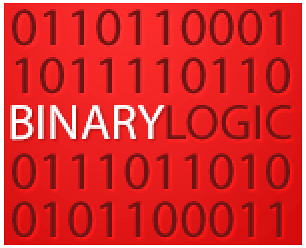


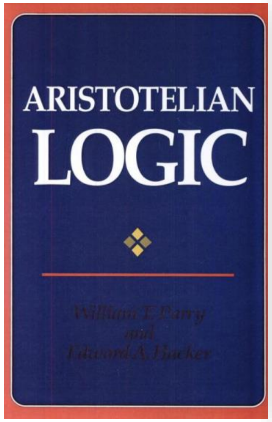
BINARY LOGIC

Boolean Algebra

Today the computing (information) technology is based on **Binary logic**

Binary (True-False) logic

- The Greek philosopher Aristotle (384-322 BC) founded a system of logic based on two types of propositions: **True** and **False**. This lead to the four foundational laws of logic:
  - **Law of Identity**: ("A" is "A" or ("A" = "A");
  - **Law of Non-contradiction**: ("A" is not "non-A");
  - **Law of the Excluded Middle**: (Something is either "A" or "non-A");
  - **Law of Rational Inference**…
    - all letters are characters
    - A is a Letter
    - A is a Character



Aristotle gestures to the earth, representing his belief in knowledge through empirical observation and experience, while holding a copy of his Nicomachean Ethics in his hand, while **Plato** gestures to the heavens, representing his belief in The Forms.

Centuries later…

Mathematicians (Leibniz, Boole, …) and Engineers (Shannon, Shestakov) extended the Aristotelian Logic to symbolic logic to algebra of logic to logic circuits …

Gottfried Wilhelm von LEIBNIZ (1646-1716)

