Counting Techniques (Chapter 6)

1

Overview

- Section 6.1: Counting Basics
  - Sum Rule
  - Product Rule
  - Inclusion-Exclusion
- Section 6.2
  - Basic pigeonhole principle
  - Generalized pigeonhole principle
- Section 6.3
  - r-permutations: $P(n,r)$
  - r-combinations: $C(n,r)$
  - Anagrams
  - Cards and Poker

Counting Basics

Counting techniques are important in programming design.

EG: How large an array *or* hash table *or* heap is needed?

EG: What is the average case complexity of quick-sort?

Answers depend on being able to count.

Counting is useful in the gambling arena also.

EG: What should your poker strategy be?

3

Counting Basics Set Cardinalities

Interested in answering questions such as:

- How many bit strings of length n are there?
- How many ways are there to buy 13 different bagels from a shop that sells 17 types?
- How many bit strings of length 11 contain a streak of one type of bit of exact length 7?
- How many ways can a dating service match 13 men to 17 women?

COMMON THEME: convert to set cardinality problems so each question above is about counting the number of elements in some set:

Q: What is the corresponding set in each case? 4

Counting Basics
Set Cardinalities

A: The set to measure the cardinality of is…

- How many bit strings of length n are there?
  - {bit strings of length n}
- How many ways are there to buy 13 different bagels from a shop that sells 17 types?
  - $\{S \subseteq \{1, 2, \ldots 17\} \mid |S| = 13\}$
- How many bit strings of length 11 contain a streak of one type of bit of exact length 7?
  - {length 11 bit strings with 0-streak of length 7}

  $\cup$ {length 11 bit strings with 1-streak of length 7}
- How many ways to match 13 M to 17 W ?
  - $\{ f :\{1,2,\ldots,13\} \rightarrow \{1,2,\ldots,17\} \mid f \text{ is 1-to-1}\}$

5

Product Rule

As counting problems can be turned into set cardinality problems, useful to express counting principles set theoretically.

Product-Rule: For finite sets A, B:

$$|A \times B| = |A| \cdot |B|$$

Q: How many bit strings of length n are there?

6

Product Rule

A: 2^n.

Proof : Let S = {bit strings of length n}.

S is in 1-to-1 correspondence with $\underbrace{\mathbf{B}\times\mathbf{B}\times\mathbf{B}\times\cdots\times\mathbf{B}}_{n \text{ times}}$ where $\mathbf{B}=\{0,1\}$

Consequently the product rule implies:

$$|S|=|\mathbf{B}\times\mathbf{B}\times\mathbf{B}\times\cdots\times\mathbf{B}|$$

$$=|\mathbf{B}|\times|\mathbf{B}|\times|\mathbf{B}|\times\cdots\times|\mathbf{B}|=|\mathbf{B}|^n=2^n$$

Cardinality of Power Set

THM: $|P(\{1,2,3,\ldots,n\})| = 2^n$

Proof . The set of bit strings of length n is in 1-to-1 correspondence with the $P(\{1,2,3,\ldots,n\})$ since subsets are represented by length n bit strings.☐

Sum Rule

Next the number of length 11 bit strings with a streak of length exactly 7.

Q: Which of the following should be counted:

1. 10011001010
2. 0110111101011
3. 10000000011
4. 10000000101
5. 01111111010

Sum Rule

Next the number of length 11 bit strings with a streak of length exactly 7.

Q: Which of the following should be counted:

1. 10011001010

   No!, longest streak has length 2.
2. 01101111010**11** No! Too long.
3. 1**00000000**011 No! Streak too long.
4. 1**0000000**101 Yes!
5. 0**1111111**010 Yes!

Sum Rule

We are trying to compute the cardinality of:

{length 11 bit strings with 0-streak of length 7}

$$\cup$$

{length 11 bit strings with 1-streak of length 7}

Call the first set A and the second set B.

Q: Are A and B disjoint?

Sum Rule

A: Yes. If had both a 0-streak and a 1-streak of length 7 each, string would have length at least 14!

When counting the cardinality of a disjoint union we use:

SUM RULE: If A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

By symmetry, in our case A and B have the same cardinality. Therefore the answer would be $2|A|$.

Sum Rule

Break up

A = {length 11 bit strings with
0-streak of length exactly 7}

into more cases and use sum rule:

A_1 = {00000001***} (* is either 0 or 1)

A_2 = {100000001**}

A_3 = {*100000001*}

A_4 = {**100000001}

A_5 = {***10000000}.

Apply sum rule:

$|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$

Sum Rule

So let's count each set.

