Complex Analysis Summary

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0 Preface

This note is for people studying complex analysis, and got lost in the middle with bunch of technical explanations. I wil try my best to be succinct as possible, stating important results (mostly without proof, but a bit of justification).

Warning: This summary note is not a substitute for the lecture note. Make sure you study from lecture note!

1 Complex Plane and Möbius Maps

1.1 Complex Plane and Complex Infinity

We will be working in what's known as the *extended complex plane*. Define the symbol $\mathbb{C}_\infty := \mathbb{C} \cup \left\{ \underbrace{\infty}_{\text{Complex Infinity}} \right\}$; that is, I refer to the space of complex numbers and infinity.

Note that in $\mathbb{C}_\infty$, ∞ is different from infinity in real numbers. $\infty := \frac{1}{0}$ is a value that is not "larger" or "smaller" than any number (since we are talking about complex number...), but rather a number on a complex plane at a really far distance from origin.

It is **WRONG** to say:

- $\infty \geq a$ for any $a \in \mathbb{C}_\infty$
- $\infty \leq a$ for any $a \in \mathbb{C}_\infty$

However, it is **CORRECT**[1] to say:

- $|\infty| \geq |a|$ for any $a \in \mathbb{C}_\infty$.

∞ is not like a point on $\mathbb{C}$, but rather like a gigantic circle that you can never reach.

1.2 Möbius Maps

Definition 1.1 (Möbius Map). $\psi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a **Möbius map** if:

$$\psi(z) := \frac{az+b}{cz+d}$$

[1]Subtlety here: it seems a bit dodgy to say $\infty = \infty$, but this is matter of definition; you won't really encounter this type of "philosophical" problem in your exam.

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonsingular matrix. (This restriction removes the possibility of $\frac{0}{0}$, or trivial maps (eg: Constant function).)

One needs to be careful when defining this function at infinity, but it should be sensible.[2]

Exercise 1.1 (Composition of two Möbius map is a Möbius map)**.** Show that for two Möbius maps ψ_1, ψ_2, its composition $\psi_1 \circ \psi_2$ is also a Möbius map.

Remark 1.1. Consider the 2×2-matrix-to-Möbius-map map as follows:

$$f(A) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$$

Then it turns out that $f(AB) = f(A)f(B)$

Exercise 1.2 (Decomposition of Möbius maps)**.** It turns out that Möbius maps can be written as composition of

- translation
- dialation ("scaling by nonzero constant")
- inversion ($z \mapsto \frac{1}{z}$)

Prove this. (Hint: You can do a constructive proof.)

Möbius maps also has a very convenient property:

Exercise 1.3 (Circline to Circline)**.** Show that Möbius maps map circlines to circline. (This means a line will either map to a circle or a line, and also a circle will either map to a circle or a line.)

(Note: This is a boring long tedious proof, that probably won't be asked in exam, but don't take my word for it.)

2 Complex Differentiability

Complex differentiability is one of the highlights of the complex analysis.

[2]That said, if you are supposed to define what a Möbius map is, you are **required** to definitions involving infinity as well.

Definition 2.1 (Differentiable Function AKA Holomorphic Function)**.** Take $a \in \mathbb{C}$. Let $f : U \to \mathbb{C}$ be a function where U is a neighbourhood[3] of a. Then f is **(complex) differentiable** or **holomorphic** at a if

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and call it derivative of f at a. If f is differentiable for all points in U, then it is said to be differentiable/holomorphic on U.

Remark 2.1. Note that the definition seems to be have trivially extended from real analysis. However, there is a subtlety. The limit does not approach just from positive or negative side, but from any direction. (Figure 2.1)

Exercise 2.1 (Differentiation Rules)**.** Show that all differentiation rules from real analysis holds with holomorphic functions.

- Sum
- Product Rule
- Quotient Rule
- Chain Rule

Due to the definition of complex limits being more restrictive, a more nontrivial result follows.

Exercise 2.2 (Cauchy-Riemann Equations)**.** Let $a \in \mathbb{C}$ and U be a neighbourhood of a. $f : U \to \mathbb{C}$ be holomorphic a. Write $f(z) = u(x, y) + iv(x, y)$ where u, v are real functions and $z = x + iy$ where $x, y \in \mathbb{R}$. Then $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ all exist, and the following **Cauchy-Riemann equations** hold:

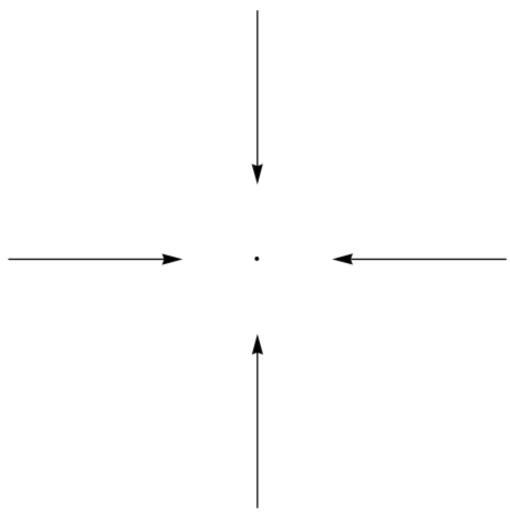
$$\begin{aligned} \partial_x u &= \partial_y v \\ \partial_x v &= -\partial_y u \end{aligned}$$

(Hint: Figure 2.1 might give you an insight.)

Remark 2.2. If Cauchy-Riemann does not hold, then it must mean that f is not holomorphic! (Consider $f(z) = \bar{z}$. Cauchy-Riemann does not hold for any point, so it is nowhere holomorphic.)

[3] Some open set containing a.

(a) Real Limit



(b) Complex Limit

Figure 2.1: Real limit (former) only concerns the approaching value from left and right side, but complex limit (latter) oncerns the approaching value from all direction.

Exercise 2.3. If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on U, and u, v are twice differentiable, deduce that u and v are *harmonic*, that is, they satisfy the Laplace equation $\Delta u = \Delta v = 0$.

Remark 2.3. It turns out complex plane reveals a lot about solving Laplace equation!

Here is another kicker:

Exercise 2.4 (Cauchy-Riemann to Holomorphic)**.** If the partial derivatives exist and are continuously differentiable, Cauchy-Riemann implies holomorphicity.

Remark 2.4. This means if you check that Cauchy-Riemann holds, you can immediately assume you can construct an analytic function!

Holomorphic functions also have Taylor expansion:

Remark 2.5 (Holomorphic functions have Taylor expansion)**.** If f is holormorphic at a, then in a neighbourhood of a, you can write

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

All the formulae for Taylor expansion holds (term-by-term differentiation, etc.)

Example 2.1 (Common Function Definitions)**.** Here are definitions for some of the functions.

$$e^z = \exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Exercise 2.5. Show that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Exercise 2.6. Show that $\exp(z + w) = \exp(z)\exp(w)$

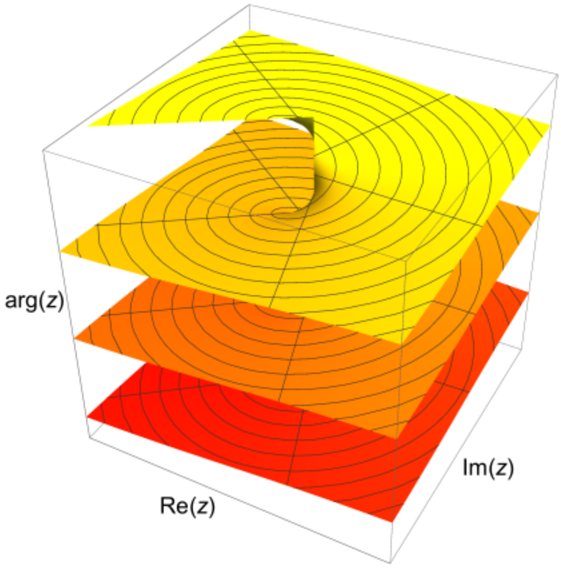


Figure 3.1: $\arg z$ being multivalued results in the need to introduce branch cut for logarithm.

3 Branch Cut

Sometimes, there is just no sensible way to define a function that it is holomorphic everywhere... Two of the unfortunate (or fortunate) functions is the logarithm and square root. We will first introduce the logarithm function.

Define logarithm function as:

$$\log z := \log |z| + i\theta$$

where θ is the argument of z.

Exercise 3.1. Verify that $\exp(\log z) = z$.

The choice of the interval for θ changes the behaviour of $\log z$ function. For example, one could take the interval to be $[0, 2\pi)$, or one could take it to be $[-\pi, \pi)$, or even just $[a, a + 2\pi)$ for some $a \in \mathbb{R}$.

The problem is that for given z, the argument of z is not unique, and if you try to define it continuously around a circle, you will find that it is not possible... (Figure 3.1). This means there needs to be some sort of contour from 0 that the function is not continuous on. This is known as a **branch cut**, and you have total freedom to choose based on what problem you want to solve.

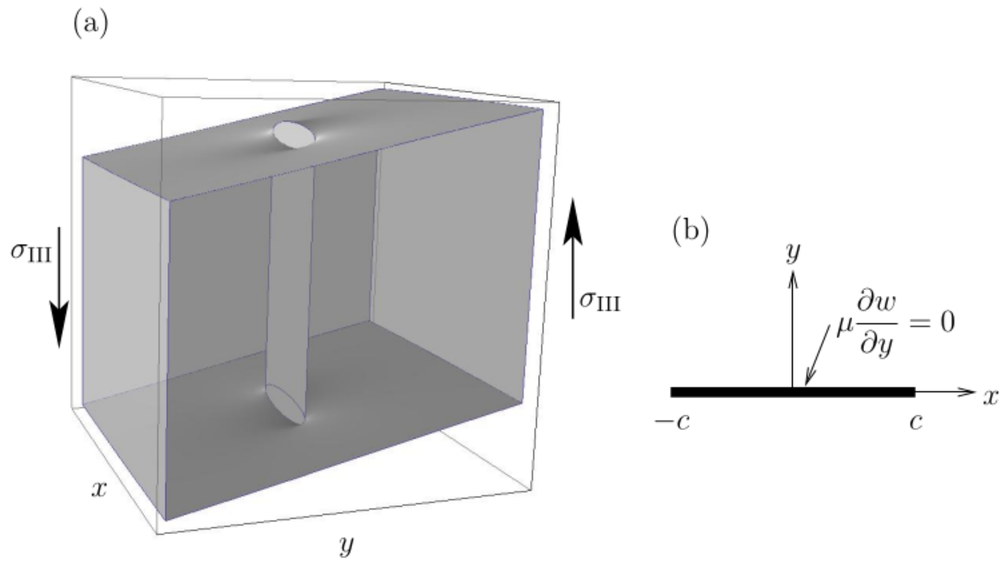


Figure 3.2: Fracture in elastic material.

Example 3.1 (Where do we use branch cut?)**.** If you want to solve a problem with a fracture in an elastic solid (Figure 3.2), one standard way to solve it is to find some holomorphic function outside of the crack $[-c, c]$ satisfying some condition. It turns out that there is no function that is holomorphic everywhere satisfying that, so you would define a "branch cut" to be the straight line $[-c, c]$ to resolve it. Then it is possible to define a function that is continuous away from the crack.

Remark 3.1. When I say I am defining a branch cut, it means I am defining the function to be continuous away from the branch cut. I state this again due to its importance: **I am defining a function**.

Branch cuts are something that honestly makes more sense once you **played around with it for a while**.

Example 3.2 (Logarithm: branch cut along positive real axis)**.** See Figure 3.3 for the diagram of a branch cut along positive real axis. $\log z := \log|z| + \theta i$ where $\theta \in [0, 2\pi)$ has a branch cut along positive real axis. Right above the positive real axis, θ takes the value 0, so $(\log x)_+ = \log|x|$. On the other hand, right below the positive real axis, θ takes the value 2π, so $(\log x)_- = \log|x| + 2\pi i$

So you might ask: *What is* $\log z$ *where* $z \in \mathbb{R}^{>0}$*?* and the answer is, you are asking the wrong question, because we can only define the "limiting value" on each side of the branch cut, NOT on the branch cut.

