) \cdot \underline{\underline{A}}^n$

Solution to the fundamental equation (5)

c) Multiply $\underline{V}_j \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V}_j$ by $\underline{\underline{A}}$ to obtain $\underline{V}_j \cdot \underline{\underline{A}} \cdot \underline{\underline{A}} = \omega_j \boxed{\underline{V}_j \cdot \underline{\underline{A}}}$

Since $\boxed{\underline{V}_j \cdot \underline{\underline{A}}} = \omega_j \cdot \underline{V}_j$ then $\underline{V}_j \cdot \underline{\underline{A}}^2 = \omega_j \boxed{\omega_j \cdot \underline{V}_j} = \omega_j^2 \cdot \underline{V}_j$

••• (proceeding in the same recursive way)

$$\underline{V}_j \cdot \underline{\underline{A}}^n = \omega_j^n \cdot \underline{V}_j$$

d) Substitute $\underline{V}_j \cdot \underline{\underline{A}}^n = \omega_j^n \cdot \underline{V}_j$ into $\boxed{\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j} = \underline{C} \cdot \underline{\underline{A}}^n = \sum_{j=0}^{N} c_j \cdot \boxed{\underline{V}_j \underline{\underline{A}}^n}$

$$\sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j = \sum_{j=0}^{N} c_j \cdot \boxed{\omega_j^n \cdot \underline{V}_j} \quad \Rightarrow \quad \boxed{\alpha_j = c_j \cdot \omega_j^n}$$

Quantity of Interest

Steady state probabilities

- **Steady state probabilities π_j:** probability of the system being in state j **asymptotically**
- **TWO ALTERNATIVE APPROACHES:**

if we found eigenvalues/vectors

1) Since $\omega_0 = 1$ and $|\omega_j| < 1, j = 1,2,\dots,N$ (ω_j = eigenvalues)

AT STEADY STATE: $\lim_{n\to\infty} \underline{P}(n) = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{\alpha_j} \cdot \underline{V}_j = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{c_j \cdot \omega_j^n} \cdot \underline{V}_j = c_0 \underline{V}_0 = \underline{\Pi}$

if we don't want to look for eigen-

2) Use the recursive equation $\underline{P}(n) = \underline{P}(n-1) \cdot \underline{\underline{A}}$

AT STEADY STATE: $\underline{P}(n) = \underline{P}(n-1) = \underline{\Pi}$

SOLVE $\underline{\Pi} = \underline{\Pi} \cdot \underline{\underline{A}}$ subject to $\sum_{j=0}^{N} \Pi_j = 1$

Example 2: wet and dry days in a town

$$\underline{\underline{A}} = \begin{array}{c} \\ dry \\ wet \end{array} \begin{array}{cc} dry & wet \\ \multicolumn{2}{c}{\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}} \end{array} \qquad \underline{C} = [1 \quad 0]$$

- *Question*: what is the probability that **one year from now** the day will be **dry**?

$n = 365 \Rightarrow$ steady state condition ($\underline{P}^{365}$ is exagerated)

$$\begin{cases} \underline{\pi} = \underline{\pi} \cdot \underline{A} \\ \sum \pi_j = 1 \end{cases} \Rightarrow [\pi_1 \ \pi_2] = [\pi_1 \ \pi_2] \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} = [0.8\pi_1 + 0.5\pi_2 \quad 0.2\pi_1 + 0.5\pi_2]$$

$$\Rightarrow \begin{cases} \pi_1 = 0.8\pi_1 + 0.5\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \pi_2 = \frac{2}{5}\pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = 5/7 = 0.714 \\ \pi_2 = 2/7 = 0.286 \end{cases}$$

$$\Rightarrow \mathbb{P}(dry) = \pi_1 = 0.714$$

First Passage Probabilities (1)

- **FIRST PASSAGE RANDOM TIME:**

Random time that the system arrives **for the first time** in state j given that it was in state i at the initial time 0

$$T_{ij} = \begin{cases} \min\{n \geq 1 \mid X(n) = j \ provided \ that \ X(0) = i\} \ if \ \exists \, n \geq 1 : X(n) = 1 \\ +\infty \text{ otherwise} \end{cases}$$

NOTICE:

$$\{T_{ij} = n\} = \{X(n) = j, X(m) \neq j, 0 < m < n \mid X(0) = i\}$$

$$\{T_{ij} < \infty\} = \{X(n) = j, n \geq 1 \mid X(0) = i\}$$

First Passage Probabilities (2)

- **FIRST PASSAGE PROBABILITY AFTER *n* TIME STEPS:**

Probability that the system arrives **for the first time** in state j **after *n* steps**, given that it was in state i at the initial time 0

$$f_{ij}(n) = P[T_{ij} = n]$$

f_{ij} gives the distribution of T_{ij}

$$=$$

$$f_{ij}(n) = P[X(n) = j, X(m) \neq j, 0 < m < n \mid X(0) = i]$$

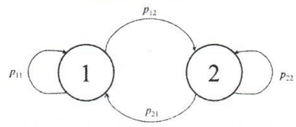
NOTICE:

$$f_{ij}(n) \neq p_{ij}(n)$$

$p_{ij}(n)$ =probability that the system reaches state j **after *n* steps** starting from state i, but **not necessarily for the first time**

