# 11. Transformadas de Laplace

1) a) $\mathcal{L}\{e^{3t}\cos(2t)\} = \dfrac{s-3}{(s-3)^2+2^2} = \dfrac{s-3}{(s-3)^2+4}$

b) $\mathcal{L}\{\cos^2(at)\} = \mathcal{L}\left\{\dfrac{1}{2} + \dfrac{\cos(2at)}{2}\right\} = \mathcal{L}\left\{\dfrac{1}{2}\right\} + \mathcal{L}\left\{\dfrac{\cos(2at)}{2}\right\} =$

$= \dfrac{1}{2}\mathcal{L}\{1\} + \dfrac{1}{2}\mathcal{L}\{\cos(2at)\} = \dfrac{1}{2}\times\dfrac{1}{s} + \dfrac{1}{2}\times\dfrac{s}{s^2+4a^2} = \dfrac{1}{2s} + \dfrac{s}{2(s^2+4a^2)}$

c) $\mathcal{L}\{\sin(5t)\cos(2t)\} = \mathcal{L}\left\{\dfrac{1}{2}\left(\sin(3t)+\sin(7t)\right)\right\} = \begin{pmatrix}\text{fórmula}\\ \text{acima}\end{pmatrix}$

$= \dfrac{1}{2}\mathcal{L}\{\sin(3t)\} + \dfrac{1}{2}\mathcal{L}\{\sin(7t)\} = \dfrac{1}{2}\times\dfrac{3}{s^2+9} + \dfrac{1}{2}\times\dfrac{7}{s^2+49} = \dfrac{1}{2}\left(\dfrac{7}{s^2+49}+\dfrac{3}{s^2+9}\right)$

d) $\mathcal{L}\{t^2\sin t\} = (-1)^2\left(\mathcal{L}\{\sin t\}\right)'' = \left(\dfrac{1}{s^2+1}\right)'' = \left(\left(\dfrac{1}{s^2+1}\right)'\right)' = \left(\dfrac{-2s}{(s^2+1)^2}\right)' =$

$= \dfrac{-2(s^2+1)^2 + 8s^2(s^2+1)}{(s^2+1)^4} = \dfrac{-2s^2-2+8s^2}{(s^2+1)^3} = \dfrac{6s^2-2}{(s^2+1)^3}$

e) $\mathcal{L}\{t^3e^{-t}\} = \dfrac{3!}{(s+1)^4} = \boxed{\dfrac{6}{(s+1)^4}}$

$\boxed{\mathcal{L}\{t\cos(bt)\} = (-1)^1\times\left(\mathcal{L}\{\cos(bt)\}\right)' =}$

$= -\left(\dfrac{s}{s^2+b^2}\right)' = -\dfrac{s^2+b^2-2s^2}{(s^2+b^2)^2} = \dfrac{s^2-b^2}{(s^2+b^2)^2}$

f) $\mathcal{L}\{t\cos(bt)e^{at}\} =$ Aplicando a translação:

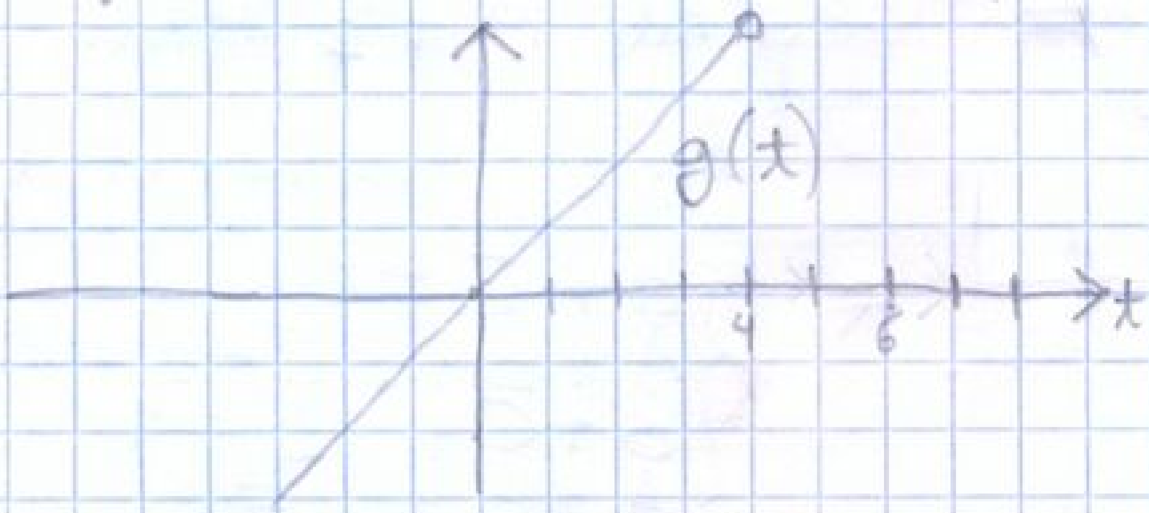
$= \dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$.

