MARKOV PROCESS:

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space $\qquad (E, \mathcal{E})$ measurable space

Stochastic process: $(X_t)_t$ collection of random variables $X_t : \Omega \to E$

Markov process: $(X_t)_t$ stochastic process with Markov property:

$$\mathbb{P}\{X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m, \dots, X_{t_1} \in E_1\} = \mathbb{P}\{X_{t_{m+1}} \in E_{m+1} \mid X_{t_m} \in E_m\}$$

$$\forall t_1 < t_2 < \dots < t_{m+1} \quad \forall E_1, \dots, E_{m+1} \subset \mathcal{E} \quad \forall m \in \mathbb{N}.$$

TRANSITION PROBABILITIES

$s<t \quad p_{ij}(t,s) = \mathbb{P}\{X_t = j \mid X_s = i\}$

TIME-HOMOGENEITY

$p_{ij}(t,s)$ depends only on $t-s \quad \forall s<t$

TRANSITION MATRIX

$(p_{ij})_{ij \in E}$, where $p_{ij} = p_{ij}^{(1)}$ are the 1-step transition probabilities.

PROPERTIES OF THE TRANSITION MATRIX

1) $0 \le p_{ij} \le 1 \quad \forall i,j \in E$
2) $\sum_{j \in E} p_{ij} = 1 \quad \forall i \in E$

FORMULAS RELATED TO THE TRANSITION PROBABILITIES

- $p_{ij}^{(2)} = \sum_{k \in E} p_{ik} p_{kj}$
- $\mathbb{P}\{X_{t_n} = j_n, X_{t_{n-1}} = j_{n-1}, \dots, X_{t_1} = j_1, X_0 = j_0\} = \mathbb{P}\{X_0 = j_0\}\, p_{j_0 j_1}^{(t_1)}\, p_{j_1 j_2}^{(t_2 - t_1)} \cdots p_{j_{n-1} j_n}^{(t_n - t_{n-1})}$

ACCESSIBLE STATE

$i,j \in E \quad j$ accessible from i if $\exists n \ge 0$ s.t. $\mathbb{P}\{X_n = i \mid X_0 = j\} = p_{ij}^{(n)} > 0$.

COMMUNICATING STATES

$i,j \in E \quad j$ and i communicate if one is accessible from the other.

CLASS OF STATES

$C \subseteq E$ is a class of states if all states in C communicate and they do not communicate with the states in $E \setminus C$.

IRREDUCIBILITY

$(X_n)_{n \ge 0}$ is an irreducible Markov Chain if all states communicate.

RECURRENCE/TRANSIENCE

$(X_n)_{n \ge 0}$ discrete time Markov Chain.

A state $i \in E$ is called recurrent if $\mathbb{P}\{\bigcup_{1 \le n < +\infty} \{X_n = i\} \mid X_0 = i\} = 1$,

namely the probability of returning to i in finite time (starting from i) is 1;

if not, i is called transient.

FIRST ENTRANCE TIME

$$T_i = \begin{cases} \min\{n \mid X_n = i\} & \text{if } \{n \ge 1 \mid X_n = i\} \ne \emptyset \\ +\infty & \text{otherwise} \end{cases} \qquad \text{first entrance/visit time in } i \in E.$$

RENEWAL EQUATION

Notation: $f_{ji}^{(n)} = \mathbb{P}\{T_i = n \mid X_0 = j\}$

Proposition: $p_{ij}^{(n)} = \sum_{\nu=1}^{n} f_{ij}^{(\nu)} p_{jj}^{(n-\nu)}$

Interpretation: the probability of going from i to j in n steps can be seen as the sum on ν of the probability of going from i to j for the first time in ν steps times the probability of returning to j after some walk.

THEOREM [RECURRENCE/AVERAGE TIME SPENT]

$\forall i \in E$ the following conditions are equivalent: (a) i is recurrent
(b) $\sum_{n \ge 0} p_{ii}^{(n)} = +\infty$

Interpretation: (a) I will surely return to i

(b) $\sum_{n \ge 0} p_{ii}^{(n)} = \mathbb{E}[\sum_{n \ge 0} \mathbb{1}_{\{X_n = i\}}]$ average time spent in i

$\Rightarrow$ the average time spent in i is infinite.

COROLLARY [MUTUAL RECURRENCE/TRANSIENCE OF COMMUNICATING STATES]

Two communicating states $i,j\in E$ are "both recurrent" or "both transient".

COROLLARY [TRANSIENCE/TRANSITION PROBABILITIES]

If j is transient, then $\sum_{n\geq 0} p_{ij}^{(n)} < \infty$. In particular $\lim_{n\to\infty} \mathbb{P}_i\{X_n=j\}=0$.

COROLLARY [EXISTENCE OF A RECURRENT STATE FOR FINITE STATE MC]

If the set of states E is finite, then there exists at least a recurrent state.

INVARIANT DISTRIBUTIONS

$(X_n)_{n\geq 0}$ MC with transition matrix $(p_{ij})_{i,j\in E}$

$\pi=(\pi_i)_{i\in E}$ probability density on E: $0\leq \pi_i \leq 1$ $\forall i\in E$, $\sum_{i\in E}\pi_i=1$.

π is an invariant distribution if $X_{n+1}\sim\pi$ whenever $X_n\sim\pi$ $\forall n$.

THEOREM [EXISTENCE OF AN INVARIANT DENSITY FOR FINITE STATE MC]

If the set of states E is finite, then there exists at least one invariant density π.

Remark: it is not necessarily unique.

Remark 2: an infinite state MC may not have invariant distributions.

THEOREM [EXISTENCE AND UNICITY OF THE INVARIANT DISTRIBUTION]

If limits $\lim_{n\to\infty} p_{ij}^{(n)}$ exist, are >0 and depend only on j, then the Markov chain admits a unique invariant distribution π and

$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \int f\,d\pi = \sum_{j\in E} f(j)\pi_j$ for any f bounded.

PERIOD OF A STATE

The period of a state $i\in E$ is defined as $MCD\{n\geq 1 \mid p_{ii}^{(n)}>0\}$

If the period is 1, the state is called aperiodic.

PROPERTIES ABOUT PERIODICITY

- States of the same class have the same period
- For irreducible MC, all states have the same period and one can talk about periodic/aperiodic Markov chains.

STOPPING TIME

$T:\Omega \to \mathbb{N}$ random variable is a stopping time of the Markov chain $(X_n)_{n\geq 0}$ if $\forall n$ the event $\{T\leq n\}$ belongs to the σ-algebra generated by $X_n,\dots,X_0$, namely the σ-algebra generated by events $\{X_n=i_n, X_{n-1}=i_{n-1},\dots,X_0=i_0\}$, where $i_n, i_{n-1},\dots,i_0$ are arbitrary states in the countable set of states.

FIRST EXIT TIME

$$U_j=\begin{cases} \inf\{n\geq 1 \mid X_n\neq j\} & \text{if } \{n\geq 1 \mid X_n\neq j\}\neq\emptyset \\ +\infty & \text{otherwise}\end{cases}$$ first exit time from $j\in E$.

PROPOSITION [LAW OF THE FIRST EXIT TIME]

$(X_n)_{n\geq 0}$ Markov chain, $(p_{ij})_{i,j\in E}$ transition matrix

If $0<p_{jj}<1$ then the stopping time U_j has geometric density with parameter $1-p_{jj}$ and the random variable X_{U_j} satisfies $\mathbb{P}_j\{X_{U_j}=k\}=\frac{p_{jk}}{1-p_{jj}}$ $\forall j\in E$.

THEOREM [RESTARTED MARKOV CHAIN]

$(X_n)_{n\geq 0}$ Markov chain, T stopping time (a.s. finite)

$$\Rightarrow Y_n(\omega)=\begin{cases} X_{T(\omega)+n} & \text{if } T(\omega)<+\infty \\ \text{arbitrary} & \text{if } T(\omega)=+\infty\end{cases}$$ (process restarted at the random time T)

$(Y_n)_{n\geq 0}$ is a Markov chain with the same transition matrix of $(X_n)_{n\geq 0}$.

