7. Primitivação por decomposição em frações simples

a) $\int \frac{x+1}{x^3+x^2-6x}\,dx = \int \frac{x+1}{x(x^2+x-6)}\,dx =$

CA: $x^2+x-6=0 \Leftrightarrow x=\frac{-1\pm\sqrt{25}}{2} \Leftrightarrow$
$\Leftrightarrow x=\frac{-1\pm5}{2} \Leftrightarrow x=2 \vee x=-3$

$= \int \frac{x+1}{x(x+3)(x-2)}$ $*$ $\Big| \frac{x+1}{x(x+3)(x-2)} = \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-2}$

$x+1 = A(x+3)(x-2)+Bx(x-2)+Cx(x+3) \Leftrightarrow$

$\Leftrightarrow x+1 = A(x^2+x-6)+Bx^2-2Bx+Cx^2+3Cx \Leftrightarrow$

$\Leftrightarrow x+1 = Ax^2+Ax-6A+Bx^2-2Bx+Cx^2+3Cx$

Método dos coeficientes indeterminados:

$$\begin{array}{l} [x^2] \\ [x^1] \\ [x^0] \end{array} \begin{cases} A+B+C=0 \\ A-2B+3C=1 \\ -6A=1 \end{cases}$$

$A=-\frac{1}{6}$ $\quad B=\frac{1}{6}-C$

$-\frac{1}{6}-2\left(\frac{1}{6}-C\right)+3C=1 \Leftrightarrow$

$\Leftrightarrow -\frac{1}{2}+2C+3C=1 \Leftrightarrow 5C=\frac{3}{2} \Leftrightarrow C=\frac{3}{10}$

$B=\frac{1}{6}-\frac{3}{10}=-\frac{8}{60}=-\frac{2}{15}$

$* = \int\left(-\frac{1}{6x}-\frac{2}{15x+45}+\frac{3}{10x-20}\right)dx = -\frac{1}{6}\int\frac{1}{x}dx-\frac{2}{15}\int\frac{1}{x+3}dx+\frac{3}{10}\int\frac{1}{x-2}dx$

$= -\frac{1}{6}\ln(x)-\frac{2}{15}\ln(x+3)+\frac{3}{10}\ln(x-2)+c,\ c\in\mathbb{R}$

b) $\int \frac{x^4-x^3-3x^2-2x+2}{(x^3+x^2-2)x}\,dx = \int \frac{x^4-x^3-3x^2-2x+2}{x^4+x^3-2x}\,dx =$

CA:
$$\begin{array}{rrrrr|l} x^4 & -x^3 & -3x^2 & -2x & +2 & x^4+x^3-2x \\ \hline -x^4 & -x^3 & -3x^2 & +2x & +2 & 1 \\ \hline & -2x^3 & -3x^2 & & +2 & \end{array}$$

$= \int\left(1+\frac{-2x^3-3x^2+2}{x(x^3+x^2-2)}\right)dx =$

$= \int 1\,dx + \int \frac{-2x^3-3x^2+2}{x(x-1)(x^2+2x+2)}\,dx \overset{*1}{=}$

CA:
$$\begin{array}{c|cccc} & 1 & 1 & 0 & -2 \\ 1 & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

$x^3+x^2-2 = (x-1)(x^2+2x+2)$

$\frac{-2x^3-3x^2+2}{x(x-1)(x^2+2x+2)} = \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+2x+2}$

$-2x^3-3x^2+2 = A(x-1)(x^2+2x+2)+Bx(x^2+2x+2)+Cx(x^2-x)+D(x^2-x) \Leftrightarrow$

$\Leftrightarrow -2x^3-3x^2+2 = A(x^3+x^2-2)+Bx^3+2Bx^2+2Bx+Cx^3-Cx^2+Dx^2-Dx \Leftrightarrow$

$\Leftrightarrow -2x^3-3x^2+2 = Ax^3+Ax^2-2A+Bx^3+2Bx^2+2Bx+Cx^3-Cx^2+Dx^2-Dx \Leftrightarrow$

$$\begin{cases} A+B+C=-2 \\ A+2B-C+D=-3 \\ 2B-D=0 \\ -2A=2 \end{cases}$$

CA:

