POLITECNICO DI MILANO

CRC

Introduction to Monte Carlo Simulation

The experimental view

Enrico Zio

CONTENTS

- Sampling Random Numbers
- Simulation of system transport
- Simulation for reliability/availability analysis of a component
- Examples

The History of Monte Carlo Simulation

Buffon	1707-88
⬇	
Kelvin	1824-1907
⬇	
Gosset (Student)	1908
⬇	
Fermi, von Neumann, Ulam	1930-40's
⬇	
Neutron transport	1950's
⬇	
System Transport (RAMS)	1990's

SAMPLING RANDOM NUMBERS

Example: Exponential Distribution

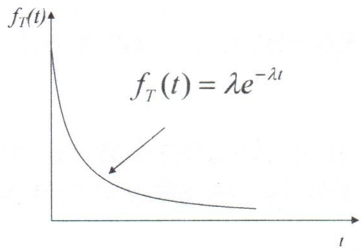
Probability density function:

$$f_T(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

Expected value and variance:

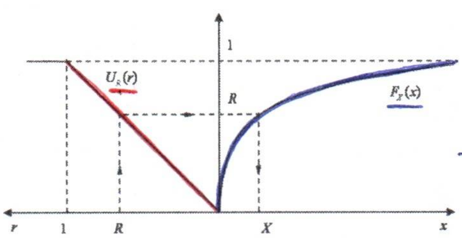
$$E[T] = \frac{1}{\lambda}$$

$$Var[T] = \frac{1}{\lambda^2}$$



How can we sample random numbers that follow this distribution?

Sampling Random Numbers from $F_X(x)$



INVERSE TRANSFORM TECHNIQUE: to sample random values from a cumulative distr.

Sample R from $U_R(r)$ and find X:

$$X = F_X^{-1}(R)$$

→ we start from the assumpt. that we know how to sample from $\mathcal{U}([0,1])$

Example: Exponential distribution

$$F_X(x) = 1 - e^{-\lambda x}$$

$$R = F_X(x) = 1 - e^{-\lambda x}$$

$$\Downarrow$$

$$X = F_X^{-1}(R) = -\frac{1}{\lambda}\ln(1-R)$$

when we sample a random value from $\mathcal{U}([0,1])$ we take it as a value of the cumulative distribution function F_X (which we know $\in [0,1]$)
Then we go on the curve, we invert it and we find the value of X corresponding.

X is a random variable since R is a random variable

Sampling from discrete distributions

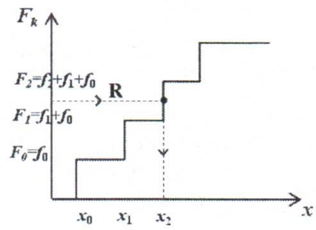
(always through inverse transform)

The random variable X has discrete domain

$$\Omega = \{x_0, x_1, \ldots, x_k, \ldots\}$$

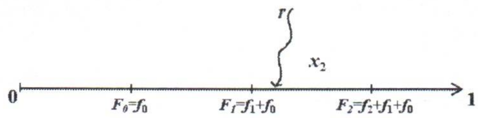
$$F_k = P\{X \le x_k\} = \sum_{i=0}^{k} P[X = x_i]$$

sample an $R \sim U[0,1)$



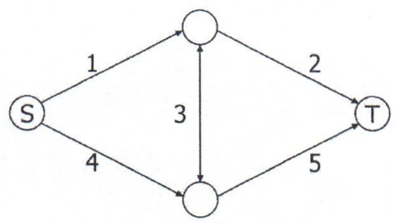
It's like we're partitioning the [0,1]-y segment with values $F_0, F_1, F_2, \ldots, F_k, \ldots$

Graphically:



! *

Failure probability estimation: example



Arc number i	Failure probability P_i
1	0.050
2	0.025
3	0.050
4	0.020
5	0.075

minimal cutsets
{1, 4}
{2, 5}
{1, 3, 5}
{4, 3, 2}

- **1- Calculate the analytic solution for the failure probability of the network, i.e., the probability of no connection between nodes S and T** (we can solve this through fault tree, finding the minimal cutsets and computing their probs.)

- **2- Repeat the calculation with Monte Carlo simulation**
We suppose to have ("buy") 10k of these networks. With the written probs. we have the arcs functioning. Depending on the configuration of success/failure we may have a functioning/failed network. The failure prob. estimation is #not functioning/10.000

(solutions at the end)

SIMULATION OF SYSTEM TRANSPORT

Monte Carlo simulation for system reliability

PLANT = system of N_c suitably connected components.

COMPONENT = a subsystem of the plant (pump, valve,...) which may stay in different exclusive (multi)states (nominal, failed, stand-by,...). Stochastic transitions from state-to-state occur at stochastic times.

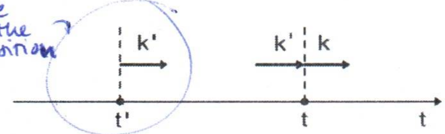
STATE of the PLANT at t = the set of the states in which the N_c components stay at t. The states of the plant are labeled by a scalar which enumerates all the possible combinations of all the component states.

PLANT TRANSITION = when any one of the plant components performs a state transition we say that the plant has performed a transition. The time at which the plant performs the n-th transition is called t_n and the plant state thereby entered is called k_n.

PLANT LIFE = stochastic process.

Stochastic Transitions: Governing Probabilities

current state resulted from the very last transition ?



To characterize the process we need to provide the probabilities of the two variables that are involved: the time at which we'll have the next transition and the state that will be reached upon the transition occurrence.

- $T(t\,/t';\,k')dt$ = conditional probability of a transition at $t \in dt$, given that the preceding transition occurred at t' and that the state thereby entered was k'.
- $C(k\;/\;k';\,t)$ = conditional probability that the plant enters state k, given that a transition occurred at time t when the system was in state k'. Both these probabilities form the "trasport kernel":

modelling in **time** (when is the next trans. ?)

modelling in **state-change** (which is the next state ?)

$$\boxed{K(t;\,k\;/\;t';\,k')dt = T(t\;/\;t';\,k')dt\; C(k\;/\;k';\,t)}$$

– probability that the next transition of the system will occur at time t and in this transition the system will end in the state k given that the previous transition has occurred at time t' and the system left in the state k'!
This gives the full stochastic description of the process of transition.

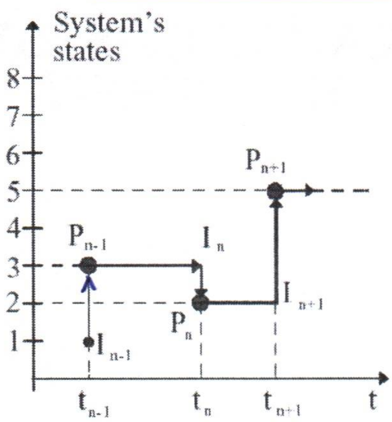
- $\psi(t;\,k)$ = ingoing transition density or probability density function (pdf) of a system transition at t, resulting in the entrance in state k

The overall probability (unconditionally) is
$\Psi(t;k)$ = probability that the system has a transition at time t in which it enters at state k

Plant life: random walk

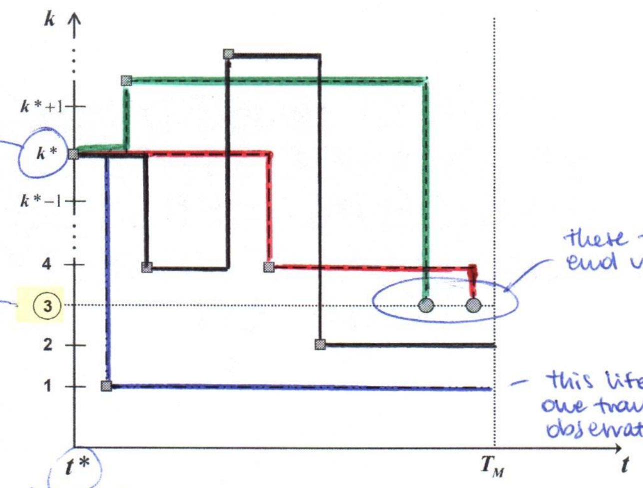
Random walk = realization of the system life generated by the underlying state-transition stochastic process.

We can picture this process in the PHASE-SPACE of the system (where the 2 variables are time t and the system state (represented as ordered numbers))



Phase Space

Here we represent 4 possible life histories of the system:



initial condition for all the systems

FAILURE STATE

these two life histories end up in failure

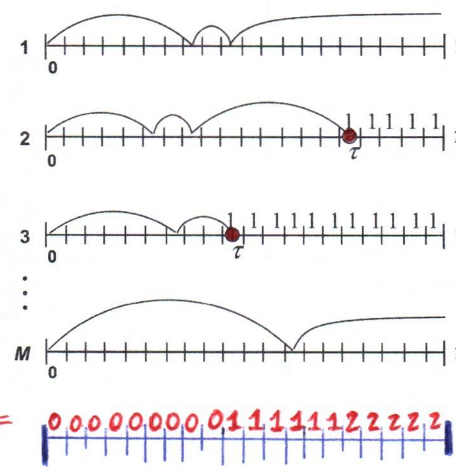
– this life history has just one transition during the observational period

initial time

We may conclude that the system has a 50% probability of failure. The more life histories we have, the more accurate the probability is.

Example: System Reliability Estimation

instead of the phase space we use: (TIMELINE)



$C^R(t) = C^R(t) \quad t \in [0, T_M]$

$C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M]$

$C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M]$

$C^R(t) = C^R(t) \quad t \in [0, T_M]$

repeated simulations

$C^R(t)$ = 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 2 2 2 2 2

given only {1, 2, 3, M}

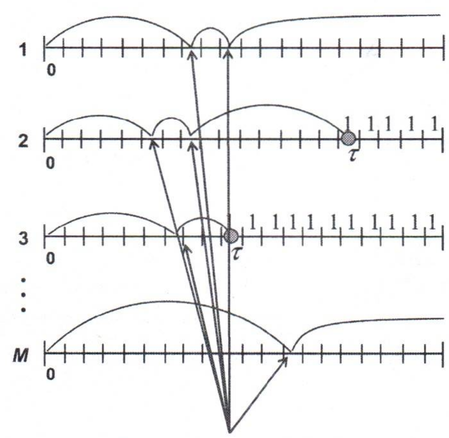
$$\boxed{\hat{F}_T(t) = \frac{C^R(t)}{M}}$$

UNRELIABILITY OF THE SYSTEM (= cumulative distr. of the failure time of the system)

$C^R(t)$ counts how many systems have failed before (≤) t.

