12. Polinómio de Taylor com Resto de Lagrange.

① a) $f(x) = e^x \; ; \; f(0) = 1$

$f'(x) = e^x \; ; \; f'(0) = 1$

$f''(x) = e^x \; ; \; f''(0) = 1$

$\vdots$

$f^{(k)}(x) = e^x \; ; \; f^{(k)}(0) = 1$

Calcular polinómio de Taylor:

$$P_{m,0}(x) = \sum_{k=0}^{m} \frac{1}{k!} \times x^k =$$

$$= \sum_{k=0}^{m} \frac{x^k}{k!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots$$

Calcular Resto de Lagrange: $R_{m,0}(x) = \frac{e^c}{(m+1)!} x^{m+1}$, com $x < c < 0$ ou $0 < c < x$

$$f(x) = \sum_{k=0}^{m} \frac{x^k}{k!} + \frac{e^c}{(m+1)!} x^{m+1}$$

b) $f(x) = \cos(x) \; ; \; f(0) = 1$

$f'(x) = -\sin(x) \; ; \; f'(0) = 0$

$f''(x) = -\cos(x) \; ; \; f''(0) = -1$

$f'''(x) = \sin(x) \; ; \; f'''(0) = 0$

$f^{IV}(x) = \cos(x) \; ; \; f^{IV}(0) = 1$

Calcular polinómio de Taylor:

$$\cos(x) = f(0) + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

$$\cos(x) = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Calcular Resto de Lagrange: (da série) $\quad P_{m,0} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}$

$R_{m,0}(x) = \lim_{m \to +\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 0$, pois o fatorial cresce mais depressa que x^{2k}.

Logo, $f(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}$

$$P_{m,\frac{\pi}{2}} = \sum_{k=0}^{m} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!} \left(x - \frac{\pi}{2}\right)^k$$

c) $f(x) = \sin(x) \; ; \; f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos(x) \; ; \; f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin(x) \; ; \; f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos(x) \; ; \; f'''\left(\frac{\pi}{2}\right) = 0$

$f^{IV}(x) = \sin(x) \; ; \; f^{IV}\left(\frac{\pi}{2}\right) = 1$

*(continuação escrita em cima)

Calcular polinómio de Taylor:

$$\sin(x) = 1 + 0 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + 0 + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} + \dots$$

$$\sin(x) = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \frac{\left(x - \frac{\pi}{2}\right)^6}{6!} + \dots$$

$$P_{m,\frac{\pi}{2}} = \sum_{k=0}^{m} (-1)^k \frac{\left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}$$

$$R_m = \lim_{m \to +\infty} (-1)^m \frac{\left(x - \frac{\pi}{2}\right)^{2m}}{(2m)!} = 0$$

$$f(x) = \sum_{k=0}^{m} (-1)^k \frac{\left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}$$

(continuação dos exercícios de AMAT que acabaram na zona norte do caderno).

d) $f(x) = \ln(x)$; $f(2) = \ln(2)$

$f'(x) = \frac{1}{x}$; $f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2}$; $f''(2) = -\frac{1}{2^2}$

$f'''(x) = \frac{2}{x^3}$; $f'''(2) = \frac{2}{2^3}$

$f^{IV}(x) = \frac{2\times(-3)}{x^4}$; $f^{IV}(2) = \frac{2\times(-3)}{2^4}$

$f^{V}(x) = \frac{2\times(-3)\times 4}{x^5}$; $f^{V}(2) = \frac{2\times(-3)\times 4}{2^5}$

$$P_{n,2} = \sum_{k=0}^{n} \frac{f^{(k)}(2)}{k!}(x-2)^k$$

$$f(x) = \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{2^2\, 2!} + \frac{2(x-2)^3}{2^3\times 3!} + \frac{2\times(-3)}{2^4\times 4!}(x-2)^4 + \ldots$$

$$P_{n,2} = \ln(2) + \sum_{k=1}^{n} (-1)^{k+1} \times \frac{(x-2)^k}{k!} \times \frac{(k-1)!}{2^k}$$

$$R_{n,2} = \lim_{k\to+\infty} (-1)^{k+1} \times \frac{(x-2)^k}{k!} \times \frac{(k-1)!}{2^k} = \lim_{k\to+\infty} (-1)^{k+1} \times \frac{(x-2)^k}{k\, 2^k} = 0 \quad (\ldots?)$$

e) $f(x) = \frac{1}{x}$; $f(2) = \frac{1}{2}$

$f'(x) = -\frac{1}{x^2}$; $f'(2) = -\frac{1}{2^2}$

$f''(x) = \frac{2}{x^3}$; $f''(2) = \frac{2}{2^3}$

$f'''(x) = \frac{2\times(-3)}{x^4}$; $f'''(2) = \frac{2\times(-3)}{2^4}$

$f^{IV}(x) = \frac{2\times(-3)\times 4}{x^5}$; $f^{IV}(2) = \frac{2\times(-3)\times 4}{2^5}$

$$P_{n,2} = \sum_{k=0}^{n} \frac{f^{(k)}(2)}{k!}(x-2)^k$$

$$f(x) = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{2(x-2)^2}{2^3\, 2!} + \frac{2\times(-3)(x-2)^3}{2^4\times 3!} + \ldots$$

$$P_{n,2} = \sum_{n=0}^{n} (-1)^k \times \frac{(x-2)^k}{k!} \times \frac{(k-1)!}{2^{k+1}}$$

$$R_{n,2} = \lim_{n\to+\infty} (-1)^k \times \frac{(x-2)^k}{k!} \times \frac{(k-1)!}{2^{k+1}} = 0 \quad (\ldots?)$$

f) (não demos).