

Continuous Optimisation Theorems

Lemma 1. Let $f \in \mathcal{C}$, $x \in \mathbb{R}^n$, and $s \in \mathbb{R}^n$ with $s \neq 0$. Then

$$\nabla f(x)^T s < 0 \implies f(x + \alpha s) < f(x) \quad \forall \alpha > 0 \text{ suff. small}$$

Lemma (Exact linesearch for quadratics). For $q(x) = g^T x + \frac{1}{2} x^T H x$, $\phi(\alpha) := q(x + \alpha s)$,

$$\alpha = -\frac{\nabla f(x)^T s}{s^T H s}$$

Definition (Armijo Condition). Choose $\beta \in (0, 1)$.

$$f(x^k + \alpha^k s^k) \leq f(x^k) + \beta \alpha^k \nabla f(x^k)^T s^k$$

Lemma 2. Let $f \in \mathcal{C}^1(\mathbb{R}^n)$ with ∇f Lipschitz continuous with L , then Armijo condition at k^{th} satisfied for all $\alpha \in [0, \alpha_{\max}^k]$ where

$$\alpha_{\max}^k = \frac{(\beta - 1) \nabla f(x^k)^T s^k}{L \|s^k\|^2}$$

Lemma 3. Let $f \in \mathcal{C}^1(\mathbb{R}^n)$ with ∇f Lipschitz continuous with L , then at k^{th} iteration,

$$\alpha^k \geq \min \{ \alpha_{(0)}, \tau \alpha_{\max}^k \}$$

Theorem 4 (Convergence of GLM). Let $f \in \mathcal{C}^1(\mathbb{R}^n)$ bounded below by f_{low} , and ∇f Lipschitz continuous. Then either

$$\exists l \geq 0 \text{ s.t. } \nabla f(x^l) = 0$$

or

$$\lim_{k \rightarrow \infty} \min \left\{ \frac{|\nabla f(x^k)^T s^k|}{\|s^k\|}, |\nabla f(x^k)^T s^k| \right\} = 0$$

Theorem 6 (Exact Linesearch Convergence). $f \in \mathcal{C}^2$, x^* local minimizer of f with $\nabla^2 f(x^*)$ positive definite between λ_{\max}^* and λ_{\min}^* . With SD-e, if $x^k \rightarrow x^*$ as $k \rightarrow \infty$, then x^k converges linearly to x^* :

$$\rho \leq \frac{\kappa(x^*) - 1}{\kappa(x^*) + 1} =: \rho_{SD}$$

where $\kappa(x^*) = \frac{\lambda_{\max}^*}{\lambda_{\min}^*} = \kappa(\nabla^2 f(x^*))$.

Definition (Newton's Method).

$$s^k := -(\nabla^2 f(x^k))^{-1} \nabla f(x^k)$$

Theorem 7 (Local Convergence of (Pure) Newton's Method). • $f \in \mathcal{C}^2(\mathbb{R}^n)$

- $\nabla f(x^*) = 0$
 - $\nabla^2 f(x^*)$ nonsingular.
 - $\nabla^2 f$ locally Lipschitz continuous at x^*
- Then, $x^k \rightarrow x^*$ at quadratic rate.

Theorem 9 (Convergence of Newton bArmijo). • $f \in \mathcal{C}^2(\mathbb{R}^n)$ bounded below.

