Complex Analysis Summary

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0 Preface

This note is for people studying complex analysis, and got lost in the middle with bunch of technical explanations. I wil try my best to be succinct as possible, stating important results (mostly without proof, but a bit of justification).

Warning: This summary note is not a substitute for the lecture note. Make sure you study from lecture note!

1 Complex Plane and Möbius Maps

1.1 Complex Plane and Complex Infinity

We will be working in what's known as the *extended complex plane*. Define the symbol $\mathbb{C}_\infty \coloneqq \mathbb{C} \cup \left\{ \underbrace{\infty}_{\text{Complex Infinity}} \right\}$; that is, I refer to the space of complex numbers and infinity.

Note that in $\mathbb{C}_\infty$, ∞ is different from infinity in real numbers. $\infty \coloneqq \frac{1}{0}$ is a value that is not "larger" or "smaller" than any number (since we are talking about complex number...), but rather a number on a complex plane at a really far distance from origin.

It is **WRONG** to say:

- $\infty \geq a$ for any $a \in \mathbb{C}_\infty$
- $\infty \leq a$ for any $a \in \mathbb{C}_\infty$

However, it is **CORRECT**[1] to say:

- $|\infty| \geq |a|$ for any $a \in \mathbb{C}_\infty$.

∞ is not like a point on $\mathbb{C}$, but rather like a gigantic circle that you can never reach.

1.2 Möbius Maps

Definition 1.1 (Möbius Map)**.** $\psi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a **Möbius map** if:

$$\psi(z) \coloneqq \frac{az+b}{cz+d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonsingular matrix. (This restriction removes the possibility of $\frac{0}{0}$, or trivial maps (eg: Constant function).)

One needs to be careful when defining this function at infinity, but it should be sensible.[2]

[1] Subtlety here: it seems a bit dodgy to say $\infty = \infty$, but this is matter of definition; you won't really encounter this type of "philosophical" problem in your exam.

[2] That said, if you are supposed to define what a Möbius map is, you are **required** to definitions involving infinity as well.

Exercise 1.1 (Composition of two Möbius map is a Möbius map)**.** Show that for two Möbius maps ψ_1, ψ_2, its composition $\psi_1 \circ \psi_2$ is also a Möbius map.

Remark 1.1. Consider the 2×2-matrix-to-Möbius-map map as follows:

$$f(A) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$$

Then it turns out that $f(AB) = f(A)f(B)$

Exercise 1.2 (Decomposition of Möbius maps)**.** It turns out that Möbius maps can be written as composition of

- translation
- dialation ("scaling by nonzero constant")
- inversion ($z \mapsto \frac{1}{z}$)

Prove this. (Hint: You can do a constructive proof.)

Möbius maps also has a very convenient property:

Exercise 1.3 (Circline to Circline)**.** Show that Möbius maps map circlines to circline. (This means a line will either map to a circle or a line, and also a circle will either map to a circle or a line.)

(Note: This is a boring long tedious proof, that probably won't be asked in exam, but don't take my word for it.)

2 Complex Differentiability

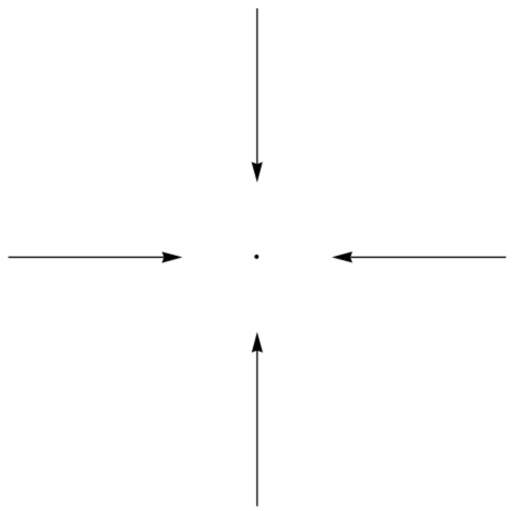
Complex differentiability is one of the highlights of the complex analysis.

Definition 2.1 (Differentiable Function AKA Holomorphic Function)**.** Take $a \in \mathbb{C}$. Let $f : U \to \mathbb{C}$ be a function where U is a neighbourhood[3] of a. Then f is **(complex) differentiable** or **holomorphic** at a if

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and call it derivative of f at a. If f is differentiable for all points in U, then it is said to be differentiable/holomorphic on U.

(a) Real Limit



(b) Complex Limit

Figure 2.1: Real limit (former) only concerns the approaching value from left and right side, but complex limit (latter) oncerns the approaching value from all direction.

Remark 2.1. Note that the definition seems to be have trivially extended from real analysis. <u>However, there is a subtlety</u>. The limit does not approach just from positive or negative side, but from any direction. (Figure 2.1)

Exercise 2.1 (Differentiation Rules)**.** Show that all differentiation rules from real analysis holds with holomorphic functions.

- Sum
- Product Rule
- Quotient Rule
- Chain Rule

Due to the definition of complex limits being more restrictive, a more nontrivial result follows.

