D. Derivada da função composta (regra da cadeia)

a) V = volume do cubo (cm^3) $\quad$ t = tempo (min) $\quad$ a = aresta (cm)

$$\left.\frac{dV}{dt}\right|_{a=20\,cm} = 300\ cm^3/min \qquad \left.\frac{da}{dt}\right|_{a=20\,cm} = ?$$

Regra da Cadeia: $\quad V = a^3 \qquad \frac{dV}{da} = 3a^2$

$$\frac{dV}{dt} = \frac{dV}{da} \times \frac{da}{dt} \Leftrightarrow \frac{dV}{dt} = 3a^2 \times \frac{da}{dt}$$ ← caso geral (qualquer instante)

caso particular: (a = 20 cm)

$$300 = 3a^2 \times \frac{da}{dt} \Leftrightarrow \frac{da}{dt} = \frac{100}{a^2} \Leftrightarrow \frac{da}{dt} = \frac{100}{20^2} = \frac{100}{400} = \frac{1}{4} = 0{,}25\ cm/min$$

b) V = volume do balão esférico (m^3) $\quad$ t = tempo (s) $\quad$ D = diâmetro (m)

(qualquer instante)

$$\frac{dV}{dt} = 1\ m^3/s \qquad \left.\frac{dD}{dt}\right|_{t=2} = ?$$

caso geral:

$$\frac{dD}{dt} = \frac{dD}{dV} \times \frac{dV}{dt} \qquad V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 \Leftrightarrow$$

$$\Leftrightarrow V = \frac{4\pi D^3}{24} \Leftrightarrow V = \frac{\pi D^3}{6} \Leftrightarrow D = \sqrt[3]{\frac{6V}{\pi}} \qquad \frac{dD}{dV} = \frac{\frac{6}{\pi}}{3\sqrt[3]{\left(\frac{6V}{\pi}\right)^2}} =$$

$$= \frac{2}{\pi\sqrt[3]{\frac{(6V)^2}{\pi^2}}} = \frac{2}{\sqrt[3]{(6V)^2\pi}}$$ ← caso geral

caso particular ($V = 2 m^3$)

$$\frac{dD}{dV} = \frac{2}{\sqrt[3]{12^2\pi}} \qquad \frac{dD}{dt} = 1 \times \frac{2}{\sqrt[3]{12^2\pi}} = \frac{2}{\sqrt[3]{12^2\pi}} \quad \text{c.q.p.}$$

—//—

A = área superficial (m^2) $\quad t$ = tempo (s) $\quad A = 4\pi r^2$

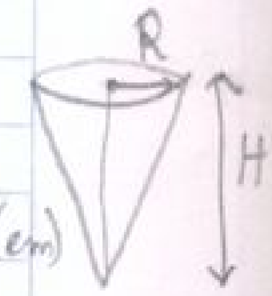
$$\frac{dA}{dt} = \frac{dA}{dD} \times \frac{dD}{dt} \qquad A = 4\pi\left(\frac{D}{2}\right)^2 = \frac{4\pi D^2}{4} = \pi D^2$$

$D = \sqrt[3]{\frac{6V}{\pi}}$, pelo caso anterior. $\quad D = \sqrt[3]{\frac{12}{\pi}}$

$$\frac{dA}{dD} = 2\pi D = 2\pi \sqrt[3]{\frac{12}{\pi}}$$

$$\frac{dA}{dt} = 2\pi \sqrt[3]{\frac{12}{\pi}} \times \frac{2}{\sqrt[3]{12^2\pi}} = \frac{4\pi \sqrt[3]{\frac{12}{\pi}}}{\sqrt[3]{12^2\pi}} = 4\pi \times \sqrt[3]{\frac{\frac{12}{\pi}}{12^2\pi}} = 4\pi \times \sqrt[3]{\frac{12}{12^2\pi^2}} =$$

$$= 4\pi \times \sqrt[3]{\frac{1}{12\pi^2}} = 4\sqrt[3]{\frac{\pi^3}{12\pi^2}} = 4\sqrt[3]{\frac{\pi}{12}} \quad \text{c.q.p.}$$



e) V = volume do cone (cm^3) $\quad t$ = tempo (s) $\quad h$ = altura (cm) $\quad r$ = raio (cm)

$$\frac{dV}{dt} = 2\, cm^3/s \qquad \frac{dh}{dt} = ? \qquad \text{Regra da Cadeia: } \frac{dh}{dt} = \frac{dh}{dV} \times \underbrace{\frac{dV}{dt}}_{2\,cm^3/s}$$

