Entropy Rates of Hidden Markov Processes emerge from Blackwell's Trapdoor Channel

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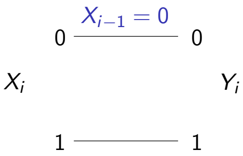
Focus of Talk

1. Blackwell's Trapdoor Channel Feedback Capacity
   - Numerical Calculations (and "chemical channel" generalization)
   - Analytic Solution
   - Zero-error Communication Scheme

2. Hidden-Markov Processes
   - Not So Hidden Markov Processes

Channel Capacity Puzzle

What is the capacity of this two-state channel where the state is a function of the previous input?



(Observation: Markov inputs lead to Hidden-Markov channel outputs.)

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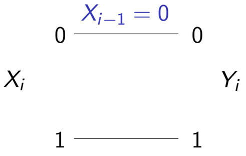
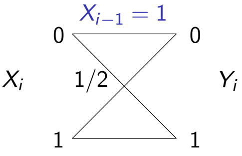


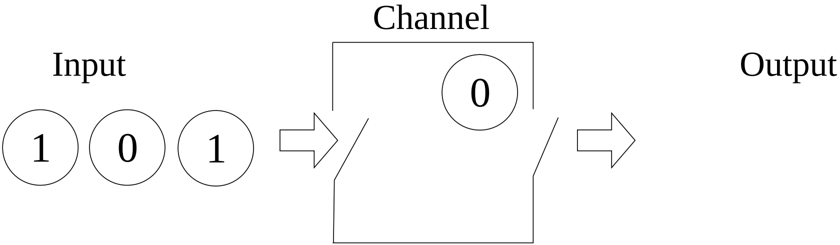


Alternative Characterization:

$$\begin{aligned} Z_i &\sim \text{i.i.d. Bern}(1/2), \\ Z^n &\perp X^n, \\ Y_i &= X_i + (X_{i-1} Z_i) \bmod 2. \end{aligned}$$

(Observation: Markov inputs lead to Hidden-Markov channel outputs.)

The Trapdoor Channel



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$s_0 = 0$

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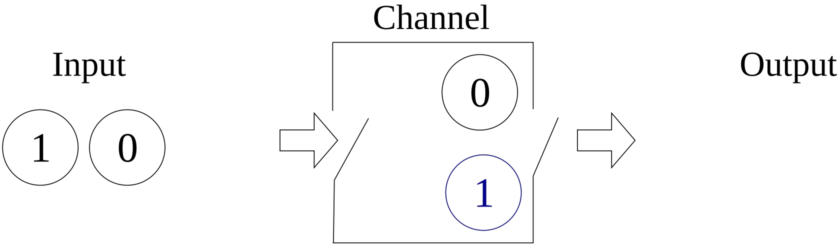

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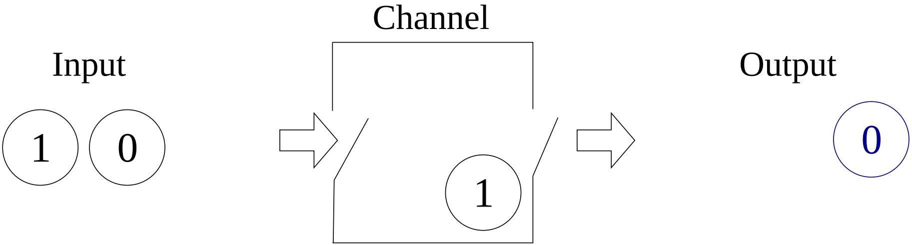

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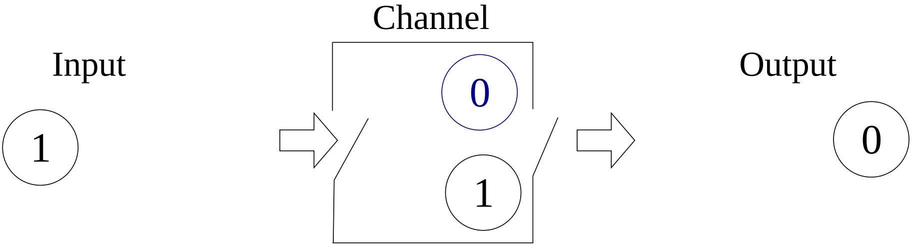

