Distribuições conjuntas de probabilidade

5.1) i)

Resultados possíveis	X	Y
E E E	2	0
C E E	1	0
E C E	1	1
E E C	2	1
E C C	1	2
C E C	1	1
C C E	0	1
C C C	0	2

X	Y	$p_{X,Y}(x,y)$
2	0	$\frac{1}{8}$
1	0	$\frac{1}{8}$
1	1	$\frac{1}{4}$
2	1	$\frac{1}{8}$
1	2	$\frac{1}{8}$
0	1	$\frac{1}{8}$
0	2	$\frac{1}{8}$
2	2	0
0	0	0

ii) $p_{Y|X}(Y|X=1) = \dfrac{p_{X,Y}(X=1,Y)}{p_X(X=1)}$ $\qquad p_X(X=1) = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$

$p_X(X=1) = \sum_y p_{X,Y}(x,y)$

$Y=0 \Rightarrow p_{X,Y}(X=1,0) = \frac{1}{8}$ $\qquad p_{Y|X}(0|X=1) = \dfrac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

$Y=1 \Rightarrow p_{X,Y}(X=1,1) = \frac{1}{4}$ $\qquad p_{Y|X}(1|X=1) = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$Y=2 \Rightarrow p_{X,Y}(X=1,2) = \frac{1}{8}$ $\qquad p_{Y|X}(2|X=1) = \dfrac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

Y	$p_{Y\|X}(Y\|X=1)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

iii) $\rho_{X,Y} = \dfrac{\gamma_{X,Y}}{\sigma_X \times \sigma_Y}$ $\qquad E(X) = 2\times\left(\frac{1}{8}+\frac{1}{8}+0\right) + 1\times\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{8}\right) + 0\times\left(\frac{1}{8}+\frac{1}{8}+0\right) = 1$

$E(Y) = 2\times\left(\frac{1}{8}+\frac{1}{8}+0\right) + 1\times\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{8}\right) + 0\times\left(\frac{1}{8}+\frac{1}{8}+0\right) = 1$

$Var(X) = (0-1)^2\times\left(\frac{1}{8}+\frac{1}{8}\right) + (1-1)^2\times\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{8}\right) + (2-1)^2\times\left(\frac{1}{8}+\frac{1}{8}+0\right) = 0{,}5$

$Var(Y) = (0-1)^2\times\left(\frac{1}{8}+\frac{1}{8}\right) + (1-1)^2\times\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{8}\right) + (2-1)^2\times\left(\frac{1}{8}+\frac{1}{8}+0\right) = 0{,}5$

$\sigma_X = \sigma_Y = \sqrt{0{,}5} = 0{,}707$

$\gamma_{X,Y} = \sum_x \sum_Y (x-1)\times(y-1)\times p_{X,Y}(x,y) = (2-1)\times(0-1)\times\frac{1}{8} + (0-1)\times(2-1)\times\frac{1}{8} = -0{,}25$

$\rho_{X,Y} = \dfrac{-0{,}25}{0{,}707\times 0{,}707} = -0{,}5$

Z

5.2 i) $f_X(x) = \int_0^x f_{X,Y}(x,y)\,dy = \int_0^x 2\,dy = [2y]_0^x = 2x \quad f_X(x) = 2x$

$f_Y(y) = \int_y^1 f_{X,Y}(x,y)\,dx = \int_y^1 2\,dx = [2x]_y^1 = 2 - 2y = 2(1-y)$

ii) $E(x) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_0^1 x \times 2x\,dx = \int_0^1 2x^2\,dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$

$E(y) = \int_{-\infty}^{+\infty} y \times f_Y(y)\,dy = \int_0^1 2y - 2y^2\,dy = \left[y^2 - \frac{2y^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

iii) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y} \quad f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$

iv) $\gamma_{X,Y} = \int_0^1 \int_y^1 \left(x - \frac{2}{3}\right) \times \left(y - \frac{1}{3}\right) \times 2\,dx\,dy = \int_0^1 \int_y^1 \left(xy - \frac{x}{3} - \frac{2y}{3} + \frac{2}{9}\right) \times 2\,dx\,dy =$

