Assignment 4: Fourier Series Expansion

Paul Monroy
University of California, Santa Barbara
Santa Barbara, CA 93106
paulmonroy@ucsb.edu

Abstract

Fourier series expansion is the idea of deconstructing a periodic signal into many signals and adding them up. I will analyze problems regarding the Fourier Series expansion and how it relates to the discrete version: DTFT.

A. Introduction

Fourier series expansion has been commonly introduced in the engineering classes in the form of an orthogonal decomposition, along with properties and applications. A different perspective is presented in the form of the modulation property of Fourier analysis. This is the most effective way to lead into the concept of frequency-shift keying (FSK). We will also see how the DTFT relates to the Fourier Series expansion. We also go into modulation. Waveform modulation is a simple Fourier transform property, which has made profound and long-lasting impacts.

B. Problem 1: Showing the that the basis functions of Fourier series expansion are mutually orthogonal

Attached is my work for the problem. We start the definition of orthogonality of complex functions

$$< f(t), g(t) > = \frac{1}{T}\int_0^T f(t)g^*(t)dt$$

In part a of the problem, we are asked to prove orthogonality of basis functions $e^{jn\omega_0 t}$. In Solutions part 1a, we define the Fourier Series expansion. In 1b we denote $\Phi_\mathrm{n}(t) = e^{jn\omega_0 t}$ where n can take on any integer value from -∞ to +∞, as seen in 1c and 1d.

In part 2, we once again define the inner product in 2a and implement it with our $\Phi_\mathrm{n}(t)$ in part 2b. In 2c, we carry out the definite integral and end up with the result seen

$$inner\ product = \frac{e^{j2\pi(n-m)} - 1}{j(\frac{2\pi}{T})(n-m)}$$

In the case n ≠ m, we see that n-m turns into another integer, since n and m are integers. Therefore, $e^{j2\pi(n-m)} = 1$ and the inner product becomes 0. If the inner product is 0, then it implies orthogonality. This is seen in 2d.

In 2e for the case n = m, we see that the inner product is 1. We therefore conclude that $\Phi_\mathrm{n}(t) = e^{jn\omega_0 t}$ is orthogonal for any n such that the n is not the same as itself (seen in 2f).

In part b of the problem, we are asked to do the same procedure for $e^{jn\theta}$ from DTFT. In step 3a, we define the basis functions $\Phi_\mathrm{n}(\theta) = e^{jn\theta}$ and $\Phi_\mathrm{m}(\theta) = e^{jm\theta}$ where n and m could be any integer. The inner product is carried out in step 3b and get the result

$$\frac{1}{2\pi}\left(\frac{e^{j2\pi(n-m)} - 1}{j(n-m)}\right)$$

In 3c, for n ≠ m, the inner product goes to 0 because n-m is any integer. Therefore, $e^{j2\pi(n-m)} = 1$ and the inner product becomes 0.

In step 3d, we see the case where n = m. In this case, the inner product becomes 1. The result can be seen in step 3e. In step 3f, we see the reasoning for orthogonality of the basis functions.

C. Problem 2: Using 7-digit perm number to produce a 7-point sequence {F_n}, where n = 0,1,…,6. Using my perm number: 4010278, we do the problem.

Part a of the problem asks us to use { F_n } as the Fourier series coefficients to formulate and plot the periodic function f(t) for the interval (0, T). The coefficients are seen in part 1a of the solutions in my attached work. In step 2a-c, I carry out the Fourier Series expansion using the coefficients and end up with the f(t) function seen in step 2c.

In **Figure 1**, n = 2 and F_2 = 1 and the plot is of $e^{j2\omega_0 t}$.