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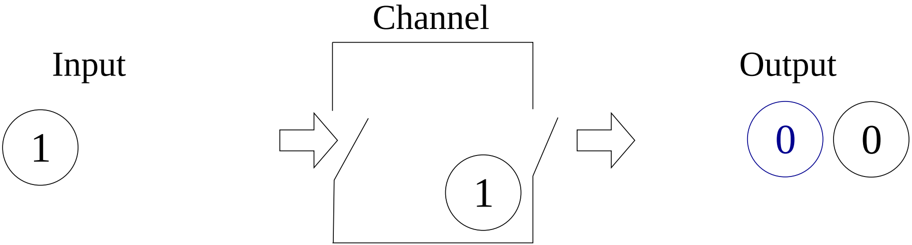

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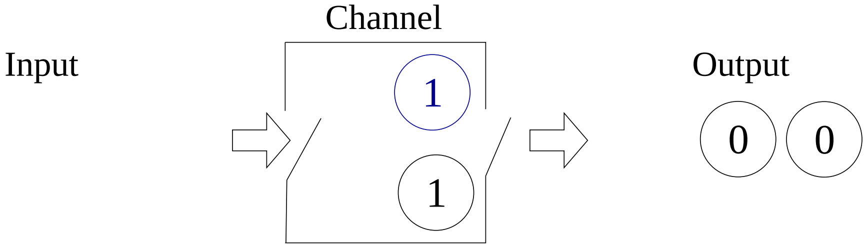

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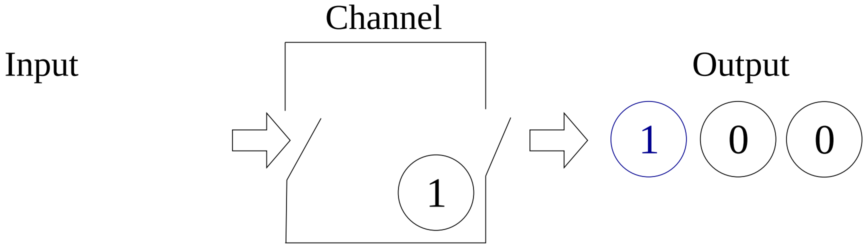

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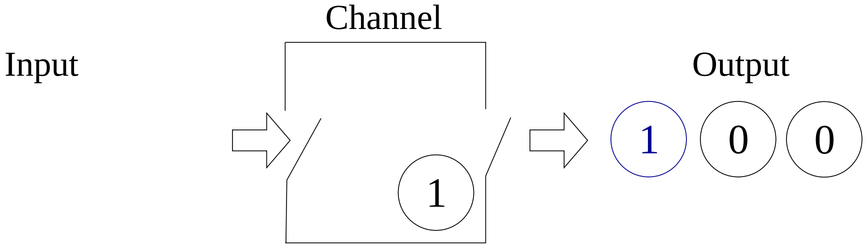

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Biochemical Interpretation [Berger 71]

The Chemical Channel

Balls are not equally likely to exit the channel.

x_t	s_{t-1}	$p(y_t = 0 \mid x_t, s_{t-1})$	$p(y_t = 1 \mid x_t, s_{t-1})$
0	0	1	0
0	1	p_1	$1 - p_1$
1	0	p_2	$1 - p_2$
1	1	0	1

Special cases:

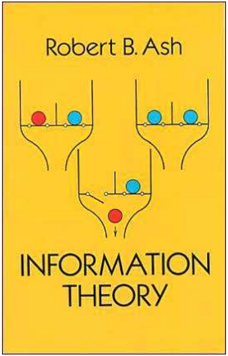
Trapdoor channel: $p_1 = p_2 = \frac{1}{2}$.

New vs. Old: $p_1 = 1 - p_2 = p_{switch}$.

'0' vs. '1': $p_1 = p_2 = p_{zero}$.

The Trapdoor Channel

Introduced by David Blackwell in 1961.
[Ash 65], [Ahlswede & Kaspi 87], [Ahlswede 98], [Kobayashi 02 & 03].



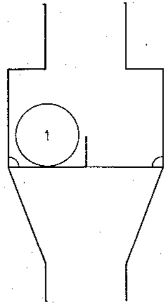


Fig. 7.1 A simple two-state channel.

