D Project Description

D.1 Background and Motivation

The next century will see unprecedented changes to the climate system with extensive socioeconomic repercussions. To understand these changes, models have been built to represent the global climate system. However, the disparity between global and regional scales has made it difficult to diagnose changes to regional climate which occur due to shifts in the whole Earth system: Restrictions on computational cost have inherently limited the finest resolution of our uniform-resolution climate models to 50-100 kilometers, far beyond the scale necessary for capturing many facets of regional climate. Many issues important for regional planning remain out of scope for current climate modeling systems, including changes in the behavior of mesoscale storm systems, pressure blocking events driven by topography (responsible for heat waves and cold spells), mountain snowpack, wildfires, topographically-driven precipitation, watershed-level hydrology, urban development and agriculture. As stated in the National Center for Atmospheric Research (NCAR) 2009-2014 strategic plan, "human-induced global climate change has been largely accepted as real, but information about temperature changes are not at a sufficiently fine scale for planning regional adaptation and mitigation." Consequently, the development of high-resolution regional climate simulations at horizontal resolutions at or below 10 km is among the top NCAR imperatives (NCAR, 2009). Nonetheless, a number of key issues still remain before these models can be pushed to finer scales:

1. New numerical methods must be designed which are accurate and efficient at high horizontal resolutions on **massively parallel computers and hybrid architectures**.

2. Numerical methods must be developed for the **non-hydrostatic fluid equations** that account for the large aspect ratios (> 100) between horizontal and vertical atmospheric scales.

3. Discretizations must be developed which are accurate over highly variable and steep topography (known as the **horizontal pressure gradient problem**.)

4. A **global multi-resolution test bed** must be developed for understanding the behavior and performance of physical parameterizations over a range of horizontal resolutions.

In order to address these issues, this proposal focuses on a category of arbitrary-order local numerical methods known as **hybrid finite element methods (HFEMs)**, which hybridize continuous and discontinuous finite element formulations. It is the central hypothesis of this proposal that *a framework consisting of HFEMs and block-adaptive mesh refinement can be used to solve the problems enumerated above.* The overarching objectives are as follows:

1. Analyze the wave-like behavior of the 1D and 2D HFEMs for different choices of order-of-accuracy, prognostic variable staggering and choice of flux reconstruction function.

2. Investigate the use of an **arbitrary order-of-accuracy vertical discretization** via hybrid finite element methods.

3. Develop **high-order implicit-explicit (IMEX) time discretizations** to improve the accuracy of coupling between horizontal and vertical discretizations.

4. Develop a novel and robust HFEM that is applicable for **non-hydrostatic modeling** on the global scale and extend this approach to a **multi-resolution framework using block-based mesh-refinement methods**.

The central hypothesis of this proposal has been formulated based on past success of massively parallel finite element methods (FEMs) (Dennis *et al.*, 2011), the analysis of staggered methods performed by Thuburn and Woollings (2005), an understanding of the wave-like properties of numerical methods built from Ullrich and Jablonowski (2011), the analysis of low-order HFEM performed by Cotter and Shipton (2012) and preliminary analysis of HFEM discussed in section D.5. In addition to the central hypothesis stated above, we further hypothesize the following:

1. The C-grid HFEM (see section D.5) with exact mass matrix will provide optimal wave-like behavior, but the method with inexact mass matrix will be more computationally efficient.

2. A high-order vertical discretization combined with a high-order horizontal-vertical coupler will greatly reduce errors associated with the pressure gradient problem.

3. Overall HFEM will reduce errors in the atmospheric dynamical core (the component of atmospheric models responsible for solving the equations of fluid motion), particularly for phenomena with relatively short spatial scales.

The remainder of this proposal is as follows: Broader impacts and intellectual merit are described in section D.2. The Tempest model, which will be the foundation for software development as part of this proposal, is described in section D.3. The pressure gradient problem is explained in detail in section D.4. HFEMs are described in section D.5. The proposed research project is found in section D.6 and the research plan and timeline in section D.7. Future work is described in section D.8.

D.2 Broader Impacts and Intellectual Merit

The goal of this project is to improve the state-of-the-art for global climate and weather forecasting models by improving the treatment of the underlying dynamics within these models and developing methods which are capable of being used at high horizontal resolutions. This work has substantial societal consequences:

- High-resolution and multi-resolution methods are directly applicable to regional scale climate problems: As mentioned above, changes in the behavior of mesoscale storm systems, pressure blocking events driven by topography (responsible for heat waves and cold spells), mountain snowpack, wildfires, topographically-driven precipitation, watershed-level hydrology, urban development and agriculture are all regional scale issues that are poorly addressed in current generation global climate modeling systems.

- This technology is important for building global atmospheric models which are effective in regions of sharply varying topography, such as at the periphery of the California central valley.

- The improved wave-capturing properties of these methods will have the potential to capture phenomena which are highly dependent on wave-like motion, such as the Madden-Julian Oscillation (MJO).

- These advances have the potential to improve the technology that underlies meteorological models. This particular framework can further be used for tropical cyclone forecasting by improving global resolution over the major global hurricane basins.

- The multi-resolution framework can be used to test physical parameterizations in a multi-scale environment, to verify realistic performance of these parameterizations across scales.

- The development of HFEM for simulating the atmosphere could lead to their adoption in other fields, such as aerospace or computational biology.

This project further seeks to improve the partnership between the National Center for Atmospheric Research (NCAR) and UC Davis by a mutual exchange of expertise on atmospheric dynamics. Further, as a consequence of this work, a new next-generation numerical framework will be available for simulating atmospheric dynamics. The PI intends to release this product (including all source code and documentation of implementation details) for use in future research as open source, and later integrate this work into the Community Earth System Model (CESM) (Hurrell *et al.*, 2013) for use in operational climate simulations. The code for this project will be released under the Lesser GNU Public License (LGPL), which allows largely unrestricted access to the technology developed as part of this effort. This project is further aimed at funding the thesis work of a graduate student through completion of their degree.

