Assignment 1: Sampling and Aliasing

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Abstract

Sampling and aliasing have both been used to ease digital communications. In this report, I will analyze how sampling and aliasing affects frequency migration from an input frequency to an output frequency. I will also analyze the modulation of a time domain signal.

A. Introduction

Sampling and aliasing have both been used to ease digital communications and transfer information from analog to digital signals. We are able to take a continuous signal, sample it at a rate Δt, convert it to frequency spectrum, low-pass and reconstruct the signal. I will analyze two problems dealing with migration of input frequency to an output frequency and the modulation of a time-domain signal that is sampled and low-passed to get an output signal.

B. Problem 1: Mapping input frequency ω_x to an output frequency ω_y.

We start with a simple coherent signal f(t) is in the form

$$f(t) = e^{j\omega_x t}$$

The Fourier spectrum of this signal consists of one single peak at $\omega = \omega_x$. This coherent waveform is first sampled, with sample spacing Δt. Subsequently, it is reconstructed through a lowpass filter with cutoff frequencies at $\pm\,\omega_o/2$, where

$$\omega_o = 2\pi/\Delta$$

The signal is reconstructed in the form of

$$g(t) = e^{j\omega_y t}$$

And its Fourier spectrum shows a single peak at $\omega = \omega_y$. If the sampling and reconstruction are conducted adequately, the frequency remains unchanged, $\omega_y = \omega_x$. If aliasing occurs, the resultant frequency will be different than the frequency of the input waveforms.

The conversion the frequency from ω_x to ω_y is a mapping process. We start by using a reference marker at the left edge of the lowpass filter at $\omega = -\omega_0/2$. The distance from ω_x to $-\omega_0/2$ is $\omega_x + \omega_0/2$. Modulating this number by ω_0 and looking at the remainder, it will be in between 0 and ω_0. We remove the marker by adding our reference i.e. subtracting $\omega_0/2$. The process is shown in the source code and the correspondence of the mapping can be seen in **Figure 1**.

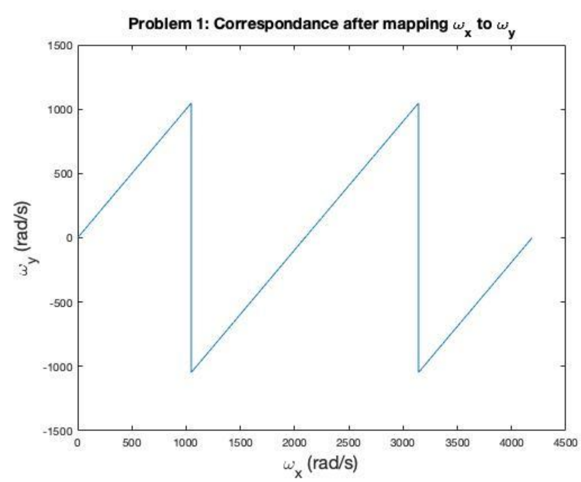

Figure 1: Correspondence of ω_x to ω_y

We see that at $\omega_0/2$, the direction of rotation reverses and goes with the same speed in the opposite direction

C. Problem 2: Output of modulated time-domain signal

We are to extend the following function to a lowpass periodic function

$$g(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t}$$

The frequency spectrum spans from $-N\omega_x$ to $+N\omega_x$ for a total bandwidth of $B = 2N\omega_x < \omega_0$.

We then modulate each of the 2N+1 components with an extra single-frequency term $e^{jk_n\omega_0 t}$ to form another function

$$g'(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t}\, e^{jk_n\omega_0 t}$$

Where k_n is an arbitrary integer.

If we sample g'(t) with sample spacing Δt and then lowpass the sampled function with a filter with cutoff frequencies at $\pm\ \omega_0/2$.

Attached is the handwritten work for this project. After sampling the signal, that is $g'(t) = g'(m\Delta t)$, we see that because in **Step 4 $\boldsymbol{\omega_0 = 2\pi/\Delta t}$**, we get the result in **Step 5**. As mentioned in the problem statement, $B = 2N\omega_x < \omega_0$ which takes into account the lowpass filter and we essentially recover our signal

$$g(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t}$$

D. Summary

Here we covered the topic of sampling which is used widely to convert analog signals to digital in order to more easily transmit and receive information. Through sampling, we sample the transmitted signal's frequency spectrum and lowpass the sampled signal with a frequency with edges at $\pm\omega_0/2$.

In Problem 1, we analyze this topic by seeing how the mapping from a signal with frequency ω_x to output frequency ω_y happens with the use of a reference frequency. In figure 1, we see that at $\pm\ \omega_0/2$, we have aliasing and therefore we end up with the waveform that is seen (reverses direction with same speed). An example of this can be seen with the wagon wheel effect, where a wheel will spin in one direction and reverse once it hits a certain frequency.

In Problem 2, we have a time domain signal and modulate it to become g'(t). After sampling and reconstructing the signal, we see that we have recovered our original signal g(t). We were able to do so because our signal was a sum of multiple signals as given in the problem statement. Therefore, the bandwidth is $2N\omega_x$ and it is given to be less than $\omega_0 = 2\pi/\Delta t$, hence we are able to reconstruct our original signal.

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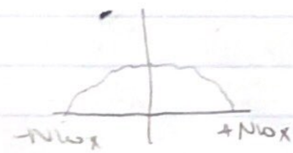
(1) given $g(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t}$ $\forall$ n integer



given:
$B = 2N\omega_x < \omega_0$

(2) Modulated signal

$$g'(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t} e^{jk_n \omega_0 t}$$

where k_n integer

(3) Sampling $g'(t) = g'(m\Delta t)$

$$g'(m\Delta t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x m\Delta t} e^{jk_n \omega_0 m\Delta t}$$

$\forall\ \omega_0 = 2\pi/\Delta t$

(4) $g'(m\Delta t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x m\Delta t} e^{jk_n \frac{2\pi}{\Delta t} m\Delta t}$

$= \sum_{n=-N}^{N} G_n e^{jn\omega_x m\Delta t} e^{jk_n 2\pi m}$

$\because$ m is any integer, $\therefore$ $k_n \cdot m$ is any integer

$\because$ any integer $m \cdot k_n$ in $e^{jk_n 2\pi m}$ gives **1**

(5) $g'(m\Delta t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x m\Delta t}$

b/c $B = 2N\omega_x < \omega_0$, we have already taken into account a low-pass filter

(6) $g'(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t} = g(t)$