# 10. Equações diferenciais

1) a) $(x^2-x)\frac{dy}{dx} = y^2+y \Leftrightarrow \frac{1}{y^2+y}dy = \frac{1}{x^2-x}dx \Leftrightarrow$

$\Leftrightarrow \int \frac{1}{y(y+1)}dy = \int \frac{1}{x(x-1)}dx \Leftrightarrow$

CA:
$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \quad 1 = Ay + A + By$

$\begin{cases} A+B=0 \\ A=1 \end{cases} \quad B=-1 \quad \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$

$\Leftrightarrow \int\left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \int\left(-\frac{1}{x} + \frac{1}{x-1}\right)dx \Leftrightarrow$

CA:
$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \quad 1 = Ax - A + Bx$

$\begin{cases} A+B=0 \\ -A=1 \end{cases} \quad B=1 \quad A=-1 \quad \frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}$

$\Leftrightarrow \int \frac{1}{y}dy - \int \frac{1}{y+1}dy = -\int \frac{1}{x}dx + \int \frac{1}{x-1}dx \Leftrightarrow$

$\Leftrightarrow \ln(y) - \ln(y+1) = -\ln(x) + \ln(x-1) + c \Leftrightarrow$

$\Leftrightarrow \ln\left(\frac{y}{y+1}\right) = \ln\left(\frac{cx-c}{x}\right) \Leftrightarrow$

$\Leftrightarrow \frac{y}{y+1} = c\left(\frac{x-1}{x}\right) \Leftrightarrow y = \frac{c(x-1)(y+1)}{x}$

b) $y' - \underbrace{2\cos x \sin x}_{P(x)} \times y = \underbrace{e^{\sin^2 x}}_{Q(x)}$

Fator integrante: $e^{\int -2\cos x \sin x \, dx} =$

$\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

$= e^{-2\int \cos x \sin x \, dx} = e^{-2\int u \, du} = e^{-2\left(\frac{u^2}{2}\right)} =$

$= e^{-u^2} = e^{-\sin^2 x}$

Multiplicar ambos os termos pelo f.i.:

$y' e^{-\sin^2 x} - 2\cos x \sin x \, e^{-\sin^2 x} y = e^{\sin^2 x} \cdot e^{-\sin^2 x} \Leftrightarrow$

$\Leftrightarrow y' e^{-\sin^2 x} - 2\cos x \sin x \, e^{-\sin^2 x} y = 1 \Leftrightarrow$

$\Leftrightarrow \frac{d}{dx}\left(y e^{-\sin^2 x}\right) = 1 \Leftrightarrow d\left(y e^{-\sin^2 x}\right) = 1\,dx \Leftrightarrow \int d\left(y e^{-\sin^2 x}\right) = \int 1\,dx \Leftrightarrow$

$\Leftrightarrow y e^{-\sin^2 x} = x + c \Leftrightarrow y = (x+c)e^{\sin^2 x} \Leftrightarrow y = \frac{ex + e}{e^{\cos^2 x}}$

c) $y' = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \Leftrightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \Leftrightarrow \frac{1}{\sqrt{1-y^2}}dy = \frac{1}{\sqrt{1-x^2}}dx \Leftrightarrow$

$\Leftrightarrow \int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{1}{\sqrt{1-x^2}}dx \Leftrightarrow \arcsin(y) = \arcsin(x) + c \Leftrightarrow$

$\Leftrightarrow y = \sin(\arcsin(x) + c)$

$$\begin{cases} u = \tan x \\ du = \sec^2 x\,dx \end{cases} \quad \begin{cases} v = \tan y \\ dv = \sec^2 y\,dy \end{cases}$$

**d)** $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0 \Leftrightarrow$

$\Leftrightarrow \sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy \Leftrightarrow$

$\Leftrightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy \Leftrightarrow \int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy \Leftrightarrow$

$\Leftrightarrow \int \frac{1}{u} du = -\int \frac{1}{v} dv \Leftrightarrow \ln(u) = -\ln(v) + c \Leftrightarrow$

$\Leftrightarrow \ln(\tan x) = -\ln(\tan y) + \ln(c) \Leftrightarrow \tan x = \frac{c}{\tan y} \Leftrightarrow$