This project will further tie into the Dynamical Core Model Intercomparison Project (DCMIP), which is a major international effort that focuses on the components of atmospheric models that are responsible for solving the equations of motion. The PI is currently a lead organizer of this effort. DCMIP has previously run a two-week summer school and workshop in the summer of 2008 and 2012, with the next workshop planned for 2016. These workshops have historically attracted over a dozen international modeling groups and over 30 graduate students to facilitate discussion of new numerical methods and provide an intensive educational experience for participants in global modeling. The next workshop will specifically focus on developing numerical methods for simulating at high resolution, multi-resolution modeling and the coupling of model dynamics and physics. It is expected that the graduate student and postdoc funded as part of this proposal will represent and showcase this work at the DCMIP workshop.

D.3 The Tempest Model

Tempest is a numerical atmospheric modeling framework (`https://github.com/paullric/tempestmodel`) (Ullrich, 2014) that uses a finite element discretization of the non-hydrostatic equations of motion. It has been designed and built by the PI to improve the ease of experimenting with numerical discretizations in an Earth-system context. It currently supports Discontinuous Galerkin (DG), Spectral Element (SE) and Flux Reconstruction (FR) (Huynh, 2007) methods using a horizontally-explicit vertically-implicit (HEVI) discretization, with coupling via Strang splitting (Ullrich and Jablonowski, 2012b). The equations of motion are based on a terrain-following height coordinate and the simulation grid is either Cartesian or a cubed-sphere (Figure 1) (Ullrich *et al.*, 2010; Ullrich and Jablonowski, 2012a). The model is parallelized via MPI. Figure 2 shows the results at day 10 of the baroclinic wave test of Ullrich *et al.* (2013) as simulated by the Tempest model. It is eventually the goal that the Tempest model will be incorporated into the Community Earth System Model (Hurrell *et al.*, 2013). This proposal aims to use the Tempest framework as a foundation for software development and experimentation.

Figure 1: The cubed-sphere grid.

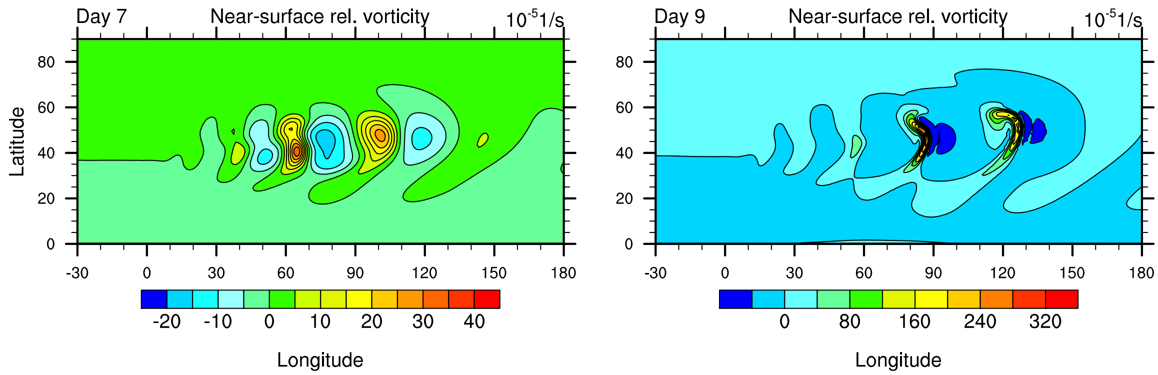

Figure 2: A developing non-hydrostatic baroclinic instability simulated in the Tempest framework. Relative vorticity shown.

Traditional global atmospheric models use the hydrostatic approximation, which assumes that the vertical atmosphere is perpetually in a state of balance between pressure gradient and buoyancy forces. In this case, vertical momentum is no longer tracked and is instead diagnosed from other state variables. However, this approximation breaks down when the horizontal resolution is finer than roughly 10 km, and so the next-generation of global atmospheric models have instead turned to using the non-hydrostatic Euler equations (with shallow-atmosphere approximation):

$$\frac{\partial u^\alpha}{\partial t} + u^i \nabla_i u^\alpha + \frac{1}{\rho}\left[g^{\alpha\alpha}\nabla_\alpha p + g^{\alpha\beta}\nabla_\beta p\right] + f(\mathbf{k}\times\mathbf{u})^\alpha = -\frac{g^{\alpha\xi}}{\rho}\nabla_\xi p, \tag{1}$$

$$\frac{\partial u^\beta}{\partial t} + u^i \nabla_i u^\beta + \frac{1}{\rho}\left[g^{\alpha\alpha}\nabla_\alpha p + g^{\alpha\beta}\nabla_\beta p\right] + f(\mathbf{k}\times\mathbf{u})^\beta = -\frac{g^{\beta\xi}}{\rho}\nabla_\xi p, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + u^\alpha \frac{\partial \theta}{\partial \alpha} + u^\beta \frac{\partial \theta}{\partial \beta} = -u^\xi \frac{\partial \theta}{\partial \xi}, \tag{3}$$

$$\frac{\partial u^\xi}{\partial t} + u^\alpha \nabla_\alpha u^\xi + u^\beta \nabla_\beta u^\xi + u^\xi \nabla_\xi u^\xi = -\frac{1}{\rho}\nabla^\xi p - g_c, \tag{4}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \alpha}(J\rho u^\alpha) + \frac{1}{J}\frac{\partial}{\partial \beta}(J\rho u^\beta) = -\frac{1}{J}\frac{\partial}{\partial \xi}(J\rho u^\xi). \tag{5}$$

