problem2

December 8, 2020

2. Build a Perceptron [multilayer or single layer as the case may be] to recognize a certain area of the plane. That is, the Perceptron should output a "1" if the input vector lies in the shaded region.

`a. Determine the vector of coefficients W for a single layer perceptron of the form in Figure`

Figure 2:

In Figure 2, all points (x_1, x_2) such that $x_1 < 1$ are shaded and therefore in Class 1, all other points are in Class 2. The weight vector w must be perpendicular to the discriminant line and point towards Class 1, so we know w points in the $(-1, 0)$ direction. w also must have a bias term, so w has the form $w = (w_0, -1, 0)$. We will also append a 1 to each input point so x has the form $x = (1, x_1, x_2)$.

The activation function is a step function, so, if $x \cdot w < 0$, x will be assigned to Class 0; likewise, if $x \cdot w > 0$, x will be assigned to Class 1. Therefore we must choose the bias term w_0 such that $x \cdot w = 0$ at the discriminant boundary $x_1 = 1$.

$$
\begin{aligned}
x \cdot w =& 0 && (1)\\
(1, 1, x_2) \cdot (w_0, -1, 0) =& 0 && (2)\\
w_0 - 1 =& 0 && (3)\\
w_0 =& 1 && (4)
\end{aligned}
$$

Therefore $w = (1, -1, 0)$ is the final weight vector for the classes in Figure 2.

Figure 3:

In Figure 3, all points (x_1, x_2) such that $x_2 > x_1 - 1$ are shaded and therefore in Class 1, all other points are in Class 2. The weight vector w must be perpendicular to the discriminant line and point towards Class 1, so we know w points in the $(-1, -1)$ direction. w also must have a bias term, so w has the form $w = (w_0, -1, 1)$. We will also append a 1 to each input point so x has the form $x = (1, x_1, x_2)$.

The activation function is a step function, so, if $x \cdot w < 0$, x will be assigned to Class 0; likewise, if $x \cdot w > 0$, x will be assigned to Class 1. Therefore we must choose the bias term w_0 such that $x \cdot w = 0$ at the discriminant boundary $x_2 = x_1 - 1$. I'll use the point $(x_1, x_2) = (1, 0) \implies x = (1, 1, 0)$ to solve for the bias term.

$$
\begin{align}
x \cdot w =& 0 \tag{5}\\
(1,1,0) \cdot (w_0, -1, 1) =& 0 \tag{6}\\
w_0 - 1 =& 0 \tag{7}\\
w_0 =& 1 \tag{8}
\end{align}
$$

Therefore $w = (1, -1, 1)$ is the final weight vector for the classes in Figure 3.

b. Determine coefficients W and V in the 2-layer Perceptron of the form in Figure 4 to recogni
HINT: The shaded region in Figure 5 equals the intersection of the regions of Figures 2 & 3.

Because Class 1 in Figure 5 is simply the intersection of Class 1 from Figures 2 & 3, we are essentially implementing the "and" function and can therefore re-use the weight vectors we calculated in part a (called w_1 and w_2).

$$
W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}
$$

Each point x is multiplied by the weight vectors/matrix and then evaluated by the step funciton to produce the values of the hidden nodes, z_1 and z_2. These z values will either be 0 or 1. z_1 will be 1 if the point x is in Class 1 of Figure 2; z_2 will be 1 if the point x is in Class 1 of Figure 3. Therefore points in Class 1 of Figure 5 (which is the intersection of Figures 2 & 3) will have both $z_1 = 1$ and $z_2 = 1$. We must choose V such that $(1, 1)$ is assigned to Class 1, but all other possible z points ($(1, 0)$, $(0, 0)$, and $(0, 1)$) are not.

Because the hidden layer weights vector V has a biased term, the variables have the form $z = (1, z_1, z_2)$ and $V = (V_0, V_1, V_2)$. Therefore we must choose the weight vector V such that $z \cdot V = 1$ at $(1, 1)$ and 0 at all other possible points. Technically any vector pointing into the first quadrant would work, but I will chose $(1, 1)$ to be the direction of V (without the bias term yet). Now I must choose the bias V_0 such that:

$$
\begin{align}
(1,1,1) \cdot (V_0, 1, 1) > 0 &\implies V_0 + 2 > 0 \tag{9}\\
(1,0,1) \cdot (V_0, 1, 1) < 0 &\implies V_0 + 1 < 0 \tag{10}\\
(1,1,0) \cdot (V_0, 1, 1) < 0 &\implies V_0 + 1 < 0 \tag{11}\\
(1,0,0) \cdot (V_0, 1, 1) < 0 &\implies V_0 < 0 \tag{12}\\
& \tag{13}
\end{align}
$$

These conditions are satisfied by any V_0 between -2 and -1, but I will choose $V_0 = -1.5$. Therefore the final weight vectors for Figure 5 are:

$$
W = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} V = \begin{bmatrix} -1.5 & 1 & 1 \end{bmatrix}
$$

