The Foundations: Logic and Proofs

Chapter 1, Part I: Propositional Logic

With Question/Answer Animations

Chapter Summary

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
- Proofs
  - Rules of Inference
  - Proof Methods
  - Proof Strategy

Propositional Logic Summary

- The Language of Propositions
  - Connectives
  - Truth Values
  - Truth Tables
- Applications
  - Translating English Sentences
  - System Specifications
  - Logic Puzzles
  - Logic Circuits
- Logical Equivalences
  - Important Equivalences
  - Showing Equivalence
  - Satisfiability

Propositional Logic

Section 1.1

Section Summary

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth Tables

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Trenton is the capital of New Jersey.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

Propositional Logic

- Constructing Propositions
  - Propositional Variables: p, q, r, s, …
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

Compound Propositions: Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- **Example**: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

- In English "or" has two distinct meanings.
  - "Inclusive Or" - In the sentence "Students who have taken CS202 or Math120 may take this class," we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
  - "Exclusive Or" - When reading the sentence "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.
- These implications are perfectly fine, but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates. "
  - "If the moon is made of green cheese then I'm on welfare."
  - "If 1 + 1 = 3, then your grandma wears combat boots."

Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Conditional: why F→F is True

Remember, all of these are mathematical constructs, not attempts to mimic English. Mathematically, p should imply q whenever it is possible to derive q from p by using valid arguments. For example consider the mathematical analog of no. 4:

$$\text{If } 0 = 1 \text{ then } 3 = 9.$$

Q: Is this true mathematically?

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)
3. 3 = 6 (multiplied both sides by 3)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)
3. 3 = 6 (multiplied both sides by 3)
4. 0 = 3 (multiplied no. 1 by 3)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)
3. 3 = 6 (multiplied both sides by 3)
4. 0 = 3 (multiplied no. 1 by 3)

5. 3 = 9 (added no. 3 and no. 4) *QED*

Conditional: why F→F is True

As we want the conditional to make sense in the semantic context of mathematics, we better define it as we have!

Other questionable rows of the truth table can also be justified in a similar manner.

Different Ways of Expressing $p \rightarrow q$

- **if** p**, then** q
- **if** p, q
- q **unless** $\neg p$
- q **if** p
- q **whenever** p
- q **follows from** p
- p **implies** q
- p **only if** q
- q **when** p
- q **when** p
- p **is sufficient for** q
- q **is necessary for** p

- **a necessary condition for** p **is** q
- **a sufficient condition for** q **is** p

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements .
  - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
  - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$

Example: Find the converse, inverse, and contrapositive of "Raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:
  - **p is necessary and sufficient for q**
  - **if p then q , and conversely**
  - **p iff q**

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are ***e**quivalent* if they always have the same truth value.
- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	*q*	¬ *p*	¬ *q*	*p*→*q*	¬*q*→ ¬ *p*
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Problem

- How many rows are there in a truth table with n propositional variables?

Solution: 2^n We will see how to do this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
If the intended meaning is $p \vee (q \rightarrow \neg r)$
then parentheses must be used.

Applications of Propositional Logic

Section 1.2

Applications of Propositional Logic: Summary

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits
- AI Diagnosis Method (Optional)

Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - p: I go to Harry's
  - q: I go to the country.
  - r: I will go shopping.

If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

One Solution: Let *a*, *c*, and *f* represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$a \rightarrow (c \lor \neg f)$$

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let *p* denote "The automated reply can be sent" and *q* denote "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let p denote "The diagnostic message is not stored in the buffer." Let q denote "The diagnostic message is retransmitted" The specification can be written as: $p \vee q$, $p \rightarrow q$, $\neg p$. When p is false and q is true all three statements are true. So the specification is consistent.

- What if "The diagnostic message is not retransmitted is added."

  Solution: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

Logic Puzzles

Raymond Smullyan (Born 1919)

- An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.