Here α and β are arbitrary horizontal coordinates with basis vectors $\mathbf{i}$ and $\mathbf{j}$, ξ is a vertical coordinate with unit basis vector $\mathbf{k}$, the index i spans each coordinate, g^{ij} denotes the contravariant metric, $J = \sqrt{\det g_{ij}}$ is the metric Jacobian, ∇_i denotes the covariant derivative along the i^{th} coordinate, g_c is gravity, f is the Coriolis parameter, ρ is the density, $\mathbf{u} = u^\alpha \mathbf{i} + u^\beta \mathbf{j} + u^\xi \mathbf{k}$ is the vector velocity and θ is the potential temperature. Einstein summation notation is used for repeated indices. These equations are closed via the equation of state

$$p = p_0 \left(\frac{R_d(\rho\theta)}{p_0}\right)^{c_p/c_v},$$

where R_d is the ideal gas constant, p_0 is a reference pressure and c_p and c_v denote the specific heat capacity of dry air at constant pressure and constant volume. Observe that (1)-(4) are given in a non-conservative form; this formulation is generally desirable over the conservative formulation (where momentum and $\rho\theta$ are prognostic variables) since this form can more readily conserve quantities relevant to atmospheric motion, such as angular momentum and potential enstrophy, and (depending on the discretization) can lead to a more accurate treatment of wave-like motions (Thuburn and Woollings, 2005).

The atmosphere consists of a "thin shell" around the Earth (the atmosphere is roughly 100 km thick, relative to the Earth's radius of 6371 km) and so requires special considerations when being modeled. Vertical discretizations typically use $20-30$ grid points over the whole atmosphere, with a resolution of approximately 100 m in the boundary layer, whereas even a high resolution horizontal grid at 10 km is 100 times coarser that the vertical. Since the vertical coordinate is geometrically stiff, non-hydrostatic models must use an implicit treatment for the geometrically stiff terms (Ullrich and Jablonowski, 2012b). Hence, terms that are handled using an implicit operator appear on the right-hand-side of equations (1)-(5).

This equation set has been studied in detail for over a century, and a wide range of mature numerical methods are now available for modeling its solutions. Finite element methods (FEM), which include both the spectral element (SE) method and the discontinuous Galerkin (DG) method, have a long history in the field of computational fluid dynamics stretching back to the 1970s (see Patera (1984); Maday and Patera (1989); Bassi and Rebay (1997); Cockburn and Shu (1998)). Recent advances in FEM motived by Huynh (2007) also no longer impose the need for a conservative equation set when formulated with a DG discretization. However, FEM have only recently been adopted by the atmospheric modeling community (Taylor *et al.*, 1997; Giraldo *et al.*, 2002; Giraldo and Rosmond, 2004; Fournier *et al.*, 2004; Nair *et al.*, 2005; Giraldo and Restelli, 2008; Kelly and Giraldo, 2012). The primary motivation for the adoption of FEM in global atmospheric codes has been the need for methods which perform well on very large-scale parallel systems (those with hundreds of thousands to millions of computing cores). To its advantage, FEM can provide near-optimal scalability on parallel computing systems with minimal bandwidth requirements (that is, it minimizes the message size needed to convey information between neighboring elements) and so, unlike many global spectral and semi-Lagrangian methods, are tenable for use on these systems. The need for scalable numerical methods is essential with the rapid expansion of massively-parallel supercomputers in the past decade. FEM can also be formulated to have many desirable mathematical properties, including mass and energy conservation, high-order accuracy, as well as discrete preservation of the adjoint and annihilator properties of the gradient, divergence and curl operators. Further, recent work on FEM has led to techniques for effective positivity and monotonicity filtering (Guba *et al.*, 2013b). Consequently, the Community Atmosphere Model (CAM), for the first time, has adopted the spectral element method of Taylor and Fournier (2010) as the default dynamical core. Many other modeling groups are now working on adopting finite element methods as the basis for their dynamical cores, including the NUMA model of Kelly and Giraldo (2012). FEM are also likely candidates for adoption on the U.K. Met Office dynamical core and the upcoming model from the Korean Institute of Atmospheric Prediction Systems (KIAPS).

D.4 The Pressure Gradient Problem

Since the sharpest gradients in topography are typically adjacent to local peaks, an increase in horizontal resolution is expected to lead to steeper slopes. A simple example of this problem is depicted in Figure 3, where an isolated Utah mountain is plotted with a variable horizontal resolution. As the horizontal mesh is refined, the maximum slope also increases in steepness to $32°$ at the finest resolution. Under a HEVI discretization, an initially balanced flow over such a mountain will only remain balanced if the pressure gradient terms on the left-hand-side of the

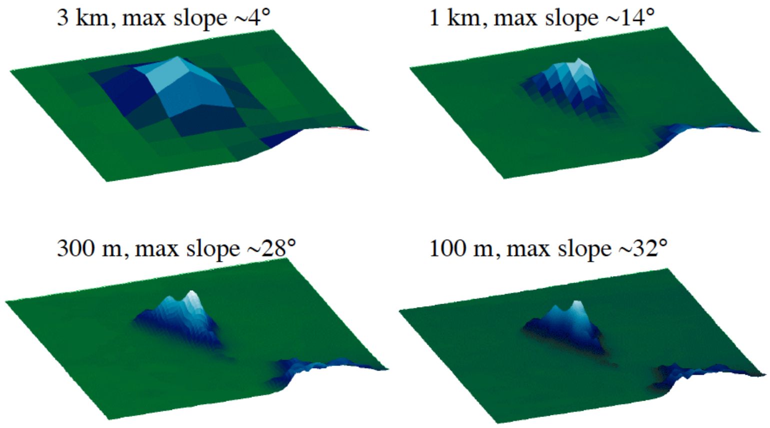

Figure 3: The averaged slope of topography increases monotonically with increased horizontal resolution. Here an image of an isolated Utah mountain is shown with variable horizontal resolution. Image courtesy of Dr. Tina Chow, UC Berkeley.

horizontal momentum equations (1)-(2) (those treated via an explicit method) and those on the right-hand-side (treated with an implicit method) cancel each other exactly. However, a low-order discretization and coupling strategy between the horizontal and vertical discrete pressure gradient terms (that is, standard dimensional splitting) is insufficient to maintain balance. Figure 4 shows the vertical velocity in the presence of rapidly varying bottom topography, and highlights the effects of inaccurate evaluation of the horizontal pressure gradient term for models with a terrain-following coordinate. Since typical vertical velocities for large-scale weather systems are ~ 0.01 m s^{-1}, these perturbations represent an error of nearly 50%.

The issue of the pressure gradient problem has significant implications for global atmospheric modeling, with efforts to tackle this problem stretching back as far as the late 1970s (Janjic, 1977; Mihailović and Janjić, 1986). As global models reach to finer spatial scales, the pressure gradient problem leads to an increasingly polluted dynamical solution near steep topography. The typical solution to this problem is easily considered crude: A diffusive filter is repeatedly applied to topography data until the maximum slope fits within a specified tolerance and there is no sign of numerical noise associated with steep topography. This approach has the obvious consequence of severely damaging the quality of the solution near topography, and leading to an effective grid resolution much coarser than the grid would allow. Topographically driven precipitation, which is particularly important for estimating mountain snow-pack, is weakened substantially under this approach. Further, atmospheric pressure blocking, which is responsible for stationary pressure systems and the development of heat waves / cold spells, is suppressed under topographical smoothing. Other approaches which address mitigation of this problem include the use of cut-cells (Steppeler *et al.*, 2013), coordinate surface smoothing (Klemp, 2011), reconstruction of pressure along horizontal coordinate surfaces Zängl (2012), or formulations which preserve consistency with the continuous operator (Lin, 1997). In the context of high-order FEM, this proposal advocates removing errors by improving the accuracy of the coupling between the horizontal and vertical discrete pressure gradient operators.

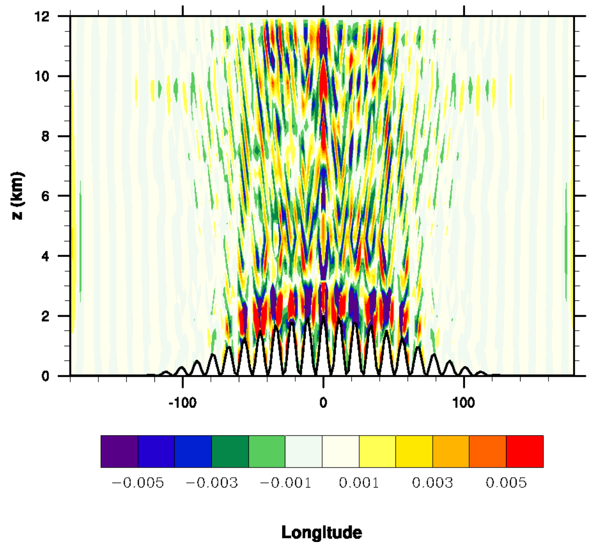

Figure 4: Vertical velocity for a balanced flow over a rapidly varying mountain range. Because of inaccurate evaluation of the horizontal pressure gradient term, models with terrain-following coordinates can generate significant spurious numerical errors in the presence of steep topography. Here a hydrostatically balanced atmosphere initially at rest quickly generates a train of spurious internal mountain waves in response to numerical errors.

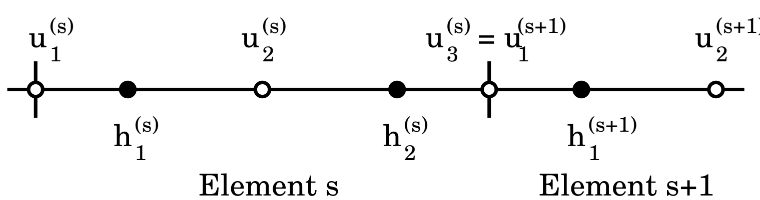

Figure 5: An example 3rd order horizontal staggering of 1D velocity (u) and height (h) state variables for a SE/DG HFEM for the shallow water equations. Here u nodes are placed at Gauss-Lobatto quadrature points and h nodes are placed at Gaussian quadrature points. The superscript denotes the element number and the subscript denotes the sub-grid-scale index.

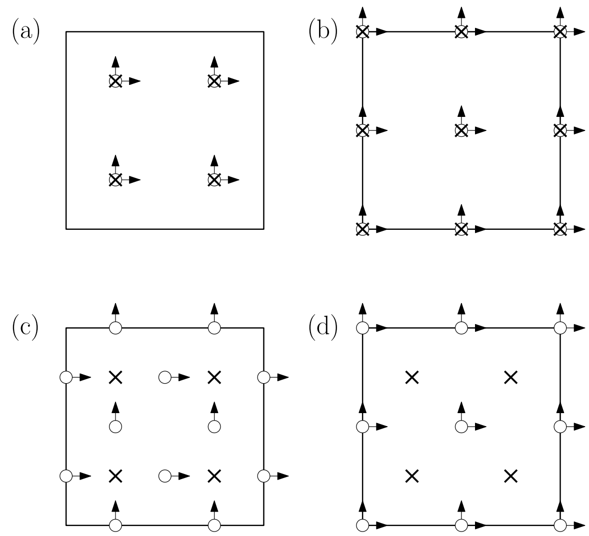

Figure 6: Four options for the placement of scalar ($\times$) and velocity variables (arrows): (a) nodal discontinuous Galerkin, (b) nodal Spectral Element, (c) C-grid HFEM, (d) B-grid HFEM.

D.5 Hybrid Finite Element Methods (HFEM)

In the past year there has been growing interest in the atmospheric modeling community in the use of Hybrid Finite Element methods (HFEM). In the context of nodal FEM, HFEM represent a class of methods where scalar (density, pressure, potential temperature) and velocity variables are stored at different nodal points within an element. One such example of a 3rd order 1D hybrid SE/DG method is depicted in Figure 5. Four possible arrangements of prognostic variables are depicted in Figure 6 for a 2D HFEM. Closely related to HFEMs are mixed finite element methods, which have been used for elliptic problems since the 1970s by Raviart and Thomas (1977). HFEMs are particularly desirable since they generalize the staggered grid approach which has been a staple of atmospheric models since the work of Arakawa and Lamb (1977). Staggered grid models are the standard in the ocean modeling community, and have been the foundation for many current and previous generation atmospheric models. Staggered grids are used in the CAM finite-volume dynamical core (Lin, 2004), the previous default dynamical core in CAM, and its successor the Geophysical Fluid Dynamics Laboratory (GFDL) finite-volume cubed-sphere model (Putman and Lin, 2009). They have also been adopted in the Weather Research and Forecasting (WRF) model (Skamarock *et al.*, 2005), which is the standard in regional weather forecasting, and its spiritual successor the global Method for Prediction Across Scales (MPAS) model (Skamarock *et al.*, 2012).

