§1 Linear System $\dot{\mathbf{x}} = A\mathbf{x}$ $\qquad \mathbf{x}(0) = \mathbf{x}_0 \Longrightarrow \mathbf{x}(t) = e^{tA}\mathbf{x}_0$

(If $A = BCB^{-1}$, $e^{tA} = Be^{tC}B^{-1} = B\text{diag}(e^{\lambda_1 t}, \cdots, \lambda_n t)B^{-1}$)

<u>2D</u> $\lambda_1\lambda_2 < 0$ Saddle $\qquad \lambda_1\lambda_2 > 0$ Node

$\lambda_1 = \lambda_2$ and $\dim E_\lambda = 1$ Degenerate node ($y1 = y_{10}e^{\lambda t} + y_{20}te^{\lambda t}$, $y_2 = y_{20}e^{\lambda t}$): Secular, but stability still determined by sign.

$\lambda_1 = a + ib$, $\lambda_2 = a - ib \Longrightarrow a = 0$ Center $\qquad a \neq 0$ Focus (Spiral)

Hyperbolic System: All eigval $\neq 0$

Eigenspace is invariant.

<u>Lin Subsp</u>: For $A\mathbf{w}_j = \lambda_j\mathbf{w}_j$, $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$, $\lambda_j = a_j + ib_j$ (For nonsemisimple, $(A - \lambda I)^k\mathbf{w} = 0$)

$E^s = \text{Span}(\mathbf{u}_j, \mathbf{v}_j | a_j < 0)$ $\qquad E^c = \text{Span}(\mathbf{u}_j, \mathbf{v}_j | a_j = 0)$ $\qquad E^u = \text{Span}(\mathbf{u}_j, \mathbf{v}_j | a_j > 0)$

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§2 Nonlinear Systems

<u>Orbit</u> of $\mathbf{x}_0$: curve $\Gamma_{\mathbf{x}_0} \subset E$: $\Gamma_{\mathbf{x}_0} = \{\mathbf{x}(t; \mathbf{x}_0) | t \in \mathbb{R}\}$

<u>Flow</u> is the map $\varphi_t : E \to E$ s.t. $\varphi_t(\mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}_0)$

$\mathbf{p} \in E$ is <u>ω-limit pt</u> of $\varphi_t(\mathbf{x})$: $\exists t_1 < t_2 < \cdots < t_n \to \infty$ s.t. $\lim_{i\to\infty} \phi_{t_i} = \mathbf{p}$ $\qquad$ cf) α-lim pt

Closed invariant set A <u>Attracting set</u> if $\exists$ neighborhood U of A s.t. $\varphi_t(U) \subset U$ and $A = \cap_{t>0}\varphi_t(U)$

<u>Domain of attraction</u> $D(A) = \cup_{t\leq 0}\varphi_t(U)$ (AKA set of all IC s.t. have A as ω-lim set)

<u>Attractor</u>: Attracting set with a dense orbit.

<u>Poincaré-Bendixson Thrm</u>: $E \subset \mathbb{R}^2$ open, $\mathbf{f} \in C^1(E)$. If $D \subset E$ compact (closed bdd) s.t. $\mathbf{x}(t) \in D$ for all $t \geq 0$ where $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then orbit is either a limit cycle or approaches a limit cycle, or approaches an equilibrium.

Fixed point $\mathbf{x}_0$ <u>Lyapunov stable</u>: $\forall\epsilon > 0 \exists\delta > 0$ s.t. $\forall\mathbf{x} \in B_\delta(\mathbf{x}_0)$ and $t \geq 0$, $\varphi_t(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0)$

Fixed point $\mathbf{x}_0$ <u>asympt. stable</u>: (i) Lyapunov and (ii) $\exists\delta > 0$ s.t. $\varphi_t(\mathbf{x}) \to \mathbf{x}_0$ as $t \to \infty$ $\forall x \in B_\delta(\mathbf{x}_0)$

Lyapunov Function $V(\mathbf{x}_0) = 0, V(\mathbf{x}) > 0 \forall\mathbf{x} \in W \setminus \{\mathbf{x}_0\}$ $\qquad$ Tip: $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \dot{\mathbf{x}} = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$

(i) $\dot{V}(\mathbf{x}) \leq 0$ $\forall\mathbf{x} \in W \setminus \{\mathbf{x}_0\} \Longrightarrow$ Stable

(ii) $\dot{V}(\mathbf{x}) < 0$ $\forall\mathbf{x} \in W \setminus \{\mathbf{x}_0\} \Longrightarrow$ Asymptotic Stable

(iii) $\dot{V}(\mathbf{x}) > 0$ $\forall\mathbf{x} \in W \setminus \{\mathbf{x}_0\} \Longrightarrow$ Unstable

