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Exercise 1.

Given a 2^{nd} order system in state space with matrices :

$$F = \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \qquad G = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = [0]$$

(strictly proper system)

a. Write the system of difference equations

Theory : State Space Representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) & \leftarrow \text{ state equation} \\ y(t) = Hx(t) + Du(t) & \leftarrow \text{ output equation} \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad n \text{ is the system order}$$

$y(t), u(t) \in \mathbb{R}$ (SISO)

$$n = 2, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

$$\begin{cases} x_1(t+1) = 2x_2(t) + \frac{1}{2}u(t) \\ x_2(t+1) = \frac{1}{2}x_1(t) + 3x_2(t) + \frac{1}{2}u(t) \\ y(t) = x_1(t) \end{cases}$$

b. Compute the transfer function

Theory: transfer function

$y(t) = W(z)\,u(t) \implies \xrightarrow{u(t)} \boxed{W(z)} \xrightarrow{y(t)}$

↳ digital filter

I method : apply the z transformation to the system of difference equations ; $(x(t+1) = z\,x(t))$

$$\begin{cases} z x_1(t) = 2x_2(t) + 1/2\, u(t) & (1) \\ z x_2(t) = 1/2\, x_1(t) + 3x_2(t) + \frac{1}{2}u(t) & (2) \\ y(t) = x_1(t) & (3) \end{cases}$$

$(2) \longrightarrow x_2(t) \longrightarrow (1)$

$$\implies (z-3)x_2(t) = \frac{1}{2}x_1(t) + \frac{1}{2}u(t)$$

$$\implies x_2(t) = \frac{1}{2(z-3)}x_1(t) + \frac{1}{2(z-3)}u(t)$$

$$\implies z x_1(t) = \frac{1}{(z-3)}x_1(t) + \frac{1}{(z-3)}u(t) + \frac{1}{2}u(t)$$

$$\implies \frac{z^2-3z-1}{z-3}x_1(t) = \frac{1 + 1/2\,z + 3/2}{z-3}u(t)$$

$$\implies x_1(t) = \frac{1/2\,(z-1)}{z^2-3z-1}u(t)$$

$$\implies y(t) = \underbrace{\boxed{\frac{1/2\,(z-1)}{z^2-3z-1}}}_{\longrightarrow W(z)} u(t)$$

II method : transformation formula (preferred method)

Theory : $W(z) = H(zI - F)^{-1}G + D$

$$F = \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \quad G = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

Step **1.** compute $zI - F$:

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} = \begin{bmatrix} z & -2 \\ -1/2 & z-3 \end{bmatrix}$$

Step **2.** $\det(zI - F)$:

$$\det(zI - F) = \det\left(\begin{bmatrix} z & -2 \\ -1/2 & z-3 \end{bmatrix}\right) = z(z-3) - 1 = z^2 - 3z - 1$$

Step **3.** $(zI - F)^{-1}$:

$$(zI - F)^{-1} = \frac{1}{z^2 - 3z - 1} \begin{bmatrix} z-3 & 2 \\ 1/2 & z \end{bmatrix}$$

Step **4.** matrix multiplication

$$W(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{z^2 - 3z - 1} \begin{bmatrix} z-3 & 2 \\ 1/2 & z \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

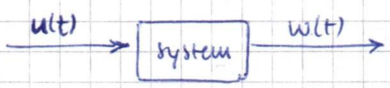
$$= \frac{1}{z^2 - 3z - 1} \begin{bmatrix} z-3 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

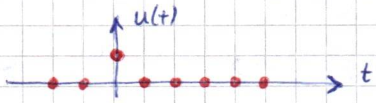
$$= \frac{1/2\, z - 3/2 + 1}{z^2 - 3z - 1}$$

$$= \frac{1/2\,(z-1)}{z^2 - 3z - 1}$$

c. Compute the first 4 samples of the impulse response

Theory : Impulse Response (IR) :





← the output to this is called "Impulse Response"

4 methods :

x(t), w(t) { 1. directly from the system of difference eq. } from the state space
w(t) { 2. matrix multiplication formula

3. long division
4. geometric series trick } transfer function

In this exercise we'll use method 3.

Theory : convolution of the input with the input response (IR)

$$y(t) = w(0)\,u(t) + w(1)\,u(t-1) + \dots = \sum_{k=0}^{\infty} w(k)\,u(t-k)$$

$$y(t) = \underbrace{\left(w(0) + w(1)\,z^{-1} + \dots\right)}_{W(z)} u(t)$$

$$W(z) = \frac{B(z)}{A(z)} = w(0) + w(1) + w(2) + \dots$$

long division

In our case: $W(z) = \dfrac{1/2\,(z-1)}{z^2 - 3z - 1}$ (not the best form)

$\Longrightarrow \quad W(z) = \dfrac{1/2\, z^{-1} - 1/2\, z^{-2}}{1 - 3z^{-1} - z^{-2}}$ (**best form**)

$$
\begin{array}{rrrrr|l}
\frac{1}{2} z^{-1} & -\frac{1}{2} z^{-2} & & & & 1 - 3z^{-1} - z^{-2} \\
-\frac{1}{2} z^{-1} & +\frac{3}{2} z^{-2} & +\frac{1}{2} z^{-3} & & & \frac{1}{2} z^{-1} + z^{-2} + \frac{7}{2} z^{-3} + \frac{23}{2} z^{-4} \\
\hline
/ & z^{-2} & +\frac{1}{2} z^{-3} & & & \\
 & -z^{-2} & +3z^{-3} & +z^{-4} & & \\
\hline
 & / & \frac{7}{2} z^{-3} & +z^{-4} & & \\
 & & -\frac{7}{2} z^{-3} & +\frac{21}{2} z^{-4} & +\frac{7}{2} z^{-5} & \\
\hline
 & & / & \frac{23}{2} z^{-4} & +\frac{7}{2} z^{-5} &
\end{array}
$$

$\Longrightarrow \quad W(z) = \frac{1}{2} z^{-1} + z^{-2} + \frac{7}{2} z^{-3} + \frac{23}{2} z^{-4} + \dots$

$\Longrightarrow \quad w(0) = 0, \quad w(1) = \frac{1}{2}, \quad w(2) = 1, \quad w(3) = \frac{7}{2}, \quad w(4) = \frac{23}{2}$

d. Check the system observability and reachability

Theory: observability and

$\mathcal{S}$ is observable $\Longleftrightarrow$ the observability matrix O is full rank,

$$O = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

$\mathcal{S}$ is reachable $\Longleftrightarrow$ the reachability matrix R is full rank

$$R = [\, G \quad FG \quad F^2G \quad \cdots \quad F^{n-1}G \,]$$

$n = 2$:

$O = \begin{bmatrix} H \\ HF \end{bmatrix}$: $\quad O[1,:] = [\,1 \quad 0\,]$

$\quad O[2,:] = [\,1 \quad 0\,] \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} = [\,0 \quad 2\,]$

$O = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$: $\quad \det(O) = 2 \neq 0 \Longrightarrow O$ full rank

$\Longrightarrow \mathcal{S}$ is observable

$R = [\,G \quad FG\,]$ $\quad R[:,1] = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$\quad R[:,2] = \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}$

$R = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 7/4 \end{bmatrix}$ $\quad \det(R) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8} \neq 0 \Longrightarrow R$ full rank

$\Longrightarrow \mathcal{S}$ is reachable

e. Compute the Henkel matrix
1. starting from the impulse response
2. starting from O and R

Theory: Henkel matrix from IR (1.)

$$H_n = \begin{bmatrix} w(1) & w(2) & w(3) & \cdots & w(n-1) & w(n) \\ w(2) & w(3) & w(4) & \cdots & w(n) & w(n+1) \\ \vdots & & & & & \\ \vdots & & & & & \\ w(n) & w(n+1) & w(n+2) & \cdots & w(2n-2) & w(2n-1) \end{bmatrix}$$

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 1 & 7/2 \end{bmatrix}$$

Theory: Henkel matrix from O and R (O_n, R_n)

$$H_n = O_n R_n$$

$$O = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 7/4 \end{bmatrix} \longrightarrow H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/2 & 1 \\ 1/2 & 7/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 1 & 9/2 \end{bmatrix}$$

Exercise 2.

Given the impulse response (IR):

$$w(t) = \begin{cases} 0 & t \leq 1 \\ (-2)^{2-t} & t > 1 \end{cases}$$

a. Compute the transfer function.

$w(0) = w(1) = 0$ ($w(0) = 0 \Rightarrow$ the system is strictly proper)

$w(2) = (-2)^{2-2} = 1$

$w(3) = (-2)^{-1} = -\frac{1}{2}$

$w(4) = (-2)^{-2} = +\frac{1}{4}$

$w(5) = (-2)^{-3} = -\frac{1}{8}$

$w(6) = (-2)^{-4} = +\frac{1}{16}$

$$\Rightarrow y(t) = u(t-2) - \frac{1}{2}u(t-3) + \frac{1}{4}u(t-4) - \dots$$

$$= \left(z^{-2} - \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4} - \dots\right)u(t)$$

$$= \left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \dots\right)z^{-2}u(t)$$

$$= \underbrace{\sum_{k=0}^{a}\left(-\frac{1}{2}z^{-1}\right)^k}_{} z^{-2}u(t)$$

geometric series: $\sum_{k=0}^{+\infty} a^k = \frac{1}{1-a}$

$$= \frac{1}{1+\frac{1}{2}z^{-1}} z^{-2}u(t)$$

$$\Rightarrow W(z) = \frac{z^{-2}}{1+\frac{1}{2}z^{-1}}$$

b. Write the state-space representation in control form

Theory:

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1}}{1\cdot z^n + a_1 z^{n-1} + \dots + a_n}$$ (strictly proper system)

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & 1 & 0 \\ 0 & & & 0 & 1 \\ -a_n & & \cdots & & -a_1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad H = [b_{n-1} \cdots b_0]$$

$D = 0$ (we can check it for instance with checking $w(0) = 0$)

$$W(z) = \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} = \frac{1}{z^2+\frac{1}{2}z}$$

$\Longrightarrow n=2: \ a_1 = \frac{1}{2}, \quad a_2 = 0, \quad b_0 = 0, \quad b_1 = 1$

$\Longrightarrow F = \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = [0]$

A possible check is to compute from this ($\uparrow$) the transfer function ($W(z) = H(zI-F)^{-1}G + D$)

c. Write the system of difference equations and apply the change of variable:

$$\tilde{x} = Tx, \qquad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$n=2, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = -1/2\, x_2(t) + u(t) \\ y(t) = x_1(t) \end{cases}$$

Theory: change of variables

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Du(t) \end{cases}$$

$$\tilde{x}(t) = Tx(t) \Longrightarrow \begin{cases} \tilde{x}(t+1) = \tilde{F}\tilde{x}(t) + \tilde{G}u(t) \\ y(t) = \tilde{H}\tilde{x}(t) + \tilde{D}u(t) \end{cases}$$

$x(t) = T^{-1}\tilde{x}(t)$

$$\Longrightarrow \begin{cases} T^{-1}\tilde{x}(t+1) = FT^{-1}\tilde{x}(t) + Gu(t) \\ y(t) = HT^{-1}\tilde{x}(t) + Du(t) \end{cases}$$

$$\Longrightarrow \begin{cases} \tilde{x}(t+1) = TFT^{-1}\tilde{x}(t) + TGu(t) \\ y(t) = HT^{-1}\tilde{x}(t) + Du(t) \end{cases}$$

$y(t)$ and $u(t)$ don't change, so the two systems are EQUIVALENT (so the transfer functions are the same.)

$$\Longrightarrow \tilde{F} = TFT^{-1}, \quad \tilde{G} = TG, \quad \tilde{H} = HT^{-1}, \quad \tilde{D} = D$$

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{F} = TFT^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \dots = \begin{bmatrix} 1/4 & -1/4 \\ 3/4 & -3/4 \end{bmatrix}$$

$$\tilde{G} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\tilde{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$\tilde{D} = 0$$

d. Compute the transfer function starting from the new representation.

$$W(z) = \tilde{H}(zI - \tilde{F})^{-1}\tilde{G} + \tilde{D}$$

Step 1.

$$zI - \hat{F} = \begin{bmatrix} z - 1/4 & 1/4 \\ -3/4 & z + 3/4 \end{bmatrix}$$

Step 2.

$$\det(zI - \hat{F}) = (z - 1/4)(z + 3/4) + 3/16$$
$$= z^2 + (3/4 - 1/4)\, z - 3/16 + 3/16$$
$$= z^2 + \tfrac{1}{2} z$$

Step 3.

$$(zI - \hat{F})^{-1} = \frac{1}{z^2 + \frac{1}{2} z} \begin{bmatrix} z + 3/4 & -1/4 \\ 3/4 & z - 1/4 \end{bmatrix}$$

Step 4.

$$W(z) = \frac{1}{z^2 + \frac{1}{2} z} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z + 3/4 & -1/4 \\ 3/4 & z - 1/4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1/2\, z + 1/2 - 1/2\, z + 1/2}{z^2 + 1/2\, z}$$

$$= \frac{1}{z^2 + 1/2}$$

it's the same as the other one from the other system (this is a check!)

e. Compute H_3 (Hankel matrix of order 3) and check that $H_3 = O_3 R_3$

extended observability and reachability matrices

$$H_3 = \begin{bmatrix} w(1) & w(2) & w(3) \\ w(2) & w(3) & w(4) \\ w(3) & w(4) & w(5) \end{bmatrix}$$

always from $w(1)$ (even if $w(0) \neq 0$)

$$= \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -1/2 & 1/4 \\ -1/2 & 1/4 & -1/8 \end{bmatrix}$$

$$O_3 = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} \quad (3\times 2) \qquad R = \begin{bmatrix} G & FG & F^2G \end{bmatrix} \quad (2\times 3)$$

$$O_3[1,:] = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$O_3[2,:] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$O_3[3,:] = HF^2 = (HF)F = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \end{bmatrix}$$

$$R_3[:,1] = G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$R_3[:,2] = FG = \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$
$$R_3[:,3] = F \cdot FG = \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$$

$$O_3 \cdot R_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -1/2 & 1/4 \\ -1/2 & 1/4 & -1/8 \end{bmatrix} = H_3$$

I

f. Check that $F = O_3[1:n, :]^{-1} O_3[2:n+1, :]$.

$$O_3[1:n, :] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow O_3[1:n, :]^{-1} = I$$

$$O_3[2:n+1, :] = \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix}$$

$$\Longrightarrow F = O_3[2:n+1, :] = \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} \qquad (\checkmark)$$

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Exercise 3.

