6. Primitivação por partes

a) $\int x \cos x \, dx =$ $\quad \begin{cases} u = x \\ du = 1\,dx \end{cases} \quad \begin{cases} V = \sin x \\ dV = \cos x \, dx \end{cases}$

$= x \sin x - \int \sin x \, dx = x \sin x - (-\cos x) + C = x \sin x + \cos x + C, C \in \mathbb{R}$

b) $\int x^2 \sin x \, dx =$ $\quad \begin{cases} u = x^2 \\ du = 2x\,dx \end{cases} \quad \begin{cases} V = -\cos x \\ dV = \sin x \, dx \end{cases}$

$= -x^2 \cos(x) - \int -2x \cos(x) \, dx = -x^2 \cos(x) + 2 \int x \cos(x) \, dx =$ $\begin{cases} u = x \\ du = 1dx \end{cases} \begin{cases} V = \sin x \\ dV = \cos x \, dx \end{cases}$

$= -x^2 \cos(x) + 2 \left(x \sin(x) - \int \sin(x) \, dx \right) = -x^2 \cos(x) + 2 \left(x \sin x - (-\cos x) \right) + C =$

$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C, C \in \mathbb{R}$

c) $\int \sin^2(x) \, dx = \int \sin x \times \sin x \, dx =$ $\Bigg| \quad \begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases} \quad \begin{cases} V = -\cos x \\ dV = \sin x \, dx \end{cases}$

$= -\sin x \cos x - \int -\cos^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx =$

$= -\sin x \cos x + \int (1 - \sin^2 x) dx = -\sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx =$

$= -\sin x \cos x + x - \int \sin^2(x) \, dx$

Logo, $\int \sin^2(x) \, dx = -\sin x \cos x + x - \int \sin^2(x) \, dx \Leftrightarrow$

$\Leftrightarrow 2 \int \sin^2(x) \, dx = -\sin x \cos x + x \Leftrightarrow$

$\Leftrightarrow \int \sin^2(x) \, dx = \dfrac{x - \sin x \cos x}{2} + C, C \in \mathbb{R}$

d) $\int \cos^5 x \, dx = \int \cos x \cos^4 x \, dx = \int \cos x (\cos^2 x)^2 \, dx =$

$= \int \cos x (1 - \sin^2 x)^2 \, dx =$ Seja $u = \sin x$ $\quad du = \cos x \, dx$

$= \int (1-u^2)^2 \, du = \int (1 - 2u^2 + u^4) du = \int 1 \, du - 2\int u^2 \, du + \int u^4 \, du =$

$= u - 2\left(\frac{1}{3}u^3\right) + \frac{1}{5}u^5 + c = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c = \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c, c \in \mathbb{R}$

e) $\int \cos^4 x \, dx = \int \cos^3 x \times \cos x \, dx =$ $\begin{cases} u = \cos^3 x \\ du = 3\cos^2 x \times (-\sin x) = -3\cos^2 x \sin x \end{cases}$

$= \cos^3 x \sin x - \int -3\cos^2 x \sin^2 x =$ $\begin{cases} V = \sin x \\ dV = \cos x \, dx \end{cases}$

$= \cos^3 x \sin x + 3\int \cos^2 x \sin^2 x =$

Seja $u = 2x$ $\quad du = 2\,dx$

$= \cos^3 x \sin x + 3\int \frac{\sin^2(2x)}{4} dx = \cos^3 x \sin x + \frac{3}{4}\int \sin^2(2x) \, dx =$

$= \cos^3 x \sin x + \frac{3}{4}\int \frac{\sin^2(u)}{2} du = \cos^3 x \sin x + \frac{3}{8}\int \sin^2(u) \, du =$

$= \cos^3 x \sin x + \frac{3}{8}\int \frac{1-\cos(2u)}{2} du = \cos^3 x \sin x + \frac{3}{16}\int 1 - \cos(2u) \, du =$

$= \cos^3 x \sin x + \frac{3}{16}\left(u - \frac{\sin(2u)}{2}\right) + c = \cos^3 x \sin x + \frac{3}{16}\left(2x - \frac{\sin(4x)}{2}\right) + c =$

$= \cos^3 x \sin x + \frac{3}{8}x - \frac{3\sin(4x)}{32} + c, c \in \mathbb{R}$

f) $\int e^x \cos x \, dx =$ $\begin{cases} u = \cos x \\ du = -\sin x \end{cases}$ $\begin{cases} V = e^x \\ dV = e^x \, dx \end{cases}$

