Induction (Chapter 5)

Announcement:

- HW5 is due now
- HW6 has been posted

Proofs

1. Direct proof

   $P \rightarrow Q$

2. Indirect proof

   $\neg Q \rightarrow \neg P$ $(P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P)$

3. Proof by contradiction

   $(P \wedge \neg Q) \rightarrow (\neg P \vee Q)$

Rational Numbers an Easier Characterization

Recall the set of rational numbers **Q** = the set of numbers with decimal expansion which is periodic past some point (I.e. repeatinginginginginging…)

Easier characterization

Q = { p/q | p,q are integers with $q \neq 0$ }

Prove that the sum of any irrational number with a rational number is irrational:

Reductio Ad Absurdum Example –English!

You don't have to use a sequence of formulas. Usually an *English* proof is preferable! EG:

Suppose that claim is false. So [x is rational and y irrational] [$=P$] and [$x+y$ is rational] [$=\neg Q$]. But $y = (x+y) - x$. The difference of rational numbers is rational since
a/b – c/d = (ad-bc)/bd.

Therefore [y must be rational] [implies $\neg P$]. This contradicts the hypotheses so the assumption that the claim was false was incorrect and the claim must be **true**. □

Proofs Disrefutation

Disproving claims is often much easier than proving them.

Claims are usually of the form $\forall k\ P(k)$. Thus to disprove, enough to find one k –called a ***counterexample***– which makes $P(k)$ false.

Disrefutation by Counterexample

Disprove: The product of irrational numbers is irrational.

1. Let $x = \sqrt{2}$ and $y = \dfrac{1}{\sqrt{2}}$. Both are irrational.

2. Their product $xy = \sqrt{2} \cdot \dfrac{1}{\sqrt{2}} = 1$ is rational.

Overview

Mathematical Induction Proofs

- Well Ordering Principle
- Simple Induction
- Strong Induction (Second Principle of Induction)
- Program Correctness
  - Correctness of iterative Fibonacci program

Mathematical Induction

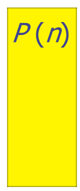
Suppose we have a sequence of propositions which we would like to prove:

$P(0), P(1), P(2), P(3), P(4), \ldots P(n), \ldots$

EG: $P(n) =$

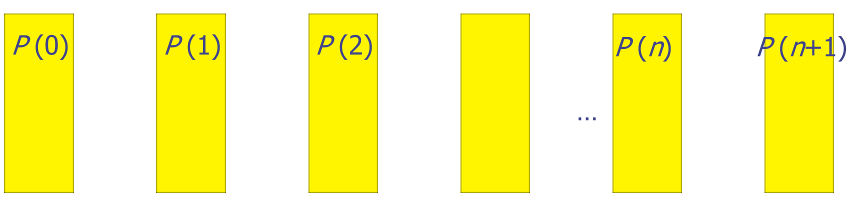
"The sum of the first n positive odd numbers is the n^{th} perfect square"

We can picture each proposition as a domino:



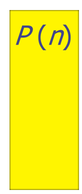
Mathematical Induction

So sequence of propositions is a sequence of dominos.



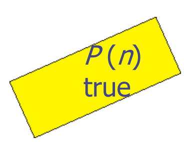
Mathematical Induction

When the domino falls, the corresponding proposition is considered true:



Mathematical Induction

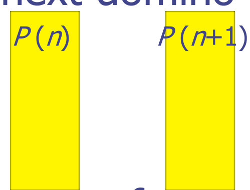
When the domino falls (to right), the corresponding proposition is considered true:



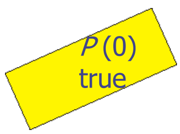
Mathematical Induction

Suppose that the dominos satisfy two constraints.

1) Well-positioned: If any domino falls (to right), next domino (to right) must fall also.