The 1st example leaves $C^R(t)$ as it is ($C^R(t) = C^R(t)$) since the system doesn't fail in the period $[0, T_M]$. The 2nd and 3rd example add both +1 when the system fails (τ).

Example: System Reliability Estimation



Events at components level, which do not entail system failure

$C^R(t) = C^R(t) \quad t \in [0, T_M]$

$C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M]$

$C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M]$

$C^R(t) = C^R(t) \quad t \in [0, T_M]$

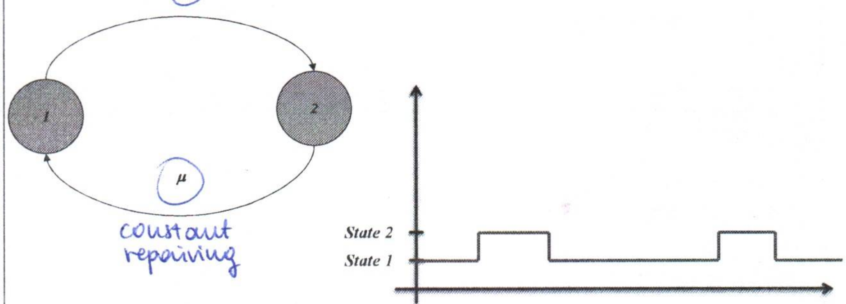
$$\hat{F}_T(t) = \frac{C^R(t)}{M}$$

SIMULATION OF COMPONENT STOCHASTIC STATE TRANSITION PROCESS FOR AVAILABILITY / RELIABILITY ESTIMATION

One component with exponential distribution of the failure time

We consider only 2 states:
(1): the component is on (functioning)
(2): the component is failed

constant failure rate



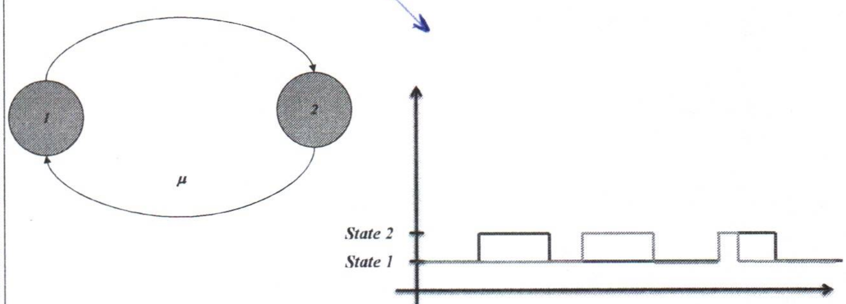
constant repairing

State X=1 → ON

State X=2 → OFF

We buy a lot of these components (all starting from the functioning state) and we observe them.

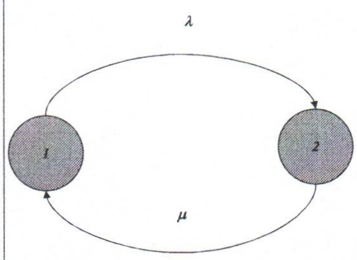
One component with exponential distribution of the failure time

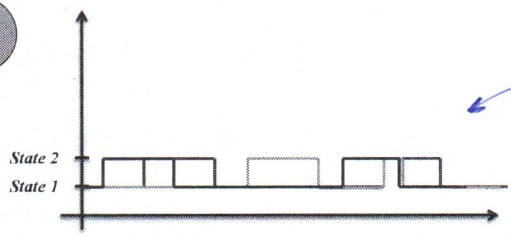


State X=1 → ON

State X=2 → OFF

One component with exponential distribution of the failure time



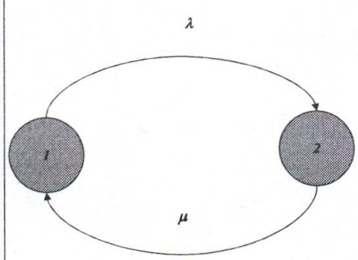


These are the observed life histories of all the components

State X=1 → ON

State X=2 → OFF

One component with exponential distribution of the failure time

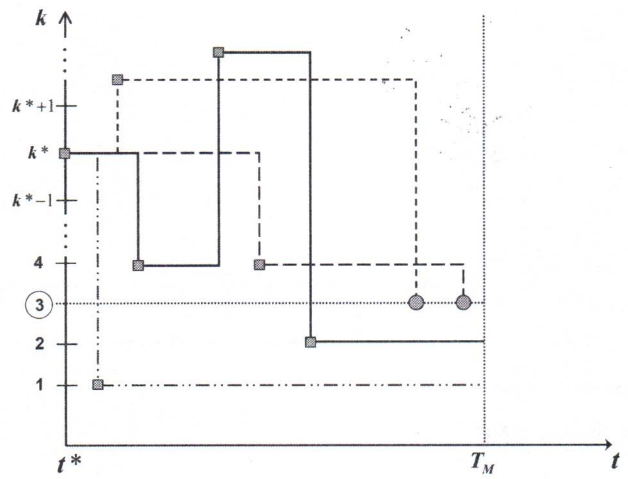


	values
λ	$3\cdot 10^{-3}\ h^{-1}$
μ	$25\cdot 10^{-3}\ h^{-1}$

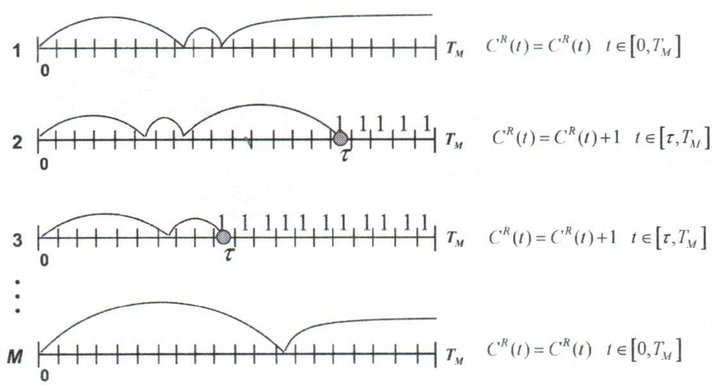
Given these values, what is the unavailability of the component? ($C^R(t)$, $\hat{F}_T(t)$?)

SIMULATION OF SYSTEM STOCHASTIC STATE TRANSITION PROCESS FOR AVAILABILITY / RELIABILITY ESTIMATION

Phase Space



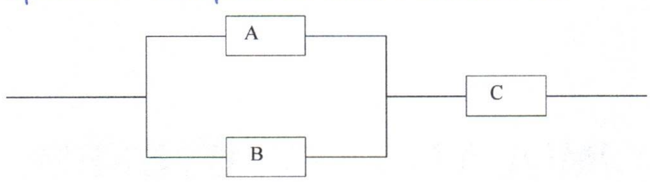
Example: System Reliability Estimation



$$\hat{F}_T(t) = \frac{C^R(t)}{M}$$

Indirect Monte Carlo: Example (1)

3 components: 2 in parallel and then one out



Components' times of transition between states are exponentially distributed

($\lambda^i_{j_i \to m_i}$ = rate of transition of component i going from its state j_i to the state m_i)

For components A/B:

States:
1: perfectly functioning
2: partially functioning
3: completely failed

Initial \ Arrival	1	2	3
1	-	$\lambda^{A(B)}_{1\to 2}$	$\lambda^{A(B)}_{1\to 3}$
2	$\lambda^{A(B)}_{2\to 1}$	-	$\lambda^{A(B)}_{2\to 3}$
3	$\lambda^{A(B)}_{3\to 1}$	$\lambda^{A(B)}_{3\to 2}$	-

Indirect Monte Carlo: Example (2)

For component C:

States:
1 : perfectly functioning
⋮
4 : completely failed

		Arrival			
		1	2	3	4
Initial	1	-	$\lambda^C_{1\to 2}$	$\lambda^C_{1\to 3}$	$\lambda^C_{1\to 4}$
	2	$\lambda^C_{2\to 1}$	-	$\lambda^C_{2\to 3}$	$\lambda^C_{2\to 4}$
	3	$\lambda^C_{3\to 1}$	$\lambda^C_{3\to 2}$	-	$\lambda^C_{3\to 4}$
	4	$\lambda^C_{4\to 1}$	$\lambda^C_{4\to 2}$	$\lambda^C_{4\to 3}$	-

- The components are initially (t=0) in their nominal states (1,1,1)
- One minimal cut set of order 1 (C in state 4:(*,*,4)) and one minimal cut set of order 2 (A and B in 3: (3,3,*)).

We need to simulate the system process!

Analog Monte Carlo Trial

SAMPLING THE TIME OF TRANSITION

The rate of transition of component A(B) out of its nominal state 1 is:

$$\lambda_1^{A(B)} = \lambda_{1\to 2}^{A(B)} + \lambda_{1\to 3}^{A(B)}$$

- The rate of transition of component C out of its nominal state 1 is:

$$\lambda_1^C = \lambda_{1\to 2}^C + \lambda_{1\to 3}^C + \lambda_{1\to 4}^C$$

- The rate of transition of the system out of its current configuration (1, 1, 1) is:

$$\lambda^{(1,1,1)} = \lambda_1^A + \lambda_1^B + \lambda_1^C \implies$$

the system has a constant transition rate of leaving the current configuration
($\implies T(t|t_0,(1,1,1))$ is exponential with $\lambda^{(1,1,1)}$)

- We are now in the position of sampling the first system transition time t_1, by applying the inverse transform method:

$$t_1 = t_0 - \frac{1}{\lambda^{(1,1,1)}}\ln(1-R_t)$$

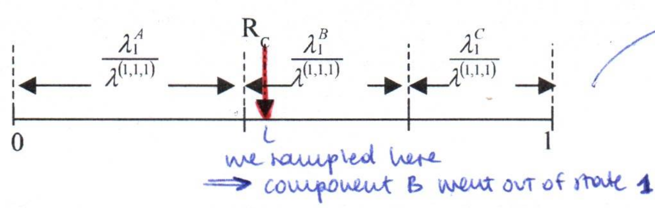
where $R_t \sim U[0,1)$

Sampling the Kind of Transition (1)

- Assuming that $t_1 < T_M$ (otherwise we would proceed to the successive trial), we now need to determine which transition has occurred, i.e. which component has undergone the transition and to which arrival state.
- The probabilities of components A, B, C undergoing a transition out of their initial nominal states 1, given that a transition occurs at time t_1, are:

$$\frac{\lambda_1^A}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^B}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^C}{\lambda^{(1,1,1)}}$$

- Thus, we can apply the inverse transform method to the discrete distribution



we partition [0,1] with the previous probabilities and we obtain an interval for A, one for B and one for C. We sample $\sim \mathcal{U}([0,1])$ and it'll be the component s.t. the sampling fell.

