Euler Equations: $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)$ RTT: $\frac{d}{dt}\iiint_{V(t)} f dV = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})dV$

(i) $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$ (ii) $\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g}$ (iii) $\rho c_v\frac{DT}{Dt} = -p\nabla\cdot\mathbf{u} + pq$ (iv) $p = \rho RT$, $pV = MRT$

(v) $\gamma \coloneqq \frac{c_p}{c_v} = 1 + \frac{R}{c_v}$, $\rho q = \frac{p}{\gamma-1}\frac{D}{Dt}\left(\log\left(\frac{p}{\rho^\gamma}\right)\right)$, $q = T\frac{DS}{DT}$ ($S = S_0 + c_v\log\left(\frac{p}{\rho^\gamma}\right)$)

Incompressible: $\frac{D\rho}{Dt} = 0 \implies \nabla\cdot\mathbf{u} = 0$ Irrotational: $\nabla\wedge\mathbf{u} = 0 \implies \mathbf{u} = \nabla\phi$ where $\nabla^2\phi = 0$

Bernoulli: $\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + \chi = F(t)$ (Often $\chi = gz = g\eta$)

Linear Wave Propagation From $\rho = \rho_0 + \rho'$, $\mathbf{u} = \mathbf{0} + \mathbf{u}'$, $p = p_0 + p'$

$\implies p' = c_0^2\rho'$ ($c_0^2 \coloneqq \frac{dp}{d\rho}(\rho_0)$),

(ii) $\implies \rho_0\mathbf{u'}_t = -\nabla p' = -c_0^2\nabla\rho'$ ($\implies \frac{\partial}{\partial t}(\nabla\wedge\mathbf{w}) = \mathbf{0}$)

(i) $\implies {\rho'}_t + \rho_0\nabla^2\phi = 0$ (With prev, it implies $\phi_{tt} = c_0^2\nabla^2\phi$)

For $p'(0,t) = Ae^{-i\omega t}$, $p'(x,t) = f(x)e^{-i\omega t}$

Conditions: Radiation condition, Boundedness

Laplacian: 2D: $\nabla^2 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)$ 3D: $\nabla^2 f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$

Mach: $M \coloneqq \frac{U}{c_0}$ (Seek steady state (no dep on t)) $\phi_{xx} + \phi_{yy} \overset{\text{elim}p',\ \text{expr in }\rho'}{=\cdots} \implies (1-M^2)\phi_{xx} + \phi_{yy} = 0$ ($U\mathbf{e}_x$ background flow)

Stokes Wave $\nabla^2\phi = 0$

KBC: $\frac{D}{Dt}(z-\eta) = 0 \implies \phi_z = \eta_t + \phi_x\eta_x + \phi_y\eta_y \overset{\text{Lin.}}{\implies} \phi_z = \eta_t$ at $z = 0$

DBC: Bernoulli $\implies \phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \overset{\text{Lin.}}{\implies} \phi_t + g\eta = 0$ at $z = 0$

No Flux: $\mathbf{u}\cdot\mathbf{n} = \mathbf{v}_B\cdot\mathbf{n}$

Seek $\phi(x,z,t) = f(z)e^{i(kx-\omega t)}$, nontrivial solution.

Flowing fluid: (BC) $\phi_z = \eta_t + U\eta_x$, $\phi_t + U\phi_x + g\eta = 0$ at $z = 0$

Two fluids: KBC: $\eta_t = \frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z}$ at $z = 0$, DBC: $\rho_1\left(\frac{\partial\phi_1}{\partial t} + g\eta\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + g\eta\right)$ at $z = 0$

Surface tension: DBC: $\rho_1(\frac{\partial\phi_1}{\partial t} + g\eta) - \rho_2(\frac{\partial\phi_2}{\partial t} + g\eta) = \gamma\frac{\partial^2\eta}{\partial x^2}$ at $z = 0$

Internal Gravity, Strat. fluid $\frac{D\rho}{Dt} = 0 \iff \nabla\cdot\mathbf{u} = 0$ Seek perturb: $\mathbf{u} = \mathbf{0}$, $\rho = \rho_(z)$, $p = p_0(z) \implies p_0(z) = p_a - g\int_0^z\rho_0(\zeta)d\zeta$ (From momentum equ.)