$= e^x \cos x - \int -e^x \sin x \, dx = e^x \cos x + \int e^x \sin x \, dx =$ $\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$ $\begin{cases} V = e^x \\ dV = e^x dx \end{cases}$

$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

Logo, $\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx \Leftrightarrow$

$\Leftrightarrow 2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x \Leftrightarrow \int e^x \cos x \, dx = \frac{e^x \cos x + e^x \sin x}{2} +$

$+ c, c \in \mathbb{R}$

g) $\int x^2 e^x \, dx = \quad \begin{cases} u = x^2 \\ du = 2x \, dx \end{cases} \quad \begin{cases} v = e^x \\ dv = e^x \, dx \end{cases}$

$= x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x e^x \, dx = \quad \begin{cases} u = x \\ du = 1 \, dx \end{cases} \quad \begin{cases} v = e^x \\ dv = e^x \, dx \end{cases}$

$= x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right) = x^2 e^x - 2x e^x + 2 \int e^x \, dx =$

$= x^2 e^x - 2x e^x + 2e^x + c, \; c \in \mathbb{R}$

h) $\int \ln(1+x^2) \, dx = \quad \begin{cases} u = \ln(1+x^2) \\ du = \frac{2x}{1+x^2} \, dx \end{cases} \quad \begin{cases} v = x \\ dv = 1 \, dx \end{cases}$

$= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} \, dx = x \ln(1+x^2) - 2 \int \frac{x^2}{1+x^2} \, dx =$

$= x \ln(1+x^2) - 2 \int \frac{x^2+1-1}{1+x^2} \, dx = x \ln(1+x^2) - 2 \int \left(1 - \frac{1}{1+x^2}\right) dx =$

$= x \ln(1+x^2) - 2 \left[\int 1 \, dx - \int \frac{1}{1+x^2} \, dx \right] = x \ln(1+x^2) - 2 \left[x - \arctan(x) \right] =$

$= x \ln(1+x^2) - 2x + 2 \arctan(x) + c, \; c \in \mathbb{R}$

i) $\int x \ln^2(x) \, dx = \quad \begin{cases} u = \ln^2(x) \\ du = \frac{2\ln(x)}{x} \, dx \end{cases} \quad \begin{cases} v = \frac{x^2}{2} \\ dv = x \, dx \end{cases}$

$= \frac{x^2 \ln^2(x)}{2} - \int \frac{2x^2 \ln(x)}{2x} \, dx = \frac{x^2 \ln^2(x)}{2} - \int x \ln(x) \, dx = \quad \begin{cases} u = \ln(x) \\ du = \frac{1}{x} \end{cases} \quad \begin{cases} v = \frac{x^2}{2} \\ dv = x \end{cases}$

$= \frac{x^2 \ln^2(x)}{2} - \left[\frac{x^2 \ln(x)}{2} - \int \frac{x^2}{2x} \, dx \right] = \frac{x^2 \ln^2(x)}{2} - \frac{x^2 \ln(x)}{2} + \int \frac{x}{2} \, dx =$

$= \frac{x^2 \ln^2(x) - x^2 \ln(x)}{2} + \frac{1}{4} x^2 + c = \frac{2x^2 \ln(x)\left(\ln(x) - 1\right) + x^2}{4} + c, \; c \in \mathbb{R}$

$\frac{x^3}{x+1} = x^2 - \frac{x^2}{x+1}$

j) $\int x^2 \ln(1+x) \, dx = \quad \begin{cases} u = \ln(1+x) \\ du = \frac{1}{1+x} \end{cases} \quad \begin{cases} v = \frac{x^3}{3} \\ dv = x^2 \end{cases}$

$\begin{array}{r|l} x^3 & x+1 \\ -x^3 - x^2 & x^2 \\ \hline -x^2 & \end{array}$

$= \frac{x^3 \ln(1+x)}{3} - \int \frac{x^3}{3(1+x)} \, dx = \frac{x^3 \ln(1+x)}{3} - \frac{1}{3} \int \frac{x^3}{1+x} \, dx =$

$$= \frac{x^3\ln(1+x)}{3} - \frac{1}{3}\int\left(x^2 - \frac{x^2}{x+1}\right)dx = \frac{x^3\ln(1+x)}{3} - \frac{1}{3}\int x^2\,dx + \frac{1}{3}\int\frac{x^2}{x+1}\,dx \overset{(1)}{=}$$

