# Phys 230A – QFT – Lec01

UCLA, Fall 2014

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# 1 Introduction

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Main books for course

- Peskin & Schroeder
- Srednicki
- There's also Schwartz, which is less formal and more conceptual
- There's also Zee, which is more fun and casual and conceptual
- And of course Weinberg volumes I, II, III
- Other books to check out: Ryder, Ramond

## 1.1 Why QFT?

In non-relativistic QM, you have a fixed number of particles. Given $n$ particles, we have the wavefunction $\psi(x_1, x_2, \ldots, x_n, t)$ with the Schroedinger equation $H\psi = i\hbar(\partial\psi/\partial t)$ and the probability density $|\psi|^2$. This treatment is not necessarily non-relativistic, in fact it could be relativistic if given the proper Hamiltonian.

However, non-relativistic QM cannot treat particle creation/annihilation e.g. pair production. This is because we need an infinite number of degrees of freedom. We must introduce a field $\phi(x^\mu)$, which provides a degree of freedom per spacetime point. But we can have fields and particles classically, so how is this different? Classically, they are distinct; in QFT, particles are the quanta of the fields. What exactly does this mean? We will see that particles are localized eigenstates of the Hamiltonian with a definite mass and momentum.

## 1.2 Quantization of free scalar field

We will use the metric convention $\eta_{\mu\nu} = (+, -, -, -)$ and units $\hbar = c = 1$. Start with a classical variable: $\phi(x^\mu)$. We want a Lorentz invariant equation of motion for $\phi$. Denote a Lorentz transform $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$. It preserves the dot product: $\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$, thus $\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$.

We can show that $\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$, or in other words $\Lambda^\top\eta\Lambda = \eta$. Furthermore, $\Lambda^{-1} = \eta\Lambda^\top\eta$, so

$$(\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\alpha}\Lambda^\beta{}_\alpha\eta_{\beta\nu} = \Lambda_\nu{}^\mu. \tag{1}$$

Then we find

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu\frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu\frac{\partial}{\partial x^\nu} \tag{2}$$

and thus

$$\eta^{\mu\nu}\frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'^\nu} = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \tag{3}$$

Now we can define the Lorentz invariant D'Alembertian operator: $\Box = \partial^\mu\partial_\mu$. We can also write down the Klein-Gordon equation

$$\Box\phi(x) + m^2\phi(x) = 0. \tag{4}$$

It IS Lorentz invariant — if $\phi(x^\mu)$ is a solution, then so is $\phi'(x^\mu) = \phi(\Lambda^\mu{}_\nu x^\nu)$. Is this the most general equation? It is the most general Lorentz invariant, <u>scalar</u>, <u>linear</u>, and with at most <u>two derivatives</u>. Why isn't there another term like $\alpha\Box^2\phi$? Theoretically it could be there but in real life the higher coefficients are very small.

We can do some dimensional analysis in mass units: $[m] = 1$, $[m^2] = 2$, $[\partial/\partial x] = 1$ (recall Compton wavelength has units $[\hbar/mc] = -1$), and $[\Box] = 2$. Notice that in our higher order derivative version of the KG equation, we would have $[\alpha] = -2$. Since the deviations are not seen in experiments, we must have $\alpha \sim \Lambda^{-2}$ where $\Lambda$ is the UV cutoff, which is some very large energy in modern theories and so the terms are highly suppressed.

## 1.3 Variational principles, functional derivatives

Consider a Newtonian particle: $m\ddot{x} = -dV/dx$. Then the action is given by $S = \int_{t_1}^{t_2} dt L(x, \dot{x})$ where $L = (1/2)m\dot{x}^2 - V(x)$. The action principle of course says that Newton's 2nd law holds iff $\delta S/\delta x(t) = 0$ subject to $x(t_1) = x_1$, $x(t_2) = x_2$.

