1

Problem 1: Discrete optimization model (7.5 points)

The 118 Operations Center of the Niguarda Hospital, which coordinates the emergency medical services for the city of Milan and its province, must decide where to locate k ambulances so as to answer as well as possibile to the emergency calls/requests.

We assume to know:

- the set $S = \{1, \ldots, m\}$ of the candidate sites where an ambulance can be stationed while waiting to be assigned a mission, with $|S| > k$,
- the set $C = \{1, \ldots, n\}$ of the points which represent the possible emergency call locations (estimated based on demographic and historical data),
- the time t_{ij} (in minutes) that the ambulance needs to go from candidate site i to the call location j, for each pair $i \in S$ and $j \in C$.

In this simplified version of the problem we consider the answer to any next single call. Moreover, we suppose that every candidate site can host at most one ambulance and that for the given S, C and times t_{ij} a feasible solution exists.

a) Give an Integer Linear Programming formulation for the problem of determining in which candidate sites to locate the ambulances and which ambulance to send to each call location so as to minimize the maximum response time to the next single call (i.e., the time needed for the ambulance to arrive at the call location).

b) To partially account for the fact that an ambulance may already be busy (be assigned to another mission not yet completed), we have to identify two ambulances for each possible call location $j \in C$: a first ambulance that must be able to arrive at the call location within 8 minutes and a second ambulance that will be sent only in case the first ambulance is already busy.

Give an Integer Linear Programming formulation for the problem of determining at which candidate sites to locate the ambulances so as to minimize the maximum time needed for the second ambulance to arrive at any possible next single call location, while guaranteeing that in the case of any next single call an ambulance can arrive within 8 minutes.

2

Problem 2: Optimality conditions for constrained nonlinear optimization (8 points)

Consider the following nonlinear optimization problem

$$\begin{array}{ll} \min & (x_1 - 2)^2 + x_2 \\ \text{s.t.} & x_1^2 \le x_2 \\ & x_1 + x_2 \le 2. \end{array}$$

i) Draw the feasible region.

ii) At which points of the feasible region is the constraint qualification assumption satisfied? Motivate the answer.

iii) Give a precise statement of the first order optimality conditions.

iv) Explain why the above optimality conditions are necessary/sufficient or both.

v) Determine all candidate points that satisfy the above optimality conditions and identify the global optimal solution.

vi) Write the Lagrangian dual of the above problem and indicate the connection between the primal problem and the dual problem under consideration. Motivate the answer.

DO exercise

DO model

NLO exercise

4

Problem 4: Nonlinear optimization method (7.5 points)

a) Describe the method presented in the course for solving the so-called quadratic programming problems, that is, the constrained nonlinear optimization problems with a quadratic objective function subject to linear inequality and equality constraints.

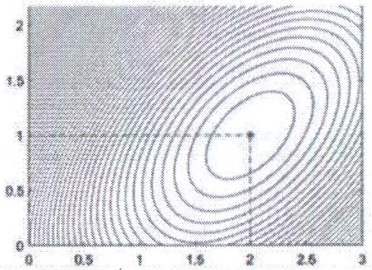
b) Discuss the properties of the method, and indicate the advantages and disadvantages.

c) Consider the simple quadratic program

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 - x_1 x_2 - 3x_1 & \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 & (1) \\ & x_1 \geq 0 & (2) \\ & x_2 \geq 0 & (3) \end{aligned}$$

and describe/sketch the first three iterations of the algorithm starting from $\mathbf{x}_0 = (0,0)^t$. In case of choice in the initialization, clearly mention the choice you made.

Note that the level curves of the objective function are as follows:



3

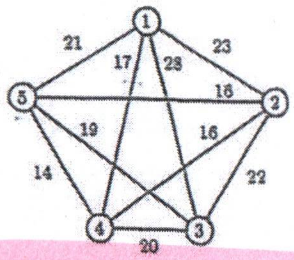
Problem 3: Discrete optimization method (8 points)

Consider the Symmetric Traveling Salesman problem (STSP): given an undirected graph $G = (V, E)$ with a c_e associated to each edge $e \in E$, determine an Hamiltonian cycle, i.e., a cycle that visits each node exactly once, of minimum total cost.