Given the system in state-space:

$$\begin{cases} x(t+1) = \frac{1}{2} x(t) + 2u(t) \\ y(t) = 3x(t) \end{cases}$$

$$\longrightarrow F = \frac{1}{2}, \quad G = 2, \quad H = 3, \quad D = 0$$

a. Compute the first **5** samples of the IR.

Method **1**: directly from the system of difference equations

$$u(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}, \qquad x(0) = 0 \quad \text{(if not specified)}$$

t	x(t)	y(t)
0	$x(0) = 0$	$y(0) = 3x(0) = 0$
1	$x(1) = \frac{1}{2}x(0) + 2u(0) = 2$	$y(1) = 3x(1) = 6$
2	$x(2) = \frac{1}{2}x(1) + \cancel{2u(1)} = 1$	$y(2) = 3x(2) = 3$
3	$x(3) = \frac{1}{2}x(2) + \cancel{2u(2)} = 1/2$	$y(3) = 3x(3) = 3/2$
4	$x(4) = \frac{1}{2}x(3) + \cancel{2u(3)} = 1/4$	$y(4) = 3x(4) = 3/4$
5	$x(5) = \frac{1}{2}x(4) + \cancel{2u(4)} = 1/8$	$y(5) = 3x(5) = 3/8$

$$\Longrightarrow w(0) = 0, \quad w(1) = 6, \quad w(2) = 3, \quad w(3) = 3/2, \quad w(4) = 3/4, \quad w(5) = 3/8$$

(this method allows us to know the state (internal)) (but it can be a long method)

Method **4**: geometric series trick

$$W(z) = H(zI - F)^{-1} G + D$$
$$= 3\left(z - \frac{1}{2}\right)^{-1} 2$$
$$= \frac{6}{z - \frac{1}{2}} = \frac{6z^{-1}}{1 - \frac{1}{2}z^{-1}} = 6z^{-1}\left(\sum_{k=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^k\right)$$
$$= 6z^{-1}\left(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \frac{1}{16}z^{-4} + \dots\right)$$
$$= \underbrace{6z^{-1}}_{w(1)} + \underbrace{3z^{-2}}_{w(2)} + \underbrace{\frac{3}{2}z^{-3}}_{w(3)} + \underbrace{\frac{3}{4}z^{-4}}_{w(4)} + \underbrace{\frac{3}{8}z^{-5}}_{w(5)} + \dots \qquad (w(0) = 0)$$

(This method saves a lot of computation: if we have a system of order 2/3 this is the recommended method)

b. Compute the second order Henkel matrix and check that it is not full rank (and justify why).

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

$H_2(2,:) = \frac{1}{2} H(1,:) \Longrightarrow$ since the two rows are not linearly independent, H_2 is not full rank

$$H_2 = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}, \quad \text{rank}(H_2) < 2$$

Justification: consider the Henkel matrix of order i: if $i > n \Rightarrow \text{rank}(H_i) = n$.
In our case $i = 2$, $n = 1 \Rightarrow \text{rank}(H_2) = 1$.

c. Identify the system matrices using the 4SID method.

Step 1: Identify the system order n

In this case $n = 1$

Step 2: Build H_{n+1}: (H_2)

$$H_2 = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

Step 3: Find a factorization $O_{n+1} R_{n+1} = H_{n+1}$
(where O_{n+1} is a $(n+1) \times n$ matrix
R_{n+1} is a $n \times (n+1)$ matrix)

In our case: O_{n+1} (2×1), R_{n+1} (1×2):

$$\begin{bmatrix} \ \\ \ \end{bmatrix} \begin{bmatrix} \ & \ \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

- put n independent rows of H_{n+1} in R_{n+1}: (we can put $\forall$ row, not necessarily the first)

$$\begin{bmatrix} \ \\ \ \end{bmatrix} \begin{bmatrix} 6 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

- fill the rows of O_{n+1} such that $H_{n+1} = O_{n+1} R_{n+1}$

$$\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \begin{bmatrix} 6 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

Step 4: Matrix extraction:

$$\hat{F} = O_{n+1}(1:n,:)^{-1} O_{n+1}(2:n+1,:) = \frac{1}{2}$$
$$\hat{G} = R_{n+1}(:,1) = 6$$
$$\hat{H} = O_{n+1}(1,:) = 1$$
$$\hat{D} = 0$$ (because the system is strictly proper $(w(0) = 0)$)

$\Longrightarrow \hat{F} = \frac{1}{2}, \quad \hat{G} = 6, \quad \hat{H} = 1, \quad \hat{D} = 0$ (different from F, G, H, D given, but it's ok, they're both represent the same syst.)

d. Compute the transfer function starting from the identified matrices.

$$\hat{W}(z) = \hat{H}(zI - \hat{F})^{-1} \hat{G} + \hat{D}$$
$$= 1 \cdot \frac{1}{z - \frac{1}{2}} \cdot 6$$
$$= \frac{6}{z - \frac{1}{2}} = \frac{6z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

same transfer function as the one generated from F, G, H, D $\Rightarrow$ the two (system matrices)'s representate the same system

Exercise 4.

Given the transfer function:

$$W(z) = \frac{(z+2)}{(z+\frac{1}{2})(z+2)}$$

Theory: Remark on zero-pole cancellation.

$$W_1(z) = \frac{(z+2)}{(z+\frac{1}{2})(z+2)} = \frac{z+2}{z^2+\frac{5}{2}z+1}$$

$$W_2(z) = \frac{1}{(z+\frac{1}{2})}$$

represent the same I/O relationship but considering $W_1(z)$ we see that it's a second order system and it's unstable ($\exists$ pole outside the unitary circle).
If we consider $W_2(z)$ we may conclude that the system is a first order stable system (**WRONG**)
$\longrightarrow$ do not cancel N/D common terms

a. Compute the state-space representation in control form.

$$W(z) = \frac{z+2}{z^2+\frac{5}{2}z+1} \qquad b_0 = 1,\quad b_1 = 2,\quad a_1 = \frac{5}{2},\quad a_2 = 1$$

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -5/2 \end{bmatrix},\quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad H = \begin{bmatrix} 2 & 1 \end{bmatrix},\quad D = 0$$

(strictly proper because the order of N is < the order of D) !!

b. Check observability and reachability. $\longrightarrow$

we expect that is not obs. or not reach. because where there is a zero-pole cancellation it means that the I/O is not representing everything of the system ($\Rightarrow$ there is something of the system that the I/O repr. cannot catch)

$$n = 2 \Longrightarrow \mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix},\quad R = [G \quad FG]$$

$$\mathcal{O}(1,:) = [2 \quad 1]$$

$$\mathcal{O}(2,:) = [2 \quad 1]\begin{bmatrix} 0 & 1 \\ -1 & -5/2 \end{bmatrix} = [-1 \quad -\tfrac{1}{2}]$$

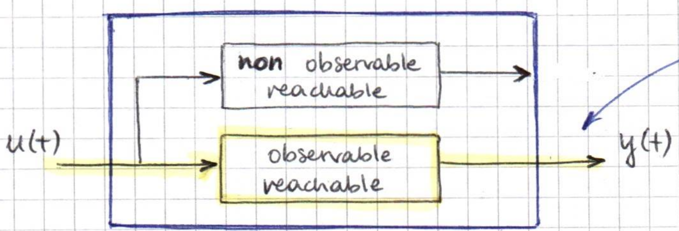
$$\Longrightarrow \mathcal{O} = \begin{bmatrix} 2 & 1 \\ -1 & -1/2 \end{bmatrix} \Longrightarrow \mathcal{O}(2,:) = -\tfrac{1}{2}\mathcal{O}(1,:) \Longrightarrow \text{rank}(\mathcal{O}) < 2$$

$\longrightarrow$ the system S is <u>not</u> observable

$$R(:,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$R(:,2) = \begin{bmatrix} 0 & 1 \\ -1 & -5/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5/2 \end{bmatrix}$$

$$\Longrightarrow R = \begin{bmatrix} 0 & 1 \\ 1 & -5/2 \end{bmatrix} \Longrightarrow \text{rank}(R) = 2 \Longrightarrow S \text{ is reachable}$$



this is the only part that the transfer function can represent

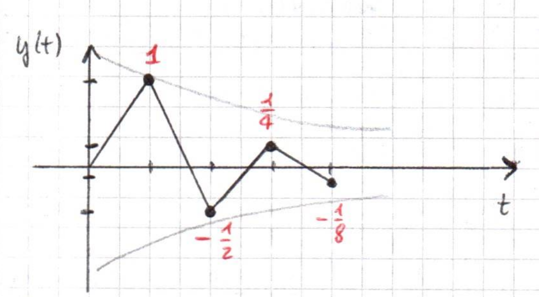
c. Compute the first 4 samples of the Impulse Response and plot the state and output evolution.

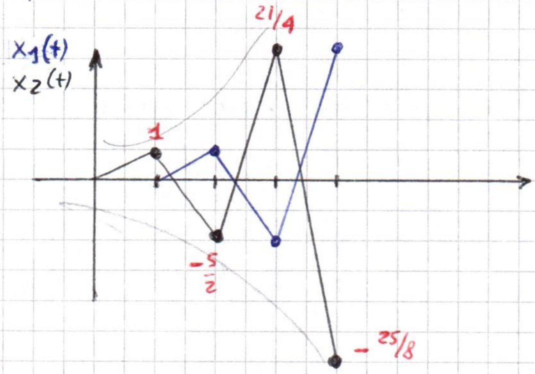
Method 1: $\longrightarrow$ $x(t)$, $y(t)$

(we need to use Method 1 because it's the only one that provides the state evolution)

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = -x_1(t) - \frac{5}{2}x_2(t) + u(t) \\ y(t) = 2x_1(t) + x_2(t) \end{cases} \qquad x(0) = 0, \qquad u(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

t	$x(t)$	$y(t)$
0	$\begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \end{cases}$	$y(0) = \cancel{2x_1(0)} + \cancel{x_2(0)} = 0$
1	$\begin{cases} x_1(1) = x_2(0) = 0 \\ x_2(1) = -x_1(0) - 5/2\, x_2(0) + u(0) = 1 \end{cases}$	$y(1) = \cancel{2x_1(1)} + x_2(1) = 1$
2	$\begin{cases} x_1(2) = x_2(1) = 1 \\ x_2(2) = -x_1(1) - 5/2\, x_2(1) + u(1) = -5/2 \end{cases}$	$y(2) = 2x_1(2) + x_2(2) = -\frac{1}{2}$
3	$\begin{cases} x_1(3) = x_2(2) = -5/2 \\ x_2(3) = -x_2(2) - 5/2\, x_2(2) + u(2) = 21/4 \end{cases}$	$y(3) = 2x_1(3) + x_2(3) = \frac{1}{4}$
4	$\begin{cases} x_1(4) = x_2(3) = 21/4 \\ x_2(4) = -x_2(3) - 5/2\, x_2(3) + u(3) = -85/8 \end{cases}$	$y(4) = 2x_1(4) + x_2(4) = -\frac{1}{8}$





If we look at the I/O relationship it seems like the system is stable because the output tends to 0 when $t \to \infty$ but if we look at the state it's clear that the system's states are unstable (so there is an internal part of the system which is unstable)

d. Identify the system matrices using 4SID method.

Step 1: Identify the system order n

$\text{rank}(H_i) = n \quad i \geq n \quad$ where n is the system order

$\left.\begin{array}{l} \text{rank}(H_n) = n \\ \text{rank}(H_{n+1}) = n \end{array}\right\} \leftarrow$ the rank stops increasing

Theory

$H_1 = w(1) = 1, \quad \text{rank}(H_1) = 1$

$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}, \quad \text{rank}(H_2) = 1$ (since: $H_2(2,:) = -\frac{1}{2} H_2(1,:)$)

$\Longrightarrow \left.\begin{array}{l} \text{rank}(H_1) = 1 \\ \text{rank}(H_2) = 1 \end{array}\right\} \Longrightarrow n = 1$

Step 2: Build H_{n+1}:

$H_2 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$

Step 3: Find a factorization $O_{n+1} R_{n+1} = H_{n+1}$

$\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} \; & \; \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$

- put n independent rows of H_{n+1} in R_{n+1}

$\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$

- fill O_{n+1} such that $O_{n+1} R_{n+1} = H_{n+1}$

$\Longrightarrow O_{n+1} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$

Step 4: Matrix extraction

$\hat{F} = -\frac{1}{2}, \quad \hat{G} = 1, \quad \hat{H} = 1, \quad \hat{D} = 0$

(the system is strictly proper since $w(0) = 0$)

e. Compute the transfer function (starting from $\hat{F}, \hat{G}, \hat{H}, \hat{D}$) (it's a check)

$$\hat{W}(z) = \hat{H}(zI - \hat{F})^{-1}\hat{G} + \hat{D} = \frac{1}{z + \frac{1}{2}}$$

Exercise 5

Given the Impulse Response:

$w(0) = 0, \quad w(1) = 0, \quad w(2) = 2, \quad w(3) = 0, \quad w(4) = 1, \quad w(5) = 0$

a. Identify the system order. (= Identify when the rank of H_n stops increasing)

$H_1 = w(1) = 0 \Longrightarrow \text{rank}(H_1) = 0$

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \Longrightarrow \text{rank}(H_2) = 2$$

$$H_3 = \begin{bmatrix} w(1) & w(2) & w(3) \\ w(2) & w(3) & w(4) \\ w(3) & w(4) & w(5) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longrightarrow H_3(3,:) = \frac{1}{2} H_3(1,:) \Longrightarrow \text{rank}(H_3) = 2$$

$\Longrightarrow \text{rank}(H_3) = \text{rank}(H_2) = 2$

$\Longrightarrow$ the system order is $n = 2$

b. Compute the transfer function.

