I. WORKFLOW

Goal of the following workflow:
Gouverning equations:

$$(\nabla^2 - \phi^2)u = 0 \qquad \text{in } \Omega \tag{1}$$

$$\begin{aligned} u^+(s) &= U(s) + l u_n^+(s) \qquad \text{on } \Gamma \\ u^-(s) &= U(s) - l u_n^-(s) \qquad \text{on } \Gamma \end{aligned} \tag{2}$$

with $\phi^2 = \frac{\nu}{D_2}$ and $l = \frac{D_2}{\kappa}$.

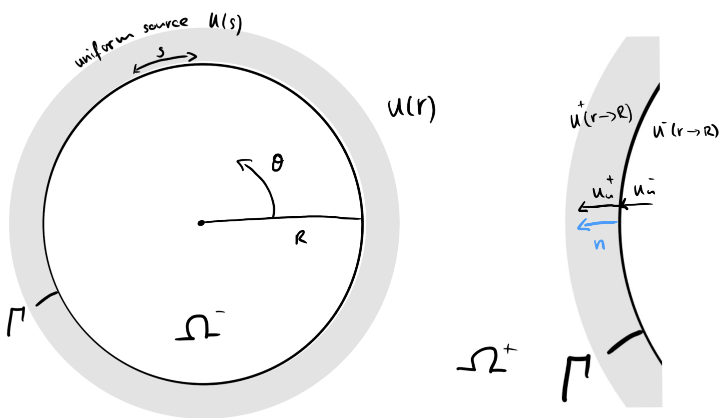
The oxygen flux through the channel is $j_s = -D_1 R(s) U'(s)$ and the flux in the direction of the outside pointing normal from the channel in $j_n = -D_2(u_n^+ - u_n^-)$ The flux conservation law in the one-dimensional channel and into the two-dimensional tissue given by $j_s' = j_n$ gives us:

$$-D_1(R(s)U'(s))' = -D_2(u_n^+(s) - u_n^-(s)) \tag{3}$$

CONTENTS

II. LAPLACE PROBLEM

A. Analytical solution to circle with uniform source



Consider a one-dimensional circular perfusing channel with arc length s embedded in a two-dimensional tissue domain Ω. The channel forms the interface boundary Γ which splits the domain into Ω^+ and Ω^+. In the following problem, the channel has a constant oxygen concentration $U(s)$ which serves as a uniform source of oxygen on Γ. Oxygen can diffuse into the tissue domain Ω.
To start with the simplest problem possible, the tissue domain is considered passive, meaning it does not take up oxygen, so $\phi^2 = 0$ and thus reduces our problem to the interior Laplace problem:

$$\nabla^2 u(r,\theta) = 0 \qquad \text{in } \Omega \tag{4}$$

$$U(s) = const. = C \qquad \text{inside the 1D channel} \tag{5}$$

In a two-dimensional tissue space ($n = 2$) the following radial solution (for $r = |x|$) for the Laplacian leverages it's symmetric nature and reduces our PDE problem giving us two regimes for $r > R$ and $r < R$ in which we can find a solution:

$$u(r) = \begin{cases} u^+ : c_1 \ln(\frac{1}{r}) + c_2 & \text{for } r > R \\ u^- : c_3 & \text{for } r < R \end{cases} \tag{6}$$

where for $x \in R^n, |x| \neq 0$ is a solution of Laplace's equation in $R^n - \{0\}$. We can compute the derivatives $u'(r)$ and $u''(r)$ for both regimes:

$$u'(r) = \begin{cases} u_n^+ : -c_1/r & \text{for } r > R \\ u_n^- : 0 & \text{for } r < R \end{cases} \tag{7}$$

We can solve for c_1, c_2, c_3, and C by computing the sum and difference of boundary condition 2 on Γ and the flux conservation boundary condition 3 (to find a non-zero solution we add a -1 to the right side of 3):

$$\begin{aligned} 2U(\theta) &= u^+ + u^- - l(u_n^+ - u_n^-) & \text{for } BC^+ + BC^- \\ 0 &= u^+ - u^- - l(u_n^+ + u_n^-) & \text{for } BC^+ - BC^- \\ -D_1 RC'' &= -D_2(u_n^+ - u_n^-) - 1 & \text{for } BC\ 2 \end{aligned} \tag{8}$$

