1 Banach Spaces

1.1 Normed Spaces

7.9. Write the definition of **normed space** and provide examples.
What is the **metric space induced** by a given normed space?

Let X be a vector space.
A norm on X is a function $||\cdot|| : X \to [0, +\infty)$ such that:
(i) $\forall x \in X$: $||x|| \geq 0$, $||x|| = 0 \Leftrightarrow x = 0$
(ii) $\forall \alpha \in \mathbb{R}$, $x \in X$: $||\alpha\, x|| = |\alpha| \cdot ||x||$
(iii) $\forall x, y \in X$: $||x + y|| \leq ||x|| + ||y||$
The property (iii) is called triangular inequality.
A norm satisfies also the inverse triangular inequality, namely:
$\forall x, y \in X$: $\big|\, ||x|| - ||y||\, \big| \leq ||x - y||$.
The pair $(X, ||\cdot||)$ is called normed space.

$\mathbb{R}^N$	$\lVert x\rVert_p = \left(\sum_{i=1}^N \lvert x_i\rvert^p\right)^{\frac{1}{p}}$
$\mathbb{R}^N$	$\lVert x\rVert_\infty = \max_{i=1,..,N} \lvert x_i\rvert$
$C^0([a,b])$	$\lVert f\rVert_\infty = \max_{x\in[a,b]} \lvert f(x)\rvert$
$L^1(X, \mathcal{A}, \mu)$	$\lVert f\rVert_1 = \int_X \lvert f(x)\rvert\, dx$
$L^\infty(X, \mathcal{A}, \mu)$	$\lVert f\rVert_\infty = \operatorname{ess\,sup}_X \lvert f\rvert$
$\ell^p = \{x = \{x^{(k)}\}_k \subseteq \mathbb{R} : \sum_{j=1}^\infty \lvert x^{(k)}\rvert^p < +\infty\}$	$\lVert x\rVert_p = \left(\sum_{k=1}^\infty \lvert x^{(k)}\rvert^p\right)^{\frac{1}{p}}$
$\ell^\infty = \{x = \{x^{(k)}\}_k \subseteq \mathbb{R} : \sup_{k\in\mathbb{N}} \lvert x^{(k)}\rvert < +\infty\}$	$\lVert x\rVert_\infty = \sup_{k\in\mathbb{N}} \lvert x^{(k)}\rvert$
$C^k([a,b])$	$\lVert f\rVert_{\infty,k} = \sum_{i=0}^k \lVert f^{(i)}\rVert_\infty$
$BV([a,b])$	$\lVert f\rVert_{BV} = \lVert f\rVert_1 + V_a^b(f)$
$AC([a,b])$	$\lVert f\rVert_{AC} = \lVert f\rVert_1 + \lVert f'\rVert_1$
$L^p(X, \mathcal{A}, \mu)\quad p \in [1, \infty)$	$\lVert f\rVert_p = \left(\int_X \lvert f(x)\rvert^p\, dx\right)^{\frac{1}{p}}$

Given a normed space $(X, ||\cdot||)$, the function $d(x, y) := ||x - y||$ is a distance on X.
Thus, we say that (X, d), with $d(x, y) = ||x - y||$, is the metric space induced by a normed space.
Starting from a norm we can always define a metric and from a metric we can always define the induced metric space.

1.2 Sequences and Series

7.10. In a normed space, write the definition of: **convergent sequence, Cauchy sequence, bounded sequence.**
Which are the relations among these notions? Show that if $x_n \to x$, then $||x_n|| \to ||x||$ as $n \to +\infty$.

Let $(X, ||\cdot||)$ be a normed space, $\{x_n\}_n \subseteq X$, $x \in X$.
Then:
- the sequence $\{x_n\}_n$ converges to x, and we write $x_n \xrightarrow[n\to\infty]{} x$, if: $||x_n - x|| \xrightarrow[n\to\infty]{} 0$
- the sequence $\{x_n\}_n$ is Cauchy if: $\forall \varepsilon > 0\ \exists \bar{n} \in \mathbb{N}$: $||x_n - x_m|| < \varepsilon\ \forall n, m > \bar{n}$
- the sequence $\{x_n\}_n$ is bounded if $\exists M > 0$: $||x_n|| < M\ \ \forall n \in \mathbb{N}$

convergent
$\Downarrow\ \cancel{\Uparrow}$
Cauchy
$\Downarrow\ \cancel{\Uparrow}$
bounded

Let $\{x_n\}_n \subseteq X$ be a sequence. The following relations hold:
- $\{x_n\}_n$ is Cauchy $\Rightarrow$ $\{x_n\}_n$ is bounded
- $\{x_n\}_n$ is Cauchy $\not\Leftarrow$ $\{x_n\}_n$ is bounded
- $\{x_n\}_n$ is convergent $\Rightarrow$ $\{x_n\}_n$ is Cauchy
- $\{x_n\}_n$ is convergent $\not\Leftarrow$ $\{x_n\}_n$ is Cauchy
- $\{x_n\}_n$ is convergent $\Leftarrow$ $\{x_n\}_n$ is Cauchy, $\exists \{x_{n_k}\}_k \subseteq \{x_n\}_n$ convergent
- $\{x_n\}_n$ is convergent $\Rightarrow$ $\{x_n\}_n$ is bounded
- $\{x_n\}_n$ is convergent $\not\Leftarrow$ $\{x_n\}_n$ is bounded

Consider now $\{x_n\}_n \subseteq X$, $x \in X$.
If $x_n \xrightarrow[n\to\infty]{} x \Rightarrow ||x_n|| \xrightarrow[n\to\infty]{} ||x||$.
In fact, a norm satisfies the inverse triangular inequality, namely:
$\forall x, y \in X$: $\big|\, ||x|| - ||y||\, \big| \leq ||x - y||$.
Therefore:
$$\big|\, ||x_n|| - ||x||\, \big| \leq \underbrace{||x_n - x|| \xrightarrow[n\to\infty]{} 0}_{x_n \xrightarrow[n\to\infty]{} x}$$
Thus, we can conclude that $||x_n|| \xrightarrow[n\to\infty]{} ||x||$.

7.11. Write the definition of **series** in a normed space. Is it true that if $\sum_{n=0}^\infty ||x_n|| < \infty$, then $\sum_{n=0}^\infty x_n$ is convergent?

Let $(X, ||\cdot||)$ be a normed space, $\{x_n\}_n \subseteq X$.
Moreover, $\forall n \in \mathbb{N}$ we define:
$s_n := x_0 + x_1 + .. + x_n = \sum_{k=0}^n x_k$
Then $\{s_n\}_n$ is called sequence of partial sums, or series.
The series $\{s_n\}_n$ is said to be convergent if there exists $x \in X$ such that $s_n \xrightarrow[n\to\infty]{} x$.
Then, we say that $x = \sum_{k=0}^\infty x_k$ is the sum of the series.
We recall that the convergence $s_n \xrightarrow[n\to\infty]{} x$ is intended w.r.t. the norm of the normed space X.
In a generic normed space, the following hold:
- $\sum_{n=1}^\infty x_n$ convergent $\Rightarrow$ $\sum_{n=1}^\infty ||x_n||$ convergent
- $\sum_{n=1}^\infty x_n$ convergent $\not\Leftarrow$ $\sum_{n=1}^\infty ||x_n||$ convergent

However, if the normed space $(X, ||\cdot||)$ is complete, namely if every Cauchy sequence in X is convergent, then also the second implication holds.

1.3 Banach Spaces

7.12. What is a **complete normed space**? Write the definition of **Banach space**, provide examples.

Let $(X, ||\cdot||)$ be a normed space, $\{x_n\}_n \subseteq X$, $x \in X$.
The sequence $\{x_n\}_n$ is convergent, and we write $x_n \xrightarrow[n\to\infty]{} x$, if: $||x_n - x|| \xrightarrow[n\to\infty]{} 0$.
The sequence $\{x_n\}_n$ is Cauchy if: $\forall \varepsilon > 0\ \exists \bar{n} \in \mathbb{N}$: $||x_n - x_m|| < \varepsilon\ \forall n, m > \bar{n}$.
Every convergent sequence is Cauchy but, in general, not every Cauchy sequence in convergent.
The normed space $(X, ||\cdot||)$ is said to be complete if every Cauchy sequence in X is convergent.
A complete normed space is called Banach space.

$\mathbb{R}^N$	$\|\|x\|\|_p = (\sum_{i=1}^N \|x_i\|^p)^{\frac{1}{p}}$
$\mathbb{R}^N$	$\|\|x\|\|_\infty = \max_{i=1,\dots,N} \|x_i\|$
$C^0([a,b])$	$\|\|f\|\|_\infty = \max_{x\in[a,b]} \|f(x)\|$
$L^1(X, \mathcal{A}, \mu)$	$\|\|f\|\|_1 = \int_X \|f(x)\|\, dx$
$L^\infty(X, \mathcal{A}, \mu)$	$\|\|f\|\|_\infty = \text{ess sup}_X \|f\|$
$\ell^p = \{x = \{x^{(k)}\}_k \subseteq \mathbb{R} : \sum_{j=1}^\infty \|x^{(k)}\|^p < +\infty\}$	$\|\|x\|\|_p = (\sum_{k=1}^\infty \|x^{(k)}\|^p)^{\frac{1}{p}}$
$\ell^\infty = \{x = \{x^{(k)}\}_k \subseteq \mathbb{R} : \sup_{k\in\mathbb{N}} \|x^{(k)}\| < +\infty\}$	$\|\|x\|\|_\infty = \sup_{k\in\mathbb{N}} \|x^{(k)}\|$
$C^k([a,b])$	$\|\|f\|\|_{\infty,k} = \sum_{i=0}^k \|\|f^{(i)}\|\|_\infty$
$BV([a,b])$	$\|\|f\|\|_{BV} = \|\|f\|\|_1 + V_a^b(f)$
$AC([a,b])$	$\|\|f\|\|_{AC} = \|\|f\|\|_1 + \|\|f'\|\|_1$
$L^p(X, \mathcal{A}, \mu)\quad p \in [1, \infty)$	$\|\|f\|\|_p = (\int_X \|f(x)\|^p\, dx)^{\frac{1}{p}}$

1.4 Separability

7.13. Show that $C^0([a,b])$ is separable.

(Recall) Let (X, d) be metric space. *(the norm is not required)*
Then, X is separable if $\exists\, A \subset X$ countable and dense in X, namely $\overline{A} = X$.

Theorem. $C^0([a,b])$ is separable.

proof.

Let us denote with P the set of polynomials and with $P_\mathbb{Q}$ the set of polynomials with rational coefficients.
By the Stone-Weierstrass theorem we know that P is dense in $C^0([a,b])$, namely:
$\forall f \in C^0([a,b])\ \forall \varepsilon > 0\ \exists p \in P$ such that: $||f - p||_\infty < \frac{\varepsilon}{2}$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a polynomial $r \in P_\mathbb{Q}$ such that:
$\Rightarrow ||p - r||_\infty < \frac{\varepsilon}{2}$
$\Rightarrow ||f - r||_\infty \leq ||f - p||_\infty + ||p - r||_\infty < \varepsilon$
$\Leftrightarrow \forall f \in C^0([a,b])\ \forall \varepsilon > 0\ \exists r \in P_\mathbb{Q}$ such that: $||f - r||_\infty < \varepsilon$
namely, $P_\mathbb{Q}$ is dense in $C^0([a,b])$.
Therefore, since $P_\mathbb{Q}$ is **countable** then $C^0([a,b])$ is separable. ■

1.5 Completeness

8.1. State and prove the criterion for completeness of a normed space.

Theorem. **Criterion for completeness** *(characterization of Banach spaces)*
Let $(X, ||\cdot||)$ be a normed space. Then:
X is a Banach space $\Leftrightarrow$ $[\forall \{x_n\}_n \subseteq X$ such that $\sum_{n=1}^\infty ||x_n||$ converges $\Rightarrow \underbrace{\sum_{n=1}^\infty x_n \text{ converges}}_{\text{w.r.t. the norm of the normed space } X}]$.

proof.

($\Rightarrow$) We recall the Cauchy criterion for series of real numbers:
$\sum_{i=1}^\infty a_i$ is convergent $\Leftrightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall m, n > N$: $\sum_{i=m+1}^n a_i < \varepsilon$.
Therefore, since $\sum_{n=1}^\infty ||x_n||$ converges:
$\forall \varepsilon > 0\ \exists \bar{n} \in \mathbb{N}$ such that $\forall m, n > \bar{n}$: $\sum_{i=m+1}^n ||x_i|| < \varepsilon$.
In other terms, recalling that $s_n := \sum_{i=1}^n x_i$:
$\Rightarrow s_n - s_m = \sum_{i=m+1}^n x_i$
$\Rightarrow ||s_n - s_m|| \leq \sum_{i=m+1}^n ||x_i|| < \varepsilon$.
This means that $\{s_n\}_n$ is a Cauchy sequence in X and, since X is a Banach space:
$\exists x \in X$ such that $s_n \xrightarrow[n\to\infty]{} x$
with $x = \sum_{i=1}^\infty x_i$, which means that $\sum_{i=1}^\infty x_i$ converges.

($\Leftarrow$) Let $\{y_n\}_n$ be a Cauchy sequence in X.
By definition of Cauchy, we can extract a subsequence $\{y_{n_k}\}_k \subset \{y_n\}_n$ such that:
$||y_{n_{k+1}} - y_{n_k}|| < \frac{1}{k^2}\quad \forall k \in \mathbb{N}$.
Therefore, since $\sum_{k=1}^\infty \frac{1}{k^2} < \infty \Rightarrow \sum_{k=1}^\infty ||y_{n_{k+1}} - y_{n_k}||$ converges.
Let us denote $x_k := y_{n_{k+1}} - y_{n_k}$:
$\Rightarrow \sum_{k=1}^\infty ||x_k||$ converges
$\Rightarrow \sum_{k=1}^\infty x_k$ converges (by assumptions)
$\Rightarrow \exists s \in X$: $s = \lim_{m\to\infty} \sum_{k=1}^m x_k = \lim_{m\to\infty} \sum_{k=1}^m (y_{n_{k+1}} - y_{n_k}) = \lim_{m\to\infty} y_{n_{m+1}} - y_{n_1}$
$\Rightarrow \exists s \in X$: $\lim_{m\to\infty} y_{n_{m+1}} = s + y_{n_1}$
$\Leftrightarrow \{y_{n_m}\}_m$ converges
Since $\{y_n\}_n$ is Cauchy and $\{y_{n_m}\}_m \subset \{y_n\}_n$ converges, we know that also $\{y_n\}_n$ converges.
Hence, any $\{y_n\}_n$ Cauchy sequence in X converges.
$\Rightarrow X$ is complete and thus, by definition, X is a Banach space. ■

Def. **Open/Closed ball**

Let $(X, ||\cdot||)$ be a normed space. Then $\forall x_0 \in X$, $r > 0$ we define:

$B_r(x_0) := \{x \in X : ||x - x_0|| < r\}$ open ball

$\overline{B}_r(x_0) := \{x \in X : ||x - x_0|| \leq r\}$ closed ball

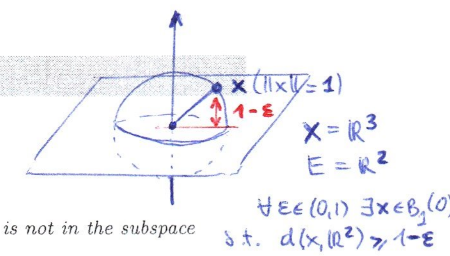
Moreover, we denote the closure of $B_r(x_0)$ as $\overline{B_r(x_0)}$.

Prop. If (X, d) is a generic metric space then $\overline{B_r(x_0)} \subsetneq \overline{B}_r(x_0)$.

If $(X, ||.||)$ is a Banach space then $\overline{B_r(x_0)} = \overline{B}_r(x_0)$.

1.6 Compactness

8.2. State and prove the **Riesz Lemma.**



Lemma. Riesz Lemma

Let $(X, ||\cdot||)$ be a normed space, $E \subsetneq X$ closed subspace.

Then $\forall \varepsilon \in (0, 1)$ $\exists x \in X$ such that:

$||x|| = 1$, $dist(x, E) := \inf_{y \in E} ||x - y|| \geq 1 - \varepsilon$.

If we consider a closed subspace E, then $\forall\varepsilon$ we can find an unitary element which is not in the subspace and which is such that its distance with the subspace is $\geq 1 - \varepsilon$.

proof.

Let $y \in X \setminus E$.

Since E is closed $d := dist(y, E) > 0$.

Let $\varepsilon \in (0, 1)$. Then we can choose $z \in E$ such that:

$d \leq ||y - z|| \leq \frac{d}{1-\varepsilon}$.

We know that such z exists because $d = dist(y, E) := \inf_{\xi \in E} ||\xi - y||$ and $d \leq \frac{d}{1-\varepsilon}$.

Let $x := \frac{y-z}{||y-z||}$.

Then $\forall \xi \in E$:

$$||x - \xi|| = \left|\left| \frac{y-z}{||y-z||} - \xi \right|\right| = \frac{1}{||y-z||} ||\, y - z - \xi\, ||y - z||\, || = \frac{1}{||y-z||} \underbrace{||\, y - \underbrace{(z + \xi\, ||y - z||)}_{\in E\ (1)}\, ||}_{> d\ (2)} \geq \frac{d}{||y-z||} \geq 1 - \varepsilon.$$

(1) since $z, \xi \in E$, $||y - z|| \in \mathbb{R}$ and E is a subspace

(2) since the distance between y and any other element of E is bigger than $d := \inf_{\xi \in E} ||y - \xi||$ ∎

8.3. State and prove the **Riesz Theorem.**

Theorem. Riesz Theorem

Let $(X, ||\cdot||)$ be a normed space. If the closed ball $\overline{B}_1(0)$ is compact $\Rightarrow$ $dim(X) < \infty$.

proof.

We'll prove that if $dim(X) = \infty$ then the ball $\overline{B}_1(0)$ cannot be compact.

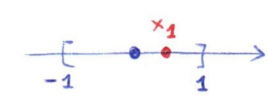
To do so, we'll focus on a characterization of compactness:

$\overline{B}_1(0)$ is compact $\Leftrightarrow$ $\overline{B}_1(0)$ is sequentially compact, namely if any *bounded* sequence has a convergent subsequence.

- Let $x_1 \in \overline{B}_1(0)$ and $Y_1 := span\{x_1\}$.

  In a normed space any vector subspace of finite dimension is closed, thus Y_1 is closed.

  If $X = Y_1$ $\Rightarrow$ $dim(X) = 1 < \infty$ and the thesis is proved.



- If $X \neq Y_1$ then we can apply Riesz lemma with $\varepsilon = \frac{1}{2}$ to conclude that $\exists x_2 \in \overline{B}_1(0)$ such that:

  $||x_2 - x_1|| \geq \frac{1}{2}$.

  This is because, since Y_1 is a closed subspace of X then $\exists x_2 \in X$ such that:

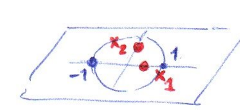
  $dist(x_2, Y_1) := \inf_{\xi \in Y_1} ||x_2 - \xi|| \geq 1 - \varepsilon$.

  Choosing $\varepsilon = \frac{1}{2}$ we get $||x_2 - \xi|| \geq 1 - \varepsilon = \frac{1}{2}$ $\forall \xi \in Y_1$, which holds also for $x_1 \in Y_1$.

  We now define $Y_2 := span\{x_1, x_2\}$.

  Since Y_2 is of finite dimension then it is closed.

  If $X = Y_2$ $\Rightarrow$ $dim(X) = 2 < \infty$ and the thesis is proved.

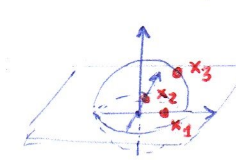


- If $X \neq Y_2$ then we can apply Riesz lemma with $\varepsilon = \frac{1}{2}$ to conclude that $\exists x_3 \in \overline{B}_1(0)$ such that:

  $||x_3 - x_i|| \geq \frac{1}{2}$ $i = 1, 2$.

  Again, $Y_3 = span\{x_1, x_2, x_3\}$ is of finite dimension and therefore it is closed.

  If $X = Y_3$ $\Rightarrow$ $dim(X) = 3 < \infty$ and the thesis is proved.



- We iterate the procedure.

  If $dim(X) = \infty$ the process can be iterated infinitely many times, generating a sequence $\{x_n\}_n \subseteq \overline{B}_1(0)$ such that:

  $||x_i - x_j|| \geq \frac{1}{2}$ $\forall i, j \in \mathbb{N}$, $i \neq j$.

  The whole sequence $\{x_n\}_n$ is contained in $\overline{B}_1(0)$ and so it is bounded ($||x_n|| \leq 1$ $\forall n \in \mathbb{N}$).

  However, there cannot be a convergent subsequence.

  In fact, if $\exists \{x_{n_k}\}_k \subseteq \{x_n\}_n$ such that $\{x_{n_k}\}_k$ converges then $\{x_{n_k}\}_k$ has to be Cauchy. However the whole sequence is such that every element distances at least $\frac{1}{2}$ from the other, so there cannot exist a Cauchy sequence.

  Thus, $\overline{B}_1(0)$ is **not** (sequentially) compact. ∎

Prop. From Riesz theorem it follows that:

$E \subseteq X$ closed and bounded is compact $\Leftrightarrow$ $dim(X) < \infty$.

Prop. Ascoli-Arzelà theorem states:

$F \subset C^0(X)$ is bounded, closed and equicontinuous $\Rightarrow$ F is compact.

The hypothesis of equicontinuity is the reason we can conclude that F is compact even though it is a subset of a space of infinite dimension ($C^0(X)$).

1.7 Equivalent Norms

8.4. Write the definition of **equivalent norms.**
Provide examples of equivalent norms and of norms that are not equivalent.

Let $(X, ||\cdot||)$, $(X, ||\cdot||_\#)$ be two normed spaces.
Then, we say that the two norms $||\cdot||$ and $||\cdot||_\#$ are equivalent if $\exists\, m,\ M > 0$ such that:
$m\,||x|| \le ||x||_\# \le M\,||x|| \quad \forall x \in X.$

Examples:
- In $C^1([a,b])$ the norms $||f||_{\infty,1} = ||f||_\infty + ||f'||_\infty$ and $||f||_\# = |f(a)| + ||f'||_\infty$ are equivalent.
- In $AC([a,b])$ the norms $||f||_{AC} = ||f||_1 + ||f'||_1$ and $||f||_\# = |f(a)| + ||f'||_1$ are equivalent.
- In $BV([a,b])$ the norms $||f||_{BV} = ||f||_1 + V_a^b(f)$ and $||f||_\# = |f(a)| + V_a^b(f)$ are equivalent.
- In $C^0([a,b])$ the norms $||\cdot||_\infty$ and $||\cdot||_1$ are **not** equivalent.
  In fact $(C^0([a,b]), ||\cdot||_\infty)$ is complete, while $(C^0([a,b]), ||\cdot||_1)$ is not complete.
  If two norms are equivalent then properties hold or not w.r.t. both norms.