- Give two Integer Linear Programming (ILP) formulations for the problem. How many constraints do they contain?
- Describe the Lagrangian relaxation method for STSP base on the 1-trees.

  After recalling the definition of a 1-tree, explain which ILP formulation we start from and which group of constraints is relaxed.

  Explain how we can solve: i) the Lagrangian subproblems and ii) the Lagrangian dual problem.
- Apply one iteration of the method to the following instance starting from multiplier vector $\mathbf{u}_0^t = (-1, 0, -1, 3, -1)$ and considering 1-trees with node 2 as special node.



3

Problem 3: Integer programming method (11,5 points)

- Describe the general idea of the column generation method for integer linear programming problems, including all components.
- Describe the method for the 1-D cutting stock problem (CSP) or for another discrete optimization problem. Given a set of large rolls of width W and demands for b_i small rolls of width w_i, with $i \in I = \{1, \ldots, m\}$, the 1-D cutting stock problem consists in deciding how to cut the large rolls into small rolls so as to minimize the number of large rolls used, while satisfying the customer demand.
- Briefly discuss the main advantages and drawbacks of the method as well as a possibile extension.

1

Problem 1: Model (8 points)

A company has to cut n disks from a rectangular metal sheet of length L and width W. The i-th disk has a given radius r_i, with $1 \leq i \leq n$. To have some tolerance in cutting, when disks are cut there must be a distance d between them. The same requirement of a tolerance d must also hold for each disk with respect to the boundaries of the metal sheet.

The goal of the company is to place the disks as efficiently as possibile. Unused parts of the metal sheet at the edges with a rectagular form, can be used in another process and these parts have a value of v per area unit.

Give a nonlinear optimization formulation for the problem of determining the placement of the disks so as to maximize the value of the unused rectangular parts.

Hint: Define variables corresponding to the sizes of the unused rectangular parts.

Temi d'esame – Ottimizzazione Discreta

Problema 1

Sia il problema del Commesso Viaggiatore Simmetrico: dato un grafo non orientato $G = (V, E)$ con un costo c_e associato ad ogni lato $e \in E$, determinare un ciclo hamiltoniano, ovvero un ciclo che passi esattamente una volta per ogni nodo, di costo totale minimo.

- Dare due formulazioni di programmazione lineare intera per questo problema, indicando l'ordine di grandezza del numero di vincoli rispetto al numero di nodi del grafo.
- Quale relazione esiste tra i rilassamenti lineari (continui) delle due formulazioni? Motivare la risposta.

Problema 2

Consideriamo un insieme di n siti candidati in cui attivare un deposito con un costo fisso f_j per ogni deposito j, $1 \leq j \leq n$, e m clienti distribuiti sul territorio. Sia c_{ij} il profitto quando tutta la domanda del cliente i viene soddisfatta dal deposito j, $1 \leq i \leq m$ e $1 \leq j \leq n$. Il problema di localizzazione (uncapacitated facility location problem) consiste nel decidere quali depositi attivare e come soddisfare la domanda di ogni cliente in modo da massimizare il profitto totale (guadagni meno costi fissi).

- Dare due formulazioni di programmazione lineare intera del problema.
- Indicare quale formulazione è più stringente, spiegando perché.
- Descrivere il metodo del rilassamento lagrangiano per questo problema in cui si rilassa i vincoli di domanda. Spiegare come si può risolvere il sottoproblema del rilassamento lagrangiano.
- Applicare il metodo all'istanza con $m = 6$, $n = 5$, la seguente matrice di profitto e i

$$(c_{ij}) = \begin{pmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{pmatrix}$$

costi fissi $f = (2, 4, 5, 3, 3)$. Partire dal vettore di moltiplicatori $\mathbf{u}_0^t = (5, 6, 3, 2, 6, 4)$.

Problema 3

Descrivere l'idea generale dei metodi di taglio per i problemi di ottimizzazione discreta. Illustrarla con un esempio: descrivere un problema specifico ed almeno una classe di disuguaglianze valide, spiegando come si possono generare. Indicare come si possono integrare la generazione di tagli con il Branch and Bound e quali sono i vantaggi.

Problema 4

Descrivere la tecnica del rilassamento lagrangiano per i problemi di ottimizzazione discreta. Illustrarla con un esempio. Spiegare quanto è stringente il bound ottenuto e come si risolve il duale lagrangiano.

