

ExpEcon Methods: Resampling Methods Permutation Tests & Bootstrapping

ECON 8877

P.J. Healy

First version thanks to Sam Stelnicki

Updated 2026-01-28

Permutation Tests

Example: The Permutation Test

How it works:

<https://www.jwilber.me/permutationtest/>

Permutation Test

- Fisher (1935), Pitman (1937,1938)
- Resampling method where we use our data in different orders (without replacement) to test for differences between populations
- Example was for sample means
- Could do exact same for sample medians, modes, variances...
 - Any statistic of a sample!

Assumptions & Properties

- Only assumption: observations are exchangeable
 - Joint dist'n: $F(Y_1, Y_2, Y_3) = F(Y_3, Y_1, Y_2)$
 - Same marginals, “symmetric” correlation
 - True if treatments are randomly assigned!
- Permutation test is always **valid**
- The issues are **power** and **exactness**
 - e.g., outliers can affect resampled distributions

Two-Sample Framework

Chung and Romano [2013]

- Sample 1: X_1, \dots, X_m i.i.d. from P
- Sample 2: Y_1, \dots, Y_n i.i.d. from Q
- Let $Z = (Z_1, \dots, Z_N) = (X_1, \dots, X_m, Y_1, \dots, Y_n)$, $N = m + n$
- Model/hypothesis: $(P, Q) \in \mathcal{P}$
- Important example: $\bar{\mathcal{P}} = \{(P, Q) : P = Q\}$
- Permutations: $\pi : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$, $\pi \in \mathbf{G}_N$
- Let $Z_\pi = (Z_{\pi(1)}, \dots, Z_{\pi(N)})$
- Test statistic: $T_{m,n}(Z)$ (eg, $T_{m,n}(Z) = \frac{1}{m} \sum_{i=1}^m X_i - \frac{1}{n} \sum_{i=1}^n Y_i$)
- $T_{m,n}(Z_\pi)$ calculated after permuting via π
- Order all $T_{m,n}(Z_\pi)$: $T_{m,n}^{(1)} \leq T_{m,n}^{(2)} \leq \dots \leq T_{m,n}^{(N!)}$
- Given α , threshold ranking is $k^* = (1 - \alpha)N!$
- Permutation test function:

$$\phi(Z) = \begin{cases} 1 & T_{m,n}(Z) > T_{m,n}^{(k^*)} \\ 0 & T_{m,n}(Z) < T_{m,n}^{(k^*)} \end{cases}$$

Two-Sample Framework

- If $(P, Q) \in \bar{\mathcal{P}} = \{(P, Q) : P = Q\}$ then the test is exact:

$$E_{P,Q}[\phi(X_1, \dots, X_m, Y_1, \dots, Y_n)] = \alpha$$

- But what if we assume $(P, Q) \in \mathcal{P}_0 \supset \bar{\mathcal{P}}$?
 - Permuted data no longer has the same distribution as original
- Test may not even be asymptotically exact
- Example: $\mathcal{P}_0 = \{(P, Q) : \mu(P) = \mu(Q)\}$, $T_{m,n}(Z) = \sqrt{N}(\bar{X}_m - \bar{Y}_n)$
- Romano [1990]: Rejection rate higher than α even with $N \rightarrow \infty$ unless
 1. $m/n \rightarrow 1$ as $N \rightarrow \infty$, or
 2. variances of P and Q are equal
- Unbalanced samples: Rejecting the null might actually be due to different variances, not different means

Chung and Romano [2013] Correction

Chung and Romano [2013] offer a correction:

$$S_{m,n}(Z) = \frac{T_{m,n}(Z)}{V_{m,n}}$$

where

$$V_{m,n}(Z) = \sqrt{\frac{N}{m} \hat{\sigma}_m^2(X_1, \dots, X_m) + \frac{N}{n} \hat{\sigma}_n^2(Y_1, \dots, Y_n)}$$

For testing difference in means:

$$S_{m,n} = \frac{\sqrt{N}(\bar{X}_m - \bar{Y}_n)}{\frac{N}{m} S_X^2 + \frac{N}{n} S_Y^2}$$

where

$$S_X^2 = \frac{1}{m-1} \sum_{i=1}^m (X_i - \bar{X}_m)^2$$

Why? Distribution of $T_{m,n}(Z_\pi)$ is asymptotically normal with mean 0

Chung and Romano [2013]

- Different adjustments for different statistics
- Based on variance of the large-sample distribution, which is approximately normal
- Always divide $T_{m,n}$ by an estimator of that asymptotic variance
- Paper gives guidance for testing medians

Randomization Tests

- You run ALL possible combinations of the data, rather than just a random subset
- Permutation test is a subset of randomization test

- Runs permutation tests for 53 different published AEA papers
- Finds 13-22% fewer significant results than the methods used in the papers
- This increases to 33-49% for multiple effects

Young (2019)

- Runs permutation tests on regression coefficients of the previous paper
- Uses the Wald statistic and t-statistics
- Also runs bootstrap and jackknives for all of these papers

- Key Takeaways: the design of experiments is really important for whether the p-values of resampling vs. statistical testing are similar
- Lots of treatments and interactions allows for more sensitivity to outliers and creates more volatility causing these methods to vary greatly

Bootstrapping

Bootstrapping - Efrom (1979)

Goal: estimate a parameter of a distribution. e.g. median

- Resample your collected n -sized data *with replacement* to produce M samples of n -sized data
- Each data point in your original sample has $\frac{1}{n}$ chance of being chosen
- Plot the distribution of observed parameter values.
- Estimate: mean of bootstrap dist'n
- Standard error: std. deviation of bootstrap dist'n
- Confidence interval: 5th to 95th quantile
- Completely non-parametric

Bootstrapping Assumptions

- For the standard bootstrap method, observations are assumed to be independent
 - Block bootstrapping was developed to deal with correlated data
- Sample data needs to resemble the population its drawn from and sufficiently large
- Do not need to know the real distribution
- Sufficiently large: enough data to get to around 200 samples, but it's better to run as many bootstrap samples until your statistics converge

Bootstrapping Consistency

- As long as the bootstrap variance converges, we have convergence of the entire distribution
 - It seems like the only case where it won't is if the variance is infinite

Using Bootstrapped Estimates

- Bootstrapping itself is not a statistical test, but rather just estimating different parts of a distribution
- You can then use these estimates in a hypothesis test

Bootstrapping Mean

- Compute sample mean. Is this truly the population mean?
- Step 1: Bootstrap some samples of the data
- Step 2: Find the mean of each of these samples
- Step 3: Plot these from smallest to largest to get a distribution of the bootstrapped mean
- Step 4: Find the confidence interval of these means and that's your estimate

Bootstrapping Standard Errors

- Sample standard error may not be enough to give you insight to a statistical test - Monte Carlo Simulation
- Step 1: Bootstrap some samples of data
- Step 2: Calculate the statistic of interest that you want standard errors for i.e. mean
- Step 3: Calculate the standard deviation of each of these statistics
- Step 4: As the number of bootstrapped samples grows, this will become the bootstrapped standard error

Standard Deviation of Bootstrapped Statistics Calculation

$$\hat{\sigma}_B = \left(\frac{\sum_{b=1}^B [\hat{\theta}^*(b) - \hat{\theta}^*(\cdot)]^2}{B-1} \right)^{1/2} \text{ where } \hat{\theta}^*(\cdot) = \frac{\sum_{b=1}^B \hat{\theta}^*(b)}{B}$$

$$\text{As } B \rightarrow \infty, \hat{\sigma}_B \rightarrow \sigma$$

Bootstrapping Difference of Two Sample Means

- You have two samples: X and Y, with n and m observations. You want to know if the means are the same.
- Step 1: Compute $t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}}$
- Step 2: Compute x' and y' , where $x'_i = x_i - \bar{x} + \bar{z}$ and $y'_i = y_i - \bar{y} + \bar{z}$ where \bar{z} is the mean of the joint sample
- Step 3: Draw bootstrapped sample of x' and y' and use those to compute test statistic
- Step 4: p-value = $\frac{\sum_{i=1}^B I[t_i > t]}{B}$
- Basically a permutation test, but sampling with replacement

When to Use?

- When the sample size is too small for clear analysis
- When you do not have a clear understanding of the underlying population distribution

Example

| | | | | | | | | |
|----------------|----|---|---|----|---|---|----|----|
| Original Data: | 13 | 8 | 1 | 11 | 7 | 4 | 15 | 12 |
|----------------|----|---|---|----|---|---|----|----|

Mean: 8.875, std err: 1.6844

Example

| | | | | | | | | |
|----------------|----|----|----|----|----|----|----|----|
| Original Data: | 13 | 8 | 1 | 11 | 7 | 4 | 15 | 12 |
| Bootstrap 1: | 11 | 11 | 15 | 15 | 8 | 11 | 11 | 4 |
| Bootstrap 2: | 4 | 15 | 1 | 4 | 4 | 8 | 13 | 11 |
| Bootstrap 3: | 12 | 1 | 7 | 8 | 15 | 1 | 7 | 4 |
| Bootstrap 4: | 12 | 12 | 7 | 8 | 8 | 1 | 15 | 1 |
| Bootstrap 5: | 15 | 8 | 12 | 1 | 8 | 1 | 7 | 11 |

Means: 10.75, 7.5, 6.875, 8, 7.875 SE: 1.4911

- Stata: `bootstrap, reps(N): X Y Z`
- Matlab: `bootstrp(N,@stat,X,Y)`

```
a = [13 8 1 11 7 4 15 12];  
a_mean = mean(a);  
a_stderror = std(a)/sqrt(length(a));  
  
[a_bootstrap,a_bootstrapdata] = bootstrp(5,@mean,a);  
a_bootstrapSE = std(a_bootstrap);
```

Jackknifing

- Earlier resampling method than bootstrapping, where we no longer use the whole sample for resampling
- Rather, we remove one datapoint and calculate whatever statistic we want using the $n-1$ observations
 - We do this for all possible samples of $n-1$, so we find a statistic removing every possible observation once

Jackknifing Assumptions

- Normally distributed data
 - Small sample sizes may not be normal
- Our resampled values are necessarily correlated

Bootstrap vs. Jackknife

- Jackknife gives a more conservative estimate of standard error, but usually it's not as accurate as the bootstrap
- Jackknife gives same results every time, whereas bootstraps can change

References

EunYi Chung and Joseph P. Romano. Exact and asymptotically robust permutation tests. *The Annals of Statistics*, 41(2):484–507, April 2013. ISSN 0090-5364, 2168-8966. doi: 10.1214/13-AOS1090.

Joseph P. Romano. On the Behavior of Randomization Tests without a Group Invariance Assumption. *Journal of the American Statistical Association*, 85(411):686–692, September 1990. ISSN 0162-1459. doi: 10.1080/01621459.1990.10474928.