8.5. Show that in a normed space of finite dimension, all norms are equivalent.

Theorem. Let $(X, ||\cdot||)$ be a normed space. If $dim(X) < \infty$ then all norms on X are equivalent.
proof.

We'll give the proof in the case of $X = \mathbb{R}^n$, $||\cdot||_1$ and $||\cdot|| =$ any norm.
Therefore, we need to prove that $\exists\, m,\ M > 0$ such that: $m\,||x||_1 \le ||x|| \le M\,||x||_1 \quad \forall x \in X$.
We notice that if $x = 0$ then $||x||_1 = ||x|| = 0$ and so both inequalities hold.

1. In $\mathbb{R}^n$ we consider the canonical basis $\{e_j\}_{j=1}^n$:
   (M) $\Rightarrow \forall x \in X \ \exists \{\alpha_j\}_{j=1}^n \subset \mathbb{R}$ such that: $x = \sum_{j=1}^n \alpha_j\, e_j$
   $\Rightarrow ||x||_1 = \sum_{j=1}^n |\alpha_j|$
   $\Rightarrow ||x|| \le \sum_{j=1}^n |\alpha_j|\,||e_j|| \le \underbrace{\left(\max_{j=1,\dots,n} ||e_j||\right)}_{M := \max_{j=1,\dots,n} ||e_j||} \sum_{j=1}^n |\alpha_j| = M\,||x||_1.$
2. Now we show that $\exists\, m > 0$ such that: $||x|| \ge m\,||x||_1$.
   (m) We define the function $\phi : X \to \mathbb{R}_+$ as:
   $\phi(x) := ||x||$.
   Since ϕ is a norm then it satisfies the inverse triangular inequality and so, $\forall x,\ x_0 \in X$ it holds:
   $|\phi(x) - \phi(x_0)| = \big|\,||x|| - ||x_0||\,\big| \le ||x - x_0|| \le M\,||x - x_0||_1$
   $\Rightarrow \phi$ is Lipschitz in X
   $\Rightarrow \phi$ is continuous in X.
   We define $K := \{x \in \mathbb{R}^n : ||x||_1 = 1\}$, which is compact since $dim(X) = n < \infty$.
   Since K is compact and ϕ is continuous, by Weierstrass theorem:
   $\exists\, m := \min_K \phi$.
   Moreover, since $\phi : X \to \mathbb{R}_+$ then $m \ge 0$.
   Therefore, $\forall x \in X \setminus \{0\}$:
   $\frac{||x||}{||x||_1} = \left|\left| \frac{x}{||x||_1} \right|\right| = \underbrace{\phi\left(\frac{x}{||x||_1}\right) \ge \min_{x \in K} \phi(x)}_{\left|\left|\frac{x}{||x||_1}\right|\right|_1 = 1 \Rightarrow \frac{x}{||x||_1} \in K} =: m.$
   If $m = 0$ then $\exists\, \bar{x} \in K$ (so that $||\bar{x}||_1 = 1$) such that $\phi(\bar{x}) = ||\bar{x}|| = 0$.
   But this is a contraddiction because if $||\bar{x}|| = 0$ then $\bar{x} = 0$ and so it cannot be that $||\bar{x}||_1 = 1$.
   Thus $m > 0$ and so:
   $\frac{||x||}{||x||_1} \ge m \Leftrightarrow ||x|| \ge m\,||x||_1.$ ∎

2 Lebesgue Spaces

9.1. Write the definitions of $\mathcal{L}^p$ and L^p. Show that L^p is a vector space (and its preliminary lemma).

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $p \in [1, \infty]$.
We define:
$\mathcal{L}^p(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : f \text{ is measurable}, \int_X |f|^p\, d\mu < \infty\} \qquad p \in [1, \infty)$
$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} : f \text{ is measurable}, \operatorname{ess\,sup}_X |f| < \infty\} \qquad p = \infty$
We say that two functionfs $f,\ g$ are in relation $f \sim g \Leftrightarrow f = g$ a.e. in X, which is an equivalence relation.
We consider $\mathcal{L}^p(X, \mathcal{A}, \mu)$ and $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$ and we define:
$L^p(X, \mathcal{A}, \mu) := \frac{\mathcal{L}^p(X, \mathcal{A}, \mu)}{\sim}, \qquad L^\infty(X, \mathcal{A}, \mu) := \frac{\mathcal{L}^\infty(X, \mathcal{A}, \mu)}{\sim}.$
The elements of $L^p(X, \mathcal{A}, \mu)$ and $L^\infty(X, \mathcal{A}, \mu)$ are equivalence classes defined by the equivalence relation.
However, to simplify we'll say $f \in L^p(X, \mathcal{A}, \mu)$ or $f \in L^\infty(X, \mathcal{A}, \mu)$.

Lemma. Let $p \in [1, \infty)$ and $a \ge 0,\ b \ge 0$. Then: $(a+b)^p \le 2^{p-1}(a^p + b^p)$.
proof.

We need to prove that $\left(\frac{a+b}{2}\right)^p \le \frac{a^p}{2} + \frac{b^p}{2}. \quad (*)$
For every $p \ge 1$ the function $x \mapsto x^p$ is convex, where a function $\varphi : \mathbb{R} \to \mathbb{R}$ is convex if:
$\varphi(t\,x + (1-t)\,y) \le t\,\varphi(x) + (1-t)\,\varphi(y) \quad \forall x, y \in \mathbb{R},\ \forall t \in [0,1].$
Choosing $x = a$, $y = b$ and $t = \frac{1}{2}$ and applying the definition of convexity we get $(*)$. ∎

Lemma. Let $p \in [1, \infty]$. Then $L^p(X, \mathcal{A}, \mu)$ is a vector space.

proof

If $f,\, g \in L^p(X,\mathcal{A},\mu) \;\Rightarrow\; f,\, g$ are finite a.e. in X.
Let $\lambda \in \mathbb{R} \;\Rightarrow\; f + \lambda\, g$ is well defined (since $f,\, g$ are finite a.e.) and measurable (since $f,\, g$ are measurable).
Consider $p \in [1,\infty)$:

$$\int_X |f + \lambda\, g|^p\, d\mu \underbrace{\le 2^{p-1}\Big(\int_X |f|^p\, d\mu + |\lambda|^p \int_X |g|^p\, d\mu \Big)}_{\text{by previous lemma}} \underbrace{< \infty}_{f,g\,\in L^p} .$$

If $p = \infty$:

$$\operatorname{ess\,sup}_X |f + \lambda\, g| \le \operatorname{ess\,sup}_X |f| + \lambda\, \operatorname{ess\,sup}_X |g| \underbrace{< \infty}_{f,g\,\in L^\infty} .$$

■

9.2. Write the definitions of **conjugate numbers**. Show the **Young inequality**.

We say that two real numbers $p, q \in [1,\infty]$ are conjugate if either:

- $p, q \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$
- $p = 1,\; q = \infty$
- $p = \infty,\; q = 1$

Lemma. **Young inequality**

Let $p, q \in (1,\infty)$ be conjugate and $a > 0,\, b > 0$. Then: $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

proof.

We define the function $\varphi(x) = e^x$, which is convex.
We write: $ab = e^{\log(a)}\, e^{\log(b)} = e^{\frac{1}{p}\log(a^p)}\, e^{\frac{1}{q}\log(b^q)}$.
Since φ is convex:

$$\varphi(t\,x + (1-t)\,y) \le t\,\varphi(x) + (1-t)\,\varphi(y) \quad \forall x, y \in \mathbb{R}_+,\;\; \forall t \in [0,1].$$

Now we consider $t = \frac{1}{p} \in (0,1)$ and $1 - t = 1 - \frac{1}{p} = \frac{1}{q}$.
Moreover, we take $x = \log(a^p),\; y = \log(b^q)$.
Then:

$$ab = e^{\log(a)}\, e^{\log(b)} = e^{\frac{1}{p}\log(a^p)}\, e^{\frac{1}{q}\log(b^q)} = e^{\frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)} \le \frac{1}{p}\, e^{\log(a^p)} + \frac{1}{q}\, e^{\log(b^q)} = \frac{a^p}{p} + \frac{b^q}{q}.$$

■

9.3. Show the **Holder inequality**.

Consider $(X,\mathcal{A},\mu)$ measure space and $p \in [1,\infty]$.
Then, if $f \in L^p(X,\mathcal{A},\mu)$ we define:

$$||f||_p := \Big(\int_X |f|^p\, d\mu \Big)^{1/p} \qquad p \in [1,\infty)$$
$$||f||_\infty := \operatorname{ess\,sup}_X |f| \qquad p = \infty$$

Theorem. **Holder inequality**

Let $f,\, g \in \mathcal{M}(X,\mathcal{A})$ and let $p, q \in [1,\infty]$ be conjugate. Then: $||f\,g||_1 \le ||f||_p\, ||g||_q$.

proof.

(i) Case of $p, q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$.
Since the only requirements is that $f,\, g \in \mathcal{M}(X,\mathcal{A})$, we distinguish three cases.

1. If $||f||_p\, ||g||_q = \infty$ then the inequality holds for any value of $||f\,g||_1$ (even ∞).
2. If $||f||_p\, ||g||_q = 0$ then either $f = 0$ a.e. or $g = 0$ a.e. $\Rightarrow\; f\,g = 0$ a.e. $\Rightarrow\; ||f\,g||_1 = 0$.
3. Now we suppose that $||f||_p$ and $||g||_q$ are finite and different from zero.
   We recall the Young inequality:
   $p, q \in (1,\infty)$ conjugate, $a > 0,\, b > 0 \;\Rightarrow\; ab \le \frac{a^p}{p} + \frac{b^q}{q}$.
   Then, for all $x \in X$ we can apply the Young inequality:

$$\underbrace{\frac{|f(x)|}{||f||_p}}_{a}\, \underbrace{\frac{|g(x)|}{||g||_q}}_{b} \le \underbrace{\frac{1}{p}\, \frac{|f(x)|^p}{||f||_p^p}}_{\frac{a^p}{p}} + \underbrace{\frac{1}{q}\, \frac{|g(x)|^q}{||g||_q^q}}_{\frac{b^q}{q}}$$

   Hence, we obtain:

$$\frac{1}{||f||_p\, ||g||_q} \int_X |f\,g|\, d\mu \le \frac{1}{p}\, \underbrace{\frac{\int_X |f|^p\, d\mu}{||f||_p^p}}_{=1} + \frac{1}{q}\, \underbrace{\frac{\int_X |g|^q\, d\mu}{||g||_q^q}}_{=1} = \frac{1}{p} + \frac{1}{q} = 1$$

   Therefore:

$$||f\,g||_1 =: \int_X |f\,g|\, d\mu \le ||f||_p\, ||g||_q$$

(ii) Case of $p = 1,\; q = \infty$.
We have that $|g| \le ||g||_\infty$ a.e. in X and so $|f\,g| \le |f|\, ||g||_\infty$ a.e. in X.
Therefore:

$$||f\,g||_1 = \int_X |f\,g|\, d\mu \le ||g||_\infty \int_X |f|\, d\mu = ||f||_1\, ||g||_\infty .$$

■

9.4. Show the **Minkowski inequality**.

Consider $(X,\mathcal{A},\mu)$ measure space and $p \in [1,\infty]$.
Then, if $f \in L^p(X,\mathcal{A},\mu)$ we define:

$$||f||_p := \Big(\int_X |f|^p\, d\mu \Big)^{1/p} \qquad p \in [1,\infty)$$
$$||f||_\infty := \operatorname{ess\,sup}_X |f| \qquad p = \infty$$

Theorem. **Minkowski inequality**

Let $f,\, g \in L^p(X,\mathcal{A},\mu)$ and let $p \in [1,\infty]$. Then: $||f + g||_p \le ||f||_p + ||g||_p$.

proof.

(i) Case of $p = 1$.
By the triangular inequality, we have:

$$||f + g||_1 = \int_X |f + g|\, d\mu \le \int_X (|f| + |g|)\, d\mu = ||f||_1 + ||g||_1 .$$

(ii) Case of $p = \infty$.
By the triangular inequality, we have:
$||f+g||_\infty = \operatorname{ess\,sup}_X |f+g| \le \operatorname{ess\,sup}_X(|f|+|g|) \le \operatorname{ess\,sup}_X |f| + \operatorname{ess\,sup}_X |g| = ||f||_\infty + ||g||_\infty$.

(iii) Case of $p, q \in (1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$.
By the triangular inequality, we have:

$$||f+g||_p^p = \int_X |f+g|^p \, d\mu$$
$$= \int_X |f+g|\,|f+g|^{p-1} \, d\mu$$
$$\le \int_X |f|\,|f+g|^{p-1} \, d\mu + \int_X |g|\,|f+g|^{p-1} \, d\mu$$

Let us denote with q the conjugate of p, namely $q = \frac{p}{p-1}$.
Then, by Holder inequality:

$$\int_X |f|\,|f+g|^{p-1} \, d\mu \le ||f||_p \, \big|\big|\,|f+g|^{p-1}\,\big|\big|_q$$
$$\int_X |g|\,|f+g|^{p-1} \, d\mu \le ||g||_p \, \big|\big|\,|f+g|^{p-1}\,\big|\big|_q$$

Moreover, since $(p-1)\,q = p$:

$$\big|\big|\,|f+g|^{p-1}\,\big|\big|_q = \left(\int_X |f+g|^{(p-1)q} \, d\mu\right)^{\frac{1}{q}} = \left(\int_X |f+g|^p \, d\mu\right)^{\frac{1}{q}} = \left(||f+g||_p^p\right)^{\frac{1}{q}} = ||f+g||_p^{\frac{p}{q}}.$$

Therefore:

$$||f+g||_p^p \le \left(||f||_p + ||g||_p\right) ||f+g||_p^{\frac{p}{q}}$$
$$\Rightarrow \ ||f+g||_p^{p-\frac{p}{q}} \le ||f||_p + ||g||_p$$

$\Rightarrow$ since p and q are conjugate it follows that $p - \frac{p}{q} = 1$

$$\Rightarrow \ ||f+g||_p \le ||f||_p + ||g||_p. \qquad \blacksquare$$

9.5. Show that L^p is a normed space.

1. $L^p(X,\mathcal{A},\mu)$ is a normed space with $||f||_p := \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$ $\quad p \in [1,\infty)$.
2. $L^\infty(X,\mathcal{A},\mu)$ is a normed space with $||f||_\infty := \operatorname{ess\,sup}_X |f|$ $\quad p = \infty$.

proof.

1. The function $||\cdot||_p : L^p(X,\mathcal{A},\mu) \to [0,\infty)$ is well defined and such that:
   - $\forall f \in L^p(X,\mathcal{A},\mu)$: $||f||_p \ge 0$
   - $\forall f \in L^p(X,\mathcal{A},\mu)$: $||f||_p = 0 \Leftrightarrow \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} = 0 \Leftrightarrow f = 0$ a.e. in $X \Leftrightarrow f = 0$ in $L^p(X,\mathcal{A},\mu)$
   - $\forall f \in L^p(X,\mathcal{A},\mu)$, $\forall \alpha \in \mathbb{R}$: $||\alpha f||_p = \left(\int_X |\alpha f|^p \, d\mu\right)^{\frac{1}{p}} = |\alpha| \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} = |\alpha|\,||f||_p$
   - $\forall f,g \in L^p(X,\mathcal{A},\mu)$: $||f+g||_p \le ||f||_p + ||g||_p$, due to Minkowski inequality

   Therefore, $||\cdot||_p$ is a norm and $(L^p(X,\mathcal{A},\mu), ||\cdot||_p)$ a normed space.
2. The function $||\cdot||_\infty : L^\infty(X,\mathcal{A},\mu) \to [0,\infty)$ is well defined and such that:
   - $\forall f \in L^\infty(X,\mathcal{A},\mu)$: $||f||_\infty \ge 0$
   - $\forall f \in L^\infty(X,\mathcal{A},\mu)$: $||f||_\infty = 0 \Leftrightarrow \operatorname{ess\,sup}_X |f| = 0 \Leftrightarrow |f| = 0$ a.e. in X
     $\Leftrightarrow f = 0$ a.e. in $X \Leftrightarrow f = 0$ in $L^p(X,\mathcal{A},\mu)$
   - $\forall f \in L^\infty(X,\mathcal{A},\mu)$, $\forall \alpha \in \mathbb{R}$: $||\alpha f||_\infty = \operatorname{ess\,sup}_X |\alpha f| = |\alpha| \operatorname{ess\,sup}_X |f| = |\alpha|\,||f||_\infty$
   - $\forall f,g \in L^\infty(X,\mathcal{A},\mu)$: $||f+g||_\infty \le ||f||_\infty + ||g||_\infty$, due to Minkowski inequality

   Therefore, $||\cdot||_\infty$ is a norm and $(L^\infty(X,\mathcal{A},\mu), ||\cdot||_\infty)$ a normed space. $\blacksquare$

9.6. Show the inclusion of L^p spaces. Which hypothesis is essential? Justify the answer.

Let $(X,\mathcal{A},\mu)$ be a measure space and suppose $\mu(X) < \infty$.
If $1 \le p \le q \le \infty$ then $L^q(X,\mathcal{A},\mu) \subseteq L^p(X,\mathcal{A},\mu)$.

proof.

The thesis follows if we show that there is a constant $C = C(X,p,q) > 0$ such that:
$||f||_p \le C\,||f||_q \quad \forall f \in L^q(X,\mathcal{A},\mu)$.
Indeed, if this is the case then $f \in L^q(X,\mathcal{A},\mu) \Rightarrow f \in L^p(X,\mathcal{A},\mu)$.

(i) Suppose $q = \infty$.

$$\Rightarrow \ ||f||_p^p = \int_X |f|^p \, d\mu \le ||f||_\infty^p \, \mu(X)$$
$$\Rightarrow \ ||f||_p \le \underbrace{(\mu(X))^{\frac{1}{p}}}_{:=C} ||f||_\infty = C\,||f||_\infty.$$

(ii) Suppose now $q < \infty$.
Then, it always exists $s \in (1,\infty)$ such that $p\,s = q$, so that $\frac{1}{s} = \frac{p}{q}$.
Let r be the conjugate of s, namely: $\frac{1}{r} = 1 - \frac{1}{s} = \frac{q-p}{q}$.
Then:

$$\Rightarrow \ ||f||_p^p = \int_X |f|^p \, d\mu = \int_X 1\,|f|^p \, d\mu \le \left(\int_X 1^r \, d\mu\right)^{\frac{1}{r}} \left(\int_X |f|^{p\,s} \, d\mu\right)^{\frac{1}{s}} = (\mu(X))^{\frac{q-p}{q}} \left(\int_X |f|^q \, d\mu\right)^{\frac{p}{q}}$$
$$\Rightarrow \ ||f||_p \le \underbrace{(\mu(X))^{\frac{q-p}{qp}}}_{:=C} \left(\int_X |f|^q \, d\mu\right)^{\frac{1}{q}} = C\,||f||_q. \qquad \blacksquare$$

The hypothesis that $\mu(X) < \infty$ is essential.
If $\mu(X) = \infty$, in general, the previous inclusion is false.
Consider $X = (1,\infty)$ (so that $\mu(X) = \infty$), the measure space $(X,\mathcal{L}(X),\lambda)$ and the function $f : (1,\infty) \to \mathbb{R}$ defined as:
$f(x) = \frac{1}{x}$.
Then $f \notin L^1(X,\mathcal{L}(X),\lambda)$ but $f \in L^2(X,\mathcal{L}(X),\lambda)$ since $\int_1^\infty \frac{1}{x^2} \, dx < \infty$.

2.1 Completness of L^p

9.7. Show the completeness of L^p spaces.

(Recall) **Theorem.** **Criterion for completeness** *(characterization of Banach spaces)*
Let $(X, ||\cdot||)$ be a normed space.
Then X is a Banach space
$\Leftrightarrow \left[\forall \{f_n\}_n \subseteq X \text{ such that } \sum_{n=1}^{\infty} ||f_n|| \text{ converges } \Rightarrow \underbrace{\sum_{n=1}^{\infty} f_n \text{ converges}}_{\text{w.r.t. the norm of the normed space } X}\right].$

Theorem. $L^p(X, \mathcal{A}, \mu)$ is a Banach space $\forall p \in [1, \infty]$.
proof.

applying DCT on $\{f_n\} = \{s_n - s\}$, $f = 0$

(i) Case of $p \in [1, \infty)$. (charact. of completeness in terms of series)
We want to prove that:
$\forall \{f_n\}_n \subseteq L^p(X, \mathcal{A}, \mu)$ if $\underbrace{\sum_{n=1}^{\infty} ||f_n||_p \text{ converges}}_{\lim_k \sum_{n=1}^{k} ||f_n||_p < \infty}$ then $\underbrace{\sum_{n=1}^{\infty} f_n \text{ converges in } L^p(X, \mathcal{A}, \mu)}_{\lim_k \int_X \left|\sum_{n=1}^{k} f_n - \sum_{n=1}^{\infty} f_n\right|^p d\mu = 0}$.
If this is the case then the thesis follows by the **criterion of completeness**.

1. We define:
$g_k(x) := \sum_{n=1}^{k} |f_n(x)| \quad \forall k \in \mathbb{N}$
$g(x) := \sum_{n=1}^{\infty} |f_n(x)|$

1. define g_k, g

Due to Minkowski inequality we have:
$$||g_k||_p \leq ||f_1||_p + .. + ||f_k||_p = \sum_{n=1}^{k} ||f_n||_p \leq \underbrace{\sum_{n=1}^{\infty} ||f_n||_p}_{\text{the series converges by assumption}} := M \in \mathbb{R}_+ \qquad (*)$$
We observe that g_k is measurable $\forall k \in \mathbb{N}$ and $\{g_k\}_k \nearrow$, thus:
1. $|g_k|^p$ is measurable $\forall k \in \mathbb{N}$
2. $\{|g_k|^p\}_k \nearrow$
3. $g_k \xrightarrow[k\to\infty]{} g$ a.e. in X

Therefore, we can apply **MCT**:
$\lim_{k\to\infty} \int_X |g_k|^p \, d\mu = \int_X \lim_{k\to\infty} |g_k|^p \, d\mu = \int_X |g|^p \, d\mu$
From $(*)$ it follows that $||g_k||_p^p \leq M^p$ and so:
$\Rightarrow \int_X |g|^p \, d\mu = \lim_{k\to\infty} \int_X |g_k|^p \, d\mu = \lim_{k\to\infty} ||g_k||_p^p \leq M^p$
$\Rightarrow g \in L^p(X, \mathcal{A}, \mu)$

2. by MCT and Minkowski: $g \in L^p$

$\Rightarrow g$ is finite a.e. in X
$\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ converges absolutely a.e. in X
$\Rightarrow$ $\boldsymbol{\sum_{n=1}^{\infty} f_n(x)}$ **converges a.e. in** $\boldsymbol{X}$

3. $\sum_{n=1}^{\infty} f_n(x)$ converges

2. We define:
$s_k(x) := \sum_{n=1}^{k} f_n(x) \quad \forall k \in \mathbb{N}$
$s(x) := \sum_{n=1}^{\infty} f_n(x)$

4. define s_k, s (s well defined because of 3.)

