Finite Difference: $D_x^{\pm}u(x_i) = u'(x_i) + O(h) \qquad D_x^+ D_x^- u(x_i) = u''(x_i) + O(h^2)$

$(V,W)_h \coloneqq \sum_{i=1}^{N-1} hV_iW_i$ Sum. by parts: $(-D_x^+D_x^-U,V)_h = (D_x^-U, D_x^-V]$ if $(U,V)_0 = (U,V)_N = 0$

L_2 norm $||U||_h = (U,U)^{1/2}$ Sob. norm $||U||_{1,h} = (||U||_h^2 + ||D_x^-U||_h^2)^{1/2}$

Discrete Poincaré $||V||_h^2 \le c_*||D_x^-V]|_h^2$ (1D $c_* = \frac{1}{2}$)

1D Elliptic $-u'' + c(x)u = f(x)$, $u(0) = 0 = u(1) \qquad Au = (-D_x^+D_x^- + c)u \qquad c \ge 0$

$(AV,V)_h \ge ||D_x^-V]|_h^2$ (Uniqueness) & $(AV,V)_h \ge \frac{1}{c_*}||V||_h^2 \Longrightarrow (1+c_*)(AV,V)_h \ge ||V||_h^2 + ||D_x^-V]|_h^2$

$\Longrightarrow (AV,V)_h \ge c_0||V||_{1,h}^2 \Longrightarrow ||U||_{1,h} \le \frac{1}{c_0}||f||_h$ (Stability)

Global error $e_i \coloneqq u(x_i) - U_i$ for $i = 0, \cdots, N$

$Ae_i = Au(x_i) - f(x_i) = u''(x_i) - D_x^+D_x^-u(x_i) \Longrightarrow Ae_i = \varphi_i = O(h^2)$

By stability result, $||u-U||_{1,h} = ||e||_{1,h} \le \frac{1}{c_0}||\varphi||_h$

Since $|\varphi_i| \le \frac{1}{12}h^2||u^{(4)}(x)||_C$, we have $||u-U||_{1,h} \le \frac{1}{8}||u^{(4)}||_{C[0,1]}$ (Convergence)

"Stability + Consistency $\Longrightarrow$ Convergence"

2D Elliptic $\Delta u + c(x,y)u = f(x,y)$ in Ω and $u = 0$ on $\partial\Omega$

$AU_{i,j} = -(D_x^+D_x^-U_{i,j} + D_y^+D_y^-U_{i,j}) + c_{i,j}U_{i,j} = f_{i,j} \qquad U_{i,j} = 0$ for Γ_h "5-pt + scheme"

$(V,W)_h = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} h^2V_{i,j}W_{i,j}$

Sum by Parts $\Longrightarrow (AV,V)_h \ge ||D_x^-V]|_x^2 + ||D_y^-V]|_y^2$ (Uniqueness)

(Poincaré) $||V||_h^2 \le c_*(||D_x^-V]|_x^2 + ||D_y^-V]|_y^2)$ (2D $c_* = \frac{1}{4}$)

SBP + Poincaré: $(AV,V)_h \ge c_0||V||_{1,h}^2$ $(c_0 = (1+c_*)^{-1}) \Longrightarrow ||U||_{1,h} \le \frac{1}{c_0}||f||_h$ (Stability)

Global error $e_{i,j} \coloneqq u(x_i,y_j) - U_{i,j}$ Consistency Error $\varphi_{i,j} \coloneqq Au_{i,j} - f_{i,j}$

Since $\varphi_{i,j} = \frac{h^2}{12}(u_{xxxx}(\xi_i,y_j) + u_{yyyy}(x_i,\eta_j))$ and $Ae_{i,j} = \varphi_{i,j}$, $e = 0$ on Γ_h

$||u-U||_{1,h} \le \frac{5h^2}{48}(||u_{xxxx}||_{C(\bar\Omega)} + ||u_{yyyy}||_{C(\bar\Omega)})$ (Convergence)

Nonaxiparallel: $h_{i+1} = x_{i+1} - x_i$, $h_i = x_i - x_{i-1}$, $\hbar_i = \frac{1}{2}(h_{i+1} + h_i)$

Discrete Max. Principle Useful for uniqueness, and max norm stability (cts dep on BD).

$-D_x^+D_x^-U_{i,j} - D_y^+D_y^-U_{i,j} = f_{i,j}$ & $U_{i,j} = g_{i,j}$ on Γ, $f \le 0$

$\Longrightarrow (\cdot)U_{i,j} = (\cdot)U_{i+1,j} + (\cdot)U_{i-1,j} + (\cdot)U_{i,j-1} + (\cdot)U_{i,j+1} + f_{i,j}$

Contradiction: For $(f < 0)$ Replace RHS $U_{i\pm1,j\pm1} \to U_{i,j}$ then $(\cdot) < (\cdot)$ (without f), but the expr on both sides are equal.