A "simple two-state channel." - Blackwell

Ideas for Communication without Feedback

- Repeat each bit three time: $R = 1/3$ bit.

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- Repeat each bit twice: $R = 1/2$ bit. [Ahlswede & Kaspi 87]
- $C \approx 0.572$ bits per channel use. [Kobayashi & Morita 03]

Communication Setting (with feedback)

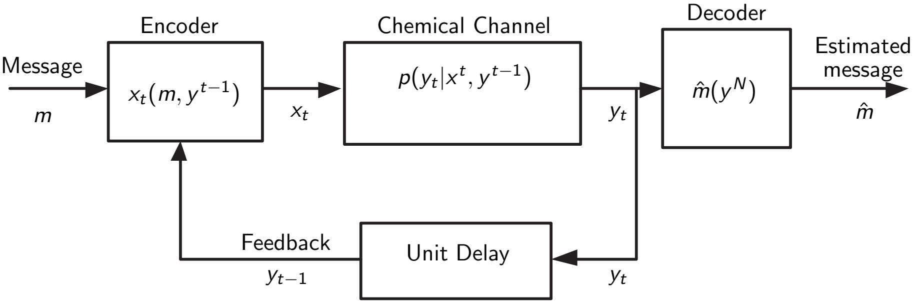

Figure: Communication with feedback

Feedback Capacity of FSC

Lower and upper bounds:

$$
\begin{aligned}
C_{FB} &\geq \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^{N}} \min_{s_0} I(X^N \to Y^N|s_0) \\
C_{FB} &\leq \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^{N}} \max_{s_0} I(X^N \to Y^N|s_0)
\end{aligned}
$$

[Permuter, Weissman & Goldmith ISIT06]

Directed Information

Mutual Information

$$I(X^n; Y^n) = \sum_{i=1}^{n} I(X^{n}; Y_i | Y^{i-1})$$

Directed Information was defined by Massey in 1990.

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Intuition [Massey 05]:

$$I(X^n;Y^n) \;=\; I(X^n \rightarrow Y^n) + I(Y^{n-1} \rightarrow X^n)$$

Feedback Capacity of Unifilar, Strongly Connected, FSC

Chemical channel has two other properties of interest.

1. Unifilar [Ziv 85]: State is deterministic function of past state, input, and output.
2. Strongly connected: Any state s_t can be reached with positive probability from any other state s_{t-1}.

Consequence

Initial state doesn't matter; upper and lower bounds become equal.

$$C_{FB} = \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} I(X^N \to Y^N)$$

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&= \sup_{\{p(x_t|s_{t-1},y^{t-1})\}_{t\geq 1}} \liminf_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} I(X_t, S_{t-1}; Y_t|Y^{t-1})
\end{aligned}
$$

Dynamic Programming (infinite horizon, average reward)

Variable Assignments

State: $\beta_t = p(s_t|y^t)$

Action: $u_t = p(x_t|s_{t-1})$

Disturbance: $w_t = y_{t-1}$

Dynamic Programming Requirements

State evolution:

$$\beta_t = F(\beta_{t-1}, u_t, w_t)$$

Reward function per unit time:

$$g(\beta_{t-1}, u_t) = I(X_t, S_{t-1}; Y_t|\beta_{t-1})$$

Similar work: [Yang, Kavčić & Tatikonda 05], [Chen & Berger 05]

Dynamic Programming (infinite horizon, average reward)

Dynamic Programming Operator T

The dynamic programming operator T is given by

$$T \circ J(\beta) = \sup_{u \in \mathcal{U}} \left(g(\beta, u) + \int P_w(dw|\beta, u) J(F(\beta, u, w)) \right).$$

Bellman Equation

If there exist a function $J(\beta)$ and constant ρ that satisfy

$$J(\beta) = T \circ J(\beta) - \rho$$

then ρ is the optimal infinite horizon average reward.

Feedback Capacity of Chemical Channel (20 iterations)

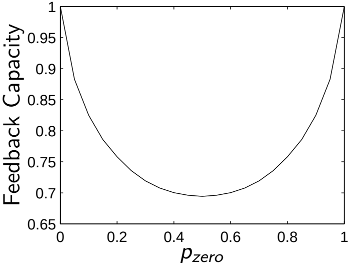

Figure: Feedback capacity of chemical channel as functions of two parameters.

