THE COPPERBELT UNIVERSITY
SCHOOL OF MATHEMATICS AND NATURAL SCIENCES
DEPARTMENT OF PHYSICS
PH 110-INTRODUCTORY PHYSICS LECTURE NOTES

CHAPTER 3
KINEMATICS

Kinematics is the branch of mechanics that describes the motion of objects without consideration of the causes of motion. In this chapter, we study the motion of objects in relation to time. The moving object of concern is either a point-like object or an object that can be viewed to move like a particle.

3.1 Motion

Everything in the universe is in motion, from the tiniest particles within atoms, to the largest galaxies of stars. An object is said to be in motion if it continuously changes its position with respect to surroundings. The line joining all positions of a particle is called a path.

3.2 Types of motion

Motion can be categorized into three types; **one dimensional**, **two dimensional** and **three dimensional.** Motion of an object in a straight line is called one dimensional motion, for example, a car moving on a straight road or a freely falling object. Motion of an object in a plane is called two-dimensional motion, for example a boat on a lake or car making a circular turn. An object that is moving in space is said to execute three-dimensional motion, for example, an aero plane or a flying insect.

3.3 Motion in one dimension

3.3.1 Origin, unit and sense of passage time

In any process involving time, some instant of time is assigned the value of zero time. To represent the passage of time along a line, zero time is taken as the origin. The unit of time may be second, minute or hour as per convenience.

3.3.2 Origin, unit and direction for position measurement

To represent the motion of a body along a straight line, some point O is taken as the origin and some unit of length is chosen. At each time t, the position of the object is given a real number positive or negative written as $x(t)$. This is called position coordinate.

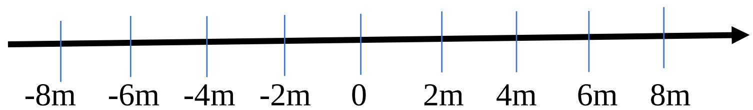

Figure 3.1 *Representation of motion of a body along a straight line.*

The distance measured to the right of the origin is taken as positive and the distance measured to the left of the origin is taken as negative, see figure 3.1. In case of motion along the vertical line the distance covered above the origin is taken positive while that below the origin is taken negative. Position is measured in metres, kilometres , etc.

3.3.3 Distance and Displacement

(a) Distance

The actual length of the path traversed by a particle in a certain interval of time is called distance travelled by that particle. It is a scalar quantity. The SI unit of distance is *m* and its dimension is *L*

(b) displacement

Displacement is defined as the change of position of a particle from an initial position x_i to a final position x_f, along a given direction. Thus,

$$\Delta x = x_f - x_i$$

Displacement can also be defined as the shortest distance from the initial position to the final position of the particle. It is a vector quantity. Just like distance, the SI unit of displacement is *m* and its dimension is *L*

(c) Comparison between distance and displacement

(i) For a moving particle distance can never be negative or zero while displacement can be *i.e.* Distance > 0 but displacement > = or < 0.
(ii) For motion between two points displacement is single-valued while distance depends on the actual path and so can have many values.
(iii) For a moving particle, distance can never decrease with time while displacement can. Decrease in displacement with time means the particle is moving towards the initial position.
(iv) In general, the magnitude of displacement is less or equal to the distance covered by the particle.

3.3.4 Speed and velocity

(a) Speed

Rate of distance covered with time is called speed. It is a scalar quantity. The SI unit of speed is ms^{-1} and its dimensions are LT^{-1}. The following are types of speeds;

(i) Uniform speed

When a particle covers equal distances in equal intervals of time, (no matter how small the intervals are) then it is said to be moving with uniform speed.

(ii) Non-uniform (variable) speed

In non-uniform speed, a particle covers unequal distances in equal intervals of time.

(iii) Average speed

Average speed is defined as the ratio of the total distance covered to the time interval. Mathematically, it is given by

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Time interval}}$$

(iv) Instantaneous speed

Instantaneous speed is the speed of the particle at a particular instant. It is the average speed for infinitesimally small-time interval (as the time interval Δt approaches zero).

(b) Velocity

Velocity is defined as rate of change of position or rate of displacement with time. It is a vector quantity. The SI unit of velocity is ms^{-1} and its dimensions are LT^{-1}. The following are types of velocities;

(i) Uniform velocity

A particle is said to move with uniform velocity if its velocity at every instant is the same. In this case, the magnitude as well as the direction of its velocity remains the same and this is possible only when the particle moves in the same straight line without reversing its direction

(ii) Non-uniform velocity

A particle is said to have non-uniform velocity, if either of magnitude or direction of the velocity changes (or both changes).

(iii) Average velocity

The average velocity of a particle is defined as the ratio of the displacement to the time interval. Mathematically, it is given by

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time interval}} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

Where x_f and x_i, are final and initial positions of the particle respectively, and t_f and t_i, are the final and initial times respectively.

(iv) Instantaneous velocity

The velocity of a particle at any instant of its motion is called instantaneous velocity. It is defined as the limiting value of the average velocity as the time interval Δt approaches zero. Mathematically, the magnitude of the instantaneous velocity (instantaneous speed) is expressed as:

$$v = \lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t} \right) = \frac{dx}{dt} = \dot{x}$$

In calculus notation, the above limit is called the derivative of x with respect to t. Thus:

$$v = \frac{dx}{dt} \Leftrightarrow x_f - x_i = \int_{t_i}^{t_f} v\, dt \equiv \text{Area under } v - t \text{ graph}$$

(a) Comparison between average speed and average velocity

(i) Average speed is a scalar quantity while average velocity is a vector quantity, both having same units (*m/s*) and dimensions (LT^{-1}).

