REAL AND FUNCTIONAL ANALYSIS A.Y. 2020-2021
- EXAM -
08-09-2021

Exercise 1. Consider the functions $f, g : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}, \qquad g(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}.$$

(1) Determine whether f is of bounded variation in $[0,1]$ by using the definition of BV functions.
(2) Is g differentiable a.e. in $[0,1]$? If so, determine whether the a.e. derivative g' belongs to $L^1([0,1])$.
(3) Is g absolutely continuous in $[0,1]$? Is g of bounded variation in $[0,1]$?

Solution Ex. 1

(1) f is not of bounded variation in $[0,1]$. Indeed, for $n \in \mathbb{N}$, $n \ge 2$, consider for instance the following partition P_n of $[0,1]$:

$$P_n = \{x_k\}_{k=0}^n, \qquad x_k = \begin{cases} 1, & k = 0 \\ \frac{1}{\sqrt{k\pi}}, & k = 1, \dots, n-1 \\ 0, & k = n \end{cases}.$$

Then

$$f(x_k) = \begin{cases} \cos 1, & k = 0 \\ \frac{(-1)^k}{k\pi}, & k = 1, \dots, n-1 \\ 0, & k = n \end{cases},$$

so that we get

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})| \ge \sum_{k=2}^{n-1} |f(x_k) - f(x_{k-1})| = \sum_{k=2}^{n-1} \left(\frac{1}{k\pi} + \frac{1}{(k-1)\pi} \right)$$
$$\ge \sum_{k=2}^{n-1} \frac{1}{k\pi}.$$

In conclusion

$$V_0^1(f) \ge \sup_n \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \ge \sum_{k=2}^{\infty} \frac{1}{k\pi} = \infty.$$

(2) Certainly g is differentiable in $(0,1)$ hence differentiable a.e. in $[0,1]$. For $x \in (0,1)$ we have

$$g'(x) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right),$$

in particular $|g'(x)| \le 2x + 1$, hence $g' \in L^1([0,1])$.

(3) We show that g is absolutely continuous in $[0,1]$ and thus g is also of bounded variation since $AC([0,1]) \subset BV([0,1])$. By the previous item, to prove that g is absolutely continuous it remains to show that the Calculus formula for g holds, i.e. we need to prove that

$$g(x) - g(0) = \int_0^x g'(t)\,dt, \qquad \forall x \in [0,1].$$

Take $0 < a < 1$, noting that $g \in C^1([a,1])$ then by the standard fundamental theorem of calculus in $[a,1]$ we have

$$g(x) - g(a) = \int_a^x g'(t)\,dt, \qquad \forall x \in [a,1]$$

We want to take the limit as $a \to 0^+$ on both sides of the equality above. First notice that $g(a) \to g(0)$ as $a \to 0^+$, since g is continuous at 0. Moreover, since $g' \in L^1([0,1])$ then the map

$$a \mapsto \int_a^x g'(t)\,dt$$

is absolutely continuous (hence continuous) in $[0,1]$, so that $\lim_{a\to 0^+} \int_a^x g'(t)\,dt = \int_0^x g'(t)\,dt$. We get

$$g(x) - \lim_{a\to 0^+} g(a) = \lim_{a\to 0^+} \int_a^x g'(t)\,dt$$

thus

$$g(x) - g(0) = \int_0^x g'(t)\,dt, \qquad \forall x \in [0,1].$$

Exercise 2. Consider the following subspace of $H = L^2([-1,1])$

$$V = \left\{ u \in H : \int_{-1}^{1} u(x)\,dx = \int_{-1}^{1} (6x-2)\,u(x)\,dx = 0 \right\}.$$

(1) Prove that V is a closed subspace of H.
(2) Find $V^\perp$ and build an orthonormal basis for $V^\perp$.
(3) Given $f \in H$ defined by $f(x) = 5x^3$, determine the projection $P_V(f)$ of f on V.

Solution Ex. 2

(1) Notice that H is an Hilbert space and that V is the orthogonal of the set $E \subset H$, where

$$E = \{1,\, 6x-2\}.$$

It is immediate to show that $E^\perp = V$ is a closed subspace of H.

(2) Let $W := \operatorname{span} E$, i.e. the vector subspace of H generated by the elements of the set E above. Notice that W is closed as well being finite-dimensional, hence $V^\perp = (W^\perp)^\perp = \overline{W} = W$. We thus construct an orthonormal basis for W from the (Hamel) base $\{u_1, u_2\}$, where $u_1 = 1$, $u_2 = 6x-2$, by using the the Gram-Schmidt algorithm. First, we normalize u_1, by defining

$$v_1 := \frac{u_1}{\|u_1\|_H} = \frac{1}{\sqrt{2}}.$$

Since v_1 and u_2 are not orthogonal, we apply the algorithm and define

$$\tilde{v}_2 := u_2 - \langle u_2,\, v_1\rangle_H\, v_1 = 6x - 2 - \frac{1}{2}\int_{-1}^{1} (6x-2)\,dx = 6x - 2 - \frac{1}{2}(0-4) = 6x.$$

Now, by construction, v_1 and $\tilde{v}_2$ are orthogonal. We normalize $\tilde{v}_2$:

$$\|\tilde{v}_2\|_H^2 = \int_{-1}^{1} (6x)^2\,dx = 36\left[\frac{x^3}{3}\right]_{-1}^{1} = 24.$$

