Assignment 8: Digital Highpass Filter

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Abstract

The purpose of this report is to understand and implement a procedure for designing a digital high-pass Butterworth filter from an analog low-pass filter. The procedure consists of frequency transformations where we can jump from analog to digital and low-pass to high-pass.

A. Introduction

Butterworth filters are one of the most widely applied techniques of filter design. They are a basis for filters, especially lowpass filters. They are a very good approximation for lowpass filters and serve as a very good starting point for designing digital and analog filters, as well as changing to a highpass one.

B. Method and results

The general formula for the lowpass Butterworth Filter is

$$|H_n(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}}$$

Where n is the order of the filter and ω_0 is the filter cutoff frequency. We are given the following criteria for a lowpass Butterworth filter.

a. Maximum passband attenuation: α_{max} = 0.5 dB
b. Passband frequency: $\Omega_p = 3\pi/4$
c. Min. stopband attenuation: α_{min} = 20dB
d. Stopband frequency: $\Omega_s = \pi/2$

We start off by determining the order of the filter. We do this by relating the allowed attenuation in the passband and stopbands.

In the passband, $|\omega| \leq \omega_p$, the frequency response provides relatively high gains. In practice, the gain with the passband is not unity. Thus, to be practical, a small attenuation is allowed. Typically, the frequency response is bounded between zero and – α_{max} dB, where α_{max} denotes the maximum allowable attenuation within the passband.

$$0 \geq 10log\frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}} \geq -\alpha_{max}$$

The signal is reconstructed in the form of

$$0 \leq \left(\frac{\omega}{\omega_0}\right)^{2n} \leq 10^{-\frac{\alpha_{max}}{10}} - 1$$

Thus, for the frequency components within the passband, the basic requirement is

$$0 \leq \omega \leq \omega_0 \left(10^{\frac{\alpha_{max}}{10}} - 1\right)^{\frac{1}{2n}}$$

The transfer function is monotonically decreasing. Therefore, the maximum attenuation occurs at the passband frequency. Thus, the relationship can be simplified further,

$$\omega_p \leq \omega_0 \left(10^{\frac{\alpha_{max}}{10}} - 1\right)^{\frac{1}{2n}}$$

In the passband, $|\omega| \leq \omega_p$, a similar numerical relationship can be established. By definition, frequency response gives high attenuation for the removal or reduction of the frequency components. In practice, although the frequency response is not zero, large attenuation is required. Typically, a minimum attenuation, α_{min} dB, is assigned as part of the design specifications. This concept translates into the relationship

$$10log\frac{1}{1+\left(\frac{\omega}{\omega_0}\right)^{2n}} \leq -\alpha_{min}$$

Through similar steps, we end up with

$$\omega_s \geq \omega o \left(10^{\frac{\alpha_{min}}{10}}-1\right)^{\frac{1}{2n}}$$

If we relate this two as such

$$\frac{\omega_s}{\omega_p} \geq \left(\frac{10^{\frac{\alpha_{min}}{10}}-1}{10^{\frac{\alpha_{max}}{10}}-1}\right)^{\frac{1}{2n}}$$

and solve for *n*

$$n \geq \frac{\log\left(\frac{10^{\frac{\alpha_{min}}{10}}-1}{10^{\frac{\alpha_{max}}{10}}-1}\right)}{2\log\left(\frac{\omega_s}{\omega_p}\right)}$$

We need to solve for ω_s and ω_p. We know through frequency transformations,

$$Z = e^{j\theta}, z = e^{j\Omega}$$
$$e^{j\Omega} = -e^{j\theta} = e^{j(\theta\pm\pi)}$$
$$\theta = \Omega \pm \pi$$

According to the bilinear transform,

$$s = \beta\frac{z-1}{z+1}$$

The frequency conversion is

$$\omega = \beta\tan\left(\frac{\theta}{2}\right)$$

We are given Ω_s and Ω_p, so we have what we need to find ω_s and ω_p.

$$\omega_s = \tan\left(\frac{\tan(\pi-\Omega_s)}{2}\right) = 1\,rad/s$$

$$\omega_p = \tan\left(\frac{\tan(\pi-\Omega_p)}{2}\right) = 0.414\,rad/s$$

Using this to solve for *n*, we get that n = 3.8, which we can round to 4.

The upper and lower bounds of the cutoff frequency can be determined

$$\frac{\omega_p}{\left(10^{\frac{\alpha_{max}}{10}}-1\right)^{\frac{1}{2n}}} \leq \omega_0 \leq \frac{\omega_s}{\left(10^{\frac{\alpha_{min}}{10}}-1\right)^{\frac{1}{2n}}}$$
$$0.5388 \leq \omega_0 \leq 0.5630$$

I chose the mean of these two: $\omega_0 = 0.5509$ rad/s.

