$\underline{\text{Newton}}$ $\mathbf{F} = m\ddot{\mathbf{r}} = \dot{\mathbf{p}}$ $\quad$ $\underline{\text{Work}}$: $W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = T(t_2) - T(t_1)$ $\quad$ $\underline{\text{Strong N3}}$: $\mathbf{F}_{12} || (\mathbf{r}_1 - \mathbf{r}_2)$

$\underline{\text{Angular}}$: $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ $\quad$ $\boldsymbol{\tau} = \mathbf{r} \wedge \mathbf{F} = \dot{\mathbf{L}}$ (Origin) (Vanishes if $\mathbf{F} || \mathbf{r}$)

$\underline{\text{Kepler}}$: $V(\mathbf{r}) = -\frac{\kappa}{r}$ $\quad$ $\mathbf{F} = -\nabla V = \left(-\frac{\kappa}{r^3}\right) \mathbf{r}$

$\underline{\text{Principle of Least Action}}$: $S(\mathbf{q}(t)) \coloneqq \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt$ $\quad$ $L \coloneqq T - V$

$\frac{\partial}{\partial \epsilon} S[\mathbf{q}(t) + \epsilon \mathbf{u}(t)]|_{\epsilon=0} = 0 \Longrightarrow \underline{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{0}}$ (LE)

$L_2(\mathbf{q}, \dot{\mathbf{q}}, t) = L_1(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{d}{dt} f(\mathbf{q}, t) \Longrightarrow S_2 = S_1 + f\left(\mathbf{q}^{(2)}, t\right) - f\left(\mathbf{q}^{(1)}, t\right)$

$\underline{\text{Invar under Coord Change}}$: Let $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$, $\tilde{L}\left(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t\right) \equiv L\left(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}\left(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t\right), t\right)$ Note $\dot{q}_a = \sum_{b=1}^n \frac{\partial q_a}{\partial \tilde{q}_b} \dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t}$ Then $\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} = \sum_{b=1}^n \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_b}\right) - \frac{\partial L}{\partial q_b}\right] \frac{\partial q_b}{\partial \tilde{q}_a}$

$\underline{\text{Holonomic Constraints}}$: Constraint: $f_A(\mathbf{x}, t) = 0$, $\hat{L}(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\lambda}, t) = L_0(\mathbf{x}, \dot{\mathbf{x}}, t) + \sum_{A=1}^{d-n} \lambda_A f_A(\mathbf{x}, t)$

$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_i}\right) - \frac{\partial L_0}{\partial x_i} = \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial x_i}$ for $i = 1, \cdots, d$ $\quad$ $f_a(\mathbf{x}, t) = 0$ for $A = 1, \cdots, d-n$

Noether & Symmetry

$\frac{\partial}{\partial \epsilon} L(\mathbf{q}(t) + \epsilon \mathbf{u}(t), \dot{\mathbf{q}} + \epsilon \dot{\mathbf{u}}(t), t)|_{\epsilon=0} = \frac{d}{dt} f(\mathbf{q}(t)\dot{\mathbf{q}}(t), t)$ $\quad$ ($\boldsymbol{\rho}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = \mathbf{u}(t)$ Generator)

$\Longrightarrow F \coloneqq \sum_{a=1}^n \frac{\partial L}{\partial \dot{q}_a} \rho_a - f$ conserved $\quad$ ($\dot{F} \equiv 0$)

$\underline{\text{Time-translation}}$ $\frac{\partial L}{\partial t} = 0 \overset{\boldsymbol{\rho} = \dot{\mathbf{q}}, f = L}{\Longrightarrow} \frac{\partial}{\partial \epsilon}(\cdot)|_{\epsilon=0} = \frac{d}{dt} L \Longrightarrow H \coloneqq \sum_{a=1}^n \frac{\partial L}{\partial \dot{q}_a} \dot{q}_a - L$ conserved

$\underline{\text{Ignorable Coords}}$ $\frac{\partial L}{\partial q_i} = 0 \overset{\boldsymbol{\rho} = \mathbf{e}_i, f \equiv 0}{\Longrightarrow} p_i \coloneqq \frac{\partial L}{\partial \dot{q}_i}$ conserved

$\underline{\text{Rota. Invar.}}$ $\mathbf{r} \to R\mathbf{r}$ ($R = I + \epsilon \Omega + O(\epsilon^2)$) $\Longrightarrow \mathbf{r} \to \mathbf{r} + \epsilon \mathbf{n} \wedge \mathbf{r} \Longrightarrow \mathbf{L}$ conserved