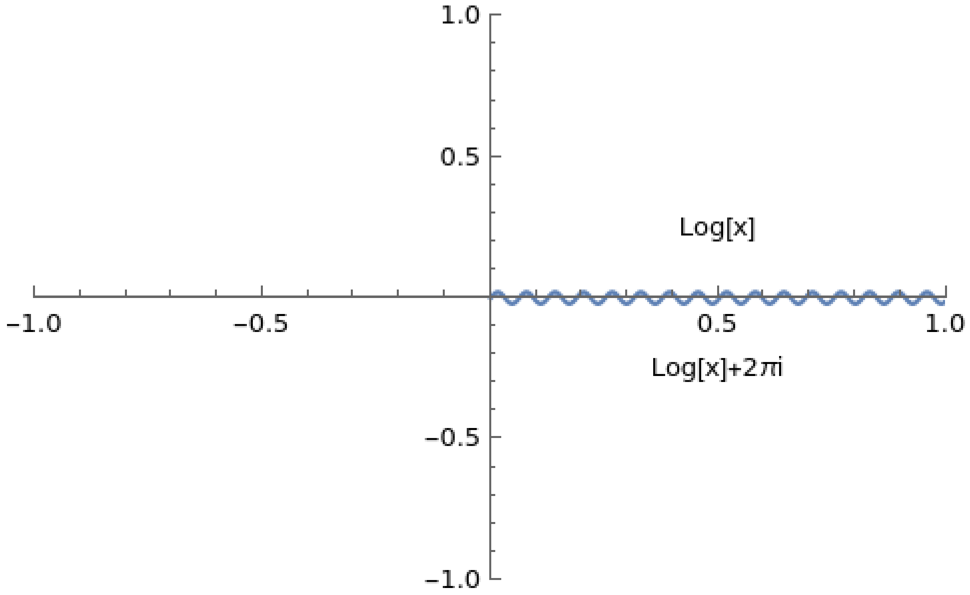


Figure 3.3: Logarithm function with branch cut along the positive real axis

Example 3.3 (Logarithm: branch cut along positive real axis)**.** See Figure 3.4. This time take $\theta \in [-\pi, \pi)$. On the right side of the branch cut, we have $\theta = \pi$, so $(\log(yi))_+ = \log y + \pi i$, whereas on the left side, we have $\theta = -\pi$, so $(\log(yi))_- = \log y - \pi i$.

Exercise 3.2 (Square Root)**.** Consider the definition of square root as given:

$$z^{1/2} := |z|^{1/2} e^{i\theta/2}$$

(Note that I've just taken "half" as exponent in the polar form.)

Define the branch cut along the negative real axis and evaluate $i^{1/2}$ in this branch. Define the branch cut along the negative imaginary axis and evaluate it again.

Remark 3.2. In both square roots and logarithms, there was a point which the branch cut naturally starts from. These points are called **branch point**; these are the points that you cannot avoid having a branch cut.

Remark 3.3. There is absolutely no need for a branch cut to be a straight line, and in some cases, it is more natural to define the branch cut in some other way (out of scope, however) Figure 3.5 is a classic example of non-straight branch.

Exercise 3.3. Try defining some branch cuts of $(1+z)^{1/2}$. What is the branch point in this case?

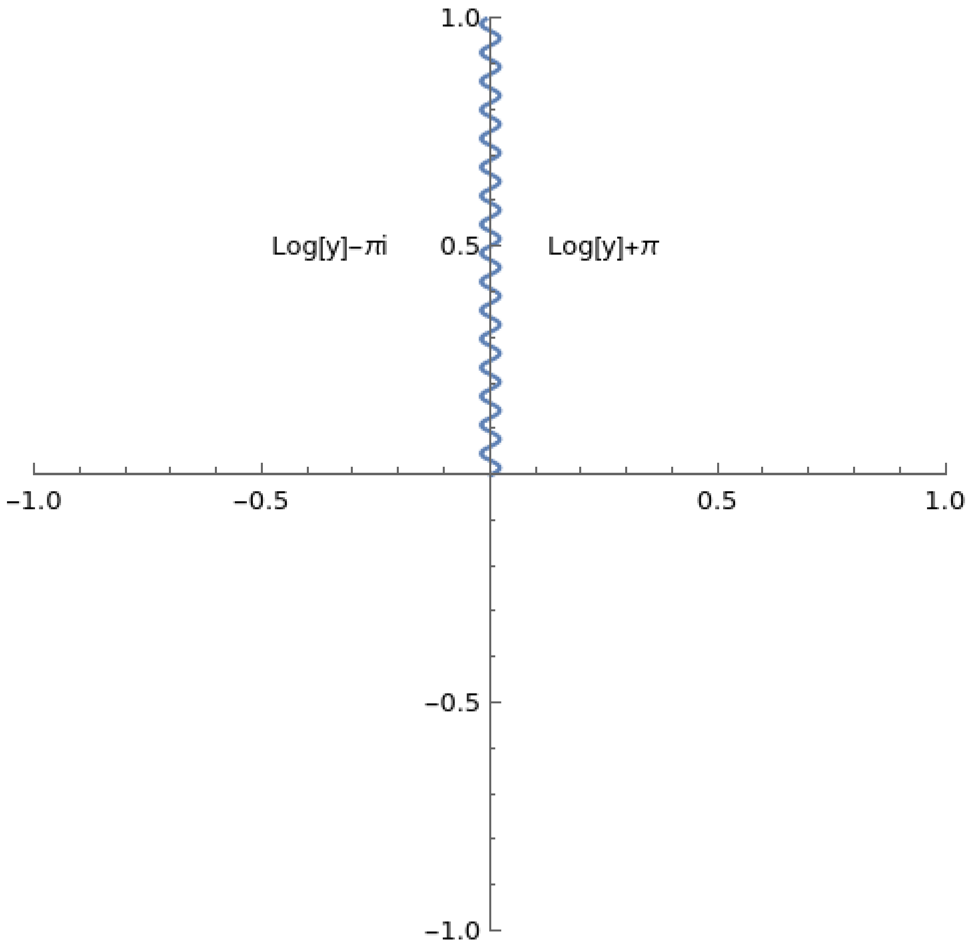


Figure 3.4: Logarithm function with branch cut along the positive imaginary axis **Edit:** π on the right side should be πi

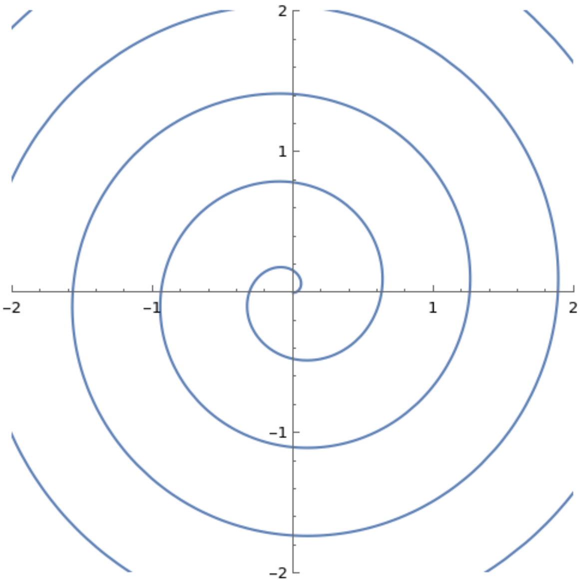


Figure 3.5: A possible branch from a branch point (taken to be origin); a bit more complicated to describe, and often times, drawing this diagram might just be sufficient.

Example 3.4 (Square Root Branch Cut of Elastic Crack)**.** Suppose you want to define the branch of the function $(z^2-1)^{1/2}$ such that it is holomorphic away from the branch cut $[-1,1]$.

Consider writing the function in a following way:

$$\left(z^2-1\right)^{1/2} = (z-1)^{1/2}(z+1)^{1/2}$$

Now note that $(z-1)^{1/2}$ and $(z+1)^{1/2}$ need branch cuts. One way to define them is through:

$$\begin{aligned}(z-1)^{1/2} &= r_1^{1/2}e^{i\theta_1/2}\\ (z+1)^{1/2} &= r_2^{1/2}e^{i\theta_2/2}\end{aligned}$$

where $\theta_1, \theta_2 \in [-\pi, \pi)$ (See Figure 3.6[4]) Hence, we get our branch cut between -1 and 1:

$$\left(z^2-1\right)^{1/2} = r_1^{1/2}r_2^{1/2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$$

Exercise 3.4. Show that the branch cut defined on Example 3.4 has different limiting values across the branch cut. **Again, the angle range given is not a mistake!**

[4] The angle range is not a mistake, even though it might seem a bit unintuitive.

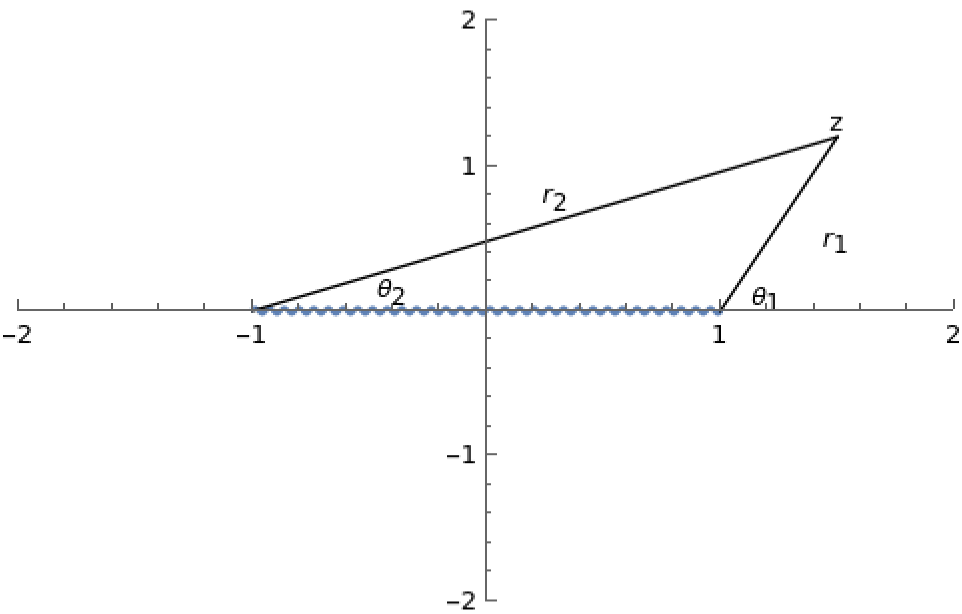


Figure 3.6: $r_1, r_2, \theta_1, \theta_2$ as shown. The point on the upper right is the z. Branch cut is at $[-1, 1]$.

Exercise 3.5. Suppose the angle ranges are given to be $\theta_1 \in [0, 2\pi)$, $\theta_2 \in [-\pi, \pi)$. Verify that the branch cuts are $(-\infty, -1] \cup [1, \infty) \subset \mathbb{R}$; that is, $(z^2 - 1)^{1/2}$ has jump discontinuity across those intervals.

(Hint: Sketch of the branch cut diagram is given as Figure 3.7)

Exercise 3.6. Let $f(z) = (k - i)^{1/2}$ with the branch cut at $i\,[1, \infty)$ (positive imaginary axis starting from i, the branch point). Show that $f(0) = e^{-i\pi/4}$.

4 Paths and Integration

You may have seen paths (AKA curves or lines) and integral along them in multivariable calculus. The complex analysis version is similar, but from a different perspective.

4.1 Paths

Definition 4.1 (Path)**.** Let $a < b$. $\gamma : [a, b] \to \mathbb{C}$ is a **path** if it is a continuous function. It is **closed** if $\gamma(a) = \gamma(b)$, that is, if the endpoints coincide.

