Phys 221A – Quantum Mechanics – Lec01

UCLA, Fall 2014

Monday 6^{th} October, 2014

1 Introduction

Basic info

- Professor: Yaroslav Tserkovnyak, condensed matter (experimentalist?). Email: yaroslav@physics.ucla Office hours: Thursday 11-12, Knudsen 6-137C. No midterm. Grade: 35% HW, 65% final. Homeworks will be assigned Wednesdays, due Wednesday the next week.
- TA: Shahriar, office hours Tuesday 1-2 at Knudsen 3-111.

What the course is **not**:

- Will not motivate QM. You should have motivation by now.
- Will not be overly mathematically rigorous. Only need a certain level of physical rigor.
- Will not give historical or philosophical perspective.

What the course will cover:

1. Intro
   (a) Hilbert spaces
   (b) Quantum numbers
   (c) Uncertainty principle
   (d) Correspondence principle — establishes link between QM and classical mechanics
2. Quantum dynamics
   (a) Schroedinger vs Heisenberg picture
   (b) Feynman path integrals — establishes connection with principle of least action, Lagrangian mechanics
   (c) Gauge transformations — establishes connection with E&M
3. Rotations

(a) Spin and orbital angular momentum

(b) Addition of angular momentum

(c) Bell inequalities — relatively recent. Show that there are measurements that prove classical description is incomplete.

221B will cover symmetries, approximations, scattering theory. 221C will cover applications with many (identical) particles and fundamentals needed for quantum field theory.

1.1 Hilbert spaces

A Hilbert space G is a complete inner product space. One example is Euclidean space $\mathbb{E}^n$, that is, $\mathbb{R}^n$ with the standard inner product $\langle x|\, y\rangle = \mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^n x_i y_i$. Of course $\mathbb{R}^n$ itself is a vector space. We will define what completeness means later.

We should define the inner product. The inner product $\langle x|\, y\rangle$ between vectors x and y in G is a complex number such that:

1. $\langle x|\, y\rangle = \langle y|\, x\rangle^*$,
2. Linearity: $\langle x|\, \alpha_1 y_1 + \alpha_2 y_2\rangle = \alpha_1 \langle x|\, y_1\rangle + \alpha_2 \langle x|\, y_2\rangle$ for any scalars $\alpha_1, \alpha_2 \in \mathbb{C}$, and
3. Positivity: $\langle x|\, x\rangle \geq 0$ with equality iff $x = 0$.

Define the distance

$$d(x,y) = d(y,x) = \|x-y\| = \sqrt{\langle x-y|\, x-y\rangle}. \tag{1}$$

Need to prove this is a good distance function. In particular, we need the triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$. Mathematicians would call this a metric space. Can prove the triangle inequality with the Cauchy-Schwartz inequality:

$$|\langle x|\, y\rangle| \leq \|x\| \cdot \|y\|\,. \tag{2}$$

We can prove this as follows. Define

$$z = x - y\frac{\langle y|\, x\rangle}{\langle y|\, y\rangle}, \tag{3}$$

then

$$x = z + y\frac{\langle y|\, x\rangle}{\langle y|\, y\rangle} \tag{4}$$

so $\|x\|^2 = \|z\|^2 + \|y\|^2 (\dots)^2$ by Pythagorean theorem, thus

$$\|x\|^2 \|y\|^2 = \|z\|^2 \|y\|^2 + |\langle y|\, x\rangle|^2 \tag{5}$$

and we see equality occurs iff $z = 0$ i.e. $x \parallel y$

Now we can define completeness of a metric space G. Define a Cauchy sequence $x_i \in G$, $i = 1, 2, \dots$ as a sequence such that for any $\delta > 0$ there exists an integer $N > 0$ so that

$\|x_n - x_m\| < \delta$ for all $m, n > N$. The space G is complete if every Cauchy sequence converges. Good, now we've defined a Hilbert space.

Now, the universe itself is a Hilbert space. Or rather, physical realizations of the universe are elements of a Hilbert space. A quantum mechanical state ψ is represented by a vector in a Hilbert space. We will be interested in the preparation, measurement, and evolution of this state.

As $\psi(t)$ evolves, it must remain in G. Furthermore, the physical state ψ must be properly normalized: $\langle\psi|\,\psi\rangle = 1$ at all times. Dirac's bracket notation provides us with a good way of dealing with elements of the Hilbert space and the dual space.

- $|\psi\rangle \in G$ is the ket, the representation in the Hilbert space
- $\langle\psi| \in G$ is the bra, the representation in the dual space. The bra is a linear map $\langle\psi| : G \to \mathbb{C}$ mapping kets $|\phi\rangle \mapsto \langle\psi|\,\phi\rangle$.

For each Hilbert space G there is a dual (Hilbert) space G^*. Bras, the elements of G^*, are defined by their action on the kets in G. Note that we have

$$\langle c_1\psi_1 + c_2\psi_2|\,\phi\rangle = c_1^* \langle\psi_1|\,\phi\rangle + c_2^* \langle\psi_2|\,\phi\rangle\,, \tag{6}$$

so that

$$\langle c_1\psi_1 + c_2\psi_2| = c_1^* \langle\psi_1| + c_2^* \langle\psi_2|\,. \tag{7}$$

A Hilbert space can be spanned by an orthonormal basis $|\alpha_i\rangle$, $i = 1, 2, \ldots$, i.e. $\langle\alpha_i|\,\alpha_j\rangle = \delta_{ij}$. We can find an orthonormal basis via the Gram-Schmidt method as follows. Suppose we have a complete set of linearly independent states $|u_i\rangle$. Define a normal vector $e_1 = u_1/\|u_1\|$. Next define $v_2 = u_2 - P_{e_1}(u_2)$ where we use the projection operator $P_e(v) = |e\rangle\langle e|\,v\rangle$, and finally $e_2 = v_2/\|v_2\|$. We can see that e_1 and e_2 form an orthonormal set so we continue this process ad infinitum to obtain the full orthonormal basis, i.e. $v_{i+1} = u_{i+1} - P_{e_i}(u_{i+1})$ and $e_{i+1} = v_{i+1}/\|v_{i+1}\|$ for all $i > 1$.

Any state ψ can be written in the basis: $|\psi\rangle = \sum_i c_i |\alpha_i\rangle$. Note that the coefficients can be obtained by taking the inner product with the corresponding orthonormal basis vector: $\langle\alpha_i|\,\psi\rangle = \sum_j c_j\delta_{ij} = c_i$. Since $c_i = \langle\alpha_i|\,\psi\rangle$, we can write $|\psi\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\,\psi\rangle$, thus we find

$$\sum_i |\alpha_i\rangle\langle\alpha_i| = 1. \tag{8}$$

We call this the closure relation.