Politecnico di Milano

Optimization – Ottimizzazione Discreta – Complementi di R. O.
Prof. E. Amaldi

Written exam of September 1, 2015

Problem 1 [OPT and OD] (Points: 7 OPT and 8 OD)

Consider a transportation problem with a single type of product, m production plants ($1 \leq i \leq m$) and n clients ($1 \leq j \leq n$).

Let

- p_i be the production capacity of plant i (maximum amount that can be produced),
- d_j be the demand of client j,
- f_{ij} be the fixed transportation cost from plant i to client j,
- t^1_{ij} be the unit transportation cost from plant i to client j for up to 100 units of product, and t^2_{ij} be the unit transportation cost from plant i to client j for more than 100 units, with $t^1_{ij} > t^2_{ij}$,
- q_{ij} be the maximum amount that can be shipped from plant i to client j.

To avoid excessive splitting of the deliveries, each client can be served from at most k plants.

Give a mixed integer programming formulation for the problem of determining a transportation plan which minimizes the total transportation costs, while satisfying all client demands and plant capacities.

Problem 2 [OPT and CRO] (Points: 8 OPT and 11 CRO)

Consider the following nonlinear optimization problem

$$\begin{array}{ll} \min & (x_1-2)^2 + x_2 \\ \text{s.t.} & x_1^2 \leq x_2 \\ & x_1 + x_2 \leq 2. \end{array}$$

i) Draw the feasible region.

ii) At which points of the feasible region is the constraint qualification assumption satisfied? Motivate the answer.

iii) State the optimality conditions, indicate whether they are necessary/sufficient or both. Explain why.

iv) Determine all candidate points that satisfy the above optimality conditions and identify the global optimal solution.

v) Write the Lagrangian dual of the above problem and indicate the relationship between the primal problem and its Lagrangian dual problem. Motivate the answer.

Problem 3 [OPT and CRO] (Points: 7 OPT and 12 CRO)

Describe the methods for solving constrained nonlinear optimization problems with a quadratic objective function subject to linear constraints, that is, the so-called quadratic programming problems. First consider the case with only linear equality constraints and then the case with also linear inequality constraints. Discuss the properties of the methods. Describe an application that can be formulated as a quadratic program.

Problem 4 [OPT and OD] (Points: 8 OPT and 11 OD)

Consider the Symmetric Traveling Salesman problem (STSP): given an undirected graph $G = (V, E)$ with a c_e associated to each edge $e \in E$, determine an Hamiltonian cycle, i.e., a cycle that visits each node exactly once, of minimum total cost.

a
- Give two Integer Linear Programming (ILP) formulations for the problem. How many constraints do they contain?

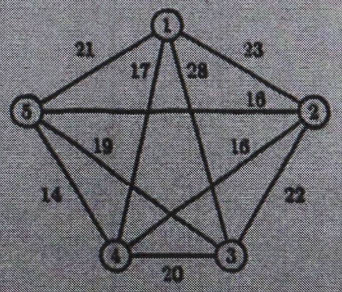
b
- Describe the Lagrangian relaxation method for STSP based on the 1-trees:

  Which ILP formulation do we start from and which constraints are relaxed?

  Define i) the Lagrangian subproblems and ii) the Lagrangian dual problem, and explain how each of them can be solved.

c
- Apply one iteration of the method to the following instance starting from the Lagrange multiplier vector $\mathbf{u}_0^t = (-1, 0, -1, 3, -1)$ and considering 1-trees with node 2 as special node.

  4



Optimization - Ottimizzazione Discreta - Complementi di R. O.

Prof. E. Amaldi

Written exam of September 28, 2016

Problem 1 [OPT and OD] (Points: 7 OPT and 8 OD)

The manager of a water reservoir with a single source and n clients has to plan which canals to build for the distribution of water and has to determine, for each client, the amount of water that can be extracted from the basin.

