Inverse methods in optical design

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Introductory Problem

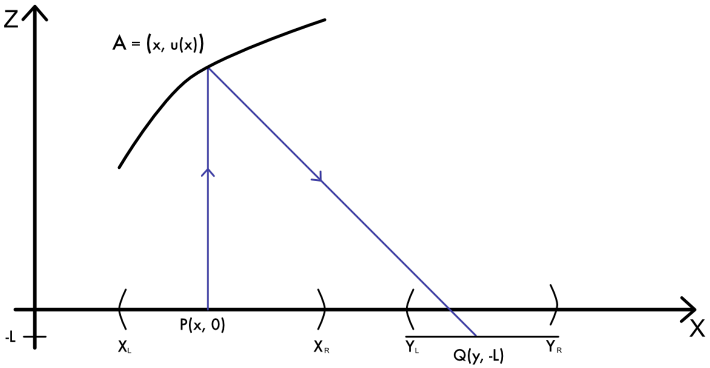


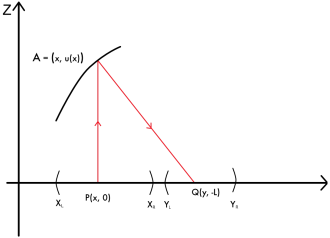
Figure: $V(x, y) = d(P, A) + d(A, Q)$

Optical Path Length

Let the path length be

$$V := \underbrace{d(P,A)}_{u(x)} + d(A,Q)$$

$$V = u(x) + \sqrt{(x-y)^2 + (L+u(x))^2}$$



Differentiating $\frac{\partial V}{\partial x}$, only one term depending on y remains:

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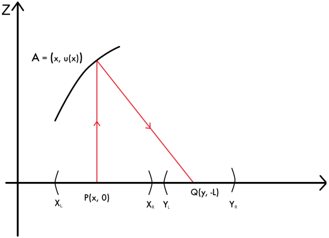

Differentiating $\frac{\partial V}{\partial x}$, only one term depending on y remains:

$$0 = u'(x) + \frac{(x-y) + (L+u(x))u'(x)}{\sqrt{(x-y)^2 + (L+u(x))^2}}$$

The function m

Let $m : [x_L, x_R] \to [y_L, y_R],\ m(x) := y.$

Energy conservation law:

$$\int_{x_l}^{x} E(\xi)\, d\xi = \pm \int_{m(x_l)}^{m(x)} G(y)\, dy$$
$$\int_{x_l}^{x} E(\xi)\, d\xi = \pm \int_{x_l}^{x} G(m(\xi)) \cdot m'(\xi)\, dx$$

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To find $m(x)$, we numerically solve the ODE

$$m'(x) = \pm \frac{E(x)}{G(m(x))}, \quad m(x_l) = \begin{cases} y_L, & \text{positive } \pm \\ y_R, & \text{negative } \pm \end{cases}$$

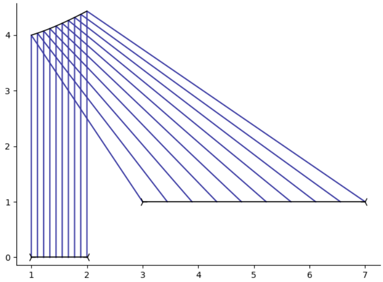
The main equation

Now that we have an equation $y = m(x)$, we can insert it into the main equation:

$$u'(x) = \frac{m(x) - x}{\sqrt{(x - m(x))^2 + (L + u(x))^2} + L + u(x)}$$

$\Longrightarrow$ solve $V\big(x,\, m(x)\big)$ numerically!

Test problems



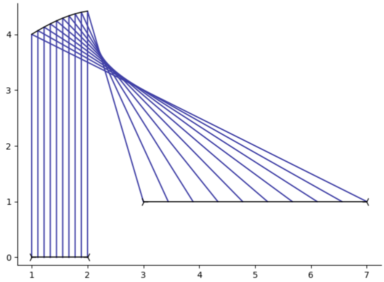


Figure: Computed solutions for $E(x) = 1$, $G(y) = 1$, $S = [1, 2]$, $T = [3, 7]$, $L = 1$ and $u(1) = 4$