2) First domino has fallen to right



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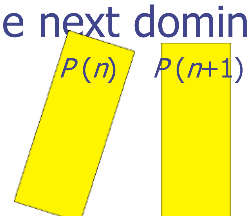

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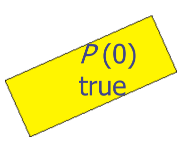

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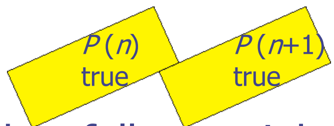

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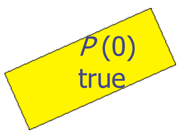
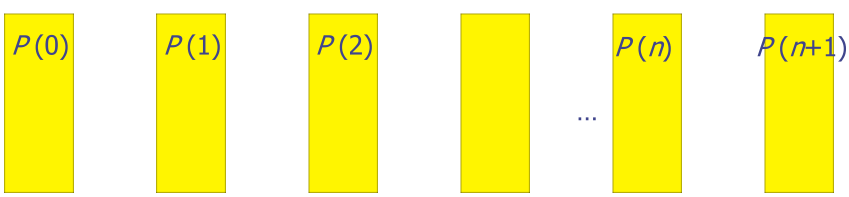


Mathematical Induction

Then can conclude that all the dominos fall!



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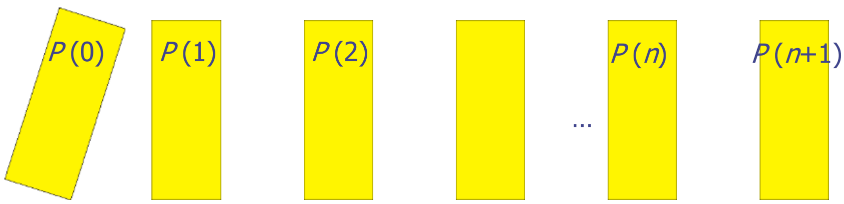

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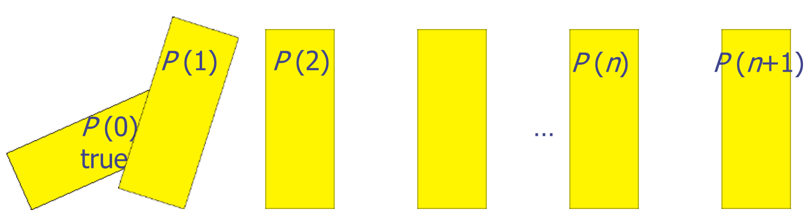

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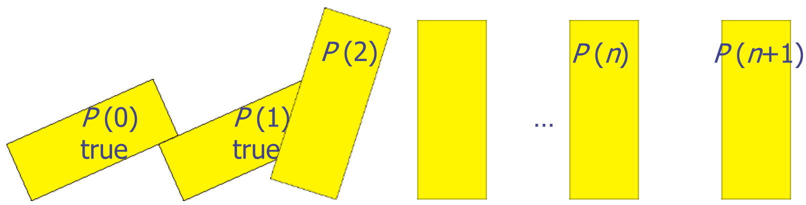

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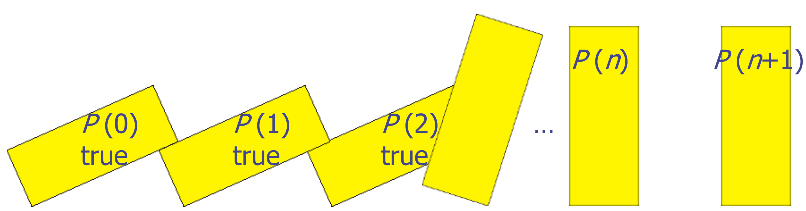

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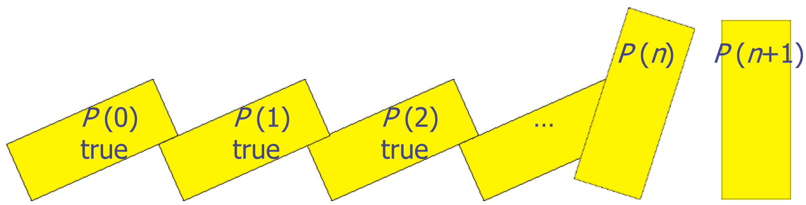

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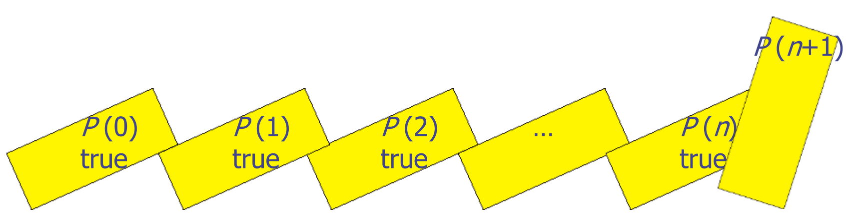