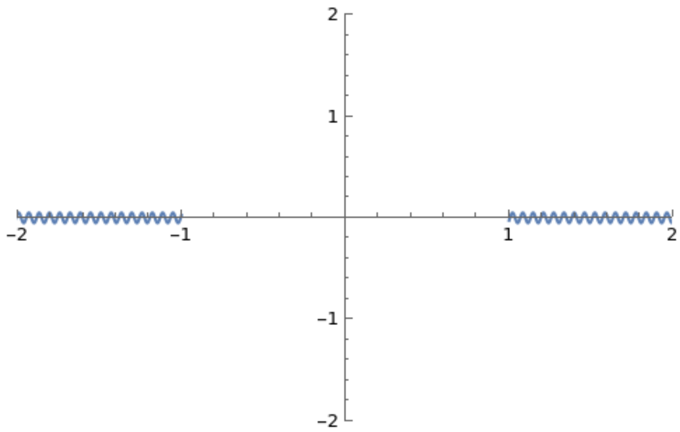


Figure 3.7: Branch cut diagram for $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [-\pi, \pi)$

Tangent vector of a curve was an important concept in curves in MVC. Similarly, one could define the notion of it in complex analysis by introducing the "derivative".

Definition 4.2 (Differentiability of Path)**.** Path $\gamma : [a, b] \to \mathbb{C}$ is **differentiable** at t_0 if its real and imaginary parts are differentiable at t_0, which is equivalent to saying

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists. If so, we write this limit as $\gamma'(t_0)$.

If $\gamma'(t)$ is continuous, then we say the path is in C^1.

Exercise 4.1 ("Tangent")**.** Show that $\gamma'(t)$ (if it exists) characterizes the tangent direction of the path γ. (Hint: Turn it into an MVC problem!) Explain what happens at t_0 if $\gamma'(t_0) = 0$.

Example 4.1. Consider

$$\gamma(t) := \begin{cases} t^2 & -1 \le t \le 0 \\ it^2 & 0 \le t \le 1 \end{cases}$$

This is a piecewise C^1 path. See Figure 4.1 Note that $\gamma'(0) = 0$, so there is no "tangent" at the sharp corner.

Example 4.2 (Circle)**. One of the most important paths!** You can parameterize a circle with center z_0 and radius r by:

$$\gamma(t) = z_0 + re^{it}$$

where $t \in [0, 2\pi]$. (You could also do $\gamma(t) = z_0 + re^{2\pi it}$ where $t \in [0, 1]$)

Note that the direction is counterclockwise.

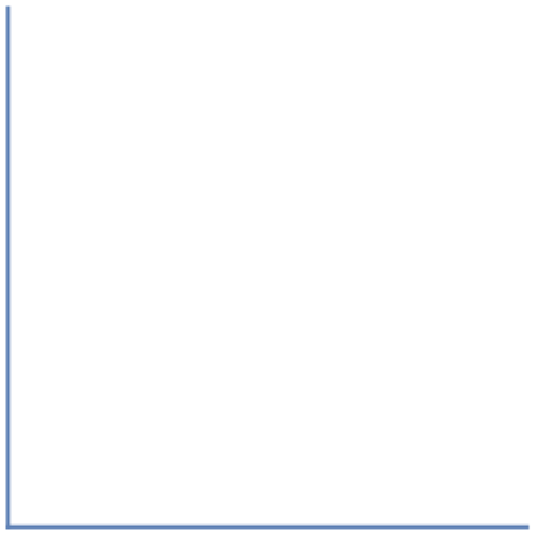


Figure 4.1: Path given by piecewise C^1 path.

4.2 Complex Path Integral

Definition 4.3 (Complex Integral)**.** Given a complex function $F(t) = x(t) + iy(t)$, one could define the integral over $t \in [a, b]$ to be

$$\int_a^b F(t)\,\mathrm{d}t := \int_a^b x(t)\,\mathrm{d}t + i\int_a^b y(t)\,\mathrm{d}t$$

Exercise 4.2. Prove that, just like in real analysis, you can bound by the integral of the modulus of integrand; that is:

$$\left|\int_a^b F(t)\,\mathrm{d}t\right| \leq \int_a^b |F(t)|\,\mathrm{d}t$$

Definition 4.4 (Path Integral)**.** Given piecewise C^1 path $\gamma : [a, b] \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$, then

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f\left(\gamma(t)\right)\gamma'(t)\,\mathrm{d}t$$

Remark 4.1. For remembering the definition, you can treat it as substitution rule in real integral:

$$z = \gamma(t)$$
$$\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$$

Example 4.3 (Terms in Taylor Around a Circle[5]). Given unit circle around origin (counterclockwise) as the path, let's compute the path integral of $f(z) = z^n$ where $n \in \mathbb{Z}$.

$$\begin{aligned}\int_\gamma z^n \,\mathrm{d}z &= \int_0^{2\pi} \left(e^{it}\right)^n e^{it} i \,\mathrm{d}t \\ &= \int_0^{2\pi} e^{it(n+1)} i \,\mathrm{d}t \\ &= \begin{cases} \frac{1}{n+1}\left[e^{it(n+1)} i\right]_0^{2\pi} & (n \neq -1) \\ \int_0^{2\pi} i \,\mathrm{d}t & (n = -1) \end{cases} \\ &= \begin{cases} 0 & (n \neq -1) \\ 2\pi i & (n = -1) \end{cases}\end{aligned}$$

Exercise 4.3 (Path integral is well-defined). Show that if $\gamma : [a, b] \to \mathbb{C}$ and $\tilde{\gamma} : [c, d] \to \mathbb{C}$ are equivalent paths (of same orientation), then for any continuous function,

$$\int_\gamma f(z) \,\mathrm{d}z = \int_{\tilde{\gamma}} f(z) \,\mathrm{d}z$$

(Hint: since γ and $\tilde{\gamma}$ are equivalent paths, there exists a bijective $s : [c, d] \to [a, b]$ with $s'(t) > 0$ such that $s(c) = a, s(d) = b$.)

Definition 4.5 (Length). If $\gamma : [a, b] \to \mathbb{C}$ is a C^1 path, then **length** of γ is defined as

$$\ell(\gamma) := \int_a^b |\gamma'(t)| \,\mathrm{d}t$$

Remark 4.2. This is very similar to the length defined in MVC...

Remark 4.3 (Path Integral Properties). Given functions f, g and paths γ, η,

- Linearity
  - $\int_\gamma (\alpha f(z) + \beta g(z)) \,\mathrm{d}z = \alpha \int_\gamma f(z) \,\mathrm{d}z + \beta \int_\gamma g(z) \,\mathrm{d}z$
- Opposite Orientation: If γ^- is traversal of γ in the opposite direction,
  - $\int_\gamma f(z) \,\mathrm{d}z = -\int_{\gamma^-} f(z) \,\mathrm{d}z$
- Additivity: If $\gamma \star \eta$ is concatenation of the two paths,
  - $\int_{\gamma \star \eta} f(z) \,\mathrm{d}z = \int_\gamma f(z) \,\mathrm{d}z + \int_\eta f(z) \,\mathrm{d}z$

[5] Come back to this after you do residue theorem as well!

- Estimation Lemma (γ^* is the image of the path)
  - $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma^*} |f(z)|\ell(\gamma)$
  - Useful for having an upper bound.

Exercise 4.4. Prove the estimation lemma.

Here is also a theorem that resembles the FTC

Exercise 4.5. $F(z)$ is called **primitive** of f if $F'(z) = f(z)$. Suppose $\gamma : [a,b] \to U$ is a piecewise C^1 path in U, then prove

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

Also the fact that zero derivative means constant:

Exercise 4.6. Let $f : U \to \mathbb{C}$ with $f'(z) = 0$ for all $z \in U$, then show f is constant.

5 Cauchy's Theorem

One awesome theorem by Mr. Cauchy here!

Theorem 5.1 (Cauchy's Theorem)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function over U. Let γ be a closed path in U such that interior lies entirely in U. Then*

$$\int_\gamma f(z)\,dz = 0$$

The proof of this is quite tedious... but the idea is that you prove this theorem when γ is triangular path, then generalize it to star-like domain, then generalize it further!

Example 5.1. Take γ to be a counterclockwise unit circle around the origin. For any polynomial $f(z)$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

by Cauchy's theorem, because polynomial is holomorphic inside γ. (In fact, try verifying this by computing it!)

Example 5.2 (**WRONG** Use of Cauchy's Theorem)**.** Again, take γ to be a counterclockwise unit circle around the origin. This time take $f(z) = z^{-2}$. You will find that

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

but this is NOT by Cauchy's theorem, as $f(z)$ here is not holomorphic inside γ. (Rather, it is just a consequence of it having a primitive that does not cross a branch cut.)

In fact, you will get zero integral for $f(z) = \frac{1}{z^n}$ for integer $n \geq 2$, but these are not by Cauchy's theorem.

What about $f(z) = \frac{1}{z}$ though? (See Example 4.3 which might be relevant; this is, as mentioned before, related to residue calculus which is coming!)

Theorem 5.2 (Cauchy's Integral Formula)**.** *Suppose $f : U \to \mathbb{C}$ is holomorphic on an open set U containing disc $\bar{B}(a, r)$. Then for all $w \in B\,(a, r)$,*

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w}\, dz$$

where γ is $\partial B(a, r)$ (counterclockwise).

Remark 5.1. Cauchy's integral formula is also an example of residue theorem (which will be covered later)!!! (If you are only interested in knowing theorems, then it means residue calculus is all you need at the end of the day. . .) Note that integrand is holomorphic in $U \setminus \{w\}$.

Definition 5.1 (Winding Number)**.** Consider a closed curve γ and assume z_0 is not on the curve. Then it is possible to write the curve function (parameterized over $t \in [0, 1]$ as $\gamma(t) = z_0 + |\gamma(t) - z_0|e^{2\pi i a(t)}$ for some $a(t)$ real continuous function.[6] (In fact (for simplicity consider $z_0 = 0$) if $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)} = |\gamma(t)|e^{2\pi i b(t)}$, then $a(t) = b(t) + n$ for some $n \in \mathbb{Z}$.) **Winding number** of the curve γ around z_0 is defined as:

$$I\,(\gamma, z_0) \coloneqq a(1) - a(0) \in \mathbb{Z}$$

An explicit form of winding number of the curve, if it is piecewise C^1, can be written as the following:

$$I\,(\gamma, z_0) \coloneqq \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

[6] Writing out a complex number as a polar form centered at z_0.

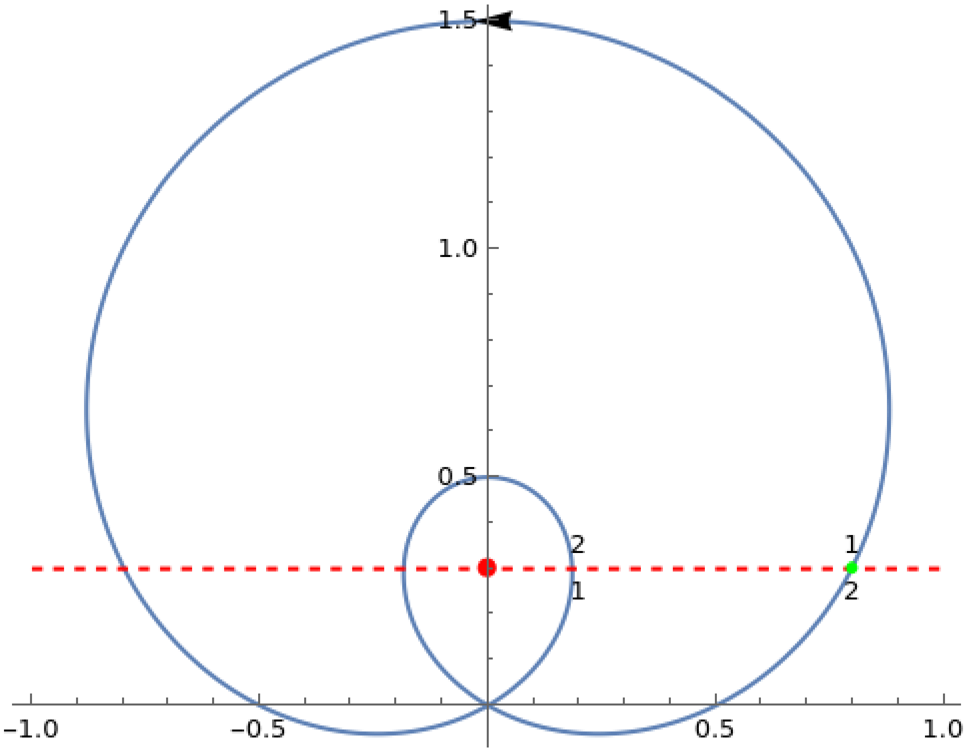


Figure 5.1: How to count the winding number

Example 5.3. Name says it all, winding number counts how many times a curve winds around a point. (Positive means anticlockwise winding and negative means clockwise winding.) See Figure 5.1. Start from the green point (on the side labeled 1), and follow the curve in the direction of the arrow (anticlockwise). Once you get to the inner point also labeled 1, it means you've counted one winding. Continue from the point inner labeled 2 until you get to the green point (on the side labeled 2). You've now counted the second winding.

