Real and Functional Analysis

Master Degree Program in Mathematical Engineering, a.y. 2021/22, group M-Z
Prof. Fabio Punzo

Theory Questions

Sheet n.1

Answers can be given in English or in Italian.

1) Write the definitions of: sequence of sets $\{E_n\}$; increasing and decreasing sequence of sets $\{E_n\}$; $\limsup_{n\to+\infty} E_n$, $\liminf_{n\to+\infty} E_n$, $\lim_{n\to+\infty} E_n$.

2) Write the definitions of: cover (or covering) of a set; subcover; open cover of a set; finite subcover.

3) Write the definitions of: equivalence relation, equivalence class, quotient set.

4) Write the definitions of: order relation, partially ordered set, totally ordered set (or chain). Provide examples of partially ordered sets and of totally ordered sets.

5) Write the definition of equipotent sets. Write the definition of cardinality of a set. State the Schröder-Bernstein theorem and the Cantor theorem.

6) Write the definitions of: infinite set, finite set, countable set, uncountable set. Provide examples.

7) Write the continuum hypothesis and the axiom of choice.

8) Write the definitions of: maximal element; upperbound; sup of a subset of a partially ordered set. State the Zorn's lemma. Which is the relation between the axiom of choice and Zorn's lemma?

9) Write the definitions of: dense set, separable metric space, nowhere dense set, set of first category, set of second category. Provide an example of a nowhere dense and one of a set of first category.

10) Write the definition of sequence of nested balls. State the theorem of nested balls (or the Cantor intersection theorem).

11) State and prove the Baire category theorem and its corollary. (*The proof includes also the preliminary lemma.*)

12) Write the definitions of: compact metric space; sequentially compact metric space, totally bounded metric space. Explain how these properties are related.

13) Write the $\varepsilon-\delta$ definition of equicontinuous subset F of $C^0(X)$, where X is a compact metric space. Explain from which parameters δ depends. In particular, write the definition when $F = \{f_n\}_{n\in\mathbb{N}}$.

14) State and prove the Ascoli-Arzelà theorem.

15) Write the statement of the Ascoli-Arzelà theorem when the subset of $C^o(X)$ is a sequence $\{f_n\}$.

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Theory Questions

Sheet n.2

Answers can be given in English or in Italian.

1) Write the definitions of: algebra, $\sigma-$algebra, measurable space, measurable set. Show that if $\mathcal{A}$ is a $\sigma-$algebra and $\{E_k\} \subset \mathcal{A}$, then $\bigcap_{k=1}^{+\infty} E_k \in \mathcal{A}$.

2) State the theorem concerning the existence of the $\sigma-$algebra generated by a given set. Give an idea of the proof.

3) Write the definition of the Borel $\sigma-$algebra in a metric space. Provide classes of Borel sets. Characterize $\mathcal{B}(\mathbb{R}), \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(\mathbb{R}^N)$.

4) Write the definitions of: measure, finite measure, $\sigma-$finite measure, measure space, probability space. Provide some examples of measures.

5) State and prove the theorem regarding properties of measures. For what concerns continuity w.r.t. a descending sequence $\{E_k\}$, show that the hypothesis $\mu(E_1) < +\infty$ is essential.

6) Write the definitions of: sets of zero measure; negligible sets. What is meant by saying that a property holds a.e.? Provide typical properties that can be true a.e. .

7) Write the definition of complete measure space. State the theorem concerning the existence of the completion of a measure space. Give just an idea of the proof.

8) Write the definition of outer measure. State and prove the theorem concerning generation of outer measure on a general set X, starting from a set $K \in \mathcal{P}(X)$, containing $\emptyset$, and a function $\nu : K \to \overline{\mathbb{R}}_+, \nu(\emptyset) = 0$. Intuitively, which is the meaning of (K, ν)?

9) What is the Caratheodory condition? How can it be stated in an equivalent way? Prove it.

10) Can it exist a set of zero outer measure, which does not fulfill the Caratheodory condition? Prove it.

11) State the theorem concerning generation of a measure as a restriction of an outer measure.

