Complex Analysis Summary

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0 Preface

This note is for people studying complex analysis, and got lost in the middle with bunch of technical explanations. I wil try my best to be succinct as possible, stating important results (mostly without proof, but a bit of justification).

Warning: This summary note is not a substitute for the lecture note. Make sure you study from lecture note!

1 Complex Plane and Möbius Maps

1.1 Complex Plane and Complex Infinity

We will be working in what's known as the *extended complex plane*. Redefine the symbol $\mathbb{C} \coloneqq \mathbb{C} \cup \left\{ \underbrace{\infty}_{\text{Complex Infinity}} \right\}$; that is, whenever I mention $\mathbb{C}$, I refer to the space of complex numbers and infinity.

Note that in $\mathbb{C}$, ∞ is different from infinity in real numbers. $\infty \coloneqq \frac{1}{0}$ is a value that is not "larger" or "smaller" than any number (since we are talking about complex number...), but rather a number on a complex plane at a really far distance from origin.

It is **WRONG** to say:

- $\infty \geq a$ for any $a \in \mathbb{C}$
- $\infty \leq a$ for any $a \in \mathbb{C}$

However, it is **CORRECT**[1] to say:

[1]Subtlety here: it seems a bit dodgy to say $\infty = \infty$, but this is matter of definition; you won't really encounter this type of "philosophical" problem in your exam.

- $|\infty| \geq |a|$ for any $a \in \mathbb{C}$.

∞ is not like a point on $\mathbb{C}$, but rather like a gigantic circle that you can never reach.

1.2 Möbius Maps

Definition 1.1 (Möbius Map)**.** $\psi : \mathbb{C} \to \mathbb{C}$ is a **Möbius map** if:

$$\psi(z) := \frac{az+b}{cz+d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonsingular matrix. (This restriction removes the possibility of $\frac{0}{0}$, or trivial maps (eg: Constant function).)

One needs to be careful when defining this function at infinity, but it should be sensible.[2]

Exercise 1.1 (Composition of two Möbius map is a Möbius map)**.** Show that for two Möbius maps ψ_1, ψ_2, its composition $\psi_1 \circ \psi_2$ is also a Möbius map.

Remark 1.1. Consider the 2×2-matrix-to-Möbius-map map as follows:

$$f(A) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$$

Then it turns out that $f(AB) = f(A)f(B)$

Exercise 1.2 (Decomposition of Möbius maps)**.** It turns out that Möbius maps can be written as composition of

- translation
- dialation ("scaling by nonzero constant")
- inversion ($z \mapsto \frac{1}{z}$)

Prove this. (Hint: You can do a constructive proof.)

Möbius maps also has a very convenient property:

Exercise 1.3 (Circline to Circline)**.** Show that Möbius maps map circlines to circline. (This means a line will either map to a circle or a line, and also a circle will either map to a circle or a line.)

(Note: This is a boring long tedious proof, that probably won't be asked in exam, but don't take my word for it.)

[2] That said, if you are supposed to define what a Möbius map is, you are **required** to definitions involving infinity as well.