$= 2\int_0^1 \left[\frac{x^2 y}{2} - \frac{x^2}{6} - \frac{2yx}{3} + \frac{2x}{9}\right]_y^1 dy = 2\int_0^1 \frac{y}{2} - \frac{1}{6} - \frac{2y}{3} + \frac{2}{9} - \left(\frac{y^3}{2} - \frac{y^2}{6} - \frac{2y^2}{3} + \frac{2y}{9}\right) dy$

$= 2\int_0^1 -\frac{y^3}{2} + \frac{y^2}{6} + \frac{2y^2}{3} - \frac{2y}{3} - \frac{2y}{9} + \frac{y}{2} - \frac{1}{6} + \frac{2}{9}\,dy = 2\left[-\frac{y^4}{8} + \frac{y^3}{18} + \frac{2y^3}{9} - \frac{2y^2}{6} - \frac{2y^2}{18} + \frac{y^2}{4} - \frac{y}{6} + \frac{2y}{9}\right]_0^1 =$

$= 2\left[-\frac{1}{8} + \frac{1}{18} + \frac{2}{9} - \frac{2}{6} - \frac{2}{18} + \frac{1}{4} - \frac{1}{6} + \frac{2}{9}\right] = \frac{1}{36}$

5.3 $T = \phi(R,C) = \frac{1}{100} \times R^3 \times C^3 \quad E(T) = E(\phi(R,C)) \simeq \phi(E(R), E(C)) =$

$= \frac{1}{100} \times E(R)^3 \times E(C)^3 = \frac{1}{100} \times (10^6)^3 \times (10^{-6})^3 = 0{,}01$

$\frac{d\phi}{dR} = \frac{3}{100} \times R^2 \times C^3 \quad \frac{d\phi}{dC} = \frac{3}{100} \times R^3 \times C^2 \quad \left[\frac{d\phi}{dR}\right]^2_{\substack{R=10^6 \\ C=10^{-6}}} = \left(\frac{3}{100} \times (10^6)^2 \times (10^{-6})^3\right)^2 = 9 \times 10^{-16}$

$\left[\frac{d\phi}{dC}\right]^2_{\substack{R=10^6 \\ C=10^{-6}}} = \left(\frac{3}{100} \times (10^6)^3 \times (10^{-6})^2\right)^2 = 9 \times 10^8 \quad Var(R) = (0{,}03 \times 10^6)^2 = 9 \times 10^8$

$Var(C) = (0{,}05 \times 10^{-6})^2 = 2{,}5 \times 10^{-15} \quad Cov(R,C) = 0$ (são independentes).

$Var(T) = Var(\phi(R,C)) \approx \left[\frac{d\phi}{dR}\right]^2_{\substack{R=10^6 \\ C=10^{-6}}} \times Var(R) + \left[\frac{d\phi}{dC}\right]^2_{\substack{R=10^6 \\ C=10^{-6}}} \times Var(C) =$

$= 9 \times 10^{-16} \times 9 \times 10^8 + 9 \times 10^8 \times 2{,}5 \times 10^{-15} = 3{,}06 \times 10^{-6} \quad \sigma(T) = \sqrt{3{,}06 \times 10^{-6}} = 1{,}75 \times 10^{-3}$

5.4 t: "teor de humidade" T: "peso total do carvão" S: "peso do carvão seco".

$S = \frac{100 - t}{100} \times T = \left(1 - \frac{t}{100}\right) \times T \leftarrow$ combinação não linear

$E(S) = E(\phi(t,T)) \approx \left(1 - \frac{E(t)}{100}\right) \times E(T) = \left(1 - \frac{8}{100}\right) \times 1000 = 920$ toneladas

$\left[\frac{dS}{dt}\right]^2_{\substack{E(t)=8 \\ E(T)=1000}} = \left[-\frac{T}{100}\right]^2_{\substack{E(t)=8 \\ E(T)=1000}} = \left(-\frac{1000}{100}\right)^2 = 100 \quad \left[\frac{dS}{dT}\right]^2_{\substack{E(t)=8 \\ E(T)=1000}} = \left[1 - \frac{t}{100}\right]^2_{E(t)=8} = \left(1 - \frac{8}{100}\right)^2 = 0{,}8464$

$Var(S) = 100 \times Var(t) + 0{,}8464 \times Var(T) = 100 \times 0{,}5^2 + 0{,}8464 \times 5^2 = 46{,}16$

$\sigma_S = \sqrt{46{,}16} = 6{,}79$ toneladas.