The water distribution system can be described as a directed graph $G = (N \cup \{0\}, A)$ where the node 0 is the source and the other n nodes are the clients (i.e., $N = \{1, ..., n\}$). Arcs $(i, j) \in A$ represent possible canals between pairs of nodes; their direction depends on the slope of the terrain. For each possible canal $(i, j) \in A$ the building cost c_{ij} and the maximum flow capacity u_{ij} in gallons are known. Each client $i \in N$ needs between b_i and B_i gallons of water for his business and earns g_i dollars for each gallon of water received. To ensure market fairness the authority requires that the total earnings of any client cannot exceed by more than 10% the earnings of the client(s) with lowest earnings.

To prevent hydro-geological instability: the flow of each canal in $E \subset A$ (if built) must be at least α times its capacity, where $0 \leq \alpha < 1$. Moreover, if both canals (v_1, v_2) and $(v_3, v_4) \in A$ are built then either (w_1, w_2) or $(w_3, w_4) \in A$ can not be built.

The goal of the manager is to minimize the total building cost, while providing the service in a fair way. Give a mixed integer programming formulation for this problem.

Problem 2 [OPT and CRO] (Points: 8 OPT and 11 CRO)

Consider the following nonlinear optimization problem

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & (x_1 - 1)^3 + (x_2 - 2) \leq 0 \\ & (x_1 - 1)^3 - (x_2 - 2) \leq 0 \\ & x_1 \geq 0. \end{aligned}$$

i) Draw the feasible region and indicate at which points of the feasible region the constraint qualification assumption is satisfied. Motivate the answer.

ii) State the optimality conditions. Are they necessary/sufficient or both? Explain why.

iii) Determine all candidate points that satisfy the above optimality conditions and identify the global optimal solution. Motivate the answer.

iv) Write the Lagrangian dual of the above problem and indicate the relationship between the primal problem and its Lagrangian dual problem.

Problem 3 [OPT and CRO] (Points: 7 OPT and 12 CRO)

Describe the Newton method for unconstrained nonlinear optimization problems. Discuss the properties of the method, stating the main results and indicating the advantages and disadvantages. Describe the general idea of quasi-Newton methods and that of the Davidon-Fletcher-Powell (DFP) method, emphasizing the advantages with respect to the Newton method.

?

Problem 4 [OPT and OD] (Points: 8 OPT and 11 OD)

Consider the maximum independent set problem in graphs that arises (as subproblem) in a wide variety of applications in which we wish to determine the maximum number of activities that are compatible based on a list of pairs of incompatible activities.

Maximum independent set problem: Given an undirected graph $G = (V, E)$, determine an independent subset of nodes $S \subseteq V$ (i.e., a subset of nodes such that $\{i, j\} \notin E$ for each pair of nodes i, j in S) containing the largest number of nodes.

i) Give an integer programming formulation of the problem.

ii) Describe two types of valid inequalities based on particular subsets of nodes and explain why they are valid inequalities.

iii) Copy the graph drawn on the blackboard and determine all the valid inequalities (of both types) which are not included in the formulation of i).

iv) For one of the two classes of valid inequalities the separation problem can be solved in polynomial time. Describe the separation problem and explain how it can be solved. Illustate with an example.

TDE OTTIMIZZAZIONE 21/07/15

Domanda 1 118 calls

$J = \{1, \ldots, m\}$ set of candidate sites $|J| \leq K$ for the ambulances

$C = \{1, \ldots, v\}$ set of possible call locations

t_{ij} time (in minutes) $i \to j$ $i \in J$, $j \in C$

1) 1 ambulance per call site

Max 1 ambulance per candidate site

ILP formulation

2) 2 ambulances per call site

1^{st} must arrive in less than 8 min

if 1^{st} is busy, 2^{nd} goes

ILP formulation

Question 2

$\min \ (x+2)^2 + (y+2)^2$

s.t. $x^2 + y^2 \leq 2$

$y \leq 1$

1) Feasible region

2) Constraint qualification

3) Optimal conditions. Nec/Suff

4) Candidate points. Global optimum.

5) Lagrangian dual. Link between the 2.

Question 3

Progr. pb. with quadratic objective function and linear constraints.

(only equalities and then also inequalities).

Methods to solve them

Problem 4

Symmetric Traveling Salesman problem

$G = (V, E)$ undirected graph

c_e cost of arc E

1) 2 ILP formulations.

