Cardinality of Sets

Section 2.5

Section Summary

- Cardinality
- Countable Sets
- Computability

Cardinality and Countability

Up to now cardinality has been the number of elements in a finite sets. Really, cardinality is a much deeper concept. Cardinality allows us to generalize the notion of number to infinite collections and it turns out that many type of infinities exist.

EG:

- {☺ ☹}
- { ,  }
- {Ø , {Ø,{Ø,{Ø}}} }

These all share "2-ness".

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Cardinality and Countability

For finite sets, can just count the elements to get cardinality. *Infinite* sets are harder.

First Idea: Can tell which set is bigger by seeing if one contains the other.

- $\{1, 2, 4\} \subset \mathbf{N}$
- $\{0, 2, 4, 6, 8, 10, 12, \ldots\} \subset \mathbf{N}$

So set of even numbers ought to be smaller than the set of natural number because of *strict* containment.

Q: Any problems with this?

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Cardinality and Countability

A: Set of even numbers is obtained from **N** by multiplication by 2. I.e.

{even numbers} = 2•**N**

For finite sets, since multiplication by 2 is a one-to-one function, the size doesn't change.

EG: {1,7,11} – ×2 → {2,14,22}

Another problem: set of even numbers is disjoint from set of odd numbers. Which one is bigger?

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Cardinality

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

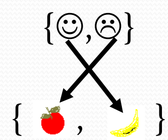
- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality and Countability – Finite Sets

DEF: Two sets A and B have the same ***cardinality*** if there's a bijection

$f : A \rightarrow B$

For finite sets this is the same as the old definition:



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Cardinality and Countability – Infinite Sets

But for infinite sets…

…there are **surprises**.

DEF: If S is finite or has the same cardinality as **N**, S is called ***countable***.

Notation, the Hebrew letter Aleph is often used to denote infinite cardinalities. Countable sets are said to have cardinality $\aleph_0$.

Intuitively, countable sets can be counted in the sense that if you allocate 1 second to count each member, eventually any particular member will be counted after a finite time period. Paradoxically, you won't be able to count the *whole set* in a finite time period!

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Countability – Examples

Q: Why are the following sets countable?

1. {0,2,4,6,8,…}
2. {1,3,5,7,9,…}
3. {1,3,5,7, $100^{100^{100^{100^{100}}}}$ }
4. **Z**

Countability – Examples

1. {0,2,4,6,8,…}: Just set up the bijection $f(n) = 2n$
2. {1,3,5,7,9,…} : Because of the bijection $f(n) = 2n + 1$
3. {1,3,5,7, $100^{100^{100^{100^{100}}}}$ } has cardinality 5 so is therefore countable
4. **Z**: This one is more interesting. Continue on next page:

Countability of the Integers

Let's try to set up a bijection between **N** and **Z**. One way is to just write a sequence down whose pattern shows that every element is hit (onto) and none is hit twice (one-to-one). The most common way is to alternate back and forth between the positives and negatives. I.e.:

0,1,-1,2,-2,3,-3,…

It's possible to write an explicit formula down for this sequence which makes it easier to check for bijectivity:

$$a_i = -(-1)^i \left\lfloor \frac{i+1}{2} \right\rfloor$$

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Demonstrating Countability. Useful Facts

Because $\aleph_0$ is the smallest kind of infinity, it turns out that to show that a set is countable one can either demonstrate an injection into **N** or a surjection from **N**.

THM: Suppose A is a set. If there is an one-to-one function $f : A \rightarrow$ **N**, or there is an onto function $g :$ **N** $\rightarrow A$ then A is countable.

The proof requires the principle of mathematical induction, which we'll get to at a later date.

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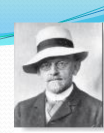
Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1),\ a_2 = f(2), \ldots,\ a_n = f(n), \ldots$

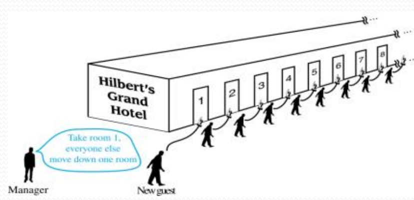
Hilbert's Grand Hotel



David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

Showing that a Set is Countable

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots \end{array}$$

Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$. ◀

Showing that a Set is Countable

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots\ldots$

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

◀

The Positive Rational Numbers are Countable

- **Definition**: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution:The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. →

The Positive Rational Numbers are Countable

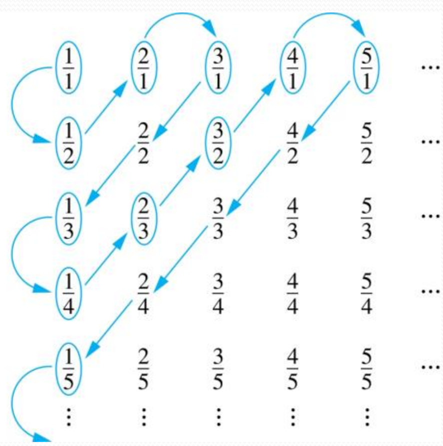
First row $q = 1$.
Second row $q = 2$.
etc.

Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

Terms not circled are not listed because they repeat previously listed terms



1, ½, 2, 3, 1/3,1/4, 2/3, ...

