EINDHOVEN UNIVERSITY OF TECHNOLOGY

Inverse methods in optical design

ECMI modeling week, Szeged, Hongary, July 2023

Martijn Anthonissen

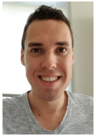
Computational Illumination Optics group at TU/e

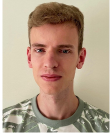

Jan ten Thije Boonkkamp


Wilbert IJzerman
Signify Research


Martijn Anthonissen


Antonio Barion


Maikel Bertens

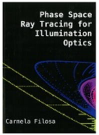

Roel Hacking


Willem Jansen

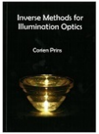

Vì Kronberg


Teun van Roosmalen

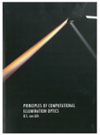

Sanjana Verma



Carmela Filosa

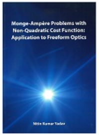


Corien Prins



Bart van Lith

Lotte Romijn

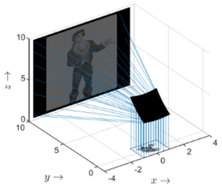


Nitin Yadav

TU/e

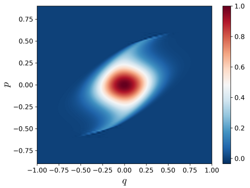
Computational Illumination Optics at TU/e

Track 1: Freeform design



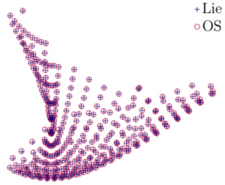
- Compute optical surfaces that convert given source into desired target distribution
- Freeform surfaces
- Fully nonlinear PDE of Monge-Ampère type

Track 2: Improved direct methods



- Ray tracing: iterative procedure to compute final design. Slow convergence
- New methods based on Hamiltonian structure of system and advanced numerical schemes for PDEs

Track 3: Imaging optics

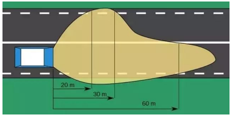


- Make a very precise image of an object, minimizing aberrations
- Description with Lie transformations

Inverse methods in illumination optics

Design of optical systems for illumination purposes

- LED lighting
- Road lights
- Car headlights

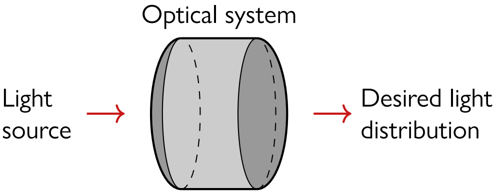


Industry standard: ray tracing

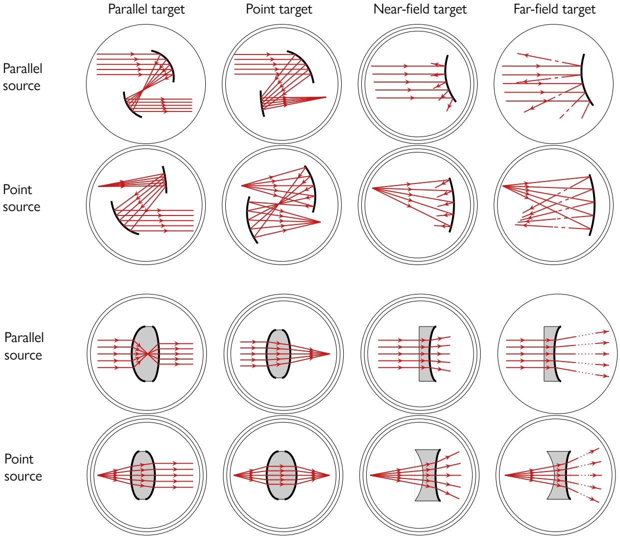
- Easy to implement
- Slow convergence
- Design, ray trace, change design, ray trace, …

Inverse methods

- Directly compute required optical system
- Need solving PDE of Monge-Ampère type
- Avoid iterations and manual optimization