Hence, we define the versor $v_2 = \frac{6x}{\sqrt{24}} = \frac{\sqrt{6}}{2}x$. We conclude that $\{v_1, v_2\}$ is an orthonormal basis for $V^\perp = W$, with

$$v_1 = \frac{1}{\sqrt{2}}, \qquad v_2 = \frac{\sqrt{6}}{2}x.$$

(3) Since $H = V \oplus V^\perp$, then f can be written as $f = P_V(f) + P_{V^\perp}(f)$, thus

$$P_V(f) = f - P_{V^\perp}(f).$$

We have $P_{V^\perp}(f) = \langle f,\, v_1\rangle_H\, v_1 + \langle f,\, v_2\rangle_H\, v_2$, where

$$\langle f,\, v_1\rangle_H = 0,$$

$$\langle f,\, v_2\rangle_H = \frac{5\sqrt{6}}{2}\int_{-1}^{1} x^4\,dx = \sqrt{6}.$$

Thus, $P_{V^\perp}(f) = 3x$ and

$$P_V(f) = f - P_{V^\perp}(f) = 5x^3 - 3x.$$

REAL AND FUNCTIONAL ANALYSIS

13/7/2021

Q1. State the monotone convergence and the dominated convergence theorems in the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$, where μ_c is the counting measure.

Q2. Let $(\Omega, \mathcal{M}, \mu)$ be a complete measure space and consider a sequence of measurable functions $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$. Suppose there exists $g \in L^1(\Omega, \mathcal{M}, \mu)$ such that $f_n \leq g$ a.e. in Ω. Then **prove** that (hint: use Fatou's lemma)

$$\int_\Omega \limsup_{n\to\infty} f_n d\mu \geq \limsup_{n\to\infty} \int_\Omega f_n d\mu.$$

Q3. Let X be an infinite dimensional Banach space. **Show** a concrete example of linear compact (but not finite rank) operator $K : X \to X$ which is injective but not surjective. **Prove** that a linear compact operator $K : X \to X$ can never be surjective.

Q4. Let $k \in L^\infty([0,1]^2)$ (Lebesgue measure). Set

$$a(u,v) = \int_{[0,1]^2} k(x,y)u(x)v(y)dxdy$$

for all $u, v \in H = L^2([0,1])$. **State** sufficient conditions on k which ensure that for all $L \in H^*$ there is a unique $u \in H$ such that

$$a(u,v) = Lv, \qquad \forall v \in H.$$

Does the validity of this property ensure that $a(\cdot,\cdot)$ is a scalar product in H?

ALL THE ANSWERS MUST BE JUSTIFIED

E1. Consider the measure space $([1,+\infty), \mathcal{L}([1,+\infty)))$, with the Lebesgue measure, and the sequence of functions

$$f_n(x) = \frac{\sin(nx)}{nx + x^2} + \frac{n}{nx^{3/2}+1}, \quad x \in [1,+\infty), \quad n \in \mathbb{N}.$$

(1) Study the convergence a.e. of the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $[1,+\infty)$ and prove that, for any $n \in \mathbb{N}$, the function f_n is integrable in $[1,+\infty)$.

(2) Study the convergences in $L^1([1,+\infty))$ and in measure of $\{f_n\}_{n\in\mathbb{N}}$ in $[1,+\infty)$.

(3) Does $\lim_{n\to+\infty} \int_1^{+\infty} f_n(x)\,dx$ exist finite?

SOLUTION

(1) $\frac{\sin(nx)}{nx+x^2} \to 0$ as $n \to +\infty$, while $\frac{n}{nx^{3/2}+1} \to \frac{1}{x^{3/2}}$ as $n \to +\infty$. So, $f_n(x)$ converges to $f(x) = \frac{1}{x^{3/2}}$ for any $x \in [1,+\infty)$, as $n \to +\infty$. Hence f is the limit a.e. of $\{f_n\}_{n\in\mathbb{N}}$ in $[1,+\infty)$.

Since

$$|f_n(x)| \leq \left|\frac{\sin(nx)}{nx+x^2}\right| + \left|\frac{n}{nx^{3/2}+1}\right| \leq \frac{1}{x^2} + \frac{n}{nx^{3/2}} \leq \frac{1}{x^2} + \frac{1}{x^{3/2}},$$

and the function at the right hand side is integrable in $[1, +\infty)$, then so is f_n for any $n \in \mathbb{N}$.

(2) Since, by previous computations, $|f_n(x) - f(x)| \leq \frac{1}{x^2} + \frac{2}{x^{3/2}} =: g(x)$, with g integrable on $[1, +\infty)$, then we can apply Dominated convergence theorem to get the convergence in $L^1([1, +\infty))$ of $\{f_n\}_{n\in\mathbb{N}}$. Thus, by Chebyshev's inequality, we have also the convergence in measure.

(3) Again by Dominated Convergence Theorem, the answer is affirmative, moreover we can compute the value explicitly

$$\lim_{n\to+\infty} \int_1^{+\infty} f_n(x)\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x^{3/2}}\,\mathrm{d}x = 2.$$

E2. Let $X = C^0([0,1])$ with the supremum norm. Let $g(t) = \dfrac{e^t - 1}{t}$ for any $t \in (0, 1]$. Then consider the operator T defined by

$$T(u)(t) = \begin{cases} g(t)\,u(t), & t \in (0,1], \\ u(0), & t = 0, \end{cases}$$

for all $u \in X$.

