Computing Residue

For a function $f(z)$ with isolated singularity at $z = z_0$, it is possible to write a Laurent exansion around $z = z_0$:

$$f(z) \sim \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \qquad \text{in an annulus around } z = z_0 \tag{1}$$

c_{-1} is what is known as the **residue**.

Often, functions are given in the form $f(z) = \frac{g(z)}{h(z)}$, where h would have roots at which they become the poles of f. Strategies for computing the residue at $z = z_0$ is as follows:

1. Decompose f as a product of a function that are holomorphic around $z = z_0$, and a function that results in a singularity, such as $f(z) = g(z) \times \frac{1}{h(z)}$, where h would have roots at $z = z_0$.
2. For the function holomorphic around $z = z_0$, just evaluate it at $z = z_0$, because around $z = z_0$, the approximation is its evaluation.
3. For the function with a pole, try Taylor expanding the denominator, then pull out a factor such that the dominant term is 1.

$$\begin{aligned} \frac{1}{h(z)} &= \frac{1}{h(z_0) + h'(z_0)(z - z_0) + \cdots} \\ &= \frac{1}{h'(z_0)(z - z_0) + \cdots} && \because z = z_0 \text{ is the root of } h(z) \\ &= \frac{1}{h'(z_0)(z - z_0)} \times \underbrace{\frac{1}{1 + \cdots}}_{(*)} \end{aligned}$$

4. Notice that the non-dominant terms are "small" near $z = z_0$. Now recall the geometric series formula:

$$\frac{1}{1 - r} = 1 + r + r^2 + \cdots \qquad \text{for } |r| < 1 \text{, in particular, } |r| \ll 1 \tag{2}$$

   Note that you can apply this to $(*)$ to get the denominator as a series involving no denominators.

5. Expand and gather terms that are order -1. That is the term that gives residue.

Now, if the pole at $z = z_0$ is **simple**, then a faster way is to just multiply by $(z - z_0)$, then take limit:

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} f(z)(z - z_0) \tag{3}$$

This follows from substituting (1) into the limit on the RHS.

1 Examples

Example 1. *Find the residue of $f(z) = \frac{\sin z}{\cos z}$ at $z = \frac{\pi}{2}$.*

<u>*Method 1*</u>*: Note that $f(z)$ has a simple pole at $z = \frac{\pi}{2}$. This means residue is just:*

$$\operatorname*{Res}_{z=\frac{\pi}{2}} f(z) = \lim_{z\to\frac{\pi}{2}} f(z)(z - \frac{\pi}{2}) = -1$$

Note that the evaluation of the limit is often done by noticing the definition of the derivative:

$$\lim_{z\to\frac{\pi}{2}} \frac{z - \frac{\pi}{2}}{\cos z} = \frac{1}{((\cos z)')\,|_{z=\frac{\pi}{2}}}$$

<u>*Method 2*</u>*: You can also compute the Laurent series. Because $\sin z$ is holomorphic at $z = \frac{\pi}{2}$, you don't need to take Laurent series of $\sin z$; you only need to take the Laurent series of $\frac{1}{\cos z}$. Let's walk through the five-step method as before:*

1. *Decomposition: $f(z)$ can be written as:*

$$f(z) = \sin z \times \frac{1}{\cos z}$$

2. *$\sin z$ being holomorphic at $z = \frac{\pi}{2}$ means $\sin z$ does not have z^{-n} ($n > 1$) terms, so you can take Taylor series, but you actually only need the constant term for computing residue:*

$$\sin\frac{\pi}{2} = 1$$

3. *Expand out $\frac{1}{\cos z}$ at $z = \frac{\pi}{2}$:*

$$\begin{aligned}\frac{1}{\cos z} &= \frac{1}{-\left(z-\frac{\pi}{2}\right) + \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^3 + \cdots} \\ &= \frac{1}{-\left(z-\frac{\pi}{2}\right)} \times \underbrace{\frac{1}{1 - \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^2 + \cdots}}_{(\star)}\end{aligned}$$

4. *Since we are inspect around $\frac{\pi}{2}$, the subleading term in $(\star)$ is "small", so we may use the geometric series expansion:*

$$\begin{aligned}\frac{1}{\cos z} &= \frac{1}{-\left(z-\frac{\pi}{2}\right)} \times \underbrace{\frac{1}{1 - \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^2 + \cdots}}_{(\star)} \\ &= \frac{1}{-\left(z-\frac{\pi}{2}\right)}\left(1 + \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^2 + \cdots\right)\end{aligned}$$

5. *Hence,*

$$\frac{\sin z}{\cos z} = \sin z \times \left(-\left(z-\frac{\pi}{2}\right)^{-1} + O\left(z-\frac{\pi}{2}\right)\right)$$

Hence, (by evaluating $\sin z$ at $z = \frac{\pi}{2}$,) residue is -1. Note that the full Laurent series can be acquired by Taylor exapnding $\sin z$ around $z = \frac{\pi}{2}$ as well.

Example 2. *Find the residues of $f(z) = \frac{1}{(z+1)^2(z-1)}$ at $z = -1$ and $z = 1$.*

We will go through this one a bit quicker. Residue at $z = 1$: $f(z)$ has a simple pole at $z = 1$, so

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} f(z)(z-1) = \frac{1}{4}$$

Residue at $z = -1$: There is a double pole at $z = -1$. You can construct Laurent expansion:

$$\begin{aligned}
f(z) &= \frac{1}{(z+1)^2} \times \frac{1}{z-1} \\
&= \frac{1}{(z-1)^2} \times \frac{1}{-2 + (z+1)} && \because \textit{Taylor of } z-1 \textit{ at } z=-1 \\
&= \frac{1}{(z+1)^2} \times \frac{1}{-2} \times \frac{1}{1 - \frac{z+1}{2}} && \textit{Get 1 in the leading term} \\
&= \frac{1}{(z+1)^2} \times \frac{1}{-2} \times \left(1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \cdots\right) \\
&= -\frac{1}{2}\left((z+1)^{-2} + \frac{1}{2}(z+1)^{-1} + \cdots\right)
\end{aligned}$$

So

$$\operatorname*{Res}_{z=-1} f(z) = -\frac{1}{4}$$

Example 3. *Define a branch cut at the positive real axis for $\sqrt{z}$. Find the residue of $f(z) = \frac{\sqrt{z}}{z+i}$ at $z = -i$.*

Let $z = re^{i\theta}$ where $r = |z|$ and $\theta \in (0, 2\pi)$, and define the branch cut as:

$$\sqrt{z} := r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$$

Now note that $\sqrt{z}$ is holomorphic away from the positive real axis, and specifically holomorphic at $z = -i$, hence it can be considered separately.

$$\begin{aligned}
f(z) &= \frac{\sqrt{z}}{z+i} \\
&= \sqrt{z} \times \frac{1}{z+i}
\end{aligned}$$

So

$$\operatorname*{Res}_{z=-i} f(z) = \sqrt{-i} \times 1 = e^{\frac{3\pi}{4}i}$$

For more examples of residues, see `https://math.libretexts.org/Bookshelves/Analysis/Complex_Variables_with_Applications_(Orloff)/09%3A_Residue_Theorem/9.04%3A_Residues`