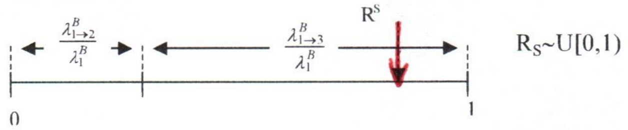
Now we know who, but we still need to know where the component goes:

Sampling the Kind of Transition (2)

- Given that at t_1 component B undergoes a transition, its arrival state can be sampled by applying the inverse transform method to the set of discrete probabilities

$$\left\{ \frac{\lambda^B_{1\to 2}}{\lambda^B_1}, \frac{\lambda^B_{1\to 3}}{\lambda^B_1} \right\}$$

of the mutually exclusive and exhaustive arrival states



- As a result of this first transition, at t_1 the system is operating in configuration (1,3,1).
- The simulation now proceeds to sampling the next transition time t_2 with the updated transition rate

$$\lambda^{(1,3,1)} = \lambda^A_1 + \lambda^B_3 + \lambda^C_1$$

Sampling the Next Transition

- The next transition, then, occurs at

$$t_2 = t_1 - \frac{1}{\lambda^{(1,3,1)}} \ln(1 - R_t)$$

where $R_t \sim U[0,1)$.

- Assuming again that $t_2 < T_M$, the component undergoing the transition and its final state are sampled as before by application of the inverse trasform method to the appropriate discrete probabilities.
- The trial simulation then proceeds through the various transitions from one system configuration to another up to the mission time T_M.

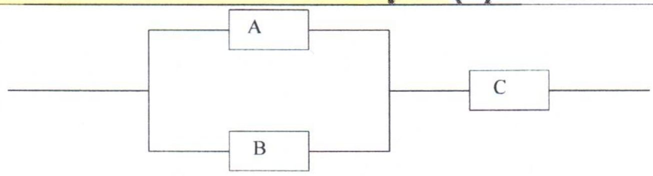
Unreliability and Unavailability Estimation

- When the system enters a failed configuration (*,*,4) or (3,3,*), where the * denotes any state of the component, tallies are appropriately collected for the unreliability and instantaneous unavailability estimates (at discrete times $t_j \in [0, T_M]$);
- After performing a large number of trials M, we can obtain estimates of the system unreliability and instantaneous unavailability by simply dividing by M, the accumulated contents of $C^R(t_j)$ and $C_A(t_j)$, $t_j \in [0, T_M]$

Another way to simulate the process:

based directly on the component description

Direct Monte Carlo: Example (1)



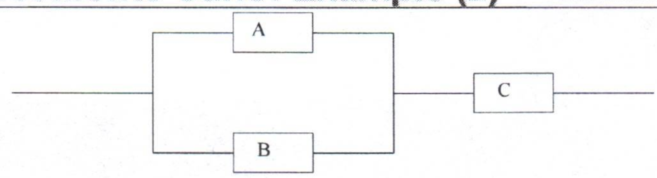
For any arbitrary trial, starting at t=0 with the system in nominal configuration (1,1,1) we would sample all the transition times:

$$t^i_{1\to m_i} = t_0 - \frac{1}{\lambda^i_{1\to m_i}} \ln(1 - R^i_{t,1\to m_i}) \qquad \left.\begin{array}{lll} i = A, B, C & \\ m_i = 2,3 & \text{for } i = A, B \\ m_i = 2,3,4 & \text{for } i = C \end{array}\right\}$$

where $R^i_{t,1\to m_i} \sim U[0,1)$

These transition times would then be ordered in ascending order from t_{min} to $t_{max} \leq T_M$. Let us assume that t_{min} corresponds to the transition of component A to state 3 of failure. The current time is moved to $t_1 = t_{min}$ in correspondence of which the system configuration changes, due to the occurring transition, to (3,1,1) still operational.

Direct Monte Carlo: Example (2)

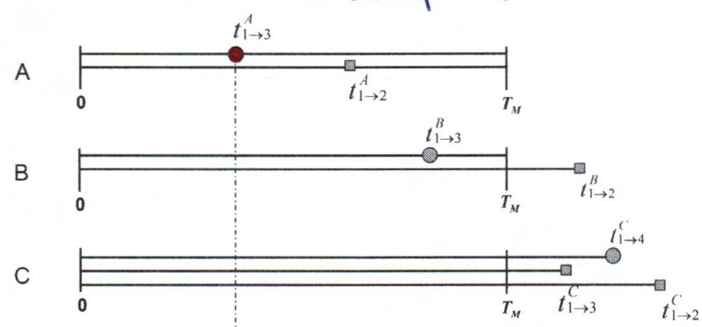


These transition times would then be ordered in ascending order from t_{min} to $t_{max} \leq T_M$.

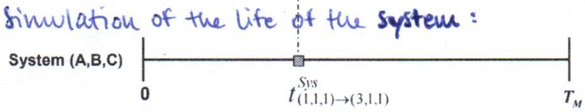
Let us assume that t_{min} corresponds to the transition of component A to state 3 of failure. The current time is moved to $t_1 = t_{min}$ in correspondence of which the system configuration changes, due to the occurring transition, to (3,1,1) still operational.

Example (1)

Simulation of the lives of the 3 components:



Simulation of the life of the system:



Example (2)

The new transition times of component A are then sampled

$$t^{A}_{3\to m_A} = t_1 - \frac{1}{\lambda^{A}_{3\to m_A}} \ln(1 - R^{A}_{t,3\to m_A}) \qquad k = 1,2 \qquad R^{A}_{t,3\to m_A} \sim U[0,1)$$

and placed at the proper position in the timeline of the succession of occurring transitions

- The simulation then proceeds to the successive times in the list, in correspondence of which a system transition occurs.
- After each transition, the timeline is updated with the times of the transitions that the component which has undergone the last transition can do from its new state.
- During the trial, each time the system enters a failed configuration, tallies are collected and in the end, after M trials, the unreliability and unavailability estimates are computed.

If A moves its state then we don't need to re-sample also for B and C (if they're ⊥ from A)

If there are correlation then we need to recompute.
For example: if A fails B is overcharged (since they're in parallel) and so λ_B changes ⇒ we need to resample B also

