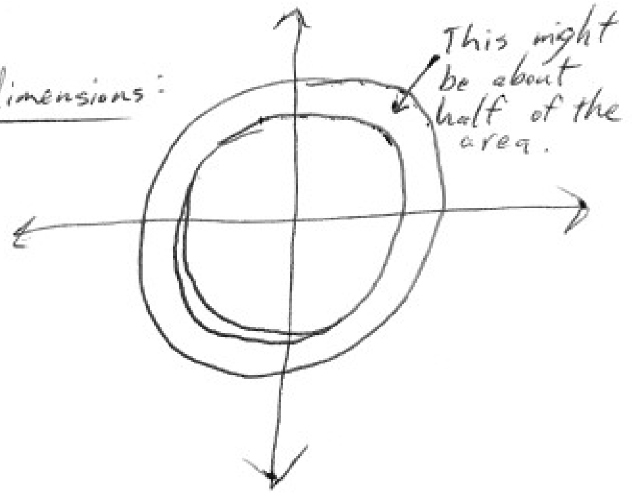
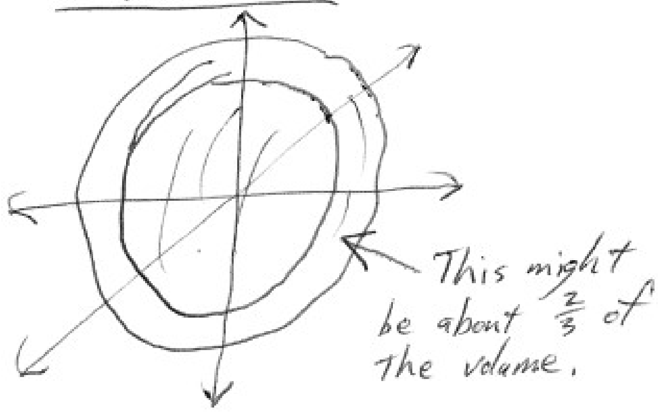
3-8-05

Today in my Information Theory class Dr. Cover was telling us some interesting features of working in multiple dimensions. It started when he mentioned that a sphere in a high-dimensional space will have all of its mass in its shell. Let me explain.

2 dimensions:



3 dimensions:

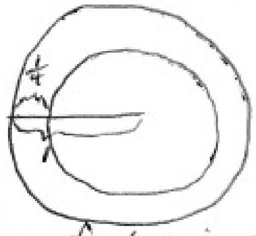


Let's start slow and with numbers:
Let's use sphere's of radius 1.

1 dimensional sphere:

Length of outer portion is $\frac{1}{4}$th of total length.

2 dimensional sphere:



Area of outer region $= \pi\left(1-\left(\frac{3}{4}\right)^2\right)$
Area of whole region $= \pi$
Fraction in outer region: $\frac{7}{16}$
Almost half!

3D Sphere:

Fraction in outer region: $1-\left(\frac{3}{4}\right)^3 = \frac{37}{64}$

N Dimension:

Fraction $= 1-\left(\frac{3}{4}\right)^n$

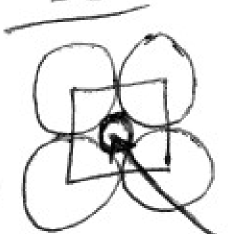
$$\lim_{n\to\infty}\left(1-\left(\frac{3}{4}\right)^n\right) = 1$$

There, that was a warm up. It's an interesting result that's useful for information theory though.
The real question is on the next page.

As a tangent during our class today Dr. Cover wanted us to recognize that intuition doesn't always serve us well when dealing with multiple dimensions. To illustrate his point he asked the following question (he actually warned us not to vote when he gave us the three choices for the answer because he knew we'd be wrong).

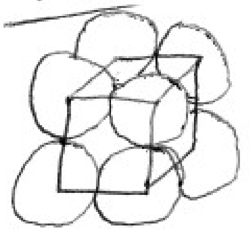
Consider the following pattern in which you are interested in the volume of the inner sphere:

2D



The above is a unit square, and the rest is just as it seems. We are interested in the area at the center (the circle in the center)

3D



Again a unit cube and we're interested in the volume of the sphere that would fit tightly in the center.

Now assume you continue extending this problem to the ~~n the~~ n^{th} dimension. What is the limit of the volume of that sphere in the middle as n goes to infinity. (By the way, the usual volume and distance equations work for n dimensions) You have three choices for the answer.

a. 0
b. 1
c. ∞

Nice try. Now go ahead and prove yourself wrong by calculating what the volume really is.