5.5 i) $E(X_i)=\mu$ $Var(X_i)=\sigma^2$ Como $X_i(1,2,\dots,m)$ são independentes,

$$Cov(X_i-\overline{X},\overline{X})=E\left[(X_i-\overline{X}-\mu_{X_i-\overline{X}})\times(\overline{X}-\mu_{\overline{X}})\right]$$

$$\mu_{\overline{X}}=E\left(\frac{1}{N}\sum_{i=1}^{N}X_i\right)=\frac{1}{N}\sum_{i=1}^{N}E(X_i)=\frac{1}{N}\times N\mu=\mu$$

$$\mu_{X_i-\overline{X}}=E(X_i-\overline{X})=E(X_i)-E(\overline{X})=\mu-\mu=0$$

Logo, $Cov(X_i-\overline{X},\overline{X})=E\left[(X_i-\overline{X})\times(\overline{X}-\mu)\right]=E\left[X_i\overline{X}-\mu X_i-\overline{X}^2+\mu\overline{X}\right]=$

$$=E(X_i\overline{X})-\mu^2+\mu^2-E(\overline{X}^2)=E(X_i\overline{X})-E(\overline{X}^2)$$

$$E(X_i\overline{X})=E\left(X_i\times\frac{1}{N}\sum_{j=1}^{N}X_j\right)=\frac{1}{N}\left(E(X_i^2)+(N-1)\times\overset{j\neq i}{E}(X_iX_j)\right)$$

$$Var(X_i)=E\left[(X_i-\mu)^2\right]=E(X_i^2)-2\mu E(X_i)+\mu^2=E(X_i^2)-\mu^2=\sigma^2\Leftrightarrow$$

$$\Leftrightarrow E(X_i^2)=\sigma^2+\mu^2$$

$$Cov(X_iX_j)=E\left[(X_i-\mu)(X_j-\mu)\right]=E(X_iX_j)-\mu^2-\mu^2+\mu^2=E(X_iX_j)-\mu^2=0\Leftrightarrow$$

$$\Leftrightarrow E(X_iX_j)=\mu^2$$

Logo, $E(X_i\overline{X})=\frac{1}{N}\left(\sigma^2+\mu^2+(N-1)\times\mu^2\right)=\frac{1}{N}\sigma^2+\mu^2$

$$E(\overline{X}^2)=E\left(\left(\frac{1}{N}\sum_{i=1}^{N}X_i\right)^2\right)=\frac{1}{N^2}\left[N\times E(X_i^2)+N\times(N-1)\times E(X_iX_j)\right]=$$

$$=\frac{1}{N}(\sigma^2+\mu^2)+\frac{N-1}{N}\times\mu^2=\frac{1}{N}\sigma^2+\mu^2$$

Logo, $Cov(X_i-\overline{X},\overline{X})=0$. $(X_i-\overline{X})$ não está correlacionado com $\overline{X}$, uma vez que $\overline{X}$ tem uma posição central em relação aos X_i's, pelo que qualquer desvio positivo é compensado por um desvio negativo.

ii) $Cov(X_i-\overline{X},X_i)=E\left[(X_i-\overline{X}-\mu_{X_i-\overline{X}})\times(X_i-\mu_{X_i})\right]=E\left[(X_i-\overline{X})\times(X_i-\mu)\right]=$

$$=E\left[X_i^2-\mu X_i-X_i\overline{X}+\overline{X}\mu\right]=E(X_i^2)-\mu^2-E(X_i\overline{X})+\mu E(\overline{X})=$$