(continuação dos exercícios na zona verde do caderno (MDIS))

# Continuação dos exercícios de AMAT.

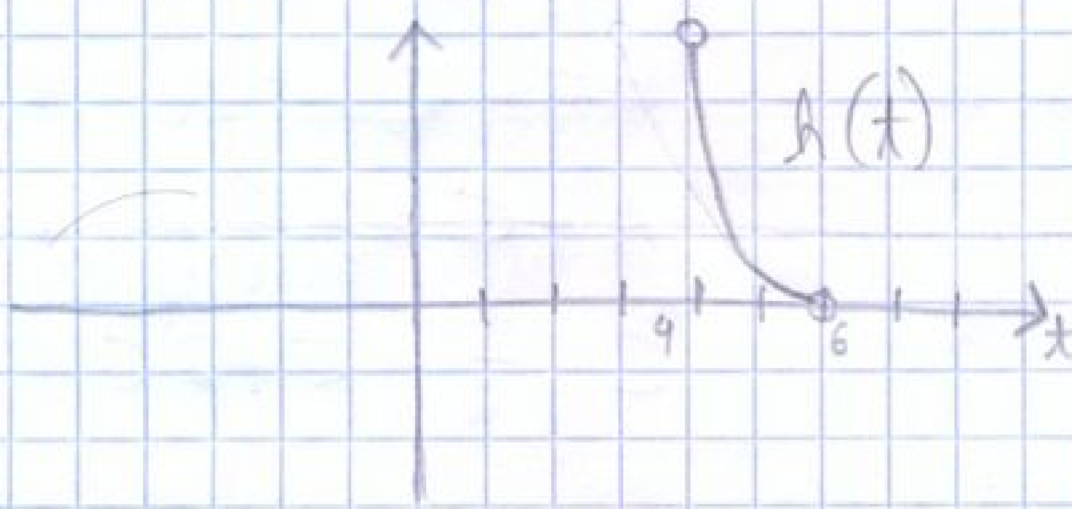
g) $f(t) = t[1-u(t-4)] + (t-6)^2[u(t-4)-u(t-6)]$