(1) After verifying that T is well defined from X to X, prove that T is linear and bounded and compute its operator norm.
(2) Does T have eigenvalues (same definition as in the Hilbert case)? Justify the answer.
(3) Is T injective? Is T surjective? Justify the answers.
(4) Is T compact? Justify the answer.

SOLUTION

(1) For any $u \in X$, we have $Tu \in X$. Indeed $t \mapsto \frac{e^t-1}{t}u(t)$ is continuous in $[0,1]$ and $\frac{e^t-1}{t} \to 1$ as $t \to 0^+$, so that $\lim_{t\to 0^+} \frac{e^t-1}{t}u(t) = u(0)$.
T is trivially linear. Concerning boundedness,

$$\|Tu\|_\infty = \max_{t\in[0,1]} |Tu(t)| \leq \left\| \frac{e^t - 1}{t} \right\|_\infty \|u\|_\infty = (e-1)\|u\|_\infty,$$

so T is a bounded operator from X to X. By taking $\bar{u} = 1 \in X$, we have $\|\bar{u}\|_\infty = 1$ and $\|T\bar{u}\|_\infty = e - 1$, hence $\|T\|_{\mathcal{B}} = e - 1$.

(2) $\lambda \in \mathbb{R}$ is an eigenvalue of T if $Tu = \lambda u$ for some $X \ni u \neq 0$, i.e. $Tu(t) = \lambda u(t)$ for any $t \in [0,1]$, for some $X \ni u \neq 0$. Since $Tu(0) = u(0)$, the only candidate to be an eigenvalue for T is $\lambda = 1$. However, we notice that $\frac{e^t-1}{t} > 1$ for $t \in (0,1]$ and the only possibility to have $Tu(t) = u(t)$ for any $t \in [0,1]$ is to take $u = 0$. So T has no eigenvalues.

(3) T is injective: if $u_1, u_2 \in X$ are such that $g(t)u_1(t) = g(t)u_2(t)$ for all $t \in (0,1]$, then $u_1 = u_2$ since $g(t) \neq 0$ for all $t \in (0,1]$.
T is also surjective: given any $v \in X$, we should show that there exists $u \in X$ such that $v = Tu$, i.e. we should prove that any $v \in X$ can be written as

$$v(t) = \begin{cases} g(t)u(t), & t \in (0,1] \\ u(0), & t = 0 \end{cases}$$

for some $u \in X$. To this end, given $v \in X$, it is sufficient to take

$$u(t) = \begin{cases} \dfrac{v(t)}{g(t)}, & t \in (0,1] \\ v(0), & t = 0 \end{cases}$$

and prove that $u \in X$: since both v and g are continuous and $g \neq 0$ in $(0,1]$, then the function $t \mapsto \frac{v(t)}{g(t)}$ is continuous in $(0,1]$; moreover, $\frac{v(t)}{g(t)} \to v(0)$ as $t \to 0^+$ so that $u \in X$.

(4) T is not compact, since it is a linear and continuous bijective operator defined on an infinite dimensional Banach space X.

NB Q1-Q4$=4/30$ each; **E1-E2**$=7/30$ each.

REAL AND FUNCTIONAL ANALYSIS

21/6/2021

Q1. Let $E \subset \mathbb{R}$ be a set with non-empty interior (Euclidean topology). Is it true that E contains a subset which is not Lebesgue measurable? Is it true that a subset of $\mathbb{R}$ with empty interior can contain a subset which is not Lebesgue measurable?

Q2. Suppose that $f \in AC([a,b])$. **Prove** that

$$V_a^b(f) \leq \|(f')^+\|_{L^1([a,b])} + \|(f')^-\|_{L^1([a,b])}.$$

Is it true that the above inequality also holds for any $f \in BV([a,b])$?

Q3. Let $(\Omega, \mathcal{M}, \mu)$ be a complete measure space. If $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}, \mu)$ is bounded and $f_n \to f$ in $L^1(\Omega, \mathcal{M}, \mu)$, **prove** that $f_n \to f$ in any $L^p(\Omega, \mathcal{M}, \mu)$ with $p \in (1, \infty)$.

Q4. Let H be an infinite dimensional separable Hilbert space and let $\{u_n\}_{n\in\mathbb{N}}$ be an o.n.b. in H. Consider the subspace V_N generated by the first N elements of the o.n.b. Define the orthogonal projector $P_N : H \to V_N$ and describe its spectrum. Is it true that $\|P_N\|_{\mathcal{B}(H)} = 1$ for all $N \in \mathbb{N}$? Is it possible to construct an infinite-dimensional proper subspace V of H such that the orthogonal projector $P : H \to V$ has an eigenvalue of finite multiplicity?

ALL THE ANSWERS MUST BE JUSTIFIED

E1. Let $g(x) = \dfrac{x}{1+x^2}$ and let $f_n(x) = g(nx)\chi_{[-n,n]}(x)$ for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$. For the sequence $\{f_n\}_{n\in N}$ study

(1) the convergences a.e. and uniform a.e. in $\mathbb{R}$.
(2) the convergence in $L^1(\mathbb{R})$ and in measure.
(3) the convergence in $L^p(\mathbb{R})$ for all $p \in (1, +\infty)$.

SOLUTION

We have $f_n(x) = \frac{nx}{1+n^2x^2}\chi_{[-n,n]}(x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

(1) $\chi_{[-n,n]}(x) \to 1$ as $n \to +\infty$, for any $x \in \mathbb{R}$. Hence, f_n converges pointwisely everywhere to 0 as $n \to +\infty$, in particular the a.e. limit of $\{f_n\}_{n\in\mathbb{N}}$ exists and is $f = 0$.