$A=-1 \quad D=2B \quad -1+2B-C+2B=-3 \Leftrightarrow$

$\Leftrightarrow C=4B+2 \quad -1+B+4B+2=-2 \Leftrightarrow$

$\Leftrightarrow 5B=-3 \Leftrightarrow B=-\frac{3}{5} \quad D=2\left(-\frac{3}{5}\right)=-\frac{6}{5}$

$C=4\left(-\frac{3}{5}\right)+2=-\frac{12}{5}+2=-\frac{2}{5}$

$$\frac{-2x^3-3x^2+2}{x(x-2)(x^2+2x+2)}=$$

$$=-\frac{1}{x}-\frac{3}{5}\times\frac{1}{x-1}+\frac{-\frac{2}{5}x-\frac{6}{5}}{x^2+2x+2}=-\frac{1}{x}-\frac{3}{5}\times\frac{1}{x-1}-\frac{2}{5}\times\frac{x+3}{x^2+2x+2}$$

$$* = x+\int\left(-\frac{1}{x}-\frac{3}{5}\times\frac{1}{x-1}-\frac{2}{5}\times\frac{x+3}{x^2+2x+2}\right)dx = x-\int\frac{1}{x}dx-\frac{3}{5}\int\frac{1}{x-1}dx-\frac{2}{5}\int\frac{x+3}{x^2+2x+2}dx=$$

$$=x-\ln(x)-\frac{3}{5}\times\ln(x-1)-\frac{1}{5}\int\frac{2x+2+4}{x^2+2x+2}dx+c=$$

$$=x-\ln(x)-\frac{3}{5}\times\ln(x-1)-\frac{1}{5}\int\frac{2x+2}{x^2+2x+2}dx-\frac{4}{5}\int\frac{1}{x^2+2x+2}dx+c=$$

$$=x-\ln(x)-\frac{3}{5}\times\ln(x-1)-\frac{1}{5}\ln(x^2+2x+2)-\frac{4}{5}\int\frac{1}{1+(x+1)^2}dx+c=$$

$$=x-\ln(x)-\frac{3}{5}\times\ln(x-1)-\frac{1}{5}\times\ln(x^2+2x+2)-\frac{4}{5}\arctan(x+1)+c$$

e) $\int\frac{x^4-x^3-x-1}{x^3-x^2}dx=$

CA:

$$\begin{array}{l|l} x^4-x^3-x-1 & x^3-x^2 \\ \hline -x^4+x^3 & x \\ 0 \quad 0 \quad -x-1 & \end{array}$$

$$\frac{x^4-x^3-x-1}{x^3-x^2}=x-\frac{x+1}{x^3-x^2}$$

$$=\int\left(x-\frac{x+1}{x^3-x^2}\right)dx=\int x\,dx-\int\frac{x+1}{x^3-x^2}dx \; *$$

CA:

$$\frac{x+1}{x^3-x^2}=\frac{x+1}{x^2(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}$$

$x+1=A(x-1)+B(x^2-x)+Cx^2 \Leftrightarrow$

$\Leftrightarrow x+1=Ax-A+Bx^2-Bx+Cx^2$

Pelo método dos coeficientes indeterminados:

$$\begin{cases} B+C=0 \\ A-B=1 \\ -A=1 \end{cases}$$

$A=-1 \quad -1-B=1 \Leftrightarrow B=-2 \quad -2+C=0 \Leftrightarrow C=2$

$$\frac{x+1}{x^3-x^2}=-\frac{1}{x^2}-\frac{2}{x}+\frac{2}{x-1}$$

$$* = \frac{x^2}{2}-\int\left(-\frac{1}{x^2}-\frac{2}{x}+\frac{2}{x-1}\right)dx=\frac{x^2}{2}+\int\frac{1}{x^2}dx-2\int\frac{1}{x}dx+2\int\frac{1}{x-1}dx=$$

$$=\frac{x^2}{2}-\frac{1}{x}+2\ln(x)-2\ln(x-1)+c,\ c\in\mathbb{R}$$

d) $\int \frac{x^3+2}{x^3-1}\,dx = \int \left(1+\frac{3}{x^3-1}\right)dx =$

$= \int 1\,dx + 3\int \frac{1}{x^3-1}\,dx = x + 3\int \frac{1}{x^3-1}\,dx \overset{(1)}{}$

CA:

$$\begin{array}{rrrr|l} x^3 & +0x^2 & +0x & +2 & x^3-1 \\ -x^3 & \cancel{} & \cancel{} & +1 & 1 \\ \hline 0 & 0 & 0 & 3 & \end{array}$$

