Exercícios resolvidos 2:

160 $C = A + B = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 1 \\ 6 & -1 & 4 \\ 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 1 \\ 5 & -1 & 5 \\ 2 & 5 & 10 \end{bmatrix}$

161 $B + C = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \\ 6 & -5 \end{bmatrix}$

$$\begin{array}{ccc|cc|l} & & & 1 & 2 & B \\ & & & -1 & 3 & \\ & & & 5 & -2 & \\ \hline 1 & -4 & 2 & 15 & -14 & \\ -1 & 4 & -2 & -15 & 14 & \\ \hline A & & & & & \end{array}$$

$\underset{2\times 3}{A} \times \underset{3\times 2}{B}$ prod. definido

matriz produto 2×2

$AB = \begin{bmatrix} 15 & -14 \\ -15 & 14 \end{bmatrix}$

$$\begin{array}{cc|ccc|l} & & 1 & -4 & 2 & A \\ & & -1 & 4 & -2 & \\ \hline 1 & 2 & -1 & 4 & -2 & \\ -1 & 3 & -4 & 16 & -8 & \\ 5 & -2 & 7 & -28 & 14 & \\ \hline B & & & & & \end{array}$$

$\underset{3\times 2}{B} \times \underset{2\times 3}{A}$ prod. definido.

matriz produto 3×3

$BA = \begin{bmatrix} -1 & 4 & -2 \\ -4 & 16 & -8 \\ 7 & -28 & 14 \end{bmatrix}$

$$\begin{array}{ccc|cc|l} & & & 2 & 2 & C \\ & & & 1 & -1 & \\ & & & 1 & -3 & \\ \hline 1 & -4 & 2 & 0 & 0 & \\ -1 & 4 & -2 & 0 & 0 & \\ \hline A & & & & & \end{array}$$

$\underset{2\times 3}{A} \times \underset{3\times 2}{C}$ prod. definido

matriz produto 2×2

$AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{array}{cc|ccc|l} & & 1 & -4 & 2 & A \\ & & -1 & 4 & -2 & \\ \hline 2 & 2 & 0 & 0 & 0 & \\ 1 & -1 & 2 & -8 & 4 & \\ 1 & -3 & 4 & -16 & 8 & \\ \hline C & & & & & \end{array}$$

$\underset{3\times 2}{C} \times \underset{2\times 3}{A}$ prod. definido

matriz produto 3×3

$CA = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -8 & 4 \\ 4 & -16 & 8 \end{bmatrix}$

$2B = 2\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \\ 10 & -4 \end{bmatrix}$ $\quad 3C = 3\begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & -3 \\ 3 & -9 \end{bmatrix}$

$2B - 3C = \begin{bmatrix} 2 & 4 \\ -2 & 6 \\ 10 & -4 \end{bmatrix} - \begin{bmatrix} 6 & 6 \\ 3 & -3 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -5 & 9 \\ 7 & 5 \end{bmatrix}$

$$\begin{array}{ccc|cc|l} & & & -4 & -2 & 2B-3C \\ & & & -5 & 9 & \\ & & & 7 & 5 & \\ \hline 1 & -4 & 2 & 30 & -28 & \\ -1 & 4 & -2 & -30 & 28 & \\ \hline A & & & & & \end{array}$$

$\underbrace{A}_{2\times 3}\,(\underbrace{\underbrace{2B}_{3\times 2} - \underbrace{3C}_{3\times 2}}_{3\times 2})$

pode-se agrupar

$A(2B - 3C) = \begin{bmatrix} 30 & -28 \\ -30 & 28 \end{bmatrix}$

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$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 1 & -1 & 5 \end{bmatrix}$$

$$\begin{array}{c|c} & \begin{matrix} 1 & 3 & 2 & 4 \\ 4 & 1 & -1 & 5 \end{matrix}\ B \\ \hline A\ \begin{matrix} 1 & 2 \\ -1 & 3 \end{matrix} & \begin{matrix} 9 & 5 & 0 & 14 \\ 11 & 0 & -5 & 11 \end{matrix} \end{array}$$

$$\underbrace{A \times B}_{\substack{2\times 2 \quad 2\times 4 \\ \text{prod. definido}}}$$

matriz produto 2×4

$$AB = \begin{bmatrix} 9 & 5 & 0 & 14 \\ 11 & 0 & -5 & 11 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 6 \end{bmatrix}$$

$$\begin{array}{c|c} & \begin{matrix} 3 \\ -2 \\ 2 \\ 6 \end{matrix}\ C \\ \hline \begin{matrix} 9 & 5 & 0 & 14 \\ 11 & 0 & -5 & 11 \end{matrix} & \begin{matrix} 101 \\ 89 \end{matrix} \\ AB & \end{array}$$

$$\underbrace{(AB)C}_{\substack{2\times 4 \quad 4\times 1 \\ \text{prod. definido}}}$$

matriz produto 2×1

$$(AB)C = \begin{bmatrix} 101 \\ 89 \end{bmatrix}$$

$$\begin{array}{c|c} & \begin{matrix} 3 \\ -2 \\ 2 \\ 6 \end{matrix}\ C \\ \hline \begin{matrix} 5 & 3 & -1 & 2 \end{matrix} & 19 \\ D & \end{array}$$

$$\underbrace{D \times C}_{\substack{1\times 4 \quad 4\times 1 \\ \text{prod. definido}}}$$

matriz produto 1×1

$$DC = \begin{bmatrix} 19 \end{bmatrix}$$

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a)

$$\begin{array}{c|c} & \begin{matrix} b_{11} & b_{12} \\ 0 & 0 \end{matrix}\ B \\ \hline A\ \begin{matrix} 0 & 1 \\ 0 & 2 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix} \quad (b_{11}, b_{12} \in \mathbb{R})$$

b)

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 \\ 0 & 2 \end{matrix} \\ \hline \begin{matrix} -2b_{12} & b_{12} \\ -2b_{22} & b_{22} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}$$

$$B = \begin{bmatrix} -2b_{12} & b_{12} \\ -2b_{22} & b_{22} \end{bmatrix} \quad (b_{12}, b_{22} \in \mathbb{R})$$

