Math 217 – Geometry and Physics – Lec01

UCLA, Fall 2014

Friday 3rd October, 2014

1 Introduction

1.1 Books for reference

1. T. Frankel – The Geometry of Physics
2. Bott, Tu – Differential Forms in Algebraic Topology
3. Lawson, Michelson – Spin Geometry
4. J. Roe – Elliptic Operators, Topology, and Asymptotic Methods
5. Berline, Getzler, Vergne – Heat Kernel & Dirac operators

1.2 History

$$\int_M \text{geometry} \neq \text{Topological} \tag{1}$$

$$\varepsilon(TM) \qquad \chi(M)$$

1. Chern, 1946: Gauss-Bonnet theorem.
2. Hodge Theory (Analysis of elliptic PDEs)
3. Hirzebruch, 1950: Riemann-Roch Signature (Algebraic geometry, topology)
4. Grothendieck (GRR), 1958-59: K-theory
5. Atiyah-Hirzebruch: topological K-theory, topological Riemann-Roch
6. Atiyah-Singer: index formula, Dirac operator ($\hat{A}(M) = \operatorname{Ind} D$)
7. Mckean-Singer formula: heat-Kernel of elliptic operators
8. Atiyah-Boti-Patodi, 1978
9. Witten, Alveraz, Gaume, 1980: Heat kernel proof (Getzler), QFT (adiabatic limit)
10. Applications

2 Manifolds

2.1 Basics

Denote the **manifold** $M = \cup_{i \in I} U_i$ with **coordinate maps** $\varphi_i : U_i \to \mathbb{R}^n$ which are homeomorphisms / coordinate covers. Recall that a manifold is a topological space (Hausdorff) and locally Euclidean.

$$\text{(insert canonical diagram of coordinate maps)} \tag{2}$$

If all the $\varphi_j \circ \varphi_i^{-1} \in C^\infty$, then it is a **smooth manifold**.

2.2 Examples

1. The circle $M = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

$$\text{(insert diagram of } S^1 \text{ with four charts)} \tag{3}$$

with charts

$$U_1 = \{x > 0\} \xrightarrow{\varphi_1} \mathbb{R}^1, \qquad \varphi_1(p) = y, \tag{4}$$
$$U_2 = \{y > 0\} \xrightarrow{\varphi_2} \mathbb{R}^1, \qquad \varphi_2(p) = x, \tag{5}$$
$$U_3 = \{x < 0\} \xrightarrow{\varphi_3} \mathbb{R}^1, \qquad \varphi_3(p) = y, \tag{6}$$
$$U_4 = \{y < 0\} \xrightarrow{\varphi_4} \mathbb{R}^1, \qquad \varphi_4(p) = x. \tag{7}$$

On $U_1 \cap U_2$, we have the coordinate change $y = \sqrt{1 - x^2}$, $x = \sqrt{1 - y^2}$.

2. The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

$$\text{(insert diagram of } S^2\text{)} \tag{8}$$

with charts

$$U_+ = \{z \neq -1\} \xrightarrow{\varphi_+} \mathbb{R}^2 \tag{9}$$
$$U_- = \{z \neq 1\} \xrightarrow{\varphi_-} \mathbb{R}^2 \tag{10}$$

$$\varphi_+(x, y, z) = (\frac{x}{1+z}, \frac{y}{1+z}) \tag{11}$$
$$\varphi_-(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z}). \tag{12}$$

The inverse of the coordinate maps can be written

$$\varphi_+^{-1} : \mathbb{R}^2 \longrightarrow U_+ \tag{13}$$
$$(x_1, x_2) \mapsto (\frac{2x_1}{1+\rho^2}, \frac{2x_2}{1+\rho^2}, \frac{1-\rho^2}{1+\rho^2}) \tag{14}$$

$$\varphi_-^{-1} : \mathbb{R}^2 \longrightarrow U_- \tag{15}$$

$$(x_1, x_2) \mapsto (\frac{2x_1}{1+\rho^2}, \frac{2x_2}{1+\rho^2}, \frac{\rho^2 - 1}{1+\rho^2}) \tag{16}$$

where $\rho^2 = x^2 + y^2$. Then

$$\varphi_- \circ \varphi_+^{-1} : \varphi_+(U_+ \cap U_-) \to \varphi_-(U_+ \cap U_p) \tag{17}$$

$$(x_1, x_2) \mapsto (x_1/\rho, x_2/\rho) \tag{18}$$

3. The real projective space $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* = S^n/\mathbb{Z}_2$, i.e. $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$, $\lambda \in \mathbb{R}^*$

4. The complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* = S^{2n+1}/S^1$, i.e. $z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, $\lambda \in \mathbb{C}^*$, say with charts: $U_i = \{z_i \neq 0\}$,

$$\varphi_i : U_i \longrightarrow \mathbb{C}^n \tag{19}$$

$$(z_0, \ldots, z_n) \mapsto (z_0/z_i, \ldots, z_n/z_i) \tag{20}$$

For example, $\mathbb{CP}^1 \cong S^2$.

2.3 More on manifolds

Denote a **tangent bundle** $TM = \cup_{x \in M} T_x M$ with **canonical map** $\pi : TM \to M$. TM has a manifold structure.

$f : M \to N$ is a **smooth map** if $f \circ \varphi_i^{-1}$ is smooth for all i. If f is smooth, then define the differential or pushforward $df = f_* : TM \to TN$ such that for all $x \in M$,

$$f_*(x) : T_x M \longrightarrow T_{f(x)} M \tag{21}$$

$$v \mapsto f_*(v) = \frac{d}{dt} f(\alpha(t)) \Big|_{t=0} \tag{22}$$

where $\alpha : (-\epsilon, \epsilon) \to M$, $\alpha(0) = x$, and $\alpha'(0) = v$.

We define a **smooth section** as a map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$. For example, a vector field on M is a standard example of a smooth section.

Using coordinates on chart U, $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ for all $f \in C^\infty(M)$ such that $f : M \to \mathbb{R}$. Then $\mathcal{L}_X f = X(f) = f_*(X)$.

Lie bracket: $[X, Y]f = X(Yf) - Y(Xf)$. Antisymmetric, Jacobi (???)

Given a vector field X, let $\varphi : (a_x, b_x) \to M$ be a curve s.t. $\varphi(0) = x$, $\partial\varphi/\partial t = X \circ \varphi$ on (a_x, b_x) maximal interval

If $(a_x, b_x) = (-\infty, \infty) = \mathbb{R}$, φ or X is complete