# Continuous-time Markov chains

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# 1 Transition probabilities

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $I$ a finite or countable set that will be considered, when necessary, as a measurable space endowed with the $\sigma$-algebra $\mathcal{P}(I)$ of all subsets of $I$.

**Definition 1.1** *A family $(X_t)_{t\geq 0}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $I$ is a* Markov chain *if*

$$\mathbb{P}\left\{X_{t_{n+1}} = j \mid X_{t_n} = i_n, \ldots, X_{t_1} = i_1\right\} = \mathbb{P}\left\{X_{t_{n+1}} = j \mid X_{t_n} = i_n\right\} \tag{1}$$

*for all $n$, $i_1, i_2, \ldots, i_n, j \in I$, $t_1 < \cdots < t_{n+1}$.*

Identity (1) is called *Markov property.*

**Definition 1.2** Transition probabilities *$p_{ij}(s,t)$ on a time interval $[s,t]$ are defined by*

$$p_{ij}(s, t+s) = \mathbb{P}\left\{X_{t+s} = j \mid X_s = i\right\}.$$

*The Markov chain is called* time-homogeneous *if $p_{ij}(s, t+s) = p_{ij}(0,t)$ for all $i, j \in I$, $t, s \geq 0$.*

We will consider time-homogeneous Markov chains and denote by $p_{ij}(t)$ transition probabilities $p_{ij}(0,t)$.

The family $(P_t)_{t\geq 0}$ of (possibly infinite) matrices $P_t$

$$(P_t)_{ij} = p_{ij}(t)$$

is called *Markovian semigroup.* Denote by $\mathbb{1}$ the matrix $(\delta_{ij})_{i,j\in S}$ with $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.

**Theorem 1.3** *The Markovian semigroup $(P_t)_{t\geq 0}$ has the following properties:*

*(a)* $P_0 = \mathbb{1}$,

*(b)* $(p_{ij}(t))_{t\geq 0}$ *is a stochastic matrix for all $t \geq 0$,*

*(c) the Chapman-Kolmogorov $P_{t+s} = P_t P_s$ equation holds for all $t, s \geq 0$.*

*Proof.* Condition $(a)$ clearly holds by definition.

In order to check $(b)$ it suffices to note that $p_{ij}(t) \geq 0$ and

$$\sum_{k\in I} p_{ik}(t) = \sum_{k\in I} \mathbb{P}\left\{X_t = k \mid X_0 = i\right\} = \mathbb{P}\left\{\Omega \mid X_0 = i\right\} = 1.$$

Finally, by the total probability formula and the Markov property Markov (1), for all $t, s \geq 0$, we have

$$\begin{aligned}
p_{ij}(t+s) &= \mathbb{P}\{\, X_{t+s} = j \mid X_0 = i \,\} \\
&= \sum_{k\in S} \mathbb{P}\{\, X_{t+s} = j, X_s = k \mid X_0 = i \,\} \\
&= \sum_{k\in S} \frac{\mathbb{P}\{\, X_{t+s} = j, X_s = k, X_0 = i \,\}}{\mathbb{P}\{\, X_0 = i \,\}} \\
&= \sum_{k\in S} \frac{\mathbb{P}\{\, X_{t+s} = j \mid X_s = k, X_0 = i \,\}\mathbb{P}\{\, X_s = k, X_0 = i \,\}}{\mathbb{P}\{\, X_0 = i \,\}} \\
&= \sum_{k\in S} \mathbb{P}\{\, X_{t+s} = j \mid X_s = k \,\} \frac{\mathbb{P}\{\, X_s = k, X_0 = i \,\}}{\mathbb{P}\{\, X_0 = i \,\}} \\
&= \sum_{k\in S} \mathbb{P}\{\, X_{t+s} = j \mid X_s = k \,\}\mathbb{P}\{\, X_s = k \mid X_0 = i \,\} \\
&= \sum_{k\in S} p_{ik}(s)p_{kj}(t).
\end{aligned}$$

This shows that property $(c)$ holds. □

Condition $(c)$ is called *semigroup property.*

Computations of probabilities of events determined by a continuous-time Markov chain can be done using the transition semigroup. This semigroup determines, in particular, finite-dimensional, distributions. Indeed, for all $n \geq 1$ and all $t_1 < t_2 < \cdots < t_n$, $i_0, i_1, \ldots, i_n \in I$, we have

$$\mathbb{P}\{\, X_{t_n} = i_n, \ldots, X_{t_1} = i_1, X_0 = i \,\} = p_{i_0 i_1}(t_1)p_{i_1 i_2}(t_2 - t_1) \ldots p_{i_{n-1} i_n}(t_n - t_{n-1}).$$

For all $\omega \in \Omega$, $X_t(\omega)$ represents the state visited by the Markov chain at time $t$ when the "atomic event" $\{\omega\}$ occurs. Functions

$$t \to X_t(\omega)$$

defined on $[0, +\infty[$, have values in a set made up of isolated points, and so they are not continuous. In "good", regular, say, models, however, they are piecewise continuous and so they can be assumed to be right continuous just by considering their right limits at discontinuity points. Moreover, for reasons that will become clear later, a fixed time $t$ for which the event $\{X_t \neq \lim_{s\to t} X_s\}$ has positive probability does not exists.

From now on we will consider Markov chains with the properties:

- continuity in probability, i.e. $\lim_{s\to t} \mathbb{P}\{X_s \neq X_t\} = 0$,
- almost sure right-continuity with left-hand limits.

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The above assumptions are not restrictive for all practical purposes.

**Proposition 1.4** *If the Markov chain $(X_t)_{t\geq 0}$ is continuous in probability, functions $t \to p_{ij}(t)$ are continuous.*

*Proof.* Denote by $\mathbb{P}_i\{\,\cdot\,\}$ the conditional probability $\mathbb{P}\{\,\cdot \mid X_0 = i\,\}$. For all $t, s \geq 0$ (such that $t + s \geq 0$) we have

$$\begin{aligned}
|p_{ij}(t+s) - p_{ij}(t)| &= |\mathbb{P}_i\{\,X_{t+s} = j\,\} - \mathbb{P}_i\{\,X_t = j\,\}| \\
&= \big|\mathbb{P}_i\{\,X_{t+s} = j, X_t \neq j\,\} + \mathbb{P}_i\{\,X_{t+s} = j, X_t = j\,\} \\
&\quad -\mathbb{P}_i\{\,X_{t+s} \neq j, X_t = j\,\} - \mathbb{P}_i\{\,X_{t+s} = j, X_t = j\,\}\big| \\
&= |\mathbb{P}_i\{\,X_{t+s} = j, X_t \neq j\,\} - \mathbb{P}_i\{\,X_{t+s} \neq j, X_t = j\,\}| \\
&\leq 2\mathbb{P}_i\{\,X_{t+s} \neq X_t\,\}
\end{aligned}$$

The conclusion is now immediate. □

## 2 Transition rates

Assuming continuity of transition probabilities one can prove the following

**Theorem 2.1** *If transition probabilities are continuous, then the following limit exist*

$$q_{ij} = \lim_{t\to 0^+} \frac{p_{ij}(t)}{t}, \qquad \textit{per } i \neq j, \qquad q_{ii} = \lim_{t\to 0^+} \frac{p_{ii}(t) - 1}{t}, \tag{2}$$

*moreover, we have $0 \leq q_{ij} < \infty$ for $i \neq j$ and $q_{ii} \leq 0$ (possibly $q_{ii} = -\infty$).*

*Numbers $q_{ij}$ are called* transition rates.