$IR \longrightarrow \{\hat{F}, \hat{G}, \hat{H}, \hat{D}\} \longrightarrow \hat{W}(z)$

Step 1: Identify the system order.: $n = 2$

Step 2: H_{n+1}:

$$H_3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 3: Find a factorization: $O_{n+1} R_{n+1} = H_{n+1}$

O_3 is a (3×2) matrix, R_3 is a (2×3) matrix

$$\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \begin{bmatrix} \; \\ \; \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- put n independent rows of H_{n+1} in R_{n+1}

$$\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(we could also do: $\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$)

- Fill the rows of O_{n+1} such that $O_{n+1} R_{n+1} = H_{n+1}$

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ \begin{matrix} 1/2 & 0 \end{matrix} \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

it's useful to have the identity here (it's suggested to take the first two rows of H_{n+1} if they're $\not\parallel$)

Step 4 : Matrix extraction :

$$\hat{F} = \boxed{O_{n+1}(1:n, :)^{-1}} \, O_{n+1}(2:n+1, :)$$

since we have to invert a matrix, it's really useful to have $O_{n+1} = \begin{bmatrix} I \\ \vdots \end{bmatrix}$, that's why we should always pick the first **n** rows from that when we build R_{n+1} (if they're linearly ⫫)

$$= \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}$$

$$\hat{G} = R_{n+1}(:, 1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\hat{H} = O_{n+1}(1, :) = [1 \;\; 0]$$

$\hat{D} = 0$ (the system is strictly proper : $w(0) = 0$)

$$\Longrightarrow \hat{W}(z) = \hat{H}(zI - \hat{F})^{-1}\hat{G} + \hat{D}$$

$$zI - F = \begin{bmatrix} z & -1 \\ -1/2 & z \end{bmatrix}$$

$$(zI - F)^{-1} = \frac{1}{z^2 - 1/2}\begin{bmatrix} z & 1 \\ 1/2 & z \end{bmatrix}$$

$$\Longrightarrow \hat{W}(z) = \frac{1}{z^2 - 1/2}[1 \;\; 0]\begin{bmatrix} z & 1 \\ 1/2 & z \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{z^2 - 1/2}[z \;\; 1]\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{2}{z^2 - 1/2}$$

C. Compute the first 4 samples of the IR.

Method 2 : Matrix multiplication formula

Theory : Matrix multiplication formula

$$w(t) = \begin{cases} 0 & t = 0 \\ HF^{t-1}G & t \geq 1 \end{cases} \longrightarrow$$ valid only for **strictly proper systems**

$$\Longrightarrow w(0) = 0$$

$$w(1) = HG = [1 \;\; 0]\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$$

$$w(2) = HFG = [1 \;\; 0]\begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \overbrace{[0 \;\; 1]}^{HF}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2$$

$$w(3) = HF^2G = HF \cdot FG = [0 \;\; 1]\begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \overbrace{[\tfrac{1}{2} \;\; 0]}^{HF^2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$$

$$w(4) = HF^3G = HF^2FG = [\tfrac{1}{2} \;\; 0]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = [0 \;\; \tfrac{1}{2}]\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 1$$

$$\longrightarrow w(0) = 0, \; w(1) = 0, \; w(2) = 2, \; w(3) = 0, \; w(4) = 1$$

Exercise 6

Given $y(t) = \dfrac{z^{-1} + \frac{1}{5} z^{-2}}{1 + \frac{1}{2} z^{-1}} u(t)$

a. Identify the system order from the impulse response

$$W(z) = \left(z^{-1} + \frac{1}{5} z^{-2}\right) \underbrace{\left(\frac{1}{1+\frac{1}{2} z^{-1}}\right)}_{= \sum_{k=0}^{+\infty} \left(-\frac{1}{2} z^{-1}\right)^k}$$

$$= \left(z^{-1} + \frac{1}{5} z^{-2}\right)\left(\sum_{k=0}^{+\infty} \left(-\frac{1}{2} z^{-1}\right)^k\right)$$

$$= \left(z^{-1} + \frac{1}{5} z^{-2}\right)\left(1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} - \frac{1}{8} z^{-3} + \frac{1}{16} z^{-4} - \frac{1}{2} z^{-} + \dots\right)$$

$$= z^{-1} + \frac{1}{5} z^{-2} - \frac{1}{2} z^{-2} - \frac{1}{10} z^{-3} + \frac{1}{4} z^{-3} + \frac{1}{20} z^{-4} - \frac{1}{8} z^{-4} - \frac{1}{40} z^{-5} + \frac{1}{16} z^{-5} + \dots$$

$w(0) = 0$, $w(1) = 1$, $w(2) = \frac{1}{5} - \frac{1}{2} = \frac{3}{10}$, $w(3) = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$, $w(4) = -\frac{3}{40}$, $w(5) = \frac{3}{80}$

advise: let's stop at $w(5)$ so we can compute H_3, in case $\text{rank}(H_3) = 3$ we'll procede

Henkel matrices:

$H_1 = w(1) = 1 \implies \text{rank}(H_1) = 1$

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 1 & -3/10 \\ -3/10 & 3/20 \end{bmatrix} \implies \text{rank}(H_2) = 2$$

$$H_3 = \begin{bmatrix} w(1) & w(2) & w(3) \\ w(2) & w(3) & w(4) \\ w(3) & w(4) & w(5) \end{bmatrix} = \begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \\ 3/20 & -3/40 & 3/80 \end{bmatrix}$$

$H_3(3,:) = -\frac{1}{2} H_3(2,:) \implies \text{rank}(H_3) = 2$

$$\implies \left.\begin{array}{l} \text{rank}(H_2) = 2 \\ \text{rank}(H_3) = 2 \end{array}\right\} \implies n = 2$$

b. Find the state space representation of the system

- Control form (from the transfer function)
- 4SID method ⟵ we use this

Step 1: identify the system order

$n = 2$ (from a.)

Step 2: compute $H_{n+1} = H_3$

$$H_3 = \begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \\ 3/20 & -3/40 & 3/80 \end{bmatrix}$$

Step 3: factorization of H_3: $O_{n+1} \cdot R_{n+1} = H_{n+1}$

$$\underset{\substack{O_3 \\ (3\times 2)}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}} \underset{\substack{R_3 \\ (2\times 3)}}{\begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \end{bmatrix}} = \begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \\ 3/20 & -3/40 & 3/80 \end{bmatrix}$$

- put n independent rows of H_3 in R_3 (•)
- fill the rows of O_3 s.t. $O_3 \cdot R_3 = H_3$ (•)

it's convenient, if we can, to put the first n rows of H_{n+1} (if they've lin. indep.)

Step 4 : Matrix extraction

$$\hat{F} = O_{n+1}(1:n, :)^{-1} O_{n+1}(2:n+1, :)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix}$$

For check if this is correct we can compute the transfer funct. from this and check if it's the same of the one given at the beginning

$$\hat{G} = R_{n+1}(:, 1) = \begin{bmatrix} 1 \\ -3/10 \end{bmatrix}$$

$$\hat{H} = O_{n+1}(1, :) = [1 \quad 0]$$

$\hat{D} = 0$ (the system is strictly proper ($w(0) = 0$))

Exercise 7

Given $W(z) = \dfrac{z}{z-a}$ $\quad a \in \mathbb{R} \setminus \{0\}$

a. Compute the first 7 samples of the impulse response.

$$W(z) = \frac{1}{1 - az^{-1}} = \sum_{k=0}^{+\infty} (az^{-1})^k$$
$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + a^4z^{-4} + a^5z^{-5} + a^6z^{-6} + a^7z^{-7} + \dots$$

$\Rightarrow$ $\underbrace{w(0) = 1}$, $w(1) = a$, $w(2) = a^2$, $w(3) = a^3$, $w(4) = a^4$, $w(5) = a^5$, $w(6) = a^6$, $w(7) = a^7$

$\longrightarrow$ the system is not strictly proper

b. Identify the system order strarting from the Impulse Response. (same procedure even if $w(0) \neq 0$)

$H_1 = w(1) = a \quad \Longrightarrow \quad \text{rank}(H_1) = 1 \quad (a \neq 0)$

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix} \Longrightarrow H_2(2, :) = a\, H_2(1, :)$$

$\longrightarrow$ $\left.\begin{array}{l} \text{rank}(H_1) = 1 \\ \text{rank}(H_2) = 1 \end{array}\right\} \Longrightarrow n = 1$

c. Find the state-space representation.

We'll use 4SID.

Step 1. System order : $n = 1$

Step 2. H_{n+1} :

$$H_2 = \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix}$$

Step 3. Factorization :

$$\begin{bmatrix} 1 \\ a \end{bmatrix} [a \quad a^2] = \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix} \qquad O_2 \cdot R_2 = H_2$$

Step 4 : Matrix extraction :

$$\hat{F} = O_{n+1}(1:n, :)^{-1} O_{n+1}(2:n+1, :) = a$$
$$\hat{G} = R_{n+1}(:, 1) = a$$
$$\hat{H} = O_{n+1}(1, :) = 1$$

$\hat{D}$?

$\hat{D} \neq 0$ because $w(0) \neq 0$ (the system is not strictly proper)

Theory: matrix $\hat{D}$ identification

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Du(t) \end{cases}$$

Consider the impulse response: $u(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$, $x(0) = 0$

$$t=0: \quad y(0) = H\underbrace{x(0)}_{0} + D\underbrace{u(0)}_{1} = D$$

$$\Rightarrow \hat{D} = w(0)$$

In our case $\hat{D} = 1$

$$\Rightarrow \hat{F} = a, \quad \hat{G} = a, \quad \hat{H} = 1, \quad \hat{D} = 1$$

d. Compute the transfer function. (from $\hat{F}, \hat{G}, \hat{D}, \hat{H}$ to see if it's the same as the given one at the beginning)

$$W(z) = H(zI - F)^{-1}G + D$$
$$= 1 \cdot \frac{1}{z-a} a + 1$$
$$= \frac{z}{z-a}$$

KALMAN FILTER

20/05

Exercise 1

Given the system:

$$\begin{cases} x(t+1) = \frac{2}{5} x(t) + v_1(t) \\ y(t) = 3x(t) + v_2(t) \end{cases} \qquad \begin{matrix} v_1 \sim WN\left(0, \frac{123}{125}\right) \\ v_2 \sim WN(0, 1) \end{matrix} \qquad v_1 \perp v_2$$

a. Compute the 1-step Kalman Predictor.

Theory: 1-step Kalman predictor

$$y(t) \longrightarrow \boxed{\text{1-step Kalman pred.}} \longrightarrow \hat{x}(t+1|t)$$

$$\begin{cases} \hat{x}(t+1|t) = F\hat{x}(t|t-1) + K(t)\,e(t) & \text{state prediction} \\ \hat{y}(t|t-1) = H\hat{x}(t|t-1) & \text{output prediction} \\ e(t) = y(t) - \hat{y}(t|t-1) & \text{innovation} \\ K(t) = (FP(t)H^T + V_{12})(HP(t)H^T + V_2)^{-1} & \text{filter gain} \\ P(t+1) = FP(t)F^T + V_1 - (FP(t)H^T + V_{12})(HP(t)H^T + V_2)^{-1}(FP(t)H^T + V_{12})^T \end{cases}$$

DRE
(Difference Riccati Eq.)

$$\begin{cases} \hat{x}(1|0) = x_0 & \text{initial guess} \\ P(1) = P_0 & \text{(how much we trust the initial guess)} \end{cases}$$

$n=2$, $F = \frac{2}{5}$, $H = 3$, $V_1 = \frac{123}{125}$, $V_2 = 1$, $V_{12} = 0$
(since $v_1 \perp v_2$)

Step 1: DRE

$$P(t+1) = \frac{4}{25}P(t) + \frac{123}{125} - \frac{(6/5\ P(t))^2}{9P(t)+1}$$

$$P(t+1) = \frac{(4/25\ P(t) + 123/125)(9P(t)+1) - 36/25\ P(t)^2}{9P(t)+1}$$

$$= \frac{36/25\ P(t)^2 + 4/25\ P(t) + 9\cdot 123/125\ P(t) + 123/125 - 36/25\ P(t)}{9P(t)+1}$$

$$= \frac{(20 + 9\cdot 123)\ (t) + 123}{125\,(9P(t)+1)}$$

Step 2: Compute the filter gain $K(t)$

$$K(t) = (FP(t)H^T + V_{12})[HP(t)H^T + V_2]^{-1}$$

$$= \frac{6/5\ P(t)}{9P(t)+1}$$

Step 3: Write the system of equations

$$\begin{cases} \hat{x}(t+1|t) = \frac{2}{5}\hat{x}(t|t-1) + K(t)\,e(t) \\ \hat{y}(t|t-1) = 3\hat{x}(t|t-1) \\ e(t) = y(t) - \hat{y}(t|t-1) \\ K(t) = \dfrac{6/5\ P(t)}{9P(t)+1} \\ P(t+1) = \dfrac{(20+9\cdot 123)P(t) + 123}{125\,(9P(t)+1)} \end{cases}$$

simplifying:

$$\hat{x}(t+1|t) = \frac{2}{5}\hat{x}(t|t-1) + \frac{6/5\ P(t)}{9P(t)+1}\left(y(t) - 3\hat{x}(t|t-1)\right)$$

$$\begin{cases} \hat{x}(t+1|t) = \left(\dfrac{2}{5} - \dfrac{6/5\ P(t)}{9P(t)+1}\cdot 3\right)\hat{x}(t|t-1) + \dfrac{6/5\ P(t)}{9P(t)+1}\,y(t) \\ P(t+1) = \dfrac{(20+9\cdot 123)P(t) + 123}{125\,(9P(t)+1)} \\ \hat{y}(t|t-1) = 3\hat{x}(t|t-1) \end{cases}$$

answer to **a.**

Notice: $\hat{x}(t+1|t) = (F - K(t)H)\,\hat{x}(t|t-1) + K(t)\,y(t)$

b. Compute the steady state Kalman predictor (1-step K-predictor)