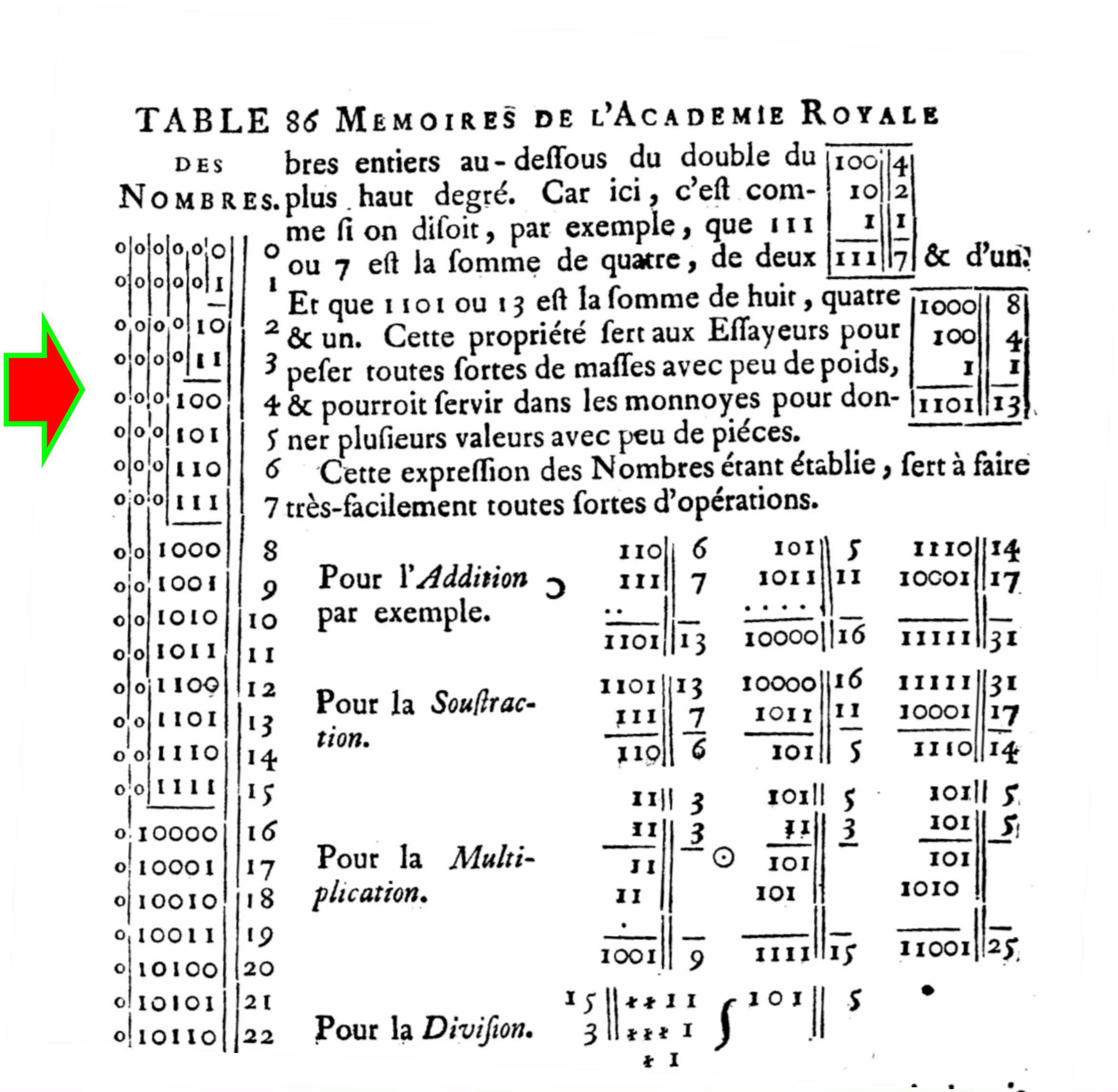


TABLE DES NOMBRES. — 86 MEMOIRES DE L'ACADEMIE ROYALE

bres entiers au-deſſous du double du plus haut degré. Car ici, c'eſt comme ſi on diſoit, par exemple, que 111 ou 7 eſt la ſomme de quatre, de deux & d'un. Et que 1101 ou 13 eſt la ſomme de huit, quatre & un. Cette propriété ſert aux Eſſayeurs pour peſer toutes ſortes de maſſes avec peu de poids, & pourroit ſervir dans les monnoyes pour donner pluſieurs valeurs avec peu de piéces.

100	4
10	2
1	1
111	7

1000	8
100	4
1	1
1101	13

Cette expreſſion des Nombres étant établie, ſert à faire très-facilement toutes ſortes d'opérations.

0 0 0 0 0 0	0
0 0 0 0 0 1	1
0 0 0 0 10	2
0 0 0 0 11	3
0 0 0 100	4
0 0 0 101	5
0 0 0 110	6
0 0 0 111	7
0 0 1000	8
0 0 1001	9
0 0 1010	10
0 0 1011	11
0 0 1100	12
0 0 1101	13
0 0 1110	14
0 0 1111	15
0 10000	16
0 10001	17
0 10010	18
0 10011	19
0 10100	20
0 10101	21
0 10110	22

Pour l'*Addition* par exemple.

110	6	101	5	1110	14
111	7	1011	11	10001	17
1101	13	10000	16	11111	31

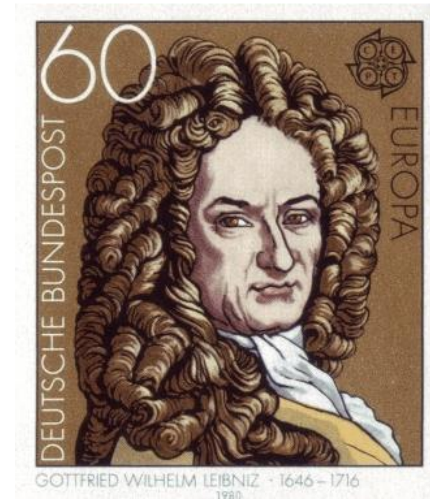
Pour la *Souſtraction.*

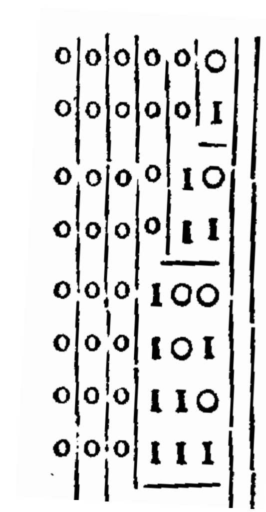
1101	13	10000	16	11111	31
111	7	1011	11	10001	17
110	6	101	5	1110	14

Pour la *Multiplication.*

11	3	101	5	101	5
11	3	11	3	101	5
11		101		101	
11		101		1010	
1001	9	1111	15	11001	25