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Theory Questions

Sheet n.3

Answers can be given in English or in Italian.

1) Show that the measure induced by an outer measure on the $\sigma-$algebra of all sets fulfilling the Carathéodory condition is complete.

2) Describe the construction of the Lebesgue measure in $\mathbb{R}$ and in $\mathbb{R}^n$.

3) Prove that any countable subset $E \subset \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$.

4) Show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.

5) Prove that the translate of a measurable set is measurable.

6) Write the excision property and prove it.

7) Define the Vitali set. Prove that any measurable bounded set $E \subset \mathbb{R}$ with $\lambda(E) > 0$ contains a subset which is not Lebesgue measurable.

8) Write the definition of measurable function. Show the measurability of the composite function.

9) Characterize measurability of functions and prove it.

10) Write the definitions of: a) Borel measurable functions; b) Lebesgue measurable functions.

11) Prove that continuous functions are both Borel and Lebesgue measurable.

12) Characterize Lebesgue measurability of functions and prove it.

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Theory Questions

Sheet n.4

Answers can be given in English or in Italian.

1) Are there disjoint subsets $A, B \subset \mathbb{R}$ such that $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$? Justify the answer.

2) Establish and show all equivalent statements to the fact that $f : X \to \overline{\mathbb{R}}$ is measurable.

3) Let $f, g \in \mathcal{M}(X, \mathcal{A})$. What can we say about measurability of $\{f < g\}, \{f \leq g\}, \{f = g\}$? Justify the answer.

4) Let $\{f_n\} \subset \mathcal{M}(X, \mathcal{A})$. Show that $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n \in (X, \mathcal{A})$. Can they exist two functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $\max\{f, g\} \notin \mathcal{M}(X, \mathcal{A})$? Why?

5) Let $f, g \in \mathcal{M}(X, \mathcal{A})$. Show that $f + g, fg \in (X, \mathcal{A})$.

6) Prove that A is measurable if and only if χ_A is a measurable function.

7) Prove or disprove the following statements:

$$(a) \quad f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow f_{\pm} \in \mathcal{M}_{+}(X, \mathcal{A});$$

$$(b) \quad f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow |f| \in \mathcal{M}(X, \mathcal{A}).$$

8) Write the definition of simple function. What is its canonical form? How can we characterize measurability of a simple function? Write the definition of step function.

9) State and give a sketch of the proof of the Simple Approximation Theorem.

10) Write the definitions of $\operatorname{esssup}_X f$ and $\operatorname{essinf}_X f$. State their properties and prove some of them.

11) What is $\mathcal{L}^\infty$? Which is the relation between functions finite a.e. and essentially bounded functions? Justify the answer.

12) Write the definitions of the Lebesgue integral of a nonnegative measurable simple function over X and over a measurable subset $E \subseteq X$. Write the main properties of the integral and prove some of them.

13) Let $s \in \mathcal{S}_{+}(X, \mathcal{A})$. For any $E \in \mathcal{A}$, let $\varphi(E) := \int_E s d\mu$. Prove that φ is a measure.

14) Write the two possible equivalent definitions of Lebesgue integral of a measurable nonnegative function.

15) State and prove the Chebychev inequality.

16) Define the Cantor set. State its main properties and prove some of them.

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Theory Questions

Sheet n.5

Answers can be given in English or in Italian.

1) Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f d\mu < \infty$. Show that f is finite a.e. in X.

2) Prove the vanishing lemma for functions $f \in \mathcal{M}_+(X, \mathcal{A})$.

3) State and prove the Monotone Convergence Theorem.

4) State and prove the theorem concerning integration of series with general terms $f_n \in \mathcal{M}_+(X, \mathcal{A})$.

5) State and prove the Fatou's Lemma.

6) Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Show that $\nu(E) := \int_E f d\mu$ is a measure; state and prove its properties.

7) Let $f, g \in \mathcal{M}_+(X, \mathcal{A})$ with $f = g$ a.e. in X. Show that $\int_X f d\mu = \int_X g d\mu$.

