

Complex Analysis Summary

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0 Preface

This note is for people studying complex analysis, and got lost in the middle with bunch of technical explanations. I will try my best to be succinct as possible, stating important results (mostly without proof, but a bit of justification).

Warning: This summary note is not a substitute for the lecture note. Make sure you study from lecture note!

1 Complex Plane and Möbius Maps

1.1 Complex Plane and Complex Infinity

We will be working in what's known as the *extended complex plane*. Define the symbol $\mathbb{C}_\infty := \mathbb{C} \cup \underbrace{\{\infty\}}_{\text{Complex Infinity}}$; that is, I refer to the space of complex numbers and infinity.

Note that in \mathbb{C}_∞ , ∞ is different from infinity in real numbers. $\infty := \frac{1}{0}$ is a value that is not “larger” or “smaller” than any number (since we are talking about complex number...), but rather a number on a complex plane at a really far distance from origin.

It is **WRONG** to say:

$$\circ \infty \geq a \text{ for any } a \in \mathbb{C}_\infty$$

$$\circ \infty \leq a \text{ for any } a \in \mathbb{C}_\infty$$

However, it is **CORRECT**¹ to say:

$$\circ |\infty| \geq |a| \text{ for any } a \in \mathbb{C}_\infty.$$

∞ is not like a point on \mathbb{C} , but rather like a gigantic circle that you can never reach.

1.2 Möbius Maps

Definition 1.1 (Möbius Map). $\psi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is a **Möbius map** if:

$$\psi(z) := \frac{az + b}{cz + d}$$

¹Subtlety here: it seems a bit dodgy to say $\infty = \infty$, but this is matter of definition; you won't really encounter this type of “philosophical” problem in your exam.

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonsingular matrix. (This restriction removes the possibility of $\frac{0}{0}$, or trivial maps (eg: Constant function).)

One needs to be careful when defining this function at infinity, but it should be sensible.²

Exercise 1.1 (Composition of two Möbius map is a Möbius map). Show that for two Möbius maps ψ_1, ψ_2 , its composition $\psi_1 \circ \psi_2$ is also a Möbius map.

Remark 1.1. Consider the 2×2 -matrix-to-Möbius-map map as follows:

$$f(A) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$$

Then it turns out that $f(AB) = f(A)f(B)$

Exercise 1.2 (Decomposition of Möbius maps). It turns out that Möbius maps can be written as composition of

- translation
- dialation (“scaling by nonzero constant”)
- inversion ($z \mapsto \frac{1}{z}$)

Prove this. (Hint: You can do a constructive proof.)

Möbius maps also has a very convenient property:

Exercise 1.3 (Circline to Circline). Show that Möbius maps map circlines to circline. (This means a line will either map to a circle or a line, and also a circle will either map to a circle or a line.)

(Note: This is a boring long tedious proof, that probably won’t be asked in exam, but don’t take my word for it.)

2 Complex Differentiability

Complex differentiability is one of the highlights of the complex analysis.

²That said, if you are supposed to define what a Möbius map is, you are **required** to definitions involving infinity as well.

Definition 2.1 (Differentiable Function AKA Holomorphic Function). Take $a \in \mathbb{C}$. Let $f : U \rightarrow \mathbb{C}$ be a function where U is a neighbourhood³ of a . Then f is **(complex) differentiable** or **holomorphic** at a if

$$f'(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists, and call it derivative of f at a . If f is differentiable for all points in U , then it is said to be differentiable/holomorphic on U .

Remark 2.1. Note that the definition seems to be have trivially extended from real analysis. However, there is a subtlety. The limit does not approach just from positive or negative side, but from any direction. (Figure 2.1)

Exercise 2.1 (Differentiation Rules). Show that all differentiation rules from real analysis holds with holomorphic functions.

- Sum
- Product Rule
- Quotient Rule
- Chain Rule

Due to the definition of complex limits being more restrictive, a more nontrivial result follows.

Exercise 2.2 (Cauchy-Riemann Equations). Let $a \in \mathbb{C}$ and U be a neighbourhood of a . $f : U \rightarrow \mathbb{C}$ be holomorphic a . Write $f(z) = u(x, y) + iv(x, y)$ where u, v are real functions and $z = x + iy$ where $x, y \in \mathbb{R}$. Then $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ all exist, and the following **Cauchy-Riemann equations** hold:

$$\begin{aligned}\partial_x u &= \partial_y v \\ \partial_x v &= -\partial_y u\end{aligned}$$

(Hint: Figure 2.1 might give you an insight.)

Remark 2.2. If Cauchy-Riemann does not hold, then it must mean that f is not holomorphic! (Consider $f(z) = \bar{z}$. Cauchy-Riemann does not hold for any point, so it is nowhere holomorphic.)

³Some open set containing a .



(a) Real Limit



(b) Complex Limit

Figure 2.1: Real limit (former) only concerns the approaching value from left and right side, but complex limit (latter) concerns the approaching value from all direction.

Exercise 2.3. If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on U , and u, v are twice differentiable, deduce that u and v are *harmonic*, that is, they satisfy the Laplace equation $\Delta u = \Delta v = 0$.

Remark 2.3. It turns out complex plane reveals a lot about solving Laplace equation!

Here is another kicker:

Exercise 2.4 (Cauchy-Riemann to Holomorphic). If the partial derivatives exist and are continuously differentiable, Cauchy-Riemann implies holomorphicity.

Remark 2.4. This means if you check that Cauchy-Riemann holds, you can immediately assume you can construct an analytic function!

Holomorphic functions also have Taylor expansion:

Remark 2.5 (Holomorphic functions have Taylor expansion). If f is holomorphic at a , then in a neighbourhood of a , you can write

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

All the formulae for Taylor expansion holds (term-by-term differentiation, etc.)

Example 2.1 (Common Function Definitions). Here are definitions for some of the functions.

$$\begin{aligned} e^z = \exp z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ \sin z &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

Exercise 2.5. Show that

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

Exercise 2.6. Show that $\exp(z + w) = \exp(z) \exp(w)$



Figure 3.1: $\arg z$ being multivalued results in the need to introduce branch cut for logarithm.

3 Branch Cut

Sometimes, there is just no sensible way to define a function that it is holomorphic everywhere. . . Two of the unfortunate (or fortunate) functions is the logarithm and square root. We will first introduce the logarithm function.

Define logarithm function as:

$$\log z := \log |z| + i\theta$$

where θ is the argument of z .

Exercise 3.1. Verify that $\exp(\log z) = z$.

The choice of the interval for θ changes the behaviour of $\log z$ function. For example, one could take the interval to be $[0, 2\pi)$, or one could take it to be $[-\pi, \pi)$, or even just $[a, a + 2\pi)$ for some $a \in \mathbb{R}$.

The problem is that for given z , the argument of z is not unique, and if you try to define it continuously around a circle, you will find that it is not possible. . . (Figure 3.1). This means there needs to be some sort of contour from 0 that the function is not continuous on. This is known as a **branch cut**, and you have total freedom to choose based on what problem you want to solve.