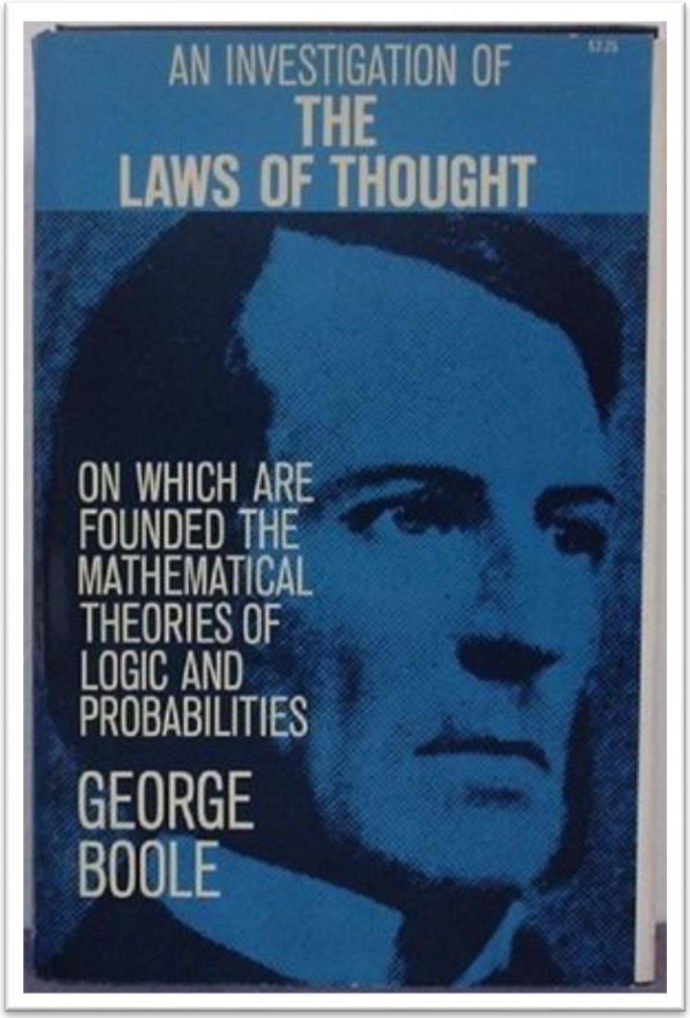
Pour la *Diviſion.* 15 / 3 ... 101 5





0 0 0 0 0 0	
0 0 0 0 0 1	
0 0 0 0 10	
0 0 0 0 11	
0 0 0 100	
0 0 0 101	
0 0 0 110	
0 0 0 111	

George Boole, Mathematician, 1815-1864



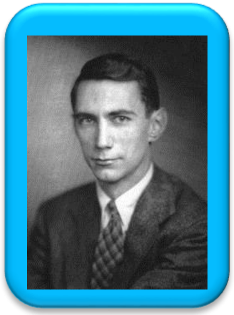
«The Mathematical Analysis of Logic»

«The Laws of Thought »

Symbolic logic | **Algebra of logic** | **Boolean algebra**

Claude Shannon, Victor Ivanovich Shestakov

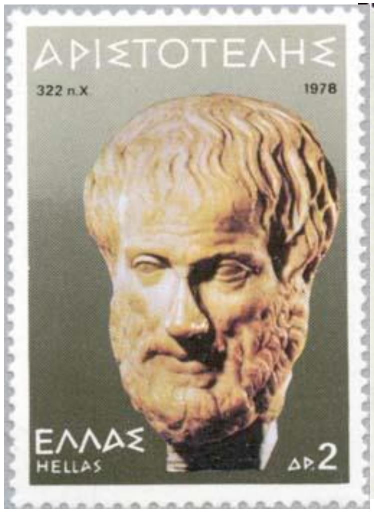
- **Claude Shannon** (1916-2001): «*A symbolic analysis of relay and switching circuits*», Thesis (M.S.E.E)-Massachusetts Institute of Technology, 1940.



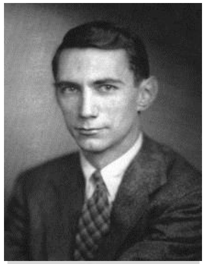
- **Victor Ivanovich Shestakov** (1907-1987): «*Mathematical logic and foundations*», Ph.D. Dissertation-Moscow State University, 1939.

