Complex Analysis Summary

Paul Joo-Hyun Kim

June 24, 2023

0 Preface

This note is for people studying complex analysis, and got lost in the middle with bunch of technical explanations. I wil try my best to be succinct as possible, stating important results (mostly without proof, but a bit of justification).

Warning: This summary note is not a substitute for the lecture note. Make sure you study from lecture note!

1 Complex Plane and Möbius Maps

1.1 Complex Plane and Complex Infinity

We will be working in what's known as the *extended complex plane.* Define the symbol $\mathbb{C}_\infty := \mathbb{C} \cup \left\{ \underbrace{\infty}_{\text{Complex Infinity}} \right\}$; that is, I refer to the space of complex numbers and infinity.

Note that in $\mathbb{C}_\infty$, ∞ is different from infinity in real numbers. $\infty := \frac{1}{0}$ is a value that is not "larger" or "smaller" than any number (since we are talking about complex number...), but rather a number on a complex plane at a really far distance from origin.

It is **WRONG** to say:

- $\infty \geq a$ for any $a \in \mathbb{C}_\infty$
- $\infty \leq a$ for any $a \in \mathbb{C}_\infty$

However, it is **CORRECT**[1] to say:

[1] Subtlety here: it seems a bit dodgy to say $\infty = \infty_\infty$, but this is matter of definition; you won't really encounter this type of "philosophical" problem in your exam.

- $|\infty| \geq |a|$ for any $a \in \mathbb{C}_\infty$.

∞ is not like a point on $\mathbb{C}$, but rather like a gigantic circle that you can never reach.

1.2 Möbius Maps

Definition 1.1 (Möbius Map)**.** $\psi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a **Möbius map** if:

$$\psi(z) := \frac{az+b}{cz+d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonsingular matrix. (This restriction removes the possibility of $\frac{0}{0}$, or trivial maps (eg: Constant function).)

One needs to be careful when defining this function at infinity, but it should be sensible.[2]

Exercise 1.1 (Composition of two Möbius map is a Möbius map)**.** Show that for two Möbius maps ψ_1, ψ_2, its composition $\psi_1 \circ \psi_2$ is also a Möbius map.

Remark 1.1. Consider the 2×2-matrix-to-Möbius-map map as follows:

$$f(A) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$$

Then it turns out that $f(AB) = f(A)f(B)$

Exercise 1.2 (Decomposition of Möbius maps)**.** It turns out that Möbius maps can be written as composition of

- translation
- dialation ("scaling by nonzero constant")
- inversion ($z \mapsto \frac{1}{z}$)

Prove this. (Hint: You can do a constructive proof.)

Möbius maps also has a very convenient property:

Exercise 1.3 (Circline to Circline)**.** Show that Möbius maps map circlines to circline. (This means a line will either map to a circle or a line, and also a circle will either map to a circle or a line.)

(Note: This is a boring long tedious proof, that probably won't be asked in exam, but don't take my word for it.)

[2] That said, if you are supposed to define what a Möbius map is, you are **required** to definitions involving infinity as well.

2 Complex Differentiability

Complex differentiability is one of the highlights of the complex analysis.

Definition 2.1 (Differentiable Function AKA Holomorphic Function)**.** Take $a \in \mathbb{C}$. Let $f : U \to \mathbb{C}$ be a function where U is a neighbourhood[3] of a. Then f is **(complex) differentiable** or **holomorphic** at a if

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and call it derivative of f at a. If f is differentiable for all points in U, then it is said to be differentiable/holomorphic on U.

Remark 2.1. Note that the definition seems to be have trivially extended from real analysis. <u>However, there is a subtlety</u>. The limit does not approach just from positive or negative side, but from any direction. (Figure 2.1)

Exercise 2.1 (Differentiation Rules)**.** Show that all differentiation rules from real analysis holds with holomorphic functions.

- Sum
- Product Rule
- Quotient Rule
- Chain Rule

Due to the definition of complex limits being more restrictive, a more nontrivial result follows.