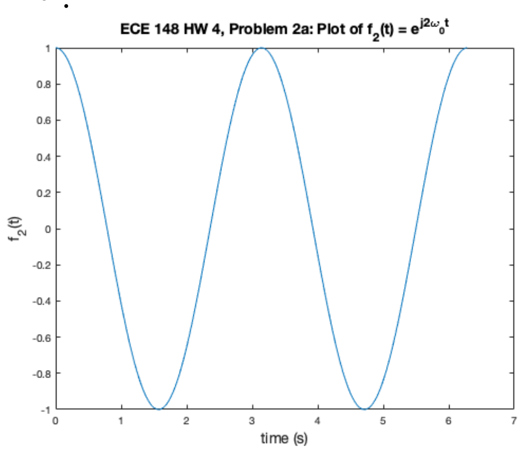

Figure 1

In **Figure 2**, n = 4 and F_4 = 2 and the plot is of $2e^{j2\omega_0 t}$.

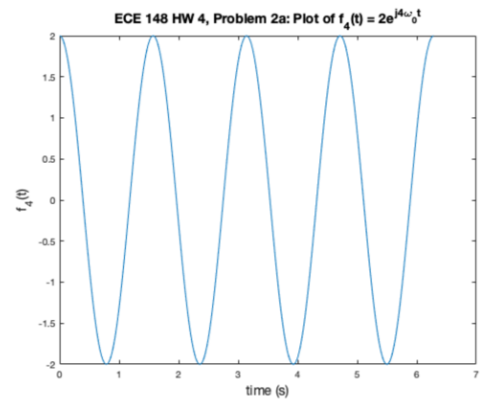

Figure 2

In **Figure 3**, n = 5 and F_5 = 7 and the plot is of $7e^{j2\omega_0 t}$.

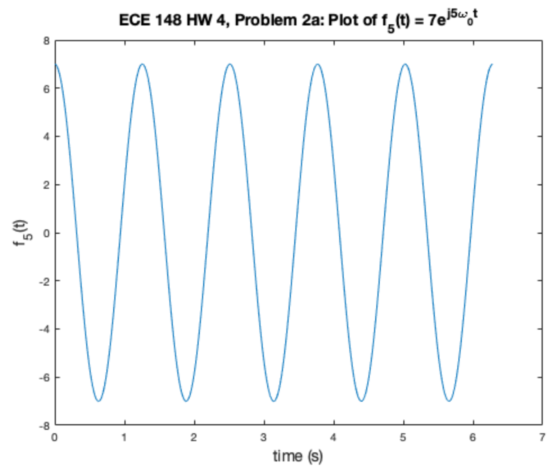

Figure 3

In **Figure 4**, n = 6 and F_6 = 8 and the plot is of $8e^{j2\omega_0 t}$.

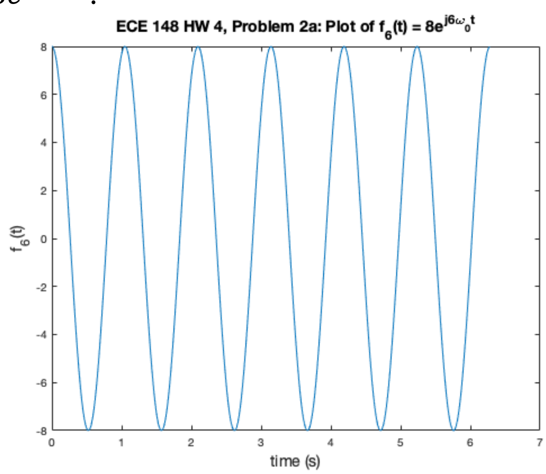

Figure 4

In **Figure 5,** we see the transformation from adding instances of the Fourier series expansion and seeing the final f(t).