Applied the Algebra of logic → Logic Circuits

Logic … logic circuits



George Boole



- Aristotle (400 B.C) : Logic (True and False)
- Muslim mathematicians (middle ages) → survived Aristotelian and other manuscripts
- Leibniz (1679-1701): Aristotelian logic → Mathematical Logic
- Boole (1854): Gave a meaning to Mathematical Logic → Algebra of Logic
- Claude Shannon (1937) and Victor Ivanovich Shestakov (1935): Applied the Algebra of logic → Logic Circuits

Basic Theorems of Boolean algebra

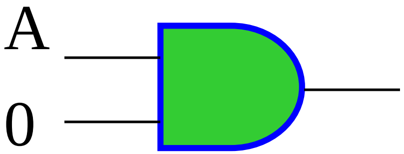
Boolean Theorems

- Single Variable: $f(A)$
- Multiple variable: $f(A, B, C, \ldots)$.

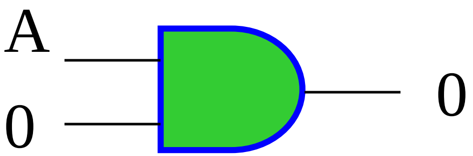
SingleVariable Boolean Theorems

$$f(A) = A \bullet 0$$

Operation with zero (1); A•0 = ?

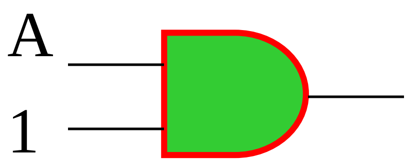


Operation with zero (1); A•0 = ?

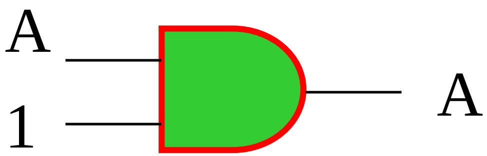


A	0	Output
0	0	0
1	0	0

Operation with one (2); A•1 = ?

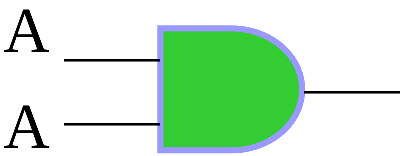


Operation with one (2); A•1 = ?

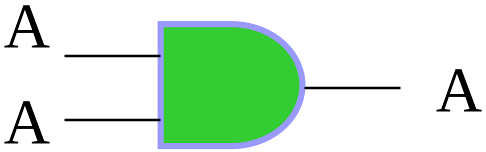


A	1	Output
0	1	0
1	1	1

Idempotent theorem (3) ; A•A = ?

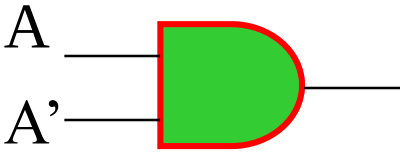


Idempotent theorem (3) ; A•A = ?

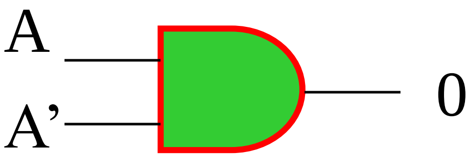


A	A	Output
0	0	0
1	1	1

Complementary (4) ; A•A' = ?

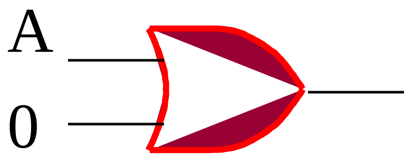


Complementary (4) ; A•A' = ?

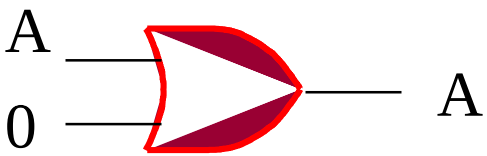


A	A'	Output
0	1	0
1	0	0

Operation with zero (5) ; A + 0 = ?

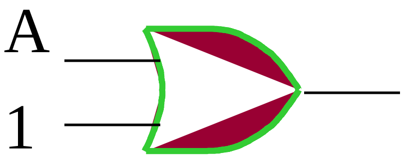


Operation with zero (5) ; A + 0 = A

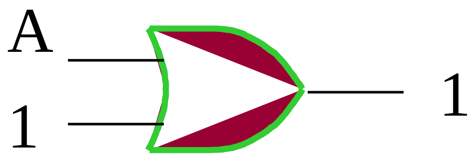


A	0	Output
0	0	0
1	0	1

Operation with one (6) ; A + 1 = ?