THEOREM [STRONG MARKOV PROPERTY]

$(X_n)_{n\geq 0}$ Markov chain, $(p_{ij})_{i,j\in E}$ transition matrix, T stopping time

$(X_{T\wedge n})_{n\geq 0}$ $(T\wedge n)(\omega)=\min\{T(\omega),n\}$ is the Markov chain stopped at time T

$(Y_n)_{n\geq 0}=(X_{T+n})_{n\geq 0}$ is the Markov chain restarted at time T

The stopped MC and the restarted MC are independent w.r.t.

$\mathbb{P}\{\cdot \mid X_T=i, T<+\infty\}$

THEOREM [NUMBER OF VISITS OF A RECURRENT STATE]

The number of visits of a recurrent state is infinite almost surely

PROOF: number of visits in state i: $N_i = \sum_{n\geq 1} \mathbb{1}_{\{X_n = i\}}$

times of visits: $T_i^{(1)} = \inf\{n \geq 1 \mid X_n = i\}$ time of 1st visit

$T_i^{(k+1)} = \inf\{n > T_i^{(k)} \mid X_n = i\}$ time of $(k+1)^{th}$ visit.

By definition: i recurrent $\iff P_i\{T_i^{(1)} < +\infty\} = 1 \iff P_i\{N_i \geq 1\} = 1$.

$T_i^{(2)}$ is the first visit time in i for the Markov chain restarted from time $T_i^{(1)}$, but the transition matrix is the same, then $P_i\{T_i^{(2)} < +\infty\} = 1$.

By induction: $P_i\{T_i^{(k)} < +\infty\} = 1 \ \forall k \iff P_i\{N_i \geq k\} = 1 \quad \forall k$

$\Rightarrow P_i\{N_i = +\infty\} = 1$.

THEOREM [PROBABILITY OF STAYING FOREVER IN TRANSIENT STATES]

$\mathcal{T}$: set of transient states $\qquad U_i = P_i\{\bigcap_{1\leq n < +\infty} \{X_n \in \mathcal{T}\}\}$

Remark: if i is transient, U_i is the probability of wandering in transient states.

$\Rightarrow (U_i)_{i\in\mathcal{T}}$ is the biggest solution with $0 \leq U_i \leq 1 \ \forall i \in \mathcal{T}$ of the system of equations:

$U_i = \sum_{j\in\mathcal{T}} p_{ij} U_j$

Remark: generally there is no unique solution, but if $\mathcal{T}$ is finite, the only solution is $U_i = 0$.

PROOF: define $u_i^{(n)} = P_i\{X_n \in \mathcal{T}, \ldots, X_1 \in \mathcal{T}\}$ as the probability of staying in transient states from time 1 to n.

$(u_i^{(n)})_{n\geq 1} = (\{X_n \in \mathcal{T}, \ldots, X_1 \in \mathcal{T}\})_{n \geq 1}$ is a decreasing sequence.

Call $u_i = \lim_{n\to\infty} u_i^{(n)} = \lim_{n\to\infty} P_i\{X_n \in \mathcal{T}, \ldots, X_1 \in \mathcal{T}\} = P_i\{\bigcap_{n\geq 1}\{X_n \in \mathcal{T}, \ldots, X_1 \in \mathcal{T}\}\}$.

$u_i^{(n+1)} = P_i\{X_{n+1} \in \mathcal{T}, \ldots, X_1 \in \mathcal{T}\} = \sum_{j\in\mathcal{T}} P_i\{X_{n+1} \in \mathcal{T}, \ldots, X_2 \in \mathcal{T}, X_1 = j\} =$

$= \sum_{j\in\mathcal{T}} \underbrace{P_i\{X_{n+1} \in \mathcal{T}, \ldots, X_2 \in \mathcal{T} \mid X_1 = j\}}_{= P_j\{X_{n+1}\in\mathcal{T}, \ldots, X_2 \in \mathcal{T}\} = u_j^{(n)}} \underbrace{P_i\{X_1 = j\}}_{= p_{ij}}$

$\Rightarrow u_i^{(n+1)} = \sum_{j\in\mathcal{T}} u_j^{(n)} p_{ij}$

Applying the limits to both sides and knowing that $u_i^{(n)} \downarrow u_i$,

we find $u_i = \sum_{j\in\mathcal{T}} p_{ij} u_j \quad \forall i \in \mathcal{T}$.

Now we prove it is the biggest $[0,1]$-valued solution.

Suppose $(v_i)_{i\in\mathcal{T}}$ to be another $[0,1]$-valued solution: $v_i = \sum_{j\in\mathcal{T}} p_{ij} v_j$

Then $u_i^{(1)} = P_i\{X_1 \in \mathcal{T}\} = \sum_{j\in\mathcal{T}} p_{ij} \geq \sum_{j\in\mathcal{T}} p_{ij} v_j = v_i \Rightarrow u_i^{(1)} \geq v_i$

By induction: suppose $u_i^{(n)} \geq v_i \ \forall i \in \mathcal{T}$

then $u_i^{(n+1)} = \sum_{j\in\mathcal{T}} p_{ij} u_j^{(n)} \geq \sum_{j\in\mathcal{T}} p_{ij} v_j = v_i \Rightarrow u_i^{(n+1)} \geq v_i$

By considering the limit of $u_i^{(n)}$, which decreases to u_i, we find $u_i \geq v_i$ $\forall i \in \mathcal{T}$, thus u_i is the biggest solution.

We also prove the remark: $\mathcal{T}$ finite $\Rightarrow u_i = 0 \ \forall i \in \mathcal{T}$

$u_i = \sum_{j\in\mathcal{T}} p_{ij} u_j = \sum_{j,k\in\mathcal{T}} p_{ij} p_{jk} u_k \leq \sum_{k\in\mathcal{T}} \left(\sum_{j\in E} p_{ij} p_{jk}\right) u_k = \sum_{k\in\mathcal{T}} p_{ik}^{(2)} u_k$

By iterating n times: $u_i = \sum_{k_1,\ldots,k_n\in\mathcal{T}} p_{ik_1} \cdot p_{k_1k_2} \cdots p_{k_{n-1}k_n} u_{k_n} \leq \sum_{\substack{k_1,\ldots,k_{n-1}\in E \\ k_n\in\mathcal{T}}} (\ldots) u_{k_n} = \sum_{k_n\in\mathcal{T}} p_{ik_n}^{(n)} u_{k_n}$

$\Rightarrow \forall n \quad u_i \leq \sum_{j\in\mathcal{T}} p_{ij}^{(n)} u_j \leq \sum_{j\in\mathcal{T}} p_{ij}^{(n)}$

Since $\forall j$ j is transient, $p_{ij}^{(n)} \longrightarrow 0$ as $n \to +\infty \Rightarrow u_i = 0 \ \forall i \in \mathcal{T}$.

EXAMPLE [GAMBLER'S RUIN AGAINST A BANK]

$(X_n)_{n\geq 0}$ Markov chain, where X_n:= "player's capital at time n" and $E = \mathbb{N}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

$p + q = 1, \ p, q > 0$

Classes: $\{0\}$ recurrent $p_{00}^{(n)} = 1, \sum_{n\geq 1} p_{00}^{(n)} = +\infty$

$\{1,2,3,\ldots\}$ transient $\Rightarrow \mathcal{T} = \{1,2,3,\ldots\}$

$u_i = \sum_{j\geq 1} p_{ij} u_j \quad \Rightarrow \quad u_0 = 0$

$u_1 = p u_2$

$u_i = p u_{i+1} + q u_{i-1} \quad i > 1$

The roots of the characteristic equation of the difference equation associated to the previous are 1 and $\frac{q}{p}$: for $p \neq q$ the solution of the difference equation is $u_i = k_1 + k_2 \left(\frac{q}{p}\right)^i$.

By applying the first boundary condition: $u_0 = 0 \iff k_2 = -k_1$.