Overall, the winding number around the red point is 2.

If the direction of the arrow were reversed (so that it is clockwise), you would've ended up with -2 as the winding number.

Remark 5.2. The explicit integral representation of the winding number is an application of Cauchy's integral formula to $f(z) \equiv 1$, which again I remark is an application of the residue theorem!

Exercise 5.1. Prove (or justify) why the explicit form involving the integral should be the winding number (if you are justifying, you could use residue calculus).

Exercise 5.2 (Cauchy's Integral Formula for Derivatives)**.** If $f : U \to \mathbb{C}$ is holomorphic on open set U, then for any $z_0 \in U$, $f(z)$ is equal to its Taylor series at z_0, and it converges on any open disk centered at z_0 lying in U, and derivatives are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

Remark 5.3. This implies that "analytic = holomorphic"! They can be used interchangeably! This also means <u>holomorphic functions are infinitely differentiable</u>.

Exercise 5.3 (Liouville's Theorem)**.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire[7] function. If f is bounded, then f is a constant function.

Remark 5.4. A part of fundamental theorem of algebra can be proven using Liouville's theorem!

Example 5.4 (Wiener-Hopf Method)**. THIS IS ABSOLUTELY OUT OF SCOPE FOR PART A**, but demonstrates how powerful Liouville's theorem can be!

[7] Holomorphic on $\mathbb{C}$

Suppose you want to solve the following problem for smooth bounded $f(x)$ for $x \in \mathbb{R}$:

$$\int_0^\infty K(x-t)f(t)\,\mathrm{d}t = f(x) \qquad \text{for } x \geq 0$$

where $K(x) = e^{-|x|}$ for $x \in \mathbb{R}$.

After a bit of trickery[8] (in Applied Complex Variables in part C), you end up needing to solve the following for $\hat{f}_+$ and $\hat{h}_-$ (or at least one of the two unknown functions):

$$\frac{1-k^2}{k+i}\hat{f}_+(k) = (k-i)\,\hat{h}_-(k) \qquad \text{for } 0 < \Im(k) < 1$$

Wait, you have to solve for $\hat{f}_+$ for $\hat{g}_-$ from a single equation? The magic here is that the LHS is holomorphic on $\Im(k) > 0$, and the RHS is holomorphic on $\Im(k) < 1$. Also from more complicated analysis of the problem, we know that $\hat{f}_+(k) = O\left(k^{-1}\right)$ and $\hat{h}_-(k) = O\left(k^{-1}\right)$.

Define $E(k)$ as

$$E(k) = \begin{cases} \frac{1-k^2}{k+i}\hat{f}_+(k) & (\Im(k) > 0) \\ (k-i)\hat{h}_-(k) & (\Im(k) < 1) \end{cases}$$

(Note that on the intersection $0 < \Im(k) < 1$, either of the definitions work, due to the given problem.) $E(k)$ is an entire function by construction, and because $\hat{f}_+(k) = O\left(k^{-1}\right)$ and $\hat{h}_- = O\left(k^{-1}\right)$, it is bounded at (complex infinity), so by Liouville's theorem $E(k) \equiv C$ for some constant C. Hence, (in the definition of Wiener-Hopf method, f_+ is defined to be $f_+(x) = f(x)\mathbb{I}_{x>0}$, where $\mathbb{I}$ is the indicator function.

$$\hat{f}_+(k) = \frac{C\,(k+i)}{1-k^2}$$

Taking inverse Fourier transform,

$$f(x) = f_+(x) = \frac{C}{2\pi}\int_\Gamma \frac{(k+i)\,e^{-ikx}}{1-k^2}\,\mathrm{d}k$$

where Γ is a contour from $-\infty + ci$ to $+\infty + ci$ for any $c \in (0,1)$.

It turns out that in most cases, the region at which a function ceases to be holomorphic are isolated most of the time, and sometimes, not even non-holomorphic even if you haven't defined them!

[8] In a nutshell, one takes (complex) Fourier transform of the entire problem

Exercise 5.4 (Riemann's Removable Singularity Theorem)**.** Suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then show f extends to a holomorphic function on all of U.

Example 5.5. Consider $f(z) = \frac{\sin z}{z}$. From the definition, it seems like $f(z)$ might not be holomorphic at $z = 0$, but $f(z)$ is bounded around $z = 0$ (since $\lim_{z\to 0} f(z) = 1$), and in a neighbourhood away from $z = 0$, so by Riemann's removable singularity theorem, $f(z)$ is holomorphic at $z = 0$.

Note that if it is holomorphic at $z = 0$, then it also means it is infinitely differentiable at $z = 0$.

Remark 5.5 (General Strategy for Checking Holomorphicity)**.** Given $f(z)$,

1. Identify which points might be problematic.
2. Check boundedness (often via checking if limit exists).
3. If bounded, then it is definitely holomorphic, otherwise not holomorphic at that point.

Exercise 5.5.

- Given $f(z) = \frac{\sin z}{\cos z}$, is it holomorphic at $z = \frac{\pi}{2}$?
- What about $f(z) = \frac{\left(z-\frac{\pi}{2}\right)\sin z}{\cos z}$?
- Is $f(z) = \frac{1}{(z+1)^2(z-1)}$ holomorphic at $z = i$? What about at $z = 1$? What about at $z = -1$?
- Define branch cut of $\sqrt{z}$ along the positive real axis. Determine if $f(z) = \frac{(z+i)\sqrt{z}}{z}$ is holomorphic at $z = -i$ and $z = i$, and if holomorphic, evaluate at those points.

Here is a kind-of converse to Cauchy's theorem.

Theorem 5.3 (Morera's Theorem)**.** *Suppose $f : U \to \mathbb{C}$ is a continuous function on open $U \subset \mathbb{C}$. If for any closed path γ in U, $\int_\gamma f(z)\,dz$, then f is holomorphic.*

6 Identity Theorem, Isolated Zeros, and Singularities

I've mentioned it before, but we deal with isolated singularities often, that is, the isolated points which some function f is not holomorphic.

Definition 6.1 (Pole and Order)**.** Suppose $f(z)$ is holomorphic on $U \setminus \{a\}$ for some $a \in \mathbb{C}$. The minimum $n \in \mathbb{Z}^{n\geq 0} \cup \{+\infty\}$ such that $(z-a)^n f(z)$ is holomorphic on U is called **pole** of **order** n of f at $z = a$.

- If $n = 0$, then we say a is a **removable singularity**[9] as before.
- If $n = 1$, then we say a is a **simple pole**.
- If $1 \leq n < \infty$, then we say a is a **pole of order** n.
- If $n = \infty$, then we say a is an **essential singularity**.

Example 6.1.

- $\frac{\sin z}{z}$ has a removable singularity at $z = 0$.
- $\frac{1}{z}$ has a simple pole at $z = 0$.
- $\frac{1}{z^2}$ has a pole of order 2 at $z = 0$.
- $\frac{1}{\cos z}$ has simple poles at $z = \frac{\pi}{2} \pm n\pi$ for $n \in \mathbb{Z}$.
- $e^{\frac{1}{z}}$ has essential singularity at $z = 0$.

Remark 6.1 (Determining the Order, Laurent Series, and Residue)**.** For tips on determining the order of a singularity, refer to `https://github.com/pauljoohyunkim/OxfordMathematicsNotesForPoorSouls/blob/main/2nd%20Year/Metric%20Spaces%20and%20Complex%20Analysis/Residue%20Calculus/Computing_Residue.pdf`.

But, here are the main points:

1. Identify where the singularities are (in most cases, it is the zeros of the denominator of the functions you are given).
2. Taylor expand the denominator (and possibly the numerator).
3. Use the formula $\frac{1}{1-r} = 1 + r + r^2 + \cdots$ for $|r| < 1$ to transform denominator to a multiplication by an infinite series.

What you are basically computing is "Laurent series" which gives all the info about the order of the pole and residue, which will be introduced very shortly.

Remark 6.2. The idea of isolated singularity is also very important to residue calculus.

[9] You can think of this as not having any singularity at all, in fact; if we have removable singularity, it means we can "get rid of" it by analyticity.

Theorem 6.1 (Identity Theorem)**.** *Let U be a domain, and f_1 and f_2 are holomorphic on U. If $S = \{z \in U | f_1(z) = f_2(z)\}$ has a limit point in U, then $S = U$, and hence $f_1(z) = f_2(z)$.*

Remark 6.3. In a way, you could think of it as, if two functions f_1 and f_2 agree on limiting points on a "dense set", then they must be equal on the domain.

One could think intuitionistically by noting that if f is holomorphic at z_0, then f is equal to its Taylor series in a neighbourhood around that point.

Exercise 6.1. Suppose f has a pole of order m at z_0. Show that there exists some $r > 0$ such that for all $z \in B(z_0, r) \setminus \{z_0\}$,

$$f(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

that is, f is equal to its **Laurent series** in any sufficiently small annulus around z_0. (Hint: What is the definition of a pole of order m?)

Remark 6.4. If f has essential singularity at z_0, then Laurent series goes from negative infinity to positive infinity:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Definition 6.2 (Principal Parts and Residue)**.** Given f in its Laurent series around z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

(where $c_m = 0$ for $m \leq -k - 1$ if pole of order k) the **principal part** of f at z_0 is

$$\sum_{n=-\infty}^{-1} c_n (z - z_0)^n$$

and **residue** of f at z_0 is

$$\mathrm{Res}_{z=z_0} f(z) = c_{-1}$$

Remark 6.5. You might wonder why the principal part and residue are defined this way.

Principal part is the only part that results in f having a "singularity", so for singularity analysis, one only needs to look at the principal part.

Residue is actually really cool. Consider counterclockwise integral of $f(z)$ (assume holomorphic in the unit circle except at 0) around the unit circle around 0.

$$\begin{aligned}\int_{|z|=1} f(z)\,\mathrm{d}z &= \int_{|z|=1} \sum_{n=-\infty}^{\infty} c_n z^n \,\mathrm{d}z \\ &= \int_{|z|=1} \frac{c_{-1}}{z}\,\mathrm{d}z \\ &= 2\pi i \operatorname{Res}_{z=0} f(z)\end{aligned}$$

where the second equality comes from Exercise 4.3. This means if you know the residues of $f(z)$ at its singularities, then one could compute closed-curve integrals of $f(z)$ easily, and this is the motivation for the residue calculus.

Example 6.2 (Computing Residues)**.** (As mentioned before) For examples of computing residues step-by-step, refer to `https://github.com/pauljoohyunkim/OxfordMathematicsNotesForPoorSouls/blob/main/2nd%20Year/Metric%20Spaces%20and%20Complex%20Analysis/Residue%20Calculus/Computing_Residue.pdf`

Conceptually, it is important that you recall that you can write $f(z)$ as its Laurent series around an annulus centered at its singularity.