A_1 = {00000001***}. There are 3 *'s, each with 2 choices, so product rule gives $|A_1| = 2^3 = 8$

A_2 = {100000001**}. There are 2 *'s. Therefore, $|A_2| = 2^2 = 4$

A_3 = {*100000001*}, A_4 = {**100000001}

Similarly: $|A_2| = |A_3| = |A_4| = 4$

A_5 = {***10000000}.

$|A_1| = |A_5| = 8$

$|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| = 8+4+4+4+8 = 28.$

Therefore answer is **56**.

Counting Functions

How many ways to match 13M to 17W ?

{ f:{1,2,…,13} → P {1,2,…,17} | f is 1-to-1}

Use product rule thoughtfully.

1. 17 possible output values for $f(1)$
2. 16 values remain for $f(2)$

………………………………

i. $17-i+1$ values remain for $f(i)$

………………………………

13. 17-13+1=5 values remain for $f(13)$

ANS: $17\cdot 16\cdot 15\cdot 14\cdot \ldots \cdot 7\cdot 6\cdot 5 = 17! / 4!$

Q: In general how many 1-to-1 functions from size k to size n set?

Counting Functions

A: The number of 1-to-1 functions from a size k set to a size n set is

$$n! / (n-k)!$$

As long as k is no larger than n. If $k > n$ there are no 1-to-1 functions.

Q: How about general functions from size k sets to size n sets?

Counting Functions

A: The number of functions from a size k set to a size n set is

$$n^k$$

Inclusion-Exclusion

The principle of Inclusion-Exclusion generalized the sum rule to the case of non-empty intersection:

INCLUSION-EXCLUSION: If A and B are sets, then

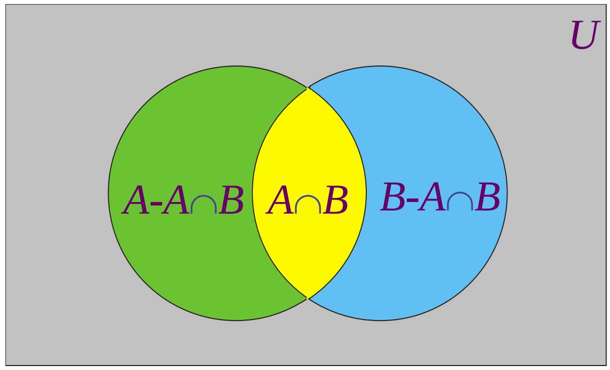
$$|A \cup B| = |A| + |B| - |A \cap B|$$

This says that when counting all the elements in A or B, if we just add the the sets, we have double-counted the intersection, and must therefore subtract it out.

Inclusion-Exclusion

Visualize.

Diagram gives proof Inclusion-Exclusion principle:

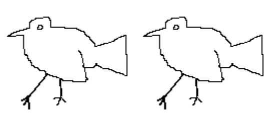


$$
\begin{aligned}
|A \cup B| &= |A - A\cap B| + |A \cap B| + |B - A\cap B| \\
&= (|A - A\cap B| + |A\cap B|) + (|B - A\cap B| + |A \cap B|) - |A\cap B| \\
&= |A| + |B| - |A \cap B|
\end{aligned}
$$

6.2 The Pigeonhole Principle

The pigeonhole principle is the (rather obvious) statement:

If there are n+1 pigeons, which must fit into n pigeonholes then some pigeonhole contains 2 or more pigeons.



Less obvious: Why this would ever be useful… Principle often applied in surprising ways.

Pigeonhole Principle

EG: Given 12 or more numbers between 0 and 1 inclusive, there are 2 (or more) numbers *x,y* whose strictly within 0.1 of each other.

Proof. Any pigeonhole principle application involves discovering who are the pigeons, and who are the pigeonholes. In our case:

Pigeons: The 12 numbers

Pigeonholes: The sets

[0,0.1), [0.1,0.2), [0.2,0.3), … ,[0.9,1), {1}

There are 11 pigeonholes so some *x,y* fall in one of these sets, and have difference < 0.1

Pigeonhole Principle

Harder Example: In a party of 2 or more people, there are 2 people with the same number of friends in the party. (Assuming you can't be your own friend and that friendship is mutual.)

Pigeonhole Principle

Harder Example: In a party of 2 or more people, there are 2 people with the same number of friends in the party. (Assuming you can't be your own friend and that friendship is mutual.)

Proof.

Pigeons –the n people (with $n > 1$).

Pigeonholes –the possible number of friends. i.e. the set $\{0,1,2,3,\ldots n\text{-}1\}$

Pigeonhole Principle

The proof proceeds with 2 cases.