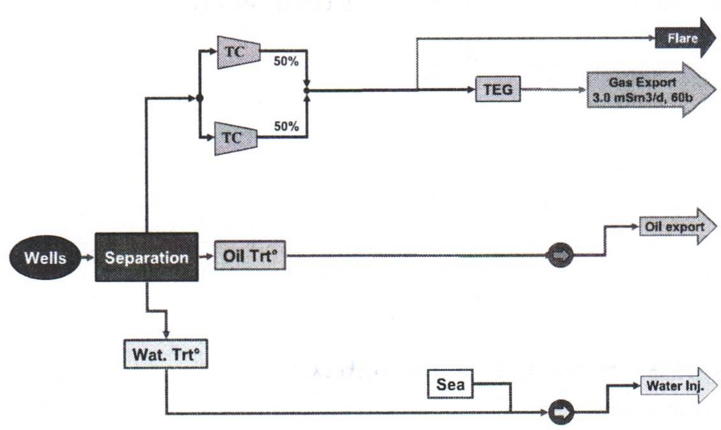
POLITECNICO DI MILANO

PRODUCTION AVAILABILITY EVALUATION OF AN OFFSHORE INSTALLATION

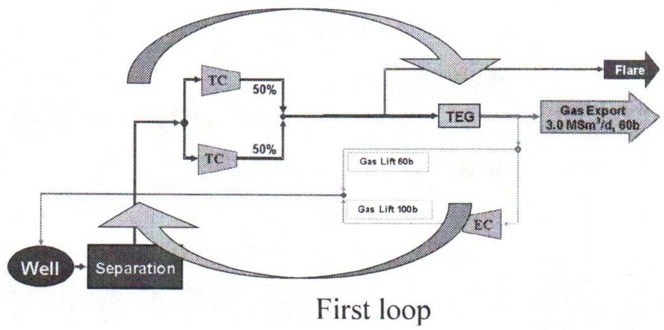
A real example of Indirect Simulation

CENTRALE PARIS Supélec

System description: basic scheme

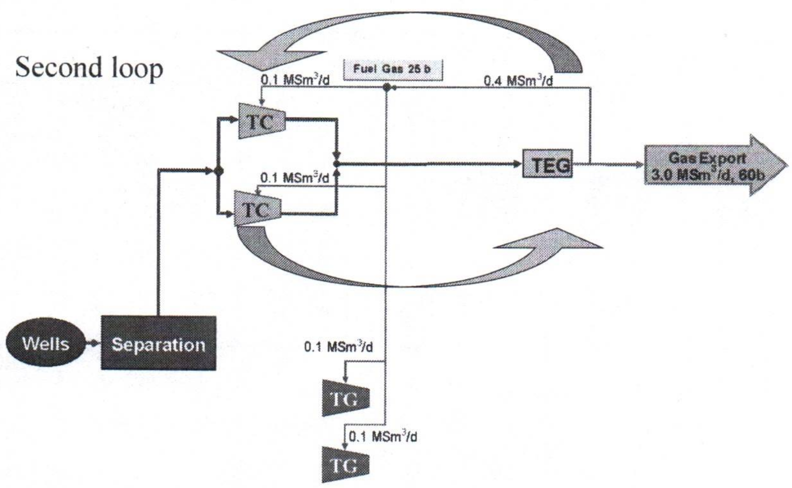


System description: gas-lift

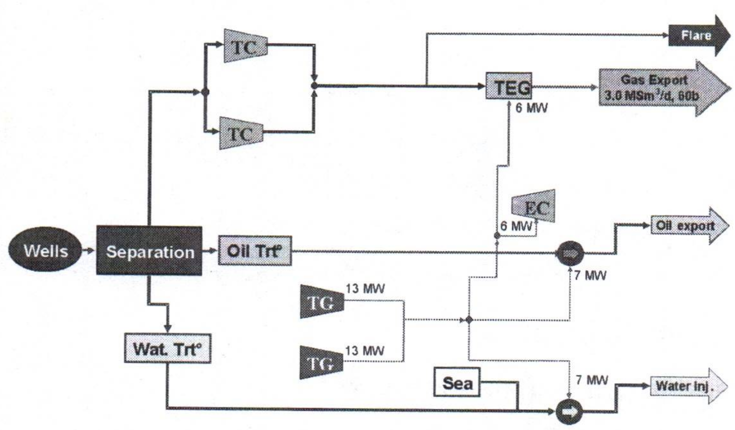


Gas-lift pressure	Production of the Well
100	100%
60	80%
0	60%

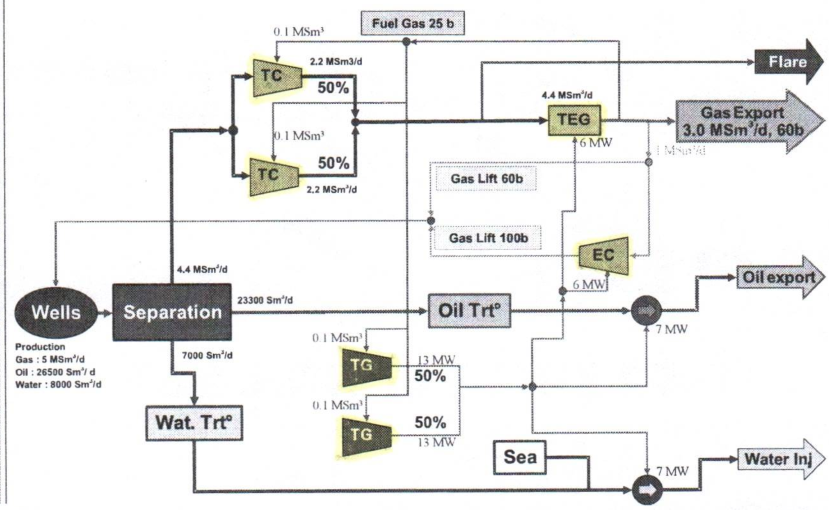
System description:
fuel gas generation and distribution



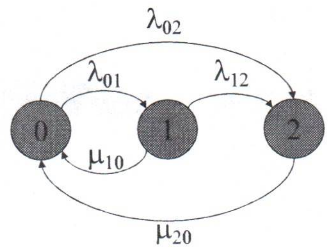
System description:
electricity power production and distribution



The offshore production plant



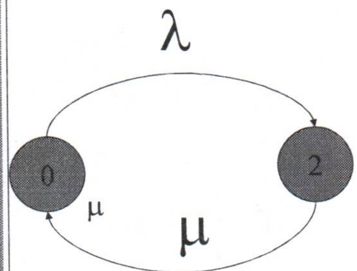
Component failures and repairs: TCs and TGs



	TC	TG
λ_{01}	$0.89 \cdot 10^{-3}\ h^{-1}$	$0.67 \cdot 10^{-3}\ h^{-1}$
λ_{02}	$0.77 \cdot 10^{-3}\ h^{-1}$	$0.74 \cdot 10^{-3}\ h^{-1}$
λ_{12}	$1.86 \cdot 10^{-3}\ h^{-1}$	$2.12 \cdot 10^{-3}\ h^{-1}$
μ_{10}	$0.033\ h^{-1}$	$0.032\ h^{-1}$
μ_{20}	$0.048\ h^{-1}$	$0.038\ h^{-1}$

State 0 = as good as new
State 1 = degraded (no function lost, greater failure rate value)
State 2 = critical (function is lost)

Component failures and repairs: EC and TEG



	EC	TEG
λ	$0.17 \cdot 10^{-3}\ h^{-1}$	$5.7 \cdot 10^{-5}\ h^{-1}$
μ	$0.032\ h^{-1}$	$0.333\ h^{-1}$

State 0 = as good as new

State 2 = critical (function is lost)

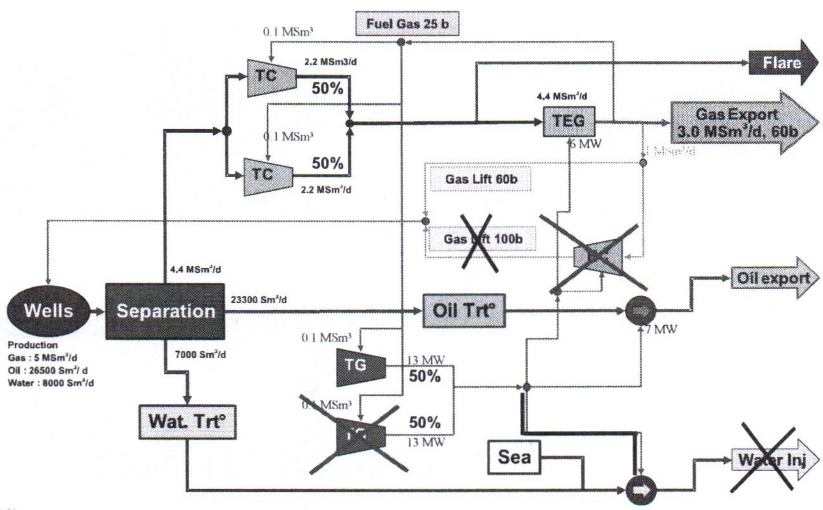
Production priority

When a failure occurs, the system is reconfigured to minimise (in order):

- the impact on the export oil production
- the impact on export gas production

➢ **The impact on water injection does not matter**

Production priority: example



Maintenance policy: reparation

Only 1 repair team

Priority levels of failures:

1. Failures leading to total loss of export oil (both TG's or both TC's or TEG)
2. Failures leading to partial loss of export oil (single TG or EC)
3. Failures leading to no loss of export oil (single TC failure)

Maintenance policy: preventive maintenance

- Only 1 preventive maintenance team
- The preventive maintenance takes place only if the system is in perfect state of operation

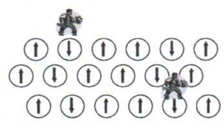
	Type of maintenance	Frequency [hours]	Duration [hours]
Turbo-Generator and Turbo-Compressors	Type 1	2160 (90 days)	4
	Type 2	8760 (1 year)	120 (5 days)
	Type 3	43800 (5 years)	672 (4 weeks)
Electro Compressor	Type 4	2666	113

MARKOV APPROACH

Number of components = 6
Number of states for component = 2 or 3 ⇒ $2^2 \cdot 3^4 =$ **324** plant states

+

1 repair team ⇒ 129 new plant states



+

1 maintenance team ⇒ Non homogeneous Markov chain

Markov approach too complex

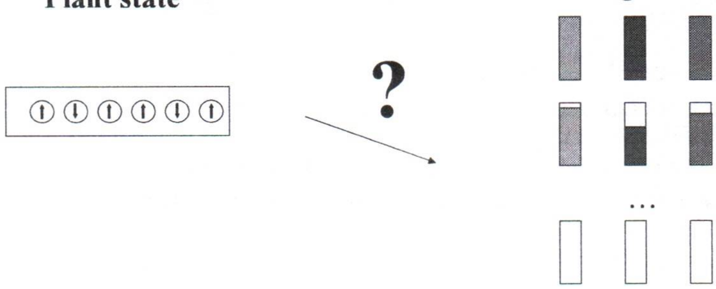
MONTE CARLO APPROACH

MONTE CARLO APPROACH

Production levels

Plant state

oil gas water



Associate a production level to each of the 453 plant states ⇒ too long, error prone

A systematic procedure

7 different production levels

⇩

6 different system faults

⇩

6 fault trees

⇩

6 families of mcs

Production Level	Gas [kSm^3/d]	Oil [k m^3/d]	Water [m^3/d]	mcs	MCS
0=(100%)	3000	23.3	7000		
1	900	23.3	7000	X5, X6	X5,X6
2	2700	21.2	0	X3, X4	X2X3,X2X4
3	1000	21.2	0	X3X5, X3X6, X4X5, X4X6	X2X3X5, X2X3X6, X2X4X5, X2X4X6
4	2600	21.2	6400	X2	X2
5	900	21.2	6400	X2X5, X2X6	X2X5, X2X6
6	0	0	0	X1, X3X4, X5X6	X1X2X3X4X5X6

Numerical results

Case A: corrective maintenance and no preventive maintenance (T_{miss}= 1· 10^3 hours, trials=10^6)

CPU time ≈ 15 min

Case B: perfect system (no failures) and preventive maintenance (T_{miss}= 10^4 hours, trials=10^5)

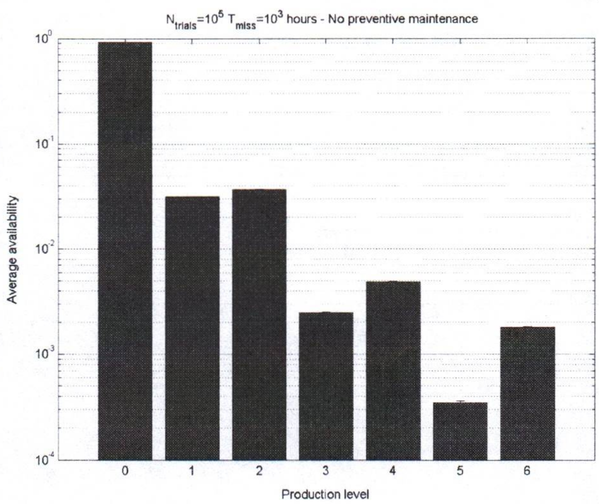
CPU time ≈ 12 min

Case C: corrective and preventive maintenance (T_{miss}=5·10^5 hours, trials=10^5)