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Oscillation: $\mathbf{F} \coloneqq -\frac{\partial V}{\partial \mathbf{q}}$, $T_{\text{quad}} = \frac{1}{2} \sum_{a,b} T_{ab}(\mathbf{0}) \dot{q}_a \dot{q}_b + O(q^3)$, $V_{\text{quad}} = \cancel{V(\mathbf{0})} + \frac{1}{2} \sum_{a,b} V_{q_a q_b}(\mathbf{0}) q_a q_b + O(q^3)$

Critical Point: $\mathbf{F} = 0 \Longleftrightarrow V_{\mathbf{q}} = \mathbf{0}$ $\quad$ $L_{\text{quad}} = \frac{1}{2} \sum_{a,b=1}^n \mathcal{T}_{ab} \dot{q}_a \dot{q}_b - \frac{1}{2} \sum_{a,b=1}^n \mathcal{V}_{ab} q_a q_b$

$\underline{\text{Quad EL}}$: $\Longrightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = \sum_b \frac{d}{dt}\left(T_{ab} \dot{q}^b\right) - \sum_{b,c} \frac{\partial T_{bc}}{\partial q^a} \dot{q}^b \dot{q}^c + \frac{\partial V}{\partial q^a} \Longleftrightarrow \sum_b \mathcal{T}_{ab} \ddot{q}_b = -\sum_b \mathcal{V}_{ab} q_b$

$\Longleftrightarrow \mathcal{T} \ddot{\mathbf{q}} = -\mathcal{V} \mathbf{q} \Longleftrightarrow \ddot{\mathbf{q}} = -\mathcal{T}^{-1} \mathcal{V} \mathbf{q} \overset{\mathbf{q}(t) = f(t)\boldsymbol{\alpha}}{\Longrightarrow} \cdots$

$\mathcal{T}, \mathcal{V}$: $T = \frac{1}{2} \dot{\mathbf{q}}^T \mathcal{T} \dot{\mathbf{q}}$ $\quad$ $\mathcal{V} = \frac{1}{2} \mathbf{q}^T \mathcal{V} \mathbf{q}$

$\underline{\text{Char Equ}}$: $\det(\lambda \mathcal{T} - \mathcal{V}) = 0 \Longrightarrow \lambda > 0$ (stable), $\lambda < 0$ (unstable) $\quad$ $\omega = \sqrt{\lambda}$, $\alpha \in \ker(\lambda \mathcal{T} - \mathcal{V})$

Rigid Body Coriolis: $\hat{D}\mathbf{r} = D\mathbf{r} + \boldsymbol{\omega} \wedge \mathbf{r} \overset{\text{Fixed in S}}{=} \boldsymbol{\omega} \wedge \mathbf{r}$

$\hat{\mathbf{a}} = \hat{D}^2(\mathbf{r} + \mathbf{x}) = \mathbf{a} + (D\boldsymbol{\omega}) \wedge \mathbf{r} + 2\boldsymbol{\omega} \wedge D\mathbf{r} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) + \mathbf{A}$ $\quad$ ($\mathbf{x}$: pos of O from $\hat{O}$, $\mathbf{A} = \hat{D}^2 \mathbf{x}$)

$\mathbf{v} = \hat{D}(\mathbf{r} + \mathbf{x}) = \boldsymbol{\omega} \wedge \mathbf{r} + \mathbf{v}_O$ $\quad$ $M = \int_R \rho\, dV$ $\quad$ $\mathbf{r}_G = \frac{1}{M} \int_R \mathbf{r} \rho(\mathbf{r})\, dV$

$\mathbf{P} = \int_R \rho \mathbf{v}\, dV = \int_R \rho(\mathbf{r})(\boldsymbol{\omega} \wedge \mathbf{r} + \mathbf{v}_O)\, dV \overset{\text{if } O = G}{=} \int_R \rho(\mathbf{r}) \mathbf{v}_G\, dV = M\mathbf{v}_G$

About $\mathbf{r}_G$, $\mathbf{L} = \int_R \rho(\mathbf{r})\, \mathbf{r} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})\, dV$ $\quad$ $\mathbf{L} = \mathcal{I} \boldsymbol{\omega}$ $\quad$ $\mathcal{I}_{ij} = \int_R \rho(\mathbf{r})(\mathbf{r} \cdot \mathbf{r} \delta_{ij} - r_i r_j)\, dV$

$$\mathcal{I} = \int_R \rho(\mathbf{r}) \begin{pmatrix} y^2 + z^2 & -xy & -zx \\ -xy & x^2 + z^2 & -yz \\ -zx & -yz & x^2 + y^2 \end{pmatrix} dV$$