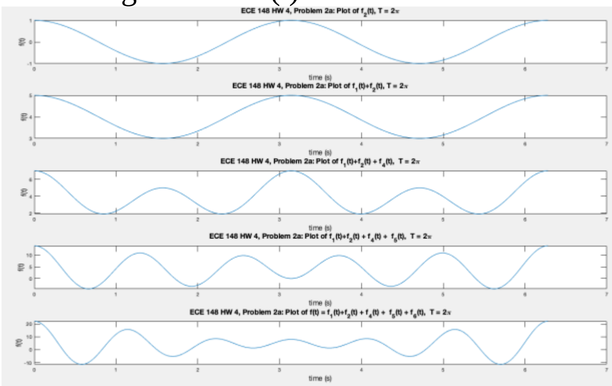

Figure 5

In **Figure 6,** we see the final f(t) after adding all instances of different n. This is the sum of all $e_{jn\omega 0}$. This helps us visualize how any function can be decomposed into various smaller functions of sinusoids.

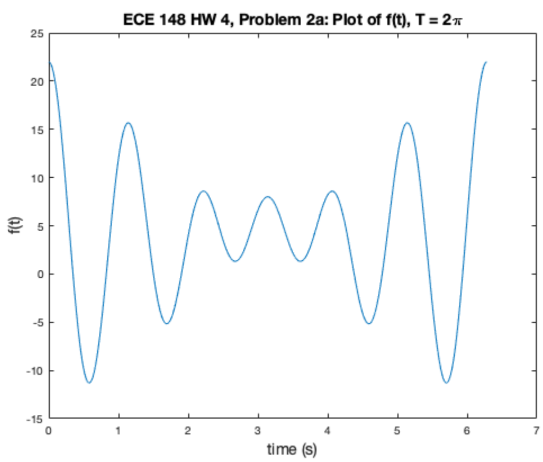

Figure 6

Part b of the problem asks us to use { F_n } to formulate and plot the DTFT spectrum for the interval (0, 2π). We use the definition of DTFT

$$X\left(e^{jn\theta}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta}$$

Where x[n] is our coefficients that we used in the previous part. This is seen in steps 3a and b. In step 3d, we see the final $X\left(e^{jn\theta}\right)$.

In **Figure 7**, we see that n = 2, x[2] = 1, and $X_2\left(e^{j\theta}\right) = e^{j2\theta}$.

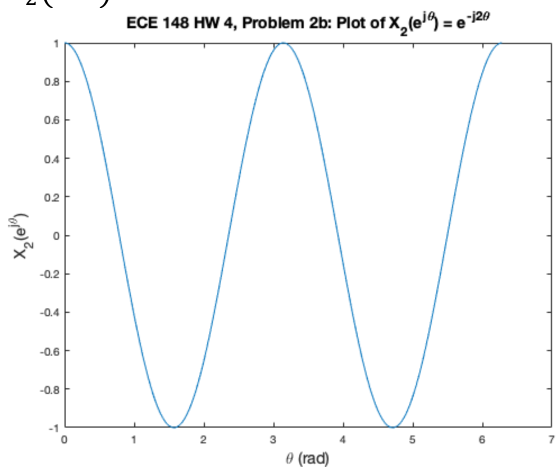

Figure 7

In **Figure 8**, we see that n = 4, x[4] = 2, and $X_4\left(e^{j\theta}\right) = 2e^{j4\theta}$.

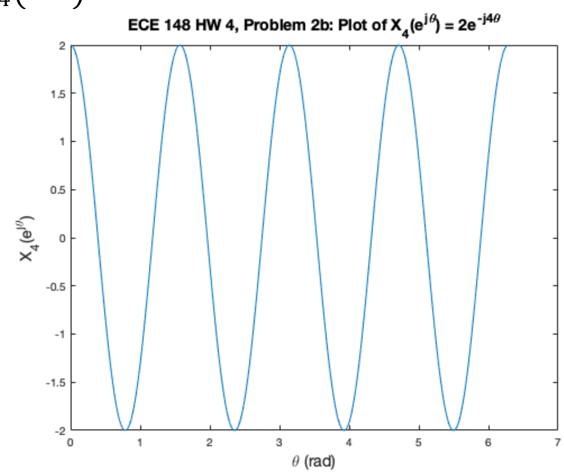

Figure 8

In **Figure 9**, we see that n = 5, x[5] = 7, and $X_5\left(e^{j\theta}\right) = 7e^{-j5\theta}$.