Mathematical Induction

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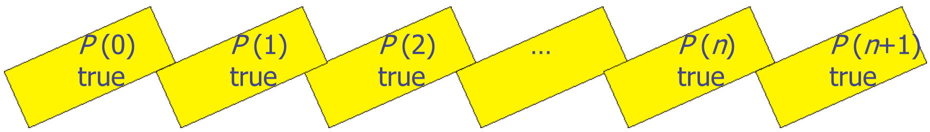


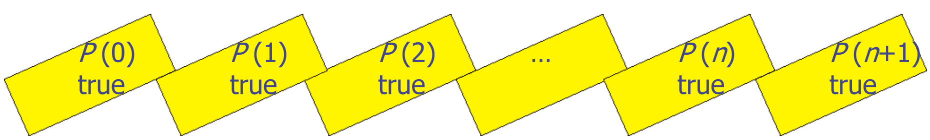
Mathematical Induction

Principle of Mathematical Induction:

If:

1) [***basis***] $P(0)$ is true

2) [***induction***] $\forall n\ \ P(n) \rightarrow P(n+1)$ is true



Then:

$\forall n\ P(n)$ is true

This formalizes what occurred to dominos.

Mathematical Induction Example

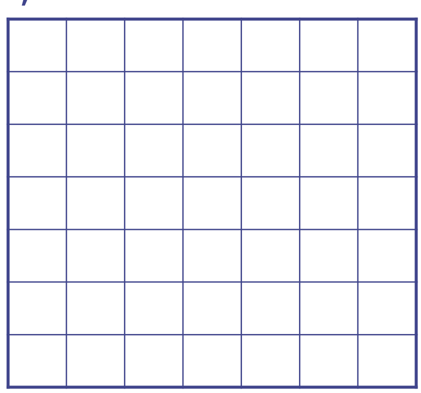
EG: Prove $\forall n \geq 0\ P(n)$ where

$P(n)$ = "The sum of the first n positive odd numbers is the n^{th} perfect square."

$$= \quad \sum_{i=1}^{n}(2i-1) = n^2$$

Mathematical Induction Example.

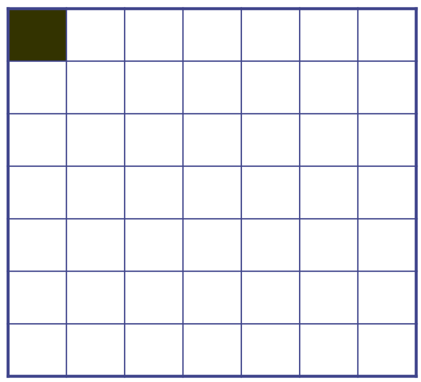
Geometric interpretation. To get next square, need to add next odd number:



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

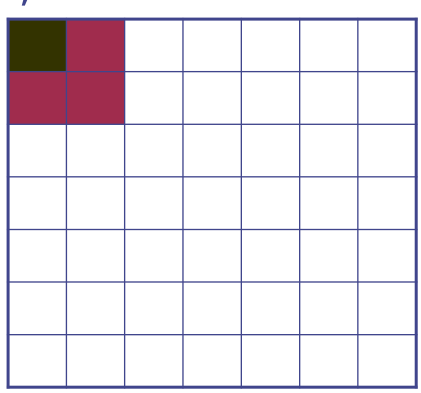
1



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

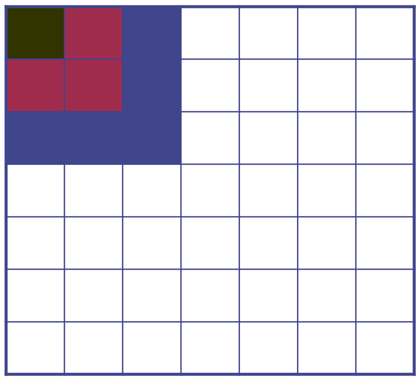
1
+3



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

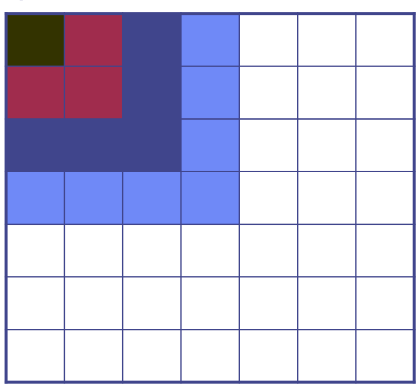
1
+3
+5



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

1
+3
+5
+7



Mathematical Induction Example.