Parallel Axis Thrm: $\mathcal{I}_{ij}^{(Q)} = \mathcal{I}_{ij}^{(G)} + M((\mathbf{c} \cdot \mathbf{c})\delta_{ij} - c_i c_j)$ where $\mathbf{c} = \overrightarrow{GQ}$

$T = \frac{1}{2} M |\mathbf{v}_G|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}$ (Note $\frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \sum_{i=1}^3 I_i \omega_i^2$) $\quad$ $\dot{\mathbf{e}}_i = \boldsymbol{\omega} \wedge \mathbf{e}_i$

$\mathbf{L} = \sum_{i=1}^3 I_i \omega_i \mathbf{e}_i$ $\quad$ ($\mathbf{e}_i$ axis of inertia)

No external force, torque: $\dot{\mathbf{L}} = \mathbf{0} \Longrightarrow$ (Euler Equ) $\begin{cases} I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 = 0 \\ I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1 = 0 \\ I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 = 0 \end{cases}$

$$\begin{cases} T = \frac{1}{2} \sum_{i=1}^3 I_i \omega_i^2 \equiv \text{const.} \\ \mathbf{L} \cdot \mathbf{L} = L^2 = \sum_{i=1}^3 I_i^2 \omega_i^2 \equiv \text{const.} \end{cases} \overset{\omega_2 = f(\omega_1), \omega_3 = g(\omega_1)}{\Longrightarrow} \begin{cases} 2I_3 T - L^2 = I_1(I_3 - I_1)\omega_1^2 + I_2(I_3 - I_2)\omega_2^2 \\ 2I_2 T - L^2 = I_1(I_2 - I_1)\omega_1^2 + I_3(I_2 - I_1)\omega_3^2 \\ I_1^2 \dot{\omega}_1^2 = (I_2 - I_3)^2 \omega_2^2 \omega_3^2 \end{cases}$$

$\underline{\text{Euler Angle Rep of } \boldsymbol{\omega}} = \begin{pmatrix} \dot{\theta} \sin\psi - \dot{\varphi} \sin\theta \cos\psi \\ \dot{\theta} \cos\psi + \dot{\varphi} \sin\theta \sin\psi \\ \dot{\psi} + \dot{\varphi} \cos\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

Rigid Body

$T = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$ $\quad$ (I_1, I_2, I_3 w.r.t G) $\quad$ $\mathbf{v}_G = (\dot{x}, \dot{y}, \dot{z})$

$V = \int_R \rho(\mathbf{r})\, gz\, dV = MGz_G$

Lagrange Top: $T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$ $\quad$ (I_1, I_2, I_3 w.r.t G) $\quad$ $\mathbf{v}_O = \mathbf{0}$ $\quad$ $V = MGl \cos\theta$

Hamiltonian Mechanics Legendre Transform: $g(s) \coloneqq sx(s) - f(x(s))$

Lagrange to Hamiltonian: L w.r.t $\dot{\mathbf{q}} \overset{LT}{\Longrightarrow} H(\mathbf{q},\mathbf{p},t) = \sum_{a=1}^{n} p_a \dot{q}_a - L(\mathbf{q},\dot{\mathbf{q}},t)|_{\dot{\mathbf{q}}=\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t)}$

Hamiltonian Equation (I): $\dot{\mathbf{p}} = -H_{\mathbf{q}} \quad \dot{\mathbf{q}} = H_{\mathbf{p}}$

$\frac{d}{dt} f = \sum_{a=1}^{n} (f_{q_a}\dot{q}_a + f_{p_a}\dot{p}_a) + f_t \overset{\text{Hamiltonian Equ}}{\Longrightarrow} \sum_{a=1}^{n} (f_{q_a}H_{p_a} - f_{p_a}H_{q_a}) + f_t = \{f,H\} + \frac{\partial f}{\partial t}$

Hamiltonian Equation (II): $\dot{p}_a = \{p_a, H\} = -H_{q_a} \quad \dot{q}_a = \{q_a, H\} = H_{p_a}$

Canonical: $\{q_a, q_b\} = 0 = \{p_a, p_b\} \qquad \{q_a, p_b\} = \delta_{ab}$

<u>Poisson's Thrm</u>: $\dot{f} = 0 = \dot{g} \Longrightarrow \frac{d}{dt}\{f,g\} = 0$