(ii) Average speed or velocity depends on time interval over which it is defined.

(iii) For a given time interval, average velocity is single-valued while average speed can have many values depending on the path followed.

(iv) If after motion a particle comes back to its initial position, then the average velocity is zero (as displacement is zero), but the average speed is greater than zero.

(v) For a moving particle, average speed can never be negative or zero (unless $t \to \infty$) while average velocity can be.

(b) Comparison between instantaneous speed and instantaneous velocity

(i) Instantaneous velocity is always tangential to the path followed by the particle

(ii) A particle may have constant instantaneous speed but variable instantaneous velocity.

(iii) The magnitude of instantaneous velocity is equal the instantaneous speed.

(iv) If a particle is moving with constant velocity, then its average velocity and instantaneous velocity are always equal.

3.3.5 Acceleration

Acceleration is defined as the time rate of change of velocity of an object. It is a vector quantity. Its direction is the same as that of change in velocity. The SI unit of acceleration is ms^{-2} and its dimensions are LT^{-2}. If the velocity of a particle is increasing, it is said to be accelerated (positive acceleration). Similarly, if the velocity of the particle is decreasing it is said to move with deceleration or retardation (negative acceleration).

The following are types of acceleration;

(a) Uniform acceleration

If the magnitude and direction of the acceleration of a particle at every instant of its motion remains constant, it is said to move with uniform acceleration**.** If a particle is moving with uniform acceleration, this does not necessarily imply that the particle is moving in straight line, *e.g.*, projectile motion.

(b) Non-uniform acceleration

A particle is said to have non-uniform acceleration, if magnitude or direction or both, change during motion.

(c) Average acceleration

The average acceleration of a particle in a specified interval of time is defined as the ratio of change in velocity to the interval of time. Mathematically,

$$\text{Average acceleration } = \frac{\text{Change in velocity}}{\text{Time interval}} = \frac{\Delta v}{\Delta t} = \frac{v_f - vi_i}{t_f - t_i}$$

Where v_f and v_i, are final and initial velocities of the particle respectively, and t_f and t_i, are the final and initial times respectively.

(d) Instantaneous acceleration

The instantaneous acceleration of a particle is defined as the limiting value of the average acceleration $\frac{\Delta v}{\Delta t}$ when time interval Δt approaches zero. Mathematically, it is expressed as:

$$v = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t}\right) = \frac{dv}{dt} = \dot{v}$$

Since $v = \frac{dx}{dt}$; then,

$$a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$$

In calculus notation, the above limit is called the derivative of v with respect to *t*. Thus:

$$a = \frac{dv}{dt} \Leftrightarrow v_f - v_i = \int_{t_i}^{t_f} a \, dt$$

3.3.6 Uniform motion

A particle is said to possess or execute uniform motion if its velocity is constant at every instant. The following are the characteristics of uniform motion;

- The velocity of a particle is constant.
- The acceleration of a particle is zero.
- The path of a particle is a straight line.

(a) Formula for uniform motion

Since the velocity is uniform, the displacement *s* at any instant or interval of time *t* is given by

$$s = v\Delta t \Leftrightarrow v = \frac{s}{\Delta t}$$

Since $s = \Delta x = x_f - x_i$, then

$$x_f - x_i = v(t_f - t_i)$$

If x_o is the position of the particle at $t = 0$ and x is the position of the particle at time t, then;

$$x = x_o + vt$$

(b) Position-time graph of uniform motion

The position-time graph for uniform motion is a straight line. The gradient or slope of the graph gives the velocity of the particle.

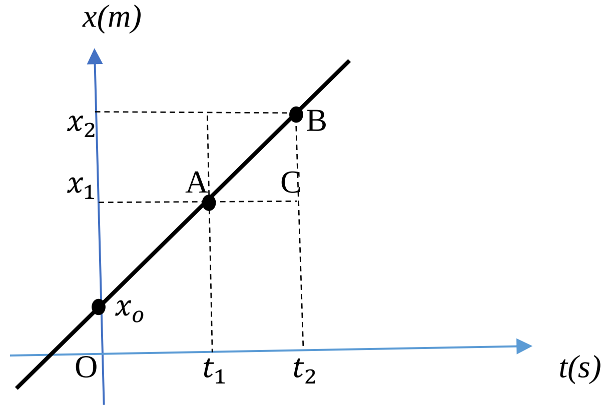

Figure 3.2 *Position-time* $(x - t)$ *graph for uniform motion of a particle moving in a straight line*

From the graph, the slope or gradient gives the velocity of the particle

$$Slope = \frac{BC}{AC} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

(c) Velocity-time graph for uniform motion

The velocity-time graph for uniform motion is a straight line parallel to the time axis. The region under the velocity-time graph between t_1 and t_2 represents the displacement in that interval of time.

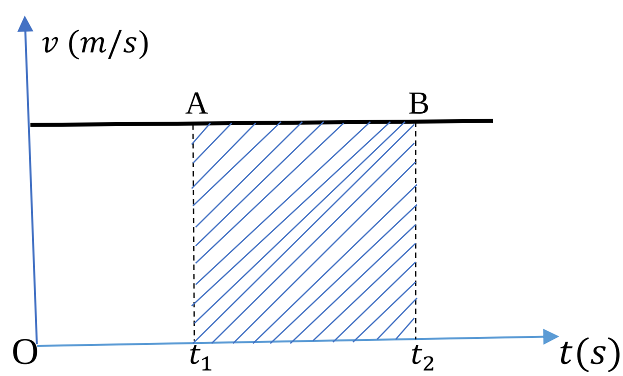

Figure 3.3 *Velocity-time* $(v - t)$ *graph for uniform motion of a particle moving in a straight line*