$=\sigma^2+\mu^2-\mu^2-\frac{1}{N}\sigma^2-\mu^2+\mu^2=\left(1-\frac{1}{N}\right)\sigma^2$ Quando $N\geqslant 2$, a covariância é positiva e será tanto maior quanto maior for N.

iii) $Cov(X_i-\overline{X},X_j-\overline{X})=E\left[(X_i-\overline{X}-0).(X_j-\overline{X}-0)\right]=$

$$=E\left[X_iX_j-X_i\overline{X}-X_j\overline{X}+\overline{X}^2\right]=E(X_iX_j)-E(X_i\overline{X})-E(X_j\overline{X})+E(\overline{X}^2)=$$

$$=\mu^2-2\left(\frac{1}{N}\sigma^2+\mu^2\right)+\left(\frac{1}{N}\sigma^2+\mu^2\right)=\mu^2-\frac{1}{N}\sigma^2-\mu^2=-\frac{\sigma^2}{N}$$

A soma dos desvios $\sum_{i=1}^{N}(X_i-\overline{X})$ é sempre nula. Logo, se $X_i-\overline{X}>0$, os restantes desvios $X_j-\overline{X}$ têm um valor esperado negativo. Então, esta covariância será sempre negativa. No entanto, tende a anular-se quando $N\rightarrow+\infty$, pois a compensação do desvio $X_i-\overline{X}$ será diluída por um número infinito de desvios $X_j-\overline{X}$, pelo que o valor esperado destes desvios tenderá para zero.

5.6 i) $p_X(x) = \sum_y p_{X,Y}(x,y) =$

X=1000	X=2000	X=3000	X=4000
0,25	0,55	0,15	0,05

$X=1000 \Rightarrow p_X(x) = 0,20+0,04+0,01+0 = 0,25$
$X=2000 \Rightarrow p_X(x) = 0,10+0,36+0,09+0 = 0,55$
$X=3000 \Rightarrow p_X(x) = 0+0,05+0,10+0 = 0,15$
$X=4000 \Rightarrow p_X(x) = 0+0+0+0,05 = 0,05$

$p_Y(y) = \sum_x p_{X,Y}(x,y) =$

Y=1000	Y=2000	Y=3000	Y=4000
0,30	0,45	0,20	0,05

$Y=1000 \Rightarrow p_Y(y) = 0,20+0,10 = 0,30$
$Y=2000 \Rightarrow p_Y(y) = 0,04+0,36+0,05 = 0,45$
$Y=3000 \Rightarrow p_Y(y) = 0,01+0,09+0,10 = 0,20$
$Y=4000 \Rightarrow p_Y(y) = 0+0+0+0,05 = 0,05$

ii) $$P_{X|Y}(X=2000|Y=2000) = \frac{P_{X,Y}(x=2000, y=2000)}{P_Y(y=2000)} = \frac{0,36}{0,45} = 0,8 = 80\%$$

iii) $E(X) = \sum_x x \times p(x) = 0,25\times1000 + 0,55\times2000 + 0,15\times3000 + 0,05\times4000 = 2000€$

$E(Y) = \sum_y y\times p(y) = 0,30\times1000 + 0,45\times2000 + 0,20\times3000 + 0,05\times4000 = 2000€$.

$Var(X) = \sum_x (x-\mu_x)^2 \times p(x) = (1000-2000)^2\times0,25 + (3000-2000)^2\times0,15 + (4000-2000)^2\times0,05 =$
$= 600\,000 \qquad \sigma_x = \sqrt{600\,000} = 774,6€$

$Var(Y) = \sum_y (y-\mu_y)^2\times p(y) = (1000-2000)^2\times0,30 + (3000-2000)^2\times0,20 + (4000-2000)^2\times0,05$
$= 700\,000 \qquad \sigma_y = \sqrt{700\,000} = 836,7€$

iv) As variáveis são independentes se e só se $\forall x,y$: $P_{X,Y}(x,y) = p_X(x)\times p_Y(y)$