Exercise 2.2 (Cauchy-Riemann Equations)**.** Let $a \in \mathbb{C}$ and U be a neighbourhood of a. $f : U \to \mathbb{C}$ be holomorphic a. Write $f(z) = u(x, y) + iv(x, y)$ where u, v are real functions and $z = x + iy$ where $x, y \in \mathbb{R}$. Then $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ all exist, and the following **Cauchy-Riemann equations** hold:

$$\partial_x u = \partial_y v$$
$$\partial_x v = -\partial_y u$$

(Hint: Figure 2.1 might give you an insight.)

Remark 2.2. If Cauchy-Riemann does not hold, then it must mean that f is not holomorphic! (Consider $f(z) = \bar{z}$. Cauchy-Riemann does not hold for any point, so it is nowhere holomorphic.)

Exercise 2.3. If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on U, and u, v are twice differentiable, deduce that u and v are *harmonic*, that is, they satisfy the Laplace equation $\Delta u = \Delta v = 0$.

Remark 2.3. It turns out complex plane reveals a lot about solving Laplace equation!

Here is another kicker:

Exercise 2.4 (Cauchy-Riemann to Holomorphic)**.** If the partial derivatives exist and are continuously differentiable, Cauchy-Riemann implies holomorphicity.

[3]Some open set containing a.

Remark 2.4. This means if you check that Cauchy-Riemann holds, you can immediately assume you can construct an analytic function!

Holomorphic functions also have Taylor expansion:

Remark 2.5 (Holomorphic functions have Taylor expansion)**.** If f is holomorphic at a, then in a neighbourhood of a, you can write

$$f(z) = \sum_{n=0}^{\infty} c_n \left(z - a\right)^n$$

All the formulae for Taylor expansion holds (term-by-term differentiation, etc.)

Example 2.1 (Common Function Definitions)**.** Here are definitions for some of the functions.

$$\begin{aligned}
e^z = \exp z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\cos z &:= \sum_{n=0}^{\infty} \left(-1\right)^n \frac{z^{2n}}{(2n)!} \\
\cos z &:= \sum_{n=0}^{\infty} \left(-1\right)^n \frac{z^{2n+1}}{(2n+1)!}
\end{aligned}$$

Exercise 2.5. Show that

$$\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i}
\end{aligned}$$

Exercise 2.6. Show that $\exp\left(z + w\right) = \exp(z)\exp(w)$

3 Branch Cut

Sometimes, there is just no sensible way to define a function that it is holomorphic everywhere... Two of the unfortunate (or fortunate) functions is the logarithm and square root. We will first introduce the logarithm function.

Define logarithm function as:

$$\log z := \log|z| + i\theta$$

where θ is the argument of z.

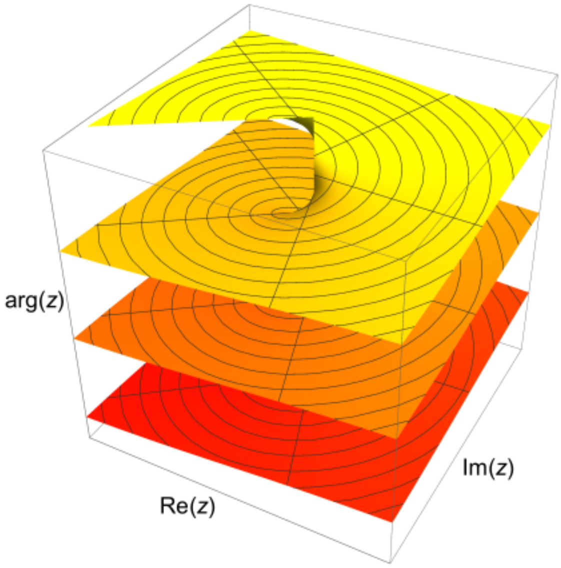


Figure 3.1: $\arg z$ being multivalued results in the need to introduce branch cut for logarithm.

Exercise 3.1. Verify that $\exp(\log z) = z$.

The choice of the interval for θ changes the behaviour of $\log z$ function. For example, one could take the interval to be $[0, 2\pi)$, or one could take it to be $[-\pi, \pi)$, or even just $[a, a + 2\pi)$ for some $a \in \mathbb{R}$.

The problem is that for given z, the argument of z is not unique, and if you try to define it continuously around a circle, you will find that it is not possible. . . (Figure 3.1). This means there needs to be some sort of contour from 0 that the function is not continuous on. This is known as a **branch cut**, and you have total freedom to choose based on what problem you want to solve.

Example 3.1 (Where do we use branch cut?)**.** If you want to solve a problem with a fracture in an elastic solid (Figure 3.2), one standard way to solve it is to find some holomorphic function outside of the crack $[-c, c]$ satisfying some condition. It turns out that there is no function that is holomorphic everywhere satisfying that, so you would define a “branch cut” to be the straight line $[-c, c]$ to resolve it. Then it is possible to define a function that is continuous away from the crack.

Remark 3.1. When I say I am defining a branch cut, it means I am defining the function to be continuous away from the branch cut. **I am not the value of the function** at the function.