Test problems

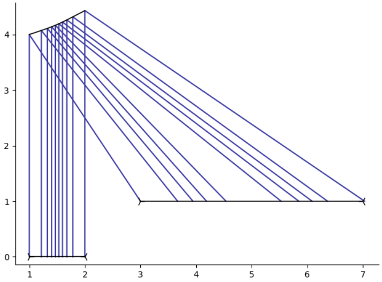
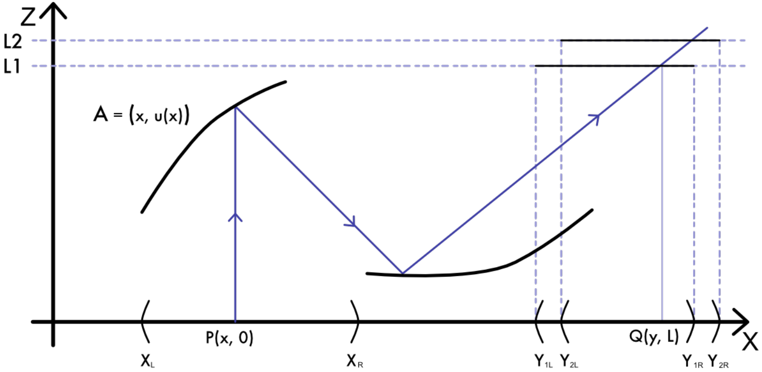


Figure: Computed solution for $E(x) = (e^{5(x-1.5)} + e^{-5(x-1.5)})^{-1}$, $G(y) = 2 - \big||2y - 10| - 2\big|$, $S = [1, 2]$, $T = [3, 7]$, $L = 1$ and $u(1) = 4$

Parallel source

with two reflectors

Parallel source, two reflectors, two target densities



Computing the m function

To compute $y_2 = m(y_1)$ we use the energy conservation law again

$$\int_{y_l}^{y} G_1(\eta)d\eta = \int_{y_l}^{y} G_2(m(\eta))m'(\eta)d\eta$$

Computing the m function

To compute $y_2 = m(y_1)$ we use the energy conservation law again

$$\int_{y_l}^{y} G_1(\eta)d\eta = \int_{y_l}^{y} G_2(m(\eta))m'(\eta)d\eta$$

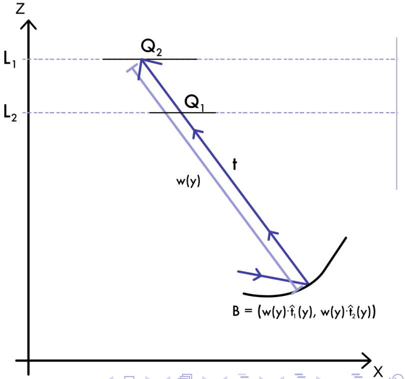
and obtain the ODE

$$m'(y) = \pm\frac{G_1(y)}{G_2(m(y))}, \quad m(y_{1,L}) = \begin{cases} y_{2,L}, & \pm \text{ positive} \\ y_{2,R}, & \pm \text{ negative} \end{cases}$$

The t vector

Using the function m, we can compute the final ray direction $\hat{\mathbf{t}} = \frac{\mathbf{t}}{||\mathbf{t}||}$ as

$$\mathbf{t}(y) = \begin{bmatrix} m(y) - y \\ L_2 - L_1 \end{bmatrix}$$



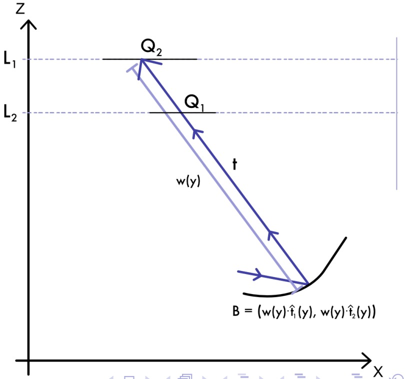
The t vector

Using the function m, we can compute the final ray direction $\hat{\mathbf{t}} = \frac{\mathbf{t}}{||\mathbf{t}||}$ as

$$\mathbf{t}(y) = \begin{bmatrix} m(y) - y \\ L_2 - L_1 \end{bmatrix}$$

and the coordinates of B

$$B = \begin{bmatrix} -w(y)\hat{t}_1(y) + y \\ -w(y)\hat{t}_2(y) + L_1 \end{bmatrix}$$



The $\tilde{m}$ function

To find $y = m(x)$ we solve the ODE

$$\tilde{m}'(x) = \pm \frac{E(x)}{G_1(\tilde{m}(x))}, \quad \tilde{m}(x_l) = \begin{cases} y_{1,L}, & \pm \text{ positive} \\ y_{1,R}, & \pm \text{ negative} \end{cases}$$

Optical path length

$$V(x,y) = \underbrace{d(P,A)}_{u(x)} + \underbrace{d(A,B)}_{d} + \underbrace{d(B,Q_1)}_{w(y)}$$

Considering that

$$\frac{\partial V}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = \hat{t}_1(y)$$

We can now compute $V(x,y) = V(y)$ by solving the second equation!