First Passage Probabilities (3)

EXAMPLE



- Probability of going from state 1 to state 1 in 1 step for the first time :

  $f_{11}(1) = p_{11}$

- Probability that the system, starting from state 1, will return to the same state 1 for the first time after n steps : we have to jump in 2 at the first step, remain there for $n-2$ steps and then return in 1 at the last step :

  $f_{11}(n) = p_{12}\, p_{22}^{n-2}\, p_{21}$

- Probability that the system will arrive for the first time in state 2 after n steps : we have to remain in 1 for $n-1$ steps and then go to 2 : $f_{12}(n) = p_{11}^{n-1}\, p_{12}$

First Passage Probabilities (4)

- RELATIONSHIP WITH TRANSITION PROBABILITIES

$$f_{ij}(1) = p_{ij}(1) = p_{ij}$$

$$f_{ij}(2) = p_{ij}(2) - f_{ij}(1) \cdot p_{jj}$$

Probability that the system reaches state j at step 2, given that it was in i at 0

Probability that the system reaches state j for the first time at step 1 (starting from i at 0) and that it remains in j at the successive step

$$f_{ij}(3) = p_{ij}(3) - f_{ij}(1) \cdot p_{jj}(2) - f_{ij}(2) \cdot p_{jj}$$

...

$$\boxed{f_{ij}(k) = p_{ij}(k) - \sum_{l=1}^{k-1} f_{ij}(k-l)\, p_{jj}(l)}$$ (Renewal Equation)

Recurrent, transient and absorbing states (1)

- **DEFINITIONS:**

- First passage probability that the system goes to state j **within m steps** given that it was in i at time 0:

  $q_{ij}(m) = \sum_{n=1}^{m} f_{ij}(n)$ = sum of the probabilities of the **mutually exclusive events** of reaching j for the first time after $n = 1, 2, 3, \ldots, m$ steps

- Probability that the system **eventually** reaches state j from state i: (sooner or later)

  $q_{ij}(\infty) = \lim_{m \to \infty} q_{ij}(m)$

- Probability that the system **eventually** returns to the initial state:

  $f_{ii} = q_{ii}(\infty)$

Recurrent, transient and absorbing states (2)

- State i is **recurrent** if the system starting at such state will **surely** return to it **sooner or later** (i.e., in finite time):

  $$f_{ii} = q_{ii}(\infty) = 1$$

  - For recurrent states $\Pi_i \neq 0$

- State i is **transient** if the system starting at such state has a **finite probability** of **never** returning to it:

  $$f_{ii} = q_{ii}(\infty) < 1$$

  - For these states, at steady state $\Pi_i = 0$

we **cannot** have a **finite Markov process** in which **all states** are **transients** because eventually it will leave them and **somewhere** it must go **at steady state**

- State i is **absorbing** if the system cannot leave it once it enters: $p_{ii} = 1$

Sojourn Time in a state

- **Sojourn time S_i**: time spent in a state i
- Recalling that:

  $p_{ii} =$ probability that the system "moves to" i in one step, given that it was in i

  $1 - p_{ii} =$ probability that the system exits i in one step, given that it was in i

$$\mathbb{P}(S_i = n) = p_{ii}^n(1 - p_{ii})$$

= probability that the system stays in the state i for n steps $(p_{ii}{}^n)$ and then leaves $[(1-p_{ii})]$

$$S_i \sim \text{Geom}(1 - p_{ii})$$

$\mathbb{E}\{S_i\}$ = **average number of steps** before the system **exits** state = $\frac{1}{1-p_{ii}}$ = expected sojourn time

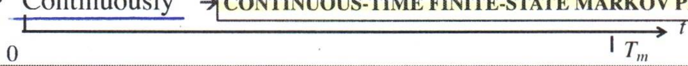
Markov Processes: Basic Elements-Mathematical Representation

- The **random** process of system transition in **time** can be described by an **integer random variable** $X(t)$

  $X(t) = 5$ → the system occupies **state number 5** at time t

- The **stochastic process** may be **observed** at:
  - Discrete times → **DISCRETE-TIME FINITE-STATE MARKOV CHAIN**

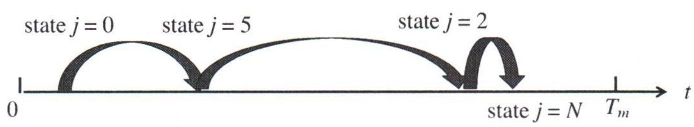
  - Continuously → **CONTINUOUS-TIME FINITE-STATE MARKOV PROCESS**



Continuous-Time Finite-State Markov Processes (CTFSMP)

The conceptual model: Continuous-Time

- The stochastic process is **observed continuously** and **transitions** are assumed to **occur continuously in time**



