

# **ExpEcon Methods: MLE, Finite Mixture Models, & Model Selection**

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ECON 8877

P.J. Healy

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1. Model estimation via MLE: how to code it Finite mixture models
2. Model selection: Cross-validation vs. BIC vs. AIC

# Maximum Likelihood Estimation

## Likelihood function

- $y$ : random variable,  $\theta$ : set of parameters
- $f(\mathbf{y}|\theta)$ : pdf,  $\theta$  identifies possible DGPs (true models)
- The joint density of  $n$  i.i.d. observations from this process

$$f(y_1 \dots y_n | \theta) = \prod_{i=1}^n f(y_i | \theta)$$

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$$f(y_1 \dots y_n | \theta) = \prod_{i=1}^n f(y_i | \theta) = L(\theta | \mathbf{y})$$

- $L(\theta | \mathbf{y})$ : function of the unknown parameter vector,  $\theta$ ,  
given observed data  $\mathbf{y}$

# Maximum Likelihood Estimation

Example: Behavioral game theory models

- 2-player,  $3 \times 3$  game
- $S_i$  set of strategies,  $s_i \in S_i$
- $\sigma_i(s_i)$ :  $i$ 's probability of playing  $s_i$  in mixed strategy  $\sigma_i(\cdot)$
- $u_i(\sigma_1, \sigma_2) = \sum_{(s_1, s_2)} \sigma_1(s_1) \sigma_2(s_2) u_i(s_1, s_2)$
- Some models use deterministic best response:

$$BR_i(\sigma_j) = \arg \max_{s_i} u_i(s_i, \sigma_j)$$

(assume unique for simplicity here)

- Others assume noisy behavior, like logistic response:

$$LR_i(\sigma_j | \lambda_i)(s_i) = \frac{\exp(\lambda u_i(s_i, \sigma_j))}{\sum_{s'_i \in S_i} \exp(\lambda u_i(s'_i, \sigma_j))}$$

# Maximum Likelihood Estimation

## Model 1: Level- $k$ with logistic trembles

- Observed data: a played strategy
- 3 parameters:  
 $k$  (hierarchy level),  $\epsilon$  (prob. tremble),  $\lambda$  (precision parameter)
- Base model ( $k$ ):  $\sigma_i^{LK}(\cdot | k = 0) = U[S_i]$  is uniform, then  $\forall k > 0$

$$\sigma_i^{LK}(s_i | k) = 1 \text{ iff } s_i = BR_i \left( \sigma_j^{LK}(\cdot | k - 1) \right)$$

- Problem: deterministic model. Zero likelihood possible.
- Solution ( $\epsilon, \lambda$ ): modify the model with logistic trembles:

$$f^{LK}(s_i | \epsilon, \lambda, k) = (1 - \epsilon) \mathbb{1}_{\{s_i = BR_i(\sigma_j^{LK}(\cdot | k - 1))\}} + \epsilon \cdot LR_i(\sigma_j^{LK}(\cdot | (k - 1)) | \lambda)(s_i)$$

# Maximum Likelihood Estimation

## Example

### QRE

- One parameter:  $\lambda$
- Model: defined by fixed point. For each  $i$ ,

$$\sigma_i^{QRE}(s_i|\lambda) = LR_i(\sigma_j^{QRE}(\cdot|\lambda)|\lambda, S_i)(s_i)$$

$$f^{QRE}(s_i|\lambda) = \sigma_i^{QRE}(s_i|\lambda)$$

Data:  $s_i^g$  for games  $g \in \mathbf{G} = \{1, 2, 3 \dots, G\}$ .

$$f^{LK}(\mathbf{s}|\epsilon, \lambda, k) = \prod_{g \in \mathbf{G}} f^{LK}(s_i^g|\epsilon, \lambda, k) = L^{LK}(\epsilon, \lambda, k|\mathbf{s})$$

and

$$f^{QRE}(\mathbf{s}|\lambda) = \prod_{g \in \mathbf{G}} f^{QRE}(s_i^g|\lambda) = L^{QRE}(\lambda|\mathbf{s})$$

# Maximum Likelihood Estimation

Maximum Likelihood Estimation of a single model (eg, noisy LK)

1. Let's allow different parameters for each subject. So, fix  $i$
2. Set a 3D grid of  $(\epsilon, \lambda, k)$  values
3. For each point on that grid calculate

$$\begin{aligned} f^{LK}(\mathbf{s}|\epsilon, \lambda, k) &= \prod_{g \in \mathbf{G}} f^{LK}(s_i^g|\epsilon, \lambda, k) \\ &= \sum_{g \in \mathbf{G}} \log(f^{LK}(s_i^g|\epsilon, \lambda, k)) \end{aligned}$$

4. MLE estimate:  $(\epsilon^*, \lambda^*, k^*)$  with highest value

Model Selection:

- Which model has higher likelihood at its MLE parameter?
- Problem: unfair advantage having more parameters
- Solutions: penalize MLE by subtracting param. penalty
  - Akaike Info Criterion (AIC) vs. Bayesian Info Criterion (BIC)



# Maximum Likelihood Estimation

## Example

Game 1	T	M	B	Game 2	T	M	B	Game 3	T	M	B
T	25	30	100	T	30	50	100	T	10	100	40
M	40	45	65	M	40	45	10	M	0	70	50
B	31	0	40	B	35	60	0	B	20	50	60