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§3 Local Analysis

From linearized system around origin,

$W^s(0) = \left\{(x, y) \in \mathbb{R}^2 | \varphi_t(x, y) \to 0 \text{ as } t \to \infty\right\}$, $W^u(0) = \left\{(x, y) \in \mathbb{R}^2 | \varphi_t(x, y) \to 0 \text{ as } t \to -\infty\right\}$

<u>Center manifold</u>: (i) Tangent to E^c at $\mathbf{x}_0$ (ii) of class C^r (where $f \in C^R(E)$) (iii) invar under flow. (Interested in smoothest center manifold)

<u>Shadowing Theorem</u>: $(\mathbf{x}_0, \mathbf{y}_0)$ close enough to origin, $(\mathbf{x}(t), \mathbf{y}(t))$ solution $\Longrightarrow \exists\tilde{\mathbf{x}}(t)$ on center man. s.t. $\begin{cases} \mathbf{x}(t) = \tilde{\mathbf{x}}(t) + O(e^{-\gamma t}) \\ \mathbf{y}(t) = \mathbf{h}\left(\tilde{\mathbf{x}}(t)\right) + O(e^{-\gamma t}) \end{cases}$ for some const $\gamma > 0$.

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§4. Bifurcations

<u>Topological equivalence</u> Two vector field $\mathbf{f}$ and $\mathbf{g}$ and resp. flows $\varphi_t(\mathbf{x})$ and $\psi_t(\mathbf{x})$ are top.equi. if $\exists$ homeomorphism (1-1,cts,cts inv) $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ and $\tau(t, \mathbf{x}) \to \mathbb{R}$ strict. inc. on t s.t. $\tau(t + s, \mathbf{x}) = \tau(s, \mathbf{x}) + \tau(t, \varphi_s(\mathbf{x}))$

<u>Structural Stability</u>: VF $\mathbf{f}$ structurally stable if $\forall$ ctsly diff. VF $\mathbf{v}$, $\exists\epsilon_v > 0$ s.t. $\mathbf{f}$ is top.equiv to $\mathbf{f} + \epsilon\mathbf{v} \forall 0 < \epsilon < \epsilon_v$

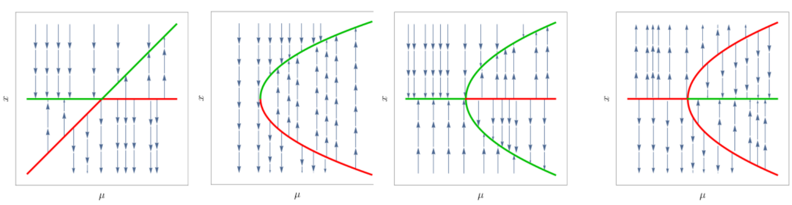
<u>Bifurcation Pt</u>: μ_c point in param. space where f is not structurally stable.

<u>Bifurcation</u>: Change in structure of the solution.

Step-by-Step: Find fixed point. Compute Jacobian (w.r.t μ as a const) around a fixed pt and set it to zero. Check Hopf.

Examples (i) $\dot{x} = \mu x - x^2$ (Transcritical, fp:$x = 0, \mu$) (ii) $\dot{x} = \mu - x^2$ (Saddle, fp:$x = \pm\mu^{1/2}$)

(iii) $\dot{x} = \mu x - x^3$ (Sup.crit pitch, fp:$x = 0, \pm\mu^{1/2}$) (iv) $\dot{x} = -\mu x + x^3$ (Sub.crit pitch, fp:$x = 0, \pm\mu^{1/2}$)



Transcritical $\qquad$ Saddle $\qquad$ supercritical pitchfork $\qquad$ subcritical pitchfork

§5 Local Analysis of Maps $\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n)$

$E^s = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j || \lambda_j| < 1)$ $\quad E^c = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j || \lambda_j| = 1)$ $\quad E^u = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j || \lambda_j| > 1)$

Stability: Fixed pt $\mathbf{x}_0$ s.t. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$ for all $n \in \mathbb{Z}^+$

Asympt Stability: $\mathbf{x}_0$ s.t. stable and $\exists \delta > 0$ s.t. $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$, $\mathbf{G}^{(n)}(\mathbf{x}) \to \mathbf{x}_0$ as $n \to \infty$

$\forall \mathbf{x} \in W^s_{loc}$, $\mathbf{G}^{(n)} \to \mathbf{x}_0$ as $n \to \infty$ $\quad \forall \mathbf{x} \in W^u_{loc}$, $\mathbf{G}^{(n)} \to \mathbf{x}_0$ as $n \to -\infty$