Trapdoor channel feedback capacity found at $p_{zero} = 0.5$ and $p_{switch} = 0.5$.

Coincidence

Trapdoor channel: $C_{FB} \approx 0.694$ bits

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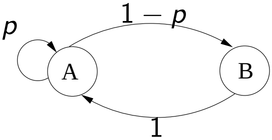
<u>Homework Question</u>

Entropy rate. Find the maximum entropy rate of the following two-state Markov chain:



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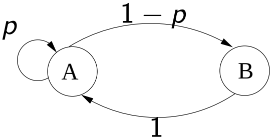

Solution: (Golden Ratio: $\phi = \frac{\sqrt{5}+1}{2}$)

$$p^\star = \phi - 1 = \frac{1}{\phi}$$

$$H(\mathcal{X}) = \log \phi = 0.6942\ldots \text{ bits}$$

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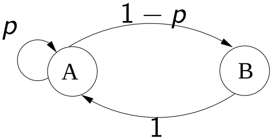

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Rename channel input: $\tilde{x}_t = x_t \oplus s_{t-1}$.
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This is like the puzzle at the beginning (don't repeat 1's into the channel).

Decoding Example

Decoding rules

$$\tilde{x}_t = 0 \quad \Rightarrow \quad \tilde{x}_{t-1} = \tilde{y}_t$$

$$\tilde{x}_t = 1 \quad \Rightarrow \quad \tilde{x}_{t-1} = 0$$

$\tilde{x}^n$: $\quad\quad\quad\quad 0$

$\tilde{y}^n$: 1 1 1 1 0 0 1

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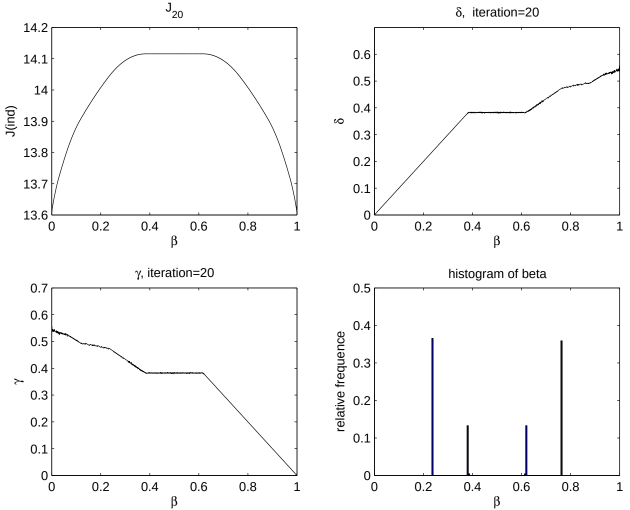
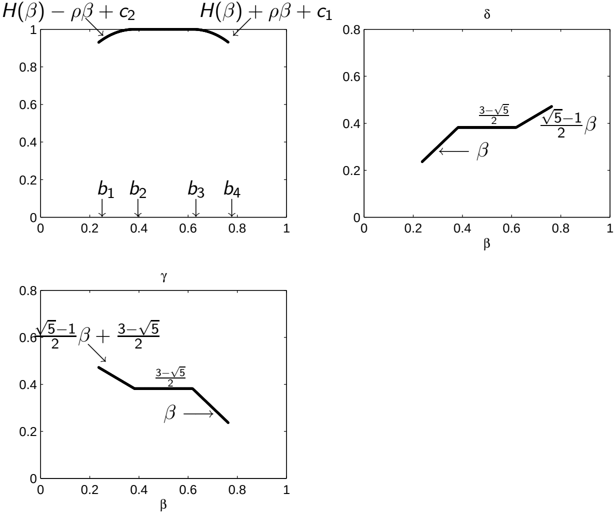
Dynamic Programming 20th Value iteration



Bellman Equation: $J(\beta) = T \circ J(\beta) - \rho$.

Conjectured Solution to the Bellman Equation



Bellman Equation: $J(\beta) = T \circ J(\beta) - \rho$.

Proven Feedback Capacity

Bellman Equation is satisfied.