I) There is a pigeonhole which isn't hit. In that case, left with n -1 remaining pigeonholes, but n pigeons, so done.

Pigeonhole Principle

II) The proof proceeds with 2 cases.
Every pigeonhole hit.

In particular, the friendship numbers n-1 as well as 0 were hit.

Thus someone is friends with everyone while someone else is friends with no-one. This is a contradiction so this case cannot happen! →←

□ 25

Generalized Pigeonhole Principle

If N objects are placed into k boxes, there is at least one box containing $\lceil N/k \rceil$ objects.

Specialize to $N = n+1$ and $k = n$, gives at least $\lceil (n+1)/n \rceil = 2$ objects in one box, which is the regular pigeonhole principle.

Q: Suppose that NYC has more than 7,000,000 inhabitants and that human heads contain at most 500,000 hairs. Find a guaranteed minimum number of people in NYC that all have the same number of hairs on their heads.

26

Generalized Pigeonhole Principle

A: $\lceil 7{,}000{,}000 \,/\, 500{,}001 \rceil = 14$

27

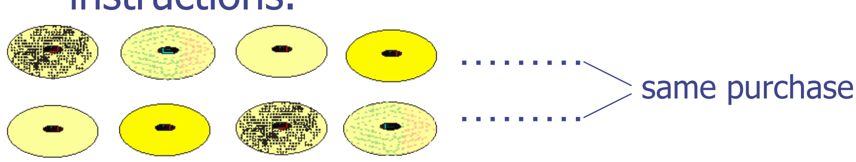
Permutations and Combinations

Several ways of interpreting question

"How many ways are there to buy 13 different bagels out of 17 types?"

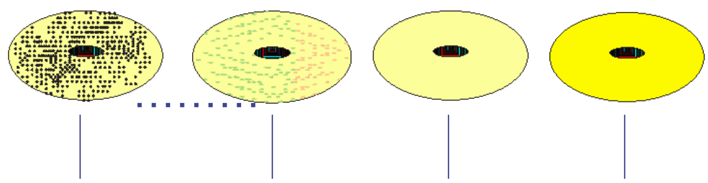
1st Interpretation: order matters

2nd Interpretations: order doesn't matter so only care about set of bagels that have at the end, not about the purchasing instructions.



Permutations and Combinations

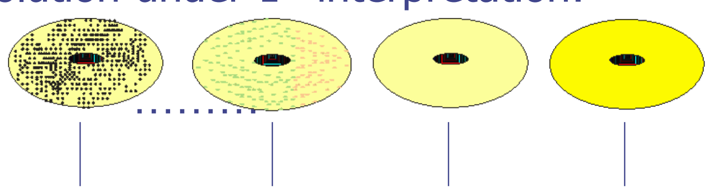
Usually 2nd interpretation is taken, but in other situations, 1st interpretation makes more sense. EG: Suppose the bagels were destined to the 13 alphabetized children in a kindergarten.



Al Bina Charmaine Darlene

First Interpretation: Number of *r*-permutations

Solution under 1st interpretation:



Al Bina Charmaine Darlene

17 · 16 · 15 · 14 · … · 6·5

$= 17^{\underline{13}}$ where we introduce the ***falling power*** notation:

$$x^{\underline{r}} = P(x,r) = x(x-1)(x-2)(x-3)\cdots(x-r+1)$$

Number of *r*-permutations

Abstractly, $P(n,r)$ is defined as the cardinality of:

{ordered *r*-tuples with distinct coord's from {1,2,3, …, *n*}}

= {***r-permutations*** of the set {1,2,3, …, *n*}} (by definition)

EG: $P(3,2)$ is the number of 2-tuples with distinct coordinates taken from {1,2,3}, so is the cardinality of:

$$\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$$

Which is $6 = 3\cdot 2$.

THM: The abstract definition of $P(n,r)$ is calculated with the formula $P(n,r) = n^{\underline{r}}$.

Proof by bagels!!!! •

Number of *r*-permutations Counting 1-to-1 Functions

Q: Express the number of one-to-one functions from $\{1,2,3,\ldots,m\}$ to $\{1,2,3,\ldots,n\}$ by using the falling power.

Number of r-permutations Counting 1-to-1 Functions

A: From last time, this is just given by $n^{\underline{m}}$ for $m <= n$ and is 0 otherwise. Can see this directly, since a 1-to-1 function can be thought of as an m-tuple with no repeated elements.

Second Interpretation: Number of r-combinations

In the second interpretation of bagel counting problem, order doesn't matter.

Answer will be $C(17,13)$ vs. $P(17,13)$.