Computing w

- Two different equations for d in dependence of x and y respectively motivate the hope for cancellations:

$$d^2 = (V - u - w)^2 \text{ and } d^2 = d(A, B)^2$$

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- Eventually, we derive

$$w = \frac{-(u-V)^2 + (y-x)^2 + (L_1 - u)^2}{2(u - V + \hat{t}_1(y-x) + \hat{t}_2(L_1 - u))}$$

Computing u

We compute the derivative for the first equation with respect to x and isolate the term u' to obtain

$$u' = \frac{x - y + w\hat{t}_1}{w - V + L_1 - w\hat{t}_2}$$

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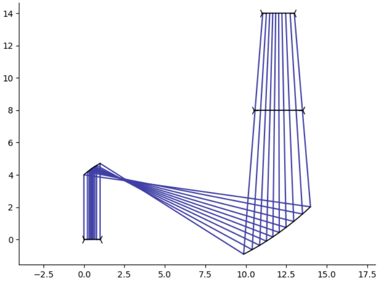
We can substitute every occurence of y, w, $\hat{t}$ and V and reach something in the form

$$u' = f(x, u)$$

Test problems



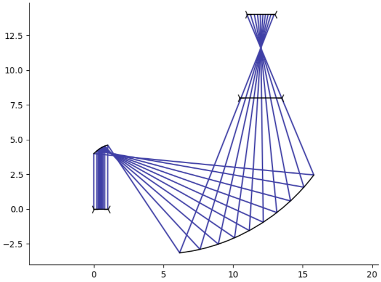


Figure: Computed solutions for $E(x) = (e^{5(x-1.5)} + e^{-5(x-1.5)})^{-1}$, $G(y) = 1$, $S = [0, 1]$, $T_1 = [10, 13]$, $u(1) = 4$ and $w(10) = 6$

Test problems

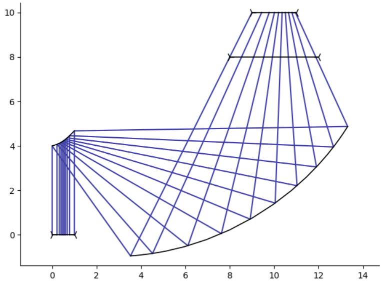
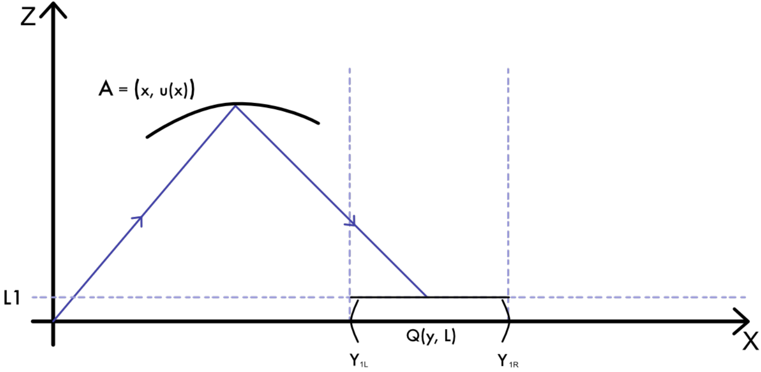


Figure: Computed solution for $E(x) = (e^{5(x-1.5)} + e^{-5(x-1.5)})^{-1}$, $G_1(y) = 1$, $G_2(y_2) = 2 + y_2$ $S = [0, 1]$, $T_1 = [7, 13]$, $u(1) = 4$ and $w(7) = 8$