Exercise 6.2 (Residue of f/g)**.** Suppose f, g are holomorphic functions and g has a zero at z_0. Show that

$$\operatorname{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

(Hint: Obviously, consider Taylor expansion, like I told you!)

Exercise 6.3 (Residue of pole of order m)**.** (More general than the previous exercise) Suppopse f has a pole of order m at z_0. Show that

$$\operatorname{Res}_{z=z_0}(f) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left((z-z_0)^m f(z)\right)$$

(Hint: Consider Laurent series of f.)

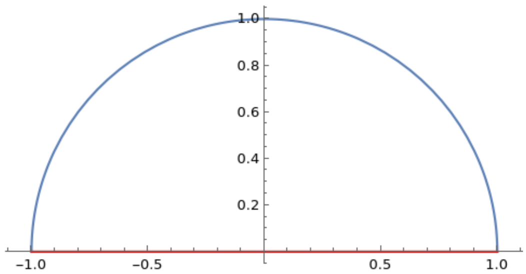
7 Homotopies, Simply-Connected Domains, and Cauchy's Theorem

Definition 7.1 (Homotopy and Simply-Connectedness)**.** Suppose that U is an open set in $\mathbb{C}$ and $a, b \in U$, and that $\eta : [0,1] \to U$, $\gamma : [0,1] \to U$ are paths in U such that $\gamma(0) = \eta(0) = a$ and $\gamma(1) = \eta(1) = b$.

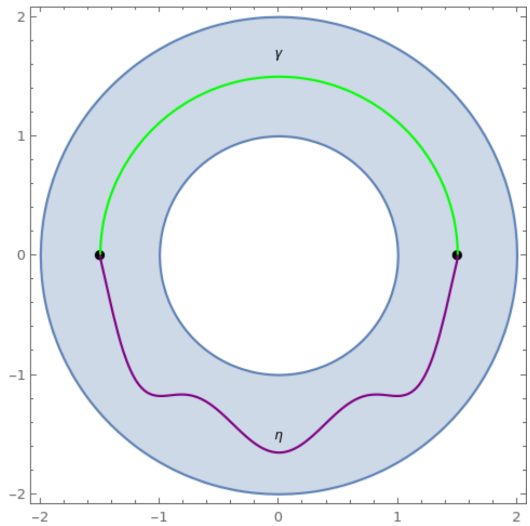
γ andd η are **homotopic** if there exists a continuous function $h : [0,1] \times [0,1] \to U$ such that,

$$\begin{aligned} h(0,s) &= a \\ h(1,s) &= b \\ h(t,0) &= \gamma(t) \\ h(t,1) &= \eta(t) \end{aligned}$$

See Figure 7.1 for examples of homotopicity and non-homotopicity. We say



(a) These two curves are homotopic in $\mathbb{C}$ as you can continuously deform one to get the other.



(b) These two curves are not homotopic in the annulus as you cannot continuously deform one to get the other.

Figure 7.1: To talk about homotopy, one must give the two curves and a region.

U is **simply-connected** if two curves in U are always homotopic.

Exercise 7.1. You can think of simply-connected regions to be regions without holes (though I guess it's an oversimplification.)

Exercise 7.2 (Deformation Theorem I)**.** Let U be simply-connected domain. Let $f : U \to \mathbb{C}$ be holomorphic on U. If $\gamma, \tilde{\gamma}$ are paths in U from $a \in \mathbb{C}$ to $b \in \mathbb{C}$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\tilde{\gamma}} f(z)\,\mathrm{d}z$$

(Hint: It helps to draw! It becomes basically a one-line argument by the use of Cauchy's theorem.)

Exercise 7.3 (Deformation Theorem II)**.** (Nonexaminable? Maybe? idk) For closed curves, there are no "ends", so you can deform it how ever you want so long as it does not cross any non-holomorpic region.

8 Argument Principle

We say function is **meromorphic** if it is holomorphic except at a few isolated singularities.

Let's start with a simple (verification) exercise.

Exercise 8.1. Suppose $f : U \to \mathbb{C}$ is meromorphic and has zero of order k or a pole of order k at $z_0 \in U$. Then show that $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue k or $-k$ respectively.

Here is the statement of the argument principle

Exercise 8.2 (Argument Principle)**.** Suppose $f : U \to \mathbb{C}$ is a meromorphic function on U. Assume $B(a, r) \subset U$. Let N denote the number of zeros (counted with multiplicity) and P denote the number of poles (counted again with multiplicity) of f inside $B(a, r)$, and f has neither zeros nor poles on $\partial B(a, r)$, then show that

$$N - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

where γ is a path with image $\partial B(a, r)$.

Moreover, $N - P$ is also the winding number of the path $\Gamma = f \circ \gamma$.

Remark 8.1. You should now be convinced that you can think of pole as a reverse zero if you have not already…

Remark 8.2. For understanding of the argument principle, I found this resource to be incredibly helpful: `https://youtu.be/79-ESkh5_f0` Highly recommend the videos by the channel as well for complex analysis to get a grasp of some of the concepts!

Here is another result proven using argument principle.

Exercise 8.3 (Rouché's Theorem)**.** Suppose f, g are holomorphic functions on open set $U \subset C$ and $\bar{B}(a, r) \subset U$. If $|f(z)| > |g(z)|$ for all $z \in \partial B(a, r)$ then $f, f + g$ both have the same change in argument around $\partial B(a, r)$, and hence the same number of zeros in $B(a, r)$ (counted with multiplicities).

Remark 8.3. Apprently Rouché's theorem is useful for counting number of zeros of a function f.

Example 8.1. (An example from the lecture note) Let $P(z) = z^4 + 5z + 2$. On circle $|z| = 2$, $|z|^4 = 16 > 5.2 + 2 \geq |\underbrace{5z + 2}_{g(z)}|$. Then $P - g = z^4$ and P has the same number of roots in $B(0, 2)$, so by Rouché's theorem, all four roots of $P(z)$ fall in $B(0, 2)$. Taking $|z| = 1$, $|g(z)| \geq 5 - 2 = 3 > |z^4| = 1$, so $P(z)$ and $g(z)$ have the same number of roots in $B(0, 1)$, so $P(z)$ has one root of modulus less than 1, and 3 of modulus between 1 and 2.

Other theorems that sound quite trivial can now be proven:

Exercise 8.4 (Open Mapping Theorem)**.** Suppose $f : U \to \mathbb{C}$ is holomorphic and nonconstant on U. Then for any open set $V \subset U$, $f(V)$ is also open.

Exercise 8.5 (Inverse Function Theorem)**.** Suppose $f : U \to \mathbb{C}$ be injective and holomorphic, and $f'(z) \neq 0$ for all $z \in U$. If $g : f(U) \to U$ is the inverse of f, then g is holomorphic, and $g'(w) = \frac{1}{f'(g(w))}$.

9 Residue Theorem

You are now at the heart of complex analysis! All the buildup for one of the most important results in complex analysis... I know it's tiring, but this is one of the sections that is almost guaranteed to be on the exam, and you can actually prepare for it by practice!

Exercise 9.1 (Residue Theorem)**.** Suppose U is an open set in $\mathbb{C}$ and γ is a closed curve whose inside is contained in U (so that $\forall z \notin U$, $I(\gamma, z) = 0$.)

Then if $S \subset U$ is a finite set such that $S \cap \gamma^*$ and f is a holomorphic function on $U \setminus S$,

Here is a very important theorem. (Don't worry if it looks complicated. In practice, you will be working with very nice functions.)

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{a\in S} I(\gamma, a)\,\mathrm{Res}_{z=a}(f)$$

Remark 9.1. In most cases, winding number $I(\gamma, a)$ will be 1 (or -1 if you go clockwise), so the RHS in most cases becomes $\pm\sum_{a\in S}\mathrm{Res}_{z=a}(f)$

Remark 9.2. Explicit form of the winding number, Cauchy's integral formula, and the argument principle can be understood as a case of residue theorem! Armed with residue theorem and deformation theorem (Exercise 7.2, 7.3), you can compute a vast types of integrals really easily!

Remark 9.3. We could've done Example 4.3 using residue theorem really quickly!

Exercise 9.2. Verify the following result:

$$\int_\gamma \frac{\sin z}{\cos z}\,\mathrm{d}z = 2\pi i$$

where γ is a clockwise circular path of radius 0.2 around $\pi/2$ (Note: Beware of the sign!)

Exercise 9.3. Verify the following result:

$$\int_\gamma \frac{1}{(z+1)^2(z-1)}\,\mathrm{d}z = -\frac{i\pi}{2}$$

where $\gamma(t) = -1 + \frac{1}{5}e^{it}$ over $t \in [0, 2\pi]$.

9.1 Residue Calculus

Being able to compute complex integrals very easily means one could now compute real integrals that does not have an easy antiderivative.

Example 9.1. Suppose we wish to compute

$$\int_0^{2\pi} \frac{1}{1+3\cos^2(t)}\,\mathrm{d}t$$

If we consider the path $t \mapsto e^{it}$ (a unit circle), and let $z = e^{it}$, and note that

$$\cos t = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

(since z is on the unit circle) we can turn the real integral into complex integral by the transforming the integrand

$$\frac{1}{1+3\cos^2(t)} = \frac{1}{1+3\times\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2} = \frac{4z^2}{3+10z^2+3z^4}$$

and noting $\mathrm{d}z = iz\,\mathrm{d}t$:

$$\int_0^{2\pi} \frac{1}{1+3\cos^2(t)}\,\mathrm{d}t = \int_\gamma \frac{-4iz}{3+10z^2+3z^4}\,\mathrm{d}z$$

By residue theorem, the complex integral is simply π, so

$$\int_0^{2\pi} \frac{1}{1+3\cos^2(t)}\,\mathrm{d}t = \pi$$

Remark 9.4. One question you might have is how I came up with the path to be a unit circle. Whenever you see an integral involving trig function such as $\sin t$ or $\cos t$, with integration range $[0, 2\pi]$ you know that you can write it as some combination of $z, \bar{z}$, but it is convenient to take the curve to be unit circle so that $\bar{z} = \frac{1}{z}$, so that we don't have to worry about the conjugate symbol.

At the end of the day, it is up to how much you've practiced.

Remark 9.5 (Some Categorized Examples of Residue Calculus)**.** Refer to part C Applied Complex Variables lecture note (`https://courses.maths.ox.ac.uk/pluginfile.php/22180/mod_resource/content/1/C5_6LectureNotes.pdf`), section 1.9 for some VERY DOABLE examples. Note that the entire first section is in fact a highly condensed summary of relevant materials from complex analysis!

Example 9.2. Suppose we want to compute

$$I := \int_0^\infty \frac{1}{1+x^2+x^4}\,\mathrm{d}x$$

Note that I can write this as

$$I = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{1+x^2+x^4}\,\mathrm{d}x$$

It is difficult (if impossible) to get a simple form for the antiderivative of the integrand. This means we have to use complex integrals.