*If the set of states $I$ is finite, then $q_{ii} > -\infty$ and the following systems of differential equations hold*

$$\frac{d}{dt} p_{ij}(t) = \sum_{k\in S} p_{ik}(t) q_{kj}, \qquad p_{ij}(0) = \delta_{ij} \tag{3}$$

$$\frac{d}{dt} p_{ij}(t) = \sum_{k\in S} q_{ik} p_{kj}(t), \qquad p_{ij}(0) = \delta_{ij}. \tag{4}$$

The above systems (3) (resp. (4)) are called *forward Kolmogorov equations* (resp. *backward Kolmogorov equations*). It can be shown that solutions (with the same initial condition) coincide in several "good" models (for instance when $I$ is finite or, in a more general situation, when $\sup_i(-q_{ii}) < +\infty$).

When $I$ is finite it is easy to check that condition $\sum_j p_{ij}(t) = 1$ for all $i$ is equivalent to

$$\sum_{j\in S} q_{ij} = 0. \tag{5}$$

**Example 2.2** *Two-state Markov chain.* The most general transition rates matrix of a two state Markov chain, by (5) is

$$Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

with $a, b > 0$ (except trivial cases). As a consequence, transition probabilities are

$$(p_{i1}(t), p_{i2}(t)) = (\delta_{i1}, \delta_{i2})\, \mathrm{e}^{tQ}$$

where $\mathrm{e}^{tQ}$ is the exponential of the matrix $Q$ namely, with simple (albeit somewhat complex) computations

$$\mathrm{e}^{tQ} = \frac{1}{a+b}\begin{pmatrix} b + ae^{-t(b+a)} & a(1-e^{-t(b+a)}) \\ b(1-e^{-t(b+a)}) & a + be^{-t(b+a)} \end{pmatrix}$$

In this way, we computed the transition matrix $P_t = \mathrm{e}^{tQ}$.

In the case where $I$ is infinite, one must be more cautious because, even if transition rates $q_{ii}$ are finite, one of the two systems may have a unique solution and the other one may not or, even both may not have a unique solution. However we have the following

**Theorem 2.3** *If*

$$\sup_{i\in S}(-q_{ii}) < +\infty, \tag{6}$$

*then the backward and forward Kolmogorov equations admit transition probabilities $(p_{ij}(t))_{t\geq 0}$ as their unique solution.*

*Proof.* Consider first, for simplicity, the case when $I$ is finite and the assumption obviously holds. Let $Q$ be the transition rate matrix $(q_{ij})_{i,j\in S}$, and, for all $t \geq 0$ let $Q = \mathrm{e}^{tQ}$ be the exponential of the matrix $tQ$. Easy computations show that

$$\frac{d}{dt}\mathrm{e}^{tQ} = Q\mathrm{e}^{tQ} = \mathrm{e}^{tQ}Q$$

namely, matrix elements $(\mathrm{e}^{tQ})_{ij}$ satisfy (4) and (3). Solution of linear systems (4) and (3) is clearly unique.

In the case when $I$ is infinite it suffices to consider the linear operator $Q$ on the Banach space $\ell^\infty(I)$ of bounded functions on $I$ with norm

$$|f|_\infty = \sup_{j\in S}|f(j)|.$$

By the assumption $\sup_{i\in S}(-q_{ii}) < +\infty$, the operator $Q$ is bounded and his norm is

$$\|Q\|_\infty = \sup_{f\in\ell^\infty(S),\, |f|_\infty\leq 1} |Qf|_\infty.$$

For all $f \in \ell^\infty(I)$ with $|f|_\infty \le 1$ and $j \in I$ we have

$$\begin{aligned} |(Qf)(j)| &= \left| \sum_k q_{jk} f(k) \right| \\ &\le -q_{jj}|f(j)| + \sum_{k \ne j} q_{jk}|f(k)| \\ &\le |f|_\infty \left( -q_{jj} + \sum_{k \ne j} q_{jk} \right) \\ &= -2q_{jj}|f|_\infty. \end{aligned}$$

By taking the sup on $j$ we find

$$\|Q\|_\infty \le 2 \sup_{j \in S}(-q_{jj}) < +\infty.$$

We can define the exponential of the operator $tQ$ as

$$\mathrm{e}^{tQ} = \sum_{n \ge 0} \frac{t^n Q^n}{n!}$$

because the series converges for the uniform norm on linear operators on $\ell^\infty(S)$. The proof can now be completed as in the case where $I$ is finite. □

**Definition 2.4** *The transition rate matrix* $Q = (q_{ij})_{i,j \in S}$ *is also called the* generator *(or* infinitesimal generator*) of the Markovian semigroup* $(P(t))_{t \ge 0}$.

It is worth to notice that, by the differentiability of transition probabilities $p_{ij}(t)$, and time homogeneity of the Markov chain, the probability of jump from a state $i$ to a state $j$ in a "small" time interval $[t, t+h]$ satisfies

$$\mathbb{P}\{ X_{t+h} = j \mid X_t = i \} = \begin{cases} p_{ij}(h) = q_{ij}h + o(h) & \text{se } i \ne j, \\ p_{ii}(h) = 1 + q_{ii}h + o(h) & \text{se } i = j, \end{cases} \tag{7}$$

($o(h)$ is an infinitesimum of order bigger than $h$, i.e. a function such that $\lim_{h \to 0} o(h)/h = 0$).

When constructing models, a Markov chain is often defined starting from transition rates $q_{ij}$.

# 3 Exit times from a state

The study of exit times from a state clarifies both the mean of transition rates and the Markov property when one keeps in mind that the exponential distribution is the only continuous distribution which is *memoryless*.

The *first exit time* from a state $i \in S$ is

$$T_i(\omega) = \inf\{ t \ge 0 \mid X_t(\omega) \ne i \} \tag{8}$$

(with the convention that $\inf \emptyset = +\infty$). Functions $T_i$ are clearly random variables with values in $[0, +\infty]$ (in other words, they are measurable with respect to the suitable $\sigma$-algebras).

**Theorem 3.1** *If $-\infty < q_{ii} < 0$ the random variable $T_i$ has exponential distribution $\mathcal{E}(-q_{ii})$ with parameter $-q_{ii}$.*

*Proof.* For all $n \geq 1$, we have the inclusion

$$\{ T_i > t \} \subseteq \{ X_{t/2^n} = i, \ldots, X_{(2^n-1)t/2^n} = i, X_t = i \}$$

and, more precisely,

$$\{ T_i > t \} = \bigcap_{n \geq 1} \{ X_{t/2^n} = i, \ldots, X_{(2^n-1)t/2^n} = i, X_t = i \}$$

Since the right-hand side sequence is non decreasing, by (7), we have

$$\begin{aligned} \mathbb{P}_i \{ T_i > t \} &= \lim_{n \to \infty} \mathbb{P} \{ X_{t/2^n} = i, \ldots, X_{(2^n-1)t/2^n} = i, X_t = i \} \\ &= \lim_{n \to \infty} (p_{ii}(t/2^n))^{2^n} = \lim_{n \to \infty} \left( 1 + \frac{tq_{ii}}{2^n} + o(t/2^n) \right)^{2^n} = \mathrm{e}^{q_{ii}\,t}. \end{aligned}$$

This shows that the reandom variable $T_i$ has exponential distribution with parameter $-q_{ii}$. $\square$

We can now show that, after leaving a state $i$, the Markov chain jumps in another state $j$ with probability $q_{ij}/(-q_{ii})$. We denote by $\mathbb{P}_i$ the conditional probability $\mathbb{P}\{\,\cdot\,|X_0 = i\}$.