$\Leftrightarrow \tan y = \frac{c}{\tan x} \Leftrightarrow y = \arctan\left(\frac{c}{\tan x}\right)$

**e)** $y' + \underbrace{3}_{P(x)} y = \underbrace{e^{-3x} x^{-2}}_{Q(x)}$ F.I: $e^{\int P(x) dx} = e^{\int 3 dx} = e^{3x}$

Multiplicando ambos os membros pelo f. i. :

$y' e^{3x} + y 3 e^{3x} = e^{-3x} x^{-2} \cdot e^{3x} \Leftrightarrow \frac{d}{dx}\left(y e^{3x}\right) = x^{-2} \Leftrightarrow$

$\Leftrightarrow d\left(y e^{3x}\right) = x^{-2} dx \Leftrightarrow \int d\left(y e^{3x}\right) = \int x^{-2} dx \Leftrightarrow$

$\Leftrightarrow y e^{3x} = -\frac{1}{x} + c \Leftrightarrow y = e^{-3x}\left(c - \frac{1}{x}\right)$

**2 a)** $y' = \frac{2}{x} y - 1 \Leftrightarrow y' \underbrace{-\frac{2}{x}}_{P(x)} y = \underbrace{-1}_{Q(x)}$ F.I: $e^{\int P(x) dx} = e^{\int -\frac{2}{x} dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln(x)}$

Multiplicando o f.i. em ambos os termos:

$y' e^{-2\ln(x)} - \frac{2}{x} e^{-2\ln(x)} y = -e^{-2\ln(x)} \Leftrightarrow \frac{d}{dx}\left(y e^{-2\ln(x)}\right) = -e^{-2\ln(x)} \Leftrightarrow$

$\Leftrightarrow d\left(y e^{-2\ln(x)}\right) = -e^{-2\ln(x)} dx \Leftrightarrow \int d\left(y e^{-2\ln(x)}\right) = -\int e^{-2\ln(x)} dx \Leftrightarrow$

$\Leftrightarrow y e^{-2\ln(x)} + c = -\int e^{\ln\left(\frac{1}{x^2}\right)} dx \Leftrightarrow y e^{-2\ln(x)} = -\int \frac{1}{x^2} dx \Leftrightarrow$

$\Leftrightarrow y e^{-2\ln(x)} = -\left(-\frac{1}{x}\right) + c \Leftrightarrow y e^{-2\ln(x)} = \frac{1}{x} + c \Leftrightarrow$

$\Leftrightarrow y e^{\ln(x^{-2})} = \frac{1}{x} + c \Leftrightarrow y\left(\frac{1}{x^2}\right) = \frac{1}{x} + c \Leftrightarrow y = x + cx^2, c \in \mathbb{R}$

**b)** $3 = 1 + c \times 1^2 \Leftrightarrow 1 + c = 3 \Leftrightarrow c = 2$

R: $y = x + 2x^2$

3 a) $(ax^2+b)^{\frac{1}{2}}y' - xy^3 = 0 \Leftrightarrow \frac{dy}{dx} = \frac{xy^3}{\sqrt{(ax^2+b)}} \Leftrightarrow$ $\left\{\begin{array}{l} u = ax^2+b \\ du = 2ax\,dx \end{array}\right.$

$\Leftrightarrow \frac{1}{y^3}dy = \frac{x}{\sqrt{ax^2+b}}dx \Leftrightarrow \int y^{-3}dy = \frac{1}{2a}\int \frac{2ax}{\sqrt{ax^2+b}}dx \Leftrightarrow$

$\Leftrightarrow \frac{y^{-2}}{-2} + c = \frac{1}{2a}\int \frac{1}{\sqrt{u}}du \Leftrightarrow -\frac{1}{2y^2} + c = \frac{1}{2a}\int u^{-\frac{1}{2}}du \Leftrightarrow$