Note that the integral is over the range $[-R, R]$, and we see that at infinity, the integrand is like $O\left(z^{-4}\right)$ as $z \to \infty$ (where $z \in \mathbb{C}$), which seems to indicate that the integrand decays at a sufficient rate (and hence be zero at infinity... All of this is hand-wavy, but it will make more sense as you practice)

Consider the following complex integral:

$$\tilde{I}_R := \int_{\gamma_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z = \underbrace{\int_{y_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z}_{\to 2I} + \int_{b_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z$$

where γ_R is as shown in Figure 9.1 (concatenation of the yellow and blue contours) and y_R, b_R are yellow and blue paths respectively of the full contour. (Note that $\int_{y_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{1+x^2+x^4}\,\mathrm{d}x$)

By estimation lemma, the contribution from b_R tends to zero as $R \to 0$:

$$\left|\int_{b_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z\right| \leq \sup_{z\in b_R^*} \left|\frac{1}{1+z^2+z^4}\right| \pi R \leq \frac{\pi R}{R^4-R^2-1} \to 0$$

where in the last inequality, I used the (reverse) triangle inequality. This implies $\tilde{I}_R \to \int_{y_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z$ as $R \to \infty$. This means all we now have to do is to compute I_R using residue theorem, which turns out to be:

$$\tilde{I}_R = 2\pi i \left(\mathrm{Res}_{z=\omega}\left(\frac{1}{1+z^2+z^4}\right) + \mathrm{Res}_{z=\omega^2}\left(\frac{1}{1+z^2+z^4}\right)\right) = \pi/\sqrt{3}$$

where $\omega := e^{i\pi/3}$

Hence, putting it all together,

$$\pi/\sqrt{3} = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^2+z^4}\,\mathrm{d}z$$

So ultimately,

$$I = \frac{\pi}{2\sqrt{3}}$$

Remark 9.6 (PRO TIP: Behaviour at Infinity)**.** (<u>The outlined technique is only for sanity checks! Not to be used in the actual solution!</u> The only technique you are given in lecture and hence can be used is the Jordan's lemma. (See Exercise 9.4)) If you can infer the behaviour of integral at infinity, the whole process of turning real integral into a complex one might

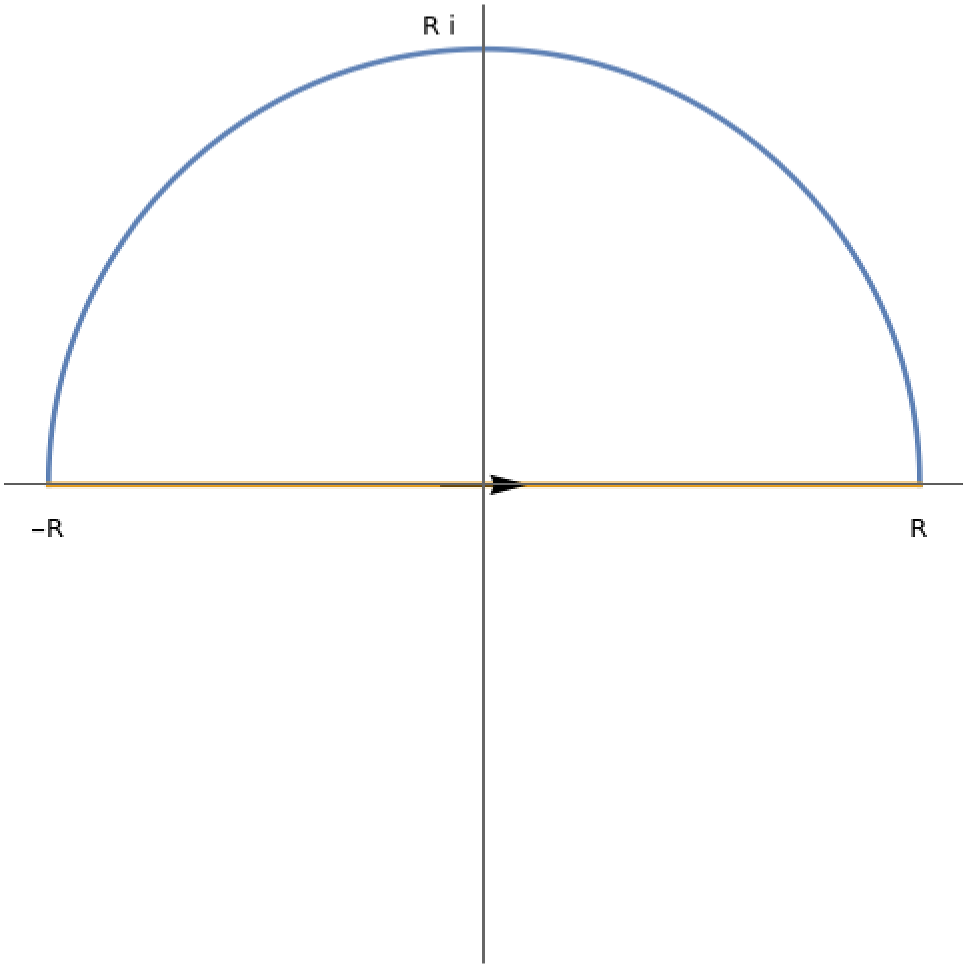


Figure 9.1: Contour for $\tilde{I}_R$

just become more natural. Suppose we have integral $\int_{\Gamma_R} f(z)\,\mathrm{d}z$ such that $f(z) = O\left(\frac{1}{R^n}\right)$, and Γ_R is a circular arc that "goes to" infinity[10] as $R \to \infty$.

Bounding the contribution:

$$\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| \leq \sup_{z \in \Gamma_R^*} |f(z)| \ell(\Gamma_R^*) \leq \frac{C_1}{R^n}\left(C_2 R\right) = CR^{1-n}$$

where

- the first inequality is from estimation lemma.
- the second inequality is from the definition of big-O notation and the fact that Γ_R is a part of a circle, so we know that $\ell\left(\Gamma_R^*\right) = O\left(R\right)$.

[10] Just like the semicircle from the example.

This means if $n > 1$, $\int_{\Gamma_R} f(z)\,\mathrm{d}z$ vanishes as $R \to 0$.

Example 9.3 (Behaviour at Infinity)**.** For the following $f(z)$, the integral $\int_{\Gamma_R} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$ where Γ_R is a part of a circle of radius R that the following $f(z)$ satisfy $f(z) = O\left(\frac{1}{z^p}\right)$ for some $p > 1$.

- $\frac{1}{z+z^2}$
- $\frac{1}{z^{3/2}+z^2}$
    - Γ_R here would have to exclude the branch cut!
- $z^n e^{-z}$ for any $n \in \mathbb{Z}$
- (WRONG EXAMPLE) $\frac{\sin z}{z^2}$
    - Note that $|\sin z| \leq 1$ is not true in complex analysis!!!

Here is a result that is given in lectures that you ARE allowed to use.

Exercise 9.4 (Jordan's Lemma)**.** Let $f : \mathbb{H} \to \mathbb{C}_\infty$ be a meromorphic function on $\mathbb{H} := \{z \in \mathbb{C} | \Im(z) > 0\}$. Suppose $f(z) \to 0$ as $z \to \infty$ in $\mathbb{H}$, then if $\gamma_R(t) := Re^{it}$ for $t \in [0, \pi]$, you have

$$\int_{\gamma_R} f(z) e^{i\alpha z}\,\mathrm{d}z \to 0$$

as $R \to 0$ for any $\alpha > 0$.

Remark 9.7. The pro tip (Remark 9.6) covers way more general cases, but you will only need Jordan's lemma in the exam!

Example 9.4 (Integral in the Method of Stationary Phase)**. This is material from Part B Waves and Compressible Flow and Part C Perturbation Methods** and the exact derivation is not expected in Part A. <u>Howeverm, the relevant bit is the computation of one complex integral.</u>

<u>Method of stationary phase</u> is a way of approximating an integral of the following form as $x \to \infty$:

$$I(x) = \int_a^b f(t) e^{ix\psi(t)}\,\mathrm{d}t$$

where $\psi'(t) \neq 0$ for $a < t < c$ and $c < t < b$ for some $c \in [a, b]$, and $\psi''(c) \neq 0$.

The leading order approximation by method of stationary phase[11] turns out to be:

$$I(x) = \underbrace{\frac{f(c)e^{ix\psi(c)}}{x^{1/2}} \int_{-\infty}^{\infty} e^{is^2\psi''(c)/2}\, \mathrm{d}s}_{\text{Approximation}} + \underbrace{O\left(\frac{1}{x\epsilon}\right)}_{\text{Correction Term}}$$

where the correction term decays as $x \to \infty$ (enforced by choosing $\epsilon > 0$ in some range). To derive the explicit form of the approximation term, one needs to be able to compute:

$$\tilde{I} := \int_{-\infty}^{\infty} e^{is^2\psi''(c)/2}\, \mathrm{d}s$$

Note that by the "evenness" of the integrand, one can write:

$$\tilde{I} = 2\int_{0}^{\infty} e^{is^2\psi''(c)/2}\, \mathrm{d}s$$

Consider now the following complex integral:

$$J := \underbrace{\int_R e^{is^2\psi''(c)/2}\, \mathrm{d}s}_{=\tilde{I}/2} + \int_B e^{is^2\psi''(c)/2}\, \mathrm{d}s - \int_G e^{is^2\psi''(c)/2}\, \mathrm{d}s$$

where R, B, G refer to the paths specified in red, blue, green in Figure 9.2, respectively. [12] The directions paths are:

- R path goes from the origin to the right infinity.
- B path goes around anticlockwise.
- G path goes from the origin to infinity at $\pi/4$ angle.

[11] In a nutshell, you can derive this by Taylor expanding the integrand at $t = c$ and arguing that the contribution around a small interval around c dominates the contributions from rest of the domain.

[12] You might wonder **why this path is chosen**; it is because as s varies from 0 to infinity, the integrand $e^{is^2\psi''(c)/2}$ "oscillates" in some sense, and is not an easy integral to evaluate. On the other hand, along the G path, by parameterizing it as $s = e^{\pi/4}s$, effectively, we now consider a new integrand of the form e^{-kt^2}, which is a Gaussian integral which we know the value to. By relating these two, and the knowledge of complex analysis, we can indirectly compute the former integral. Note that this is in fact NOT an obvious choice, and some super clever people thought of it!

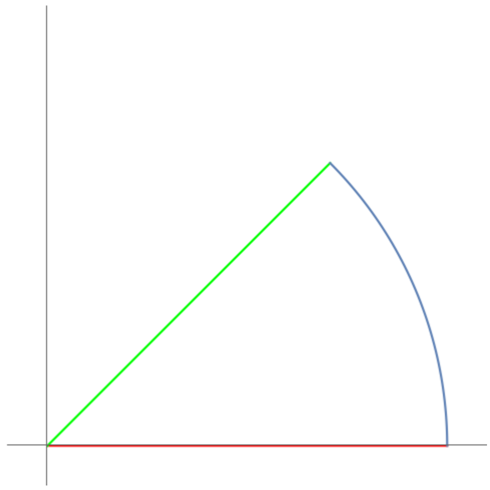


Figure 9.2: Integration contour for method of stationary phase for $\psi''(c) > 0$. For $\psi''(c) < 0$, one could consider the same path reflected across the x axis.

(Here I have made a deliberate choice to make G path start from origin rather than going from infinity to origin for simplicity in dealing with signs later.)

First, Because the integrand does not have any poles inside the closed curve, by Cauchy's theorem, $J = 0$.

Second, By Jordan's lemma, $\int_B e^{is^2\psi''(c)/2}\,\mathrm{d}s \to 0$ as the radius goes to infinity.

Hence, putting these two information, we deduce that as radius diverges:

$$2 \times \int_G e^{is^2\psi''(c)/2}\,\mathrm{d}s \to 2\int_{t=0}^{\infty} e^{-\psi''(c)t^2/2}e^{i\pi/4}\,\mathrm{d}t = \tilde{I}$$

where the second integral comes from substitution $s = e^{i\pi/4}t$.

We finally have $\tilde{I}$ in terms of a Gaussian integral. I invite you to finish the computation and verify that:

$$I(x) = \frac{\sqrt{2\pi}f(c)e^{ix\psi(c)}e^{i\pi/4}}{x^{1/2}\sqrt{\psi''(c)}} + O\left(\frac{1}{x\epsilon}\right)$$

Remark 9.8. In the previous example of method of stationary phase, if one wants to be less rigorous (for the intuition's sake), one could think that the integral contribution of the arc is nearly zero due to the exponential decay (in the positive upper half plane), and think of G path as a "deformation" from R curve (noting the homotopicity since no pole in the integrand, and complex infinity is treated as just a point) hence by deformation theorem

$$\int_R e^{is^2\psi''(c)/2}\,\mathrm{d}s = \int_G e^{is^2\psi''(c)/2}\,\mathrm{d}s$$

9.2 Keyhole Contour: When there is a "constant-factor" branch cut

It is straightforward to compute contour integral if it only involves poles; just use the residue theorem (and/or deformation theorem) and you are all good.