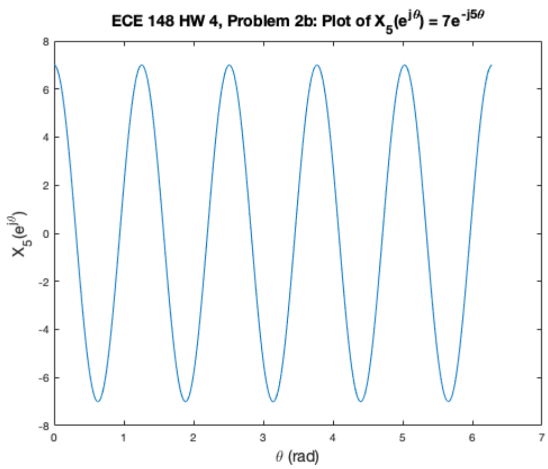

Figure 9

In **Figure 10**, we see that n = 6, x[6] = 8, and $X_6\left(e^{j\theta}\right) = 8e^{j6\theta}$.

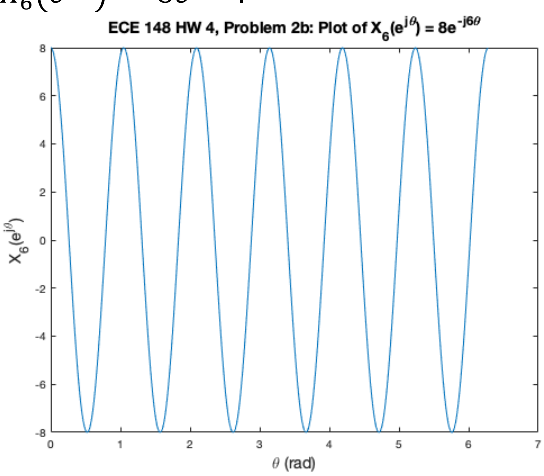

Figure 10

In **Figure 11,** we see the transformation from adding instances of the DTFT. We also see the final $X\left(e^{j\theta}\right)$ a lot better in **Figure 12.**