Trapdoor channel feedback capacity:

$$C_{FB} = \log \phi = 0.6942\ldots \text{ bits}$$

A Zero-error Communication Scheme

$\tilde{x}^n$	flag	index
0010100010100101		

1. Message maps to unique sequence without repeating 1's.

A Zero-error Communication Scheme

$\tilde{x}^{n+1}$	flag	index
00101000101001010		

1. Message maps to unique sequence without repeating 1's.
2. Concatenate with 0.

A Zero-error Communication Scheme

$\tilde{x}^{n+1}$	flag	index
0010100010100101**0**	xxx	

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3. Indicate if "inconsistency" was observed.

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Number of messages

How many binary sequences of length n without repeating 1's?
Fibonachi sequence: $f_n \doteq \phi^n$.

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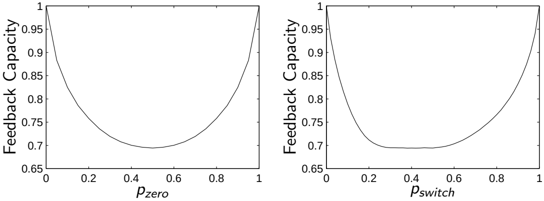
Feedback Capacity Summary

- Chemical Channel



- Trapdoor channel

$$C_{FB} = \log \phi = 0.6942\ldots \text{ bits}$$

- Zero-error communication scheme

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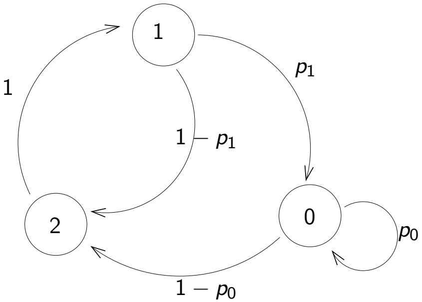
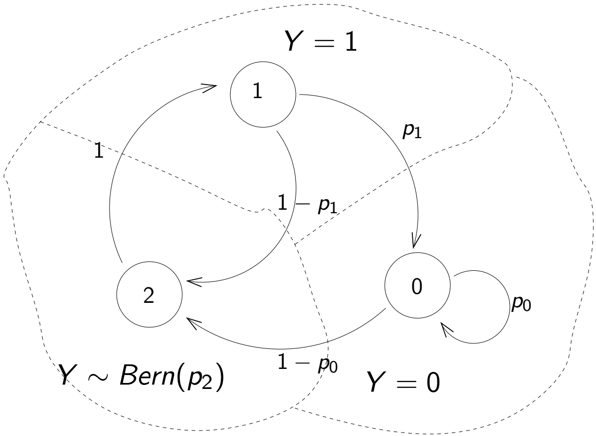
Markov Process X_n

(related to the capacity achieving input to the trapdoor channel)



Hidden Markov Process Y_n

(related to the output of the trapdoor channel)



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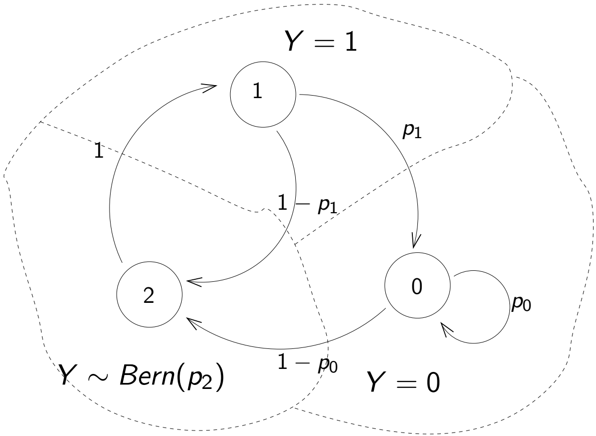

$H(\mathcal{Y}) = \mu_0 H(p_0) + \mu_1 H(p_1) + \mu_2 H(p_2)$, where μ is the stationary distribution of X_n ($\mu_1 = \mu_2 = (1 - p_0)/(2(1 - p_0) + p_1), \mu_0 = 1 - \mu_1 - \mu_2$).

