Contents

1

Brusselator

The Brusselator is characterized by the reactions

$$
\begin{align}
A &\to X \tag{1.1}\\
2X + Y &\to 3X \tag{1.2}\\
B + X &\to Y + D \tag{1.3}\\
X &\to E \tag{1.4}
\end{align}
$$

and the rate equations are

$$
\begin{align}
\frac{\mathrm{d}}{\mathrm{d}t}\{X\} &= \{A\} + \{X\}^2\{Y\} - \{B\}\{X\} - \{X\} \tag{1.5}\\
\frac{\mathrm{d}}{\mathrm{d}t}\{Y\} &= \{B\}\{X\} - \{X\}^2\{Y\} \tag{1.6}
\end{align}
$$

2

Damped spring

The damped spring is described by the simple equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx + -c\dot{x} \tag{2.1}$$

where k is the spring constant and c is the damping coefficient. We can trivially write this as the following 1st-order ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \tag{2.2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -kx + -cv \tag{2.3}$$

3

Double pendulum

3.1 Lagrangian

$$L = \frac{1}{6}ml^2\left[\dot{\theta}_2^2 + 4\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + \frac{1}{2}mgl(3\cos\theta_1 + \cos\theta_2) \tag{3.1}$$

3.2 Equations of motion

The momenta are

$$p_{\theta_1} = \frac{\partial L}{\partial\dot{\theta}_1} = \frac{1}{6}ml^2\left[8\dot{\theta}_1 + 3\dot{\theta}_2\cos(\theta_1-\theta_2)\right] \tag{3.2}$$

$$p_{\theta_2} = \frac{\partial L}{\partial\dot{\theta}_2} = \frac{1}{6}ml^2\left[2\dot{\theta}_2 + 3\dot{\theta}_1\cos(\theta_1-\theta_2)\right] \tag{3.3}$$

which can be inverted to

$$\dot{\theta}_1 = \frac{6}{ml^2}\frac{2p_{\theta_1} - 3\cos(\theta_1-\theta_2)p_{\theta_2}}{16 - 9\cos^2(\theta_1-\theta_2)} \tag{3.4}$$

$$\dot{\theta}_2 = \frac{6}{ml^2}\frac{8p_{\theta_2} - 3\cos(\theta_1-\theta_2)p_{\theta_1}}{16 - 9\cos^2(\theta_1-\theta_2)} \tag{3.5}$$

Finally,

$$\dot{p}_{\theta_1} = \frac{\partial L}{\partial\theta_1} = -\frac{1}{2}ml^2\left[\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + 3\frac{g}{l}\sin\theta_1\right] \tag{3.6}$$

$$\dot{p}_{\theta_2} = \frac{\partial L}{\partial\theta_2} = -\frac{1}{2}ml^2\left[-\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + \frac{g}{l}\sin\theta_2\right] \tag{3.7}$$

4

Duffing equation

The Duffing equation is given by the second-order ODE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \delta\frac{\mathrm{d}x}{\mathrm{d}t} + \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{4.1}$$

which we can cast into the first-order ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \tag{4.2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \gamma\cos(\omega t) - \delta v - \alpha x - \beta x^3 \tag{4.3}$$

5

Lorenz system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x) \tag{5.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y \tag{5.2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta x \tag{5.3}$$

- $\sigma = 10, \beta = 8/3, \rho = 28$

6

Lotka-Volterra equations

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Van der Pol oscillator

8

2-D Van der Pol oscillator

9

Symmetric top

9.1 Moments of inertia

$$I_1 = I_2 = \frac{3}{20} m \left(r^2 + \frac{l^2}{4} \right) \tag{9.1}$$

$$I_3 = \frac{3}{mr^2} \tag{9.2}$$

9.2 Lagrangian

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\varphi} \ \cos\theta \right)^2 - mgl \cos\theta \tag{9.3}$$

9.3 Equations of motion

The Euler-Lagrange equation gives

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{9.4}$$

Fill in derivation later...

Solving for $\ddot{\theta}$,

$$\ddot{\theta} = \frac{\dot{\varphi}^2 \sin\theta \cos\theta (I_1 - I_3) - I_3 \dot{\varphi} \dot{\psi} \sin\theta + mgl \sin\theta}{I_1} \tag{9.5}$$

Fill in derivation later...

Solving for $\ddot{\varphi}$,

$$\ddot{\varphi} = \frac{2(I_3 - I_1) \dot{\varphi} \dot{\theta} \sin\theta \cos\theta - I_3 \dot{\varphi} \dot{\theta} \sin\theta \cos\theta + I_3 \dot{\psi} \dot{\theta} \sin\theta}{I_1 \sin^2\theta} \tag{9.6}$$

Fill in derivation later…

Solving for $\ddot{\psi}$,

$$\ddot{\psi} = \frac{\cos\theta\left[\frac{(I_1\sin^2\theta+I_3\cos^2\theta)\dot{\varphi}\dot{\theta}\sin\theta}{\cos\theta} - 2(I_3-I_1)\dot{\varphi}\dot{\theta}\sin\theta\cos\theta - I_3\dot{\psi}\dot{\theta}\sin\theta\right]}{I_1\sin^2\theta} \tag{9.7}$$