1.B. Derivação da função inversa

37 $f'(x) = \left(\arcsin\left(\frac{1}{1+x^2}\right)\right)'$ $\begin{cases} y = \arcsin(u) \\ u = \sin(y) \end{cases}$ $u = \frac{1}{1+x^2}$

Pelo teorema da função composta: (regra de derivação em cadeia)

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\frac{du}{dx} = -\frac{2x}{(1+x^2)^2}$ $\frac{dy}{du} = \frac{1}{\frac{du}{dy}}$ $\frac{du}{dy} = \cos y$

$$\frac{dy}{du} = \frac{1}{\cos y} = \frac{1}{\sqrt{\sin^2 y - 1}} = \frac{1}{\sqrt{u^2 - 1}} = \frac{1}{\sqrt{\frac{1}{(1+x^2)^2} - 1}} = \frac{1}{\sqrt{\frac{1-(1+x^2)^2}{(1+x^2)^2}}}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{1-(1+x^2)^2}}{1+x^2}} \times \left(-\frac{2x}{(1+x^2)^2}\right) = \frac{1+x^2}{\sqrt{1-(1+x^2)^2}} \times \left(-\frac{2x}{(1+x^2)^2}\right) =$$

$$= \frac{1}{\sqrt{2x^2 + x^4}} \times -\frac{2x}{(1+x^2)} = \frac{1}{x\sqrt{2+x^2}} \times \left(-\frac{2x}{(1+x^2)}\right) = \frac{-2}{(1+x^2)\sqrt{2+x^2}}$$

38 $f'(x) = \left(\frac{\arctan(x)}{x^2}\right)'$ $\begin{cases} y = \arctan(x) \\ x = \tan y \end{cases}$ $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2(y)} =$

$$= \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2} \qquad f'(x) = \frac{\left(\frac{1}{1+x^2}\right)x^2 - 2x\arctan(x)}{x^4} = \frac{\frac{1}{1+x^2}x - 2\arctan(x)}{x^3} =$$

$$= \frac{x - 2\arctan(x) - 2x^2\arctan(x)}{(1+x^2)\,x^3} = \frac{x - 2\arctan(x) - 2x^2\arctan(x)}{x^3 + x^5}$$

39 $f'(x) = \left(x^7\left(\frac{\pi}{2} - \arctan(x)\right)\right)'$ $y = \arctan(x)$ $x = \tan y$

Pelo teorema da função inversa: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ $\frac{dx}{dy} = \sec^2(y)$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

$$f'(x) = 7x^6\left(\frac{\pi}{2} - \arctan(x)\right) - \frac{x^7}{1+x^2}$$

(40) $y = \arctan\left(\frac{1}{x}\right)$ $\quad \frac{1}{x} = \tan(y) \Leftrightarrow x = \frac{1}{\tan(y)}$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\left(\frac{1}{\tan y}\right)'} = \frac{1}{\frac{-\sec^2 y}{\tan^2 y}} = -\frac{\tan^2 y}{\sec^2 y} = -\frac{\tan^2 y}{1+\tan^2 y} =$$

$$= -\frac{\left(\frac{1}{x}\right)^2}{1+\left(\frac{1}{x}\right)^2} = -\frac{\frac{1}{x^2}}{1+\frac{1}{x^2}} = -\frac{\frac{1}{x^2}}{\frac{x^2+1}{x^2}} = -\frac{1}{x^2+1}$$

$$f'(x) = \left(\frac{\arctan\left(\frac{1}{x}\right) - \left(\frac{1}{x}\right)}{\cos\left(\frac{1}{x}\right) - 1}\right)' = \frac{\left(-\frac{1}{x^2+1} + \frac{1}{x^2}\right)\left(\cos\left(\frac{1}{x}\right) - 1\right) - \left(\frac{1}{x^2} \times \sin\left(\frac{1}{x}\right)\right)\left(\arctan\left(\frac{1}{x}\right) - \frac{1}{x}\right)}{\left(\cos\left(\frac{1}{x}\right) - 1\right)^2}$$

(41) $y = \arctan(u)$ $\quad u = \tan y$ $\quad u = \frac{x}{x+1}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{du}{dx} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