Figure 3.2: Fracture in elastic material.

Example 3.1 (Where do we use branch cut?). If you want to solve a problem with a fracture in an elastic solid (Figure 3.2), one standard way to solve it is to find some holomorphic function outside of the crack $[-c, c]$ satisfying some condition. It turns out that there is no function that is holomorphic everywhere satisfying that, so you would define a “branch cut” to be the straight line $[-c, c]$ to resolve it. Then it is possible to define a function that is continuous away from the crack.

Remark 3.1. When I say I am defining a branch cut, it means I am defining the function to be continuous away from the branch cut. I state this again due to its importance: **I am defining a function.**

Branch cuts are something that honestly makes more sense once you **played around with it for a while.**

Example 3.2 (Logarithm: branch cut along positive real axis). See Figure 3.3 for the diagram of a branch cut along positive real axis. $\log z := \log |z| + \theta i$ where $\theta \in [0, 2\pi)$ has a branch cut along positive real axis. Right above the positive real axis, θ takes the value 0, so $(\log x)_+ = \log |x|$. On the other hand, right below the positive real axis, θ takes the value 2π , so $(\log x)_- = \log |x| + 2\pi i$

So you might ask: *What is $\log z$ where $z \in \mathbb{R}^{>0}$?* and the answer is, you are asking the wrong question, because we can only define the “limiting value” on each side of the branch cut, NOT on the branch cut.

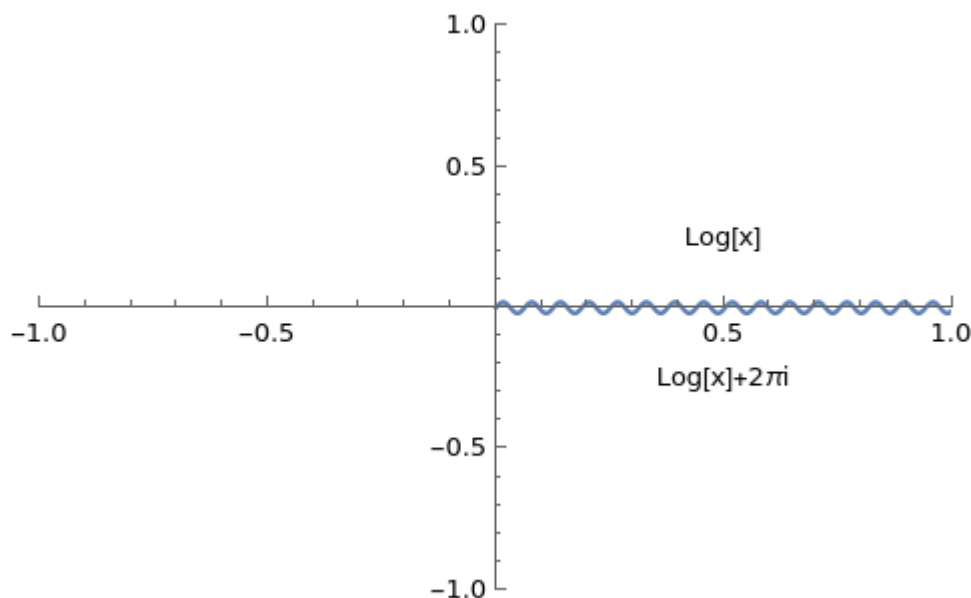


Figure 3.3: Logarithm function with branch cut along the positive real axis

Example 3.3 (Logarithm: branch cut along positive real axis). See Figure 3.4. This time take $\theta \in [-\pi, \pi)$. On the right side of the branch cut, we have $\theta = \pi$, so $(\log(yi))_+ = \log y + \pi i$, whereas on the left side, we have $\theta = -\pi$, so $(\log(yi))_- = \log y - \pi i$.

Exercise 3.2 (Square Root). Consider the definition of square root as given:

$$z^{1/2} := |z|^{1/2} e^{i\theta/2}$$

(Note that I've just taken "half" as exponent in the polar form.)

Define the branch cut along the negative real axis and evaluate $i^{1/2}$ in this branch. Define the branch cut along the negative imaginary axis and evaluate it again.

Remark 3.2. In both square roots and logarithms, there was a point which the branch cut naturally starts from. These points are called **branch point**; these are the points that you cannot avoid having a branch cut.

Remark 3.3. There is absolutely no need for a branch cut to be a straight line, and in some cases, it is more natural to define the branch cut in some other way (out of scope, however) Figure 3.5 is a classic example of non-straight branch.

Exercise 3.3. Try defining some branch cuts of $(1+z)^{1/2}$. What is the branch point in this case?



Figure 3.4: Logarithm function with branch cut along the positive imaginary axis **Edit:** π on the right side should be πi

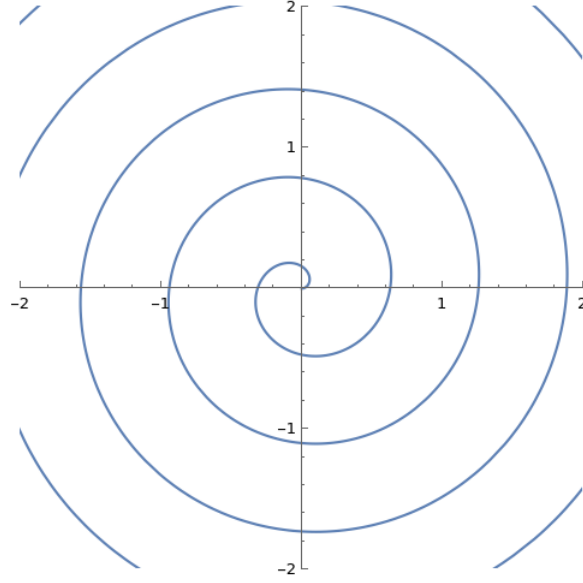


Figure 3.5: A possible branch from a branch point (taken to be origin); a bit more complicated to describe, and often times, drawing this diagram might just be sufficient.

Example 3.4 (Square Root Branch Cut of Elastic Crack). Suppose you want to define the branch of the function $(z^2 - 1)^{1/2}$ such that it is holomorphic away from the branch cut $[-1, 1]$.

Consider writing the function in a following way:

$$(z^2 - 1)^{1/2} = (z - 1)^{1/2} (z + 1)^{1/2}$$

Now note that $(z - 1)^{1/2}$ and $(z + 1)^{1/2}$ need branch cuts. One way to define them is through:

$$\begin{aligned} (z - 1)^{1/2} &= r_1^{1/2} e^{i\theta_1/2} \\ (z + 1)^{1/2} &= r_2^{1/2} e^{i\theta_2/2} \end{aligned}$$

where $\theta_1, \theta_2 \in [-\pi, \pi)$ (See Figure 3.6⁴) Hence, we get our branch cut between -1 and 1:

$$(z^2 - 1)^{1/2} = r_1^{1/2} r_2^{1/2} e^{i(\frac{\theta_1 + \theta_2}{2})}$$

Exercise 3.4. Show that the branch cut defined on Example 3.4 has different limiting values across the branch cut. **Again, the angle range given is not a mistake!**

⁴The angle range is not a mistake, even though it might seem a bit unintuitive.