Here we focus on the case $p > q$ in order to have a drift towards $+\infty$ and that the probability of wandering forever in transient states is strictly positive:

$\lim_{i\to+\infty} u_i = k_1 \Rightarrow$ in order that u_i is the biggest solution, $k_1 = 1$

$\Rightarrow u_i = \left(1 - \left(\frac{q}{p}\right)^i\right)$ probability that the gambler becomes infinitely rich

$\left(\frac{q}{p}\right)^i$ probability of absorption in 0, i.e. ruin.

THEOREM [ABSORPTION PROBABILITY IN A RECURRENT CLASS]

$(v_i)_{i\in\mathcal{T}}$, the probability of absorption in a recurrent class C, is the smallest $[0,1]$-valued solution of $\quad v_i = \sum_{j\in C} p_{ij} + \sum_{j\in\mathcal{T}} p_{ij} v_j$

Remark: if $\mathcal{T}$ is finite then the solution is unique.

Interpretation: the absorption probability can be seen as the probability to be absorbed directly in C in one step plus the probability to move to a transient state times its absorption probability

PROOF: we define $v_i^{(n)} = \mathbb{P}_i\{X_n\in C, X_{n-1}\notin C, \dots, X_1\notin C\}$ as the probability of absorption at time n.

Remark: $v_i^{(1)} = \sum_{j\in C} p_{ij}$

$\underline{n>1}$: $v_i^{(n)} = \sum_{j\in\mathcal{T}} \mathbb{P}_i\{X_n\in C, X_{n-1}\notin C, \dots, X_1 = j\}$ (total probability formula)

$= \sum_{j\in\mathcal{T}} \mathbb{P}_i\{X_n\in C, X_{n-1}\notin C, \dots, X_2\notin C \mid X_1 = j\}\,\mathbb{P}_i\{X_1=j\} =$ (definition of conditional probability)

$= \sum_{j\in\mathcal{T}} v_j^{(n-1)} p_{ij}$

The event "absorption in C" can be written as $\bigcup_{n\geq 1}\{X_n\in C\}$.

Moreover $\bigcup_{n\geq1}\{X_n\in C\} = \bigcup_{n\geq1}\{X_n\in C, X_{n-1}\notin C, \dots, X_1\notin C\}$

We observe that

$v_i^{(n)} = \mathbb{P}_i\{\bigcup_{1\leq k\leq n}\{X_k\in C\}\}$

$v_i^{(n+1)} - v_i^{(n)} = \mathbb{P}_i\{X_{n+1}\in C, X_n\notin C, \dots, X_1\notin C\} \geq 0$

$\Rightarrow (v_i^{(n)})_{n\geq1}$ is a nondecreasing sequence, thus we call $v_i = \sup_{n\geq1} v_i^{(n)}$.

Applying the limit for $n\to+\infty$ to the previous equation and knowing that $v_i^{(n)}\uparrow v_i$, we find $v_i = \underbrace{\sum_{j\in C} p_{ij}}_{n=1} + \underbrace{\sum_{n>1}\sum_{j\in\mathcal{T}} v_j^{(n-1)} p_{ij}}_{n>1} = \sum_{j\in C} p_{ij} + \sum_{j\in\mathcal{T}} p_{ij} v_j$.

Now we prove $(v_i)_{i\in\mathcal{T}}$ is the smallest $[0,1]$-valued solution.

Let $(x_i)_{i\in\mathcal{T}}$ be another $[0,1]$-valued solution: $x_i = \underbrace{\sum_{j\in C} p_{ij}}_{v_i^{(1)}} + \sum_{j\in\mathcal{T}} p_{ij} x_j$

Then $x_i \geq \sum_{j\in C} p_{ij} = v_i^{(1)}$

Suppose $x_i \geq \sum_{k=1}^{n} v_i^{(k)}$ $(*)$, then

$x_i = \sum_{j\in C} p_{ij} + \sum_{j\in\mathcal{T}} p_{ij} x_j \geq \sum_{j\in C} p_{ij} + \sum_{j\in\mathcal{T}} p_{ij} \sum_{k=1}^{n} v_j^{(k)} = \sum_{k=1}^{n+1} v_i^{(k)}$

So, by induction, we showed that $\forall n$ $(*)$ holds.

Thus it holds for $n\to\infty$: $x_i \geq v_i \quad \forall i\in\mathcal{T}$.

Remark in this proof $\sum_{n\geq1} v_i^{(n)} \uparrow v_i$

EXAMPLE [GAMBLER'S RUIN AGAINST A BANK]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ 0 & q & 0 & p & \cdots \end{pmatrix}$$

We consider the case $p>q$ because for $q\geq p$ the probability of ruin is 1.

$\{0\}$ absorbing $\quad \{1,2,3,\dots\}$ transient

ruin $\leftrightarrow$ absorption in 0

The convention is $v_0 = 1$; thus $v_i = p v_{i+1} + q v_{i-1} \quad i\geq 1$

General solution: $v_i = k_1 + k_2\left(\frac{q}{p}\right)^i$

$v_0 = 1 \iff k_1 + k_2 = 1$

$i\to+\infty \Rightarrow v_i \to k_1 \Rightarrow$ the smallest solution is with $k_1 = 0$

$\Rightarrow v_i = \left(\frac{q}{p}\right)^i$

THEOREM [MEAN ABSORPTION TIME IN RECURRENT CLASSES]

Suppose that the set of states E is finite and there is a unique recurrent class C. Then the mean absorption time w_i in C is finite and satisfies

$w_i = 1 + \sum_{j\in\mathcal{T}} p_{ij} w_j$

Interpretation: the mean time for absorption in C is the time for the 1-step absorption (1) plus the mean time for absorption in C after moving to a different transient state j.

Useful lemma: Let V be an $\mathbb{N}$-valued r.v., then $\mathbb{E}[V] = \sum_{n=0}^{\infty} \mathbb{P}\{V>n\}$.

Proof: $\mathbb{E}[V] = \sum_{m=1}^{\infty} m\,\mathbb{P}\{V=m\} = \sum_{m=1}^{\infty}\sum_{k=1}^{m} \mathbb{P}\{V=m\} = \sum_{k=1}^{\infty}\sum_{m=k}^{\infty} \mathbb{P}\{V=m\} = \sum_{k=1}^{\infty} \mathbb{P}\{V\geq k\} = \sum_{n=0}^{\infty} \mathbb{P}\{V>n\}$

PROOF: Step 1: check that w_i are finite by: $\exists\, 0<c<1$ s.t. $\mathbb{P}\{V>n\}\leq c^n$, where V is the absorption time in C

$\mathbb{P}_i\{V>n\} = \mathbb{P}_i\{X_n\in\mathcal{T}, \dots, X_1\in\mathcal{T}\} = u_i^{(n)}$ probability of staying in transient states from time 1 to n.

$u_i^{(n+1)} = \sum_{j\in\mathcal{T}} p_{ij} u_j^{(n)}$

$u_i^{(n+2)} = \sum_{j,k\in\mathcal{T}} p_{ij} p_{jk} u_k^{(n)} \leq \sum_{k\in\mathcal{T}}\left(\sum_{j\in\mathcal{T}} p_{ij} p_{jk}\right) u_k^{(n)} = \sum_{k\in\mathcal{T}} p_{ik}^{(2)} u_k^{(n)}$

By iterating $\quad u_i^{(n+m)} \leq \sum_{j\in\mathcal{T}} p_{ij}^{(m)} u_j^{(n)}$

If i,j are transient, then $\lim_{m\to+\infty} p_{ij}^{(m)} = 0$

If $\mathcal{T}$ is finite, we can fix m s.t. $\sum_{j\in\mathcal{T}} p_{ij}^{(m)} < 1 \ \forall i$ and we set

$M = \max_{i\in\mathcal{T}} \left\{\sum_{j\in\mathcal{T}} p_{ij}^{(m)}\right\}$ that will be <1.