Theory: Asymptotic Kalman predictor

- $K(t) \xrightarrow{t\to\infty} \bar{K}$
- $(F - KH)$ asymptotically stable

Note that: $K(t) = f(P(t))$: $(K(t) \to \bar{K})$ if $(P(t) \to \bar{P})$

If $(P(t) \to \bar{P}) \Rightarrow P(t+1) = P(t) = \bar{P}$

$$\Rightarrow \bar{P} = F\bar{P}F^T + V_1 - (F\bar{P}H^T + V_{12})^{-1}(HPH^T + V_2)^{-1}(F\bar{P}H^T + V_{12})^T$$

ARE
(Algebraic Riccati Eq.)

$$\Rightarrow \bar{K} = (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}$$

So we have to check: if:

- $\bar{P}$ exists
- $P(t) \longrightarrow \bar{P}$
- $F - \bar{K}H$ is asymptotically stable

Method 1: Graphical method (only if we have a 1^{st} order system)

Step 1: Compute the ARE solution $\bar{P}$

$$\bar{P} = \frac{(20 + 123 \cdot 9)\bar{P} + 123}{125\,(9\bar{P} + 1)}$$

$$\bar{P}\,125\,(9\bar{P} + 1) = (20 + 123 \cdot 9)\,\bar{P} + 123$$

$$1125\,\bar{P}^2 + 125\,\bar{P} - 1127\,\bar{P} - 123 = 0$$

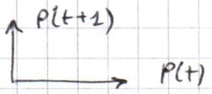
$$1125\,\bar{P}^2 - 1002\,\bar{P} - 123 = 0 \implies \bar{P} = \begin{cases} 1 \\ \cancel{-\frac{41}{375}} \end{cases}$$

it can't be since $P(t)$ is positive def.

$\Rightarrow$ The ARE has one and only one positive definite solution

Step 2: Draw $P(t+1) = f(P(t))$

$$P(t+1) = \frac{1127\,P(t) + 123}{1125\,P(t) + 125}$$



- Horizontal asymptote:

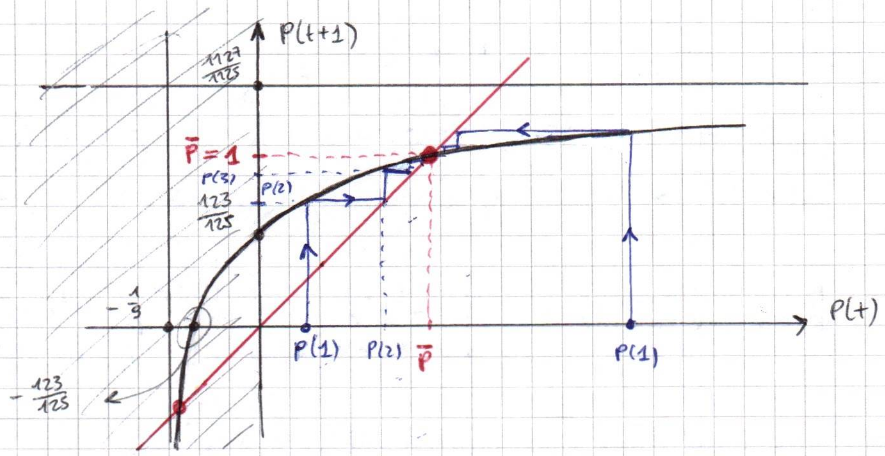
$$P(t) \to \infty \implies P(t+1) \to \frac{1127}{1125}$$

- Vertical asymptote:

$$1125\,P(t) + 125 = 0 \implies P(t) = -\frac{125}{1125} = -\frac{1}{9} \qquad P(t+1) \to \infty$$

- Axes intersection:

$$\begin{cases} P(t) = 0 \\ P(t+1) = 123/125 \end{cases} \qquad \begin{cases} P(t) = -123/1127 \\ P(t+1) = 0 \end{cases}$$



The DRE converges to $\bar{P}$ $\forall P_0 \geq 0$

Step 3: Compute $\bar{K}$ and check the asymptotic stability

$$\bar{K} = \frac{6/5\,\bar{P}}{9\bar{P} + 1} = \frac{6}{50} = \frac{3}{25}$$

The asymptotic KP (Kalman Predictor) is asymptotic stable (A.S.) if the matrix $F - \bar{K}H$ (which defines the dynamic of KP) has all the eigenvalues inside the unitary circle (strictly inside)

$$F - \bar{K}H = \frac{2}{5} - \frac{3}{25} \cdot 3 = \frac{1}{25}$$

$eig(F - \bar{K}H) \Rightarrow \det(\lambda I - (F - \bar{K}H)) = 0 \longrightarrow \lambda - \frac{1}{25} = 0$

$\Rightarrow \lambda = \frac{1}{25}$: $|\lambda| < 1 \Rightarrow \bar{K}$ is s.t. $F - \bar{K}H$ has all the eigenvalues strictly inside the unitary circle

$\Rightarrow$ Steady state of KP:

$$\begin{cases} \hat{x}(t+1|t) = \frac{1}{25}\,\hat{x}(t|t-1) + \frac{3}{25}\,y(t) \\ \hat{y}(t|t-1) = 3\hat{x}(t|t-1) \end{cases}$$ solution of **b.**

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Method **2**: Use the theorems

Theory: Asymptotic theorems

Theorem 1:

If
- the system is asymptotically stable
- $V_{12} = 0$

$\Rightarrow$
- ARE has one and only one positive semidefinite solution $\bar{P} \geq 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- The corresponding $\bar{K}$ is such that $(F - \bar{K}H)$ has all eigenvalues strictly inside the unitary circle

Theorem 2:

If
- $V_{12} = 0$
- (F, H) is observable
- (F, Γ) is reachable $\quad (\Gamma\Gamma^T = V_1)$

$\Rightarrow$
- ARE has one and only one positive definite solution $\bar{P} > 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- The corresponding $\bar{K}$ is such that $(F - \bar{K}H)$ has all the eigenvalues strictly inside the unitary circle

let's check the theorems:

Theorem 1:
- $V_{12} = 0$? Yes, $V_1 \perp V_2$
- S is asymptotically stable ? $F = \frac{2}{5}$

$eig(F) \rightarrow \det(\lambda I - F) = 0 \Longrightarrow \lambda - \frac{2}{5} = 0 \rightarrow \lambda = \frac{2}{5}$

$|\lambda| < 1 \Longrightarrow S$ is asymptotically stable

Theorem 2: (not necessary, we already proved the $\exists$ of KP)
- $V_{12} = 0$? Yes
- (F, H) observable ? $\quad O = H = 3$, $\text{rank}(O) = 1 = n \Rightarrow$ Yes
- (F, Γ) reachable ? $\quad \Gamma\Gamma^T = 123/125 \longrightarrow \Gamma = \sqrt{123/125}$

$R = \Gamma \longrightarrow \text{rank}(R) = 1 = n \Longrightarrow$ Yes

$\longrightarrow$
- ARE has one and only one positive definite solution
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- $\bar{K}$ is such that all the eigenvalues of $(F - \bar{K}H)$ are strictly inside the unitary circle

From here we have to compute $\bar{P}$, $\bar{K}$ and KP: (theorems only prove existence)

$\bar{P} = \dots = 1$

$\bar{K} = \dots = 3/25$

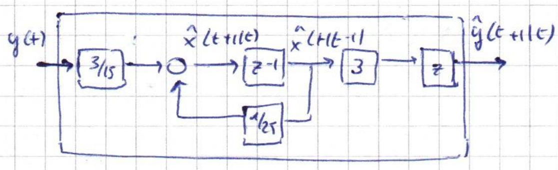
$$\begin{cases} \hat{x}(t+1|t) = \frac{1}{25}\,\hat{x}(t|t-1) + \frac{3}{25}\,y(t) \\ \hat{y}(t|t-1) = 3\hat{x}(t|t-1) \end{cases}$$

SISO single input single output: $\tilde{F} = \frac{1}{25}$, $\tilde{G} = \frac{3}{25}$, $\tilde{H} = 3$

c. Compute the transfer function from the measured output to the 1-step steady state predicted output

$$y(t) \longrightarrow \boxed{W(z)} \longrightarrow \hat{y}(t+1|t)$$

equivalent to



$$\hat{y}(t|t-1) = \tilde{H}(zI-\tilde{F})^{-1}\tilde{G}\, y(t)$$

$$= \frac{9/25}{z - 1/25}\, y(t)$$

$$\hat{y}(t+1|t) = z\,\hat{y}(t|t-1) = \frac{9/25 \cdot z}{z - 1/25}\, y(t)$$

$$W(z) = \frac{9/25\, z}{z - 1/25}$$

d. Compute the variance of the steady state 1-step output predictor error $e(t) = y(t) - \hat{y}(t|t-1)$:

$$\begin{aligned}
Var(e(t)) &= var\left(y(t) - \hat{y}(t|t-1)\right) \\
&= \mathbb{E}\left[\left(y(t) - \hat{y}(t|t-1)\right)^2\right] \\
&= \mathbb{E}\left[\left(3x(t) + v_2(t) - 3\hat{x}(t|t-1)\right)^2\right] \\
&= \mathbb{E}\left[\left(3\left(x(t) - \hat{x}(t|t-1)\right) + v_2(t)\right)^2\right] \\
&= 9\,\mathbb{E}\left[\left(x(t) - \hat{x}(t|t-1)\right)^2\right] + \mathbb{E}\left[v_2^2(t)\right] + 6\,\cancel{\mathbb{E}\left[x(t)\, v_2(t)\right]} \quad 1. \\
&\quad - 6\,\cancel{\mathbb{E}\left[\hat{x}(t|t-1)\, v_2(t)\right]} \quad 2. \\
&= 9\, P(t) + V_2
\end{aligned}$$

variance of the state prediction error (by def.)

1. Represents the correlation between $x(t)$ and $v_2(t)$: $v_2(t)$ cannot influence $x(t)$ $\longrightarrow$ $x(t) \perp v_2(t)$

2. Represents the correlation between $v_2(t)$ and $\hat{x}(t|t-1)$: $v_2(t)$ cannot influence $\hat{x}(t|t-1)$ $\Longrightarrow$ $\hat{x}(t|t-1) \perp v_2(t)$

$\Longrightarrow$ $Var(e(t)) = 9\,P(t) + 1$

$\Longrightarrow$ At steady state : $Var(e(t)) = 10$

f. Find the steady state 3-steps predictor and compute the transfer function from $y(t)$ to $\hat{y}(t+3|t)$

Theory: K step predictor

$$\hat{x}(t+1|t) \longrightarrow \boxed{F^{k-1}} \longrightarrow \hat{x}(t+k|t)$$

$$\begin{cases} \hat{x}(t+k|t) = F^{k-1}\,\hat{x}(t+1|t) \\ \hat{y}(t+k|t) = H\,\hat{x}(t+k|t) \end{cases}$$

$k = 3$, $\quad F^2 = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$

$$\hat{x}(t+1|t) = \frac{1}{25}\hat{x}(t|t-1) + \frac{3}{25}y(t)$$

$$\begin{cases} \hat{x}(t+3|t) = \frac{4}{25}\hat{x}(t+1|t) \\ \hat{y}(t+3|t) = 3\hat{x}(t+3|t) \end{cases}$$

$$y(t) \longrightarrow \boxed{\text{1-step KP}} \xrightarrow{\hat{x}(t+1|t)} \boxed{F^2} \xrightarrow{\hat{x}(t+3|t)} \boxed{H} \xrightarrow{\hat{y}(t+3|t)}$$

$\longrightarrow W(z)$

$$\hat{x}(t+1|t) = \frac{1}{25}\underbrace{\hat{x}(t|t-1)}_{z^{-1}\hat{x}(t+1|t)} + \frac{3}{25}y(t)$$

$$\left(1-\frac{1}{25}\right)\hat{x}(t+1|t) = \frac{3/25}{1-1/25\,z^{-1}}\;y(t)$$

$$\hat{x}(t+3|t) = \frac{4}{25}\;\frac{3/25}{1-1/25\,z^{-1}}\;y(t)$$

$$\hat{y}(t+3|t) = 3\cdot\frac{4}{25}\;\frac{3/25}{1-z^{-1}\,1/25}\;y(t) \quad\Longrightarrow\quad W(z) = \frac{36/625}{1-z^{-1}\,1/25}$$

g. Find the state equation of the Kalman filter at steady state
(till now we considered Kalman predictor)

