$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

5. Primitivação por substituição trigonométrica

(caso 3)

a) $\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{x^2-2^2}}{x}\,dx =$ Seja $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$

$= \int \frac{\sqrt{(2\sec\theta)^2-4}}{\cancel{2\sec\theta}} \times \cancel{2\sec\theta}\tan\theta\,d\theta = \int\sqrt{4(\sec^2\theta-1)}\times\tan\theta\,d\theta =$

$= \int 2\sqrt{\tan^2\theta}\times\tan\theta\,d\theta = 2\int\tan^2\theta\,d\theta = 2\int(\sec^2\theta-1)\,d\theta =$

$= 2(\tan\theta-\theta)+c =$

CA: $x = 2\sec\theta \Leftrightarrow \theta = \operatorname{arcsec}\left(\frac{x}{2}\right)$

$x = 2\left(\sqrt{\tan^2\theta+1}\right) \Leftrightarrow \frac{x^2}{4} = \tan^2\theta+1 \Leftrightarrow$

$\Leftrightarrow \tan\theta = \sqrt{\frac{x^2-4}{4}} = \frac{\sqrt{x^2-4}}{2}$

$= 2\left(\frac{\sqrt{x^2-4}}{2} - \operatorname{arcsec}\left(\frac{x}{2}\right)\right)+c =$

$= \sqrt{x^2-4} - 2\operatorname{arcsec}\left(\frac{x}{2}\right)+c,\ c\in\mathbb{R}$

(caso 1)

b) $\int\sqrt{x(6-x)}\,dx = \int\sqrt{6x-x^2}\,dx = \int\sqrt{6x-x^2+9-9}\,dx = \int\sqrt{9-(x-3)^2}\,dx =$

$x-3 = 3\sin\theta \qquad dx = 3\cos\theta\,d\theta$

$= \int\sqrt{9-(3\sin\theta)^2}\times 3\cos\theta\,d\theta = \int\sqrt{9-9\sin^2\theta}\times3\cos\theta\,d\theta = \int\sqrt{9(1-\sin^2\theta)}\times3\cos\theta\,d\theta =$

$= \int 3\sqrt{\cos^2\theta}\times3\cos\theta\,d\theta = \int 3\cos\theta\times3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta = 9\int\cos^2\theta\,d\theta =$

$= 9\int\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta = 9\left[\int\frac{1}{2}\,d\theta + \int\frac{1}{2}\cos(2\theta)\,d\theta\right] = 9\left[\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right]+c =$

$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta)+c =$

CA: $x-3 = 3\sin\theta \Leftrightarrow \theta = \arcsin\left(\frac{x-3}{3}\right)$

$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{x-3}{3}\right)\frac{\sqrt{9-(x-3)^2}}{3}$

$\cos\theta = \sqrt{1-\left(\frac{x-3}{3}\right)^2} = \frac{\sqrt{3^2-(x-3)^2}}{3}$

$= \frac{9}{2}\arcsin\left(\frac{x-3}{3}\right) + \frac{9}{4}\times\frac{2}{9}(x-3)\sqrt{9-(x-3)^2}+c =$

$= \frac{9}{2}\arcsin\left(\frac{x-3}{3}\right) + \frac{1}{2}(x-3)\sqrt{9-(x-3)^2}+c,\ c\in\mathbb{R}.$

c) $\int\frac{x^3}{\sqrt{2-x^2}}\,dx = \int\frac{x^3}{\sqrt{(\sqrt{2})^2-x^2}}\,dx =$ (caso 1)

$x = \sqrt{2}\sin\theta \Leftrightarrow \sin\theta = \frac{x}{\sqrt{2}}$

$dx = \sqrt{2}\cos\theta\,d\theta$

$\cos\theta = \sqrt{1-\frac{x^2}{2}}$

$= \int\frac{(\sqrt{2}\sin\theta)^3}{\sqrt{2-(\sqrt{2}\sin\theta)^2}}\times\sqrt{2}\cos\theta\,d\theta = \int\frac{2\sqrt{2}\sin^3\theta}{\sqrt{2-2\sin^2\theta}}\times\sqrt{2}\cos\theta\,d\theta =$

$= \int\frac{2\sqrt{2}\sin^3\theta}{\sqrt{2(1-\sin^2\theta)}}\times\sqrt{2}\cos\theta\,d\theta = \int\frac{4\sin^3\theta\cos\theta}{\sqrt{2\cos^2\theta}}\,d\theta = \int\frac{4\sin^3\theta\cos\theta}{\sqrt{2}\cos\theta}\,d\theta =$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ $\frac{1}{\sec\theta} = \cos\theta$