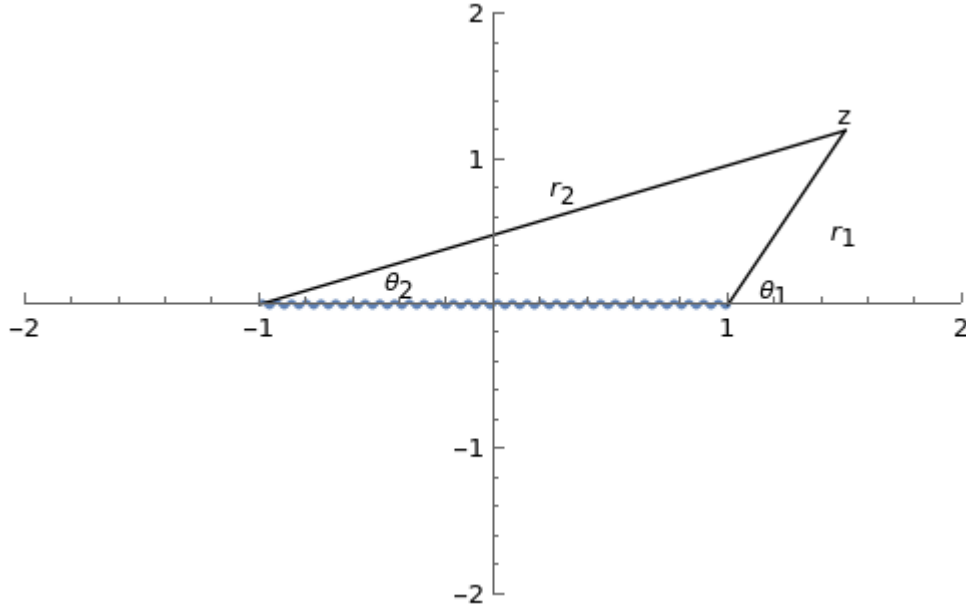


Figure 3.6: $r_1, r_2, \theta_1, \theta_2$ as shown. The point on the upper right is the z . Branch cut is at $[-1, 1]$.

Exercise 3.5. Suppose the angle ranges are given to be $\theta_1 \in [0, 2\pi)$, $\theta_2 \in [-\pi, \pi)$. Verify that the branch cuts are $(-\infty, -1] \cup [1, \infty) \subset \mathbb{R}$; that is, $(z^2 - 1)^{1/2}$ has jump discontinuity across those intervals.

(Hint: Sketch of the branch cut diagram is given as Figure 3.7)

Exercise 3.6. Let $f(z) = (k - i)^{1/2}$ with the branch cut at $i[1, \infty)$ (positive imaginary axis starting from i , the branch point). Show that $f(0) = e^{-i\pi/4}$.

4 Paths and Integration

You may have seen paths (AKA curves or lines) and integral along them in multivariable calculus. The complex analysis version is similar, but from a different perspective.

4.1 Paths

Definition 4.1 (Path). Let $a < b$. $\gamma : [a, b] \rightarrow \mathbb{C}$ is a **path** if it is a continuous function. It is **closed** if $\gamma(a) = \gamma(b)$, that is, if the endpoints coincide.

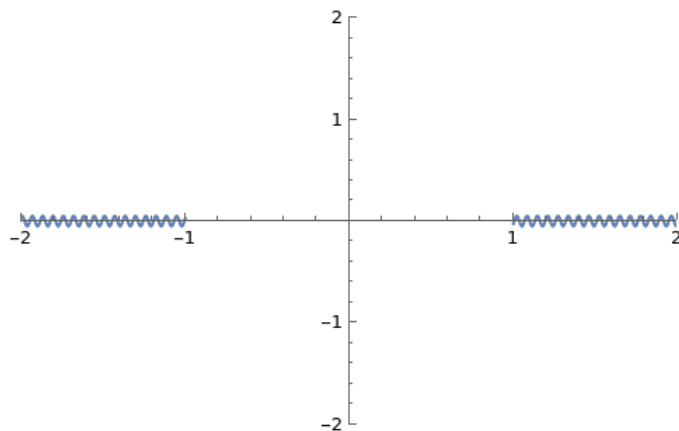


Figure 3.7: Branch cut diagram for $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [-\pi, \pi)$

Tangent vector of a curve was an important concept in curves in MVC. Similarly, one could define the notion of it in complex analysis by introducing the “derivative”.

Definition 4.2 (Differentiability of Path). Path $\gamma : [a, b] \rightarrow \mathbb{C}$ is **differentiable** at t_0 if its real and imaginary parts are differentiable at t_0 , which is equivalent to saying

$$\lim_{t \rightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists. If so, we write this limit as $\gamma'(t_0)$.

If $\gamma'(t)$ is continuous, then we say the path is in C^1 .

Exercise 4.1 (“Tangent”). Show that $\gamma'(t)$ (if it exists) characterizes the tangent direction of the path γ . (Hint: Turn it into an MVC problem!) Explain what happens at t_0 if $\gamma'(t_0) = 0$.

Example 4.1. Consider

$$\gamma(t) := \begin{cases} t^2 & -1 \leq t \leq 0 \\ it^2 & 0 \leq t \leq 1 \end{cases}$$

This is a piecewise C^1 path. See Figure 4.1 Note that $\gamma'(0) = 0$, so there is no “tangent” at the sharp corner.

Example 4.2 (Circle). **One of the most important paths!** You can parameterize a circle with center z_0 and radius r by:

$$\gamma(t) = z_0 + re^{it}$$

where $t \in [0, 2\pi]$. (You could also do $\gamma(t) = z_0 + re^{2\pi it}$ where $t \in [0, 1]$)

Note that the direction is counterclockwise.



Figure 4.1: Path given by piecewise C^1 path.

