# Real and Functional Analysis

Master Degree Program in Mathematical Engineering, a.y. 2021/22, group M-Z
Prof. Fabio Punzo

## Suggested Learning Material

### Bibliography


- *[Exercises]* M. Muratori, F. Punzo, N. Soave *Esercizi Svolti di Analisi Reale e Funzionale*, Esculapio (2021)


Note that the exercises from this book are only suggested. Obviously, it is possibile to find useful exercises elsewhere.

*(*) theorem with proof; (**) theorem with a partial proof*

### Week 13-17 September 2021

*Covered Topics*

*Set Theory.* Power set, collection of sets, sequence of sets; lim inf, lim sup, lim of sequence of sets, cover and subcover of sets, characteristic function of a set. Relations, equivalence and order relations. Equipotent sets, cardinality, Schröder-Bernstein theorem, Cantor theorem. Infinite and finite sets, countable and uncountable sets, continuum hypothesis. Axiom of choice. Partially ordered sets, chains, maximal element, upper bounds, Zorn's lemma.

*Metric and Topological Spaces.* Distance function, metric spaces. Examples: $\mathbb{R}^n$, $\ell^p$, $C^k([a,b])$. Balls, interior, boundary and exterior points, open and closed sets, accumulation points (or cluster points), closure of a set. Sequences, Cauchy sequences, convergent sequences. Completeness. Theorem of nested balls. Dense set, separable spaces. Nowhere dense set. Sets of first and second category. Baire's category theorem (*), corollary (*). Compact metric spaces, sequentially compact metric spaces, totally bounded metric spaces. Characterization of compact metric spaces. Ascoli-Arzelà theorem (*), corollary (*).

**Lecture Notes** (on WeBeep): 14 + 13 + 8 pages.

### Theory Questions

Sheet n. 1 (on WeBeep)

### Exercises

*Chapter 1*, Ex. 1-3, 4, 7, 12, 13, 15.

**Week 20-24 September 2021**

*Covered Topics*

Completion of a metric space. Functions (or mapping) between two metric spaces, continuous functions, Lipschitz functions.

*Measure.* Algebra, $\sigma-$algebra, measurable space, measurable sets, $\sigma-$algebra generated by a set, Borel $\sigma-$algebra, Borel sets, generation of $\mathcal{B}(\mathbb{R})$ and of $\mathcal{B}(\overline{\mathbb{R}})$. Measure, finite measure, $\sigma-$finite measure, measure space. Properties of a measure (additivity, monotonicity, $\sigma-$additivity, continuity) (*), Borel-Cantelli Lemma (*). Sets of zero measure, negligible sets, properties true almost everywhere. Complete measure space, completion of a measure space (**). Outer measure $\mu^*$. Generation of an outer measure (*). Carathéodory's condition and $\mu^*-$measurable sets, its equivalent form (*). If a set has zero outer measure, then it is $\mu^*-$measurable (*). Measure induced by an outer measure.

**Lecture Notes** (on WeBeep): 13 + 7 + 13 pages.

**Theory Questions**

Sheet n. 2 (on WeBeep)

**Exercises**

*Chapter 1*, Ex. 5, 6, 16. *Chapter 2*, Ex. 1-5, 16-18.

**Week 27 September-1st October 2021**

*Covered Topics*

Lebesgue measure in $\mathbb{R}$ and in $\mathbb{R}^N$. Every countable set is Lebesgue-measurable (*), every Borel set is Lebesgue measurable (*), the translate of a measurable set is measurable (*). Excision property (*). Regularity of the Lebesgue measure (**). Every measurable set of positive measure contains a non-measurable subset (Vitali set) (*).

**Lecture Notes** (on WeBeep): 12 + 14 pages.

**Theory Questions**

Sheet n. 3 (on WeBeep)

**Exercises**

*Chapter 2*, Ex. 6, 8-10, 12, 14, 21.

**Week 4-8 October 2021**

*Covered Topics*

The ternary Cantor set and its properties. Measurability for real valued functions (*), measurability of $\sup_n f_n$, $\inf_n f_n$, $\max\{f,g\}$, $\min\{f,g\}$, $\limsup_n f_n$, $\liminf_n f_n$ (*). Measurability of $f+g, fg$ (*). Measurability of $f^{\pm}$ and of $|f|$ (*). Simple functions, step functions. The simple approximation theorem (**); $\operatorname{esssup} f, \operatorname{essinf} f$, their properties (*), essentially bounded functions, $\mathcal{L}^{\infty}$. The Vitali-Lebesgue function.

*The Lebesgue Integral.* Integral of nonnegative simple functions, properties. Measure defined by the integral of a nonnegative simple function (*). Integral of nonnegative measurable functions, properties. Chebychev's inequality (*).