Theory: Kalman Filter

$$y(t) \longrightarrow \boxed{\text{K.F.}} \xrightarrow{\hat{x}(t|t)}$$

if F is non singular : $\hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)$

otherwise, if $V_{12}=0$: $\hat{x}(t|t) = F\hat{x}(t-1|t-1) + K_F(t)\,e(t)$

$$\hat{y}(t|t-1) = H\hat{x}(t|t-1)$$

$$e(t) = y(t) - \hat{y}(t|t-1)$$

$$K_F(t) = P(t)H^T\left(HP(t)H^T+V_2\right)^{-1}$$

$$P(t+1) = FP(t)F^T + V_1 - \left(FP(t)H^T\right)\left(HP(t)H^T+V_2\right)^{-1}\left(FP(t)H^T\right)^T$$

$$F = \frac{2}{5}, \quad F^{-1} = \frac{5}{2} \quad\Longrightarrow\quad \hat{x}(t|t) = \frac{5}{2}\hat{x}(t+1|t)$$

but we need $\hat{x}(t|t) = f(\hat{x}(t-1|t-1))$:

$$\hat{x}(t+1|t) = \frac{1}{25}\underbrace{\hat{x}(t|t-1)}_{F\hat{x}(t-1|t-1)} + \frac{3}{25}y(t)$$

$$\hat{x}(t|t) = \frac{5}{2}\left[\frac{1}{25}\cdot\frac{2}{5}\hat{x}(t-1|t-1) + \frac{3}{25}y(t)\right]$$

$$\hat{x}(t|t) = \frac{1}{25}\hat{x}(t-1|t-1) + \frac{3}{10}y(t)$$

h. Compute the variance of the state estimation error using the filter at steady state

$$\operatorname{Var}\left(x(t) - \hat{x}(t|t)\right)$$

Theory:

$$P(t) = \operatorname{Var}\left(x(t) - \hat{x}(t|t-1)\right)$$

$$P(t+1) = \operatorname{var}\left(x(t+1) - \hat{x}(t+1|t)\right)$$

$$P(t+1) = \mathrm{Var}(x(t+1) - \hat{x}(t+1|t))$$
$$= \mathbb{E}[(x(t+1) - \hat{x}(t+1|t))^2]$$
$$= \mathbb{E}[(\tfrac{2}{5}x(t) + v_1(t) - \tfrac{2}{5}\hat{x}(t|t))^2]$$
$$= \frac{4}{25}\underbrace{\mathbb{E}[(x(t) - \hat{x}(t|t))^2]}_{\mathrm{Var}(x(t) - \hat{x}(t|t))} + \underbrace{\mathbb{E}[v_1(t)^2]}_{V_1 = \frac{123}{125}} + \cancel{\frac{4}{5}\mathbb{E}[(x(t) - \hat{x}(t|t))\, v_1(t)]}$$

because $v_1(t)$ influences $x(t+1)$ so $x(t) \perp v_1(t)$ and $\hat{x}(t|t) \perp v_1(t)$ $(\hat{x}(t|t) \sim y(t) = f(x(t)))$

$$\mathrm{Var}(x(t) - \hat{x}(t|t)) = \frac{25}{4}\left(P(t+1) - \frac{123}{125}\right)$$

At steady state: $P(t+1) \longrightarrow \bar{P}$

$$\mathrm{Var}(x(t) - \hat{x}(t|t)) = \frac{25}{4}\left(\bar{P} - \frac{123}{125}\right) = \frac{1}{10}$$

Theory:

$\mathrm{Var}(x(t) - \hat{x}(t|t-1)) = \bar{P}$ at steady state

$\underbrace{\mathrm{Var}(x(t) - \hat{x}(t|t))}_{\text{error of the prediction with data till } t} \leq \underbrace{\mathrm{Var}(x(t) - \hat{x}(t|t-1))}_{\text{here } \hat{x}(t|t-1) \text{ stops at } t-1 \Rightarrow \text{less info}}$

Exercise 2

Given the system:

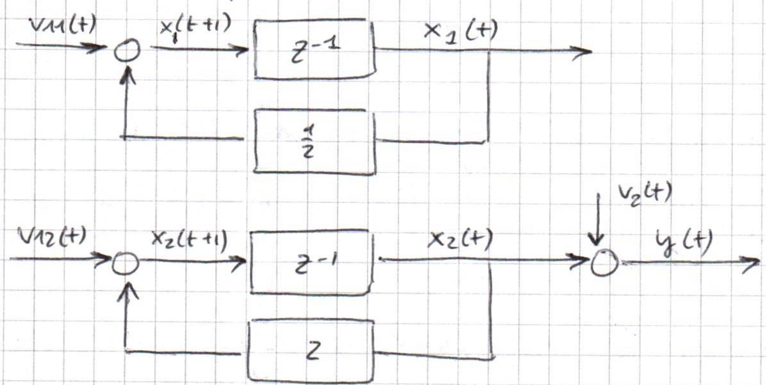
$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + v_{11}(t) \\ x_2(t+1) = 2x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$$

$v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$v_2(t) \sim WN(0, 1)$

$v_1 \perp v_2$ $\quad \left(V_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$

a. Draw the system block scheme



notice that we have 2 autonomous subsystems

b. Is the asymptotic state predictor error bounded?
(1-step state predictor)

$e_x(t) = x(t) - \hat{x}(t|t-1)$ is bounded?

$\boxed{\text{bounded} \Leftrightarrow \text{KP 1-step steady state exists}}$!

(Check if the asymptotic KP exists)

Let's check the theorems:

Theorem 1:
- $V_{12} = [0 \;\; 0]^T$? Yes $(v_1 \perp v_2)$
- S is asymptotically stable?

$$\det(\lambda I - S) = 0 \Rightarrow \det\left(\begin{bmatrix} \lambda - \frac{1}{2} & 0 \\ 0 & \lambda - 2 \end{bmatrix}\right) = (\lambda - \tfrac{1}{2})(\lambda - 2) = 0$$

$\Rightarrow \lambda = \frac{1}{2},\ \lambda = 2$ outside the unitary circle $\Rightarrow$ S is **not** asymp. stable

Theorem 2 :

- $V_{12} = [0 \;\; 0]^T$? Yes
- (F, H) observable ?

$$O = \begin{bmatrix} H \\ HF \end{bmatrix}$$

$$O(1,:) = H = [0 \;\; 1]$$

$$O(2,:) = HF = [0 \;\; 1] \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} = [0 \;\; 2]$$

$$O = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \longrightarrow \text{rank}(O) = 1 \Longrightarrow \text{not observable}$$

Considering the whole system, neither theorem 1 nor theorem 2 is valid.

$$\begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + v_{11}(t) \\ x_2(t+1) = 2 x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases} \quad \begin{matrix}] \; A \\ ] \; B \end{matrix}$$

Let's check one subsystem at time : (since they're $\perp\!\!\!\perp$)

A : $\begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + v_{11}(t) \\ y_A(t) = 0 \end{cases}$ (fictitious output)

$F_A = \frac{1}{2}$, $H_A = 0$, $V_{1A} = 1$, $V_{2A} = 0$, $V_{2A} \perp V_{11} \Rightarrow V_{12A} = 0$

Theorem 1 :

- $V_{12} = 0$
- S is asymptotically stable $\quad (\lambda = \frac{1}{2})$

$\Longrightarrow$ the theorem 1 is valid

B : $\begin{cases} x_2(t+1) = 2 x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$

* since F is unstable we procede directly with theorem 2

$F_B = 2$*, $H_B = 1$, $V_{1B} = 1$, $V_{2B} = 1$, $V_{12B} = 0$

$(V_{2B} = 1) \Longrightarrow \Gamma_B = 1$

Theorem 2 :

- $V_{12} = 0$
- (F, H) observable ? $\quad O = H = 1$, rank$(O) = 1 = n$
  $\Longrightarrow (F, H)$ observable
- (F, Γ) reachable ? $\quad R = \Gamma = 1$, rank$(R) = 1 = n$
  $\Longrightarrow (F, \Gamma)$ reachable

$\Longrightarrow$ the theorem 2 is valid

For theorem 1 it is possible to build an asymptotic stable asymptotic kalman predictor of $x_1(t)$:

$$x_1(t) - \hat{x}_1(t|t-1) < \infty \qquad \forall t$$

For theorem 2 it is possible to build an asymptotic stable asymptotic kalman predictor of $x_2(t)$:

$$x_2(t) - \hat{x}_2(t|t-1) < \infty \qquad \forall t$$

Moreover :

$$e_x(t) = x(t) - \hat{x}(t|t-1) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(t|t-1) \\ \hat{x}_2(t|t-1) \end{bmatrix} = \begin{bmatrix} x_1(t) - \hat{x}_1(t|t-1) \\ x_2(t) - \hat{x}_2(t|t-1) \end{bmatrix} < \infty$$

The asymptotic state prediction error is bounded.

c. Compute the steady state 1-step Kalman predictor (since we know that it exist)

Method 1: Consider separately the 2 subsystems
(we can do it because they don't interact)

Subsystem **A**:

$$\begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + v_{11}(t) \\ y_A(t) = 0 \end{cases}$$

$$\bar{K}_A = (F_A \bar{P}_A \cancel{H_A^T} + \cancel{V_{12_A}})(\cancel{H_A} \bar{P}_A \cancel{H_A^T} + \cancel{V_{2A}})^{-1} = 0$$

Remember: $\hat{x}(t+1|t) = F\hat{x}(t|t-1) + \underbrace{\bar{K} e(t)}$

now $\bar{K}_A = 0$ so we can't correct $\hat{x}(t+1|t)$ (the only way we can adjust our prediction is with the error, here we don't have it (or better, the gain is 0), so the best prediction is replicate the system dynamic)

$$\hat{x}_1(t+1|t) = \frac{1}{2}\hat{x}_1(t|t-1)$$

Subsystem **B**:

$$\begin{cases} x_2(t+1) = 2x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$$

$$\bar{P} = F\bar{P}F^T + V_1 - (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}(F\bar{P}H^T + V_{12})^T$$

$$\bar{P} = 4\bar{P} + 1 - \frac{4\bar{P}^2}{\bar{P}+1}$$

$$\Rightarrow [..] \rightarrow \bar{P}^2 - 4\bar{P} - 1 = 0 \Rightarrow \bar{P}_{1/2} = 2 \pm \sqrt{5}$$

$$\Rightarrow \bar{P} = 2+\sqrt{5} \quad (\text{since } 2-\sqrt{5} < 0)$$

$$\Rightarrow \bar{K}_B = (F_B \bar{P} H_B^T + V_{12B})(H_B \bar{P} H_B^T + V_2)^{-1} = \frac{2(2+\sqrt{5})}{2+\sqrt{5}+1} = \frac{4+2\sqrt{5}}{3+\sqrt{5}}$$

$$\hat{x}_2(t+1|t) = 2\hat{x}_2(t|t-1) + \frac{4+2\sqrt{5}}{3+\sqrt{5}}(y(t) - \hat{y}(t|t-1))$$

$$\hat{y}(t|t-1) = \hat{x}_2(t|t-1)$$

$\Rightarrow$ Total Kalman predictor:

$$\begin{cases} \hat{x}_1(t+1|t) = \frac{1}{2}\hat{x}_1(t|t-1) \\ \hat{x}_2(t+1|t) = 2\hat{x}_2(t|t-1) + \frac{4+2\sqrt{5}}{3+\sqrt{5}}(y(t) - \hat{y}(t|t-1)) \\ \hat{y}(t|t-1) = \hat{x}_2(t|t-1) \end{cases}$$

(equivalently) $\rightarrow$ (matrix form)

$$\begin{cases} \hat{x}(t+1|t) = \overbrace{\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}}^{F \text{ of the original system}} \hat{x}(t|t-1) + \overbrace{\begin{bmatrix} 0 \\ \frac{4+2\sqrt{5}}{3+\sqrt{5}} \end{bmatrix}}^{\bar{K}} (y(t) - \hat{y}(t|t-1)) \\ \hat{y}(t|t-1) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{H \text{ of the original system}} \hat{x}(t|t-1) \end{cases}$$

Method 2: Consider the entire system
(only method we can use when we can't split the system)

ARE:

$$\bar{P} = F\bar{P}F^T + V_1 - (F\bar{P}H^T - V_{12})(H\bar{P}H^T + V_2)^{-1}(F\bar{P}H^T + V_{12})^T$$

We have to find:

$\bar{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ that satisfies ARE. $(\Longrightarrow \alpha, \beta, \gamma\,?)$

Replacing:

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} - \underbrace{\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}_{\underbrace{\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \beta \\ \gamma \end{bmatrix}}_{\begin{bmatrix} 1/2\,\beta \\ 2\gamma \end{bmatrix}}} \underbrace{\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\right)^{-1}}_{\frac{1}{\gamma+1}} \underbrace{\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^T}_{\begin{bmatrix} \frac{1}{2}\beta & 2\gamma \end{bmatrix}} + I$$

$$= \underbrace{\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}}_{\underbrace{\begin{bmatrix} 1/2\,\alpha & 1/2\,\beta \\ 2\beta & 2\gamma \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}}_{\begin{bmatrix} 1/4\,\alpha & \beta \\ \beta & 4\gamma \end{bmatrix}}} - \underbrace{\begin{bmatrix} 1/2\,\beta \\ 2\gamma \end{bmatrix} \frac{1}{\gamma+1} \begin{bmatrix} \frac{1}{2}\beta & 2\gamma \end{bmatrix}}_{\frac{1}{\gamma+1}\begin{bmatrix} \frac{1}{4}\beta^2 & \beta\gamma \\ \beta\gamma & 4\gamma^2 \end{bmatrix}} + I$$

(we forgot about V_1, which is the identity matrix)

$$= \begin{bmatrix} 1/4\,\alpha + 1 & \beta \\ \beta & 4\gamma + 1 \end{bmatrix} - \frac{1}{\gamma+1}\begin{bmatrix} 1/4\,\beta^2 & \beta\gamma \\ \beta\gamma & 4\gamma^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4\,\alpha + 1 - \frac{1}{\gamma+1}(1/4\,\beta^2) & \beta - \frac{1}{\gamma+1}(\beta\gamma) \\ \beta - \frac{1}{\gamma+1}(\beta\gamma) & 4\gamma + 1 - \frac{1}{\gamma+1}(4\gamma^2) \end{bmatrix}$$

$$\Longrightarrow \begin{cases} \alpha = \frac{1}{4}\alpha + 1 - \frac{1}{\gamma+1}\left(\frac{1}{4}\beta^2\right) & (1) \\ \beta = \beta - \frac{1}{\gamma+1}(\beta\gamma) & (2) \\ \gamma = 4\gamma + 1 - \frac{1}{\gamma+1}(4\gamma^2) & (3) \end{cases}$$

