Phys 230A – QFT – Lec02

UCLA, Fall 2014

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1 More on scalar fields

Take our action

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{1}$$

then the variation $\delta S = 0$ of the action implies the Euler-Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{2}$$

For free scalar field we have the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \tag{3}$$

giving the KG equation $+m^2\phi = 0$.

1.1 Canonical (Hamiltonian) formulation

The canonical momentum is given by $\Pi_\phi = \partial \mathcal{L}/\partial \dot{\phi}$. The Hamiltonian is given by the integral of the Hamiltonian density,

$$H = \int d^3x \mathcal{H}(\phi, \nabla\phi, \Pi_\phi). \tag{4}$$

The Hamiltonian density is given by $\mathcal{H} = \Pi_\phi \dot{\phi} - \mathcal{L}$ where $\dot{\phi} = \dot{\phi}(\phi, \Pi_\phi)$. More fully written as functions we have

$$\mathcal{H}(\phi, \nabla\phi, \Pi_\phi) = \Pi_\phi \dot{\phi}(\phi, \Pi_\phi) - \mathcal{L}(\phi, \nabla\phi, \dot{\phi}(\phi, \Pi_\phi)). \tag{5}$$

Of course we have Hamilton's equations

$$\dot{\phi} = \frac{\delta H}{\delta \Pi_\phi}, \qquad \dot{\Pi}_\phi = -\frac{\delta H}{\delta \phi} \tag{6}$$

Let's check equivalence to Euler-Lagrange. We have

$$\frac{\delta H}{\delta \Pi_\phi} = \dot{\phi} + \Pi_\phi \frac{\partial \dot{\phi}}{\partial \Pi_\phi} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \dot{\phi}}{\partial \Pi_\phi} = \dot{\phi} \tag{7}$$

where the second and third terms cancel by definitions. Furthermore, we find

$$\frac{\delta H}{\delta \phi} = \Pi_\phi \frac{\partial \dot{\phi}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \dot{\phi}}{\partial \phi} - \int \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \frac{\delta(\partial_i \phi)}{\delta \phi} d^3x \tag{8}$$

and we see that the first and third terms cancel and we can rewrite the last term as

$$+\int \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \frac{\delta \phi}{\delta \phi} d^3x = \partial_i \frac{\delta \mathcal{L}}{\partial(\partial_i \phi)} \tag{9}$$

which is just $-\dot{\Pi}_\phi$. So we see the equivalence of Hamiltonian and Lagrangian mechanics in this formalism:

$$\begin{aligned} 0 &= \dot{\Pi}_\phi + \frac{\delta H}{\delta \phi} &(10)\\ &= \dot{\Pi}_\phi - \frac{\partial \mathcal{L}}{\partial \phi} + \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} &(11)\\ &= \frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}) + \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} - \frac{\delta \mathcal{L}}{\partial \phi} &(12)\\ &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi}. &(13) \end{aligned}$$

Now if we have the action

$$S = \int d^4x(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)) \tag{14}$$

we can write the Hamiltonian

$$H = \int d^3x(\frac{1}{2}\Pi_\phi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + V(\phi)). \tag{15}$$

Then we can write the variation

$$\begin{aligned} \delta H(\phi, \Pi_\phi) &= \int d^3x(\frac{\delta H}{\delta \phi}\delta\phi + \frac{\delta H}{\delta \Pi_\phi}\delta Pi_\phi) &(16)\\ &= \int d^3x(\frac{\delta H}{\delta \phi}\dot{\phi} + \frac{\delta H}{\delta \Pi_\phi}\dot{\Pi}_\phi)dt &(17) \end{aligned}$$

and thus

$$\begin{aligned} \frac{dH}{dt} &= \int d^3x(\frac{\delta H}{\delta \phi}\dot{\phi} + \frac{\delta H}{\delta \Pi_\phi}\dot{\Pi}_\phi) &(18)\\ &= -\dot{\Pi}_\phi \dot{\phi} + \dot{\phi}\dot{\Pi}_\phi &(19)\\ &= 0. &(20) \end{aligned}$$

1.2 Canonical quantization

In QM, we start with a canonical pair (p, q) and impose the commutation relation $[q(t), p(t)] = i$. Then the time derivatives are given by

$$\dot{q}(t) = i[H, q(t)], \qquad \dot{p}(t) = i[H, p(t)] \tag{21}$$

in fact this works for any observable.