- Suppose that a subject plays M, T, B
- Let  $\lambda \in \{0.01, 0.05, 1\}$
- $\log(L^{QRE}(0.01|(M, T, B))) = \log(0.3628) + \log(0.3914) + \log(0.3311)$
- $\log(L^{QRE}(0.05|(M, T, B))) = \log(0.5391) + \log(0.5355) + \log(0.3518)$
- $\log(L^{QRE}(1|(M, T, B))) = \log(1) + \log(0.7024) + \log(0.9999)$
- Thus, in this example  $\hat{\lambda} = 1$

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  - For example, Georganas et al., (2015) show that a cognitive hierarchy is not persistent across classes of games
  - Suggests that estimating with *one* hierarchy cannot be valid.
  - Another example at the population level, people might have different risk preferences, etc.,

# Finite Mixture Model

## "Mix" models

- $m = 1, 2, \dots, M$  denotes model
- $f(\mathbf{y}|\psi) = \sum_{m=1}^M \pi_m f_m(\mathbf{y}|\theta_m)$ , where  $\psi = (\{\theta_m\}_{m=1}^M, \pi_1, \pi_2, \dots, \pi_M)$
- Usually,  $f_m(\mathbf{y}, \theta_m)$  (called component density) are taken to belong to the same parametric family.
  - There are special cases where component densities are taken to be different (nonstandard mixture)
- posterior probability that data is drawn from model  $m$ , given observed data  $\mathbf{y}$  is  $\pi_m \cdot \frac{f_m(\mathbf{y}|\theta_m)}{f(\mathbf{y}|\psi)}$
- A parametric family of densities is primitive. Each component has distinct values
- Using MLE to fitting mixture distributions  $\pi$  (Most commonly used way)

# Model Selection

## Going back to Level- $k$ and QRE Example

- Suppose that an experimenter wants to compare which model better explains data
- They can horse-race models
- Using MLE?
  - For level- $k$ , find the ML estimates  $(\hat{\epsilon}, \hat{\lambda}, \hat{k})$   
plug in those values to the level- $k$  model's likelihood function
  - For QRE, find the ML estimate  $\hat{\lambda}$   
plug in  $\hat{\lambda}$  to the QRE's likelihood function
  - Compare the likelihood values of two models and pick the model that gives the higher value

# Model Selection

What is the problem with ML approach?

- level- $k$  has three parameters, while QRE has only one parameter
- level- $k$  has more “flexibility”
- Consider a weird model with  $\infty$  numbers of parameters
  - $\infty$  flexibility
  - Can explain any behavior in the data
  - Then, this model “wins” just because it has more flexibility, not because it is true DGP.
- Need for fixing the problem of over-fitting due to large # of parameters



# Model Selection

How to penalize over-fitting due to large numbers of parameters?

- AIC (Akaike information criterion)
- BIC (Bayesian information criterion)
- Cross-Validation

# Model Selection: AIC, BIC

- $AIC = 2k \ln(n) - 2 \ln(\hat{L})$
- $BIC = k \ln(n) - 2 \ln(\hat{L})$

where

- $\hat{L}$  = the maximized value of the likelihood function of the model
  - From observed data, get ML estimates, and plug into the ML function
- $n$  = number of observed data point
  - In our example, the number of games subjects played
- $k$  = number of parameters
  - In level- $k$  model,  $k=3$   
In QRE,  $k = 1$

## Model Selection: AIC, BIC

- $AIC = 2k \ln(n) - 2 \ln(\hat{L})$
- $BIC = k \ln(n) - 2 \ln(\hat{L})$
- The preferred model is the one with the minimum AIC/BIC value
- The second term gives benefits to the model with goodness-of-fit
- The first term gives a penalty to the number of parameters
- Those can be only used for linear-models
- Can be used only when  $n \gg k$

# Model Selection: Cross-Validation

- $k$ -fold cross-validations
- Divide data into  $k$  sub-samples
  - For example, 12 data points, 4 sub-samples that include 3 data points each
- $k - 1$  sub-samples = training data
  - Fit the data to a model (MLE, MSE ...)
- one sub-sample = testing data
  - Using the fitted parameters from training data, test the model i.e., Plug in the estimated parameter to the goodness-of-fit function used for training (MLE, MSE, ...)
- Repeat this for  $K$  times
- Extreme case of  $K$  fold cross-validation is leave-one-out cross-validation that  $K = n$ , where  $n$  = number of data points

# Model Selection: Cross-Validation

How does Cross-Validation penalize the number of parameters?

Consider the following example..

- Suppose that subjects played four games
- For three games, a subject's choices coincide with the level- $k$  level-1's predictions
- Then  $\hat{\epsilon} = 0$  and log-likelihood function value is 0.
- Suppose that the subject did not play level-1's predicted strategy.
- Then for the testing data (fourth game),  $\hat{\epsilon} = 0$  results in  $-\infty$
- For QRE, less likely to over-fit (since it has only one parameter).  
Less likely to have  $-\infty$  for testing data

## Model Selection: Cross-Validation

- Cross-Validation penalizes the number of parameters internally.
- Over-fitting due to a higher number of parameters penalizes deviation from the prediction a lot in the testing data
- No restriction for models being tested; does not have to be linear

# Model Selection

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- NO!
- The penalization can be "too" severe
- See Healy & Park (2023) for suggestions :)