$(2) \Longrightarrow \beta\gamma = 0 \Longrightarrow \beta = 0$ (since if $\gamma = 0 \Longrightarrow (3):\ 1 = 0$)

$\Longrightarrow \bar{P} = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} \geq 0 \Longrightarrow \alpha, \gamma \geq 0$ (so we have to neglect α and $\gamma < 0$)

$(3) \Longrightarrow \gamma(\gamma+1) = (4\gamma+1)(\gamma+1) - 4\gamma^2$

$\Longrightarrow [..] \Longrightarrow \gamma^2 - 4\gamma - 1 = 0$

$\Longrightarrow \gamma_{1/2} = 2 \pm \sqrt{5} \Longrightarrow \gamma = 2 + \sqrt{5}$

$(1) \Longrightarrow \alpha = \frac{1}{4}\alpha + 1 \longrightarrow \alpha = \frac{4}{3}$

$$\Longrightarrow \bar{P} = \begin{bmatrix} 4/3 & 0 \\ 0 & 2+\sqrt{5} \end{bmatrix}$$

Now $\bar{K}$:

$$\bar{K} = \underbrace{(F\bar{P}H^T + V_{12})}_{A}\underbrace{(H\bar{P}H^T + V_2)^{-1}}_{B}$$

$$A = F\bar{P}H^T + V_{12} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4/3 & 0 \\ 0 & 2+\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2+\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 4+2\sqrt{5} \end{bmatrix}$$

$$B = \left(\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 4/3 & 0 \\ 0 & 2+\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \right)^{-1} = \frac{1}{3+\sqrt{5}}$$

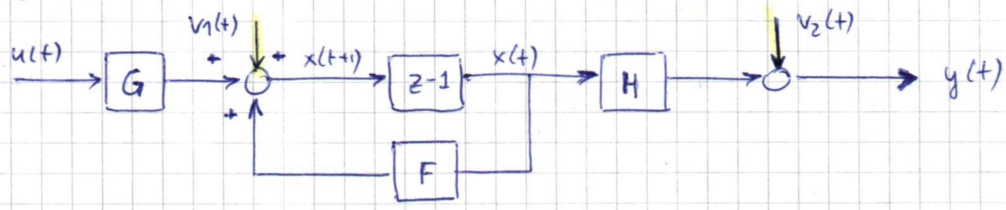
$$\Longrightarrow \quad \bar{K} = \begin{bmatrix} 0 \\ \frac{4+2\sqrt{5}}{3+\sqrt{5}} \end{bmatrix}$$

$$\Longrightarrow \begin{cases} \hat{x}(t+1|t) = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \hat{x}(t|t-1) + \begin{bmatrix} 0 \\ \frac{4+2\sqrt{5}}{3+\sqrt{5}} \end{bmatrix} (y(t) - \hat{y}(t|t-1)) \\ \hat{y}(t|t-1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}(t|t-1) \end{cases}$$

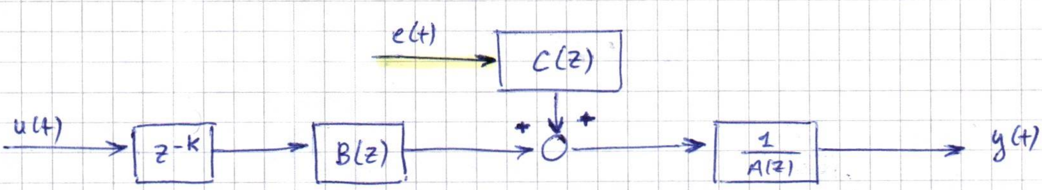
Exercise 3 (theoretical exercise)

Note:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) + v_1(t) \\ y(t) = Hx(t) + v_2(t) \end{cases}$$



$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t)$$



Are equivalent? Yes
The difficulty is pass from two noises to one.

Consider the system:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + \frac{1}{4}u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \qquad \begin{matrix} v_1 \sim WN(0,1) \\ v_2 \sim WN(0,1) \end{matrix} \qquad v_1 \perp v_2$$

a. Find the equivalent model in the form of an ARMAX:

$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

$$\begin{cases} z\,x(t) = \frac{1}{2}x(t) + \frac{1}{4}u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases}$$

$$\Longrightarrow \quad x(t) = \frac{1/4}{z - 1/2}\,u(t) + \frac{1}{z - 1/2}\,v_1(t)$$

Now we replace in $y(t)$:

$$\Longrightarrow \quad y(t) = \frac{1/4}{z-1/2}\,u(t) + \frac{1}{z-1/2}\,v_1(t) + v_2(t)$$

$$\frac{1/4}{1-1/2\,z^{-1}}\,u(t-1) + \underbrace{\boxed{\frac{z^{-1}}{1-1/2\,z^{-1}}\,v_1(t) + v_2(t)}}_{d(t)}$$

$$y(t) = \frac{B(z)}{A(z)}\,u(t-k) + \underbrace{\boxed{\frac{C(z)}{A(z)}\,e(t)}}_{\eta(t)}$$

$(k=1)$

We want to find an equivalent noise representation (equivalent in frequency domain : $\Gamma_d(\omega) = \Gamma_\eta(\omega)$).

same spectrum

$$d(t) = \frac{z^{-1}\,v_1(t)}{1-1/2\,z^{-1}} + v_2(t) \qquad \hat{v}_1(t) := v(t-1) \sim WN(0,1)$$

the WN has the same statistical properties, at each time is a $WN(0,\lambda^2)$

$$\Gamma_d(\omega) = \left|\left(\frac{1}{1-1/2\,z^{-1}}\right)_{z=e^{j\omega}}\right|^2 \Gamma_{\hat{v}_1}(\omega) + \Gamma_{v_2}(\omega)$$

Recall :

$$|a+be^{j\omega}|^2 = (a+be^{j\omega})(a+be^{-j\omega})$$

$$\Gamma_d(\omega) = \left(\frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}\right)_{z=e^{j\omega}} + 1$$

$$= \left(\frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} + 1\right)_{z=e^{j\omega}}$$

$$= \left(\frac{1+1-\frac{1}{2}z^{-1}-\frac{1}{2}z+1/4}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}\right)_{z=e^{j\omega}}$$

$$= \left(\frac{9/4-\frac{1}{2}(z^{-1}+z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}\right)_{z=e^{j\omega}}$$

$$\eta(t) = \frac{C(z)}{A(z)}\,e(t) \qquad e \sim WN(0,\lambda^2)$$

$$A(z) = 1 - \frac{1}{2}z^{-1}$$

$C(z)$? We consider a parametric polynomial

$$\Longrightarrow \quad \eta(t) = \frac{1+c_0 z^{-1}}{1-\frac{1}{2}z^{-1}}\,e(t) \qquad e \sim WN(0,\lambda^2) \qquad c_0,\ \lambda^2\ ?$$

$$\Gamma_\eta(\omega) = \left|\left(\frac{1+c_0 z^{-1}}{1-\frac{1}{2}z^{-1}}\right)_{z=e^{j\omega}}\right|^2 \lambda^2$$

$$= \left(\frac{(1+c_0 z^{-1})(1+c_0 z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}\right)_{z=e^{j\omega}} \lambda^2$$

$$= \left(\frac{(1+c_0 z^{-1})(1+c_0 z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}\,\lambda^2\right)_{z=e^{j\omega}}$$

$$\Gamma_y(\omega) = \left(\frac{\lambda^2(1+c_0^2) + \lambda^2 c_0 (z^{-1}+z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} \right)_{z=e^{j\omega}}$$

Now we impose:

$$\Gamma_d(\omega) = \Gamma_y(\omega)$$

$$\Longrightarrow \frac{9}{4} - \frac{1}{2}(z^{-1}+z) = \lambda^2(1+c_0^2) + \lambda^2 c_0 (z^{-1}+z)$$

$$\Longrightarrow \begin{cases} \lambda^2(1+c_0^2) = \frac{9}{4} & (1) \\ \lambda^2 c_0 = -\frac{1}{2} & (2) \end{cases}$$

(1) with (2) in it $\Longrightarrow (1+c_0^2)\frac{1}{2c_0} = -\frac{9}{4} \Longrightarrow c_0^2 + \frac{9}{2}c_0 + 1 = 0$

$\Longrightarrow c_0 = \begin{cases} 4.236 \longrightarrow \text{this makes } \frac{C(z)}{A(z)} \text{ to be not in a canonical form} \\ -0.236 \longrightarrow \text{we chose this} \end{cases}$

(2) $\Longrightarrow \lambda^2 = 2.13$

$$\Longrightarrow y(t) = \frac{1/4}{1-\frac{1}{2}z^{-1}} u(t-1) + \frac{1-0.236z^{-1}}{1-\frac{1}{2}z^{-1}} e(t) \qquad e(t) \sim WN(0, 2.13)$$

MINIMUM VARIANCE CONTROL

28/05

Exercise 1

Consider the following ARMAX system:

$$y(t) = \frac{1}{2}y(t-1) + u(t-1) + e(t) + \frac{1}{3}e(t-1) \qquad e(t) \sim WN(0,1)$$

a. Check the assumptions for the design of the MVC.

Theory: Assumptions for MVC

$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

- $b_0 \neq 0$
- $B(z)$ is minimum phase $\rightarrow$ if it has all the roots in the unitary circle
- $C(z)/A(z)$ is in canonical form

In our case: $y(t) = \frac{1}{1-z^{-1}\frac{1}{2}} u(t-1) + \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} e(t)$

$B(z) = 1$
$A(z) = 1 - \frac{1}{2}z^{-1}$
$C(z) = 1 + \frac{1}{3}z^{-1}$
$k = 1$

- $b_0 = 1 \neq 0 \Longrightarrow$ ✓
- $B(z)$ has no roots $\longrightarrow$ $B(z)$ is minimum phase ✓
- $\frac{C(z)}{A(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{z+\frac{1}{3}}{z-\frac{1}{2}}$
  - zero relative degrees ✓
  - $C(z)$ and $A(z)$ are monic ✓
  - $C(z)$ and $A(z)$ are coprime:
    $\left.\begin{matrix} C(z)=0 \Rightarrow z = -1/3 \\ A(z)=0 \Rightarrow z = 1/2 \end{matrix}\right\} \Longrightarrow$ they're coprime ✓
  - $C(z)$ and $A(z)$ have all the roots strictly inside the unit circle $\Rightarrow$ ✓

$\Longrightarrow C(z)/A(z)$ is in canonical form ✓

b. Compute the k-step predictor.

In our case $k = 1$.

Theory: k-step predictor

$$\hat{y}(t|t-k) = \frac{B(z)E(z)}{C(z)} u(t-k) + \frac{\tilde{R}(z)}{C(z)} y(t-k)$$

$$\begin{array}{c|l} C(z) & A(z) \\ \cline{2-2} \vdots & E(z) \\ \hline z^{-k}\tilde{R}(z) & \end{array}$$

$$\left(z^{-k}\tilde{R}(z) := R(z) \right)$$

$$\frac{C(z)}{A(z)} = E(z) + \frac{\tilde{R}(z) z^{-k}}{A(z)}$$

1-step predictor:

$$\hat{y}(t|t-1) = \frac{B(z)}{C(z)} u(t-k) + \frac{C(z) - A(z)}{C(z)} y(t)$$

$$C(z) - A(z) = \cancel{1} + \frac{1}{3} z^{-1} \cancel{-1} + \frac{1}{2} z^{-1}$$
$$= \frac{5}{6} z^{-1}$$

$$\hat{y}(t|t-1) = \frac{1}{1 + \frac{1}{3} z^{-1}} u(t-1) + \frac{5/6 \, z^{-1}}{1 + \frac{1}{3} z^{-1}} y(t)$$

$$= \frac{1}{1 + \frac{1}{3} z^{-1}} u(t-1) + \frac{5/6}{1 + \frac{1}{3} z^{-1}} y(t-1)$$

c. Compute the minimum variance control (MVC)

Theory: MVC

Objective: find $u(t)$ s.t. $y(t) \simeq y^{\circ}(t)$

$$u(t) = \arg\min_{u(t)} \left\{ \mathbb{E}\left[(y(t) - y^{\circ}(t))^2 \right] \right\}$$
$$= \arg\min_{u(t)} \left\{ \mathbb{E}\left[(\hat{y}(t|t-k) - y^{\circ}(t))^2 \right] \right\}$$

The solution is given by: $\hat{y}(t|t-k) = y^{\circ}(t)$

$$\frac{B(z)E(z)}{C(z)} u(t-k) + \frac{\tilde{R}(z)}{C(z)} y(t-k) = y^{\circ}(t)$$

time shift of k-steps:

$$\frac{B(z)E(z)}{C(z)} u(t) + \frac{\tilde{R}(z)}{C(z)} y(t) = \underline{y^{\circ}(t+k)}$$

we don't know the future of $y^{\circ}(\cdot)$
we assume a constant reference for the next k steps: $y^{\circ}(t+k) = y(t)$

$$\Longrightarrow u(t) = \frac{1}{B(z)E(z)} \left(C(z) y^{\circ}(t) - \tilde{R}(z) y(t) \right)$$

$$\hat{y}(t|t-1) = \frac{1}{1 + \frac{1}{3} z^{-1}} u(t-1) + \frac{5/6}{1 + \frac{1}{3} z^{-1}} y(t-1) = y^{\circ}(t)$$