Linearize: (i) $\rho'_t + w'\rho'_0(z) = 0$ (ii) $u'_x + v'_y + w'_z = 0$ (iii) $\rho_0 u'_t = -p'_x$ (iv) $\rho_0 v'_t = -p'_y$ (v) $\rho_0 w'_t = -p'_z - \rho' g$

$\overset{\text{(iii)},\text{(iv)}\to\text{(ii)}}{\implies} -p'_{xx} - p'_{yy} = \rho_0(u'_x + v'_y)_t = -\rho_0 w'_{zt}$, $\overset{\text{(i)}\implies\text{(v)}}{\implies} \rho_0 w'_{tt} = -p'_{zt} - gp'_t = -p'_{zt} + gw'\rho'_0(z)$

$\overset{\text{Elim }p'}{\implies} (w'_{xx} + w'_{yy} + w'_{zz})_{tt} = \frac{g}{\rho_0}\rho'_0(z)(w'_{xx} + w'_{yy} - \frac{1}{g}w'_{ztt})$

Theory for Linear Waves $\phi_{tt} = c_0^2\nabla^2\phi$

$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + (\frac{\omega}{c_0})^2 f = 0 \overset{\xi=kr,k=\frac{\omega}{c_0}}{\implies} \xi^2\frac{d^2f}{d\xi^2} + \xi\frac{df}{d\xi} + \xi^2 f = 0 \overset{|f|<\infty}{\implies} f = AJ_0(\xi)$

Take $\phi(x,y,z,t) = f(x)g(y)h(z)e^{-i\omega t}$

Fourier Transform $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$

IFT: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk$, $\hat{f}(k) = \hat{g}(k)\hat{h}(k) \implies f(x) = (g*h)(x) = \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$

$\widehat{\frac{d^n f}{dx^n}} = (ik)^n\hat{f}$

Method of Stationary Phase $\eta(Vt,t)$, eval asymptotically, then $V = \frac{x}{t}$.

$I(t) = \int_a^b f(k)e^{i\psi(k)t}dt$: $\psi'(k)$ simple zero at k_*, then $I(t) \sim f(k_*)e^{i(\psi(k_*)t\pm\frac{\pi}{4})}\sqrt{\frac{2\pi}{|\psi''(k_*)|t}}$

Derivation: $I(t) = \int_a^{k_*-\epsilon} f(k)e^{i\psi(k)t}dk + \int_{k_*+\epsilon}^b f(k)e^{i\psi(k)t}dk + \int_{k_*-\epsilon}^{k_*+\epsilon} f(k)e^{i\psi(k)t}dk$

First and second terms $\sim O(\frac{1}{t})$ by Riemann-Lebesgue. By approxing $f(k) \sim f(k_*)$, $\psi(k) \sim \psi(k_*) + \frac{\psi''(k_*)}{2}(k-k_*)^2$, $I(t) \sim \int_{k_*-\epsilon}^{k_*+\epsilon} f(k_*)e^{i\psi(k_*)t}\exp(\frac{it\psi''(k_*)}{2}(k-k_*)^2)dk + O(\frac{1}{t})$. With $k = k_* + s\sqrt{\frac{2}{t\psi''(k_*)}}$ for $\psi''(k_*) > 0$, $I(t) \sim f(k_*)e^{i\psi(k_*)t}\sqrt{\frac{2}{\psi''(k_*)t}}\int_{-\epsilon\sqrt{\psi''(k_*)/2}}^{\epsilon\sqrt{\psi''(k_*)/2}} e^{is^2}ds + O(\frac{1}{t})$. Contour integral, $I(t) \overset{t\to\infty}{\sim} (1+i)f(k_*)e^{i\psi(k_*)t}\sqrt{\frac{\pi}{\psi''(k_*)t}} + O(\frac{1}{t})$. Similarly with $\psi''(k_*) < 0$. General result can be written.

For $\psi(k) = kV \mp \omega t$, Group velocity: $c_g(k) \coloneqq \frac{d\omega}{dk}$ Phase Velocity: $c_p(k) \coloneqq \frac{\omega}{k}$

Tip: Think about evenness, oddness when diff.