**Lecture Notes** (on WeBeep): 17 + 14 pages.

**Theory Questions**

Sheet n. 4 (on WeBeep)

**Exercises**

*Chapter 2*, Ex. 7, 8, 11, 20, 22-28.

**Week 11-15 October 2021**

*Covered Topics*

Finite integral and a.e. finite functions (*). Vanishing lemma for nonnegative measurable functions (*). Monotone convergence theorem (*). Fatou's lemma (*). Integration of series of nonnegative functions (*). Measure defined by the integral of a nonnegative measurable function (*). Integrals and sets of zero measure (*). Integrable functions, the Lebesgue integral. The set $\mathcal{L}^1$. Membership to $\mathcal{L}^1$ of $f, f^{\pm}, |f|$ (*). $\mathcal{L}^1$ is a vector space (*). Vanishing lemma for $\mathcal{L}^1$ functions (*). Functions equal a.e. have the same integral (*). Lebesgue's dominated convergence theorem (*), its alternative form (*). Integration of series (*). Comparison between Riemann and Lebesgue integrals. Radon-Nikodym's theorem. Uniqueness of the derivative of a measure (*), properties of the derivative. The spaces $L^1$ and $L^{\infty}$. $L^1$ and $L^{\infty}$ are metric spaces (*).

*Types of Convergence and Product Measure.* Various types of convergence for sequence of functions: point-wise convergence, uniform convergence, a.e. convergence, convergence in $L^1$ and in $L^\infty$, convergence in measure. Two equivalent formulations of convergence a.e. (*).

**Lecture Notes** (on WeBeep): 16 + 17+ 15 pages.

**Theory Questions**

Sheet n. 5 (on WeBeep)

**Exercises**

*Chapter 3*, Ex. 1-46, 52, 53, 61, 62, 66.

## Week 18-22 October 2021

*Covered Topics*

Convergence in measure implies convergence a.e. up to subsequences. On space of finite measure, convergence a.e. implies convergence in measure (*), the converse implication is not true. Convergence in $L^1$ implies convergence in measure (*), the converse implication is not true. Product measure space. $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ is the completion of $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$. Tonelli theorem, Fubini theorem.

*Functions of Bounded Variations and Absolutely Continuous Functions.* Lebesgue points. First fundamental theorem of calculus (*), total variation of a function, functions of bounded variation.

**Lecture Notes** (on WeBeep): 15 + 14 pages.

**Theory Questions**

Sheet n. 6 (on WeBeep)

**Exercises**

*Chapter 3*, Ex. 47-51, 54-60. *Chapter 4*, Ex. 1-4, 31-33.

## Week 25-29 October 2021

*Covered Topics*

The space $BV([a,b])$. Properties of the total variation, positive and negative variations. Characterization of $BV$ functions (Jordan decomposition). Monotone functions are differentiable a.e. , properties of the derivative of monotone functions (*). Absolutely continuous functions, the space $AC$. Absolute continuity of the integral (*), the integral function is absolutely continuous (*). Characterization of $AC$ functions. Second fundamental theorem of calculus (*).

*Banach Spaces.* Normed spaces. Sequences, bounded sequences, Cauchy sequences, convergent sequences. Series. Dense set, separable space. Stone-Weierstrass theorem. $C^0([a,b])$ is separable (*). Completeness, Banach spaces. Examples: $\mathbb{R}^N, C^k([a,b]), L^1$, $L^\infty$, $\ell^p$, $BV([a,b])$, $AC([a,b])$.

**Lecture Notes** (on WeBeep): 14 + 14 pages.

### Theory Questions

Sheet n. 7 (on WeBeep)

### Exercises

*Chapter 4*, Ex. 5-13, 19-25.

## Week 1-5 November 2021

*Covered Topics*

Exercises on Real Analysis.

Completeness of normed spaces, Banach spaces. Examples: $\mathbb{R}^N, C^k([a,b]), L^1$, $L^\infty$, $\ell^p$, $BV([a,b])$, $AC([a,b])$. Completeness and convergence of series (*). Compactness. Riesz lemma (*), Riesz theorem (*). Equivalent norms, all norms are equivalent when the dimension is finite (*).

**Lecture Notes** (on WeBeep): 16 + 12 pages.

### Theory Questions

Sheet n. 8 (on WeBeep)

### Exercises

*Chapter 5*, Ex. 1, 2, 6, 9, 12, 14, 15.

## Week 15-19 November 2021

*Covered Topics*

*Lebesgue Spaces.* The sets $\mathcal{L}^p$ and $L^p$, $L^p$ is a vector space (*), Young's inequality (*), Hölder's inequality (*), Minkowski inequality (*). Inclusion of $L^p$ spaces (*), $L^p$ is a Banach space (Riesz-Fisher theorem) (*). Convergence in $L^p$ implies convergence in measure (*). Lusin theorem. Simple functions with support of finite measure are dense in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$ (*), $C_c^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$ (*), $L^p$ is separable for $p \in [1, +\infty)$ (*).

