MEASURE

$(X, \mathcal{A})$ measurable space, $\mu : \mathcal{A} \to [0, +\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) finite additive : $\forall \{E_k\}_{k=1}^{n} \subseteq \mathcal{A}$ disjoint :

$$\mu(\cup_{k=1}^{n} E_k) \leq \sum_{k=1}^{n} \mu(E_k)$$

(iii) continuous along increasing sequences :

$\forall \{E_n\}_n \subseteq \mathcal{A}$ s.t. $E_n \subseteq E_{n+1}$:

$$\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$$

$\Longrightarrow$ μ is a measure

MEASURABLE FUNCTIONS $(\mathcal{M}(X, \mathcal{A}))$

$f : (X, \mathcal{A}) \to (X', \mathcal{A}')$ measurable $\iff f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{A}'$

- continuous
- $\sup f_n$, $\inf f_n$
- $\limsup f_n$, $\liminf f_n$, $\lim f_n$
- $\max\{f, g\}$, $\min\{f, g\}$
- $f_\pm$, $|f|$ $\quad (f \in \mathcal{M}(X, \mathcal{A}) \iff f_\pm \in \mathcal{M}(X, \mathcal{A})$, $\quad f \in \mathcal{M}(X, \mathcal{A}) \underset{\not\Longleftarrow}{\Longrightarrow} |f| \in \mathcal{M}(X, \mathcal{A}))$
- $f + g$, $f \cdot g$
- $A \in \mathcal{A} \iff \mathbb{1}_A \in \mathcal{M}(X, \mathcal{A})$
- $f \in \mathcal{M}(X, \mathcal{A}) \iff \{f > \alpha\} \in \mathcal{A} \quad \forall \alpha \in \mathbb{R}$
- BOREL MEASURABLE :

  $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ Borel measurable

  $\iff f^{-1}(E) \in \mathcal{B} \quad \forall E \in \mathcal{B}' \,/\, \forall E \in \mathcal{C}' = \{E \subseteq X' : E \text{ open}\}$
- LEBESGUE MEASURABLE :

  $f : (X, \mathcal{L}) \to (X', \mathcal{B}')$ Lebesgue measurable

  $\iff f^{-1}(E) \in \mathcal{L} \quad \forall E \in \mathcal{B}' \,/\, \forall E \in \mathcal{C}' = \{E \subseteq X' : E \text{ open}\}$
- $f : X \to \mathbb{R} \iff f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

  measurable $\iff f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{B}(\mathbb{R})$

MCT

$(X, \mathcal{A}, \mu)$ measure space.

$\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$, $f \in \mathcal{M}_+(X, \mathcal{A})$: (a.e.)

(i) $f_n \leq f_{n+1} \quad \forall n \in \mathbb{N}$ in X (a.e.)

(ii) $f_n \xrightarrow[n\to\infty]{} f$ pointwise in X (a.e.)

$$\Longrightarrow \int_X f_n \, d\mu \xrightarrow[n\to\infty]{} \int_X f \, d\mu$$

Corollary

$(X, \mathcal{A}, \mu)$ measure space.

$\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$.

$$\Longrightarrow \int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, d\mu\right)$$

Fatou's

$(X, \mathcal{A}, \mu)$ measure space.
$\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$.

$$\Longrightarrow \int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \left(\int_X f_n \, d\mu\right)$$

DCT

$(X, \mathcal{A}, \mu)$ measure space.
$\{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$, $f \in \mathcal{M}(X, \mathcal{A})$:

$$f_n \xrightarrow[n\to\infty]{} f \quad \text{a.e. in } X$$

$\exists g \in L^1(X, \mathcal{A}, \mu)$:

$$|f_n| \leq g \quad \text{a.e. in } X \quad \forall n \in \mathbb{N}$$

$$\Longrightarrow \begin{cases} 1. & f_n, f \in L^1 \quad \forall n \\ 2. & \int_X |f_n - f| \, d\mu \xrightarrow[n\to\infty]{} 0 \\ 3. & \int_X f_n \, d\mu \xrightarrow[n\to\infty]{} \int_X f \, d\mu \end{cases}$$

Corollary

$(X, \mathcal{A}, \mu)$ measure space.
$\{f_n\}_n \subseteq \mathcal{M}(X, \mathcal{A})$:

$$\sum_{n=1}^{\infty} \left(\int_X |f_n| \, d\mu\right) < \infty$$

$$\Longrightarrow \begin{cases} 1. & \sum_{n=1}^{\infty} f_n \text{ conv. a.e. in } X \\ 2. & \int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, d\mu\right) \end{cases}$$

Conv. of $f_n \to f$ in L^p?

$$\Longrightarrow \begin{cases} f_n \longrightarrow f \text{ a.e.} \\ |f_n - f|^p \leq g \in L^1 \end{cases} \quad (\text{or: } |f_n - f|^p \leq |g|^p \in L^1)$$

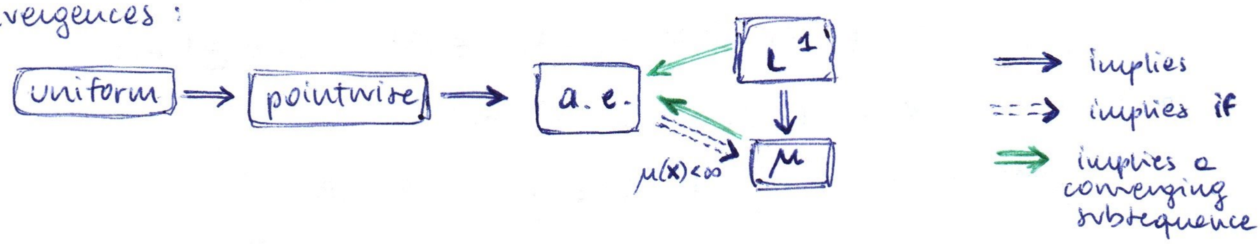
$\Longrightarrow$ MCT : $f_n \to f$ in L^p

LIMITS (TRICKS)

- $\lim\limits_{n\to\infty} \int_0^n f(t)\,dt \Longrightarrow f_n(t) = f(t) \cdot \mathbb{1}_{[0,n]}(t)$

  monotone ? Y $\Longrightarrow$ MCT
  N $\Longrightarrow$ DCT

- convergences :



BV

- f monotone $\Longrightarrow f \in BV$
  with total variation $= |f(b) - f(a)|$

- f **not** bounded $\Longrightarrow f \notin BV$

- $f \notin BV$?
  - $\exists f'$ a.e. but $f' \notin L^1 \Longrightarrow f \notin BV$
  - Find a partition $\{x_k\}_{k=0}^{n}$ s.t.

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \geq (\cdot\cdot) \xrightarrow[n\to\infty]{} \infty$$

$V_a^b(f) \geq \sum_{k=0}^{n-1} |\cdot\cdot| \quad \forall n \in \mathbb{N}$ and so:

$$\left(V_a^b(f) \geq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \xrightarrow[n\to\infty]{} \infty \right)$$

AC

- $f \notin AC$?
  - $f \notin C$ (continuous)
  - $f \notin BV$ ($AC \subsetneq BV$: Vitali function $\in BV, \notin AC$)

- $f \in \text{Lip} \Longrightarrow f \in AC$

- $f \in AC \Longleftrightarrow$
  0. $f \in C$ (f is continuous)
  1. f differentiable a.e. ($\exists f'$ a.e.) in $[a,b]$
  2. $f' \in L^1([a,b])$
  3. $f(x) = f(a) + \int_a^x f'(t)\,dt \quad \forall x \in [a,b]$

To prove 3.

If $\underline{f \in C^1([\xi,b])}$ $\forall \xi \in (a,b]$:

$$f(x) = f(\xi) + \int_\xi^x f'(t)\,dt \quad \forall a < \xi \le x \le b$$

If $\underline{f' \in L^1([a,b])}$, as $\xi \to a$:

$$f(x) = f(a) + \int_a^x f'(t)\,dt \quad \forall a \le x \le b$$

To prove 2.

$$|f'(x)| \le g(x) \in L^1 \Longrightarrow f' \in L^1$$

GENERAL HINTS

- $f \notin BV$:

$$f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases} \qquad x_n = \frac{1}{\left(n\pi + \frac{\pi}{2}\right)^{1/\beta}}$$

$(x_0 = 1,\ x_N = 0,\ n = 1, .., N-1)$

$$f(x_n) = \frac{(-1)^n}{\left(n\pi + \frac{\pi}{2}\right)^{\alpha/\beta}}$$

$$|f(x_{n+1}) - f(x_n)| = .. = \left(\frac{2}{\pi}\right)^{\alpha/\beta}\left(\frac{1}{(2n+3)^{\alpha/\beta}} + \frac{1}{(2n+1)^{\alpha/\beta}}\right) \ge \left(\frac{2}{\pi}\right)^{\alpha/\beta}\frac{1}{(2n+1)^{\alpha/\beta}}$$

$$\sum_{n=1}^{N-2} |f(x_{n+1}) - f(x_n)| \ge \left(\frac{2}{\pi}\right)^{\alpha/\beta} \sum_{n=1}^{N-2} \frac{1}{(2n+1)^{\alpha/\beta}} \xrightarrow[N\to\infty]{} \infty \quad \left(\frac{\alpha}{\beta} \le 1\right)$$

$$f(x) = \begin{cases} x^\alpha \cos\left(\frac{1}{x^\beta}\right) & x \in (0,1] \\ 0 & x = 0 \end{cases} \qquad x_n = \frac{1}{(n\pi)^{1/\beta}}$$