This proposal uses a differential formulation of HFEM, where the differential operators on Gauss-Legendre (GL) and Gauss-Lobatto-Legendre (GLL) nodes (that is, those typically associated with nodal DG and SE discretizations, respectively) arise naturally from the flux reconstruction methods of Huynh (2007). This method can be applied in conjunction with the direct stiffness summation operation (which averages co-located state values on GLL nodes) when continuity is desired. An operator for stabilization / diffusion, which is necessary for the application of these methods to long-term climate simulations, will be developed based on the work of Guba *et al.* (2013a). It is well-known that an optimal choice of grid staggering can greatly improve the treatment of atmospheric waves (Randall, 1994), but it is not immediately clear how staggering of variables should be handled

Method	Shortest Resolved Wave Order 3	Order 4
Finite Volume	12.53 Δx	9.98 Δx
Spectral Volume	11.94 Δx	9.16 Δx
Discontinuous Galerkin	10.01 Δx	7.87 Δx
Spectral Element	8.33 Δx	8.87 Δx
Hybrid Finite Element	6.68 Δx	5.56 Δx

Table 1: Shortest wavelength which is resolved to at least 0.5% error in both the diffusive and dispersive error components, obtained from analysis of the 1D linear wave equation, for several methods of third- and fourth-order accuracy.

in the context of FEM. However, recent efforts by Boffi and Gastaldi (2009) and Cotter and Shipton (2012) have investigated a standard low-order 1D HFEM staggering and generally found that HFEM retain the desirable properties of FEM (parallel scalability, mass/energy conservation and a discrete analogue of the gradient, divergence and curl operators). Further, when the staggering is chosen correctly, these methods also retain the beneficial properties of staggered grid methods. The PI has also investigated the ability of several standard numerical methods to properly capture the diffusive and dispersive properties of the 1D linear wave equation, and the results suggest that HFEM greatly outperforms competing methods (see Table 1 and Ullrich (2013)).

D.6 Detail of the Proposed Research

The overarching theme of this research is the development of a next-generation multi-resolution non-hydrostatic atmospheric modeling environment based on (optimal) FEM/HFEM which addresses known issues in pushing atmospheric models to high resolutions. This work will also address the horizontal pressure gradient problem, which is a notorious issue in atmospheric models in the vicinity of steep topography.

There is clear utility to the geophysics community in the adoption of new and improved numerical methods, and HFEM has the potential to greatly enhance the capability of current atmospheric dynamical cores to accurately and efficiently simulate the atmospheric equations of motion. This research will be addressed in four parts, corresponding to the four objectives of the proposal:

(a) Implementation and analysis of 1D and 2D arbitrary-order HFEM (§D.6.1),

(b) Implementation and analysis of a HFEM vertical discretization (§D.6.2),

(c) Implementation and analysis of high-order coupling strategies using implicit-explicit Runge-Kutta methods (§D.6.3)

(d) Implementation and analysis of a HFEM horizontal discretization in the context of block-based mesh refinement (§D.6.4)

Each of these items can be pursued independently, and each is expected to be the basis for one or two future publications. This research is expected to be carried out over a period of three years.

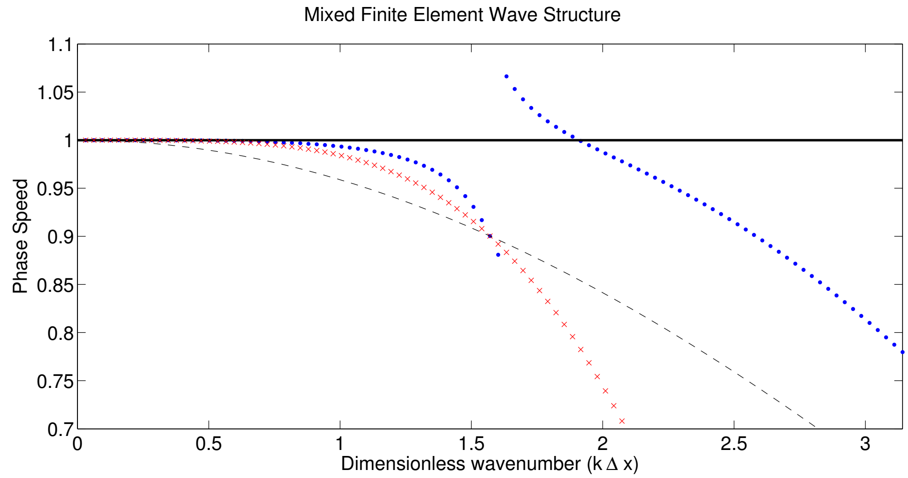

Figure 7: A dispersion analysis of the right-going mode of the 1D third-order hybrid finite element method depicting normalized phase speed as a function of dimensionless wavenumber. The solid line denotes the exact phase speed, normalized to 1. The dotted blue curve depicts the HFEM phase speed, the dashed line depicts the phase speed of a standard staggered second-order scheme and the red crosses ($\times$) depict the phase speed of the third-order SE method. Note that since the SE method is unstaggered, the phase speed must go to zero at $k\Delta x = \pi$. Both HFEM and SE have the same number of degrees of freedom per element, but HFEM boasts the best long-wavelength ($k < \pi/2$) and short-wavelength ($k > \pi/2$) properties. The presence of the spectral gap at $k\Delta x = \pi/2$ will be addressed in this work.

