

Linear Algebra I & II Summary

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Preface

Note that this is not your actual lecture note, and I do not condone only sticking to this as your resource. On the other hand, if you are reading this for your intuition, you are at the right place.

This document is based on 2019 linear algebra I and 2018 linear algebra II lecture notes by Vicky Neale and Ulrike Tillmann. You can check them out at <https://courses.maths.ox.ac.uk/> in the Archive tab.

Most examples here will include the ones from these lecture notes.

Note that this summary note will gloss over quite many of the proofs, but will try to give geometric intuition behind them.

Part I

Linear Algebra I

1 Linear Equations and Matrices

You've probably already seen linear system of equations like the following:

$$\begin{cases} 3x + 5y = -1 \\ 4x - y = 10 \end{cases} \quad (1.1)$$

$$\begin{cases} 2x + y = 0 \\ 4x + 2y = 1 \end{cases} \quad (1.2)$$

$$\begin{cases} 8x - 7y + 6z = 59 \\ x + 2z = 9 \\ 3x + 2y - z = 11 \end{cases} \quad (1.3)$$

Note that (1.1) and (1.3) are uniquely solvable, but (1.2) does not have solution.

You've probably also seen examples of ones with infinite solutions (because for example, you have more variables than "effective" equations).

You can analyze them by hand, but is there a systematic approach to analyzing them? Say, you are telling a computer to solve these. What are your options?

Definition 1.1 (Matrix). For $m, n \geq 1$, an $m \times n$ matrix is a rectangular array with m rows and n columns with entries from \mathbb{R} (or \mathbb{C} or other fields).

Remark 1.1. Always count the number of rows, then number of columns.

Remark 1.2. Notation: For matrix $A \in \mathbb{R}^{m \times n}$, $A_{i,j}$ represents the entry at row i and column j .

Example 1.1 (Matrices). Here are examples of 3×2 matrices:

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -2/3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

Definition 1.2 (Vector). An $n \times 1$ matrix is called a **column vector**. A $1 \times n$ matrix is called a **row vector**.

Example 1.2 (Vector). Here are examples of a column vector and a row vector respectively:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$(1 \quad 3 \quad 2 \quad 5)$$

Remark 1.3. I highly recommend using variables for denoting column vectors by default. For example $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ If you get a row vector, always denote it with a transpose: $v^T = (1 \quad 2)$ This is because by convention, we will use left multiplication by a matrix far more often than right multiplication.

Definition 1.3 (Square Matrices). A matrix with the same number of rows and columns is called a **square matrix**.

1.1 Matrix Multiplication and Transpose

Addition and subtractions: $A + B$ are defined naturally by their entry-wise operation. (Note that A and B have to have the same dimensions!) Multiplication of matrices A and B are defined for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times \ell}$ as the following

Definition 1.4 (Matrix Multiplication).

$$(AB)_{i,j} = \sum_{k=1}^n (A_{i,k}) (B_{k,j})$$

Note that the resulting AB has dimension $m \times \ell$.

Example 1.3 (Matrix Multiplication). Consider the following two matrices:

$$A = \underbrace{\begin{pmatrix} 3 & 1 & 2 \\ 4 & 5 & -1 \end{pmatrix}}_{2 \times 3}$$

$$B = \underbrace{\begin{pmatrix} 10 \\ 15 \\ -5 \end{pmatrix}}_{3 \times 1}$$

Then:

$$AB = \underbrace{\begin{pmatrix} 3 & 1 & 2 \\ 4 & 5 & -1 \end{pmatrix}}_{2 \times 3} \underbrace{\begin{pmatrix} 10 \\ 15 \\ -5 \end{pmatrix}}_{3 \times 1} = \underbrace{\begin{pmatrix} 3 \times 10 + 1 \times 15 + 2 \times (-5) \\ 4 \times 10 + 5 \times 15 + (-1) \times (-5) \end{pmatrix}}_{2 \times 1}$$

Remark 1.4. One way to remember if matrix multiplication is well-defined is to note that: $m \times \underline{n}$ and $\underline{n} \times \ell$ results in $m \times \ell$.

Remark 1.5. The way matrix multiplication is defined may not be intuitive. However, it is a natural way to capture *linear transforms*.