$\frac{x^3+2}{x^3-1} = 1 + \frac{3}{x^3-1}$

$$\begin{array}{c|cccc} & 1 & 0 & 0 & -1 \\ 1 & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$

$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$1 = A(x^2+x+1) + Bx(x-1) + C(x-1) \Leftrightarrow$

$\Leftrightarrow 1 = Ax^2+Ax+A+Bx^2-Bx+Cx-C$

$$\begin{array}{l} [x^2] \\ [x^1] \\ [x^0] \end{array}\begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=1 \end{cases}$$

$A=-B$

$C=2B$

$-B-2B=1 \Leftrightarrow$

$\Leftrightarrow B=-\frac{1}{3}$

$C=2\left(-\frac{1}{3}\right)=-\frac{2}{3}$ $\quad A=\frac{1}{3}$

$\overset{(1)}{=} x + 3\int\left(\frac{1}{3(x-1)} + \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\right)dx =$

$= x + 3\int \frac{1}{3}\times\frac{1}{x-1}\,dx + 3\int -\frac{1}{3}\times\frac{x+2}{x^2+x+1}\,dx$

$= x + \int\frac{1}{x-1}\,dx - \int\frac{x+2}{x^2+x+1}\,dx = x + \ln(x-1) + c - \frac{1}{2}\int\frac{2x+1+3}{x^2+x+1}\,dx =$

$= x + \ln(x-1) + c - \frac{1}{2}\int\frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int\frac{1}{x^2+x+1}\,dx =$

$= x + \ln(x-1) - \frac{1}{2}\ln(x^2+x+1) + c - \frac{3}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx =$

$= x + \ln(x-1) - \frac{1}{2}\ln(x^2+x+1) + c - \frac{3}{2}\times\frac{4}{3}\int\frac{1}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1}\,dx =$

$= x + \ln(x-1) - \frac{1}{2}\ln(x^2+x+1) + c - \frac{12}{6}\int\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx =$

$= x + \ln(x-1) - \frac{1}{2}\ln(x^2+x+1) + c - 2\times\frac{\sqrt{3}}{2}\int\frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx =$

$= x + \ln(x-1) - \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c,\ c\in\mathbb{R}$

e) $\int\frac{2x^2+3}{(x^2+1)^2}\,dx \overset{(1)}{=}$

$\frac{2x^2+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}$

$2x^2+3 = Ax+B+Cx(x^2+1)+Dx^2+D \Leftrightarrow$

$\Leftrightarrow 2x^2+3 = Ax+B+Cx^3+Cx+Dx^2+D$

$$\begin{array}{l} [x^3] \\ [x^2] \\ [x^1] \\ [x^0] \end{array}\begin{cases} C=0 \\ D=2 \\ A+C=0 \\ B+D=3 \end{cases}$$

$A=0$

$B+2=3 \Leftrightarrow B=1$

$= \int\left(\frac{1}{(x^2+1)^2} + \frac{2}{x^2+1}\right)dx =$

$= \int\frac{1}{(x^2+1)^2}\,dx + 2\int\frac{1}{x^2+1}\,dx =$

$= \int\frac{1}{(x^2+1)^2}\,dx + 2\arctan(x) + c \overset{(2)}{=}$

$$\begin{cases} x = \tan\theta \Leftrightarrow \theta = \arctan(x) \\ dx = \sec^2\theta \, d\theta \end{cases}$$

$$\overset{(2)}{=} \int \frac{1}{(\tan^2\theta+1)^2} \sec^2\theta \, d\theta + 2\arctan(x) + c = \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta \, d\theta + 2\arctan(x) + c =$$

$$= \int \frac{1}{\sec^2\theta} d\theta + 2\arctan(x) + c = \int \cos^2\theta \, d\theta + 2\arctan(x) + c =$$

$$= \int \frac{1+\cos(2\theta)}{2} d\theta + 2\arctan(x) + c = \int \frac{1}{2} d\theta + \frac{1}{2}\int \cos(2\theta) d\theta + 2\arctan(x) + c =$$

$$= \frac{\theta}{2} + \frac{1}{2} \times \frac{1}{2} \times \sin(2\theta) + \arctan(x) + c = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + 2\arctan(x) + c =$$