3.3.7 Uniformly accelerated motion

A particle is said to execute uniformly accelerated motion if its acceleration is uniform at every instant of its motion. The following are the characteristics of uniformly accelerated motion;

- The velocity of the particle is non-uniform, i.e. it changes with time
- The acceleration of a particle is constant (either positive or negative).
- The path of a particle is a straight line

(a) Equations of uniformly accelerated motion

These are the various relations between v_0, v, a, t and s for the moving particle, where the notations are used as:

$v_0 \equiv$ Initial velocity of the particle at a time $t = 0$

$v \equiv$ Final velocity of the particle at time t seconds

$a \equiv$ Acceleration of the particle

$s \equiv$ Distance travelled by the particle in time t seconds

(i) Velocity-time relation

By definition, the acceleration of the particle is given by;

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - vi_i}{t_f - t_i} = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - 0} = \frac{v - v_0}{t}$$

Rearranging the above equation gives;

$$\boldsymbol{v = v_0 + at}$$

(ii) Position-time relation

By definition, displacement s of the particle is given by;

Displacement=Average velocity × time

$$s = \bar{v}t$$

The average velocity $\bar{v}$ of the particle is given by

$$\bar{v} = \frac{v + v_0}{2}$$

Then;

$$s = \left(\frac{v + v_0}{2}\right)t$$

The final velocity v of the particle is given by

$$v = v_0 + at$$

Then,

$$s = \left(\frac{v_0 + at + v_0}{2}\right)t$$
$$s = \left(\frac{2v_0 + at}{2}\right)t$$
$$\boldsymbol{s = v_0 t + \frac{1}{2}at^2}$$

If x is the position coordinate of the particle at time t seconds and x_0 is the position coordinate of the particle at time $t = 0$, then the displacement is given by;

$$s = \Delta x = x - x_0$$

Then, the above relation becomes

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$
$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

(iii) Position-velocity relation

Since the velocity is given by

$$v = v_0 + at$$

Then,

$$t = \frac{v - v_0}{a}$$

The displacement is therefore given by

$$\text{Displacement=Average velocity} \times \text{time}$$
$$s = \left(\frac{v + v_0}{2}\right)t$$
$$s = \left(\frac{v + v_0}{2}\right)\left(\frac{v - v_0}{a}\right)$$
$$s = \frac{v^2 - vv_0 + vv_0 - {v_0}^2}{2a}$$

$$s = \frac{v^2 - {v_0}^2}{2a}$$

$$\boldsymbol{v^2 = {v_0}^2 + 2as}$$

If x is the position coordinate of the particle at time t seconds and x_0 is the position coordinate of the particle at time $t = 0$, then the displacement is given by;

$$s = \Delta x = x - x_0$$

Then, the above relation becomes

$$v^2 = {v_0}^2 + 2a(x - x_0)$$

(a) Position-time graph for uniformly accelerated motion

The position time graph of the particle executing uniformly accelerated is a curve in the form;

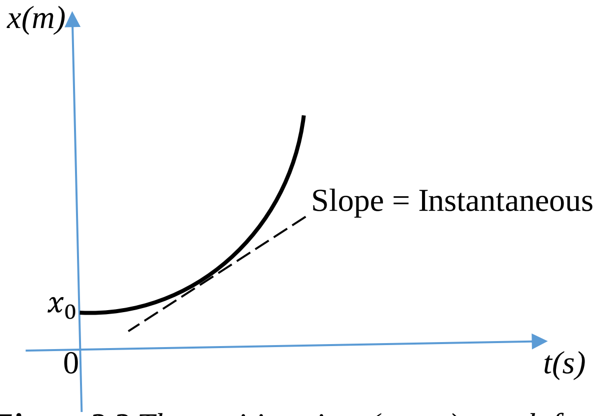

Figure 3.3 *The position-time $(x - t)$ graph for uniformly accelerated motion of a particle moving in a straight line. The gradient or slope of a tangent at any point of the curve represents the instantaneous velocity at that point.*

(b) Velocity-time graph for uniformly accelerated motion

The velocity time graph of a particle executing uniformly accelerated motion is a straight line and the slope of the graph gives the uniform acceleration of the particle.

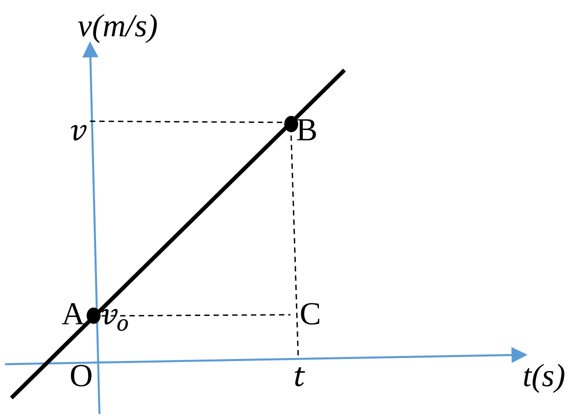

Figure 3.4 *The velocity-time $(v - t)$ graph for uniformly accelerated motion of a particle moving in a straight line. The gradient or slope of a graph represents uniform acceleration.*

From the graph, the slope or gradient gives the uniform acceleration of the particle

$$\text{Slope} = \frac{BC}{AC} = \frac{v - v_0}{t - 0} = \frac{v - v_0}{t} = \frac{\Delta v}{\Delta t} = a$$

The area under the graph of the of the velocity-time graph represents the displacement of the particle.

(c) Acceleration-time graph for uniformly accelerated motion

The acceleration-time graph for a particle executing uniformly accelerated motion is a straight line parallel to the time axis.

