Phys 220A – Classical Mechanics – Lec01

UCLA, Fall 2014

Thursday 2$^{\text{nd}}$ October, 2014

1 Introduction

Lecture will be Tuesday – Thursday

- Ask questions! I will try to make this as interactive as possible
- I post scans of my notes before class

Homework

- Problem solving is the most important part of learning
- Invest serious amount of time in solving the problem yourself
- It's okay to discuss problems with other students but I would advise on trying to solve them yourself first

Exams

- Checks and gives feedback on your understanding
- Preparation for comps

Help

- During lecture
- Office hours
- TA office hours
- email
- your peers

Goal of course

- Review of basic undergraduate mechanics
- A more mathematically sophisticated look at Lagrangian and Hamiltonian mechanics
- Provide conceptual underpinning for stat mech and QM
- Solving more complicated problems

[Need to fix: Time of midterm, time for office hours]

Topics

- Review of Newtonian mechanics
- **Lagrangian mechanics**
- small oscillations
- **Hamiltonian mechanics**
- Rigid bodies
- **Relativistic mechanics**
- Fluid mechanics
- Integrability and chaos

2 Review of Newtonian mechanics

Newtonian mechanics is an idealization.

1. Newtonian dynamics deals with point like masses $m_1, \ldots, m_N$. Their positions are given by vectors $\mathbf{x}_i(t)$ in Euclidean space $\mathbb{R}^3$ (as functions of time)

2. One defines a special class of coordinate systems, so-called inertial frames of reference, in which Newton's laws take the same form.

   The Galilean principle of relativity is that two inertial frames of references are related by

   (a) Translation of the origin in $\mathbb{R}^3 \times \mathbb{R}$

   $$\mathbf{x} \to \mathbf{x} + \mathbf{x}_0, \qquad t \to t + t_0 \tag{1}$$

   (b) Rotation (time independent)

   $$\mathbf{x} \to R\mathbf{x}, \qquad R \in SO(3) \tag{2}$$

(c) Galilean boost — relates 2 frames of reference which move with relative constant velocity

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}t, \qquad t' = t. \tag{3}$$

Note that these three operations form a group, the so called Galilean group (which happens to be the Poincare group from special relativity in the limit $c \to \infty$. Principle of relativity says that physical laws take the same form in all inertial frames.) Note also that this is a 10 dimensional group!

3. Newton's force law

- Point particles are acted upon by other point particles
- The motion of point particles in the future is determined by the knowledge of the position and velocity of all particles at a fixed time t (not acceleration a, or jerk, snap, crackle, or pop).
- Define the momentum $\mathbf{p} = m\dot{\mathbf{x}}$. (Note that the definition of $\mathbf{p}$ holds in all frames, but needs to be modified in relativistc mechanics.) The theory is defined by the system of ordinary differential equations:

$$\dot{\mathbf{p}}_i = \mathbf{F}_i(\mathbf{x}_j, \dot{\mathbf{x}}_j, t). \tag{4}$$

  This is a system of $3N$ 2nd order differential equations. We also need $2 \times 3N$ initial conditions $\mathbf{x}_i(t_0)$, $\dot{\mathbf{x}}_i(t_0)$.
- Newton's 3rd law (action and reaction): If m_1 exerts a force $\mathbf{F}_{21}$ on m_2 then m_2 exerts a force $\mathbf{F}_{12} = -\mathbf{F}_{21}$ on m_1.
- Newton's law of gravity: point masses m_1 and m_2 exert a gravitational force

$$\mathbf{F}_{12} = -G_N m_1 m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \tag{5}$$

  where $G_N \approx 6.673 \times 10^{-11}\ \mathrm{m}^3/(\mathrm{kg} \cdot \mathrm{s}^3)$ is the Newtonian gravitational constant.

Comments

- Forces are instantaneous ($c \to \infty$) — idealization not true for E&M, relativity.
- In classical mechanics $\mathbf{x}$ and $\mathbf{p}$ can both be measured to arbitrary precision ($\hbar \to 0$) — not true in QM.

Applications

- Consequences of Newton's laws
- Conservative forces
- Important examples of physical systems

  1. Harmonic oscillator
  2. motion in central potential
  3. particle in electromagnetic field

3 Some simple consequences of Newton's laws

(Should all be familiar from undergraduate.)

