Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$\exists x\, J(x)$

Solution 2: But if *U* is all people, then translate as

$\exists x\, (S(x) \land J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people.
- The two premises are: $\forall x Man(x) \rightarrow Mortal(x)$

  $Man(Socrates)$
- The conclusion is: $Mortal(Socrates)$
- Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in calculus" and the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken calculus."

  Symbolically $\neg \forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists\, x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg \exists\, x\, J(x)$ and $\forall\, x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x \; (S(x) \wedge M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x \, (S(x) \rightarrow (M(x) \vee C(x)))$$

Some Fun with Translating from English into Logical Expressions

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): x is a thingamabob

  Translate "Everything is a fleegle"

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  "Nothing is a snurd."

  Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

  Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"All fleegles are snurds."

Solution: $\forall x\,(F(x) \rightarrow S(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\,(F(x) \wedge T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "No snurd is a thingamabob."

  Solution: $\neg\exists x\ (S(x) \land T(x))$ What is this equivalent to?

  Solution: $\forall x\ (\neg S(x) \lor \neg T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "If any fleegle is a snurd then it is also a thingamabob."

  Solution: $\forall x\ ((F(x) \land S(x)) \rightarrow T(x))$

System Specification Example

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."
- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message m is larger than y megabytes."
  - Let $C(m)$ denote "Mail message m will be compressed."
  - Let $A(u)$ represent "User u is active."
  - Let $S(n, x)$ represent "Network link n is state x.
- Now we have: $\forall m(L(m,1) \rightarrow C(m))$

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

- The first two are called *premises* and the third is called the *conclusion*.
  1. "All lions are fierce."
  2. "Some lions do not drink coffee."
  3. "Some fierce creatures do not drink coffee."
- Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.
  1. $\forall x\, (P(x) \rightarrow Q(x))$
  2. $\exists x\, (P(x) \wedge \neg R(x))$
  3. $\exists x\, (Q(x) \wedge \neg R(x))$
- Later we will see how to prove that the conclusion follows from the premises.

Some Predicate Calculus Definitions (*optional*)

- An assertion involving predicates and quantifiers is *valid* if it is true
  - for all domains
  - every propositional function substituted for the predicates in the assertion.

  Example: $\forall x \neg S(x) \leftrightarrow \neg \exists x S(x)$
- An assertion involving predicates is *satisfiable* if it is true
  - for some domains
  - some propositional functions that can be substituted for the predicates in the assertion.

  Otherwise it is *unsatisfiable*.

  Example: $\forall x (F(x) \leftrightarrow T(x))$ not valid but satisfiable

  Example: $\forall x (F(x) \wedge \neg F(x))$ unsatisfiable

More Predicate Calculus Definitions (*optional*)

- The *scope* of a quantifier is the part of an assertion in which variables are bound by the quantifier.

  Example: $\forall x (F(x) \vee S(x))$ *x* has wide scope

  Example: $\forall x (F(x)) \vee \forall y (S(y))$ *x* has narrow scope

Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $\forall x \exists y(x + y = 0)$

  where the domains of x and y are the real numbers.
- We can also think of nested propositional functions:

  $\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x\, \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x\, \forall y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y\, P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y\ \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,
Define $P(x,y) : x \cdot y = 0$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\ P(x,y)$
   Answer: True
4. $\exists x\ \exists\ y\ P(x,y)$
   Answer: True

Questions on Order of Quantifiers

Example 2: Let *U* be the real numbers,
Define *P(x,y)* : *x / y* = 1
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False
2. $\forall x \exists y P(x,y)$
   Answer: True
3. $\exists x \forall y\ P(x,y)$
   Answer: False
4. $\exists x\ \exists\ y\ P(x,y)$
   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x,y*) is true for every pair *x,y*.	There is a pair *x, y* for which *P*(*x,y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x,y*) is true.	There is an x such that *P*(*x,y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x,y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x, y* for which *P*(*x,y*) is true.	*P*(x,y) is false for every pair *x,y*