CPU time ≈ 20 h

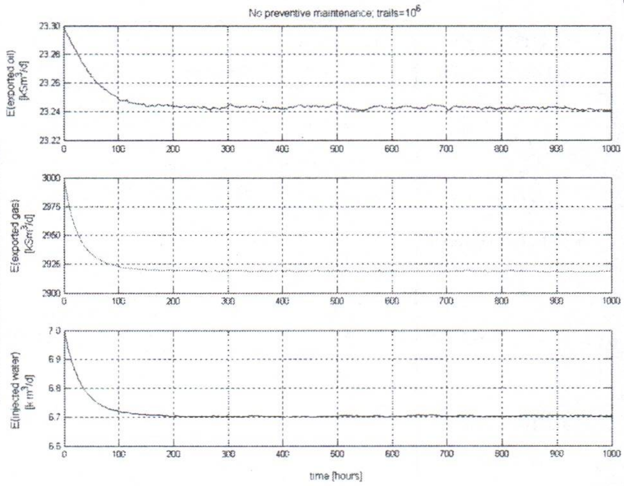
Case A: no preventive maintenances

Production level	Average availability
0	9.23E-1
1	3.13E-2
2	3.67E-2
3	2.47E-3
4	4.88E-3
5	3.50E-4
6	1.79E-3



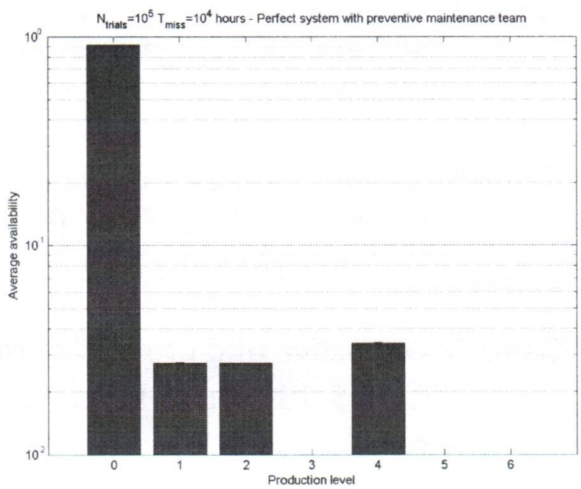
Case A: no preventive maintenances

	Asymptotic values
Oil [k m³/d]	23.24
Gas [k Sm³/d]	2918
Water [k m³/d]	6.703



Case B: perfect system and preventive maintenances

Production level	Average availability
0	9.12E-1
1	2.73E-2
2	2.72E-2
3	0.00
4	3.40E-2
5	0.00
6	0.00



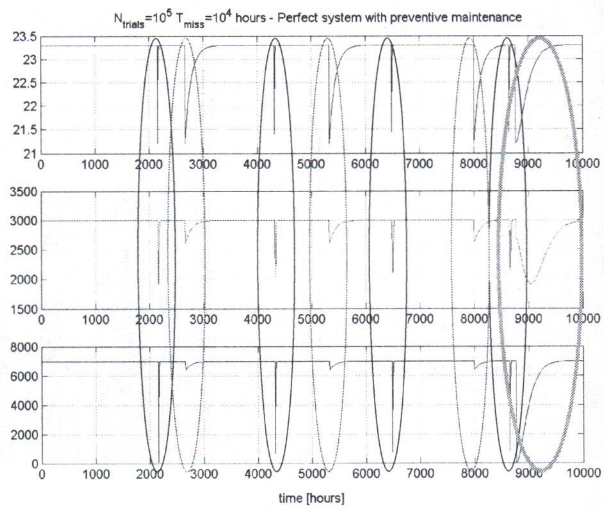
Case B: perfect system and preventive maintenances

P.Maintenance Type 1 (TC,TG)

P.Maintenance Type 2 (EC)

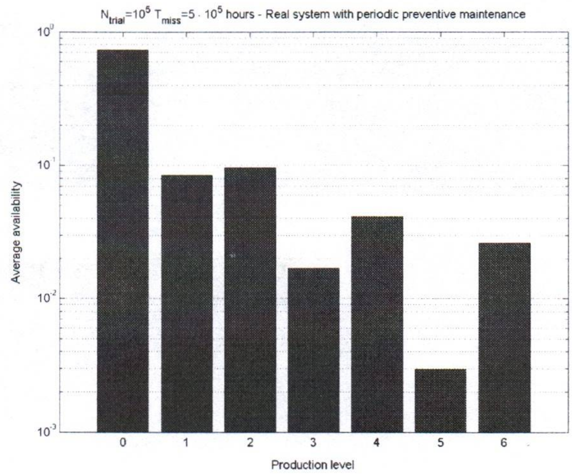
P.Maintenance Type 3 (TC,TG)

	Mean	Std
Oil [k m³/d]	23.230	0.263
Gas [k Sm³/d]	2929	194.0
Water [k m³/d]	6.811	0.883



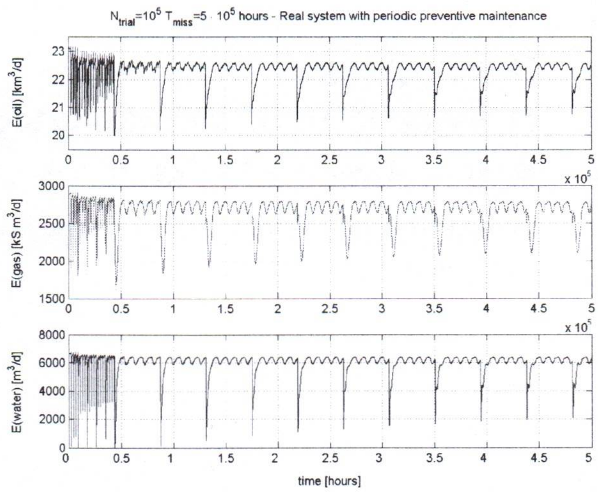
Case C: real system with preventive maintenances

Production level	Average availability
0	8.13E-1
1	5.68E-2
2	6.58E-2
3	1.19E-2
4	3.55E-2
5	2.34E-3
6	1.50E-2



Case C: real system with preventive maintenances

	Mean	Std
Oil [k m³/d]	22.60	0.42
Gas [k Sm³/d]	2687	194.3
Water [k m³/d]	6.04	0.76

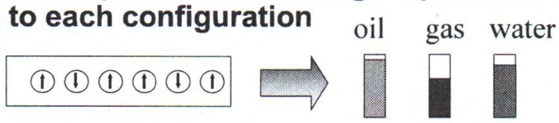


Conclusions

- **Complex multi-state system with maintenance and operational loops**

MC simulation

- **Systematic procedure to assign a production level to each configuration**



- **Investigation of effects maintenance on production**

POLITECNICO DI MILANO

CRC

Introduction to Monte Carlo Simulation

The theroetical view

Enrico Zio

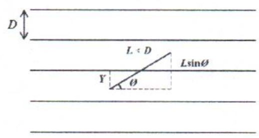
CONTENTS

- Sampling
- Evaluation of definite integrals
- Simulation of system transport
- Simulation for reliability/availability analysis

SAMPLING

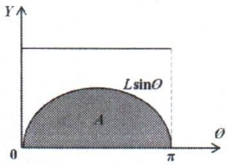
Buffon's needle

Buffon considered a set of parallel straight lines a distance D apart onto a plane and computed the probability P that a needle of length $L < D$ randomly positioned on the plane would intersect one of these lines.



$$P = P\{Y \le L\sin\varnothing\}$$

We can evaluate P analytically:



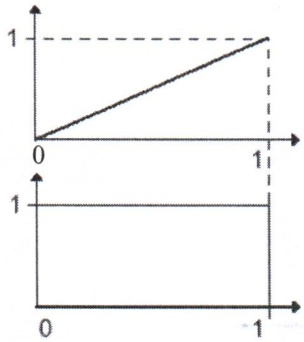
$$f_Y(y) = \frac{1}{D} \qquad y \in [0, D]$$

$$f_\varnothing(\varphi) = \frac{1}{\pi} \qquad \varphi \in [0, \pi]$$

$$P = \iint_A \frac{dy}{D} \cdot \frac{d\varphi}{\pi} = \frac{L/D}{\pi/2}$$

simulating we get an approximation of P and so the only ? is π ⇒ we get an approx. of π

Sampling (pseudo) Random Numbers Uniform Distribution



$$\text{cdf}: \quad U_R(r) = P\{R \le r\} = r$$

$$\text{pdf}: \quad u_R(r) = \frac{dU_R(r)}{dr} = 1$$

Sampling (pseudo) Random Numbers Uniform Distribution

$$R \sim U[0,1)$$

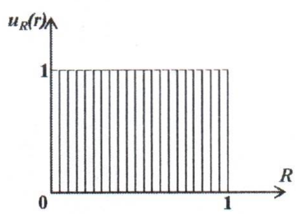
(an example of:) a method of sampling values from U[0,1])

$$\boxed{x_i = (ax_{i-1} + c) \bmod m}$$

where $a, c \in [0, m-1]$

$m \gg 1$

$$r_i = \frac{x_i}{m}$$



Example: $a = 5, c = 1, m = 16$

$$x_0 = 2 \Rightarrow r_0 = \frac{2}{16}$$

$$x_1 = (5 \cdot 2 + 1) \bmod 16 = 11 \Rightarrow r_1 = \frac{11}{16}$$

...

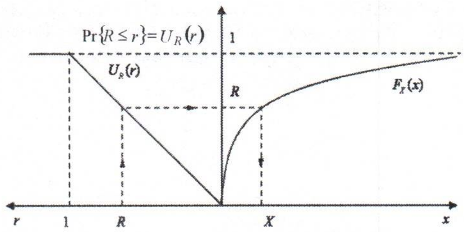
$$x_{15} = 13 \Rightarrow r_{15} = \frac{13}{16}$$

$$x_{16} = 2$$

from here we got the same values (M is the period because after M iterations we've back at the start)

we try to keep M as large as possible

Sampling (pseudo) Random Numbers Generic Distribution



Sample R from $U_R(r)$ and find X:

$$X = F_X^{-1}(R)$$

Question: which distribution does X obey?

$$P\{X \le x\} = P\{F_X^{-1}(R) \le x\}$$

Application of the operator F_x to the argument of P above yields

$$P\{X \le x\} = P\{R \le F_X(x)\} = F_X(x)$$

$\mathbb{P}(R<r)=r$ since $R \sim \mathcal{U}([0,1])$

Summary:

From an $R \sim U_R(r)$ we obtain an $X \sim F_X(x)$

Example: Exponential Distribution

- Markovian system with two states (good, failed)
- hazard rate, λ = constant

$$F_T(t) = P\{T \le t\} = 1 - e^{-\lambda t}$$

- cdf $\quad f_T(t) \cdot dt = P\{t \le T < t + dt\} = \lambda e^{-\lambda t} \cdot dt$