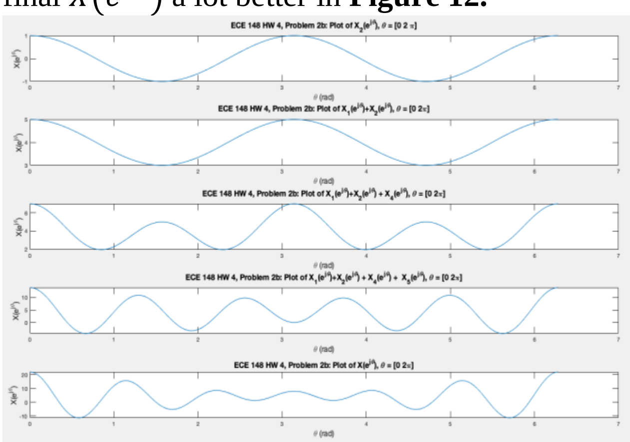
Figure 11

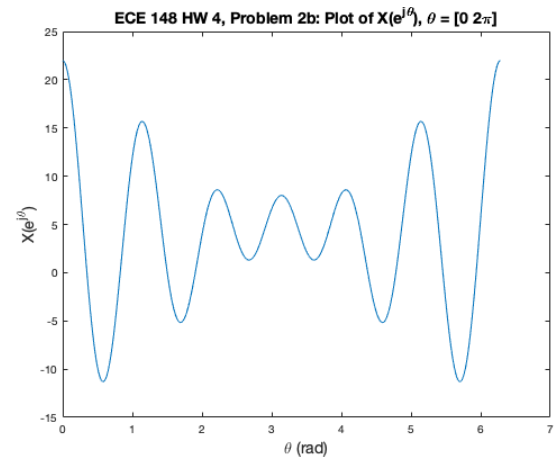

Figure 12

Part c is a summarization of the two methods we have just done. Looking at each iteration of n, they look very similar in shape, especially since the period in both is 2π. Comparing **Figure 6** and **Figure 12**, we see that they are similar in appearance as well, only just changed in units. As mentioned in lecture, Fourier Series and DTFT are like cousins because they relate so much. The Fourier Series expansion has a periodic function f(t) with period T, frequency ω_0/T, and coefficients F_n. Similarly, DTFT has periodic function $X\left(e^{j\theta}\right)$, period 2π, frequency 1, and coefficients x[n]. By setting Fourier series expansion period to 2π, both of these are basically identical.

D. Problem 3

In part **a** of the problem, we consider a real periodic signal, with period T. The signal has 6 harmonics, for n=0,1,…,5.

$$f(t) = \sum_{n=0}^{5} a_n \cos\left(n\omega_0 t + \theta_n\right)$$

Then we take 16 uniform samples within one period to form a short 16-point sequence {f(m)}. Subsequently, we take a 16-point DFT of the sequence F(k) = DFT {f(m). We are to identify and list the 16-point DFT spectrum F(k) in terms of a_n and θ_n.

The work is seen in the attached document. In step 1a, we define and expand the function f(t). In step 1b, we have the parameters for the problem. In step 2a, we define how we will implement the parameters into the function f(t). We see that we can define the term within the summation is $f_n(t) = a_n \cos(n\omega_0 t + \theta_n)$. In 2b, we implement $f_n\left(\frac{mT}{M}\right)$ to get $f_n[m]$. We use Euler's formula for sines and cosines. In step 2c, we get the DFT of $e^{j2\pi nm/M}$ and get the finite geometric series form. We apply the approximation in step 2d and get the result Mδ[k-n], which means it is M-periodic. Finally, since $e^{j\theta_n}$ is a scalar, we can insert Mδ[k-n], back into the summation to get $F_n[k]$ in step 2e. The final answer is the summation of all $F_n[k]$ as seen in the 2e.

In part b of the problem, we repeat the process for

$$g(t) = \sum_{n=1}^{5} b_n \sin(n\omega_0 t + \phi_n)$$

The steps are the same and are seen in the attached work.

E. Summary

We have analyzed how Fourier series expansions have orthogonal bases. That is, any function $e^{jn\omega_0 t}$ is orthogonal if it is not the same value of n. The definition of Fourier Series expansion means that it has orthogonal bases so it makes sense that all signals $f_n(t)$ that make up a function f(t) are orthogonal.

We then analyzed how Fourier series relates to DTFT as they are within the same family and do the same thing. However, they belong to different domains (time and frequency, respectively). The summary in Problem 2 explains this. It is interesting to see how they both can provide similar results based on a certain choice of period.

Finally, we see in problem 3 that sampling a signal in the time-domain gives us a frequency spectrum of delta functions that are evenly spaced out. Because we want M samples, we see that the frequency spectrum consists of M delta functions.

Problem 1

(1)

a.) $f(t) = \sum_{n=\infty}^{\infty} F_n e^{jnw_0 t}$

b.) $f(t) = \sum_{n=-\infty}^{\infty} F_n \cdot \Phi_n(t)$

c.) $\{\Phi_n(t)\}$: orthogonal basis functions

$\Phi_0(t) = 1$

$\Phi_1(t) = e^{jw_0 t}$

$\Phi_2(t) = e^{j2w_0 t}$

$\vdots$

d.) Say n is current interation, m is some other iteration

ex: $n=1$, $m=n+2=3$

$\Phi_n(t) = e^{jnw_0 t}$

$\Phi_m(t) = e^{jmw_0 t}$

(2) Definition of inner product

a.) $\langle f, g \rangle = \frac{1}{T}\int_0^T f(t) g^*(t)$

b.) $\langle \Phi_n(t), \Phi_m(t) \rangle = \frac{1}{T}\int_0^T \Phi_n(t)\Phi_m^*(t)$