For $f \le 0$, $V_{i,j} \coloneqq U_{i,j} + \frac{\epsilon}{4}({x_i}^2 + {y_j}^2)$, consider $\max_{(x_i,y_j)\in\Gamma_h} U_{i,j}$

Tip: $\psi_{i,j} = (x_i - \frac{1}{2})^2 + (y_j - \frac{1}{2})^2 \Longrightarrow \mathcal{L}_h\psi_{i,j} = -4(=-2n) \Longrightarrow \mathcal{L}_he_{i,j} = \tau_{i,j}$. Let $\tau \coloneqq \max|\tau_{i,j}|$. Then $\phi_{i,j} = e_{i,j} \mp \frac{\tau}{2n}\psi_{i,j}$, $\mathcal{L}_h(\phi_{i,j}) \ge (\le) 0$. Use min/max principle with $e = 0$ on Γ. (Convergence)

Eigen 1D: $-D_x^+D_x^-U_i + cU_i = \Lambda U_i$ with $U_0 = 0 = U_N$

$\Longrightarrow U_i = \sin k\pi x_i$ with $c + 8 \le \Lambda = c + \frac{4}{h^2}\sin^2\frac{k\pi h}{2} \le c + \frac{4}{h^2}$ (Symm 3diag $A \in \mathcal{M}_{n\times n}$ s.t. $AU = F$)

Eigen 2D: $-D_x^+D_x^-U_{i,j} - D_y^+D_y^-U_{i,j} + cU_{i,j} = \Lambda U_{i,j}$ with homogen BC.

$\Longrightarrow U_{i,j} = \sin k\pi x_i + \sin l\pi y_j$ with $\Lambda = c + \frac{4}{h^2}(\sin^2\frac{k\pi h}{2} + (\sin^2\frac{l\pi h}{2}) \in [c+16, c+\frac{8}{h^2}]$

Iter Method: $U^{(j+1)} \coloneqq U^{(j)} - \tau(AU^{(j)} - F)$ with lim sol U s.t. $U = U - \tau(AU - F)$ (A sym, + eigval)

$\Longrightarrow U - U^{(j+1)} \coloneqq (I - \tau A)(U - U^{(j)}) \Longrightarrow U - U^{(j)} = (I - \tau A)^j(U - U^{(0)})$

To ensure convergence $||I - \tau A|| \le \min_{\tau>0}\max\{|1-\tau\beta|, |1-\tau\alpha|\} < 1 \Longrightarrow \tau = \frac{2}{\alpha+\beta}$ for best convergence.

Parabolic $u_t = u_{xx} + f$ on $\{(x,t) \in (-\infty,\infty)\times[0,T]\}$

θ-scheme: $\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} + \theta\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} \qquad \theta = 0{,}1$ (expl,impl resp)

Consistency Error:

$T_j^m = \frac{u_j^{m+1} - u_j^m}{\Delta t} - (1-\theta)\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} - \theta\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2}$ for $j = 0, \pm1, \pm2$, $m = 0, \cdots, M-1$

$T_j^m = \begin{cases} O(\Delta x^2 + \Delta t^2), & \theta = \frac{1}{2} \\ O(\Delta x^2 + \Delta t), & \theta \ne \frac{1}{2} \end{cases}$ from Taylor expansion around x_j, $t_{m+\frac{1}{2}}$

Practical Stability: $||U^m||_{l_2} \le ||U^0||_{l_2}$ where $||U^m||_{l_2} = (\Delta x\sum_{j=-\infty}^{\infty}|U_j^m|^2)^{\frac{1}{2}}$

Semidiscrete FT: $\hat{U}(k) = \Delta x\sum_{j=-\infty}^{\infty} U_je^{-ikx_j}$ for $k \in [-\pi/\Delta x, \pi/\Delta x]$

Semidiscrete IFT: $U_j = \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x}\hat{U}(k)e^{ikj\Delta x}dk$

Discrete Parseval: $||U||_{l_2} = \frac{1}{\sqrt{2\pi}}||\hat{U}||_{L_2}$ Pf: $||\hat{U}||^2_{L_2} = \int \hat{U}(k)\overline{\hat{U}(k)}dk = \int \hat{U}(k)\Delta x \sum U_j e^{ikj\Delta x}dk = \sum U_j \Delta x 2\pi \frac{1}{2\pi}\int \hat{U}(k)e^{ikj\Delta x}dk = \sum U_j \Delta x 2\pi U_j = ||U||^2_{l_2}2\pi$