Then $u_i^{(n)} \le \max_{j\in\mathcal{T}} \{u_j^{(n-m)} \sum_{j\in\mathcal{T}} p_{ij}^{(m)}\} \le M \max_{j\in\mathcal{T}}\{u_j^{(n-m)}\}$

We maximize on i: $\max_{i\in\mathcal{T}}\{u_i^{(n)}\} \le M \max_{j\in\mathcal{T}}\{u_j^{(n-m)}\}$

The probability of staying forever in transient states is exponentially decreasing.

Indeed, iterate: $\max_{i\in\mathcal{T}}\{u_i^{(n)}\} \le M^2 \max_{j\in\mathcal{T}}\{u_j^{(n-m-m)}\} \le \dots \le M^{\frac{n}{m}} \max_{j\in\mathcal{T}}\{u_j^{(n-\frac{n}{m}\cdot m)}\}$

$\Rightarrow P_i\{V>n\} \le \max_{i\in\mathcal{T}}\{u_i^{(n)}\} \le M^{\frac{n}{m}} < M^{\frac{n}{m}-1} = M^{-1}(M^{\frac{1}{m}})^n$

$\Rightarrow E_i\{V\} = \sum_{n=0}^{\infty} P_i\{V>n\} \le M^{-1}\sum_{n=0}^{\infty}(M^{\frac{1}{m}})^n = (M(1-M^{\frac{1}{m}}))^{-1} < +\infty$

<u>Step 2</u>: check that $w_i = 1 + \sum_{j\in\mathcal{T}} p_{ij} w_j$

$$E_i[V] = \sum_{n=1}^{\infty} n\, P_i\{V=n\} =$$

$$P_i\{V=n, X_1\in C\} = \begin{cases} \emptyset & \text{if } n>1 \\ \{X_1\in C\} & \text{if } n=1 \end{cases}$$

$$= \sum_{n=1}^{\infty} n\, P_i\{V=n, X_1\in\mathcal{T}\} + \sum_{n=1}^{\infty} n\, P_i\{V=n, X_1\in C\} =$$

$$= \sum_{n=2}^{\infty} n \sum_{j\in\mathcal{T}} P_i\{V=n, X_1=j\} + \sum_{j\in C} p_{ij} =$$

$$= \sum_{n=2}^{\infty} n \sum_{j\in\mathcal{T}} P_i\{V=n \mid X_1=j\}\, p_{ij} + \sum_{j\in C} p_{ij} =$$

(by time homogeneity and Markov property: $P_i\{V=n \mid X_1=j\} = P_j\{V=n-1\}$)

$$= \sum_{n=1}^{\infty} (n+1) \sum_{j\in\mathcal{T}} P_j\{V=n\}\, p_{ij} + \sum_{j\in C} p_{ij} =$$

$$= \sum_{j\in\mathcal{T}} \underbrace{\sum_{n=1}^{\infty} n\, P_j\{V=n\}}_{=E_j[V]=w_j} p_{ij} + \underbrace{\sum_{j\in\mathcal{T}} p_{ij} \underbrace{\sum_{n=1}^{\infty} P_j\{V=n\}}_{=1} + \sum_{j\in C} p_{ij}}_{=1}$$

$$\Rightarrow w_i = 1 + \sum_{j\in\mathcal{T}} p_{ij} w_j$$

<u>EXAMPLE [COUPON COLLECTOR PROBLEM]</u>

You want to collect most of the N pictures in the collection: at time n you buy an envelope containing a random picture.

Model: $(X_n)_{n\ge0}$ Markov chain, where $X_n :=$ n° of different pictures collected at time n $(X_0=0)$

Transition probabilities:

$$\left.\begin{aligned} &P\{X_{n+1}=K+1 \mid X_n=K\} = \frac{N-K}{N} \\ &P\{X_{n+1}=K \mid X_n=K\} = \frac{K}{N} \\ &P\{X_{n+1}=j \mid X_n=K\} = 0 \quad \forall j\notin\{K,K+1\} \end{aligned}\right\} \text{ for } 1\le K\le N-1$$

$P\{X_{n+1}=1 \mid X_n=0\} = 1$, $P\{X_{n+1}=j \mid X_n=0\} = 0 \ \ \forall j\ne1$

$P\{X_{n+1}=N \mid X_n=N\} = 1$, $P\{X_{n+1}=j \mid X_n=N\} = 0 \ \ \forall j\ne N$

Transition matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & & \\ 0 & \frac{1}{N} & \frac{N-1}{N} & 0 & \cdots & & \\ 0 & 0 & \frac{2}{N} & \frac{N-2}{N} & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & \cdots & 0 & \frac{N-1}{N} & \frac{1}{N} \\ & & & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Classes of states: $\mathcal{T} = \{0,1,2,\dots,N-1\}$ $\qquad C = \{N\}$

Call w_i the mean time for reaching N starting from i:

$$\begin{cases} w_i = 1 + p_{ii} w_i + p_{i,i+1} w_{i+1} & i = 0,1,2,\dots,N-1 \\ w_N = 0 \end{cases}$$

$\rightarrow w_i = 1 + \frac{i}{N} w_i + \frac{N-i}{N} w_{i+1} \iff \frac{N-i}{N} w_i = 1 + \frac{N-i}{N} w_{i+1}$

$\iff w_i = \frac{N}{N-i} + w_{i+1} = \frac{N}{N-i} + \frac{N}{N-i-1} + w_{i+2} = \dots = \frac{N}{N-i} + \frac{N}{N-i-1} + \dots + \frac{N}{N-(N-1)}$

$\Rightarrow w_0 = \sum_{K=0}^{N-1} \frac{N}{N-K} = (h=N-K) = N \sum_{h=1}^{N} \frac{1}{h} \underset{N \text{ large}}{\approx} N \log N$

and $\sum_{h=1}^{N} \frac{1}{h} = 1 + \sum_{h=2}^{N} \frac{1}{h} \le 1 + \int_1^{N-1} \frac{dx}{x} = 1 + \log(N-1)$

<u>EXAMPLE [GAMBLER'S RUIN AGAINST A BANK]</u>

We now consider the fair case of the game, namely where $p=q=\frac{1}{2}$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

We are interested in computing the mean ruin time w.r.t. $P_i\{\cdot \mid \bigcap_{n=1}^{\infty}\{X_n\ne N\}\}$; we can do this only if we exclude the possibility to reach the state N from state $N-1$.

$$\begin{cases} w_0 = 0 \\ w_i = 1 + \frac{1}{2} w_{i-1} + \frac{1}{2} w_{i+1} & 1\le i\le N-2 \\ w_{N-1} = 1 + w_{N-2} \end{cases}$$

$\longrightarrow$ By solving the difference equation and by imposing the two boundary conditions, we find: $w_i = -i^2 + 2(N-1)i$

THEOREM [TRANSIENCE CRITERION]

$(X_n)_{n\geq 0}$ irreducible Markov chain (with countable space E) is transient if and only if there exists a bounded non constant solution of:

$\circledast$ $\sum_{k\in E} p_{jk} y_k = y_j$ for all $j\in E$ but one at most.

PROOF: ($\Rightarrow$) Suppose $(X_n)_{n\geq 0}$ transient and denote by e the unique state for which $\circledast$ possibly doesn't hold.

Consider the transformed Markov chain in which e is absorbing; its transition matrix will be: $\tilde{p}_{ij} = \begin{cases} \delta_{ij} & \text{if } i=e \\ p_{ij} & \text{if } i\neq e \end{cases}$

Since the given Markov chain is transient (the transformed one is not), there exists $i\in E$ s.t. $\tilde{v}_i := \mathbb{P}_i\{T_e < +\infty\} < 1$, where T_e is the first entrance time in e, because otherwise if $\mathbb{P}_i\{T_e<+\infty\}=1$ $\forall i$, we would find $\mathbb{P}_e\{T_e<+\infty\} = p_{ee} + \sum_{k\neq e} p_{ek} \underbrace{\mathbb{P}_k\{T_e<+\infty\}}_{=1\ \forall k} = p_{ee} + \sum_{k\neq e} p_{ek} = 1$, which contraddicts the transience of $(X_n)_{n\geq 0}$.

