Induction (Chapter 5)

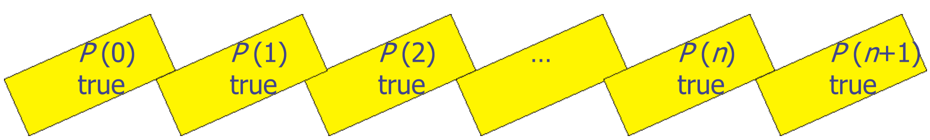
Mathematical Induction

Principle of Mathematical Induction:

If:

1) [***basis***] $P(0)$ is true

2) [***induction***] $\forall n \; P(n) \rightarrow P(n+1)$ is true



Then:

$\forall n \; P(n)$ is true

This formalizes what occurred to dominos.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

Solution:

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$.
- INDUCTIVE STEP: Assume true for $P(k)$.

The inductive hypothesis is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Under this assumption,

$$\begin{aligned} 1+2+\ldots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

◀

Proof of Induction
Well Ordering Property

A fundamental axiom about the natural numbers:

Well Ordering Property: Any *non-empty* subset S of **N** has a smallest element!

Q1: What's the smallest element of the set $\{\ 16.99+1/n \mid n \in \mathbf{Z}^+ \}$?

Q2: How about $\{ \lfloor 16.99+1/n \rfloor \mid n \in \mathbf{Z}^+ \}$?

Proof of Induction Principle
Well Ordering Property

A1: $\{ 16.99+1/n \mid n \in \mathbf{Z}^+ \}$ doesn't have a smallest element (though it does have limit-point 16.99)! Well-ordering principle does not apply to subsets of **R**.

A2: 16 is the smallest element of $\{ \lfloor 16.99+1/n \rfloor \mid n \in \mathbf{Z}^+ \}$.

(EG: set $n = 101$)

Well Ordering Property
All Numbers are Cool

"THM": All natural numbers are interesting.

EG: 0, 1, 2, … interesting, everything else too!

Proof by contradiction: Assume that there are uninteresting numbers in **N**. Consider the set S of such numbers. By the well ordering principle, there is a number u which is the smallest uninteresting number. But being the smallest uninteresting number is pretty darn interesting. Therefore, u is interesting, contradicting that fact that it is uninteresting. Therefore S must be empty, and all numbers must therefore be interesting.

Proof of Induction Principle

Proof by contradiction. Suppose that the basis assumption $-P(0)$ – and induction assumption – $\forall n\ P(n) \rightarrow P(n+1)$ – hold, yet it is not the case that the conclusion $-\forall n\ P(n)$ – holds.

Let S be the set of all numbers for which $P(n)$ is false.

By assumption S is non-empty, so well ordering principle gives a smallest number m in S.

By assumption, $P(0)$ is true, so $m>0$.

Since m is the smallest number for which $P(m)$ is false, and is non-zero, $P(m-1)$ must be true.

By assumption $P(m-1) \rightarrow P(m)$ is true, so as LHS of conditional is true, by definition of conditional, RHS is true.

Thus, $P(m)$ is true, contradicting fact that $m \in S$.

This shows that assumption that S is non-empty was false, and $\forall n\ P(n)$ must therefore be true.

Induction
Geometric Example

Let's come up with a formula for the (maximum) number of intersection points in a plane containing n lines.