CA:

$$-\frac{1}{3}\int x^2\,dx = -\frac{1}{3}\left(\frac{x^3}{3}\right) = -\frac{x^3}{9}$$

$$\begin{array}{r|l} x^2 & x+1 \\ \hline -x^2-x & x \\ -x & \end{array} \qquad \frac{x^2}{x+1} = x - \frac{x}{x+1}$$

$$\overset{(1)}{=} \frac{x^3\ln(1+x)}{3} - \frac{x^3}{9} + \frac{1}{3}\int x - \frac{x}{x+1}\,dx = \frac{x^3\ln(1+x)}{3} - \frac{x^3}{9} + \frac{1}{3}\int x\,dx - \frac{1}{3}\int\frac{x}{x+1}\,dx \overset{(2)}{=}$$

CA:

$$\frac{1}{3}\int x\,dx = \frac{1}{3}\left(\frac{x^2}{2}\right) = \frac{x^2}{6}$$

$$\begin{array}{r|l} x & x+1 \\ \hline -x-1 & 1 \\ -1 & \end{array} \qquad \frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$\overset{(2)}{=} \frac{x^3\ln(1+x)}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{1}{3}\int 1 - \frac{1}{x+1}\,dx = \frac{x^3\ln(1+x)}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{1}{3}\int 1\,dx + \frac{1}{3}\int\frac{1}{x+1}$$

$$= \frac{x^3\ln(1+x)}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + \frac{\ln(x+1)}{3} + c,\ c \in \mathbb{R}$$

K) $\int \arcsin\left(\frac{x}{\sqrt{2}}\right)dx =$ $\qquad u = \frac{x}{\sqrt{2}} \qquad du = \frac{\sqrt{2}}{2}\,dx$

$$= \frac{2}{\sqrt{2}}\int \arcsin\left(\frac{x}{\sqrt{2}}\right) \times \frac{\sqrt{2}}{2}\,dx = \frac{2\sqrt{2}}{2}\int \arcsin(u)\,du =$$

$$= \sqrt{2}\int \arcsin(u)\,du = \qquad \begin{cases} u = \arcsin(u) \\ du = \frac{1}{\sqrt{1-u^2}}\,du \end{cases} \quad \begin{cases} v = u \\ dv = 1\,du \end{cases}$$

$$= \sqrt{2} \times \left(u\arcsin(u) - \int\frac{u}{\sqrt{1-u^2}}\,du\right) = \quad \text{Seja } v = 1-u^2 \Leftrightarrow u = \sqrt{1-v}, \quad dv = -2u\,du$$

$$= \sqrt{2}\left(u\arcsin(u) - \int\frac{u \times (-2u)}{(-2u)\sqrt{1-u^2}}\,du\right) = \sqrt{2}\left(u\arcsin(u) + \frac{1}{2}\int\frac{1}{\sqrt{v}}\,dv\right.$$

$$= \sqrt{2}\left(u\arcsin(u) + \frac{1}{2}\int v^{-\frac{1}{2}}\,dv\right) = \sqrt{2}\left(u\arcsin(u) + \frac{1}{2}\left(\frac{v^{\frac{1}{2}}}{\frac{1}{2}}\right)\right) =$$

$$= \sqrt{2}\left(u\arcsin(u) + \sqrt{v}\right) + c = \sqrt{2}\left(u\arcsin(u) + \sqrt{1-u^2}\right) + c =$$

$$= \sqrt{2} \times \left(\frac{x}{\sqrt{2}} \times \arcsin\left(\frac{x}{\sqrt{2}}\right) + \sqrt{1 - \frac{x^2}{2}}\right) + c = x\arcsin\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2-x^2} + c =$$

$$= x\arcsin\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2-x^2} + c,\ c \in \mathbb{R}$$

$$\begin{array}{r|l} x^2 & x^2+1 \\ -x^2-1 & 1 \\ -1 & \end{array}$$

l) $\int x \arctan x \, dx = \quad \begin{cases} u = \arctan x \\ du = \frac{1}{1+x^2}\,dx \end{cases} \quad \begin{cases} v = \frac{x^2}{2} \\ dv = x\,dx \end{cases}$

$= \frac{x^2 \arctan(x)}{2} - \int \frac{x^2}{2(1+x^2)}\,dx = \frac{x^2\arctan(x)}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx =$