Then, we observe that:
(1) $s_k \xrightarrow[k\to\infty]{} s$ a.e. in X
(2) $|s_k - s|^p \leq \left|\sum_{n=k+1}^{\infty} f_n\right|^p \leq \Big(\underbrace{\sum_{n=k+1}^{\infty} |f_n|}_{\leq g}\Big)^p \leq \underbrace{g^p \in L^1(X, \mathcal{A}, \mu)}_{\text{since } g \in L^p(X,\mathcal{A},\mu)}$

Therefore, can apply **DCT**:
$\Rightarrow \lim_{k\to\infty} \int_X |s_k - s|^p \, d\mu = 0$
$\Leftrightarrow s_k \xrightarrow[k\to\infty]{} s$ in $L^p(X, \mathcal{A}, \mu)$
$\Leftrightarrow$ $\boldsymbol{\sum_{n=1}^{\infty} f_n}$ **converges in** $\boldsymbol{L^p(X, \mathcal{A}, \mu)}$.

5. by DCT: $\sum_{n=1}^{\infty} f_n$ conv. in L^p

(ii) Case of $p = \infty$. (definition of complete space: every Cauchy sequence converges)
We want to prove that:
$\forall \{f_n\}_n \subseteq L^\infty(X, \mathcal{A}, \mu)$ Cauchy $\exists f \in L^\infty(X, \mathcal{A}, \mu)$ such that $f_n \xrightarrow[n\to\infty]{} f$ in $L^\infty(X, \mathcal{A}, \mu)$.

1. Let $\{f_n\}_n \subseteq L^\infty(X, \mathcal{A}, \mu)$ be a Cauchy sequence.
Then, $\forall n, m \in \mathbb{N}$ we define:
$A_n := \{x \in X : |f_n(x)| > ||f_n||_\infty\}$
$B_{nm} := \{x \in X : |f_n(x) - f_m(x)| > ||f_n - f_m||_\infty\}$
We recall that $||f||_\infty = \text{ess sup}_X |f| = \inf\{M \geq 0 : \mu(\{x \in X : |f(x)| > M\}) = 0\}$.
Therefore, since $f_n, f_n - f_m \in L^\infty(X, \mathcal{A}, \mu)$ $\forall n, m \in \mathbb{N}$:
$\mu(A_n) = \mu(B_{nm}) = 0$.
We define:
$E := \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n,m\in\mathbb{N}} B_{nm}\right)$
Then $\boldsymbol{\mu(E) = 0}$.

1. A_n, B_{nm}, E $\mu(E) = 0$

2. Since $\{f_n\}_n$ is Cauchy then $\forall \varepsilon > 0$ $\exists \bar{n} \in \mathbb{N}$ such that:
$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty < \varepsilon \quad \forall n, m > \bar{n} \;\; \forall x \in E^c \qquad (**)$
$\Rightarrow \{f_n(x)\}_n$ is a Cauchy sequence of real numbers $\forall x \in E^c$
$\Rightarrow \exists f$ measurable such that $f_n \xrightarrow[n\to\infty]{} f$ pointwise in E^c

2. $\exists f : f_n \to f$ pointwise in E^c

Moreover, since $\{f_n\}_n$ is Cauchy then it is bounded:
$\Leftrightarrow \exists M > 0$: $||f_n||_\infty = \text{ess sup}_X |f_n| \leq M \quad \forall n \in \mathbb{N}$
$\Rightarrow |f(x)| := \lim_{n\to\infty} |f_n(x)| \leq M$ for a.a. $x \in X$
$\Rightarrow$ $\boldsymbol{f \in L^\infty(X, \mathcal{A}, \mu)}$

3. $f \in L^\infty$

3. Starting from $(**)$ and letting $m \to \infty$ we obtain:
$|f_n(x) - f(x)| < \varepsilon \quad \forall n > \bar{n} \;\; \forall x \in E^c$
$\Leftrightarrow f_n \xrightarrow[n\to\infty]{} f$ uniformly in E^c.
Since the set on which we do not have convergence (E) is such that $\mu(E) = 0$:
$\Rightarrow$ $\boldsymbol{f_n \xrightarrow[n\to\infty]{} f}$ **in** $\boldsymbol{L^\infty(X, \mathcal{A}, \mu)}$.

4. $f_n \to f$ in L^∞ (since $\mu(E) = 0$)

■

2.2 Convergence in L^p

9.8. Show that convergence in L^p implies convergence in measure.

Let $p \in [1, \infty]$ and let $f_n, f \in L^p(X, \mathcal{A}, \mu)$ $\forall n \in \mathbb{N}$.
Then $f_n \xrightarrow[n\to\infty]{} f$ in $L^p(X, \mathcal{A}, \mu)$ $\Leftrightarrow$ $||f_n - f||_p \xrightarrow[n\to\infty]{} 0$ $\Leftrightarrow$ $\int_X |f_n - f|^p \, d\mu \xrightarrow[n\to\infty]{} 0$.

Prop. Let $p \in [1, \infty]$ and let $f_n, f \in L^p(X, \mathcal{A}, \mu)$ $\forall n \in \mathbb{N}$.
If $f_n \xrightarrow[n\to\infty]{} f$ in $L^p(X, \mathcal{A}, \mu)$ $\Rightarrow$ $f_n \xrightarrow[n\to\infty]{} f$ in measure.

proof.

We recall that $f_n \xrightarrow[n\to\infty]{} f$ in measure $\Leftrightarrow$ $\forall \varepsilon > 0$ $\forall \sigma > 0$ $\exists \bar{n} \in \mathbb{N} : \mu(\{|f_n - f| \geq \varepsilon\}) < \sigma$ $\forall n \geq \bar{n}$.
Suppose by contraddiction that $f_n \not\xrightarrow[n\to\infty]{} f$ in measure.
Then $\exists \varepsilon > 0$, $\sigma > 0$ such that:
$\mu(\{|f_n - f| \geq \varepsilon\}) \geq \sigma$ for infinitely many n in $\mathbb{N}$.
Thus, if $p \in [1, \infty)$ we have:
$||f_n - f||_p^p = \int_X |f_n - f|^p \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f|^p \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon^p \, d\mu = \varepsilon^p \, \mu(\{|f_n - f| \geq \varepsilon\}) \geq \varepsilon^p \, \sigma > 0$
for infinitely many n in $\mathbb{N}$.
$\Rightarrow$ $f_n \not\xrightarrow[n\to\infty]{} f$ in L^p, which is a contraddiction.
Instead, if $p = \infty$ we have:
$||f_n - f||_\infty = \operatorname{ess\,sup}_X |f_n - f| \geq \operatorname{ess\,sup}_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| \geq \operatorname{ess\,sup}_{\{|f_n - f| \geq \varepsilon\}} \varepsilon = \varepsilon > 0$
for infinitely many n in $\mathbb{N}$.
$\Rightarrow$ $f_n \not\xrightarrow[n\to\infty]{} f$ in L^∞, which is a contraddiction.

■

9.9. Show that if $f_n \to f$ in L^p as $n \to \infty$ ($p \in [1, +\infty]$), then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \to f$ a.e. in X as $k \to \infty$.

Prop. Let $p \in [1, \infty]$ and let $f_n, f \in L^p(X, \mathcal{A}, \mu)$ $\forall n \in \mathbb{N}$.
If $f_n \xrightarrow[n\to\infty]{} f$ in $L^p(X, \mathcal{A}, \mu)$ $\Rightarrow$ $\exists \{f_{n_k}\}_k \subseteq \{f_n\}_n$ such that: $f_{n_k} \xrightarrow[k\to\infty]{} f$ a.e. in X.

proof.

If $p = \infty$ then it is obvious, since convergence in L^∞ implies convergence a.e. for $\{f_n\}_n$.
Consider $p \in [1, \infty)$.
If $f_n \xrightarrow[n\to\infty]{} f$ in L^p then $f_n \xrightarrow[n\to\infty]{} f$ in measure.
If $f_n \xrightarrow[n\to\infty]{} f$ in measure then $\exists \{f_{n_k}\}_k \subseteq \{f_n\}_n$ such that $f_{n_k} \xrightarrow[k\to\infty]{} f$ a.e. in X. ■

2.3 Separability of L^p

$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$

9.10. State the **Luisin theorem**.

Let $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$.
We define the support of g as:
$\operatorname{supp}(g) := \overline{\{x \in \Omega : g(x) \neq 0\}}$.
Moreover, we define the set of continuous functions with compact support, namely:
$C_c^0(\mathbb{R}) := \{f \in C^0(\mathbb{R}) : \operatorname{supp}(f) \text{ is compact}\}$.

Theorem. **Luisin**
Consider $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and let $\Omega \in \mathcal{L}(\mathbb{R})$ be such that $\lambda(\Omega) < \infty$.
Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and such that $f \equiv 0$ in Ω^c.
Then $\forall \varepsilon > 0$ $\exists g \in C_c^0(\mathbb{R})$ such that:
(i) $\lambda(\{x \in \mathbb{R} : f(x) \neq g(x)\}) < \varepsilon$
(ii) $\sup_{\mathbb{R}} |g| \leq \sup_{\mathbb{R}} |f|$

We can find a continuous function with compact support, g, which is close to f in a suitable way:
- *the set where the two functions are different has small measure (as we want)*
- *the supremum of $g \leq$ the supremum of f*

9.11. Show that the set of simple functions with supports of finite measure is dense in L^p ($p \in [1, +\infty)$).

(Recall) **Theorem.** **Simple Approximation Theorem**
Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$.
Then, there exists a sequence $\{s_n\}_n$ of simple functions such that:
$s_n \xrightarrow[n\to\infty]{} f$ pointwise in X, namely $s_n \xrightarrow[n\to\infty]{} f$ $\forall x \in X$.
Moreover:
(i) $f \in \mathcal{M}(X, \mathcal{A})$ $\Rightarrow$ $\{s_n\}_n \subseteq \mathcal{S}(X, \mathcal{A}) = \{s : X \to \mathbb{R} \text{ measurable simple functions}\}$
(ii) $f \geq 0$ $\Rightarrow$ $\{s_n\}_n \nearrow$, $0 \leq s_n \leq f$
(iii) f bounded $\Rightarrow$ $s_n \xrightarrow[n\to\infty]{} f$ uniformly in X

Consider $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$.
We define the set of simple functions with compact support, namely:
$\tilde{\mathcal{S}}(\mathbb{R}) := \{s : \mathbb{R} \to \mathbb{R} \text{ simple function}, \lambda(\operatorname{supp}(s)) < \infty\}$.
Then, two characterization of $\tilde{\mathcal{S}}(\mathbb{R})$ are the equivalent:
$s \in \tilde{\mathcal{S}}(\mathbb{R})$ $\Leftrightarrow$ s is simple and $s \neq 0$ on a set of finite measure
$\Leftrightarrow$ s is simple and $s \in L^p(\mathbb{R})$ for $p \in [1, \infty)$

Theorem. Let $p \in [1, \infty)$. Then $\tilde{\mathcal{S}}(\mathbb{R})$ is dense in $L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$.

proof.

We have to prove that for any $f \in L^p(\mathbb{R})$ there exists a sequence of elements of $\tilde{S}(\mathbb{R})$ converging to f.
Let $f \in L^p(\mathbb{R})$ such that $\boldsymbol{f \geq 0}$ a.e. in $\mathbb{R}$.
Because of the simple approximation theorem:
$\Rightarrow \exists \{s_n\}_n \subseteq S(\mathbb{R})$ such that $0 \leq s_n \leq f$, $\{s_n\}_n \nearrow$ and $s_n \xrightarrow[n\to\infty]{} f$ a.e. in $\mathbb{R}$.
Since $0 \leq s_n \leq f$ and $f \in L^p(\mathbb{R})$ then $\{s_n\}_n \subseteq L^p(\mathbb{R})$.
Thus, since $\{s_n\}_n$ are simple functions and $\{s_n\}_n \subseteq L^p(\mathbb{R})$, by a characterization of $\tilde{S}(\mathbb{R})$ we conclude that $\{s_n\}_n \subseteq \tilde{S}(\mathbb{R})$.
Then, we can apply the DCT to obtain:
$s_n \xrightarrow[n\to\infty]{} f$ in $L^p(\mathbb{R})$
which is the thesis.
If f is **sign-changing** then we can argue as above for f_+ and f_-.
Then, the thesis follows since $f = f_+ - f_-$. ∎

$\Downarrow$

9.12. Show that $C_c^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ ($p \in [1, +\infty)$).

Theorem. Let $p \in [1, \infty)$. Then $C_c^0(\mathbb{R})$ is dense in $L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$.
proof.

We have to prove that for any $f \in L^p(\mathbb{R})$ and $\forall \varepsilon > 0$ $\exists g \in C_c^0(\mathbb{R})$ such that the distance between g and f is $< \varepsilon$.
Let $f \in L^p(\mathbb{R})$, $\varepsilon > 0$.
Since the set of simple functions with compact support is dense in $L^p(\mathbb{R})$:
$\Rightarrow \exists s \in \tilde{S}(\mathbb{R})$ such that: $||f - s||_p \leq \varepsilon$.
By Luisin theorem $\exists g \in C_c^0(\mathbb{R})$ such that $\lambda(\{g \neq s\}) < \varepsilon$, $\sup_{\mathbb{R}} |g| \leq \sup_{\mathbb{R}} |s|$.
The last condition implies that $||g||_\infty \leq ||s||_\infty$.
Now we aim to prove that $||f - g||_p$ is small as we want:

$$\begin{aligned}
||f-g||_p &\leq ||f-s||_p + ||s-g||_p \\
&< \varepsilon + \left(\int_{\mathbb{R}} |g-s|^p \, d\lambda\right)^{\frac{1}{p}} \\
&= \varepsilon + \left(\int_{\{g\neq s\}} |g-s|^p \, d\lambda\right)^{\frac{1}{p}} \\
&\leq \varepsilon + \left(\int_{\{g\neq s\}} (|g|+|s|)^p \, d\lambda\right)^{\frac{1}{p}} \\
&\leq \varepsilon + \left[(||g||_\infty + ||s||_\infty)^p \int_{\{g\neq s\}} d\lambda\right]^{\frac{1}{p}} \\
&= \varepsilon + (||g||_\infty + ||s||_\infty)\,[\lambda(\{g \neq s\})]^{\frac{1}{p}} \\
&< \varepsilon + 2\,||s||_\infty \, \varepsilon^{\frac{1}{p}}
\end{aligned}$$ ∎

$\Downarrow$

9.13. Show that $L^p(\mathbb{R})$ is separable ($p \in [1, +\infty)$).

Theorem. Let $p \in [1, \infty)$. Then $L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is separable.
proof.

Let $f \in L^p(\mathbb{R})$, $\varepsilon > 0$.
Since the set of continuous functions with compact support is dense in $L^p(\mathbb{R})$:
$\Rightarrow \exists g \in C_c^0(\mathbb{R})$ such that: $||f - g||_p < \frac{\varepsilon}{2}$.
Moreover, since the support of g is compact then $\exists \bar{n} \in \mathbb{N}$ such that $\text{supp}(g) \subset [-\bar{n}, \bar{n}]$.
We proved that the set of polynomials with rational coefficients is dense in $C^0([a,b])$ and, since it is a countable set, $C^0([a,b])$ is separable.
Therefore, also $C^0([-\bar{n}, \bar{n}])$ is separable, and so there exist a polynomial Q with rational coefficients such that:

$$||g - Q\,\mathbb{1}_{[-\bar{n},\bar{n}]}||_{L^\infty(\mathbb{R})} = ||g - Q||_{L^\infty([-\bar{n},\bar{n}])} < \underbrace{\frac{\varepsilon}{2(2\bar{n})^{1/p}}}_{\substack{Q \text{ and } g \text{ are as close as we want,} \\ \text{we choose this constant}}}$$

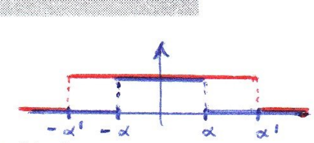
Therefore:

$$\begin{aligned}
||f - Q\,\mathbb{1}_{[-\bar{n},\bar{n}]}||_p &\leq ||f-g||_p + ||g - Q\,\mathbb{1}_{[-\bar{n},\bar{n}]}||_p \\
&< \frac{\varepsilon}{2} + \left(\int_{[-\bar{n},\bar{n}]} |g-Q|^p \, d\lambda\right)^{1/p} \\
&< \frac{\varepsilon}{2} + \left(\lambda([-\bar{n},\bar{n}])\,||g-Q||^p_{L^\infty([-\bar{n},\bar{n}])}\right)^{1/p} \\
&= \frac{\varepsilon}{2} + \left(2\bar{n}\,||g-Q||^p_{L^\infty([-\bar{n},\bar{n}])}\right)^{1/p} \\
&< \frac{\varepsilon}{2} + (2\bar{n})^{1/p} \frac{\varepsilon}{2(2\bar{n})^{1/p}} = \varepsilon
\end{aligned}$$

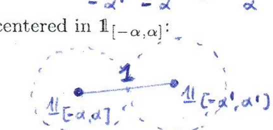
Hence, the set of all functions of the form $Q\,\mathbb{1}_{[-\bar{n},\bar{n}]}$, with Q polynomial with rational coefficients, is dense in $L^p(\mathbb{R})$.
Moreover, since this set is countable we can conclude that $L^p(\mathbb{R})$ is separable. ∎

10.1. Show that $L^\infty(\mathbb{R})$ is not separable.

Theorem. $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is not separable.
proof.



Let $\alpha > 0$ and consider $\{\mathbb{1}_{[-\alpha,\alpha]}\}_{\alpha>0} \subset L^\infty(\mathbb{R})$, which is then an uncountable family.
If $\alpha, \alpha' > 0$ are such that $\alpha \neq \alpha'$ $\Rightarrow$ $||\,\mathbb{1}_{[-\alpha,\alpha]} - \mathbb{1}_{[-\alpha',\alpha']}\,||_\infty = 1$.
Consider now in $L^\infty(\mathbb{R})$ the uncountable family $\{B_\alpha\}_{\alpha>0}$ where B_α is the ball of radius $\frac{1}{2}$ centered in $\mathbb{1}_{[-\alpha,\alpha]}$:
$B_{\frac{1}{2}}(\mathbb{1}_{[-\alpha,\alpha]}) := B_\alpha = \{f \in L^\infty(\mathbb{R}) : ||\,\mathbb{1}_{[-\alpha,\alpha]}\} - f\,||_\infty < \frac{1}{2}\}$
Then, $B_\alpha \cap B_{\alpha'} = \varnothing \quad \forall \alpha \neq \alpha'$. $\forall \alpha' \neq \alpha$:



Consider $Z \subset L^\infty(\mathbb{R})$ such that Z is dense in $L^\infty(\mathbb{R})$.
Since Z is dense then $\forall \alpha > 0$: $B_\alpha \cap Z \neq \varnothing$.
However, since $B_\alpha \cap B_{\alpha'} = \varnothing$ $\forall \alpha \neq \alpha'$ and since $\{B_\alpha\}_\alpha$ is uncountable, it must be that Z is uncountable.
Therefore, if there is a dense set in $L^\infty(\mathbb{R})$ then the set is uncountable.
Thus, $L^\infty(\mathbb{R})$ cannot be separable since there cannot exist a countable dense set. ∎

> ***Prop.*** The same results concerning separability hold if we consider $\Omega \subseteq \mathbb{R}^n$ open.
> Namely $L^p(\Omega, \mathcal{L}(\Omega), \lambda)$ is separable $\forall p \in [1, \infty)$ and $L^\infty(\Omega, \mathcal{L}(\Omega), \lambda)$ is not separable.

2.4 ℓ^p Spaces

10.2. How ℓ^p and L^p are related?

Let $p \in [1, \infty]$. Then: $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$.
Let $\{a_n\}_n \subset \mathbb{R}$. It is possible to define $f : \mathbb{N} \to \mathbb{R}$ as $f \equiv \{a_n\}_n$ and so:

$||f||_p = \left(\int_{\mathbb{N}} |f|^p \, d\mu^\#\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} \quad p \in [1, \infty)$

$||f||_\infty = \text{ess sup}_{n \in \mathbb{N}} |f(n)| = \sup_{n \in \mathbb{N}} a_n \quad p = \infty$

The space ℓ^p is separable for $p \in [1, \infty)$.
In fact, the set c_0 defined as:

$c_0 := \{a_n = \{a_n^{(k)}\}_k \subset \mathbb{Q} : a_n^{(k)} = 0 \ \forall k > n, \, n \in \mathbb{N}\}$

is countable and dense in ℓ^p.
However, ℓ^∞ is not separable.

	$k=1$	$k=2$	$k=3$		
$n=1$	a_1	0	0	0	..
$n=2$	a_1	a_2	0	0	..
$n=3$	a_1	a_2	a_3	0	..

$a_i \in \mathbb{Q}$

10.3. State the **Jensen inequality**.

Theorem. **Jensen inequality**
Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$, $f \in L^1(X, \mathcal{A}, \mu)$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex.
Then:

$\varphi\left(\int_X f \, d\mu\right) \leq \int_X (\varphi \circ f) \, d\mu.$

For example, if we consider the function $\varphi(x) = e^x$ *then it follows:* $e^{\int_X f \, d\mu} \leq \int_X e^f \, d\mu$.

3 Linear Operators

10.4. Write the definitions of: **linear operator, bounded operator, functional, continuous operator.**

Let X, Y be two vector spaces.
A mapping $T : X \to Y$ is a linear operator if:

$T(\alpha \, x_1 + \beta \, x_2) = \alpha \, T(x_1) + \beta \, T(x_2) \quad \forall x_1, x_2 \in X, \ \forall \alpha, \beta \in \mathbb{R}.$

In particular, if $Y = \mathbb{R}$ then T is called functional.
Namely, a functional is a linear operator $T : X \to \mathbb{R}$.
The image of zero by means of any linear operator is always zero: $T(0) = T(0 \cdot x) = 0 \cdot T(x) = 0$.