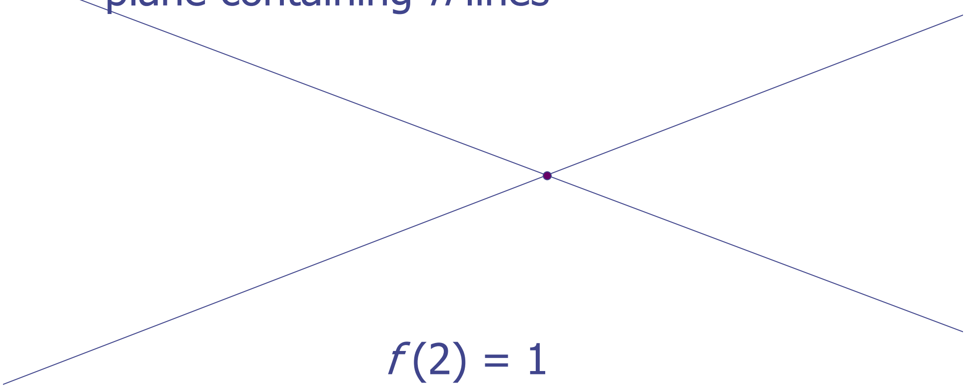
Induction
Geometric Example

The number of intersections points in a plane containing *n* lines

$f(1) = 0$

Induction
Geometric Example

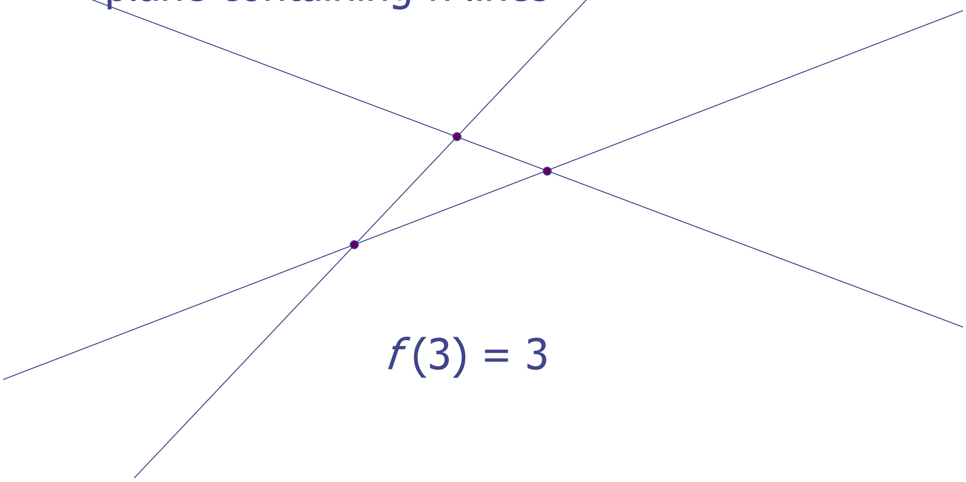
The number of intersections points in a plane containing *n* lines



$f(2) = 1$

Induction
Geometric Example

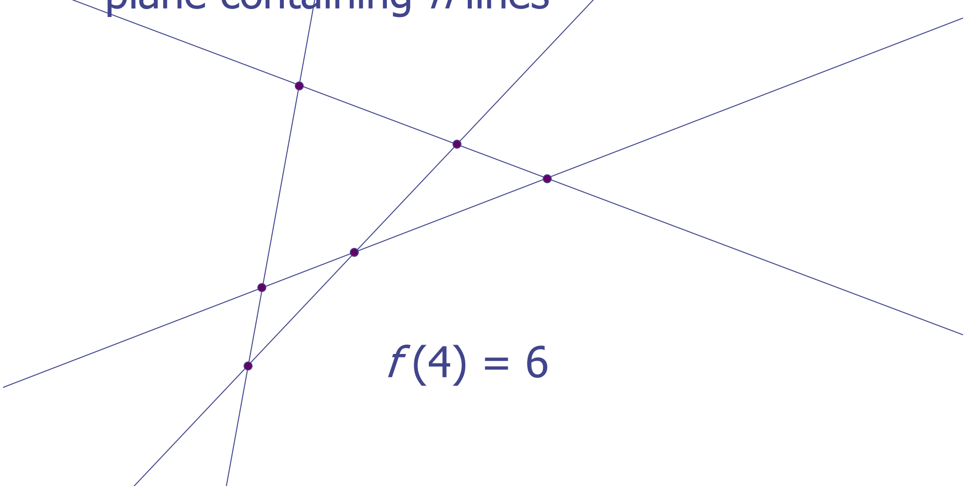
The number of intersections points in a plane containing *n* lines



$f(3) = 3$

Induction
Geometric Example

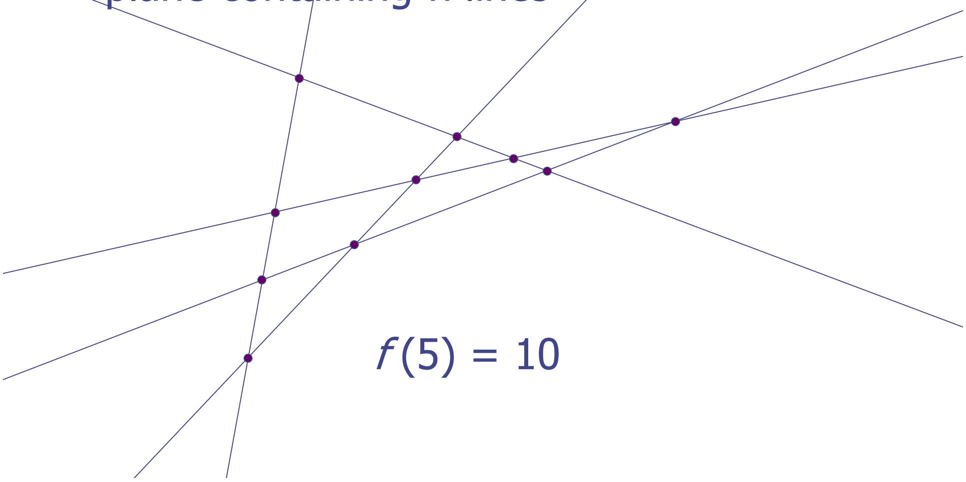
The number of intersections points in a plane containing *n* lines