$= \frac{x^2\arctan(x)}{2} - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx = \frac{x^2\arctan(x)}{2} - \frac{1}{2}\int 1\,dx + \frac{1}{2}\int\frac{1}{x^2+1}\,dx =$

$= \frac{x^2\arctan(x)}{2} - \frac{x}{2} + \frac{\arctan(x)}{2} + c = \frac{x^2\arctan(x) + \arctan(x) - x}{2} + c, c \in \mathbb{R}$

m) $\int \frac{x \arctan(x)}{\sqrt{1+x^2}}\,dx = \quad \begin{cases} u = \arctan(x) \\ du = \frac{1}{1+x^2}\,dx \end{cases} \quad \begin{cases} v = \sqrt{1+x^2} \\ dv = \frac{x}{\sqrt{1+x^2}}\,dx \end{cases}$

$= \sqrt{1+x^2}\arctan(x) - \int \frac{\sqrt{1+x^2}}{1+x^2}\,dx = \sqrt{1+x^2}\arctan(x) - \int \frac{1}{\sqrt{1+x^2}}\,dx =$ (caso 2)

Seja $x = \tan u \Leftrightarrow u = \arctan(x)$ $\quad dx = \sec^2 u \, du$ $\quad \arctan(x)^2 + 1 = \operatorname{arcsec}^2(x) \Leftrightarrow \arctan(x) = \sqrt{\operatorname{arcsec}^2(x) - 1}$

$= \sqrt{1+x^2}\arctan(x) - \int \frac{\sec^2 u}{\sqrt{1+\tan^2 u}}\,du = \sqrt{1+x^2}\arctan(x) - \int\frac{\sec^2 u}{\sqrt{\sec^2 u}}\,du =$

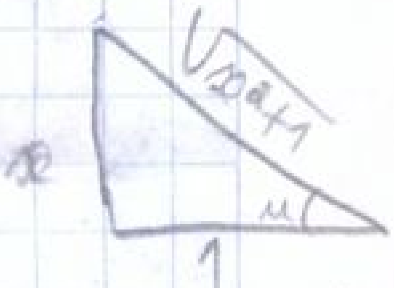
$= \sqrt{1+x^2}\arctan(x) - \int \frac{\sec^2 u}{\sec u}\,du = \sqrt{1+x^2}\arctan(x) - \int \sec(u)\,du =$

$= \sqrt{1+x^2}\arctan(x) - \int \frac{\sec(u)(\sec(u)+\tan(u))}{\sec(u)+\tan(u)} = \sqrt{1+x^2}\arctan(x) - \int\frac{\sec^2(u)+\sec(u)\tan(u)}{\tan(u)+\sec(u)}$

Seja $v = \sec(u) + \tan(u)$ $\quad dv = (\sec^2(u) + \sec(u)\tan(u))\,du$

$= \sqrt{1+x^2}\arctan(x) - \int \frac{1}{v}\,dv = \sqrt{1+x^2}\arctan(x) - \ln|v| + c =$

$= \sqrt{1+x^2}\arctan(x) - \ln|\sec(u) + \tan(u)| =$



$= \sqrt{1+x^2}\arctan(x) - \ln|\sqrt{x^2+1} + x| + c, c \in \mathbb{R}$

$\cos u = \frac{1}{\sqrt{x^2+1}}$

$\sec u = \frac{1}{\cos u} = \sqrt{x^2+1}$

m) $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \quad \left| \begin{array}{l} u = 1+x^2 \\ x = \sqrt{u-1} \end{array} \right. \quad du = 2x\,dx$

$\displaystyle = \int \frac{x^2\,x}{\sqrt{1+x^2}}\,dx = \int \frac{(u-1)\,x}{\sqrt{1+u-1}\;2x}\,du = \int \frac{u-1}{\sqrt{u}\times 2}\,du = \int \frac{u-1}{2\sqrt{u}}\,du =$

$\displaystyle = \int \frac{u}{2\sqrt{u}}\,du - \int \frac{1}{2\sqrt{u}}\,du = \frac{1}{2}\int \sqrt{u}\,du - \frac{1}{2}\int u^{-\frac{1}{2}}\,du =$

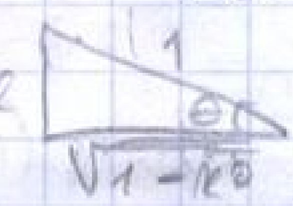
$\displaystyle = \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) = \frac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} = \frac{u\sqrt{u}}{3} - \sqrt{u} =$