Point source

with one reflector

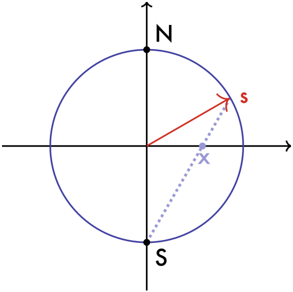
Point source with one reflector



South pole stereographic projection

- Traditional cartesian coordinates or radians are impractical $\implies$ choose stereographic projection
- The use of this parameter x gives us the $\mathbf{s}$ vector $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ with $||\mathbf{s}|| = 1$ and

$$s_1 = \frac{2x}{1+x^2} \qquad s_2 = \frac{1-x^2}{1+x^2}$$



Computing the m function and defining $u(x)$

- $y = m(x)$ is a very similar ODE:

$$m'(x) = \pm\frac{E(x)||s'(x)||}{G(m(x))}, \quad m(x_L) = \begin{cases} y_L, \pm \text{ positive} \\ y_R, \pm \text{ negative} \end{cases}$$

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- $y = m(x)$ is a very similar ODE:

$$m'(x) = \pm \frac{E(x)||s'(x)||}{G(m(x))}, \quad m(x_L) = \begin{cases} y_L, \pm \text{ positive} \\ y_R, \pm \text{ negative} \end{cases}$$

- $u(x) := d(0, A) = ||A(x)||$ the distance between source and A with new coordinates $A\Big(u(x)\hat{s}_1(x), u(x)\hat{s}_2(x)\Big)$

The main equation

- Starting with the optical path length

$$
\begin{aligned}
V &= u(x) + d(A, Q) = V(y) \\
V &= u + \sqrt{u^2 + y^2 - 2yu\hat{s}_1 + L^2 + 2u\hat{s}_2 L}
\end{aligned}
$$

The main equation

- Starting with the optical path length

$$V = u(x) + d(A, Q) = V(y)$$
$$V = u + \sqrt{u^2 + y^2 - 2yu\hat{s}_1 + L^2 + 2u\hat{s}_2 L}$$

- After computing the derivative in respect to x we get the final equation

$$u' = \frac{yu\hat{s}_1' - u\hat{s}_2' L}{\sqrt{u^2 + y^2 - 2yu\hat{s}_1 + L^2 + 2u\hat{s}_2 L} + u - 2y\hat{s}_1 + \hat{s}_2 L}$$

Test problems

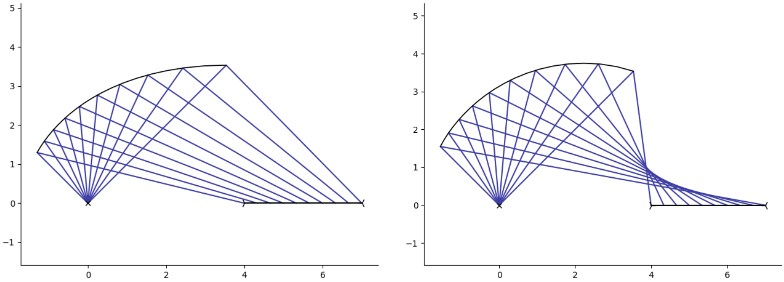
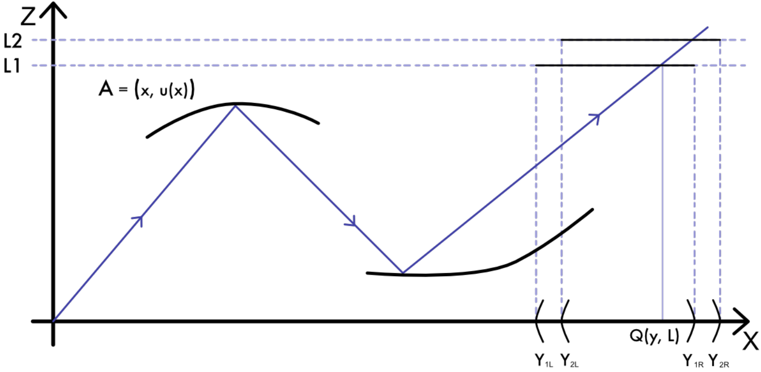


Figure: Computed solutions for $E(x) = 1$, $G(y) = 1$, $\theta = [\frac{\pi}{4}, \frac{3\pi}{4}]$, $T = [4, 7]$ and $u_0 = 6$