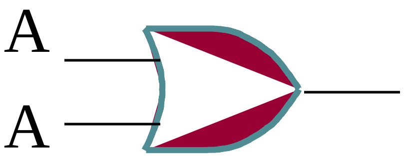


Operation with one (6) ; A + 1 = 1

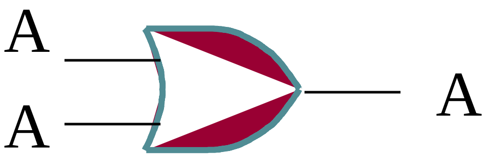


A	1	Output
0	1	1
1	1	1

Idempotent (7) ; A + A = ?

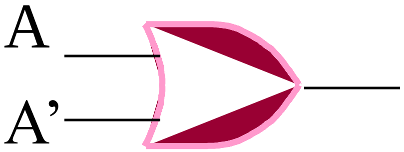


Idempotent (7) ; A + A = A

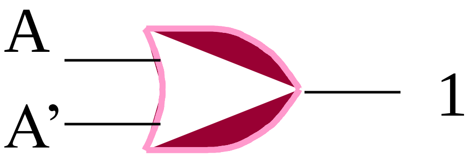


A	A	Output
0	0	0
1	1	1

Complementary (8) ; A + A' = ?

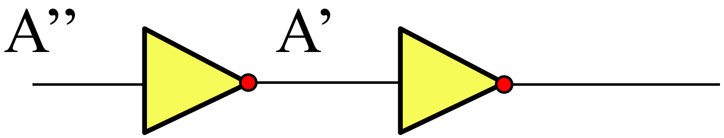


Complementary (8) ; A + A' = 1

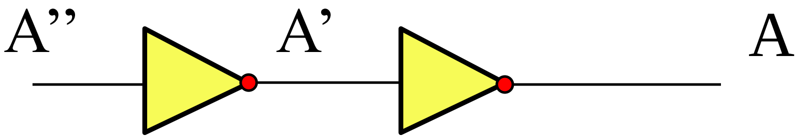


A	A'	Output
0	1	1
1	0	1

Involution theorem (9); A’’ = ?



Involution theorem (9); A’’ = A



A’’	A’	Output
0	1	0
1	0	1

The 9 basic Boolean theorems

- $A \cdot 0 = 0$
- $A \cdot 1 = A$
- $A \cdot A = A$
- $A \cdot A' = 0$
- $A + 0 = A$
- $A + 1 = 1$
- $A + A = A$
- $A + A' = 1$

$A' = \overline{A}$

- $(A')' = A$

MultiVariable Boolean theorems

$f(A,B) = A + B$

Multivariable theorems(1)

- Commutative Laws:
  - ❖ $A+B = B+A$
  - ❖ $A \bullet B = B \bullet A$

Multivariable theorems(2)

- Associative Laws:
  - ❖ $A+(B+C) = (A+B)+C = A+B+C$
  - ❖ $A \bullet (B \bullet C) = (A \bullet B) \bullet C = A \bullet B \bullet C$

Multivariable theorems(3)

- Distributed Law over Multiplication
  - ❖ $(D+A) \bullet (B+C) = D \bullet B + D \bullet C + A \bullet B + A \bullet C$
  - ❖ $A \bullet (B+C) = A \bullet B + A \bullet C$

Multivariable theorems(3)

- Distributed Law over Addition
  - $A+(B\bullet C) = (A+B)\bullet(A+C)$

Since it is not obvious…

- $A+(B \bullet C) \stackrel{?}{=} (A+B) \bullet (A+C)$

- Prove it … (5 minutes)

Proof …

$A+(B\bullet C) = (A+B)\bullet(A+C)$

Distribute …

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

… A●A

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= A \quad +A\bullet C+A\bullet B+B\bullet C$$