The conceptual model: Finite State Space

- The random process of system transition between states in time is described by a **stochastic process** $\{X(t);\ t \geq 0\}$
- $X(t) :=$ system state at time t
  - $X(3.6) = 5$: the system is in state number 5 at time $t = 3.6$

OBJECTIVE:
Computing the probability that the system is in a given state as a function of time, for all possible states

$$P[X(t) = j], t \in [0, T_m], j = 0, 1, \ldots, N$$

The conceptual model: Markov Assumption

- **IN GENERAL STOCHASTIC PROCESSES:**

the **probability** of a **future** state of the system usually depends on its **entire life history**

$$P[X(t+v)=j \mid X(t)=i, X(u)=x(u), 0 \le u < t]$$
$$(i = 0, 1, \dots, N, j = 0, 1, \dots, N)$$

- **IN MARKOV PROCESSES:**

the **probability** of a **future** state of the system **only** depends on its **present state**

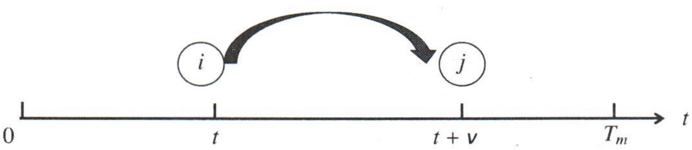
$$P[X(t+v)=j \mid X(t)=i, \cancel{X(u)=x(u), 0 \le u < t}]$$
$$=$$
$$P[X(t+v)=j \mid X(t)=i]$$
$$(i = 0, 1, \dots, N, j = 0, 1, \dots, N)$$

THE PROCESS HAS "NO MEMORY"

The conceptual model: Transition Probabilities

- **Transition probability** that the system in state i at time t moves to state j at time $t + v$

$$p_{ij}(t, t+v) = P[X(t+v)=j \mid X(t)=i], t, v > 0$$
$$(i = 0, 1, \dots, N, j = 0, 1, \dots, N)$$



- If the **transition probability** depends on the **interval v** and **not** on the **individual times** t and $t + v$
  - the probabilities are **stationary**
  - the Markov process is **homogeneous** in time

$$p_{ij}(t, t+v) = P[X(t+v)=j \mid X(t)=i] = p_{ij}(v)$$

The conceptual model: Transition Rates

HYPOTHESIS:

- The time interval $v = dt$ is **small** such that **only one** event (i.e., one **stochastic transition**) can occur within it

$$p_{ij}(v) = p_{ij}(dt) = P[X(t+dt)=j \mid X(t)=i]$$

$=$ (Taylor1^{st} order expansion)

$$\alpha_{ij} \cdot dt + \theta(dt), \quad \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0$$

value that multiplied by dt gives the one step probability where the step is dt

α_{ij} = **transition rate** from state i to state j

The conceptual model: The Transition Probability Matrix (1)

$$p_{ij}(dt) = \alpha_{ij} \cdot dt + \theta(dt), \quad \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0$$

$$p_{ii}(dt) = 1 - \sum_{j \neq i} p_{ij}(dt) = 1 - dt \cdot \sum_{j \neq i} \alpha_{ij} + \theta(dt)$$

- **In analogy with the discrete-time formulation:**

Discrete-time transition probability matrix

$$\underline{\underline{A}} = \begin{array}{c|c} i/j & \begin{matrix} 0 & 1 & \dots & N \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \dots \\ N \end{matrix} & \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0N} \\ p_{10} & p_{11} & \cdots & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & \cdots & p_{NN} \end{pmatrix} \end{array}$$

Continuous-time transition probability matrix

$$\underline{\underline{A}}^* = \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt & \dots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{\substack{j=0 \\ j \neq 1}}^{N} \alpha_{1j} & \dots & \alpha_{1N} \cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

The conceptual model: The Transition Probability Matrix (2)

- **In analogy with the discrete-time formulation:**

$$\underline{P}(t+dt) = \underline{P}(t) \cdot \underline{\underline{A}}^*$$

$$[P_0(t+dt) P_1(t+dt) \dots P_N(t+dt)]$$
$$=$$
$$[P_0(t) P_1(t) \dots P_N(t)] \cdot \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt & \dots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{\substack{j=0 \\ j \neq 1}}^{N} \alpha_{1j} & \dots & \alpha_{1N} \cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

- First-equation:

probability of being in 0 at time t + dt $= P_0(t+dt) = \left[1 - dt \sum_{j=1}^{N} \alpha_{0j}\right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \dots + \alpha_{N0} P_N(t) dt$ = probability of remaining in 0 + P of being somewhere else at time t and moving to 0