3.1 Work and kinetic energy

- For a single particle trajectory along path C_{12} going from points $1 \to 2$ subject to force $\mathbf{F}$, the work is given by

$$
\begin{aligned}
W_{12} &= \int_{C_{12}} \mathbf{F} \cdot d\mathbf{x} && (6)\\
&= \int_{C_{12}} m\dot{\mathbf{v}} \cdot \frac{d\mathbf{x}}{dt} dt && (7)\\
&= \int_{C_{12}} m\dot{\mathbf{v}} \cdot \mathbf{v} dt && (8)\\
&= \int_{C_{12}} \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}^2\right) dt && (9)\\
&= T_2 - T_1 && (10)
\end{aligned}
$$

  where $T = \frac{1}{2}m\mathbf{v}^2$ is the kinetic energy.

- For N particle system: same expression, just have

$$
W_{12} = \sum_i \int \mathbf{F}_i \cdot d\mathbf{x}_i, \qquad T = \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \tag{11}
$$

3.2 Conservative Forces & energy conservation

- For dissipative forces like friction $\mathbf{F} = -k\mathbf{v}$ the work W_{12} depends on the path C_{12}. For a longer path C'_{12} the work will be larger, $W'_{12} > W_{12}$.

- For a conservative force the work only depends on the initial and final points. So a conservative force should not explicitly depend on time.

For the time being, let's assume that $\mathbf{F}$ is independent of $\mathbf{v}$. Take two distinct paths C and $C' \neq C$ from point 1 to point 2. If $W_{12} = W'_{12}$ then $W_{12} - W'_{12} = 0$ so we have

$$
\oint_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = 0 \tag{12}
$$

where C is the closed path $C - C'$. In other words, work along a closed path is zero. Using Stoke's theorem we have

$$
\oint_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_D (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d^2\mathbf{S} \tag{13}
$$

for any surface D with boundary $\partial D = C$, so that we have $\boldsymbol{\nabla} \times \mathbf{F} = 0$. Thus for an $\mathbf{F}$ with a simply connected domain, we have

$$
\boldsymbol{\nabla} \times \mathbf{F} = 0 \quad \text{for all } \mathbf{x} \qquad \Longrightarrow \qquad \mathbf{F} = -\boldsymbol{\nabla} V. \tag{14}
$$

Furthermore, we find

$$W_{12} = \int_{C_{12}} \mathbf{F} \cdot d\mathbf{x} = -\int_{C_{12}} d\mathbf{x} \cdot \boldsymbol{\nabla} V = -(V_2 - V_1) \tag{15}$$

thus we have finally

$$T_2 - T_1 = -(V_2 - V_1) \qquad \text{or} \qquad T_1 + V_1 = T_2 + V_2 \tag{16}$$

so the total energy $E = T + V$ is conserved.

Note that we excluded velocity dependent forces (like friction) since they are generally non-conservative. Any important exception is the Lorentz force $\mathbf{F} = e\mathbf{v} \times \mathbf{B}$. Since $\mathbf{F} \perp \mathbf{v} = d\mathbf{x}/dt$, the work done by the Lorentz force is zero so the force is conservative. However it seems that the potential is zero — one has to generalize to a velocity dependent potential, so we have

$$U = e\phi - e\mathbf{v} \times \mathbf{A} \quad \text{and} \quad \mathbf{F} = -\boldsymbol{\nabla}_{\mathbf{x}} U - \frac{d}{dt} \boldsymbol{\nabla}_{\dot{\mathbf{x}}} U. \tag{17}$$

(See D'Hoker for a more details discussion.)

4 System of Particles

Consider a system of N particles with mass m_i. We have Newton's equation for each particle, $m_i \ddot{\mathbf{x}}_i = \mathbf{F}_i$, where the forces $\mathbf{F}_i = \mathbf{F}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ are functions of the positions (and possibly the velocities). We can again split the forces into interparticle and external forces,

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}}. \tag{18}$$

We define the center of mass

$$\mathbf{X} = \frac{1}{M} \sum_i m_i \mathbf{x}_i, \qquad M = \sum_i m_i. \tag{19}$$

Then Newton's equation for the center of mass is given by

$$M\ddot{\mathbf{X}} = \sum_i m_i \ddot{\mathbf{x}}_i = \sum_i \mathbf{F}_i = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_i \sum_{j \neq i} \mathbf{F}_{ij} \tag{20}$$

$$= \sum_i \mathbf{F}_i^{\text{ext}} + \frac{1}{2} \sum_i \sum_{j < i} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) \tag{21}$$

$$= 0 \tag{22}$$

where the last equality comes from Newton's 3rd law.