We now study uniform convergence. Since u.a.e. convergence implies a.e. convergence, then the uniform a.e. limit of $\{f_n\}_{n\in\mathbb{N}}$, if it exists, should be equal to 0. Since f_n are continuous, then

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \left| f_n\left(\frac{1}{n}\right)\right| = \frac{1}{2},$$

thus $\lim_{n\to+\infty} \sup_{x\in\mathbb{R}} |f_n(x)| = 1/2 \neq 0$. We conclude that $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly.

(2) *Convergence in $L^1(\mathbb{R})$:*

$$\begin{aligned}\int_{\mathbb{R}} |f_n(x) - f|\,dx &= \int_{-n}^{n} \left|\frac{nx}{1+n^2x^2}\right| dx\\ &= \frac{1}{n}\int_{-n^2}^{n^2} \left|\frac{y}{1+y^2}\right| dy\\ &= \frac{2}{n}\int_{0}^{n^2} \frac{y}{1+y^2}\,dy = \frac{1}{n}[\ln(1+y^2)]_0^{n^2}\\ &= \frac{1}{n}\ln(1+n^4).\end{aligned}$$

Since $\ln(1+n^4) \sim \ln(n^4) = 4\ln(n)$ as $n \to +\infty$, we get $\lim_{n\to+\infty} \|f_n - f\|_{L^1} = 0$, hence $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{R})$ (to $f = 0$). This implies immediately that $\{f_n\}_{n\in\mathbb{N}}$ converges also in measure (to $f = 0$) by Chebyshev's inequality.

(3) *Convergence in $L^p(\mathbb{R})$ for $p \in (1, +\infty)$:*

$$\begin{aligned}\|f_n - f\|_{L^p}^p &= \int_{\mathbb{R}} \left|\frac{nx}{1+n^2x^2}\right|^p \chi_{[-n,n]}(x)\,dx\\ &= \frac{1}{n}\int_{-n^2}^{n^2} \left|\frac{y}{1+y^2}\right|^p dy\\ &= \frac{1}{n}\int_{\mathbb{R}} \left|\frac{y}{1+y^2}\right|^p \chi_{[-n^2,n^2]}(y)\,dy\\ &= \frac{1}{n}\int_{\mathbb{R}} h_n(y)\,dy,\end{aligned}$$

where $h_n(y) := \left|\frac{y}{1+y^2}\right|^p \chi_{[-n^2,n^2]}(y)$. Notice that

$$|h_n(y)| \le \left|\frac{y}{1+y^2}\right|^p =: g(y),$$

and $g(y) \sim \frac{1}{|y|^p}$ as $y \to \pm\infty$, so g is integrable in $(-\infty, -1) \cup (1, +\infty)$ (since $p > 1$), and $|h_n|$ is bounded in $[-1, 1]$. We can thus apply the Dominated Convergence Theorem to get

$$\lim_{n\to+\infty} \int_{\mathbb{R}} h_n(y)\,dy = \int_{\mathbb{R}} \lim_{n\to+\infty} h_n(y)\,dy = \int_{\mathbb{R}} \left|\frac{y}{1+y^2}\right|^p dy < +\infty.$$

Thus

$$\lim_{n\to+\infty} \|f_n - f\|_{L^p}^p = \lim_{n\to+\infty} \frac{1}{n}\int_{\mathbb{R}} h_n(y)\,dy = 0,$$

so $\{f_n\}_{n\in\mathbb{N}}$ converges also in $L^p(\mathbb{R})$ for any $p \in (1, +\infty)$.

E2. Consider the following subspace of $H = L^2([-1, 1])$

$$V = \{u \in H \,:\, u(x) = a + bx^2,\ x \in [-1, 1],\ \text{for some } a, b \in \mathbb{R}\}.$$

Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in [-1, 0),\\ 0 & \text{if } x = 0,\\ 1/2 & \text{if } x \in (0, 1].\end{cases}$$

(1) Justify why $f \in H$ and why V is a closed subspace of H.
(2) Build an orthonormal basis for V.
(3) Find the projection of f on V.

SOLUTION

(1) f belongs to H indeed $\|f\|_H < +\infty$. V is a subspace of H and it is finite-dimensional, indeed it is the vector space generated by the Hamel basis $\{u_1, u_2\} \subset H$, with $u_1(x) = 1$ and $u_2(x) = x^2$. Being finite-dimensional, V is closed.
(2) We construct an orthonormal basis for V from the (Hamel) base $\{u_1, u_2\}$. First, we normalize u_1, by defining
$$v_1 := \frac{u_1}{\|u_1\|_H} = \frac{1}{\sqrt{2}}.$$
Since v_1 and u_2 are not orthogonal, we apply the Gram-Schmidt algorithm to u_2 to get
$$\tilde{v}_2 := u_2 - \langle u_2, v_1\rangle_H \, v_1 = x^2 - \frac{1}{2}\int_{-1}^{1} x^2 \, dx = x^2 - \frac{1}{3}.$$
Now, by construction, v_1 and $\tilde{v}_2$ are orthogonal. It remains to normalize $\tilde{v}_2$:
$$\|\tilde{v}_2\|_H^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 + \frac{1}{9} - \frac{2}{3}x^2\right) dx = \cdots = \frac{8}{45}.$$
We conclude that $\{v_1, v_2\}$, with
$$v_1 := \frac{1}{\sqrt{2}}, \quad v_2 := \frac{\tilde{v}_2}{\|\tilde{v}_2\|_H^2} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$
is an orthonormal basis for V.
(3) The projection of f on V is given by $P_V f := \langle f, v_1\rangle_H \, v_1 + \langle f, v_2\rangle_H \, v_2$, where
$\langle f, v_1\rangle_H = 0$,
$$\langle f, v_2\rangle_H = \sqrt{\frac{45}{8}}\int_{-1}^{1} f(x)\left(x^2 - \frac{1}{3}\right) dx = \sqrt{\frac{45}{8}}\left(\int_{-1}^{1} f(x)x^2 \, dx - \frac{1}{3}\int_{-1}^{1} f(x)\, dx\right)$$
$$= \sqrt{\frac{45}{8}}\left(\int_{-1}^{0} x^3 \, dx + \frac{1}{2}\int_{0}^{1} x^2 \, dx\right) = -\frac{1}{12}\sqrt{\frac{45}{8}} = -\frac{\sqrt{5}}{8\sqrt{2}}.$$
Hence, $P_V f = -\dfrac{15}{32}\left(x^2 - \dfrac{1}{3}\right)$.