Let $(X, || \cdot ||_X)$, $(Y, || \cdot ||_Y)$ be two normed spaces.
We say that the linear operator $T : X \to Y$ is bounded if $\exists \, M > 0$ such that:

$||T(x)||_Y \leq M \, ||x||_X \quad \forall x \in X.$

Moreover, we say that the linear operator $T : X \to Y$ is continuous in $x_0 \in X$:

$\Leftrightarrow \ \forall \{x_n\}_n \subset X$ such that $x_n \xrightarrow[n \to \infty]{} x_0$ in $X \Rightarrow T(x_n) \xrightarrow[n \to \infty]{} T(x_0)$ in Y

$\Leftrightarrow \ \forall \{x_n\}_n \subset X$ such that $||x_n - x_0||_X \xrightarrow[n \to \infty]{} 0 \Rightarrow ||T(x_0) - T(x_n)||_Y \xrightarrow[n \to \infty]{} 0$

10.5. State and prove the theorem about the characterization of linear, bounded operators.

Let $(X, || \cdot ||_X)$, $(Y, || \cdot ||_Y)$ be two normed space.
A linear operator $T : X \to Y$ is said to be Lipschitz if $\exists \, L > 0$ such that:

$||T(x) - T(y)||_Y \leq L \, ||x - y||_X \quad \forall x, y \in X.$

Theorem. Let $(X, || \cdot ||_X)$, $(Y, || \cdot ||_Y)$ be two normed spaces and let $T : X \to Y$ be a linear operator.
Then, the following statements are equivalent:
(i) T is bounded
(ii) T is Lipschitz
(iii) T is continuous at $x_0 = 0 \in X$

proof.

(i) $\Rightarrow$ (ii) Since T is bounded:

$||T(x) - T(y)||_Y = ||T(x - y)||_Y \leq M \, ||x - y||_X \quad \forall x, y \in X.$

(ii) $\Rightarrow$ (iii) Consider $\{x_n\}_n \subset X$ such that $x_n \xrightarrow[n \to \infty]{} 0$.
Since $\forall \, T$ linear operator $T(0) = 0$ and since T is Lipschitz:

$||T(x_n)||_Y = ||T(x_n) - T(0)||_Y \leq M \, ||x_n - 0||_X \xrightarrow[n \to \infty]{} 0$

$\Rightarrow ||T(x_n)||_Y \xrightarrow[n \to \infty]{} 0$

$\Rightarrow T(x_n) \xrightarrow[n \to \infty]{} 0 = T(0)$

(iii) $\Rightarrow$ (i) Suppose by contraddiction that T is not bounded.
Then, there exists $\{x_n\}_n \subset X$ with $x_n \neq 0$ such that:

$||T(x_n)||_Y \geq n \, ||x_n||_X.$

Let $z_n := \frac{x_n}{n \, ||x_n||_X}$, so that $T(z_n) = \frac{1}{n \, ||x_n||_X} T(x_n)$:

$\Rightarrow ||z_n||_X = \left|\left| \frac{x_n}{n \, ||x_n||_X} \right|\right|_X = \frac{1}{n} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow z_n \xrightarrow[n \to \infty]{} 0$

$\Rightarrow ||T(z_n)||_Y = \frac{1}{n \, ||x_n||_X} ||T(x_n)||_Y \geq \frac{1}{n \, ||x_n||_X} n \, ||x_n||_X = 1$

$\Rightarrow T(z_n) \not\xrightarrow[n \to \infty]{} T(0) = 0$, while $z_n \xrightarrow[n \to \infty]{} 0$

Therefore T is not continuous at $x_0 = 0$, which is a contraddiction. ∎

10.6. Prove or disprove the following statement. Let X, Y be normed spaces, $T: X \to Y$ be a linear operator. T is continuous in X iff it is continuous at x_0.

Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be two normed spaces.
A linear operator $T: X \to Y$ is said to be continuous in X if T is continuous at any $x_0 \in X$.

Prop. Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be two normed spaces and let $T: X \to Y$ be a linear operator.
Then, T is continuous in X $\Leftrightarrow$ T is continuous in $x_0 = 0 \in X$.

proof.

($\Rightarrow$) If T is continuous in X then T is continuous at any $x_0 \in X$, also in $x_0 = 0 \in X$.
($\Leftarrow$) Since T is a linear operator:
T continuous at $x_0 = 0 \in X$ $\Leftrightarrow$ T is Lipschitz.
Moreover, if T is Lipschitz then T is continuous in X.
In fact $\forall x_0 \in X$, $\forall \{x_n\}_n \subset X$ such that $x_n \xrightarrow[n\to\infty]{} x_0$:
$||T(x_n) - T(x_0)||_Y \le L\,||x_n - x_0||_X \xrightarrow[n\to\infty]{} 0$
$\Rightarrow T(x_n) \xrightarrow[n\to\infty]{} T(x_0)$. ■

10.7. Prove that $\mathcal{L}(X, Y)$ is a vector space. Which is the standard norm on it?

Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be two normed spaces.
We define:

$$\begin{aligned}\mathcal{L}(X,Y) &:= \{T : X \to Y : T \text{ is linear and continuous}\}\\ &= \{T : X \to Y : T \text{ is linear and bounded}\}\end{aligned}$$

(boundedness is equivalent to continuity at $x_0 = 0 \in X$, which is equivalent to continuity in X)
If $X = Y$ then we write $\mathcal{L}(X,Y) \equiv \mathcal{L}(X)$.

Prop. $\mathcal{L}(X,Y)$ is a vector space.
proof.

Consider $T, S \in \mathcal{L}(X,Y)$, $\lambda \in \mathbb{R}$.
Then $\forall x_1, x_2 \in X$, $\forall \alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}(T + \lambda S)(\alpha x_1 + \beta x_2) &= T(\alpha x_1 + \beta x_2) + \lambda S(\alpha x_1 + \beta x_2)\\ &= \alpha T(x_1) + \beta T(x_2) + \lambda\alpha S(x_1) + \lambda\beta S(x_2)\\ &= \alpha(T(x_1) + \lambda S(x_1)) + \beta(T(x_2) + \lambda S(x_2))\\ &= \alpha(T + \lambda S)(x_1) + \beta(T + \lambda S)(x_2)\end{aligned}$$

$\Rightarrow T + \lambda S$ is linear
Moreover, since $T, S \in \mathcal{L}(X,Y)$ then T and S are bounded, namely $\exists M_1, M_2 > 0$ such that:
$||T(x)||_Y \le M_2\,||x||_X \quad \forall x \in X$
$||S(x)||_Y \le M_2\,||x||_X \quad \forall x \in X$
$\Rightarrow ||(T + \lambda S)(x)||_Y \le ||T(x)||_Y + |\lambda|\,||S(x)||_Y \le (M_1 + |\lambda|\,M_2)\,||x||_X \quad \forall x \in X$
$\Rightarrow T + \lambda S$ is bounded (and so, continuous)
$\Rightarrow \mathcal{L}(X,Y)$ is closed w.r.t. linear combinations of its elements
$\Rightarrow \mathcal{L}(X,Y)$ is a vector space. ■

On $\mathcal{L}(X,Y)$ we define the norm $||T||_{\mathcal{L}(X,Y)} := \sup_{x\in X,\,||x||_X \le 1} ||T(x)||_Y$.
Therefore $(\mathcal{L}(X,Y), ||\cdot||_{\mathcal{L}(X,Y)})$ is a normed space.
Notice that $||T||_{\mathcal{L}(X,Y)} \neq \infty$ because, since $T \in \mathcal{L}(X,Y)$:
$\Rightarrow \exists M > 0$: $||T(x)||_Y \le M\,||x||_X \quad \forall x \in X$
$\Rightarrow \exists M > 0$: $||T(x)||_Y \le M \quad \forall x \in X$ *such that* $||x||_X \le 1$
$\Rightarrow \exists M > 0$: $||T||_{\mathcal{L}(X,Y)} := sup_{x\in X,\,||x||_X \le 1} ||T(x)||_Y \le M$

Prop. From the definition of $||\cdot||_{\mathcal{L}(X,Y)}$ it follows that: $||T(x)||_Y \le ||T||_{\mathcal{L}(X,Y)}\,||x||_X \quad \forall x \in X$.

10.8. The norm on $\mathcal{L}(X,Y)$ satisfies two important equalities. Write and show them.

Prop. $||T||_{\mathcal{L}} := \sup_{x\in X,\,||x||_X \le 1} ||T(x)||_Y \overset{1.}{=} \sup_{x\in X,\,||x||_X = 1} ||T(x)||_Y \overset{2.}{=} \sup_{x\in X,\,x\neq 0} \frac{||T(x)||_Y}{||x||_X}$.
proof.

1. Of course:
   $\sup_{x\in X,\,||x||_X\le 1} ||T(x)||_Y \ge \sup_{x\in X,\,||x||_X=1} ||T(x)||_Y$. (1)
   Consider now $||x||_X \le 1$ such that $x \neq 0$:
   $\Rightarrow ||T(x)||_Y = ||x||_X\,||T(\frac{x}{||x||_X})||_T \le ||T(\frac{x}{||x||_X})||_Y$
   which means that $||T(x)||_Y$ is less than $||T(\cdot)||_Y$ evaluated on an element of norm 1 (since $||\frac{x}{||x||_X}||_X = 1$).
   Therefore:
   $\sup_{x\in X,\,||x||_X\le 1} ||T(x)||_Y \le \sup_{x\in X,\,||x||_X=1} ||T(x)||_Y$. (2)
   Combining (1) and (2):
   $||T||_{\mathcal{L}} = \sup_{x\in X,\,||x||_X=1} ||T(x)||_Y$.
2. $\forall x \in X \setminus \{0\}$ we have:
   $\frac{||T(x)||_Y}{||x||_X} = ||T(\frac{x}{||x||_X})||_Y$
   which means that $\frac{||T(x)||_Y}{||x||_X}$ is equivalent to $||T(\cdot)||_Y$ evaluated on an element of norm 1 (since $||\frac{x}{||x||_X}||_X = 1$).
   Therefore:
   $\sup_{x\in X,\,x\neq 0} \frac{||T(x)||_Y}{||x||_X} = \sup_{x\in X,\,||x||_X=1} ||T(x)||_Y = ||T||_{\mathcal{L}}$. ■

10.9. Show that $\mathcal{L}(X,Y)$ is a Banach space, provided that Y is a Banach space.

Theorem. Let $(X, ||\cdot||_X)$ be a normed space and let $(Y, ||\cdot||_Y)$ be a Banach space.
Then $(\mathcal{L}(X,Y), ||\cdot||_{\mathcal{L}})$ is a Banach space.

proof.

We already know that $(\mathcal{L}(X,Y), ||\cdot||_{\mathcal{L}})$ is a normed space. We just need to prove completeness.
We'll procede by definition, namely we'll prove that any Cauchy sequence in $(\mathcal{L}(X,Y), ||\cdot||_{\mathcal{L}})$ converges.

- Let $\{T_n\}_n \subset \mathcal{L}(X,Y)$ be a Cauhcy sequence, 1 that is:
  $\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}$: $||T_m - T_n||_{\mathcal{L}} < \varepsilon \ \forall n, m > n_\varepsilon$.
  Then, $\forall x \in X$:
  $||T_m(x) - T_n(x)||_Y \leq ||T_m - T_n||_{\mathcal{L}} \, ||x||_X < \varepsilon \, ||x||_X \quad \forall n, m > n_\varepsilon$.
  This means that $\{T_n(x)\}_n \subset Y$ is a Cauchy sequence and, since $(Y, ||\cdot||_Y)$ is a Banach space:
  $\Rightarrow \boxed{\exists y = y(x) \in Y \colon T_n(x) \xrightarrow[n\to\infty]{} y \text{ in } Y.}$ 2
- To underly the dependence of y on x, we define the operator $\boxed{T : X \to Y \text{ as: } T(x) := y \ \forall x \in X.}$ 3
  We claim that this operator $T \in \mathcal{L}(X,Y)$.
  - T is linear.
    In fact, since T_n is linear, we have that $\forall x_1, x_2 \in X$ and $\forall \alpha, \beta \in \mathbb{R}$:
    $T_n(\alpha\, x_1 + \beta\, x_2) = \alpha\, T_n(x_1) + \beta\, T_n(x_2)$.
    Since $\{T_n(x)\}_n$ conveceerges in Y $\forall x \in X$ then $\exists y_1, \, y_2, \, \bar{y} \in Y$ such that:
    $$\begin{aligned} T_n(\alpha\, x_1 + \beta\, x_2) &\xrightarrow[n\to\infty]{} \bar{y} \\ \alpha\, T_n(x_1) &\xrightarrow[n\to\infty]{} \alpha\, y_1 \\ \beta\, T_n(x_2) &\xrightarrow[n\to\infty]{} \beta\, y_2 \end{aligned}$$
    But, by the definition of T we can write:
    $$\begin{aligned} T_n(\alpha\, x_1 + \beta\, x_2) &\xrightarrow[n\to\infty]{} \bar{y} = T(\alpha\, x_1 + \beta\, x_2) \\ \alpha\, T_n(x_1) &\xrightarrow[n\to\infty]{} \alpha\, y_1 = \alpha\, T(x_1) \\ \beta\, T_n(x_2) &\xrightarrow[n\to\infty]{} \beta\, y_2 = \beta\, T(x_2) \end{aligned}$$
    Therefore:
    $\lim_{n\to\infty} T_n(\alpha\, x_1 + \beta\, x_2) = \lim_{n\to\infty} \alpha\, T_n(x_1) + \lim_{n\to\infty} \beta\, T_n(x_2)$
    $\Leftrightarrow \ \bar{y} = \alpha\, y_1 + \beta\, y_2$
    $\Leftrightarrow \ T(\alpha\, x_1 + \beta\, x_2) = \alpha\, T(x_1) + \beta\, T(x_2)$
    *Here we are **not** proving that $T_n \xrightarrow[n\to\infty]{} T$, we are using T just as something to refer to y.*
  - T is bounded.
    In fact, $\forall x \in X$, $\forall \varepsilon > 0$, $\forall m > n > n_\varepsilon$ we have:
    $$\begin{aligned} ||T(x) - T_n(x)||_Y &= ||T(x) \pm T_m(x) - T_n(x)||_Y \\ &\leq ||T(x) - T_m(x)||_Y + ||T_m(x) - T_n(x)||_Y \\ &\leq ||T(x) - T_m(x)||_Y + \varepsilon\, ||x||_X \quad (*) \end{aligned}$$
    Since $x \in X$ then $\exists y \in Y$ such that $T_m(x) \xrightarrow[m\to\infty]{} y$ in Y, namely: $||T_m(x) - y||_Y \xrightarrow[m\to\infty]{} 0$.
    Then, by the definition of T we can write $y = T(x)$ and so: $||T_m(x) - T(x)||_Y \xrightarrow[m\to\infty]{} 0$.
    Therefore, letting $m \to \infty$, $(*)$ becomes:
    $$||T(x) - T_n(x)||_Y \leq \varepsilon\, ||x||_X \quad (**)$$
    $$\begin{aligned} \Rightarrow \ ||T(x)||_Y &= ||T(x) \pm T_n(x)||_Y \\ &\leq ||T(x) - T_n(x)||_Y + ||T_n(x)||_Y \\ &\leq \varepsilon\, ||x||_X + \underbrace{||T_n||_{\mathcal{L}}}_{\substack{\leq M \\ \{T_n\}_n \text{ is Cauchy} \\ \text{so it is bounded}}} ||x||_X \\ &\leq (\varepsilon + M)\, ||x||_X \end{aligned}$$
  Since T is linear and bounded then $\boxed{T \in \mathcal{L}(X,Y).}$ 4
- We are now left to prove that $T_n \xrightarrow[n\to\infty]{} T$ in $\mathcal{L}(X,Y)$.
  Namely that T is not just an object we use to refer to y, but it is the limit of T_n in $\mathcal{L}(X,Y)$.
  Starting from $(**)$, that is $||T(x) - T_n(x)||_Y \leq \varepsilon\, ||x||_X$:
  $\Rightarrow \ ||T - T_n||_{\mathcal{L}} \leq \varepsilon \quad \forall n > n_\varepsilon$
  $\Leftrightarrow \boxed{T_n \xrightarrow[n\to\infty]{} T \text{ in } \mathcal{L}(X,Y)}$ 5
  $\Rightarrow \ \mathcal{L}(X,Y)$ is complete and so it is a Banach space. ∎

3.1 Uniform Bounded Principle (UBP)

Def. **Kernel of an opeartor**
Let $T \in \mathcal{L}(X,Y)$. The kernel of T is defined as $Ker(T) := \{x \in X : T(x) = 0\}$.
Prop. $Ker(T)$ is a vector space and it is closed.
Prop. T is injective $\Leftrightarrow$ $Ker(T) = \{0\}$.

10.10. Write the definitions of: **invertible operator, isometry, embedding.**

Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be two normed spaces.
An operator $T \in \mathcal{L}(X,Y)$ is said to be invertible if it is surjective and injective.
The normed spaces $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ are isomorphi if $\exists T \in \mathcal{L}(X,Y)$ invertible with $T^{-1} \in \mathcal{L}(Y,X)$.
An operator $T \in \mathcal{L}(X,Y)$ is said to be an isometry if it preserves the norm, that is:
$||T(x)||_Y = ||x||_X \quad \forall x \in X$.
Consider now $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ normed spaces such that $X \subseteq Y$ and define the mapping $J : X \to Y$ as:
$J(x) = x \quad \forall x \in X$.
If $J \in \mathcal{L}(X,Y)$ then J is called embedding and we write $X \hookrightarrow Y$.
Moreover, if $J \in \mathcal{L}(X,Y)$ is an embedding then $\exists M > 0$: $||x||_Y = ||J(x)||_Y \leq M\, ||x||_X \ \forall x \in X$.
Therefore, we can establish a relation between the X-norm of x and the Y-norm of x.

10.11. State and prove the **UBP** (or BS theorem).

Theorem. **Uniform Bounded Principle (UBP) - Banach-Steinhaus theorem**
Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces.
Let $\mathcal{F} \subset \mathcal{L}(X,Y)$ be a family of linear bounded operators.
Assume that $\mathcal{F}$ is pointwise bounded, namely:

(*PB condition*) $\forall x \in X\ \exists M_x > 0$: $\sup_{T\in\mathcal{F}} ||T(x)||_Y \leq M_x$.

Then $\mathcal{F}$ is uniformly bounded, namely:

(*UB condition*) $\exists K > 0$: $\sup_{T\in\mathcal{F}} ||T||_{\mathcal{L}} = \sup_{T\in\mathcal{F}} \left(\sup_{x\in X,\, ||x||_X\leq 1} ||T(x)||_Y\right) \leq K$.

In the PB condition, for any $x \in X$ we keep x fixed and we take the supremum of $||T(x)||_Y$ over all $T \in \mathcal{F}$.
In the UB condition we consider the supremum of $||T||_{\mathcal{L}}$ over $T \in \mathcal{F}$, it does not depend on x anymore.
In conclusion, if there is pointwise boundness then there is automatically uniform boundness.

proof.

- We define $\forall n \in \mathbb{N}$ the set:
  $\boxed{C_n := \{x \in X : ||T(x)||_Y \leq n\ \forall T \in \mathcal{F}\}.}$ 1
  Then, C_n is closed.
  Indeed, let $\{x_m\}_m \subset C_n$ be such that $x_m \xrightarrow[m\to\infty]{} x_0 \in X$.
  In order to prove that C_n is closed we need to prove that $x_0 \in C_n$.
  Because of continuity of T (since $T \in \mathcal{F} \subset \mathcal{L}(X,Y)$):
  $\Rightarrow T(x_m) \xrightarrow[m\to\infty]{} T(x_0)$
  $\Rightarrow ||T(x_m)||_Y \xrightarrow[m\to\infty]{} ||T(x_0)||_Y$
  But, since $\{x_m\}_m \subset C_n$:
  $\Rightarrow ||T(x_m)||_Y \leq n \quad \forall m \in \mathbb{N}$
  $\Rightarrow ||T(x_0)||_Y \leq n$
  $\Rightarrow x_0 \in C_n$
  $\Rightarrow \boxed{C_n \text{ is closed.}}$ 2
- Because of the *(PB condition)*, that is:
  $\forall x \in X\ \exists M_x > 0$: $\sup_{T\in\mathcal{F}} ||T(x)||_Y \leq M_x$
  it must that $\forall x\ \exists n \in \mathbb{N}$ large enough for which $x \in C_n$, and thus: $\boxed{X = \bigcup_{n=1}^{\infty} C_n.}$ 3
- Since X is a complete normed space, by Baire's theorem $\boxed{X \text{ is of second category.}}$ 4
  Therefore, since X is a countable union of sets (C_n), it must be that:
  $\boxed{\exists n_0 \in \mathbb{N}\text{: } int(C_{n_0}) \neq 0.}$ 5
  If not, $int(C_n) = 0\ \forall n \in \mathbb{N}$ and X would be the countable union of nowhere dense sets (namely, of first category).
  Thus, if $int(C_{n_0}) \neq 0$ we can find a closed ball of radius ε centered in a point $x_0 \in C_{n_0}$ entirely contained in C_{n_0}:
  $\boxed{\overline{B(x_0,\varepsilon)} \subset C_{n_0}.}$ 6
- If we consider $z \in X$ such that $\boxed{||z||_X \leq \varepsilon}$ 7 then $z + x_0 \in \overline{B(x_0,\varepsilon)} \subset C_{n_0}$ and so:
  $$\boxed{||T(z)||_Y} = ||T(z) \pm T(x_0)||_Y \leq ||T(z) + T(x_0)||_Y + ||T(x_0)||_Y = \underbrace{||T(z+x_0)||_Y}_{z+x_0\, \in C_{n_0}} + \underbrace{||T(x_0)||_Y}_{x_0\, \in C_{n_0}} \boxed{\leq 2\, n_0} \quad \forall T \in \mathcal{F}$$
  7 7
- Then, $\forall x \in X \setminus \{0\}$:
  $||T(x)||_Y = \frac{||x||_X}{\varepsilon}\, ||T(\frac{\varepsilon\, x}{||x||_X})||_Y \quad (*)$
  If we call $z := \frac{\varepsilon\, x}{||x||_X}$ then $||z||_X = ||\frac{\varepsilon\, x}{||x||_X}||_X = \varepsilon$ and we already proved that $||T(z)||_Y \leq 2\, n_0$ if $||z||_X \leq \varepsilon$.
  Therefore, starting from $(*)$:
  $\Rightarrow ||T(x)||_Y \leq \frac{||x||_X}{\varepsilon}\, 2\, n_0$
  $\Rightarrow ||T||_{\mathcal{L}} \leq \frac{2\, n_0}{\varepsilon} =: K \quad \forall T \in \mathcal{F}$ 8
  $\Rightarrow \sup_{T\in\mathcal{F}} ||T||_{\mathcal{L}} \leq K.$ ∎

↓

10.12. From the UBP it is possible to infer an important property of operators defined by means of a pointwise limit. What is that? Justify the answer.

Corollary. Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces, $\{T_n\}_n \subset \mathcal{L}(X,Y)$.
Assume that $\forall x \in X\ \exists T(x) := \lim_{n\to\infty} T_n(x)$.
Then $T \in \mathcal{L}(X,Y)$.
If a sequence of bounded operators converges pointwise, then the limit operator is bounded.

proof.