Exercise 1.1 (Associativity of Matrix Multiplication). Show that for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times \ell}$, and $C \in \mathbb{R}^{\ell \times p}$, then $(AB)C = A(BC)$. Is it true that $AB = BA$ in general?

Exercise 1.2 (Diagonal matrices form a ring). Show that for **diagonal matrices** $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ such that $A_{i,j} = B_{i,j} = 0$ for $i \neq j$, AB should also be a diagonal matrix.

Exercise 1.3 (Triangular matrices form a ring). Show that for **upper triangular** $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ such that $A_{i,j} = B_{i,j} = 0$ for $i > j$, AB should also be an upper triangular matrix.

Exercise 1.4. How many diagonal $A \in \mathbb{R}^{n \times n}$ are there such that

$$A^2 = I$$

where I is the **identity matrix**, a diagonal matrix with only 1s in the diagonal entry.

i++i

Definition 1.5 (Transpose). For $A \in \mathbb{R}^{m \times n}$, **transpose** $A^T \in \mathbb{R}^{n \times m}$ is defined as

$$(A^T)_{i,j} = A_{j,i}$$

Example 1.4 (Transpose).

$$\begin{pmatrix} 3 & 0 & 1 \\ 4 & 2 & 5 \end{pmatrix}^T = \begin{pmatrix} 3 & 4 \\ 0 & 2 \\ 1 & 5 \end{pmatrix}$$

Remark 1.6. • Taking transpose of a row vector turns it into a column vector.

• Taking a transpose of a column vector turns it into a row vector.

• For any $A \in \mathbb{R}^{m \times n}$, $(A^T)^T = A$

Exercise 1.5 (Symmetric Triangular Matrix is Diagonal). Show that if an upper triangular matrix $A \in \mathbb{R}^{n \times n}$ satisfies $A^T = A$, then it must be a diagonal matrix.

Exercise 1.6 (Antisymmetric Matrix Does Not Have Diagonal Entries). Show that if $A \in \mathbb{R}^{n \times n}$ satisfies $A^T = -A$, then $A_{i,i} = 0$ for all $i = 1, \dots, n$.

Exercise 1.7 (Transpose of Product of Matrix). Show that for any $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times \ell}$,

$$(AB)^T = B^T A^T$$

Using this result, show that product of two symmetric matrices ($A^T = A$) is also a symmetric matrix.

2 System of Simultaneous Linear Equations to Matrix Equations

We've seen a system like (1.1), (1.2), and (1.3).

We can turn the three examples of linear system of equations to the following **matrix equations** of the form $\underbrace{A}_{\text{Matrix}} \underbrace{x}_{\text{Column Vector}} = \underbrace{b}_{\text{Column Vector}} :$

$$\begin{pmatrix} 3 & 5 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \end{pmatrix} \quad (2.1)$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.2)$$

$$\begin{pmatrix} 8 & -7 & 6 \\ 1 & 0 & 2 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 59 \\ 9 \\ 11 \end{pmatrix} \quad (2.3)$$

One could do even better and skip writing the name of the variables:

$$\left(\begin{array}{cc|c} 3 & 5 & -1 \\ 4 & -1 & 10 \end{array}\right) \quad (2.4)$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 1 \end{array}\right) \quad (2.5)$$

$$\left(\begin{array}{ccc|c} 8 & -7 & 6 & 59 \\ 1 & 0 & 2 & 9 \\ 3 & 2 & -1 & 11 \end{array}\right) \quad (2.6)$$

(the bar is decorative to show that the system is augmented.)

Our task is to solve it, that is, we wish to put $(A|b)$ into the form $(I|\tilde{b})$. This is unfortunately not possible to do¹.

For example, the following system is over-determined (more equations² than variables):

$$\begin{cases} 3x + y = -4 \\ 2x - 5y = 3 \\ x + 2y = 8 \end{cases}$$

and the following is under-determined (less equations than variables):

$$\begin{cases} x + 2y = 5 \end{cases}$$

Also (1.2) is an inconsistent system, as the two equations violate each other. As you could see, there are many ways things can go wrong.

A systematic approach to make analysis easier by performing “row operations” is the **Gaussian elimination**.

3 EROs and Gaussian Elimination

¹A colloquial saying is that with n variables to solve, you need n independent equations.

²Conditions