$C(17,13)$ measures the number of distinct size-13 *subsets* in a set of size 13, as opposed to size 13 *permutations*. In general: $C(n,r)$ is defined as the cardinality of:

$\{$unordered size-r subsets of $\{1,2,3, \ldots, n\}\}$

$= \{$***r-combinations*** of the set $\{1,2,3, \ldots, n\}\}$

Second Interpretation: Number of *r*-combinations

Let's prove the previously mentioned formula:

$$C(n,r) = n! / (r! \cdot (n-r)!)$$

Simpler to express this with falling power:

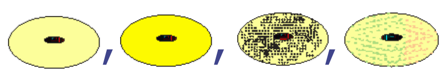
$$C(n,r) = n^{\underline{r}} / r!$$

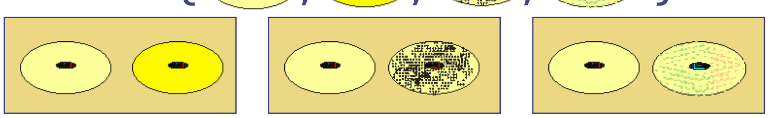
First check to see if we know how to apply formula:

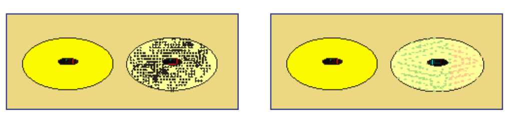
Q: How many different subsets of 2 bagels are there in set of 4?

Second Interpretation: Number of *r*-combinations

A: Subsets of 2 bagels in a set of 4.

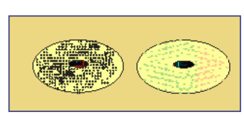
{  }





Answer = 6.

Indeed $C(4,2) = (4\cdot3)/(2\cdot1)$ $= 12/2 = \mathbf{6}$



Division Rule

Derivation of the formula will use a new counting rule:

DIVISION RULE: Suppose S is a finite set partitioned into cells of cardinality d. Then the number of cells is $|S| / d$.

Q: A treasure of 187 gold coins consists of bags consisting of exactly 17 coins each. How many bags are there?

Division Rule

A: 187 / 17 = 11

r-combinations Derivation of Formula

The claim $C(n,r) = n^{\underline{r}} / r!$ can be restated as:

$$C(n,r) = P(n,r) / r!$$

Suggests connection between *r*-combinations and *r*-permutations: there are $r!$ *r*-permutations for each *r*-combination.

EG: 3-combination {1,2,5} gives rise to 3! = 6 different 3-permutations: (1,2,5),(1,5,2),(2,1,5),(2,5,1),(5,1,2),(5,2,1).

Division rule gives formula: think of each *r*-combo as a cell containing $r!$ *r*-permutations.

r-combinations

Back to our bagel example. Under 2nd interpretation, the number of ways to buy 13 different bagels from 17 types is

$$C(17,13)$$

$$= \frac{17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{17\cdot 16\cdot 15\cdot 14}{4\cdot 3\cdot 2\cdot 1} = C(17,4) = 2380$$

r-combinations

Previous $C(17,13) = C(17,4)$ is special case of:

General formula $C(n,r) = C(n,n-r)$.

Proof: Just use formula, or the inherent symmetry of Pascal's triangle.

String Counting Example

EG: How many length 38 ASCII strings are there s.t.

- All characters are distinct
- The characters 'a' 'b' and 'c' appear in order
- (and assume that |ASCII| = 128)

String Counting Example

Method for computing answer:

- Decide where to put (a,b,c). There are 38 spots, out of which 3 must be chosen together, therefore $C(38,3)$ choices.
- After 1 has been chosen, 35 spots remain. So there are 125=128-3 choices for the first, 124 for the second, etc. Therefore, $125^{\underline{35}}$ choices.
- Product rule applies to (1) and (2) to obtain the final answer:

$$C(38,3) \cdot 125^{\underline{35}}$$

Anagrams

An ***anagram*** of a string is a rearrangement of the letters in the string.

EG, ignoring white-space and capitals an anagram on "William Gates" is:

"Will I tame gas"

-also allow random anagrams such as

"liiwlegamast"

Q: How many different anagrams are there on the string "williamgates"?