Stability Analysis: $U_j^m = \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x}\hat{U}^m(k)dk$ Tip: $1-\cos\varphi = 2\sin^2\frac{\varphi}{2}$, $e^{ik\Delta x}+e^{-ik\Delta x}-2 = 2(-1+\cos k\Delta x) = -4\sin^2\frac{k\Delta x}{2}$

$\hat{U}_j^m(k) = \lambda(k)\hat{U}^m(k)$, Stable iff $|\lambda(k)| \leq 1$

Von Neumann: On $[0,T]$, $\exists C = C(T)$ s.t. $||U^m||_{l_2} \leq C||U^0||_{l_2}$

Lemma: $\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^m(k)$, $|\lambda(k)| \leq 1 + C_0\Delta t$ for some $C_0 \implies$ Neumann Stable, PT$\implies$NS

θ-scheme has $\lambda(k) = \frac{1-4(1-\theta)\mu\sin^2\frac{k\Delta x}{2}}{1+4\theta\mu\sin^2\frac{k\Delta x}{2}}$

$\implies$ Uncond. PS if $\theta \in [1/2, 1]$ and Cond. PS if $\theta \in [0, 1/2)$ where stbl iff $2(1-2\theta)\mu \leq 1$

θ-scheme IBVP θ-scheme $\implies \left(1-\theta\mu\delta^2\right)U_j^{m+1} = \left(1+(1-\theta)\mu\delta^2\right)U_j^m$ $(\delta^2 U_j := U_{j+1} - 2U_j + U_{j-1})$

$(I-\theta\mu A)\,\mathbf{U}_j^{m+1} = (I+(1-\theta)\mu A)\,\mathbf{U}^m + \theta\mu\mathbf{F}^{m+1} + (1-\theta)\mu\mathbf{F}^m$ $(A \in \mathcal{M}^{(J-1)^2}$ 3diag $1, -2, 1$ entry$)$

with $\mathbf{U}^m := (U_1^m, \cdots, U_{J-1}m)^T$, $\mathbf{F}^m := (A(t_m), 0, \cdots, 0, B(t_m))^T$

Discrete max. principle Given $0 \leq \theta \leq 1$, $\mu(1-\theta) \leq \frac{1}{2}$ (More strct cond)

Pf) $(1+2\theta\mu)\,U_J^{m+1} = \theta\mu\left(U_{j+1}^{m+1} + U_{j-1}^{m+1}\right) + (1-\theta)\,\mu\left(U_{j+1}^m + U_{j-1}^m\right) + (1-2(1-\theta)\mu)\,U_j^m (\dagger)$

Suppose max at U_j^{m+1}. For $U^* = \max\left(U_{j\pm1}^{m+1/2\pm1/2}, U_j^m\right)$, replace $U_{j\pm1}^{m\pm1} \to U^*$, then $\leq \implies U_j^{m+1} \leq U^*$, but $U^* \leq U_j^{m+1}$

Convergence in max norm: $(\dagger) \implies (\dagger\dagger)$RHS $-$ LHS $= 0$ Replace $U \to u \implies (\cdot) = \Delta t T_J^m$

$(1+2\theta\mu)\,e_J^{m+1} = \theta\mu\left(e_{j+1}^{m+1} + e_{j-1}^{m+1}\right) + (1-\theta)\,\mu\left(e_{j+1}^m + e_{j-1}^m\right) + (1-2(1-\theta)\mu)\,e_j^m + \Delta t T_j^m$

$E^m := \max_{0\leq j\leq J}|e_j^m|$ (Analog T_j^m) $\implies E^{m+1} \leq E^m + \Delta t T^m$

$\xRightarrow{Telescope} E^m \leq T\max_{0\leq m\leq M-1}\max_{1\leq j\leq J-1}|T_j^m|$ (Noting $E^0 = 0$)

$\max_{0\leq m\leq M}\max_{0\leq j\leq J}|u_j^m - U_j^m| \leq T\max_{0\leq m\leq M-1}\max_{1\leq j\leq J-1}|T_j^m|$ (Use consistency error)

Hypberbolic $u_{tt} - c^2u_{xx} = f(x,t)$ on $(x,t) \in (a,b)\times(0,T]$;

$u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$; $u(a,t) = u(b,t) = 0$ Energy inequality for analysis. $(\cdot, u_t)$