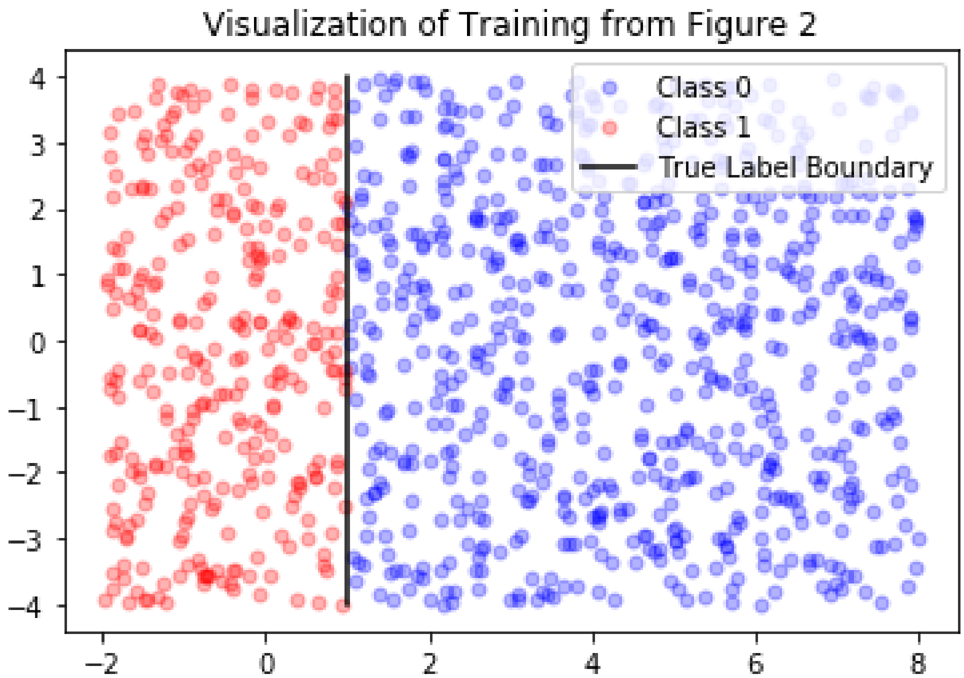
Here are some visualizations of my results:

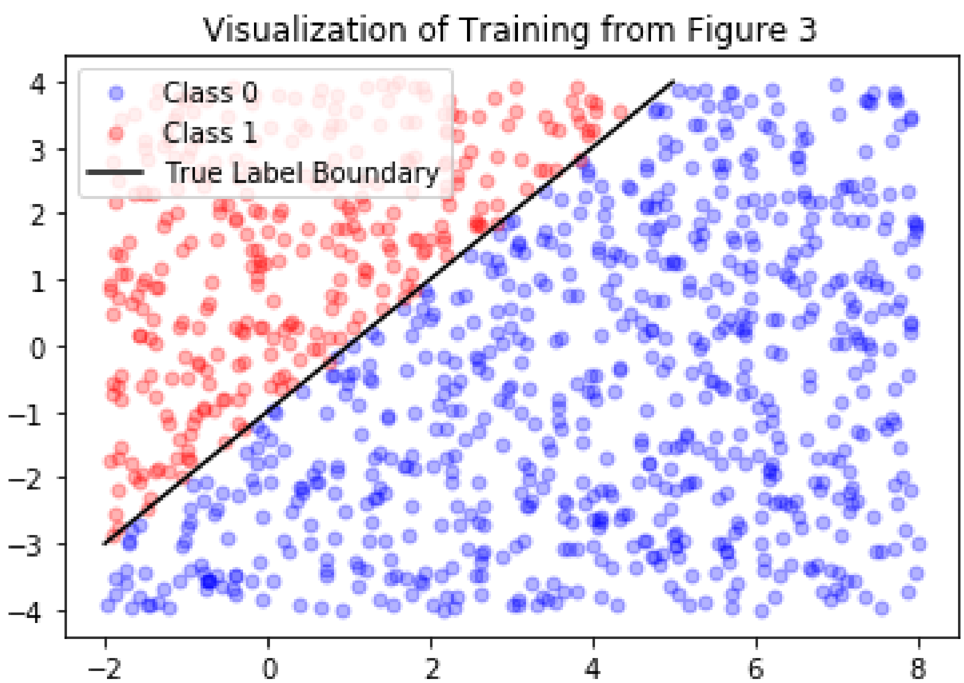
[38]:
```
import numpy as np
import matplotlib.pyplot as plt
```