$(x_0 = 1,\ x_N = 0,\ n = 1, .., N-1)$

$$\sum_{n=1}^{N-2} |f(x_{n+1}) - f(x_n)| \ge \left(\frac{1}{\pi}\right)^{\alpha/\beta} \sum_{n=1}^{N-2} \frac{1}{n^{\alpha/\beta}} \xrightarrow[N\to\infty]{} \infty \quad \left(\frac{\alpha}{\beta} \le 1\right)$$

$$f(x) = \begin{cases} e^{1/x} & x \in (0,1] \\ 0 & x = 0 \end{cases} \qquad x_n = \frac{1}{n}$$

$(x_0 = 1,\ x_N = 0,\ n = 1, .., N-1)$

$$\sum_{n=1}^{N-2} |f(x_{n+1}) - f(x_n)| = \sum_{n=1}^{N-2} |e^{n+1} - e^n| = \sum_{n=1}^{N-2} (e^{n+1} - e^n) = e^{N-1} - e \xrightarrow[N\to\infty]{} \infty$$

- $\sum_{n=0}^{+\infty} (-1)^n \frac{1}{(n+1)^2} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{12}$ $\quad\quad$ $\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\log(1-x)$

  $\sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

  $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$

  $\sum_{n=0}^{+\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

$\sin(\bullet) \sim \bullet$
$\ln(1+\bullet) \sim \bullet$
$e^{\bullet} - 1 \sim \bullet$
$1 - \cos(\bullet) \sim \frac{1}{2} \bullet^2$
$\tan(\bullet) \sim \bullet$
$\mathrm{atan}(\bullet) \sim \bullet$

- $\int_0^1 x^k \log(x)\, dx = -\frac{1}{(1+k)^2}$

  $\log(1+x) \le x \quad \forall x \in (0, \infty)$

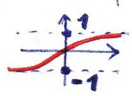
  $\int \log(t)\, dt = t(\log(t) - 1)$

  $\int \log(t+k)\, dt = (t+k)\log(t+k) - t$ $\quad\quad$ $\int_0^{\infty} x^n e^{-x} dx = \Gamma(n+1) = n!$

- $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$  $(\sinh(x))' = \cosh(x)$

  $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ 

$(\cosh(x))' = \sinh(x)$

  $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ 

$(\tanh(x))' = 1 - \tanh^2(x)$

  $\coth(x) = \frac{1}{\tanh(x)}$ $\quad\quad$ $(\coth(x))' = 1 - \coth^2(x)$

  $(\sin(x))' = \cos(x)$

  $(\cos(x))' = -\sin(x)$

  $(\mathrm{atan}(x))' = \frac{1}{1+x^2}$ $\quad\quad$ $\left((\arcsin(x))' = \frac{1}{\sqrt{1-x^2}},\ (\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}\right)$

- $f \in C([a,b]) \implies f \in L^1([a,b]) \iff \begin{cases} f \in \mathcal{M}(X,\mathcal{A}) \\ \int_X |f|\, d\mu < \infty \end{cases}$

  $(f \in C([a,b]) \implies f$ bounded $\implies \exists M > 0 : |f| \le M$
  $\implies \int_{[a,b]} |f|\, d\mu \le M \mu([a,b]) < \infty)$

  $\implies f \in L^\infty([a,b])$

  $(f \in C([a,b]) \implies f$ bounded $\implies \exists M > 0 : |f| \le M$
  $\implies \operatorname{ess\,sup}_{[a,b]} |f| \le M < \infty)$

- **HINT:** if it's asked L^1 convergence and then $\lim_{n\to\infty} \int \dots$ then we probably have to use convergence results. on the $\lim_n$. if $\lim_{n\to\infty} \int$ is asked before L^1 convergence then we probably have to evaluate $\lim_n \int ..$ manually.
  (moreover, in the second case probably we do **not** have conv in L^1)

- **HINT:** "compute $V_a^b(f)$" $\equiv$ f monotone (almost always)

- **HINT:** "$f \in BV$? By def." $\equiv$ $f \notin BV$ (almost always)

L^p

- $x_n \xrightarrow[n\to\infty]{} x_0$ in ℓ^p ?
  1. Find x_0 (pointwise limit)
  2. check that $x_0 \in \ell^p$
  3. prove that $x_n \longrightarrow x_0$ in ℓ^p

- INTERPOLATION INEQUALITIES:
  - $f \in L^p(X) \cap L^q(X) \quad 1 \le p < q \le \infty$

$$\Longrightarrow \begin{cases} f \in L^r(X) & \forall r \in [p,q] \\ \|f\|_r \le \|f\|_p^{1-t} \|f\|_q^{t} & \left(t: \frac{1}{r} = \frac{t}{q} + \frac{1-t}{p}\right) \ (q<\infty) \\ \|f\|_r \le \|f\|_p^{\frac{p}{r}} \|f\|_\infty^{1-\frac{p}{r}} & (q=\infty) \end{cases}$$