8) Write the definition of: integrable functions; Lebesgue integral; $\mathcal{L}^1(X, \mathcal{A}, \mu)$.

9) Let $f : X \to \overline{\mathbb{R}}$. How the integrability of f is related to that of $f_\pm$ and of $|f|$? Justify the answer. Show that if $f \in \mathcal{L}^1$, then $\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.

10) Prove that $\mathcal{L}^1$ is a vector space.

11) Let $f \in \mathcal{L}^1$ be such that $\int_E f d\mu = 0 \; \forall E \in \mathcal{A}$. Show that $f = 0$ a.e. in X.

12) Let $f \in \mathcal{L}^1, g \in \mathcal{M}, f = g$ a.e. in X. Show that $g \in \mathcal{L}^1$ and $\int_X g d\mu = \int_X f d\mu$.

13) State and prove the Lebesgue theorem.

14) Describe the relations between Riemann and Lebesgue integrals.

15) State and prove the theorem concerning integration for series with general terms $f_n \in \mathcal{L}^1$.

16) Write the definition of Radon-Nikodym derivative of a measure ν w.r.t. a measure μ. When ν is said to be absolutely continuous w.r.t. μ?

17) State the Radon-Nikodym theorem. Prove the uniqueness of the derivative of a measure.

18) Write the definitions of L^1 and of L^∞. Show that they are metric spaces. Are $\mathcal{L}^1$ and $\mathcal{L}^\infty$ metric spaces?

19) For a sequence of functions $\{f_n\} \subset \mathcal{M}$, write the definitions of: pointwise convergence; uniform convergence; almost everywhere convergence; convergence in L^1; convergence in L^∞; convergence in measure.

20) Give two equivalent formulations of a.e. convergence.

21) Explain how the Rademacher sequence (also called the typewriter sequence) is defined. In particular, write explicitely $I_1, I_2, \ldots, I_{14}$.

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Theory Questions

Sheet n.6

Answers can be given in English or in Italian.

1) By means of a counterexample, show that in general $f_n \to f$ in measure does not imply that $f_n \to f$ a.e..

2) Prove that convergence in measure implies convergence a.e. up to subsequences.

3) Under which hypothesis on X, does convergence a.e. imply convergence in measure? Show this property. What happens if one omits the key assumption on X?

4) Show that convergence in L^1 implies convergence in measure.

5) Show that convergence in L^1 implies convergence a.e. up to subsequences.

6) Does convergence in measure imply convergence in L^1? Does convergence a.e. imply convergence in L^1? Justify the answer.

7) Write the definitions of: product measurable space, section of a measurable set. What is the product measure? Why is the definition well-posed?

8) Is the product measure space complete? Justify the answer. Which is the relation between $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ and $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$?

9) State the Tonelli theorem.

10) State the Fubini theorem. By means of a counterexample, show that it is not possible to omit the hypothesis $f \in L^1$.

11) Write the definition of Lebesgue point. What is about the measure of the set of points that are not Lebesgue points for a function $f \in L^1$?

12) State and prove the First Fundamental Theorem of Calculus for $f \in L^1$.

13) Let $f : [a,b] \to \mathbb{R}$. Write the definitions of: variation of f relative to a partition of $[a,b]$; total variation of f over $[a,b]$; function of bounded variation.

14) Let $f : [a,b] \to \mathbb{R}$ be monotone. Why $f \in BV([a,b])$? Show that if $f \in BV([a,b])$, then f is bounded.

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Theory Questions

Sheet n.7

Answers can be given in English or in Italian.

1) What is the Jordan decomposition of a BV function? Why a function of bounded variation is differentiable a.e.?

2) Let $f : [a, b] \to \mathbb{R}$ be increasing. What can we say about f' and $\int_{[a,b]} f' d\lambda$? Justify the answer.

3) Can there exists a function $f \in BV([a, b])$ with $f' \notin L^1([a, b])$? Justify the answer.

4) Write the definition of absolutely continuous function. Show that an absolutely continuous function is also uniformly continuous, but the viceversa is not true; furthermore, a Lipschitz function is absolutely continuous, but the viceversa is not true.