Then $\tilde{v}_i<1$ for some i and $\tilde{v}_e = 1$, then $(v_i)_{i\in E}$ is $[0,1]$-valued (bounded) non-constant. Moreover, $(\tilde{v}_i)_{i\in E}$ are the absorption probabilities in e for the transformed Markov chain, therefore

$\tilde{v}_i = \tilde{p}_{ie} + \sum_{k\neq e} \tilde{p}_{ik}\tilde{v}_k$ $(*)$

But $\tilde{p}_{ik} = p_{ik}$ $\forall k\neq e$ (if $i\neq e$), $\tilde{v}_e = 1$ $\Rightarrow \tilde{v}_i = \sum_{k\in E} p_{ik}\tilde{v}_k$

Thus $(*)$ and $\circledast$ are equivalent and satisfied.

($\Leftarrow$) Suppose $\triangle$ $\sum_{k\in E} p_{ik} y_k = y_i$ $\forall i\in E\setminus\{e\}$ where y_i is bounded non-constant.

Then, if we consider $(\tilde{p}_{ij})_{i,j\in E}$ as before, we can show by contraddiction that y_i would be constant if the Markov chain was recurrent.

From $\triangle$ we get $\sum_{k\in E} \tilde{p}_{ik} y_k = y_i$ $\forall i\in E$

By iteration: $\sum_{k,h\in E} (\tilde{p}_{ik}\tilde{p}_{kh}) y_h = \sum_{h\in E}\left(\sum_{k\in E}\tilde{p}_{ik}\tilde{p}_{kh}\right) y_h = \sum_{h\in E} \tilde{p}^{(2)}_{ih} y_h = y_i$ $\forall i\in E$

$\Rightarrow \sum_{k\in E} \tilde{p}^{(n)}_{ik} y_k = y_i$

If the Markov chain was recurrent then the transformed Markov chain would converge to the state e, namely $\lim_{n\to\infty} \tilde{p}^{(n)}_{ie} = 1$ $\forall i\in E$.

In fact, $\lim_{n\to\infty} \tilde{p}^{(n)}_{je} = \lim_{n\to\infty} \mathbb{P}_j\{\tilde{X}_n = e\} = \lim_{n\to\infty} \mathbb{P}_j\{T_e \leq n\} = 1 \Leftrightarrow$ recurrence.

Then $\forall j\neq e$, $|y_j - y_e| = \lim_{n\to\infty} |y_j - \tilde{p}^{(n)}_{je} y_e| = \left(\sum_{k\in E}\tilde{p}^{(n)}_{jk} y_k = y_j \Rightarrow \sum_{k\neq e}\tilde{p}^{(n)}_{jk} y_k + \tilde{p}^{(n)}_{je} y_e = y_j\right)$

$= \lim_{n\to\infty} |y_j - (y_j - \sum_{k\neq e}\tilde{p}^{(n)}_{jk} y_k)| = \lim_{n\to\infty} \sum_{k\neq e} \tilde{p}^{(n)}_{jk} y_k$

But $|y_j - y_e| = \lim_{n\to\infty}\sum_{k\neq e}\tilde{p}^{(n)}_{jk} y_k \leq \sup_k |y_k| \lim_{n\to\infty}\sum_{k\neq e}\tilde{p}^{(n)}_{jk}$

$\leq \sup_k |y_k| \lim_{n\to\infty} (1 - \tilde{p}^{(n)}_{je}) = 0$

$\Rightarrow y_j = y_e$ $\forall j$ $\Rightarrow (y_j)_{j\in E}$ constant $\Rightarrow$ contraddiction.

So the Markov chain is transient.

THEOREM [RECURRENCE CRITERION]

Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain.

If there exists a family $(y_j)_{j\in E}$ such that $\sum_{k\in E} p_{jk} y_k \leq y_j$ for all $j\in E$ but one at most, and $\lim_{n\to\infty} y_n = +\infty$, then the Markov chain is recurrent.

EXAMPLE [RANDOM QUEUE AT A COUNTER]

- The counter serves one person at each unit time;
- the new arrivals are random A_n st. $\mathbb{P}\{A_n = k\} = a_k$, $0<a_k<1$, $\sum_{k=1}^{\infty} a_k = 1$, at time n;
- the number of customers arriving at different time instants are independent of each other.

Model: $(X_n)_{n\geq 0}$ Markov chain, where $X_n :=$ number of customers in the system at time n.

Transition matrix: $\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \\ 0 & a_0 & a_1 & a_2 & \cdots \\ & & \vdots & & \end{pmatrix}$ where $a_0>0$, $0<a_0+a_1<1$.

The Markov chain is irreducible.

Transience in this case means that the queue explodes (at finite time, $\mathbb{P}\{X_n > M\}$ is high); on the other hand recurrence means that the system works (with some fluctuations).

The average number of customers arriving at each unit time is $\lambda = \sum_{k\geq 1} a_k \cdot k$.

Intuitively: - $\lambda>1$ means I can not serve everybody $\Rightarrow$ transient

- $\lambda\leq 1$ means that I can $\Rightarrow$ recurrent

$\boxed{\lambda>1}$ we look for solutions of $\sum_{k\geq 0} p_{jk} y_k = y_j$ $\forall j>0$ (0 is the state we exclude) of the form $y_k = \xi^k$ with $0<\xi<1$

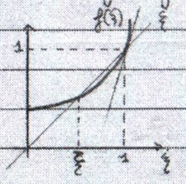
In fact, $(y_k)_{k\geq 0}$ would be a bounded non-constant solution.

$\rightarrow \sum_{k \geq j-1} a_{k-j+1} \xi^k = \xi^j$ ⊡

⊡ $\Leftrightarrow \sum_{k \geq j-1} a_{k-j+1} \xi^{k-j+1} = \xi \qquad (h = k-j+1)$

$\Leftrightarrow \sum_{h \geq 0} a_h \xi^h = \xi$

We define $f(\xi) = \sum_{h \geq 0} a_h \xi^h$: f is an analytic function written as a power series of ξ; for $|\xi| \leq 1$ the series is convergent.



$f(1) = \sum_{h \geq 0} a_h = 1$

$a_h \geq 0 \ \forall h \Rightarrow \xi \to f(\xi)$ is increasing in $[0,1]$

By Abel's lemma, $f'(1) = \sum_{h \geq 1} h a_h = \lambda$ is the slope of the tangent to f in $\xi = 1$.

Thus, for $\lambda > 1$, $\exists \bar{\xi} \in (0,1)$ s.t. $f(\bar{\xi}) = \bar{\xi}$

So $y_k = \bar{\xi}^k$ is a bounded non constant solution of the previous system: the Markov chain is transient.

$\boxed{\lambda \leq 1}$ Now we look for an unbounded solution of $\sum_{k \geq 0} p_{jk} y_k = y_j \ \forall j > 0$.

Here we choose $y_k = k$

$\sum_{k \geq 0} p_{jk} y_k = \sum_{k \geq j-1} a_{k-j+1} \cdot k = \binom{h = k-j+1}{k = h+j-1} = \sum_{h \geq 0} (h+j-1) a_h = \sum_{h \geq 0} h a_h + (j-1) \sum_{h \geq 0} a_h =$

$= \lambda + (j-1) = (\lambda - 1) + j = (\lambda - 1) + y_j \leq y_j$ for $\lambda \leq 1$

Thus the Markov chain is recurrent

1 transient $\rightarrow \lambda$

recurrent

If we ask for which values of λ the Markov chain admits an invariant distribution, we would answer $\lambda < 1$, because for $\lambda < 1$ the Markov chain is fast recurrent while for $\lambda = 1$ it is slow recurrent.

THEOREM [SUFFICIENT CONDITION FOR THE EXISTENCE OF INVARIANT DISTRIBUTIONS]

Let $(X_n)_{n \geq 0}$ be an irreducible Markov chain.