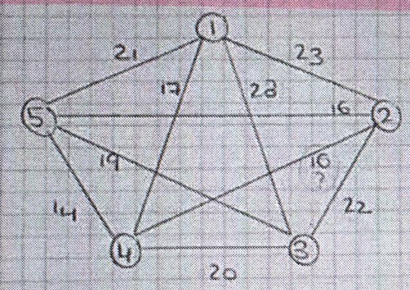
Number of constraints of each?

2) Lagrangian relaxation of one of the 2 ILP formulations

Explain: • Lagrangian subproblem

• Lagrangian constraint (?)

3)



Compute 1^{st} iter. with $\underline{u}_0 = (-1, 0, -1, 3, -1)$

TDE 11-02-14

X **Prob 1:**

Problema di trasporto 1 prodotto m impianti n clienti

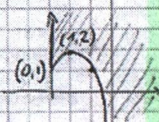
- P_i = max q.tità prodotto nell'impianto i
- d_j = domanda cliente j
- f_{ij} = costo fisso trasporto da i a j
- t_{ij} = costo unitario " da i a j

Inoltre un cliente può essere fornito al massimo da k impianti

Progran. matem. lin. per min costi soddisfando vincoli di domanda e di capacità

X **Prob 2:**

$$\min \quad (x_1+1)^2 + x_2$$
$$\text{st} \quad -(x_1-1)^2 - x_2 \le -2$$
$$x_1 \ge 0$$



- disegna reg ammissibile
- quali pti soddisfano QV ?
- enunciare le condiz di ottimalità e specificare se sono necess e/o suff
- elencare tutti i punti candidati e il punto di min. globale
- scrivere il pb duale e enunciare le proprietà (spiegare che relazione tra il primale)

duale Lagrangiano

TDE 18-07-13

? **Pb 1:** silos cilindrico

magazzino 10 x 20 base

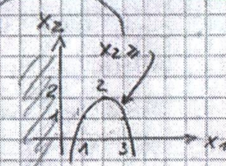
materiale per silos 200 m^2

$\Longrightarrow$ max volume silos soggetto ai vincoli

Scrivere modello

OK **Pb 2:**

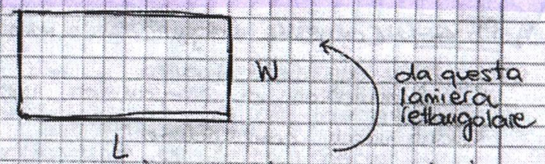
$$\min \quad (x_1+1)^2 + x_2$$
$$\text{sv} \quad$$
$$x_1 \ge 0$$



- regione delle sol. ammissibili
- pti della reg amm che soddisfa condiz qualifica vincoli
- enuncia condizioni di ottimalità (specificare se sono condiz nec e/o suff.)
- quali sono i pti candidati a diventare minimo globale
- scrivere il pb duale Lagrangiano e proprietà della fz duale

TDE 11-09-13

Pb1:



devo ritagliare n coperchi rotondi raggio r_i $i = 1:n$, distanza minima tra coperchio e bordo e tra coperchi = d

Le parti rettangolari che avanzano (lungo il bordo) hanno valore p €/m^2

Modello programmaz. non lineare ~~per massimizzare~~ per tagliare gli n coperchi massimizzando valore parti recuperate

Pb 2: $f(x_1, x_2) = x_1 + x_2$

reg ammiss. $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 2, \ x_2 \geq 0\}$

- Quali pti di X soddisfano condiz. di qualifica dei vincoli?
- enunciare condiz. di ottimalità e di che condiz. si tratta. ↳ per il pb di minimizzare f su X
- Scrivere tutti pti di minimo e di massimo
- Pb duale del pb di minimizzazione di f. Che relazione c'è tra primale e duale?

Pb 3: descrivere Metodo penalità quadratica per pb ottimizz. non lineare vincolata. Enunciare, commentando, i principali risultati ottenuti. Descrivere l'idea del metodo di Lagrange aumentato e quali sono i vantaggi rispetto alla penalità quadratica

Pb 4 Commesso viaggiatore simmetrico

Scrivi 2 formulazioni di programmaz. intera (o misto intera). Quanti vincoli comprendono?