- ∇f Lipschitz continuous.
 - Newton's method + bArmijo linesearch.
 - For all $k \geq 0$, eigenvalues of $\nabla^2 f(x^k)$ at iterates by positive and uniformly bounded below, away from zero, independently of k .
- Then either

$$\exists l \geq 0 \text{ s.t. } \nabla f(x^l) = 0$$

or

$$\|\nabla f(x^k)\| \rightarrow 0$$

as $k \rightarrow \infty$

Definition (Secant Approximation). $B^k \approx \nabla^2 f(x^k)$ where

$$\underbrace{\nabla f(x^{k+1}) - \nabla f(x^k)}_{\gamma^k} = B^{k+1} \underbrace{(x^{k+1} - x^k)}_{\delta^k}$$

Definition (SR1). $B^{k+1} := B^k + u^k (u^k)^T$ where

$$u^k = \frac{\gamma^k - B^k \delta^k}{\rho^k}$$

where $(\rho^k)^2 := (\gamma^k - B^k \delta^k)^T \delta^k > 0$

Definition (BFGS). $B^{k+1} := B^k + u^k (u^k)^T + v^k (v^k)^T$ where

$$u^k (u^k)^T = \frac{1}{\gamma^T \delta^k} \gamma^k (\gamma^k)^T$$

$$v^k (v^k)^T = -\frac{B^k \delta^k (B^k \delta^k)^T}{(\delta^k)^T B^k \delta^k}$$

where $(\rho^k)^2 := (\gamma^k - B^k \delta^k)^T \delta^k > 0$

Definition (Gauss-Newton). For nonlinear least-squares (NLS):

$$f(x) := \frac{1}{2} \sum_{j=1}^m (r_j(x))^2 = \frac{1}{2} \|r(x)\|^2$$

where $r : \mathbb{R}^n \rightarrow \mathbb{R}^m$,

$$\nabla^2 f(x) = J(x)^T J(x) + \underbrace{\sum_{j=1}^m r_j(x) \nabla^2 r_j(x)}_{\text{Negligible}} \text{ suggests}$$

$$J(x^k)^T J(x^k) s^k = -J(x^k)^T r(x^k)$$

Definition (TR Decrease Param). $\rho^k := \frac{f(x^k) - f(x^k + s^k)}{f(x^k) - m_k(s^k)}$

Definition (Cauchy Point). $\alpha_c^k := \arg \min_{\alpha > 0} m_k(-\alpha \nabla f(x^k))$ subject to $\|\alpha \nabla f(x^k)\| \leq \Delta_k$

Theorem 11 (GTR Global Convergence).

- $f \in \mathcal{C}^2(\mathbb{R}^n)$ bounded below.
- ∇f Lipschitz continuous.
- $m_k(s^k) \leq m_k(s_c^k)$ for all k .

Then either

$$\exists k \geq 0 \text{ s.t. } \nabla f(x^k) = 0$$

or

$$\lim_{k \rightarrow \infty} \|\nabla f(x^k)\| = 0$$

Theorem 12 (Cauchy Model Decrease). *GTR with Cauchy decrease* $m_k(s^k) \leq m_k(s_c^k)$ for all k ,

$$f(x^k) - m_k(s^k) \geq f(x^k) - m_k(s_c^k) \geq \frac{1}{2} \|\nabla f(x^k)\| \min \left\{ \Delta_k, \frac{\|\nabla f(x^k)\|}{\|\nabla^2 f(x^k)\|} \right\}$$

Theorem 13 (Lower bound on TR radius). $f \in \mathcal{C}^2(\mathbb{R}^n)$, ∇f Lipschitz, Cauchy decrease. Suppose $\exists \epsilon > 0$ such that $\|\nabla f(x^k)\| \geq \epsilon$ for all k , then

$$\exists c \in (0, 1) \text{ indep of } k \text{ s.t. } \Delta_k \geq \frac{c}{L} \epsilon$$

Theorem 14 (At least one limit point is stationary). $f \in \mathcal{C}^2(\mathbb{R}^n)$, ∇f Lipschitz, Cauchy decrease. Then either $\exists k \geq 0$ s.t. $\nabla f(x^k) = 0$ or

$$\liminf_{k \rightarrow \infty} \|\nabla f(x^k)\| = 0$$

Theorem 15 (Global Minimizer of TR Subproblem).