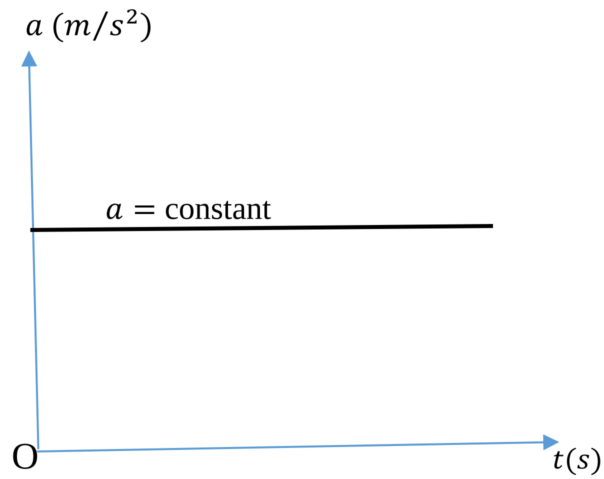

Figure 3.5 *The acceleration-time $(a - t)$ graph for uniformly accelerated motion of a particle moving in a straight line.*

3.3.8 Relative velocity

We consider the body to be at rest when its position does not change with respect to fixed object in its surrounding. Usually, we refer to the state of motion or rest with respect to the earth, but the earth is moving with respect to the sun. Thus, we cannot say a body to be at absolute rest or in a state of absolute motion.

The relative velocity of a body *A* with respect to another body *B* is the velocity that body *A* would appear to have to an observer situated on body *B* moving along with it.

Consider two bodies *A* and *B* be moving along a straight line in the same direction, with constant velocities v_A and v_B. Then, the velocity of *A* with respect to *B* is equal to $v_A - v_B$. If *B* is moving in the opposite direction, the relative velocity of *A* with respect to *B* is equal to $v_A + v_B$.

3.3.9 Motion of body under gravity (Free fall)

All bodies near the Earth's surface are attracted towards the centre of the earth by the force of gravity. This force produces a vertical uniform acceleration called acceleration due to gravity, denoted by the letter 'g'. It is independent of the mass of the body and varies from place to place. Its mean value on Earth's surface is 9.8 m/s^2.

Any object that is being acted upon by the force of gravity is said to be in the state of free fall. The following are two important motion characteristics of free-falling objects;

- Free-falling objects do not encounter air resistance
- All free-falling objects accelerate downwards on earth at a rate of 9.8 m/s^2 (Approximated)

Like any moving object executing uniformly accelerated motion, the motion of an object in free fall can be described by equation of kinematics. In this case, the three equations of uniformly accelerated motion become;

$$v = v_0 + gt$$
$$y = y_0 + v_0 t + \frac{1}{2} g t^2$$
$$v^2 = {v_0}^2 + 2g(y - y_0)$$

3.4 Motion in two dimensions

In this section, we study the motion of a particle in a plane. Motion of a particle in a straight line is called rectilinear motion. A combination of two rectilinear motions along different directions gives motion along a plane. Some common examples of motion in a plane are motion of projectiles, motion of charged particles in electric field, satellites, etc.

3.4.1 Position, displacement, velocity, and acceleration vectors

(a) Position

We describe the position of a particle in two dimensions with a position vector $\vec{r}$ denoted by

$$\vec{r} = x\vec{i} + y\vec{j}$$

Where x and y are rectangular coordinates (x, y). In three dimensions, the position vector becomes $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

(b) Displacement

If $\vec{r}_f$ and $\vec{r}_i$ are the final and initial positions of a particle respectively at time t_f and t_i respectively, then the displacement of a particle is

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

(c) Velocity

The average velocity of the particle in two dimensions is therefore given by

$$\langle\vec{v}\rangle = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

In two dimensions, the instantaneous velocity is given by

$$\vec{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{r}}{\Delta t}\right)$$

In calculus notation, the above limit is called the derivative of $\vec{r}$ with respect to t, and is written as

$$\vec{v} = \frac{d\vec{r}}{dt} \quad \equiv \quad \vec{r}_f - \vec{r}_i = \int_{t_i}^{t_f} \vec{v}\ dt$$

Since $\vec{r} = x\vec{i} + y\vec{j}$, then

$$\vec{v} = \frac{d}{dt}\left(x\vec{i} + y\vec{j}\right) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

or

$$\vec{v} = v_x\vec{i} + v_y\vec{j}$$

where the two components of the velocity vector are given by:

$$v_x = \frac{dx}{dt}$$

$$v_y = \frac{dy}{dt}$$

In a three-dimensional study, the velocity vector can be written in the general form $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$.

(d) Acceleration

For a two-dimensional motion, the average acceleration is given by:

$$\langle\vec{a}\rangle = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

and the instantaneous acceleration is defined as:

$$\vec{a} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{v}}{\Delta t}\right)$$

In calculus notation, we obtain:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Since $\vec{v} = \frac{d\vec{r}}{dt}$

then,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2} \quad \equiv \quad \vec{v}_f - \vec{v}_i = \int_{t_i}^{t_f} \vec{a}\ dt$$

In unit vector notation, we use $\vec{v} = v_x\vec{i} + v_y\vec{j}$, so that:

$$\vec{v} = \frac{d}{dt}\left(v_x\vec{i} + v_y\vec{j}\right) = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j}$$

or

$$\vec{a} = a_x\vec{i} + a_y\vec{j}$$

where the two components of the acceleration vector are given by:

$$a_x = \frac{dv_x}{dt}$$

$$a_y = \frac{dv_y}{dt}$$

In a three-dimensional study, the acceleration vector can be written in the general form $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$.