NB Q1-Q4= 4/30 each ; **E1-E2**= 7/30 each.

REAL AND FUNCTIONAL ANALYSIS

18/2/2021

Q1. Can $\chi_{[0,+\infty)}$ be equal a.e. in $\mathbb{R}$ to a continuous function? If $f : \mathbb{R} \to \mathbb{R}$ is equal a.e. to a continuous function is it true that f is necessarily a.e. continuous?

Q2. Is it true that a closed set $E \subset \mathbb{R}$ of Lebesgue measure zero is nowhere dense? If E has positive Lebesgue measure then can it be nowhere dense?

Q3. Let $(X, \|\cdot\|)$ be a Banach space. Consider the subset $C := \{x \in X : \|x\| = 1\}$. Establish the Baire category of C w.r.t. the induced topology and w.r.t. to the X-topology. Is it true that C is compact in the X-topology?

Q4. Let H be a separable Hilbert space of infinite orthogonal dimension. Is it true that any finite rank (linear) symmetric operator T is a projector on a finite dimensional subspace of H? Does 0 always belong to the spectrum of T?

ALL THE ANSWERS MUST BE JUSTIFIED

E1. Let $\alpha > 0$ and consider the following series defined for $x \in [0, +\infty)$,

$$\sum_{k=1}^{+\infty} \frac{1}{x^\alpha + k^\alpha}.$$

(1) Find the values of $\alpha > 0$ for which the above series converges for any $x \in [0, +\infty)$. For such values of α, define the function $F_\alpha : [0, +\infty) \to \mathbb{R}$ by setting

$$F_\alpha(x) := \sum_{k=1}^{+\infty} \frac{1}{x^\alpha + k^\alpha}, \qquad x \in [0, +\infty).$$

(2) For which $\alpha > 0$ the following identity holds?

$$\int_0^{+\infty} F_\alpha(x)\,\mathrm{d}x = \lim_{n\to+\infty} \sum_{k=1}^{n} \int_0^{+\infty} \frac{1}{x^\alpha + k^\alpha}\,\mathrm{d}x.$$

(3) For which $\alpha > 0$ we have that F_α is integrable on $[0, +\infty)$?
(4) Find the values of $\alpha > 0$ such that $F_\alpha \in BV([0,1])$ and calculate $V_0^1(F_\alpha)$.

SOLUTION

(1) The series converges if and only if $\alpha > 1$ by asymptotic comparison with the harmonic series. So, the function F_α is defined for $\alpha > 1$.

(2) For $\alpha > 1$, set

$$f_k^\alpha(x) = \frac{1}{x^\alpha + k^\alpha}, \qquad s_n^\alpha(x) = \sum_{k=1}^{n} f_k^\alpha(x).$$

Since $f_k^\alpha > 0$, $\{s_n^\alpha\}_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative measurable functions, pointwise convergent to F_α. Hence, by Monotone Convergence Theorem (Beppo-Levi)

$$\int_0^{+\infty} F_\alpha(x)\,\mathrm{d}x = \lim_{n\to+\infty} \int_0^{+\infty} s_n^\alpha(x)\,\mathrm{d}x = \lim_{n\to+\infty} \sum_{k=1}^{n} \int_0^{+\infty} f_k^\alpha(x)\,\mathrm{d}x$$

for any $\alpha > 1$.

(3) Recalling that
$$\int_0^{+\infty} f_k^\alpha(x)\,\mathrm{d}x = \frac{1}{k^\alpha}\int_0^{+\infty}\frac{\mathrm{d}x}{(x/k)^\alpha+1} = \frac{1}{k^\alpha}\int_0^{+\infty}\frac{k\,\mathrm{d}y}{y^\alpha+1} = \frac{1}{k^{\alpha-1}}\int_0^{+\infty}\frac{\mathrm{d}y}{y^\alpha+1},$$
by comparison with the power of $1/y$, the last integral converges for $\alpha > 1$. So, for $\alpha > 1$ we get
$$\begin{aligned}\int_0^{+\infty} F_\alpha(x)\,\mathrm{d}x &= \lim_{n\to+\infty}\sum_{k=1}^{n}\int_0^{+\infty} f_k^\alpha(x)\,\mathrm{d}x\\ &= \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k^{\alpha-1}}\int_0^{+\infty}\frac{\mathrm{d}y}{y^\alpha+1} = \int_0^{+\infty}\frac{\mathrm{d}y}{y^\alpha+1}\sum_{k=1}^{+\infty}\frac{1}{k^{\alpha-1}},\end{aligned}$$
and the last series converges if and only if $\alpha-1>1$ thus $\alpha>2$. Hence F_α is integrable on $[0,+\infty)$ if and only if $\alpha > 2$.