- $f \in C(I)$, I compact $\Longrightarrow$ $\operatorname{ess\,sup}_I |f| = \sup_I |f|$

(useful when $f_n \not\to f_0$ in L^∞ $\left(\|f_n - f_0\|_\infty = \sup_I |f_n - f_0| \ge (\cdot\cdot) \longrightarrow \infty \right)$)

- $f_n \not\to f_0$ in L^∞ but we cannot do $\|f_n - f_0\|_\infty \ge (\cdot\cdot) \to \infty$?

if $\underline{\lambda(X) < \infty}$ $\Longrightarrow$ $[\, L^\infty(X) \subseteq L^p(X) \quad p \in [1,\infty] \,]$

$\Longrightarrow$ $[\, f_n \to f_0$ in $L^\infty \Longrightarrow f_n \to f_0$ in $L^p \quad p \in [1,\infty] \,]$

$\Longrightarrow$ If $\exists p \in [1,\infty)$: $f_n \not\to f_0$ in $L^p \Longrightarrow f_n \not\to f_0$ in L^∞

- HOLDER : $\|fg\|_1 \le \|f\|_p \|g\|_q$

  MINKOWSKI : $\|f+g\|_p \le \|f\|_p + \|g\|_p$

LINEAR OPERATORS $\quad T: X \to Y$

- T bounded $\Longleftrightarrow$ $\exists M > 0$: $\|T(x)\|_Y \le M \|x\|_X \quad \forall x \in X$

  $\|T\|_{\mathcal{L}} \le M$

- $\|T\|_{\mathcal{L}}$?
  1. $\|T(x)\|_{\mathcal{Y}} \le M \|x\|_X \Longrightarrow \|T\|_{\mathcal{L}} \le M$

  2. a. $\underline{\exists x \in X}$: $\|x\|_X = 1$, $\|T(x)\|_Y = M$

     $\Longrightarrow \|T\|_{\mathcal{L}} = M$

  2. b. $\underline{\exists \{x_n\}_n \subseteq X}$: $\|x_n\|_X = 1$, $\|T(x_n)\|_Y \ge M$

     $\Longrightarrow \|T\|_{\mathcal{L}} = M$

     $\left(\|T(x_n)\|_Y \ge (\cdot\cdot) \xrightarrow[n\to\infty]{} M \right)$

- $T: X \to Y$ injective $\Longleftrightarrow$ $[\, \forall x, y \in X : x \neq y \Longrightarrow T(x), T(y) \in Y : T(x) \neq T(y) \,]$

  surjective $\Longleftrightarrow$ $[\, \forall y \in Y \ \exists x \in X : T(x) = y \,]$

  (How to procede? Solve $T(x) = y$ and check $x \in X$)

- $\|T(x)\|_X \leq \|T\|_{\mathcal{L}} \|x\|_X$

- $\left.\begin{array}{l}(X, \|\cdot\|_X) \text{ normed space} \\ (Y, \|\cdot\|_Y) \text{ Banach space}\end{array}\right\} \Longrightarrow (\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}})$ Banach

- $\left.\begin{array}{l}(X, \|\cdot\|_X) \text{ Banach} \\ (Y, \|\cdot\|_Y) \text{ Banach}\end{array}\right\} \Longrightarrow \left[\left.\begin{array}{l}\{T_n\}_n \subset \mathcal{L}(X,Y) \\ \exists T(x) = \lim_{n\to\infty} T_n(x) \ \forall x\end{array}\right\} \Longrightarrow T \in \mathcal{L}(X,Y)\right]$

- $f \in L^p \Longrightarrow \forall T \in (L^p)^* \ \exists! g \in L^q : \left\{\begin{array}{l} T_g(f) = \int_X fg\, d\mu \\ \|T_g\|_{\mathcal{L}} = \|g\|_q \end{array}\right.$
  $p \in [1,\infty)$

- $x \in \ell^p \Longrightarrow \forall T \in (\ell^p)^* \ \exists! y \in \ell^q : \left\{\begin{array}{l} T(x) = \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \\ \|T\|_{\mathcal{L}} = \|y\|_q \end{array}\right.$
  $p \in [1,\infty)$

WEAK CONV.