Fazer $u = \cos\theta$ $du = -\sin\theta \, d\theta$

$$= \int \frac{4\sqrt{2}\sin^3\theta}{2} d\theta = \int 2\sqrt{2}\sin^3\theta \, d\theta = 2\sqrt{2}\int \sin^3\theta \, d\theta =$$

$$= 2\sqrt{2}\int \sin^2\theta \sin\theta \, d\theta = -2\sqrt{2}\int (1-\cos^2\theta)\times(-\sin\theta \, d\theta) =$$

$$= -2\sqrt{2}\int (1-u^2)\,du = -2\sqrt{2}\left(u - \frac{1}{3}u^3\right) + c = -2\sqrt{2}\,u + \frac{2\sqrt{2}}{3}u^3 + c =$$

$$= -2\sqrt{2}\cos\theta + \frac{2\sqrt{2}}{3}\cos^3\theta + c = -2\sqrt{2}\left(\sqrt{1-\frac{x^2}{2}}\right) + \frac{2\sqrt{2}}{3}\times\sqrt{\left(1-\frac{x^2}{2}\right)^3} + c =$$

$$= -2\sqrt{2-x^2} + \frac{2\sqrt{2}}{3}\times\left(1-\frac{x^2}{2}\right)\sqrt{1-\frac{x^2}{2}} + c = -2\sqrt{2-x^2} + \frac{2\sqrt{2-x^2}\left(1-\frac{x^2}{2}\right)}{3} + c =$$

$$= \frac{(2-x^2)\sqrt{2-x^2}}{3} - 2\sqrt{2-x^2} + c, \; c \in \mathbb{R}$$

d) $\int \frac{\sqrt{x^2-9}}{x^2}dx = \int \frac{\sqrt{x^2-3^2}}{x^2}dx =$ (caso 3)

$x = 3\sec\theta \Leftrightarrow \sec\theta = \frac{x}{3}$

$dx = 3\sec\theta\tan\theta \, d\theta$

$\sec^2\theta = 1+\tan^2\theta \Leftrightarrow \tan^2\theta = \sec^2\theta - 1$

$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{9\sec^2\theta}\times 3\sec\theta\tan\theta \, d\theta = \int \frac{3\sqrt{\sec^2\theta-1}}{9\sec^2\theta}\times 3\sec\theta\tan\theta \, d\theta =$$

$$= \int \frac{\sqrt{\tan^2\theta}}{3\sec\theta}\times 3\tan\theta \, d\theta = \int \frac{3\tan^2\theta}{3\sec\theta}d\theta = \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}d\theta =$$

$$= \int \sec\theta \, d\theta - \int \frac{1}{\sec\theta}d\theta = \ln|\sec\theta + \tan\theta| - \int \cos\theta \, d\theta =$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c =$$

CA: $\tan^2\theta = \sec^2\theta - 1 \Leftrightarrow \tan\theta = \sqrt{\frac{x^2}{9}-1}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{x}{3}} = \frac{3}{x}$

$\sin^2\theta + \cos^2\theta = 1 \Leftrightarrow \sin\theta = \sqrt{1-\frac{9}{x^2}}$

$$= \ln\left|\frac{x}{3} + \sqrt{\frac{x^2}{9}-1}\right| - \sqrt{\frac{x^2-9}{x^2}} + c =$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + c = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + c, \; c\in\mathbb{R}$$

e) $\int \frac{1}{x^3\sqrt{x^2-9}}dx = \int \frac{1}{x^3\sqrt{x^2-3^2}}dx =$ (caso 3)

$x = 3\sec\theta \Leftrightarrow \theta = \operatorname{arcsec}\left(\frac{x}{3}\right)$

$dx = 3\sec\theta\tan\theta \, d\theta$

$$= \int \frac{1}{27\sec^3\theta\sqrt{9\sec^2\theta - 9}}\times 3\sec\theta\tan\theta \, d\theta = \int \frac{1}{27\sec^3\theta\times 3\sqrt{\sec^2\theta-1}}\times 3\sec\theta\tan\theta \, d\theta =$$

$$= \int \frac{1}{81\sec^3\theta\tan\theta}\times 3\sec\theta\tan\theta \, d\theta = \int \frac{1}{27\sec^2\theta}d\theta = \int \frac{1}{27}\times\frac{1}{\sec^2\theta}d\theta$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$= \int \frac{1}{27} \times \cos^2\theta \, d\theta = \frac{1}{27}\int \cos^2\theta \, d\theta = \frac{1}{27}\int \frac{\cos(2\theta)+1}{2} d\theta =$$

$$= \frac{1}{27}\int\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta = \frac{1}{27}\left[\int \frac{1}{2} d\theta + \int \frac{1}{2}\cos(2\theta) d\theta =\right.$$

$$= \frac{1}{27}\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + c = \frac{1}{54}\theta + \frac{1}{108}\sin(2\theta) + c =$$

$$= \frac{\operatorname{arcsec}\left(\frac{x}{3}\right)}{54} + \frac{1}{108} \times \frac{6\sqrt{x^2-9}}{x^2} + c =$$

$$= \frac{\operatorname{arcsec}\left(\frac{x}{3}\right)}{54} + \frac{\sqrt{x^2-9}}{18x^2} + c, \; c \in \mathbb{R}$$