If we can find two sequences $(y_j)_j, (x_j)_j$ both unbounded ($\lim_{j \to +\infty} y_j = \lim_{j \to +\infty} x_j = +\infty$) s.t. $\sum_{k \geq 0} p_{jk} y_k \leq y_j - x_j$, then the Markov chain admits a unique invariant density.

EXAMPLE [RANDOM QUEUE AT A COUNTER]

We introduce an additional assumption $\sum_{k \geq 0} k^2 a_k < +\infty$, i.e. the arrivals have a finite second order moment. We call it m_2.

Consider $y_j = j^2$ ($\lim_{j \to +\infty} y_j = +\infty$)

$\Rightarrow \sum_{k \geq 0} p_{jk} y_k = \sum_{k \geq j-1} a_{k-j+1} k^2 = \binom{h = k-j+1}{k = h+j-1} = \sum_{h \geq 0} a_h (h+j-1)^2 =$

$= \sum_{h \geq 0} a_h h^2 + (j-1)^2 \sum_{h \geq 0} a_h + 2(j-1) \sum_{h \geq 0} h a_h =$

$= m_2 + 2(j-1)\lambda + (j-1)^2 = m_2 + 2j\lambda - 2\lambda + j^2 - 2j + 1 =$

$= j^2 - 2(1-\lambda)j + (1 - 2\lambda + m_2) = y_j - (\underbrace{2(1-\lambda)j - (1 - 2\lambda + m_2)}_{=: x_j}) = y_j - x_j$

But $x_j \to +\infty$ as $j \to +\infty$ for $\lambda < 1$

$\sum_{k \geq 0} p_{jk} y_k = y_j - x_j \quad \Rightarrow$ the Markov chain admits a unique invariant density for $\lambda < 1$.

SOJOURN TIME

$\mathcal{T}$: set of transient states, $\mathcal{S} \subseteq \mathcal{T}$

The total/sojourn time spent in $\mathcal{S}$ is $T_{\mathcal{S}} := \sum_{n \geq 0} \mathbb{1}_{\{X_n \in \mathcal{S}\}}$

Remark: if $\mathcal{S}$ contains some recurrent states, $T_{\mathcal{S}} = +\infty$ with probability 1.

MOMENT GENERATING FUNCTION OF SOJOURN TIMES

The moment generating function of a random variable T is defined as:

$M_i(z) = E_i[z^T] \quad \forall i \in E$

If $\mathcal{S}$ is finite, $M_i(z)$ is well defined for $|z| < 1$.

PROPERTIES OF THE MOMENT GENERATING FUNCTION OF $T_{\mathcal{S}}$

- $m_i'(z) = \mathbb{E}_i\left[\frac{d}{dz} e^{T \log z}\right] = \mathbb{E}\left[\frac{1}{z} z^T\right] = \mathbb{E}[z^{T-1}]$
- Remark: $m_i(z) = \sum_{k \geq 0} z^k \mathbb{P}_i\{T=k\} \to$ it can be seen as Taylor series, where $\mathbb{P}_i\{T=0\}$ is the constant, $\mathbb{P}_i\{T=1\}$ is the coefficient multiplying the first order term, and so on.

- $m_i(z) = 1$ if i is recurrent $\leftrightarrow T=0$ almost surely.
- $m_i(z) = z \sum_j p_{ij}\, m_j(z)$ if $i \in \mathcal{S}$ $\leftrightarrow z^T = z \cdot z^{T-1}$
- $m_i(z) = \sum_j p_{ij}\, m_j(z)$ if $i \in \mathcal{T} \setminus \mathcal{S} \leftrightarrow$ no z, I start outside $\mathcal{S}$

Short proof: $\cdot$ if $i \in \mathcal{T}$, $T_{\mathcal{S}} = \mathbb{1}\{X_0 \in \mathcal{S}\} + \underbrace{\sum_{n \geq 1} \mathbb{1}\{X_n \in \mathcal{S}\}}_{=\tilde{T}}$

$$\mathbb{E}_i[z^T] = \sum_{j \in E} \mathbb{E}_i[z^T | X_1 = j]\, \mathbb{P}_i\{X_1 = j\} =$$
$$= \sum_{j \in E} \mathbb{E}_i[z^{\mathbb{1}\{X_0 \in \mathcal{S}\}} z^{\tilde{T}} | X_1 = j]\, \mathbb{P}_i\{X_1 = j\} =$$
$$= z^{\mathbb{1}\{i \in \mathcal{S}\}} \cdot \sum_{j \in E} \mathbb{E}_i[z^{\tilde{T}} | X_1 = j]\, p_{ij} =$$
$$= z^{\mathbb{1}\{i \in \mathcal{S}\}} \cdot \sum_{j \in E} \mathbb{E}[z^{\tilde{T}} | X_1 = j, X_0 = i]\, p_{ij} = \text{(Markov property)}$$
$$= z^{\mathbb{1}\{i \in \mathcal{S}\}} \cdot \sum_{j \in E} \mathbb{E}_j[z^T]\, p_{ij}$$

$\Rightarrow$ if $i \in \mathcal{S}$: $m_i(z) = z \sum_{j \in E} p_{ij}\, m_j(z)$

$\Rightarrow$ if $i \in \mathcal{T} \setminus \mathcal{S}$: $m_i(z) = \sum_{j \in E} p_{ij}\, m_j(z)$

$\cdot$ if $i \in C$, $\mathbb{P}_i\{T_{\mathcal{S}} = 0\} = 1 \Rightarrow m_i(z) = \mathbb{E}[z^0] = 1$.

EXAMPLE [GAMBLER'S RUIN AGAINST A BANK]

We consider the symmetric case ($p = q = \frac{1}{2}$) with finite space state $E = \{0,1,2,3\}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We want to compute the moment generating function of the total time spent in $\mathcal{S} = \{1\}$ starting from state 1.

$$\begin{cases} m_0(z) = m_3(z) = 1 \\ m_1(z) = z\left(\frac{1}{2} m_0(z) + 0\, m_1(z) + \frac{1}{2} m_2(z) + 0\, m_3(z)\right) = z\left(\frac{1}{2} + \frac{1}{2} m_2(z)\right) \\ m_2(z) = \left(0\, m_0(z) + \frac{1}{2} m_1(z) + 0 \cdot m_2(z) + \frac{1}{2} m_3(z)\right) = \frac{1}{2} + \frac{1}{2} m_1(z) \end{cases}$$

$$\Rightarrow m_1(z) = \frac{3/4\, z}{1 - z/4}, \quad m_2(z) = \frac{1 + z/2}{2(1 - z/4)}$$

CONTINUOUS TIME DISCRETE STATE MARKOV CHAINS

Markov property: $\mathbb{P}\{X_{t_{n+1}} = j | X_{t_n} = i_n, \dots, X_{t_0} = i_0\} = \mathbb{P}\{X_{t_{n+1}} = j | X_{t_n} = i_n\}$ $\forall n, \forall j, i_n, \dots, i_0 \in E$ $\forall 0 \leq t_0 < \dots < t_{n+1}$

Time homogeneity: $\mathbb{P}\{X_{t+s} = j | X_s = i\} = \mathbb{P}\{X_t = j | X_0 = i\}$

Transition semigroup: $P_t = (p_{ij}(t))_{i,j \in E}$ where $p_{ij}(t) = \mathbb{P}\{X_t = j | X_0 = i\}$

Properties of P_t:
- $0 \leq p_{ij}(t) \leq 1$ $\forall i,j \in E$ $\forall t > 0$
- $\sum_{j \in E} p_{ij}(t) = 1$ $\forall i \in E$ $\forall t > 0$
- $P_{t+s} = P_t P_s$ (semigroup property) $\forall t,s > 0$

TRANSITION RATES

In "regular situations", we know that the following limits exist:

$$q_{ij} := \lim_{t \to 0^+} \frac{p_{ij}(t)}{t} \quad i \neq j \quad \text{and} \quad q_{ii} := \lim_{t \to 0^+} \frac{-1 + p_{ii}(t)}{t}$$

They are the derivatives of transition probabilities in 0 (since $p_{ij}(0) = 0$ $i \neq j$, $p_{ii}(0) = 1$) and they are called transition rates. Moreover $q_{ij} \geq 0$, $q_{ii} \leq 0$.