For an isolated (or closed) system we have $\sum_i \mathbf{F}_i^{\text{ext}} = 0$. Then the center of mass behaves like a free particle,

$$\mathbf{X}(t) = \frac{\mathbf{P}}{M} t + \mathbf{X}(0), \qquad \mathbf{P} = \sum_i \mathbf{p}_i. \tag{23}$$

This gives us 3 conserved quantities,

$$\mathbf{X}(0) = \mathbf{X}(t) - \frac{\mathbf{P}}{M}t. \tag{24}$$

Similarly we can show that the total momentum

$$\mathbf{P}_{\text{tot}} = \sum_i \mathbf{p}_i \tag{25}$$

is conserved, giving us 3 more conserved quantities, and the total angular momentum

$$\mathbf{L}_{\text{tot}} = \sum_i \mathbf{x}_i \times \mathbf{p}_i \tag{26}$$

is conserved, giving yet 3 more conserved quantities.

4.1 Work and energy

Notice that if we write $\widetilde{\mathbf{x}}_i = \mathbf{x}_i - \mathbf{X}$, the total kinetic energy is given by

$$T = \sum_i \frac{1}{2} m \mathbf{x}_i^2 = \frac{1}{2} \sum_i m_i \dot{\widetilde{\mathbf{x}}}_i^2 + \frac{1}{2} M \dot{\mathbf{X}}^2. \tag{27}$$

The work energy theorem works as before,

$$T_2 - T_1 = \sum_i \int \mathbf{F}_i \cdot d\mathbf{x}_i. \tag{28}$$

For conservative forces we have

$$\mathbf{F}_i^{\text{ext}} = -\boldsymbol{\nabla}_i V^{\text{ext}}. \tag{29}$$

The contribution to the RHS of (28) from external forces is given by

$$\sum_i V_{i,(1)}^{\text{ext}} - \sum_i V_{i,(2)}^{\text{ext}}. \tag{30}$$

Internal forces contribute to the RHS too. We have

$$\mathbf{F}_{ij} = -\boldsymbol{\nabla}_i V_{ij}. \tag{31}$$

If V_{ij} is of the form $V_{ij} = V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$ then we find

$$\mathbf{F}_{ij} = -\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} V_{ij}'(|\mathbf{x}_i - \mathbf{x}_j|) \tag{32}$$

and hence $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, as expected from Newton's third law, and we also know that the force is in the direction of $x_i - \mathbf{x}_j$. Then we find that the interparticle contribution to the

RHS of (28) is given by

$$\sum_i \sum_{j\neq i} \int \mathbf{F}_i \cdot d\mathbf{x}_i = -\sum_i \sum_{j\neq i} \int \boldsymbol{\nabla}_i V_{ij} \cdot d\mathbf{x}_i \tag{33}$$

$$= -\frac{1}{2} \sum_i \sum_{j\neq i} \int \boldsymbol{\nabla}_i V_{ij} \cdot (d\mathbf{x}_i - d\mathbf{x}_j) \tag{34}$$

$$= -\frac{1}{2} \sum_i \sum_{j\neq i} \int \boldsymbol{\nabla}_i V_{ij} \cdot d\mathbf{r}_{ij} \tag{35}$$

$$= -\sum_i \left(-\frac{1}{2} \sum_{j\neq i} V_{ij}^{(2)} - \frac{1}{2} \sum_{j\neq i} V_{ij}^{(1)} \right). \tag{36}$$

Thus combining the kinetic and potential terms for initial and final states we find the total energy

$$E = T + \sum_i V_i^{\text{ext}} + \frac{1}{2} \sum_i \sum_{j\neq i} V_{ij} \tag{37}$$

is yet another conserved quantity.

We have found a lot of conserved quantities for isolated systems:

- 3 for the total momentum $\mathbf{P}_{\text{tot}}$,
- 3 for the total angular momentum $\mathbf{L}_{\text{tot}}$,
- 1 for the total energy E,
- 3 for $\mathbf{X} - (\mathbf{P}/M)t$,

giving us a total of *10 conserved quantities*. This is no accident: remember that the Galilean group has 10 dimensions — this is the essence of Noether's theorem!