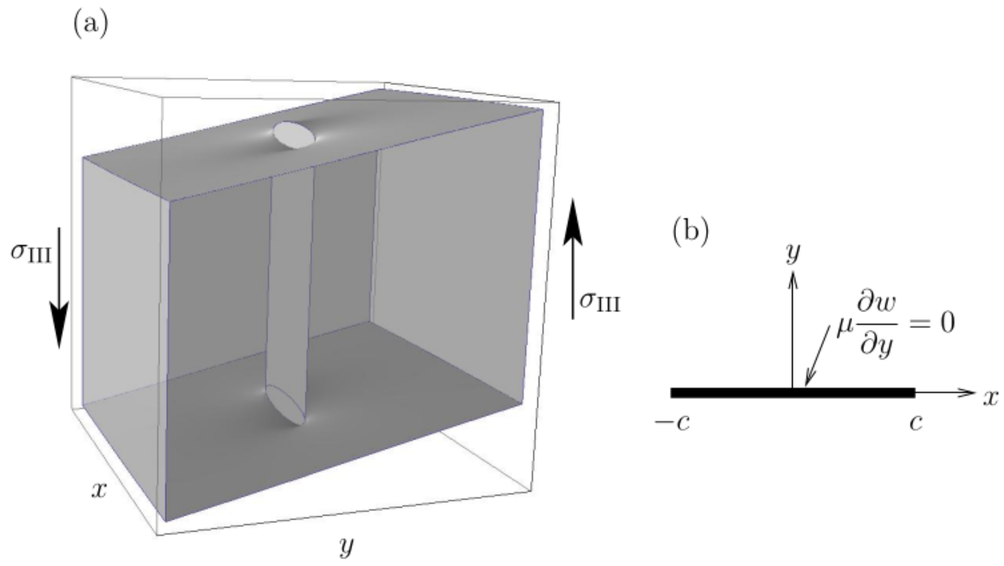


Figure 3.2: Fracture in elastic material.

Branch cuts are something that honestly makes more sense once you **played around with it for a while**.

Example 3.2 (Logarithm: branch cut along positive real axis)**.** See Figure 3.3 for the diagram of a branch cut along positive real axis. $\log z := \log|z| + \theta i$ where $\theta \in [0, 2\pi)$ has a branch cut along positive real axis. Right above the positive real axis, θ takes the value 0, so $(\log x)_+ = \log|x|$. On the other hand, right below the positive real axis, θ takes the value 2π, so $(\log x)_- = \log|x| + 2\pi i$

So you might ask: *What is* $\log z$ *where* $z \in \mathbb{R}^{>0}$*?* and the answer is, you are asking the wrong question, because we can only define the "limiting value" on each side of the branch cut, NOT on the branch cut.

Example 3.3 (Logarithm: branch cut along positive real axis)**.** This time take $\theta \in [-\pi, \pi)$. On the right side of the branch cut, we have $\theta = \pi$, so $(\log(yi))_+ = \log y + \pi i$, whereas on the left side, we have $\theta = -\pi$, so $(\log(yi))_- = \log y - \pi$.

Exercise 3.2 (Square Root)**.** Consider the definition of square root as given:

$$z^{1/2} := |z|^{1/2} e^{i\theta/2}$$

(Note that I've just taken "half" as exponent in the polar form.)

Define the branch cut along the negative real axis and evaluate $i^{1/2}$ in this branch. Define the branch cut along the negative imaginary axis and evaluate it again.

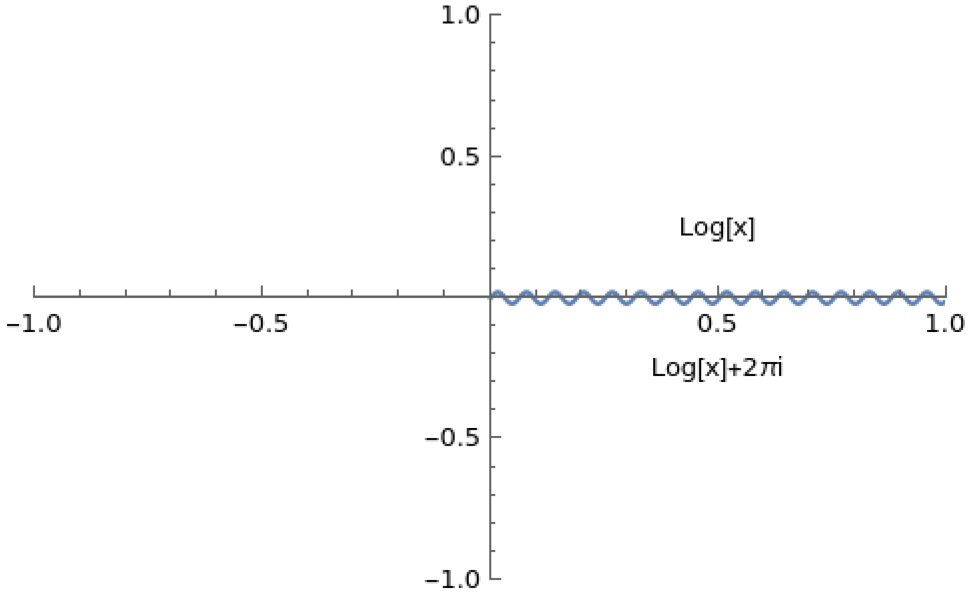


Figure 3.3: Logarithm function with branch cut along the positive real axis

Remark 3.2. In both square roots and logarithms, there was a point which the branch cut naturally starts from. These points are called **branch point**; these are the points that you cannot avoid having a branch cut.