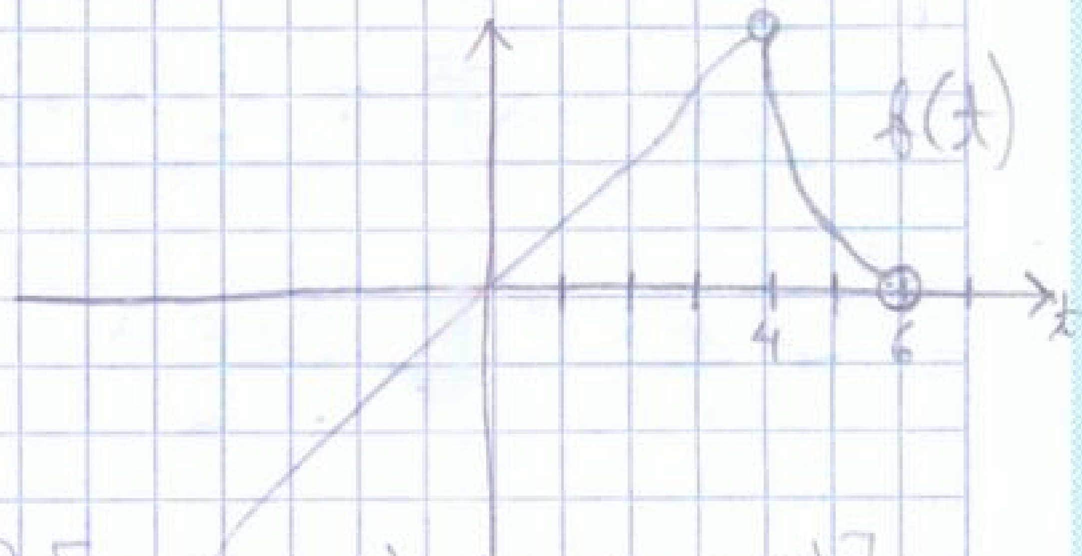
Seja $g(t) = t[1-u(t-4)]$ e $h(t) = (t-6)^2[u(t-4)-u(t-6)]$




$$g(t) = \begin{cases} t & \text{se } t<4 \\ 0 & \text{se } t>4 \end{cases}$$

$$h(t) = \begin{cases} 0 & \text{se } t<4 \\ (t-6)^2 & \text{se } 4<t<6 \\ 0 & \text{se } t>6 \end{cases}$$




$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t[1-u(t-4)] + (t-6)^2[u(t-4)-u(t-6)]\} =$$

$$= \mathcal{L}\{t - t\,u(t-4)\} + \mathcal{L}\{(t-6)^2u(t-4) - (t-6)^2u(t-6)\} =$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{t\,u(t-4)\} + \mathcal{L}\{(t-6)^2u(t-4)\} - \mathcal{L}\{(t-6)^2u(t-6)\} =$$

$$= \frac{1}{s^2} - \mathcal{L}\{(t-4+4)u(t-4)\} + \mathcal{L}\{(t-6+2-2)u(t-4)\} - \mathcal{L}\{(t-6)^2u(t-6)\} =$$

$$= \frac{1}{s^2} - \mathcal{L}\{(t-4)u(t-4)\} - 4\mathcal{L}\{u(t-4)\} + \mathcal{L}\{(t-4)u(t-4)\} - 2\mathcal{L}\{u(t-4)\}$$
$$- \mathcal{L}\{(t-6)^2u(t-6)\} =$$

$$= \frac{1}{s^2} - 6\mathcal{L}\{u(t-4)\} - \mathcal{L}\{(t-6)^2u(t-6)\} =$$

$$= \frac{1}{s^2} - 6e^{-4s}\mathcal{L}\{1\} - e^{-6s}\times\mathcal{L}\{t^2\} = \frac{1}{s^2} - \frac{6e^{-4s}}{s} - \frac{2e^{-6s}}{s^3}$$

2 a) $\mathcal{L}^{-1}\{(s-2)^{-2}\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = te^{2t}$

b) $\mathcal{L}^{-1}\left\{\frac{7}{(s-1)^3} + \frac{1}{(s+1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{6+1}{(s-1)^3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-2^2}\right\} =$

$$= 3\mathcal{L}^{-1}\left\{\frac{2!}{(s-1)^3}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{(s-1)^3}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2-2^2}\right\} =$$

$$= 3t^2e^t + \frac{1}{2}t^2e^t + \frac{1}{2}\sinh(2t)\times e^{-t} = \frac{7}{2}t^2e^t + \frac{1}{2}e^{-t}\sinh(2t)$$

c) $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} =$

$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1}\right\} =$

$= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} =$

$= -\frac{1}{2} \times t e^{-t} + \frac{1}{2} \times \sin t =$

$= -\frac{t e^{-t}}{2} + \frac{\sin t}{2}$

CA:

$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{Cs+D}{s^2+1}$

$A(s^2+1) + B(s+1)(s^2+1) + Cs(s+1)^2 + D(s+1)^2$

$As^2 + A + B(s^3+s^2+s+1) + Cs^3 + 2Cs^2 + Cs + Ds^2 + 2Ds + D = s$

$$\begin{cases} B + C = 0 \\ A + B + 2C + D = 0 \\ B + C + 2D = 1 \\ A + B + D = 0 \end{cases}$$

$B = -C \quad 2D = 1 \Leftrightarrow D = \frac{1}{2}$

$A - C = -\frac{1}{2} \Leftrightarrow A = C - \frac{1}{2}$

$C - \frac{1}{2} - C + 2C + \frac{1}{2} = 0 \Leftrightarrow$

$\Leftrightarrow 2C = 0 \Leftrightarrow C = 0 \quad B = 0$

$A = -\frac{1}{2}$

d) $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+16}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2} \times e^{-\pi s}\right\} = \frac{1}{4}\sin(4t)\,u(t-\pi) =$

$= \begin{cases} 0 & \text{se } t < \pi \\ \frac{1}{4}\sin(4t) & \text{se } t \geq \pi \end{cases}$

e) (não demos).

③ a) $y'' + 4y' + 4y = e^{-x}$ $\quad \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{e^{-x}\} \Leftrightarrow$

$\Leftrightarrow s^2Y(s) - s\,y(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \frac{1}{s+1} \Leftrightarrow$

$\Leftrightarrow s^2Y(s) - 1 + 4sY(s) + 4Y(s) = \frac{1}{s+1} \Leftrightarrow Y(s)(s^2+4s+4) = \frac{1}{s+1} + 1 \Leftrightarrow$

$\Leftrightarrow Y(s) = \frac{s+2}{(s+1)(s^2+4s+4)} \Leftrightarrow$ $\quad$ CA: $s^2+4s+4 = 0 \Leftrightarrow s = \frac{-4 \pm \sqrt{0}}{2} \Leftrightarrow s = -2$

$\Leftrightarrow Y(s) = \frac{s+2}{(s+1)(s+2)^2} \Leftrightarrow Y(s) = \frac{1}{(s+1)(s+2)}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} \Leftrightarrow$

$\Leftrightarrow y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \Leftrightarrow$

$\Leftrightarrow y(x) = e^{-x} - e^{-2x}$ $\quad$ R: $y = e^{-x} - e^{-2x}$

CA: $\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$A(s+2) + B(s+1) = 1$

$As + 2A + Bs + B = 1$

$\begin{cases} A + B = 0 \\ 2A + B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -B \\ -B = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$

b) $y''+4y'+3y=0 \qquad \mathcal{L}\{y''\}+4\mathcal{L}\{y'\}+3\mathcal{L}\{y\}=\mathcal{L}\{0\} \Leftrightarrow$

$\Leftrightarrow s^2Y(s)-s\,y(0)-y'(0)+4[sY(s)-y(0)]+3Y(s)=0 \Leftrightarrow$

$\Leftrightarrow s^2Y(s)-1+4sY(s)+3Y(s)=0 \Leftrightarrow Y(s)(s^2+4s+3)=1 \Leftrightarrow$

$\Leftrightarrow Y(s)=\dfrac{1}{s^2+4s+3} \Leftrightarrow$

CA: $s^2+4s+3=0 \Leftrightarrow s=\dfrac{-4\pm\sqrt{4}}{2} \Leftrightarrow s=-1 \vee s=-3$

$\Leftrightarrow Y(s)=\dfrac{1}{(s+1)(s+3)} \Leftrightarrow$

CA: $\dfrac{1}{(s+1)(s+3)}=\dfrac{A}{s+1}+\dfrac{B}{s+3}$

$A(s+3)+B(s+1)=1 \Leftrightarrow As+3A+Bs+B=1$

$\begin{cases} A+B=0 \\ 3A+B=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ -2B=1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$