4.2 Complex Path Integral

Definition 4.3 (Complex Integral). Given a complex function $F(t) = x(t) + iy(t)$, one could define the integral over $t \in [a, b]$ to be

$$\int_a^b F(t) dt := \int_a^b x(t) dt + i \int_a^b y(t) dt$$

Exercise 4.2. Prove that, just like in real analysis, you can bound by the integral of the modulus of integrand; that is:

$$\left| \int_a^b F(t) dt \right| \leq \int_a^b |F(t)| dt$$

Definition 4.4 (Path Integral). Given piecewise C^1 path $\gamma : [a, b] \rightarrow \mathbb{C}$ and $f : \mathbb{C} \rightarrow \mathbb{C}$, then

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt$$

Remark 4.1. For remembering the definition, you can treat it as substitution rule in real integral:

$$\begin{aligned} z &= \gamma(t) \\ dz &= \gamma'(t) dt \end{aligned}$$

Example 4.3 (Terms in Taylor Around a Circle⁵). Given unit circle around origin (counterclockwise) as the path, let's compute the path integral of $f(z) = z^n$ where $n \in \mathbb{Z}$.

$$\begin{aligned} \int_{\gamma} z^n dz &= \int_0^{2\pi} (e^{it})^n e^{it} i dt \\ &= \int_0^{2\pi} e^{it(n+1)} i dt \\ &= \begin{cases} \frac{1}{n+1} [e^{it(n+1)} i]_0^{2\pi} & (n \neq -1) \\ \int_0^{2\pi} i dt & (n = -1) \end{cases} \\ &= \begin{cases} 0 & (n \neq -1) \\ 2\pi i & (n = -1) \end{cases} \end{aligned}$$

Exercise 4.3 (Path integral is well-defined). Show that if $\gamma : [a, b] \rightarrow \mathbb{C}$ and $\tilde{\gamma} : [c, d] \rightarrow \mathbb{C}$ are equivalent paths (of same orientation), then for any continuous function,

$$\int_{\gamma} f(z) dz = \int_{\tilde{\gamma}} f(z) dz$$

(Hint: since γ and $\tilde{\gamma}$ are equivalent paths, there exists a bijective $s : [c, d] \rightarrow [a, b]$ with $s'(t) > 0$ such that $s(c) = a, s(d) = b$.)

Definition 4.5 (Length). If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a C^1 path, then **length** of γ is defined as

$$\ell(\gamma) := \int_a^b |\gamma'(t)| dt$$

Remark 4.2. This is very similar to the length defined in MVC...

Remark 4.3 (Path Integral Properties). Given functions f, g and paths γ, η ,

◦ Linearity

$$* \int_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz$$

◦ Opposite Orientation: If γ^- is traversal of γ in the opposite direction,

$$* \int_{\gamma} f(z) dz = - \int_{\gamma^-} f(z) dz$$

◦ Additivity: If $\gamma \star \eta$ is concatenation of the two paths,

$$* \int_{\gamma \star \eta} f(z) dz = \int_{\gamma} f(z) dz + \int_{\eta} f(z) dz$$

⁵Come back to this after you do residue theorem as well!

◦ Estimation Lemma (γ^* is the image of the path)

$$* \left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in \gamma^*} |f(z)| \ell(\gamma)$$

* Useful for having an upper bound.

Exercise 4.4. Prove the estimation lemma.

Here is also a theorem that resembles the FTC

Exercise 4.5. $F(z)$ is called **primitive** of f if $F'(z) = f(z)$. Suppose $\gamma : [a, b] \rightarrow U$ is a piecewise C^1 path in U , then prove

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a))$$

Also the fact that zero derivative means constant:

Exercise 4.6. Let $f : U \rightarrow \mathbb{C}$ with $f'(z) = 0$ for all $z \in U$, then show f is constant.

5 Cauchy's Theorem

One awesome theorem by Mr. Cauchy here!

Theorem 5.1 (Cauchy's Theorem). *Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function over U . Let γ be a closed path in U such that interior lies entirely in U . Then*

$$\int_{\gamma} f(z) dz = 0$$

The proof of this is quite tedious...but the idea is that you prove this theorem when γ is triangular path, then generalize it to star-like domain, then generalize it further!

Example 5.1. Take γ to be a counterclockwise unit circle around the origin. For any polynomial $f(z)$,

$$\int_{\gamma} f(z) dz = 0$$

by Cauchy's theorem, because polynomial is holomorphic inside γ . (In fact, try verifying this by computing it!)

Example 5.2 (WRONG Use of Cauchy's Theorem). Again, take γ to be a counterclockwise unit circle around the origin. This time take $f(z) = z^{-2}$. You will find that

$$\int_{\gamma} f(z) dz = 0$$

but this is NOT by Cauchy's theorem, as $f(z)$ here is not holomorphic inside γ . (Rather, it is just a consequence of it having a primitive that does not cross a branch cut.)

In fact, you will get zero integral for $f(z) = \frac{1}{z^n}$ for integer $n \geq 2$, but these are not by Cauchy's theorem.

What about $f(z) = \frac{1}{z}$ though? (See Example 4.3 which might be relevant; this is, as mentioned before, related to residue calculus which is coming!)

Theorem 5.2 (Cauchy's Integral Formula). *Suppose $f : U \rightarrow \mathbb{C}$ is holomorphic on an open set U containing disc $\bar{B}(a, r)$. Then for all $w \in B(a, r)$,*

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz$$

where γ is $\partial B(a, r)$ (counterclockwise).

Remark 5.1. Cauchy's integral formula is also an example of residue theorem (which will be covered later)!!! (If you are only interested in knowing theorems, then it means residue calculus is all you need at the end of the day...) Note that integrand is holomorphic in $U \setminus \{w\}$.

Definition 5.1 (Winding Number). Consider a closed curve γ and assume z_0 is not on the curve. Then it is possible to write the curve function (parameterized over $t \in [0, 1]$ as $\gamma(t) = z_0 + |\gamma(t) - z_0|e^{2\pi i a(t)}$ for some $a(t)$ real continuous function.⁶ (In fact (for simplicity consider $z_0 = 0$) if $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)} = |\gamma(t)|e^{2\pi i b(t)}$, then $a(t) = b(t) + n$ for some $n \in \mathbb{Z}$.)

Winding number of the curve γ around z_0 is defined as:

$$I(\gamma, z_0) := a(1) - a(0) \in \mathbb{Z}$$

An explicit form of winding number of the curve, if it is piecewise C^1 , can be written as the following:

$$I(\gamma, z_0) := \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

⁶Writing out a complex number as a polar form centered at z_0 .



Figure 5.1: How to count the winding number

Example 5.3. Name says it all, winding number counts how many times a curve winds around a point. (Positive means anticlockwise winding and negative means clockwise winding.) See Figure 5.1. Start from the green point (on the side labeled 1), and follow the curve in the direction of the arrow (anticlockwise). Once you get to the inner point also labeled 1, it means you’ve counted one winding. Continue from the point inner labeled 2 until you get to the green point (on the side labeled 2). You’ve now counted the second winding.

Overall, the winding number around the red point is 2.

If the direction of the arrow were reversed (so that it is clockwise), you would’ve ended up with -2 as the winding number.

Remark 5.2. The explicit integral representation of the winding number is an application of Cauchy’s integral formula to $f(z) \equiv 1$, which again I remark is an application of the residue theorem!

Exercise 5.1. Prove (or justify) why the explicit form involving the integral should be the winding number (if you are justifying, you could use residue calculus).

Exercise 5.2 (Cauchy’s Integral Formula for Derivatives). If $f : U \rightarrow \mathbb{C}$ is holomorphic on open set U , then for any $z_0 \in U$, $f(z)$ is equal to its Taylor series at z_0 , and it converges on any open disk centered at z_0 lying in U , and derivatives are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Remark 5.3. This implies that “analytic = holomorphic”! They can be used interchangeably! This also means holomorphic functions are infinitely differentiable.