However, what if you have a branch cut? The problem here is that, the branch cut that you define may need to extend to infinity, which means you might not be able to construct a complex integral with a closed loop enclosing a branch point.

The resolution here is to use a closed curve avoiding the branch cut around branch points. These are called **keyhole contours**.

Example 9.5. Suppose you want to compute:

$$I := \int_0^\infty \frac{x^{1/2}}{1+x^2}\,\mathrm{d}x$$

The intuition should say you should try to consider a complex integral of the form:

$$\tilde{I} := \int_\Gamma \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z$$

where Γ is a closed curve to be chosen later.

Because $z^{1/2}$ is a multifunction, one needs to choose a branch cut. For computation of I, it involves the contribution of the integrand on the domain $[0,\infty)$. If one chooses the branch cut along the domain, that is, the positive x axis, $z^{1/2} := \frac{r}{2}e^{i\theta/2}$ ($\theta \in [0, 2\pi]$) has a sign difference across the branch cut[13]; this is a good thing, since you want to be able to end up with an equation of the form:

$$k\int_0^\infty (\mathit{Integrand})\,\mathrm{d}x = \ell$$

where k, ℓ are some constants.[14]

Choose Γ as Figure 9.3. Then,

$$\tilde{I} = \int_{\gamma_+} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z + \int_{\gamma_R} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z - \int_{\gamma_-} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z - \int_{\gamma_\epsilon} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z$$

where

- γ_+ refers to the straight contour above the positive x axis from "near origin" to the right.

[13]check this!

[14]This is analogous to using IBP twice to evaluate an integral, where you end up turning the problem into a linear equation.

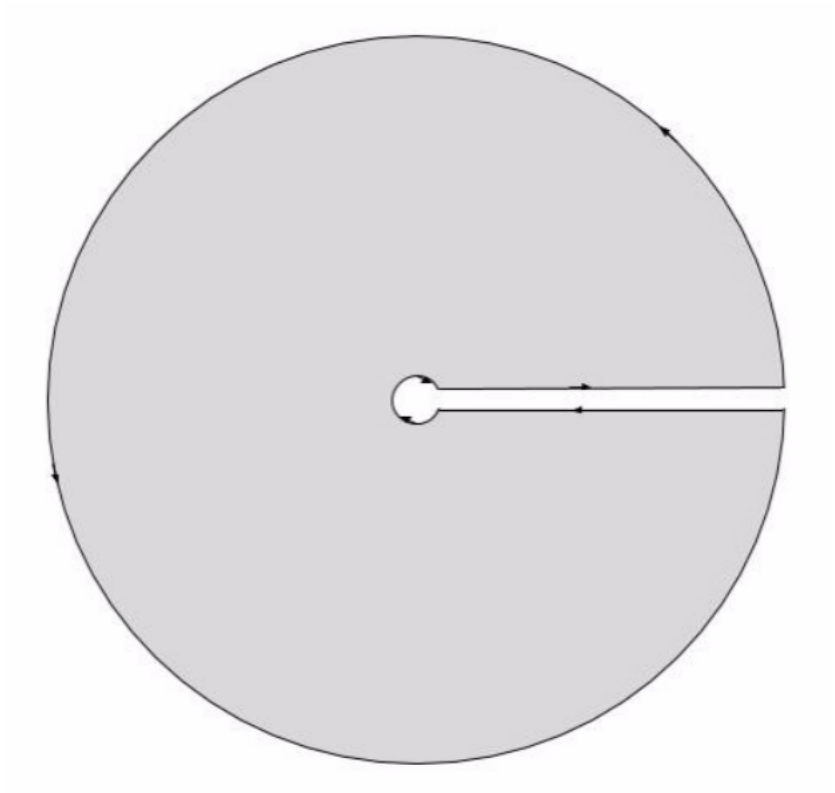


Figure 9.3: Keyhole contour with branch cut in the positive x axis.

  * On this contour, $\frac{z^{1/2}}{1+z^2} = \frac{x^{1/2}}{1+x^2}$

- γ_R is the large anticlockwise arc of radius R.
- γ_- refers to the straight contour below the positive x axis from "near origin" to the right.
  * On this contour, $\frac{z^{1/2}}{1+z^2} = -\frac{x^{1/2}}{1+x^2}$
- γ_ϵ is the small anticlockwise arc of radius ϵ around the origin.

(Pay attention to the direction of the contour and all the signs!) Considering the values of the $\frac{z^{1/2}}{1+z^2}$ on $\gamma_\pm$, note that:

$$\tilde{I} = \int_0^{\tilde{R}} \frac{x^{1/2}}{1+x^2}\,\mathrm{d}x + \int_{\gamma_R} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z + \int_0^{\tilde{R}} \frac{x^{1/2}}{1+x^2}\,\mathrm{d}x - \int_{\gamma_\epsilon} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z$$

where $\tilde{R}$ is the x value of the right-end of the $\gamma_\pm$, which is not important.

We will be taking the limits as $R \to \infty$ and $\epsilon \to 0^+$.

First, by residue theorem, one can compute $\tilde{I} = 2\pi i \left(\frac{1}{2}e^{-\pi i/4} + \frac{1}{2}e^{5\pi i/4}\right) = \pi\sqrt{2}$.

Second, Note that

$$\left|\int_{\gamma_R} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z\right| \le \sup_{z\in\gamma_R}\left|\frac{z^{1/2}}{1+z^2}\right| \times 2\pi R \to 0$$

where we used estimation lemma, reverse triangle inequality, and sandwich lemma to show that the contribution from $\int_{\gamma_R} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z$ decays as $R \to \infty$. **If you weren't sure if the contribution was zero a priory though**, read my pro tip at Remark 9.6; since $\left|\frac{z^{1/2}}{1+z^2}\right| \sim \frac{R^{1/2}}{1+R^2} \sim \frac{1}{R^{3/2}} \ll O\left(\frac{1}{R}\right)$ as $R \to \infty$, immediately, you should've known that the contribution is zero.

Third, similarly, we note

$$\int_{\gamma_\epsilon} \frac{z^{1/2}}{1+z^2}\,\mathrm{d}z \le \sup_{z\in\gamma_\epsilon}\left|\frac{z^{1/2}}{1+z^2}\right| 2\pi\epsilon \to 0$$

so again, you get no contribution from this integral[15] in the limit $\epsilon \to 0$.

Hence, putting all these together, we deduce

$$2\int_0^\infty \frac{x^{1/2}}{1+x^2}\,\mathrm{d}x = \pi\sqrt{2}$$

[15] I haven't thought much into if the trick in Remark 9.6 could work with curve going to zero, but it is worth investigating on your own I guess...

, so we get

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}$$

Remark 9.9. The fact that there was a factor (of negative one) difference across the branch cut is what allowed us to use the keyhole contour!

Example 9.6. Let's go through the computation of the followwing integral quickly!

$$I := \int_0^\infty \frac{x^{1/2}\log x}{(1+x)^2}\,\mathrm{d}x$$

Remember from discussion of branch cuts of $\log z$ that it doesn't have a constant factor difference across the branch cuts, but a constant addition across the branch cuts; this is not a good thing, as it's like using IBP twice and accidentally removing the integral you needed to compute... Fortunately, in this case, $z^{1/2}$ also provides branch cut, which has a factor difference.

Using the same keyhole contour (Figure 9.3), define

$$\tilde{I} := \int_{\gamma_+} \frac{z^{1/2}\log z}{(1+z)^2}\,\mathrm{d}z + \int_{\gamma_R} \frac{z^{1/2}\log z}{(1+z)^2}\,\mathrm{d}z - \int_{\gamma_-} \frac{z^{1/2}\log z}{(1+z)^2}\,\mathrm{d}z - \int_{\gamma_\epsilon} \frac{z^{1/2}\log z}{(1+z)^2}\,\mathrm{d}z$$

First, Note that $\frac{z^{1/2}\log z}{(1+z)^2} = \pm\frac{x^{1/2}\log x}{(1+x)^2}$ on $\gamma_\pm$.

Second, Because as $R \to \infty$,

$$\left|\frac{z^{1/2}\log z}{(1+z)^2}\right| \sim \frac{R^{1/2}\log R}{R^2}$$

by Remark 9.6, contribution from γ_R is zero.

Third, similarly,

$$\int_{\gamma_\epsilon} \frac{z^{1/2}\log z}{(1+z)^2}\,\mathrm{d}z \le \sup_{z\in\gamma_\epsilon}\left|\frac{z^{1/2}\log z}{(1+z)^2}\right| 2\pi\epsilon \to 0$$

so no contribution from γ_ϵ.[16]

Fourth computing $\tilde{I}$ directly using the residue theorem,

$$\tilde{I} = \pi^2 + 2\pi$$

Hence, putting all the info together (taking real part) deduce

$$I = \pi$$

As a byproduct, by taking imaginary part instead, you also deduce

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,\mathrm{d}x = \pi/2$$

[16]For those who want to see in more detail as to showing the limit going to zero, it might be helpful to note that $\epsilon\sqrt{\epsilon}\log\epsilon = \sqrt{\epsilon}\frac{\log\epsilon}{1/\epsilon}$, and use l'Hôpital on $\frac{\log\epsilon}{1/\epsilon}$.

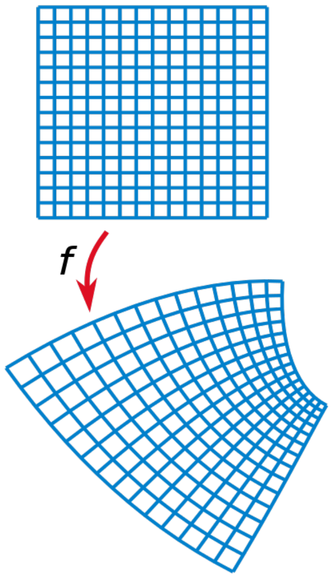


Figure 10.1: Conformal maps are functions that locally preserve the angles.

10 Conformal Mapping

Many of the holomorphic functions that we deal with are “conformal”.

Definition 10.1 (Conformal Map)**.** If $\gamma : [-1, 1] \to \mathbb{C}$ is C^1 path such that $\gamma'(t) \neq 0$ for all t, we first define **tangent line** of the curve at t as

$$\{\gamma(t) + s\gamma'(t) | s \in \mathbb{R}\}.$$

Then given two such C^1 paths, define the **angle of the paths** at the intersection as the angle between the respective tangent lines.

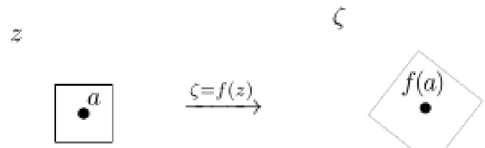
f is a **conformal map** at z_0 if the angle between two C^1 curves at z_0 is preserved under the map f, that is, it is equal to the angle at $f(z_0)$ of the images of the two curves under f.

See from Figure 10.1 that the right angles in the grid are preserved in the image as well (locally).

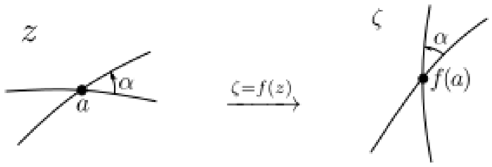
Here is why conformal maps are related to complex analysis.

Remark 10.1 (Nonconstant holomorphic functions are conformal)**.** If $f : U \to \mathbb{C}$ is holomorphic on U and $f'(z) \neq 0$ for all $z \in U$, then it is conformal on U.

The intuition is that, locally nonconstant holomorphic functions are translation, dialation and rotation. Take $a \in U$. Locally, one could Taylor expand



(a) Local behaviour of holomorphic function



(b) Any angle between curves preserved.

Figure 10.2: Local behaviour of holomorphic functions suggests that they are conformal.

in a neighborhood:

$$f(z) = f(a) + f'(a)(z-a) + \frac{1}{2}f''(a)\,(z-a)^2 + O\left((z-a)^3\right)$$

$f(a)$ is the image of a, which takes care of translation.