Implicit: $\frac{U_j^{m+1}-2U_j^m+U_j^{m-1}}{(\Delta t)^2} - c^2\frac{U_{j+1}^{m+1}-2U_j^{m+1}+U_{j-1}^{m+1}}{(\Delta t)^2} = f(x_j, t_{m+1})$ for $j = 1, \cdots, J-1$, $m = 1, \cdots, M-1$

$U_j^0 = u_0(x_j)$ $U_j^1 = U_j^0 + \Delta t u_1(x_j)$ $U_0^m = 0 = U_J^m$

Note $(A-B, A) = \frac{1}{2}(||A||^2 - ||B||^2) + \frac{1}{2}||A-B||^2$ (Analog $(A-B, A]$)

Take inner product of scheme with $\frac{U^{m+1}-U^m}{\Delta t}$, and define $\mathcal{M}^2(U^m) := ||\frac{U^{m+1}-U^m}{\Delta t}||^2 + c^2||D_x^- U^{m+1}]|^2$

Unconditional Stability: $\mathcal{M}^2(U^m) \leq e^2\mathcal{M}(U^0) + 2e^2T\sum_{k=1}^m \Delta t||f(\cdot, t_{k+1})||^2$

Consistency: $T_j^{m+1} := \frac{u_j^{m+1}-2u_j^m+u_j^{m-1}}{(\Delta t)^2} - c^2\frac{u_{j+1}^{m+1}-2u_j^{m+1}+u_{j-1}^{m+1}}{(\Delta x)^2}$ $T_j^1 := \frac{u_j^1-u_j^0}{\Delta t} - u_1(x_j)$

$\implies |T_j^{m+1}| \leq \frac{1}{12}c^2(\Delta x)^2M_{4x} + \frac{5}{3}\Delta t M_{3t}$ where $M_{nx} := \max_{(x,t)\in[a,b]\times[0,T]}|\frac{\partial^n u}{\partial x^n}(x,t)|$ (Analog M_{nt})

Global Error $e_j^m := u(x_j, t_m) - U_j^m \implies \frac{e_j^{m+1}-2e_j^m+e_j^{m-1}}{(\Delta t)} - c^2\frac{e_{j+1}^{m+1}-2e_j^{m+1}+e_{j-1}^{m+1}}{(\Delta x)^2} = T_j^{m+1}$, $e_j^1 = e_j^0 + \Delta t T_j^1$

Explicit

Conditionally stable: $c\frac{\Delta t}{\Delta x} \leq c_0 < 1$

1st Hyperbolic $u_t + b(x)u_x + c(x,t)u = f(x,t)$ $u(x,t) = 0$ (inflow) $u(x,0) = u_0(x)$

Explicit: $u_t + bu_x = f(x,t)$ on $x \in (0,1)$, $t \in (0,T]$ $b > 0 \implies u(0,t) = 0$ with $t \in (0,T]$

$\frac{U_j^{m+1}-U_j^m}{\Delta t} + bD_x^-U_j^m = f(x_j, t_m) \iff U_j^{m+1} = (1-\mu)U_j^m + \mu U_{j-1}^m + \Delta t f_j^m$ with $0 \leq \mu = \frac{b\Delta t}{\Delta x} \leq 1$

Stability: $|U_j^{m+1}| \leq \max_{0\leq j\leq J}|U_j^m| + \Delta t\max_{0\leq j\leq J}|f_j^m| \implies ||U^{m+1}|_\infty \leq ||U^m||_\infty + \Delta t f(\cdot, t_m)_\infty$

$\implies \max_{1\leq k\leq M}||U^k||_\infty \leq ||U^0||_\infty + \sum_{m=0}^{M-1}\Delta t||f(\cdot, t_m)||_\infty$

Nonlinear Hyperbolic $u_t + (f(u))_x = 0$ $\mathbb{R}\times(0,\infty)$, $f(0) = f'(0) = 0$, $f''(s) \geq 0$, $|f'(s)| \leq f'(|s|)$

Use $f'(u) = [f'(u)]_+ + [f'(u)]_- \implies u_t + [f'(u)]_+u_x + [f'(u)]_-u_x = 0$

Scheme: $\frac{U_j^{m+1}-U_j^m}{\Delta t} + [f'(U_j^m)]_+D_x^-U_j^m + [f'(U_j^m)]_-D_x^-U_j^m = 0$

Stability: CFL $\frac{f'(||U^k||_\infty)\Delta t}{\Delta x} \leq 1 \implies ||U^{m+1}||_\infty \leq ||U^m||_\infty$

If u has cts bdd 2nd derivatives w.r.t x and t, then $\max_{1\leq m\leq M}||u^m - U^m||_\infty = O(\Delta x + \Delta t)$