Math 217 – Geometry and Physics – Lec02

UCLA, Fall 2014

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1 More on manifolds

1.1 Remarks

1. Partition of unity: $M = \cup_{\alpha\in I} U_\alpha$ where U_α coordinate charts. There exists $\rho_\alpha : M \to \mathbb{R}$, $\rho_\alpha \geq 0$ smooth with $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ and $\sum_\alpha \rho_\alpha = 1$. $\omega = \sum_\alpha \rho_\alpha \omega$, $\mathrm{supp}(\rho_\alpha\omega) \subset U_\alpha$.
2. Sheaf theory. local $\implies$ global (Cohomology)
3. Characteristic classes. local computations $\implies$ global invariants.
4. $M(= \cup_{\alpha\in I} U_\alpha) \xrightarrow{f} N(= \cup_\beta V_\beta)$ is smooth iff $\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$ [is smooth?]. Where $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, $\psi_\beta : V_\beta \to \mathbb{R}^n$.
5. $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1} = \cdots = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C} \cup \{\rho_+\}$ [rho sub what?]

Given a tangent bundle TM we have a vector field (smooth section) $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and differential form $\omega = \sum_i a_i dx_i$, where $\langle dx_i, \frac{\partial}{\partial x_j}\rangle = \delta_{ij}$. Furthermore, a differential p-form $\eta = \sum_I a_I dx_I \in A^p(M)$, where $I = \{1 \leq i_1 < \cdots < i_p \leq n\}$ and $1 \leq p \leq n$, is a section on $\Lambda^p T^* M$. $p = \deg \eta$.

The exterior product $\Lambda^p T^* M \xrightarrow{\pi} M$.

Define $A(M) = A^*(M) = \oplus_{p=0}^n A^p(M)$. Then $\eta \wedge \eta' = (-1)^{pq} \eta' \wedge \eta$.

M is orientable iff there exists a nowhere vanishing n-form ω on M^n.

$d : A^p(M) \to A^{p+1}(M)$, $d\eta = \sum_I da_I \wedge dx_I$.

$da_I = \sum_{j=1}^n \frac{\partial a_I}{\partial x_j} dx_j$.

1.2 Exercise

1. $d^2 = 0$. $A^p(M) \xrightarrow{d} A^{p+1}(M) \xrightarrow{d} A^{p+2}(M)$. $\eta = p$.
2. Leibniz rule: $d(\eta \wedge \eta') = d\eta \wedge \eta' + (-1)^p \eta \wedge d\eta'$.
3. $\int_M \omega = \sum_{\alpha\in I} \int_{U_\alpha} (\rho_\alpha \omega) = \int_M (\sum_\alpha \rho_\alpha)\omega$.
   $\int_{U_\alpha} (\rho_\alpha\omega) = \int_{\varphi(U_\alpha)\subseteq\mathbb{R}^n} (\varphi_\alpha^{-1})^* (\rho_\alpha\omega)$

1.3 Stokes

$\int_M d\eta = \int_{\partial M} \eta$ where $\eta \in A^{n-1}(M)$ implies that $\int_M d\eta = 0$ if M closed compact.

Consider: $f : M \to M'$ a smooth map. Then the pullback $f^*\eta'$ of $\eta' = \sum_I b_I(y) dy_I \in A(M')$ is given by

$$f^*\eta'(x) = \sum_I b_I(f(x)) df_I. \tag{1}$$

1.4 exercise

1. Chain rule: $f^* \circ d_{M'} = d_M f^*$
2. $f^*(\eta_1 \wedge \eta_2) = (f^*\eta_1) \wedge (f^*\eta_2)$

1.5 de Rham Cohomology

$$H^p_{\mathrm{dR}}(M, \mathbb{R}) = \frac{\operatorname{Ker} d}{\operatorname{Im} d}\Big|_{A^p(M)} \tag{2}$$

where of course $\operatorname{Im} d \subseteq \operatorname{Ker} d$. Then

$$0 \to A^0(M) \xrightarrow{d} A^1(M) \to \ldots \xrightarrow{d} A^p(M) \xrightarrow{d} \ldots \xrightarrow{d} A^n(M) \to 0 \tag{3}$$

1.6 Singular homology, Cohomology

Famous theorem of de Rham: $H^p_{\mathrm{dR}}(M, \mathbb{R}) \cong H^p_{\mathrm{sing}}(M, \mathbb{R})$ when M is a compact, closed manifold. We will also see later that it is $\cong \operatorname{Ker} \Delta_d$, where Δ_d is the elliptic operator.

Cohomology ring: $H^*_{\mathrm{dR}}(M) = \oplus_{p=0}^n H^p_{\mathrm{dR}}(M)$.

$$H^p_{\mathrm{dR}}(M) \qquad \times H^q_{\mathrm{dR}}(M) \qquad \to H^{p+q}_{\mathrm{dR}}(M) \tag{4}$$

$$[\eta] = [\eta + d\alpha] \qquad [\eta'] = [\eta + d\alpha'] \qquad \mapsto [\eta \wedge \eta'] \tag{5}$$

1.7 Poincare duality

$$f^* : H^*_{\mathrm{dR}}(M') \to H^*_{\mathrm{dR}}(M) \tag{6}$$

$$[\eta'] \mapsto [f^*\eta'] \tag{7}$$

$$H^p_{\mathrm{dR}}(M) \qquad \times H^{n-p}_{\mathrm{dR}}(M) \qquad \to \mathbb{R} \tag{8}$$

$$[\eta] = [\eta + d\alpha] \qquad [\eta'] = [\eta'] \qquad \mapsto \int_M \eta \wedge \eta' \tag{9}$$

1.8 Betti numbers

$$H^{n-}_{\mathrm{dR}}(M,\mathbb{R}) \cong H^{p}_{\mathrm{dR}}(M,\mathbb{R}) \cong H^{p}_{\mathrm{dR}}(M,\mathbb{R}) \implies b_p(M) = b_{n-p}(M) \tag{10}$$

where

$$b_p(M) = \dim_{\mathbb{R}} H^p_{\mathrm{dR}}(M,\mathbb{R}) \tag{11}$$

is the pth Betti number of M.

Euler number: $\chi(M) = \sum_{p=0}^{n}(-1)^p b_p(M) = \int_M^{2n} e(TM)$ by Gauss-Bonnet, where $e(TM)$ is the Euler characteristic class.

Corollary: $\dim_{\mathbb{R}} M = \text{odd} \implies \chi(M) = 0$

Conjecture (Chern): Let M be an affine-flat compact closed manifold. Then $\chi(M) = 0$.