Also note that $z-a$ can be understood as a vector from a to z on $\mathbb{C}$, and multiplication by $f'(a) = |f'(a)|e^{i\arg f'(a)}$ is dialation by $|f'(a)|$ and rotation by $\arg f'(a)$. (See Figure 10.2)

This also suggests that if a conformal map f over complex plane satisfies $f'(a) \neq 0$ it is locally biholomorphic.

Example 10.1 (Square Map)**.** Consider $f(z) = z^2$. Note that $f'(z) = 2z$, so it is holomorphic on $\mathbb{C}\setminus\{0\}$. To see what happens, consider writing $z = re^{i\theta}$. Then $f(z) = r^2e^{i(2\theta)}$. Effectively, it opened up a region, "pivoting" from the origin (Figure 10.3). Note that the origin is not holomorphic; as you can see, it doubled the angle there.

Remark 10.2. Note that $f'(a) \neq 0$ is when "holomorphic $\implies$ conformal" breaks down; this is because $f'(a) = 0$ means it is a dialation by factor zero, meaning, it is "degenerate".

Remark 10.3. Often times, when dealing with conformal maps, exactly where each point gets mapped to is not something to worry about; rather you'll be working out the image of some region under a holomorphic function.

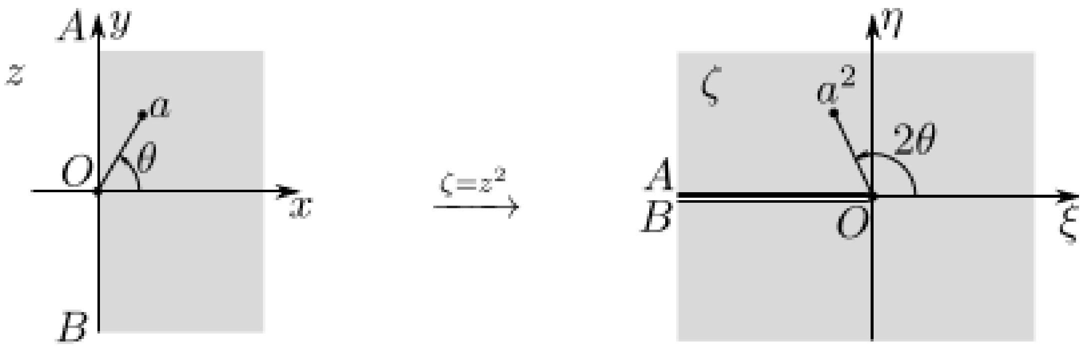


Figure 10.3: Squaring map opens up a region like a fan (doubling the angle at the origin) and squares the modulus of each point.

Open mapping theorem might be useful; this suggests that you can usually just attempt to parameterize the boundaries, and see where the boundaries map to.

Exercise 10.1 (Möbius maps are conformal)**. First**, show that a Möbius map is conformal. (Trivial, but need to check it is nowhere constant.)

Then, show that for z_1, z_2, z_3 and w_1, w_2, w_3, two sets of pairwise distinct complex numbers (including infinity), there exists a unique Möbius map f such that

$$\forall i = 1, 2, 3 \ f(z_i) = w_i$$

(Constructive proof, but one needs to consider cases involving infinity)

Remark 10.4. Recall that Möbius maps map circlines to circlines. Noting that infinity only exists in lines, Möbius maps are especially easy to deal with for conformal map stuff!

Often times, you will also be dealing with maps involving 0, 1, and ∞ often.

Example 10.2 (Upper Half Plane to Unit Disk)**.** Suppose we wish to find a conformal map that map upper half plane $\mathbb{H}$ to $B(0, 1)$, a unit disk. (See Figure 10.4) We can try Möbius map, since we know lines can be mapped to a circle. Attempting to map the real axis to the unit circle via the following:

$$\begin{aligned} 0 &\mapsto 1 \\ 1 &\mapsto i \\ \infty &\mapsto -1 \end{aligned}$$

Note that I have deliberately made the choice of image points to be 1, i, and -1, as they are clearly on the circle, and are simple numbers.

First, consider that $\infty \mapsto -1$ implies that the ratio of the leading order in the Möbius map must be -1:

$$f(z) = -\frac{z+a}{z+b}$$

Second, 0 maps to 1, so

$$f(z) = -\frac{z+a}{z-a}$$

Finally, 1 maps to i, so

$$f(z) = -\frac{z-i}{z+i}$$

This function maps the real axis to the unit circle. We don't know if this maps to the inside or the outside of the unit circle yet, however! We could find this out by trying where i, a point in the upper half plane, lands.

Since $f(i) = 0$, it must be that $f(z)$ maps the upper half plane to the inside of the circle, so we are done! (Figure 10.4)

Example 10.3. Let's do the same as before, but choosing a slightly different information. Suppose we take:

$$\begin{aligned} 0 &\mapsto 1 \\ 1 &\mapsto -i \\ \infty &\mapsto -1 \end{aligned}$$

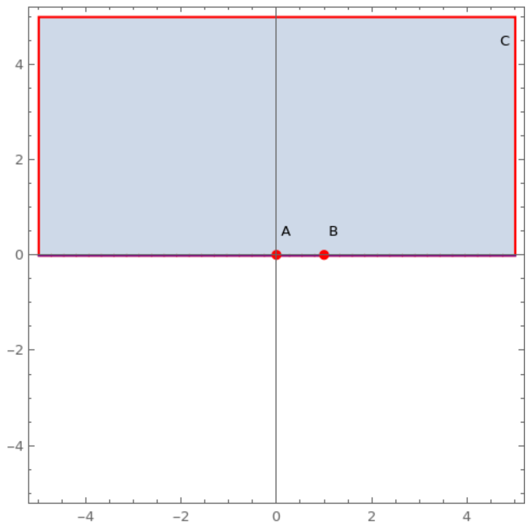
The Möbius mapping that achieves this is:

$$f(z) = -\frac{z+i}{z-i}$$

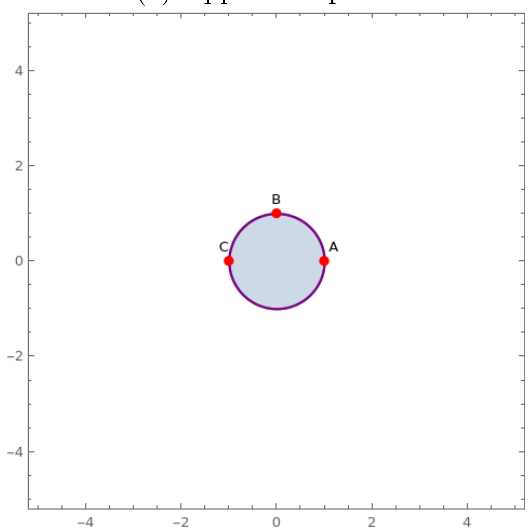
This would again map the real axis to the unit circle. However this time, testing with a value in the upper half plane, $f(i/2) = 3$, which is not in the unit disk. It seems that we have mapped the upper half plane to the outside of the circle. (Figure 10.5) To correct this, we use the inversion map $g : z \mapsto \frac{1}{z}$, as it preserves the unit circle where it is, but maps infinity to zero and vise versa.

The overall conformal map that we constructed is a composition of these two.

$$g \circ f(z) = -\frac{z-i}{z+i}$$

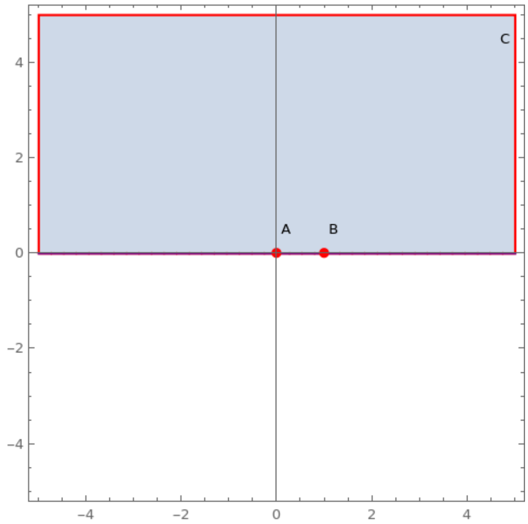


(a) Upper half plane

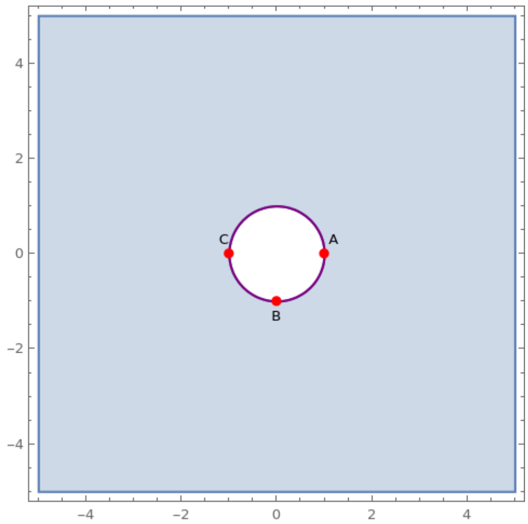


(b) Unit disk

Figure 10.4: Mapping upper half plane to the unit disk; values 0, 1, and ∞ chosen to be mapped to 1, i, and -1.



(a) Upper half plane



(b) Outer region of the unit circle

Figure 10.5: Mapping upper half plane to the unit circle; values 0, 1, and ∞ chosen to be mapped to 1, $-i$, and -1.

Remark 10.5. There is no guarantee that the conformal map you generate will be unique. In fact, there is three degrees according to the full statement of Riemann mapping theorem (which shall be stated later.)

Exercise 10.2 (Exponential Map)**.** Check that $z \mapsto e^{\pi z}$ maps the strip of width π:

$$R := \{z \in \mathbb{C} | \Im(z) \in (0, \pi)\}$$

to the upper half plane.

Exercise 10.3. Investigate what the map $z \mapsto z^\alpha$ does to a unit fan of angle and orientation of your choosing, anchored at the origin. (Hint: $\alpha = 2$ is the squaring map from before. The answer is in the next remark.)

Exercise 10.4. What does the map $z \mapsto e^{i\theta} z$ do? (The answer is in the next remark.)

Remark 10.6 (Useful Maps)**.** Now we have a set of maps that are useful for constructing a conformal map from one region to another

- Möbius maps (circlines to circlines)
- Translation: $z \mapsto z + c$
- Dialation: $z \mapsto kz$ where $k \in \mathbb{R} \setminus \{0\}$
- Rotation: $z \mapsto e^{i\theta} z$
- Exponential: $z \mapsto e^z$ (Horizontal Width-π strip to half plane)
- Constant-exponential: $z \mapsto z^\alpha$ (Opens up a fan by α times, with the reference along the positive real axis.)

Refer to `https://github.com/pauljoohyunkim/OxfordMathematicsNotesForPoorSouls/blob/main/2nd%20Year/Metric%20Spaces%20and%20Complex%20Analysis/Conformal%20Maps.pdf` for a cheat sheet of most of these.

Just a definition that might pop up here:

Definition 10.2 (Conformally Equivalent)**.** If there is a bijective conformal transformation between U and V in the complex plane, they are called **conformally equivalent**.

In the case of simply-connected domains, it turns out that most are conformally equivalent to each other!

Remark 10.7 (Riemann's Mapping Theorem)**.** Let U be open connected simply-connected proper subset of $\mathbb{C}$. Then for any $z_0 \in U$ there is a unique bijective conformal transformation $f : U \to B\,(0, 1)$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

This means all open connected simply-connected proper subsets are conformally equivalent!

Remark 10.8. Why did Riemann's mapping theorem have to include "proper"? Suppose we seek conformal map f mapping $\mathbb{C}$ to a unit circle. Then it must mean that $f(\mathbb{C}) \subset B\,(0, 1)$, which means f is a bounded function. By Liouville, f must be a constant function, but a constant function is not conformal.