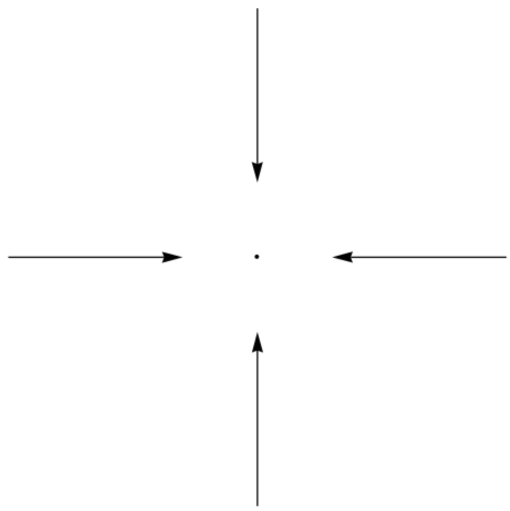
Exercise 2.2 (Cauchy-Riemann Equations)**.** Let $a \in \mathbb{C}$ and U be a neighbourhood of a. $f : U \to \mathbb{C}$ be holomorphic a. Write $f(z) = u(x, y) + iv(x, y)$ where u, v are real functions and $z = x + iy$ where $x, y \in \mathbb{R}$. Then $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ all exist, and the following **Cauchy-Riemann equations** hold:

$$\partial_x u = \partial_y v$$
$$\partial_x v = -\partial_y u$$

(Hint: Figure 2.1 might give you an insight.)

[3]Some open set containing a.

(a) Real Limit



(b) Complex Limit

Figure 2.1: Real limit (former) only concerns the approaching value from left and right side, but complex limit (latter) oncerns the approaching value from all direction.

Remark 2.2. If Cauchy-Riemann does not hold, then it must mean that f is not holomorphic! (Consider $f(z) = \bar{z}$. Cauchy-Riemann does not hold for any point, so it is nowhere holomorphic.)

Exercise 2.3. If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on U, and u, v are twice differentiable, deduce that u and v are *harmonic*, that is, they satisfy the Laplace equation $\Delta u = \Delta v = 0$.

Remark 2.3. It turns out complex plane reveals a lot about solving Laplace equation!

Here is another kicker:

Exercise 2.4 (Cauchy-Riemann to Holomorphic)**.** If the partial derivatives exist and are continuously differentiable, Cauchy-Riemann implies holomorphicity.

Remark 2.4. This means if you check that Cauchy-Riemann holds, you can immediately assume you can construct an analytic function!

Holomorphic functions also have Taylor expansion:

Remark 2.5 (Holomorphic functions have Taylor expansion)**.** If f is holormorphic at a, then in a neighbourhood of a, you can write

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

All the formulae for Taylor expansion holds (term-by-term differentiation, etc.)

Example 2.1 (Common Function Definitions)**.** Here are definitions for some of the functions.

$$\begin{aligned} e^z = \exp z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ \cos z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

Exercise 2.5. Show that

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Exercise 2.6. Show that $\exp(z + w) = \exp(z)\exp(w)$

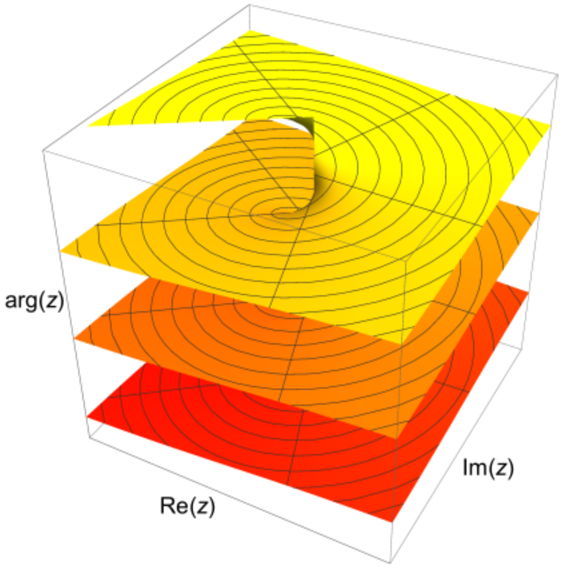


Figure 3.1: $\arg z$ being multivalued results in the need to introduce branch cut for logarithm.

3 Branch Cut

Sometimes, there is just no sensible way to define a function that it is holomorphic everywhere... Two of the unfortunate (or fortunate) functions is the logarithm and square root. We will first introduce the logarithm function.

Define logarithm function as:

$$\log z := \log |z| + i\theta$$

where θ is the argument of z.

The choice of the interval for θ changes the behaviour of $\log z$ function. For example, one could take the interval to be $[0, 2\pi)$, or one could take it to be $[-\pi, \pi)$, or even just $[a, a + 2\pi)$ for some $a \in \mathbb{R}$.