**Proposition 3.2** *For all $i \neq j$ we have*

$$\mathbb{P}_i \{ X_{T_i} = j \} = \frac{q_{ij}}{-q_{ii}}.$$

*Proof.* Consider the non-decreasing sequence $(S_n)_{n \geq 1}$ of discrete stopping times

$$S_n = \sum_{k=0}^{\infty} (k+1) 2^{-n} 1_{\{k2^{-n} < T_i \leq (k+1)2^{-n}\}}$$

that converges almost surely to $T_i$ since $T_i \leq S_n \leq T_i + 2^{-n}$ for all $n$. By right continuity of trajectories, the sequence of random variables $(X_{S_n})_{n \geq 1}$ converges almost surely to $X_{T_i}$ and so

$$\mathbb{P}_i \{ X_{T_i} = j \} = \lim_{n \to \infty} \mathbb{P}_i \{ X_{S_n} = j \}.$$

Moreover, for all $n \geq 1$, by the Markov property and exponential distribution $\mathcal{E}(-q_{ii})$ of $T_i$,

$$\begin{aligned}
\mathbb{P}_i\{X_{S_n} = j\} &= \sum_{k=0}^{\infty} \mathbb{P}_i\{X_{(k+1)2^{-n}} = j, T_i > k2^{-n}\} \\
&= \sum_{k=0}^{\infty} \mathbb{P}_i\{X_{(k+1)2^{-n}} = j \mid T_i > k2^{-n}\} \mathbb{P}_i\{T_i > k2^{-n}\} \\
&= \sum_{k=0}^{\infty} \mathbb{P}_i\{X_{(k+1)2^{-n}} = j \mid X_{k2^{-n}} = i\} \mathbb{P}_i\{T_i > k2^{-n}\} \\
&= \sum_{k=0}^{\infty} p_{ij}(2^{-n}) \left(e^{q_{ii}2^{-n}}\right)^k = p_{ij}(2^{-n}) \left(1 - e^{q_{ii}2^{-n}}\right)^{-1}.
\end{aligned}$$

Since $i \neq j$, we have $p_{ij}(2^{-n}) = q_{ij}2^{-n} + o_{ij}(2^{-n})$ so that

$$\mathbb{P}_i\{X_{S_n} = j\} = q_{ij}(1 + o_{ij}(1)) \frac{2^{-n}}{1 - e^{q_{ii}2^{-n}}}.$$

The conclusion follows taking the limit as $n \to \infty$. □

The above results motivate the introduction of the *discrete skeleton* or *jump chain* of a continuous-time Markov chain $(X_t)_{t \geq 0}$ which is defined by means of the sequence $(T_n)_{n \geq 0}$ of jump times defined by

$$\begin{aligned}
T_0 &= 0, \\
T_n &= \inf\{t \geq T_{n-1} \mid X_t \neq X_{T_{n-1}}\}
\end{aligned}$$

with the usual convention $\inf \emptyset = +\infty$. The discrete skeleton is the family of random variables $(Y_n)_{n \geq 0}$ given by $Y_0 = X_0$, and for $n \geq 1$ (with the convention $T_0 = 0$)

$$Y_n = \begin{cases} X_{T_n} & \text{if } \; T_n < \infty, \\ X_{T_{n-1}} & \text{if } \; T_n = \infty. \end{cases}$$

We state the following result without proof

**Theorem 3.3** *The discrete skeleton $(Y_n)_{n \geq 0}$ is a discrete-time homogeneous Markov chain with state space $I$ and transition probabilities given by*

$$p_{ij} = \begin{cases} q_{ij}/(-q_{ii}) & \text{if } q_{ii} \neq 0 \text{ and } i \neq j, \\ 0 & \text{if } q_{ii} \neq 0 \text{ and } i = j \text{ or } q_{ii} = 0 \text{ and } i \neq j, \\ 1 & \text{if } q_{ii} = 0 \text{ and } i = j. \end{cases}$$

In view of the above results one can think of a continuous-time homogeneous Markov chain just as a discrete time homogeneous Markov chain, the discrete skeleton, with exponentially distributed time intervals between jumps only depending on the current state. The intuitive picture of the random phenomena (if we neglect events with zero probability) is as follows: we start from a state $i$, wait an exponential time with parameter $-q_{ii}$ and then jump to another state $j$ with probability $q_{ij}/(-q_{ii})$.

# 4 The Poisson process

The Poisson process is the simplest example of a continuous time Markov chain with state space $\mathbb{N}$. It is usually defined by means of an independent sequence of random variables $(S_n)_{n\geq 1}$ with exponential distribution $\mathcal{E}(\lambda)$ $(\lambda > 0)$, each $S_n$ representing the sojourn time in the state $n$.

Define the sequence of jump times $(T_n)_{n\geq 1}$ by

$$T_1 = S_1, \qquad T_n = S_1 + \cdots + S_n.$$

**Definition 4.1** *The family of random variables $(N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} 1\!\!1_{\{T_n \leq t\}}$$

*is a Poisson process with parameter $\lambda$.*

From the above definition it is clear that the random variable $N_t$ counts the number of jumps occurred before time $t$. As a consequence we have the following identities

$$\{N_t \geq n\} = \{T_n \leq t\}, \quad \{N_t = n\} = \{T_n \leq t, T_{n+1} > t\}$$

that turn out to be useful for computing probabilities.

**Proposition 4.2** *If $(N_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda$ then the random variable $N_t$ has Poisson distribution $\mathcal{P}(\lambda t)$ for all $t > 0$.*

*Proof.* It suffices to recall that $T_n$ has Gamma distribution $\Gamma(n, \lambda)$ and compute

$$\begin{aligned}
\mathbb{P}\{N_t = n\} &= \mathbb{P}\{T_n \leq t, T_{n+1} > t\} \\
&= \mathbb{P}\{T_n \leq t\} - \mathbb{P}\{T_{n+1} \leq t\} \\
&= \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda x} dx - \int_0^t \frac{\lambda^{n+1} x^n}{n!} \mathrm{e}^{-\lambda x} dx.
\end{aligned}$$

