Complex Analysis Summary

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June 24, 2023

0 Preface

This note is for people studying complex analysis, and got lost in the middle with bunch of technical explanations. I wil try my best to be succinct as possible, stating important results (mostly without proof, but a bit of justification).

Warning: This summary note is not a substitute for the lecture note. Make sure you study from lecture note!

1 Complex Plane and Möbius Maps

1.1 Complex Plane and Complex Infinity

We will be working in what's known as the *extended complex plane.* Define the symbol $\mathbb{C}_\infty := \mathbb{C} \cup \left\{ \underbrace{\infty}_{\text{Complex Infinity}} \right\}$; that is, I refer to the space of complex numbers and infinity.

Note that in $\mathbb{C}_\infty$, ∞ is different from infinity in real numbers. $\infty := \frac{1}{0}$ is a value that is not "larger" or "smaller" than any number (since we are talking about complex number...), but rather a number on a complex plane at a really far distance from origin.

It is **WRONG** to say:

- $\infty \geq a$ for any $a \in \mathbb{C}_\infty$
- $\infty \leq a$ for any $a \in \mathbb{C}_\infty$

However, it is **CORRECT**[1] to say:

[1]Subtlety here: it seems a bit dodgy to say $\infty = \infty_\infty$, but this is matter of definition; you won't really encounter this type of "philosophical" problem in your exam.

- $|\infty| \geq |a|$ for any $a \in \mathbb{C}_\infty$.

∞ is not like a point on $\mathbb{C}$, but rather like a gigantic circle that you can never reach.

1.2 Möbius Maps

Definition 1.1 (Möbius Map)**.** $\psi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a **Möbius map** if:

$$\psi(z) := \frac{az+b}{cz+d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonsingular matrix. (This restriction removes the possibility of $\frac{0}{0}$, or trivial maps (eg: Constant function).)

One needs to be careful when defining this function at infinity, but it should be sensible.[2]

Exercise 1.1 (Composition of two Möbius map is a Möbius map)**.** Show that for two Möbius maps ψ_1, ψ_2, its composition $\psi_1 \circ \psi_2$ is also a Möbius map.

Remark 1.1. Consider the 2×2-matrix-to-Möbius-map map as follows:

$$f(A) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}$$

Then it turns out that $f(AB) = f(A)f(B)$

Exercise 1.2 (Decomposition of Möbius maps)**.** It turns out that Möbius maps can be written as composition of

- translation
- dialation ("scaling by nonzero constant")
- inversion ($z \mapsto \frac{1}{z}$)

Prove this. (Hint: You can do a constructive proof.)

Möbius maps also has a very convenient property:

Exercise 1.3 (Circline to Circline)**.** Show that Möbius maps map circlines to circline. (This means a line will either map to a circle or a line, and also a circle will either map to a circle or a line.)

(Note: This is a boring long tedious proof, that probably won't be asked in exam, but don't take my word for it.)

[2]That said, if you are supposed to define what a Möbius map is, you are **required** to definitions involving infinity as well.

2 Complex Differentiability

Complex differentiability is one of the highlights of the complex analysis.

Definition 2.1 (Differentiable Function AKA Holomorphic Function)**.** Take $a \in \mathbb{C}$. Let $f : U \to \mathbb{C}$ be a function where U is a neighbourhood[3] of a. Then f is **(complex) differentiable** or **holomorphic** at a if

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and call it derivative of f at a. If f is differentiable for all points in U, then it is said to be differentiable/holomorphic on U.

Remark 2.1. Note that the definition seems to be have trivially extended from real analysis. However, there is a subtlety. The limit does not approach just from positive or negative side, but from any direction. (Figure 2.1)

Exercise 2.1 (Differentiation Rules)**.** Show that all differentiation rules from real analysis holds with holomorphic functions.

- Sum
- Product Rule
- Quotient Rule
- Chain Rule

Due to the definition of complex limits being more restrictive, a more nontrivial result follows.

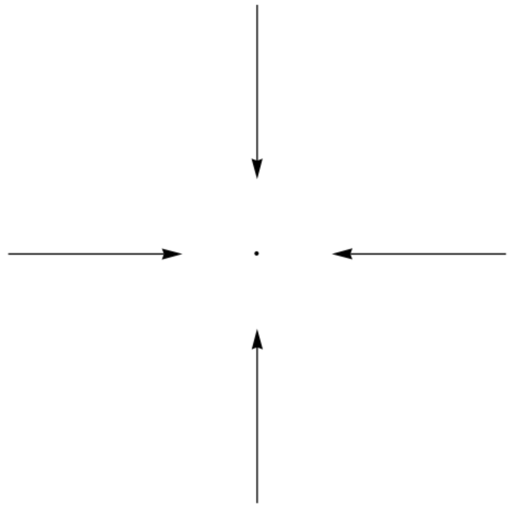
Exercise 2.2 (Cauchy-Riemann Equations)**.** Let $a \in \mathbb{C}$ and U be a neighbourhood of a. $f : U \to \mathbb{C}$ be holomorphic a. Write $f(z) = u(x, y) + iv(x, y)$ where u, v are real functions and $z = x + iy$ where $x, y \in \mathbb{R}$. Then $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ all exist, and the following **Cauchy-Riemann equations** hold:

$$\partial_x u = \partial_y v$$
$$\partial_x v = -\partial_y u$$

(Hint: Figure 2.1 might give you an insight.)

[3] Some open set containing a.

(a) Real Limit



(b) Complex Limit

Figure 2.1: Real limit (former) only concerns the approaching value from left and right side, but complex limit (latter) oncerns the approaching value from all direction.

Remark 2.2. If Cauchy-Riemann does not hold, then it must mean that f is not holomorphic! (Consider $f(z) = \bar{z}$. Cauchy-Riemann does not hold for any point, so it is nowhere holomorphic.)

Exercise 2.3. If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on U, and u, v are twice differentiable, deduce that u and v are *harmonic*, that is, they satisfy the Laplace equation $\Delta u = \Delta v = 0$.

Remark 2.3. It turns out complex plane reveals a lot about solving Laplace equation!

Here is another kicker:

Exercise 2.4 (Cauchy-Riemann to Holomorphic)**.** If the partial derivatives exist and are continuously differentiable, Cauchy-Riemann implies holomorphicity.

Remark 2.4. This means if you check that Cauchy-Riemann holds, you can immediately assume you can construct an analytic function!

Holomorphic functions also have Taylor expansion:

Remark 2.5 (Holomorphic functions have Taylor expansion)**.** If f is holormorphic at a, then in a neighbourhood of a, you can write

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

All the formulae for Taylor expansion holds (term-by-term differentiation, etc.)

Example 2.1 (Common Function Definitions)**.** Here are definitions for some of the functions.

$$e^z = \exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Exercise 2.5. Show that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Exercise 2.6. Show that $\exp(z + w) = \exp(z)\exp(w)$

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