The conceptual model: The Transition Probability Matrix (3)

$$P_0(t+dt) = \left[1 - dt \sum_{j=1}^{N} \alpha_{0j}\right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \dots + \alpha_{N0} P_N(t) dt$$

subtract $P_0(t)$ on both sides

$$P_0(t+dt) - P_0(t) = P_0(t) - P_0(t) - \sum_{j=1}^{N} \alpha_{0j} P_0(t) dt + \alpha_{10} P_1(t) dt + \dots + \alpha_{N0} P_N(t) dt$$

divide by dt

$$\frac{P_0(t+dt) - P_0(t)}{dt} = -\sum_{j=1}^{N} \alpha_{0j} P_0(t) + \alpha_{10} P_1(t) + \dots + \alpha_{N0} P_N(t)$$

let $dt \to 0$

$$\lim_{dt \to 0} \frac{P_0(t+dt) - P_0(t)}{dt} = \frac{dP_0}{dt} = -\sum_{j=1}^{N} \alpha_{0j} \cdot P_0(t) + \alpha_{10} \cdot P_1(t) + \dots + \alpha_{N0} \cdot P_N(t)$$

The conceptual model: The Transition Probability Matrix (4)

- Extending to the other equations:

$$\frac{d\underline{P}}{dt} = \underline{P}(t)\cdot \underline{\underline{A}}\ ,\ \underline{\underline{A}} = \begin{pmatrix} \underbrace{-\sum_{j=1}^{N}\alpha_{0j}}_{\alpha_{00}} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & \underbrace{-\sum_{\substack{j=0\\j\neq 1}}^{N}\alpha_{1j}}_{\alpha_{11}} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

TRANSITION RATE MATRIX

System of **linear**, **first-order differential equations** in the unknown state probabilities

$$P_j(t),\ j = 0,1,2,\dots,N,\ t \geq 0$$

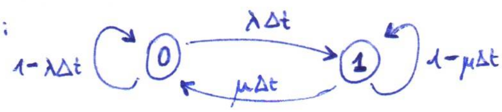
Example 3: one component/one repairman-Markov Diagram and transition rate matrix

ASSUMPTION: exponential failure/repair times distributions

Discrete states = 0 → component working
1 → component failed

Transition rates = λ → rate of failure (i.e., from 0 to 1)
= μ → rate of repair (i.e., from 1 to 0)

Markov diagram:



Transition Rate matrix:

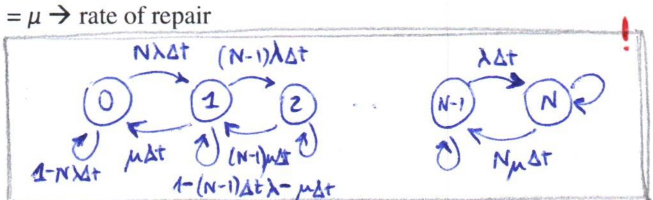
$$\underline{\underline{A}} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Example 4: system with *N* identical components and *N* repairman available

SYSTEM CHARACTERISTICS:

- The system is made of *N* **identical** components
- **Each** component can be in **two** states: **working** or **failed**
- The **transition rates** are **constant** = λ → rate of failure
= μ → rate of repair
- ***N* repairmen** are available

System states:
- 0: none failed
- 1: one failed, N-1 function
- 2: two failed, N-2 function
- ⋮
- N-1: N-1 failed, 1 function
- N: all component failed

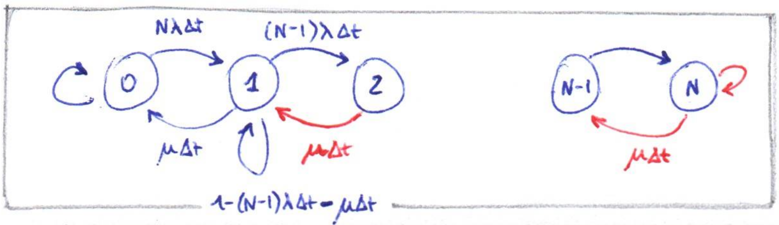


It can be repaired only one at the time / it can break only one at the time (Δt is very small)

Example 5: system with N identical components and 1 repairman available

SYSTEM CHARACTERISTICS:

- The system is made of N **identical** components
- **Each** component can be in **two** states: **working** or **failed**
- The **transition rates** are **constant** $= \lambda \rightarrow$ rate of failure
  $= \mu \rightarrow$ rate of repair
- **One** repairman is available



— = changes w.r.t. the previous case

Solution to the Fundamental Equation

Solution to the fundamental equation of CTFSMP

$$\begin{cases} \dfrac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{\underline{A}} \\ \underline{P}(0) = \underline{C} \end{cases} \quad \text{where} \quad \underline{\underline{A}} = \begin{pmatrix} -\sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j \neq 1}}^{N} \alpha_{1j} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