$f(4) = 6$

Induction
Geometric Example

The number of intersections points in a plane containing n lines



$f(5) = 10$

Induction
Geometric Example

The number of intersections points in a plane containing n lines. Denote this number by $f(n)$. We have:

$n =$ 1, 2, 3, 4, 5

$f(n) =$ 0, 1, 3, 6, 10

Q: Come up with a conjectured formula for $f(n)$. Can be in terms of previous values (in recursive notation).

Induction
Geometric Example

A: $f(n) = f(n\text{-}1) + n - 1$

Q: How do you find a closed formula?

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to n=1:

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to n=1:

1. $f(n) = f(n-1) + n-1$

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to n=1:

1. $f(n) = f(n-1) + n-1$

2. Therefore, $f(n-1) = f(n-2) + n-2$

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to $n=1$:

1. $f(n) = f(n-1) + n-1$
2. Therefore, $f(n-1) = f(n-2) + n-2$
3. Plug in (2) into (1) to get: $f(n) = f(n-2) + n-2 + n-1$

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to $n=1$:

1. $f(n) = f(n-1) + n-1$
2. Therefore, $f(n-1) = f(n-2) + n-2$
3. Plug in (2) into (1) to get: $f(n) = f(n-2) + n-2 + n-1$

4. Repeat this process, plugging in for $f(n-2)$:
   $f(n) = f(n-3) + n-3 + n-2 + n-1$

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to n=1:

1. $f(n) = f(n\text{-}1) + n–1$
2. Therefore, $f(n\text{-}1) = f(n\text{-}2) + n–2$
3. Plug in (2) into (1) to get: $f(n) = f(n\text{-}2) + n–2 + n–1$
4. Repeat this process, plugging in for $f(n\text{-}2)$:
   $f(n) = f(n\text{-}3) + n\text{-}3 + n–2 + n–1$
5. Pattern arises after repeating this i times:
   $f(n) = f(n\text{-}i) + n\text{-}i + \ldots + n\text{-}3 + n–2 + n–1$

Induction
Geometric Example

A: Repeatedly insert recursive formula for lower and lower values of n until get down to n=1:

1. $f(n) = f(n\text{-}1) + n–1$
2. Therefore, $f(n\text{-}1) = f(n\text{-}2) + n–2$
3. Plug in (2) into (1) to get: $f(n) = f(n\text{-}2) + n–2 + n–1$
4. Repeat this process, plugging in for $f(n\text{-}2)$:
   $f(n) = f(n\text{-}3) + n\text{-}3 + n–2 + n–1$
5. Pattern arises after repeating this i times:
   $f(n) = f(n\text{-}i) + n\text{-}i + \ldots + n\text{-}3 + n–2 + n–1$
6. To get to $n = 1$, plug in $i = n – 1$:
   $f(n) = f(1) + 1 + 2 + \ldots + n\text{-}3 + n–2 + n–1$
   $$= 0 + \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$

Induction
Geometric Example

Induction step: Assume $n > 1$. What is the maximum number of intersection points of n lines? Remove one line.

$n-1$ lines remain. By induction, we may assume that the maximal number intersections of these lines is $(n-1)(n-2)/2$. Consider adding back the n^{th} line. This line intersects at most all the n-1 other lines. For the maximal case, the line can be arranged to intersect all the other lines, by selecting a slope different from all the others. E.g. consider the following:

Induction
Geometric Example

…shew. But that's not the end of the story. This was just the intuitive derivation of the formula, not the proof.