For the simple harmonic oscillation, we have

$$S = \int dt(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2) \tag{22}$$

so that

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \qquad H = p\dot{q} - L = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{23}$$

Then we have

$$\dot{q} = i[H, q] = p/m, \qquad \dot{p} = i[H, p] = -m\omega^2 q \tag{24}$$

We will be working in the Heisenberg picture where the operators evolve in time rather than the Schroedinger picture where the wavefunctions evolve in time. So we will have coordinate $\psi = \psi(q)$, $p(0) = -i\partial/\partial q$. In QFT will have $\psi[\phi]$.

We also have of course the ladder operators

$$a = \frac{ip}{\sqrt{2m\omega}} + \sqrt{m\omega/2}q \tag{25}$$

$$a^\dagger = -\frac{ip}{\sqrt{2m\omega}} + \sqrt{m\omega/2}q \tag{26}$$

where $[a, a^\dagger] = 1$. We write the Hamiltonian

$$H = \frac{1}{2}\omega(a^\dagger a + aa^\dagger) = (a^\dagger a + \frac{1}{2})\omega. \tag{27}$$

So we have ground state $a\left|0\right\rangle = 0$ with $H\left|0\right\rangle = \frac{1}{2}\omega\left|0\right\rangle$. The excited states are written

$$\left|n\right\rangle = N(a^\dagger)^n \left|0\right\rangle \tag{28}$$

$$H\left|n\right\rangle = (n + 1/2)\omega\left|n\right\rangle. \tag{29}$$

Furthermore we have $\dot{a} = i[H, a] = -i\omega a$ so that

$$a(t) = a(0)e^{-i\omega t}, \qquad a^\dagger(t) = a^\dagger(0)e^{i\omega t}. \tag{30}$$

Finally note that we can write

$$q(t) = \frac{1}{\sqrt{2m\omega}}(a(t) + a^\dagger(t)) \tag{31}$$

$$= \frac{1}{\sqrt{2m\omega}}(a(0)e^{-i\omega t} + a^\dagger(0)e^{i\omega t}). \tag{32}$$

So we can now port this all over to scalar fields where $q \to \phi$ and $p \to \Pi_\phi$.

1.3 Quantizing the scalar field

Given the action

$$S = \int d^4x(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2) \tag{33}$$

we have $\Pi_\phi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$. Then the hamiltonian is written

$$\int d^3x(\frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2) \tag{34}$$

and we impose the commutation relation

$$[\phi(\mathbf{x},t), \Pi_\phi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}). \tag{35}$$

(A key idea to note is that the Hamiltonian is manifestly unitary but not Lorentz invariant, while the Lagrangian is manifestly Lorentz invariant but not unitary.)

Now we can compute

$$\dot{\phi} = i[H,\phi], \quad \text{and} \quad \dot{\Pi}_\phi = i[H,\Pi_\phi]. \tag{36}$$

We see that the first equation reproduces $\dot{\phi} = \Pi_\phi$. For the second equation, we have

$$\begin{aligned}
\dot{\Pi}_\phi &= i\int d^3y[\frac{1}{2}(\nabla\phi(y))^2 + \frac{1}{2}m^2\phi^2(\mathbf{y},t), \Pi_\phi(\mathbf{x},t)] && (37)\\
&= i\int d^3y\left\{\nabla\phi(\mathbf{y})\cdot\nabla_y[\phi(\mathbf{y},t),\Pi_\phi(\mathbf{x},t)] + m^2\phi(\mathbf{y},t)[\phi(\mathbf{y},t),\Pi_\phi(\mathbf{x},t)]\right\} && (38)\\
&= -\int d^3y\left\{\nabla\phi(\mathbf{y},t)\cdot\nabla_y\delta^3(\mathbf{y}-\mathbf{x}) + m^2\phi(\mathbf{y},t)\delta^3(\mathbf{y}-\mathbf{x})\right\} && (39)\\
&= \nabla^2\phi(\mathbf{x},t) - m^2\phi(\mathbf{x},t). && (40)
\end{aligned}$$

Notice that since $\dot{\Pi}_\phi = \ddot{\phi}$, this just reproduces the KG equation for a free field, $\Box\phi + m^2\phi = 0$.

1.4 Mode expansion

First we'll do a spatial Fourier expansion.