Flow past thin wing Wing: $x \in [-a, a]$, $y = f_\pm(x)$. BG: $\mathbf{u} = U\mathbf{e}_x \Longrightarrow (1 - M^2)\phi_{xx} + \phi_{yy} = 0$

BC1: $\mathbf{u} \cdot \mathbf{n} = 0 \Longleftrightarrow (U + \phi_x, \phi_y) \cdot (f'_\pm, -1) = 0 \overset{\text{Lin.}}{\Longrightarrow} \phi_y = Uf'_\pm$ on $y = 0_\pm$, $|x| < a$

BC2: Normal velo. and Pressure cts $\Longrightarrow [\phi_x]_-^+ = [\phi_y]_-^+ = 0$ on $y = 0$, $|x| > a$

Lift $L = -\rho U \Gamma$ where $\Gamma = \phi(a, 0_-) - \phi(a, 0_+)$

Subsonic $M < 1$: Cond: $\nabla\phi \to \mathbf{0}$ as $|\mathbf{x}| \to \infty$

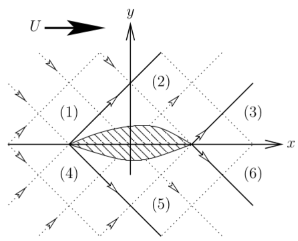
Define $Y = \beta y$, $\Phi = \beta\phi$, $\beta = \sqrt{1 - M^2}$, then (i) $\Phi_{xx} + \Phi_{YY} = 0$, (ii) $\Phi_Y = Uf'_\pm$ at $Y = 0_\pm$, $|x| < a$

(iii) $[\Phi_x]_-^+ = [\Phi_Y]_-^+ = 0$ at $Y = 0$, $|x| > a$, (iv) $\Phi_x, \Phi_y \to 0$ as $x^2 + Y^2 \to \infty$

If wing sym ($f_+ = f_-$), $\Phi_{xx} + \Phi_{YY} = 0$ on $Y > 0$, $\Phi_Y = U\eta(x)$ on $Y = 0$ ($\eta = f'_\pm \mathbf{1}_{|x|<a}$), decay

Acquire $\hat{\Phi}_Y = U\hat{\eta}e^{-|k|Y}$ by FT in x, then use conv. thrm.

Supersonic $M > 1$: Gen sol: $\phi(x, y) = F(x - By) + G(x + By)$ where $B = \sqrt{M^2 - 1}$



(Causality) Upstream undisturbed: $\phi \overset{x \to -\infty}{\to} 0$ (Char: $F(x - By) =$ const., $G(x + By) =$ const. on $x \mp By =$ const. resp) $\Longrightarrow F = C$, $G = -C$ (where $C =$ const.) on char from $x = -\infty \Longrightarrow \phi \equiv 0$ (1),(4)

On (2), char $x + By =$ const. from $x = -\infty \Longrightarrow G = -C$, and cond on $y = 0_+ \Longrightarrow -BF'(x) = Uf'_+(x)$

$\Longrightarrow \phi_+(x, y) = -\frac{U}{B}\left(f_+(x - By) - f_+(-a)\right)$ (2) with continuity with (1)

Similar: $\phi_-(x, y) = \frac{U}{B}\left(f_-\left(x + By\right) - f_-(-a)\right)$ (5)

$\phi = C_1 = -U\left(f_+(a) - f_+(-a)\right)/B$ in (3) by continuity with (2)

$\phi = C_2 = U\left(f_-(a) - f_-\left(-a\right)\right)$ in (6) by continuity with (5)

Nonlinear Waves (i) $\frac{D\rho}{Dt} = -\rho u_x$ (ii) $\frac{Du}{Dt} = -\frac{1}{\rho}p_x$ (iii) $\rho c_v \frac{DT}{Dt} = -pu_x$ (iv) $p = \rho RT$ (v) $\gamma = 1 + \frac{R}{c_v}$

1D Gas Dynamics If ρ_0, p_0 (init, uniform), then $\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}$ Pf) $\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{\rho^{2\gamma}}\left(\frac{Dp}{Dt}\rho^\gamma - p\gamma\rho^{\gamma-1}\frac{D\rho}{Dt}\right) =$
$\frac{1}{\rho^\gamma}\left(\frac{Dp}{Dt} - \frac{p\gamma}{\rho}\frac{D\rho}{Dt}\right) \overset{\text{(iv)}}{\Longrightarrow} \rho^\gamma \frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = R\rho\frac{DT}{Dt} + RT\frac{D\rho}{Dt} - \gamma RT\frac{D\rho}{Dt} \overset{(1),(3)}{=} -\frac{Rpu_x}{c_v} + (\gamma - 1)\rho RTu_x \overset{\gamma-1=\frac{R}{c_v}}{=} 0$