•pdf

$$R \equiv F_R(r) = F_T(t) = 1 - e^{-\lambda t}$$

•Sampling a failure time T

$$\Downarrow$$

$$\boxed{T = F_T^{-1}(R) = -\frac{1}{\lambda}\ln(1-R)}$$

Example: Weibull Distribution

•hazard rate, λ = constant

• cdf
$$F_T(t) = P\{T \le t\} = 1 - e^{-\beta t^\alpha}$$

pdf
$$f_T(t) \cdot dt = P\{t \le T < t + dt\} = \alpha\beta t^{\alpha-1} e^{-\beta t^\alpha} \cdot dt$$

•Sampling a failure time T

$$R \equiv F_R(r) = F_T(t) = 1 - e^{-\beta t^\alpha}$$

$$\Downarrow$$

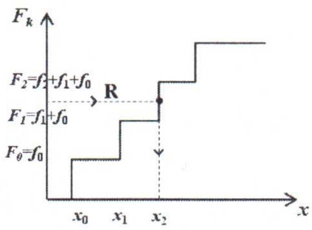
$$\boxed{T = F_T^{-1}(R) = \left(-\frac{1}{\beta}\ln(1-R)\right)^{\frac{1}{\alpha}}}$$

Sampling by the Inverse Transform Method: Discrete Distributions

$$\Omega = \{x_0, x_1, \dots, x_k, \dots\}$$

$$F_k = P\{X \le x_k\} = \sum_{i=0}^{k} P[X = x_i]$$

sample an $R \sim U[0,1)$

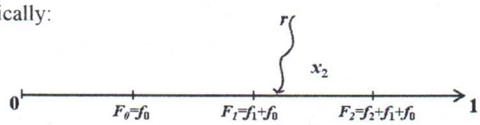


$$P[F_{k-1} < R \le F_k] = F_R(F_k) - F_R(F_{k-1})$$

$$R \sim U[0,1) \text{ and } F_R(r) = r$$

$$\Rightarrow P[F_{k-1} < R \le F_k] = F_k - F_{k-1} = f_k = P[X = x_k]$$

Graphically:



Sampling by the Rejection Method: von Neumann Algorithm

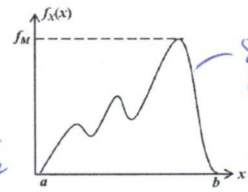
Sometimes we may have a distribution s.t. it's difficult to invert F_X.
To sample from complicated density functions:

- Given a pdf $f_X(x)$ limited in (a,b), let

$$h(x) = \frac{f_X(x)}{f_M}$$

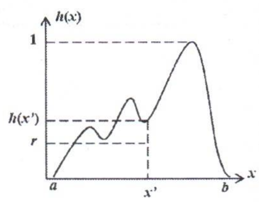
1. we normalize the distribution dividing by the maximum value

so that $0 \le h(x) \le 1, \forall x \in (a,b)$



Suppose that we want to sample from this distribution

- The operative procedure to sample a realization of X from $f_X(x)$:



2. • sample $X' \sim U(a,b)$, the tentative value for X, and calculate $h(X')$
3. • sample $R \sim U[0,1)$. If $R <= h(X')$ the value X' is accepted; else start again.

Sampling by the Rejection Method: von Neumann Algorithm

- More generally:

$$X \sim f_X(x) = g_{X'}(x) \cdot H(x)$$

$$B_H : \max_x H(x)$$

$$h(x) = \frac{H(x)}{B_H}, \quad 0 \le h(x) \le 1$$

- The operative procedure:
  - sample $X' \sim g_{X'}(x)$, and calculate $h(X')$
  - sample $R \sim U[0,1)$. If $R <= h(X')$ the value X' is accepted; else start again.

we wanted to generalize this (instead of imposing $X' \sim U([0,1])$
(Notice that $g_X(x)$ is generic but we have to know how to sample from it)

What is the distribution of the random variable that is build with this procedure?

- We show that the accepted value is actually a realization of X sampled from $f_X(x)$

proof.

1. $$P[X' \le x \mid \text{accepted}] = \frac{P[X' \le x \cap \text{accepted}]}{P[\text{accepted}]} = \frac{P[X' \le x \cap R \le h(X')]}{P[\text{accepted}]}$$

Sampling by the Rejection Method: von Neumann Algorithm

2. $P\left[z \le X' \le z+dz \cap \text{accepted}\right] = P\left[z \le X' \le z+dz\right] P\left[R \le h(z)\right] =$
$$= g_{X'}(z)dz \cdot h(z)$$
because $R \sim \mathcal{U}([0,1])$, $\mathbb{P}(R \le r) = r$

3. $P\left[X' \le x \cap R \le h(X')\right] = \int_{-\infty}^{x} g_{X'}(z)dz \cdot h(z)$

4. $P\left[\text{accepted}\right] = \int_{-\infty}^{\infty} g_{X'}(z)dz \cdot h(z) =$
$$= \frac{1}{B_H}\int_{-\infty}^{\infty} g_{X'}(z)dz \cdot H(z) = \frac{1}{B_H}\int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{B_H}$$
$f_X(x)$ is a density

Sampling by the Rejection Method: von Neumann Algorithm

$$P\left[X' \le x \mid \text{accepted}\right] = \frac{P\left[X' \le x \cap R \le h(x')\right]}{P[\text{accepted}]} = \frac{\int_{-\infty}^{x} g_{X'}(z)dz \cdot h(z)}{\frac{1}{B_H}}$$
$$= \int_{-\infty}^{x} g_{X'}(z)dz \cdot H(z) = \int_{-\infty}^{x} f_X(z)dz = F_X(x)$$

- The efficiency of the method is given by the probability of accepted:

$$\varepsilon = P[\text{accepted}] = \int_{-\infty}^{\infty} g_{X'}(z)h(z)dz = \frac{1}{B_H}$$

Sampling by the Rejection Method: von Neumann Algorithm

Example

- Sample from the pdf:

$$f_X(x) = \frac{2}{\pi}\cdot\frac{1}{(1+x)\sqrt{x}} \qquad 0 \le x \le 1$$

Sampling by the Rejection Method: von Neumann Algorithm

Example

- Let the pdf:

$$f_X(x) = \frac{2}{\pi} \cdot \frac{1}{(1+x)\sqrt{x}} \qquad 0 \le x \le 1$$

- consider $X' = R^2,\ R \sim U[0,1)$
- the cd of the r.v. X' is:

$$G_{X'}(x) = P[X' \le x] = P\left[R^2 \le x\right] = P\left[R \le \sqrt{x}\right] = \sqrt{x}$$

- the corresponding pdf:

$$g_{X'}(x) = \frac{dG_{X'}(x)}{dx} = \frac{1}{2\sqrt{x}}$$

- $\Rightarrow f_X(x) = g_{X'}(x) \cdot H(x) = \frac{1}{2\sqrt{x}} \cdot \left(\frac{4}{\pi} \cdot \frac{1}{(1+x)}\right)$

$$\Rightarrow B_H = \frac{4}{\pi};\ h(x) = \frac{H(x)}{B_H} = \frac{1}{1+x} \qquad 0 \le x \le 1$$

Sampling by the Rejection Method: von Neumann Algorithm

Example

- The operative procedure:
  - sample $R_1 \sim U[0,1) \Rightarrow X' = R_1^2$ and $h(X') = \frac{1}{1+R_1^2}$
  - sample $R_2 \sim U[0,1)$. If $R_2 \le h(X')$ accept $X = X'$; else start again
- The efficiency of the method is:

$$\varepsilon = \frac{1}{B_H} = \frac{\pi}{4} = 78.5\%$$

CONTENTS

- **Sampling**
- **Evaluation of definite integrals**
- **Simulation of system transport**
- **Simulation for reliability/availability analysis**

EVALUATION OF DEFINITE INTEGRALS

MC Evaluation of Definite Integrals (1D)

Analog Case

We want to evaluate G:

$$G = \int_a^b g(x) f(x) dx$$

if $X \sim f(x)$ we can treat it as $\mathbb{E}[g(X)] = G$

$$f(x) \equiv \text{pdf} \quad \rightarrow \quad f(x) \geq 0 \quad ; \quad \int f(x) dx = 1$$

MC analog dart game: sample x from $f(x)$

- the probability that a shot hits $x \in dx$ is $f(x)dx$
- the award is $g(x)$

Consider N trials with result $\{x_1, x_2, \ldots, x_n\}$: the average award is

$$G_N = \frac{1}{N} \sum_{i=1}^{N} g(x_i)$$

MC Evaluation of Definite Integrals (1D)

Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2} x\right) dx = \left[\frac{2}{\pi} \sin\left(\frac{\pi}{2} x\right)\right]_0^1 = \frac{2}{\pi}$$

We can see $g(x) = \cos\left(\frac{\pi}{2} x\right)$, $f(x) = 1 \, \mathbb{1}_{[0,1]}(x)$

MC Evaluation of Definite Integrals (1D)
Example

Consider the Weibull Distribution:

$$F_T(t) = 1 - e^{-\beta t^{\alpha}}, \qquad f_T(t) = \alpha\beta t^{\alpha-1} e^{-\beta t^{\alpha}}$$

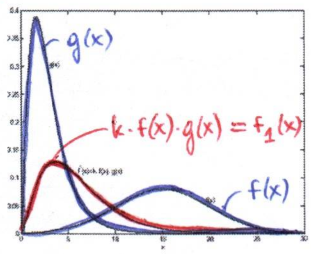
With $\alpha = 1.5, \beta = 1$

1. Sample N = 1000 values from $f_T(t)$
2. Verify that the 1000 sample are distributed according to $f_T(t)$
3. Provide an estimate G_N of $\int_0^{+\infty} t \cdot f_T(t)dt$
4. Estimate the variance of G_N
5. Draw your conclusion considering that:

$$\int_0^{+\infty} t \cdot f_T(t)dt = \Gamma(5/3) = 0.90275$$

MC Evaluation of Definite Integrals (1D)
Biased Case — VARIANCE-REDUCTION TECHNIQUE

How can we apply it to reliability? for example f(x) can be the distribution of the fail of a component and g(x) can be the damage (distr.)
→ $E[g(x)]$ average damage



The expression for G may be written

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x) \right] f_1(x)dx \equiv \int_D g_1(x) f_1(x) dx$$

We "change" the density from which we sample because the awards ($g(x)$) are most high where $f(x)$ is very low
⟹ we change $f(x)$ to $f_1(x)$ (and so, also $g(x)$ to $g_1(x)$)

MC biased dart game: sample x from $f_1(x)$

- the probability that a shot hits $x \in dx$ is $f_1(x)dx$
- the award is

$$g_1(x) = \frac{f(x)}{f_1(x)} g(x) \quad \Rightarrow \quad G_{1N} = \frac{1}{N} \sum_{i=1}^{N} g_1(x_i)$$

changing $f_1(x)$ (and so $g_1(x)$) we also change the precision in the result (approximation). The optimal $f_1(x)$ (so that the variance of the estimator is minimum) depends on the integral of $g(x) \cdot f(x)$ ⟹ we still haven't solved the problem! That's why we proceede with parametric functions which have similar shapes to $g(x) \cdot f(x)$ and then we try to minimize the parameters (so that the overall variance is minimized).
(CHECK THE BOOK)

MC Evaluation of Definite Integrals (1D)
Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx$$

The pdf $f_1^*(x)$ is: $f_1^*(x) = a - bx^2$ $\qquad a = \frac{3}{2} = 1.5$

SIMULATION OF SYSTEM TRANSPORT

Monte Carlo simulation for system reliability

PLANT = system of Nc suitably connected components.