D.6.1 Analysis of 1D and 2D Hybrid Finite Element Methods

To a close approximation, the atmosphere is in a state of geostrophic and hydrostatic balance. The dynamic character of the atmosphere is governed by a slow adjustment process which gives rise to wave motion over all scales. Accurate treatment of these waves is important to capture departures from geostrophic balance, and to ensure that the adjustment process is correct and free from spurious numerical errors. As shown by Lauritzen *et al.* (2010), an accurate treatment of the equations of motion is also important to avoid errors due to grid imprinting. The capability of a numerical method to capture wave-like motion in atmospheric models is typically evaluated using dispersion analysis. This mathematical technique decomposes the discrete response in the atmospheric model into diffusive and dispersive errors introduced by the discretization. Diffusive errors are typically associated with an unphysical loss of wave energy from the system and dispersive errors are associated with unphysical ringing, corresponding to an incorrect treatment of individual wave speeds. A key paper by Randall (1994) used dispersion analysis to demonstrate the superior performance of staggered grids for modeling geophysical flows. More recently, Ainsworth and Wajid (2009) (and later Melvin *et al.* (2012)) applied dispersion analysis to the SE method in the context of geophysical motions. They found that, although the SE method possessed good long-wave characteristics, the unstaggered nature of the scheme led to a short-wavelength regime with unphysical backwards energy propagation. There is currently no thorough analysis of HFEM for use in atmospheric models. Preliminary results shown in Figure 7 suggest that the HFEM method possesses superior wave properties for the 1D wave equation, with the exception of a small gap at wavenumber $\pi/2$. A thorough dispersion analysis of this method, extending preliminary work to 2D

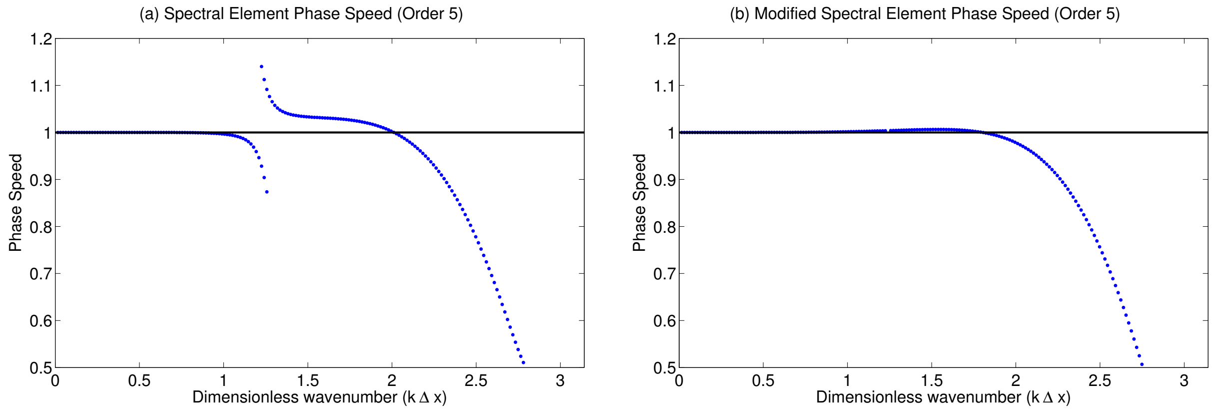

Figure 8: (a) Dispersion relation for the unmodified SE method of order 5 showing a spectral gap and (b) Dispersion relation for the SE method using a modified basis where the spectral gap has been removed. The solid line depicts the exact phase speed.

geophysical problems, would be incredibly beneficial for determining any outlying issues with the formulation of the method prior to its use in atmospheric modeling. Further, this analysis would provide guidance in choosing a HFEM formulation that preserves desirable geophysical properties, including geostrophic balance on the f-plane. A thorough analysis must address several outstanding questions:

- The "spectral gap" is a phenomenon that often appears when a numerical method adopts a non-uniform grid to simulate the linear wave equation. It is associated with an unphysical jump in the phase speed (see Figure 7) which can lead to inaccurate treatment of certain wave numbers. This gap can be fixed by either partial mass lumping (Staniforth *et al.*, 2012), the application of dissipation (such as upwinding, as in the discontinuous Galerkin approach), or by an improved choice of the functional space within an element (modified basis method). The modified basis method has been verified by the PI to successfully remove the spectral gap for the SE method and improve the accuracy of the method (see Figure 8 and Ullrich (2013)). This project aims to study how the choice of dissipative operator and basis functions affects the spectra of the HFEM approach, and if the spectral gap can be eliminated for the linear 1D and 2D shallow-water equations using a similar approach. It is hypothesized that the spectral gap can be completely eliminated at all orders-of-accuracy through a combination of judiciously chosen dissipative operator and/or an improved choice of basis functions.

- An extension of the HFEM analysis to 2D is important to understand the behavior of HFEM in a context which is relevant to geophysical motions. In particular, this project is interested in an extension of the 1D analysis to the 2D linear shallow-water equations with non-zero Coriolis parameter f. It is not currently understood if high-order HFEM will share similar dispersion properties to the staggered finite-difference schemes which were analyzed by Randall (1994). Further, it is known that the arrangement of nodal points can have a significant effect on the dispersive properties of the underlying method. Consequently, this effort aims to compare and contrast the different arrangements of nodal points (see Figure 6) and their influence on the wave-capturing properties of the underlying HFEM. It is hypothesized that the C-grid arrangement of Figure 6c will produce an optimal dispersion relationship.

- Another issue that requires investigation is the behavior of HFEM at refinement boundaries. Atmospheric models that support variable resolution are becoming increasingly important in addressing issues related to extreme weather and regional climate change (Skamarock *et al.*, 2012). However, as pointed out by Ullrich and Jablonowski (2011), in the presence of grid refinement staggered grid methods cannot distinguish between incident waves and artificially reflected waves. As a consequence, energy in the incident mode tends to be reflected at grid refinement interfaces, leading to potential contamination of the solution in a region of enhanced resolution. To further understand this behavior, an analysis of HFEM in the presence of grid refinement is needed.