164 a)

$$\begin{array}{c|c} & \begin{matrix} 3 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 3 & 2 \end{matrix}\ B \\ \hline \begin{matrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 3 & 1 \end{matrix} & \begin{matrix} 8 & 1 & 6 \\ 0 & 5 & 9 \\ 8 & 6 & 15 \end{matrix} \\ A & \end{array}$$

$$AB = \begin{bmatrix} 8 & 1 & 6 \\ 0 & 5 & 9 \\ 8 & 6 & 15 \end{bmatrix} \qquad (AB)^T = \begin{bmatrix} 8 & 0 & 8 \\ 1 & 5 & 6 \\ 6 & 9 & 15 \end{bmatrix}$$

b)

$$A^T = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

R: $\boxed{(AB)^T = B^T A^T}$

$$\begin{array}{c|c} & \begin{matrix} 3 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 2 \end{matrix}\ B^T \\ \hline \begin{matrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{matrix} & \begin{matrix} 7 & 7 & -3 \\ 6 & 16 & 13 \\ 1 & 5 & 5 \end{matrix} \\ A^T & \end{array}$$

$$A^T B^T = \begin{bmatrix} 7 & 7 & -3 \\ 6 & 16 & 13 \\ 1 & 5 & 5 \end{bmatrix}$$

$$\begin{array}{c|c} & \begin{matrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{matrix}\ A^T \\ \hline \begin{matrix} 3 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 2 \end{matrix} & \begin{matrix} 8 & 0 & 8 \\ 1 & 5 & 6 \\ 6 & 9 & 15 \end{matrix} \\ B^T & \end{array}$$

$$B^T A^T = \begin{bmatrix} 8 & 0 & 8 \\ 1 & 5 & 6 \\ 6 & 9 & 15 \end{bmatrix}$$

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a)

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mm}\end{bmatrix}\quad B=\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1m}\\ b_{21} & b_{22} & \cdots & b_{2m}\\ \vdots & \vdots & & \vdots\\ b_{m1} & b_{m2} & \cdots & b_{mm}\end{bmatrix}\quad A+B=\begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m}\\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m}\\ \vdots & \vdots & & \vdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mm}+b_{mm}\end{bmatrix}$$

$$(A+B)^T=\begin{bmatrix}a_{11}+b_{11} & a_{21}+b_{21} & a_{m1}+b_{m1}\\ a_{12}+b_{12} & a_{22}+b_{22} & a_{m2}+b_{m2}\\ \cdots & \cdots & \cdots\\ a_{1m}+b_{1m} & a_{2m}+b_{2m} & a_{mm}+b_{mm}\end{bmatrix}\quad A^T=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{22} & \cdots & a_{m2}\\ a_{1m} & a_{2m} & \cdots & a_{mm}\end{bmatrix}$$

$$B^T=\begin{bmatrix}b_{11} & b_{21} & \cdots & b_{m1}\\ b_{12} & b_{22} & \cdots & b_{m2}\\ \vdots & \vdots & & \vdots\\ b_{1m} & b_{2m} & \cdots & b_{mm}\end{bmatrix}\quad A^T+B^T=\begin{bmatrix}a_{11}+b_{11} & a_{21}+b_{21} & \cdots & a_{m1}+b_{m1}\\ a_{12}+b_{12} & a_{22}+b_{22} & \cdots & a_{m2}+b_{m2}\\ \vdots & \vdots & & \vdots\\ a_{1m}+b_{1m} & a_{2m}+b_{2m} & \cdots & a_{mm}+b_{mm}\end{bmatrix}\quad \boxed{(A+B)^T=A^T+B^T}$$

b)

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mm}\end{bmatrix}\quad cA=\begin{bmatrix}ca_{11} & ca_{12} & \cdots & ca_{1m}\\ ca_{21} & ca_{22} & \cdots & ca_{2m}\\ ca_{m1} & ca_{m2} & \cdots & ca_{mm}\end{bmatrix}\quad (cA)^T=\begin{bmatrix}ca_{11} & ca_{21} & \cdots & ca_{m1}\\ ca_{12} & ca_{22} & \cdots & ca_{m2}\\ \vdots & \vdots & & \vdots\\ ca_{1m} & ca_{2m} & \cdots & ca_{mm}\end{bmatrix}$$

$$A^T=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{22} & \cdots & a_{m2}\\ \vdots & \vdots & & \vdots\\ a_{1m} & a_{2m} & \cdots & a_{mm}\end{bmatrix}\quad cA^T=\begin{bmatrix}ca_{11} & ca_{21} & \cdots & ca_{m1}\\ ca_{12} & ca_{22} & \cdots & ca_{m2}\\ \vdots & \vdots & & \vdots\\ ca_{1m} & ca_{2m} & \cdots & ca_{mm}\end{bmatrix}\quad \boxed{(cA)^T=cA^T}$$

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a)

$$\begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}c\\ a\\ b\\ d\end{bmatrix}\qquad \begin{bmatrix}c\\ a\\ b\\ d\end{bmatrix}=\begin{bmatrix}1\\ 9\\ 6\\ 5\end{bmatrix}$$

R: $a=9 \wedge b=6 \wedge c=1 \wedge d=5$

b)

a	b	c	d	a	c	2a+b+d	b
1	4	9	2	1	9	8	4
				1	0	2	0
				0	0	1	1
				0	1	0	0
				0	0	1	0

$$\begin{bmatrix}a & c & 2a+b+d & b\\ 1 & 9 & 8 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 6 & 6\\ 1 & 9 & 8 & 4\end{bmatrix}$$

$a=1 \quad c=0 \quad 2a+b+d=6 \quad b=6$

$2(1)+(6)+d=6 \Leftrightarrow d=6-8 \Leftrightarrow d=-2$

R: $a=1 \wedge b=6 \wedge c=0 \wedge d=-2$

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$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

a	b	0	0
c	d	0	0
		a	b
		c	d

$a^2+bc=0 \Leftrightarrow a^2=-bc$

$ab+bd=0$

$ac+cd=0$

$bc+d^2=0 \Leftrightarrow d^2=-bc$

Logo, $a^2=d^2 \Rightarrow a=d \vee a=-d$

$b(a+d)=0 \Leftrightarrow b=0 \vee a=-d \qquad c(a+d)=0 \Leftrightarrow c=0 \vee a=-d$

R: $A=\begin{bmatrix}-d & b\\ c & d\end{bmatrix}$, e $d^2=-bc$; $b,c,d \in \mathbb{R}$

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$$\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

$$2c_1 - c_3 = 7 \Leftrightarrow c_3 = 2c_1 - 7$$

$$2c_2 - c_4 = 6 \Leftrightarrow c_4 = 2c_2 - 6$$

$$-2c_1 + 3c_3 = 9 \Leftrightarrow -2c_1 + 3(2c_1 - 7) = 9 \Leftrightarrow$$

$$\Leftrightarrow 4c_1 - 21 = 9 \Leftrightarrow 4c_1 = 30 \Leftrightarrow c_1 = \frac{15}{2}$$

$$-2c_2 + 3c_4 = 8 \Leftrightarrow -2c_2 + 3(2c_2 - 6) = 8 \Leftrightarrow$$

$$\Leftrightarrow 4c_2 = 26 \Leftrightarrow c_2 = \frac{13}{2} \qquad C = \begin{bmatrix} \frac{15}{2} & \frac{13}{2} \\ 8 & 7 \end{bmatrix}$$

$$c_3 = 2\left(\frac{15}{2}\right) - 7 = 8$$

$$c_4 = 2\left(\frac{13}{2}\right) - 6 = 7$$

$$\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

$$2d_1 - 2d_2 = 7 \Leftrightarrow 2d_1 = 2d_2 + 7$$

$$-d_1 + 3d_2 = 6 \Leftrightarrow d_1 = 3d_2 - 6$$

$$2(3d_2 - 6) = 2d_2 + 7 \Leftrightarrow 4d_2 = 19 \Leftrightarrow d_2 = \frac{19}{4}$$