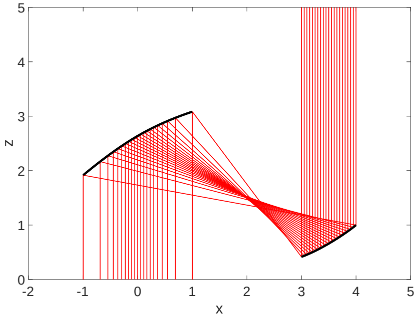


Sixteen basic systems



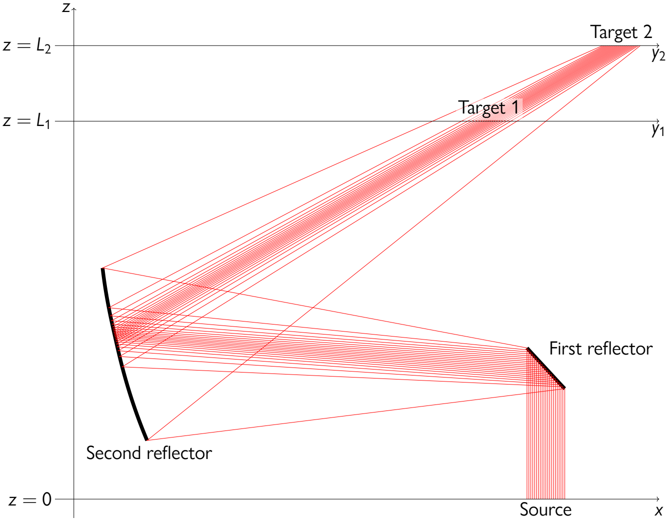
TU/e

Example: Parallel-to-parallel reflector 2D



- Source: parallel beam with Gaussian light distribution
- Desired at target: parallel beam with uniform distribution
- Find the two freeform reflector surfaces

The challenge



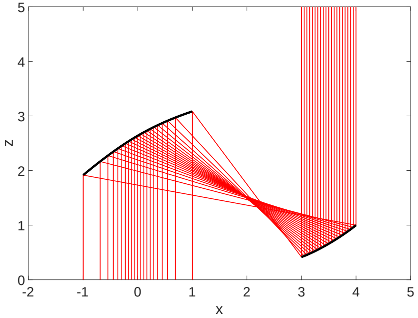
- Source: parallel beam with given light distribution
- Target 1: given desired light distribution
- Target 2: given desired light distribution
- Find the two freeform reflector surfaces

Approach

- Parallel-to-near-field reflector (one freeform reflector surface)
- This challenge in two steps
  1: target 1 to target 2
  2: source to target 1
- Point source
- Lens instead of reflector

Parallel-to-parallel reflector 2D

SMA



- Source: parallel beam with Gaussian light distribution
- Desired at target: parallel beam with uniform distribution
- Find the two freeform reflector surfaces

TU/e

Mathematical model: optimal transport formulation

SMA

- Path of a ray
  - Leaves source S at $P = (x, 0)$
  - Hits first reflector at $A = (x, u(x))$
  - Hits second reflector at $B = (y, L - w(y))$
  - Arrives at target T at $Q = (y, L)$
- **Optical path length**

$$V = u(x) + d(A, B) + w(y)$$

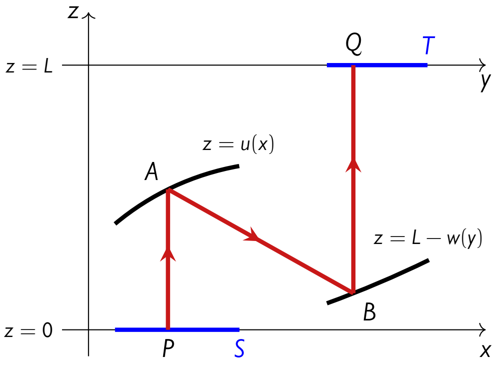
- Let $\beta := V - L$. Then

$$u(x) + w(y) = \tfrac{1}{2}\beta + L - \frac{(y-x)^2}{2\beta}$$

$$=: c(x, y)$$

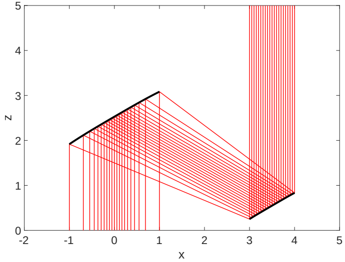
$c(x, y)$ is called cost function

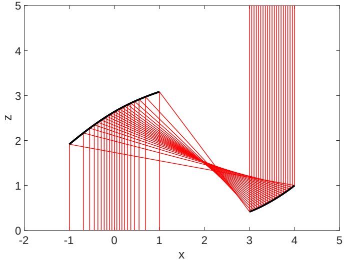
- Differentiate to x: **optical mapping** $y = m(x) = x + \beta u'(x)$