$$\underbrace{(H + \lambda^* I)}_{\text{positive semidef}} s^* = -g$$

with $\lambda^* \geq 0$, $\lambda^* (\|s^*\| - \Delta) = 0$, and $\|s^*\| \leq \Delta$

Definition (KKT of (CP)).

$$\begin{aligned} \nabla f(\hat{x}) &= J_E(x)^T \hat{y} + J_I(x)^T \hat{\lambda} \\ c_E(\hat{x}) &= 0 \\ c_I(\hat{x}) &\geq 0 \\ \hat{\lambda}_i &\geq 0 \\ \hat{\lambda}_i c_i(\hat{x}) &= 0 \end{aligned}$$

Definition (Lagrangian of (CP)).

$$\begin{aligned}\mathcal{L}(x, y, \lambda) &:= f(x) - y^T c_E(x) - \lambda^T c_I(x) \\ \nabla_x \mathcal{L}(x, y, \lambda) &= \nabla f(x) - J_E(x)^T y - J_I(x)^T \lambda\end{aligned}$$

so KKT implies $\nabla_x \mathcal{L}(\hat{x}, \hat{y}, \hat{\lambda}) = 0$

Theorem 16 (First Order Necessary Condition for (CP)). x^* local minimizer implies x^* KKT under one of the conditions:

- Slater: $\exists x$ s.t. $c_E(x) = Ax - b = 0$ and $c_I(x) > 0$
- LICQ: $\nabla c_i(x)$ linearly indep.

Definition (Convex Programming Problem). $f(x)$ is a convex function, $c_i(x)$ is a concave function for all $i \in I$, and $c_E(x) := Ax - b$

Theorem 18 (Sufficient Optimality Conditions for Convex Problem). (CP) be convex programming problem, then KKT implies global minimizer.

Theorem 19 (Second-order Necessary Conditions). (CP) satisfies some CQ. Let x^* be a local minimizer, and (y^*, λ^*) are Lagrange multipliers of KKT at x^* .

Then,

$$s^T \nabla_{xx}^2 \mathcal{L}(x^*, y^*, \lambda^*) s \geq 0 \quad \forall s \in F(\lambda^*)$$

Definition (Quadratic Penalty Function).

$$\Phi_\sigma(x) = f(x) + \frac{1}{2\sigma} \|c(x)\|^2$$

Theorem 21 (Global Convergence of Penalty Method). Apply basic quadratic penalty method. Assume $f, c \in \mathcal{C}^1$, $y_i^k := -c_i(x^k)/\sigma^k$ for $i = 1, 2, \dots, m$, and

$$\|\nabla \Phi_{\sigma^k}(x^k)\| \leq \epsilon^k$$

where $\epsilon^k \rightarrow 0$.

Also assume, $x^k \rightarrow x \rightarrow x^*$ and $\nabla c_i(x^*)$ are linearly indep (LICQ).

Then, x^* is KKT, and $y^k \rightarrow y^*$, the vector of Lagrange multipliers of constraints.

Definition (Augmented Lagrangian).

$$\Phi(x, u, \sigma) = f(x) - u^T c(x) + \frac{1}{2\sigma} \|c(x)\|^2$$

Theorem 22 (Global Convergence of Augmented Lagrangian). Assuming $f, c \in \mathcal{C}^1$ in

$$\min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } c(x) = 0$$

let $y^k = u^k - \frac{c(x^k)}{\sigma^k}$ for some $u^k \in \mathbb{R}^m$, and assume that

$$\|\nabla \Phi(x^k, u^k, \sigma^k)\| \leq \epsilon^k$$

where $\epsilon^k \rightarrow 0, k \rightarrow \infty$. Additionally assume $x^k \rightarrow x^*$ where $\nabla c_i(x^*)$ are linearly independent.

Then, $y^k \rightarrow y^*$ as $k \rightarrow \infty$ with y^* satisfying $\nabla f(x^*) - J(x^*)^T y^* = 0$