$$d_1 = 3\left(\frac{19}{4}\right) - 6 \Leftrightarrow d_1 = \frac{57}{4} - 6 = \frac{33}{4}$$

$$2d_3 - 2d_4 = 9 \Leftrightarrow 2d_3 = 2d_4 + 9$$

$$-d_3 + 3d_4 = 8 \Leftrightarrow d_3 = 3d_4 - 8$$

$$2(3d_4 - 8) = 2d_4 + 9 \Leftrightarrow 4d_4 = 25 \Leftrightarrow d_4 = \frac{25}{4}$$

$$d_3 = 3\left(\frac{25}{4}\right) - 8 = \frac{75}{4} - 8 = \frac{43}{4}$$

$$D = \begin{bmatrix} \frac{33}{4} & \frac{19}{4} \\ \frac{43}{4} & \frac{25}{4} \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad A^4 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A^2 = 2A - I$$

$$2A = 2\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} \qquad 2A - I = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$A^3 = 3A - 2I$$

$$3A = 3\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 3 \end{bmatrix} \qquad 3A - 2I = \begin{bmatrix} 3 & 0 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

R: $A^n = nA - (n-1)I = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$ $(n \geq 0)$

(171)

$$\begin{array}{ccc|ccc} & & & a & b & c \\ & & & d & e & f \\ & & & g & h & i \\ 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 3 & 0 & 2 & 3a+2g & 3b+2h & 3c+2i \end{array}$$

A, B

$$\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 3 & 0 & 2 \\ a & b & c & a+3c & b & 2c \\ d & e & f & d+3f & e & 2f \\ g & h & i & g+3i & h & 2i \end{array}$$

B, A

Matrizes que se comutam $\longrightarrow AB = BA$.

$a = a+3c \Leftrightarrow 3c = 0 \Leftrightarrow c = 0$ $\quad d = d+3f \Leftrightarrow 3f = 0 \Leftrightarrow f = 0$

$3a+2g = g+3i \Leftrightarrow g = 3i-3a$ $\quad 3b+2h = h \Leftrightarrow h = -3b$

R: $B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{bmatrix} \wedge e = 3g - 3a \wedge f = -3b \wedge a,b,c,d,e,f,g \in \mathbb{R}$.

(172) a) $A^T = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 2 \\ 3 & 8 & 4 \end{bmatrix}$ $\quad 2A - A^T = 2\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 8 \\ 2 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 2 \\ 3 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 7 & 2 & 14 \\ 1 & -4 & 4 \end{bmatrix}$.

b) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 8 \\ 2 & 2 & 4 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longleftrightarrow \left.\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right\}$ 3 linhas

$r(A) = 3$ Como $r(A)$ é igual à menor dimensão da matriz (3×3), conclui-se que A é regular (não singular), isto é, admite inversa.

(173) Sabe-se que $A^n = nA - (n-1)I \Rightarrow A^m = mA - (m-1)I$

$A^nA^m = (nA - (n-1)I)(mA - (m-1)I) = (nA - nI + I)(mA - mI + I) =$

$= nAmA - nAmI + nAI - mAnI + mnII - nII + mAI - mII + II =$

$= -2mnAI + mnA^2 + nA + mA + mnI - nI - mI + I =$

$= nA + mA - nI - mI - 2mnA + mnI + I + mn(2A - (2-1)I) =$

$= nA + mA - nI - mI + I + mnI - 2mnA + 2mnA - Imn =$

$= nA + mA - nI - mI + I$

$A^{n+m} = (n+m)A - (n+m-1)I = nA + mA - nI - mI + I$.

Logo, $\boxed{A^nA^m = A^{m+n}}$

174) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -1 \\ 1 & -1 & 9 & -9 \\ 2 & 6 & 5 & 12 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & -3 & 6 & -13 \\ 0 & 2 & -1 & 4 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \longleftrightarrow$

$\longleftrightarrow \left.\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right\}$ 4 linhas $\longrightarrow r(A) = 4$

175) a) A tem dimensão $3\times 4 \rightarrow r(A) \leq 3$

$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 4 & 3 & 4 \\ 1 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow$

$\longleftrightarrow \left.\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right\}$ 3 linhas $\quad r(A) = 3$

b) $(3B)^T = \left(3\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & 3 & 2 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 3 & 9 & -3 & 0 \\ -3 & 9 & 6 & 3 \end{bmatrix}^T = \begin{bmatrix} 6 & 3 & -3 \\ 3 & 9 & 9 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

$A/2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 4 & 3 & 4 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & 1 & 1 \\ 2 & \frac{3}{2} & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad C = (3B)^T + A/2 = \begin{bmatrix} 6 & 3 & -3 \\ 3 & 9 & 9 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & 1 & 1 \\ 2 & \frac{3}{2} & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} =$

$= \begin{bmatrix} \frac{13}{2} & \frac{7}{2} & -\frac{5}{2} \\ \frac{9}{2} & 10 & 10 \\ 2 & -\frac{3}{2} & 8 \\ \frac{1}{2} & 0 & \frac{7}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 13 & 7 & -5 \\ 9 & 20 & 20 \\ 4 & -3 & 16 \\ 1 & 0 & 7 \end{bmatrix}$

A matriz quadrada de ordem n, A:

- A é singular $\Leftrightarrow$ não é regular $\Leftrightarrow$ não tem inversa $\Leftrightarrow |A| = 0 \Leftrightarrow$ $\Leftrightarrow r(A) < n$
- A é regular $\Leftrightarrow$ não é singular $\Leftrightarrow$ tem inversa $\Leftrightarrow |A| \neq 0 \Leftrightarrow$ $\Leftrightarrow r(A) = n$

176 a) Para provar que A tem inversa, podemos mostrar que $r(A) = 3$ (maior ordem), ou que $|A| \neq 0$.

1 - Calcular a característica da matriz:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & 0 & 2 \end{bmatrix} \longleftrightarrow \left.\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -4 \end{bmatrix}\right\} 3 \text{ linhas} \quad r(A) = 3$$

(tem inversa)

2 - Calcular o determinante da matriz:

- pelo método de Sarrus (só para matrizes 3×3)

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 1 & 0 & 2 \\ 2 & 2 & 2 \end{matrix} = 4 + 0 + 0 - 12 - 0 - 0 = 4 - 12 = -8$$

$|A| = -8$ $\quad |A| \neq 0 \Rightarrow A$ tem inversa.

- pelo método da condensação:

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & 0 & 2 \end{vmatrix} \longleftrightarrow \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -4 \end{vmatrix} = 1 \times 2 \times (-4) = -8 \quad |A| = -8$$

$|A| \neq 0 \Rightarrow A$ tem inversa.