$\Leftrightarrow -\frac{1}{2y^2} + c = \frac{2\sqrt{u}}{2a} \Leftrightarrow -\frac{1}{2y^2} = \frac{\sqrt{ax^2+b}}{a} + c \Leftrightarrow y = \sqrt{-\frac{a}{2\sqrt{ax^2+b}} + c}$

b) $y' + \frac{2y}{x} = \frac{y^3}{x^3}$ $\left\{\begin{array}{l} u = y^{1-3} = y^{-2} \\ u' = -2y^{-3}y' \end{array}\right.$

Multiplicando ambos os termos por $-2y^{-3}$:

$-2y^{-3}y' + \frac{2y}{x}\times(-2y^{-3}) = \frac{y^3}{x^3}\times(-2y^{-3}) \Leftrightarrow -2y^{-3}y' - \frac{4y^{-2}}{x} = -\frac{2}{x^3} \Leftrightarrow$

$\Leftrightarrow u' \underbrace{-\frac{4}{x}}_{P(x)} u = \underbrace{-\frac{2}{x^3}}_{Q(x)} \rightarrow$ E.D.O. de 1.ª ordem linear.

CA: f. i: $e^{\int P(x)dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\ln(x)} = e^{\ln(x^{-4})} = x^{-4} = \frac{1}{x^4}$

Multiplicando ambos os termos pelo f. i:

$\frac{1}{x^4}u' - \frac{4}{x}\times\frac{1}{x^4}u = -\frac{2}{x^3}\times\frac{1}{x^4} \Leftrightarrow x^{-4}u' - 4x^{-5}u = -2x^{-7} \Leftrightarrow$

$\Leftrightarrow \frac{d}{dx}(ux^{-4}) = -2x^{-7} \Leftrightarrow d(ux^{-4}) = -2x^{-7}dx \Leftrightarrow \int d(ux^{-4}) = -2\int x^{-7}dx \Leftrightarrow$

$\Leftrightarrow ux^{-4} = -2\times\frac{x^{-6}}{-6} + c \Leftrightarrow ux^{-4} = \frac{x^{-6}}{3} + c \Leftrightarrow$

$\Leftrightarrow \frac{1}{y^2} = \frac{x^{-2}}{3} + cx^4 \Leftrightarrow y^2 = \frac{3}{x^{-2}+cx^4} \Leftrightarrow y = \sqrt{\frac{3}{x^{-2}+cx^4}}$

c) $xy' + x^2 - y = 0 \Leftrightarrow y' + x - \frac{1}{x}y = 0 \Leftrightarrow y' \underbrace{-\frac{1}{x}}_{P(x)} y = \underbrace{-x}_{Q(x)}$ ← E.D.O 1.ª ordem

f.i: $e^{\int P(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln(\frac{1}{x})} = \frac{1}{x}$

Multiplicando ambos os termos por $\frac{1}{x}$:

$\frac{1}{x}y' - \frac{1}{x^2}y = -1 \Leftrightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = -1 \Leftrightarrow \int d\left(\frac{y}{x}\right) = \int -1\,dx \Leftrightarrow$

$\Leftrightarrow \frac{y}{x} = -x + c \Leftrightarrow y = -x^2 + cx$

d) $3y'+y=(1-2x)y^4 \Leftrightarrow y'+\underbrace{\frac{1}{3}}_{P(x)}y=\underbrace{\frac{1-2x}{3}}_{Q(x)}y^4 \Leftrightarrow$ $\begin{cases} u=y^{1-4}=y^{-3} \\ u'=-3y^{-4}y' \end{cases}$

Multiplicando ambos os termos por $-3y^{-4}$:

$-3y^{-4}y'-yy^{-4}=-(1-2x) \Leftrightarrow u'\underbrace{-}_{P(x)}u=\underbrace{2x-1}_{Q(x)}$ f.i: $e^{\int P(x)dx}=e^{-\int 1dx}=e^{-x}$

Multiplicando ambos os termos por $e^{-x}$

$u'e^{-x}-ue^{-x}=(2x-1)e^{-x} \Leftrightarrow \frac{d}{dx}(ue^{-x})=(2x-1)e^{-x} \Leftrightarrow$ $\begin{cases} t=x \\ dt=1dx \end{cases}$ $\begin{cases} v=-e^{-x} \\ dv=e^{-x}dx \end{cases}$