- Since T_n is linear, we have that $\forall x_1, x_2 \in X$ and $\forall \alpha, \beta \in \mathbb{R}$:
  $T_n(\alpha\, x_1 + \beta\, x_2) = \alpha\, T_n(x_1) + \beta\, T_n(x_2)$.
  By assumption, we know that $\forall x \in X\ \exists T(x) := \lim_{n\to\infty} T_n(x)$:
  $$\begin{aligned} T_n(\alpha\, x_1 + \beta\, x_2) &\Rightarrow \lim_{n\to\infty} T_n(\alpha\, x_1 + \beta\, x_2) &&=: T(\alpha\, x_1 + \beta\, x_2) \\ \alpha\, T_n(x_1) &\Rightarrow \alpha \lim_{n\to\infty} T_n(x_1) &&=: \alpha\, T(x_1) \\ \beta\, T_n(x_2) &\Rightarrow \beta \lim_{n\to\infty} T_n(x_2) &&=: \beta\, T(x_2) \end{aligned}$$
  Therefore:
  $\lim_{n\to\infty} T_n(\alpha\, x_1 + \beta\, x_2) = \lim_{n\to\infty} \alpha\, T_n(x_1) + \lim_{n\to\infty} \beta\, T_n(x_2) \Leftrightarrow T(\alpha\, x_1 + \beta\, x_2) = \alpha\, T(x_1) + \beta\, T(x_2)$
  $\Rightarrow$ ***T* is linear.**
- By assumptions $\forall x \in X\ \{T_n(x)\}_n$ is convergent and so it is bounded:
  $\Leftrightarrow \forall x \in X\ \exists M_x > 0$: $||T_n(x)||_Y \leq M_x \quad \forall n \in \mathbb{N}$
  $\Rightarrow \forall x \in X\ \exists M_x > 0$: $\sup_{n\in\mathbb{N}} ||T_n(x)||_Y \leq M_x$.
  Therefore, we can apply UBP theorem and conclude that $\exists M > 0$: $\sup_{n\in\mathbb{N}} ||T_n||_{\mathcal{L}} \leq M$.
  $\Rightarrow ||T_n(x)||_Y \leq M\, ||x||_X \quad \forall x \in X,\ \forall n \in \mathbb{N}$
  Moreover, since $T_n(x) \xrightarrow[n\to\infty]{} T(x)$:
  $\Rightarrow ||T_n(x)||_Y \xrightarrow[n\to\infty]{} ||T(x)||_Y$
  $\Rightarrow ||T(x)||_Y := \lim_{n\to\infty} ||T_n(x)||_Y \leq M\, ||x||_X \quad \forall x \in X$
  $\Rightarrow$ ***T* is bounded.** ∎

3.2 Open Mapping Theorem (OMT)

10.13. Write the definition of **open mapping**. State the **OMT**.

Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be normed spaces.
A mapping $T : X \to Y$ is said to be an open mapping if for any open set $G \subset X$ also $T(G) \subset Y$ is open.
An open mapping transports open sets into open sets.

Theorem. Open Mappin Theorem (OMT)
Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces, $T \in \mathcal{L}(X, Y)$ surjective.
Then T is an open mapping.

$\Downarrow$

10.14. State and prove the **IBM theorem.**

Corollary. Inverse Bounded Mapping (IBM)
Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces, $T \in \mathcal{L}(X, Y)$ bijective.
Then $T^{-1} \in \mathcal{L}(Y, X)$.

proof.

Since T is bijective then $\exists \boldsymbol{T^{-1}} : Y \to X$ and it **is linear**.
We know that:
T^{-1} is continuous $\Leftrightarrow$ $(T^{-1})^{-1}(E) \subset Y$ is open $\forall E \subset X$ open.
Since T is bijective, it is in particular surijective and so we can apply OMT:
$\Rightarrow$ T is an open mapping
$\Leftrightarrow$ $T(E) \subset Y$ is open $\forall E \subset X$ open
$\Leftrightarrow$ $T(E) = (T^{-1})^{-1}(E) \subset Y$ is open $\forall E \subset X$ open
$\Leftrightarrow$ $\boldsymbol{T^{-1}}$ **is continuous**. ■

$\Downarrow$

10.15. By the IBM theorem we can infer an important property about equivalent norms on Banach spaces. What is that? Justify the answer.

Corollary. Let $(X, ||\cdot||)$, $(X, ||\cdot||_*)$ be Banach spaces.
Suppose that $\exists M > 0$: $||x||_* \leq M\,||x|| \quad \forall x \in X$.
Then $||\cdot||$ and $||\cdot||_*$ are equivalent, namely $\exists m > 0$: $m\,||x|| \leq ||x||_* \quad \forall x \in X$.

proof.

We consider the identity operator $I : (X, ||\cdot||) \to (X, ||\cdot||_*)$ defined as $I(x) = x$.
$\Rightarrow$ $||I(x)||_* := ||x||_* \leq M\,||x|| \quad \forall x \in X$.
$\Rightarrow$ I is bounded.
Moreover, since I is the identity operator then I is bijective.
Therefore, we can apply IBM:
$\Rightarrow$ $I^{-1} : (X, ||\cdot||_*) \to (X, ||\cdot||)$, defined as $I^{-1}(x) = x$, is linear and bounded
$\Rightarrow$ $\exists m' > 0$: $||I^{-1}(x)|| := ||x|| \leq m'\,||x||_* \quad \forall x \in X$
$\Rightarrow$ $\exists m := \frac{1}{m'} > 0$: $m\,||x|| =: \frac{1}{m'}\,||x|| \leq ||x||_* \quad \forall x \in X$. ■

3.3 Closed Graph Theorem

10.16. Write the definitions of: **closed operator, graph of an operator.**
Show that an operator is linear and closed iff its graph is closed.

Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be normed spaces.
A linear operator $T : X \to Y$ is said to be closed if:

$$\left.\begin{array}{rl} x_n \xrightarrow[n\to\infty]{} x & \text{in } X \\ T(x_n) \xrightarrow[n\to\infty]{} y & \text{in } Y \end{array}\right\} \Rightarrow \quad T(x) = y$$

If x_n converges to x in X and the image of x_n through T converges to some y in Y then it must be that $y = T(x)$.
Notice that if $T \in \mathcal{L}(X, Y)$ then T is closed.
In fact, by definition of continuity we have that $x_n \xrightarrow[n\to\infty]{} x \Rightarrow T(x_n) \xrightarrow[n\to\infty]{} T(x)$.
However, without further assumptions, T closed $\not\Rightarrow$ $T \in \mathcal{L}(X, Y)$.
We define the graph of the linear operator $T : X \to Y$ as:
graph$(T) := \{(x, T(x)) : x \in X\} \subseteq X \times Y$.
Notice that if $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ are Banach spaces then $(X \times Y, \underbrace{||x||_X + ||y||_Y}_{:=\text{ graph norm}})$ is a Banach space.

Prop. Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be normed spaces and let $T : X \to Y$ be an operator.
Then: T is linear and closed $\Leftrightarrow$ graph(T) is closed.

proof.

Let $\{(x_n, T(x_n))\}_n \subset$ graph(T) be such that $(x_n, T(x_n)) \xrightarrow[n\to\infty]{} (x, y) \in X \times Y$.
In view of the definition of the graph norm, this implies that: $x_n \xrightarrow[n\to\infty]{} x$, $T(x_n) \xrightarrow[n\to\infty]{} y$.
($\Rightarrow$) Since T is closed:
$\Rightarrow$ $y = T(x)$
$\Rightarrow$ $y \in T(X)$
$\Leftrightarrow$ $(x, y) = (x, T(x)) \in$ graph(T)
$\Rightarrow$ graph(T) is closed. (= if a sequence ⊆ graph(T) conv. to something ⇒ something ∈ graph(T))
($\Leftarrow$) Since graph(T) is closed:
$\Rightarrow$ $(x, y) \in$ graph(T)
$\Rightarrow$ $y = T(x)$, otherwise it cannot belong to graph(T)
$\Rightarrow$ T is closed. ■

10.17. State and prove the **closed graph theorem.**

Theorem. **Closed graph**

Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces and let $T : X \to Y$ be a linear closed operator.
Then $T \in \mathcal{L}(X, Y)$.
Note that: "$T : X \to Y$ linear closed operator" $\Leftrightarrow$ "$graph(T)$ is closed".

proof.

- We consider the graph norm: $||x||_* = ||x||_X + ||T(x)||_Y$.
  This is a norm on X.
  Moreover, $(X, ||\cdot||_*)$ is a Banach space. ← the assumption that T is closed is needed here (+)
  In fact, let $\{x_n\}_n \subseteq X$ be a Cauchy sequence w.r.t. $||\cdot||_*$.
  Then, since both $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ are Banach spaces:
  $\Rightarrow x_n \xrightarrow[n\to\infty]{} x$ in X, $T(x_n) \xrightarrow[n\to\infty]{} y$ in Y.
  Moreover, since T is a closed operator:
  $\Rightarrow T(x) = y$.
  Therefore:
  $||x_n - x||_* = ||x_n - x||_X + ||T(x_n) - T(x)||_Y \xrightarrow[n\to\infty]{} 0$
  which means that $x_n \xrightarrow[n\to\infty]{} x$ in $(X, ||\cdot||_*)$ and so $(X, ||\cdot||_*)$ is Banach.
- Since $||\cdot||_Y$ is a norm, it must be that $||T(x)||_Y \geq 0 \ \forall x \in X$ and so:
  $||x||_* = ||x||_X + ||T(x)||_Y \geq ||x||_X \quad \forall x \in X$.
  Therefore, by a corollary of the IBM theorem:
  $\exists\, M > 0$: $||x||_X + ||T(x)||_Y = ||x||_* \leq M\, ||x||_X \quad \forall x \in X$.
  We notice in particular that it must be $M \geq 1$.
  $\Rightarrow ||T(x)||_Y \leq (M-1)\, ||x||_X \quad \forall x \in X$
  $\Rightarrow T$ is bounded
  $\Rightarrow T \in \mathcal{L}(X, Y)$, since it is linear by assumptions. ∎

4 Duality and Reflexivity

4.1 Dual Spaces

10.18. Write the definition of **dual space.** Write equivalent conditions to $L \in X^*$.

Let $(X, ||\cdot||_X)$ be a normed space.
We define the dual space of X as:
$X^* := \mathcal{L}(X, \mathbb{R}) = \{T : X \to \mathbb{R} : T$ linear and continuous operator$\}$.
We can equip X^* with the standard norm and obtain the normed space $(X^*, ||\cdot||_*)$, where:
$\forall L \in X^*$: $||L||_* := ||L||_{\mathcal{L}(X,\mathbb{R})} = \sup_{x\in X,\, ||x||_X \leq 1} |L(x)|$.
If $(X, ||\cdot||_X)$ is a Banach space then also $(X^*, ||\cdot||_*)$ is a Banach space.

Prop. **Characterization of $L \in X^*$**

Let $(X, ||\cdot||_X)$, be a normed space and let $L : X \to \mathbb{R}$ be a linear operator such that $L \neq 0$.
Then, the following statements are equivalent:
(i) $L \in X^* = \mathcal{L}(X, \mathbb{R})$ ($\Rightarrow$ L is also bounded)
(ii) $Ker(L) := \{x \in X : L(x) = 0\}$ is closed
(iii) $Ker(L)$ is not dense in X

10.19. Exhibit an example of $T \in (L^p)^*$.

Let $(X, \mathcal{A}, \mu)$ be a measure space and consider $L^p(X, \mathcal{A}, \mu)$.
Let p and q be conjugate exponents.
Consider $g \in L^q(X, \mathcal{A}, \mu)$ and define the operator $T_g : L^p \to \mathbb{R}$ as:
$T_g(f) := \int_X f\, g\, d\mu \quad \forall f \in L^p$.
Then, T_g is obviously linear, since the integral is linear.
Is the operator T_g well defined?
Are we sure that if we take $g \in L^q$ and $f \in L^p$ then we can compute $\int_X f\, g\, d\mu$ and this integral will be a real number?
We observe that, because of the Holder inequality:
$|T_g(f)| = \left|\int_X f\, g\, d\mu\right| \leq \left|\int_X |f\, g|\, d\mu\right| \leq ||f||_p\, ||g||_q \quad \forall f \in L^p. \quad (*)$
And so, the operator T_g is well defined, that is $T_g \in \mathcal{L}(L^p, \mathbb{R}) = (L^p)^*$.
Moreover, from $(*)$ we conclude that the operator is bounded and it holds: $||T_g||_* \leq ||g||_q$.
The norm $||T_g||_*$ is a supremum and we found $||T_g||_* \leq ||g||_q$.
If we find a function f such that $|T_g(f)|$ is exactly $||g||_q$ then the supremum is a maximum and the norm is $||T_g||_* = ||g||_q$.
For $g \neq 0$ we define:
$$\tilde{f} := \frac{|g|^{q-2}\, g}{||g||_q^{q-1}}$$
Then:
$$T_g(\tilde{f}) = \int_X \tilde{f}\, g\, d\mu = \int_X \frac{|g|^{q-2}\, g}{||g||_q^{q-1}}\, g\, d\mu = \frac{\int_X |g|^q\, d\mu}{||g||_q^{q-1}} = \frac{||g||_q^q}{||g||_q^{q-1}} = ||g||_q$$
We found a function that realizes the supremum.
$\Rightarrow ||T_g||_* = ||g||_q$.
Therefore, $T_g \in \mathcal{L}(L^p, \mathbb{R}) = (L^p)^*$ and it is such that $||T_g||_* = ||g||_q$.

10.20. What is $(L^p)^*$ for $p \in [1, \infty)$?

Theorem. **Characterization of $(L^p)^*$ for $p \in (1, \infty)$** (Riesz representation theorem)

Let $(X, \mathcal{A}, \mu)$ be a complete measure space.
Let $p \in (1, \infty)$ and let q be its conjugate, namely $\frac{1}{p} + \frac{1}{q} = 1$.
Then, for any $T \in (L^p)^*$ $\exists!\, g \in L^q$ such that:
$T(f) = \int_X f\, g\, d\mu \quad \forall f \in L^p$.
Moreover:
$||T||_{(L^p)^*} = ||g||_{L^q}$.
Every element of $(L^p)^*$ can be written using a function $g \in L^q$ (for each element $\exists!\, g \in L^q$).

The same result holds when $p = 1$ and $q = \infty$, provided that μ is σ-finite.
Therefore, we will denote:
$(L^p)^* = L^q \quad \frac{1}{p} + \frac{1}{q} = 1,\ p \in (1, \infty)$
$(L^1)^* = L^\infty$
However, we notice that $(L^\infty)^* \neq L^1$.

$(X, \mathcal{A}, \mu)$ must be complete.
For example $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is complete
and so $(L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda))^* = L^q(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$
same for $\Omega \subseteq \mathbb{R}$ or $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$

4.2 Hahn-Banach Theorems

10.21. Write the definition of **sublinear functional**.

Let X be a vector space.
A mapping $p : X \to \mathbb{R}$ is said to be a sublinear functional if:
(i) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X$
(ii) $p(\alpha\, x) = \alpha\, p(x) \quad \forall x \in X,\ \forall \alpha \in \mathbb{R}_+$

11.1. State and prove the **Hahn-Banach Theorem** in the **dominated extension form**.

Theorem. **Hahn-Banach - dominated extension form**

Let X be a vector space, Y a subspace of X and p a sublinear functional on X.
Let $f : Y \to \mathbb{R}$ be a linear continuous functional dominated by p, namely:
$f \in Y^*$: $f(y) \leq p(y) \quad \forall y \in Y$.
Then, there exists $F \in X^*$ such that:
(i) F is an extension of f to the whole X, namely $F(y) = f(y) \ \forall y \in Y$
(ii) F is dominated by p, namely $F(x) \leq p(x) \ \forall x \in X$
It is possible to extend $f \in Y^*$ to $F \in X^*$ if f is dominated by a sublinear functional.
Notice that X and Y are just vector spaces, not normed spaces.

proof.

(Recall)
- A set P with an order relation $\leq$ is called partially ordered set.
- If $\forall x, y \in P$ it's either $x \leq y$ or $y \leq x$ then P is totally ordered/chain.
- $M \in P$ is a maximal element if: $M \leq x$ for some $x \in P \Rightarrow x = M$.
- Let $A \subseteq P$ and $u \in P$. If $\forall a \in A$: $a \leq u \Rightarrow u$ is called upper bound.
- *(Zorn's lemma).* In a partially ordered set P, if every chain has an upper bound then P has a maximal element.

- Let $\mathcal{F}$ be the set of all pairs (Y_α, f_α) where:
  (i) Y_α subspace of X, $Y_\alpha \supseteq Y$
  (ii) f_α is an extension of f to Y_α
  (iii) $f_\alpha(x) \leq p(x) \quad \forall x \in Y_\alpha$
  We observe that $\mathcal{F} \neq \varnothing$ because by hypothesis $(Y, f) \in \mathcal{F}$.
- We can define an order relation, denoted with $\leq$, as:
  $$(Y_\alpha, f_\alpha) \leq (Y_\beta, f_\beta) \ \Leftrightarrow \ \begin{cases} Y_\alpha \subseteq Y_\beta \\ f_\beta \text{ is an extension of } f_\alpha \text{ to } Y_\beta \end{cases}$$
  Then, $(\mathcal{F}, \leq)$ is a partially ordered set.

1. pass from the subset Y to the whole X through a family of pairs $(\mathcal{F})$

- In $(\mathcal{F}, \leq)$ every chain has an upper bound.
  Indeed, let $\mathcal{C} := \{(Y_\alpha, f_\alpha) : \alpha \in A\} \subset \mathcal{F}$ be a chain.
  Let $\tilde{Y} := \bigcup_{\alpha \in A} Y_\alpha$, which means that $\tilde{Y}$ is a subspace of X, and define $\tilde{f} : \tilde{Y} \to \mathbb{R}$ such that:
  $\forall y \in \tilde{Y}\ \exists \alpha \in A$ such that $y \in Y_\alpha \Rightarrow \tilde{f}(y) := f_\alpha(y)$.
  Then:
  (i) $\tilde{f}$ is linear
  (ii) $(\tilde{Y}, \tilde{f}) \in \mathcal{F}$
  (iii) $(\tilde{Y}, \tilde{f})$ is an upper bound for $\mathcal{C}$
  Indeed $\forall (Y_\alpha, f_\alpha) \in \mathcal{C}$: $Y_\alpha \subseteq \bigcup_{\alpha \in A} Y_\alpha =: \tilde{Y}$ and $\tilde{f}$ is an extension of f_α to $\tilde{Y} \Rightarrow (Y_\alpha, f_\alpha) \leq (\tilde{Y}, \tilde{f})$.
- By Zorn's lemma, there exists a maximal element (Z, g) of $\mathcal{F}$.
  This means that if $(U, h) \in \mathcal{F}$ and $(Z, g) \leq (U, h)$ then $Z = U$ and $g = h$.

2. use Zorn's lemma to show that this family has a maximal element

- To get the thesis it is sufficient to prove that $Z = X$, that is: g is the extension of f to X.
  - Let assume by contraddiction that $Z \subsetneq X$.
  - Let $x_0 \in X \setminus Z$ and let $W := \{z + t\,x_0 : z \in Z,\ t \in \mathbb{R}\}$.
  - On this set W we define a function $\phi : W \to \mathbb{R}$ as:
    $\phi(w) = g(z) + t\,c$
    where $c \in \mathbb{R}$ is fixed.
    We observe that ϕ is an extension of g to W.
    If we have an element of Z then it is also an element of W with $t = 0$ and so $\phi(z) = g(z)\ \forall z \in Z$.
  - If it is possible to select $c \in \mathbb{R}$ such that:
    $\phi(w) \leq p(w) \quad \forall w \in W$
    then $(W, \phi) \in \mathcal{F}$ and $(Z, g) \leq (W, \phi)$, which is in contraddiction with the maximality of (Z, g).
    In fact $W \supseteq Z \supseteq Y$ and ϕ is an extension of g on W (where g is an extension of f).
    Therefore, if such $c \in \mathbb{R}$ then $(W, \phi) \in \mathcal{F}$ and $(Z, g) \leq (W, \phi)$.

3. show that the maximal set where the extended functional is defined coincides with X.

– *Claim*: such $c \in \mathbb{R}$ exists:

- For $z_1, z_2 \in Z$:

$$\underbrace{g(z_1) - g(z_2) = g(z_1 - z_2)}_{g \text{ is linear}} \leq p(z_1 - z_2) = p(z_1 \pm x_0 - z_2) \underbrace{\leq p(z_1 + x_0) + p(-x_0 - z_2)}_{p \text{ is sublinear}}$$

$$\Rightarrow \underbrace{-p(-x_0 - z_2) - g(z_2)}_{\text{independent of } z_1} \leq \underbrace{p(z_1 + x_0) - g(z_1)}_{\text{independent of } z_2}$$

$\Rightarrow \exists c \in \mathbb{R}$ such that $\forall z \in Z$:

$$-p(-x_0 - z) - g(z) \leq c \leq p(z + x_0) - g(z) \qquad (*)$$

- Consider now $w := z + t\,x_0 \in W$.

If $t = 0 \Rightarrow w = z$ and $\phi(w) = g(z) \leq p(z) = p(w) \Rightarrow \phi(w) \leq p(w)$.

If $t > 0$ then $\frac{z}{t} \in Z$, since Z is a subspace.

Exploiting the second inequality of $(*)$:

$\Rightarrow c \leq p\left(\frac{z}{t} + x_0\right) - g\left(\frac{z}{t}\right)$

$\Rightarrow t\,c \leq p(z + t\,x_0) - g(z)$

$\Rightarrow \phi(w) = g(z) + t\,c \leq p(z + t\,x_0) = p(w)$.

$\Rightarrow \phi(w) \leq p(w)$.

If $t < 0$ the same reasoning holds exploiting the first inequality of $(*)$.

Therefore, the claim is proved and so we get a contraddiction.

Hence, it must be $Z = X$. ■

11.2. State and prove the **Hahn-Banach Theorem** in the **continuous extension form.**

Theorem. **Hahn-Banach - continuous extension form**

Let $(X, ||\cdot||_X)$ be a normed space and let Y be a subspace of X.

Let $f : Y \to \mathbb{R}$ be a linear continuous functional, namely $f \in Y^*$.

Then, there exists $F \in X^*$ such that:

(i) F is an extension of f to the whole X, namely $F(y) = f(y)\ \forall y \in Y$

(ii) $||F||_{X^*} = ||f||_{Y^*}$

Here we require that X and Y are normed spaces.

In return, we obtain an equality involving F, instead of an inequality.

proof.

- Define $p : X \to \mathbb{R}$ such that:

  $p(x) := ||f||_{Y^*}\,||x||_X \quad \forall x \in X$.