- pelo método dos cofatores, por ex. segundo a coluna 2 (método de Laplace)

$$|A| = 2 \times A_{22} = 2 \times (-1)^{\overset{2+2}{4}} \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2(2-6) = -8$$

$|A| = -8$

$|A| \neq 0 \Rightarrow A$ tem inversa.

Agora, podemos calcular a inversa da matriz:

1 - Pelo método de Gauss - Jordan:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & -2 & -2 & 1 & 0 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{array}\right] \longleftrightarrow$$

$$\longleftrightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -1 & 0 & 1 \\ 0 & 4 & 0 & -1 & 2 & -1 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{2}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

$\downarrow$ I

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 0 \\ -1 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

Verificação: $A^{-1} . A = I$

$$\begin{array}{ccc|ccc} & & & 1 & 0 & 2 \\ & & & 2 & 2 & 2 \\ & & & 3 & 0 & 2 \\ \hline -2 & 0 & 2 & 4 & 0 & 0 \\ -1 & 2 & -1 & 0 & 4 & 0 \\ 3 & 0 & 1 & 0 & 0 & 4 \end{array} /4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

✓

2 - Pela matriz adjunta ou matriz dos cofatores

$$A^{-1}=\frac{1}{|A|}\left[\text{Adj } A\right]^T = \frac{1}{-8}\begin{bmatrix} \begin{vmatrix}0&2\\0&2\end{vmatrix} & -\begin{vmatrix}2&2\\3&2\end{vmatrix} & \begin{vmatrix}2&2\\3&0\end{vmatrix} \\ -\begin{vmatrix}0&2\\0&2\end{vmatrix} & \begin{vmatrix}1&2\\3&2\end{vmatrix} & -\begin{vmatrix}1&0\\3&0\end{vmatrix} \\ \begin{vmatrix}0&2\\2&2\end{vmatrix} & -\begin{vmatrix}1&2\\2&2\end{vmatrix} & \begin{vmatrix}1&0\\2&2\end{vmatrix}\end{bmatrix}^T = -\frac{1}{8}\begin{bmatrix}4&2&-6\\0&-4&0\\-4&2&2\end{bmatrix}^T =$$

$$= -\frac{1}{8}\begin{bmatrix}4&0&-4\\2&-4&2\\-6&0&2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}&0&\frac{1}{2}\\-\frac{1}{4}&\frac{1}{2}&-\frac{1}{4}\\\frac{3}{4}&0&-\frac{1}{4}\end{bmatrix} \qquad A^{-1}=\frac{1}{4}\begin{bmatrix}-2&0&2\\-1&2&-1\\3&0&-1\end{bmatrix}$$

b) $$\begin{vmatrix}1&1&1\\3&1&2\\0&3&3\end{vmatrix} = 1\times(-1)^2\times\begin{vmatrix}1&2\\3&3\end{vmatrix} + 3\times(-1)^3\times\begin{vmatrix}1&1\\3&3\end{vmatrix} = -3+0=-3$$

$|B|=-3 \qquad |B|\neq 0 \Longrightarrow$ B tem inversa

$$B^{-1}=\frac{1}{|B|}\left[\text{Adj } B\right]^T = -\frac{1}{3}\begin{bmatrix}-3&-9&9\\0&3&-3\\1&1&-2\end{bmatrix}^T = -\frac{1}{3}\begin{bmatrix}-3&0&1\\-9&3&1\\9&-3&-2\end{bmatrix} = \begin{bmatrix}1&0&-\frac{1}{3}\\3&-1&-\frac{1}{3}\\-3&1&\frac{2}{3}\end{bmatrix}$$

177 a) $|A| = \begin{vmatrix}2&1\\7&3\end{vmatrix} = 2\times3-7\times1 = 6-7=-1 \qquad |A|=-1$

$$\begin{bmatrix}1&3&-1\\4&2&1\\3&1&-2\end{bmatrix} \longleftrightarrow \begin{bmatrix}1&3&-1\\0&-10&5\\0&-8&1\end{bmatrix} \longleftrightarrow 5\begin{bmatrix}1&3&-1\\0&-2&1\\0&-8&1\end{bmatrix} \longleftrightarrow 5\begin{bmatrix}1&3&-1\\0&-2&1\\0&0&-3\end{bmatrix} \quad |B| = 5\times\begin{vmatrix}1&3&-1\\0&-2&1\\0&0&-3\end{vmatrix} =$$

$$=5\times6=30$$

b) $$B^T=\begin{bmatrix}1&3&-1\\4&2&1\\3&1&-2\end{bmatrix}^T = \begin{bmatrix}1&4&3\\3&2&1\\-1&1&-2\end{bmatrix}$$

$$\begin{matrix}1&4&3\\3&2&1\end{matrix}$$

$|B^T| = -4+9-4+6-1+24=30$

$|B^T|=30=|B|$

c) $$|B| = \begin{vmatrix}1&3&-1\\4&2&1\\3&1&-2\end{vmatrix} = -4-4+9+6-1+24=30$$

$$\begin{matrix}1&3&-1\\4&2&1\end{matrix}$$

$|B|=30$

(178) $A.B = B.A \Leftrightarrow (A.B).A^{-1} = A^{-1}.(B.A) \Leftrightarrow (A.A^{-1}).(B.A^{-1}) = (A^{-1}.B).(A^{-1}.A) \Leftrightarrow$
$\Leftrightarrow I.(B.A^{-1}) = (A^{-1}.B).I \Leftrightarrow B.A^{-1} = A^{-1}.B$

Logo, se A e B comutam entre si, então o mesmo sucede com A^{-1} e B.

(179) Para provar que $(A.B)^{-1} = B^{-1}.A^{-1}$, temos que mostrar que:
$(A.B).(B^{-1}.A^{-1}) = I$ e $(B^{-1}.A^{-1}).(A.B) = I$ Prova-se a primeira (a segunda é análoga)
$(A.B).(B^{-1}.A^{-1}) = A.(B.B^{-1}).A^{-1} = (A.A^{-1}).I = I.I = I$ Logo, $\boxed{(A.B)^{-1} = B^{-1}.A^{-1}}$

(180) Suponhamos que A possui duas inversas, A^{-1} e B^{-1}, então:
$A.A^{-1} = I$ e $A.B^{-1} = I$
$A.A^{-1} = A.B^{-1} \Leftrightarrow A^{-1}.(A.A^{-1}) = A^{-1}.(A.B^{-1}) \Leftrightarrow (A^{-1}.A).A^{-1} = (A^{-1}.A).B^{-1} \Leftrightarrow$
$\Leftrightarrow I.A^{-1} = I.B^{-1} \Leftrightarrow A^{-1} = B^{-1}$ Logo, se A é matriz não singular, então a sua inversa é única.