Anagrams
Using Product Rule

A: Alphabetize "williamgates" to get:

"aaegiillmstw" (length = 12)

- Choose 2 spaces to put the a's: $C(12,2)$
- Choose 1 space of remaining for e: $C(10,1)$
- Choose 1 of remaining 9 for g: $C(9,1)$
- Choose 2 of remaining 8 for i: $C(8,2)$
- Choose 2 of remaining 6 for l: $C(6,2)$
- Choose 1 of remaining 4 for m: $C(4,1)$
- Choose 1 of remaining 3 for s: $C(3,1)$
- Choose 1 of remaining 2 for t: $C(2,1)$
- Only one choice remains for w ($C(1,1) = 1$)

Anagrams
Using Product Rule

Use product rule obtaining answer:

$C(12,2)C(10,1)C(9,1)C(8,2)C(6,2)C(4,1)C(3,1)\ C(2,1)$

$= 66\cdot 10\cdot 9\cdot 28\cdot 15\cdot 4\cdot 3\cdot 2$

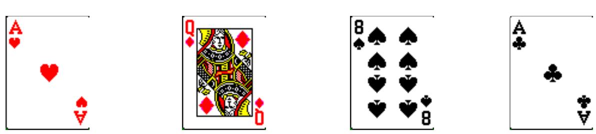
$= 59{,}875{,}200$

Cards and Poker

Analyzing the effectiveness of card-playing strategies is an important pedagogical case-study in discrete mathematics.

Types of Cards
The 4 Suits

Each deck of playing cards has 52 cards (not including Jokers). These 52 cards break up into 4 basic types of cards called ***suits*** :

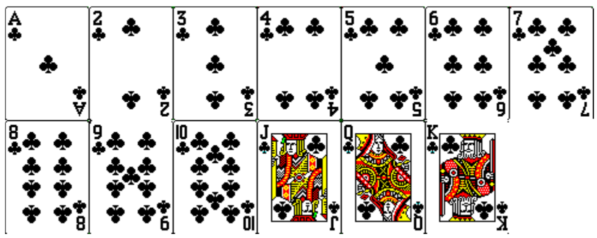


Hearts, Diamonds, Spades, Clubs

Types of Cards Within a Single Suit (clubs)

Each suit consists of 13 card ***values*** or ***ranks:***

Ace, 2, 3, 4, 5, 6, 7



8, 9, 10, jack,queen,king

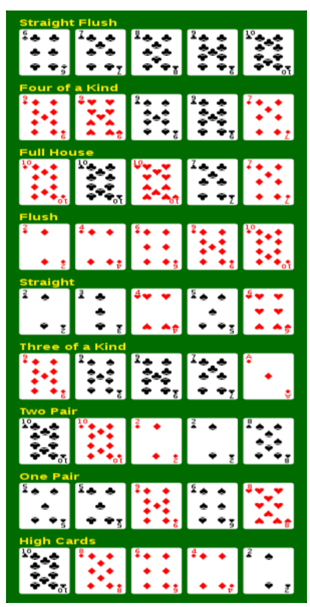
Cards and Poker

In poker a ***hand*** consists of a set of 5 cards. There are various hand configurations that have their own special names.

Glossary of Poker Hands

- **Straight flush**
  Five cards in sequence in the same suit. A royal straight flush (A-K-Q-J-10 in same suit) is the highest non-joker poker hand possible.
- **Four of a kind**
  Four cards of the same rank.
- **Full house**
  Three of a kind and a pair. When matching full houses, the one with the higher three of a kind wins.
- **Flush**
  Five cards of the same suit.
- **Straight**
  Any five cards in sequence but not all of the same suit.
- **Three of a kind**
  Three of the same rank with two unmatched cards.
- **Two pairs**
  Two cards of one rank with two cards of a different rank with one dissimilar card. When matching pairs occurs between players, the one with the higher fifth card wins.
- **One pair**
  Any two cards of the same rank.

Cards and Poker Examples



Cards and Poker Examples

Q: How many possible hands are there?

Cards and Poker Examples

A: Same as the number of 5-element subsets in a set of cardinality 52. I.e. $C(52,5) = 2{,}598{,}960$

Q: How many different hands constitute a full house?

Cards and Poker Examples

A: Same as pair + 3 of a *different* kind

- Number of pairs: $13 \cdot C(4,2)$
  - Choose a rank: 13
  - Choose 2 of 4 suits: $C(4,2)$
- Number of 3 of a kind: $12 \cdot C(4,3)$
  - Choose a *different* rank: 12
  - Choose 3 of 4 suits: $C(4,3)$

Answer: $13 \cdot C(4,2) \cdot 12 \cdot C(4,3) = 3744$

Exercises

- How many straight hands are there in poker?

Straight

Any five cards in sequence but not all of the same suit.

Exercises

The straight can start on any one of
A,K,J,Q,T,9,8,7,6,5 and go down.

This makes 10 base straight sequences. Each card in the sequence can be any of the four suits.

So the total number of straights is $10 * 4^5 = 10240$.