Math 217 – Geometry and Physics – Lec03

UCLA, Fall 2014

Wednesday 8^{th} October, 2014

1 Stuff

Seminal paper: Witten — Supersymmetry and Morse Theory (J. Diff. Geometry, 1982)

1.1 Chern conjecture

Let's go over the Chern conjecture from last time. If $M = M^{2n}$ is a compact closed manifold, then

$$\chi(M) = \int_M \varepsilon(TM). \tag{1}$$

Given the metric g on TM, we have the Levi-Civita connection ∇^g. Then we have $R^{\nabla^g} = (\nabla^g)^2$ where R is skew-symmetric, and $\left|Pf(R^{\nabla^g})\right| = \varepsilon(TM) \in H^{2n}(M)$. [not this this is right, it was fast and messy on the board.] If $R^{\nabla^g} = 0$ then $\chi(M) = 0$ where $\chi(M) = \sum_{i=0}^{2n}(-1)^i b_i$, $b_i = \dim_{\mathbb{R}} H^i(M)$

The Chern conjecture is based on that. Conjecture: If M^{2n} is an affine flat closed compact manifold, then $\chi(M) = 0$. Let's define what affine flat means. Write $M = \cup_{\alpha\in I} U_\alpha$. Then it is affine flat $\iff$ $\varphi_\beta \circ \varphi_\alpha^{-1}(x_\alpha) = A_{\alpha\beta}x_\alpha + B_{\alpha\beta}$ where $A_{\alpha\beta}, B_{\alpha\beta}$ are constant matrices $\iff$ there exists an affine connection $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ where $\nabla(fs) = f\nabla s + f\nabla s$.

The Chern conjecture is known in 2 dimensions (the torus). Not sure if known in any other dimensions. (See Sullivan-Kostant-Milnor for special cases.)

1.2 Poincare duality

Says that $H^p \cong H^{n-p}$. Exercise (Bott-Tu):

1. $M = \mathbb{R}^n$.

$$H^k(\mathbb{R}^n) = \begin{cases} 0, & k = 0, \\ \mathbb{R}, & k \neq 0 \end{cases} \qquad H_c^k(\mathbb{R}^n) = \begin{cases} 0, & k \neq n \\ \mathbb{R}, & k = n. \end{cases} \tag{2}$$

2. $M = \cup_{\alpha\in I} U_\alpha$.

We write $[\Delta] \in H_p(M, \mathbb{R}) \cong H^{n-p}(M, \mathbb{R})$ where $\Delta \subseteq M$ is a p-cycle in M. Then we can write

$$\int_\Delta \omega : H^p_{\mathrm{dR}}(M) \to \mathbb{R} \tag{3}$$

$$[\omega] \mapsto \int_\Delta \omega \tag{4}$$

for a p-form ω. Write $\Delta = \sum_i a_i \underbrace{\Delta_i}_{=\varphi_i(\tilde{\Delta}_i)}$ where $\tilde{\Delta}_i \subseteq$ [something] $\xrightarrow{\varphi_i} M$

[missed a LOT here on whatever crap he was talking about]

1.3 Lie groups

Definition: A Lie group G is a smooth manifold with a group structure in the following sense:

1. Multiplication $G \times G \to G$ taking $(g, g') \mapsto gg'$, and
2. Inverse $G \to G$ taking $g \mapsto g^{-1}$

Examples: $\mathbb{C}, \mathbb{C}, \mathbb{Q}, \mathbb{Q}_p, \ldots$

Write $M_n(K) = n \times n$ matrices with entries in K. The group with elements $g = (a_{ij})_{n\times n} \in GL_n(\mathbb{R})$ with $\det g \neq 0$ is called the general linear group. Any subgroup submanifold is also a Lie group.

Definition: A Lie subgroup H of G is a subgroup which is also a submanifold. If $H = \mathbb{R}$ then $i : \mathbb{R} \to G$ defines a one-parameter subgroup.

Examples: $SL_n(\mathbb{R}), O_n(\mathbb{R}), SO_n(\mathbb{R}), U_n, SU_n$

Definition: A Lie group homomorphism $\psi : G \to G'$ is a map satisfying $\psi(gg') = \psi(g)\psi(g')$. If ψ is a diffeomorphism then it is called an isomorphism.

Furthermore, define

- $R_g : G \to G$ mapping $R_g(g') = g'g$ for all $g' \in G$
- $L_g : G \to G$ mapping $L_g(g') = gg'$ for all $g' inG$.

Definition: A vector field X on G is left invariant if $L_g X = X$