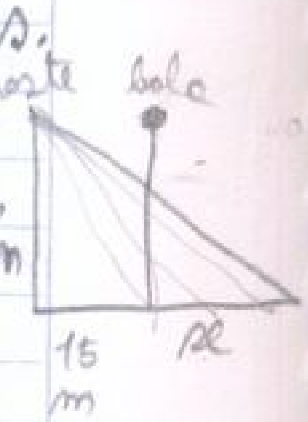
$$V = \frac{1}{3} A_{base} \times h \qquad A_{base} = \pi r^2 \qquad \begin{matrix} H & — & R \\ h & — & r \end{matrix} \Leftrightarrow h = \frac{rH}{R} \Leftrightarrow r = \frac{Rh}{H}$$

$$V = \frac{1}{3} \times \pi \times \left(\frac{Rh}{H}\right)^2 \times h \Leftrightarrow V = \frac{\pi R^2 h^3}{3H^2} \qquad \frac{dV}{dh} = \frac{\pi R^2}{3H^2} \times 3h^2 = \frac{\pi R^2 h^2}{H^2}$$

Pelo teorema da função inversa:

$$\frac{dh}{dV} = \frac{1}{\frac{dV}{dh}} = \frac{1}{\frac{\pi R^2 h^2}{H^2}} = \frac{H^2}{\pi R^2 h^2} \qquad \frac{dh}{dt} = \frac{H^2}{\pi R^2 h^2} \times 2 = \frac{2H^2}{\pi R^2 h^2}$$

$$\frac{h}{H} = 0{,}5 \Leftrightarrow h = \frac{H}{2} \qquad \frac{dh}{dt} = \frac{2H^2}{\frac{\pi R^2 H^2}{4}} = \frac{8H^2}{\pi R^2 H^2} = \frac{8}{\pi R^2}\ cm/s.$$



d) s = altura da bola (m) $\quad V$ = velocidade $\quad t$ = tempo (s)

$$\frac{dx}{dt} = \frac{dx}{ds} \times \frac{ds}{dt} \qquad s = 0{,}5gt^2 \qquad \frac{ds}{dt} = 0{,}5g \times 2t = gt$$

$$\begin{matrix} 20 & ——— & 20 - s \\ x + 15 & ——— & x \end{matrix} \Leftrightarrow x = \frac{(20-s)(x+15)}{20} \Leftrightarrow x = \frac{20x + 300 - sx - 15s}{20}$$

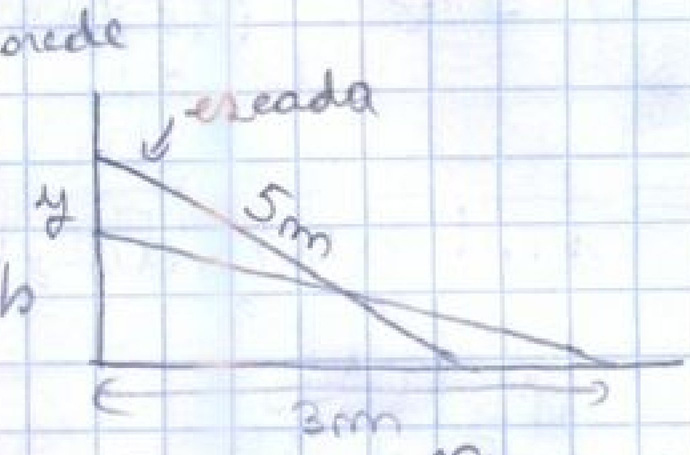
$$\Leftrightarrow x = \frac{x(20-s) + 300 - 15s}{20} \Leftrightarrow 20x - x(20-s) = 300 - 15s \Leftrightarrow$$

$$\Leftrightarrow x(20 - 20 + s) = 300 - 15s \Leftrightarrow x = \frac{300 - 15s}{s} \Leftrightarrow x = \frac{300}{s} - 15$$

$$\frac{dx}{ds} = \frac{-300}{s^2} \qquad \frac{dx}{dt} = -\frac{300}{s^2} \times gt = -\frac{300gt}{s^2} = -\frac{300gt}{0{,}25 \times g^2 \times t^4} =$$

$$= -\frac{300}{0{,}25\, g t^3} = -\frac{1200}{g t^3} \qquad \left.\frac{dx}{dt}\right|_{t=0{,}5} = -\frac{1200}{10\times(0{,}5)^3} = -960\ \text{m/s}$$

t = tempo (s)