$= \frac{1}{T}\int_0^T e^{jnw_0 t} e^{-jmw_0 t} dt = \frac{1}{T}\int_0^T e^{j(n-m)w_0 t} dt$

c.) $\frac{1}{T}\left[\frac{e^{j(\frac{2\pi}{T})(n-m)t}}{j(\frac{2\pi}{T})k}\right]_0^T = \frac{e^{j2\pi/T(n-m)T} - 1}{j(2\pi/T)(n-m)} = \frac{e^{j2\pi(n-m)} - 1}{j(2\pi/T)n-m}$

d.) if $n \neq m$, $e^{j2\pi(n-m)} = 1$ b/c $n-m$ is an integer, and $e^{j2\pi k}$ where k is an integer equals 1

Inner product is therefore 0

$\frac{\pi}{\omega} = \_\omega$

e.) if $m=n$

inner products $\frac{1}{T}\int_0^T dt = \frac{1}{T}[t]_0^T = \frac{1}{T}[T-0] = 1$

f.) therefore any basis function $\phi_n(t) = e^{jn\omega_0 t}$ where $n = \ldots -1, 0, 1, 2 \ldots$ is orthogonal for any n that is not itself, or in our example, where $n \neq m$

PART B

(3.) a.) $\Phi_n(\theta) = e^{jn\theta}$

$\Phi_m(\theta) = e^{jm\theta}$

b.) $\langle \Phi_n(\theta), \Phi_m(\theta) \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{jn\theta} e^{-jm\theta} \, d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} e^{j(n-m)\theta} \, d\theta$

$= \frac{1}{2\pi}\left[\frac{e^{j(n-m)\theta}}{j(n-m)}\right]_0^{2\pi} =$

$= \frac{1}{2\pi}\left[\frac{e^{j(n-m)2\pi}}{j(n-m)} - \frac{1}{j(n-m)}\right]$

$= \frac{1}{2\pi}\left[\frac{e^{j(n-m)2\pi} - 1}{j(n-m)}\right] =$

c.) if $n \neq m$,

inner product $= \frac{1}{2\pi} \frac{e^{j2\pi(n-m)} - 1}{j(n-m)}$

$e^{j2\pi(n-m)} = 1$ for any integer n, m

$\therefore$ inner product $= 0$

d.) if $n = m$

inner product $= \frac{1}{2\pi}\int_0^{2\pi} e^{j(n-m)\theta} \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} \cdot \frac{1}{2\pi}$

$= 1$

e.) $\therefore$ inner product $= \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

f.) for any basis function $e^{jn\theta}$ for any n, they are orthogonal to each other is $n \neq m$ (inner product is 0 implies orthogonality

Problem 2

(4.) Perm: 4010278

a.) $F_0 = 4$ $F_4 = 2$

$F_1 = 0$ $F_5 = 7$

$F_2 = 1$ $F_6 = 8$

$F_3 = 0$

PART A

(2) a) $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

b) $f(t) = F_0 e^{j(0)} + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + F_3 e^{j3\omega_0 t} + F_4 e^{j4\omega_0 t} + F_5 e^{j5\omega_0 t}$
$+ F_6 e^{j6\omega_0 t}$

c.) $f(t) = 4 + 0 + e^{j2\omega_0 t} + 0 + 2e^{j4\omega_0 t} + 7e^{j5\omega_0 t} + 8e^{j6\omega_0 t}$