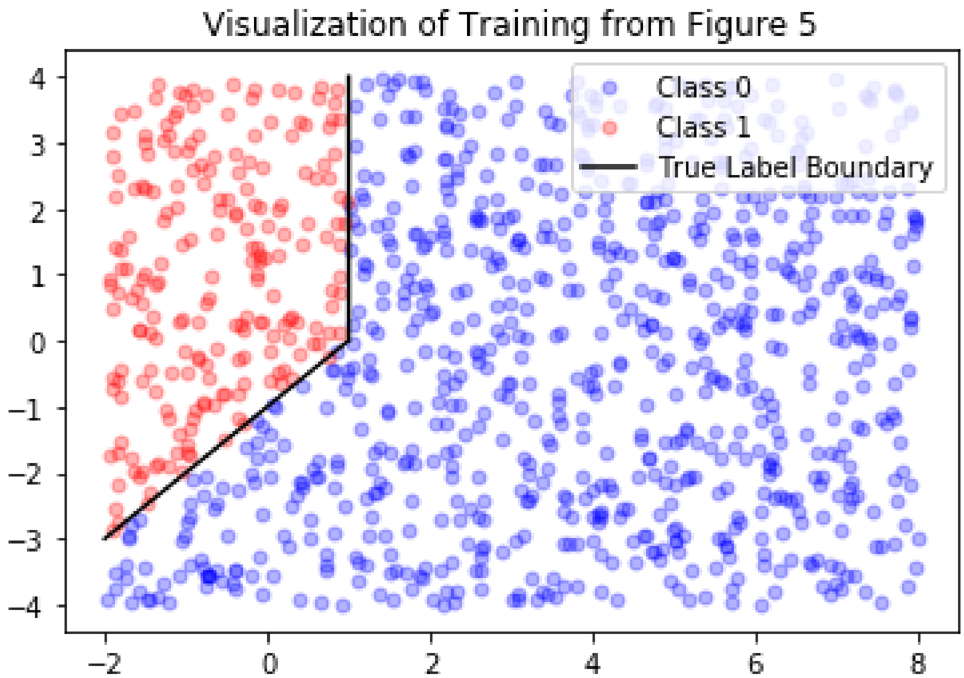
```
def step(x):
    return np.round((np.sign(x)+1)/2.)

w1 = np.array([1,-1,0])
w2 = np.array([1,-1,1])
v = np.array([-1.5,1,1])

n = 1000
X = np.empty([n,3])
Z = np.empty([n,3])
for i in range(n):
    X[i] = np.array([1, np.random.random()*10-2, np.random.random()*8-4])
    Z[i] = [1, step(X[i].dot(w1)), step(X[i].dot(w2))]
```

[43]:

```
plt.plot(X[step(X.dot(w1))==0][:,1], X[step(X.dot(w1))==0][:,2], 'o',␣
 ↪markersize = 5, color = "blue", alpha = .3, label = "Class 0")
plt.plot(X[step(X.dot(w1))==1][:,1], X[step(X.dot(w1))==1][:,2], 'o',␣
 ↪markersize = 5, color = "red", alpha = .3, label = "Class 1")
plt.plot(np.ones(10000), np.arange(-4,4,8/10000.), color = "black", label =␣
 ↪"True Label Boundary")
plt.title("Visualization of Training from Figure 2")
plt.legend()
plt.show()

plt.plot(X[step(X.dot(w2))==0][:,1], X[step(X.dot(w2))==0][:,2], 'o',␣
 ↪markersize = 5, color = "blue", alpha = .3, label = "Class 0")
plt.plot(X[step(X.dot(w2))==1][:,1], X[step(X.dot(w2))==1][:,2], 'o',␣
 ↪markersize = 5, color = "red", alpha = .3, label = "Class 1")
plt.plot(np.arange(-2,5,.01), np.arange(-2,5,.01)-1, color = "black", label =␣
 ↪"True Label Boundary")
plt.title("Visualization of Training from Figure 3")
plt.legend()
plt.show()

plt.plot(X[step(Z.dot(v))==0][:,1], X[step(Z.dot(v))==0][:,2], 'o', markersize␣
 ↪= 5, color = "blue", alpha = .3, label = "Class 0")
plt.plot(X[step(Z.dot(v))==1][:,1], X[step(Z.dot(v))==1][:,2], 'o', markersize␣
 ↪= 5, color = "red", alpha = .3, label = "Class 1")
plt.plot(np.arange(-2,1,.01), np.arange(-2,1,.01)-1, color = "black", label =␣
 ↪"True Label Boundary")
plt.plot(np.ones(10000), np.arange(0,4,4/10000.), color = "black")
plt.title("Visualization of Training from Figure 5")
plt.legend()
plt.show()
```


Visualization of Training from Figure 2
Class 0
Class 1
True Label Boundary
4
3
2
1
0
−1
−2
−3
−4
−2
0
2
4
6
8


Visualization of Training from Figure 3
Class 0
Class 1
True Label Boundary
4
3
2
1
0
−1
−2
−3
−4
−2
0
2
4
6
8



Visualization of Training from Figure 5
Class 0
Class 1
True Label Boundary
4
3
2
1
0
−1
−2
−3
−4
−2
0
2
4
6
8

[ ]: