Constrained Preference Elicitation

The five-minute version

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Introduction

A principal wants to learn *something* about the preferences of an agent, but *not* the whole ordering
(Why not? complexity, costs, privacy, etc.)

Example: NYC school match: only list favorite 12 schools

Which properties of preferences can be elicited in an incentive compatible way?

Example:

$X = \{x, y, z\}$. Let xyz denote $x \succ y \succ z$, *e.g.* Assume strict prefs.

All orderings:

$$\{xyz, xzy, zxy, zyx, yzx, yxz\}$$

A simple elicitation mechanism:
Pick from $\{x, y\}$
Paid what you choose

Example:

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All orderings:

$$\{\underbrace{\{xyz, xzy, zxy\}}_{\text{pick } x}, \underbrace{\{zyx, yzx, yxz\}}_{\text{pick } y}\}$$

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All orderings:

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Pick from $\{x, y\}$

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Example:

$X = \{x, y, z\}$. Let xyz denote $x \succ y \succ z$, *e.g.* Assume strict prefs.

All orderings:

$$\overbrace{\{\underbrace{\color{red}{\{xyz, xzy, zxy\}}}_{\text{a "type"}}, \underbrace{\color{blue}{\{zyx, yzx, yxz\}}}_{\text{a "type"}}\}}^{\text{a "type space"}}$$

A simple elicitation mechanism:
Pick from $\{\color{red}{x}, \color{blue}{y}\}$
Paid what you choose

This type space is *elicitable*. Truth FOSD's lie.

$$\{xyz, xzy, zxy, zyx, yzx, yxz\}$$

Mechanism:

Pick from $\{x, y\}$ *and* from $\{x, z\}$

We randomly pick *one* of your answers and pay it to you

$$\{\underbrace{\{xyz, xzy\}}_{\text{pick } x,x}, \underbrace{\{zxy\}}_{\text{pick } x,z}, \underbrace{\{zyx, yzx\}}_{\text{pick } y,z}, \underbrace{\{yxz\}}_{\text{pick } y,x}\}$$

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This type space is *elicitable*. Truth FOSD's lie.

$$\{\underbrace{\{xyz, yxz\}}_{\text{dislike } z}, \underbrace{\{xzy, zxy\}}_{\text{dislike } y}, \underbrace{\{yzx, zyx\}}_{\text{dislike } x}\}$$

$$\{\underbrace{\{xyz, yxz\}}_{\text{dislike } z}, \underbrace{\{xzy, zxy\}}_{\text{dislike } y}, \underbrace{\{yzx, zyx\}}_{\text{dislike } x}\}$$

There are **no** menus that generate this type space.

Generated by top *two* elements of X

$$\{\underbrace{\{xyz, yxz\}}_{\text{dislike } z}, \underbrace{\{xzy, zxy\}}_{\text{dislike } y}, \underbrace{\{yzx, zyx\}}_{\text{dislike } x}\}$$

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But it is elicitable!

$$\{\underbrace{\{xyz, yxz\}}_{\text{dislike } z}, \underbrace{\{xzy, zxy\}}_{\text{dislike } y}, \underbrace{\{yzx, zyx\}}_{\text{dislike } x}\}$$

There are **no** menus that generate this type space.
Generated by top *two* elements of X

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Mechanism:
Announce least favorite,
get paid 50-50 lottery over the other two options.

Results

Main Results:

Generated by top k elements $\Rightarrow$ elicitable $\Rightarrow$ "convex"

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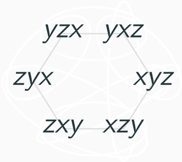

Results

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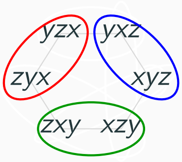
Generated by top k elements $\Rightarrow$ elicitable $\Rightarrow$ "convex"



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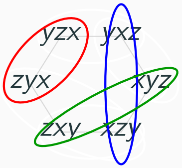

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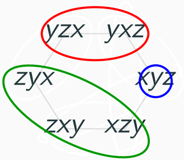

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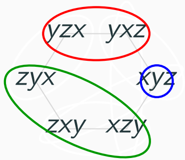

We get complete characterization when:

i. Restrict to neutral type spaces, or

ii. Pay in acts, not lotteries (no objective probabilities)

Questions/Discussion?

Comparison to mechanism design/social choice

Mechanism Design (MD) vs. Constrained Elicitation (CE):

1. Principal's goal is easier
   - MD: Implement desirable outcome. IC $=$ constraint.
   - CE: Don't care about outcomes. $\exists$ *any* IC mechanism?
2. Incentive compatibility is harder
   - MD: Only require weak IC.
   - CE: Require strict IC.
3. Limited info is harder
   - MD: Can elicit entire $\succeq$
   - CE: Can't learn more than prescribed.

