Sequences and Summations

Section 2.4

Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences (*optional*)

Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set *S*.

- The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from {0,1,2,…..} to *S*. We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$

where the *initial term a* and the *common ratio r* are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

   $$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

   $$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

   $$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$

where the *initial term a* and the *common difference d* are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?
[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots$. and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$f_2 = f_1 + f_0 = 1 + 0 = 1,$
$f_3 = f_2 + f_1 = 1 + 1 = 2,$
$f_4 = f_3 + f_2 = 2 + 1 = 3,$
$f_5 = f_4 + f_3 = 3 + 2 = 5,$
$f_6 = f_5 + f_4 = 5 + 3 = 8.$

Solving Recurrence Relations

- Finding a formula for the nth term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$
$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$
$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$
$$\vdots$$
$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\;\;\vdots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

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Financial Application

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

:

$P_n = (1.11)P_{n-1} = (1.11)^n P_0 \quad = \quad (1.11)^n\ 10{,}000$

$P_n = (1.11)^n\ 10{,}000$ (Can prove by induction, covered in Chapter 5)

$P_{30} = (1.11)^{30}\ 10{,}000 = \$228{,}992.97$

Special Integer Sequences (*opt*)

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.
- Some questions to ask?
  - Are there repeated terms of the same value?
  - Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - Can you obtain a term by combining the previous terms in some way?
  - Are they cycles among the terms?
  - Do the terms match those of a well known sequence?

Questions on Special Integer Sequences (*opt*)

Example 1: Find formulae for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16

Solution: Note that the denominators are powers of 2. The sequence with $a_n = 1/2^n$ is a possible match. This is a geometric progression with $a = 1$ and $r = ½$.

Example 2: Consider 1,3,5,7,9

Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is $a_n = 2n + 1$. This is an arithmetic progression with $a = 1$ and $d = 2$.

Example 3: 1, -1, 1, -1,1

Solution: The terms alternate between 1 and -1. A possible sequence is $a_n = (-1)^n$. This is a geometric progression with $a = 1$ and $r = -1$.

Useful Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Guessing Sequences (*optional*)

Example: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: Note the ratio of each term to the previous approximates 3. So now compare with the sequence 3^n . We notice that the nth term is 2 less than the corresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$.

Integer Sequences (*optional*)

- Integer sequences appear in a wide range of contexts. Later we will see the sequence of prime numbers (Chapter 4), the number of ways to order n discrete objects (Chapter 6), the number of moves needed to solve the Tower of Hanoi puzzle with n disks (Chapter 8), and the number of rabbits on an island after n months (Chapter 8).
- Integer sequences are useful in many fields such as biology, engineering, chemistry and physics.
- On-Line Encyclopedia of Integer Sequences (OESIS) contains over 200,000 sequences. Began by Neil Stone in the 1960s (printed form). Now found at http://oeis.org/Spuzzle.html

Integer Sequences (*optional*)

- Here are three interesting sequences to try from the OESIS site. To solve each puzzle, find a rule that determines the terms of the sequence.
- Guess the rules for forming for the following sequences:
  - 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, ...
    - Hint: Think of adding and multiplying by numbers to generate this sequence.
  - 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
    - Hint: Think of the English names for the numbers representing the position in the sequence and the Roman Numerals for the same number.
  - 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
    - Hint: Think of the English names for numbers, and whether or not they have the letter 'e.'
- The answers and many more can be found at http://oeis.org/Spuzzle.html

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \quad \sum\nolimits_{j=m}^{n} a_j \quad \sum\nolimits_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit* *m* and ending with its *upper limit* *n*.

Summations

- More generally for a set *S*:

$$\sum_{j \in S} a_j$$

- **Examples**:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation (*optional*)

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod\nolimits_{j=m}^{n} a_j \qquad \prod\nolimits_{m \le j \le n} a_j$$

represents $a_m \times a_{m+1} \times \cdots \times a_n$

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

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Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$ From previous slide.

$$= \sum_{k=1}^{n+1} ar^k$$ Shifting the index of summation with $k = j + 1$.

$$= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a)$$ Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + (ar^{n+1} - a)$$ Substituting S for summation formula

$$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$ if r ≠1

$$S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$$ if r = 1

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.	
Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)