(181) a) $\begin{vmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & -1 \\ 5 & -1 & 3 & 0 \\ -1 & 5 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & -1 & 3 & 0 \\ 2 & -1 & 1 & -1 \\ 5 & -1 & 3 & 0 \\ -1 & 5 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & -1 & 3 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 5 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix}$

Como a matriz possui uma linha nula, então, segundo as propriedades dos determinantes, $|A| = 0$.

b) $\begin{vmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{vmatrix} = \begin{vmatrix} 12 & -12 & -12 & 12 \\ 5 & 10 & 11 & 8 \\ 4 & -4 & -4 & 4 \\ 4 & 15 & 14 & 1 \end{vmatrix} = \begin{vmatrix} 12 & -12 & -12 & 12 \\ 5 & 10 & 11 & 8 \\ 0 & 0 & 0 & 0 \\ 4 & 15 & 14 & 1 \end{vmatrix} =$

Como a matriz possui uma linha nula, então, segundo as propriedades dos determinantes, $|A| = 0$.

$|A| = |A^T|$ c) $\begin{vmatrix} 1 & 6 & -5 & 3 & 5 \\ 7 & 3 & -8 & 2 & 4 \\ -1 & 2 & 3 & -1 & 3 \\ 0 & 5 & 2 & 1 & 8 \\ 2 & -3 & 6 & -5 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 6 & -5 & 5 & 5 \\ 7 & 3 & -8 & 4 & 4 \\ -1 & 2 & 3 & 3 & 3 \\ 0 & 5 & 2 & 8 & 8 \\ 2 & 3 & 6 & 0 & 0 \end{vmatrix}$

Como a matriz possui duas colunas iguais, $|A| = 0$.

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$(|A| = |A^T|)$

$$Q = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 0 & 8 & 2 \\ 3 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 3 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 8 & 0 \\ 3 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 3 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{vmatrix}} = \frac{2\begin{vmatrix} 1 & 0 & 3 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 3 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{vmatrix}} = 2$$

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$$S = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 2 \\ -1 & 0 & -2 \end{vmatrix} = 0$$

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$$|A| = \begin{vmatrix} 3 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 3 & -2 & 1 & 3 \\ 1 & 1 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & 2 \\ 3 & -2 & 1 & 3 \\ 3 & 2 & 4 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -5 & -5 & -3 \\ 0 & -1 & -2 & -3 \end{vmatrix} =$$

$$= -\begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 7 \end{vmatrix} = 1 \times (-1) \times (-1) \times 7 = 7$$

→ método da condensação

185 a)

$$Q = \frac{|A|}{|B|} = \frac{\begin{vmatrix} 1 & 5 & a & 0 & f \\ 2 & 6 & b & 0 & g \\ 3 & 7 & c & 2 & h \\ 4 & 8 & d & 3 & i \\ 5 & 9 & e & 1 & j \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 2 & 5 & 4 \\ 5 & 7 & 6 & 9 & 8 \\ 0 & 2 & 0 & 1 & 3 \\ a & c & b & e & d \\ f & h & g & j & i \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ a & b & c & d & e \\ 0 & 0 & 2 & 3 & 1 \\ f & g & h & i & j \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 2 & 3 & 1 \\ a & b & c & d & e \\ f & g & h & i & j \end{vmatrix}} =$$

$$= \frac{\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ a & b & c & d & e \\ 0 & 0 & 2 & 3 & 1 \\ f & g & h & i & j \end{vmatrix}}{-\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ a & b & c & d & e \\ 0 & 0 & 2 & 3 & 1 \\ f & g & h & i & j \end{vmatrix}} = \frac{1}{-1} = -1$$

R: $Q = \frac{|A|}{|B|} = -1$

b) $Q=\dfrac{|A|}{|B|}=\dfrac{\begin{vmatrix}5&6&1&0&0\\2&1&-1&0&1\\3&1&a&2&1\\4&1&-1&2&3\\a&1&2&3&0\end{vmatrix}}{\begin{vmatrix}1&2&-1&1&0\\6&5&1&0&0\\1&3&a&1&2\\1&a&2&0&3\\1&4&-1&3&2\end{vmatrix}}=\dfrac{\begin{vmatrix}5&6&1&0&0\\2&1&-1&0&1\\3&1&a&2&1\\4&1&-1&2&3\\a&1&2&3&0\end{vmatrix}}{-\begin{vmatrix}6&5&1&0&0\\1&2&-1&1&0\\1&3&a&1&2\\1&a&2&0&3\\1&4&-1&3&2\end{vmatrix}}=$

$=\dfrac{\begin{vmatrix}5&6&1&0&0\\2&1&-1&0&1\\3&1&a&2&1\\4&1&-1&2&3\\a&1&2&3&0\end{vmatrix}}{\begin{vmatrix}5&6&1&0&0\\2&1&-1&1&0\\3&1&a&1&2\\a&1&2&0&3\\4&1&-1&3&2\end{vmatrix}}=\dfrac{\begin{vmatrix}5&6&1&0&0\\2&1&-1&0&1\\3&1&a&2&1\\4&1&-1&2&3\\a&1&2&3&0\end{vmatrix}}{-\begin{vmatrix}5&6&1&0&0\\2&1&-1&1&0\\3&1&a&1&2\\4&1&-1&3&2\\a&1&2&0&3\end{vmatrix}}=\dfrac{\begin{vmatrix}5&6&1&0&0\\2&1&-1&0&1\\3&1&a&2&1\\4&1&-1&2&3\\a&1&2&3&0\end{vmatrix}}{\begin{vmatrix}5&6&1&0&0\\2&1&-1&0&1\\3&1&a&2&1\\4&1&-1&2&3\\a&1&2&3&0\end{vmatrix}}=\dfrac{1}{1}=1$

186 $|C|=4\times(-1)^4\times\begin{vmatrix}2&3\\1&1\end{vmatrix}+5\times(-1)^5\times\begin{vmatrix}6&2\\1&1\end{vmatrix}+7\times(-1)^6\times\begin{vmatrix}6&2\\2&3\end{vmatrix}=$
$=4\times(-1)-5\times4+7\times14=-4-20+98=74$ R: $|C|=74$

187 Aplicando o teorema de Laplace à 2ª linha:

$|B|=1\times(-1)^2\times\begin{vmatrix}1&3&2&6&3\\0&10&4&2&2\\0&0&3&1&2\\0&0&2&3&5\\0&0&1&1&7\end{vmatrix}$ ← Aplicando o teorema de Laplace à 1ª coluna

$|B|=1\times1\times1\times1^2\times\begin{vmatrix}10&4&2&2\\0&3&1&2\\0&2&3&5\\0&1&1&7\end{vmatrix}$ ← Aplicando o teorema de Laplace à 1ª coluna

$|B|=10\times(-1)^2\times\begin{vmatrix}3&1&2\\2&3&5\\1&1&7\end{vmatrix}=10\times\dfrac{1}{2}\begin{vmatrix}6&2&4\\2&3&5\\1&1&7\end{vmatrix}=5\times|C|$ c.q.p.