Energy conservation

SMA





- $S = (-1, 1)$, $T = (3, 4)$
  Source light distribution: $E = E(x)$, $x \in S$, Gaussian
  Target light distribution: $G = G(y)$, $y \in T$, uniform
- Energy conservation:
$$\int_{-1}^{x} E(\xi)\,d\xi = \pm \int_{m(-1)}^{m(x)} G(y)\,dy = \pm \int_{-1}^{x} G(m(\xi))\,m'(\xi)\,d\xi$$
- Differentiate to x:
$$m'(x) = \pm \frac{E(x)}{G(m(x))}$$
  Solve ODE for m
- Differentiate $u(x) + w(y) = c(x, y)$ to x, substitute $y = m(x)$:
$$u'(x) = \frac{\partial c}{\partial x}(x, m(x))$$
  Solve ODE for u
- Second reflector: $w(m(x)) = c(x, m(x)) - u(x)$

TU/e

Parallel-to-parallel reflector, 2D and 3D

SMA

2D

- Optical mapping:

$$y = m(x) = x + \beta u'(x) = \phi'(x)$$

- Optimal transport formulation:

$$u(x) + w(y) = c(x, y)$$

$$c(x, y) = \tfrac{1}{2}\beta + L - \frac{(y - x)^2}{2\beta}$$

- Energy conservation:

$$m'(x) = \phi''(x) = \pm \frac{E(x)}{G(m(x))}$$

3D: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$

- Optical mapping:

$$\mathbf{y} = \mathbf{m}(\mathbf{x}) = \mathbf{x} + \beta \nabla u(\mathbf{x}) = \nabla \phi(\mathbf{x})$$

- Optimal transport formulation:

$$u(\mathbf{x}) + w(\mathbf{y}) = c(\mathbf{x}, \mathbf{y})$$

$$c(\mathbf{x}, \mathbf{y}) = \tfrac{1}{2}\beta + L - \frac{\|\mathbf{y} - \mathbf{x}\|^2}{2\beta}$$

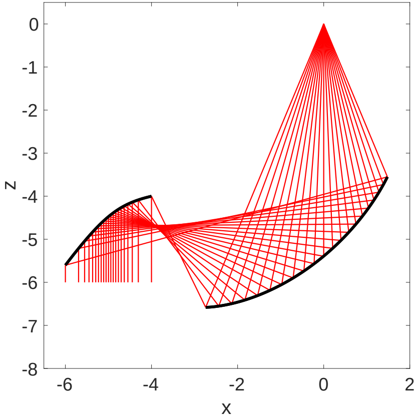
- Energy conservation:

$$\det(D\mathbf{m}(\mathbf{x})) = \frac{\partial^2 \phi}{\partial x_1^2}\frac{\partial^2 \phi}{\partial x_2^2} - \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right)^2 = \pm \frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))}$$

Standard Monge-Ampère equation
Second-order nonlinear PDE

Parallel-to-point reflector 2D

GMA



- Source: parallel beam with Gaussian light distribution
- Point target with uniform distribution
- Find the two freeform reflector surfaces

TU/e

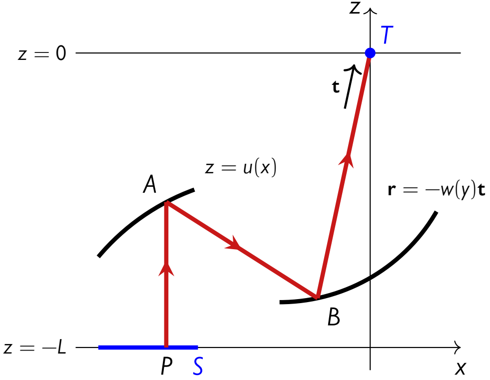
Mathematical model: optimal transport formulation