The problem is that for given z, the argument of z is not unique, and if you try to define it continuously around a circle, you will find that it is not possible... (Figure 3.1). This means there needs to be some sort of contour from 0 that the function is not continuous on. This is known as a **branch cut**, and you have total freedom to choose based on what problem you want to solve.

Example 3.1 (Where do we use branch cut?)**.** If you want to solve a problem with a fracture in an elastic solid (Figure 3.2), one standard way to solve it is to find some holomorphic function outside of the crack $[-c, c]$ satisfying

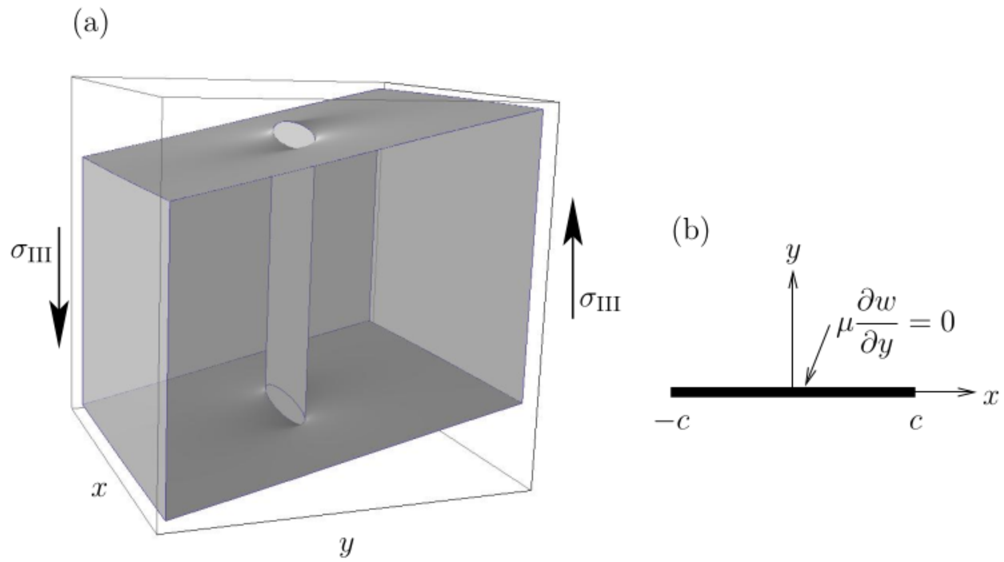


Figure 3.2: Fracture in elastic material.

some condition. It turns out that there is no function that is holomorphic everywhere satisfying that, so you would define a “branch cut” to be the straight line $[-c, c]$ to resolve it. Then it is possible to define a function that is continuous away from the crack.

Remark 3.1. When I say I am defining a branch cut, it means I am defining the function to be continuous away from the branch cut. **I am not the value of the function** at the function.

Branch cuts are something that honestly makes more sense once you **played around with it for a while**.

Example 3.2 (Logarithm: branch cut along positive real axis)**.** See Figure 3.3 for the diagram of a branch cut along positive real axis. $\log z := \log|z| + \theta i$ where $\theta \in [0, 2\pi)$ has a branch cut along positive real axis. Right above the positive real axis, θ takes the value 0, so $(\log x)_+ = \log|x|$. On the other hand, right below the positive real axis, θ takes the value 2π, so $(\log x)_- = \log|x| + 2\pi i$

So you might ask: *What is* $\log z$ *where* $z \in \mathbb{R}^{>0}$*?* and the answer is, you are asking the wrong question, because we can only define the “limiting value” on each side of the branch cut, NOT on the branch cut.

Example 3.3 (Logarithm: branch cut along positive real axis)**.** This time take $\theta \in [-\pi, \pi)$. On the right side of the branch cut, we have $\theta = \pi$,