3.4.2 Projectile motion

If the force acting on a particle is oblique with initial velocity then the motion of the particle is called projectile motion. A body which is in flight through the atmosphere but it is not being propelled by any fuel is called projectile.

(a) Assumptions of projectile motion

The following are the assumptions of projectile motion:

- There is no resistance due to air.
- The effect due to curvature of the earth is negligible.
- The effect due to the rotation of the earth is negligible.
- For all points of the trajectory, the acceleration due to gravity 'g' is constant in magnitude and direction.

(b) Principles of physical independence of motions

The following are principles of physical independence of projectile motions

- The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts; horizontal motion and vertical motion. These two motions take place independent of each other. This is called the principle of physical independence of motions.

- The velocity of the particle can be resolved into two mutually perpendicular components; horizontal and vertical components.
- The horizontal component of the velocity remains unchanged throughout the flight. The force of gravity continuously affects the vertical component of the velocity.
- The horizontal motion is a uniform motion and the vertical motion is a uniformly accelerated motion.

(c) Types of projectile motion

There are two types of projectile; horizontal projectile motion and oblique projectile motion.

(i) Horizontal Projectile

Here we consider a body projected horizontally from a certain height 'h' vertically above the ground with initial velocity v_0. If friction is considered to be absent then there is no other horizontal force which can affect the horizontal motion. The horizontal velocity therefore remains constant. See figure 3.6

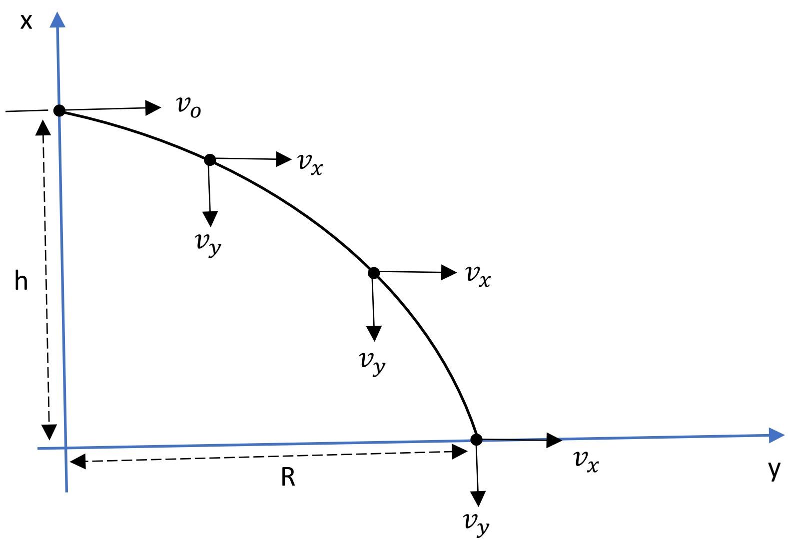

Figure 3.6 *The path of a projectile projected horizontally from a height 'h' vertically above the ground.*

When the particle is released, it moving with uniform velocity in horizontal direction and uniform acceleration (g) in the vertical direction. The displacement as a function of time in each case is therefore given by:

In horizontal direction: $x = v_x t = v_0 t$

In vertical direction: $y = \frac{1}{2} g t^2$; Since $v_{0y} = 0$

From the above two equations, we can obtain the cartesian equation by eliminating the parameter 't'.

$$x = v_0 t \quad \Leftrightarrow \quad t = \frac{x}{v_0}$$

$$y = \frac{1}{2}gt^2 = \frac{1}{2}g\left(\frac{x}{v_0}\right)^2 = \frac{1}{2}g\frac{x^2}{{v_0}^2}$$

$$y = \left(\frac{g}{2{v_0}^2}\right)x^2$$

This equation is an equation of a parabola since it is in the form $y = ax^2$.

The velocity at any point of the curve is given by:

$$\vec{v} = v_x\vec{i} + v_y\vec{j}$$

where $v_x = v_0$ and $v_y = v_{0y} + gt = gt$; since $v_{0y} = 0$

(ii) Oblique projectile

In oblique projectile motion, the horizontal component of the velocity and acceleration (g) remains constant while the vertical component of the velocity changes. Velocity is maximum at the point of projection while minimum at highest point.

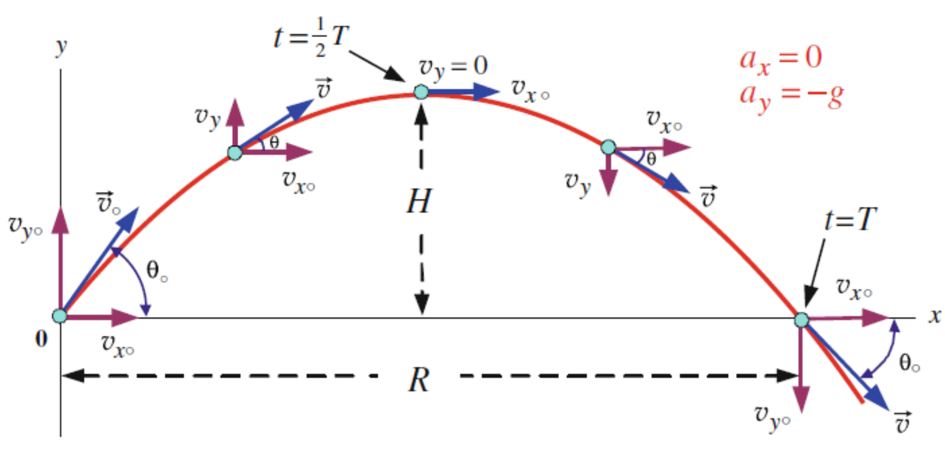

Figure 3.7 *The path of a projectile launched from the origin with initial velocity that makes an angle* θ_0 *with the x axis.*

The initial velocity can be resolved into horizontal and vertical components as follows:

Horizontal component: $v_{0x} = v_0 \cos\theta$

Vertical component: $v_{0y} = v_0 \sin\theta$

The displacement as a function of time in each case is therefore given by:

In horizontal direction: $x = v_{0x}t = v_0 \cos\theta\, t$

In vertical direction: $y = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_0 \sin\theta\, t - \frac{1}{2}gt^2$

The velocity at any time 't' in the horizontal direction is $v_{0x} = v_0 \cos\theta$.