(4) Notice that F_α is a monotone (decreasing) function, hence it is of bounded variation in any interval $[a,b]$, with $0\le a\le b<+\infty$, thus in particular in $[0,1]$. The total variation $V_0^1(F_\alpha)$ is given by
$$V_0^1(F_\alpha) = F_\alpha(0)-F_\alpha(1) = \sum_{k=1}^{\infty}\left(\frac{1}{k^\alpha}-\frac{1}{1+k^\alpha}\right) = \sum_{k=1}^{\infty}\frac{1}{k^\alpha(1+k^\alpha)}.$$

E2. Let $H=\ell^2$ and for any $n\in\mathbb{N}\setminus\{0\}$ let $T_n:\ell^2\to\ell^2$ be defined by
$$T_n(x) = (y_k)_{k\in\mathbb{N}\setminus\{0\}},\ \forall x=(x_k)_{n\in\mathbb{N}\setminus\{0\}}\in\ell^2,\quad \text{where } y_k := \begin{cases}\frac{1}{3}x_k, & k<n\\ x_{k+1}, & k\ge n.\end{cases}$$

(1) Prove that, for any fixed $n\in\mathbb{N}\setminus\{0\}$, T_n is a linear and continuous operator.
(2) For any $n\in\mathbb{N}\setminus\{0\}$, compute the operator norm $\|T_n\|_{\mathcal{B}(\ell^2)}$.
(3) Study injectivity and surjectivity of the operator T_3.
(4) For any $x\in\ell^2$, show that $T_n(x)\to\frac{x}{3}$ in ℓ^2 as $n\to+\infty$.
(5) Does the sequence $\{T_n\}_{n\in\mathbb{N}\setminus\{0\}}$ converge in the space $\mathcal{B}(\ell^2)$? If so, find its limit. If not, justify your answer.

SOLUTION

(1) For any $n\in\mathbb{N}\setminus\{0\}$, it's easy to check that T_n is linear. Also, T_n is continuous (i.e. bounded). Indeed for any $x=(x_k)_{k\in\mathbb{N}\setminus\{0\}}\in\ell^2$ we have
$$\|T_n(x)\|_{\ell^2}^2 = \frac{1}{9}\sum_{k=1}^{n-1}x_k^2+\sum_{k=n}^{\infty}x_{k+1}^2 \le \sum_{k=1}^{n-1}x_k^2+\sum_{k=n+1}^{\infty}x_k^2 \le \|x\|_{\ell^2}^2.$$
In particular, we get $\|T_n\|_{\mathcal{B}(\ell^2)}\le 1$ for any $n\in\mathbb{N}\setminus\{0\}$.

(2) For any $n\in\mathbb{N}\setminus\{0\}$, take $\mathrm{e}_n=(e_n)_{k\in\mathbb{N}\setminus\{0\}}$ defined by
$$\mathrm{e}_n = (e_n)_k = \begin{cases}1 & \text{if } k=n+1,\\ 0 & \text{otherwise,}\end{cases}$$
and notice that $\mathrm{e}_n\in\ell^2$ and $\|e_n\|_{\ell^2}=1$ for any $n\in\mathbb{N}\setminus\{0\}$. We have
$$(T_n(\mathrm{e}_n))_k = \begin{cases}1 & \text{if } k=n,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, $\|T_n(e_n)\|_{\ell^2} = 1$ and so $\|T_n\|_{\mathcal{B}(\ell^2)} = 1$ for any $n \in \mathbb{N} \setminus \{0\}$.

(3) For any $x = (x_k)_{k\in\mathbb{N}\setminus\{0\}} \in \ell^2$, we have

$$T_3(x) = \left(\frac{1}{3}x_1, \frac{1}{3}x_2, x_4, x_5, \dots\right).$$

T_3 is not injective, indeed if we take for instance $\bar{x} = (0, 0, a, 0, \dots) \in \ell^2$ and $\bar{y} = (0, 0, b, 0, \dots) \in \ell^2$, with $a \neq b$, then we have $\bar{x} \neq \bar{y}$ but $T_3(\bar{x}) = T_3(\bar{y})$.
On the other hand, it's easy to check that T_3 is surjective.

(4) For any $x = (x_k)_{k\in\mathbb{N}\setminus\{0\}} \in \ell^2$, we argue that $T_n(x)$ strongly converges to $\frac{1}{3}x$ in ℓ^2 as $n \to +\infty$. Indeed,

$$\begin{aligned}
\|T_n x - \frac{1}{3}x\|_{\ell^2}^2 &= \sum_{k=n}^{\infty}\left|x_{k+1} - \frac{1}{3}x_k\right|^2 \leq \sum_{k=n}^{\infty}\left(|x_{k+1}| + \frac{1}{3}|x_k|\right)^2 \\
&\leq 2\sum_{k=n}^{\infty}|x_{k+1}|^2 + \frac{2}{9}\sum_{k=n}^{\infty}|x_k|^2 \\
&\leq \frac{20}{9}\sum_{k=n}^{\infty}|x_k|^2,
\end{aligned}$$

where we used Young's inequality to pass from the first line to the second one. Since $x \in \ell^2$, the right hand side is the reminder of a convergent series and thus it tends to zero as $n \to +\infty$. Hence, $\|T_n x - \frac{1}{3}x\|_{\ell^2} \to 0$, as $n \to +\infty$, and so $T_n(x)$ converges strongly (and so also weakly) to $\frac{1}{3}x$ for any $x \in \ell^2$.