5) Let $f \in \mathcal{M}_+(X, \mathcal{A})$ be such that $\int_X f d\mu < +\infty$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \in \mathcal{A}$ with $\mu(E) < \delta$ there holds $\int_E f d\mu < \varepsilon$.

6) Show that if $f \in L^1([a, b])$, then $F(x) := \int_{[a,x]} f d\lambda$ is absolutely continuous in $[a, b]$.

7) Which is the relation between the spaces $BV([a, b])$ and $AC([a, b])$?

8) State and prove the Second Fundamental Theorem of Calculus.

9) Write the definition of normed space and provide examples. What is the metric space induced by a given normed space?

10) In a normed space, write the definitions of: convergent sequence; Cauchy sequence; bounded sequence. Which are the relations among these notions? Show that if $x_n \to x$, then $\|x_n\| \to \|x\|$ as $n \to +\infty$.

11) Write the definition of series in a normed space. Is it true that if $\sum_{n=0}^{+\infty} \|x_n\| < \infty$, then $\sum_{n=0}^{+\infty} x_n$ is convergent?

12) What is a complete normed space? Write the definition of Banach space, provide examples.

13) Show that $C^0([a, b])$ is separable.

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Theory Questions

Sheet n.8

Answers can be given in English or in Italian.

1) State and prove the criterion for completeness of a normed space.

2) State and prove the Riesz Lemma.

3) State and prove the Riesz Theorem.

4) Write the definition of equivalent norms. Provide examples of equivalent norms and of norms that are not equivalent.

5) Show that in a normed space of finite dimension, all norms are equivalent.

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Theory Questions

Sheet n.9

Answers can be given in English or in Italian.

1) Write the definitions of $\mathcal{L}^p$ and L^p. Show that L^p is a vector space (and its preliminary lemma).

2) Write the definition of conjugate numbers. Show the Young inequality.

3) Show the Holder inequality.

4) Show the Minkowski inequality.

5) Show that L^p is a normed space.

6) Show the inclusion of L^p spaces. Which hypothesis is essential? Justify the answer.

7) Show the completeness of L^p spaces.

8) Show that convergence in L^p implies convergence in measure.

9) Show that if $f_n \to f$ in L^p as $n \to \infty$ ($p \in [1, +\infty]$), then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. in X as $k \to \infty$.

10) State the Lusin theorem.

11) Show that the set of simple functions with supports of finite measure is dense in L^p ($p \in [1, +\infty)$).

12) Show that $C_c^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ ($p \in [1, +\infty)$).

13) Show that $L^p(\mathbb{R})$ is separable ($p \in [1, +\infty)$).

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Theory Questions

Sheet n.10

Answers can be given in English or in Italian.

1) Show that $L^\infty(\mathbb{R})$ is not separable.

2) How ℓ^p and L^p are related?

3) State the Jensen inequality.

4) Write the definitions of: linear operator; bounded operator; functional; continuous operator.

5) State and prove the theorem about the characterization of linear, bounded operators.

6) Prove or disprove the following statement. Let X, Y be normed spaces, $T : X \to Y$ be a linear operator. T is continuous in X iff it is continuous at $x_0 = 0$.

7) Prove that $\mathcal{L}(X, Y)$ is a vector space. Which is the standard norm on it?

8) The norm on $\mathcal{L}(X, Y)$ satisfies two important equalities. Write and show them.

9) Show that $\mathcal{L}(X, Y)$ is a Banach space, provided that Y is a Banach space.

10) Write the definitions of: invertible operator; isometry; embedding.

11) State and prove the UBP (or BS theorem).

12) From the UBP it is possibile to infer an important property of operators defined by means of a point-wise limit. What is that? Justify your answer.

13) Write the definition of open mapping. State the OMT.

14) State and prove the IBM theorem.

15) By the IBM theorem we can infer an important property about equivalent norms on Banach spaces. What is that? Justify your answer.

16) Write the definitions of: closed operator; graph of an operator. Show that an operator is linear and closed iff its graph is closed.

17) State and prove the closed graph theorem.