$\Leftrightarrow \int d(ue^{-x})=\int(2xe^{-x})dx-\int e^{-x}dx \Leftrightarrow ue^{-x}+c=2\int xe^{-x}dx+e^{-x} \Leftrightarrow$

$\Leftrightarrow ue^{-x}+c=-2xe^{-x}-2\int -e^{-x}dx+e^{-x} \Leftrightarrow ue^{-x}=-2xe^{-x}-2e^{-x}+e^{-x}+c \Leftrightarrow$

$\Leftrightarrow ue^{-x}=-2xe^{-x}-e^{-x}+c \Leftrightarrow y^{-3}=-2x-1+ce^{x} \Leftrightarrow y=\sqrt[3]{\frac{1}{ce^{x}-2x-1}}$

e) $xy'=y(\ln y-\ln x+1) \Leftrightarrow \frac{dy}{dx}=\frac{y}{x}(\ln y-\ln x+1) \Leftrightarrow$

Verificar se é homogénea: $\Leftrightarrow \frac{dy}{dx}=\frac{y}{x}\left(\ln\left(\frac{y}{x}\right)+1\right) \Leftrightarrow *$

$\frac{d(\lambda y)}{d(\lambda x)}=\frac{\lambda y}{\lambda x}\left(\ln\frac{\lambda y}{\lambda x}+1\right) \Leftrightarrow \frac{dy}{dx}=\frac{y}{x}\left(\ln\left(\frac{y}{x}\right)+1\right)$, logo é homogénea

$\begin{cases} u=\frac{y}{x} \\ y=ux \end{cases}$ $* \Leftrightarrow \frac{d(ux)}{dx}=\frac{ux}{x}\left(\ln\left(\frac{ux}{x}\right)+1\right) \Leftrightarrow$

$\Leftrightarrow \frac{du}{dx}x+\frac{dx}{dx}u=u(\ln(u)+1) \Leftrightarrow \frac{du}{dx}x=u\ln(u)+u-u \Leftrightarrow$

$\Leftrightarrow \frac{du}{dx}x=u\ln(u) \Leftrightarrow \int\frac{1}{u\ln(u)}du=\int\frac{1}{x}dx \Leftrightarrow$ $\begin{cases} v=\ln(u) \\ dv=\frac{1}{u}du \end{cases}$

$\Leftrightarrow \int\frac{1}{v}dv=\ln(x)+c \Leftrightarrow \ln(v)=\ln(x)+c \Leftrightarrow \ln(\ln(u))=\ln(x)+c \Leftrightarrow$

$\Leftrightarrow \ln\left(\ln\left(\frac{y}{x}\right)\right)=\ln(x)+c \Leftrightarrow \ln\left(\ln\left(\frac{y}{x}\right)\right)=\ln(xc) \Leftrightarrow$

$\Leftrightarrow \ln\left(\frac{y}{x}\right)=xc \Leftrightarrow \ln\left(\frac{y}{x}\right)=\ln(e^{xc}) \Leftrightarrow \frac{y}{x}=e^{xc} \Leftrightarrow y=xe^{xc}$

f) $\frac{1}{y}dx-\frac{x}{y^2}dy=0 \Leftrightarrow \frac{1}{y}dx=\frac{x}{y^2}dy \Leftrightarrow \int\frac{1}{y}dy=\int\frac{1}{x}dx \Leftrightarrow$

$\Leftrightarrow \ln(y)=\ln(x)+c \Leftrightarrow \ln(y)=\ln(xc) \Leftrightarrow y=xc$

R: $y=xc$

**g)** $(x+y)\,dx+(x+2y)\,dy=0 \Leftrightarrow (x+y)\,dx=-(x+2y)\,dy \Leftrightarrow$

$\Leftrightarrow \dfrac{dy}{dx}=-\dfrac{x+y}{x+2y} \overset{(1)}{\Leftrightarrow}$ Verificar se é homogénea:

$\dfrac{d(\lambda y)}{d(\lambda x)}=-\dfrac{\lambda x+\lambda y}{\lambda x+2\lambda y} \Leftrightarrow \dfrac{\lambda\,dy}{\lambda\,dx}=-\dfrac{\lambda(x+y)}{\lambda(x+2y)} \Leftrightarrow \dfrac{dy}{dx}=-\dfrac{x+y}{x+2y}$ ✓

$\begin{cases} u=\dfrac{y}{x} \\ y=ux \end{cases} \overset{(1)}{\Leftrightarrow} \dfrac{d(ux)}{dx}=-\dfrac{x+ux}{x+2ux} \Leftrightarrow \dfrac{du}{dx}x+\dfrac{dx}{dx}u=-\dfrac{x(1+u)}{x(1+2u)} \Leftrightarrow$

$\Leftrightarrow \dfrac{du}{dx}x=-\dfrac{1+u}{1+2u}-u \Leftrightarrow \dfrac{du}{dx}x=\dfrac{-1-u-u-2u^2}{1+2u} \Leftrightarrow \dfrac{du}{dx}x=\dfrac{-1-2u-2u^2}{1+2u} \Leftrightarrow$

$\Leftrightarrow \displaystyle\int\frac{1+2u}{-1-2u-2u^2}\,du=\int\frac{1}{x}\,dx \Leftrightarrow -\frac{1}{2}\int\frac{-4u-2}{-1-2u-2u^2}\,du=\ln(x)+c \Leftrightarrow$

$\Leftrightarrow -\frac{1}{2}\ln(-1-2u-2u^2)=\ln(xc) \Leftrightarrow \ln\left(\dfrac{1}{\sqrt{-2u^2-2u-1}}\right)=\ln(xc) \Leftrightarrow$

$\Leftrightarrow \dfrac{1}{\sqrt{-2u^2-2u-1}}=xc \Leftrightarrow -2u^2-2u-1=\dfrac{1}{x^2c^2} \Leftrightarrow$

$\Leftrightarrow -2\dfrac{y^2}{x^2}-2\dfrac{y}{x}-1=\dfrac{1}{x^2}c \Leftrightarrow -2y^2-2yx-x^2=c \Leftrightarrow$

$\Leftrightarrow 2y^2+2yx+x^2=c \Leftrightarrow y^2+yx+\dfrac{x^2}{2}=c$

**4) a)** $\dfrac{dy}{dx}=\dfrac{2x-y}{x-2y} \overset{(1)}{\Leftrightarrow}$ Verificar se é homogénea:

$\dfrac{d(\lambda y)}{d(\lambda x)}=\dfrac{2\lambda x-\lambda y}{\lambda x-2\lambda y} \Leftrightarrow \dfrac{\lambda\,dy}{\lambda\,dx}=\dfrac{\lambda(2x-y)}{\lambda(x-2y)} \Leftrightarrow \dfrac{dy}{dx}=\dfrac{2x-y}{x-2y}$ ✓

$\begin{cases} u=\dfrac{y}{x} \\ y=ux \end{cases} \overset{(1)}{\Leftrightarrow} \dfrac{d(ux)}{dx}=\dfrac{2x-y}{x-2y} \Leftrightarrow \dfrac{du}{dx}x+\dfrac{dx}{dx}u=\dfrac{2x-ux}{x-2ux} \Leftrightarrow$

$\Leftrightarrow \dfrac{du}{dx}x=\dfrac{2x-ux-ux+2u^2x}{x-2ux} \Leftrightarrow \dfrac{du}{dx}x=\dfrac{2u^2x-2ux+2x}{x-2ux} \Leftrightarrow$

$\Leftrightarrow \dfrac{du}{dx}x=\dfrac{x(2u^2-2u+2)}{x(1-2u)} \Leftrightarrow \dfrac{du}{dx}=\dfrac{2u^2-2u+2}{1-2u}\times\dfrac{1}{x} \Leftrightarrow$

$\Leftrightarrow \int \frac{1-2u}{2u^2-2u+2} du = \int \frac{1}{x} dx \Leftrightarrow -\frac{1}{2}\int \frac{-4u-2}{2u^2-2u+2} du = \ln(x) + c \Leftrightarrow$