  Then, p is a sublinear functional on X, indeed:

  $p(x+y) = ||f||_{Y^*}\,||x+y||_X \leq ||f||_{Y^*}\,(||x||_X + ||y||_Y) = p(x) + p(y) \quad \forall x, y \in X$

  $p(\alpha\,x) = ||f||_{Y^*}\,||\alpha\,x||_X = ||f||_{Y^*}\,|\alpha|\,||x||_X = |\alpha|\,p(x) = \alpha\,p(x) \quad \forall x \in X,\ \forall \alpha \in \mathbb{R}_+$

  Moreover, since $f \in Y^*$:

  $\Rightarrow |f(y)| \leq ||f||_{Y^*}\,||y||_X \quad \forall y \in Y$

  $\Rightarrow f(y) \leq p(y) \quad \forall y \in Y$

- It is possible to apply Hahn-Banach theorem in the dominated extension form to get $F \in X^*$ such that:

  (1) F is an extension of f

  (2) $F(x) \leq p(x) \quad \forall x \in X$

- Since F is an extension of f, that is (1), we get:

$$||f||_{Y^*} = \underbrace{\sup_{x \in Y,\, x \neq 0} \frac{|f(x)|}{||x||_X} \leq \sup_{x \in X,\, x \neq 0} \frac{|f(x)|}{||x||_X}}_{\text{sup taken on a bigger set}} = ||F||_{X^*}$$

- Moreover, by (2):

  $\Rightarrow |F(x)| \leq |p(x)| = ||f||_{Y^*}\,||x||_X$

  $\Rightarrow ||F||_{X^*} \leq ||f||_{Y^*}$

- Having both sides of inequality we conclude that $||f||_{Y^*} = ||F||_{X^*}$. ■

4.3 Consequences of Hahn-Banach Theorem – we consider the continuous extension form

11.3. State and prove two corollaries of the Hahn-Banach Theorem.

Corollary 1. Let $(X, ||\cdot||)$ be a normed space and let $x_0 \in X \setminus \{0\}$.

Then $\exists\, L_{x_0} \in X^*$ such that:

- $||L_{x_0}||_{X^*} = 1$
- $L_{x_0}(x_0) = ||x_0||$

If $x_0 \neq 0$ we can construct an operator of norm 1 such that at x_0 the value is exactly the norm of x_0.

proof.

Consider $Y := \{\lambda\,x_0 : \lambda \in \mathbb{R}\} \equiv \text{span}\{x_0\}$, so that Y is a subspace of X.

Define $L_0 : Y \to \mathbb{R}$ such that:

$L_0(\lambda\,x_0) := \lambda\,||x_0||$

which is linear and bounded, namely $L_0 \in Y^*$.

By Hahn-Banach theorem (continuous extension form) $\exists\,\tilde{L}_0 : X \to \mathbb{R}$ such that:

- $\tilde{L}_0 \in X^*$
- $||\tilde{L}_0||_{X^*} = ||L_0||_{Y^*} = \sup_{\lambda x_0 \in Y,\, ||\lambda x_0|| = 1} |\lambda\,||x_0||| = 1$
- $\tilde{L}_0(x_0) = L_0(x_0) = ||x_0|| \quad (\lambda = 1)$

Then, the proof is complete setting $L_{x_0} := \tilde{L}_0$. ■

The point is: we want a functional with some specific properties on a special point x_0. Therefore, we can construct a functional on the span of that special point and then we can extend this functiona through the Hahn-Banach theorem.

Corollary 2. Let $(X, ||\cdot||)$ be a normed space and let $y, z \in X$ be such that $L(y) = L(z)\ \forall L \in X^*$.
Then $y = z$.
The elements of X^* separate the elements of X.

proof.

Suppose by contraddiction that $\exists\, y, z \in X$ such that $y \neq z$ and $L(y) = L(z)\ \forall L \in X^*$.
Define now $x := y - z \neq 0$:
$\Rightarrow\ \boldsymbol{L(x)} = L(y-z) = L(y) - L(z) = \boldsymbol{0}\quad \boldsymbol{\forall L \in X^*}$
Therefore, by the previous corollary:
(*since* $x \neq 0$ *then* $\exists\, L_x \in X^* : ||L_x||_{X^*} = 1,\ L_x(x) = ||x||$)
$\Rightarrow\ \boldsymbol{\exists\, L_x \in X^*}$: $\boldsymbol{L_x(x)} = ||x|| \boldsymbol{\neq 0}$ (since $x \neq 0$)
Therefore, we get a contraddiction. ■

11.4. Give a sufficient condition for separability of X.

Corollary 3. Let $(X, ||\cdot||)$ be a normed space, Y a closed subspace of X and $x_0 \in X \setminus Y$.
Then $\exists\, f \in X^*$ with $||f||_{X^*} = 1$ such that:
- $f(y) = 0\ \forall y \in Y$
- $f(x_0) = dist(x_0, Y) := \inf_{\xi \in Y} ||x_0 - \xi||$

We can always construct an operator which is not zero, but it is zero on a closed subspace.

By means of this corollary we can state the following theorem.
Theorem. Let $(X, ||\cdot||)$ be a normed space. If X^* is separable $\Rightarrow X$ is separable.

4.4 The Dual of L^∞

11.5. Show that the dual of L^∞ is not L^1.

Let $\Omega \subseteq \mathbb{R}^n$ be measurable and consider $L^1(\Omega, \mathcal{L}(\Omega), \lambda)$.
Let $g \in L^1$ and define $L_g : L^\infty \to \mathbb{R}$ as:
$L_g(f) := \int_\Omega f\, g\, d\lambda \quad \forall f \in L^\infty$.
Then L_g is linear, since the integral is linear.
Moreover, by Holder inequality:
$\Rightarrow\ |L_g(f)| \leq |\int_\Omega |f\, g|\, d\lambda| \leq ||f||_\infty\, ||g||_1$
$\Rightarrow\ ||L_g||_{L^\infty} \leq ||g||_1$
Consider now $\tilde{f} := \text{sign}(g) \in L^\infty$:
$\Rightarrow\ |L_g(\tilde{f})| = |\int_\Omega g\, \text{sign}(g)\, d\lambda| = |\int_\Omega |g|\, d\lambda| = ||g||_1$
$\Rightarrow\ ||L_g||_{(L^\infty)^*} = ||g||_1$
However, $\exists\, L \in (L^\infty)^*$ such that L is not of the form L_g with $g \in L^1$.
proof.

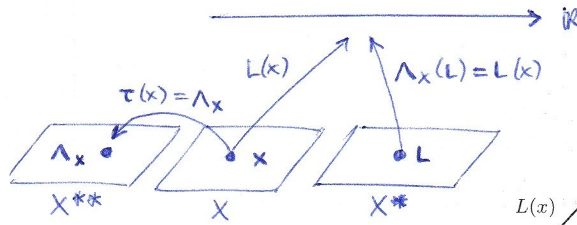
Consider $(C_c^0(\mathbb{R}^n), ||\cdot||_\infty)$ and define $L_0 \in (C_c^0(\mathbb{R}^n))^*$ as:
$L_0(f) := f(0) \quad \forall f \in C_c^0(\mathbb{R}^n)$.
We know that $C_c^0(\mathbb{R}^n)$ is a subspace of L^∞.
Therefore, by Hahn-Banach theorem $\exists\, L \in (L^\infty(\mathbb{R}^n))^*$ which is an extension of L_0.
We claim that $\nexists\, g \in L^1(\mathbb{R}^n)$ such that:
$L(f) = \int_{\mathbb{R}^n} g\, f\, d\lambda \quad \forall f \in L^\infty(\mathbb{R}^n)$.
Suppose by contraddiction that such g exists:
$\Rightarrow\ L(f) = L_0(f) = \int_{\mathbb{R}^n} g\, f\, d\lambda = f(0) \quad \forall f \in C_c^0(\mathbb{R}^n)$.
Among all $f \in C_c^0(\mathbb{R}^n)$ there are f such that $f(0) = 0$. For those f, the above equation would be:
$L(f) = L_0(f) = \int_{\mathbb{R}^n} g\, f\, d\lambda = f(0) = 0$
which implies that $g\, f = 0$ a.e. in $\mathbb{R}^n$, but, for the arbitrariety of f it must be that $g = 0$ a.e. in $\mathbb{R}^n$.
If $g = 0$ a.e. in $\mathbb{R}^n\ \Rightarrow\ L \equiv 0$, namely:
$\int_{\mathbb{R}^n} f\, g\, d\lambda = 0 \quad \forall f \in L^\infty(\mathbb{R}^n)$
which is a contraddiction because $\exists\, f \in C_c^0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ such that $f(0) \neq 0$ and so:
$0 \neq f(0) = L_0(f) = L(f) = \int_{\mathbb{R}^n} g\, f\, d\lambda = 0$. ■

Therefore, the dual of L^∞ is **not** L^1.

4.5 Reflexive Spaces

11.6. Introduce the **canonical map**. Show that it is an isometry.

Let $(X, ||\cdot||_X)$ be a normed space and let X^* be its dual.
The dual of X^*, namely $(X^*)^* \equiv X^{**}$, is called the bidual or second dual of X.
For each $x \in X$ we can define $\Lambda_x : X^* \to \mathbb{R}$ such that:
$\Lambda_x(L) = L(x) \quad \forall L \in X^*$.
Then Λ_x is linear and it holds:
$|\Lambda_x(L)| = |L(x)| \leq ||L||_{X^*}\, ||x||_X \quad (1)$
Therefore, Λ_x is linear and bounded on X^*, that is $\Lambda_x \in X^{**}$.
Furthermore, from (1) we can deduce:
$||\Lambda_x||_{X^{**}} \leq ||x||_X \quad (2)$
It follows that we can define a mapping $\tau : X \to X^{**}$ such that:
$\tau(x) = \Lambda_x \quad \forall x \in X$.
The mapping τ is called canonical map.
Notice that, from (2) we can conclude that:
$||\Lambda_x||_{X^{**}} = ||\tau(x)||_{X^{**}} \leq ||x||_X$.



$$
\begin{tikzcd}
 & & \mathbb{R} & \\
X^{**} & X \arrow[l, "\tau(x)=\Lambda_x"'] \arrow[ur, "L(x)"] & & X^* \arrow[ul, "\Lambda_x(L)=L(x)"'] \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\
\Lambda_x & x & & L
\end{tikzcd}
$$

A (generic) transformation L from $(X, ||\cdot||_X)$ to $(Y, ||\cdot||_Y)$ is an isometry if: $||L(x)||_Y = ||x||_X \ \forall x \in X$.

Theorem. The canonical map τ is linear and it is an isometry, namely $||\tau(x)||_{X^{**}} = ||x||_X \ \forall x \in X$.

proof.

- Linearity is obvious.
  Consider $x, y \in X$, $\alpha, \beta \in \mathbb{R}$:
  $\tau(\alpha\, x + \beta\, y) = \Lambda_{\alpha x + \beta y}$.
  Therefore, $\forall L \in X^*$:
  $\Lambda_{\alpha x+\beta y}(L) = L(\alpha\, x + \beta\, y) = \alpha\, L(x) + \beta\, L(y) = \alpha\, \Lambda_x(L) + \beta\, \Lambda_y(L) = (\alpha\, \Lambda_x + \beta\, \Lambda_y)(L)$
  which is equivalent to say that $\tau(\alpha\, x + \beta\, y) = \Lambda_{\alpha x+\beta y} = \alpha\, \Lambda_x + \beta\, \Lambda_y$.
- From (2) we get that:
  $||\boldsymbol{\tau(x)}||_{\boldsymbol{X^{**}}} = ||\Lambda_x||_{X^{**}} \leq ||\boldsymbol{x}||_{\boldsymbol{X}} \quad \forall x \in X$.
  By a corollary of Hahn-Banach theorem, for each $x \in X$ such that $x \neq 0$:
  $\exists\, L \in X^*$: $||L||_{X^*} = 1, \ L(x) = ||x||_X \quad (*)$
  Therefore:
  $||\boldsymbol{\tau(x)}||_{\boldsymbol{X^{**}}} = ||\Lambda_x||_{X^{**}} = \sup_{||L||_{X^*}=1} |\Lambda_x(L)| = \sup_{||L||_{X^*}=1} |L(x)| \geq ||\boldsymbol{x}||_{\boldsymbol{X}}$.
  The last inequality is justified by the fact that from $(*)$ we know that $\exists\, L \in X^*$: $||L||_{X^*} = 1, \ |L(x)| = ||x||_X$.
  Maybe there are some $\tilde{L} \in X^*$ such that $||\tilde{L}||_{X^*} = 1$ and $|\tilde{L}(x)| \geq ||x||_X$.
  Therefore, $||\tau(x)||_{X^{**}} = ||x||_X$. ■

Def. **Isomorphic normed spaces**
Two normed spaces $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ are called isomorphic if there is a linear bijection between the two which is continuous, namely:
$\exists T \in \mathcal{L}(X, Y)$: $T^{-1} \in \mathcal{L}(Y, X)$.

Corollary. The canonical map $\tau : X \to \tau(X)$ is an isometric isomorphism.
But we need to consider $\tau(X)$ instead of X^{**}.

Corollary. If $(X, ||\cdot||_X)$ is a Banach space then $\tau(X)$ is closed in X^{**}.
In fact, since X is complete and since τ is an isometry then $\tau(X)$ is complete (as the image of a complete space through an isometry). Therefore, since a complete space is closed, $\tau(X)$ is closed.

11.7. Write the definition of **reflexive space**. State the properties of reflexive spaces.

Let $(X, ||\cdot||_X)$ be a normed space and let X^{**} be its bidual.
Let $\tau : X \to X^{**}$ be the canonical map.
Then, X is said to be reflexive if $\tau(X) = X^{**}$, namely if τ is surjective.

Theorem. Let $(X, ||\cdot||_X)$ be a reflexive Banach space. Then every closed subspace of X is reflexive.

Theorem. Let $(X, ||\cdot||_X)$ be a Banach space. Then: X reflexive $\Leftrightarrow$ X^* is reflexive.

Theorem. Let $(X, ||\cdot||_X)$ be a reflexive Banach space. Then: X separable $\Rightarrow$ X^* separable.
In general X^* separable $\Rightarrow$ X separable, but not the opposite.
However, if X is a reflexive space then X separable $\Rightarrow$ X^* separable.

11.8. Write the definition of **uniformly convex** Banach space. State the **Millman-Pettis theorem**.

The Banach space $(X, ||\cdot||_X)$ is said to be uniformly convex if $\forall \varepsilon > 0 \ \exists \delta > 0$:
$\forall x, y \in X, \ ||x||_X \leq 1, \ ||y||_X \leq 1, \ ||x - y||_X > \varepsilon \ \Rightarrow \ ||\frac{x+y}{2}||_X < 1 - \delta$.

Theorem. **Millmann-Pettis**
Let $(X, ||\cdot||_X)$ be a uniformly convex Banach space. Then X is reflexive.

11.9. Show the reflexivity of L^p with $p \in (1, \infty)$. Why L^1 and L^∞ are not reflexive?

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $L^p(X, \mathcal{A}, \mu)$ for $p \in (1, \infty)$ is reflexive.

proof.

Let q be the conjugate of p. Since $p, q \in (1, \infty)$ we know that $(L^p)^* = L^q$.
Therefore $(L^p)^{**} = (L^q)^* = L^p$ and so, L^p is reflexive. ■

Here we obtained the reflexivity of L^p spaces by exploiting the representation of its dual.
There is an alternative way to prove the same result without involving the characterization of the dual of L^p.
Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $L^p(X, \mathcal{A}, \mu)$ with $p \in (1, \infty)$ is uniformly convex.
This conclusion follows from the Clarkson inequalities:

- $1 < p < 2$: $\quad ||\frac{f+g}{2}||_p^q + ||\frac{f-g}{2}||_p^q \leq \left(\frac{1}{2}\left(||f||_p^p + ||g||_p^p\right)\right)^{\frac{q}{p}} \quad f, g \in L^p$, p, q conjugate
- $p \geq 2$: $\quad ||\frac{f+g}{2}||_p^p + ||\frac{f-g}{2}||_p^p \leq \frac{1}{2}\left(||f||_p^p + ||g||_p^p\right) \quad f, g \in L^p$

Therefore, we can apply Millmann-Pettis theorem and conclude that $L^p(X, \mathcal{A}, \mu)$ with $p \in (1, \infty)$ is reflexive.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $L^1(X, \mathcal{A}, \mu)$ and $L^\infty(X, \mathcal{A}, \mu)$ are **not** reflexive.
proof.

$p = 1$ Suppose that $X = L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is reflexive.
Since X is a reflexive Banach space and it is separable:
$\Rightarrow X^* = (L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda))^* = L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is separable.
This is is a contraddiction.

$p = \infty$ Let $X = L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ and suppose that $X^* = (L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)^* = L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is reflexive.
Since X is a Banach space then X is reflexive $\Leftrightarrow X^*$ is reflexive.
$\Rightarrow X = L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is reflexive.
This is a contraddiction, we already proved that $L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is not ~~separable~~ reflexive. ■

5 Weak and Weak* Convergences

$(X, \|\cdot\|_X)$ Banach
(everywhere but 11.16)

5.1 Weak Convergence ($\{x_n\}_n$)

11.10. Write the definition of **weak convergence**. How can it be formulated in L^p and in ℓ^p?

Let $(X, \|\cdot\|_X)$ be a Banach space, $\{x_n\}_n \subset X$, $x \in X$.
The sequence $\{x_n\}_n$ weakly converges to x, and we write $x_n \xrightharpoonup[n\to\infty]{} x$, if:
$L(x_n) \xrightarrow[n\to\infty]{} L(x) \quad \forall L \in X^*$.
The latter convergence is a convergence of real numbers since $L(x_n),\ L(x) \in \mathbb{R}$.

Prop. Consider $\Omega \subseteq \mathbb{R}^n$ and $L^p(\Omega, \mathcal{L}(\Omega), \lambda)$ with $p \in [1, \infty)$. Let q be the conjugate of p and let $\{f_n\}_n \subset L^p$, $f \in L^p$.
Then, exploiting the representation theorem:
$\boxed{f_n \xrightharpoonup[n\to\infty]{} f} \Leftrightarrow L(f_n) \xrightarrow[n\to\infty]{} L(f) \quad \forall L \in (L^p)^* = L^q$
$\Leftrightarrow \boxed{\int_\Omega f_n\, g\, d\lambda \xrightarrow[n\to\infty]{} \int_\Omega f\, g\, d\lambda \quad \forall g \in L^q}$

Prop. Consider $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#)$ with $p \in [1, \infty)$. Let q be the conjugate of p and let $\{x_n\}_n \subset \ell^p$, $x \in L^p$.
Then, exploiting the representation theorem:
$\boxed{x_n \xrightharpoonup[n\to\infty]{} x} \Leftrightarrow L(x_n) \xrightarrow[n\to\infty]{} L(x) \quad \forall L \in (\ell^p)^* = \ell^q$
$\Leftrightarrow \boxed{\sum_{k=1}^\infty x_n^{(k)} y^{(k)} \xrightarrow[n\to\infty]{} \sum_{k=1}^\infty x^{(k)} y^{(k)} \quad \forall y \in \ell^q}$

11.11. Show that strong convergence imples weak convergence. Provide a counterexample for the converse implication.

Prop. Let $(X, \|\cdot\|_X)$ be a Banach space, $\{x_n\}_n \subset X$, $x \in X$. Then:
(i) $x_n \xrightarrow[n\to\infty]{} x \Rightarrow x_n \xrightharpoonup[n\to\infty]{} x$
(ii) $x_n \xrightarrow[n\to\infty]{} x \not\Leftarrow x_n \xrightharpoonup[n\to\infty]{} x$

proof.

(i) Since $x_n \xrightarrow[n\to\infty]{} x$, then $\forall L \in X^*$:
$|L(x_n) - L(x)| = |L(x_n - x)| \leq \|L\|_{X^*} \underbrace{\|x_n - x\|_X}_{\xrightarrow[n\to\infty]{} 0} \xrightarrow[n\to\infty]{} 0$
$\Rightarrow L(x_n) \xrightarrow[n\to\infty]{} L(x)$
$\Leftrightarrow x_n \xrightharpoonup[n\to\infty]{} x$.

(ii) Consider $X = \ell^2$ and define $\{e_n\}_n \subset \ell^2$ as $e_n^{(k)} = \delta_{nk}$.
Since $(\ell^2)^* = \ell^2$, we can exploit the representation theorem and conclude:
$e_n \xrightharpoonup[n\to\infty]{} x \Leftrightarrow L(e_n) \xrightarrow[n\to\infty]{} L(x) \quad \forall L \in (\ell^2)^*$
$\Leftrightarrow \sum_{k=1}^\infty e_n^{(k)} y^{(k)} \xrightarrow[n\to\infty]{} x \quad \forall y \in \ell^2 \quad (*)$
Since $\forall y = \{y^{(k)}\}_k \in \ell^2$ we have that $y^{(k)} \xrightarrow[k\to\infty]{} 0$, starting from $(*)$:
$\Rightarrow \sum_{k=1}^\infty e_n^{(k)} y^{(k)} = y^{(n)} \xrightarrow[n\to\infty]{} 0$
$\Leftrightarrow e_n \xrightharpoonup[n\to\infty]{} 0$.
However, since $\|e_n\|_{\ell^2} = 1 \ \forall n \in \mathbb{N}$ then $e_n \not\xrightarrow[n\to\infty]{} 0$ in ℓ^2. ■

11.12. Show that the weak limit is unique.

Prop. Let $(X, \|\cdot\|_X)$ be a Banach space, $\{x_n\}_n \subset X$, $x \in X$.
If $\{x_n\}_n$ converges weakly then the weak limit is unique.

proof.

Suppose by contraddiction that $\exists x_1, x_2 \in X$ such that $x_n \xrightharpoonup[n\to\infty]{} x_1$, $x_n \xrightharpoonup[n\to\infty]{} x_2$ and $x_1 \neq x_2$.
$\Rightarrow |L(x_n) - L(x_1)| \xrightarrow[n\to\infty]{} 0$, $|L(x_n) - L(x_2)| \xrightarrow[n\to\infty]{} 0 \quad \forall L \in X^*$.
These are convergences in the set of real numbers.
Convergences in the set of real numbers imply the uniqueness of the limit.
$\Rightarrow L(x_1) = L(x_2) \quad \forall L \in X^*$.
Therefore, by the Corollary 2. of the Hahn-Banach theorem we conclude that $x_1 = x_2$.
This is a contraddiction. ■

11.13. If $\{x_n\}_n$ weakly converges to x, can $\{x_n\}_n$ be unbounded?
State and prove lower semicontinuity w.r.t. weak convergence of $x \mapsto \|x\|$.

Let $(X, \|\cdot\|_X)$ be a Banach space, $\{x_n\}_n \subset X$, $x \in X$.
From the UBP we can infer that if $\{x_n\}_n \xrightharpoonup[n\to\infty]{} x \Rightarrow \{x_n\}_n$ is bounded.
Therefore, a weakly convergent sequence **cannot** be unbounded.

Prop. Let $(X, ||\cdot||_X)$ be a Banach space, $\{x_n\}_n \subset X$, $x \in X$.
If $x_n \xrightarrow[n\to\infty]{} x$ then: $\liminf_{n\to\infty} ||x_n||_X \geq ||x||_X$.
Equivalently: $x \mapsto ||x||_X$ is lower semicontinuous w.r.t. weak convergence.

proof.