Integrating by parts we find

$$\begin{aligned}
\int_0^t \frac{\lambda^{n+1} x^n}{n!} \mathrm{e}^{-\lambda x} dx &= \int_0^t \frac{\lambda^n x^n}{n!} \cdot \lambda \mathrm{e}^{-\lambda x} dx \\
&= \left[ -\frac{\lambda^n x^n}{n!} \cdot \mathrm{e}^{-\lambda x} \right]_0^t + \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda x} dx \\
&= -\mathrm{e}^{-\lambda t} \frac{\lambda^n t^n}{n!} + \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda x} dx.
\end{aligned}$$

and so

$$\mathbb{P}\{N_t = n\} = \mathrm{e}^{-\lambda t} \frac{\lambda^n t^n}{n!}.$$

□

The Poisson process has further useful properties

**Theorem 4.3** *For all $n \in \mathbb{N}$ and all times $0 < t_1 < \cdots < t_n$ the increments*

$$N_{t_1}, \quad N_{t_2} - N_{t_1}, \quad \ldots, \quad N_{t_n} - N_{t_{n-1}}$$

*are independent and*

$$N_{t_{k+1}} - N_{t_k}$$

*has Poisson distribution $\mathcal{P}(\lambda(t_{k+1} - t_k))$.*

*Proof.* We consider two increments $N_s$ and $N_{t+s} - N_s$ for simplicity. As a first step note that

$$\begin{aligned}\{N_s = m, N_{t+s} - N_s = n\} &= \{N_s = m, N_{t+s} = m + n\} \\ &= \{T_m \le s, T_{m+1} > s, T_{n+m} \le t + s, T_{n+m+1} > t + s\}\end{aligned}$$

Moreover, letting $U_{n-1} = T_{n+m} - T_{m+1} = S_{m+2} + \cdots + S_{n+m}$ and keeping into account that $T_{n+m} \le t + s$ and $T_m = x$ imply $S_{m+1} = T_{m+1} - T_m \le t + s - x$, we have

$$\begin{aligned}\mathbb{P}\{N_s = m, N_{t+s} - N_s = m\} = &\int_0^s dx \frac{\lambda^m x^{m-1}}{(m-1)!} \mathrm{e}^{-\lambda x} \int_{s-x}^{t+s-x} dy \lambda \mathrm{e}^{-\lambda y} \\ &\cdot \mathbb{P}\{U_{n-1} \le t + s - x - y, U_{n-1} + S_{n+m+1} > t + s - x - y\}\end{aligned}$$

Now, for all $r > 0$, we have also

$$\begin{aligned}\mathbb{P}\{U_{n-1} \le r, U_{n-1} + S_{n+m+1} > r\} &= \int_0^r du \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} \mathrm{e}^{-\lambda u} \int_{r-u}^{\infty} dv \lambda \mathrm{e}^{-\lambda v} \\ &= \int_0^r du \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} \mathrm{e}^{-\lambda r} \\ &= \mathrm{e}^{-\lambda r} \frac{\lambda^{n-1} r^{n-1}}{(n-1)!}\end{aligned}$$

so that

$$\begin{aligned}&\mathbb{P}\{N_s = m, N_{t+s} - N_s = m\} \\ &= \int_0^s dx \frac{\lambda^m x^{m-1}}{(m-1)!} \mathrm{e}^{-\lambda x} \int_{s-x}^{t+s-x} dy \lambda^n \mathrm{e}^{-\lambda(t+s-x)} \frac{(t+s-x-y)^{n-1}}{(n-1)!} \\ &= \lambda^n \mathrm{e}^{-\lambda(t+s)} \int_0^s dx \frac{\lambda^m x^{m-1}}{(m-1)!} \left[ -\frac{(t+s-x-y)^n}{n!} \right]_{s-x}^{t+s-x} \\ &= \frac{\lambda^n t^n}{n!} \mathrm{e}^{-\lambda(t+s)} \int_0^s dx \frac{\lambda^m x^{m-1}}{(m-1)!} \\ &= \frac{\lambda^m s^m}{m!} \mathrm{e}^{-\lambda s} \cdot \frac{\lambda^n t^n}{n!} \mathrm{e}^{-\lambda t}.\end{aligned}$$

This completes the proof. □

From the above result it is clear that $(N_t)_{t\geq 0}$ is a Markov chain. Indeed, for all $t_1 < \cdots < t_n$, $i_1, i_n, j \in \mathbb{N}$ we have

$$\begin{aligned}
&\mathbb{P}\left\{N_{t_{n+1}} = j \mid N_{t_n} = i_n, \ldots, N_{t_1} = i_1\right\} \\
&= \mathbb{P}\left\{N_{t_{n+1}} - N_{t_n} = j - i_n \mid N_{t_n} = i_n, \ldots, N_{t_1} = i_1\right\} \\
&= \mathbb{P}\left\{N_{t_{n+1}} - N_{t_n} = j - i_n\right\} \\
&= \frac{(\lambda(t_{n+1} - t_n))^{j-i_n}}{(j - i_n)!} \mathrm{e}^{-\lambda(t_{n+1}-t_n)} \\
&= \mathbb{P}\left\{N_{t_{n+1}} = j \mid N_{t_n} = i_n\right\}.
\end{aligned}$$

Moreover, it is time homogeneous since the above transition probability only depends on the length of the time interval.

The transition rate matrix is easily computed from

$$\frac{d}{dt}\mathbb{P}\{N_t = n+1 \mid N_s = n\}\Big|_{t=s} = \frac{d}{dt}\mathrm{e}^{-\lambda(t-s)}\Big|_{t=s} = \lambda$$

and so

$$q_{ij} = \begin{cases} \lambda & \text{if } j = i+1, \\ -\lambda & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

# 5 Population dynamics

In this section, we will give a somewhat sketchy introduction to the basic Markov chain models for population dynamics and random queues.

## 5.1 Pure birth processes

The state of the chain $n \in \mathbb{N}$ represents the number of individuals in a population. This number can only increase by the arrival of new individuals. Transition rates are given by a sequence $(\lambda_n)_{n\geq 0}$ of positive reals. By Theorem 3.1, $\lambda_n$ is the inverse of the mean waitinf time for the transition from $n$ to $n+1$.

The generator of the process is the matrix $Q = (q_{ij})_{i,j\in\mathbb{N}}$ with

$$q_{n\,n+1} = \lambda_n, \quad q_{n\,n} = -\lambda_n, \quad q_{n\,m} = 0 \text{ for } m \neq n,\ m \neq n+1.$$

The forward and backward Kolmogorov equations for transition probabilities $p_{nm}(t)$ with $m \geq n$ (clearly $p_{nm}(t) = 0$ se $n > m$) are

$$\begin{aligned}
p'_{nm}(t) &= -\lambda_m p_{nm}(t) + \lambda_{m-1} p_{n\,m-1}(t), \qquad p_{n\,m}(0) = \delta_{n\,m}, \\
p'_{nm}(t) &= -\lambda_n p_{n\,m}(t) + \lambda_n p_{n+1\,m}(t), \qquad p_{n\,m}(0) = \delta_{n\,m}.
\end{aligned}$$

In the special case where birth rates are constant $\lambda_n = \lambda > 0$ we recover a Poisson process.

In another remarkable case the increase rate of the population is proportional to its size and so $\lambda_n = \lambda n$. The Markov chain arising from this choice is called *Yule process*. If the initial population consists of $i$ individuals, solving inductively the forward Kolmogorov equations with the initial condition $p_{ii}(0) = 1$

$$\begin{cases} p'_{ii}(t) = -\lambda i\, p_{ii}(t), \\ p'_{ik}(t) = -\lambda k\, p_{ik}(t) + \lambda(k-1)p_{i\,k-1}(t) \text{ se } k > i \end{cases}$$

we find the probability distribution of the population $X_t$ ate time t

$$\mathbb{P}_i\{X_t = k\} = p_{ik}(t) = \binom{k-1}{k-i} \mathrm{e}^{-\lambda i t} \left(1 - \mathrm{e}^{-\lambda t}\right)^{k-i}.$$

The number of individuals at time $t$ has Pascal distribution with parameters $i$ and $\mathrm{e}^{-\lambda t}$.