Point source

with two reflectors

Point source with two reflectors



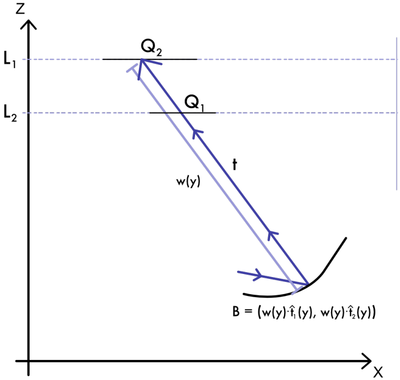
Remember: calculating m and $\hat{t}$

$$m'(y_1) = \pm \frac{G_1(y_1)}{G_2(m(y_1))}$$

$$m(y_{1,L}) = \begin{cases} y_{2,L}, & \pm \text{ positive} \\ y_{2,R}, & \pm \text{ negative} \end{cases}$$

Normalized direction vector $\hat{t}$ as

$$\hat{t} = \frac{t}{||t||} \qquad t = \begin{bmatrix} m(y_1) - y_1 \\ L_2 - L_1 \end{bmatrix}$$



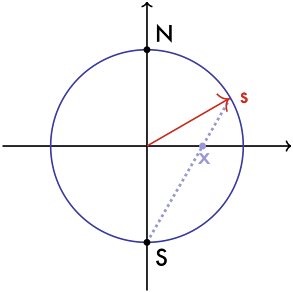
The function $\tilde{m}$ and the vector s

- express the mapping in terms of $s(x)$:

$$s(x) = \begin{bmatrix} s_1(x) \\ s_2(x) \end{bmatrix},$$

$$s_1(x) = \frac{2x}{1+x^2},$$

$$s_2(x) = \frac{1-x^2}{1+x^2}$$



- obtain ODE of familiar form

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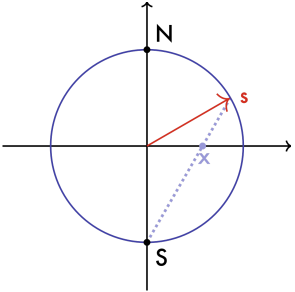

- obtain ODE of familiar form

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Equations for $V(y)$ and $w(y)$

$$\frac{\partial V}{\partial y} = \hat{t}_1(y), \quad V(y_L) = V_0$$

ODE for the function $V(y)$

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$$\frac{\partial V}{\partial y} = \hat{t}_1(y), \quad V(y_L) = V_0$$

ODE for the function $V(y)$

$$(V - u - w)^2 = d(A, B)^2 = (y - w\hat{t}_1 - u\hat{s}_1)^2 + (L_1 - w\hat{t}_2 - u\hat{s}_2)^2$$

Equations for $V(y)$ and $w(y)$

$$\frac{\partial V}{\partial y} = \hat{t}_1(y), \quad V(y_L) = V_0$$ ODE for the function $V(y)$

$$(V - u - w)^2 = d(A,B)^2 = (y - w\hat{t}_1 - u\hat{s}_1)^2 + (L_1 - w\hat{t}_2 - u\hat{s}_2)^2$$

$$w = \frac{V^2 - 2uV - y^2 + 2u\hat{s}_1 y - L_1^2 + 2L_1 u\hat{s}_2}{2(V - u - y\hat{t}_1) + u\hat{t}_1\hat{s}_1 - L_1\hat{t}_2 + u\hat{t}_2\hat{s}_2)}$$

The main equation

ODE for $u(x)$

$$u' = \frac{-u\hat{s}_1' y + wu\hat{t}_1\hat{s}_1' - L_1 u\hat{s}_2' + wu\hat{t}_2\hat{s}_2'}{-V + w + \hat{s}_1 y - w\hat{t}_1\hat{s}_1 + L_1\hat{s}_2 - w\hat{t}_2\hat{s}_2}, \qquad u(x_L) = u_0$$

Test problems

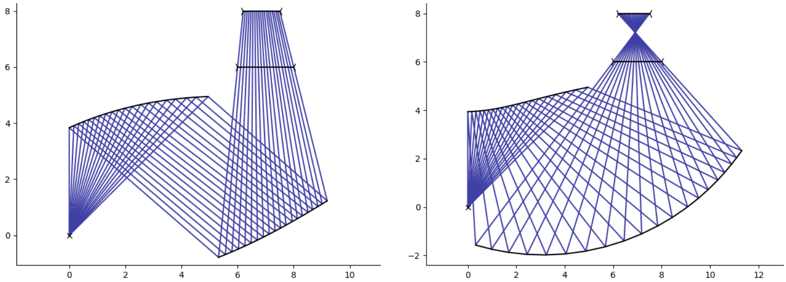


Figure: Computed solutions for $E(x) = 1$, $G_1(y_1) = 1, G_2(y_2) = 1$, $\theta = [\frac{\pi}{4}, \frac{\pi}{2}]$, $T_1 = [6, 8]$, $T_2 = [6.2, 7.5]$, $u_0 = 6$ and $w_0 = 5$