Remark 3.3. There is absolutely no need for a branch cut to be a straight line, and in some cases, it is more natural to define the branch cut in some other way (out of scope, however) Figure 3.5 is a classic example of non-straight branch.

Exercise 3.3. Try defining some branch cuts of $(1+z)^{1/2}$. What is the branch point in this case?

Example 3.4 (Square Root Branch Cut of Elastic Crack)**.** Suppose you want to define the branch of the function $(z^2-1)^{1/2}$ such that it is holomorphic away from the branch cut $[-1,1]$.

Consider writing the function in a following way:

$$\left(z^2-1\right)^{1/2} = (z-1)^{1/2}\,(z+1)^{1/2}$$

Now note that $(z-1)^{1/2}$ and $(z+1)^{1/2}$ need branch cuts. One way to define them is through:

$$\begin{aligned}(z-1)^{1/2} &= r_1^{1/2} e^{i\theta_1/2}\\ (z+1)^{1/2} &= r_2^{1/2} e^{i\theta_2/2}\end{aligned}$$

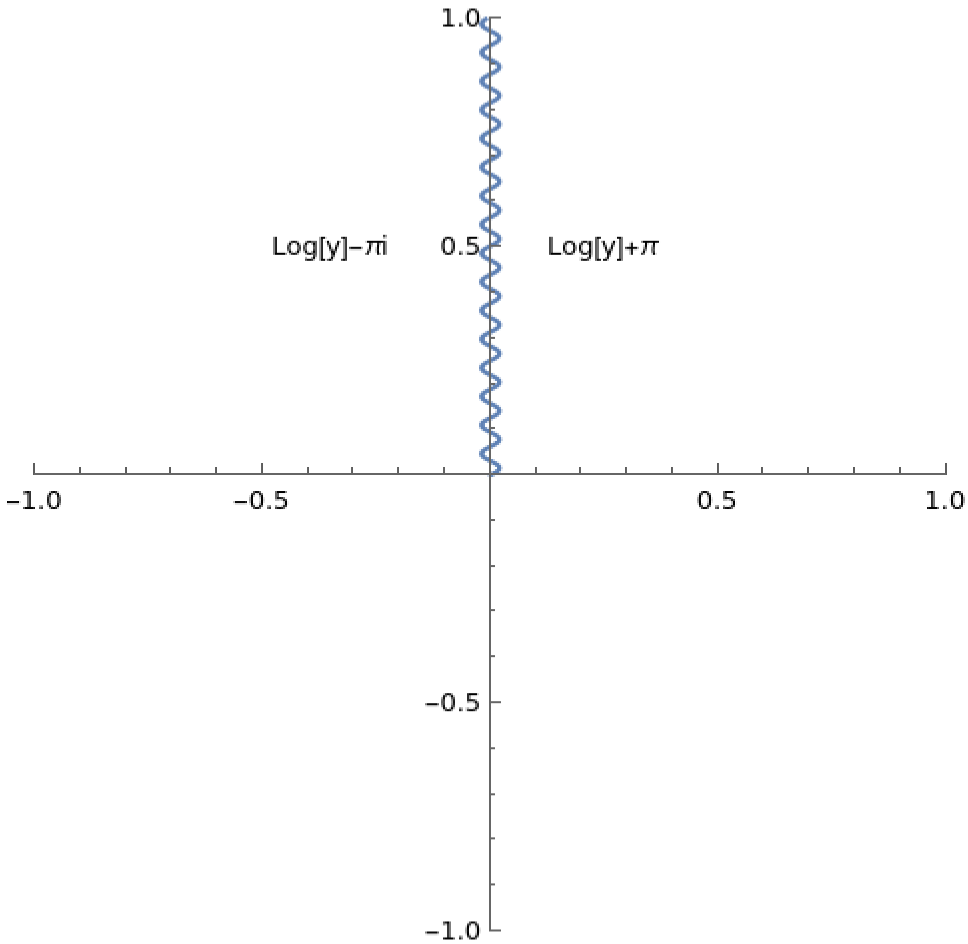


Figure 3.4: Logarithm function with branch cut along the positive imaginary axis **Edit:** π on the right side should be πi