Substituting these formulae into our equations we can derive:

$$\begin{aligned} 2C &= c_1 \ln(\frac{1}{r}) + c_2 + c_3 + lc_1/r & \text{for } BC^+ + BC^- \\ 0 &= c_1 \ln(\frac{1}{r}) + c_2 - c_3 + lc_1/r & \text{for } BC^+ - BC^- \\ 0 &= D_2 c_1/r - 1 & \text{for } BC\ 2 \end{aligned} \tag{9}$$

Considering $r = R = 1$ and choosing an arbitrary constant for $c_2 = 0$ we have:

$$u^+(1) = 0 \qquad u^-(1) = c_3 \qquad u_n^+(1) = -c_1 \qquad u_n^-(1) = 0 \tag{10}$$

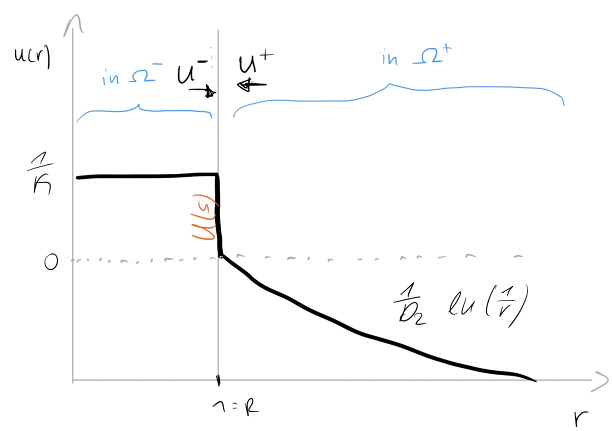
which gives us:

$$\begin{aligned} 2C &= c_3 + lc_1 && \text{for } BC^+ + BC^- \\ 0 &= lc_1 - c_3 && \text{for } BC^+ - BC^- \\ 0 &= D_2 c_1 - 1 && \text{for } BC\ 3 \end{aligned} \tag{11}$$

For $l = \frac{D_2}{\kappa}$ we get the solution:

$$c_1 = \frac{1}{D_2} \qquad c_3 = \frac{1}{\kappa} \qquad C = \frac{1}{\kappa} \tag{12}$$

We thus show that $u^-(r) = U(s)$ which satisfies our Laplace problem. The jump on the boundary Γ from $U(s)$ to $u^+(r)$ at $r = R = 1$ is $\frac{1}{\kappa}$ and decreases towards $u^+(r)$ with $\frac{1}{D_2}\ln(\frac{1}{r})$.



III. MODIFIED BESSEL PROBLEM

Guiding equations:

$$(\nabla^2 - \phi^2)u(\theta, r) = 0 \qquad \text{in } \Omega \tag{13}$$

$$U(s) = f(\theta) \qquad \text{inside the 1D channel} \tag{14}$$

For problem stated above, we can consider additional different states:

- 1. $\phi > 0$, $U(s) = const.$,
  implying that the tissue uptake rate is now positive creating a concentration gradient in both regimes of Ω,
- 2. $\phi = 0$, $U(s) = f(\theta)$,
  implying that the oxygen source is non-uniform inside the channel creating non-uniform fluxes along Γ into the tissue,
- 3. $\phi > 0$, $U(s) = f(\theta)$,
  implying that the oxygen source is non uniform on Γ and the tissue uptake rate is positive in Ω.

For (1.) and (3.) in which the tissue is regarded as an active domain with a (linear) oxygen uptake rate ν, so $\phi^2 \neq 0$, the problem is no longer satisfying Laplace equation. Through separation of variables we can decompose the PDE into two ODEs that we solve separately for to yield the modified Bessel equation, for which known solutions exists, called the modified Bessel functions of the first I_z and second K_z kind.

$$u(r, \theta) = R(r)\Theta(\theta) \tag{15}$$

$$\begin{gathered} \nabla^2(R(r)\Theta(\theta)) = \phi^2 R(r)\Theta(\theta) \\ \frac{r^2}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) - \phi^2 r^2 = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} = z^2 \end{gathered} \tag{16}$$