Derivation: *speed of sound* $c^2 := \frac{dp}{d\rho} = \frac{\gamma p_0 \rho^{\gamma-1}}{\rho_0^\gamma} \Rightarrow \rho = \left(\frac{\rho_0^\gamma}{\gamma p_0}\right)^{1/(\gamma-1)} c^{2/(\gamma-1)}$, $p = \left(\frac{\rho_0^\gamma}{\gamma p_0}\right)^{1/(\gamma-1)} \frac{c^{2\gamma/(\gamma-1)}}{\gamma}$

$\rho_t + u\rho_x + \rho u_x = 0 \Longrightarrow \frac{2}{\gamma-1}\left(c_t + uc_x\right) + cu_x = 0$, $u_t + uu_x + \frac{1}{\rho}p_x = 0 \Longrightarrow u_t + uu_x + \frac{2}{\gamma-1}cc_x = 0$

Add and subtract to get $\frac{\partial}{\partial t}\left(u \pm \frac{2c}{\gamma-1}\right) + (u \pm c)\frac{\partial}{\partial x}\left(u \pm \frac{2c}{\gamma-1}\right) = 0 \Longrightarrow \frac{d}{dt}\left(u \pm \frac{2c}{\gamma-1}\right) = 0$ on $\dot{x} = u \pm c$

Shallow water theory Rigid base $z = 0$, free surface $z = h(x, t)$ vel field $\mathbf{u} = u(x, z, t)\mathbf{e}_x + w(x, z, t)\mathbf{e}_z$

Mass Conservation $u_x + w_z = 0$, KBC: $w = 0$ on $z = 0$, $\frac{D}{Dt}(z - h) = 0 \Longrightarrow w = h_t + uh_x$ on $z = h$

Mass Cons $\overset{\text{int wrt } z}{\Longrightarrow} h_t + \left(h\bar{u}\right)_x = 0$ where $\bar{u} = \frac{1}{h}\int_0^h u dz$

Assume hydrostatic: $p = p_a + \rho g\left(h - z\right)$, irrotational: $\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$

Assume: $|w| \ll |u|$, $\bar{u} \approx u$, so $h_t + uh_x + hu_x = 0$, $u_t + uu_x + gh_x = 0$. $c = \sqrt{gh} \Longrightarrow \gamma = 2$ form.

Shock Waves Frame moving with the shock to fixed frame: $u_\pm \to u_\pm - V$

1D Gas Dynamics (i) $[pu]_-^+ = 0$ (ii) $\left[p + \rho u^2\right]_-^+ = 0$ (iii) $\left[\frac{u^2}{2} + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0$

Entropy must increase: $M = \frac{u}{c}$, $M_-^2 > 1$ and $M_+^2 < 1$ (Supersonic to subsonic)

Shallow Water Theory (i) $[hu]_-^+ = 0$ (ii) $\left[hu^2 + \frac{gh^2}{2}\right]_-^+ = 0$

$$\text{Energy Flow out of jump: } Q = \left[\underbrace{\int_0^h \left(\frac{1}{2}\rho u^2 + \rho gz\right)u dz}_{\text{KE+PE}} + \underbrace{\int_0^h (p - p_{\text{atm}})u dz}_{\text{work by pres.}}\right]_-^+$$

$= \left[\frac{\rho hu^3}{2} + \rho guh^2\right]_-^+ = \rho hu\left[\frac{u^2}{2} + gh\right]_-^+ = (\rho hu)\,\frac{g(h_- - h_+)^3}{4h_+h_-} < 0$

Weak Sol & RH Cond $\frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial x} = \mathbf{0}$, $\oint_C \mathbf{Q}dt - \mathbf{P}dx$, $V = \frac{dx}{dt} \Longrightarrow [\mathbf{Q}]_-^+ = V\left[\mathbf{P}\right]_-^+$

Gas Dynamics: $\mathbf{u} = (\rho, u, p)$, $\mathbf{P} = \left(\rho, \rho u, \rho u^2/2 + p/\left(\gamma - 1\right)\right)$, $\mathbf{Q} = \big(\rho u, \rho u^2 + p, \rho u^3/2 + \gamma pu/\left(\gamma - 1\right)\big)$

Shallow Water: $\mathbf{u} = (h, u)$, $\mathbf{P} = (h, hu)$, $\mathbf{Q} = \big(uh, hu^2 + gh^2/2\big)$