GMA

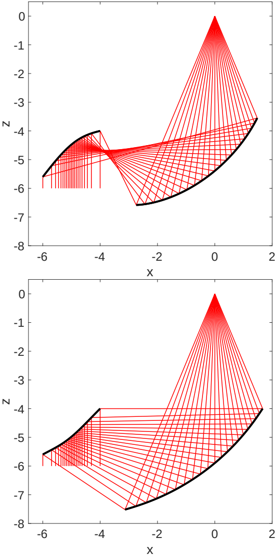
- Path of a ray
  - Leaves source S at $P = (x, -L)$
  - Hits first reflector at $A = (x, -L + u(x))$
  - Hits second reflector at $B = (-w(y)t_1, -w(y)t_2)$
  - Arrives at target $T = (0, 0)$
- Optical path length: $V = u(x) + d(A, B) + w(y)$
- Geometric relation

$$\left(-\frac{u}{\beta} - \frac{x^2}{2\beta^2} + \frac{V+L}{2\beta}\right) \cdot \left(\frac{\beta}{w}(1+y^2) - 2y^2\right) = \left(1 + \frac{xy}{\beta}\right)^2$$

- Take logarithms: $u_1(x) + u_2(y) = c(x, y)$
  Cost function is non-quadratic



Numerical method



- Energy conservation:

$$m'(x) = \pm \frac{E(x)}{G(m(x))} \frac{1 + (m(x))^2}{2}$$

  Solve ODE for m

- Differentiate $u_1(x) + u_2(y) = c(x, y)$ to x, substitute $y = m(x)$:

$$u_1'(x) = \frac{\partial c}{\partial x}(x, m(x))$$

  Solve ODE for u_1
  From u_1 compute u

- Second reflector:

$$u_2(m(x)) = c(x, m(x)) - u_1(x)$$

  From u_2 compute w

Parallel-to-point reflector, 2D and 3D

GMA

2D

- Energy conservation:

$$m'(x) = \pm \frac{E(x)}{G(m(x))} \frac{1 + (m(x))^2}{2}$$

- Optimal transport formulation:

$$u_1(x) + u_2(y) = c(x, y)$$

$$c(x, y) = \log \left(\left(1 + \frac{xy}{\beta} \right)^2 \right)$$

3D: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$

- Energy conservation:

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \pm \frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))} \frac{\left(1 + \|\mathbf{m}(\mathbf{x})\|^2\right)^2}{4}$$

- Optimal transport formulation:

$$u_1(\mathbf{x}) + u_2(\mathbf{y}) = c(\mathbf{x}, \mathbf{y})$$

$$c(\mathbf{x}, \mathbf{y}) = \log \left(\left(1 + \frac{\mathbf{x} \cdot \mathbf{y}}{\beta} \right)^2 \right)$$

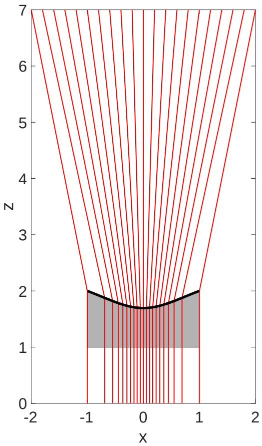
- Differentiate, substitute $\mathbf{y} = \mathbf{m}(\mathbf{x})$, differentiate

$$\underbrace{\mathrm{D}_{\mathbf{xy}} c(\mathbf{x}, \mathbf{m}(\mathbf{x}))}_{\mathbf{C}} \, \mathrm{D}\mathbf{m}(\mathbf{x}) = \underbrace{\mathrm{D}^2 u_1(\mathbf{x}) - \mathrm{D}_{\mathbf{xx}} c(\mathbf{x}, \mathbf{m}(\mathbf{x}))}_{\mathbf{P}}$$