For $\Theta(\theta)$ we derive the equation

$$\frac{d^2\Theta(\theta)}{d\theta^2} = -z^2\Theta(\theta) \tag{17}$$

that has the simple solution:

$$\Theta(\theta) = Ae^{iz\theta}. \tag{18}$$

For $R(r)$ we derive the equation:

$$\frac{\tilde{r}^2}{\tilde{R}(\tilde{r})}\frac{d}{d\tilde{r}}\left(\tilde{r}\frac{d\tilde{R}(\tilde{r})}{d\tilde{r}}\right) - (\tilde{r}^2 + z^2) = 0 \tag{19}$$

where $\tilde{r} = \phi r$ and $R(r) = R(\phi/\tilde{r}) =: \tilde{R}(\tilde{r})$ which is known as the modified Bessel equation. The solution to this equation is a linear combination of the modified Bessel functions I_z and K_z

$$\begin{gathered} \tilde{R}(\tilde{r}) = c_2 I_z(\tilde{r}) + c_1 K_z(\tilde{r}) \\ R(\phi r) = c_2 I_z(\phi r) + c_1 K_z(\phi r) \end{gathered} \tag{20}$$

Substituting these solution back into $u(r, \theta)$ we yield

$$u(r, \theta) = e^{iz\theta}(c_2 I_z(\phi r) + c_1 K_z(\phi r)) \tag{21}$$

with the three unknown constants z, c_1, and c_2.
For intuition:

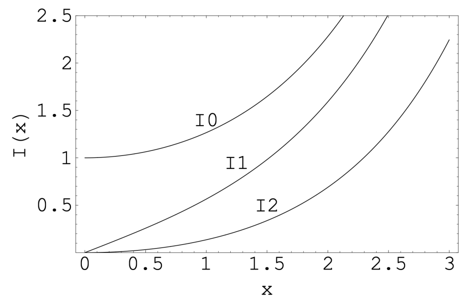


Figure 4.3: Plot of the Modified Bessel Functions of the First Kind, Integer Order

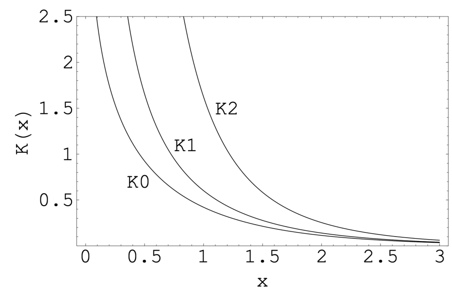


Figure 4.4: Plot of the Modified Bessel Functions of the Second Kind, Integer Order

A. Circle with uniform source, and oxygen consumption

$$(\nabla^2 - \phi^2)u(\theta, r) = 0 \qquad \text{in } \Omega \tag{22}$$

$$U(s) = f(\theta) = const. = C \qquad \text{inside the 1D channel} \tag{23}$$

$$\begin{aligned} u^+ &= C + lu_n^+ \qquad \text{on } \Gamma \\ u^- &= C - lu_n^- \qquad \text{on } \Gamma \end{aligned} \tag{24}$$

For a uniform O_2 source $U(\theta) = const. = C$ and a uniform O_2 take up rate ν in a circular geometry we loose the angular dependence in our problem, which reduces the dependence of $u(r, \theta)$ to $u(r)$ and reduces the order of the modified Bessel function to $z = 0$:

$$u(r) = c_2 I_0(\phi r) + c_1 K_0(\phi r) \tag{25}$$

The modified Bessel function of the second kind K_z diverges at $r = 0$ with the singularity being of logarithmic type, which is why we the solution to $u(r)$ is defined for K_z in the regime $r > R$ and I_z in the regime $r < R$:

$$u(r) = \begin{cases} u^+ : c_1 K_0(\phi r) & \text{for } r > R \\ u^- : c_2 I_0(\phi r) & \text{for } r < R \end{cases} \tag{26}$$

We can compute the derivative $u'(r)$ for both regimes:

$$u'(r) = \begin{cases} u_n^+ : -c_1 K_0'(\phi r) = -c_1 \phi K_1(\phi r)) & \text{for } r > R \\ u_n^- : c_2 I_0'(\phi r) = c_2 \phi I_1(\phi r)) & \text{for } r < R \end{cases} \tag{27}$$