18) Write the definition of dual space. Write equivalent conditions to $L \in X^*$.

19) Exhibit an example of $T \in (L^p)^*$.

20) What is $(L^p)^*$ for $p \in [1, +\infty)$?

20) Write the definition of sublinear functional.

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Theory Questions

Sheet n.11

Answers can be given in English or in Italian.

1) State and prove the Hahn-Banach Theorem in the dominated extension form.

2) State and prove the Hahn-Banach Theorem in the continuous extension form.

3) State and prove two corollaries of the Hanh-Banach Theorem.

4) Give a sufficient condition for separability of X.

5) Show that the dual of L^∞ is not L^1.

6) Introduce the canonical map. Show that it is an isometry.

7) Write the definition of reflexive space. State the properties of reflexive spaces.

8) Write the definition of uniformly convex Banach space. State the Milman-Pettis theorem.

9) Show the reflexivity of L^p with $p \in (1, \infty)$. Why L^1 and L^∞ are not reflexive?

10) Write the definition of weak convergence. How can it be formulated in L^p and in ℓ^p?

11) Show that strong convergence implies weak convergence. Provide a counterexample for the converse implication.

12) Show that the weak limit is unique.

13) If $\{x_n\}$ weakly converges to x, can $\{x_n\}$ be unbounded? State and prove lower semicontinuity w.r.t. weak convergence of $x \mapsto \|x\|$.

14) Show that if x_n weakly converges to x and L_n converges to L in X^*, then $L_n(x_n) \to L(x)$ as $n \to \infty$.

15) Show that $T \in \mathcal{L}(X, Y)$ is weak-weak continuous.

16) Write a sufficient condition for weak convergence in reflexive spaces.

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Theory Questions

Sheet n.12

Answers can be given in English or in Italian.

1) Write the definition of weak* convergence.

2) Which is the relation between weak and weak* convergence? Justify your answer.

3) Write the properties of weak* convergence.

4) State the Banach-Alaoglu theorem. Why can we say that from a bounded sequence in L^∞ we can extract a subsequence which weakly* converges in L^∞?

5) State and prove the corollary of the Banach-Alaoglu theorem in a separable and reflexive Banach space.

6) State the Eberlein-Smulyan theorem.

7) Write the definition of compact operator. Write the definition of operator of finite rank. How is a compact operator related to operators of finite rank? Can a compact operator defined on an infinte dimensional Banach space be bijective?

8) State the theorem about the characterization of compact operators.

9) Write the definition of pre-Hilbert and of Hilbert spaces.

10) Show the parallelogram law.

11) State and prove the theorem in Hilbert spaces about minimal distance from convex closed subsets (both the preliminary result and the corollary).

12) State and prove the theorem about orthogonal projections in Hilbert spaces.

13) Write and show properties of the orthogonal projector.

14) State and prove the Riesz theorem.

15) Write the definition of orthonormal bases in a Hilbert space.

16) Write the Bessel inequality. Write the theorem about abstract Fourier expansion in a Hilbert space and the Parseval identity.

17) Let H be a separable space, $\{\varphi_n\}$ be an orthonormal basis. Show that φ_n weakly converges to 0, but it does not converge strongly to 0.

18) State the Riesz-Fisher theorem.

19) What can we say about reflexivity of Hilbert spaces?

20) Show the formula which gives the norm of an operator on a Hilbert space.

21) Write the definition of symmetric operator on a Hilbert space. Write the formula which gives the norm of a symmetric operator on a Hilbert space.

22) Write the definition of eigenvalue, eigenvector and eigenpace for linear and bounded operators on a Hilbert space.

23) What can we say about eigenvalues of symmetric operators and of symmetric compact operators on Hilbert spaces?

24 Let H be a separable Hilbert space, $k \in \mathcal{K}(H)$. Show that $\dim V_\lambda < \infty$, provided that $\lambda \neq 0$.

25) Write the definition of spectrum and of resolvent of a symmetric operator on Hilbert spaces.

26) State the spectral theorem.

27) State the theorem about the Fredholm alternative.