In a similar way, when $\lambda_n = \lambda(n+1)$, we find

$$p_{ik}(t) = \binom{k}{i} \mathrm{e}^{-\lambda(i+1)t} \left(1 - \mathrm{e}^{-\lambda t}\right)^{k-i}.$$

## 5.2 Pure death processes

As in the case of pure birth processes the state of the Markov chain represents the number of individuals and so the set of states is $\mathbb{N}$. This number, however, can only decrease by a unit for all jumps. The transition rate matrix is now $Q = (q_{ij})_{i,j \in \mathbb{N}}$ con

$$q_{n\,n-1} = \mu_n, \quad q_{n\,n} = -\mu_n, \quad q_{n\,m} = 0 \text{ per } m \neq n,\ m \neq n-1.$$

Since the number of individuals is decreasing and non-negative, we have $p_{00}(t) = 1$ and $\mu_0 = 0$. Moreover, it is clear that, if $j > i$, then $p_{ij}(t) = 0$ for all $t \geq 0$. The backward Kolmogorov equations for other transition probabilities $p_{ij}(t)$ with $j \leq i$ and $t > 0$ are

$$\begin{cases} p'_{ii}(t) = -\mu_i\, p_{ii}(t), \\ p'_{ij}(t) = -\mu_i\, p_{ij}(t) + \mu_i p_{i-1\,j}(t) \text{ se } j < i \end{cases}$$

Solving inductively, starting from $p_{ii}(t) = \mathrm{e}^{-\mu_i t}$, one finds the probability $p_{ij}(t)$ (for $j < i$).

In the special case where $\mu_i = \mu i$ (linear rates) one finds

$$p_{ij}(t) = \binom{i}{j} \mathrm{e}^{-\mu i t} \left(\mathrm{e}^{\mu t} - 1\right)^{j-i} = \binom{i}{j} \mathrm{e}^{-\mu j t} \left(1 - \mathrm{e}^{-\mu t}\right)^{i-j}.$$

It follows that the number $X_t$ of individuals at time $t$ has binomial distribution $B(i, \mathrm{e}^{-\mu t})$.

## 5.3 Birth and death processes

The set of states is always $\mathbb{N}$ but the number of individuals of the population can now increase or decrease by a unit at each jump time. Calling $(\lambda_n)_{n \geq 0}$ e

$(\mu_n)_{n\geq 1}$ the sequences of birth and death rates, the generator of the process is the matrix $Q = (q_{ij})_{i,j\in\mathbb{N}}$ with

$$q_{n\,n+1} = \lambda_n, \quad q_{n\,n} = -(\lambda_n + \mu_n), \quad q_{n\,n-1} = \mu_n, \quad q_{n\,m} = 0 \text{ per } |m-n| > 1.$$

This process also models the length of a random queue in a system with service times and inter-arrival times exponentially distributed with suitable parameters.

# 6 Explosion

Pure birth processes exhibit a phenomenon of continuous-time Markov chains called *explosion* which is intimately related with non-uniqueness of solutions to Kolmogorov equations.

Consider a pure birth process starting from 0 and with jump rates $(\lambda_n)_{n\geq 0}$. This can be realized, likewise the Poisson process with a sequence of independent random variables $(S_n)_{n\geq 1}$ with $S_n$ exponentially distributed with parameter $\lambda_n > 0$ by calling $T_n = \sum_{k=1}^n S_k$ and defining

$$X_t = \sum_{k=1}^{n} \mathbb{1}_{\{T_k \leq t\}}.$$

If

$$\sum_{k\geq 0} \frac{1}{\lambda_k} < +\infty,$$

then the number of individuals grows so quickly that it becomes infinite in a finite time. Intuitively, this happens because, starting from 0, the number of individuals becomes 1 in time $1/\lambda_0$, 2 in a time $1/\lambda_0 + 1/\lambda_1$ ... and so on, after a time $\sum_{n\geq 1} \lambda_n^{-1}$ it becomes infinite.

More precisely, it can be shown (for instance by Kolmogorov's three-series theorem) that

$$T_n = \sum_{k=1}^{n} S_k \to \sum_{k\geq 0} \frac{1}{\lambda_k} := t_\infty \qquad \text{almost surely}$$

so that, for all $\varepsilon > 0$ and $t > t_\infty$, we can find an $m$ big enough such that $\mathbb{P}\{T_n > t\} < \varepsilon$ for all $n > m$. It follows that

$$\mathbb{P}\{X_t > n\} = \mathbb{P}\{T_n \leq t\} > 1 - \varepsilon$$

for all $n > m$, i.e. $X_t = +\infty$ with strictly positive probability.

# 7 Classes of states, recurrence and transience

Communication classes can be defined as in the discrete-time case

**Definition 7.1** *A subset $C$ of $I$ is* a class of states *if, for all $i, j \in C$ there exists $s, t \geq 0$ such that $p_{ij}(s) > 0, p_{ji}(t) > 0$.*

As an immediate consequence we have the following

**Proposition 7.2** *A Markov chain $(X_t)_{t\geq 0}$ with set of states $S$ and transition probabilities $p_{ij}(\cdot)$ is* irreducible *if, for all $i, j \in I$, there exists $t \geq 0$ such that $p_{ij}(t) > 0$.*

In other words, as in the discrete-time case, a Markov chain is irreducible if, starting from a state $i$ the probability of arrival in $j$ in a time $t$ is strictly positive.

One can identify communication classes in an equivalent way (at least for regular Markov chains whose transition probabilities are the unique solution to the backward or forward Kolmogorov equations) looking at transition rates.

**Proposition 7.3** *Let $(P_t)_{t\geq 0}$ be the transition semigroup of a continuous-time Markov chain with generator $Q = (q_{ij})_{i,j\in I}$. The Markov chain is irreducible if and only if, for all $i, j \in I$, there exists $n \geq 1$ and states $k_1, \ldots, k_n$ with $i \neq k_1, k_1 \neq k_2, \ldots, k_n \neq j$ such that*

$$q_{ik_1} q_{k_1k_2} \cdots q_{k_{n-1}k_n} q_{k_nj} > 0.$$

*Proof.* If the above condition holds for a pair $i, j \in I$ consider times $t_1 < \cdots < t_n < t$ and states $k_1, \ldots, k_n$ such that $q_{ik_1} q_{k_1k_2} \cdots q_{k_{n-1}k_n} q_{k_nj} > 0$. Since, for small $t$, $p_{kh}(t) = \delta_{hk} + q_{hk}t + o(t)$, we can find times $t_1 < \cdots < t_n < t$ such that

$$p_{ik_1}(t_1)p_{k_1k_2}(t_2 - t_1) \ldots p_{k_{n-1}k_n}(t_n - t_{n-1})p_{k_nj}(t - t_n) > 0.$$

As a consequence, we find

$$p_{ij}(t) \geq p_{ik_1}(t_1)p_{k_1k_2}(t_2 - t_1) \ldots p_{k_{n-1}k_n}(t_n - t_{n-1})p_{k_nj}(t - t_n) > 0$$

and the Markov chain is irreducible.