Let $x \in X \setminus \{0\}$.
By Corollary 1. of Hahn-Banach theorem we know that $\exists L \in X^*$ such that $||L||_{X^*} = 1$ and $L(x) = ||x||_X$.
Then, since $\{x_n\}_n$ converges weakly to x:

$||x||_X = L(x) = \lim_{n\to\infty} L(x_n) = \underbrace{\lim_{n\to\infty} |L(x_n)|}_{||x||>0}$

On the other hand:

$|L(x_n)| \leq ||L||_{X^*}\, ||x_n||_X = ||x_n||_X$

Considering the $\liminf_{n\to\infty}$ of the above expression:

$\Rightarrow \liminf_{n\to\infty} |L(x_n)| \leq \liminf_{n\to\infty} ||x_n||_X$

But, since $|L(x_n)| \xrightarrow[n\to\infty]{} L(x) = ||x||_X$:

$\Rightarrow ||x||_X = \lim_{n\to\infty} |L(x_n)| \leq \liminf_{n\to\infty} ||x_n||_X$. ■

11.14. Show that if x_n weakly converges to x and L_n converges to L in X^*, then $L_n(x_n) \to L(x)$ as $n \to \infty$.

Prop. Let $(X, ||\cdot||_X)$ be a Banach space, $\{x_n\}_n \subset X$, $x \in X$.
If $x_n \xrightarrow[n\to\infty]{} x$ in X and $L_n \xrightarrow[n\to\infty]{} L$ in X^* then: $L_n(x_n) \xrightarrow[n\to\infty]{} L(x)$.

proof.

By hypothesis:

(1) $L_n \xrightarrow[n\to\infty]{} L \;\Rightarrow\; ||L_n - L||_{X^*} \xrightarrow[n\to\infty]{} 0$

(2) $x_n \xrightarrow[n\to\infty]{} x \;\Rightarrow\; \exists M > 0$: $||x_n||_X \leq M \;\; \forall n \in \mathbb{N}$

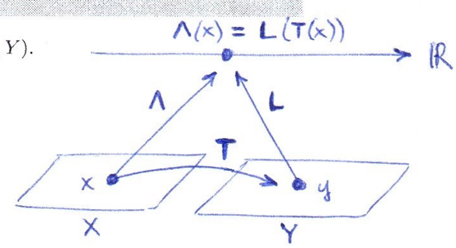
(3) $x_n \xrightarrow[n\to\infty]{} x \;\Rightarrow\; |L(x_n) - L(x)| \xrightarrow[n\to\infty]{} 0$

Therefore:

$$\begin{aligned} |L_n(x_n) - L(x)| &= |L_n(x_n) \pm L(x_n) - L(x)| \\ &\leq |L_n(x_n) - L(x_n)| + |L(x_n) - L(x)| \\ &\leq \underbrace{||L_n - L||_{X^*}}_{\substack{\xrightarrow[n\to\infty]{} 0 \\ (1)}} \underbrace{||x_n||_X}_{\substack{\leq M \\ (2)}} + \underbrace{|L(x_n) - L(x)|}_{\substack{\xrightarrow[n\to\infty]{} 0 \\ (3)}} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$ ■

11.15. Show that $T \in \mathcal{L}(X,Y)$ is weak-weak continuous.

Prop. Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces, $\{x_n\}_n \subset X$, $x \in X$ and $T \in \mathcal{L}(X,Y)$.
If $x_n \xrightarrow[n\to\infty]{} x$ then: $T(x_n) \xrightarrow[n\to\infty]{} T(x)$.
Namely, $T \in \mathcal{L}(X,Y)$ is weak-weak continuous.



proof.

Let $L \in Y^*$.
Define $\Lambda : X \to \mathbb{R}$ such that $\Lambda(x) := L(T(x))$ $\forall x \in X$.
Therefore, since $\Lambda \in X^*$ and since $x_n \xrightarrow[n\to\infty]{} x$:

$\Rightarrow \Lambda(x_n) \xrightarrow[n\to\infty]{} \Lambda(x)$

$\Leftrightarrow L(T(x_n)) \xrightarrow[n\to\infty]{} L(T(x))$ $(\forall L \in Y^*)$

$\Leftrightarrow T(x_n) \xrightarrow[n\to\infty]{} T(x)$. $(\{T(x_n)\}_n \subseteq Y \Rightarrow [T(x_n) \to T(x) \Leftrightarrow L(T(x_n)) \to L(T(x))$ ■ $\forall L \in Y^*)$

11.16. Write a sufficient condition for weak convergence in reflexive space.

Prop. Let $(X, ||\cdot||_X)$ be reflexive, $\{x_n\}_n \subset X$.
If $\{L(x_n)\}_n$ converges $\forall L \in X^*$ then: $\exists!\, x \in X$ such that $x_n \xrightarrow[n\to\infty]{} x$.

5.2 Weak* Convergence $(\{L_n\}_n)$

12.1. Write the definition of **weak* convergence**.

Let $(X, ||\cdot||_X)$ be a Banach space and let X^* be its dual.
The sequence $\{L_n\}_n$ weakly* converges to L, and we write $L_n \xrightarrow[n\to\infty]{*} L$, if:

$L_n(x) \xrightarrow[n\to\infty]{} L(x) \quad \forall x \in X$.

The latter convergence is a convergence of real numbers since $L_n(x)$, $L(x) \in \mathbb{R}$.

12.2. Which is the relation between weak and weak* convergence? Justify the answer.

If X is a Banach space then also X^* is a Banach space, and so we can define the notion of weak convergence in X^*.
What is the relation between weak convergence in X^* and weak convergence in X^*?*

Prop. Let $(X, ||\cdot||_X)$ be a Banach space and let X^* be its dual.

(i) $L_n \xrightarrow[n\to\infty]{} L$ in $X^* \;\Rightarrow\; L_n \xrightarrow[n\to\infty]{*} L$

(ii) $L_n \xrightarrow[n\to\infty]{} L$ in $X^* \;\Leftarrow\; L_n \xrightarrow[n\to\infty]{*} L$, if X is reflexive

(Recall) X reflexive $\Leftrightarrow \tau(X) = X^{**} \Leftrightarrow \forall \Lambda \in X^{**}\; \exists x \in X$: $\Lambda(L) = L(x)\; \forall L \in X^*$.
X is reflexive if we can identify through the canonical map the Banach space and its dual.
In this case, every element of the second dual can be represented in terms of an element of the space X.

proof.

(i) $L_n \xrightarrow[n\to\infty]{} L$ in $X^* \Leftrightarrow \Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L) \quad \forall \Lambda \in X^{**}$
$\Rightarrow \Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L) \quad \forall \Lambda \in \tau(X) \subseteq X^{**}$
$\Leftrightarrow L_n(x) \xrightarrow[n\to\infty]{} L(x) \quad \forall x \in X$
$\Leftrightarrow L_n \xrightarrow[n\to\infty]{*} L$ in X^*

(ii) Since X is reflexive then $\tau(X) = X^{**}$:
$L_n \xrightarrow[n\to\infty]{*} L$ in $X^* \Leftrightarrow L_n(x) \xrightarrow[n\to\infty]{} L(x) \quad \forall x \in X$
$\Leftrightarrow \Lambda(L_n) \xrightarrow[n\to\infty]{} \Lambda(L) \quad \forall \Lambda \in \tau(X) = X^{**}$
$\Leftrightarrow L_n \xrightarrow[n\to\infty]{} L$ in X^* ∎

12.3. Write the properties of weak* convergence.

Prop. Let $(X, ||\cdot||_X)$ be a Banach space and let X^* be its dual.
- If $\{L_n\}_n \subset X^*$ weakly* converges then the limit is unique.
- $L_n \xrightarrow[n\to\infty]{*} L \Rightarrow \{L_n\}_n$ is bounded
- $L_n \xrightarrow[n\to\infty]{*} L \Rightarrow \liminf_{n\to\infty} ||L_n||_{X^*} \geq ||L||_{X^*}$
  Equivalently: $L \mapsto ||L||_{X^*}$ *is lower semicontinuous w.r.t. weak* convergence.*
- If $x_n \xrightarrow[n\to\infty]{} x$ in X and $L_n \xrightarrow[n\to\infty]{*} L$ in X^* then $L_n(x_n) \xrightarrow[n\to\infty]{} L(x)$.

12.4. State the **Banach-Alaoglu theorem**. Why can we say that from a bounded sequence in L^∞ we can extract a subsequence which weakly* convergence in L^∞?

Theorem. Banach-Alaoglu
Let $(X, ||\cdot||_X)$ be a separable Banach space and let X^* be its dual.
Then, any bounded sequence $\{L_n\}_n \subset X^*$ admits a subsequence that weakly* converges to some $L \in X^*$.

Let $\Omega \subseteq \mathbb{R}^n$ be measurable and consider $(\Omega, \mathcal{L}(\Omega), \lambda)$.
In view of the Banach-Alaoglu theorem, we can conclude that every bounded sequence $\{f_n\}_n \subset L^\infty(\Omega, \mathcal{L}(\Omega), \lambda)$ admits a subsequence $\{f_{n_k}\}_k$ which weakly* converges to some $f \in L^\infty(\Omega, \mathcal{L}(\Omega), \lambda)$.

proof.

Let $\{f_n\}_n \subset L^\infty(\Omega)$ be a bounded sequence.
Then, $\forall n \in \mathbb{N}$ define $L_n : L^1(\Omega) \to \mathbb{R}$ as:
$L_n(g) := \int_\Omega f_n \, g \, d\lambda \quad \forall g \in L^1(\Omega)$. $(L_n \in (L^1(\Omega))^*)$

$X = L^1(\Omega)$
$X^* = (L^1(\Omega))^*$
$\{L_n\}_n \subseteq X^*$ bounded

Since $\{f_n\}_n$ is bounded in $L^\infty(\Omega)$:
$\Rightarrow \exists C > 0$: $||f_n||_\infty \leq C \quad \forall n \in \mathbb{N}$
$\Rightarrow |L_n(g)| \leq ||f_n||_\infty ||g||_1 \leq C ||g||_1 \quad \forall n \in \mathbb{N}$
$\Rightarrow ||L_n||_{(L^1(\Omega))^*} \leq C$
$\Leftrightarrow \{L_n\}_n$ is bounded in $(L^1(\Omega))^*$

Since $L^1(\Omega)$ is a separable Banach space and $\{L_n\}_n \subset (L^1(\Omega))^*$ is bounded, we can apply Banach-Alaoglu theorem:
$\Rightarrow \exists \{L_{n_k}\}_k \subset \{L_n\}_n$: $L_{n_k} \xrightarrow[k\to\infty]{*} L$ for some $L \in (L^1(\Omega))^*$
$\Leftrightarrow \boxed{\exists \{L_{n_k}\}_k \subset \{L_n\}_n: \ L_{n_k}(g) \xrightarrow[k\to\infty]{} L(g) \quad \forall g \in L^1(\Omega)}$ (1)

We recall the representation theorem for $(L^1(\Omega))^*$:
$\forall L \in (L^1(\Omega))^* \ \exists! f \in L^\infty(\Omega)$: $L(g) = \int_\Omega f \, g \, d\lambda \quad \forall g \in L^1(\Omega)$

Therefore, (1) becomes:
$\Rightarrow \boxed{\exists \{f_{n_k}\}_k \subset \{f_n\}_n \subset L^\infty(\Omega): \ \int_\Omega f_{n_k} \, g \, d\lambda \xrightarrow[k\to\infty]{} \int_\Omega f \, g \, d\lambda \quad \forall g \in L^1(\Omega)}$
$\Leftrightarrow \exists \{f_{n_k}\}_k \subset \{f_n\}_n \subset L^\infty(\Omega)$: $f_{n_k} \xrightarrow[k\to\infty]{*} f$. ∎

12.5. State and prove the corollary of the Banach-Alaoglu theorem in a separable and reflexive Banach space.

Corollary. Let $(X, ||\cdot||_X)$ be a separable and reflexive Banach space.
Then, any bounded sequence $\{x_n\}_n \subset X$ admits a subsequence that weakly converges to some $x \in X$.

proof.

Since X is a reflexive Banach space and it is separable then also X^* is separable.
Consider the canonical map τ.
Then $\tau(x_n) = \Lambda_{x_n} \ \forall n \in \mathbb{N}$.
$\Rightarrow \Lambda_{x_n}(L) = L(x_n)$
$\Rightarrow |\Lambda_{x_n}(L)| = |L(x_n)| \leq ||L||_{X^*} ||x_n||_X$.

Since $\{x_n\}_n$ is bounded then:
$\Rightarrow \exists C > 0$: $||x_n||_X \leq C \quad \forall n \in \mathbb{N}$
$\Rightarrow |\Lambda_{x_n}(L)| \leq C ||L||_{X^*}$
$\Rightarrow \{\Lambda_{x_n}\}_n = \{\tau(x_n)\}_n$ is bounded.

Considering X^* separable Banach space and $\{\tau(x_n)\}_n \subset X^{**}$ bounded, we can apply the Banach-Alaouglu to X^*:
$\Rightarrow \exists \{\tau(x_{n_k})\}_k$: $\tau(x_{n_k}) = \Lambda_{x_{n_k}} \xrightarrow[k\to\infty]{*} \Lambda$ for some $\Lambda \in X^{**}$
$\Leftrightarrow \boxed{\exists \{\tau(x_{n_k})\}_k: \ (\tau(x_{n_k}))(L) = \Lambda_{x_{n_k}}(L) \xrightarrow[k\to\infty]{} \Lambda(L) \quad \forall L \in X^*}$ (1)

Then, by definition of τ and since X is reflexive:
- $(\tau(x_{n_k}))(L) = \Lambda_{x_{n_k}}(L) = L(x_{n_k})$
- $\Lambda(L) = L(x)$ with $x := \tau^{-1}(\Lambda)$

And so, starting from (1):
$\Rightarrow \boxed{\exists \{x_{n_k}\}_k \subset \{x_n\}_n: \ L(x_{n_k}) \xrightarrow[k\to\infty]{} L(x) \quad \forall L \in X^*}$
$\Leftrightarrow \exists \{x_{n_k}\}_k \subset \{x_n\}_n$: $x_{n_k} \xrightarrow[k\to\infty]{} x$. ∎

12.6. State the **Eberlein-Smulyan theorem.**

Theorem. **Eberlein-Smulyan**
Let $(X, ||\cdot||_X)$ be a Banach space.
If any bounded sequence admits a weakly convergent subsequence then X is reflexive.

6 Compact Operators

12.7. Write the definition of **compact operator.** Write the definition of **operator of finite rank.**
How is a compact operator related to operators of finite rank?
Can a compact operator defined on an infinite dimensional Banach space be bijective?

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces.
A linear operator $K : X \to Y$ is said to be compact if $\forall E \subseteq X$ bounded $\overline{K(E)}$ is compact.
A linear mapping is compact if it transforms bounded subsets into subsets whose closure is compact.
We define:
$\mathcal{K}(X,Y) := \{K : X \to Y : K \text{ is compact}\}$
If $X = Y$ then we write $\mathcal{K}(X,Y) \equiv \mathcal{K}(X)$.
Prop. If $\{x_n\}_n \subset X$ is bounded then $\{K(x_n)\}_n$ has a subsequence which converges strongly in Y.
Prop. If $K : X \to Y$ is linear and compact then $K \in \mathcal{L}(X,Y)$. (K *is also bounded*)
A linear and bounded operator $T \in \mathcal{L}(X,Y)$ has finite rank if $\dim(\mathrm{Im}(T)) < \infty$.
Prop. Let $T \in \mathcal{L}(X,Y)$.
(i) $\mathrm{rank}(T)$ is finite $\Rightarrow$ T is compact
(ii) $\mathrm{rank}(T)$ is finite $\not\Leftarrow$ T is compact
Prop. An operator defined on an infinite dimensional Banach space cannot be bijective.
proof.

In the settings of $\dim(X) = \infty$, suppose by contraddiction that $\exists K \in K(X,Y)$ bijective.
By Riesz theorem we know that the closed ball $\overline{B}_1(0)$ is compact only if $\dim(X) < \infty$.
Therefore, since we are considering $\dim(X) = \infty$, $\overline{B}_1(0)$ is not compact.
Since $\overline{B}_1(0)$ is bounded, its image through a compact operator is compact and, since K is bijective, we can go back from the image obtaining that also $\overline{B}_1(0)$ is compact.
This is a contraddiction and so $\nexists K \in K(X,Y)$ bijective if $\dim(X) = \infty$. ■

In view of this, if $\dim(X) = \infty$, *surjective operators cannot be injective and viceversa.*
For example, let $X = Y = \ell^2$ *and* $x \in \ell^2$. *Then:*
- $L_r(x) = (0, x_1, x_2, ..)$ *is injective but not surjective*
- $L_\ell(x) = (x_2, x_3, x_4, ..)$ *is surjective but not injective*

Def. **Compactly embedded subspace**
Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be Banach spaces such that $Y \subset X$ is a subspace.
If the identity $I : Y \to X$ is compact, we say that Y is compactly embedded in X and we write $Y \overset{c}{\hookrightarrow} X$.

12.8. State the theorem about the characterization of compact operators.

Theorem. Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces.
(i) If $K \in K(X,Y)$ then K is weak-strong continuous, namely:
$x_n \underset{n\to\infty}{\rightharpoonup} x \Rightarrow K(x_n) \underset{n\to\infty}{\longrightarrow} K(x)$.
(ii) If X is reflexive and $K \in \mathcal{L}(X,Y)$ is weak-strong continuous then $K \in K(X,Y)$.

Corollary. Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces. Then $K(X,Y)$ is a Banach space.

7 Hilbert Spaces

7.1 Hilbert Spaces

12.9. Write the definition of **pre-Hilbert** and **Hilbert spaces.**

Let H be a vector space and let $p : H \times H \to \mathbb{R}$ be a simmetric, positive definite bilinear form, namely:
(i) $\forall x \in H$: $p(x,x) \geq 0$, $p(x,x) = 0 \Leftrightarrow x = 0$
(ii) $\forall x, y \in H$: $p(x,y) = p(y,x)$
(iii) $\forall x, y, z \in H$, $\forall \alpha, \beta \in \mathbb{R}$: $p(\alpha\, x + \beta\, y, z) = \alpha\, p(x,z) + \beta\, p(y,z)$
A simmetric, positive definite bilinear form p on $H \times H$ is called scalar (or inner) product and we set $< x, y > := p(x,y)$.
A vector space H equipped with a scalar product is called pre-Hilbert space.
Prop. Let $(H, < \cdot, \cdot >)$ be a pre-Hilbert space.
(i) Cauchy Schwartz inequality: $| < x, y > | \leq \sqrt{< x, x >}\sqrt{< y, y >}$.
(ii) $\forall x \in H$: $||x|| := \sqrt{< x, x >}$ is a norm on H.
A scalar product induces a norm and a norm induces a distance.
Therefore, a pre-Hilbert space is and normed space and a metric space.
A pre-Hilbert space $(H, < \cdot, \cdot >)$ is a Hilbert space if $(H, ||\cdot||)$ (where $||\cdot||$ is induced by $< \cdot, \cdot >$) is a Banach space or, equivalently, if (H, d) (where d is induced by $||\cdot||$, which is induced by $< \cdot, \cdot >$) is a complete metric space.

Examples:
- $(C([a,b]), <\cdot,\cdot>)$ with $<f,g> = \int_a^b f\, g\, dx$ is a pre-Hilbert space.
- $(L^2(X,\mathcal{A},\mu), <\cdot,\cdot>)$ with $<f,g> = \int_X f\, g\, d\mu$ is a Hilbert space.
- $(\ell^2, <\cdot,\cdot>)$ with $<x,y> = \sum_{k=1}^{\infty} x^{(k)}\, y^{(k)}$ is a Hilbert space.

12.10. Show the **parallelogram law**.

Theorem. **Parallelogram law**
Let $(H, <\cdot,\cdot>)$ be a pre-Hilbert space. Then $\forall x, y \in H$:
$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$.

proof.

$$\begin{aligned}\forall x,y \in H: \quad ||x+y||^2 + ||x-y||^2 &= <x+y, x+y> + <x-y, x-y> \\ &= <x,x> + 2<x,y> + <y,y> + <x,x> - 2<x,y> + <y,y> \\ &= 2||x||^2 + 2||y||^2.\end{aligned}$$ ■

7.2 Orthogonal Projections

Def. **Orthogonal elements**
Let $(H, <\cdot,\cdot>)$ be a Hilbert space. Two elements $x, y \in H$ are orthogonal if $<x,y> = 0$.

12.11. State and prove the theorem in Hilbert spaces about **minimal distance** for convex closed subsets (both the preliminary result and the corollary).

(Recall) A set S is convex if $\forall x, y \in S$ the segment joining x and y is contained in S.

Theorem. Let $(H, <\cdot,\cdot>)$ be a Hilbert space and let $\varnothing \neq S \subseteq H$ be a convex closed subset.
Then $\exists!\, h \in S$ such that:
$\min_{s \in S} ||s|| = ||h||$.
As a special case we can consider S closed subspace.

proof.

1. Let $\boxed{d := \inf_{s\in S} ||s||}$. **1**
Then it exists a $\boxed{\text{minimizing sequence } \exists \{s_n\}_n \subset S}$ **2** such that:
   - $||s_n|| \geq d$
   - $||s_n|| \xrightarrow[n\to\infty]{} d$

convexity of S

We claim that the sequence $\{s_n\}_n$ is Cauchy.
Since S is convex, $\forall m, n \in \mathbb{N}$ we have that $\frac{s_m + s_n}{2} \in S$.
Therefore, by the definition of d:
$\Rightarrow ||\frac{s_m+s_n}{2}|| = \frac{1}{2}||s_m + s_n|| \geq d$
$\Rightarrow ||s_m + s_n||^2 \geq 4d^2$.
Then, by the parallelogram law:
$||s_m - s_n||^2 = 2\,(||s_m||^2 + ||s_n||^2) - ||s_m + s_n||^2 \leq 2\,(||s_m||^2 + ||s_n||^2) - 4d^2$.
Starting from the previous expression and considering that $||s_n||^2 \xrightarrow[n\to\infty]{} d^2$:
$\Rightarrow \forall \varepsilon > 0 \;\; \exists \bar{n} \in \mathbb{N}$ such that $\forall m, n > \bar{n}$:
$||s_m - s_n||^2 \leq 2\,(d^2 + \varepsilon) + 2\,(d^2+\varepsilon) - 4d^2 = 4\varepsilon$
$\Leftrightarrow \boxed{\{s_n\}_n \text{ is a Cauchy sequence.}}$ **3**

2. Since H is a Hilbert space then H is also a complete metric space.
$\Rightarrow \{s_n\}_n$ is a Cauchy sequence in a complete metric space.
$\Rightarrow \boxed{\exists h \in H \text{ such that } ||s_n - h|| \xrightarrow[n\to\infty]{} 0.}$ **4**
$\Rightarrow h \in S$ since $\{s_n\}_n \subset S$ and S is closed.

closedness of S

We claim that $\boxed{||h|| = d = \min_{s\in S} ||s||.}$ **5**
In fact:
$|\,||s_n|| - ||h||\,| \leq ||s_n - h|| \xrightarrow[n\to\infty]{} 0$
$\Rightarrow ||s_n|| \xrightarrow[n\to\infty]{} ||h||$
$\Rightarrow \boldsymbol{||h|| \geq d}$, since $||s_n|| \geq d$.
On the other hand, by definition of limit:
$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n > n_\varepsilon$:
   - $||s_n|| < d + \varepsilon$
   - $||s_n - h|| < \varepsilon$

Therefore:
$||h|| = ||s_n - (s_n - h)|| \leq ||s_n|| + ||s_n - h|| < d + \varepsilon + \varepsilon = d + 2\varepsilon$.
Since $\varepsilon > 0$ is arbitrary we get that $\boldsymbol{||h|| \leq d}$.

