Series of Common Functions at $x = 0$:

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$$

$$\sinh z = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

$$\cosh z = 1 + \frac{z^2}{2} + \frac{z^4}{24} + \cdots$$

$$\cot z = \frac{1}{z} - \frac{z}{3} + O(z^3)$$

For expansion at $x = a$, use Taylor's expansion.

Roots of Common Functions

$$\sin z = 0 \to z = n\pi$$

$$\cos z = 0 \to z = n\pi + \frac{1}{2}\pi$$

$$\sinh z = 0 \to z = n\pi i$$

$$\cosh z = 0 \to \left(n\pi + \frac{1}{2}\pi\right) i$$

$$e^z - 1 = 0 \to z = 2n\pi i$$

$$z^n - 1 = 0 \to z = e^{i\theta} \left(\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, \frac{2(n-1)\pi}{n}\right)$$

Determining Poles

1. Identify where the poles may be. (Approximations such as $\sin x \sim x$ might help).
2. If $f(z) \to \infty$ but $f(z)(z - z_0)^m \to p$ then take least m to be the "order". (Using Mathematica might be useful.)
3. Series Expansion

Residue

Either use series expansion or residue formula.

General residue formula for order m:

$$Res_a(f) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left(f(z)(z-a)^m\right)$$

Residue formula for simple pole where $f = \frac{g}{h}$ (h gives simple pole):

$$Res_a(f) = \frac{g(a)}{h'(a)}$$

Residue Theorem

$$\oint_\gamma f(z)\,dz = 2\pi i \left(\sum_{a \in S} Res_a(f) \right)$$

(For positively oriented curve)

General Strategy for Residue Calculus

When given an integral over $[0,2\pi]$, change trigs to exponential: $\cos t = \frac{z+z^{-1}}{2}, \sin t = \frac{z-z^{-1}}{2}, dz = izdt$.

When given an integral over $\mathbb{R}$, change trigs to exponential: $\cos t, \sin t \longmapsto e^{iz}$

Jordan's Lemma

$f: \mathbb{H} \to \mathbb{C}_\infty$ meromorphic on $\mathbb{H}$. Suppose $f(z) \to 0$ as $z \to \infty$ in $\mathbb{H}$. Then, as $R \to \infty$,

$$\oint_{\gamma_R} f(z) e^{i\alpha z}\,dz \to 0$$

(For any $\alpha \in \mathbb{R}^+$. γ_R is the circular arc on $\mathbb{H}$.)

Epsilon Lemma

$f: U \to \mathbb{C}$ meromorphic with <u>simple pole</u> at $a \in U$. $\gamma_\epsilon : [\alpha, \beta] \to \mathbb{C}$ with $\gamma_\epsilon(t) = a + \epsilon e^{it}$. Then:

$$\lim_{\epsilon \to 0} \oint_{\gamma_\epsilon} f(z)\,dz = Res_a(f)(\beta - \alpha) i$$

(BE CAREFUL: FOR CLOCKWISE, need the opposite sign)

(Also, when circumventing zero, $\oint_{-R}^{-\epsilon} = \oint_{\epsilon}^{R}$ for even function.)

Strategy for Choosing Contour

If there is a pole at zero, use a half circular contour, but bypassing the zero.

If there is a branch point, take a keyhole contour bypassing the branch cut (but sometimes previous might be easier.)