$\Leftrightarrow -\frac{1}{2} \ln (2u^2-2u+2) = \ln(x) + \ln(c) \Leftrightarrow \ln\left(\frac{1}{\sqrt{2u^2-2u+2}}\right) = \ln(xc) \Leftrightarrow$

$\Leftrightarrow \frac{1}{\sqrt{2u^2-2u+2}} = xc \Leftrightarrow 2u^2-2u+2 = \frac{1}{x^2c^2} \Leftrightarrow 2u^2-2u+2 = \frac{1}{x^2} c \Leftrightarrow$

$\Leftrightarrow 2\frac{y^2}{x^2} - 2\frac{y}{x} + 2 = \frac{1}{x^2} c \Leftrightarrow 2y^2 - 2yx + 2x^2 = c \Leftrightarrow$

$\Leftrightarrow y^2 - yx + x^2 = c \quad 3^2 - 3\times 1 + 1^2 = c \Leftrightarrow 9-3+1 = c \Leftrightarrow c = 7$

R: $y^2 - yx + x^2 = 7$

b) $y' + \underbrace{3x^2}_{P(x)} y = \underbrace{e^{-x^3+x}}_{Q(x)} \qquad$ f.i: $e^{\int P(x)dx} = e^{3\int x^2 dx} = e^{3\left(\frac{x^3}{3}\right)} = e^{x^3}$

Multiplicando ambos os termos pelo fator integrante:

$y' e^{x^3} + 3x^2 e^{x^3} y = e^{-x^3+x} \cdot e^{x^3} \Leftrightarrow \frac{d}{dx}\left(y e^{x^3}\right) = e^{x} \Leftrightarrow$

$\Leftrightarrow \int d\left(y e^{x^3}\right) = \int e^{x} dx \Leftrightarrow y e^{x^3} = e^{x} + c \Leftrightarrow y = e^{x-x^3} + c e^{-x^3}$

$2 = e^{0-0^3} + c \times e^{0} \Leftrightarrow 2 = 1 + c \Leftrightarrow c = 1 \quad$ R: $y = e^{x-x^3} + e^{-x^3}$

c) $(x^2-9)y' + xy = 0 \Leftrightarrow (x^2-9)\frac{dy}{dx} = -xy \Leftrightarrow -\int \frac{1}{y} dy = \int \frac{x}{x^2-9} dx \Leftrightarrow$

$\Leftrightarrow -\ln(y) + c = \frac{1}{2}\int \frac{2x}{x^2-9} dx \Leftrightarrow \ln\left(\frac{1}{y}\right) = \frac{1}{2}\ln(x^2-9) + \ln(c) \Leftrightarrow$

$\Leftrightarrow \ln\left(\frac{1}{y}\right) = \ln\left(c\sqrt{x^2-9}\right) \Leftrightarrow \frac{1}{y} = c\sqrt{x^2-9} \Leftrightarrow y = \frac{1}{c\sqrt{x^2-9}}$

$y_0 = \frac{1}{c\sqrt{5^2-9}} \Leftrightarrow y_0 c = \frac{1}{\sqrt{16}} \Leftrightarrow c = \frac{1}{4y_0}$

R: $y = \frac{1}{\frac{1}{4y_0}\sqrt{x^2-9}} \Leftrightarrow y_0 = \frac{4y_0}{\sqrt{x^2-9}}$

Z

(5) Mostrar que $y_1(x)=x^2$ é solução da equação:

$y_1'(x) = 2x \qquad y_1''(x) = 2$

$2x^2 - 12x^2 + 10x^2 = 0 \Leftrightarrow -10x^2 + 10x^2 = 0 \Leftrightarrow 0=0$ ✓

Mostrar que $y_2(x) = x^5$ é solução da equação:

$y_2'(x) = 5x^4 \qquad y''(x) = 20x^3$

$20x^3 x^2 - 6x \times 5x^4 + 10x^5 = 0 \Leftrightarrow 20x^5 - 30x^5 + 10x^5 = 0 \Leftrightarrow 0=0$ ✓