(5) Denote by I the identity operator in $\mathcal{B}(\ell^2)$. Recalling the definition of operator norm, by the previous item we know that if T_n has a limit in $\mathcal{B}(\ell^2)$, then this limit should be $\frac{1}{3}I$. We thus need to check if $\|T_n - \frac{1}{3}I\|_{\mathcal{B}(\ell^2)} \to 0$ as $n \to +\infty$:

$$\|T_n - \frac{1}{3}I\|_{\mathcal{B}(\ell^2)} \geq \|T_n\|_{\mathcal{B}(\ell^2)} - \frac{1}{3}\|I\|_{\mathcal{B}(\ell^2)} = 1 - \frac{1}{3} = \frac{2}{3},$$

so T_n does not converge in $\mathcal{B}(\ell^2)$.

NB Q1-Q4= 4/30 each ; **E1-E2**= 7/30 each.

REAL AND FUNCTIONAL ANALYSIS

28/1/2021

Q1. Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$ where μ_c is the counting measure. If $\{a_n\}_{n\in\mathbb{N}}$ is a given sequence of non-negative real numbers, set

$$\nu(E) = \sum_{n\in E} a_n \qquad \forall E \in \mathcal{P}(\mathbb{N}).$$

Prove that ν is a measure. Does $\frac{d\nu}{d\mu_c}$ exist? If so, calculate it. Does $\frac{d\mu_c}{d\nu}$ always exist? Can we have $\nu = \mu_c$ for a suitable choice of $\{a_n\}_{n\in\mathbb{N}}$?

Q2. Suppose that $f \in BV([a,b])$ and continuous at $x_0 \in (a,b)$. Show that x_0 is a Lebesgue point for f. May f have non-Lebesgue points which are not jumps?

Q3. Let $(\Omega, \mathcal{M}, \mu)$ be a complete measure space. Is it true that $L^1(\Omega, \mathcal{M}, \mu) \cap L^\infty(\Omega, \mathcal{M}, \mu) \hookrightarrow L^p(\Omega, \mathcal{M}, \mu)$ for all $p \in [1, \infty]$?

Q4. Let H be a separable Hilbert space and $K \in \mathcal{K}(H)$ be a symmetric operator. Consider the equation $\mu x - Kx = y$ for a given $\mu \in \mathbb{R}$. If $|\mu| > \|K\|_{\mathcal{B}(H)}$, is it true that for any $y \in H$ the equation has a unique solution? Is this still true if $|\mu| = \|K\|_{\mathcal{B}(H)}$?

E1. Consider the measurable space $([-1,1], \mathcal{L}([-1,1]))$ with the Lebesgue measure. Let $f_n : [-1,1] \to \mathbb{R}$ be a sequence of functions defined by

$$f_n(x) = \frac{n+5}{1+n^2x^2}, \qquad x \in [-1,1],$$

for any $n \in \mathbb{N}$.

(1) Study the measurability and the integrability of f_n for each $n \in \mathbb{N}$.
(2) Study the convergence a.e. in $[-1,1]$ of $\{f_n\}_{n\in\mathbb{N}}$. Does $\{f_n\}_{n\in\mathbb{N}}$ converge pointwise everywhere in $[-1,1]$? Justify your answer.
(3) Study the convergence uniformly a.e., in measure and in $L^1([-1,1])$.
(4) Compute $\displaystyle\lim_{n\to+\infty} \int_{-1}^{1} f_n(x)\,dx$.

SOLUTION

(1) For any $n \in \mathbb{N}$, f_n is continuous hence measurable in $[-1,1]$. We analyze the integrability:

$$\int_{-1}^{1} f_n(x)\,dx = \left(1 + \frac{5}{n}\right) \int_{-n}^{n} \frac{1}{1+y^2}\,dy = \left(1 + \frac{5}{n}\right) [\arctan(y)]_{-n}^{n} < +\infty,$$

for any $n \in \mathbb{N}$. So f_n are integrable in $[-1,1]$ for any $n \in \mathbb{N}$.

(2) $f_n(x) \to 0$ for any $x \in [-1,1]$ with $x \neq 0$, while $f_n(0) \to +\infty$. So $\{f_n\}_{n\in\mathbb{N}}$ converges a.e. to $f = 0$ in $[-1,1]$ but it does not converge pointwisely everywhere.

(3) Since $\{f_n\}_{n\in\mathbb{N}}$ converges a.e. to $f=0$ in $[-1,1]$ and the Lebesgue measure of $[-1,1]$ is finite, then $\{f_n\}_{n\in\mathbb{N}}$ converges in measure to $f=0$. Concerning the uniform a.e. convergence, notice that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions and that $\sup_{x\in[-1,1]}|f_n(x)| = f_n(0) = n+5 \to +\infty$ as $n\to+\infty$, so $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly, nor uniformly a.e. in $[-1,1]$.