Eliciting a property

Why not elicit entire $\succeq$?

- Privacy concerns
- Future interactions
- Costs

Some related literature

- Osband (1985), Lambert-Pennock-Shoham (2008), Lambert (2018): Scoring rules to elicit properties of beliefs
- Gibbard (1977), Bahel and Sprumont (2019): Characterizing strategy-proof random mechanisms
- Carroll (2012), Saito (2013): Sufficiency of local IC constraints
- ACH (2018, 2020): Characterizing IC mechanisms in experimental environments

Framework

- X - a finite set of alternatives
  - Typical elements: $x, y, z, w, \ldots$
- O - the set of strict orders over X
  - Typical elements: $\succeq, \succeq', \ldots$

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  - Typical elements: $x, y, z, w, \dots$
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Definition
A *type space* $T = \{t_1, \dots, t_k\}$ is a partition of O.

- A *type* is any $t \in T$, so $t = \{\succeq, \succeq', ..., \succeq''\}$

- Example: $t = \{\text{all} \succeq \text{ satisfying the Independence axiom}\}$

- Notation: $t(\succeq) \in T$ is the type containing $\succeq$

Examples

$X = \{x, y, z\}$

- Entire ranking:
  $T = \{\{xyz\}, \{yxz\}, \{yzx\}, \{zyx\}, \{zxy\}, \{xzy\}\}$
- First-best:
  $T = \{\{xyz, xzy\}, \{yxz, yzx\}, \{zyx, zxy\}\}$
- Top-2:
  $T = \{\{xyz, yxz\}, \{xzy, zxy\}, \{zyx, yzx\}\}$
- Best from $\{x, y\}$:
  $T = \{\{xyz, xzy, zxy\}, \{zyx, yzx, yxz\}\}$
- Ranking of x:
  $T = \{\{xyz, xzy\}, \{zxy, yxz\}, \{yzx, zyx\}\}$

Mechanisms

$\Delta(X)$ is the set of lotteries on X

Definition
A T-mechanism is any $g : T \to \Delta(X)$.

- Why random payments?
  - With deterministic mechanisms very little can be elicited

Elicitable type spaces

Recall that p strictly FOSD q relative to $\succeq$ ($p \succ^* q$) if

$$\forall x \in X \quad p(\{y : y \succeq x\}) \geq q(\{y : y \succeq x\})$$

with strict inequality for at least one x

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g is *IC* if for every $\succeq \in O$ and every $t \neq t(\succeq)$

$$g(t(\succeq)) \succ^* g(t).$$

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Definition
A type space T is *elicitable* if there exists an IC T-mechanism.

Goal: Characterize elicitable type spaces (spoiler: we can't)

Top elements of menus

"What's your favorite thing from X'?"

- Every menu $X' \subseteq X$ corresponds to a type space:

$$\succeq, \succeq' \in t \iff \succeq, \succeq' \text{ have the same favorite item in } X'$$

Examples:

$X' = \{x, y\} \implies T = \{\{xyz, xzy, zxy\}, \{zyx, yzx, yxz\}\}$

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- The (deterministic) mechanism that pays the revealed top element in X' is IC

RPS mechanisms

- One can elicit top elements of several menus $X_1, \ldots, X_l \subseteq X$

Examples:

$$X_1 = \{x, y, z\},\ X_2 = \{x, y\}$$
$$\Longrightarrow \quad T = \{\{xyz, xzy\}, \{yzx, yxz\}, \{zxy\}, \{zyx\}\}$$

$$X_1 = \{x, y\},\ X_2 = \{x, z\},\ X_3 = \{y, z\}$$
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- The corresponding IC mechanism randomly chooses a menu and pays the announced top element
- This is widely used in experimental economics

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What else is elicitable?

Top sets of menus

The top-2 type space $T = \{\{xyz, yxz\}, \{xzy, zxy\}, \{zyx, yzx\}\}$ does not reveal top elements of menus but is elicitable

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- How? If they announce "x and y" pay x and y with equal probability, and z with less probability.
- Every $X' \subseteq X$ and k defines a type space by

  $$\succeq, \succeq' \in t \iff \succeq, \succeq' \text{ have the same top } k \text{ elements of } X'$$

- This is elicitable by paying the uniform lottery over the set of announced top-k elements
- Can elicit the top-k_i elements of $X_i \subseteq X$, $i = 1, \ldots, I$

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- Can elicit the top-k_i elements of $X_i \subseteq X$, $i = 1, \ldots, l$

Anything else??