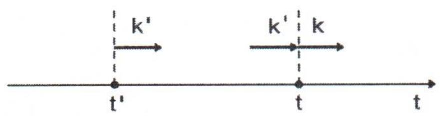
COMPONENT = a subsystem of the plant (pump, valve,...) which may stay in different exclusive (multi)states (nominal, failed, stand-by,...). Stochastic transitions from state-to-state occur at stochastic times.

STATE of the PLANT at t = the set of the states in which the Nc components stay at t. The states of the plant are labeled by a scalar which enumerates all the possible combinations of all the component states.

PLANT TRANSITION = when any one of the plant components performs a state transition we say that the plant has performed a transition. The time at which the plant performs the n-th transition is called t_n and the plant state thereby entered is called k_n.

PLANT LIFE = stochastic process.

Stochastic Transitions: Governing Probabilities



$T(t\ /t';k')dt$ = conditional probability of a transition at $t \in dt$, given that the preceding transition occurred at t' and that the state thereby entered was k'.

$C(k\ /\ k';t)$ = conditional probability that the plant enters state k, given that a transition occurred at time t when the system was in state k'.

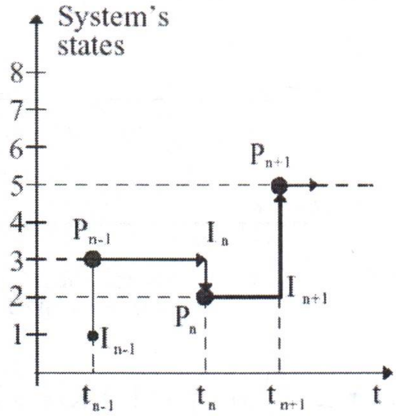
Both these probabilities form the "trasport kernel" :

$$K(t;k\ /\ t';k')dt = T(t\ /\ t';k')dt\ C(k\ /\ k';t)$$

$\psi(t;k)$ = ingoing transition density or probability density function (pdf) of a system transition at t, resulting in the entrance in state k

Plant life: random walk

Random walk = realization of the system life generated by the underlying state-transition stochastic process.



The von Neumann's Approach and the Transport Equation

unconditionally density function of entering a transition at time t which leads the system to state k

The transition density $\psi(t; k)$ is expanded in series of the partial transition densities:

partially density function

$\psi^n(t; k)$ = pdf that the system performs the n–th transition at t, entering the state k.

Then, $$\psi(t,k) = \sum_{n=0}^{\infty} \psi^n(t,k) =$$

$$\boxed{\psi(t,k) = \psi^0(t,k) + \sum_{k'} \int_{t_0}^{t} dt' \psi(t',k')K(t,k \mid t',k')}$$

BOLTZMAN TRANSPORT EQUATION

probability that the system entered at time t' the state k' and then moved to the state k at time t (i.e. $\psi(t',k') \cdot \underbrace{K(t,k|t',k')}_{\text{transport kernel}}$)

summed over all the possible k' of departure and over all the possible times at which this transition can occur (t')

Transport equation for the plant states

Monte Carlo Solution to the Transport Equation (1)

Von Neumann approach:

- Initial Conditions: $t_0=t^*$, $k_0=k^*$, $P_0 \equiv P^*$
- The subsequent transition densities in the random walk:

$$\psi^1(t_1,k_1) = K(t_1,k_1 \mid t_0,k_0)$$

$$\psi^2(t_2,k_2) = \sum_{k_1} \int_{t^*}^{t_2} \psi^1(t_1,k_1)dt_1 K(t_2,k_2 \mid t_1,k_1)$$

probability (density) of entering the state k_2 as 2nd transition which occurs at time t_2

probability (density) that we move somewhere at time t_1 (whatever time in the interval (t^*, t_2)) and from there we move at state k_2

$$\psi^n(t_n,k_n) = \sum_{k_{n-1}} \int_{t^*}^{t_n} \psi^{n-1}(t_{n-1},k_{n-1})dt_{n-1} K(t_n,k_n \mid t_{n-1},k_{n-1})$$

- Changing notation:

$t_n \to t \qquad k_{n-1} \to k'$

$t_{n-1} \to t' \qquad k_n \to k$

Monte Carlo Solution to the Transport Equation (2)

$$\boxed{\psi^n(t,k)=\sum_{k'}\int_{t^*}^{t}\psi^{n-1}(t',k')dt'K(t,k|t',k')}$$

$$\Rightarrow \psi(t,k)=\sum_{n=0}^{\infty}\psi^n(t,k)=\psi^0(t,k)+$$

$$+\sum_{k'}\int_{t^*}^{t}\underbrace{\sum_{n-1=0}^{\infty}\psi^{n-1}(t',k')}_{\psi(t',k')}dt'K(t,k|t',k')$$

$$\left(\sum_{n-1=0}^{\infty}\psi^{n-1}(t',k')=\psi(t',k')\right)$$

we took the 0 out, this is as "n=1"

$$\rightarrow \Psi(t,k)=\Psi^0(t,k)+\sum_{k'}\int_{t_0}^{t}dt'\Psi(t',k')K(t,k|t',k')$$

we need to solve this integral to get $\Psi(t,k)$ (full description of our model) $\Rightarrow$ MONTE CARLO SIMULATION

$\int g(x)f(x)dx$

Monte Carlo Solution to the Transport Equation (3)

we're in great shape since both $\Psi(t',k')$ and $K(t,k|t',k')$ are density functions. However, we don't know Ψ! So, how do we procede?

Initial Conditions: (t*, k*)

Formally rewrite the partial transition densities:

$$\psi^1(t_1,k_1)=\sum_{k_0}\int_{t^*}^{t_1}dt_0\psi^0(t_0,k_0)K(t_1,k_1|t_0,k_0)=K(t_1,k_1|t^*,k^*)$$

we know it

$$\psi^2(t_2,k_2)=\sum_{k_1}\int_{t^*}^{t_2}dt_1\psi^1(t_1,k_1)K(t_2,k_2|t_1,k_1)=$$

$$=\sum_{k_1}\int_{t^*}^{t_2}dt_1K(t_1,k_1|t^*,k^*)K(t_2,k_2|t_1,k_1)$$

we all know it

......

$$\psi^n(t,k)=\sum_{k_1,k_2,\ldots,k_{n-1}}\int_{t^*}^{t_n}dt_{n-1}\int_{t^*}^{t_{n-1}}dt_{n-2}\ldots$$

$$\ldots\int_{t^*}^{t_2}dt_1K(t_1,k_1|t^*,k^*)K(t_2,k_2|t_1,k_1)\cdots K(t,k|t_{n-1},k_{n-1})$$

we know it all (the components) (we don't have Ψ anymore)

MC Evaluation of Definite Integrals

!!!

This is f(x), the function from which we sample. That's because we sample THE FULL PATH and if at the n-th transition we have a transition at time t that leads to k then we collect our award which is 1 (because we're observing a realization of a system that has the n-th trans that occurs at time t and leads to state k). We sample many paths

(The paths are sampled step by step)

$$G=\int_a^b g(x)f(x)dx$$

$$f(x)\equiv \text{pdf} \quad\rightarrow\quad f(x)\geq 0 \quad ; \quad \int f(x)dx=1$$

•MC analog dart game: sample $x=(t1,\,k1;\,t2,\,k2;\,\ldots)$ from

whole path

$f(x)=\ K(t_1,k_1|t^*,k^*)K(t_2,k_2|t_1,k_1)\cdots K(t,k|t_{n-1},k_{n-1})$

• the probability that a shot hits $x\in dx$ is $f(x)dx$

• the award is $g(x)=1$

Consider N trials with result $\{x_1, x_2, \ldots, x_n\}$: the average award is

$$G_N=\frac{1}{N}\sum_{i=1}^{N}g(x_i)$$

CONTENTS

- Sampling
- Evaluation of definite integrals
- Simulation of system transport
- Simulation for reliability/availability analysis

SIMULATION FOR SYSTEM RELIABILITY ANALYSIS

Monte Carlo Simulation in RAMS

$$\boxed{G(t)=\sum_{k\in\Gamma}\int_0^t \psi(\tau,k)R_k(\tau,t)d\tau}$$ Expected value

- *Γ = subset of all system failure states*
- *$R_k(\tau,t)$ = 1 $\Rightarrow$ G(t) = unreliability*
- *$R_k(\tau,t)$ = prob. system not exiting before t from the state k entered at τ<t*

  $\Rightarrow$ G(t) = unavailability

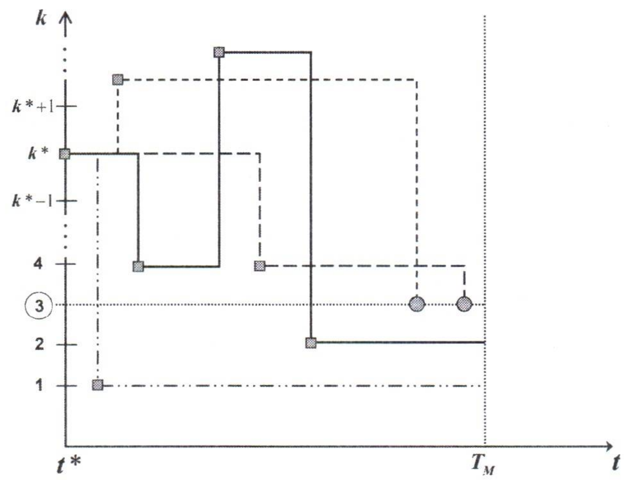
the unreliability of a system at time t is the probability that a system has failed before time t (entered a failed state before ($\leq$) time t)