Factor-out common terms

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= A \quad +A\bullet C+A\bullet B+B\bullet C$$

$$= A\bullet(1+C)+A\bullet B+B\bullet C$$

Remove: $(1+C)=1$

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= A \quad +A\bullet C+A\bullet B+B\bullet C$$

$$= A\bullet(1+C)+A\bullet B+B\bullet C$$

$$= A\bullet 1 \quad +A\bullet B+B\bullet C$$

$A \bullet 1 = A$

$$
\begin{aligned}
A+(B \bullet C) &= (A+B) \bullet (A+C) \\
&= A \bullet A + A \bullet C + A \bullet B + B \bullet C \\
&= A + A \bullet C + A \bullet B + B \bullet C \\
&= A \bullet (1+C) + A \bullet B + B \bullet C \\
&= A \bullet 1 + A \bullet B + B \bullet C
\end{aligned}
$$

Factor-out common terms

$$
\begin{aligned}
A+(B\bullet C) &= (A+B)\bullet(A+C) \\
&= A\bullet A+A\bullet C+A\bullet B+B\bullet C \\
&= A \quad +A\bullet C+A\bullet B+B\bullet C \\
&= A\bullet(1+C)+A\bullet B+B\bullet C \\
&= A \qquad +A\bullet B+B\bullet C \\
&= A\bullet(1+B)+B\bullet C
\end{aligned}
$$

Remove (1+B)=1

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= A \quad +A\bullet C+A\bullet B+B\bullet C$$

$$= A\bullet(1+C)+A\bullet B+B\bullet C$$

$$= A \qquad +A\bullet B+B\bullet C$$

$$= A\bullet(1+B)+B\bullet C$$

$A \bullet 1 = A$

$$
\begin{aligned}
A+(B\bullet C) &= (A+B)\bullet(A+C) \\
&= A\bullet A + A\bullet C + A\bullet B + B\bullet C \\
&= A + A\bullet C + A\bullet B + B\bullet C \\
&= A\bullet(1+C) + A\bullet B + B\bullet C \\
&= A + A\bullet B + B\bullet C \\
&= A\bullet 1 + B\bullet C
\end{aligned}
$$

Done …

$$A+(B \bullet C) = (A+B) \bullet (A+C)$$

$$= A \bullet A + A \bullet C + A \bullet B + B \bullet C$$

$$= A + A \bullet C + A \bullet B + B \bullet C$$

$$= A \bullet (1+C) + A \bullet B + B \bullet C$$

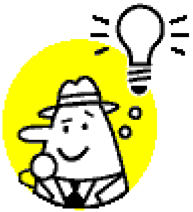
$$= A + A \bullet B + B \bullet C$$

$$= A \bullet 1 + B \bullet C$$

$$= A+(B \bullet C)$$

$A+(B\bullet C) = (A+B)\bullet(A+C)$

Another way to prove the equation?



$A+(B\bullet C) = (A+B)\bullet(A+C)$

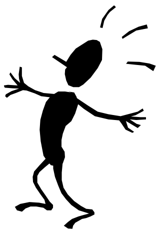
A	B	C	A+B	A+C	(A+B)(A+C)	BC	A+(BC)
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

A+(B•C) = (A+B)•(A+C)

A	B	C	A+B	A+C	(A+B)(A+C)	BC	A+(BC)
0	0	0	0				
0	0	1	0				
0	1	0	1				
0	1	1	1				
1	0	0	1				
1	0	1	1				
1	1	0	1				
1	1	1	1				

$A+(B \bullet C) = (A+B) \bullet (A+C)$

A	B	C	A+B	A+C	(A+B)(A+C)	BC	A+(BC)
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1



Perfect induction

Useful formula (UF-1)

- $A+A \bullet B = A$

- Proof …

Useful formula

- $A+A \bullet B = A$
- $A \bullet (1+B)$
- A

More useful formulas

- $A+A'B = A+B$ (UF-2)
- $A'+AB = A'+B$ (UF-3)
- $A(A+B) = A$ (UF-4)

$A + A'B = A + B$; Proof- 1

$$A + A'B = A + AB + A'B \qquad (A = A + AB)$$

$$= A + B(A + A') \qquad (A + A' = 1)$$

$$= A + B$$

$A' + AB = A' + B$; proof-2

$A' + AB = A' + A'B + AB \qquad (A' = A' + A'B)$

$= A' + B(A' + A) \qquad (A + A' = 1)$

$= A' + B$

A(A+B) = A; proof-3

$$
\begin{aligned}
A(A+B) &= AA + AB \\
&= A + AB \\
&= A(1+B) \\
&= A\,1 \\
&= A
\end{aligned}
$$