Example (based on Shapley, 1971)

$X = \{x, y, z, w\}$

Type space:

$\{xyzw, yxzw, xywz, yxwz\}$

$\{xzyw\}$, $\{xwyz\}$, $\{xzwy, xwzy\}$

$\{ywxz\}$, $\{yzxw\}$, $\{yzwx, ywzx\}$

$\{zxyw, zyxw\}$, $\{zywx, zwyx\}$, $\{zxwy, zwxy\}$

$\{wxyz, wyxz\}$, $\{wyzx, wzyx\}$, $\{wxzy, wzxy\}$

Claim

$\exists$ *IC mechanism, but type space is not generated by top sets.*

There is a close connection between IC mechanisms and convex TU cooperative games...

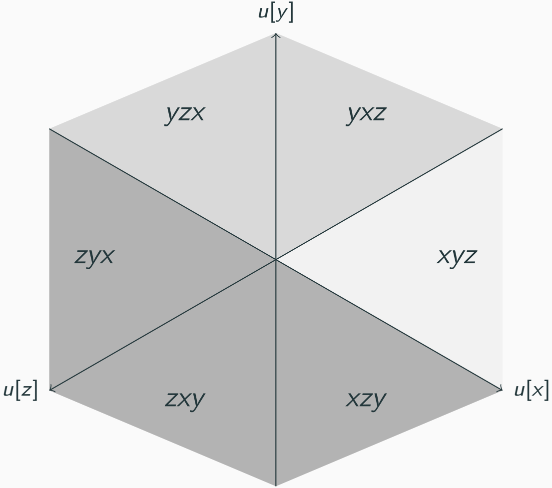
So far...

$$\{\text{all } T\}$$
$$\cup|$$
$$\{T : \text{elicitable}\}$$
$$\cup\!\!\not|$$
$$\{T : \text{generated by top sets}\}$$
$$\cup\!\!\not|$$
$$\{T : \text{generated by top elements}\}$$

A convex type space - example

Necessary condition: **convex** type space

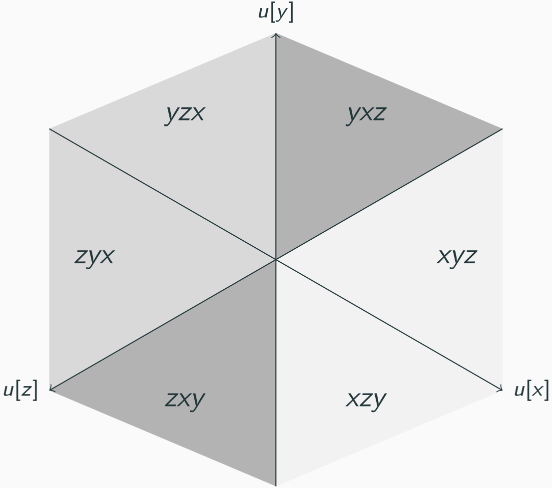
Example: $T = \{\{xyz\}, \{yxz, yzx\}, \{zyx, zxy, xzy\}\}$



A non-convex type space - example

Example of a non-convex type space:

$T = \{\{xyz, xzy\}, \{yxz, zxy\}, \{zyx, yzx\}\}$



Convexity is necessary

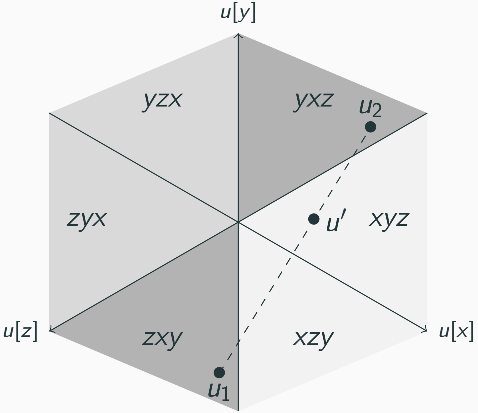
Proposition
If T is elicitable then it is convex.