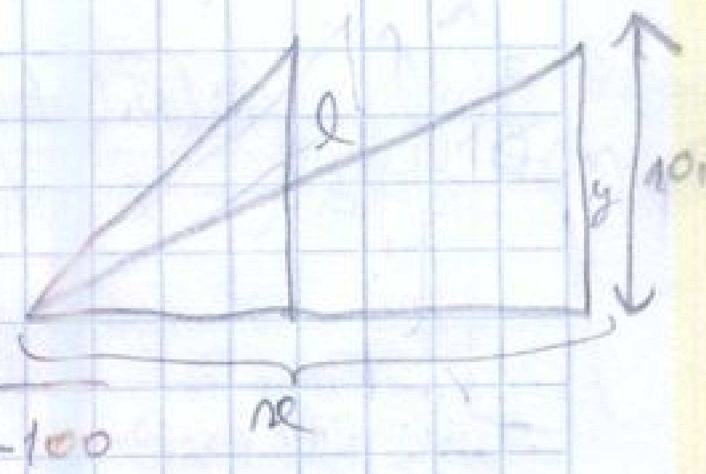
e) y = altura da escada (m) $\quad x$ = deslocamento da escada (m)

$$\left.\frac{dy}{dt}\right|_{x=3m} = ? \qquad \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \qquad \frac{dx}{dt} = 5\ \text{m/s}$$

$$y^2 + x^2 = 5^2 \Leftrightarrow y = \sqrt{25 - x^2} \qquad \frac{dy}{dx} = \frac{-2x}{2\sqrt{25-x^2}} = -\frac{x}{\sqrt{25-x^2}}$$

$$\frac{dy}{dt} = -\frac{x}{\sqrt{25-x^2}} \times 5 = -\frac{5x}{\sqrt{25-x^2}} \qquad \left.\frac{dy}{dt}\right|_{x=3m} = -\frac{15}{\sqrt{25-9}} = -\frac{15}{4} = -3{,}75\ \text{m/s}$$

R: A velocidade de 3,75 m/s



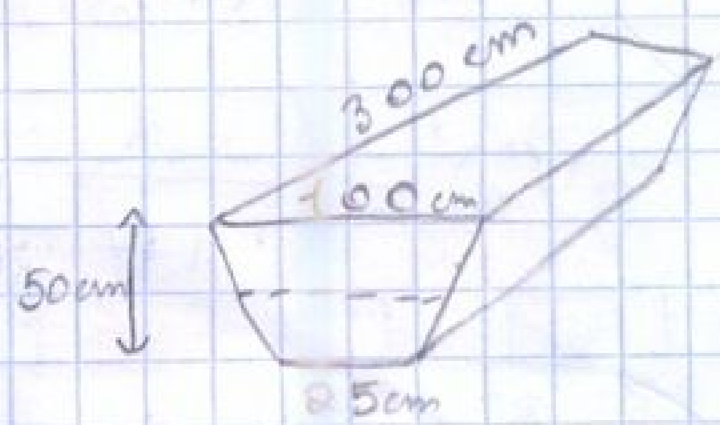
f) y = altura (m) $\quad t$ = tempo (s)

$$\frac{dx}{dt} = \frac{dx}{dl} \times \frac{dl}{dt} \qquad \frac{dl}{dt} = 0{,}2\ \text{m/s}$$

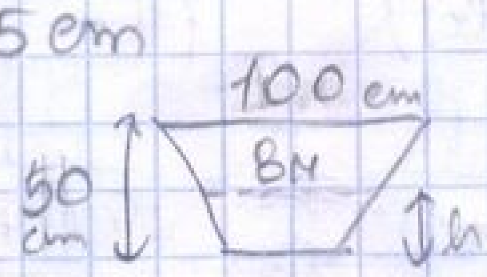
$$\frac{dx}{dl} = \frac{2l}{2\sqrt{l^2-100}} = \frac{l}{\sqrt{l^2-100}} \qquad x^2 + 10^2 = l^2 \Leftrightarrow x = \sqrt{l^2-100}$$

$$\frac{dx}{dt} = \frac{l}{\sqrt{l^2-100}} \times 0{,}2 \qquad \left.\frac{dx}{dt}\right|_{l=12{,}5m} = \frac{12{,}5}{\sqrt{12{,}5^2-100}} \times 0{,}2 = \frac{2{,}5}{7{,}5} = \frac{1}{3}\ \text{m/s}$$