$\Leftrightarrow Y(s)=\dfrac{\frac{1}{2}}{s+1}-\dfrac{\frac{1}{2}}{s+3}$

$\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s+3}\right\} \Leftrightarrow y=\frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}$

c) $y''+6y-7=0 \qquad \mathcal{L}\{y''\}+6\mathcal{L}\{y\}-7\mathcal{L}\{1\}=\mathcal{L}\{0\} \Leftrightarrow$

$\Leftrightarrow s^2Y(s)-s\,y(0)-y'(0)+6Y(s)-\dfrac{7}{s}=0 \Leftrightarrow s^2Y(s)-s+6Y(s)=\dfrac{7}{s} \Leftrightarrow$

$\Leftrightarrow Y(s)(s^2+6)=\dfrac{7+s^2}{s} \Leftrightarrow Y(s)=\dfrac{7+s^2}{s(s^2+6)} \Leftrightarrow$

CA: $\dfrac{7+s^2}{s(s^2+6)}=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+6}$

$A(s^2+6)+Bs^2+Cs=7+s^2 \Leftrightarrow$

$\Leftrightarrow As^2+6A+Bs^2+Cs=7+s^2$

$\begin{cases} A+B=1 \\ C=0 \\ 6A=7 \end{cases} \Leftrightarrow \begin{cases} B=-\frac{1}{6} \\ C=0 \\ A=\frac{7}{6} \end{cases}$

$\Leftrightarrow Y(s)=\dfrac{\frac{7}{6}}{s}-\dfrac{\frac{1}{6}\times s}{s^2+6} \Leftrightarrow$

$\Leftrightarrow \mathcal{L}^{-1}\{Y(s)\}=\frac{7}{6}\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\}-\frac{1}{6}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+6}\right\} \Leftrightarrow$

$\Leftrightarrow y=\frac{7}{6}-\frac{1}{6}\cos(\sqrt{6}\,t)$

d) $y''-y'-2y=x \qquad \mathcal{L}\{y''\}-\mathcal{L}\{y'\}-2\mathcal{L}\{y\}=\mathcal{L}\{x\} \Leftrightarrow$

$\Leftrightarrow s^2Y(s)-s\,y(0)-y'(0)-(sY(s)-y(0))-2Y(s)=\dfrac{1}{s^2} \Leftrightarrow$

$\Leftrightarrow s^2Y(s)-sY(s)-2Y(s)=\dfrac{1}{s^2} \Leftrightarrow Y(s)(s^2-s-2)=\dfrac{1}{s^2} \Leftrightarrow$

$\Leftrightarrow Y(s)=\dfrac{1}{s^2(s^2-s-2)} \Leftrightarrow$

CA: $s^2-s-2=0 \Leftrightarrow s=\dfrac{1\pm\sqrt{9}}{2} \Leftrightarrow s=2 \vee s=-1$

$\Leftrightarrow Y(s)=\dfrac{1}{s^2(s-2)(s+1)}$

CA: $\dfrac{1}{s^2(s-2)(s+1)}=\dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{C}{s-2}+\dfrac{D}{s+1}$

$A(s-2)(s+1)+Bs(s-2)(s+1)+Cs^2(s+1)+Ds^2(s-2)=1 \Leftrightarrow$

$\Leftrightarrow As^2-As-2A+Bs^3-Bs^2-2Bs+Cs^3+Cs^2+Ds^3-2Ds^2=1 \Leftrightarrow$

$\Leftrightarrow \begin{cases} B+C+D=0 \\ A-B+C-2D=0 \\ -A-2B=0 \\ -2A=1 \end{cases} \Leftrightarrow \begin{cases} D=-\frac{1}{3} \\ C=\frac{1}{12} \\ B=\frac{1}{4} \\ A=-\frac{1}{2} \end{cases}$

$C=\frac{3}{4}+2D \qquad \frac{3}{4}-\frac{2}{3}=\frac{9}{12}-\frac{8}{12}=\frac{1}{12}$

$\mathcal{L}^{-1}\{Y(s)\}=-\frac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\}+\frac{1}{12}\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} \Leftrightarrow$