To determine our unknown coefficients c_1, c_2, and C we apply our boundary conditions, computing the sum and difference of equations 24 on Γ and the flux conservation boundary condition 3 as given in 8 for $r = R = 1$:

$$\begin{aligned} 2C &= c_1 K_0(\phi) + c_2 I_0(\phi) - l(-c_1 \phi K_1(\phi)) - c_2 \phi I_1(\phi))) \qquad \text{for } BC^+ + BC^- \\ 0 &= c_1 K_0(\phi) - c_2 I_0(\phi) - l(-c_1 \phi K_1(\phi)) + c_2 \phi I_1(\phi))) \qquad \text{for } BC^+ - BC^- \\ -D_1 R C'' &= -D_2(-c_1 \phi K_1(\phi)) - c_2 \phi I_1(\phi))) \qquad \text{for } BC\ 3 \end{aligned} \tag{28}$$

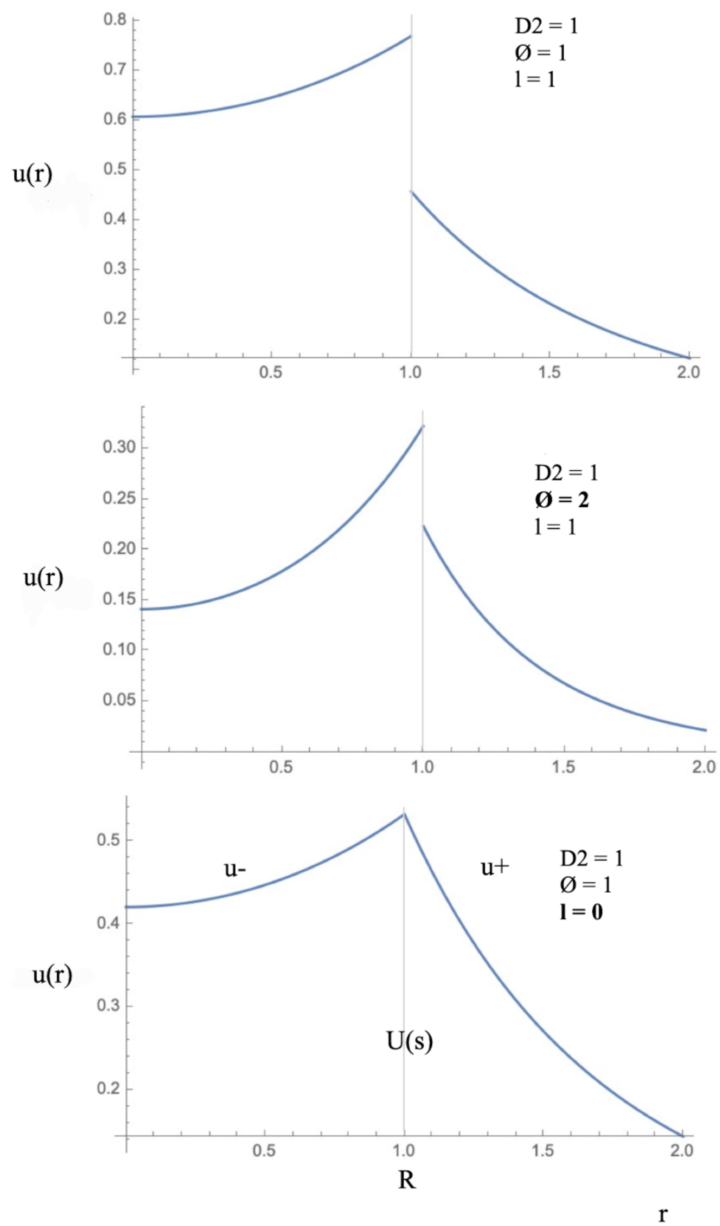
For our unknown coefficients c_1, c_2, and C we get:

$$c_1 = \frac{1}{D_2 \phi} \frac{I_0 + l\phi I_1}{I_0 K_1 + K_0 I_1 + 2l\phi K_1 I_1} \tag{29}$$

