Functions

Section 2.3

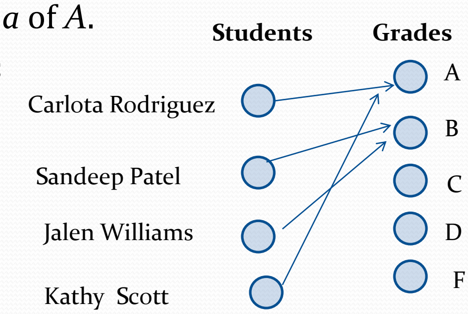
Section Summary

- Definition of a Function.
  - Domain, Cdomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function $f: A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

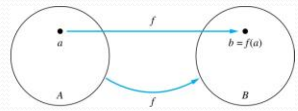
$$\forall x[x \in A \rightarrow \exists y[y \in B \land (x, y) \in f]]$$

and

$$\forall x \forall y_1 \forall y_2[[(x, y_1) \in f \land (x, y_2) \in f] \rightarrow y_1 = y_2]$$

Functions

Given a function $f: A \to B$:



- We say *f maps A* to *B or*
  f is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.
- If $f(a) = b$,
  - then *b* is called the *image* of *a* under *f*.
  - *a* is called the *preimage* of *b*.
- The range of *f* is the set of all images of points in **A** under *f*. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.
    Students and grades example.
  - A formula.
    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also inChapter 5).

Questions

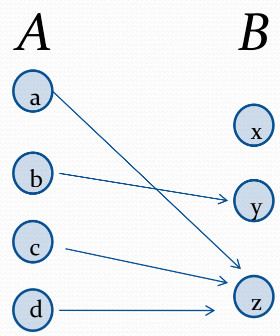
f(a) = ? z

The image of d is ? z

The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b



$f(A)$ = ? {y,z}

The preimage(s) of z is (are) ? {a,c,d}

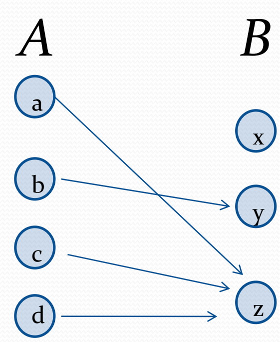
Question on Functions and Sets

- If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

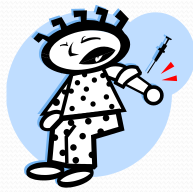
f {a,b,c,} is ? {y,z}

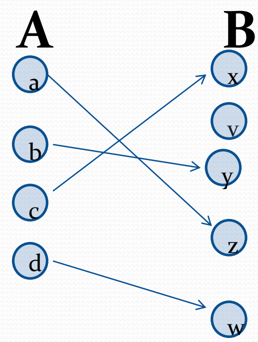
f {c,d} is ? {z}



Injections

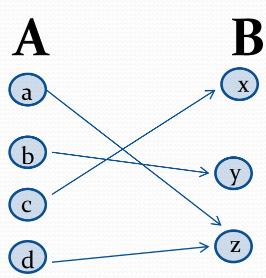
Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.





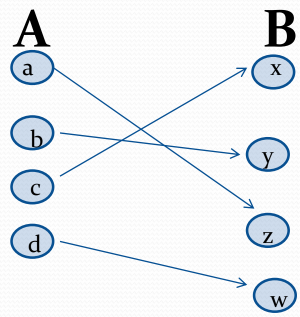
Surjections

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is onto.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that *f* is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that *f* is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

One-to-One, Onto, Bijection. Examples.

Q: Which of the following are 1-to-1, onto, a bijection? If f is invertible, what is its inverse?

1. $f : \mathbf{Z} \rightarrow \mathbf{R}$ is given by $f(x) = x^2$
2. $f : \mathbf{Z} \rightarrow \mathbf{R}$ is given by $f(x) = 2x$
3. $f : \mathbf{R} \rightarrow \mathbf{R}$ is given by $f(x) = x^3$
4. $f : \mathbf{Z} \rightarrow \mathbf{N}$ is given by $f(x) = |x|$
5. f : {people} → {people} is given by
   $f(x)$ = the father of x.

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One-to-One, Onto, Bijection. Examples.