System of **linear**, **first-order** **differential equations** in the unknown state probabilities

$$P_j(t),\ j = 0,1,2,\dots,N,\ t \geq 0$$

USE LAPLACE TRANSFORM

Solution to the fundamental equation of CTFSMP: the Lapace Transform Method

- Laplace Transform: $\tilde{P}_j(s) = L\left[P_j(t)\right] = \int_0^\infty e^{-st} P_j(t)\,dt, \; j = 0,1,\ldots,N$
- First derivative: $L\left(\frac{dP_j(t)}{dt}\right) = s\cdot\tilde{P}_j(s) - P_j(0), \quad j = 0,1,\ldots,N$
- Apply the Laplace operator to $\frac{d\underline{P}}{dt} = \underline{P}(t)\cdot\underline{\underline{A}}$

$$L\left[\frac{d\underline{P}(t)}{dt}\right] = L\left[\underline{P}(t)\cdot\underline{\underline{A}}\right]$$

First derivative ⟵ $\boxed{s\underline{\tilde{P}}(s) - \underline{C}} = \boxed{\underline{\tilde{P}}(s)\cdot\underline{\underline{A}}}$ ⟶ Linearity

$$\underline{\tilde{P}}(s) = \underline{C}\cdot\left[s\cdot\underline{\underline{I}} - \underline{\underline{A}}\right]^{-1}$$

$\underline{P}(t)$ = **inverse transform of** $\underline{\tilde{P}}(s)$

CTFSMP : steady state probabilities

- At steady state $\frac{d\underline{P}(t)}{dt} = 0 \Rightarrow \frac{d\underline{P}(t)}{dt} = \underline{P}(t)\cdot\underline{\underline{A}} = \underline{\Pi}\cdot\underline{\underline{A}} = 0$
- Solve the (linear) system: $\begin{cases} \underline{\Pi}\cdot\underline{\underline{A}} = 0 \\ \sum_{j=0}^{N}\Pi_j = 1 \end{cases}$
- It can be shown that $\boxed{\Pi_j = \frac{D_j}{\sum_{i=0}^{N} D_i}} \quad j = 0,1,2,\ldots,N$

probability of occupying j at steady state

D_j = determinant of the **square matrix** obtained from A by **deleting** the *j*-th row and column

Example 6: one component/one repairman – Solution to the fundamental equation (4)

- TIME-DEPENDENT STATE PROBABILITIES

$P_0(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\lambda+\mu)t}$ (system instantaneous availability)

$P_1(t) = \frac{\lambda}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\lambda+\mu)t}$ (system instantaneous unavailability)

- STEADY STATE PROBABILITIES

$\Pi_0 = \lim_{t\to\infty} P_0(t) = \frac{\mu *}{\mu+\lambda} = \frac{1/\lambda}{1/\mu+1/\lambda} = \frac{MTBF}{MTTR+MTBF}$

= average fraction of time the system is functioning

$\Pi_1 = \lim_{t\to\infty} P_1(t) = \frac{\lambda}{\mu+\lambda} = \frac{1/\mu}{1/\mu+1/\lambda} = \frac{MTTR}{MTTR+MTBF}$

= average fraction of time the system is down (i.e., under repair)

(*) $P_0(t)$ = system istantaneous availability (probability of being in operational state 0 at time t)

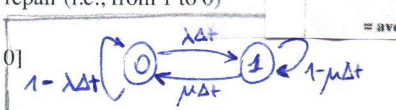
$P_1(t)$ = system istantaneous unavailability (prob. of being in failed state 1 at time t)

Example 6: one component/one repairman – Solution to the fundamental equation (1)

- Component discrete states = 0 → component working
  = 1 → component failed

- **Constant** transition rates = λ → rate of failure (i.e., from 0 to 1)
  = μ → rate of repair (i.e., from 1 to 0)

- Component is working at $t = 0$: $\underline{C} = [1 \;\; 0]$



$\underline{\underline{A}} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

Solve: $\underline{\tilde{P}}(s) = \underline{C}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}$

→ $(s\underline{\underline{I}} - \underline{\underline{A}})^{-1} = \begin{bmatrix} s+\lambda & -\lambda \\ -\mu & \mu+s \end{bmatrix}^{-1} = \frac{1}{\det[(s\underline{\underline{I}} - \underline{\underline{A}})]}\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix} = \frac{1}{s^2+s\lambda+s\mu}\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}$

$\underline{\tilde{P}}(s) = \frac{1}{s^2+s\lambda+s\mu}[1 \;\; 0]\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix} = \left[\frac{s+\mu}{s(s+\lambda+\mu)} \quad \frac{\lambda}{s(s+\lambda+\mu)}\right]$

$\begin{cases} L^{-1}\left[\frac{1}{s+a}\right] = e^{-ax} \\ L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1}{a}(1-e^{-ax}) \end{cases}$

⟹ $\underline{P}(t) = \left[\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}, \; \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}\right] = [P_0(t), P_1(t)]$ (*)

Quantity of Interest

Frequency of departure from a state

- **Unconditional** probability of arriving in state j in the next dt departing from state i at time t :

  $P[X(t+dt)=j, X(t)=i]$

  $P[X(t+dt)=j|X(t)=i]P[X(t)=i] = p_{ij}(dt)P_i(t)$

- **Frequency of departure** from state i to state j:

$$v_{ij}^{dep}(t) = \lim_{dt\to 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t) \underset{\text{(at steady state)}}{=} v_{ij}^{dep} = \alpha_{ij} \cdot \Pi_i$$