Test problems

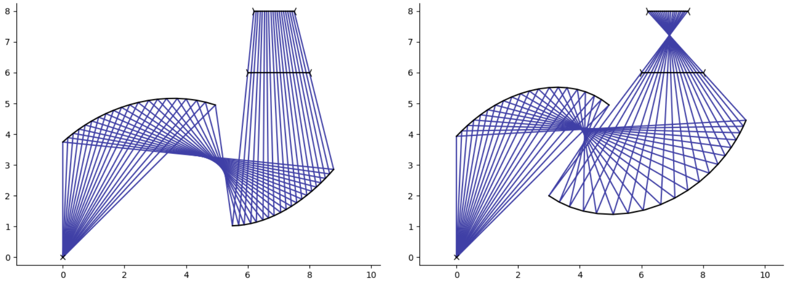
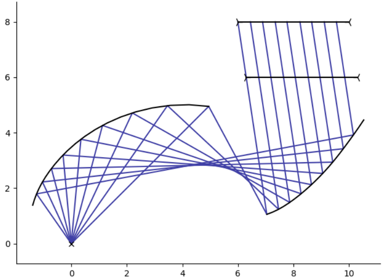


Figure: Computed solutions for $E(x) = 1$, $G_1(y_1) = 1, G_2(y_2) = 1$, $\theta = [\frac{\pi}{4}, \frac{\pi}{2}]$, $T_1 = [6, 8]$, $T_2 = [6.2, 7.5]$, $u_0 = 6$ and $w_0 = 5$

Code validation



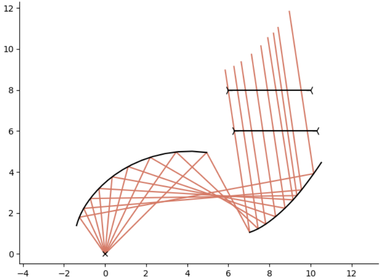
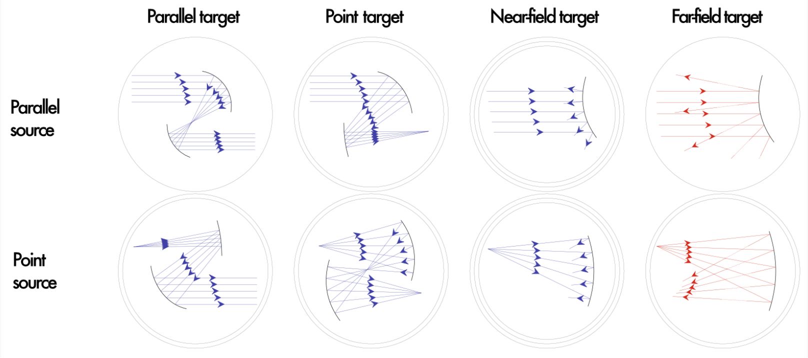


Figure: Computed solution and ray traced validation of the system

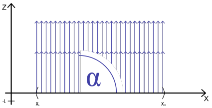
Generalized formulations of the problem

Problem classification

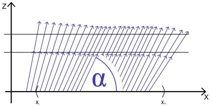


All distributions are directed density sources

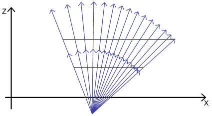
- interval light distributions can be interpreted as *light distributions with an angle*: $\sin\alpha = 1$ if parallel.



- two interval densities can be interpreted as *light distributions with an angle*: $\cos\alpha(x)$ depends on x



- all point sources can be interpreted as *parallel interval sources*: α implicitly given by the angular distribution in radians



Single mirrors and degeneration

Describe the single mirror case as a special case of the two mirror near field frame work:

$$V := u + d(A, F) + w$$

$$\begin{cases} w(y) \equiv 0, & F := \Big(m(x), L\Big) \\ w(y) \neq 0 \text{ a.e.}, & F := B = \begin{bmatrix} -w(y)\hat{t}_1(y) + y \\ -w(y)\hat{t}_2(y) + L_1 \end{bmatrix} \end{cases}$$

$\Longrightarrow$ all near field problems can be interpreted in the *two mirror, two directed densities* frame work