- **Generalized Monge-Ampère equation**

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \frac{\det(\mathbf{P})}{\det(\mathbf{C})} = \pm \frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))} \frac{\left(1 + \|\mathbf{m}(\mathbf{x})\|^2\right)^2}{4}$$

Parallel-to-near-field lens



- Source: parallel beam with Gaussian light distribution
- Near-field target with uniform distribution
- Find the single freeform lens surface

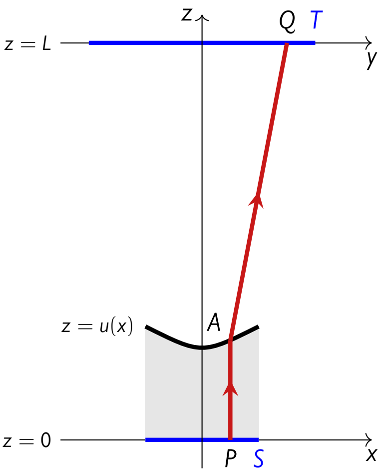
Mathematical model

GJE

- Path of a ray
  - Leaves source S at $P = (x, 0)$
  - Hits freeform lens surface at $A = (x, u(x))$
  - Arrives at target T at $Q = (y, L)$
- Optical path length:

$$
\begin{aligned}
V(y) &= n \cdot u(x) + d(A, Q) \\
&= n \cdot u(x) + \sqrt{(y - x)^2 + (L - u(x))^2}
\end{aligned}
$$

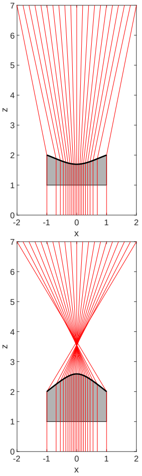
Cannot be formulated as optimal transport problem



TU/e

Numerical method

GJE



- Energy conservation:

$$m'(x) = \pm \frac{E(x)}{G(m(x))}$$

Solve ODE for m

- Geometric relation:

$$V(y) = n \cdot u(x) + \sqrt{(y-x)^2 + (L-u(x))^2}$$

differentiate to x, substitute $y = m(x)$, solve for $u'(x)$:

$$u'(x) = \frac{m(x) - x}{n \cdot \sqrt{(m(x)-x)^2 + (L-u(x))^2} + u(x) - L}$$

Solve ODE for u

TU/e

Parallel-to-near-field lens, 2D and 3D

GJE

2D

- Energy conservation:

$$m'(x) = \pm \frac{E(x)}{G(m(x))}$$

- Geometric relation:

$$V(y) = n \cdot u(x) + \sqrt{(y-x)^2 + (L-u(x))^2}$$

3D

- Energy conservation:

$$\det(\mathrm{D}\mathbf{m}(\mathbf{x})) = \pm \frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))}$$

- Geometric relation:

$$V(\mathbf{y}) = n \cdot u(\mathbf{x}) + \sqrt{\|\mathbf{y}-\mathbf{x}\|^2 + (L-u(\mathbf{x}))^2} =: H(\mathbf{x}, \mathbf{y}, u(\mathbf{x}))$$

- Differentiate to $\mathbf{x}$: $\nabla_{\mathbf{x}} \big(H(\mathbf{x}, \mathbf{y}, u(\mathbf{x})) \big) = \mathbf{0}$
- Let $\widetilde{H}(\mathbf{x}, \mathbf{y}) := H(\mathbf{x}, \mathbf{y}, u(\mathbf{x}))$, substitute $\mathbf{y} = \mathbf{m}(\mathbf{x})$, differentiate to $\mathbf{x}$:

$$\mathrm{D}_{\mathbf{xx}}\widetilde{H}(\mathbf{x}, \mathbf{m}(\mathbf{x})) + \underbrace{\mathrm{D}_{\mathbf{xy}}\widetilde{H}(\mathbf{x}, \mathbf{m}(\mathbf{x}))}_{\mathbf{C}} (\mathrm{D}\mathbf{m})(\mathbf{x}) = \mathbf{O}$$

- **Generated Jacobian equation**

$$\det\big(\mathrm{D}_{\mathbf{xx}}\widetilde{H}(\mathbf{x}, \mathbf{m}(\mathbf{x}))\big) = \pm \det(\mathbf{C}) \frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))}$$