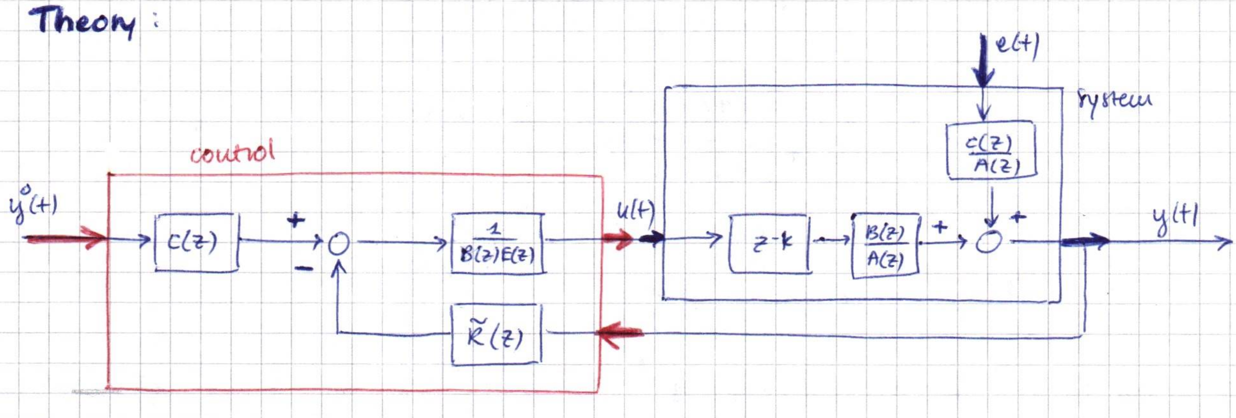
$$\frac{1}{1+\frac{1}{3}z^{-1}} u(t) + \frac{5/6}{1+\frac{1}{3}z^{-1}} y(t) = y^{\circ}(t)$$

$$\Longrightarrow \quad u(t) = \left(1+\tfrac{1}{3}z^{-1}\right) y^{\circ}(t) - \tfrac{5}{6} y(t)$$

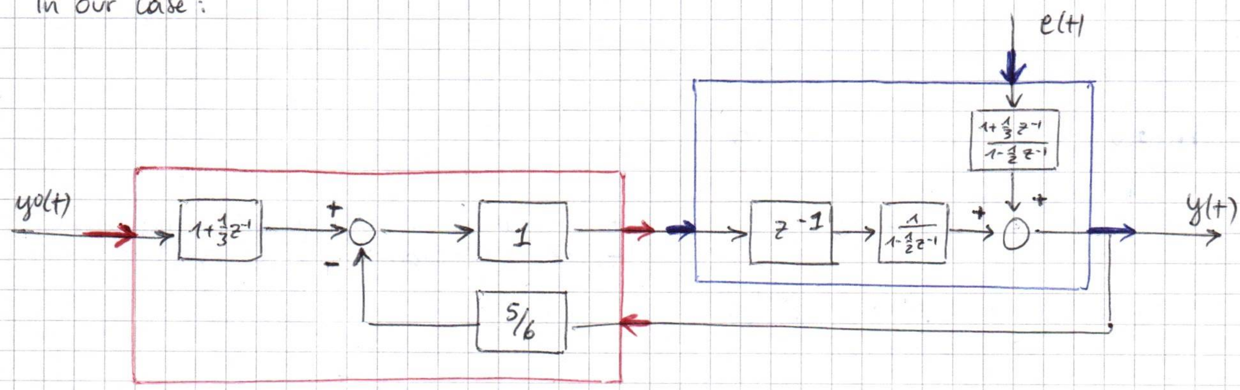
$$u(t) = 1 + \tfrac{1}{3} y^{\circ}(t-1) - \tfrac{5}{6} y(t)$$

d. Draw the closed loop block scheme.

Theory :



In our case :



e. Find the transfer function from
- $y^{\circ}(t)$ to $y(t)$
- $e(t)$ to $y(t)$

Method **1** : signal replace

$$y(t) = \frac{1}{1-\frac{1}{2}z^{-1}} u(t-1) + \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} e(t)$$

We replace the control law:

$$y(t) = \frac{1}{1-\frac{1}{2}z^{-1}} \left[\left(1+\tfrac{1}{3}z^{-1}\right) y^{\circ}(t) - \tfrac{5}{6} y(t) \right] z^{-1} + \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} e(t)$$

$$y(t) = \frac{\left(1+\frac{1}{3}z^{-1}\right) z^{-1}}{1-\frac{1}{2}z^{-1}} y^{\circ}(t) - \frac{5/6\, z^{-1}}{1-\frac{1}{2}z^{-1}} y(t) + \dots$$

$$y(t) \left(1 + \frac{5/6\, z^{-1}}{1-\frac{1}{2}z^{-1}} \right) = \frac{\left(1+\frac{1}{3}z^{-1}\right) z^{-1}}{1-\frac{1}{2}z^{-1}} y^{\circ}(t) + \dots$$

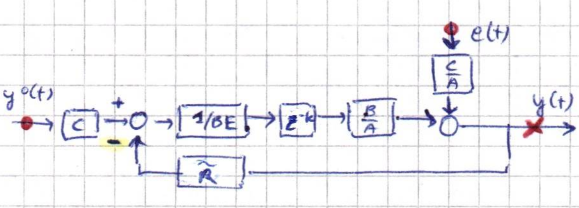
$$y(t) \left(\frac{\cancel{1+\frac{1}{3}z^{-1}}}{\cancel{1-\frac{1}{2}z^{-1}}} \right) = \frac{\cancel{\left(1+\frac{1}{3}z^{-1}\right)} z^{-1}}{\cancel{1-\frac{1}{2}z^{-1}}} y^{\circ}(t) + \frac{\cancel{1+\frac{1}{3}z^{-1}}}{\cancel{1-\frac{1}{2}z^{-1}}} e(t)$$

$$y(t) = z^{-1} y^{\circ}(t) + e(t)$$

$$\Longrightarrow \begin{cases} F_{y^{\circ}y}(z) = z^{-1} \\ F_{ey}(z) = 1 \end{cases}$$

Method 2: Block scheme

Theory:



$$W(z) = \frac{\text{DIRECT PATH}}{1 + \text{LOOP FUNCTION}}$$

$$W_{y^oy}(z) = \frac{C(z)\, z^{-k}\, \frac{B(z)}{A(z)} \cdot \frac{1}{B(z)E(z)}}{1 + \frac{1}{B(z)E(z)}\, z^{-k}\, \frac{B(z)}{A(z)}\, \tilde{R}(z)}$$

$$= \frac{\frac{C(z)}{A(z)E(z)}\, z^{-k}}{\frac{E(z)A(z) + z^{-k}\tilde{R}(z)}{E(z)A(z)}} = z^{-k}$$

$$W_{ey}(z) = \frac{\frac{C(z)}{A(z)}}{\frac{E(z)A(z) + z^{-k}\tilde{R}(z)}{E(z)A(z)}} = E(z)$$

In our case: $k = 1$, $E(z) = 1$ $\Longrightarrow$ $[\ldots]$ $\Longrightarrow$ $\begin{cases} W_{y^oy}(z) = z^{-1} \\ W_{ey}(z) = 1 \end{cases}$

f. Check the closed loop asymptotic stability.

Theory: Asymptotic stability

- loop function $L(z)$
- $\chi(z) = D_L + N_L$ (+ because the feedback is negative, − if the feedback is positive)

$$L(z) = \frac{z^{-k} B(z)}{B(z)E(z)A(z)}\, \tilde{R}(z)$$

$$B(z)E(z)A(z) + z^{-k}B(z)\tilde{R}(z) = B(z)\left(z^{-k}\tilde{R}(z) + E(z)A(z)\right) \stackrel{\downarrow}{=} B(z)\, C(z)$$

Asymp. stable $\Longleftrightarrow$ roots $(\chi(z))$ are inside the unit circle
(thanks to the assumptions $B(z)$ and $C(z)$ have all roots inside)

In our case:

$$\chi(z) = B(z)\, C(z) = 1\left(1 + \tfrac{1}{3} z^{-1}\right) = 0 \Longrightarrow z = -\tfrac{1}{3}$$

$|z| < 1 \longrightarrow$ the system is asymptotically stable

---

Exercise 2

Consider the system:

$$y(t) = \tfrac{1}{2} y(t-1) + u(t-2) + e(t-1) + 2e(t-2) \qquad e(t) \sim WN(0,1)$$

a. Find the MVC.

First check the assumptions:

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1}}\, u(t-2) + \frac{z^{-1} + 2z^{-2}}{1 - \frac{1}{2} z^{-1}}\, e(t)$$

$B(z) = 1$, $A(z) = 1 - \frac{1}{2} z^{-1}$, $C(z) = z^{-1} + 2z^{-2}$, $k = 2$

- $b_0 \neq 0$, $b_0 = 1 \implies \checkmark$
- $B(z) = 1 \implies$ minimum phase $\checkmark$
- $\dfrac{z^{-1} + 2z^{-2}}{1 - \frac{1}{2}z^{-1}} e(t) = \dfrac{z+2}{z^2 + \frac{1}{2}z} e(t)$
  - relative degree $= 0 \longrightarrow$ not true! $\times$

    $\tilde{e}(t) = e(t-1) \implies \tilde{e}(t) \sim WN(0,1)$

    $\dfrac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}} \tilde{e}(t) = \dfrac{z+2}{z-\frac{1}{2}} \tilde{e}(t) \quad \checkmark$
  - $C(z)$ and $A(z)$ are monic $\longrightarrow \checkmark$
  - $C(z)$ and $A(z)$ are coprime $\longrightarrow \checkmark$
  - $C(z)$ and $A(z)$ have all the roots strictly inside the unit circle $\times$

    $\implies$ all pass filter trick:

    $$\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}} \tilde{e}(t) = \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}} \underbrace{\frac{1+\frac{1}{2}z^{-1}}{1+2z^{-1}} \cdot 2}_{\text{all pass filter}} \tilde{e}(t)$$

    $$\underbrace{\frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} \, 2\tilde{e}(t)}_{} := \eta(t) \sim WN(0,4)$$

    this is in a canonical form $\implies \checkmark$

Now we compute the k-step predictor $(k=2)$

$$
\begin{array}{rrr|l}
1 & +\frac{1}{2}z^{-1} & & 1-\frac{1}{2}z^{-1} \\
-1 & +\frac{1}{2}z^{-1} & & \hline 1 + z^{-1} \\
\hline
/ & 1z^{-1} & & \\
 & -z^{-1} & +\frac{1}{2}z^{-2} & \\
\hline
 & / & \frac{1}{2}z^{-2} & \\
 & & \tilde{R}(z)z^{-k} &
\end{array}
$$

$\tilde{R}(z) = \frac{1}{2}$

$$\implies \hat{y}(t|t-2) = \frac{1+z^{-1}}{1+\frac{1}{2}z^{-1}} u(t-2) + \frac{\frac{1}{2}}{1+\frac{1}{2}z^{-1}} y(t-2)$$

$$\hat{y}(t|t-2) = y^o(t)$$

Shift in time domain:

$$\frac{1+z^{-1}}{1+\frac{1}{2}z^{-1}} u(t) + \frac{\frac{1}{2}}{1+\frac{1}{2}z^{-1}} y(t) = y^o(t+k) = y^o(t)$$

$$\implies u(t) = \frac{1}{1+z^{-1}} \left(\left(1+\frac{1}{2}z^{-1}\right) y^o(t) - \frac{1}{2} y(t) \right)$$

b. Draw the closed loop block scheme. (for us)

c. Check the closed loop asymptotic stability. (with canonical repr.!)

$$L(z) = \frac{z^{-k} B(z) \tilde{R}(z)}{B(z) E(z) A(z)}$$

$$\chi(z) = B(z)E(z)A(z) + z^{-k}B(z)\tilde{R}(z) \overset{!}{=} B(z)C(z)$$

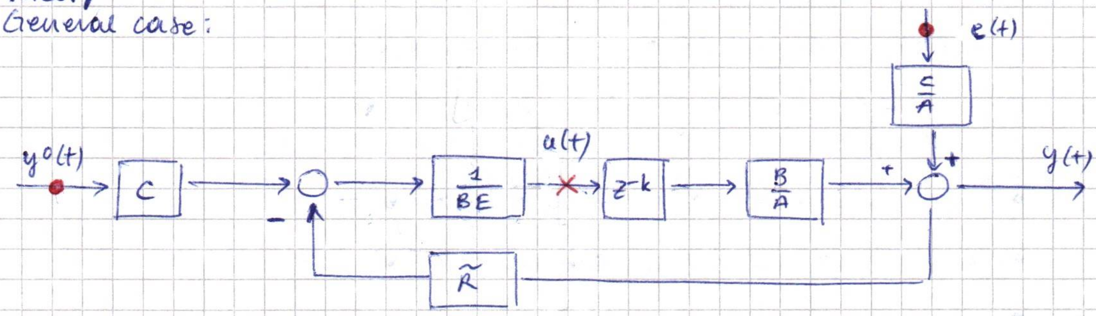
$\chi(z) = 1\left(1+\frac{1}{2}z^{-1}\right) = 0 \implies z = -\frac{1}{2}$

Since it's strictly inside the unit circle $\implies$ the closed loop is asymp. stable

d. Compute the transfer functions from:
- y° to u
- η to u

Theory:

General case:



$$W_{y^\circ u}(z) = \frac{\frac{C(z)}{B(z)E(z)}}{\frac{E(z)A(z)+z^{-k}\tilde{R}(z)}{E(z)A(z)}} = \frac{A(z)}{B(z)}$$

$$W_{eu}(z) = \frac{-\frac{C(z)\tilde{R}(z)}{B(z)E(z)A(z)}}{\frac{E(z)A(z)+\tilde{R}(z)z^{-k}}{E(z)A(z)}} = -\frac{\tilde{R}(z)}{B(z)}$$

$$W_{y^\circ u}(z) = 1 - \frac{1}{2}z^{-1}$$

$$W_{\eta u}(z) = -\frac{1}{2}$$

NOTES ON: All pass filter:

$\frac{1+az^{-1}}{1+\frac{1}{a}z^{-1}}\frac{1}{a} = T(z) =$ all pass filter: it doesn't change the spectrum of the process

RECURSIVE IDENTIFICATION

ARX IDENTIFICATION OUTLINE

- least squares (recap)
- Recursive least squares (I - II - III form) } new
- Recursive least squares with forgetting factor } new