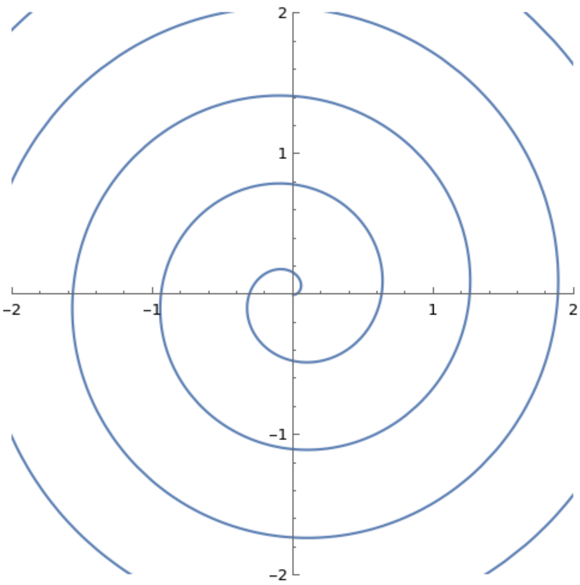


Figure 3.5: A possible branch from a branch point (taken to be origin); a bit more complicated to describe, and often times, drawing this diagram might just be sufficient.

where $\theta_1, \theta_2 \in [-\pi, \pi)$ (See Figure 3.6[4]) Hence, we get our branch cut between -1 and 1:

$$\left(z^2 - 1\right)^{1/2} = r_1^{1/2} r_2^{1/2} e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$$

Exercise 3.4. Show that the branch cut defined on Example 3.4 has different limiting values across the branch cut. **Again, the angle range given is not a mistake!**

Exercise 3.5. Suppose the angle ranges are given to be $\theta_1 \in [0, 2\pi)$, $\theta_2 \in [-\pi, \pi)$. Verify that the branch cuts are $(-\infty, -1] \cup [1, \infty) \subset \mathbb{R}$; that is, $(z^2 - 1)^{1/2}$ has jump discontinuity across those intervals.

(Hint: Sketch of the branch cut diagram is given as Figure 3.7)

Exercise 3.6. Let $f(z) = (k - i)^{1/2}$ with the branch cut at $i\,[1, \infty)$ (positive imaginary axis starting from i, the branch point). Show that $f(0) = e^{-i\pi/4}$.

4 Paths and Integration

You may have seen paths (AKA curves or lines) and integral along them in multivariable calculus. The complex analysis version is similar, but from a

[4]The angle range is not a mistake, even though it might seem a bit unintuitive.

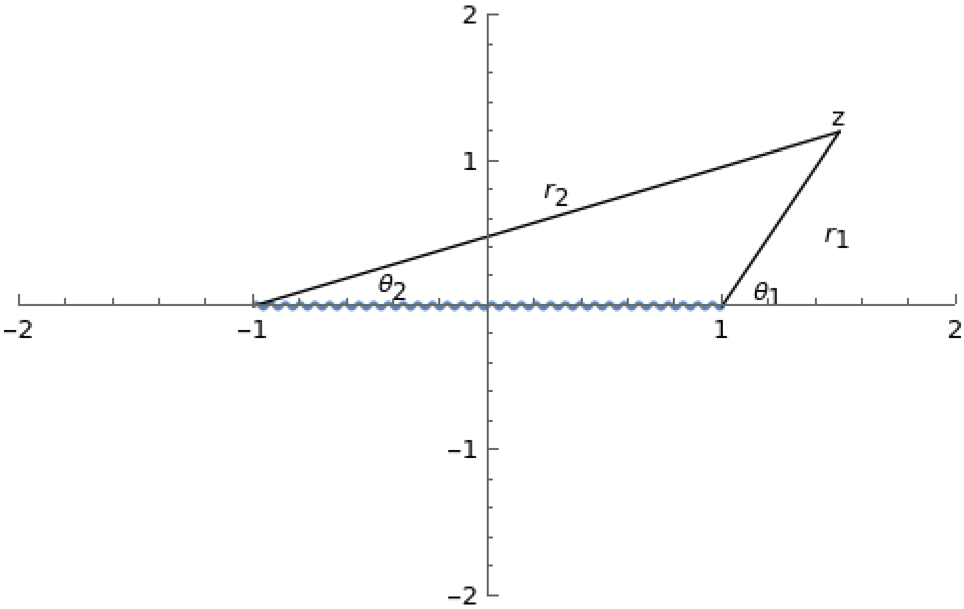


Figure 3.6: $r_1, r_2, \theta_1, \theta_2$ as shown. The point on the upper right is the z. Branch cut is at $[-1, 1]$.

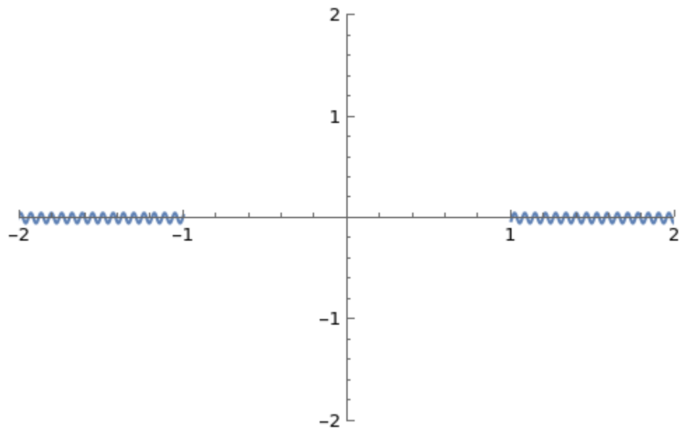


Figure 3.7: Branch cut diagram for $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [-\pi, \pi)$

different perspective.