- **Total frequency of departure** from state i to any other state j:

$$v_i(t) = \sum_{\substack{j=0 \\ j\neq i}}^{N} \alpha_{ij} \cdot P_i(t) = -\alpha_{ii} \cdot P_i(t) \underset{\text{(at steady state)}}{=} v_i = -\alpha_{ii} \cdot \Pi_i$$

Frequency of arrival to a state

- **In analogy**, considering the **arrivals** to state i from any state k:

$$v_i^{arr}(t) = \sum_{\substack{k=0 \\ k\neq i}}^{N} \alpha_{ki} \cdot P_k(t)$$

$$v_i^{arr} = \sum_{\substack{k=0 \\ k\neq i}}^{N} \alpha_{ki} \cdot \Pi_k \quad \text{(at steady state)}$$

- Since $\underline{\Pi} \cdot \underline{\underline{A}} = 0 = \sum_{k=0}^{N} \alpha_{ki} \cdot \Pi_k \Rightarrow -\alpha_{ii} \cdot \Pi_i = \sum_{\substack{k=0 \\ k\neq i}}^{N} \alpha_{ki} \cdot \Pi_k$

AT STEADY STATE:
frequency of departures from state i = frequency of arrivals to state i

System Failure Intensity

- **SYSTEM FAILURE INTENSITY W_f:**
  - **Rate** at which **system failures** occur
  - **Expected number** of system **failures** per **unit of time**
  - **Rate** of **exiting a success state** to go into one of **fault**

$$W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \to F}$$

rate of transitions from a functioning state to a non functioning one (transitions that exit a functioning state to go to a non functioning state)

S = set of success states of the system

F = set of failure states of the system

$P_i(t)$ = probability of the system being in the functioning state i at time t

$\lambda_{i \to F}$ = conditional (transition) probability of leaving success state i towards a failure state

= probability of being in a state of success ($\forall$) multiplied by the rate of going from the state of success to any of the fail states

System Repair Intensity

- **SYSTEM REPAIR INTENSIT W_r:**
  - **Rate** at which **system repairs** occur
  - **Expected number** of system **repairs** per **unit of time**
  - **Rate** of **exiting a failed state** to go into one of **success**

$$W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \to S}$$

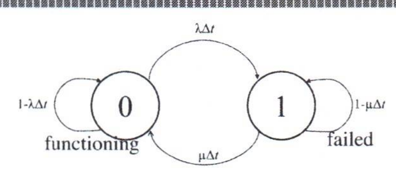
S = set of success states of the system

F = set of failure states of the system

$P_j(t)$ = probability of the system being in the failure state j at time t

$\mu_{j \to S}$ = conditional (transition) probability of leaving failure state j towards a success state

Example 7: one component/one repairman – Failure and repair intensities



S = set of success states of the system = {0}

F = set of failure states of the system = {1}

$\lambda_{i \to F} = \lambda$

$\mu_{j \to S} = \mu$

Failure/repair intensities?

$$W_f(t) = \lambda P_0(t) = \frac{\lambda\mu}{\mu+\lambda} + \frac{\lambda^2}{\mu+\lambda} e^{-(\lambda+\mu)t}$$

$$W_r(t) = \mu P_1(t) = \frac{\mu\lambda}{\mu+\lambda} + \frac{\mu\lambda}{\mu+\lambda} e^{-(\lambda+\mu)t}$$

Sojourn Time in a state (1)

83

- **Sojourn time T_i**: time spent in a state i
- Markov property and time homogeneity imply that if at time t the process is in state i, the time remaining in state i is independent of time already spent in state i

$\mathbb{P}(T_i > t + s | T_i > t) = \mathbb{P}(X(t+u) = i, 0 \le u \le s | X(u) = i, o \le u \le t) =$

$= \mathbb{P}(X(t+u) = i, 0 \le u \le s | X(t) = i)$ (by Markov property)

$= \mathbb{P}(X(u) = i, 0 \le u \le s | X(0) = i)$ (by homogeneity)

$= \mathbb{P}(T_i > s)$ **Memoryless Property**

- The only distribution satisfying the memoryless property is the **Exponential distribution**

$$T_i \sim Exp$$

Enrico Zio

POLITECNICO MILANO 1863

Sojourn Time in a state (2)

84

- **Sojourn time T_i**: time spent in a state i
- The system **remain** in state i before leaving it with **constant** rate $-\alpha_{ii}$

$$T_i \sim Exp(-\alpha_{ii})$$

- **Expected sojourn time l_i**: average time of occupancy of state i

$$l_i = \mathbb{E}\{T_i\} = \frac{1}{-\alpha_{ii}}$$

Enrico Zio

POLITECNICO MILANO 1863

Sojourn Time in a state (3)