$$= \frac{\arctan(x)}{2} + \frac{\sin(2\arctan(x))}{4} + 2\arctan(x) + c =$$

$$= \frac{5}{2}\arctan(x) + \frac{\sin(2\arctan(x))}{4} + c, \; c \in \mathbb{R}$$

b) $$\int \frac{x^2+x+2}{(x^2+2x+3)^2} dx =$$

$$\frac{x^2+x+2}{(x^2+2x+3)^2} = \frac{Ax+B}{(x^2+2x+3)^2} + \frac{Cx+D}{x^2+2x+3}$$

$$x^2+x+2 = Ax+B+Cx(x^2+2x+3)+D(x^2+2x+3)$$

$$\Leftrightarrow x^2+x+2 = Ax+B+Cx^3+2Cx^2+3Cx+Dx^2+2Dx+3D$$

$$\begin{array}{l} [x^3] \\ [x^2] \\ [x^1] \\ [x^0] \end{array} \begin{cases} C = 0 \\ 2C+D = 1 \\ A+3C+2D = 1 \\ B+3D = 2 \end{cases} \quad \begin{array}{l} \\ 2\times 0 + D = 1 \Leftrightarrow D = 1 \\ A+2 = 1 \Leftrightarrow A = -1 \\ B+3 = 2 \Leftrightarrow B = -1 \end{array}$$

$$= \int \left(\frac{-x-1}{(x^2+2x+3)^2} + \frac{1}{x^2+2x+3} \right) dx =$$

$$= -\int \frac{x+1}{(x^2+2x+3)^2} dx + \int \frac{1}{x^2+2x+3} dx =$$

$$\begin{cases} u = x^2+2x+3 \\ du = (2x+2)dx \end{cases}$$

$$= -\frac{1}{2}\int \frac{2x+2}{(x^2+2x+3)^2} dx + \int \frac{1}{(x+1)^2+2} dx = -\frac{1}{2}\int \frac{1}{u^2} du + \frac{1}{2}\int \frac{1}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1} dx =$$

$$= -\frac{1}{2} \times \frac{u^{-1}}{-1} + \frac{\sqrt{2}}{2}\int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1} dx = -\frac{1}{2u} + \frac{\sqrt{2}}{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c =$$

$$= \frac{1}{2(x^2+2x+3)} + \frac{\sqrt{2}}{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c =$$

$$= \frac{1}{2x^2+4x+6} + \frac{\sqrt{2}}{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c, \; c \in \mathbb{R}$$

$(x-2)^3 = (x-2)(x-2)^2 = (x-2)(x^2-4x+4) = (x^3-6x^2+12x-8)$

g) $\displaystyle\int \frac{x^3-1}{x^2(x-2)^3}\,dx =$

$= \displaystyle\int \frac{1}{8x^2} + \frac{3}{16x} + \frac{7}{4(x-2)^3} + \frac{5}{4(x-2)^2} - \frac{3}{16(x-2)}\,dx$

$= \displaystyle\frac{1}{8}\int\frac{1}{x^2}dx + \frac{3}{16}\int\frac{1}{x}dx + \frac{7}{4}\int\frac{1}{(x-2)^3}dx + \frac{5}{4}\int\frac{1}{(x-2)^2} - \frac{3}{16}\int\frac{1}{x-2}dx$

$\begin{cases} u = x-2 \\ du = 1\,dx \end{cases}$

$= \displaystyle -\frac{1}{8x} + \frac{3}{16}\ln(x) + \frac{7}{4}\int\frac{1}{u^3}du + \frac{5}{4}\int\frac{1}{u^2}du - \frac{3}{16}\ln(x-2)$

$= \displaystyle -\frac{1}{8x} + \frac{3}{16}\ln(x) + \frac{7}{4}\left(-\frac{1}{2u^2}\right) + \frac{5}{4}\left(-\frac{1}{u}\right) - \frac{3}{16}\ln(x-2) =$

$= \displaystyle -\frac{1}{8x} + \frac{3}{16}\ln(x) - \frac{7}{8(x-2)^2} - \frac{5}{4x-8} - \frac{3}{16}\ln(x-2) + c,\ c\in\mathbb{R}$

$\displaystyle\frac{x^3-1}{x^2(x-2)^3} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^2} + \frac{E}{x-2}$