188 $\begin{vmatrix}0&1&3&0\\1&2&-1&3\\0&K&K&K\\2&2&-1&3\end{vmatrix}=1\times(-1)^3\times\begin{vmatrix}1&3&0\\K&K&K\\2&-1&3\end{vmatrix}+2\times(-1)^5\times\begin{vmatrix}1&3&0\\2&-1&3\\K&K&K\end{vmatrix}=$

$=-\left(K\times(-1)^5\times\begin{vmatrix}1&3\\2&-1\end{vmatrix}+3\times(-1)^6\times\begin{vmatrix}1&3\\K&K\end{vmatrix}\right)-2\left(1\times(-1)^2\times\begin{vmatrix}-1&3\\K&K\end{vmatrix}+3\times(-1)^3\times\begin{vmatrix}2&3\\K&K\end{vmatrix}\right)=$
$=-(7K-6K)-2(-4K+3K)=-K+2K=K$

$$\begin{vmatrix} 1 & 0 & K & 2 \\ 2 & 1 & K & 2 \\ -1 & 3 & K & -1 \\ 3 & 0 & 0 & 3 \end{vmatrix} = 3\times(-1)^5\times\begin{vmatrix} 0 & K & 2 \\ 1 & K & 2 \\ 3 & K & -1 \end{vmatrix} + 3\times(-1)^8\times\begin{vmatrix} 1 & 0 & K \\ 2 & 1 & K \\ -1 & 3 & K \end{vmatrix} =$$

$$= -3\left(1\times(-1)^3\times\begin{vmatrix} K & 2 \\ K & -1 \end{vmatrix} + 3\times(-1)^4\times\begin{vmatrix} K & 2 \\ K & 2 \end{vmatrix}\right) + 3\times\left(1\times(-1)^2\times\begin{vmatrix} 1 & K \\ 3 & K \end{vmatrix} + K\times(-1)^4\times\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}\right) =$$

$$= -3(3K) + 3(-2K + 7K) = -9K + 15K = 6K$$

$$|A| + |B| = 1 \Leftrightarrow K + 6K = 1 \Leftrightarrow 7K = 1 \Leftrightarrow K = \frac{1}{7}$$

189 a) $$|B| = \begin{vmatrix} 2x & 2y & 2z \\ \frac{3}{2} & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2x & 2y & 2z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{2}{2}\begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= |A| = 1$$

b) $$|C| = \begin{vmatrix} 2x & 2y & 2z \\ 3x+3 & 3y & 3z+2 \\ x+1 & y+1 & z+1 \end{vmatrix} = 2\begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+1 & y+1 & z+1 \end{vmatrix} = 2\begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= 2|A| = 2\times 1 = 2$$

c) $$|D| = \begin{vmatrix} x-1 & y-1 & z-1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} =$$

$$= |A| = 1$$

190 $$F(x) = \begin{vmatrix} f(x) & g(x) \\ p(x) & q(x) \end{vmatrix} = f(x)q(x) - p(x)g(x)$$

$$F'(x) = f'(x)q(x) + f(x)q'(x) - (p'(x)g(x) + p(x)g'(x)) =$$

$$= f'(x)q(x) + f(x)q'(x) - p'(x)g(x) - p(x)g'(x) =$$

$$= f'(x)q(x) - p(x)g'(x) + f(x)q'(x) - p'(x)g(x) =$$

$$= \begin{vmatrix} f'(x) & g'(x) \\ p(x) & q(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ p'(x) & q'(x) \end{vmatrix}$$

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) \\ p(x) & q(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ p'(x) & q'(x) \end{vmatrix}$$ c.q.p.

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a) $\begin{vmatrix} -4 & 7 & -5 & 3 & 4 & -4 \\ 11 & 5 & 1 & 2 & 1 & 11 \\ 10 & 14 & 6 & -7 & 5 & 8 \\ 6 & 3 & 0 & -1 & 8 & 6 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 18 & 0 & 0 & -1 \end{vmatrix} = 2 \times (-1)^8 \times \begin{vmatrix} -4 & 7 & 3 & 4 & -4 \\ 11 & 5 & 2 & 1 & 11 \\ 10 & 14 & -7 & 5 & 8 \\ 6 & 3 & -1 & 8 & 6 \\ -1 & 0 & 0 & 0 & 1 \end{vmatrix} =$

$= 2 \times \left[(-1) \times (-1)^6 \times \begin{vmatrix} 7 & 3 & 4 & -4 \\ 5 & 2 & 1 & 11 \\ 14 & -7 & 5 & 8 \\ 3 & -1 & 8 & 6 \end{vmatrix} + 1 \times (-1)^{10} \times \begin{vmatrix} -4 & 7 & 3 & 4 \\ 11 & 5 & 2 & 1 \\ 10 & 14 & -7 & 5 \\ 6 & 3 & -1 & 8 \end{vmatrix} \right] =$

$= 2 \times \left[\begin{vmatrix} 14 & -7 & 5 & 8 \\ 5 & 2 & 1 & 11 \\ 7 & 3 & 4 & -4 \\ 3 & -1 & 8 & 6 \end{vmatrix} - \begin{vmatrix} 10 & 14 & -7 & 5 \\ 11 & 5 & 2 & 1 \\ -4 & 7 & 3 & 4 \\ 6 & 3 & -1 & 8 \end{vmatrix} \right] =$

$= 2 \times \left[8 \times (-1)^5 \begin{vmatrix} 5 & 2 & 1 \\ 7 & 3 & 4 \\ 3 & -1 & 8 \end{vmatrix} + 11 \times (-1)^6 \begin{vmatrix} 14 & -7 & 5 \\ 7 & 3 & 4 \\ 3 & -1 & 8 \end{vmatrix} - 4 \times (-1)^7 \begin{vmatrix} 14 & -7 & 5 \\ 5 & 2 & 1 \\ 3 & -1 & 8 \end{vmatrix} + 6 \times (-1)^8 \begin{vmatrix} 14 & -7 & 5 \\ 5 & 2 & 1 \\ 7 & 3 & 4 \end{vmatrix} + \right.$

$\left. + 10 \times (-1)^2 \begin{vmatrix} 5 & 2 & 1 \\ 7 & 3 & 4 \\ 3 & -1 & 8 \end{vmatrix} + 11 \times (-1)^3 \begin{vmatrix} 14 & -7 & 5 \\ 7 & 3 & 4 \\ 3 & -1 & 8 \end{vmatrix} - 4 \times (-1)^4 \begin{vmatrix} 14 & -7 & 5 \\ 5 & 2 & 1 \\ 3 & -1 & 8 \end{vmatrix} + 6 \times (-1) \begin{vmatrix} 14 & -7 & 5 \\ 5 & 2 & 1 \\ 7 & 3 & 4 \end{vmatrix} \right] =$

$= 2 \left[2 \begin{vmatrix} 5 & 2 & 1 \\ 7 & 3 & 4 \\ 3 & -1 & 8 \end{vmatrix} \right] = 4 \begin{vmatrix} 5 & 2 & 1 \\ 7 & 3 & 4 \\ 3 & -1 & 8 \end{vmatrix}$ c.q.p.

