Solutions to exercises in SICP (2e)

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Exercise 1.10

This is a pretty fun exercise in rendering Scheme procedures in mathematical notation. Personally, I find it helps with visualizing the recursive process.

Ackermann's function is defined in the question as

$$A(x, y) = \begin{cases} 0 & \text{if } y = 0 \\ 2y & \text{if } x = 0 \\ 2 & \text{if } y = 1 \\ A(x-1, A(x, y-1)) & \text{otherwise} \end{cases} \tag{1.1}$$

For each of the given expressions, we have

$$
\begin{aligned}
A(1,10) &= A(0, A(1,9)) \\
&= A(0, A(0, A(1,8))) \\
&= A(0, A(0, A(0, A(1,7)))) \\
&= A(0, A(0, A(0, A(0, A(1,6))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(1,5)))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(1,4))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(1,3)))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(1,2))))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(1,1)))))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(0,2))))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(0, A(0, 2\cdot 2)))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, A(0, 2\cdot 2\cdot 2))))))) \\
&= A(0, A(0, A(0, A(0, A(0, A(0, 2\cdot 2\cdot 2\cdot 2)))))) \\
&= A(0, A(0, A(0, A(0, A(0, 2\cdot 2\cdot 2\cdot 2\cdot 2))))) \\
&= A(0, A(0, A(0, A(0, 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2)))) \\
&= A(0, A(0, A(0, 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2))) \\
&= A(0, A(0, 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2)) \\
&= A(0, 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2) \\
&= 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 \\
&= 2^{10}
\end{aligned}
$$

In general, it appears that for $n > 0$ we have

$$A(1,n) = 2^n \tag{1.2}$$

(Recall that $A(1,0) = 0$ by definition.)

Next expression:

$$
\begin{aligned}
A(2,4) &= A(1, A(2,3)) \\
&= A(1, A(1, A(2,2))) \\
&= A(1, A(1, A(1, A(2,1)))) \\
&= A(1, A(1, A(1,2))) \\
&= A(1, A(1, 2^2)) && \text{using the result found above} \\
&= A(1, 2^{2^2}) && \text{ditto} \\
&= 2^{2^{2^2}} && \text{and again} \\
&= 2^{2^4} \\
&= 2^{16} \\
&= 65536
\end{aligned}
$$

And one more:

$$
\begin{aligned}
A(3,3) &= A(2, A(3,2)) \\
&= A(2, A(2, A(3,1))) \\
&= A(2, A(2,2)) \\
&= A(2, A(1, A(2,1)) \\
&= A(2, A(1,2)) \\
&= A(2, A(0, A(1,1))) \\
&= A(2, A(0,2)) \\
&= A(2,4) \\
&= A(1, A(2,3)) \\
&= A(1, A(1, A(2,2))) \\
&= A(1, A(1, A(1, A(2,1)))) \\
&= A(1, A(1, A(1,2))) \\
&= A(1, A(1, A(0, A(1,1)))) \\
&= A(1, A(1, A(0,2))) \\
&= A(1, A(1,4)) && \text{familiar pattern emerging...} \\
&= A(1, A(0, A(1,3))) \\
&= A(1, A(0, A(0, A(1,2)))) \\
&= A(1, A(0, A(0, A(0, A(1,1))))) \\
&= A(1, A(0, A(0, A(0,2)))) \\
&= A(1, A(0, A(0,4))) \\
&= A(1, A(0,8)) \\
&= A(1,16) \\
&= 2^{16} \\
&= 65536
\end{aligned}
$$

(Gotta say, the way the equations "fan" in and out like that is pretty neat. First it fans out 3 times, then 2, then just once on the final expansion.)

Just to check, I computed these values in **sicp_1-10_check.scm** and they are indeed correct. Phew!

The last part is essentially the same idea, but with a variable parameter n. Let's see...

$$
\begin{aligned}
f(n) &= A(0, n) \\
&= 2n
\end{aligned}
$$

$$
\begin{aligned}
g(n) &= A(1,n) \\
&= A(0, A(1, n-1)) \\
&= 2 \cdot A(1, n-1) \\
&= 2 \cdot A(0, A(1, n-2)) \\
&= 2 \cdot 2 \cdot A(1, n-2) \\
&= \dots && \text{follow the pattern\dots} \\
&= 2^{n-1} \cdot A(1,1) \\
&= 2^{n-1} \cdot 2 \\
&= 2^n && \text{as found earlier!}
\end{aligned}
$$

$$
\begin{aligned}
h(n) &= A(2,n) \\
&= A(1, A(2, n-1)) \\
&= A(1, A(1, A(2, n-2))) \\
&= A(1, A(1, A(1, A(2, n-3))))
\end{aligned}
$$

At this point, note that the levels of nesting is equal to the number m in $n-m$ at the end...

$$
\begin{aligned}
\Rightarrow h(n) &= A(1, A(1, A(1, \dots A(1, A(2,1)) \dots))) \\
&= A(1, A(1, A(1, \dots A(1,2) \dots))) && \text{We know what } A(1,n) \text{ is from earlier!} \\
&= A(1, A(1, A(1, \dots 2^2 \dots))) && \text{Now we "fold" it back in } n-1 \text{ times\dots} \\
&= 2^{2^{2^{\cdots}}} && \text{with } n \text{ "2"s}
\end{aligned}
$$

That is,

$$
\begin{aligned}
&A(2,1) = 2 \\
&A(2,2) = 2^2 = 4 \\
&A(2,3) = 2^{2^2} = 16 \\
&A(2,4) = 2^{2^{2^2}} = 65536 \quad \text{(as we found earlier)}
\end{aligned}
$$

and so on. To be honest, I have no idea how to formulate this as a "concise mathematical definition". Any suggestions welcome...