Iterative least-squares solver for 3D systems

SMA GMA GJE

- Find mapping $\mathbf{m} : S \to T$ such that

$$\det(\mathrm{D}\mathbf{m}) = \frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))}$$

$$\mathbf{m}(\partial S) = \partial T$$

- Break down in substeps
  We compute $\mathbf{P}$, $\mathbf{b}$, $\mathbf{m}$ such that

$$\mathbf{P} = \mathbf{C}\,\mathrm{D}\mathbf{m}$$

$$\det(\mathbf{P}) = \det(\mathbf{C})\,\frac{E(\mathbf{x})}{G(\mathbf{m}(\mathbf{x}))}$$

$$\mathbf{b}(\mathbf{x}) = \mathbf{m}(\mathbf{x}) \qquad \mathbf{x} \in \partial S$$

$\mathbf{b}$ maps ∂S to ∂T

- Iterative procedure:
  1. Choose an initial guess $\mathbf{m}^0$
     Let n = 0
  2. Let $J_I(\mathbf{m}, \mathbf{P}) = \frac{1}{2} \iint_S \|\mathbf{C}\,\mathrm{D}\mathbf{m} - \mathbf{P}\|^2 \, d\mathbf{x}$

     $$\mathbf{P}^{n+1} = \underset{\mathbf{P}}{\operatorname{argmin}}\; J_I(\mathbf{m}^n, \mathbf{P})$$

     Constrained minimization problem
  3. Let $J_B(\mathbf{m}, \mathbf{b}) = \frac{1}{2} \int_{\partial S} \|\mathbf{m} - \mathbf{b}\|^2 \, ds$

     $$\mathbf{b}^{n+1} = \underset{\mathbf{b}}{\operatorname{argmin}}\; J_B(\mathbf{m}^n, \mathbf{b})$$

     Projection on boundary ∂T
  4. Let $J(\mathbf{m}, \mathbf{P}, \mathbf{b}) = \alpha J_I(\mathbf{m}, \mathbf{P}) + (1 - \alpha) J_B(\mathbf{m}, \mathbf{b})$

     $$\mathbf{m}^{n+1} = \underset{\mathbf{m}}{\operatorname{argmin}}\; J(\mathbf{m}, \mathbf{P}^{n+1}, \mathbf{b}^{n+1})$$

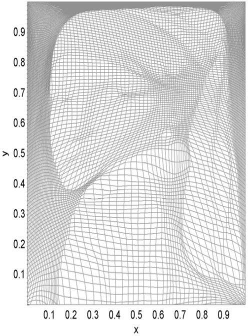
     Elliptic PDEs for m_1 and m_2 — FVM
  5. Let $n := n + 1$, go to Step 2

Girl with a pearl earring (parallel-to-far-field reflector 3D)

SMA

- Parallel source
- Uniform source distribution
- Far-field target
- Single reflector

Target



Mapping

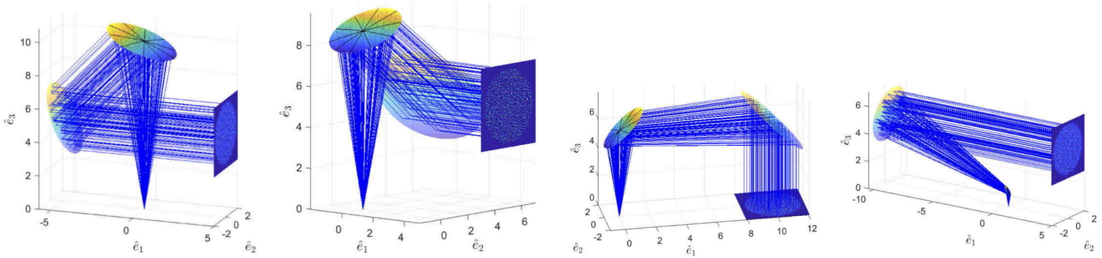
Ray trace

Prins, C. R., ten Thije Boonkkamp, J. H. M., van Roosmalen, J., IJzerman, W. L., Tukker, T. W. (2014)
A Monge-Ampère-solver for freeform reflector design
SIAM Journal on Scientific Computing, 36(3), B640-B660
https://doi.org/10.1137/130938876