KOLMOGOROV EQUATIONS

- FORWARD KOLMOGOROV EQUATIONS: $p_{ij}'(t) = \sum_{k \in E} p_{ik}(t)\, q_{kj}$
- BACKWARD KOLMOGOROV EQUATIONS: $p_{ij}'(t) = \sum_{k \in E} q_{ik}\, p_{kj}(t)$

TRANSITION RATE MATRIX

$Q = (q_{ij})_{i,j \in E}$

Properties: 1) $q_{ij} \geq 0$ $\forall i \neq j$, $q_{ii} \leq 0$

2) $\sum_{j \in E} q_{ij} = 0$ $\forall i \in E$

THEOREM [SOLUTION OF KOLMOGOROV EQUATIONS]

If $\sup_i |q_{ii}|$ is finite then (BKE) and (FKE) have the same unique solution which is $P_t = e^{tQ}$.

THEOREM [EXPONENTIAL DISTRIBUTION OF EXIT TIMES]

Let $T_i = \inf\{t > 0 | X_t \neq i\}$ be the first exit time from some state $i \in E$.

If $-\infty < q_{ii} < 0$ then $T_i \sim \exp\{-q_{ii}\}$

THEOREM [STATE VISITED WHEN LEAVING A STATE]

$\forall j \neq i \quad P_i\{X_{\tau_1} = j\} = \frac{q_{ij}}{-q_{ii}} \qquad (-\infty < q_{ii} < 0)$

DISCRETE SKELETON

We can define the stochastic matrix $(\hat{p}_{ij})_{i,j\in E}$, where

$$\hat{p}_{ij} = \begin{cases} \frac{q_{ij}}{-q_{ii}} & \text{if } q_{ii} \neq 0, i \neq j \\ 0 & \text{if } q_{ii}=0, i\neq j \text{ or } q_{ii}\neq 0, i=j \\ 1 & \text{if } q_{ii}=0, i=j \end{cases}$$

and we can associate it with a discrete time Markov chain $(Y_n)_{n\geq 0}$, called discrete skeleton of the continuous time Markov chain $(X_t)_{t\geq 0}$.

POISSON PROCESS

We define a Poisson process a Markov chain of state space $\mathbb{N}$ and transition rate matrix: $Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda & \cdots \\ \vdots & \vdots & & \end{pmatrix}$ with $\lambda > 0$. We call it $(N_t)_{t\geq 0}$.

Then $\forall t \quad P\{N_t = n \mid N_0 = 0\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad$ (i.e. $N_t \mid N_0 = 0 \sim \mathcal{P}o(\lambda t)$).

THEOREM [INDEPENDENT INCREMENTS OF A POISSON PROCESS]

Let $(N_t)_{t\geq 0}$ be a Poisson Process.

Then $\forall n \quad \forall t_1 < t_2 < \dots < t_n \quad N_{t_1}, N_{t_2}-N_{t_1}, \dots, N_{t_n}-N_{t_{n-1}}$ are independent with respect to P and $N_{t_k}-N_{t_{k-1}} \sim \mathcal{P}o(\lambda(t_k - t_{k-1}))$.

COMMUNICATION CLASSES

- j is accessible from i if $\exists t>0$ s.t. $p_{ij}(t)>0$

  Proposition: j is accessible from i if and only if $\exists n, i_1, \dots, i_n$ s.t. $q_{i i_1} \cdot \ldots \cdot q_{i_n j} > 0$.
- j and i communicate if each one is accessible from the other.
- $C \subseteq E$ is a class of states if all states in C communicate and they do not communicate with the states in $E \setminus C$.

RECURRENCE / TRANSIENCE

- A state i is recurrent if $P_i\{\omega \mid \{t\geq 0 \mid X_t(\omega) = i\}$ is unbounded$\} = 1$; otherwise if this probability is equal to 0, then i is transient.
- Theorem: the following results hold. ⓐ a state i is recurrent [transient] for the continuous time Markov chain $(X_t)_{t\geq 0}$ if and only if it is recurrent [transient] for the discrete skeleton $(Y_n)_{n\geq 0}$.

  ⓑ every state is recurrent or transient.

  ⓒ all the states of the same communication classes are all recurrent or all transient.

INVARIANT DISTRIBUTIONS

- $(\mu_i)_{i\in E}$ is an invariant density if and only if $\mu_j = \sum_{i\in E} \mu_i p_{ij}(t) \quad \forall t, \forall j$
- If we suppose $q_{ii} > -\infty$ and $(p_{ij}(t))_{ij}$ are the same unique solution of the FKE and BKE, then $(\mu_i)_{i\in E}$ s.t. $\mu_i \geq 0$ and $\sum_{i\in E}\mu_i = 1$ is an invariant density if and only if

  $0 = \sum_{i\in E} \mu_i q_{ij}$ holds $\forall j \in E$.
- If $(\pi_i)_{i\in E}$ is an invariant density for $(X_t)_{t\geq 0}$, then $(\pi_j(-q_{jj}))_{j\in E}$ (possibly normalized) is an invariant density for the discrete skeleton $(Y_n)_{n\geq 0}$.

CONDITIONAL EXPECTATION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra and X a real random variable with $\mathbb{E}[|X|] < \infty$.

We call conditional expectation of X w.r.t. $\mathcal{G}$ any $\mathcal{G}$-measurable random variable Y such that $\int_G X \, d\mathbb{P} = \int_G Y \, d\mathbb{P} \quad \forall G \in \mathcal{G}$

THEOREM [EXISTENCE OF THE CONDITIONAL EXPECTATION]

For all X $\mathcal{F}$-measurable r.v. such that $\mathbb{E}[|X|] < +\infty$ and for any $\mathcal{G} \subseteq \mathcal{F}$, there exists Y $\mathcal{G}$-measurable r.v. such that $\int_G X \, d\mathbb{P} = \int_G Y \, d\mathbb{P} \quad \forall G \in \mathcal{G}$.
It is unique up to sets $G \in \mathcal{G}$ of probability $\mathbb{P}(G) = 0$ and is denoted by $\mathbb{E}[X|\mathcal{G}]$.

PROPERTIES OF THE CONDITIONAL EXPECTATION

- Let X_1 and X_2 be two integrable random variables with $\mathbb{E}[|X_1|] < +\infty$, $\mathbb{E}[|X_2|] < +\infty$.
  - a) linearity: $\forall a, b \in \mathbb{R}$ $\mathbb{E}[aX_1 + bX_2|\mathcal{G}] = a\mathbb{E}[X_1|\mathcal{G}] + b\mathbb{E}[X_2|\mathcal{G}]$
  - b) positivity: if $X_1 \geq 0$, then $\mathbb{E}[X_1|\mathcal{G}] \geq 0$
  - c) normalization: if $X_1 = \text{constant}$, then $\mathbb{E}[X_1|\mathcal{G}] = X_1 = \text{constant}$.
  - d) projective property: if $\mathcal{H} \subseteq \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$

- Let X be a $\mathcal{F}$-measurable r.v. with $\mathbb{E}[|X|^p] < +\infty$ and Y a $\mathcal{G}$-measurable r.v. with $\mathbb{E}[|Y|^q] < +\infty$ where $\mathcal{G} \subseteq \mathcal{F}$ and $\frac{1}{p} + \frac{1}{q} = 1$.
  Then $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$.