PART B

(3) a.) $x(0) = 4$ $x(4) = 2$

$x(1) = 0$ $x(5) = 7$

$x(2) = 1$ $x(6) = 8$

$x(3) = 0$

b.) DTFT

$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$

c.) $= x(0) + x(1)e^{-j\theta} + x(2)e^{-j2\theta} + x(3)e^{-j3\theta} + x(4)e^{-j4\theta}$
$+ x(5)e^{-j5\theta} + x(6)e^{-j6\theta}$

d.) $= 4 + e^{-j2\theta} + 2e^{-j4\theta} + 7e^{-j5\theta} + 8e^{-j6\theta}$

Problem 3

(a.) $f(t) = \sum_{n=0}^{5} a_n \cos(n\omega_0 t + \theta_n)$ T-periodic, $\omega_0 = \frac{2\pi}{T}$

$= a_0 \cos(\theta_0) + a_1 \cos(\omega_0 t + \theta_1) + a_2 \cos(2\omega_0 t + \theta_2) + a_3 \cos(3\omega_0 t + \theta_3)$
$+ a_4 \cos(4\omega_0 t + \theta_4) + a_5 \cos(5\omega_0 t + \theta_5)$

b) want sequence $f[m]$

$M = 16$

Sample time $= m\frac{T}{M}$, $m = 0, \dots, M-1$

(2)

a.) $f_n[m] = f_n\left(m\frac{T}{M}\right)$

b.) $f_n(t) = \frac{a_n}{2}\left(e^{j\theta_n} e^{jn\omega_0 t} + e^{-j\theta_n} e^{-jn\omega_0 t}\right)$

$f_n[m] = \frac{a_n}{2}\left(e^{j\theta_n} e^{j2\pi nm/M} + e^{-j\theta_n} e^{-j2\pi nm/M}\right)$

c.) want DFT of $f_n[m] \to F_n[k]$

$e^{j2\pi nm/M} \xrightarrow{DFT} \sum_{m=0}^{M-1} e^{j2\pi nm/M} e^{-j2\pi km/M} = \sum_{m=0}^{M-1}\left[e^{j2\pi(n-k)/M}\right]^m$

d.) Finite geometric series

$\sum_{m=0}^{M-1} r^m = \frac{1-r^M}{1-r}$

$= \begin{cases} \frac{1-e^{j2\pi(n-k)}}{1-e^{j2\pi(n-k)/M}} = 0 & \text{if } k \neq n \\ M & \text{if } k = n \end{cases}$

$= M\delta[k-n]$ (M-periodic)

e.) $F_n[k] = M\frac{a_n}{2}\left(e^{j\theta_n}\delta[k-n] + e^{-j\theta_n}\delta[k+n]\right)$

$\boxed{F[k] = \sum_{n=0}^{5} F_n[k]}$

Part b

(3) Repeating for $g(t)$ where

a.) $$g(t) = \sum_{n=1}^{5} b_n \sin(n\omega_0 t + \phi_n)$$

b.) $M = 16$ samples $\qquad$ T-periodic: $T = \frac{2\pi}{\omega_0}$

$$t = m\left(\frac{T}{M}\right)$$

c.) set $b_n \sin(n\omega_0 t + \phi_n) = F_n(t)$

d.) $F_n(t) = b_n \cdot \frac{1}{2j}\left(e^{jn\omega_0 t} e^{j\phi_n} - e^{-jn\omega_0 t} e^{-j\phi_n}\right.$

e.) $$g_n[m] = \frac{b_n}{2j}\left[e^{j\phi_n} e^{j2\pi nm/M} - e^{-j\phi_n} e^{-j2\pi nm/M}\right]$$

f.) $e^{j2\pi nm/M} \xrightarrow{DFT} M\delta[k-n]$

Same as problem 2

g.) $g_n[m] \rightarrow$ DFT

$$G_n[k] = M\frac{b_n}{2j}\left[e^{j\phi_n}\delta[k-n] - e^{-j\phi_n}\delta[k+n]\right]$$

h.) and

$$G[k] = \sum_{n=1}^{5} G_n[k]$$