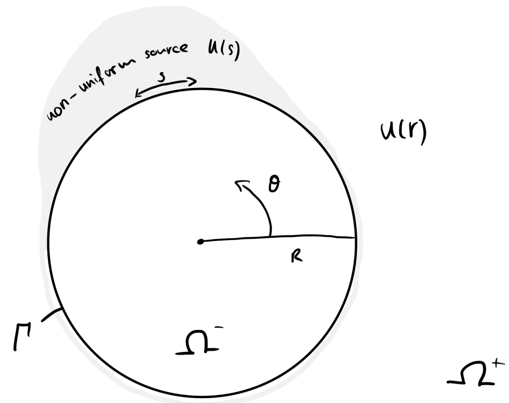
$$c_2 = \frac{1}{D_2\phi}\frac{K_0 + l\phi K_1}{I_0K_1 + K_0I_1 + 2l\phi K_1I_1} \tag{30}$$

$$C = \frac{1}{D_2\phi}\frac{(I_0 + l\phi I_1)(K_0 + l\phi K_1)}{I_0K_1 + K_0I_1 + 2l\phi K_1I_1} \tag{31}$$

(Modified) Bessel function can be (approximately) evaluated for a given order z and at a given value for r which can provide us with a better intuition and approximate solution to the system of equations stated above.



B. Circle with non-uniform source, and oxygen consumption



To test our system understand how the jump function across the boundary Γ behaves for a non-uniform function $f(\theta)$ we set up our next problem as follows:

$$(\nabla^2 - \phi^2)u(\theta, r) = 0 \qquad \text{in } \Omega \tag{32}$$

$$U(s) = f(\theta) = bumpfunction \qquad \text{inside the 1D channel} \tag{33}$$

In this example, we define $f(\theta)$ by a simple "bump" function on the circle where $f(\theta) = f(\theta + 2\pi)$.We are thus concerned with in a θ-dependent problem and need to solve for $u(\theta, r) = e^{iz\theta}(c_2 I_z(\phi r) + c_1 K_z(\phi r))$ derived in section III. We define:

$$u(\theta, r) = \begin{cases} u^+ : c_1 \sum_{z=0}^{\infty} e^{iz\theta} K_z(\phi r) & \text{for } r > R \\ u^- : c_2 \sum_{z=0}^{\infty} e^{iz\theta} I_z(\phi r) & \text{for } r < R \end{cases} \tag{34}$$

and for the corresponding derivatives.

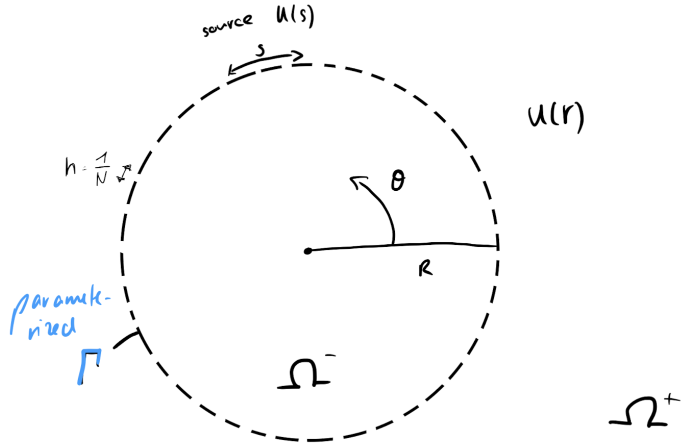
$$u'(\theta, r) = \begin{cases} u_n^+ : -c_1 \phi \sum_{z=0}^{\infty} e^{iz\theta} K_z'(\phi r) = -c_1 \phi \sum_{z=0}^{\infty} e^{iz\theta} (\frac{z}{r} K_z(\phi r) + K_{z+1}(\phi r)) & \text{for } r > R \\ u_n^- : c_2 \phi \sum_{z=0}^{\infty} e^{iz\theta} I_z'(\phi r) = c_2 \phi \sum_{z=0}^{\infty} e^{iz\theta} (\frac{z}{r} I_z(\phi r) + I_{z+1}(\phi r)) & \text{for } r < R \end{cases} \tag{35}$$

considering $r = R = 1$ we simplify to:

$$u'(\theta, 1) = \begin{cases} u_n^+ : -c_1 \phi \sum_{z=0}^{\infty} e^{iz\theta} K_z'(\phi) = -c_1 \phi \sum_{z=0}^{\infty} e^{iz\theta} (z K_z(\phi) + K_{z+1}(\phi)) & \text{for } r > R \\ u_n^- : c_2 \phi \sum_{z=0}^{\infty} e^{iz\theta} I_z'(\phi) = c_2 \phi \sum_{z=0}^{\infty} e^{iz\theta} (z I_z(\phi) + I_{z+1}(\phi)) & \text{for } r < R \end{cases} \tag{36}$$