- Let X be a $\mathcal{F}$-measurable r.v. and $\mathcal{G} \subseteq \mathcal{F}$.
  Suppose that X and $\mathcal{G}$ are independent (i.e. $\sigma(X) \perp\!\!\!\perp \mathcal{G}$). Then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

THEOREM [MARKOV CHAINS AND CONDITIONAL EXPECTATION]

- Let $(X_n)_{n\geq 0}$ be a discrete time Markov Chain, P its transition matrix and $f: E \to \mathbb{R}$ a function such that $\mathbb{E}[|f(X_n)|] < +\infty \quad \forall n$.
  Then $\mathbb{E}[f(X_{n+1})|\sigma(X_n, \dots, X_0)] = (Pf)(X_n)$, where $(Pf)(j) = \sum_{k \in E} p_{jk} f(k) \quad \forall j \in E$.
  Interpretation: the conditional expectation of the function of a r.v. depends only on the values of the function Pf at the previous state.

- Let $((X_t)_{t\geq 0}$ be a continuous time Markov chain, $(P_t)_{t\geq 0}$ its transition semigroup and $f: E \to \mathbb{R}$ a function such that $\mathbb{E}[|f(X_t)|] < +\infty \quad \forall t$
  Then $\mathbb{E}[f(X_t)|\sigma(X_u | u \leq s)] = (P_{t-s}f)(X_s) \qquad s \leq t$.

THEOREM [MIN SQUARE ERROR APPROXIMATION OF A R.V.]

Let X be a random variable with $\mathbb{E}[|X|^2] < +\infty$
Then $\mathbb{E}[(\mathbb{E}[X|\mathcal{G}])^2] < +\infty$ and $\min_{\substack{Y \, \mathcal{G}\text{-measurable} \\ \text{s.t. } \mathbb{E}[Y^2] < +\infty}} \mathbb{E}[(X-Y)^2] = \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2]$

i.e. the conditional expectation of X w.r.t. $\mathcal{G}$ is the Min Square Error approximation of X with a $\mathcal{G}$-measurable r.v.

THEOREM [FREEZING LEMMA]

Let X be a random variable and $\mathcal{G}$ a σ-algebra with $\mathcal{G} \perp\!\!\!\perp \sigma(X)$
Suppose that X has density f and that $Y_1, \dots, Y_n$ are $\mathcal{G}$-measurable r.v.
Then $\mathbb{E}[h(X, Y_1, \dots, Y_n)|\mathcal{G}] = \int_{\mathbb{R}} h(x, Y_1, \dots, Y_n) f(x) dx$

MARKOV PROCESS

$(X_t)_{t\geq 0}$ collection of random variables $X_t: \Omega \to E$ is a Markov process if $\forall n \quad \forall t_1 < t_2 < \dots < t_{m+1} \quad \forall E_1, \dots, E_{m+1} \in \mathcal{E}$
$\mathbb{P}\{X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m, \dots, X_{t_1} \in E_1\} = \mathbb{P}\{X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m\}$

TIME-HOMOGENEITY

$\mathbb{P}\{X_{t+s} \in E_2 \mid X_s \in E_1\}$ doesn't depend on $s \quad \forall s,t \geq 0$.

TRANSITION KERNELS

- (heuristic definition) $P_t(x, A) = \mathbb{P}\{X_t \in A \mid X_0 = x\}$
- (formal definition) collection of functions $(P_t)_t$, $P_t: E \times \mathcal{E} \mapsto [0,1]$, s.t.
  1) $P_t(x, \cdot)$, i.e. $A \mapsto P_t(x, A)$, is a probability measure on $\mathcal{E}$, $\forall x \in E$
  2) $P_t(\cdot, A)$, i.e. $x \mapsto P_t(x, A)$, is $\mathcal{E}$-measurable, $\forall A \subseteq \mathcal{E}$
  3) $\mathbb{P}\{X_{t+s} \in A | X_s = x\} = \int_A P_t(x, dy)$

THEOREM [MARKOV PROCESSES AND CONDITIONAL EXPECTATION]

Let $(X_t)_t$ be a Markov process (time homogeneous), P_t its transition kernel and $f: E \to \mathbb{R}$ a measurable function s.t. $\mathbb{E}[|f(X_{t+s})|] < +\infty$.
Then $\mathbb{E}[f(X_{t+s})|\sigma(X_s)] = \int_E f(y) P_t(X_s, dy) \quad \forall t, s \geq 0$

STATIONARITY

Let $(X_t)_t$ be a Markov process and μ a measure on $\mathcal{E}$.
$(X_t)_t$ is ~~stationary~~ or invariant if when $X_0 \sim \mu$ then $X_t \sim \mu$ $\forall t$.
Proposition: a measure μ is invariant if and only if $\mu(A) = \int_E \mu(dx) P_t(x,A)$ $\forall t \forall A$.

IRREDUCIBILITY

$(X_t)_t$ Markov process with values in $(E, \mathcal{E})$ and with transition kernels $(P_t)_t$ is irreducible w.r.t. a reference measure φ on $\mathcal{E}$ if
$\forall x \in E, \forall A \in \mathcal{E}$ with $\varphi(A) > 0$ $\quad \exists t > 0$ such that $P_t(x,A) > 0$.

HARRIS RECURRENCE

Consider a discrete time Markov chain with $E \subseteq \mathbb{R}^d$ and $\mathcal{E} = \mathcal{B}(E)$.
Let $N_A = \sum_{n \geq 0} \mathbb{1}_A\{X_n\} = \sum_{n \geq 0} \mathbb{1}_{\{X_n \in A\}}$ be the number of visits in $A \subset \mathcal{E}$.
Then:
1) A is called Harris recurrent if $\forall x \in A$ $\mathbb{P}_x\{N_A = +\infty\} = 1$;
2) $(X_n)_{n \geq 0}$ is Harris recurrent if $\exists \varphi$ reference measure s.t. all $A \subseteq \mathcal{E}$ with $\varphi(A) > 0$ are Harris recurrent.

MARTINGALES

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $(\mathcal{F}_t)_t$ increasing family of sub-σ-algebras (filtration): $s < t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$.
Let $(M_t)_t$ be a family of real random variables.
Then $(M_t)_t$ is a martingale if:
1) $\mathbb{E}[|M_t|] < +\infty$ $\forall t$
2) M_t is $\mathcal{F}_t$-measurable $\forall t$
3) $\forall s < t$ $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ (martingale property)

MARKOV CHAINS AND MARTINGALES

- Let $(X_n)_{n \geq 0}$ be a discrete time discrete state Markov chain and f a measurable function with $\mathbb{E}[|f(X_n)|] < +\infty$ $\forall n$; let $\mathcal{F}_m = \sigma(X_0, \dots, X_m)$.
Then:
  - $\forall m < n$ $\quad \mathbb{E}[f(X_n) | \mathcal{F}_m] = (P^{n-m} f)(X_m)$
  - define $M_n = f(X_n) - \sum_{k=0}^{n-1} \left((Pf)(X_k) - f(X_k) \right)$
  $\Rightarrow (M_n)_{n \geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n \geq 0}$.
- Let $(X_t)_{t \geq 0}$ be a continuous time discrete state Markov chain and $f: E \to \mathbb{R}$ a measurable function s.t. $\mathbb{E}[|f(X_t)|] < +\infty$ $\forall t$ and $\mathbb{E}[|(Qf)(X_t)|] < +\infty$ $\forall t$.
Then: $(M_t)_{t \geq 0}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$, where $M_t = f(X_t) - \int_0^t (Qf)(X_s)\,ds$ $\forall t \geq 0$.

MARTINGALE STOPPING THEOREM

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $(\mathcal{F}_t)_{t \geq 0}$ filtration.
We define $T: \Omega \to [0, \infty]$ as a stopping time for the filtration $(\mathcal{F}_t)_t$ if $\{T < t\} \in \mathcal{F}_t$ $\forall t$.
Then the martingale stopping theorem holds:
Let $(M_t)_t$ be a martingale (where t belongs to a discrete set) w.r.t. $(\mathcal{F}_t)_t$, then the stopped process $(M_{T \wedge t})_t$ is also a martingale and in particular
$\mathbb{E}[M_{T \wedge t}] = \mathbb{E}[M_0] = \mathbb{E}[M_t]$ $\forall t$.