3. $\boxed{h \in S \text{ must be unique.}}$ **6**
In fact, suppose by contraddiction that $\exists \tilde{h} \in S$:
   - $||\tilde{h}|| = ||h|| = d$
   - $||\tilde{h} - h|| \geq \varepsilon > 0$ *(another way to say that $\tilde{h} \neq h$)*

Since S is convex then $\frac{\tilde{h}+h}{2} \in S$.
Therefore, by the definition of d:
$\Rightarrow ||\frac{\tilde{h}+h}{2}|| = \frac{1}{2}||\tilde{h} + h|| \geq d$
$\Rightarrow ||\tilde{h} + h||^2 \geq 4d^2 \quad (*)$
Instead, by the parallelogram law:
$||\tilde{h} + h||^2 = 2\,(||\tilde{h}||^2 + ||h||^2) - ||\tilde{h} - h||^2 \leq 4d^2 - \varepsilon^2 < 4d^2 \quad (**)$
Because of $(*)$ and $(**)$ we get a contraddiction and so $h \in S$ is unique. ■

Theorem. **Minimal distance**

Let $(H, <\cdot,\cdot>)$ be a Hilbert space and let $\varnothing \neq S \subseteq H$ be a convex closed subset.
Then $\forall f \in H \;\; \exists! \, h \in S$ such that:

$||f-h|| = \min_{s\in S} ||f-s|| =: dist(f,S).$

We consider an Hilbert space and a convex closed subset S. The theorem says that for every element f in the bigger space, there exists a unique element h in the smaller subset S such that the distance between f and h realizes the distance between f and S.

proof.

Let $\tilde{S} := \{v - f : v \in S\}$.
By the previous theorem $\exists! \, \xi \in \tilde{S}$ such that:

$||\xi|| = \min_{\eta\in\tilde{S}} ||\eta||.$

Since $\xi \in \tilde{S}$ then $\exists! \, h \in S$ such that $\xi = h - f$.
Therefore $\exists! \, h \in S$, given by $h = \xi + f$, for which:

$||h-f|| = ||\xi|| = \min_{\eta\in\tilde{S}} ||\eta|| = \min_{s\in S} ||s-f|| =: dist(f,S).$ ∎

12.12. State and prove the theorem about **orthogonal projections** in Hilbert spaces.

Prop. Let $(H, <\cdot,\cdot>)$ be a pre-Hilbert space and consider a subset $S \subseteq H$.

(i) $S^{\perp} := \{f \in H : <f,g> = 0 \;\forall g \in S\}$ is a closed subset.
(ii) If S is a subspace then $S \cap S^{\perp} = \{0\}$.

Theorem. **Projection theorem**

Let $(H, <\cdot,\cdot>)$ be a Hilbert space and consider $V \subseteq H$ closed subspace.
Then $\forall f \in H \;\; \exists! \, f_1, \, f_2$ satisfying:

(i) $f_1 \in V, \;\; f_2 \in V^{\perp}$
(ii) $f = f_1 + f_2$

We have an orthogonal decomposition of f w.r.t. the subspace V.

proof.

Let $f \in H$ be fixed and let f_1 be the unique element of V such that:

$||f - f_1|| = \min_{s\in V} ||f - s||.$

The existence of such f_1 follows from the minimal distance theorem.
We set $f_2 = f - f_1$, so that $f = f_1 + f_2$, and we claim that $f_2 \in V^{\perp}$, that is:

$<f_2, s> = 0 \quad \forall s \in V.$

Indeed, since V is a subspace, $\forall s \in V$ and $\forall \lambda \in \mathbb{R}$ we get that $f_1 + \lambda s \in V$.
Therefore, since $f_1 = \min_{s\in V} ||f - s||$:

$\Rightarrow \; ||f-f_1||^2 \leq ||f - (f_1 + \lambda s)||^2 = ||f - f_1||^2 + \lambda^2 ||s||^2 - 2\lambda <f - f_1, s>$

$\Rightarrow \; \lambda^2 ||s||^2 - 2\lambda <f - f_1, s> \geq 0.$

Let $s \neq 0$ and consider $\lambda = \frac{<f-f_1,s>}{||s||^2}$:

$$\Rightarrow \quad \frac{<f-f_1,s>^2}{||s||^4} ||s||^2 - 2\frac{<f-f_1,s>}{||s||^2} <f-f_1,s> \geq 0$$

$$\Rightarrow \quad \frac{<f-f_1,s>^2}{||s||^2} \leq 0 \quad \Leftrightarrow \quad \frac{<f_2,s>^2}{||s||^2} \leq 0$$

$\Rightarrow$ it must be $= 0$ since it is a non negative quantity

$\Rightarrow <f_2, s> = 0 \quad \forall s \in V, \, s \neq 0$

$\Rightarrow \; f_2 \in V^{\perp}.$ ∎

12.13. Write and show properties of the orthogonal projector.

Let $(H, <\cdot,\cdot>)$ be a Hilbert space and let $V \subseteq H$ be a closed subspace.
Then, by means of the projection theorem we can define two mappings:

$P : H \to V \qquad P(f) = f_1 \qquad \forall f \in H$

$Q : H \to V^{\perp} \qquad Q(f) = f_2 \qquad \forall f \in H$

P is the orthogonal projector on V and Q is the orthogonal projector on $V^{\perp}$.
Moreover, from the projection theorem we can infer that:

(i) $P(f) + Q(f) = f \quad \forall f \in H$
(ii) $f \in V \;\; \Rightarrow P(f) = f, \; Q(f) = 0$
$\quad\;\; f \in V^{\perp} \Rightarrow P(f) = 0, \; Q(f) = f$
(iii) $||f - P(f)|| = dist(f, V)$ $\longrightarrow$ the projection is the element that realizes the distance
(iv) $||f||^2 = ||P(f)||^2 + ||Q(f)||^2$
(v) P and Q are linear. Moreover, from (iv) we deduce that $||P(f)|| \leq ||f||$ and $||Q(f)|| \leq ||f||$.
Therefore, P and Q are bounded and so $P, Q \in \mathcal{L}(H)$.

7.3 Duality

12.14. State and prove the **Riesz theorem**.

Theorem. **Riesz Theorem**

Let $(H, <\cdot,\cdot>)$ be a Hilbert space and let H^* be its dual.
Then $\forall F \in H^* \;\; \exists! \, g \in H$ such that:

$F(f) = <f, g> \quad \forall f \in H.$

Furhtermore: $||F||_{H^*} = ||g||_H$.

H^* is the set of all linear and continuous operators from H to $\mathbb{R}$, and it is Banach, since H is Banach. The theorem says that we can represent every element of the dual space by means of an unique element of the space H. In some sense, H and H^* are the same. The action of every element of the dual can be expressed totally by means of the scalar product with elements of H.

proof.

1. If $F \equiv 0$, namely $F(f) = 0 \ \forall f \in H$, then we take $g = 0$ and the thesis follows.
   If $F \not\equiv 0$, we define:
   $\boxed{N := \{f \in H : F(f) = 0\}}$ 1
   which is a closed subspace of H, since it is the kernel of F.
   Having a closed subspace of a Hilbert space, we can apply the projection theorem to conclude that $\boxed{\exists f_2 \in N^\perp}$ 2
   Moreover, since $F \not\equiv 0$ and so $N \subsetneq H$, we can guarantee that $f_2 \neq 0$.
   Since $N^\perp$ is a subspace, we can set:
   $\boxed{h := \frac{f_2}{F(f_2)} \in N^\perp}$ 3
   where $F(f_2) \neq 0$ in view of the definition of N and because $N \cap N^\perp = \{0\}$.
   Let $\boxed{m := f - F(f)\, h \ \ \forall f \in H.}$ 4
   $\Rightarrow \ F(m) = F(f) - F(f)\, F(h) = F(f) - F(f)\, F(\frac{f_2}{F(f_2)}) = F(f) - F(f)\, \frac{1}{F(f_2)}\, F(f_2) = F(f) - F(f) = 0.$
   Thus, $m \in N$ and since $m = f - F(f)\, h$, we get an orthogonal decomposition of f:
   $\boxed{f = \underbrace{m}_{\in N} + \underbrace{F(f)\, h}_{\in N^\perp}.}$ 5
2. Let $\boxed{g := \frac{h}{||h||^2} \in N^\perp}$ 6 since $h \in N^\perp$.
   Then, since $m \in N$ and $g \in N^\perp$:
   $\Rightarrow \ < m, g >= 0 \quad (1)$
   Moreover:
   $F(g) = \frac{F(h)}{||h||^2} = \frac{1}{F(f_2)}\frac{F(f_2)}{||h||^2} = \frac{1}{||h||^2} = ||g||^2 =< g, g > \quad (2)$
   $\Rightarrow \ f = m + F(f)\, h = m + F(f)\, g\, ||h||^2 = m + F(f)\, \frac{g}{||g||^2} \quad (3)$
   Combining together (1), (2) and (3) we get:
   $< f, g >=< m + F(f)\frac{g}{||g||^2}, g >=< m, g > + F(f)\frac{1}{||g||^2} < g, g >= F(f) \quad \forall f \in H$
   $\Rightarrow \ |F(f)| = |< f, g >| \leq ||f|| \cdot ||g|| \quad \forall f \in H$
   $\Rightarrow \ ||F||_{H^*} \leq ||g||_H.$
   However, from (2) we have that $|F(g)| = ||g||^2$:
   $\Rightarrow \boxed{||F||_{H^*} = ||g||_H.}$ 7
3. We now prove that $\boxed{\text{such } g \text{ must be unique.}}$ 8
   Suppose by contraddiction that there exists $\tilde{g} \in H$, $\tilde{g} \neq g$ such that:
   $F(f) =< f, \tilde{g} >=< f, g > \quad \forall f \in H.$
   Then:
   $0 = F(f) - F(f) =< f, g > - < f, \tilde{g} >=< f, g - \tilde{g} > \quad \forall f \in H$
   For $f = g - \tilde{g}$ we get:
   $< g - \tilde{g}, g - \tilde{g} >= 0 \ \Leftrightarrow \ g = \tilde{g}.$
   This is a contraddiction and so g is unique. ∎

We underly that in this proof we have explicitly constructed the element g and we have done it by means of consequences of orthogonal projections.

7.4 Orthonormal Bases

12.15. Write the definition of **orthonormal bases** in a Hilbert space.

Let $(H, < \cdot, \cdot >)$ be a Hilbert space.
A set $S \subseteq H$ is called orthonormal if:
(i) $f \perp g \quad \forall f, g \in S$
(ii) $||f|| = 1 \quad \forall f \in S$
An orthonormal set $S \subseteq H$ is said to be complete or an orthonormal basis if $S^\perp = \{0\}$.

Theorem. Let $(H, < \cdot, \cdot >)$ be a Hilbert space.
If H has at least two different elements then H has an orthonormal basis.
- *H separable $\Rightarrow$ Gram-Schmidt method*
- *H not separable $\Rightarrow$ Zorn's lemma*

*Notice that: H separable $\Leftrightarrow$ H has a **countable** orthonormal basis.*

Examples:
- $H = \ell^2$: $\{e_n^{(k)} = \delta_{nk}\}_n$ is an orthonormal basis.
- $H = L^2([-\pi, \pi])$: $\{\frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}\}_{n \geq 1}$ is an orthonormal basis.

12.16. Write the **Bessel inequality**. Write the theorem about **abstract Fourier expansion** in a Hilbert space and the **Parseval identity**.

Theorem. **Bessel inequality**
Let $(H, < \cdot, \cdot >)$ be a Hilbert space and let $\{\varphi_n\}_n \subset H$ be an orthonormal set.
Then $\forall f \in H$:
$\sum_{n=1}^{\infty} |< f, \varphi_n >|^2 \leq ||f||^2.$

Theorem. **Fourier expansion and Parseval identity**
Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space and let $\{\varphi_n\}_n$ be an orthonormal basis.
Then $\forall f \in H$:
$f = \sum_{n=1}^{\infty} < f, \varphi_n > \varphi_n \ .$
Moreover $\forall f, g \in H$:
$< f, g >= \sum_{n=1}^{\infty} < f, \varphi_n >< g, \varphi_n > .$
In particular, it holds the so-called Parseval identity:
$||f||^2 = \sum_{n=1}^{\infty} < f, \varphi_n >^2.$
The coefficients $< f, \varphi_n >$ are called Fourier coefficients.

12.17. Let H be a separable space and $\{\varphi_n\}_n$ an orthonormal basis.
Show that φ_n weakly converges to 0, but it does not converge strongly to 0.

(Recall) Let $(H, <\cdot,\cdot>)$ be a Hilbert space.
Due to Riesz representation theorem we obtain:

$$x_n \xrightarrow[n\to\infty]{} x \text{ in } H \Leftrightarrow L(x_n) \xrightarrow[n\to\infty]{} L(x) \quad \forall L \in H^*$$
$$\Leftrightarrow < x_n, y > \xrightarrow[n\to\infty]{} < x, y > \quad \forall y \in H$$

Corollary. Let $(H, <\cdot,\cdot>)$ be a separable Hilbert space and let $\{\varphi_n\}_n$ be an orthonormal basis. Then:

- $\varphi_n \xrightarrow[n\to\infty]{} 0$
- $\varphi_n \not\xrightarrow[n\to\infty]{} 0$

proof.

By Parseval identity, $\forall f \in H$ we get that:
$||f||^2 = \sum_{n=1}^{\infty} < f, \varphi_n >^2$.
Since $||f||^2 \in \mathbb{R}$ we have that $\sum_{n=1}^{\infty} < f, \varphi_n >^2$ converges.
$\Rightarrow < f, \varphi_n > \xrightarrow[n\to\infty]{} 0 \quad \forall f \in H$
$\Leftrightarrow \varphi \xrightarrow[n\to\infty]{} 0$.
On the other hand, $||\varphi_n|| = 1 \ \ \forall n \in \mathbb{N}$ and so, $\varphi_n \not\xrightarrow[n\to\infty]{} 0$. ∎

12.18. State the **Riesz-Fisher theorem.**

Theorem. Riesz-Fisher
Let $(H, <\cdot,\cdot>)$ be a separable Hilbert space. Then H is isomorphic and isometric to ℓ^2.
The proof is based on a mapping $\mathcal{F} : \ell^2 \to \ell^2$ *defined as* $\mathcal{F}(f) =< f, \varphi_n >= x^{(n)}$, *where* $\{\varphi_n\}_n$ *is an orthonormal basis for* H.

12.19. What can we say about reflexivity of Hilbert spaces?

Let $(H, <\cdot,\cdot>)$ be a Hilbert space.
Exploiting the parallelogram law it is possible to show that H is uniformly convex.
Then, by Millman-Pettis theorem, H is reflexive.

8 Linear Bounded Operators on Hilbert Spaces

12.20. Show the formula which gives the norm of an operator on a Hilbert space.

Let $(H, <\cdot,\cdot>)$ be a Hilbert space.
If $T \in \mathcal{L}(H)$ then $||T||_{\mathcal{L}} = \sup_{||x||=||y||=1} | < T(x), y > |$.
The other forms of $||T||_{\mathcal{L}}$ *are still valid:*
$||T||_{\mathcal{L}} = \sup_{||x||\leq 1} ||T(x)|| = \sup_{||x||=1} ||T(x)|| = \sup_{x\in X\setminus\{0\}} \frac{||T(x)||}{||x||}$.
proof.

By Cauchy-Schwartz, $\forall x, y \in H$ we get:
$| < T(x), y > | \leq ||T(x)|| \cdot ||y||$
$\Rightarrow \sup_{||x||=||y||=1} | < T(x), y > | \leq \sup_{||x||=||y||=1} ||T(x)|| \cdot ||y|| = \sup_{||x||=1} ||T(x)|| = ||T||_{\mathcal{L}}$.
For $T(x) \neq 0$ we consider $y = \frac{T(x)}{||T(x)||}$, so that $||y|| = 1$, and so:
$< T(x), y >=< T(x), \frac{T(x)}{||T(x)||} >= ||T(x)||$.
Therefore, the supremum is realized and thus:
$\sup_{||x||=||y||=1} | < T(x), y > | = ||T||_{\mathcal{L}}$. ∎

8.1 Symmetric Operators

12.21. Write the definition of **symmetric operator** on a Hilbert space.
Write the formula which gives the norm of a symmetric operator on a Hilbert space.

Let $(H, <\cdot,\cdot>)$ be a Hilbert space.
An operator $T \in \mathcal{L}(H)$ is said to be symmetric if $< T(x), y >=< x, T(y) > \ \forall x, y \in H$.
If $T \in \mathcal{L}(H)$ is symmetric then $||T||_{\mathcal{L}} = \sup_{||x||=1} | < T(x), x > |$.
proof.

- Let $\alpha := \sup_{||x||=1} | < T(x), x > |$.
  Then, by Cauchy-Schwartz, $\forall x \in H$ we get:
  $| < T(x), x > | \leq ||T(x)|| \cdot ||x||$
  $\Rightarrow \alpha =: \sup_{||x||=1} | < T(x), x > | \leq \sup_{||x||=1} ||T(x)|| := ||\boldsymbol{T}||_{\mathcal{L}}$
- We observe that, by definition of α:
  $\Rightarrow \alpha = \sup_{||x||=1} | < T(x), x > | \geq < T(\frac{\xi}{||\xi||}), \frac{\xi}{||\xi||} > \quad \forall \xi \in H$
  $\Leftrightarrow \boxed{< T(\xi), \xi > \leq \alpha ||\xi||^2 \quad \forall \xi \in H}$
  $\Rightarrow 4 < T(x), y >=< T(x+y), x+y > - < T(x-y), x-y >$
  $\leq \alpha (\, ||x+y||^2 + ||x-y||^2)$
  $= 2\alpha (\, ||x||^2 + ||y||^2)$

For $T(x) \neq 0$ we consider $y = \frac{||x||}{||T(x)||} T(x)$ and we get:

$$4 \underbrace{< T(x), T(x) >}_{||T(x)||^2} \frac{||x||}{||T(x)||} \leq 2\alpha \left(||x||^2 + \frac{||x||^2}{||T(x)||^2} ||T(x)||^2 \right)$$
$$\Rightarrow 4\,||T(x)|| \cdot ||x|| \leq 4\alpha\, ||x||^2$$
$$\Rightarrow ||T(x)|| \leq \alpha\, ||x||$$
$$\Rightarrow ||T||_{\mathcal{L}} = \sup_{||x||=1} ||T(x)|| \leq \alpha.$$

■

8.2 Eigenvalues and Eigenvectors

12.22. Write the definition of **eigenvalue, eigenvector** and **eigenspace** for a linear and bounded operator on a Hilbert space.

Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space.
We say that $\lambda \in \mathbb{R}$ is an eiegenvalue for $T \in \mathcal{L}(H)$ if $\exists\, v \in H \setminus \{0\}$ such that $T(v) = \lambda\, v$.
If so, v is called eigenvector associated with λ and
$V_\lambda := \{v \in H : T(v) = \lambda\, v\}$
is called eigenspace associated with λ.
For any λ eigenvalue of T it holds that:

(i) V_λ is closed
(ii) $|\lambda| \leq ||T||_{\mathcal{L}}$

12.23. What can we say about eigenvalues of symmetric operators and of symmetric compact operators on Hilbert spaces?

Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space.

Prop. Let $T \in \mathcal{L}(H)$ be symmetric.
Then, the eigenvectors associated with distinct eigenvalues are orthogonal.
Moreover, the eigenvalues are at most countable.

Prop. Let $K \in K(H)$ be (compact and) symmetric.
Then, $||K||_{\mathcal{L}}$ or $-||K||_{\mathcal{L}}$ is an eigenvalue of K.

12.24. Let H be a separable Hilbert space, $K \in K(H)$. Show that $\dim(V_\lambda) < \infty$, provided that $\lambda \neq 0$.

Prop. Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space and let $K \in K(H)$.
If $\lambda \neq 0$ is an eigenvalue of K then $\dim(V_\lambda) < \infty$.

proof.

Suppose by contraddiction that $\dim(V_\lambda) = \infty$.
Let $\{v_n\}_n$ be an orthonormal basis of V_λ.
Then, $v_n \xrightarrow[n\to\infty]{} 0$ but $v_n \not\xrightarrow[n\to\infty]{} 0$, since $||v_n|| = 1$.
On the other hand:
$K(v_n) = \lambda\, v_n \quad \forall n \in \mathbb{N}$
and, since K is compact then K is weak-strong continuous:
$v_n \xrightarrow[n\to\infty]{} 0 \Rightarrow K(v_n) \xrightarrow[n\to\infty]{} 0$
$\Leftrightarrow v_n \xrightarrow[n\to\infty]{} 0$
This is a contraddiction, and so $\dim(V_\lambda) < \infty$.

■

8.3 Spectral Theorem

12.25. Write the definition of **spectrum** and of **resolvent** of a symmetric operator on Hilbert spaces.

Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space.
We define the spectrum of a symmetric operator $T \in \mathcal{L}(H)$ as:
$\sigma(T) := \{\lambda \in \mathbb{R} : \lambda \text{ eigenvalue of } T\}$.
Moreover, we define the resolvent of T as $R(T) := \sigma(T)^c$.

12.26. State the **spectral theorem.**

Theorem. **Spectral theorem**
Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space and let $T \in K(H)$ be symmetric.
Then $\sigma(T)$ is either finite or it is a sequence $\{\lambda_n\}_n$ such that $\lambda_n \xrightarrow[n\to\infty]{} 0$.
Moreover, the eigenvectors can be chosen in such way that they are an orhonormal basis of H.

12.27. State the theorem about the **Fredholm alternative.**

Theorem. **Fredholm alternative**
Let $(H, < \cdot, \cdot >)$ be a separable Hilbert space and let $T \in K(H)$ be symmetric.
Then, for every $\mu \in \mathbb{R}$ such that $\mu \neq 0$ it is **either**:

(a) $\forall y \in H \;\; \exists!\, x \in H$ such that: $\mu\, x - T(x) = y$
(b) $\lambda = \mu$ is an eigenvalue of T

Considering the problem of finding $x \in H$: $\mu\, x - T(x) = y$, the theorem says that $\exists!\, x$ solution $\forall y \in H$ only if λ is not an eigenvalue of T.