85

- Total frequency of departure at steady state: $\nu_i = -\alpha_{ii} \cdot \Pi_i$
- Average time of occupancy of state: $l_i = \dfrac{1}{-\alpha_{ii}}$

$$\nu_i = -\alpha_{ii} \cdot \Pi_i = \frac{\Pi_i}{l_i}$$

$$\Pi_i = \nu_i \cdot l_i$$

The **mean** proportion of time Π_i that the system spends in state i is equal to the visit frequency to state i multiplied by the mean duration of one visit in state i

Enrico Zio

POLITECNICO MILANO 1863

System Availability

- **System instantaneous availability** at time t
  = **sum** of the **probabilities** of being in a **success** state at time t

$$p(t)=\sum_{i\in S}P_i(t)=1-q(t)=1-\sum_{j\in F}P_j(t)$$

In the Laplace domain

$$\tilde{p}(s)=\sum_{i\in S}\tilde{P}_i(s)=\frac{1}{s}-\sum_{j\in F}\tilde{P}_j(s)$$

S = set of success states of the system

F = set of failure states of the system

System Reliability

- **TWO CASES:**

$\boxed{\text{1) Non-Reparaible Systems} \rightarrow \text{No repairs allowed}}$

1) Non-Reparaible Systems
→ No repairs allowed

2) Reparaible Systems
→ Repairs allowed

System Reliability: Non-Reparaible Systems

- No repairs allowed ⇒ **Reliability = Availability** $R(t)\equiv p(t)=1-q(t)$
- In the Laplace Domain: $\tilde{R}(s)=\sum_{i\in S}\tilde{P}_i(s)=\frac{1}{s}-\sum_{j\in F}\tilde{P}_j(s)$
- **Mean Time to Failure (MTTF):**

$$MTTF=\int_0^{\infty}R(t)dt=\left[\int_0^{\infty}R(t)e^{-st}dt\right]_{s=0}=\tilde{R}(0)=\sum_{i\in S}\tilde{P}_i(0)=\left[\frac{1}{s}-\sum_{j\in F}\tilde{P}_j(s)\right]_{s=0}$$

MTTF is also the value of the Laplace transform of the reliability for s=0

System Reliability

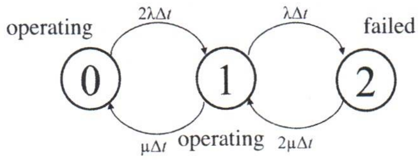
- **TWO CASES:**

1) Non-reparaible systems
→ No repairs allowed

2) Reparaible systems
→ Repairs allowed

System Reliability: Reparaible Systems (1)

1. Exclude all the failed states $j \in F$ from the transition rate matrix $\underline{\underline{A}}$ (we're considering the F states absorbing)



$S = \{0,1\}$
$F = \{2\}$

2 components
2 repairmen

$$\underline{\underline{A}} = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & \mu+\lambda & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}$$

$$\underline{\underline{A}} = \left(\begin{array}{cc|c} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ \hline 0 & 2\mu & -2\mu \end{array}\right) \Rightarrow \underline{\underline{A}}' = \begin{pmatrix} -2\lambda & 2\lambda \\ \mu & -(\mu+\lambda) \end{pmatrix}$$

The new matrix $\underline{\underline{A}}'$ contains the transition rates for transitions **only among the success states** $i \in S$

(the "reduced" system is virtually functioning continuously with no interruptions)

System Reliability: Reparaible Systems (2)

2. Solve the **reduced problem** of $\underline{\underline{A}}'$ for the probabilities $P_i^*(t),\ i \in S$ of being in these **(transient) safe states**

$$\frac{d\underline{P}^*(t)}{dt} = \underline{P}^*(t) \cdot \underline{\underline{A}}'$$

Reliability

$$R(t) = \sum_{i \in S} P_i^*(t)$$

Mean Time To Failure (MTTF)

$$MTTF = \int_0^\infty R(t)\,dt = \sum_{i \in S} \tilde{P}_i^*(0) = \tilde{R}(0)$$

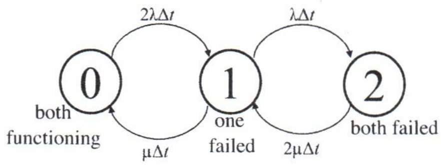
NOTICE: in the reduced problem we have only transient states $\Rightarrow \Pi_i^* = P_i^*(\infty) = 0$

we consider the failure as absorbing → {0,1} = S are transient

Example 8: system with two identical components and two repairmen available (1)

CRC

92

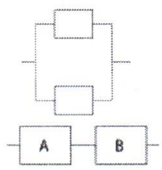


System reliability $R(t)$?
(= probability of the system of being in safe states 0 or 1 continuously from $t=0$)

- **TWO CASES:**

  a) Parallel logic (1 out of 2)

  b) Series logic (2 out of 2)

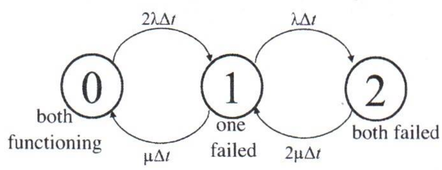


Enrico Zio POLITECNICO MILANO 1863

Example 8: system with two identical components and two repairmen available (8)