4.1 Paths

Definition 4.1 (Path)**.** Let $a < b$. $\gamma : [a, b] \to \mathbb{C}$ is a **path** if it is a continuous function. It is **closed** if $\gamma(a) = \gamma(b)$, that is, if the endpoints coincide.

Tangent vector of a curve was an important concept in curves in MVC. Similarly, one could define the notion of it in complex analysis by introducing the "derivative".

Definition 4.2 (Differentiability of Path)**.** Path $\gamma : [a, b] \to \mathbb{C}$ is **differentiable** at t_0 if its real and imaginary parts are differentiable at t_0, which is equivalent to saying

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists. If so, we write this limit as $\gamma'(t_0)$.

If $\gamma'(t)$ is continuous, then we say the path is in C^1.

Exercise 4.1 ("Tangent")**.** Show that $\gamma'(t)$ (if it exists) characterizes the tangent direction of the path γ. (Hint: Turn it into an MVC problem!) Explain what happens at t_0 if $\gamma'(t_0) = 0$.

Example 4.1. Consider

$$\gamma(t) := \begin{cases} t^2 & -1 \le t \le 0 \\ it^2 & 0 \le t \le 1 \end{cases}$$

This is a piecewise C^1 path. See Figure 4.1 Note that $\gamma'(0) = 0$, so there is no "tangent" at the sharp corner.

Example 4.2 (Circle)**. One of the most important paths!** You can parameterize a circle with center z_0 and radius r by:

$$\gamma(t) = z_0 + re^{it}$$

where $t \in [0, 2\pi]$. (You could also do $\gamma(t) = z_0 + re^{2\pi it}$ where $t \in [0, 1]$)

Note that the direction is counterclockwise.

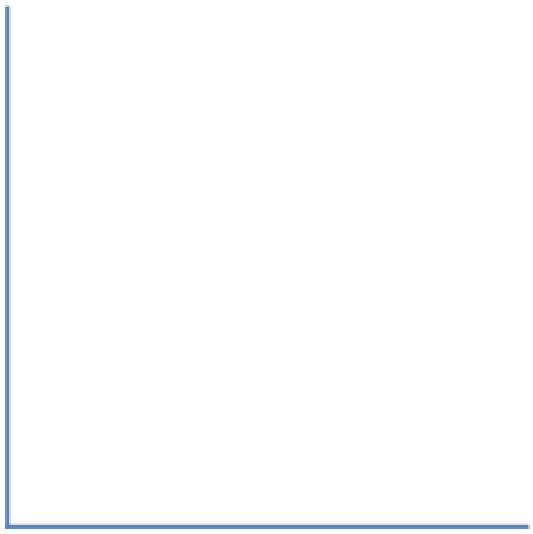


Figure 4.1: Path given by piecewise C^1 path.

4.2 Complex Path Integral

Definition 4.3 (Complex Integral)**.** Given a complex function $F(t) = x(t) + iy(t)$, one could define the integral over $t \in [a, b]$ to be

$$\int_a^b F(t)\,\mathrm{d}t \coloneqq \int_a^b x(t)\,\mathrm{d}t + i\int_a^b y(t)\,\mathrm{d}t$$

Exercise 4.2. Prove that, just like in real analysis, you can bound by the integral of the modulus of integrand; that is:

$$\left|\int_a^b F(t)\,\mathrm{d}t\right| \leq \int_a^b |F(t)|\,\mathrm{d}t$$

Definition 4.4 (Path Integral)**.** Given piecewise C^1 path $\gamma : [a, b] \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$, then

$$\int_\gamma f(z)\,\mathrm{d}z \coloneqq \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t$$

Remark 4.1. For remembering the definition, you can treat it as substitution rule in real integral:

$$\begin{aligned} z &= \gamma(t) \\ \mathrm{d}z &= \gamma'(t)\,\mathrm{d}t \end{aligned}$$

Example 4.3 (Terms in Taylor Around a Circle[5]). Given unit circle around origin (counterclockwise) as the path, let's compute the path integral of $f(z) = z^n$ where $n \in \mathbb{Z}$.

$$\begin{aligned}
\int_\gamma z^n \, \mathrm{d}z &= \int_0^{2\pi} \left(e^{it}\right)^n e^{it} i \, \mathrm{d}t \\
&= \int_0^{2\pi} e^{it(n+1)} i \, \mathrm{d}t \\
&= \begin{cases} \frac{1}{n+1} \left[e^{it(n+1)} i\right]_0^{2\pi} & (n \neq -1) \\ \int_0^{2\pi} i \, \mathrm{d}t & (n = -1) \end{cases} \\
&= \begin{cases} 0 & (n \neq -1) \\ 2\pi i & (n = -1) \end{cases}
\end{aligned}$$

Exercise 4.3 (Path integral is well-defined). Show that if $\gamma : [a, b] \to \mathbb{C}$ and $\tilde{\gamma} : [c, d] \to \mathbb{C}$ are equivalent paths (of same orientation), then for any continuous function,

$$\int_\gamma f(z) \, \mathrm{d}z = \int_{\tilde{\gamma}} f(z) \, \mathrm{d}z$$

(Hint: since γ and $\tilde{\gamma}$ are equivalent paths, there exists a bijective $s : [c, d] \to [a, b]$ with $s'(t) > 0$ such that $s(c) = a, s(d) = b$.)