Conversely, if the Markov chain is not irreducible, there exists $i, j \in I$ for which $p_{ij}(t) = 0$ for all $t \geq 0$. Let $J_a$ be the nonempty subset of states $k$ from which $j$ is accessible. Clearly, for each $k \in J_a$ we have $p_{ik}(t) = 0$ for all $t \geq 0$ (if not there exists $s > 0$ such that $p_{kj}(s) > 0$ so that $p_{ij}(t+s) \geq p_{ik}(t)p_{kj}(s) > 0$) and the same conclusion holds for all $h$ accessible from $i$. It follows that $q_{hk} = 0$ for all $k \in J_a$ and all $h$ accessible from $i$ so that the condition on transition rates breaks down. □

Pure birth and pure death processes are *not* irreducible; birth and death processes are irreducible if and only if $\lambda_n > 0$ for all $n \geq 0$ and $\mu_n > 0$ for all $n > 0$.

Recalling characterizations of recurrent and transient states for discrete-time Markov chains we define in a natural way

**Definition 7.4** *We say that a state $i$ is:* recurrent *if*

$$\mathbb{P}_i \{ t \geq 0 \mid X_t = i, \text{ is unbounded } \} = 1,$$

transient *if*

$$\mathbb{P}_i \{ t \geq 0 \mid X_t = i, \text{ is unbounded } \} = 0,$$

Recurrence and transience are determined by the discrete skeleton

**Theorem 7.5** *The following hold:*

*(a) A state $i$ is recurrent (resp. transient) for the Markov chain $(X_t)_{t\geq 0}$ if and only if it is recurrent (resp. transient) for the discrete skeleton.*

*(b) Every state is either recurrent or transient.*

*(c) States of the same class are either all recurrent or all transient.*

*(d) If $q_{ii} = 0$ or $\mathbb{P}_i\{T_i < \infty\} = 1$, then $i$ is recurrent and the total mean sojourn time $\int_0^\infty p_{ii}(t)dt$ in $i$ for $\mathbb{P}_i$ is infinite.*

*(e) If $q_{ii} < 0$ and $\mathbb{P}_i\{T_i < \infty\} < 1$, then $i$ is transient and the total mean sojourn time $\int_0^\infty p_{ii}(t)dt$ in $i$ for $\mathbb{P}_i$ is finite.*

# 8 Invariant densities and asymptotic behaviour

**Definition 8.1** *A probability density $\pi = (\pi_i)_{i\in I}$ on $I$ is called* invariant, *or also* stationary, *for a Markov chain $(X_t)_{t\geq 0}$ with transition probabilities $p_{ij}(\cdot)$ if*

$$\pi_i = \sum_{k\in S} \pi_k p_{ki}(t) \tag{9}$$

*for all $i \in S$ and $t \geq 0$.*

One can find invariant densities by means of the generator $Q = (q_{ij})_{i,j\in I}$.

**Proposition 8.2** *Suppose that $q_{ii} > -\infty$ for all $i \in I$ and that $(p_{ij}(t))_{i,j\in I}$ are the unique solution of the forward and backward Kolmogorov equations. A probability density $\pi = (\pi_i)_{i\in I}$ on $I$ is invariant if and only if*

$$\sum_{k\in S} \pi_k q_{kj} = 0, \qquad \text{for all } j \in I.$$

*Proof (case $I$ finite for simplicity).* We first check that the above condition is necessary. Differentiate

$$\pi_i = \sum_{k\in I} \pi_k p_{ki}(t)$$

and find

$$0 = \frac{d}{dt} \sum_{k\in I} \pi_k p_{kj}(t)\bigg|_{t=0} = \sum_{k\in I} \pi_k p'_{kj}(0) = \sum_{k\in S} \pi_k q_{kj}.$$

Conversely, if (9) holds, then, differentiating with respect to $t$ we have

$$\begin{aligned}\frac{d}{dt}\sum_{k\in I}\pi_k p_{kj}(t) &= \sum_{k\in I}\pi_k p'_{kj}(t)\\ &= \sum_{k,i\in I}\pi_k q_{ki} p_{ij}(t)\\ &= \sum_{i\in I}\left(\sum_{k\in I}\pi_k q_{ki}\right)p_{ij}(t) = 0\end{aligned}$$

It follows that $\sum_{k\in I}\pi_k p_{kj}(t)$ is a constant equal to its value for $t=0$, i.e $\pi_j$, and so $(\pi_i)_{i\in I}$ is an invariant density. □

**Example 8.3** *Birth and death processes: invariant density* A birth and death process with transition rates $(\lambda_n)_{n\geq 0}$ and $(\mu_n)_{n\geq 1}$ admits an invariant density if and only if one can find a solution $\pi = (\pi_i)_{i\in S}$ to equations

$$\begin{cases} 0 = -\pi_0\lambda_0 + \pi_1\mu_1, & \\ 0 = \pi_{n-1}\lambda_{n-1} - \pi_n(\lambda_n+\mu_n) + \pi_{n+1}\mu_{n+1}, & n\geq 1,\end{cases}$$

which is a probability density on $\mathbb{N}$, namely such that

$$\pi_n \geq 0 \quad \text{for all } n\in\mathbb{N} \quad \text{e} \quad \sum_{n\geq 0}\pi_n = 1.$$

You can easily check that, if transition rates $\mu_n$ are all strictly positive, and

$$z_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} \qquad n\geq 1,$$

with the initial condition $z_0 = 1$, is the unique solution of the above system.

The sequence $(z_n)_{n\geq 1}$ is clearly non-negative. If, moreover,

$$Z := 1 + \sum_{n\geq 1}\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty$$

then, defining $\pi_n = Z^{-1}z_n$, the sequence $(\pi_n)_{n\geq 0}$ is the unique invariant density for the birth and death process.

If the series is not convergent, then there is no invariant density.

**Theorem 8.4** *Transition probabilities $p_{ij}(\cdot)$ of an irreducible time-continuous Markov chain satisfy the following:*

1. $\lim_{t\to\infty} p_{ij}(t) = \pi_j$ *if there exists a (necessarily unique) invariant density* $(\pi_i)_{i\in I}$,
2. $\lim_{t\to\infty} p_{ij}(t) = 0$ *if there exists no invariant density.*

In the case when the set of states is finite one can always find an invariant density.

**Theorem 8.5** *If the set of states $I$ is finite, there exists at least an invariant density.*

*Proof.* Let $\mathcal{M}$ be the set of densities on $I$, namely

$$\mathcal{M} = \left\{ (\nu_i)_{i \in I} \mid \nu_i \geq 0, \sum_i \nu_i = 1 \right\}.$$

Clearly $\mathcal{M}$ is a closed and bounded subset of $\mathbb{R}^d$ for some $d$, therefore it is compact.

Let $\nu P_t$ be the vector $(\nu P_t)_i = \sum_j \nu_j p_{ij}(t)$ $(i \in I)$ that determines a probability density on $I$ because its components are non negative and satisfy

$$\sum_{i \in I} (\nu P_t)_i = \sum_{i,j \in I} \nu_j p_{ji}(t) = \sum_{j \in I} \nu_j \sum_{i,j \in I} p_{ji}(t) = 1.$$

In a similar way one can check that $t \geq 0$

$$\frac{1}{t} \int_0^t \nu P_s ds$$

belongs to $\mathcal{M}$, i.e. it determines a probability density on $I$.

