$$y = Ax \Rightarrow A^T y = A^T A x = x + \underbrace{(A^T A - I)}_{B} x$$

Let $A \in \mathbb{R}^{n \times p}$ & $A_{ij} \sim N(0, \frac{1}{n})$.

If $l \neq m$

$$B_{lm} = \sum_{i=1}^{n} A_{il} A_{im} \Rightarrow \mathbb{E}(B_{lm}) = 0$$ Is it clear why?

$$\mathbb{E}(B_{lm})^2 = \sum_{i=1}^{n} \sum_{i'=1}^{n} \mathbb{E}(A_{il} A_{im} A_{i'l} A_{i'm})$$

$$= \sum_{i=1}^{n} \mathbb{E}(A_{il}^2 A_{im}^2) + \underbrace{\sum_{i \neq i'} \mathbb{E}(A_{il}) \mathbb{E}(A_{im}) \mathbb{E}(A_{i'l}) \mathbb{E}(A_{i'm})}_{0} -$$

$$= n \cdot \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n} .$$

$$B_{ll} = \sum_{i=1}^{n} (A_{il}^2 - \frac{1}{n}) \Rightarrow \mathbb{E} B_{ll} = 0$$

$$\mathbb{E} B_{ll}^2 = \sum_{i=1}^{n} \sum_{i'=1}^{n} \mathbb{E}(A_{il}^2 - \frac{1}{n})(A_{i'l}^2 - \frac{1}{n})$$

$$= \sum_{i=1}^{n} \mathbb{E}(A_{il}^2 - 1/n)^2 + \underbrace{\sum_{i \neq i} \mathbb{E}(A_{il}^2 - \frac{1}{n}) \mathbb{E}(A_{i'l}^2 - \frac{1}{n})}_{0}$$

$$= \sum_{i=1}^{n} \mathbb{E}\left(A_{il}^4 - \frac{2}{n} A_{il}^2 + \frac{1}{n^2}\right) = n\left(\frac{3}{n^2} - \frac{2}{n^2} + \frac{1}{n^2}\right) = \frac{2}{n}.$$

$m \neq m'$

$$\mathbb{E}\left(B_{lm} B_{lm'}\right) = \sum_{i} \sum_{i'} \mathbb{E}\left(A_{il} A_{im} A_{i'l} A_{i'm'}\right)$$

$$= \underbrace{\sum_{i} \mathbb{E}\left(A_{il}^2 A_{im} A_{im'}\right)}_{0} + \underbrace{\sum_{i \neq i'} \mathbb{E}\left(A_{il} A_{im}\right) \mathbb{E}\left(A_{i'l} A_{i'm'}\right)}_{0}$$

$$= 0$$

$$y = x + \underbrace{(A^T A - I) x}_{\omega}$$

$$\omega_i = \sum_{j=1}^{p} B_{ij} x_j \Rightarrow \mathbb{E}\omega_i = \sum_{j=1}^{p} \mathbb{E}(B_{ij}) x_j = 0$$

$$\mathbb{E}\omega_i^2 = \sum_{j=1}^{p} \sum_{j'=1}^{p} \mathbb{E}\left(B_{ij} B_{ij'}\right) x_j x_{j'}.$$

$$= \sum_{j=1}^{p} \mathbb{E}\left(B_{ij}^2\right) x_j^2$$

$$= \mathbb{E}\left(B_{ii}^2\right) x_i^2 + \sum_{j \neq i} \mathbb{E}\left(B_{ij}^2\right) x_j^2$$

$$= \frac{2}{n} \cdot x_i^2 + \frac{1}{n} \sum x_j^2$$

$$= \frac{\|x\|^2}{n} + \frac{1}{n} x_i^2$$

If n is very large $\frac{1}{n} x_i^2 \ll 1$ & hence we

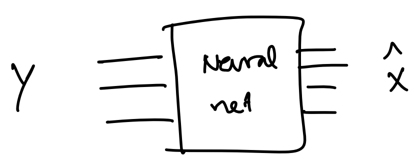
Can approximate : $\mathbb{E}\, w_i^2 \simeq \frac{\|X\|^2}{n}$.

A new experiment :

Normalize x such that $\sum x_i^2 = P$. Then compare two settings :

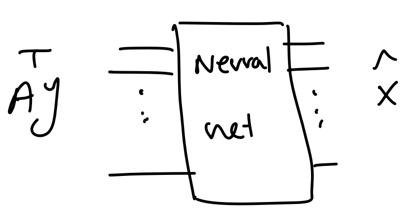
- Setting 1: Denoising problem $Y = X + w$,

  $w \sim N(0, \frac{P}{n} I)$.



$\Rightarrow$ Similar to your first experiment.

- Setting 2: $y = Ax$



$\Rightarrow$ Similar to your Second experiment

& compare the outcomes, for

$n = 0.1P$ $n = 0.2P$ $n = 0.3P$ $n = 0.4P$

How to compare? Please read about PSNR (Peak Signal to noise ratio) & use that for comparisons.