CA: $\sin(2\theta) = 2\sin\theta\cos\theta$

$\sec\theta = \frac{1}{\cos\theta} \Leftrightarrow \cos\theta = \frac{1}{\frac{x}{3}} = \frac{3}{x}$

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{x^2}} = \frac{\sqrt{x^2-9}}{x}$

$\sin(2\theta) = 2\,\frac{\sqrt{x^2-9}}{x} \times \frac{3}{x}$

f) $$\int \frac{1-x}{x\sqrt{1-x^2}} dx = \int \frac{1-x}{x\sqrt{1^2-x^2}} dx = \quad \begin{vmatrix} x = \sin\theta \\ dx = \cos\theta \, d\theta \end{vmatrix}$$

(caso 1)

$$= \int \frac{1-\sin\theta}{\sin\theta\sqrt{1-\sin^2\theta}} \times \cos\theta \, d\theta = \int \frac{(1-\sin\theta)\cos\theta}{\sin\theta\sqrt{\cos^2\theta}} d\theta = \int \frac{(1-\sin\theta)\cancel{\cos\theta}}{\sin\theta\,\cancel{\cos\theta}} d\theta =$$

$$= \int \frac{1-\sin\theta}{\sin\theta} d\theta = \int \frac{1}{\sin\theta} d\theta - \int 1 \, d\theta = \int \operatorname{cosec}\theta \, d\theta - \theta + c =$$

$$= \int \frac{\operatorname{cosec}^2\theta + \operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}\theta + \cot\theta} d\theta - \theta + c = \int \frac{\operatorname{cosec}^2\theta + \operatorname{cosec}\theta\cot\theta}{\cot\theta + \operatorname{cosec}\theta} d\theta - \theta + c$$

$$= -\int \frac{-\operatorname{cosec}^2\theta - \operatorname{cosec}\theta\cot\theta}{\cot\theta + \operatorname{cosec}\theta} - \theta + c = -\ln\left|\cot\theta + \operatorname{cosec}\theta\right| - \theta + c = (1)$$

CA: $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\qquad \cos\theta = \sqrt{1-x^2}$ $\qquad \sin\theta = x$

$\cot\theta = \frac{\sqrt{1-x^2}}{x}$ $\qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x}$ $\qquad \theta = \arcsin(x)$

$$(1) = -\ln\left|\frac{\sqrt{1-x^2}}{x} + \frac{1}{x}\right| - \arcsin(x) + c = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| - \arcsin(x) + c, \; c \in \mathbb{R}$$

g) $$\int \frac{1}{(4+x^2)^{\frac{3}{2}}} dx = \int \frac{1}{(4+x^2)\sqrt{4+x^2}} dx = \quad \begin{matrix} x = 2\tan\theta \\ dx = 2\sec^2\theta \, d\theta \end{matrix}$$

(caso 2)

$$= \int \frac{1}{(4+4\tan^2\theta)\sqrt{4+4\tan^2\theta}} \times 2\sec^2\theta \, d\theta = \int \frac{2\sec^2\theta}{8(1+\tan^2\theta)\sqrt{1+\tan^2\theta}} d\theta =$$

$$= \int \frac{2\sec^2\theta}{8\sec^2\theta\sqrt{\sec^2\theta}}\,d\theta = \int \frac{1}{4\sec\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta =$$

$$= \frac{1}{4} \times \sin\theta + c = (1)$$

C.A: $\tan\theta = \frac{x}{2}$ $\quad 1 + \frac{x^2}{4} = \frac{1}{\cos^2\theta} \Leftrightarrow \cos\theta = \frac{2}{\sqrt{x^2+4}} = \frac{2\sqrt{x^2+4}}{x^2+4}$ $\quad \frac{\sin\theta}{\cos\theta} = \frac{x}{2} \Leftrightarrow \sin\theta = \frac{2x\sqrt{x^2+4}}{2x^2+8}$