Definition 4.5 (Length). If $\gamma : [a, b] \to \mathbb{C}$ is a C^1 path, then **length** of γ is defined as

$$\ell(\gamma) := \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

Remark 4.2. This is very similar to the length defined in MVC...

Remark 4.3 (Path Integral Properties). Given functions f, g and paths γ, η,

- Linearity
  - $\int_\gamma (\alpha f(z) + \beta g(z)) \, \mathrm{d}z = \alpha \int_\gamma f(z) \, \mathrm{d}z + \beta \int_\gamma g(z) \, \mathrm{d}z$
- Opposite Orientation: If γ^- is traversal of γ in the opposite direction,
  - $\int_\gamma f(z) \, \mathrm{d}z = -\int_{\gamma^-} f(z) \, \mathrm{d}z$
- Additivity: If $\gamma \star \eta$ is concatenation of the two paths,
  - $\int_{\gamma \star \eta} f(z) \, \mathrm{d}z = \int_\gamma f(z) \, \mathrm{d}z + \int_\eta f(z) \, \mathrm{d}z$

[5] Come back to this after you do residue theorem as well!

- Estimation Lemma (γ^* is the image of the path)
  - $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma^*} |f(z)| \ell(\gamma)$
  - Useful for having an upper bound.

Exercise 4.4. Prove the estimation lemma.

Here is also a theorem that resembles the FTC

Exercise 4.5. $F(z)$ is called **primitive** of f if $F'(z) = f(z)$. Suppose $\gamma : [a, b] \to U$ is a piecewise C^1 path in U, then prove

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

Also the fact that zero derivative means constant:

Exercise 4.6. Let $f : U \to \mathbb{C}$ with $f'(z) = 0$ for all $z \in U$, then show f is constant.

5 Cauchy's Theorem

One awesome theorem by Mr. Cauchy here!

Theorem 5.1 (Cauchy's Theorem)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function over U. Let γ be a closed path in U such that interior lies entirely in U. Then*

$$\int_\gamma f(z)\,dz = 0$$

The proof of this is quite tedious...but the idea is that you prove this theorem when γ is triangular path, then generalize it to star-like domain, then generalize it further!

Exercise 5.1 (A Case of Deformation Theorem)**.** Suppose γ and $\tilde{\gamma}$ are two curves with same ends, in the same orientation (so that the starting point of γ is also the starting point of $\tilde{\gamma}$, and same goes for the ending points). If $f(z)$ is holomorphic inside interior region surrounded by $\gamma \cup \tilde{\gamma}$, then deduce from Cauchy's theorem that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\tilde{\gamma}} f(z)\,\mathrm{d}z$$

(Hint: It helps to draw! It becomes basically a one-line argument by the use of Cauchy's theorem.)

Example 5.1. Take γ to be a counterclockwise unit circle around the origin. For any polynomial $f(z)$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

by Cauchy's theorem, because polynomial is holomorphic inside γ. (In fact, try verifying this by computing it!)

Example 5.2 (**WRONG** Use of Cauchy's Theorem)**.** Again, take γ to be a counterclockwise unit circle around the origin. This time take $f(z) = z^{-2}$. You will find that

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

but this is NOT by Cauchy's theorem, as $f(z)$ here is not holomorphic inside γ. (Rather, it is just a consequence of it having a primitive that does not cross a branch cut.)

In fact, you will get zero integral for $f(z) = \frac{1}{z^n}$ for integer $n \geq 2$, but these are not by Cauchy's theorem.

What about $f(z) = \frac{1}{z}$ though? (See Example 4.3 which might be relevant; this is, as mentioned before, related to residue calculus which is coming!)

Theorem 5.2 (Cauchy's Integral Formula)**.** *Suppose $f : U \to \mathbb{C}$ is holomorphic on an open set U containing disc $\bar{B}(a, r)$. Then for all $w \in B\,(a, r)$,*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - w}\,dz$$

where γ is $\partial B(a, r)$ (counterclockwise).

Remark 5.1. Cauchy's integral formula is also an example of residue theorem (which will be covered later)!!! (If you are only interested in knowing theorems, then it means residue calculus is all you need at the end of the day...) Note that integrand is holomorphic in $U \setminus \{w\}$.

Exercise 5.2 (Cauchy's Integral Formula for Derivatives)**.** If $f : U \to \mathbb{C}$ is holomorphic on open set U, then for any $z_0 \in U$, $f(z)$ is equal to its Taylor series at z_0, and it converges on any open disk centered at z_0 lying in U, and derivatives are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$

Remark 5.2. This implies that "analytic = holomorphic"! They can be used interchangeably! This also means holomorphic functions are infinitely differentiable.