In the vertical direction, the velocity at any time 't' is given by:

$$v_y = v_{0y} - gt = v_0 \sin\theta - gt$$

The position-velocity relation in the vertical direction is given by

$$v_y{}^2 = v_{oy}{}^2 - 2g(y - y_0) = v_o{}^2 sin^2\theta - 2g(y - y_0)$$

(1) Maximum height (H)

At the maximum height, the vertical velocity is zero. Given: Initial vertical velocity $v_{0y} = v_0 \sin\theta$, final vertical velocity $v_y = 0$, acceleration $a = -g$, vertical displacement $s = H$

Then:

$$v_y{}^2 = v_{oy}{}^2 + 2gs = v_{oy}{}^2 - 2g(y - y_0)$$

$$0 = v_o{}^2 sin^2\theta - 2gH$$

$$2gH = v_o{}^2 sin^2\theta$$

$$\boldsymbol{H = \frac{v_o{}^2 sin^2\theta}{2g}}$$

(2) Time of flight (T)

It is the time taken by the projectile to return to the same horizontal level as the point of projection.

If we consider the vertical motion, we have:

$$s = v_{0y}t - \frac{1}{2}gt^2$$

Given: $v_{0y} = v_0 \sin\theta$, $s = 0$ and $t = T$, then:

$$0 = v_0 \sin\theta\, T - \frac{1}{2}gT^2 \iff gT^2 - 2v_0 \sin\theta\, T = 0$$

$$T(gT - 2v_0 \sin\theta) = 0$$

$$gT - 2v_0 \sin\theta = 0$$

$$\boldsymbol{T = \frac{2v_0 \sin\theta}{g}}$$

(3) Horizontal range (R)

It is the horizontal displacement of a particle through the point of projection. Horizontally, the particle moves with constant velocity $v_0 \cos\theta$.

Given: $x = R, t = T = \frac{2v_0 \sin\theta}{g}$, then the range is

$$x = v_0 \cos\theta\, t$$

$$R = v_0 \cos\theta\, T$$

$$R = v_0 \cos\theta \left(\frac{2v_0 \sin\theta}{g}\right)$$

$$R = \frac{2{v_0}^2 \sin\theta \cos\theta}{g}$$

$$R = \frac{{v_0}^2 2 \sin\theta \cos\theta}{g}$$

Since $\sin 2\theta = 2 \sin\theta \cos\theta$, then expression for range becomes

$$\boldsymbol{R = \frac{{v_0}^2 \sin 2\theta}{g}}$$

(4) Maximum range (R_{max})

For a given velocity of projection, the range is maximum when $2\theta = 90°$ or $\theta = 45°$

$$R_{max} = \frac{{v_0}^2 \sin 2\theta}{g} = \frac{{v_0}^2 \sin 90}{g}$$

$$\boldsymbol{R_{max} = \frac{{v_0}^2}{g}}$$

(5) Cartesian equation

The cartesian equation is obtained from the parametric equations if we eliminate the parameter *t* between them. The parametric equations are equations for displacement as a function of time. In this case we have:

$$x = v_0 \cos\theta\, t$$

$$y = v_0 \sin\theta\, t - \frac{1}{2}gt^2$$

$$t = \frac{x}{v_0 \cos\theta}$$

$$y = v_0 \sin\theta \left(\frac{x}{v_0 \cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta}\right)^2$$

$$y = \left(\frac{\sin\theta}{\cos\theta}\right)x - \frac{g}{2{v_0}^2 \cos^2\theta}x^2$$

$$\boldsymbol{y = \tan\theta\, x - \frac{g}{2{v_0}^2 \cos^2\theta}x^2}$$

The above equation is of the form $y = ax - bx^2$, which is the equation for a parabola. Hence, the path of a projectile is a parabola.

EXERCISES

1. The fastest 100 m that was ever run was in 9.58 s by Usain Bolt. What was his average velocity in
   (a) m/s?
   (b) km/min?
   (c) km/h? **[(a)10.4 m/s (b)0.63 km/min (c) 37.6 km/h]**
2. You are trying to figure out the speed a rifle can shoot. You fire at a target 100 m away and you hear the impact 0.5 seconds later. The speed of sound is 340 m/s. What was the speed of the bullet? **[485m/s]**
3. A car covers first half of the distance between two places at a speed of 40 km/h and the second half at 60 km/h. What is the average speed of the car? **[48 km/h]**
4. On a 60 km straight road, a bus travels the first 30 km with a uniform speed of 30 km/h. How fast must the bus travel the next 30 km so as to have average speed of 40 km/h for the entire trip? **[60 km/h]**
5. A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again 5 steps forward and 3 steps backward and so on. Each step is 1m and requires 1 second. Plot position-time graph for his motion. From the graph or otherwise find the time taken by him to fall in a pit 13 m away from start. **[Hint: In 8 seconds he moves forward by 2 m; total time t=37s]**
6. The position-time graph shown in figure 3.8 represents the motion of a basketball coach during the last 8 seconds of the game in overtime.
   (a)Find the average velocity in the time intervals
   (i) 0 to 2 s,
   (ii) 0 to 4s,