Exercise 5.3 (Liouville’s Theorem). Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire⁷ function. If f is bounded, then f is a constant function.

Remark 5.4. A part of fundamental theorem of algebra can be proven using Liouville’s theorem!

Example 5.4 (Wiener-Hopf Method). **THIS IS ABSOLUTELY OUT OF SCOPE FOR PART A**, but demonstrates how powerful Liouville’s theorem can be!

⁷Holomorphic on \mathbb{C}

Suppose you want to solve the following problem for smooth bounded $f(x)$ for $x \in \mathbb{R}$:

$$\int_0^\infty K(x-t)f(t) dt = f(x) \quad \text{for } x \geq 0$$

where $K(x) = e^{-|x|}$ for $x \in \mathbb{R}$.

After a bit of trickery⁸ (in Applied Complex Variables in part C), you end up needing to solve the following for \hat{f}_+ and \hat{h}_- (or at least one of the two unknown functions):

$$\frac{1-k^2}{k+i}\hat{f}_+(k) = (k-i)\hat{h}_-(k) \quad \text{for } 0 < \Im(k) < 1$$

Wait, you have to solve for \hat{f}_+ for \hat{h}_- from a single equation? The magic here is that the LHS is holomorphic on $\Im(k) > 0$, and the RHS is holomorphic on $\Im(k) < 1$. Also from more complicated analysis of the problem, we know that $\hat{f}_+(k) = O(k^{-1})$ and $\hat{h}_-(k) = O(k^{-1})$.

Define $E(k)$ as

$$E(k) = \begin{cases} \frac{1-k^2}{k+i}\hat{f}_+(k) & (\Im(k) > 0) \\ (k-i)\hat{h}_-(k) & (\Im(k) < 1) \end{cases}$$

(Note that on the intersection $0 < \Im(k) < 1$, either of the definitions work, due to the given problem.) $E(k)$ is an entire function by construction, and because $\hat{f}_+(k) = O(k^{-1})$ and $\hat{h}_- = O(k^{-1})$, it is bounded at (complex infinity), so by Liouville's theorem $E(k) \equiv C$ for some constant C . Hence, (in the definition of Wiener-Hopf method, f_+ is defined to be $f_+(x) = f(x)\mathbb{I}_{x>0}$, where \mathbb{I} is the indicator function.

$$\hat{f}_+(k) = \frac{C(k+i)}{1-k^2}$$

Taking inverse Fourier transform,

$$f(x) = f_+(x) = \frac{C}{2\pi} \int_\Gamma \frac{(k+i)e^{-ikx}}{1-k^2} dk$$

where Γ is a contour from $-\infty + ci$ to $+\infty + ci$ for any $c \in (0, 1)$.

It turns out that in most cases, the region at which a function ceases to be holomorphic are isolated most of the time, and sometimes, not even non-holomorphic even if you haven't defined them!

⁸In a nutshell, one takes (complex) Fourier transform of the entire problem

Exercise 5.4 (Riemann's Removable Singularity Theorem). Suppose $f : U \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic and bounded near z_0 , then show f extends to a holomorphic function on all of U .

Example 5.5. Consider $f(z) = \frac{\sin z}{z}$. From the definition, it seems like $f(z)$ might not be holomorphic at $z = 0$, but $f(z)$ is bounded around $z = 0$ (since $\lim_{z \rightarrow 0} f(z) = 1$), and in a neighbourhood away from $z = 0$, so by Riemann's removable singularity theorem, $f(z)$ is holomorphic at $z = 0$.

Note that if it is holomorphic at $z = 0$, then it also means it is infinitely differentiable at $z = 0$.

Remark 5.5 (General Strategy for Checking Holomorphicity). Given $f(z)$,

1. Identify which points might be problematic.
2. Check boundedness (often via checking if limit exists).
3. If bounded, then it is definitely holomorphic, otherwise not holomorphic at that point.

Exercise 5.5. \circ Given $f(z) = \frac{\sin z}{\cos z}$, is it holomorphic at $z = \frac{\pi}{2}$?

- \circ What about $f(z) = \frac{(z - \frac{\pi}{2}) \sin z}{\cos z}$?
- \circ Is $f(z) = \frac{1}{(z+1)^2(z-1)}$ holomorphic at $z = i$? What about at $z = 1$? What about at $z = -1$?
- \circ Define branch cut of \sqrt{z} along the positive real axis. Determine if $f(z) = \frac{(z+i)\sqrt{z}}{z}$ is holomorphic at $z = -i$ and $z = i$, and if holomorphic, evaluate at those points.

Here is a kind-of converse to Cauchy's theorem.

Theorem 5.3 (Morera's Theorem). Suppose $f : U \rightarrow \mathbb{C}$ is a continuous function on open $U \subset \mathbb{C}$. If for any closed path γ in U , $\int_{\gamma} f(z) dz$, then f is holomorphic.

6 Identity Theorem, Isolated Zeros, and Singularities

I've mentioned it before, but we deal with isolated singularities often, that is, the isolated points which some function f is not holomorphic.

Definition 6.1 (Pole and Order). Suppose $f(z)$ is holomorphic on $U \setminus \{a\}$ for some $a \in \mathbb{C}$. The minimum $n \in \mathbb{Z}^{n \geq 0} \cup \{+\infty\}$ such that $(z - a)^n f(z)$ is holomorphic on U is called **pole** of **order** n of f at $z = a$.

- If $n = 0$, then we say a is a **removable singularity**⁹ as before.
- If $n = 1$, then we say a is a **simple pole**.
- If $1 \leq n < \infty$, then we say a is a **pole of order** n .
- If $n = \infty$, then we say a is an **essential singularity**.

Example 6.1. ◦ $\frac{\sin z}{z}$ has a removable singularity at $z = 0$.

- $\frac{1}{z}$ has a simple pole at $z = 0$.
- $\frac{1}{z^2}$ has a pole of order 2 at $z = 0$.
- $\frac{1}{\cos z}$ has simple poles at $z = \frac{\pi}{2} \pm n\pi$ for $n \in \mathbb{Z}$.
- $e^{\frac{1}{z}}$ has essential singularity at $z = 0$.

Remark 6.1 (Determining the Order, Laurent Series, and Residue). For tips on determining the order of a singularity, refer to https://github.com/pauljoohyunkim/OxfordMathematicsNotesForPoorSouls/blob/main/2nd%20Year/Metric%20Spaces%20and%20Complex%20Analysis/Residue%20Calculus/Computing_Residue.pdf.

But, here are the main points:

1. Identify where the singularities are (in most cases, it is the zeros of the denominator of the functions you are given).
2. Taylor expand the denominator (and possibly the numerator).
3. Use the formula $\frac{1}{1-r} = 1 + r + r^2 + \dots$ for $|r| < 1$ to transform denominator to a multiplication by an infinite series.

