

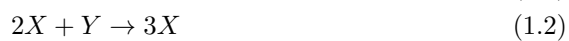
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1

Brusselator

The Brusselator is characterized by the reactions



and the rate equations are

$$\frac{d}{dt}\{X\} = \{A\} + \{X\}^2\{Y\} - \{B\}\{X\} - \{X\} \tag{1.5}$$

$$\frac{d}{dt}\{Y\} = \{B\}\{X\} - \{X\}^2\{Y\} \tag{1.6}$$

2

Damped spring

The damped spring is described by the simple equation

$$m \frac{d^2 x}{dt^2} = -kx + -c\dot{x} \tag{2.1}$$

where k is the spring constant and c is the damping coefficient. We can trivially write this as the following 1st-order ODEs:

$$\frac{dx}{dt} = v \tag{2.2}$$

$$\frac{dv}{dt} = -kx + -cv \tag{2.3}$$

3

Double pendulum

3.1 Lagrangian

$$L = \frac{1}{6}ml^2 \left[\dot{\theta}_2^2 + 4\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] + \frac{1}{2}mgl(3 \cos \theta_1 + \cos \theta_2) \quad (3.1)$$

3.2 Equations of motion

The momenta are

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{6}ml^2 \left[8\dot{\theta}_1 + 3\dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] \quad (3.2)$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{6}ml^2 \left[2\dot{\theta}_2 + 3\dot{\theta}_1 \cos(\theta_1 - \theta_2) \right] \quad (3.3)$$

which can be inverted to

$$\dot{\theta}_1 = \frac{6}{ml^2} \frac{2p_{\theta_1} - 3 \cos(\theta_1 - \theta_2)p_{\theta_2}}{16 - 9 \cos^2(\theta_1 - \theta_2)} \quad (3.4)$$

$$\dot{\theta}_2 = \frac{6}{ml^2} \frac{8p_{\theta_2} - 3 \cos(\theta_1 - \theta_2)p_{\theta_1}}{16 - 9 \cos^2(\theta_1 - \theta_2)} \quad (3.5)$$

Finally,

$$\dot{p}_{\theta_1} = \frac{\partial L}{\partial \theta_1} = -\frac{1}{2}ml^2 \left[\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) + 3\frac{g}{l} \sin \theta_1 \right] \quad (3.6)$$

$$\dot{p}_{\theta_2} = \frac{\partial L}{\partial \theta_2} = -\frac{1}{2}ml^2 \left[-\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) + \frac{g}{l} \sin \theta_2 \right] \quad (3.7)$$

4

Duffing equation

The Duffing equation is given by the second-order ODE

$$\frac{d^2x}{dt^2} + \delta \frac{dx}{dt} + \alpha x + \beta x^3 = \gamma \cos(\omega t) \quad (4.1)$$

which we can cast into the first-order ODEs:

$$\frac{dx}{dt} = v \quad (4.2)$$

$$\frac{dv}{dt} = \gamma \cos(\omega t) - \delta v - \alpha x - \beta x^3 \quad (4.3)$$

5

Lorenz system

$$\frac{dx}{dt} = \sigma(y - x) \tag{5.1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{5.2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{5.3}$$

- $\sigma = 10, \beta = 8/3, \rho = 28$

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Lotka-Volterra equations

7

Van der Pol oscillator

8

2-D Van der Pol oscillator

9

Symmetric top

9.1 Moments of inertia

$$I_1 = I_2 = \frac{3}{20}m \left(r^2 + \frac{l^2}{4} \right) \quad (9.1)$$

$$I_3 = \frac{3}{mr^2} \quad (9.2)$$

9.2 Lagrangian

$$L = \frac{1}{2}I_1 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) + \frac{1}{2}I_3 \left(\dot{\psi} + \dot{\varphi} \cos \theta \right)^2 - mgl \cos \theta \quad (9.3)$$

9.3 Equations of motion

The Euler-Lagrange equation gives

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad (9.4)$$

Fill in derivation later...

Solving for $\ddot{\theta}$,

$$\ddot{\theta} = \frac{\dot{\varphi}^2 \sin \theta \cos \theta (I_1 - I_3) - I_3 \dot{\varphi} \dot{\psi} \sin \theta + mgl \sin \theta}{I_1} \quad (9.5)$$

Fill in derivation later...

Solving for $\ddot{\varphi}$,

$$\ddot{\varphi} = \frac{2(I_3 - I_1) \dot{\varphi} \dot{\theta} \sin \theta \cos \theta - I_3 \dot{\varphi} \dot{\theta} \sin \theta \cos \theta + I_3 \dot{\psi} \dot{\theta} \sin \theta}{I_1 \sin^2 \theta} \quad (9.6)$$

Fill in derivation later...

Solving for $\ddot{\psi}$,

$$\ddot{\psi} = \frac{\cos \theta \left[\frac{(I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\varphi} \dot{\theta} \sin \theta}{\cos \theta} - 2(I_3 - I_1) \dot{\varphi} \dot{\theta} \sin \theta \cos \theta - I_3 \dot{\psi} \dot{\theta} \sin \theta \right]}{I_1 \sin^2 \theta} \quad (9.7)$$