**Lecture Notes** (on WeBeep): 15 + 12 pages.

**Theory Questions**

Sheet n. 9 (on WeBeep)

**Exercises**

*Chapter 6*, Ex. 2, 5, 7, 19-22, 24-33, 40-43, 45, 48, 52.

**Week 22-26 November 2021**

*Covered Topics*

$L^\infty(\mathbb{R})$ is not separable (*). $\ell^p$ spaces and their main properties. Jensen inequality.

*Linear Operators.* Linear operators, bounded operators, continuous operators. Characterization of linear bounded operators (*). $\mathcal{L}(X, Y)$, operator norm, equivalent definitions (*), $\mathcal{L}(X, Y)$ is a Banach space (*). Isometries, injections. Banach-Steinhaus theorem (or uniform boundedness principle) (*), corollary (*). Open mapping theorem, bounded inverse mapping theorem (*), corollary (*). Closed operators, closed operators and closed graph (*), closed graph theorem (*).

*Duality and Reflexivity.* Dual space of a normed space. Characterization of linear functionals. Duality of $L^p$ spaces for $p \in [1, +\infty)$. Sublinear functionals.

**Lecture Notes** (on WeBeep): 14 + 12 + 11 pages.

**Theory Questions**

Sheet n. 10 (on WeBeep)

**Exercises**

*Chapter 5*, Ex. 3-6, 17-20, 23-25, 27, 28, 31-34, 38, 40, 42, 44-46. *Chapter 6*, Ex. 44.

## Week 29 November-2 December 2021

*Covered Topics*

Sublinear functionals, Hahn-Banach theorem (dominated extension form) (*), Hahn-Banach theorem (continuous extension) (*), corollaries (*). Dual space and separability. The dual of $L^\infty(\mathbb{R}^N)$ is not $L^1(\mathbb{R}^N)$ (*). Bidual space, canonical embedding. The canonical embedding is linear and preserves the norm (*). Reflexive spaces. Reflexivity of $L^p$ spaces. Properties of reflexive spaces. Uniformly convex Banach spaces, Milman-Pettis theorem.

*Weak Convergence and Compact Operators.* Weak convergence in Banach spaces. Weak convergence in $L^p$ and in $\ell^p$. Strong convergence implies weak convergence (*), the weak limit is unique (*), weak convergent sequences are bounded. Lower semicontinuity of the norm w.r.t. weak convergence (*), convergence of the composition of strongly convergent operators with weakly convergent arguments (*), linear bounded operators are weak-weak continuous (*). A sufficient condition for weak convergence in reflexive spaces.

**Lecture Notes** (on WeBeep): 14 + 14 pages.

### Theory Questions

Sheet n. 11 (on WeBeep)

### Exercises

*Chapter 5*, Ex. 10, 11, 13, 16, 21, 26, 29, 38, 39, 47-50. *Chapter 6*, Ex. 1-4, 6-16, 18, 19, 23, 34-39.

## 11 December and Week 13-17 December 2021

*Covered Topics*

Weak* convergence and its properties. Banach-Alaoglu theorem, corollary (*). Eberlein-Smulyan theorem. Compact operators. Compact operators are bounded. Finite-rank operators. Characterization of compact operators (*). $K(X,Y)$ is a Banach space.

*Hilbert spaces.* Spaces with inner product, Cauchy-Schwarz inequality, Hilbert space. Examples: $L^2, \ell^2$. Parallelogram identity (*). Convex subsets. Minimal distance theorem from convex closed subsets (*). Orthogonal spaces and their properties. Projection theorem (*). Projections are linear and bounded operators (*). The dual of a Hilbert space. Riesz representation theorem (*). Orthonormal bases. Bessel inequality, abstract Fourier series and Parseval identity. In a separable Hilbert space, an orthonormal basis converges weakly, but not strongly (*). Riesz-Fisher theorem.

*Linear Operators on Hilbert spaces.* Norm of a bounded linear operator on a Hilbert space (*), symmetric operators. Norm of a symmetric operator (*). Eigenvalues, eigenvectors, eigenspaces.

Eigenvectors associated to different eigenvalues of a symmetric operator. Eigenvalues of compact symmetric linear operators on Hilbert spaces, eigenspaces of compact symmetric linear operators have finite dimension (*). Spectrum and resolvent. Spectral theorem. Fredholm's alternative.

**Lecture Notes** (on WeBeep): 13+ 13 + 14 pages.

**Theory Questions**

Sheet n. 12 (on WeBeep)

**Exercises**

*Chapter 7*, Ex. 7.1-7-17, 7.19-7.26. *Chapter 8*, Ex. 8.1-8.5, 8.7-8.12, 8.15-17, 8.20-8.23.

**Week 20-23 December 2021**

Exercises.