D.6.2 A HFEM Vertical Coordinate

There has recently been renewed interest in improving the treatment of the vertical discretization in global atmospheric models. The accurate representation of vertical wave modes is essential for models of the atmosphere, especially for those with wavelengths near the grid scale, where numerical errors typically appear. Traditionally the vertical coordinate for the non-hydrostatic fluid equations has either been discretized in the Eulerian frame via a second-order Charney-Phillips (Charney and Phillips, 1953) or Lorenz grid (Arakawa and Moorthi, 1988), or via a semi-Lagrangian approach such as Lin (2004). However, little work has been performed to develop high-order vertical coordinates due to a number of outstanding issues. First, higher-order generalizations must somehow deal with the no-flux boundary conditions at the model bottom and top without loss of accuracy, especially near the surface where accurate treatment of dynamics is paramount. Second, as observed by Thuburn and Woollings (2005), Thuburn (2006) and Toy and Randall (2007) the choice of vertical coordinate (whether height-based, mass-based or entropy-based) implies an optimal vertical staggering of prognostic variables for maintaining correct behavior for wave motions relevant to the atmosphere. Third, unstaggered discretizations (that is, discretizations where all prognostic variables are stored on model levels) possess stationary computational modes which can potentially damage the numerical solution and can degrade the quality of the solution. As in the horizontal direction, unstaggered FEM leads to waves with zero phase speed in the limit as wavelength $\lambda \to 2\Delta x$ (see Figure 7). Unlike the horizontal, these wave modes can be dramatically enhanced by an implicit treatment of the vertical at high Courant number.

This proposal will develop, analyze and implement the HFEM machinery for use as an arbitrary-order vertical discretization for the non-conservative formulation of the fluid equations, isolating terms which are relevant to vertical motion (from the right-hand-side of (1)-(5)). The HFEM prognostic variable staggerings can be easily composed in differential form using interpolation and differentiation operators built in accordance with the DG and SE discretizations that arise from the FR method (see Table 2). The use of HFEM is natural for vertical discretizations: No-flux boundary conditions are natural in the context of FEM, and the structure of individual elements (that is, the tendency for nodal values to be concentrated near element edges) lends itself readily to improved resolution in the atmospheric boundary layer. Further, HFEM inherits the mimetic properties of SE methods and so the vertical operator will automatically conserve both mass and discrete energy.

The analysis component of this proposal will generalize and extend the work of Thuburn and Woollings (2005) to understand how acoustic, inertia-gravity waves and the Rossby wave modes respond to different staggerings of prognostic variables, locations of interior nodes, choice of FR

Table 2: Composition of interpolation $\mathcal{I}$ and differentiation $\mathcal{D}$ operators for several choices of staggering. Subscript e denotes variables defined on interfaces (Gauss-Lobatto-Legendre nodes) and n represents variables defined on model levels (Gauss-Lobatto nodes). For operator $\mathcal{I}$ and $\mathcal{D}$, the subscript denotes the target (e or n) and the superscript denotes the origin. In accordance with Thuburn (2006), the pressure gradient in the vertical is reformulated as $\rho^{-1}\nabla p = \theta\nabla\Pi$, where Π is the Exner pressure ($\Pi_n(\rho,\theta) = c_p\left[R_d\rho\theta/p_0\right]^{R_d/c_v}$).

		Choice of Staggering		
Variable	Operator	DG ($\rho_n\theta_n u_n^\xi$)	HFEM-L ($\rho_n\theta_n, u_e^\xi$)	HFEM-CP ($\rho_n, u_e^\xi\theta_e$)
Π_n		$\Pi_n(\rho_n,\theta_n)$	$\Pi_n(\rho_n,\theta_n)$	$\Pi_n(\rho_n,\mathcal{I}_n^e\theta_e)$
θ	$u^\xi\dfrac{\partial\theta}{\partial\xi}$	$(u_n^\xi)\mathcal{D}_n^n\theta_n$	$(\mathcal{I}_n^e u_e^\xi)(\mathcal{D}_n^n\theta)$	$(u_e^\xi)(\mathcal{D}_e^e\theta_e)$
u^ξ	$\theta\dfrac{\partial\Pi}{\partial\xi}$	$\theta_n\mathcal{D}_n^n\Pi_n$	$(\mathcal{I}_n^e\theta_n)(\mathcal{D}_e^n\Pi_n)$	$\theta_e(\mathcal{D}_e^n\Pi_n)$
ρ	$\dfrac{1}{J}\dfrac{\partial}{\partial\xi}(J\rho u^\xi)$	$\dfrac{1}{J_n}\mathcal{D}_n^n(J_n\rho_n u_n^\xi)$	$\dfrac{1}{J_n}\mathcal{D}_n^e\left[J_e(\mathcal{I}_e^n\rho_n)u_e^\xi\right]$	$\dfrac{1}{J_n}\mathcal{D}_n^e\left[J_e(\mathcal{I}_e^n\rho_n)u_e^\xi\right]$

reconstruction function and order-of-accuracy in the context of HFEM. In total, of the prognostic variables ρ, u, v, w and θ and the diagnosed mass flux $\mathcal{F}_\rho = \rho w$ and Exner pressure Π, there are 2^5 different options for staggering on model levels and interfaces (assuming ρ, u and v are kept together). Of these options only a limited subset exhibit optimal wave-propagation properties and are free of stationary computational modes.

D.6.3 Improved Horizontal-Vertical Coupling

This research will investigate one approach for obtaining uniform high-order accuracy in both time and space by using a novel strategy for combining the horizontal, vertical and temporal discretization. The time discretization must account for the treatment of horizontal motions by an explicit discretization and vertical motions by an implicit discretization; a technique known as horizontally explicit vertically implicit (HEVI). Second-order and higher one-step methods are usually referred to as Implicit-Explicit Runge-Kutta (IMEX-RK) methods, and many such methods have been proposed in the literature (Ascher *et al.*, 1997; Kennedy and Carpenter, 2003). Recently, recognizing that high-order coupling is necessary for next-generation non-hydrostatic atmospheric models, many of these methods have been analyzed in the context of HEVI discretizations by Weller *et al.* (2013).