g) $$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} \qquad \frac{dV}{dt} = 100\ \text{cm}^3/\text{s}$$



$$V = \left(\frac{B_M + B_m}{2} \times h\right) \times 300 \qquad B_m = 25\ \text{cm}$$



$$\frac{B_M}{100} = \frac{h}{50} \Leftrightarrow B_M = 2h$$

$$V = \left(\frac{2h+25}{2} \times h\right) \times 300 = 150 \times h(2h+25) = 150 \times (2h^2 + 25h) =$$

$$= 300h^2 + 3750h \qquad \frac{dV}{dh} = 600h + 3750$$

$$\frac{dh}{dV} = \frac{1}{\frac{dV}{dh}} = \frac{1}{600h+3750} \qquad \frac{dh}{dt} = \frac{100}{600h+3750} = \frac{1}{6h+37{,}5}$$

$$\left.\frac{dh}{dt}\right|_{h=25cm} = \frac{1}{6\times25+37{,}5} = 5{,}33\times10^{-3}\ \text{cm/s}$$

h) $\dfrac{d^2y}{dx^2} = \dfrac{d(dy)}{(dx)(dx)} = \dfrac{d}{dx}\left(\dfrac{dy}{du} \times \dfrac{du}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{dy}{du}\right)\cdot\dfrac{du}{dx} + \dfrac{dy}{du}\cdot\dfrac{d}{dx}\left(\dfrac{du}{dx}\right) =$

$= \dfrac{d}{du}\left(\dfrac{dy}{du}\right)\left(\dfrac{du}{dx}\right)\left(\dfrac{du}{dx}\right) + \dfrac{dy}{du}\cdot\dfrac{d^2u}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\cdot\dfrac{d^2u}{dx^2} =$

$= \dfrac{dy}{du} \times \dfrac{d^2u}{dx^2} + \left(\dfrac{du}{dx}\right)^2 \times \dfrac{d^2y}{du^2}$ c.q.p.

i) Seja $y(u(v(x)))$ uma função regular: $(y \circ u \circ v)' = y(u(v(x)))' =$

$= \lim\limits_{x\to a}\left(\dfrac{y(u(v(x))) - y(u(v(a)))}{x - a}\right) = \lim\limits_{x\to a}\left(\dfrac{y(u(v(x))) - y(u(v(a)))}{u(v(x)) - u(v(a))} \times \dfrac{u(v(x)) - u(v(a))}{x-a}\right)$

$= \lim\limits_{x\to a}\left(\underbrace{\dfrac{y(u(v(x))) - y(u(v(a)))}{u(v(x)) - u(v(a))}}_{y'(u(v(x)))} \times \underbrace{\dfrac{u(v(x)) - u(v(a))}{v(x) - v(a)}}_{u'(v(x))} \times \underbrace{\dfrac{v(x) - v(a)}{x - a}}_{v'(x)}\right) =$

$= y'(u(v(x))) \times u'(v(x)) \times v'(x)$ Logo, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dv} \times \dfrac{dv}{dx}$ c.q.p.

j) $f'(x) = \left(\dfrac{1}{1+\frac{1}{x}}\right)' = \dfrac{\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)^2} = \dfrac{\frac{1}{x^2}}{\frac{(x+1)^2}{x^2}} = \dfrac{1}{(x+1)^2}$ $\qquad \dfrac{1}{1+\frac{1}{x}} = \dfrac{1}{\frac{x+1}{x}} = \dfrac{x}{x+1}$

$g'(x) = \left(\dfrac{1}{1+\frac{1}{f(x)}}\right)' = \dfrac{-\left(1+\frac{1}{f(x)}\right)'}{\left(1+\frac{1}{f(x)}\right)^2} = \dfrac{-\frac{\frac{1}{(x+1)^2}}{\left(\frac{x}{x+1}\right)^2}}{\left(1+\frac{1}{\frac{x}{x+1}}\right)^2} = \dfrac{-\frac{1}{x^2}}{\left(1+\frac{x+1}{x}\right)^2} =$

$= \dfrac{-\frac{1}{x^2}}{\left(\frac{2x+1}{x}\right)^2} = -\dfrac{1}{(2x+1)^2} = -\dfrac{1}{4x^2+4x+1}$