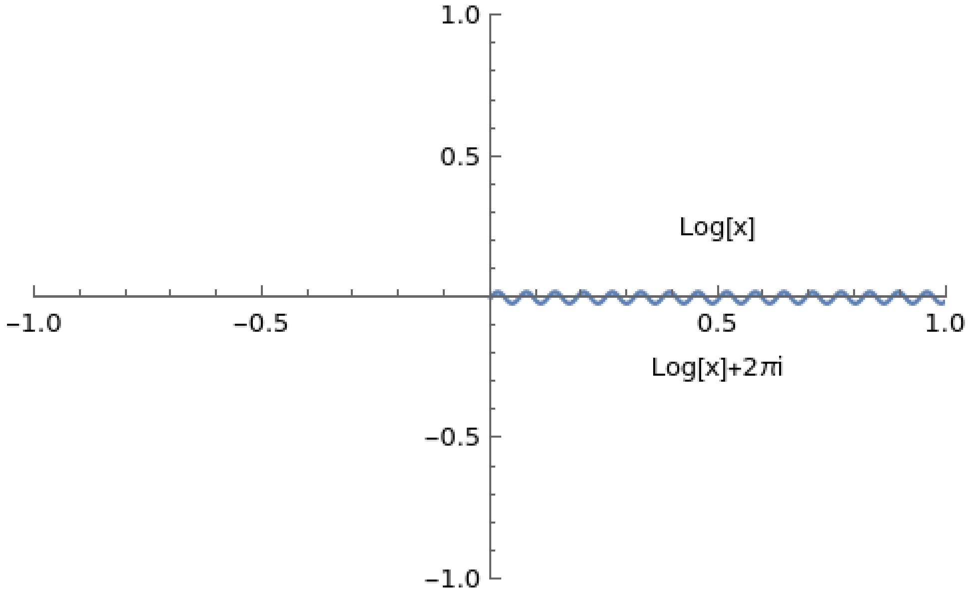


Figure 3.3: Logarithm function with branch cut along the positive real axis

so $(\log(yi))_+ = \log y + \pi i$, whereas on the left side, we have $\theta = -\pi$, so $(\log(yi))_- = \log y - \pi$.

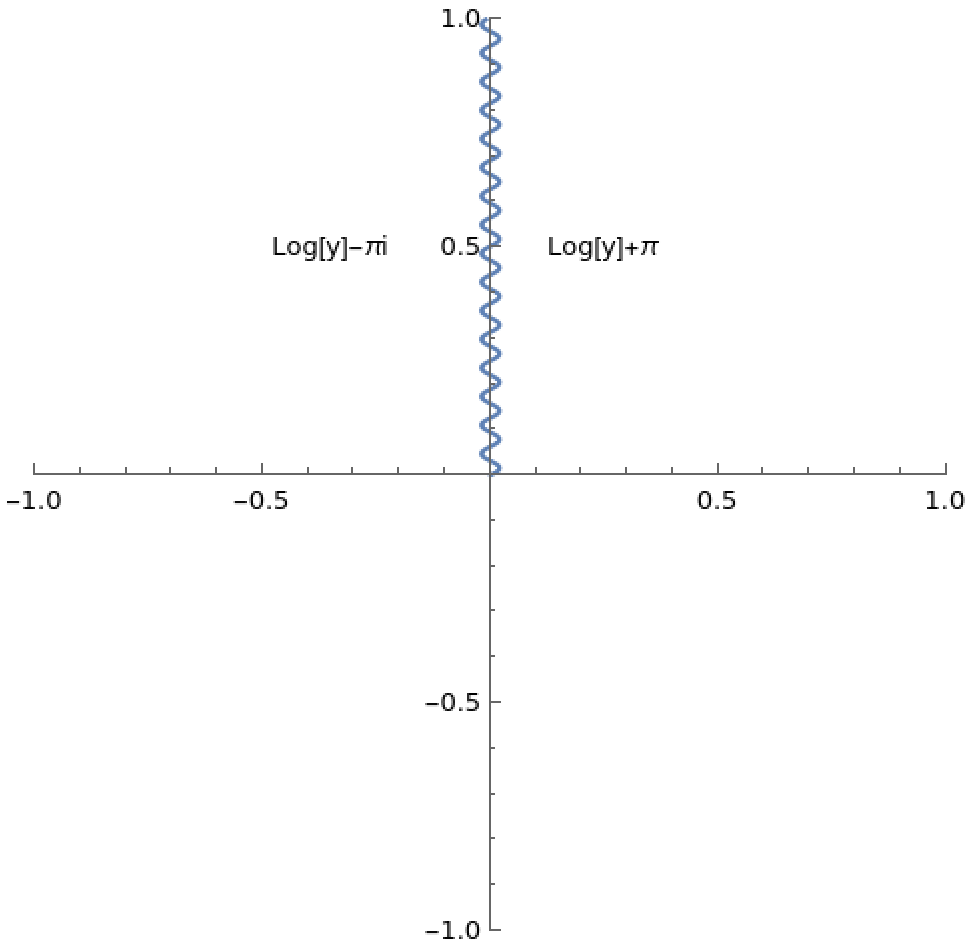


Figure 3.4: Logarithm function with branch cut along the positive imaginary axis