TU/e

Point-to-parallel reflector system 3D

GMA

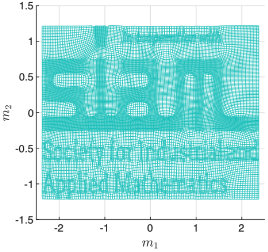


Van Roosmalen, A. H., Anthonissen, M. J. H., IJzerman, W. L., ten Thije Boonkkamp, J. H. M. (2021)
Design of a freeform two-reflector system to collimate and shape a point source distribution
Optics Express, 29(16), 25605-25625
https://doi.org/10.1364/OE.425289

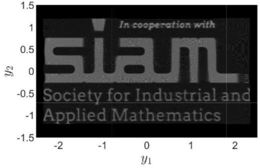
TU/e

Parallel-to-near-field reflector 3D

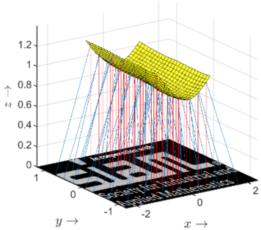
GJE



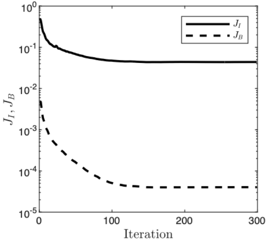
Mapping



Ray trace



Surface



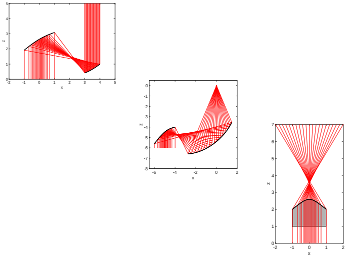
Convergence

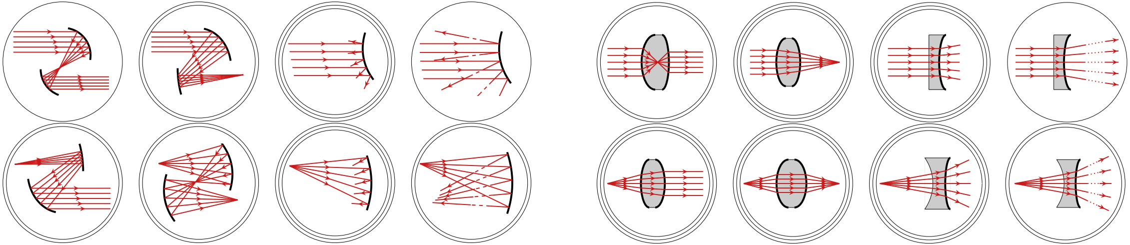
Romijn, L. B., Anthonissen, M. J. H., ten Thije Boonkkamp, J. H. M., IJzerman, W. L. (2021)
An iterative least-squares method for generated Jacobian equations in freeform optical design
SIAM Journal on Scientific Computing, 43(2), B298-B322
https://doi.org/10.1137/20M1338940

TU/e

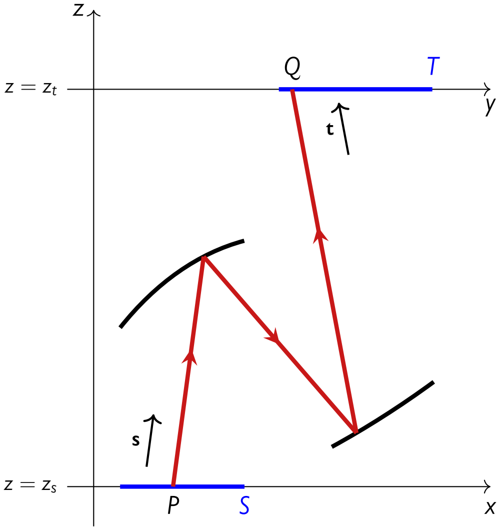
Concluding remarks

- **Model 1: Standard Monge-Ampère equation (SMA)**
  Optimal transport formulation with quadratic cost function
- **Model 2: Generalized Monge-Ampère equation (GMA)**
  Optimal transport formulation with non-quadratic cost function
- **Model 3: Generated Jacobian equation (GJE)**
  No optimal transport formulation