- $f_n \rightharpoonup f$ in $L^p \iff L(f_n) \to L(f) \quad \forall L \in (L^p)^*$
  $\iff \int_X f_n g\, d\mu \to \int_X fg\, d\mu \quad \forall g \in L^q$

- $x_n \rightharpoonup x$ in $\ell^p \iff L(x_n) \to L(x) \quad \forall L \in (\ell^p)^*$
  $\iff \sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \to \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \quad \forall y \in \ell^q$

- THEOREM:
  $\Omega \subseteq \mathbb{R}^n$ open, $f_0 \in L^p(\Omega)$, $\{f_n\}_n \subseteq L^p(\Omega)$ bounded $(p \in (1,\infty))$

  $\underline{\underline{1. = 2.}} \Longrightarrow \left\{\begin{array}{l} 1.\ f_n \rightharpoonup f_0 \text{ in } L^p \quad (\int_\Omega f_n g\, dx \to \int_\Omega f_0 g\, dx \quad \forall g \in L^q) \\ 2.\ \forall \phi \in C_c^\infty(\Omega) : \int_\Omega f_n \phi\, dx \longrightarrow \int_\Omega f_0 \phi\, dx \end{array}\right.$

- $L^p((a,b))$: $f_n \rightharpoonup 0$? $(p \in (1,\infty))$ $\underline{\{f_n\}_n \text{ bounded}}$

  1. (theorem) : $f_n \rightharpoonup f$ in $L^p \iff \boxed{\int_a^b f_n \phi\, dx} \to \int_a^b f \phi\, dx \quad \forall \phi \in C_c^\infty((a,b$

  2. $\int_a^b f_n \phi\, dx$ by integration by parts:

  $$\int_a^b f_n \phi\, dx = \underbrace{\cancel{[F_n \cdot \phi]_a^b}}_{\substack{\phi(a) = \phi(b) = 0 \\ supp(\phi) = [a',b'] \subset (a,b)}} - \int_a^b F_n \cdot \phi'\, dx$$

  3. $$\left|\int_a^b f_n \phi\, dx\right| \leq \underbrace{\boxed{\max_{x \in [a,b]} |\phi'(x)|}}_{\substack{\phi' \text{ is smooth} \\ \text{and compactly supp.} \\ \text{so it is bounded}}} \cdot \int_a^b |F_n|\, dx \leq \square(\cdot\cdot) \xrightarrow[n\to\infty]{} 0$$

$\longrightarrow f_n \rightharpoonup 0$ in $L^p((a,b))$, $p \in (1,\infty)$.

COMPACT OP. & HILBERT

- T compact ? $(T: X \to Y)$

- $\dim(\mathrm{Im}(T)) < \infty \Rightarrow T$ finite rank $\Rightarrow T$ compact

- $\exists \{T_N\}_N \subseteq K(X,Y)$:

  (usually $T_N = \text{truncated}_N(T) \Rightarrow T_N$ finite rank $\Rightarrow T_N$ compact)

  $T_N \xrightarrow[N \to \infty]{} T$ in $\mathcal{L}(X,Y)$ $\left(\Longleftrightarrow \|T_N - T\|_{\mathcal{L}} \to 0 \right)$

  (How to procede ?

  1. $T_N(x) := \{y_n\}_{n \in \mathbb{N}}$ s.t. $y_n = \dots$
  2. $\dim(\mathrm{Im}(T_N)) = N < \infty \Rightarrow T_N$ compact
  3. $\|T_N(x) - T(x)\|_Y = (\dots) \|x\|_X$
  4. $\|T_N - T\|_{\mathcal{L}} \le (\dots) \xrightarrow[N \to \infty]{} 0$

- $Y = C^0([a,b])$

  $\Longrightarrow$ [ASCOLI-ARZELÁ for $F \subset C^0([a,b])$: F bounded, closed, equicontinuous $\Longrightarrow F$ compact]

  [$T: X \to C^0([a,b])$ compact] $\Longleftrightarrow$ [$\forall E \subset X$ bounded : $\overline{T(E)}$ compact]

  So we need to prove that :

  $T(E)$ is $\underset{1.}{\underline{\text{bounded}}}$ and $\underset{2.}{\underline{\text{equicontinuous}}}$

$\boxed{1.}$

E bdd $\Longleftrightarrow \exists C_0 : \|f\|_\infty \le C_0 \quad \forall f \in E$

$T(E)$ bdd $\Longleftrightarrow \exists C_1 : \|\varphi\|_\infty \le C_1 \quad \forall \varphi \in T(E)$

$\Longleftrightarrow \exists C_1 : \|T(f)\|_\infty \le C_1 \quad \forall f \in E$

$\Longrightarrow \|T(f)\|_\infty \le (\dots) \|f\|_\infty \le (\dots) C_0 := C_1 \quad \forall f \in E$

$\Longrightarrow T(E)$ bounded

$\boxed{2.}$

$T(E)$ equicontinuous if $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$:

$\forall \varphi \in T(E) \quad \forall x,y \in [a,b] \quad |x-y| < \delta_\varepsilon \Longrightarrow |\varphi(x) - \varphi(y)| < \varepsilon$ general