(iii) 2 s to 4 s,
(iv) 4 s to 7 s,
(v) 0 to 8 s. **[(i) 5 m/s (ii) 1.25 m/s (iii) -2.5 m/s (iv) -3.3 m/s (v) 0]**

(b) Repeat part (a) for the average speed. **[5 m/s; 3.75 m/s; 2.5 m/s; 3.3 m/s; 3.75 m/s]**

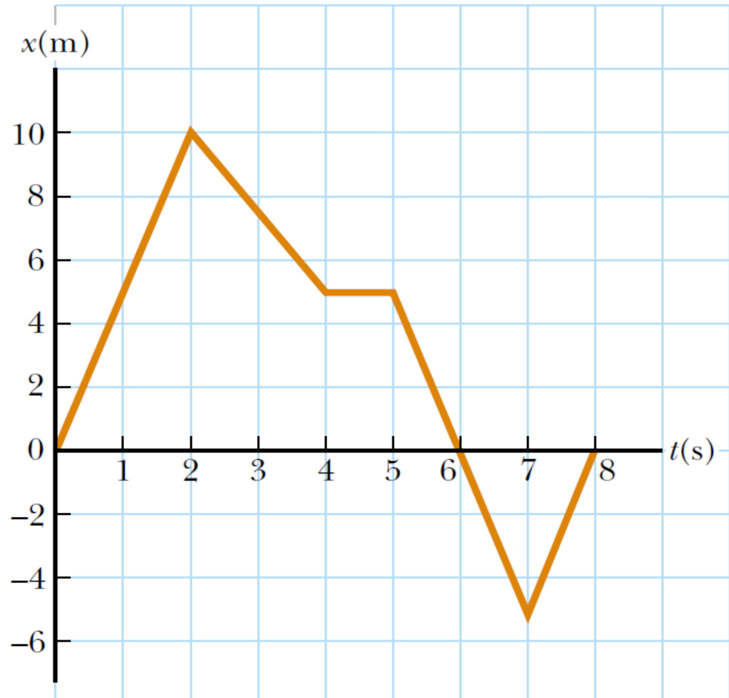

Figure 3.8 *See exercise 6*

7. A position-time graph for a particle moving along the *x* axis is shown in figure 3.9.
(a) Find the instantaneous velocity at time t = 2 seconds
(b) At what value of t is the velocity zero? **[(a) -3.8 m/s (b) 4s]**

Figure 3.9 *See exercise 7*

8. A particle moves along a straight line and O is the fixed point on that line. The displacement s metres of the particle is given by

$$s = (t - 1)(t - 5)$$

Draw a displacement-time graph for the interval of time from t=0 to t=6 s.
From the graph find,
(a) the average velocity in the interval from t=0 to t=4 s.
(b) the distance covered in the time interval from t=0 to t=4 s.
(c) the instantaneous velocity at t=4 s.
(d) the time at which the velocity is zero. **[(a) -2 m/s (b) 10 m (c) 2 m/s (d) 3 s]**

9. A particle starts from rest and accelerates as shown in the figure 3.10. Determine
(a) the particle's speed at t = 10 s and at t = 20 s.
(b) the distance traveled in the first 20 seconds. **[(a) 20 m/s; 5 m/s (b) 262 m]**

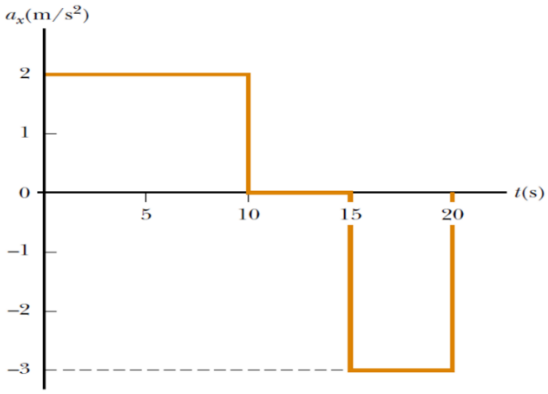

Figure 3.10 *See exercise 9*

10. A velocity–time graph for an object moving along the x direction is shown in the figure 3.11.
(a) Plot a graph of the acceleration versus time.
(b) Determine the average acceleration of the object in the time intervals
(i) t = 5 s to t = 15 s
(ii) t = 0 to t = 20 s. **[(b) (i)1.6 m/s^2; (ii)0.8 m/s^2]**

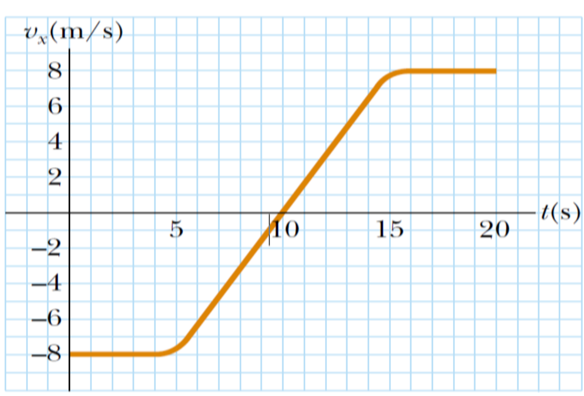

Figure 3.11 *See exercise 10*

11. The Lamborghini Murcielago can accelerate from 0 to 62.2 mi/h in a time of 3.4 seconds. Determine the acceleration of this car. **[8.18 m/s^2]**

