Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\ (C(x) \lor \exists y\ (C(y) \land F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x\ \forall y\ \forall z\ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: Some student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain, to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

   $$\forall x\ \forall\ y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w\ \forall a\ \exists f\ (P(w,f) \wedge Q(f,a))$$

Calculus in Logic (*optional*)

Example: Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.

Solution: Recall the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is "For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$."

Using quantifiers:

$$\forall \epsilon \exists \delta \forall x (0 < |\, x - a \,| < \delta \rightarrow |\, f(x) - L \,| < \epsilon)$$

Where the domain for the variables ε and δ consists of all positive real numbers and the domain for x consists of all real numbers.

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$
2. $\forall w \neg \forall a \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Return to Calculus and Logic (*Opt*)

Example : Recall the logical expression developed in the calculus example three slides back. Use quantifiers and predicates to express that $\lim_{x\to a} f(x)$ does not exist.

1. We need to say that for all real numbers L, $\lim_{x\to a} f(x) \neq L$
2. The result from the previous example can be negated to yield:
   $$\neg\forall\epsilon\exists\delta\forall x(0 < |\, x - a \,| < \delta \rightarrow |\, f(x) - L \,| < \epsilon)$$
3. Now we can repeatedly apply the rules for negating quantified expressions:

$$\begin{aligned}
&\neg\forall\epsilon\exists\delta\forall x(0 < |\, x - a \,| < \delta \rightarrow |\, f(x) - L \,| < \epsilon)\\
&\equiv \exists\epsilon\neg\exists\delta\forall x(0 < |\, x - a \,| < \delta \rightarrow |\, f(x) - L \,| < \epsilon)\\
&\equiv \exists\epsilon\forall\delta\neg\forall x(0 < |\, x - a \,| < \delta \rightarrow |\, f(x) - L \,| < \epsilon)\\
&\equiv \exists\epsilon\forall\delta\exists x\neg(0 < |\, x - a \,| < \delta \rightarrow |\, f(x) - L \,| < \epsilon)\\
&\equiv \exists\epsilon\forall\delta\exists x\neg(0 < |\, x - a \,| < \delta \wedge |\, f(x) - L \,| \geq \epsilon)
\end{aligned}$$

The last step uses the equivalence $\neg(p\rightarrow q) \equiv p\wedge\neg q$

Calculus in Predicate Logic (*optional*)

4. Therefore, to say that $\lim_{x\to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x\to a} f(x) \neq L$ can be expressed as:
   $$\forall L\exists\epsilon\forall\delta\exists x\neg(0 < |\, x - a \,| < \delta \wedge |\, f(x) - L \,| \geq \epsilon)$$

   Remember that ε and δ range over all positive real numbers and *x* over all real numbers.
5. Translating back into English we have, for every real number L, there is a real number ε > 0, such that for every real number δ > 0, there exists a real number x such that $0 < |\, x - a \,| < \delta$ and $|f(x) - L\,| \geq \varepsilon$.

Some Questions about Quantifiers (*optional*)

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

    Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.
  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

    Solution: No! The left and the right side may have different truth values for some propositional functions for *P*. Try "x + y = 0" for *P(x,y)* with *U* being the integers. The order in which the values of *x* and *y* are picked does matter.
- Can you distribute quantifiers over logical connectives?
  - Is this a valid equivalence? $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

    Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.
  - Is this a valid equivalence? $\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

    Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.

Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- Matrices
  - Matrix Arithmetic

Sets

Section 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

  $S = \{a,b,c,d\} = \{b,c,a,d\}$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a,b,c,d, \ldots\ldots,z\}$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$
- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$
- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots\ldots,99\}$
- Set of all integers less than 0:

  $S = \{\ldots., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^+$ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$
- A predicate may be used:

  $S = \{x \mid P(x)\}$
- Example: $S = \{x \mid \text{Prime}(x)\}$
- Positive rational numbers:

  $\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

$[a,b] = \{x \mid a \leq x \leq b\}$

$[a,b) = \{x \mid a \leq x < b\}$

$(a,b] = \{x \mid a < x \leq b\}$

$(a,b) = \{x \mid a < x < b\}$

closed interval [a,b]

open interval (a,b)

Set Builder Notation. Examples.

Q1: $U = \mathbf{N}$. $\{ x \mid \forall y\, (y \ge x) \} = ?$

Q2: $U = \mathbf{Z}$. $\{ x \mid \forall y\, (y \ge x) \} = ?$

Q3: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \wedge y^2 = x) \} = ?$

Q4: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \wedge y^3 = x) \} = ?$

Q5: $U = \mathbf{R}$. $\{ |x| \mid x \in \mathbf{Z} \} = ?$

Q6: $U = \mathbf{R}$. $\{ |x| \} = ?$

Set Builder Notation. Examples.

A1: $U = \mathbf{N}$. $\{ x \mid \forall y\, (y \ge x) \} = \{ 0 \}$

A2: $U = \mathbf{Z}$. $\{ x \mid \forall y\, (y \ge x) \} = \{ \}$

A3: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \wedge y^2 = x) \}$

$= \{ 0, 1, 2, 3, 4, \ldots \} = \mathbf{N}$

A4: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \wedge y^3 = x) \} = \mathbf{Z}$

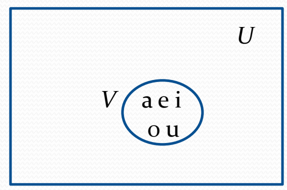
A5: $U = \mathbf{R}$. $\{ |x| \mid x \in \mathbf{Z} \} = \mathbf{N}$

A6: $U = \mathbf{R}$. $\{ |x| \}$ = non-negative reals.

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram



John Venn (1834-1923)
Cambridge, UK

Russell's Paradox

- Let *S* be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is *S* a member of itself?"
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\}, a, \{b,c\}\}$

  $\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$
- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write $A = B$ if A and B are equal sets.

  $\{1,3,5\} = \{3, 5, 1\}$

  $\{1,5,5,5,3,3,1\} = \{1,3,5\}$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff
$$\forall x(x \in A \leftrightarrow x \in B)$$
- Using logical equivalences we have that $A = B$ iff
$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$
- This is equivalent to
$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

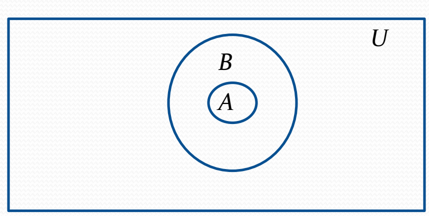
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted P(A), is called the *power set* of A.

Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

- The *ordered n-tuple* $(a_1,a_2,\ldots\ldots,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

René Descartes
(1596-1650)

Cartesian Product

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)