Observable Underlying Markov Process

Let X_n be a discrete, time-invariant Markov process and Y_n be a memoryless, time-invariant hiding of X_n (i.e. $p(y^n|x^n) = \prod_{i=1}^n p(y_i|x_i)$), and define the conditional entropy rate

$$H(\mathcal{X}|\mathcal{Y}) \triangleq \lim_{n\to\infty} \frac{1}{n} H(X^n|Y^n).$$

Observation 1

$$H(\mathcal{X}|\mathcal{Y}) = 0 \implies H(\mathcal{Y}) = H(\mathcal{X}) + H(\mathcal{Y}|\mathcal{X}).$$

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Theorem 1

$$H(\mathcal{X}|\mathcal{Y}) = 0 \quad \Longleftrightarrow \quad H(X_1^k|Y_1^k, X_0, X_{k+1}) = 0 \text{ for all } k \in \mathcal{N}.$$

Proof (part a) of Theorem 1

"$\Leftarrow$"

$$
\begin{aligned}
H(\mathcal{X}|\mathcal{Y}) &= \lim_{n\to\infty} \frac{1}{n} H(X^n|Y^n) \\
&= \lim_{n\to\infty} \frac{1}{n} \left[H(X_1^n|Y_1^n, X_0, X_{n+1}) + I(X_1^n; X_0, X_{n+1}|Y_1^n)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} I(X_1^n; X_0, X_{n+1}|Y_1^n) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \left[H(X_0) + H(X_{n+1})\right] \\
&= 0.
\end{aligned}
$$

Proof (part b) of Theorem 1

"$\Rightarrow$"

$$H(X^n|Y^n)$$

X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y
X	Y

Figure: Hidden Markov Process

Proof (part b) of Theorem 1

"$\Rightarrow$"

Conditioning on more decreases entropy.

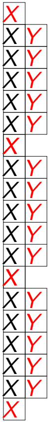

Figure: Hidden Markov Process

Proof (part b) of Theorem 1

"⇒"

Markovity allows us to separate and add the blocks.

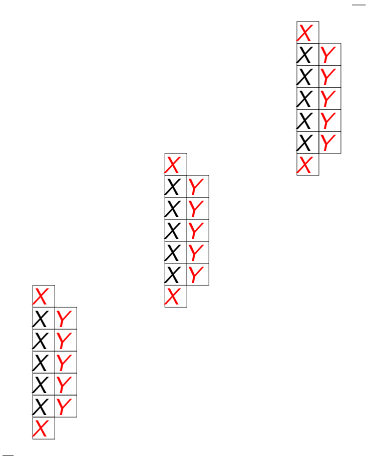

Figure: Hidden Markov Process

Proof (part b) of Theorem 1

"$\Rightarrow$"

Stationarity allows time shifts.

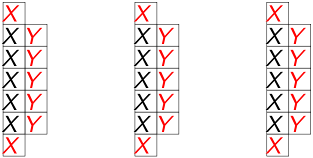

Figure: Hidden Markov Process

Proof (part b) of Theorem 1

"$\Rightarrow$"

$$
\begin{aligned}
H(X_1^k|Y_1^k, X_0, X_{k+1}) &= \frac{1}{n}\sum_{i=1}^{n} H(X_{(i-1)k'+1)}^{ik'-1}|Y_{(i-1)k'+1}^{ik'-1}, X_{(i-1)k'}, X_{ik'}) \\
&= \frac{1}{n} H(X_1^{nk'}|Y_1^{nk'}, \{X_{ik'}\}_{i=0}^{n}) \\
&= \lim_{n\to\infty} \frac{1}{n} H(X_1^{nk'}|Y_1^{nk'}, \{X_{ik'}\}_{i=0}^{n}) \\
&\leq k' \lim_{n\to\infty} \frac{1}{nk'} H(X_1^{nk'}|Y_1^{nk'}) \\
&= k' H(\mathcal{X}|\mathcal{Y}) \\
&= 0,
\end{aligned}
$$

where $k' = k + 1$.

Other Questions

- How many states are necessary to construct a non-trivial HMP that satisfies $H(X_1^k|Y_1^k, X_0, X_{k+1})$ for all k?

- Does $H(X_1^k|Y_1^k, X_0, X_{k+1})$ for all k imply that either $H(X_1|Y_1, X_0) = 0$ or $H(X_1|Y_1, X_2) = 0$?

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  - No