Pelo teorema da função inversa:

$$\frac{dy}{du} = \frac{1}{\frac{du}{dy}} = \frac{1}{(\tan y)'} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+u^2} = \frac{1}{1+\frac{x^2}{(x+1)^2}} =$$

$$= \frac{1}{\frac{x^2+2x+1+x^2}{(x+1)^2}} = \frac{(x+1)^2}{(x+1)^2 + x^2} \qquad \frac{dy}{dx} = \frac{(x+1)^2}{((x+1)^2+x^2)(x+1)^2} = \frac{1}{2x^2+2x+1}$$

$$f'(x) = \left(\log\left(\arctan\left(\frac{x}{x+1}\right)\right)\right)' = \frac{\frac{1}{2x^2+2x+1}}{\ln(10)\arctan\left(\frac{x}{x+1}\right) \times (2x^2+2x+1)} = \frac{1}{\ln(10)\arctan\left(\frac{x}{x+1}\right)(2x^2+2x+1)}$$

(42) $y = \operatorname{arcsec}(u)$ $\quad u = \sec y$ $\quad u = \ln(x)$ $\quad 1+\tan^2 y = \sec^2 y \Leftrightarrow$

$\Leftrightarrow \tan y = \sqrt{\sec^2 y - 1}$

Regra da cadeia: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\quad \frac{du}{dx} = (\ln(x))' = \frac{1}{x}$

Teorema da função inversa:

$$\frac{dy}{du} = \frac{1}{\frac{du}{dy}} = \frac{1}{(\sec y)'} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{u\sqrt{u^2-1}} = \frac{1}{\ln(x)\sqrt{\ln^2(x)-1}}$$

$$\frac{dy}{dx} = \frac{1}{\ln(x)\sqrt{\ln^2(x)-1}} \times \frac{1}{x} = \frac{1}{x\ln(x)\sqrt{\ln^2(x)-1}}$$

$$f'(x) = (\operatorname{arcsec}(\ln(x)))' = \frac{1}{x\ln(x)\sqrt{\ln^2(x)-1}}$$

(43) $y = \text{arccosec}(u)$ $u = \text{cosec}\, y$ $u = x^2$ $\text{cosec}^2 y = 1 + \text{cotan}^2 y \Leftrightarrow$

Regra da cadeia: $\Leftrightarrow \text{cotan}\, y = \sqrt{\text{cosec}^2 y - 1}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{du}{dx} = (x^2)' = 2x$$

Pelo teorema da função inversa:

$$\frac{dy}{du} = \frac{1}{\frac{du}{dy}} = \frac{1}{(\text{cosec}\, y)'} = \frac{1}{-\text{cosec}\, y\, \text{cotan}\, y} = -\frac{1}{\text{cosec}\, y \sqrt{\text{cosec}^2 y - 1}} =$$

$$= -\frac{1}{u\sqrt{u^2 - 1}} = -\frac{1}{x^2\sqrt{x^4 - 1}} \qquad \frac{dy}{dx} = -\frac{1}{x^2\sqrt{x^4-1}} \times 2x = -\frac{2}{x\sqrt{x^4-1}}$$

$y = \text{sen}(u)$
$u = \text{sen}(\text{sen}(x))$

$$\frac{dy}{du} = u' \cos u = \cos(\text{sen}(\text{sen}(x))) \times (\text{sen}(\text{sen}\, x))' =$$

$$= \cos(\text{sen}(\text{sen}(x))) \times \cos(\text{sen}(x)) \times (\text{sen}(x))' = \cos(\text{sen}(\text{sen}\, x)) \times \cos(\text{sen}(x)) \times \cos(x)$$

$$f'(x) = -\frac{2}{x\sqrt{x^4-1}} + \cos(\text{sen}(\text{sen}(x))) \times \cos(\text{sen}(x)) \times \cos(x)$$

(44) $$f'(x) = \left[(\arctan(\pi\sqrt{x}))^3\right]' = 3(\arctan(\pi\sqrt{x}))^2 \times (\arctan(\pi\sqrt{x}))'$$

$y = \arctan u$ $u = \tan y$ $u = \pi\sqrt{x}$

Pela regra da cadeia:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{du}{dx} = (\pi\sqrt{x})' = \pi \times \frac{1}{2\sqrt{x}} = \frac{\pi}{2\sqrt{x}}$$

Pelo teorema da função inversa:

$$\frac{dy}{du} = \frac{1}{\frac{du}{dy}} = \frac{1}{(\tan y)'} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+u^2} = \frac{1}{1+x\pi^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x\pi^2} \times \frac{\pi}{2\sqrt{x}} = \frac{\pi}{2\sqrt{x}(1+x\pi^2)}$$

$$f'(x) = 3(\arctan(\pi\sqrt{x}))^2 \times \frac{\pi}{2\sqrt{x}(1+x\pi^2)}$$