$x^3-1 = A(x-2)^3 + Bx(x-2)^3 + Cx^2 + Dx^2(x-2) + Ex^2(x-2)^2 \Leftrightarrow$

$\Leftrightarrow x^3-1 = A(x^3-6x^2+12x-8) + B(x^4-6x^3+12x^2-8x) + Cx^2 + Dx^3 - 2Dx^2 + Ex^2(x^2-4x+4)$

$\Leftrightarrow x^3-1 = Ax^3 - 6Ax^2 + 12Ax - 8A + Bx^4 - 6Bx^3 + 12Bx^2 - 8Bx + Cx^2 + Dx^3 - 2Dx^2 + Ex^4 - 4Ex^3 + 4Ex^2$

$\begin{array}{l|l} [x^4] & B + E = 0 \\ [x^3] & A - 6B + D - 4E = 1 \\ [x^2] & -6A + 12B + C - 2D + 4E = 0 \\ [x^1] & 12A - 8B = 0 \\ [x^0] & -8A = -1 \end{array}$

$E = -\frac{3}{16}$ $\quad A = \frac{1}{8}$ $\quad \frac{12}{8} - 8B = 0 \Leftrightarrow \frac{3}{2} - 8B = 0 \Leftrightarrow B = \frac{3}{16}$

$\frac{1}{8} - 6\left(\frac{3}{16}\right) + D - 4\left(-\frac{3}{16}\right) = 1 \Leftrightarrow$

$\Leftrightarrow \frac{1}{8} - \frac{9}{8} + D + \frac{6}{8} = 1 \Leftrightarrow D = \frac{5}{4}$

$-6\left(\frac{1}{8}\right) + 12\left(\frac{3}{16}\right) + C - 2\left(\frac{5}{4}\right) + 4\left(-\frac{3}{16}\right) = 0 \Leftrightarrow$

$\Leftrightarrow -\frac{3}{4} + \frac{9}{4} + C - \frac{10}{4} - \frac{3}{4} = 0 \Leftrightarrow C = \frac{7}{4}$

h) $\displaystyle\int \frac{x^2}{x^4-1}\,dx = \int \frac{x^2}{(x^2-1)(x^2+1)}\,dx = \int \frac{x^2}{(x-1)(x+1)(x^2+1)}\,dx =$

$= \displaystyle\int\left(\frac{1}{4}\times\frac{1}{x-1} - \frac{1}{4}\times\frac{1}{x+1} + \frac{1}{2}\times\frac{1}{x^2+1}\right)dx =$

$= \displaystyle\frac{1}{4}\int\frac{1}{x-1}dx - \frac{1}{4}\int\frac{1}{x+1}dx + \frac{1}{2}\int\frac{1}{x^2+1}dx =$

$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) + \frac{1}{2}\arctan(x) + c =$

$= \frac{1}{4}\left(\ln(x-1) - \ln(x+1)\right) + \frac{1}{2}\arctan(x) + c =$

$= \displaystyle\frac{1}{4}\ln\left(\frac{x-1}{x+1}\right) + \frac{1}{2}\arctan(x) + c,\ c\in\mathbb{R}$

$\displaystyle\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$x^2 = A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx(x^2-1) + D(x^2-1) \Leftrightarrow$

$\Leftrightarrow x^2 = Ax^3 + Ax^2 + Ax + A + Bx^3 - Bx^2 + Bx - B + Cx^3 - Cx + Dx^2 - D$

$\begin{array}{l|l} [x^3] & A + B + C = 0 \\ [x^2] & A - B + D = 1 \\ [x^1] & A + B - C = 0 \\ [x^0] & A - B - D = 0 \end{array}$

$A - B + A - B = 1 \Leftrightarrow 2(A-B) = 1 \Leftrightarrow A - B = \frac{1}{2} \Leftrightarrow A = \frac{1}{2} + B$

$C = A + B$

$D = A - B$

$D = \frac{1}{2}$ $\quad 2A + 2B = 0 \Leftrightarrow A = -B$

$B = A - \frac{1}{2} \Leftrightarrow B = -B - \frac{1}{2} \Leftrightarrow 2B = -\frac{1}{2} \Leftrightarrow$

$\Leftrightarrow B = -\frac{1}{4}$ $\quad A = \frac{1}{4}$ $\quad C = -\frac{1}{4} + \frac{1}{4} = 0$

$A = \frac{1}{4}$ $\quad B = -\frac{1}{4}$ $\quad C = 0$ $\quad D = \frac{1}{2}$