What you are basically computing is “Laurent series” which gives all the info about the order of the pole and residue, which will be introduced very shortly.

Remark 6.2. The idea of isolated singularity is also very important to residue calculus.

⁹You can think of this as not having any singularity at all, in fact; if we have removable singularity, it means we can “get rid of” it by analyticity.

Theorem 6.1 (Identity Theorem). *Let U be a domain, and f_1 and f_2 are holomorphic on U . If $S = \{z \in U \mid f_1(z) = f_2(z)\}$ has a limit point in U , then $S = U$, and hence $f_1(z) = f_2(z)$.*

Remark 6.3. In a way, you could think of it as, if two functions f_1 and f_2 agree on limiting points on a “dense set”, then they must be equal on the domain.

One could think intuitionistically by noting that if f is holomorphic at z_0 , then f is equal to its Taylor series in a neighbourhood around that point.

Exercise 6.1. Suppose f has a pole of order m at z_0 . Show that there exists some $r > 0$ such that for all $z \in B(z_0, r) \setminus \{z_0\}$,

$$f(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

that is, f is equal to its **Laurent series** in any sufficiently small annulus around z_0 . (Hint: What is the definition of a pole of order m ?)

Remark 6.4. If f has essential singularity at z_0 , then Laurent series goes from negative infinity to positive infinity:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Definition 6.2 (Principal Parts and Residue). Given f in its Laurent series around z_0 ,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

(where $c_m = 0$ for $m \leq -k - 1$ if pole of order k) the **principal part** of f at z_0 is

$$\sum_{n=-\infty}^{-1} c_n (z - z_0)^n$$

and **residue** of f at z_0 is

$$\text{Res}_{z=z_0} f(z) = c_{-1}$$

Remark 6.5. You might wonder why the principal part and residue are defined this way.

Principal part is the only part that results in f having a “singularity”, so for singularity analysis, one only needs to look at the principal part.

Residue is actually really cool. Consider counterclockwise integral of $f(z)$ (assume holomorphic in the unit circle except at 0) around the unit circle around 0.

$$\begin{aligned}\int_{|z|=1} f(z) \, dz &= \int_{|z|=1} \sum_{n=-\infty}^{\infty} c_n z^n \, dz \\ &= \int_{|z|=1} \frac{c_{-1}}{z} \, dz \\ &= 2\pi i \operatorname{Res}_{z=0} f(z)\end{aligned}$$

where the second equality comes from Exercise 4.3. This means if you know the residues of $f(z)$ at its singularities, then one could compute closed-curve integrals of $f(z)$ easily, and this is the motivation for the residue calculus.

Example 6.2 (Computing Residues). (As mentioned before) For examples of computing residues step-by-step, refer to https://github.com/pauljoohyunkim/OxfordMathematicsNotesForPoorSouls/blob/main/2nd%20Year/Metric%20Spaces%20and%20Complex%20Analysis/Residue%20Calculus/Computing_Residue.pdf

Conceptually, it is important that you recall that you can write $f(z)$ as its Laurent series around an annulus centered at its singularity.

Exercise 6.2 (Residue of f/g). Suppose f, g are holomorphic functions and g has a zero at z_0 . Show that

$$\operatorname{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

(Hint: Obviously, consider Taylor expansion, like I told you!)

Exercise 6.3 (Residue of pole of order m). (More general than the previous exercise) Suppose f has a pole of order m at z_0 . Show that

$$\operatorname{Res}_{z=z_0}(f) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z-z_0)^m f(z))$$

(Hint: Consider Laurent series of f .)

7 Homotopies, Simply-Connected Domains, and Cauchy's Theorem

Definition 7.1 (Homotopy and Simply-Connectedness). Suppose that U is an open set in \mathbb{C} and $a, b \in U$, and that $\eta : [0, 1] \rightarrow U$, $\gamma : [0, 1] \rightarrow U$ are paths in U such that $\gamma(0) = \eta(0) = a$ and $\gamma(1) = \eta(1) = b$.

γ and η are **homotopic** if there exists a continuous function $h : [0, 1] \times [0, 1] \rightarrow U$ such that,

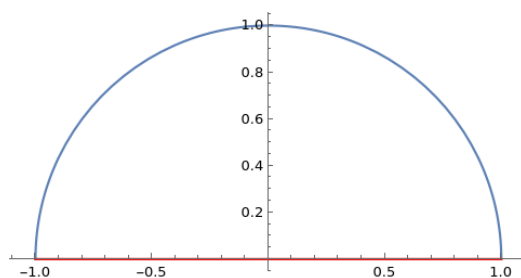
$$h(0, s) = a$$

$$h(1, s) = b$$

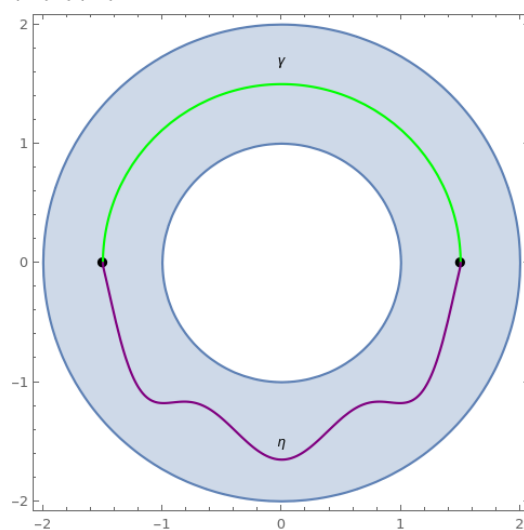
$$h(t, 0) = \gamma(t)$$

$$h(t, 1) = \eta(t)$$

See Figure 7.1 for examples of homotopicity and non-homotopicity. We say



(a) These two curves are homotopic in \mathbb{C} as you can continuously deform one to get the other.



(b) These two curves are not homotopic in the annulus as you cannot continuously deform one to get the other.

Figure 7.1: To talk about homotopy, one must give the two curves and a region.

U is **simply-connected** if two curves in U are always homotopic.

Exercise 7.1. You can think of simply-connected regions to be regions without holes (though I guess it's an oversimplification.)

Exercise 7.2 (Deformation Theorem I). Let U be simply-connected domain. Let $f : U \rightarrow \mathbb{C}$ be holomorphic on U . If $\gamma, \tilde{\gamma}$ are paths in U from $a \in \mathbb{C}$ to $b \in \mathbb{C}$, then

$$\int_{\gamma} f(z) dz = \int_{\tilde{\gamma}} f(z) dz$$

(Hint: It helps to draw! It becomes basically a one-line argument by the use of Cauchy's theorem.)

Exercise 7.3 (Deformation Theorem II). (Nonexaminable? Maybe? idk) For closed curves, there are no "ends", so you can deform it however you want so long as it does not cross any non-holomorphic region.

8 Argument Principle

We say function is **meromorphic** if it is holomorphic except at a few isolated singularities.

