Math 217 – Geometry and Physics – Lec04

UCLA, Fall 2014

Friday 10^{th} October, 2014

1 Stuff

Recall again the Gauss-Bonnet-Chern theorem: for M^{2n} a compact closed oriented manifold, we have

$$\int_M \varepsilon(TM) = \chi(M) = \sum_{i=0}^{2n} (-1)^i b_i. \tag{1}$$

Then we have

1. $H^k(M) \cong H^{2n-k}(M)$
2. Thom isomorphism: Let $\Delta \subseteq M$ be a submanifold and

$$N_\Delta \cong (TM|_\Delta)/T\Delta \tag{2}$$

the normal bundle of Δ in M. Let $T(N_\Delta)$ be the Thom class of N_Δ. Then the Poincare dual η_Δ of Δ is equal to $T(N_\Delta) \in H_c^*(N_\Delta)$ (Bott-Tu). That is, given the projection $i_{\Delta *} : N_\Delta \hookrightarrow M$ we have $i_{\Delta *} T(N_\Delta) = \eta_\Delta$.

Given a basis $\{\omega_i\}$ of $H^*(M) = \oplus_{i=0}^{2n} H^i(M)$, define a dual basis $\{\tau_i\}$ of $H^*(M)^*$ through the Poincare duality $H^k \cong (H^{2n-k})^*$ such that $\int_M \omega_i \wedge \tau_j = \delta_{ij}$.

Kuenneth Theorem states that

$$\begin{array}{ccccc} H^*(M \times M) & \cong & H^*(M) & \otimes_{\mathbb{R}} & H^*(M) \\ \{\pi^*\omega_i \wedge \rho^*\tau_j\} & & \{\omega_i\} & & \{\tau_j\} \end{array} \tag{3}$$

Denote $i : \Delta \hookrightarrow M \times M$, Δ the diagonal $\{x, x\} \in M \times M$ isomorphic to M.
Let $\eta_\Delta \in H^*(M \times M)$ be the Poincare dual of Δ. Then

$$\eta_\Delta = \sum_{ij} c_{ij} \pi^*\omega_i \wedge \rho^*\tau_j \tag{4}$$

Lemma:

$$\eta_\Delta = \sum_i (-1)^{\deg \omega_i} \pi^*\omega_i \wedge \rho^*\tau_i \tag{5}$$

Proof: We have $\pi \circ i = \text{id}, \rho \circ i = \text{id} \implies i^*\rho^*, i^*\pi^*$ are identity on cohomolgy. Then

$$\begin{aligned}\int_\Delta i^*(\pi^*\tau_k \wedge \rho^*\omega_l) &= \int_M i^*(\pi^*\tau_k \wedge \rho^*\omega_l) &(6)\\ &= \int_M (i^*\pi^*)\tau_k \wedge (i^*\rho^*)\omega_l &(7)\\ &= \int_M \tau_k \wedge \omega_l &(8)\\ &= \delta_{kl}(-1)^{\deg \tau_k \deg \omega_l}. &(9)\end{aligned}$$

Furthermore

$$\begin{aligned}\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l &= \int_{M\times M} (\pi^*\tau_k \wedge \rho^*\omega_l) \wedge \eta_\Delta &(10)\\ &= \sum_{ij} c_{ij} \int_{M\times M} (\pi^*\tau_k \wedge \rho^*\omega_l) \wedge (\pi^*\omega_i \wedge \rho^*\tau_j) &(11)\\ &= \sum_{ij} c_{ij}(-1)^{(\deg \tau_k + \deg \omega_l)\deg \omega_i} \int_{M\times M} \pi^*(\omega_i \wedge \tau_k) \wedge \rho^*(\omega_l \wedge \tau_j) &(12)\\ &= \sum_{ij} c_{ij}(-1)^{(\deg \tau_k + \deg \omega_l)\deg \omega_i} (\int_M \omega_i \wedge \tau_k)(\int_M \omega_l \wedge \tau_j) &(13)\\ &= \sum_{ij} c_{ij}(-1)^{(\deg \tau_k + \deg \omega_l)\deg \omega_i} \delta_{ik}\delta lj &(14)\\ &= c_{kl}(-1)^{(\deg \tau_k + \deg \omega_l)\deg \omega_k} &(15)\end{aligned}$$

where

$$c_{kl} = \begin{cases} 0 & k \neq l, \\ (-1)^{\deg \omega_k} & k = l. \end{cases} \qquad (16)$$

Lemma: Let T_Δ be the tangent bundle of $\Delta \hookrightarrow M \times M$. Then $T_\Delta \cong N_\Delta$.

Proof:

$$\begin{array}{ccccccccc} 0 & \to & T_\Delta & \to & T_{M\times M}|_\Delta & \to & N_\Delta & \to 0 \\ & & \cong & & \cong & & \cong & \\ 0 & \to & T_M & \to & TM \oplus TM & \to & T_M & \to 0 \end{array} \qquad (17)$$

We have

$$\int_\Delta i_\Delta^* \eta_\Delta = \int_\Delta i_\Delta^* \Phi(N_\Delta) = \int_\Delta \varepsilon(N_\Delta) = \int_M \varepsilon(TM) \qquad (18)$$

where $s : M \to E$ is a section and $s^*\Phi(E) = \varepsilon(E)$. We have

$$\begin{aligned}
\int_\Delta i_\Delta^* \eta_\Delta &= \sum_i (-1)^{\deg \omega_i} \int_\Delta i_\Delta^* (\pi^* \omega_i \wedge \rho^* \tau_i) && (19)\\
&= \sum_i (-1)^{\deg \omega_i} \int_M (i^* \pi^*) \omega_i \wedge (i^* \rho^* \tau_i) && (20)\\
&= \sum_i (-1)^{\deg \omega_i} \int_M \underbrace{\omega_i \wedge \tau_i}_{=1} && (21)\\
&= \sum_{i=0}^{2n} (-1)^i \dim H^i(M) = \chi(M). && (22)
\end{aligned}$$

2 More on Lie Algebras

Let g be a Lie algebra of a Lie group G. We have $g = T_eG$ where $e \in G$ is the identity, and this is also seen as the space of left invariant vector fields. We have the commutator taking $g \times g \to g$ defined by $X, Y \mapsto [X, Y]$. We have the identities

1. $[X, Y] = -[Y, X]$, and
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$, the Jacobi identity.

For example the Lie group SO_n has the Lie algebra $\mathfrak{so}_n$. We have $\mathfrak{so}_n \cong \Lambda^2\mathbb{R}^n$. So $\{e_1, \ldots, e_n\}$ is an orthonormal basis w.r.t. the inner product $\langle \cdot, \cdot \rangle$. And we have

$$v \wedge w(x) = \langle v, x \rangle\, w - \langle w, x \rangle\, v \quad \forall x \in \mathbb{R}^n \tag{23}$$

There is a one to one correspondence between one parameter subgroups of G and orbits of elements of g passing through e.

$$\psi : \mathbb{R} \to G \quad \leftrightarrow \quad \psi'(0) = X \in G \tag{24}$$

Geodesics of G with Killing metric on G.