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\tilde{\phi}(\mathbf{p},t). \tag{41}$$

We can rewrite the KG equation in momentum space

$$\ddot{\tilde{\phi}}(\mathbf{p},t) + (\mathbf{p}^2 + m^2)\tilde{\phi}(\mathbf{p},t) = 0 \tag{42}$$

which has solutions

$$\tilde{\phi}(\mathbf{p},t) = \tilde{\phi}(\mathbf{p},0)e^{\pm i\omega_p t}, \qquad \omega_p = \sqrt{\mathbf{p}^2 + m^2}. \tag{43}$$

Thus we can write ϕ in spatial coordinates

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\mathbf{p}} e^{-i\omega_p t + i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{i\omega_p t - i\mathbf{p}\cdot\mathbf{x}} \right). \tag{44}$$

Notice that this is a real field: $\phi^\dagger = \phi$. Then we have

$$\Pi_\phi = \dot{\phi} = -\int \frac{d^3p}{(2\pi)^3} i\sqrt{\omega_p/2} \left(a_{\mathbf{p}} e^{-i\omega_p t + i\mathbf{p}\cdot\mathbf{x}} - a_p^\dagger e^{i\omega_p t - i\mathbf{p}\cdot\mathbf{x}} \right). \tag{45}$$

Now we want to know, what is $[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger]$? We have

$$a_{\mathbf{p}} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\sqrt{\omega_p/2}\phi(\mathbf{x},0) + \frac{i}{\sqrt{2\omega_p}} \Pi_\phi(\mathbf{x},0) \right) \tag{46}$$

$$a_{\mathbf{p}}^\dagger = \int d^3x e^{+i\mathbf{p}\cdot\mathbf{x}} \left(\sqrt{\omega_p/2}\Pi_\phi(\mathbf{x},0) - \frac{i}{\sqrt{2\omega_p}} \phi(\mathbf{x},0) \right). \tag{47}$$

Then we have the commutator

$$\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] &= -\int d^3x d^3x' e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{p}'\cdot\mathbf{x}'} \frac{i}{2} \left([\phi(\mathbf{x},0), \Pi_\phi(\mathbf{x}',0)] + [\phi(\mathbf{x},0), \Pi_\phi(\mathbf{x},0)] \right) &(48)\\
&= \int d^3x e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} &(49)\\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}'). &(50)
\end{aligned}$$

and of course $[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0$.

As an exercise, show:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} \underbrace{[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]}_{(2\pi)^3\delta^3(0) = \infty\ ???}) \tag{51}$$

1.5 Spectrum

In the vacuum, $a_{\mathbf{p}} |0\rangle = 0$ for all $\mathbf{p}$ so $H|0\rangle = 0|0\rangle$. But we have

$$H a_{\mathbf{p}}^\dagger |0\rangle = [H, a_{\mathbf{p}}^\dagger] |0\rangle = \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle \tag{52}$$

where again $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. This is what we really mean when we say that particle are the quanta of the fields in QFT. Furthermore, we find

$$|\mathbf{p}, \mathbf{p}'\rangle = a_{\mathbf{p}}^\dagger a_{\mathbf{p}'}^\dagger |0\rangle = |\mathbf{p}', \mathbf{p}\rangle \tag{53}$$

so we find that scalar fields obey Bose statistics. We also have

$$H |\mathbf{p}, \mathbf{p}'\rangle = (\omega_{\mathbf{p}} + \omega_{\mathbf{p}'}) |\mathbf{p}, \mathbf{p}'\rangle. \tag{54}$$

A general state can be written

$$|\psi\rangle = |0\rangle + \int \frac{d^3p}{(2\pi)^3} f_1(\mathbf{p}) a^\dagger_{\mathbf{p}} |0\rangle + \int \frac{d^3p d^3p'}{(2\pi^3)(2\pi)^3} f_2(\mathbf{p}, \mathbf{p}') a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p}'} |0\rangle \,. \tag{55}$$

This is referred to as Fock space.

What is the energy of states in the CM frame $\mathbf{p}_{\text{tot}} = 0$?

- in vacuum: $E = 0$
- 1 particle: $E = m$
- 2 identical particles: $E = 2\sqrt{\mathbf{p}^2 + m^2}$

Gapped vs. gapless? $E < E_gap = m$ trivial. nontrivial IR dynamics? [???]