Now, we find the poles of the transfer function,

$$|H_n(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_0}\right)^{2n}} = \frac{1}{1+\left(\left(\frac{j\omega}{j\omega_0}\right)^2\right)^n}$$

Using Hermitian symmetry

$$|H_n(j\omega)|^2 = H_n(j\omega)H_n^*(j\omega)$$
$$= H_n(j\omega)H_n(-j\omega)$$

$$H_n(s)H_n(-s) = \frac{1}{1+(-1)^n\left(\frac{s}{\omega_0}\right)^{2n}}$$

Poles are where we set the denominator equal to zero

$$1+(-1)^n\left(\frac{s}{\omega_0}\right)^{2n} = 0$$

$$\left(\frac{s}{\omega_0}\right)^{2n} = (-1)^{n+1}$$

If n is odd,

$$\left(\frac{s}{\omega_0}\right)^{2n} = 1 = e^{j2k\pi}$$

$$s = \omega_0 e^{\frac{j2k\pi}{2n}} \quad k = 0,1, \dots 2n-1$$

If n is even,

$$\left(\frac{s}{\omega_0}\right)^{2n} = -1 = e^{j(2k+1)\pi}$$

$$s = \omega_0 e^{\frac{j(2k+1)\pi}{2n}} \quad k = 0,1, \dots 2n-1$$

If we group the roots of $H_n(s)H_n(-s)$

$$H_n(s)H_n(-s) = \frac{\omega_0^{2n}}{[(s-s_1)(s+s_1)] \dots [(s-s_n)(s+s_n)]}$$

$$= \frac{\omega_0^{n}}{(s-s_1)(s+s_2) \dots (s-s_n)} \cdot \frac{\omega_0^{n}}{(s+s_1)(s+s_2) \dots (s+s_n)}$$

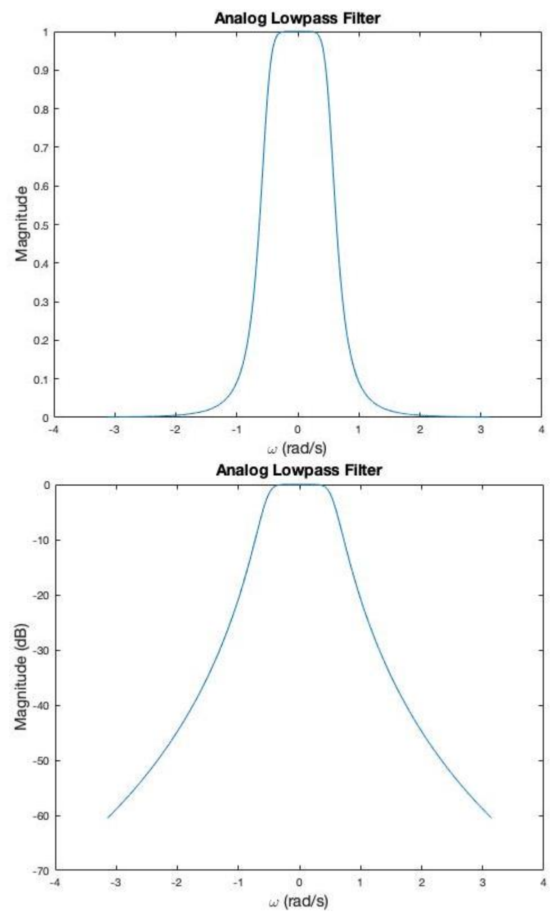
If we only take the left-hand side poles for stability reasons

$$H_n(s) = \frac{\omega_0^{n}}{(s-s_1)(s-s_2) \dots (s-s_n)}$$

n	*transfer functions H(s)*
1	$H(s) = \frac{\omega o}{s + \omega o}$
2	$H(s) = \frac{\omega_0^2}{(s^2 + 1.414\, \omega_o s + \omega_0^2)}$
3	$H(s) = \frac{\omega_0^3}{(s + \omega_o)(s^2 + \omega_o s + \omega_0^2)}$
4	$H(s) = \frac{\omega_0^4}{(s^2 + 1.848\, \omega_o s + \omega_0^2)(s^2 + 0.765\, \omega_o s + \omega_0^2)}$
5	$H(s) = \frac{\omega_0^5}{(s + \omega_o)(s^2 + 1.618\, \omega_o s + \omega_0^2)(s^2 + 0.618\, \omega_o s + \omega_0^2)}$

With the table above, we have our transfer function corresponding to our order n.

We plot this transfer function as the analog lowpass filter seen in **Figure 1**.