(6) a) $r_1 = 1 \quad r_2 = -1 \qquad (r-1)(r+1) = r^2 + r - r - 1 = r^2 - 1$

R: $y'' - y = 0$

b) $r_1 = 2 \quad r_2 = 2 \qquad (r-2)^2 = r^2 - 4r + 4$

R: $y'' - 4y' + 4y = 0$

e) $\alpha = -\frac{1}{2} \qquad \beta = 1 \qquad \lambda = -\frac{1}{2} \pm i \Leftrightarrow \lambda = -\frac{1}{2} + i \vee \lambda = -\frac{1}{2} - i$

$\left(r - \left(-\frac{1}{2} + i\right)\right)\left(r - \left(-\frac{1}{2} - i\right)\right) = \left(r + \frac{1}{2} - i\right)\left(r + \frac{1}{2} + i\right) =$

$= r^2 + \frac{r}{2} + ri + \frac{r}{2} + \frac{1}{4} + \frac{1}{2}i - ir - \frac{1}{2}i - i^2 =$

$= r^2 + r + \frac{1}{4} + 1 = r^2 + r + \frac{5}{4} \qquad$ R: $y'' + y' + \frac{5}{4}y = 0$

(7) a) $y'' - 4y = 0 \qquad \lambda^2 - 4 = 0 \Leftrightarrow \lambda^2 = 4 \Leftrightarrow \lambda = \pm\sqrt{4} \Leftrightarrow \lambda = -2 \vee \lambda = 2$

R: $y = c_1 e^{2x} + c_2 e^{-2x}$

b) $y'' + 2y' + y = 0 \quad \lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow \lambda = \frac{-2 \pm \sqrt{4-4}}{2} \Leftrightarrow \lambda = -1$

R: $y = c_1 e^{-x} + c_2 x e^{-x}$

c) $y'' - 5y' + 6y = 0 \quad \lambda^2 - 5\lambda + 6 = 0 \Leftrightarrow \lambda = \frac{5 \pm \sqrt{25-24}}{2} \Leftrightarrow \lambda = 3 \vee \lambda = 2$

R: $y = c_1 e^{3x} + c_2 e^{2x}$

**d)** $y'' + 4y' + 13y = 0 \qquad \lambda^2 + 4\lambda + 13 = 0 \Leftrightarrow \lambda = \frac{-4 \pm \sqrt{16-52}}{2} \Leftrightarrow$

$\Leftrightarrow \lambda = \frac{-4 \pm \sqrt{-36}}{2} \Leftrightarrow \lambda = \frac{-4 \pm 6i}{2} \Leftrightarrow \lambda = -2 \pm 3i$

$y = c_1 e^{-2x}\cos(3x) + c_2 e^{-2x}\sin(3x)$

**e)** $y'' - 6y' + 9y = e^{3x}\ln(x) \qquad y'' - 6y' + 9y = 0 \leftarrow$ eq. homogénea

$\lambda^2 - 6\lambda + 9 = 0 \Leftrightarrow \lambda = \frac{6 \pm \sqrt{36-36}}{2} \Leftrightarrow \lambda = 3 \qquad y = c_1 e^{3x} + c_2 x e^{3x}$

$y = c_1(x)e^{3x} + c_2(x)x e^{3x} \leftarrow$ eq. geral $\qquad \begin{cases} c_1'(x)e^{3x} + c_2'(x)xe^{3x} = 0 \\ c_1'(x)\cdot 3e^{3x} + c_2'(x)(e^{3x} + 3xe^{3x}) = e^{3x}\ln(x) \end{cases}$

$\underbrace{\begin{bmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{bmatrix}}_{W} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{3x}\ln(x) \end{bmatrix} \qquad |W| = e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{6x}$

$c_1' = \frac{\begin{vmatrix} 0 & xe^{3x} \\ e^{3x}\ln(x) & e^{3x}+3xe^{3x} \end{vmatrix}}{e^{6x}} = \frac{-e^{6x}x\ln(x)}{e^{6x}} = -x\ln(x)$

$c_2' = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & e^{3x}\ln(x) \end{vmatrix}}{e^{6x}} = \frac{e^{6x}\ln(x)}{e^{6x}} = \ln(x)$