- fare rilassam. lagrangiano con il ... degli 1-alberi. Quanti vincoli vengono rilassati?
- enunciare come si possono risolvere:
  - il sottopb lagrangiano
  - il duale
- (dà il grafo con tutti i costi, $u_0 = (\ldots)$ dice di basarsi sui 2-alberi) fare una iterazione

Pb 5: (in alternativa a pb 1)

n articoli, m macchine

s_i data, al più presto, in cui si può iniziare a lavorare l'articolo i

d_i data entro cui l'articolo i deve essere pronto

p_{ik} tempo che articolo i deve stare (in lavorazione) nella macchina k

Una macchina può lavorare 1 art. per volta.

~~Un articolo~~ La lavoraz. di un articolo non può essere interrotta e poi ripresa.

Per semplicità l'articolo segue l'ordine crescente degli indici delle macchine (da 1 a m)

- formulaz. misto intera per il pb di determinare la sequenza con cui gli articoli vanno nelle macchine minimizzando il tempo di lavorazione
- Come si può estendere la formulazione se ...da un sottoinsieme di macchine diverso dall'ordine delle macchine?

(poi diceva un'altra cosa tipo "... coppie di articoli")

gli articoli devono essere lavorati

come si modificherebbe per dare precedenza a coppie di articoli

---

o modelli?

X Pb 3: modello per problema programmazione quadratica.
Trattare prima il pb con sole uguaglianze e poi anche disuguaglianze
Proprietà, vantaggi e svantaggi.
Citare un esempio

X Pb 5: Disuguaglianze di
$X = \{ x \in (0,1)^n : \sum_{j=1}^{n} a_j x_j \leq b \}$

- cosa sono le disug. di copertura e spiegare perché sono valide per X
- probl. di separazione delle disug di copertura
- per $X = \{ \ldots x_1 + \ldots x_2 + \ldots \ldots + \ldots x_6 \leq b \}$
  scrivere tutte le disug... minimali
  e pb del rafforzamento per il pb che considera x_3 x_5 x_6.

X Pb 6: Pb di trasporto m impianti, n clienti

p_i = prodotto a disposiz. nell'impianto i
d_j = domanda cliente j
f_{ij} = costo fisso da imp i a cliente j
v_{ij} = costo unitario da i a j

un cliente può essere rifornito al massimo da k impianti.

Scrivere modello per minimizzare i costi.

A scelta tra 1 e 6

X **Prob 3**: Programmaz quadratica
Prima solo uguaglianze. Poi anche disuguag
Vantaggi svantaggi, citare un esempio

X **Prob 4**: m richieste di calcolo
n calcolatori

Pb di scegliere le richieste di calcolo e ~~[illegible]~~ su quale elaboratore allocarle massimiz il profitto

p_j = profitto del calcolo j
c_i = capacita' dell' elaboratore i
w_{ij} = capacita' necessaria per far girare la richiesta j sull' elaboratore i

e' un pb di ZAINO MULTIPLO

$$\max \sum_i \sum_j x_{ij} p_j$$
$$\text{sv} \quad \sum_j x_{ij} w_{ij} \le c_i \quad \forall i$$
$$\sum_i x_{ij} \le 1 \quad \forall j$$
$$x_{ij} \in \{0,1\}$$

penso voglia dire che ogni calcolo deve essere eseguito al max da 1 elaboratore

- Proporre 2 ~~[illegible]~~ rilassamenti lagrangi
- Scrivere i sottopb lagrangiani e proporre come risolverli
- ~~Scrive~~ Quanto sono stringenti? Motivare la risposta
- Spiegare come si potrebbe risolvere il duale lagrangiano.

\+ Metodo di Newton: vantaggi e svantaggi, risultati. Metodi quasi-Newton

X Pb.4: Rilassamento lagrangiano
Problema dello zaino:

$$\max \sum_i \sum_j x_{ij} s_{ij}$$
$$\text{s.t.} \quad \sum w_{ij} x_{ij} \le C_i$$
$$\sum x_{ij} \le 1$$
$$x_{ij} \in \{0,1\}$$

- proporre 2 rilassamenti lagrangiani
- scrivere i relativi problemi (sottoproblemi) lagrangiani e dire come si possono risolvere
- Quanto sono stringenti? Motivare
- Scrivere i duali lagrangiani e spiegare come si possono risolvere