$\Leftrightarrow y=\frac{1}{4}-\frac{1}{2}x-\frac{1}{3}e^{-x}+\frac{1}{12}e^{2x}$

**e)** $y'' - 2y' + 5y = 0 \quad \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{0\} \Leftrightarrow$

$\Leftrightarrow s^2 Y(s) - s\,y(0) - y'(0) - 2(sY(s) - y(0)) + 5Y(s) = 0 \Leftrightarrow$

$\Leftrightarrow s^2 Y(s) - 1 - 2sY(s) + 5Y(s) = 0 \Leftrightarrow Y(s)(s^2 - 2s + 5) = 1 \Leftrightarrow$

$\Leftrightarrow Y(s) = \dfrac{1}{s^2 - 2s + 5} \Leftrightarrow$ | CA: $s^2 - 2s + 5 = 0 \Leftrightarrow s = \dfrac{2 \pm \sqrt{-16}}{2}$ ← impossível

$\Leftrightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2 - 2s + 1 + 4}\right\} \Leftrightarrow y = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2 + 2^2}\right\} \Leftrightarrow$

$\Leftrightarrow y = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)^2 + 2^2}\right\} \Leftrightarrow y = \dfrac{1}{2}\sin(2t)e^{t}$

**f)** $y'' - 9y' = 5e^{-2t} \quad \mathcal{L}\{y''\} - 9\mathcal{L}\{y'\} = 5\mathcal{L}\{e^{-2t}\} \Leftrightarrow$

$\Leftrightarrow s^2 Y(s) - s\,y(0) - y'(0) - 9(sY(s) - y(0)) = \dfrac{5}{s+2} \Leftrightarrow$

$\Leftrightarrow s^2 Y(s) - s - 2 - 9sY(s) + 9 = \dfrac{5}{s+2} \Leftrightarrow$

$\Leftrightarrow Y(s)(s^2 - 9s) = \dfrac{5}{s+2} + s - 7 \Leftrightarrow Y(s)(s^2 - 9s) = \dfrac{5 + s^2 + 2s - 7s - 14}{s+2} \Leftrightarrow$

$\Leftrightarrow Y(s) = \dfrac{s^2 - 5s - 9}{s(s+2)(s-9)} \Leftrightarrow$

CA: $s^2 - 5s - 9 = 0 \Leftrightarrow s = \dfrac{5 \pm \sqrt{61}}{2}$

$\Leftrightarrow Y(s) = \dfrac{\frac{1}{2}}{s} + \dfrac{\frac{5}{22}}{s+2} + \dfrac{\frac{3}{11}}{s-9} \Leftrightarrow$

CA: $\dfrac{s^2 - 5s - 9}{s(s+2)(s-9)} = \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{s-9}$

$A(s^2 - 7s - 18) + B(s^2 - 9s) + C(s^2 + 2s) = s^2 - 5s - 9 \Leftrightarrow$

$\Leftrightarrow As^2 - 7As - 18A + Bs^2 - 9Bs + Cs^2 + 2Cs = s^2 - 5s - 9$

$\begin{cases} A + B + C = 1 \\ -7A - 9B + 2C = -5 \\ -18A = -9 \end{cases} \Leftrightarrow \begin{cases} B = \frac{1}{2} - C \\ -\frac{7}{2} - \frac{9}{2} + 9C + 2C = -5 \\ A = \frac{1}{2} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} B = \frac{1}{2} - \frac{3}{11} \\ C = \frac{3}{11} \\ A = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} B = \frac{5}{22} \\ C = \frac{3}{11} \\ A = \frac{1}{2} \end{cases}$

$\Leftrightarrow \mathcal{L}^{-1}\{Y(s)\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} + \dfrac{5}{22}\mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\} + \dfrac{3}{11}\mathcal{L}^{-1}\left\{\dfrac{1}{s-9}\right\} \Leftrightarrow$

$\Leftrightarrow y = \dfrac{1}{2} + \dfrac{3}{11}e^{9t} + \dfrac{5}{22}e^{-2t}$