Since $\mathcal{M}$ is compact, we can find a sequence of times $(t_n)_{n \geq 1}$ divergint to $+\infty$ such that the sequence of probability densities on $I$

$$\frac{1}{t_n} \int_0^{t_n} \nu P_s ds$$

converges to and element $\pi$ of $\mathcal{M}$ for $n$ to $+\infty$.

We can finally check that $\pi$ is an invariant density. For all $t > 0$ we have

$$\begin{aligned}
\pi P_t &= \lim_{n \to \infty} \left( \frac{1}{t_n} \int_0^{t_n} \nu P_s ds \right) P_t \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \nu P_{s+t} ds \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_t^{t_n + t} \nu P_s ds.
\end{aligned}$$

Writing

$$\frac{1}{t_n} \int_t^{t_n+t} \nu P_s ds = \frac{1}{t_n} \int_0^{t_n} \nu P_s ds + \frac{1}{t_n} \int_{t_n}^{t_n+t} \nu P_s ds - \frac{1}{t_n} \int_0^t \nu P_s ds$$

and noting that, for all $i \in I$,

$$\int_{t_n}^{t_n+t} (\nu P_s)_i ds \leq \sum_{j \in S} \int_{t_n}^{t_n+t} \nu_j ds \leq t$$

$$\int_0^t (\nu P_s)_i ds \leq \sum_{j \in S} \int_0^t \nu_j ds \leq t$$

since $\lim_{n\to\infty} t_n = +\infty$ we find immediately

$$\lim_{n\to\infty} \frac{1}{t_n} \int_{t_n}^{t_n+t} \nu P_s ds = 0 = \lim_{n\to\infty} \frac{1}{t_n} \int_0^t \nu P_s ds.$$

Summing up we have

$$\nu P_t = \lim_{n\to\infty} \frac{1}{t_n} \int_t^{t_n+t} \nu P_s ds = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \nu P_s ds = \pi$$

and $\pi$ is an invariant density. □

The number

$$\frac{1}{t} \int_0^t (\nu P_s)_i ds$$

represents the mean sojourn frequency in the state $i$ if the initial density is $\nu$. Indeed

$$(\nu P_s)_i = \sum_j \nu_j p_{ji}(s) = \sum_j \mathbb{P}\{ X_s = i \mid X_0 = j \}\mathbb{P}\{ X_0 = j \} = \mathbb{E}_\nu[1_{\{ X_s = i \}}]$$

so that

$$t^{-1} \int_0^t (\nu P_s)_i ds = \mathbb{E}_\nu \left[ t^{-1} \int_0^t 1_{\{ X_s = i \}} ds \right].$$

**Remark.** It is worth highlighting the relationship between invariant densities of a continuous time Markov chain and those of its discrete skeleton. If $\pi$ is a probability density satisfying $\pi Q = 0$, then, assuming, for simplicity, $q_{ii} < 0$

$$-\pi_j q_{jj} = \sum_{i\in I, i\neq j} \pi_i q_{ij} = \sum_{i\in I, i\neq j} (-\pi_i q_{ii}) \frac{q_{ij}}{-q_{ii}}$$

It follows that $(-\pi_i q_{ii})_{i\in I}$, if normalizable, is a multiple of an invariant density of the discrete skeleton. Conversely, given an invariant density $(\mu_i)_{i\in I}$ of the discrete skeleton, a similar computation shows that $\pi_i = \mu_i/(-q_{ii})$ defines an invariant density of the continuous time Markov chain.

Long time averages behave as in the discrete case. We state without proof the following

**Theorem 8.6** *(Ergodic theorem) Let $Q$ be the transition rate matrix of an irreducible Markov chain and let $\nu$ be any initial density. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_{\{X_s = i\}} ds = \frac{1}{-q_{ii}\mathbb{E}_i[T_i]}$$

*almost surely.*

# 9 Queues

A system with $k$ counters, with customers arriving at random times and served in random times as well can be described by a birth and death process under the following assumptions

- a customer arriving in the system is immediately served if there is a free counter, if not he waits until all clients arrived before him are served,
- times between the arrival of two consecutive customers are independent random variables with exponential distribution with parameter $\lambda > 0$,
- service times are independent random variables with exponential distribution with parameter $\mu > 0$.

In this situation, the state of the Markov chain represents the number $Q_t$ of customers in the system at time $t$. Clearly, transition rates for jumps $n \to n+1$ ($n \geq 0$) are all equal to $\lambda$.

Moreover, if $n \leq k$ counters are busy, the waiting time until one of them is free is the minimum of $n$ independent random variables with exponential distribution with parameter $\mu$. A simple computation shows that this random waiting time has exponential distribution with parameter $k\mu$ so that transition rates of jumps $n \to n-1$ ($n \geq 1$) are $\mu \min\{k, n\}$.

Let $X_t$ be the number of customers in the system at time $t$. The process $(X_t)_{t\geq 0}$ is a Markov chain with set of states $\mathbb{N}$ and transition matrix $(q_{mn})_{m,n\geq 0}$ given by

$$q_{mn} = \begin{cases} \lambda & \text{if} \quad n = m+1, \\ \mu \min\{k, m\} & \text{if} \quad n = m-1, \\ -(\lambda + \mu \min\{k, m\}) & \text{if} \quad n = m, \\ 0 & \text{if} \quad |n-m| > 1. \end{cases}$$

The process $(X_t)_{t\geq 0}$ is often called $M/M/k$ queue.

Letting $\nu = \lambda/\mu$, the invariant density can be explicitly computed as for birth and death processes

$$\pi_n = Z_k^{-1} \frac{\nu^n}{n!} \qquad \text{for } n = 0, 1, \ldots, k-1,$$

$$\pi_n = Z_k^{-1} \frac{\nu^n}{k!\, k^{n-k}} = \left(\frac{\nu}{k}\right)^{n-k} \pi_k \quad \text{for } n \geq k.$$

where $Z_k$ is the normalization constant. Noting that

$$\sum_{n\geq k} \frac{\nu^n}{k!\, k^{n-k}} = \frac{\nu^k}{k!} \sum_{n\geq k} \left(\frac{\nu}{k}\right)^{n-k} = \frac{\nu^k}{k!\,(1-\nu/k)}$$

we have

$$Z_k = \frac{\nu^k}{k!\,(1-\nu/k)} + \sum_{n=0}^{k-1} \frac{\nu^n}{n!},$$

in particular

$$Z_1 = (1-\nu)^{-1}, \qquad Z_2 = \frac{2+\nu}{2-\nu},$$

In this model one can compute explicitly several quality parameters of the service.

