Counting Techniques (Chapter 6)

Overview

- Section 6.1: Counting Basics
  - Sum Rule
  - Product Rule
  - Inclusion-Exclusion
- Section 6.2
  - Basic pigeonhole principle
  - Generalized pigeonhole principle
- Section 6.3
  - r-permutations: $P(n,r)$
  - r-combinations: $C(n,r)$
  - Anagrams
  - Cards and Poker

Counting Basics

Counting techniques are important in programming design.

EG: How large an array *or* hash table *or* heap is needed?

EG: What is the average case complexity of quick-sort?

Answers depend on being able to count.

Counting is useful in the gambling arena also.

EG: What should your poker strategy be?

Counting Basics
Set Cardinalities

Interested in answering questions such as:

- How many bit strings of length *n* are there?
- How many ways are there to buy 13 different bagels from a shop that sells 17 types?
- How many bit strings of length 11 contain a streak of one type of bit of exact length 7?
- How many ways can a dating service match 13 men to 17 women?

COMMON THEME: convert to set cardinality problems so each question above is about counting the number of elements in some set:

Q: What is the corresponding set in each case?

Counting Basics
Set Cardinalities

A: The set to measure the cardinality of is…

- How many bit strings of length n are there?
  - {bit strings of length n}
- How many ways are there to buy 13 different bagels from a shop that sells 17 types?
  - $\{S \subseteq \{1, 2, \ldots 17\} \mid |S| = 13\}$
- How many bit strings of length 11 contain a streak of one type of bit of exact length 7?
  - {length 11 bit strings with 0-streak of length 7}
  - $\cup$ {length 11 bit strings with 1-streak of length 7}
- How many ways to match 13 M to 17 W ?
  - $\{ f:\{1,2,\ldots,13\} \rightarrow \{1,2,\ldots,17\} \mid f \text{ is 1-to-1}\}$

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Product Rule

As counting problems can be turned into set cardinality problems, useful to express counting principles set theoretically.

Product-Rule: For finite sets A, B:

$$|A\times B| = |A|\cdot|B|$$

Q: How many bit strings of length n are there?

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Product Rule

A: 2^n.

Proof: Let S = {bit strings of length n}.

S is in 1-to-1 correspondence with

$$\underbrace{\mathbf{B}\times\mathbf{B}\times\mathbf{B}\times\cdots\times\mathbf{B}}_{n\text{ times}} \text{ where } \mathbf{B}=\{0,1\}$$

Consequently the product rule implies:

$$|S|=|\mathbf{B}\times\mathbf{B}\times\mathbf{B}\times\cdots\times\mathbf{B}|$$

$$=|\mathbf{B}|\times|\mathbf{B}|\times|\mathbf{B}|\times\cdots\times|\mathbf{B}|=|\mathbf{B}|^n=2^n$$

Cardinality of Power Set

THM: $|P(\{1,2,3,\ldots,n\})| = 2^n$

Proof. The set of bit strings of length n is in 1-to-1 correspondence with the $P(\{1,2,3,\ldots,n\})$ since subsets are represented by length n bit strings.☐

Sum Rule

Next the number of length 11 bit strings with a streak of length exactly 7.

Q: Which of the following should be counted:

1. 10011001010
2. 0110111101011
3. 10000000011
4. 10000000101
5. 01111111010

Sum Rule

Next the number of length 11 bit strings with a streak of length exactly 7.

Q: Which of the following should be counted:

1. 10011001010
   No!, longest streak has length 2.
2. 01101111010**11** No! Too long.
3. 1**00000000**011 No! Streak too long.
4. 1**0000000**101 Yes!
5. 0**1111111**010 Yes!

Sum Rule

We are trying to compute the cardinality of:

{length 11 bit strings with 0-streak of length 7}

$$\cup$$

{length 11 bit strings with 1-streak of length 7}

Call the first set A and the second set B.

Q: Are A and B disjoint?

Sum Rule

A: Yes. If had both a 0-streak and a 1-streak of length 7 each, string would have length at least 14!

When counting the cardinality of a disjoint union we use:

SUM RULE: If A and B are disjoint, then

$$|A \cup B| = |A|+|B|$$

By symmetry, in our case A and B have the same cardinality. Therefore the answer would be $2|A|$.

Sum Rule

Break up

A = {length 11 bit strings with
0-streak of length exactly 7}

into more cases and use sum rule:

A_1 = {00000001***} (* is either 0 or 1)

A_2 = {100000001**}

A_3 = {*100000001*}

A_4 = {**100000001}

A_5 = {***10000000}.

Apply sum rule:

$|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$

Sum Rule

So let's count each set.

A_1 = {00000001***}. There are 3 *'s, each with 2 choices, so product rule gives $|A_1| = 2^3 = 8$

A_2 = {100000001**}. There are 2 *'s. Therefore, $|A_2| = 2^2 = 4$

A_3 = {*100000001*}, A_4 = {**100000001}

Similarly: $|A_2| = |A_3| = |A_4| = 4$

A_5 = {***10000000}.

$|A_1| = |A_5| = 8$

$|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| =$
$8+4+4+4+8 = 28$.

Therefore answer is **56**.

Counting Functions

How many ways to match 13M to 17W ?

{ f:{1,2,…,13} → P {1,2,…,17} | f is 1-to-1}

Use product rule thoughtfully.

1. 17 possible output values for $f(1)$
2. 16 values remain for $f(2)$

………………………………

i. $17-i+1$ values remain for $f(i)$

………………………………

13. 17-13+1=5 values remain for $f(13)$

ANS: $17\cdot16\cdot15\cdot14\cdot\ldots\cdot7\cdot6\cdot5 = 17!\,/\,4!$

Q: In general how many 1-to-1 functions from size k to size n set?

Counting Functions

A: The number of 1-to-1 functions from a size k set to a size n set is

$$n\,!\,/\,(n-k)\,!$$

As long as k is no larger than n. If $k > n$ there are no 1-to-1 functions.

Q: How about general functions from size k sets to size n sets?

Counting Functions

A: The number of functions from a size k set to a size n set is

$$n^k$$

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