Strings

Example 4: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from **N** to S and hence it is a countably infinite set. ◄

The set of all Java programs is countable.

Example 5: Show that the set of all Java programs is countable.

Solution: Let S be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable. ◄

Uncountable Sets

But **R** is uncountable ("not countable")

Q: Why not ?

Uncountability of **R**

A: This is not a trivial matter. Here are some typical reasonings:

1. **R** strictly contains **N** so has bigger cardinality. What's wrong with this argument?
2. **R** contains infinitely many numbers between any two numbers. Surprisingly, this is not a valid argument. **Q** has the same property, yet *is* countable.
3. Many numbers in **R** are infinitely complex in that they have infinite decimal expansions. An infinite set with infinitely complex numbers should be bigger than **N.**

Uncountability of **R**

Last argument is the closest.

Here's the real reason: Suppose that **R** were countable. In particular, any subset of **R**, being smaller, would be countable also. So the interval [0,1] would be countable. Thus it would be possible to find a bijection from $\mathbf{Z}^+$ to [0,1] and hence list all the elements of [0,1] in a sequence.

What would this list look like?

$$r_1, r_2, r_3, r_4, r_5, r_6, r_7, \ldots$$

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Uncountability of **R**
Cantor's Diabolical Diagonal

So we have this list

$$r_1, r_2, r_3, r_4, r_5, r_6, r_7, \ldots$$

supposedly containing *every* real number between 0 and 1.

Cantor's diabolical diagonalization argument will take this supposed list, and create a number between 0 and 1 which *is not* on the list. This will contradict the countability assumption hence proving that **R** is not countable.

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.							
r_2	0.							
r_3	0.							
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.							
r_3	0.							
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	1	1	1	1	1	1
r_3	0.							
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	1	1	1	1	1	1
r_3	0.	2	5	4	2	0	9	0
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	1	1	1	1	1	1
r_3	0.	2	5	4	2	0	9	0
r_4	0.	7	8	9	0	6	2	3
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	5	1	1	1	1	1
r_3	0.	2	5	4	2	0	9	0
r_4	0.	7	8	9	0	6	2	3
r_5	0.	0	1	1	0	1	0	1
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	5	1	1	1	1	1
r_3	0.	2	5	4	2	0	9	0
r_4	0.	7	8	9	0	6	2	3
r_5	0.	0	1	1	0	1	0	1
r_6	0.	5	5	5	5	5	5	5
r_7	0.							
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	5	1	1	1	1	1
r_3	0.	2	5	4	2	0	9	0
r_4	0.	7	8	9	0	6	2	3
r_5	0.	0	1	1	0	1	0	1
r_6	0.	5	5	5	5	5	5	5
r_7	0.	7	6	7	9	5	4	4
:								
r_{evil}	0.							

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Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	7
r_2	0.	1	5	1	1	1	1	1
r_3	0.	2	5	4	2	0	9	0
r_4	0.	7	8	9	0	6	2	3
r_5	0.	0	1	1	0	1	0	1
r_6	0.	5	5	5	5	5	5	5
r_7	0.	7	6	7	9	5	4	4
:								
r_{evil}	0.	5	4	5	5	5	4	5

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Uncountability of **R**

Cantor's Diabolical Diagonal

GENERALIZE: To construct a number not on the list "r_{evil}", let $r_{i,j}$ be the j'th decimal digit in the fractional part of r_i.

Define the digits of r_{evil} by the following rule:

The j'th digit of r_{evil} is 5 if $r_{i,j} \neq 5$. Otherwise the j'th digit is set to be 4.

This guarantees that r_{evil} is an **anti-diagonal**. I.e., it does not share any elements on the diagonal. But every number on the list contains a diagonal element. This proves that it cannot be on the list and contradicts our assumption that **R** was countable so the list must contain r_{evil}. //QED

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Impossible Computation

Notice that the set of all bit strings is countable. Here's how the list looks:

0,1,00,01,10,11,000,001,010,011,100,101,110,111,0000,…

DEF: A decimal number

$$0.d_1d_2d_3d_4d_5d_6d_7\ldots$$

is said to be computable if there is a computer program that outputs a particular digit upon request.

EG:

1. 0.11111111…
2. 0.12345678901234567890…
3. 0.10110111011110….

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Impossible Computations

CLAIM: There are numbers which cannot be computed by any computer.

Proof : It is well known that every computer program may be represented by a bit-string (after all, this is how it's stored inside). Thus a computer program can be thought of as a bit-string. As there are $\aleph_0$ bit-strings yet **R** is uncountable, there can be no onto function from computer programs to decimal numbers. In particular, most numbers do not correspond to any computer program so are incomputable!

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Interesting and Amusing Story

In 2011, Google was bidding on the acquisition of a large set of patents from Nortel.

- Google's first real bid was $1,902,160,540
  - First 10 digits of Brun's constant (reciprocals of twin primes)

$$B \equiv \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots.$$

- Google upped their bid to $2,614,972,128
  - First 10 digits of Mertens' second constant

$$B_1 = \lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p} - \ln\ln x\right).$$

- Google upped their bid to $3,141,592,653
  - First 10 digits of π.

It turns out that Google lost the auction (it went for about $4.5 billion), and they've likely regretted it since.

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