$\mathbf{L} = \mathbf{r} \wedge \mathbf{p} \Longrightarrow L_i = \sum_{j,k} \epsilon_{ijk} x_j p_k$, $\{L_i, x_j\} = \sum_k \epsilon_{ijk} x_k$, $\{L_i, p_j\} = \sum_k \epsilon_{ijk} p_k$, $\{L_i, L_j\} = \sum_k \epsilon_{ijk} L_k$

$\Omega \coloneqq \begin{pmatrix} 0, \mathbb{I} \\ -\mathbb{I}, 0 \end{pmatrix} \Longrightarrow \{f,g\} = \sum_{\alpha,\beta=1}^{2n} f_{y_\alpha} \Omega_{\alpha\beta} g_{y_\beta} \qquad \mathbf{y} = (q_1, \cdots, q_n, p_1, \cdots, p_n)$

<u>Canonical Invar</u>

$\{f,g\}_y = \{f,g\}_Y \iff \Omega_{\alpha\beta} = \sum_{\gamma,\delta=1}^{2n} (Y_\alpha)_{y_\gamma} \Omega_{\gamma\delta} (Y_\beta)_{y_\delta} \iff \Omega = \mathcal{J}\Omega\mathcal{J}^T$

$(\mathcal{J}_{\alpha\beta} = \frac{\partial Y_\alpha}{\partial y_\beta}, \mathcal{J} = \begin{pmatrix} (Q_a)_{q_b} & (Q_a)_{p_b} \\ (P_a)_{q_b} & (P_a)_{p_b} \end{pmatrix})$

$\Omega = \begin{pmatrix} \{Q_a, Q_b\} & \{Q_a, P_b\} \\ \{P_a, Q_b\} & \{P_a, P_b\} \end{pmatrix} \iff$ preserves canonical Poisson bracket (ie $\{y_\alpha, y_\beta\} = \Omega_{\alpha\beta}$)

<u>Hamiltonian VF</u> $\mathcal{D}_f g \coloneqq \{f,g\} \qquad [\mathcal{D}_f, \mathcal{D}_g] \coloneqq \mathcal{D}_f\mathcal{D}_g - \mathcal{D}_g\mathcal{D}_f = \mathcal{D}_{\{f,g\}}$

$\mathbf{Y}(\mathbf{y}, s) = e^{-s\mathcal{D}_f}\mathbf{y} = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} (\mathcal{D}_f)^n \mathbf{y}$ is canonical transformation.

Generating func (1^{st} kind): $\sum_{a=1}^{n} P_a dQ_a - K dt = \sum_{a=1}^{n} p_a dq_a - H dt - dF_1(\mathbf{q}, \mathbf{Q}, t) \qquad H, K$ Hamiltonian w.r.t. $(\mathbf{q}, \mathbf{p})$ and $(\mathbf{Q}, \mathbf{P}) \iff \mathbf{p} = \frac{\partial F}{\partial \mathbf{q}} \qquad \mathbf{P} = -\frac{\partial F_1}{\partial \mathbf{Q}} \qquad K = H + \frac{\partial F_1}{\partial t}$

<u>Liouville</u>: $\operatorname{vol}(V) = \int_V d\mathbf{q}d\mathbf{p} = \int d\mathbf{Q}d\mathbf{P}|\det \mathcal{J}| \Longrightarrow$ Volume invar under Hamiltonian evolution

Hamiltonian-Jacobi Equation: $\frac{\partial S}{\partial \mathbf{q}} \leftarrow \mathbf{p}$, $\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) = 0 \Longrightarrow$ sol: $\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}$, use Ham. equ

Sep of var: H in t indep: $S(\mathbf{q}, t) = S(\mathbf{q}) - E(t) \qquad$ HJ with q_i, S_{q_i} in grouping $f(q_i, S_{q_i}) \Rightarrow S(\mathbf{q}, t) = S_1(q_1) + S_2(q_2, \cdots, q_n, t)$

Appendix

$\int_0^{2\pi} \sin^2 x dx = \pi = \int_0^{2\pi} \cos^2 x dx$

Polar Coord: $\mathbf{r} = r\mathbf{e}_r \qquad \dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r \qquad \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$

$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta$

Uniform Disk: $I_1 = \frac{Ma^2}{4} = I_2$, $I_3 = \frac{Ma^2}{2} \qquad$ Uniform Rod: $I_1 = \frac{L^2 M}{12} = I_2$, $I_3 = 0$

$\dot{q}_a = \text{const.} \Longrightarrow \exists$ frame s.t. $q_a = 0$ by Galilean boost.

Spherical Coord: $(x, y, z) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$