C. Investigation of the associated Fourier modes

IV. BIES ON CIRCLE



To solve this problem using BIEs, we need to parameterize our boundary Γ using quadrature formulations (discretization of the boundary integral). Here we define an additional speed function expressing an N-point quadrature rule that approximates an integral on Γ via a weighted sum of N function evaluations.

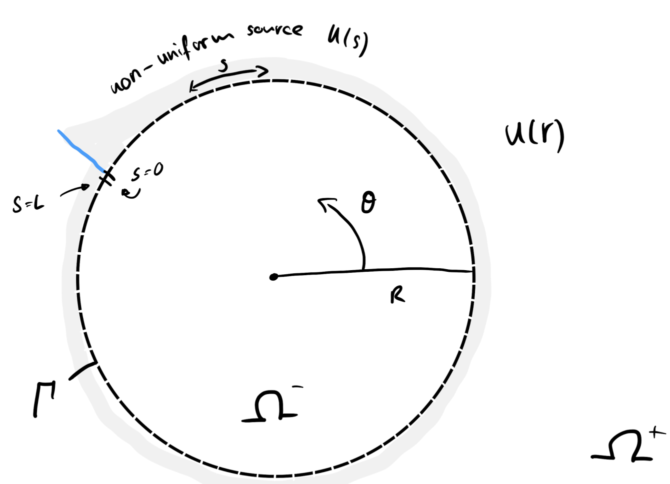
$$\int_\Gamma f(s)\,ds \approx \sum_{j=1}^{N} w_j f(s_j) \tag{37}$$

segments our boundary Γ into N segments s_j of weight w_j. Since we are concerned with 2π-periodic functions on a circular geometry $[0, 2\pi]$ in this problem, we can defined a rule of evenly weighted and spaced segments. (for more details check Alex' notes)

A. Investigation of the associated Fourier modes and comparison to analytical model

For each "source" find Fourier modes using separation of variables.

B. BIEs on circle with discontinuous source function



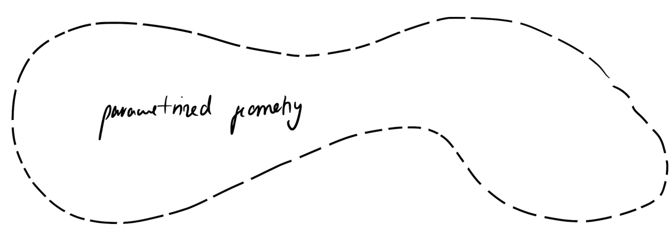
The problem we aim to solve, it is important to understand how we can handle discontinuous functions $U(s)$ on Γ given an initial value

$$U(\theta) = U_0 \Big|_{\theta=0} \tag{38}$$

and either a Dirichlet boundary condition or a Neumann boundary condition

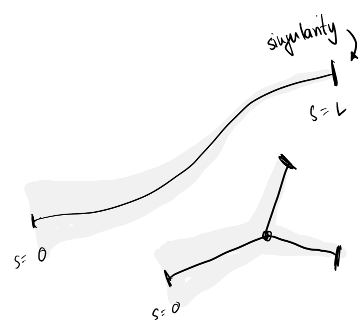
$$U(\theta) = bU_0 \Big|_{\theta=2\pi} \qquad \text{or} \qquad \frac{dU(\theta)}{d\theta} = 0 \Big|_{\theta=2\pi}. \tag{39}$$

V. BIES ON ARBITRARY CLOSED CURVE GEOMETRY



Before we will be able to solve the problem, we now first have to find a good parametrizing function that describes the arbitrary geometry we are concerned with.

VI. BIES ON ARBITRARY OPEN CURVE OR BRANCH GEOMETRY



Open curves and branched geometries pose a further problem, as the boundary is not closed, we will have to investigate the effect of the singularities faced at the end of the boundary ($s = L$) and the effect of branching/fusing of boundary elements.