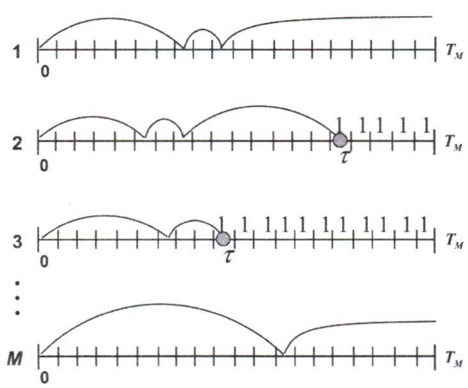
we're summing all the paths that entered a failed state before t (summing → averaging)

a system is unavailable at time t if it entered the failure state $k\in\Gamma$ at time $\tau<t$ and it stayed there until t

Monte Carlo solution of a definite integral:

expected value ≅ sample mean





$C^R(t)=C_R(t)\quad t\in[0,T_M]$

$C^R(t)=C^R(t)+1\quad t\in[\tau,T_M]$

$C^R(t)=C^R(t)+1\quad t\in[\tau,T_M]$

$C_R(t)=C^R(t)\quad t\in[0,T_M]$

$$\hat{F}_T(t)=\frac{C_R(t)}{M}$$

Monte Carlo Simulation Approaches

- Each trial of a Monte Carlo simulation consists in generating a random walk which guides the system from one configuration to another, at different times.

- During a trial, starting from a given system configuration k' at t', we need to determine when the next transition occurs and which is the new configuration reached by the system as a consequence of the transition.

- This can be done in two ways which give rise to the so called "indirect" and "direct" Monte Carlo approach.

Indirect Monte Carlo

The indirect approach consists in:

1. Sampling first the time t of a system transition from the corresponding conditional probability density $T(t|t',k')$ of the system performing one of its possible transitions out of k' entered at time t'.

2. Sampling the transition to the new configuration k from the conditional probability $C(k|t,k')$ that the system enters the new state k given that a transition has occurred at t starting from the system in state k'.

3. Repeating the procedure from k' at time t to the next transition.

Direct Monte Carlo (1)

The direct approach differs from the previous one in that the system transitions are not sampled by considering the distributions for the whole system but rather by sampling directly the times of all possible transitions of all individual components of the system and then arranging the transitions along a timeline, in accordance to their times of occurrence. Obviously, this timeline is updated after each transition occurs, to include the new possible transitions that the transient component can perform from its new state. In other words, during a trial starting from a given system configuration k' at t':

1. We sample the times of transition $t^i_{j_i\to m_i}$, $m_i = 1,2,\ldots,N_{S_i}$, of each component i, i = 1, 2,…, N_c leaving its current state j'_i and arriving to the state m_i from the corresponding transition time probability distributions $f_T^{i,j_i\to m_i}(t|t')$.

2. The time instants $t^i_{j_i\to m_i}$ thereby obtained are arranged in ascending order along a timeline from t_{min} to $t_{max} \leq T_M$

Direct Monte Carlo (2)

3. The clock time of the trial is moved to the first occurring transition time t_{min}= t* in correspondence of which the system configuration is changed, i.e. the component i* undergoing the transition is moved to its new state m_i*.

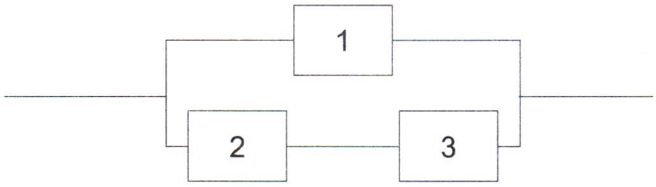
4. At this point, the new times of transition $t^{i^*}_{m_i^*\to l_i^*}$, $l_i^* = 1,2,\ldots,N_S^{i^*}$, of component i* out of its current state m_i* are sampled from the corresponding transition time probability distributions, $f_T^{i^*,m_i^*\to l_i^*}(t|t^*)$, and placed in the proper position of the timeline.

5. The clock time and the system are then moved to the next first occurring transition time and corresponding new configuration, respectively.

6. The procedure repeats until the next first occurring transition time falls beyond the mission time, i.e. $t_{min} > T_M$.

Compared to the previous indirect method, the direct approach is more suitable for systems whose components' failure and repair behaviours are represented by different stochastic distribution laws.

- Consider the following system



- Transition rates:

  Failure: $\lambda_1 = 0.001$; $\lambda_2 = 0.002$; $\lambda_3 = 0.005$;
  Repair: $\mu_1 = 0.1$; $\mu_2 = 0.15$; $\mu_3 = 0.05$;

- Estimate the **reliability** and **availability** of the system over a mission time $T_{miss} = 500$

Analytic solution

1) Minimal cut sets

$M_1=\{1,4\}$
$M_2=\{2,5\}$
$M_3=\{1,3,5\}$
$M_4=\{2,3,4\}$

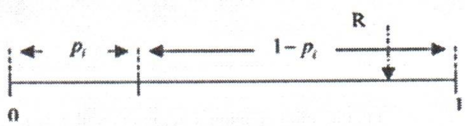
2) Network failure probability (rare event approximation):

$$P[M_1]=p_1\cdot p_4=0.05\cdot 0.02=1\cdot 10^{-3}$$
$$P[M_2]=p_2\cdot p_5=0.025\cdot 0.075=1.875\cdot 10^{-3}$$
$$P[M_3]=p_1\cdot p_3\cdot p_5=0.05\cdot 0.05\cdot 0.075=1.875\cdot 10^{-4}$$
$$P[M_4]=p_2\cdot p_3\cdot p_4=0.025\cdot 0.05\cdot 0.02=7.5\cdot 10^{-5}$$

$$P[X_T=1]\cong\sum_{j=1}^{4}P[M_j]=3.07\cdot 10^{-3}$$

Monte Carlo simulation

The state of arc *i* can be sampled by applying the inverse transform method to the set of discrete probabilities $\{p_i, 1-p_i\}$ of the mutually exclusive and exhaustive states



- **We calculate the arrival state for all the arcs of the newtork**
- **As a result of these transitions, if the system falls in any configuration corresponding to a minimal cut set, we add 1 to N_f**
- **The trial simulation then proceeds until we collect N trials. We obtain the estimation of the failure probability dividing N_f by N**

Monte Carlo simulation

- ***N* number of trials**
- **N_F number of trials corresponding to realization of a minimal cut set**
- **The failure probability is equal to N_F/N**

```
clear all;
%arc failure probabilities
p=[0.05,0.025,0.05,0.02,0.075]; nf=0;
n=1e6; %number of MC simulations
for i=1:n
   %sampling of arcs fault events
   r=rand(1,5); s=zeros(1,5); rpm=r-p;
   for j=1:5
      if (rpm(1,j)<=0)
         s(1,j)=1;
      end
   end
   %cut set check
   fault=s(1,1)*s(1,4)+s(1,2)*s(1,5)+s(1,1)*s(1,3)*s(1,5)+s(1,2)*s(1,3)*s(1,4);
   if fault >= 1
       nf=nf+1;
   end
end
pf=nf/n; %system failure probability
```

For n=10⁶, we obtain $P[X_T=1]\cong 3.04\cdot 10^{-3}$

MC Evaluation of Definite Integrals (1D)

Example

$$G=\int_0^1 \cos\left(\frac{\pi}{2}x\right)dx=\frac{2}{\pi}=0.6366198$$

By setting: $f(x)=1, \quad g(x)=\cos\left(\frac{\pi}{2}x\right)$

$$\Rightarrow \quad G=E[g(x)]$$

$$\Rightarrow \quad E\left[g^2(x)\right]=\int_0^1 \cos^2\left(\frac{\pi}{2}x\right)dx=\frac{1}{2}$$

$$Var[G_N]=\frac{1}{N}Var[g(x)]=\frac{1}{N}\left\{E\left[g^2(x)\right]-E[g(x)]^2\right\}$$

$$Var[G_N]=\frac{1}{N}\left[\frac{1}{2}-\left(\frac{2}{\pi}\right)^2\right]=\frac{1}{N}\cdot 9.47\cdot 10^{-2}$$

for $N=10^4$ histories, $x_i \sim U[0,1) \Rightarrow g(x_i)=\cos\left(\frac{\pi}{2}x_i\right)$

$$\Rightarrow G_N=0.6342,\ s^2_{G_N}=9.6\cdot 10^{-6}$$

MC Evaluation of Definite Integrals (1D)

Biased Case: Example

$$G=\int_0^1 \cos\left(\frac{\pi}{2}x\right)dx=\frac{2}{\pi}=0.6366198$$

The pdf $f_1^*(x)$ is: $f_1^*(x)=a-bx^2$

From the normalization condition:

$$\int_0^1 f_1^*(x)dx=\int_0^1 \left(a-bx^2\right)dx=1 \Rightarrow a-\frac{b}{3}=1 \Rightarrow b=3(a-1)$$

$$f_1^*(x)=a-3(a-1)x^2$$

For the minimum value $a=\frac{3}{2}=1.5$

$$G=\int_0^1 \cos\frac{\pi}{2}xdx=\int_0^1 \underbrace{\frac{\cos\frac{\pi}{2}x\cdot 1}{1.5-1.5x^2}}_{g_1(x)}\cdot\underbrace{(1.5-1.5x^2)}_{f_1(x)}dx$$

MC Evaluation of Definite Integrals (1D)

Biased Case: Example

Finally we obtain:

$$Var_1[G_{1N}]=\frac{1}{N}\left[0.406275-\left(\frac{2}{\pi}\right)^2\right]=\frac{1}{N}9.9026\cdot 10^{-4}$$

for $N=10^4$ histories, $x_i \sim f_1^* \Rightarrow g_1(x_i)=\frac{\cos\left(\frac{\pi}{2}x_i\right)}{\left(1.5-1.5x_i^2\right)}$

$$\Rightarrow G_{1N}=0.6366,\ s^2_{G_{1N}}=9{,}95\cdot 10^{-8}$$