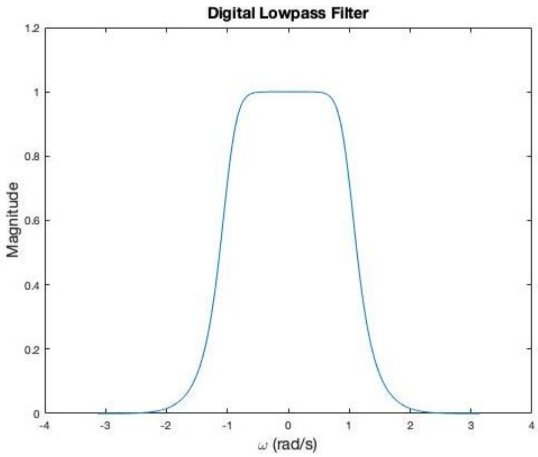


We now perform an analog to digital frequency transformation. The result will be a digital lowpass Butterworth filter. We will use the bilinear transform method and set β = 1.

$$s = \beta \frac{z-1}{z+1}$$

$$H_n(z) = \frac{\omega_0^4}{\left(\left(\frac{z-1}{z+1}\right)^2 + 1.848\omega_0 \left(\frac{z-1}{z+1}\right) + \omega_0^2\right)\left(\left(\frac{z-1}{z+1}\right)^2 + 0.765\omega_0 \left(\frac{z-1}{z+1}\right) + \omega_0^2\right)}$$

The digital lowpass filter transfer function plot is seen in **Figure 3**.



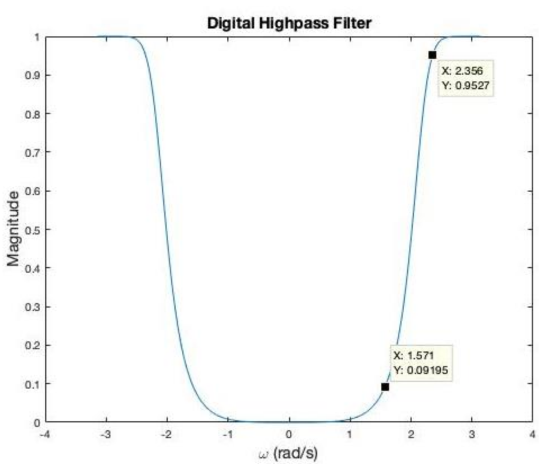
We now convert the lowpass filter to a high pass one by using the transformation Z = -z and then implementing $H_n(Z) = H_n(-z)$

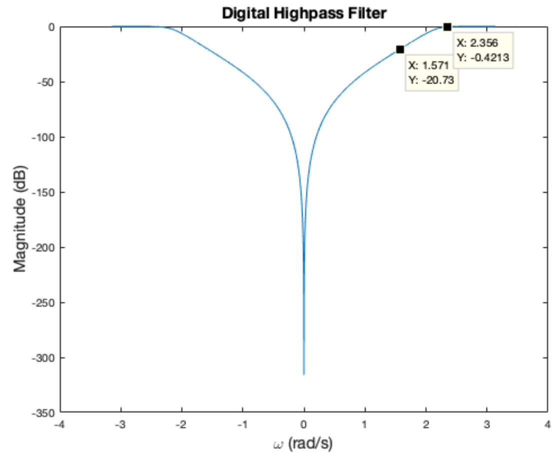
$$H_n(z) = \frac{\omega_0^4}{\left(\left(\frac{-z-1}{-z+1}\right)^2 + 1.848\omega_0\left(\frac{-z-1}{-z+1}\right) + \omega_0^2\right)\left(\left(\frac{-z-1}{-z+1}\right)^2 + 0.765\omega_0\left(\frac{-z-1}{-z+1}\right) + \omega_0^2\right)}$$

We use the transfer function equation that only takes into account the left-hand side of poles.

$$H_n(s) = \frac{\omega_0^n}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

The digital high pass filter transfer function is seen in **Figure 3** and **4**.





From **Figure 4** at the stopband frequency $\frac{\pi}{2} = 1.571$ rad/s. the stopband attenuation is -20.71 dB, which is a 3.488% difference from the 20dB we were looking for. At the passband frequency $\frac{3\pi}{4} = 2.356$ rad/s, the passband attenuation is 0.42dB, which is a 17.3913% difference.

C. Summary

The procedure consists of first finding the order of the transfer function of the filter. We then find a corresponding cutoff frequency ω_0 after finding the order of the filter. This is done by using our attenuation magnitude and frequency requirements.

Next, we found the poles of the transfer function and only use the ones that are on the left side of the jω axis because this implies stability. We now have an analog lowpass filter.

Next, we use the bilinear transform to transform from s to z to get to discrete time. We now have a digital lowpass filter.

Finally, we transform z to Z = -z and implement it within the transfer function. We now have a digital highpass filter.

As we have seen, Butterworth filters are great for implementing other filters. They are a great starting point for lowpass filters. They allow us to convert from analog to digital, and vice-versa. We are also able to go from lowpass to highpass. We can do all of these conversions with the help of frequency conversions.