Exercise 5.3 (Liouville's Theorem)**.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire[6] function. If f is bounded, then f is a constant function.

Remark 5.3. A part of fundamental theorem of algebra can be proven using Liouville's theorem!

Example 5.3 (Wiener-Hopf Method)**. THIS IS ABSOLUTELY OUT OF SCOPE FOR PART A**, but demonstrates how powerful Liouville's theorem can be!

Suppose you want to solve the following problem for smooth bounded $f(x)$ for $x \in \mathbb{R}$:

$$\int_0^\infty K(x-t) f(t)\,\mathrm{d}t = f(x) \qquad \text{for } x \geq 0$$

where $K(x) = e^{-|x|}$ for $x \in \mathbb{R}$.

After a bit of trickery[7] (in Applied Complex Variables in part C), you end up needing to solve the following for $\hat{f}_+$ and $\hat{h}_-$ (or at least one of the two unknown functions):

$$\frac{1-k^2}{k+i} \hat{f}_+(k) = (k-i)\,\hat{h}_-(k) \qquad \text{for } 0 < \Im(k) < 1$$

Wait, you have to solve for $\hat{f}_+$ for $\hat{g}_-$ from a single equation? The magic here is that the LHS is holomorphic on $\Im(k) > 0$, and the RHS is holomorphic on $\Im(k) < 1$. Also from more complicated analysis of the problem, we know that $\hat{f}_+(k) = O\left(k^{-1}\right)$ and $\hat{h}_-(k) = O\left(k^{-1}\right)$.

Define $E(k)$ as

$$E(k) = \begin{cases} \frac{1-k^2}{k+i} \hat{f}_+(k) & (\Im(k) > 0) \\ (k-i)\hat{h}_-(k) & (\Im(k) < 1) \end{cases}$$

(Note that on the intersection $0 < \Im(k) < 1$, either of the definitions work, due to the given problem.) $E(k)$ is an entire function by construction, and because $\hat{f}_+(k) = O\left(k^{-1}\right)$ and $\hat{h}_- = O\left(k^{-1}\right)$, it is bounded at (complex infinity), so by Liouville's theorem $E(k) \equiv C$ for some constant C. Hence, (in the definition of Wiener-Hopf method, f_+ is defined to be $f_+(x) = f(x)\mathbb{I}_{x>0}$, where $\mathbb{I}$ is the indicator function.

$$\hat{f}_+(k) = \frac{C\,(k+i)}{1-k^2}$$

[6] Holomorphic on $\mathbb{C}$

[7] In a nutshell, one takes (complex) Fourier transform of the entire problem

Taking inverse Fourier transform,

$$f(x) = f_+(x) = \frac{C}{2\pi} \int_\Gamma \frac{(k+i)\, e^{-ikx}}{1-k^2} \, \mathrm{d}k$$

where Γ is a contour from $-\infty + ci$ to $+\infty + ci$ for any $c \in (0,1)$.

It turns out that in most cases, the region at which a function ceases to be holomorphic are isolated most of the time, and sometimes, not even non-holomorphic even if you haven't defined them!

Exercise 5.4 (Riemann's Removable Singularity Theorem)**.** Suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then show f extends to a holomorphic function on all of U.

Example 5.4. Consider $f(z) = \frac{\sin z}{z}$. From the definition, it seems like $f(z)$ might not be holomorphic at $z = 0$, but $f(z)$ is bounded around $z = 0$ (since $\lim_{z \to 0} f(z) = 1$), and in a neighbourhood away from $z = 0$, so by Riemann's removable singularity theorem, $f(z)$ is holomorphic at $z = 0$.

Note that if it is holomorphic at $z = 0$, then it also means it is infinitely differentiable at $z = 0$.

Remark 5.4 (General Strategy for Checking Holomorphicity)**.** Given $f(z)$,

1. Identify which points might be problematic.
2. Check boundedness (often via checking if limit exists).
3. If bounded, then it is definitely holomorphic, otherwise not holomorphic at that point.

Exercise 5.5.

- Given $f(z) = \frac{\sin z}{\cos z}$, is it holomorphic at $z = \frac{\pi}{2}$?
- What about $f(z) = \frac{\left(z - \frac{\pi}{2}\right) \sin z}{\cos z}$?
- Is $f(z) = \frac{1}{(z+1)^2 (z-1)}$ holomorphic at $z = i$? What about at $z = 1$? What about at $z = -1$?
- Define branch cut of $\sqrt{z}$ along the positive real axis. Determine if $f(z) = \frac{(z+i)\sqrt{z}}{z}$ is holomorphic at $z = -i$ and $z = i$, and if holomorphic, evaluate at those points.

Here is a kind-of converse to Cauchy's theorem.

Theorem 5.3 (Morera's Theorem)**.** *Suppose $f : U \to \mathbb{C}$ is a continuous function on open $U \subset \mathbb{C}$. If for any closed path γ in U, $\int_\gamma f(z)\, dz$, then f is holomorphic.*