b) $\begin{vmatrix} a & b & b & c \\ -c & 0 & b & b \\ c & d & d & d \\ 0 & b & b & c \end{vmatrix} = - \begin{vmatrix} a & b & b & c \\ c & d & d & d \\ -c & 0 & b & b \\ 0 & b & b & c \end{vmatrix} = - \begin{vmatrix} a & b & b & c \\ c & d & d & d \\ -c & 0 & b & b \\ -a & 0 & 0 & 0 \end{vmatrix} =$

$= -\left(-a \times (-1)^5 \times \begin{vmatrix} b & b & c \\ d & d & d \\ 0 & b & b \end{vmatrix} \right) = -a \times - \begin{vmatrix} b & b & c \\ 0 & b & b \\ d & d & d \end{vmatrix} = -a \times \begin{vmatrix} 0 & b & b \\ b & b & c \\ d & d & d \end{vmatrix} =$

$= -a \times d \begin{vmatrix} 0 & b & b \\ b & b & c \\ 1 & 1 & 1 \end{vmatrix} = -a \times d \times \left(1 \times (-1)^4 \times \begin{vmatrix} b & b \\ b & c \end{vmatrix} + 1 \times (-1)^5 \times \begin{vmatrix} 0 & b \\ b & c \end{vmatrix} + 1 \times (-1)^6 \times \begin{vmatrix} 0 & b \\ b & b \end{vmatrix} \right) =$

$= -ad \left(\begin{vmatrix} b & b \\ b & c \end{vmatrix} - \begin{vmatrix} 0 & b \\ b & c \end{vmatrix} + \begin{vmatrix} 0 & b \\ b & b \end{vmatrix} \right) = -ad \left(1 - (-b^2) - b^2 \right) =$

$= -ad \left(1 + b^2 - b^2 \right) = -ad(1) = -ad$ c.q.p.

e) $\begin{vmatrix} a+3 & 1 & 2 & 1 \\ 0 & a & b & c \\ 2 & 2a+2 & 2b+1 & 2c \\ 3 & a+1 & b+2 & c+1 \end{vmatrix} \underset{l_1-l_4}{\overset{\substack{l_3-2l_2 \\ l_4-l_2}}{=}} \begin{vmatrix} a & -a & -b & -c \\ 0 & a & b & c \\ 2 & 2 & 1 & 0 \\ 3 & 1 & 2 & 1 \end{vmatrix} \overset{l_1+l_2}{=} \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & a & b & c \\ 2 & 2 & 1 & 0 \\ 3 & 1 & 2 & 1 \end{vmatrix} =$

$= a \times (-1)^2 \times \begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = a \times \Delta = a \times 1 = a$

192 $|A| = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} \underset{\substack{c_2-c_3 \\ c_3-c_4}}{\overset{c_1-c_2}{=}} \begin{array}{c} \\ \end{array}$

$$\begin{array}{cccc} & & & \text{D.L} \\ & & & \downarrow \end{array}$$

$\begin{vmatrix} a & 0 & 0 & 1 \\ -b & b & 0 & 1 \\ 0 & -c & c & 1 \\ 0 & 0 & -d & 1+d \end{vmatrix} =$

$= -\begin{vmatrix} -b & b & 0 \\ 0 & -c & c \\ 0 & 0 & -d \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & -c & c \\ 0 & 0 & -d \end{vmatrix} - \begin{vmatrix} a & 0 & 0 \\ -b & b & 0 \\ 0 & 0 & -d \end{vmatrix} + (1+d) \begin{vmatrix} a & 0 & 0 \\ -b & b & 0 \\ 0 & -c & c \end{vmatrix} =$

$= -(-bcd) + acd - (-abd) + (1+d)(abc) =$

$= bcd + acd + abd + abc + abcd =$

$= abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$ c.q.p.

193 $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} \overset{l_3-\frac{c-a}{b-a}l_2}{=}$

C.A.:

$(b-a)x = c-a \Leftrightarrow x = \frac{c-a}{b-a}$

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & 0 & \end{vmatrix}$

$(c-a)(c+a) - \frac{c-a}{b-a}(b-a)(b+a) = (c-a)(c+a) - (c-a)(b+a)$

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 1 \times (b-a) \times (c-a)(c+a-b-a) = (b-a)(c-a)(c-b)$ c.q.p.

194 $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \Leftrightarrow$ (pelo exercício anterior)

$\Leftrightarrow (y-x)(z-x)(z-y) + xyz(y-x)(z-x)(z-y) = 0 \Leftrightarrow$

$\Leftrightarrow ((y-x)(z-x)(z-y))(1+xyz) = 0 \Leftrightarrow$

$\Leftrightarrow (y-x)(z-x)(z-y) = 0 \lor xyz = -1 \Leftrightarrow$

$\Leftrightarrow y = x \lor z = x \lor z = y \lor xyz + 1 = 0 \Leftrightarrow$

$\Leftrightarrow 1 + xyz = 0 \land y \neq x \neq z \neq 0$ c.q.p.

195 a)

$$|A| = \begin{vmatrix} 1 & 6 & -9 & 2 \\ 3 & -2 & 0 & 8 \\ -4 & 4 & 6 & 5 \\ 2 & 1 & 4 & -7 \end{vmatrix} \underset{\substack{l_3+4l_1 \\ l_4-2l_1}}{\overset{l_2-3l_1}{=}} \begin{vmatrix} 1 & 6 & -9 & 2 \\ 0 & -20 & 27 & 2 \\ 0 & 28 & -30 & 13 \\ 0 & -11 & 22 & -11 \end{vmatrix} \overset{l_2 \leftrightarrow l_4\,;\ l_2 : 11}{=} -11 \begin{vmatrix} 1 & 6 & -9 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 28 & -30 & 13 \\ 0 & -20 & 27 & 2 \end{vmatrix} =$$

$$\underset{l_4-20l_2}{\overset{l_3+28l_2}{=}} -11 \begin{vmatrix} 1 & 6 & -9 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 26 & -15 \\ 0 & 0 & -13 & 22 \end{vmatrix} \overset{l_4 \leftrightarrow l_3}{=} 11 \begin{vmatrix} 1 & 6 & -9 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -13 & 22 \\ 0 & 0 & 26 & -15 \end{vmatrix} \overset{l_4+2l_3}{=} 11 \begin{vmatrix} 1 & 6 & -9 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -13 & 22 \\ 0 & 0 & 0 & 29 \end{vmatrix} =$$

$$= 11 \times (1 \times (-1) \times (-13) \times 29) = 11 \times 377 = 4147$$

b)