(1) $$= \frac{1}{4} \times \frac{x\sqrt{x^2+4}}{x^2+4} + c =$$

$$= \frac{x\sqrt{x^2+4}}{4x^2+16} + c,\ c \in \mathbb{R}$$

C.A: $(x + \frac{1}{2})^2 = x^2 + x + \frac{1}{4}$

$-(\frac{1}{2} - x)^2 = \frac{1}{4} - x + x^2$

h) $$\int x\sqrt{3+4x-4x^2}\,dx = \int x\sqrt{4\left(\frac{3}{4} + x - x^2\right)}\,dx = 2\int x\sqrt{\left(\frac{3}{4} + x - x^2\right)}\,dx =$$

(caso 1)

$$= 2\int x\sqrt{1 - \left(\frac{1}{2} - x\right)^2}\,dx =$$

$\frac{1}{2} - x = \sin\theta \Leftrightarrow x = \frac{1}{2} - \sin\theta$ $\quad dx = \cos\theta\,d\theta$

$$= 2\int\left(\frac{1}{2} - \sin\theta\right)\sqrt{1 - \sin^2\theta} \times \cos\theta\,d\theta = 2\int\left(\frac{1}{2} - \sin\theta\right)\sqrt{\cos^2\theta} \times \cos\theta\,d\theta =$$

$$= 2\int\left(\frac{1}{2} - \sin\theta\right)\cos^2\theta\,d\theta = 2\left[\int\frac{\cos^2\theta}{2}\,d\theta - \int \sin\theta\cos^2\theta\,d\theta\right] =$$

$$= 2\left[\int\frac{\cos(2\theta)+1}{4}\,d\theta - \int\sin\theta(1 - \sin^2\theta)\,d\theta\right] = 2\left[\int\frac{\cos(2\theta)}{4}\,d\theta + \int\frac{1}{4}\,d\theta - \int(\sin\theta - \sin^3\theta)\,d\theta\right]$$

$$= 2\left[\frac{1}{4}\int\cos(2\theta)\,d\theta + \frac{1}{4}\theta + c - \int\sin\theta\,d\theta + \int\sin^3\theta\,d\theta\right] =$$

$u = \cos\theta$ $\quad du = -\sin\theta\,d\theta$

$$= 2\left[\frac{1}{4} \times \frac{\sin(2\theta)}{2} + \frac{1}{4}\theta + c - (-\cos\theta) + \int(1 - \cos^2\theta)\sin\theta\,d\theta\right] =$$

$$= 2\left[\frac{\sin(2\theta)}{8} + \frac{\theta}{4} + \cos\theta + c - \int(1 - u^2)\,du =\right.$$

$$= \frac{\sin(2\theta)}{4} + \frac{\theta}{2} + 2\cos\theta + c - \left(-\frac{1}{3}u^3\right) = \frac{\sin(2\theta)}{4} + \frac{\theta}{2} + 2\cos\theta + \frac{\cos^3\theta}{3} \overset{(1)}{=}$$

C.A:

$\cos\theta = \sqrt{1 - (\frac{1}{2} - x)^2} = \sqrt{1 - (\frac{1}{4} - x + x^2)} = \sqrt{\frac{3}{4} + x - x^2}$ $\quad \sin(2\theta) = 2 \times (\frac{1}{2} - x) \times \sqrt{\frac{3}{4} + x - x^2}$

$2\cos\theta = 2\sqrt{\frac{3}{4} + x - x^2}$ $\quad \cos^3(\theta) = (\frac{3}{4} + x - x^2)\sqrt{\frac{3}{4} + x - x^2}$ $\quad \theta = \arcsin(\frac{1}{2} - x)$

(1) $$= \frac{(1 - 2x)\sqrt{\frac{3}{4} + x - x^2}}{4} + \frac{\arcsin(\frac{1}{2} - x)}{2} + 2\sqrt{\frac{3}{4} + x - x^2} + \frac{(\frac{3}{4} + x - x^2)\sqrt{\frac{3}{4} + x - x^2}}{3} =$$

$$= \frac{(3-6x)\sqrt{\frac{3}{4}+x-x^2} + 6\arcsin\left(\frac{1}{2}-x\right) + 12\sqrt{\frac{3}{4}+x-x^2} + (3+4x-4x^2)\sqrt{\frac{3}{4}+x-x^2}}{12} =$$

$$= \frac{6\arcsin\left(\frac{1}{2}-x\right) + \sqrt{\frac{3}{4}+x-x^2}\left[12+3-6x+3+4x-4x^2\right]}{12} =$$

$$= \frac{6\arcsin\left(\frac{1}{2}-x\right) + \sqrt{\frac{3}{4}+x-x^2}\left(-4x^2-2x+18\right)}{12} + c,\ c\in\mathbb{R}.$$