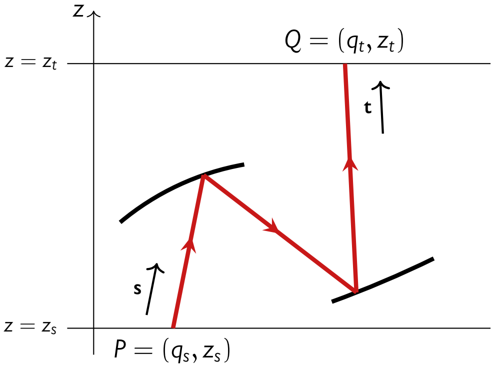


Optical path length



- Light ray starts at $P = (x, z_s)$
  ends at $Q = (y, z_t)$
- Optical path length:
$$V = \int_C n \, ds$$
- Ray has direction $\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ at P $\qquad \|\mathbf{s}\| = 1$
$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \text{ at } Q \qquad \|\mathbf{t}\| = 1$$
- Let $p_s = n s_1$
  $p_t = n t_1$
- Property:
$$V = V(x, y) \qquad p_s = -\frac{\partial V}{\partial x} \qquad p_t = \frac{\partial V}{\partial y}$$

Hamilton's characteristic functions



$p_s = ns_1$
$p_t = nt_1$

- Point characteristic: V is optical path length

$$V = V(q_s, q_t) \qquad p_s = -\frac{\partial V}{\partial q_s} \qquad p_t = \frac{\partial V}{\partial q_t}$$

- Mixed characteristic: $W = V - q_t p_t$

$$W = W(q_s, p_t) \qquad p_s = -\frac{\partial W}{\partial q_s} \qquad q_t = -\frac{\partial W}{\partial p_t}$$

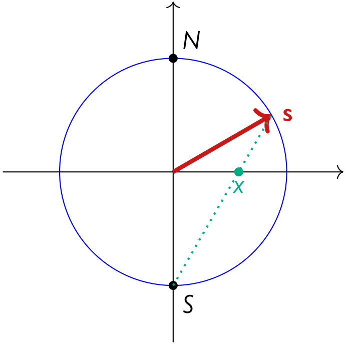
- Mixed 2nd kind: $W^* = V + q_s p_s$

$$W^* = W^*(p_s, q_t) \qquad q_s = \frac{\partial W^*}{\partial p_s} \qquad p_t = \frac{\partial W^*}{\partial q_t}$$

- Angular characteristic: $T = V + q_s p_s - q_t p_t$

$$T = T(p_s, p_t) \qquad q_s = \frac{\partial T}{\partial p_s} \qquad q_t = -\frac{\partial T}{\partial p_t}$$

Stereographic projection from the south pole



- Blue circle has radius 1, so

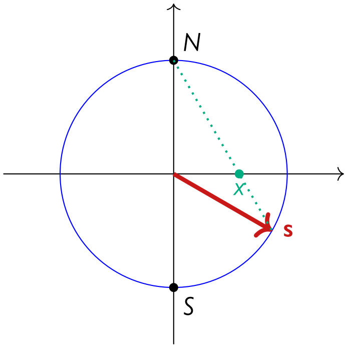
$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \qquad s_1^2 + s_2^2 = 1$$

- South pole: $S = (0, -1)$
- One can show that

$$s_1 = \frac{2x}{1+x^2}$$
$$s_2 = \frac{1-x^2}{1+x^2}$$
$$x = \frac{s_1}{1+s_2}$$

TU/e

Stereographic projection from the north pole



- Blue circle has radius 1, so

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \qquad s_1^2 + s_2^2 = 1$$

- North pole: $N = (0, 1)$
- One can show that

$$s_1 = \frac{2x}{1 + x^2}$$
$$s_2 = \frac{-1 + x^2}{1 + x^2}$$
$$x = \frac{s_1}{1 - s_2}$$

TU/e