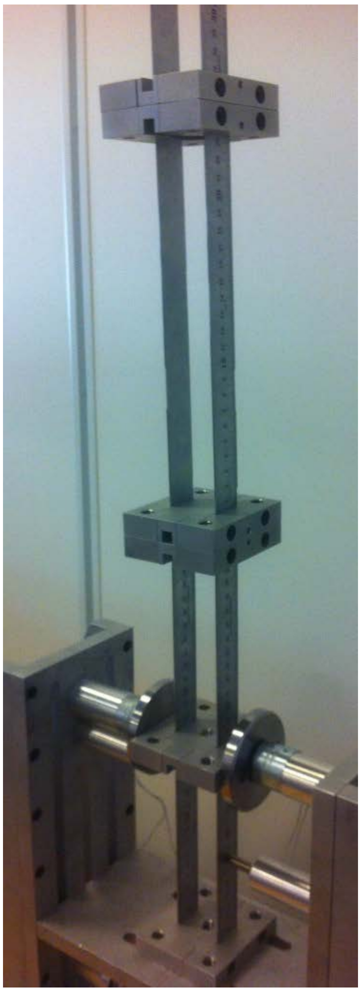
Module 5 – Similarities Transformation & Modal Analysis

Introduction – In the modules 2 and 3 you have obtained a mathematical model to describe the dynamic behavior of the electro-mechanical system presented in the figure below. The mathematical model was built based on principles and laws of Mechanics and Electromagnetism. Such a mathematical model uses 7 state variables which form the state vector **x**(t). In the module 4, using the matrices **A**, **B**, **C**, and **D**, you were able to calculate the eigenvalues and eigenvectors of the state matrix **A**. You built the matrices **Λ** with the system's eigenvalues and the modal matrix **M** (inverse of the matrix **P**) with the system's eigenvectors, and found a similar mathematical model for representing the system dynamics using the new state coordinates **z**(t), where **z**(t) = **P** **x**(t). Using the new set of state coordinates **z**(t), i.e. the modal coordinates, it was possible to decouple the system of equations responsible for describing the system dynamics. It was also possible to obtain insights into the system dynamics by analyzing the eigenvectors (normal modes), damped natural frequencies, damping ratios, and undamped natural frequencies. The goal of this new assignment (module 5) is to evaluate how changes in the positioning of actuators and sensors will affect the state matrix **A**, the control matrix **B**, and the measurement matrix **C**. This assignment will help you to understand and visualize the concept of *controllability* and *observability* which will be introduced to you later during the course.

Problem formulation – Instead of having the pair of electromagnetic actuators acting on the lowest mass, as illustrated in the figure below and described mathematically using the matrices **A**, **B**, **C**, and **D**, **image that the pair of electromagnetic actuators is repositioned acting solely on the middle mass while the displacement sensor is kept measuring the lowest mass.**



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad \mathbf{x} \in \mathbb{R}^7,\ u \in \mathbb{R}$$

$$y = \mathbf{C}\mathbf{x} + Du \quad y \in \mathbb{R}$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 & x_2 \end{bmatrix}^T$$
$$= \begin{bmatrix} i & q_1 & \dot{q}_1 & q_2 & \dot{q}_2 & q_3 & \dot{q}_3 \end{bmatrix}^T$$

$$\mathbf{A} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_i}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{K_i}{m_1} & -\frac{k_1+k_2+K_s}{m_1} & 0 & \frac{k_2}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 & \frac{k_3}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{k_3}{m_3} & 0 & -\frac{k_3}{m_3} & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C}^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D = 0$$

1) By changing the position of the pair of electromagnetic actuators to act on the middle mass, the structure of state matrix **A** will change. Recalculate the state matrix **A**.
2) The matrix **B** will also change. Recalculate the new control matrix **B**.

Using one single linearization point ($\mathbf{x_0}$ and $\mathbf{i_0}$) and the parameters provided in the modules 2 and 3.

3) Calculate **Λ** and compare with values obtained in module 4.
4) Calculate **M** and compare with values obtained in module 4.
5) Calculate **P**, i.e. the inverse of **M**.
6) Calculated **B**t and compare with values obtained in module 4. What can you conclude?
7) Calculate the new time constant τ of the electro-mechanical system.
8) Calculate the new damped natural frequencies β of the electro-mechanical system.
9) Calculate the new damping ratios ζ associated to the damped natural frequencies of the system.
10) Explain the physical meaning of the natural modes of the electro-mechanical system and their link to eigenvectors and eigenvalues (modal superposition).
11) Image now that the displacement sensor is repositioned to measure solely the movements of middle mass. Calculate the new measurement matrix C. Calculate the matrix **C**t for both cases, i.e. a) when the displacement sensor is positioned to solely measure the movements of the lowest mass and b) to solely measure the movements of the middle mass. What can you conclude?

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*Recall: The mathematical models are not unique, which you will show by building an equivalent mathematical model based on the transformation to the modal coordinates **z**(t):*

$$\boxed{\mathbf{z} = \mathbf{P}\mathbf{x} \Leftrightarrow \mathbf{x} = \mathbf{P}^{-1}\mathbf{z}}$$

where

$$\dot{\mathbf{z}}(t) = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{z}(t) + \mathbf{P}\mathbf{B}\mathbf{u}(t) \qquad \dot{\mathbf{z}}(t) = \mathbf{\Lambda}\mathbf{z}(t) + \mathbf{B}_t\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}^{-1}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) \qquad \mathbf{y}(t) = \mathbf{C}_t\mathbf{z}(t) + \mathbf{D}_t\mathbf{u}(t)$$

*and **P** and its inverse **M**, and **Λ** are the matrices composed of eigenvectors and eigenvalues of the state matrix **A**.*