ARX system:

$$\underline{y(t)} = \frac{B(z)}{A(z)} u(t-1) + \frac{1}{A(z)} e(t) \qquad e(t) \sim WN(m_e, \lambda_e^2)$$

$$B(z) = b_0 + b_1 z^{-1} + \dots + b_p z^{-p}$$

$$A(z) = 1 + a_1 z^{-1} + \dots + a_m z^{-m}$$

$$\underline{y(t)} = -a_1 y(t-1) - \dots - a_m y(t-m) + b_0 u(t-1) + \dots + b_p u(t-p-1) + e(t)$$

$$\vartheta = \begin{bmatrix} a_1 \\ \vdots \\ a_m \\ b_0 \\ \vdots \\ b_p \end{bmatrix} \qquad \varphi(t) = \begin{bmatrix} -y(t-1) \\ \vdots \\ -y(t-m) \\ u(t-1) \\ \vdots \\ u(t-p-1) \end{bmatrix} \implies \underline{y(t)} = \varphi(t)^T \vartheta + e(t)$$

Objective: (of identification) Identify ϑ starting from an available dataset:

$$\begin{Bmatrix} u(1), \dots, u(N) \\ y(1), \dots, y(N) \end{Bmatrix} \longrightarrow \boxed{\text{ID algorithm}} \longrightarrow \hat{\vartheta}$$

There are different ID algorithms:

LEAST SQUARES

$$\hat{\theta}_N = \arg\min_{\theta} \left\{ J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \underbrace{\left(y(t) - \hat{y}(t|t-1, \theta)\right)^2}_{\text{predictor based on the model}} \right\} \longrightarrow \text{PEM (Prediction Error Method)}$$

Model predictor: $\hat{y}(t|t-1, \theta)$

$$y(t) = \underbrace{\varphi(t)^T \vartheta}_{\text{completely known at time } t-1} + \underbrace{e(t)}_{\text{completely unpredictable at time } t-1}$$

$$\implies \hat{y}(t|t-1, \theta) = \varphi(t)^T \vartheta \longrightarrow \text{optimal 1-step predictor}$$

$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \left(y(t) - \varphi(t)^T \theta\right)^2 \longrightarrow \text{quadratic in } \vartheta$$

$\implies$ it is possible to find $\hat{\theta}$ (the minimizer) explicitly

$\hat{\theta}_N$ is such that $\left.\frac{\partial J_N(\theta)}{\partial \theta}\right|_{\theta = \hat{\theta}_N} = 0$

$$\longrightarrow \boxed{\begin{aligned} \hat{\theta}_N &= S(N)^{-1} \sum_{t=1}^{N} \varphi(t) y(t) \\ S(N) &= \sum_{t=1}^{N} \varphi(t) \varphi(t)^T \end{aligned}} \quad \textbf{LS}$$

but this algorithm has a drawback!

Drawback:

$$\begin{Bmatrix} u(1), \dots, u(N) \\ y(1), \dots, y(N) \end{Bmatrix} \xrightarrow{\text{new measurement available}} \begin{Bmatrix} u(1), \dots, u(N), u(N+1) \\ y(1), \dots, y(N), y(N+1) \end{Bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\hat{\theta}_N \quad \overset{?}{\dashrightarrow} \quad \hat{\theta}_{N+1}$$

we're not using $\hat{\theta}_N$,
we're losing all the computations that we have done.
it's necessary to repeat all the computations

Solution : recursive least squares

RECURSIVE LEAST SQUARES

$$\begin{matrix} \hat{\theta}_{N-1} \longrightarrow \\ \varphi(N) \longrightarrow \end{matrix} \boxed{\quad RLS \quad} \longrightarrow \hat{\theta}_N$$

Advantages:
- lower computational efforts
- we save memory allocation

RECURSIVE LEAST SQUARES: I form

Objective: $\hat{\theta}_N = f(\hat{\theta}_{N-1}, \varphi(N))$

$$\hat{\theta}_N = S(N)^{-1} \sum_{t=1}^{N} \varphi(t)\, y(t) \qquad (1)$$

From equation (1): $\displaystyle\sum_{t=1}^{N} \varphi(t)\, y(t) = \hat{\theta}_N\, S(N)$ (2)

$$\hat{\theta}_{N-1} = S(N-1)^{-1} \sum_{t=1}^{N-1} \varphi(t)\, y(t) \qquad (3)$$

From equation (3): $\displaystyle\sum_{t=1}^{N-1} \varphi(t)\, y(t) = \hat{\theta}_{N-1}\, S(N-1)$ (4)

$$\sum_{t=1}^{N} \varphi(t)\, y(t) = \left[\sum_{t=1}^{N-1} \varphi(t)\, y(t) \right] + \varphi(N)\, y(N) \qquad (5)$$

$$\longrightarrow \sum_{t=1}^{N} \varphi(t)\, y(t) = S(N-1)\, \hat{\theta}_{N-1} + \varphi(N)\, y(N) \qquad (6)$$

Notice that (6) = (2)

$$\longrightarrow S(N)\, \hat{\theta}_N = S(N-1)\, \hat{\theta}_{N-1} + \varphi(N)\, y(N) \qquad (7)$$

We need to find an expression of $S(N-1)$

$$\begin{aligned} S(N) &= \sum_{t=1}^{N} \varphi(t)\, \varphi(t)^T \\ &= \left[\sum_{t=1}^{N-1} \varphi(t)\, \varphi(t)^T \right] + \varphi(N)\, \varphi(N)^T \\ &= S(N-1) + \varphi(N)\, \varphi(N)^T \end{aligned}$$

$$\longrightarrow S(N-1) = S(N) - \varphi(N)\, \varphi(N)^T \qquad (8)$$

$$\longrightarrow S(N)\,\hat{\vartheta}_N = \left(S(N) - \varphi(N)\varphi(N)^T\right)\hat{\vartheta}_{N-1} + \varphi(N)\,y(N)$$

$$= S(N)\,\hat{\vartheta}_{N-1} - \varphi(N)\varphi(N)^T\hat{\vartheta}_{N-1} + \varphi(N)\,y(N)$$

$$\longrightarrow \hat{\vartheta}_N = \hat{\vartheta}_{N-1} + \underbrace{S(N)^{-1}\varphi(N)}_{\substack{\text{gain}\\ K(N)}}\underbrace{\left[y(N) - \varphi(N)^T\hat{\vartheta}_{N-1}\right]}_{\substack{:=\varepsilon(N)\\ \text{(prediction error)}}}$$

$$\Longrightarrow \boxed{\begin{cases} \hat{\vartheta}_N = \hat{\vartheta}_{N-1} + K(N)\,\varepsilon(N) \\ K(N) = S(N)^{-1}\varphi(N) \\ \varepsilon(N) = y(N) - \varphi(N)^T\hat{\vartheta}_{N-1} \\ S(N) = S(N-1) + \varphi(N)\varphi(N)^T \end{cases}}$$

RLS - I

Drawback:

$$S(N) - S(N-1) = \varphi(N)\varphi(N)^T \geq 0 \longrightarrow S(N) \xrightarrow{N\to\infty} \infty$$

$S(N)$ is a matrix which continuosly increases, it tends to saturate the numerical precision of the digital computing unit.

Solution: recursive least squares II form

RECURSIVE LEAST SQUARES : II form

Trick: normalization w.r.t. N

$$\longrightarrow S(N) = S(N-1) + \varphi(N)\varphi(N)^T \longrightarrow R(N) := \frac{1}{N}S(N)$$

$$\longrightarrow R(N) = \frac{1}{N}\frac{N-1}{N-1}S(N-1) + \frac{1}{N}\varphi(N)\varphi(N)^T$$

$$= \frac{N-1}{N}R(N-1) + \frac{1}{N}\varphi(N)\varphi(N)^T$$

$$\boxed{\begin{cases} \hat{\vartheta}_N = \hat{\vartheta}_{N-1} + K(N)\,\varepsilon(N) \\ K(N) = \frac{1}{N}R(N)^{-1}\varphi(N) \\ \varepsilon(N) = y(N) - \varphi(N)^T\hat{\vartheta}_{N-1} \\ R(N) = \frac{N-1}{N}R(N-1) + \frac{1}{N}\varphi(N)\varphi(N)^T \end{cases}}$$

RLS - II

Drawback: Matrix inversion at each time step $(R(N)^{-1})$

Solution: recursive least squares: III form

RECURSIVE LEAST SQUARES : III form

Lemma of matrix inversion:

Consider 4 matrices: F, G, H, K s.t.:

- $F + GHK$ makes sense (dimensionally)
- $F, H, F+GHK$ squared and invertible

$$\longrightarrow (F+GHK)^{-1} = F^{-1} - F^{-1}G\left(H^{-1} + KF^{-1}G\right)^{-1}KF^{-1}$$

In our case: $S(N)^{-1} = [\underbrace{S(N-1)}_{F} + \underbrace{\varphi(N)}_{G}\ \underbrace{1}_{H}\ \underbrace{\varphi(N)^T}_{K}]^{-1}$

$$\Longrightarrow S(N)^{-1} = \underbrace{S(N-1)^{-1}}_{\text{computed at previous step}} - S(N-1)^{-1}\varphi(N)\underbrace{[1+\varphi(N)^T S(N-1)^{-1}\varphi(N)]^{-1}}_{\text{is a scalar}}\varphi(N)^T S(N-1)^{-1}$$

define: $V(N) = S(N-1)^{-1}$

$$\longrightarrow V(N) = V(N-1) - \frac{V(N-1)\varphi(N)\varphi(N)^T V(N-1)}{1+\varphi(N)^T V(N-1)\varphi(N)}$$

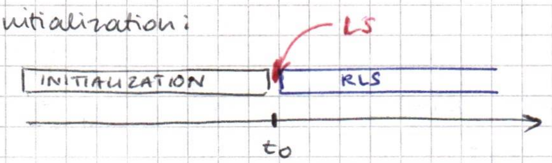
- $V(N)$ does not tend to zero
- it is no more necessary to invert a matrix, instead a scalar inversion is required

$$\begin{cases} \hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\varepsilon(N) \\ \varepsilon(N) = y(N) - \varphi(N)^T\hat{\theta}_{N-1} \\ K(N) = V(N)\varphi(N) \\ V(N) = V(N-1) - \dfrac{V(N-1)\varphi(N)\varphi(N)^T V(N-1)}{1+\varphi(N)^T V(N-1)\varphi(N)} \end{cases}$$

RLS - III

Remark on initialization: RLS is a rigorous version of **LS** (not an approximation) provided a correct initialization.

Correct initialization:



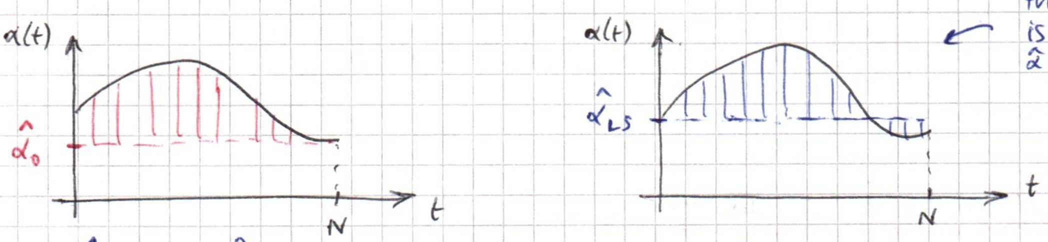
- Collect a first set of data till t_0
- compute $\hat{\theta}_{t_0}$ and $S(t_0)$ with LS
- use RLS to update $\hat{\theta}_t$ and $S(t)$

In practice:

$\hat{\theta}_0 = 0$, $S(0) = I$ $\Longrightarrow$ the error due to "wrong" initialization will expire with time

RECURSIVE LEAST SQUARES WITH FORGETTING FACTOR

Consider a time varying parameter:



this is because $\hat{\alpha}_{LS}$ is an average of $\hat{\alpha}_{LS}$ in the time

We want $\hat{\alpha}_0$, not $\hat{\alpha}_{LS}$ $\Rightarrow$ we have to forget the past!

$\hat{\alpha}_0$ is the correct estimation at time N but it doesn't minimize

$$J_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}(y(t) - \hat{y}(t|t-1,\alpha))^2$$

because it considers the entire time history of the system.

In order to identify a time-varying parameter, the RLS must be forced to forget old data.
(so the newest data must be more important than the oldest).

The solution is provided by the minimization of J_N:

$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \rho^{N-t} \left(y(t) - \hat{y}(t|t-1, \theta)\right)^2$$

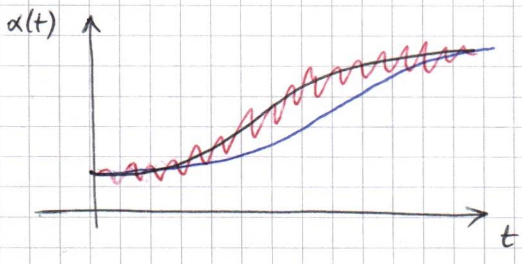
$0 \leq \rho \leq 1$
is called FORGETTING FACTOR

$\rho = 1 \rightarrow$ previous formulation
$\rho < 1 \rightarrow$ the old data have less importance w.r.t. new data

$$\begin{cases} \hat{\vartheta}_{N+1} = \vartheta_N + K(N)\,\varepsilon(N) \\ K(N) = S(N)^{-1}\varphi(N) \\ \varepsilon(N) = y(N) - \varphi(N)\,\hat{\vartheta}_{N-1} \\ S(N) = \rho\, S(N-1) + \varphi(N)\varphi(N)^T \end{cases}$$

RLS with forgetting factor

Remark on the choice of ρ:



$\rho \ll 1$ (we forget fastly) :
- high tracking speed
- low precision

$\rho \rightarrow 1$:
- low tracking speed
- high precision

Usually we chose $\rho \sim 0.95$