Let's talk about the functional derivative. A *function* $f(x_1, \ldots, x_n)$ maps $n$ numbers to a number. A *functional* maps a function with some boundary conditions to a number. Examples:

$$F[x(t)] = \int_{t_1}^{t_2} dt x^2(t), \qquad \text{or} \int_{t_1}^{t_2} dt(\frac{dx}{dt})^2, \qquad \text{or } x(t_3). \tag{5}$$

A *functional derivative* measures the change $\delta F$ in $F$ under change of the function $x(t) \to x(t) + \delta x(t)$, where $\delta x(t) \ll 1$. With an ordinary function we have

$$f(x + \delta x) = f(x) + \frac{df}{dx}\delta x + \frac{1}{2}\frac{d^2 f}{dx^2}\delta x^2 + \ldots \tag{6}$$

whereas with a functional we have

$$F[x + \delta x] = F[x] + \int dt \frac{\delta F}{\delta x(t)}\delta x(t) + \frac{1}{2}\int dt_1 dt_2 \frac{\delta^2 F}{\delta x(t_1)\delta x(t_2)}\delta x(t_1)\delta x(t_2) + \ldots. \tag{7}$$

Take for example

$$F[x] = \int_{t_1}^{t_2} dt x(t)^2. \tag{8}$$

We have

$$F[x+\delta x] = \int_{t_1}^{t_2} dt[x+\delta x]^2 \tag{9}$$

$$= \int_{t_1}^{t_2} dt[x^2 + 2x\delta x + \delta x^2] \tag{10}$$

so that, picking out the appropriate expressions from the expansion, we have

$$\frac{\delta F}{\delta x(t)} = 2x(t), \qquad \frac{\delta^2 F}{\delta x(t)\delta x(t')} = 2\delta(t-t') \tag{11}$$

since

$$\int_{t_1}^{t_2} dt\delta x(t)^2 = \int_{t_1}^{t_2} dtdt'\delta(t-t')\delta x(t)\delta x(t'). \tag{12}$$

Question:

$$\text{Does} \quad \frac{\delta x(t)}{\delta x(t')} = \delta(t-t') \quad ? \tag{13}$$

Well, using $x(t) = \int_{t_1}^{t_2} dt'\delta(t-t')x(t')$ we have $\delta x(t) = \int_{t_1}^{t_2} dt'\delta(t-t')\delta x(t')$ so the answer is yes.

Next example: $F[x(t)] = \int_{t_1}^{t_2} dt(dx/dt)^2$ with boundary conditions $x(t_i) = x_i$ and $\delta x(t_i) = 0$ for $i = 1, 2$. We have

$$F[x+\delta x] = F[x] - \int_{t_1}^{t_2} dt 2\frac{d^2x}{dt^2}\delta x + 2\frac{dx}{dt}\delta x\Big|_{t_1}^{t_2} \tag{14}$$

so $\delta F/\delta x(t) = -2d^2x/dt^2$.

We can also write equations like

$$\delta F[x] = \int dt \frac{\delta F}{\delta x}\delta x. \tag{15}$$

Recall we can write the ordinary derivative like

$$\frac{df}{dx} = \lim_{\epsilon\to 0}\frac{f(x+\epsilon) - f(x)}{\epsilon}, \tag{16}$$

we can also write something like this for functional derivatives

$$\int dt \frac{\delta F}{\delta x(t)} y(t) = \lim_{\epsilon\to 0}\frac{F[x+\epsilon y] - F[x]}{\epsilon} = \frac{d}{d\epsilon}F[x+\epsilon y]\Big|_{\epsilon=0}. \tag{17}$$

Going back to Newton's equations, writing $F[x] = \int dt V(x(t))$ we have $\delta F/\delta x(t) = V'(x(t))$. Then

$$S[x] = \int_{t_1}^{t_2} dt[\frac{1}{2}m\dot{x}^2 - V(x)] \tag{18}$$

has the functional derivative

$$\frac{\delta S}{\delta x(t)} = -m\ddot{x}(t) - V'(x(t)) \tag{19}$$

which vanishes when $m\ddot{x} = -V'(x)$ as desired.

Next let's take a spacetime field $\phi(x^\mu)$ and vary the functional

$$S[\phi] = \int_R d^4x[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)], \qquad \phi\Big|_{\partial R} = \text{fixed} \tag{20}$$

where $R$ is some spacetime region. Then

$$\delta S = \int_R d^4x[\partial^\mu\phi\partial_\mu\delta\phi - V'(\phi)\delta\phi] \tag{21}$$

$$= \int_R d^4x[-\partial_\mu\partial^\mu\phi - V'(\phi)]\delta\phi \tag{22}$$

and we see that $\delta S/\delta\phi$, the part of the integrand multiplying $\delta\phi$, which must be zero, gives us the KG equation. In other words, we have KG iff $\delta S/\delta\phi = 0$.

The action is written $S = \int d^4x\mathcal{L}(\phi, \partial_\mu\phi)$, so the Lagrangian density $\mathcal{L}$ for KG is $\mathcal{L} = (1/2)(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2)$.