Periodic orbit Γ is Lyapunov stable if $\forall \epsilon > 0$: $\exists \delta > 0$ s.t. $\varphi_t(\mathbf{x}) \in U_\epsilon(\Gamma)$ for all $t \geq 0$ and $\mathbf{x} \in U_\delta$

Periodic orbit Γ Asympt stable if Lyapunov + $\exists \delta > 0$ s.t. $d(\varphi_t(\mathbf{x}), \Gamma)$ as $t \to \infty$ for all $\mathbf{x} \in U_\delta$

Poincaré Map Traversality: $\mathbf{n} \cdot \mathbf{f}(\mathbf{x}_0) > 0$

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§6 Limit Cycles and Hopf Bifur

Poincaré-Lindstedt Method: $\ddot{x} + x = \epsilon f(x)$

Define timescale $\tau = \omega t$ s.t. $x(\tau + 2\pi, \epsilon) = x(\tau, \epsilon)$ $\quad$ (Tip: $\frac{d}{dt} = \omega \frac{d}{d\tau}$)

Translation of time: $\dot{x} = 0$ at $t = 0$ $\quad$ Amplitude of oscillation: $a = x(0, \epsilon)$

Expansion: $\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$ $\quad x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau)$ (No secular solution)

(Tip: $\cos^2\tau = \frac{1+\cos 2\tau}{2}$, $\cos^3\tau = \frac{3\cos\tau + \cos 3\tau}{4}$)

Hopf $\Re(\lambda) = 0$ at $\mu = \mu_c$ $\quad \dot{r} = d\mu r + a r^3$ $\quad \dot{\theta} = \omega + c\mu + br^2$ $\quad d = \frac{d}{d\mu}\Re\lambda(\mu)\big|_{\mu=\mu_c}$, $c = \frac{d}{d\mu}\Im\lambda(\mu)\big|_{\mu=\mu_c}$

$\dot{x} = \mu x - \omega y + f(x, y)$ $\quad \dot{y} = \omega y + \mu y + g(x, y)$

$\implies a = \frac{1}{16\omega}\left((f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy})\,\omega + f_{xy}\,(f_{xx} + f_{yy}) - g_{xy}\,(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right)$

Local Bifur

$\lambda = 1$: eg) $x + \mu - x^2$ saddle $\quad x + \mu x - x^2$ transcritical $\quad x + \mu x - x^3$ pitchfork

Bifur of Periodic Orbit: Seek Poincaré map s.t. $\delta \to \delta + \frac{\delta^2}{2} P_r r(1,0) + \mu P_\mu(1,0)$. For $\dot{r} = f(r, \theta, \mu)$, $P_\mu(1,0) = \int_0^{2\pi} f_\mu(1, \theta, 0)dt$, $P_r(1,0) = 1$, $P_{rr}(1,0) = \int_0^{2\pi} f_{rr}(1, \theta, 0)dt$

$\lambda = -1$: eg) $f(x, \mu) = -x - \mu x + x^3$ Period-doubling

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§7 Global Bifur, Homoclinic Chaos, Melnikov's Method $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t)$

f has sensitivity to IC on Λ if $\exists \epsilon > 0$ s.t. for any $p \in \Lambda$ and any neighborhood U of p, $\exists p' \in U$ and $n \in \mathbb{N}$ s.t. $|f^n(p) - f^n(p')| > \epsilon$

f topolo. transitive on Λ if for any open U,$V \subset \Lambda$, $\exists n \in \mathbb{Z}$ s.t. $f^n(U) \cap V \neq \emptyset$

Λ invar compact set for invertible f. f chaotic on Λ if $\exists$ sensitivity to IC on Λ and topo. transitive on Λ

Assumption: $\mathbf{f}(\mathbf{x}) = (f_1, f_2) = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right)$ (Hamiltonian)

Melnikov Function: $M(t_0) = \int_\infty^\infty \mathbf{f}(\mathbf{q}_0(t - t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t - t_0), t)dt$ ($\mathbf{x} = \mathbf{q}_0(t)$ for $\epsilon = 0$, homoclinic)

M has a simple zero at a point $t_0 = \tau$, then P_ϵ has a transverse homoclinic point for sufficiently small $\epsilon > 0$

If $M(t_0) > 0$ or $M(t_0) < 0$ for all t_0, then $W^s(\mathbf{x}_\epsilon) \cap W^u(\mathbf{x}_\epsilon) = \emptyset$

Eg) Duffing Oscil. $\ddot{x} = x - x^3 - \delta\dot{x} + \gamma\cos t$

$\dot{x} = y = \pm x\left(1 - \frac{x^2}{2}\right)^{\frac{1}{2}} \implies (x_0(t), y_0(t)) = \mathbf{q}_0(t) = \pm(\sqrt{2}\mathrm{sech}\ t, -\sqrt{2}\mathrm{sech}\ t\tanh t)$