Let's start with a simple (verification) exercise.

Exercise 8.1. Suppose $f : U \rightarrow \mathbb{C}$ is meromorphic and has zero of order k or a pole of order k at $z_0 \in U$. Then show that $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue k or $-k$ respectively.

Here is the statement of the argument principle

Exercise 8.2 (Argument Principle). Suppose $f : U \rightarrow \mathbb{C}$ is a meromorphic function on U . Assume $B(a, r) \subset U$. Let N denote the number of zeros (counted with multiplicity) and P denote the number of poles (counted again with multiplicity) of f inside $B(a, r)$, and f has neither zeros nor poles on $\partial B(a, r)$, then show that

$$N - P = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

where γ is a path with image $\partial B(a, r)$.

Moreover, $N - P$ is also the winding number of the path $\Gamma = f \circ \gamma$.

Remark 8.1. You should now be convinced that you can think of pole as a reverse zero if you have not already...

Remark 8.2. For understanding of the argument principle, I found this resource to be incredibly helpful: https://youtu.be/79-ESkh5_f0 Highly recommend the videos by the channel as well for complex analysis to get a grasp of some of the concepts!

Here is another result proven using argument principle.

Exercise 8.3 (Rouché's Theorem). Suppose f, g are holomorphic functions on open set $U \subset \mathbb{C}$ and $\bar{B}(a, r) \subset U$. If $|f(z)| > |g(z)|$ for all $z \in \partial B(a, r)$ then $f, f + g$ both have the same change in argument around $\partial B(a, r)$, and hence the same number of zeros in $B(a, r)$ (counted with multiplicities).

Remark 8.3. Apparently Rouché's theorem is useful for counting number of zeros of a function f .

Example 8.1. (An example from the lecture note) Let $P(z) = z^4 + 5z + 2$. On circle $|z| = 2$, $|z|^4 = 16 > 5.2 + 2 \geq \underbrace{|5z + 2|}_{g(z)}$. Then $P - g = z^4$ and P

has the same number of roots in $B(0, 2)$, so by Rouché's theorem, all four roots of $P(z)$ fall in $B(0, 2)$. Taking $|z| = 1$, $|g(z)| \geq 5 - 2 = 3 > |z^4| = 1$, so $P(z)$ and $g(z)$ have the same number of roots in $B(0, 1)$, so $P(z)$ has one root of modulus less than 1, and 3 of modulus between 1 and 2.

Other theorems that sound quite trivial can now be proven:

Exercise 8.4 (Open Mapping Theorem). Suppose $f : U \rightarrow \mathbb{C}$ is holomorphic and nonconstant on U . Then for any open set $V \subset U$, $f(V)$ is also open.

Exercise 8.5 (Inverse Function Theorem). Suppose $f : U \rightarrow \mathbb{C}$ be injective and holomorphic, and $f'(z) \neq 0$ for all $z \in U$. If $g : f(U) \rightarrow U$ is the inverse of f , then g is holomorphic, and $g'(w) = \frac{1}{f'(g(w))}$.

9 Residue Theorem

You are now at the heart of complex analysis! All the buildup for one of the most important results in complex analysis... I know it's tiring, but this is one of the sections that is almost guaranteed to be on the exam, and you can actually prepare for it by practice!

Exercise 9.1 (Residue Theorem). Suppose U is an open set in \mathbb{C} and γ is a closed curve whose inside is contained in U (so that $\forall z \notin U, I(\gamma, z) = 0$.)

Then if $S \subset U$ is a finite set such that $S \cap \gamma^*$ and f is a holomorphic function on $U \setminus S$,

Here is a very important theorem. (Don't worry if it looks complicated. In practice, you will be working with very nice functions.)

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{a \in S} I(\gamma, a) \operatorname{Res}_{z=a}(f)$$

Remark 9.1. In most cases, winding number $I(\gamma, a)$ will be 1 (or -1 if you go clockwise), so the RHS in most cases becomes $\pm \sum_{a \in S} \operatorname{Res}_{z=a}(f)$

Remark 9.2. Explicit form of the winding number, Cauchy's integral formula, and the argument principle can be understood as a case of residue theorem! Armed with residue theorem and deformation theorem (Exercise 7.2, 7.3), you can compute a vast types of integrals really easily!

Remark 9.3. We could've done Example 4.3 using residue theorem really quickly!

Exercise 9.2. Verify the following result:

$$\int_{\gamma} \frac{\sin z}{\cos z} dz = 2\pi i$$

where γ is a clockwise circular path of radius 0.2 around $\pi/2$ (Note: Beware of the sign!)

Exercise 9.3. Verify the following result:

$$\int_{\gamma} \frac{1}{(z+1)^2(z-1)} dz = -\frac{i\pi}{2}$$

where $\gamma(t) = -1 + \frac{1}{5}e^{it}$ over $t \in [0, 2\pi]$.

9.1 Residue Calculus

Being able to compute complex integrals very easily means one could now compute real integrals that does not have an easy antiderivative.

Example 9.1. Suppose we wish to compute

$$\int_0^{2\pi} \frac{1}{1 + 3 \cos^2(t)} dt$$

If we consider the path $t \mapsto e^{it}$ (a unit circle), and let $z = e^{it}$, and note that

$$\cos t = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

(since z is on the unit circle) we can turn the real integral into complex integral by the transforming the integrand

$$\frac{1}{1 + 3 \cos^2(t)} = \frac{1}{1 + 3 \times \left(\frac{1}{2} \left(z + \frac{1}{z} \right) \right)^2} = \frac{4z^2}{3 + 10z^2 + 3z^4}$$

and noting $dz = iz dt$:

$$\int_0^{2\pi} \frac{1}{1 + 3 \cos^2(t)} dt = \int_{\gamma} \frac{-4iz}{3 + 10z^2 + 3z^4} dz$$

By residue theorem, the complex integral is simply π , so

$$\int_0^{2\pi} \frac{1}{1 + 3 \cos^2(t)} dt = \pi$$

Remark 9.4. One question you might have is how I came up with the path to be a unit circle. Whenever you see an integral involving trig function such as $\sin t$ or $\cos t$, with integration range $[0, 2\pi]$ you know that you can write it as some combination of z, \bar{z} , but it is convenient to take the curve to be unit circle so that $\bar{z} = \frac{1}{z}$, so that we don't have to worry about the conjugate symbol.

At the end of the day, it is up to how much you've practiced.

Remark 9.5 (Some Categorized Examples of Residue Calculus). Refer to part C Applied Complex Variables lecture note (https://courses.maths.ox.ac.uk/pluginfile.php/22180/mod_resource/content/1/C5_6LectureNotes.pdf), section 1.9 for some VERY DOABLE examples. Note that the entire first section is in fact a highly condensed summary of relevant materials from complex analysis!