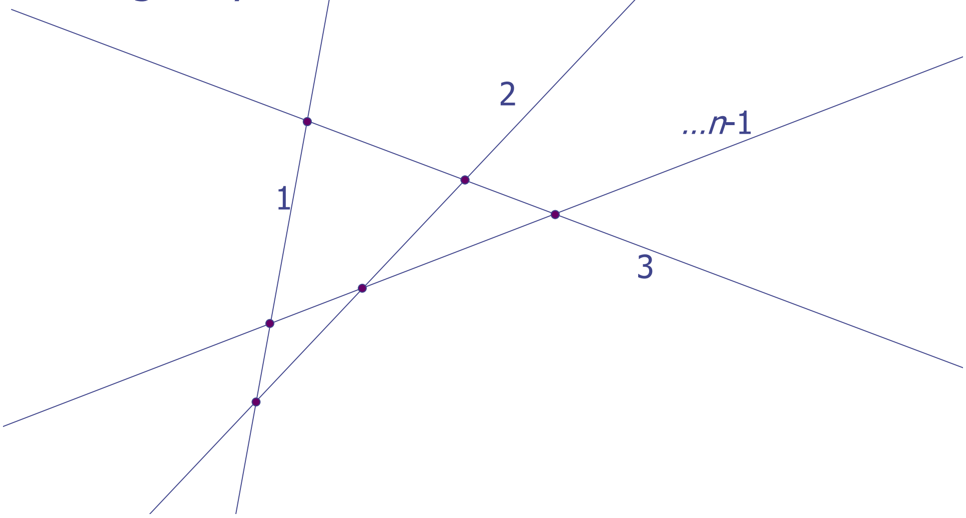
LEMMA: The maximal number of intersection points of n lines in the plane is $n(n-1)/2$.

Proof. Prove by induction.

Base case: If $n = 1$, then there is only one line and therefore no intersections. On the other hand, plugging $n = 1$ into $n(n-1)/2$ gives 0, so the base case holds.

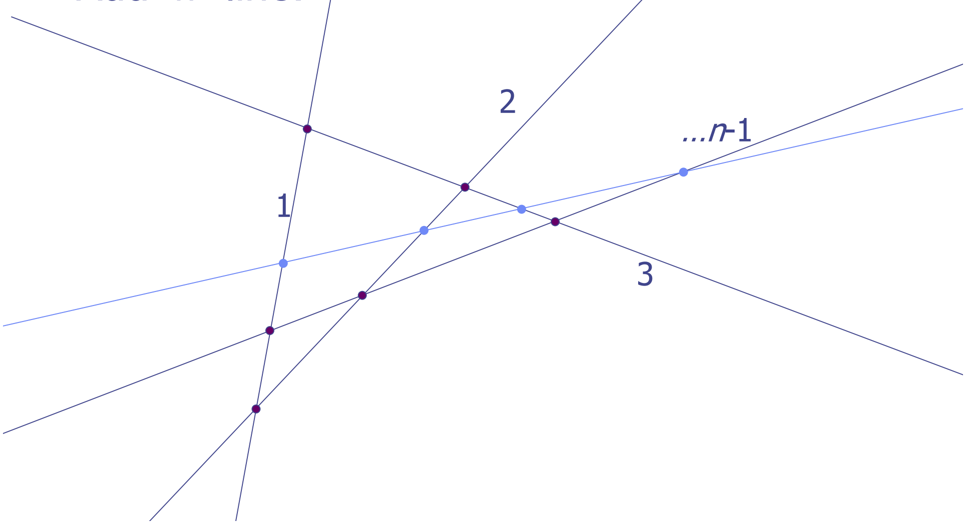
Induction
Geometric Example

Originally n-1 lines:



Induction
Geometric Example

Add n^{th} line:



Induction
Geometric Example

Therefore, the maximum number of intersection points of n lines, is the maximum number of intersections of $n-1$ lines plus the $n-1$ new intersections; this number is just

$(n-1)(n-2)/2 + n-1$

$= (n-1)((n-2)/2 + 1)$

$= (n-1)(n-2 + 2)/2 = (n-1)n/2$

which is the formula we want to prove for n.

This completes the induction step, and therefore completes the proof. $\square$

Proving Inequalities

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

- BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.
- Must show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

  $k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

Therefore $n < 2^n$ holds for all positive integers n. ◄

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

- BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\
&< (k + 1)k! \\
&= (k + 1)!
\end{aligned}
$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$. ◀

Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

Proving Divisibility Results

Example: Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Solution: Let $P(n)$ be the proposition that $n^3 - n$ is divisible by 3.

- BASIS STEP: $P(1)$ is true since $1^3 - 1 = 0$, which is divisible by 3.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k^3 - k$ is divisible by 3, for an arbitrary positive integer k. To show that $P(k + 1)$ follows:

$$
\begin{aligned}
(k + 1)^3 - (k + 1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1) \\
&= (k^3 - k) + 3(k^2 + k)
\end{aligned}
$$

By the inductive hypothesis, the first term $(k^3 - k)$ is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. So by part (i) of Theorem 1 in Section 4.1 , $(k + 1)^3 - (k + 1)$ is divisible by 3.

Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n. ◀