Por exemplo, $P_{X,Y}(x=1000, y=1000) = 0,20 \quad p_X(1000)\times p_Y(1000) = 0,25\times0,30 = 0,075$

Logo, as variáveis X e Y são dependentes.

v) $\gamma_{X,Y} = \sum_x \sum_y (x-2000)\times(y-2000)\times p_{X,Y}(x,y) =$
$= (1000-2000)\times(1000-2000)\times0,20 + (1000-2000)\times(3000-2000)\times0,01 +$
$+ (3000-2000)\times(3000-2000)\times0,10 + (4000-2000)\times(4000-2000)\times0,05 =$
$= 490\,000 \qquad \rho_{X,Y} = \frac{\gamma_{X,Y}}{\sigma_X\times\sigma_Y} = \frac{490\,000}{774,6\times836,7} = 0,756$

vi) a) $R = X+Y \leftarrow$ transformação não linear

$E(R) = E(X)+E(Y) = 2000+2000 = 4000 \quad Var(R) = Var(X)+Var(Y)+2Cov(X,Y) =$
$= 600\,000 + 700\,000 + 2\times490\,000 = 2\,280\,000 \quad \sigma_R = \sqrt{2280000} = 1510€$

b) $I = 0,20X + 0,10Y \leftarrow$ transformação não linear

$E(I) = 0,20E(X) + 0,10E(Y) = 0,20\times2000 + 0,10\times2000 = 600€$

$Var(I) = 0,20^2\times Var(X) + 0,10^2\times Var(Y) + 2\times0,20\times0,10\times Cov(X,Y) =$
$= 50\,600 \qquad \sigma_I = \sqrt{50600} = 224,9€$

(5.7) $R = \dfrac{500 \times X + 500 \times Y}{1000} = 0{,}5X + 0{,}5Y$

X \ Y	-10%	0%	10%	20%	$p_X(x)$
6%	0,10	0,10	0	0	0,20
8%	0	0,10	0,30	0,20	0,60
10%	0	0	0,10	0,10	0,20
$p_Y(y)$	0,10	0,20	0,40	0,30	

$E(X) = 0{,}20 \times 6 + 0{,}60 \times 8 + 0{,}20 \times 10 = 8\%$

$E(Y) = 0{,}10 \times (-10) + 0{,}20 \times 0 + 0{,}40 \times 10 + 0{,}30 \times 20 = 9\%$

$Var(X) = \sum_X (x - \mu_X)^2 \times p(x) = (6-8)^2 \times 0{,}20 + (8-8)^2 \times 0{,}60 + (10-8)^2 \times 0{,}20 = 1{,}6\%$

$Var(Y) = \sum_Y (y - \mu_Y)^2 \times p(y) = (-10-9)^2 \times 0{,}10 + (0-9)^2 \times 0{,}20 + (10-9)^2 \times 0{,}40 + (20-9)^2 \times 0{,}30 = 89\%$

$\sigma_X = \sqrt{1{,}6} = 1{,}265 \quad \sigma_Y = \sqrt{89} = 9{,}434 \quad E(R) = 0{,}5E(X) + 0{,}5E(Y) = 0{,}5 \times 8 + 0{,}5 \times 9 = 8{,}5\%$

$Var(R) = 0{,}5^2 \times Var(X) + 0{,}5^2 \times Var(Y) + 2 \times 0{,}5 \times 0{,}5 \times Cov(X,Y)$

$Cov(X,Y) = \gamma_{X,Y} = \sum_X \sum_Y (x-8)(y-9) \times p_{X,Y}(x,y) = (6-8)(-10-9) \times 0{,}10 + (10-8)(10-9) \times 0{,}10 +$
$+ (10-8)(20-9) \times 0{,}10 + (6-8)(0-9) \times 0{,}10 = 8$

$Var(R) = 0{,}5^2 \times 1{,}6 + 0{,}5^2 \times 89 + 2 \times 0{,}5 \times 0{,}5 \times 8 = 26{,}65 \quad \sigma_R = \sqrt{26{,}65} = 5{,}16\%$