$\forall f \in E \quad \forall x,y \in [a,b] \quad |x-y| < \delta_\varepsilon \Longrightarrow |T(f(x)) - T(f(y))| < \varepsilon$ specific

$\Rightarrow |T(f(x)) - T(f(y))| \le \|f\|_\infty |x-y| (\dots) \le C_0 \cdot \delta_\varepsilon (\dots) < \varepsilon$

if $0 < \delta_\varepsilon < \dfrac{\varepsilon}{C_0 \cdot (\dots)}$

$\Rightarrow T(E)$ equicontinuous

$$
\begin{aligned}
|T(f)(x) - T(f)(y)| &\le \left| \int_x^y \cosh(t) |f(t)| \, dt \right| \\
&\le \|f\|_\infty \left| \int_x^y \cosh(t) \, dt \right| \\
&\le \|f\|_\infty |x-y| \cosh(1) \\
&\le C_0 \cdot |x-y| \cosh(1) \\
&\le C_0 \cdot \delta_\varepsilon \cosh(1) < \varepsilon \Longrightarrow 0 < \delta_\varepsilon < \frac{\varepsilon}{C_0 \cosh(1)}
\end{aligned}
$$

- T not compact ?

- $T: X \to X$, $\mu(X) = \infty$, X Banach.
  Then, **only one** is possible:
  - (i) **T** is compact
  - (ii) **T** is a bijection

  $\Longrightarrow$ If **T** is bijective then **T** is **not** compact

---

- $\exists \{x_n\}_n$: $\left.\begin{array}{l} x_n \longrightarrow x \\ T(x_n) \not\to T(x) \end{array}\right\} \Rightarrow$ **T** is **not** compact

  check that $\{T(x_n)\}_n$ cannot contain Cauchy subsequences

  $T: \ell^2 \to \ell^2$: such that:
  $T: (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, ..) \to (0, x^{(1)}, 0, x^{(3)}, 0, ..)$
  consider:
  $$x_n^{(k)} = \begin{cases} 1 & k = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$
  $$x_1 = (1, 0, 0, 0, 0, 0, \dots)$$
  $$x_2 = (0, 0, 1, 0, 0, 0, \dots)$$
  $$x_3 = (0, 0, 0, 0, 1, 0, \dots)$$
  $$\Longrightarrow \begin{cases} \|x_n\|_2 = 1 \quad \forall n \\ x_n \longrightarrow 0, \text{ in fact:} \\ \sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} = y^{(2n-1)} \xrightarrow[n\to\infty]{} 0 \quad \forall y \in \ell^2 \end{cases}$$
  However:
  $$\|T(x_m) - T(x_n)\|_2 = \|T((0,..,\underset{2m-1}{0,1},0,..)) - T((0,..,\underset{2n-1}{0,1},0,..))\|_2$$
  $$= \|(0,..,\underset{2m}{0,1},0,..) - (0,..,\underset{2n}{0,1},0,..)\|_2$$
  $$= 2 \quad \forall m \neq n$$
  $\Longrightarrow$ $T(x_n)$ does not admit a Cauchy sub.

- $\|x\| = \sqrt{\langle x, x \rangle}$ $\quad (\langle x, x \rangle = \|x\|^2)$
- $T: H \to H$ symmetric : $\langle T(x), y \rangle = \langle x, T(y) \rangle$

- **FREDHOLM** ALTERNATIVE
  H separable, $T: H \to H$ symmetric, compact:
  $$\left[\begin{array}{l} \forall g \in H \quad \exists!\, f \in H: \\ \mu f - T(f) = g \end{array} \right] \Longleftrightarrow \left[\begin{array}{l} \lambda = 1 \text{ is } \textbf{not} \text{ an} \\ \text{eigenvalue of } \mathbf{T} \end{array} \right]$$

  if it is not easy to find $\Rightarrow$ contraddiction:
  $\exists u \in H$: $T(u) = u \Rightarrow \|T(u)\| = \|u\|$
  However: $\|T(u)\| \dots$

- $y' = -p(t)y \Longrightarrow y(t) = C e^{-\int p(t)dt}$
- $y' + p(t)y = f(t) \Longrightarrow y(t) = e^{-\int p(t)dt} \left[\int e^{\int p(t)dt} f(t)\, dt + C \right]$

$\Omega \subseteq \mathbb{R}^n$ open bounded set : $\lambda(\Omega) < \infty$.
Let $p, q \in (1, \infty)$ be conjugate.
Let $k = k(x,y) \in L^q(\Omega \times \Omega)$ be fixed. Then define $T : L^p(\Omega) \to L^q(\Omega)$:

$$T(f)(x) = \int_\Omega k(x,y) f(y)\, dy \qquad \forall f \in L^p(\Omega)$$

Prove that: (1) $T(f)(x)$ is finite a.e. in Ω and it is measurable
(2) $T \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$
(3) T is compact.