1. $f : \mathbf{Z} \rightarrow \mathbf{R}, f(x) = x^2$: none
2. $f : \mathbf{Z} \rightarrow \mathbf{Z}, f(x) = 2x$: 1-1
3. $f : \mathbf{R} \rightarrow \mathbf{R}, f(x) = x^3$: 1-1, onto, bijection, inverse is $f(x) = x^{(1/3)}$
4. $f : \mathbf{Z} \rightarrow \mathbf{N}, f(x) = |x|$: onto
5. $f(x)$ = the father of x : none

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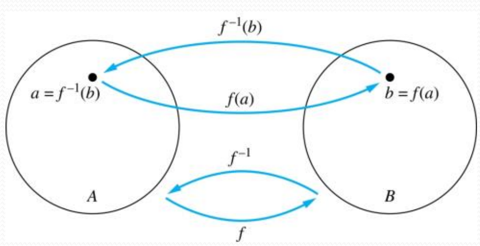
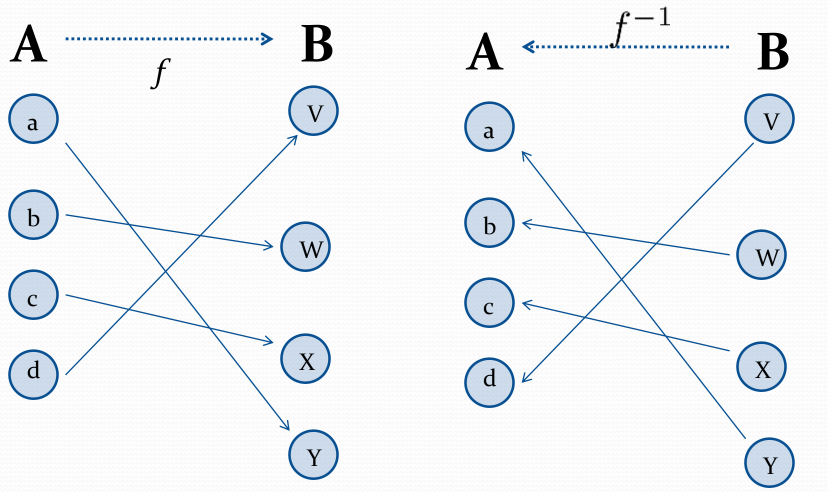
Inverse Functions

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless f is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{1} reverses the correspondence given by f, so $f^{1}(1) = c$, $f^{1}(2) = a$, and $f^{1}(3) = b$.

Questions

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

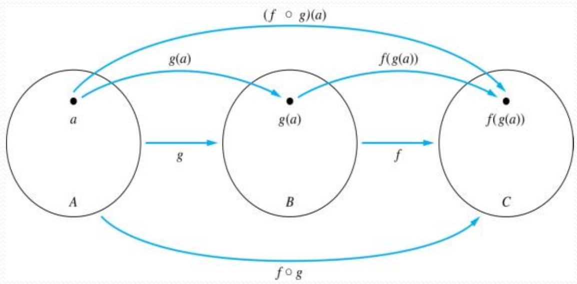
Questions

Example 3: Let f: **R** → **R** be such that $f(x) = x^2$ Is f invertible, and if so, what is its inverse?

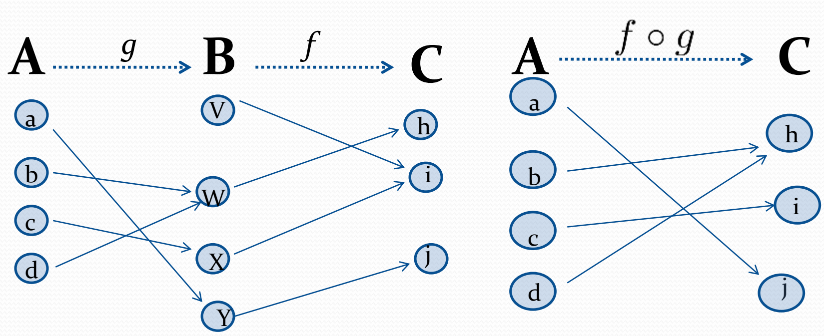
Solution: The function f is not invertible because it is not one-to-one .

Composition

- **Definition**: Let f: $B \rightarrow C$, g: $A \rightarrow B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $$f \circ g(x) = f(g(x))$$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f?

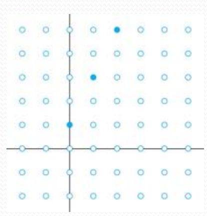
Solution:

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

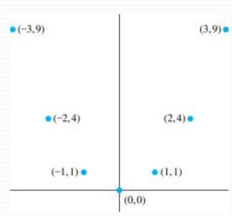
$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted

$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$