$\begin{cases} u = \ln(x) \\ du = \frac{1}{x}dx \end{cases} \quad \begin{cases} v = \frac{x^2}{2} \\ dv = x\,dx \end{cases}$

$c_1 = -\int x\ln(x) = -\left(\frac{x^2}{2}\ln(x) - \int\left(\frac{x^2}{2} \times \frac{1}{x}\right)dx\right) = -\frac{x^2}{2}\ln(x) + \int \frac{x}{2}\,dx =$

$= -\frac{x^2}{2}\ln(x) + \frac{1}{2} \times \frac{x^2}{2} = -\frac{x^2}{2}\ln(x) + \frac{x^2}{4}$

$\begin{cases} u = \ln(x) \\ du = \frac{1}{x}dx \end{cases} \quad \begin{cases} v = x \\ dv = 1\,dx \end{cases}$

$c_2 = \int \ln(x) = x\ln(x) - \int \frac{1}{x}x\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x$

$y = y_H + y_P = c_1 e^{3x} + c_2 x e^{3x} + \left(-\frac{x^2}{2}\ln(x) + \frac{x^2}{4}\right)e^{3x} + (x\ln(x) - x)xe^{3x} =$

$= c_1 e^{3x} + c_2 x e^{3x} - \frac{x^2}{2}\ln(x)e^{3x} + \frac{x^2}{4}e^{3x} + x^2\ln(x)e^{3x} - x^2 e^{3x} =$

$= c_1 e^{3x} + c_2 x e^{3x} + \frac{x^2}{2}\ln(x)e^{3x} - \frac{3}{4}x^2 e^{3x}$

**f)** $y'' + 9y = \sec(3x) \qquad y'' + 9y = 0 \rightarrow$ eq. homogénea

$\lambda^2 + 9 = 0 \Leftrightarrow \lambda^2 = -9 \Leftrightarrow \lambda = \pm\sqrt{-9} \Leftrightarrow \lambda = \pm 3i$

eq. geral

$y = c_1\cos(3x) + c_2\sin(3x) \leftarrow y_H \qquad y = c_1(x)\cos(3x) + c_2(x)\sin(3x)$

$\begin{cases} c_1'\cos(3x) + c_2'\sin(3x) = 0 \\ c_1'(-3\sin(3x)) + c_2' 3\cos(3x) = \sec(3x) \end{cases}$

no final

(continuação ex. no gene verde (MDis)) $\sin(nx)\cos(mx) = \frac{1}{2}\left[\sin((n-m)t) + \sin((n+m)t)\right]$ ← provavelmente não sai

$$\underbrace{\begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}}_{W} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(3x) \end{bmatrix}$$

$$|W| = 3\cos^2(3x) + 3\sin^2(3x) = $$
$$= 3(\cos^2(3x) + \sin^2(3x)) =$$
$$= 3 \times 1 = 3$$

$$c_1' = \frac{\begin{vmatrix} 0 & \sin(3x) \\ \sec(3x) & 3\cos(3x) \end{vmatrix}}{3} = \frac{-\sin(3x)\sec(3x)}{3} = \frac{-\sin(3x) \times \frac{1}{\cos(3x)}}{3} = \frac{-\tan(3x)}{3}$$

$$c_2' = \frac{\begin{vmatrix} \cos(3x) & 0 \\ -3\sin(3x) & \sec(3x) \end{vmatrix}}{3} = \frac{\cos(3x)\sec(3x)}{3} = \frac{\cos(3x) \times \frac{1}{\cos(3x)}}{3} = \frac{1}{3}$$

$$c_1 = -\frac{1}{3}\int \tan(3x)\,dx = -\frac{1}{3}\int \frac{\sin(3x)}{\cos(3x)}dx = \frac{1}{9}\int \frac{-3\sin(3x)}{\cos(3x)}dx = \frac{1}{9}\ln(\cos(3x))$$

$$c_2 = \int \frac{1}{3}\,dx = \frac{x}{3}$$

$$y = y_H + y_P = C_1\cos(3x) + C_2\sin(3x) + \frac{1}{9}\ln(\cos(3x))\cos(3x) + \frac{x}{3}\sin(3x)$$