Example 9.2. Suppose we want to compute

$$I := \int_0^{\infty} \frac{1}{1 + x^2 + x^4} dx$$

Note that I can write this as

$$I = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{1 + x^2 + x^4} dx$$

It is difficult (if impossible) to get a simple form for the antiderivative of the integrand. This means we have to use complex integrals.

Note that the integral is over the range $[-R, R]$, and we see that at infinity, the integrand is like $O(z^{-4})$ as $z \rightarrow \infty$ (where $z \in \mathbb{C}$), which seems to indicate that the integrand decays at a sufficient rate (and hence be zero at infinity... All of this is hand-wavy, but it will make more sense as you practice)

Consider the following complex integral:

$$\tilde{I}_R := \int_{\gamma_R} \frac{1}{1+z^2+z^4} dz = \underbrace{\int_{y_R} \frac{1}{1+z^2+z^4} dz}_{\rightarrow 2I} + \int_{b_R} \frac{1}{1+z^2+z^4} dz$$

where γ_R is as shown in Figure 9.1 (concatenation of the yellow and blue contours) and y_R, b_R are yellow and blue paths respectively of the full contour. (Note that $\int_{y_R} \frac{1}{1+z^2+z^4} dz = \int_{-R}^R \frac{1}{1+x^2+x^4} dx$)

By estimation lemma, the contribution from b_R tends to zero as $R \rightarrow 0$:

$$\left| \int_{b_R} \frac{1}{1+z^2+z^4} dz \right| \leq \sup_{z \in b_R} \left| \frac{1}{1+z^2+z^4} \right| \pi R \leq \frac{\pi R}{R^4 - R^2 - 1} \rightarrow 0$$

where in the last inequality, I used the (reverse) triangle inequality. This implies $\tilde{I}_R \rightarrow \int_{y_R} \frac{1}{1+z^2+z^4} dz$ as $R \rightarrow \infty$. This means all we now have to do is to compute I_R using residue theorem, which turns out to be:

$$\tilde{I}_R = 2\pi i \left(\operatorname{Res}_{z=\omega} \left(\frac{1}{1+z^2+z^4} \right) + \operatorname{Res}_{z=\omega^2} \left(\frac{1}{1+z^2+z^4} \right) \right) = \pi/\sqrt{3}$$

where $\omega := e^{i\pi/3}$

Hence, putting it all together,

$$\pi/\sqrt{3} = \lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{1}{1+z^2+z^4} dz$$

So ultimately,

$$I = \frac{\pi}{2\sqrt{3}}$$

Remark 9.6 (PRO TIP: Behaviour at Infinity). If you can infer the behaviour of integral at infinity, the whole process of turning real integral into a complex one might just become more natural. Suppose we have integral $\int_{\Gamma_R} f(z) dz$ such that $f(z) = O(\frac{1}{R^n})$, and Γ_R is a circular arc that “goes to” infinity¹⁰ as $R \rightarrow \infty$.

¹⁰Just like the semicircle from the example.

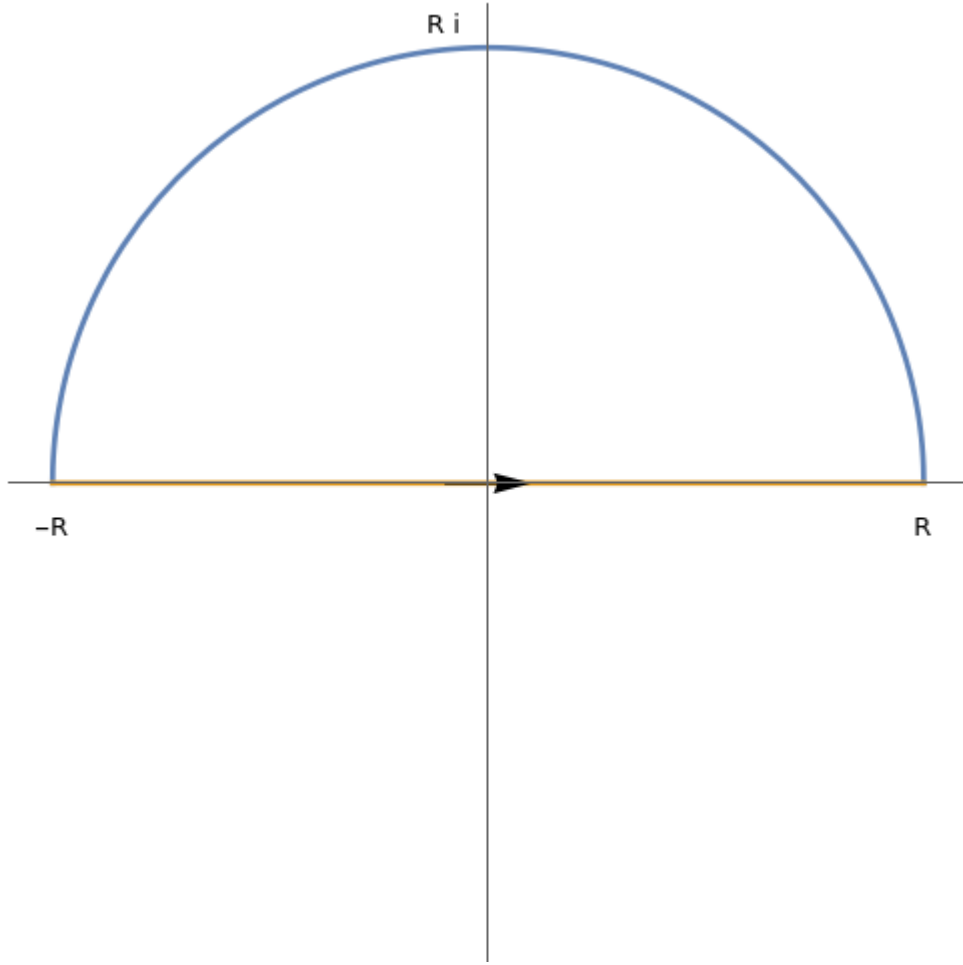


Figure 9.1: Contour for \tilde{I}_R

Bounding the contribution:

$$\left| \int_{\Gamma_R} f(z) dz \right| \leq \sup_{z \in \Gamma_R^*} |f(z)| \ell(\Gamma_R^*) \leq \frac{C_1}{R^n} (C_2 R) = C R^{1-n}$$

where

- the first inequality is from estimation lemma.
- the second inequality is from the definition of big-O notation and the fact that Γ_R is a part of a circle, so we know that $\ell(\Gamma_R^*) = O(R)$.

This means if $n > 1$, $\int_{\Gamma_R} f(z) dz$ vanishes as $R \rightarrow 0$.

Example 9.3 (Behaviour at Infinity). For the following $f(z)$, the integral $\int_{\Gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$ where Γ_R is a part of a circle of radius R (it can be the full circle of radius R as well).

- $\frac{1}{z+z^2}$

- $\frac{\sin z}{z^{3/2}+z^2}$

* Γ_R here would have to exclude the branch cut!

- $z^n e^{-z}$ for any $n \in \mathbb{Z}$