This proposal targets an issue that has not been addressed in the atmospheric science literature: Namely, can a high-order IMEX-RK method in conjunction with a high-order vertical discretization reduce errors associated with the pressure-gradient problem? (see the description of the pressure gradient problem in section D.4) The Tempest model will be used as a testbed for this study, in conjunction with either high-order FEM or HFEM. Different IMEX-RK methods will be investigated, implemented and compared in terms of accuracy and efficiency in order to determine which coupler leads to the lowest pressure gradient errors, and whether the associated cost makes this approach worthwhile.

D.6.4 HFEMs and Block-Based Mesh Refinement

HFEMs show great promise in their application to geophysical problems. Given a new numerical method, the development of an operational dynamical core is typically performed in three stages: First, a global shallow water model is developed to verify that the method performs well for problems with 2D fluid characteristics similar to those of the global atmosphere. Second, a 3D regional Cartesian-geometry climate model is developed to ensure that the method can perform well with a large horizontal-vertical aspect ratio. Third, the vertical treatment from the Cartesian model is incorporated into the shallow-water model to produce an operational 3D dynamical core. Tempest (see section D.3), which supports both Cartesian and cubed-sphere geometry, will be used as a framework for HFEM development, and will greatly ease the development at each of these stages.

The adoption of mesh refinement in the global atmospheric modeling community is fairly recent, although a number of major modeling centers are currently pushing this capability to operational general circulation models (Skamarock *et al.*, 2012; Harris and Lin, 2013; Zarzycki *et al.*, 2013). The Tempest model uses an implementation of mesh refinement based on the block-adaptive strategy of Berger and Colella (1989), with development in collaboration with the Chombo research group at LBNL (Colella *et al.*, 2000). Since only density and tracers will be conserved (which are discretized with the DG operators), an implementation of HFEM in the block-based mesh refinement framework will be a fairly straightforward modification of the existing methodology.

Once dynamical core development is complete, verification must be performed to ensure consistency with analytical test cases, and intercomparison with other dynamical cores. The standard test case suite in the shallow-water regime was established by Williamson *et al.* (1992). More recently, a test case suite for 3D atmospheric dynamical cores has been proposed as part of the Dynamical Core Model Intercomparison Project (DCMIP), documented in Ullrich *et al.* (2012). These test suites further provide the opportunity for intercomparison of the HFEM dynamical core with other operational dynamical cores via DCMIP and other scientific publications. Once verification is complete, future efforts will tackle the addition of moisture and physical parameterizations to the dynamical core.

This project will benefit greatly from the PI's past experience in atmospheric model development as author of the MCore high-order finite-volume modeling environment (see, for instance, Ullrich *et al.* (2010); Ullrich and Jablonowski (2012b,a)). The end goal of this work will be to develop a fully functional open-source dynamical core which is available for general research use by the atmospheric modeling community.

D.7 Research Plan

This project seeks funding for one postdoctoral researcher for two years and one graduate student researcher for three years. Over the course of their employment, the postdoctoral researcher will work with the PI on the implementation of the HFEM framework using the Tempest codebase, development of test problems and additional assistance with analysis of the HFEM. Simultaneously, the graduate student researcher will be in charge of the 1D and 2D analysis of HFEM, with 2D analysis carrying over into the second year. In the second year, the graduate student researcher will work to test out the various permutations of HFEM and study the role of the vertical discretization and coupling mechanism on model accuracy in the presence of steep topography. Finally, in their third year, the graduate student researcher will be responsible for validation and verification of the 3D HFEM framework code, including applications to idealized scenarios in atmospheric dynamics. The anticipated schedule for this project is given below:

Year 1	· 1D analysis of arbitrary-order HFEM, removal of spectral gap · 2D linear shallow-water analysis of arbitrary-order HFEM · Implementation and testing of HFEM vertical discretization
Year 2	· Continued analysis of 2D arbitrary-order HFEM · Implementation and testing of HFEM horizontal discretization · Validation / verification testing of global shallow-water HFEM code
Year 3	· Analysis of arbitrary-order HFEM in presence of grid refinement · Validation / verification testing of 3D Cartesian HFEM code · Validation / verification testing of 3D global HFEM code · Application of HFEM framework to idealized atmospheric dynamics experiments

We anticipate that at least five major peer-reviewed publications will arise from this work, including two covering the analysis of HFEM (for each of the 1D and 2D formulation) and three covering the implementation of the HFEM framework (for each of the vertical discretization, shallow-water model and 3D multi-resolution global model). Further, this work will be presented at major scientific meetings, including SIAM Geosciences, the meetings of the American Geophysical Union, the conference for Partial Differential Equations on the Sphere and the Dynamical Core Model Intercomparison Project 2016 Workshop and Summer School.

D.8 Implications of the Proposal for Future Applications

The work of this proposal has the potential to drive further innovations and form the basis for future research:

- The HFEM framework that is produced from this work will be used to study dynamical relationships in the tropical atmosphere, following the work of Biello and Majda (2005, 2010, 2012). The highly accurate dispersion properties of this method will assist in testing asymptotic relationships governing wave interactions.
- The HFEM framework will be eventually coupled to a stochastic cloud model, following the work of Khouider *et al.* (2005, 2010) to study cloud-dynamics interactions and the viability of stochastic cloud models in a global modeling context.
- The development of a global HFEM ocean model will be pursued based on the work of this proposal. This effort would be beneficial for several reasons: Staggered grids are ubiquitous in the global ocean modeling community. Consequently, the adoption of HFEM for ocean modeling would be able to leverage existing technologies, while improving the spatial accuracy of existing methods. This approach would also have similar near-optimal dispersive properties as HFEM for the atmosphere. This work could quickly lead to the use of a coupled atmosphere-ocean model based on HFEM that would preserve numerical consistency between the atmosphere and ocean model component.