CRC

10

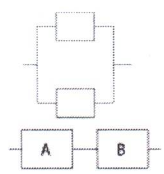


$$\underline{\underline{A}} = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}$$

- **TWO CASES:**

  a) Parallel logic (1 out of 2)

  b) Series logic (2 out of 2)



Enrico Zio POLITECNICO MILANO 1863

a) Exclude all failed states: $\underline{\underline{A}} = \left[\begin{array}{cc|c} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ \hline 0 & 2\mu & -2\mu \end{array}\right] \Longrightarrow \underline{\underline{A}}' = \begin{bmatrix} -2\lambda & 2\lambda \\ \mu & -(\mu+\lambda) \end{bmatrix}$

Then we solve the reduced problem.
In time domain:

$$\begin{cases} \dfrac{d\underline{p}^*}{dt} = \underline{p}^*(t)\cdot\underline{\underline{A}}' \\ \underline{p}^*(0) = [1 \;\; 0] \end{cases} \Longrightarrow \frac{d\underline{p}^*}{dt} = \underline{p}^*(t)\begin{bmatrix} -2\lambda & 2\lambda \\ \mu & -(\lambda+\mu) \end{bmatrix}$$

in Laplace domain:

$$\tilde{\underline{p}}^*(s) = \tilde{\underline{p}}^*(0)\,(s\underline{\underline{I}}-\underline{\underline{A}}')^{-1} \longrightarrow \tilde{\underline{p}}^*(s) = [1 \;\; 0](s\underline{\underline{I}}-\underline{\underline{A}}')^{-1}$$

$$(s\underline{\underline{I}}-\underline{\underline{A}}')^{-1} = \frac{1}{(s+2\lambda)(s+\mu+\lambda)-2\lambda\mu}\begin{bmatrix} s+\mu+\lambda & 2\lambda \\ \mu & s+2\lambda \end{bmatrix} = \frac{1}{(s-w_0)(s-w_1)}\begin{bmatrix} s+\mu+\lambda & 2\lambda \\ \mu & s+2\lambda \end{bmatrix}$$

where $w_{0,1} = \dfrac{-3\lambda \pm \sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}$

$$\Longrightarrow \tilde{\underline{p}}^*(s) = \underline{C}^*(s\underline{\underline{I}}-\underline{\underline{A}}')^{-1} = \frac{1}{(s-w_0)(s-w_1)}[1 \;\; 0]\begin{bmatrix} s+\mu+\lambda & 2\lambda \\ \mu & s+2\lambda \end{bmatrix} = \frac{1}{(s-w_0)(s-w_1)}\left[s+\mu+\lambda \quad 2\lambda\right]$$

(a) System reliability: $\tilde{R}(s) = \tilde{P}_0(s) + \tilde{P}_1(s)$ (laplace domain)
and since:

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-ax} \quad , \quad L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1}{a}(1-e^{-ax})$$

$\Longrightarrow$ System reliability: (in time domain)

$$R(t) = \frac{w_0 e^{w_1 t} - w_1 e^{w_0 t}}{w_0 - w_1}$$

Mean Time To failure: $MTTF = \tilde{R}(0) = \Sigma_i P_i^*(0)$?
starting from: $\tilde{\underline{P}}^*(s) = \underline{C}^* (s\underline{\underline{I}} - \underline{\underline{A}}')^{-1}$

$\Longrightarrow$ $MTTF = \underline{C}^* (-\underline{\underline{A}}')^{-1} \cdot \underline{w}^T$ with $\underline{w} = [1\ 1 \ldots 1]^T$

$$= [1\ 0]\begin{bmatrix} 2\lambda & -2\lambda \\ -\mu & \mu+\lambda \end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= [1\ 0]\frac{1}{2\lambda(\lambda+\mu)-2\lambda\mu}\begin{bmatrix} \mu+\lambda & 2\lambda \\ \mu & 2\lambda \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2\lambda^2}[\mu+\lambda\ \ 2\lambda]\begin{bmatrix}1\\1\end{bmatrix} = \frac{3}{2\lambda} + \frac{\mu}{2\lambda^2}$$

b) Since they're in series, the only operating state is 0 (!)
When we create the subproblem:

$$\underline{\underline{A}} = \left[\begin{array}{c|cc} -2\lambda & 2\lambda & 0 \\ \hline \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{array}\right] \Longrightarrow \underline{\underline{A}}' = [-2\lambda]$$

In time domain:

$$\begin{cases} \frac{d\underline{P}^*}{dt} = \underline{P}^* \cdot \underline{\underline{A}}' \\ \underline{P}^*(0) = \underline{C}^* \end{cases} \longrightarrow \begin{cases} \frac{dP_0^*}{dt} = -2\lambda P_0^* \\ P_0^*(0) = 1 \end{cases} \Longrightarrow R(t) = P_0^*(t) = e^{-2\lambda t}$$

(prob. of never leaving 0)