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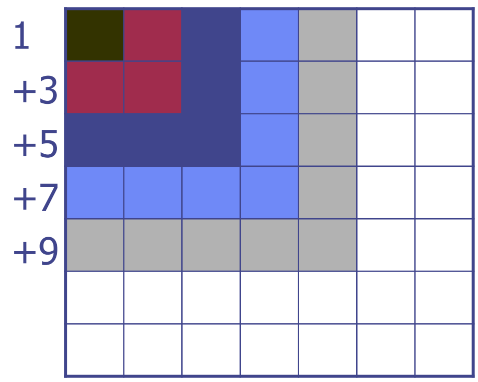

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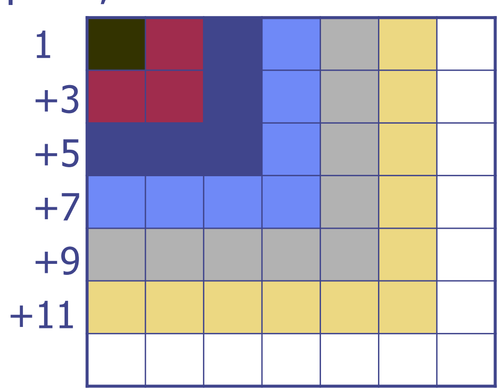
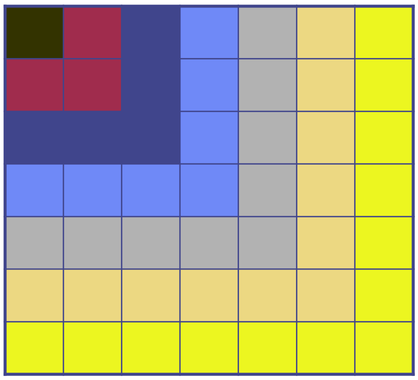


Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

1
+3
+5
+7
+9
+11
+13



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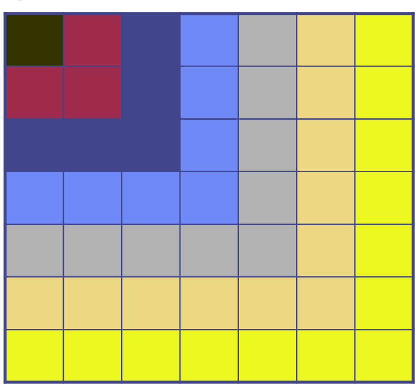

$=7^2$

Mathematical Induction Example

$$\sum_{i=1}^{n}(2i-1)=n^2$$

Every induction proof has two parts, the basis and the induction step.

1) Basis: Show that the statement holds for *n* = 0 (or whatever the smallest case is). Usually the hardest thing about the base case is understanding what is meant when *n*=0 (or smallest case). In our case, plugging in 0, we would like to show that:

$$\sum_{i=1}^{0}(2i-1)=0^2$$

This seems confusing. RULE: The sum of nothing is 0. So apply rule to get 0=0. ✓

Mathematical Induction Example

$$\sum_{i=1}^{n}(2i-1)=n^2$$

2) Induction: Show that if statement holds for *n,* then statement holds for *n*+1. For formulas, this amounts to playing around with formula for *n* and algebraically deriving the formula for *n*+1 (in this case, go in reverse):

$$\sum_{i=1}^{n+1}(2i-1)=\sum_{i=1}^{n}(2i-1)+[2(n+1)-1]$$

$$=n^2+[2n+1] \qquad \text{(induction hypothesis)}$$

$$=(n+1)^2 \text{ ✓}$$

This completes proof. □

Proof of Induction Well Ordering Property

A fundamental axiom about the natural numbers:

Well Ordering Property: Any *non-empty* subset S of **N** has a smallest element!

Q1: What's the smallest element of the set $\{ 16.99+1/n \mid n \in \mathbf{Z}^+ \}$?

Q2: How about $\{ \lfloor 16.99+1/n \rfloor \mid n \in \mathbf{Z}^+ \}$?

Proof of Induction Principle Well Ordering Property

A1: $\{ 16.99+1/n \mid n \in \mathbf{Z}^+ \}$ doesn't have a smallest element (though it does have limit-point 16.99)! Well-ordering principle does not apply to subsets of **R**.

A2: 16 is the smallest element of $\{ \lfloor 16.99+1/n \rfloor \mid n \in \mathbf{Z}^+ \}$.

(EG: set $n = 101$)