Mean number of customers in the system in the stationary regime

$$\begin{aligned}
\mathbb{E}_\pi [X_t] &= \sum_{n\geq 1} n\pi_n \\
&= \sum_{n=1}^{k} n\pi_n + \sum_{n>k} n \left(\frac{\nu}{k}\right)^{n-k} \pi_k \\
&= \sum_{n=1}^{k} n\frac{\nu^n}{n!}\pi_0 + \pi_k \sum_{n>k} (n-k) \left(\frac{\nu}{k}\right)^{n-k} + k\pi_k \sum_{n>k} \left(\frac{\nu}{k}\right)^{n-k} \\
&= \nu \sum_{n=1}^{k} \frac{\nu^{n-1}}{(n-1)!}\pi_0 + \pi_k \sum_{m>0} m \left(\frac{\nu}{k}\right)^{m} + \frac{\nu\pi_k}{1-\nu/k}
\end{aligned}$$

Recalling that the expectation of a geometric random variable with parameter $p$ is $1/p$ and so

$$\sum_{m>0} m\left(\frac{\nu}{k}\right)^m = \frac{\nu/k}{1-\nu/k}\sum_{m>0} m\left(1-\frac{\nu}{k}\right)\left(\frac{\nu}{k}\right)^{m-1} = \frac{\nu/k}{(1-\nu/k)^2},$$

we have

$$\begin{aligned}
\mathbb{E}[X_t] &= \nu \sum_{n=0}^{k-1} \frac{\nu^n}{n!}\pi_0 + \nu \sum_{n\geq k} \pi_n - \nu \sum_{n\geq k} \pi_n + \frac{\pi_k \nu/k}{(1-\nu/k)^2} + \frac{\nu\pi_k}{1-\nu/k} \\
&= \nu - \nu\pi_k \sum_{n\geq k} \left(\frac{\nu}{k}\right)^{n-k} + \frac{\pi_k \nu/k}{(1-\nu/k)^2} + \frac{\nu\pi_k}{1-\nu/k} \\
&= \nu + \frac{\pi_k \nu/k}{(1-\nu/k)^2}
\end{aligned}$$

In particular, for $k = 1, 2$, the average length of the queue is

$M/M/1 \qquad \nu + \frac{\nu^2}{1-\nu}$

$M/M/2 \qquad \nu + \frac{\nu^3}{4-\nu^2}$

The time spent in the system by a customer arriving at time $t$ is

$$W_t = S + \sum_{n\geq k} Y_{n-k+1} 1_{\{X_t = n\}}$$

where $S$ is the service time and $Y_{n-k+1}$ is the waiting time in the queue before finding one of the $k$ counters free. This event will occur when, in case on his arrival $X_t \geq k$, when $n-k+1$ customers are served.

The random variable $S$ has exponential distribution $\mathcal{E}(\mu)$. Moreover, when all the $k$ counters are busy, the waiting time for one of them to become free is the minimum of $n-k+1$ independent exponential random variables $\mathcal{E}(\mu)$ and so, has exponential distribution $\mathcal{E}(k\mu)$. The random variable $Y_{n-k+1}$, as the sum of $n-k+1$ independent exponential random variables with parameter $k\mu$ has distribution $\Gamma(n-k+1, k\mu)$; in particular, its expectation is $(n-k+1)/k\mu$.

It follows that

$$\begin{aligned}
\mathbb{E}_\pi\left[Y_{n-k+1}1_{\{X_t=n\}}\right] &= \mathbb{E}_\pi\left[Y_{n-k+1} \mid 1_{\{X_t=n\}}\right]\mathbb{P}_\pi\{X_t=n\} \\
&= \pi_n\mathbb{E}_n\left[Y_{n-k+1}\right] \\
&= \pi_n\frac{n-k+1}{k\mu}
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}_\pi[W_t] &= \frac{1}{\mu} + \sum_{n\geq k}\pi_n\frac{n-k+1}{k\mu} \\
&= \frac{1}{\mu} + \frac{\pi_k}{k\mu}\sum_{n\geq k}(n-k+1)\left(\frac{\nu}{k}\right)^{n-k} \\
&= \frac{1}{\mu} + \frac{\pi_k}{k\mu}\frac{1}{(1-\nu/k)^2}
\end{aligned}$$

In this way, we have found the *Little's law*

$$\mathbb{E}_\pi[X_t] = \lambda\mathbb{E}_\pi[W_t]$$

We can compute in a similar way other quantities like the *probability of congestion* (an arriving customer finds all the counters busy)

$$\mathbb{P}_\pi\{X_t\geq k\} = \sum_{n=k}^{\infty}\pi_n = \pi_k\sum_{n=k}^{\infty}\left(\frac{\nu}{k}\right)^{n-k} = \frac{\pi_k}{1-\nu/k} = \frac{\nu^k}{k!(1-\nu/k)Z_k}.$$

*Law of the waiting time in the queue $V_t$.* The random variable $V_t$ is a typical example of a random variable which is neither continuous nor discrete. Indeed

$$\mathbb{P}_\pi\{V_t=0\} = Z_k^{-1}\sum_{n=0}^{k-1}\frac{\nu^n}{n!}$$

and the continuous part is absolutely continuous

$$
\begin{aligned}
\mathbb{P}_\pi\{0 < V_t \le x\} &= \sum_{n=k}^{\infty} \mathbb{P}_n\{0 < V_t \le x\}\pi_n \\
&= \pi_k \sum_{n=k}^{\infty} \left(\frac{\nu}{k}\right)^{n-k} \mathbb{P}_n\{0 < Y_t \le x\} \\
&= \pi_k \sum_{n=k}^{\infty} \left(\frac{\nu}{k}\right)^{n-k} \int_0^x \frac{(k\mu)^{n-k+1} y^{n-k}}{(n-k)!} \mathrm{e}^{-k\mu y}\, dy \\
&= k\mu\pi_k \int_0^x \mathrm{e}^{-(k\mu-\lambda)y}\, dy \\
&= \frac{k\mu\pi_k}{k\mu-\lambda}\left(1 - \mathrm{e}^{-(k\mu-\lambda)x}\right).
\end{aligned}
$$

It follows that

$$
V_t \sim \left(\frac{1}{Z_k}\sum_{n=0}^{k-1}\frac{\nu^n}{n!}\right)\delta_0 + \frac{1}{Z_k}\frac{\nu^k}{k!}\frac{1}{1-\nu/k}\mathcal{E}(k\mu-\lambda)
$$

# 10 Exercises

**Exercise 10.1** A radioactive material consists of $n$ atoms. Each one of them decays (it is transformed into a non-radioactive material) in an exponential random time with parameter $\mu$. Compute the average time it takes for one-half of the atoms of to decay (radioactive half-life).

**Exercise 10.2** Let $(X_t)_{t\ge 0}$ be a continuous time Markov chain with states $\{1,2,3\}$ and transition rate matrix

$$
\begin{pmatrix} -a & a & 0 \\ 0 & -b & b \\ 0 & 0 & 0 \end{pmatrix}
$$

with $a, b > 0$. Suppose that $X_0 = 1$ (almost surely) and compute:

1. transition probabilities $p_{ij}(t)$ for all $i, j \in \{1,2,3\}$, $t \ge 0$,
2. the average time spent in states 1 and 2.

Suppose now that $X_t$ represents the state of health of a customer of an insurance company, more precisely 1 =healthy, 2 =sick 3 =dead. When the customer is healthy he pays an insurance premium of $p$ Euro per time unit, when he is sick he gets from the insurance company $q$ Euro per time unit, and when he dies the contract expires. What is the relationship among $a, b, p, q$ for the insurance policy to be fair (i.e. the expectations of the amount paid when healthy and received when sick are equal)? *Answ.* $p/a = q/b$