$t =$ dark gray
$t' =$ off-white

IC requires:

$$\mathbb{E}_{g(t)}(u_1) > \mathbb{E}_{g(t')}(u_1)$$
$$\mathbb{E}_{g(t)}(u_2) > \mathbb{E}_{g(t')}(u_2)$$

$$\implies \quad \mathbb{E}_{g(t)}(u') > \mathbb{E}_{g(t')}(u')$$



Some non-convex type spaces

- The ranking of x (with $|X| \geq 3$)
- The kth ranked alternative for $1 < k < |X|$, e.g. the median
- Any binary $T = \{t_1, t_2\}$, *except* $T = \{\{x \succeq y\}, \{y \succeq x\}\}$. In particular, tests of essentially any axiom of preferences

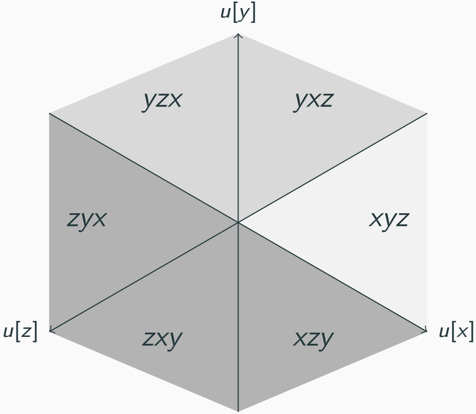
Convexity is not sufficient

$$T = \{t_1 = \{xyz\}, t_2 = \{yxz, yzx\}, t_3 = \{zyx, zxy, xzy\}\}$$

<ins>IC requires:</ins>

$$g(t_1)(x) > g(t_2)(x)$$
$$g(t_2)(x) = g(t_3)(x)$$
$$g(t_3)(x) = g(t_1)(x)$$

$$\implies \quad g(t_1)(x) > g(t_1)(x)$$



Summary

$$\begin{array}{c}
\{\text{all } T\} \\
\rotatebox[origin=c]{-90}{$\supsetneq$} \\
\{T : \text{convex}\} \\
\rotatebox[origin=c]{-90}{$\supsetneq$} \\
\{T : \text{no bad cycles}\} \\
\rotatebox[origin=c]{-90}{$\supseteq$} \\
\color{red}{\{T : \text{elicitable}\}} \\
\rotatebox[origin=c]{-90}{$\supsetneq$} \\
\{T : \text{generated by top sets}\} \\
\rotatebox[origin=c]{-90}{$\supsetneq$} \\
\{T : \text{generated by top elements}\}
\end{array}$$

Neutral type spaces

- Permutation: $\pi : X \to X$
- Let πT be T, but with every $\succeq$ permuted by π

Definition
T is *neutral* if $\pi T = T$ for every π.

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Definition
T is *neutral* if $\pi T = T$ for every π.

Proposition
Suppose T is neutral. Then the following are equivalent:

(1) T is elicitable

(2) T is convex

(3) T is generated by top sets

Neutral type spaces

$$\begin{array}{c}
\{\text{all } T\} \\
\rotatebox{90}{$\subsetneq$} \\
\{T : \text{convex}\} \\
\rotatebox{90}{$=$} \\
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\rotatebox{90}{$=$} \\
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Robust elicitation

What if the agent has subjective beliefs about the likelihood of realizations of the randomization device (or other kinds of preferences over acts)?

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- Now payments are Savage acts, not lotteries.
- Can't pay a uniform lottery b/c can't control beliefs
- This kills our ability to elicit top sets

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What if the agent has subjective beliefs about the likelihood of realizations of the randomization device (or other kinds of preferences over acts)?

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Proposition
T is elicitable with acts iff it is generated by top elements.

Elicitation under acts

$$\{\text{all } T\}$$
$$\cup\!\!\!\!\not\;\shortparallel$$
$$\{T : \text{convex}\}$$
$$\cup\!\!\!\!\not\;\shortparallel$$
$$\{T : \text{elicitable with lotteries}\}$$
$$\cup\!\!\!\!\not\;\shortparallel$$
$$\{T : \text{generated by top sets}\}$$
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$$\{T : \text{generated by top elements}\}$$
$$\shortparallel$$
$$\{T : \text{elicitable with acts}\}$$

Multiple agents

- $N = \{1, \ldots, n\}$ - agents
- T_i - agent's i type space
- $T = (T_1, \ldots, T_n)$ - a profile of type spaces
- $g : T \to \Delta(X)$ - a mechanism

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- $g : T \to \Delta(X)$ - a mechanism

Proposition
$T = (T_1, \ldots, T_n)$ *is dominant-strategy-elicitable iff each* T_i *is elicitable.*

Conclusion

- We formulate a notion of elicitability for properties of preferences
- Some necessary conditions and some sufficient conditions for elicitability, but no characterization
- We do have a characterization for neutral type spaces and for robust elicitation (acts)
- Potential extensions: Weak orders, infinite sets of alternatives, domain restrictions,...

Thank You!