$$|B| = \begin{vmatrix} 3 & 1 & 2 & 4 \\ 2 & 0 & 5 & 1 \\ 1 & -1 & -2 & 6 \\ -2 & 3 & 2 & 3 \end{vmatrix} \overset{l_1 \leftrightarrow l_3}{=} - \begin{vmatrix} 1 & -1 & -2 & 6 \\ 2 & 0 & 5 & 1 \\ 3 & 1 & 2 & 4 \\ -2 & 3 & 2 & 3 \end{vmatrix} \underset{\substack{l_3-3l_1 \\ l_4+2l_1}}{\overset{l_2-2l_1}{=}} - \begin{vmatrix} 1 & -1 & -2 & 6 \\ 0 & 2 & 9 & -11 \\ 0 & 4 & 8 & -14 \\ 0 & 1 & -2 & 15 \end{vmatrix} \overset{l_2 \leftrightarrow l_4}{=}$$

$$= + \begin{vmatrix} 1 & -1 & -2 & 6 \\ 0 & 1 & -2 & 15 \\ 0 & 4 & 8 & -14 \\ 0 & 2 & 9 & -11 \end{vmatrix} \underset{l_4-2l_2}{\overset{l_3-4l_2}{=}} \begin{vmatrix} 1 & -1 & -2 & 6 \\ 0 & 1 & -2 & 15 \\ 0 & 0 & 16 & -74 \\ 0 & 0 & 13 & -41 \end{vmatrix} \overset{l_3 \times 13}{=} \frac{1}{13} \begin{vmatrix} 1 & -1 & -2 & 6 \\ 0 & 1 & -2 & 15 \\ 0 & 0 & 208 & -962 \\ 0 & 0 & 13 & -41 \end{vmatrix} =$$

$$= -\frac{1}{13} \begin{vmatrix} 1 & -1 & -2 & 6 \\ 0 & 1 & -2 & 15 \\ 0 & 0 & 13 & -41 \\ 0 & 0 & 208 & -962 \end{vmatrix} \overset{l_4-16l_3}{=} -\frac{1}{13} \begin{vmatrix} 1 & -1 & -2 & 6 \\ 0 & 1 & -2 & 15 \\ 0 & 0 & 13 & -41 \\ 0 & 0 & 0 & -306 \end{vmatrix} = -\frac{1}{13} \times 1 \times 1 \times 13 \times (-306) = 306$$

c)

$$\text{D.L.} \rightarrow \begin{vmatrix} a & 0 & 0 & 0 & 0 \\ b & e & 0 & d & 0 \\ a & e & 0 & a & a \\ a & e & 0 & 0 & 0 \\ a & a & e & a & 0 \end{vmatrix} = a \begin{vmatrix} e & 0 & d & 0 \\ e & 0 & a & a \\ e & 0 & 0 & 0 \\ a & e & a & 0 \end{vmatrix} \leftarrow \text{D.L.} =$$

$$= a \left(e \times \begin{vmatrix} 0 & d & 0 \\ 0 & a & a \\ e & a & 0 \end{vmatrix} \right) \leftarrow \text{D.L.} = ae \left(-d \begin{vmatrix} 0 & a \\ e & 0 \end{vmatrix} \right) = a e \times (-d \times (-ae)) =$$

$$= d a^2 e^2$$

d) $|D| = \begin{vmatrix} a & 0 & b & d & 0 \\ b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & d & 0 \\ b & d & 0 & 0 & a \\ 0 & 0 & b & 0 & a \end{vmatrix} \leftarrow \text{D.L.} = -b \times \begin{vmatrix} 0 & b & d & 0 \\ 0 & 0 & d & 0 \\ d & 0 & 0 & a \\ 0 & b & 0 & a \end{vmatrix} \leftarrow \text{D.L.} =$

$= -b \times (-d) \times \begin{vmatrix} 0 & b & 0 \\ d & 0 & a \\ 0 & b & a \end{vmatrix} \leftarrow \text{D.L.} = bd\left(-b \begin{vmatrix} d & a \\ 0 & a \end{vmatrix}\right) = -db^2da = -ab^2d^2$

e) $|E| = \begin{vmatrix} -1 & 0 & 0 & 1 & 1 \\ 2 & 1 & -a & 2 & 1 \\ 1 & 2 & 1 & a & 2 \\ -2 & -1 & 2 & 1 & 2 \\ 1 & 0 & 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -a & 4 & 3 \\ 0 & 2 & 1 & a+1 & 3 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} \leftarrow \text{D.L.} = \begin{array}{c} \text{D.L.} \\ \downarrow \end{array} \begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & a+1 & 3 \\ 0 & -1 & -1 & 0 \end{vmatrix} =$

$= -\begin{vmatrix} 1 & 4 & 3 \\ 2 & a+1 & 3 \\ -1 & -1 & 0 \end{vmatrix} = -\begin{array}{c} \text{D.L.} \\ \end{array}\begin{vmatrix} 1 & 4 & 3 \\ 0 & a-7 & -3 \\ 0 & 3 & 3 \end{vmatrix} = -\begin{vmatrix} a-7 & -3 \\ 3 & 3 \end{vmatrix} = -(3a-21+9) =$

$= -3a + 12$

196 a) $|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & a & 1 \\ 1 & b & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a-1 & 0 \\ 0 & b-1 & 0 & 0 \end{vmatrix} \leftarrow \text{D.L.} =$

$= \begin{array}{c} \text{D.L.} \\ \downarrow \end{array}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & a-1 \\ 0 & b-1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a-1 \\ b-1 & 0 \end{vmatrix} = 0 - (b-1)(a-1) = -(b-1)(a-1)$

A matriz A é regular ou não singular se e só se $|A| \neq 0$, isto é:

$-(b-1)(a-1) = 0 \Leftrightarrow b = 1 \vee a = 1$ Se $a \neq 1$ e $b \neq 1$.

b) $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} = A \qquad |A^T| = |A| = -2$

$(A^{T^{-1}}) \cdot A^T = I \Leftrightarrow |A^{T^{-1}} \cdot A^T| = |I| \Leftrightarrow |A^{T^{-1}}||A^T| = 1 \Leftrightarrow |A^{T^{-1}}| = \frac{1}{|A|} = \frac{1}{-2} =$

$= -\frac{1}{2}$

$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \leftrightarrow$

$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & -1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 2 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 1 \end{array}\right] (A^T)^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 2 \\ -\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 2 & 1 & 1 \end{bmatrix}$

$(A^T)^{-1} = C^{-1}BC \Leftrightarrow (A^T)^{-1} = C^{-1}.CB \Leftrightarrow (A^T)^{-1} = I.B \Leftrightarrow (A^T)^{-1} = B \Leftrightarrow$

$\Leftrightarrow |(A^T)^{-1}| = |B|$

$$|(A^T)^{-1}| = \begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 2 \\ -\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \end{vmatrix} \leftarrow \text{D.L.} = \begin{vmatrix} 1 & -1 & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{vmatrix} \leftarrow \text{D.L.} =$$

$$= -\begin{vmatrix} 1 & -1 \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{2} \qquad |B| = -\frac{1}{2}$$