Solution: (1)

$$\left.\begin{array}{l} k \in L^q(\Omega \times \Omega) \implies k \text{ measurable (on } \Omega\times\Omega) \\ f \in L^p(\Omega) \implies f \text{ measurable (on } \Omega) \end{array}\right\} \implies k(x,y)\cdot f(y) \text{ measurable (on } \Omega\times\Omega)$$

Then, by Hölder:

$$\int_{\Omega\times\Omega} |k(x,y)|\cdot|f(y)|\, dx\,dy \leq \underbrace{\left(\int_{\Omega\times\Omega} |k(x,y)|^q dx\,dy\right)^{\frac{1}{q}}}_{\|k\|_{L^q(\Omega\times\Omega)}} \underbrace{\left(\int_{\Omega\times\Omega} |f(y)|^p dx\,dy\right)^{\frac{1}{p}}}_{\underbrace{\left(\int_\Omega \int_\Omega |f(y)|^p dy\, dx\right)^{\frac{1}{p}}}_{\underbrace{\left(\int_\Omega \|f\|_p^p\, dx\right)^{1/p}}_{\|f\|_p \cdot \mu(\Omega)^{1/p}}} \text{ (Fubini)}}$$

$$\leq \|k\|_{L^q(\Omega\times\Omega)} \cdot \|f\|_{L^p(\Omega)} \cdot \mu(\Omega)^{1/p} < \infty$$

Then, by Fubini theorem we can conclude that:

$$\psi_1(x) = \int_\Omega k(x,y) f(y)\, dy = T(f)(x) \in L^1(\Omega)$$

and so it is finite a.e. in Ω and it is measurable.

(2) T linear: let $\alpha, \beta \in \mathbb{R}$, $f, g \in L^p(\Omega)$:

$$\begin{aligned} T(\alpha f + \beta g)(x) &= \int_\Omega k(x,y)\,(\alpha f(x) + \beta g(x))\, dx \\ &= \alpha \int_\Omega k(x,y) f(x)\, dx + \beta \int_\Omega k(x,y) g(x)\, dx \\ &= \alpha T(f)(x) + \beta T(g)(x) \end{aligned}$$

T bounded:

$$\begin{aligned} \|T(f)\|_q^q = \int_\Omega |T(f)|^q d\mu &= \int_\Omega \left|\int_\Omega k(x,y) f(y)\, dy\right|^q dx \\ &\leq \int_\Omega \left(\int_\Omega |k(x,y) f(y)|\, dy\right)^q dx \\ &\leq \int_\Omega \left[\left(\int_\Omega |k(x,y)|^q dy\right)^{q \cancel{\frac{1}{q}}} \cdot \|f\|_p^q\right] dx \\ &= \|f\|_p^q \int_\Omega \int_\Omega |k(x,y)|^q dy\, dx \\ &= \|f\|_p^q\, \|k\|^q_{L^q(\Omega\times\Omega)} \quad \text{Fubini} \end{aligned}$$

$\longrightarrow$ T bounded s.t.:
$\|T\|_{\mathcal{L}} \leq \|k\|_{L^q(\Omega\times\Omega)}$

(3) T compact $\iff$ [E bounded $\implies$ $\overline{T(E)}$ compact]
$\overline{T(E)}$ compact $\iff$ $\overline{T(E)}$ sequentially compact: every bounded sequence in $\overline{T(E)}$ has a convergent subsequence

Solution (3) $T: L^p(\Omega) \to L^q(\Omega)$:

Let $\{f_n\}_n \subseteq L^p(\Omega)$ be bounded.

$\Longrightarrow$ L^p is reflexive, separable and Banach, $\{f_n\}_n$ is bounded :
BANACH-ALAOGLU corollary:

$$\exists \{f_{n_k}\} : \quad f_{n_k} \rightharpoonup f \in L^p$$

$\Longleftrightarrow$ $\exists \{f_{n_k}\}: \int_\Omega f_{n_k} g \, d\mu \to \int_\Omega f g \, d\mu \quad \forall g \in L^q(\Omega)$

$\Longrightarrow$ In particular it holds for $k(x,\bullet) \in L^q$
(since $k \in L^q(\Omega \times \Omega)$ then by Fubini $k(x,\bullet) \in L^q$) :

$$\exists \{f_{n_k}\}: \underbrace{\int_\Omega f_{n_k}(y)\, k(x,y)\, dy}_{T(f_{n_k})} \to \underbrace{\int_\Omega f(y)\, k(x,y)\, dy}_{T(f)}$$

We showed $T(f_{n_k}) \to T(f)$ pointwise (a.e.).
Moreover, it holds:

$$\underbrace{\left|\int f_{n_k}(y)\, k(x,y)\, dy\right|}_{|T(f_{n_k})|} \leq \underbrace{\|f_{n_k}\|_{L^p(\Omega)}}_{\substack{\{f_n\} \text{ is bounded} \\ \text{and so also } \{f_{n_k}\}}} \|k(x,\bullet)\|_{L^q(\Omega)} \leq C \cdot \underbrace{\|k(x,\bullet)\|_{L^q(\Omega)}}_{\in L^q(\Omega)}$$

$\Longrightarrow$ by DCT:

$$T(f_{n_k}) \to T(f) \text{ in } L^q(\Omega)$$

which means that T is compact.

RANDOM

- X not reflexive? $\Longleftarrow$ $\exists \{x_n\}_n \subset X$ bounded that does not have a weakly convergent subsequence

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- $L(x_1, x_2, x_3, 0, 0, \dots) = (x_3, x_4, x_5, \dots)$ surjective, **not** injective $(\exists \lambda)$
  $R(x_1, x_2, x_3, 0, 0, \dots) = (0, 0, x_1, x_2, x_3, \dots)$ injective, **not** surjective $(\nexists \lambda)$

- T injective $\Longleftrightarrow$ $Ker(T) = \{0\}$
  T surjective $\Longleftrightarrow$ $Im(T) = X$

- **PROJECTION** METHOD

  - it must be 2 because only L^2 are Hilbert

    $\boxed{\min \int_{-\pi}^{\pi} |\cos(x) - \alpha x - \beta|^2 dx}$

    1. $H = (L^2([-\pi, \pi]), dx)$
    $V = \{f \in L^2([-\pi,\pi]) : f(x) = \alpha x + \beta\}$
    Basis: $u_1(x) = 1, \quad u_2(x) = x$
    $\|u_1(x)\|_2^2 = \int_{-\pi}^{\pi} |u_1(x)|^2 dx = 2\pi \implies V_1(x) = \frac{1}{\sqrt{2\pi}}$
    $\|u_2(x)\|_2^2 = \int_{-\pi}^{\pi} |u_2(x)|^2 dx = \frac{2\pi^3}{3} \implies V_2(x) = \sqrt{\frac{3}{2\pi^3}}\, x$

    2. $P_V : L^2 \to V$:
    $$P_V(g) = \langle g, v_1 \rangle v_1 + \langle g, v_2 \rangle v_2$$
    $\langle g, v_1 \rangle = \int_{-\pi}^{\pi} g v_1 \, dx = 0$
    $\langle g, v_2 \rangle = \int_{-\pi}^{\pi} g v_2 \, dx = 0$
    $\implies P_V(g) = 0$

    3. $\underbrace{\|g - P_V(g)\|_2^2} = \min_{v \in L^2} \|g - v\|_2^2 = \min \int_{-\pi}^{\pi} |\cos(x) - \alpha x - \beta|^2 dx$
    $= \|g\|_2^2$
    $= \int_{-\pi}^{\pi} g^2 dx = \int_{-\pi}^{\pi} \cos^2(x)\, dx = \pi$

  - $\boxed{\min \int_0^{+\infty} |e^{-x} - \alpha x - \beta|^2 e^{-x} dx}$

    1. $H = (L^2([0, \infty)), e^{-x} dx)$
    $V = \{f \in L^2([0,\infty)) : f(x) = \alpha x + \beta\}$
    Basis: $u_1(x) = 1, \quad u_2(x) = x$
    $\|u_1(x)\|_2^2 = \int_0^{+\infty} |1|^2 e^{-x} dx = 1 \implies V_1(x) = 1$
    $V_2(x) = u_2(x) - \langle u_2, v_1 \rangle v_1$
    $\quad \stackrel{\perp}{=} x - \int_0^{\infty} x \cdot 1 \cdot e^{-x} dx = x - 1$

    2. $P_V : L^2 \to V$: $\quad P_V(g) = \langle g, v_1 \rangle v_1 + \langle g, v_2 \rangle v_2$
    $\left.\begin{array}{l} \langle g, v_1 \rangle = \int_0^{\infty} g \cdot v_1 e^{-x} dx = 1/2 \\ \langle g, v_2 \rangle = \int_0^{+\infty} g \cdot v_2 e^{-x} dx = -1/4 \end{array}\right\} \implies P_V(g) = \frac{1}{2} - \frac{1}{4}(x-1)$

    3. $\|g - P_V(g)\|_2^2 \stackrel{\perp}{=} \|e^{-x} - \frac{1}{2} + \frac{1}{4}(x-1)\|_2^2$
    $\stackrel{\perp}{=} \int_0^{\infty} |e^{-x} - \frac{1}{2} + \frac{1}{4}(x-1)|^2 e^{-x} dx = \frac{1}{48}$
    $\stackrel{\perp}{=} \min \int_0^{+\infty} |e^{-x} - \alpha x - \beta| e^{-x} dx$