Convergence in $L^1([-1,1])$:

$$\begin{aligned}\|f_n - f\|_{L^1} &= \int_{-1}^{1} |f_n(x)|\,\mathrm{d}x = \left(1+\frac{5}{n}\right)\int_{-\infty}^{+\infty} \frac{1}{1+y^2}\chi_{[-n,n]}(y)\,\mathrm{d}y \\ &=: \left(1+\frac{5}{n}\right)\int_{-\infty}^{+\infty} h_n(y)\,\mathrm{d}y.\end{aligned}$$

Notice that $|h_n(y)| \le \frac{1}{1+y^2} =: g(y)$, and g is integrable in $[-1,1]$. Moreover, $h_n(y)\to \frac{1}{1+y^2}$ a.e. in $\mathbb{R}$ as $n\to+\infty$. We can thus apply Dominated Convergence Theorem to get

$$\lim_{n\to+\infty}\int_{\mathbb{R}} h_n(y)\,\mathrm{d}y = \int_{-\infty}^{+\infty}\frac{1}{1+y^2}\,\mathrm{d}y = [\arctan y]_{-\infty}^{+\infty} = \pi.$$

Hence $\lim_{n\to+\infty}\|f_n - f\|_{L^1} = \lim_{n\to+\infty}(1+5/n)\pi = \pi \neq 0$. So $\{f_n\}_{n\in\mathbb{N}}$ does not converge in $L^1([-1,1])$.

(4) By previous computations, since $f=0$ and $f_n\ge 0$, the value of the given limit is π.

E2. Let $H = L^2(\mathbb{R})$ and $T : H\to\mathbb{R}$ be defined by

$$T(u) = \int_1^2 u(x)\,\mathrm{d}x - \int_0^1 u(x)\,\mathrm{d}x, \quad u\in H.$$

(1) Prove that T is a linear and continuous operator.
(2) For any $\alpha\in\mathbb{R}$ consider the set $E_\alpha := \{u\in H : T(u)=\alpha\}$. Find all the values of α such that E_α is a *closed subspace* of H.
(3) Let $\alpha\in\mathbb{R}$ be such that E_α is a closed subspace of H. Find an orthonormal basis for $E_\alpha^\perp$, i.e. the orthogonal of E_α.
(4) Let $\alpha\in\mathbb{R}$ be such that E_α is a closed subspace of H. Check that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) := e^{-|x-1|}$, belongs to H and find the projection of f on E_α.

SOLUTION

(1) It's easy to check that T is linear. Moreover, T is continuous. Indeed, for any $u\in H$, we have

$$\begin{aligned}|Tu| &\le \left|\int_1^2 u(x)\,\mathrm{d}x\right| + \left|\int_0^1 u(x)\,\mathrm{d}x\right| \le \int_0^2 |u(x)|\,\mathrm{d}x \\ &\le \left(\int_0^2 1\,\mathrm{d}x\right)^{1/2}\left(\int_0^2 |u(x)|^2\,\mathrm{d}x\right)^{1/2} \\ &\le \sqrt{2}\|u\|_H,\end{aligned}$$

where we've applied Hölder's inequality.

(2) On account of the fact that T is a linear continuous operator, only $\alpha=0$ makes E_α a closed *subspace* of H.

(3) For any $u\in H$, we can write

$$Tu = \int_{\mathbb{R}} u(x)\left(\chi_{[1,2]}(x) - \chi_{[0,1]}(x)\right)\mathrm{d}x,$$

so $E_0 = W^\perp$, where $W := \{\alpha\left(\chi_{[1,2]}(x) - \chi_{[0,1]}(x)\right) : \alpha \in \mathbb{R}\}$. We now determine an orthonormal basis for $W = E_0^\perp$.

Take $w_1 := \chi_{[1,2]}(x) - \chi_{[0,1]}(x)$. Therefore, we have

$$\|w_1\|_H^2 = \int_{\mathbb{R}} |\chi_{[1,2]}(x) - \chi_{[0,1]}(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \chi_{[0,2]}(x)\,\mathrm{d}x = 2.$$

Thus an orthonormal basis for $E_0^\perp$ is given by $\bar{w}_1 = \frac{1}{\sqrt{2}} w_1 = \frac{1}{\sqrt{2}}\left(\chi_{[1,2]}(x) - \chi_{[0,1]}(x)\right)$.

(4) Observe that $f \in H$. Indeed we have

$$\|f\|_H^2 = \int_{\mathbb{R}} e^{-2|x-1|}\,\mathrm{d}x = \cdots = 1 < +\infty.$$

In order to find the projection of f on E_0 we observe that

$$\int_1^2 f(x)\,\mathrm{d}x = \frac{e-1}{e} = \int_0^1 f(x)\,\mathrm{d}x,$$

so that $T(f) = 0$. Hence $f \in E_0$ and $P_{E_0}(f) = f$. Alternatively, we can argue as follows. We write $f = P_{E_0}(f) + P_{E_0^\perp}(f)$. This gives

$$P_{E_0}(f) = f - P_{E_0^\perp}(f) = f - \langle f, \bar{w}_1\rangle\, \bar{w}_1.$$

Then we observe that

$$\langle f, \bar{w}_1\rangle = \frac{1}{2}\int_1^2 e^{-(x-1)}\,\mathrm{d}x - \frac{1}{2}\int_0^1 e^{x-1}\,\mathrm{d}x = 0,$$

which implies $P_{E_0}(f) = f$, i.e. $f \in E_0$.

NB Q1-Q4= 4/30 each ; **E1-E2**= 7/30 each.