$\displaystyle = \frac{(1+x^2)\sqrt{1+x^2}}{3} - \sqrt{1+x^2} + c,\ c \in \mathbb{R}$

$\sin(2\theta) = 2\sin\theta\cos\theta = 2x\cos\theta = 2x\sqrt{1-x^2}$

o) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx =$ (caso 1) $\quad x = \sin\theta \quad dx = \cos\theta\,d\theta$

$\theta = \arcsin(x)$



$\displaystyle = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin^2\theta\,d\theta =$

$\displaystyle = \int \frac{1-\cos(2\theta)}{2}\,d\theta = \int\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right)d\theta = \int \frac{1}{2}\,d\theta - \int \frac{\cos(2\theta)}{2}\,d\theta =$

$\displaystyle = \frac{\theta}{2} - \frac{1}{2}\int \cos(2\theta)\,d\theta = \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin(2\theta)}{2}\right) = \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + c =$

$\displaystyle = \frac{\arcsin(x)}{2} - \frac{2x\sqrt{1-x^2}}{4} + c = \frac{\arcsin(x) - x\sqrt{1-x^2}}{2} + c,\ c \in \mathbb{R}$

$u = 2x \quad du = 2\,dx$

p) $\displaystyle\int \sin^2 x\cos^2 x\,dx = \int \frac{\sin^2(2x)}{4}\,dx = \frac{1}{4}\int \sin^2(2x)\,dx =$

$\displaystyle = \frac{1}{4}\int \frac{\sin^2(u)}{2}\,du = \frac{1}{8}\int \sin^2(u)\,du = \frac{1}{8}\int \frac{1-\cos(2u)}{2}\,du =$

$\displaystyle = \frac{1}{8}\left[\int \frac{1}{2}\,du - \int \frac{\cos(2u)}{2}\,du\right] = \frac{1}{8}\left[\frac{u}{2} - \frac{1}{2}\int \cos(2u)\,du\right] =$

$\displaystyle = \frac{u}{16} - \frac{1}{16}\left(\frac{\sin(2u)}{2}\right) + c = \frac{u}{16} - \frac{\sin(2u)}{32} = \frac{u}{16} - \frac{2\sin u\cos u}{32} =$

$\displaystyle = \frac{u - \sin u\cos u}{16} = \frac{2x - \sin(2x)\cos(2x)}{16} = \frac{4x - \sin(4x)}{32} + c,\ c \in \mathbb{R}$

q) $\int x \sec^2 x \, dx = \left| \begin{cases} u = x \\ du = 1\, dx \end{cases} \quad \begin{cases} V = \tan x \\ dV = \sec^2 x \, dx \end{cases} \right.$

$= x \tan x - \int \tan x \, dx = x \tan x - \int \frac{\sin x}{\cos x} dx = \left| u = \cos x ; du = -\sin x \, dx \right.$

$= x \tan x + \int \frac{1}{u} du = x \tan x + \ln|u| = x \tan x + \ln|\cos x| + c, c \in \mathbb{R}$

r) $\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx = \left| \begin{cases} u = \sec x \\ du = \sec x \tan x \, dx \end{cases} \quad \begin{cases} V = \tan x \\ dV = \sec^2 x \, dx \end{cases} \right.$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx =$

$= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx =$

$= \sec x \tan x + \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx - \int \sec^3 x \, dx$

$= \sec x \tan x + \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx - \int \sec^3 x \, dx = \left| \begin{cases} u = \tan x + \sec x \\ du = (\sec^2 x + \sec x \tan x) dx \end{cases} \right.$

$= \sec x \tan x + \int \frac{1}{u} du - \int \sec^3 x \, dx = \sec x \tan x + \ln|u| - \int \sec^3 x \, dx =$

$= \sec x \tan x + \ln|\tan x + \sec x| - \int \sec^3 x \, dx$

Logo, $\int \sec^3 x \, dx = \sec x \tan x + \ln|\tan x + \sec x| - \int \sec^3 x \, dx \Leftrightarrow$

$\Leftrightarrow 2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\tan x + \sec x| \Leftrightarrow$

$\Leftrightarrow \int \sec^3 x \, dx = \frac{\sec x \tan x + \ln|\tan x + \sec x|}{2} + c, c \in \mathbb{R}$