12. A car traveling at the speed of 54 km/h is brought to rest in one and half minute. Find the retardation and the distance travelled by the car before coming to rest. [**0.167m/s^2; 675m**]
13. A car passes a point and then 120 m away, its velocity was 21 m/s. If its acceleration was constant at 0.583 m/s^2, what was the car's velocity at that point? **[17.4 m/s]**
14. A sprinter from rest accelerates at 1.5 m/s^2 for the first 20 m but then gets tired and decelerates at -0.1 m/s^2 for the next 80 m. What is the sprinter's final velocity? **[6.64 m/s]**
15. An out of control train is hurling down the track at 35 m/s. If the conductor sees a kitten on the track 200 m in front of him, with what minimum acceleration would the train need to save the adorable kitten? **[-3.06 m/s^2]**
16. A bullet strikes a uniform plank with a velocity of 400 m/s and comes out with half velocity. What would be the velocity if the plank were only half thick? **[316.2m/s]**
17. A particle starts from a point O with an initial velocity of 2 m/s and travels along a straight line with uniform acceleration of 2 m/s^2. Two seconds later a second particle starts from rest at O and travels along the same line with an acceleration of 6 m/s^2. Find how far from O the second particle overtakes the first. **[48 m from O]**
18. A person is running at his maximum speed of 4 m/s to catch a train. When he is 6m from the door of the train, it starts moving at a constant acceleration of 1 m/s/s. How long does he take to catch the train? **[2 s]**
19. A particle moves along the x-axis according to the equation $x = 2 + 3t - t^2$, where x is in metres and t is in seconds. At $t = 3$ s, find
    (a) the position of the particle
    (b) the velocity of the particle
    (c) the acceleration of the particle. **[(a) 2 m (b) -3 m/s (c) -2 m/s^2]**
20. A particle moves along a straight line with an acceleration $a = 3t^2 - 2$, where a is in ms^{-2} and t is in seconds. Initially, the particle is at O, with velocity 2 m/s. At $t = 3$ s, find the
    (a) acceleration of the particle
    (b) velocity of the particle
    (c) position of the particle. **[(a)25 m/s^2 (b)23 m/s (c)17.25 m]**
21. Two parallel rail tracks run north-south. Train A moves north with a speed of 54 km/h and train B moves south with a speed of 90 km/h. What is the relative velocity in m/s of
    (a) B with respect to A
    (b) the ground with respect to B
    (c) a monkey running on the roof of train A against its motion (with velocity 18 km/h with respect to A) as observed by a man standing on the ground. **[-40m/s; 25m/s; 10m/s]**
22. According to Guinness, the tallest man to have ever lived was Robert Pershing of Alton, Illinois. He was last measured in 1940 to be 8.92 ft. Determine the speed which a quarter would have reached before contact with the ground if dropped from the top of his head. **[7.3 m/s]**
23. A stone is projected vertically up from the top of the tower 73.5 m with velocity 24.5 m/s. Find the time taken by the stone to reach the foot of the tower. **[7.11 seconds]**
24. A stone thrown vertically up from the top of a tower with a velocity 20m/s reaches the ground in 6 seconds. Find the height of the tower. **[56.4 m]**

25. A particle is projected vertically up from the foot of the tower 392 m high with velocity 98 m/s. At the same instant another body is dropped from the top. Find when and where they will meet. **[t=4 seconds; h=313.6 m above the foot of the tower]**
26. A ball is thrown vertically up with a speed of 14 m/s. 2 s later a second ball is dropped from the same point. Find when and where the two balls will meet. **[t=3.5 s; h=11 m]**
27. A particle moves in the *xy* plane such that, at time *t*, its displacement from a fixed point O is given by $\mathbf{r} = 4t\boldsymbol{i} + (3t-5t^2)\mathbf{j}$. Find the velocity vector at time *t* and hence find the velocity components when *t*=0. Show that the acceleration is constant. Do you notice anything significant about this acceleration? **[$\mathbf{v} = 4\boldsymbol{i} + 3\mathbf{j}$]**
28. A short fired horizontally from the top of a tower 176.4 m high hits the ground at a distance 1200 m from the foot of the tower. Find the velocity of projection. **[200 m/s]**
29. A short is fired from a gun on the top of a hill 90 m high with velocity 80 m/s at an angle 30^0 with the horizontal. Find its horizontal distance from the point of projection when it strikes the ground. **[$400\sqrt{3}$ m]**
30. You throw a rock off the cliff at 10 m/s, 20 ^{0} above the horizontal. The cliff is 25 m high. With what angle does it the rock hit the ground? **[67.2 ^{0}]**
31. You ride your bike over a ramp. The ramp is slanted at 28 ^{0}. You land 12 meters away. What was your velocity as you rode off the ramp? **[11.9 m/s]**
32. A projectile is launched on level ground. At the peak of its trajectory, it is 7.5 m high. It lands 10 m from where it was launched. At what angle was the projectile launched? **[71.6 ^{0}]**
33. A stone is thrown off a cliff. It hits the ground with an angle of 73.4 ^{0} and a speed of 65.7 m/s. If the cliff is 200 m high, how fast was the stone initially thrown? **[20.1 m/s]**
34. Prove that the time of flight T and the horizontal range R of a projectile are connected by the equation $gT^2 = 2R\tan\theta$
35. Figure (not drawn to scale) 3.12 shows a fighter plane that has a speed v_0 of 300 km/h, flying at an angle θ =15^0 below the horizontal when a decoy rocket is released. The horizontal distance between the release point and the point where the decoy strikes the ground is x =600 m. Take g = 10 m/s^2.
    (a) How long was the decoy in the air?
    (b) How high was the plane when the decoy was released? **[(a) 7.45 s, (b) 438.2m]**

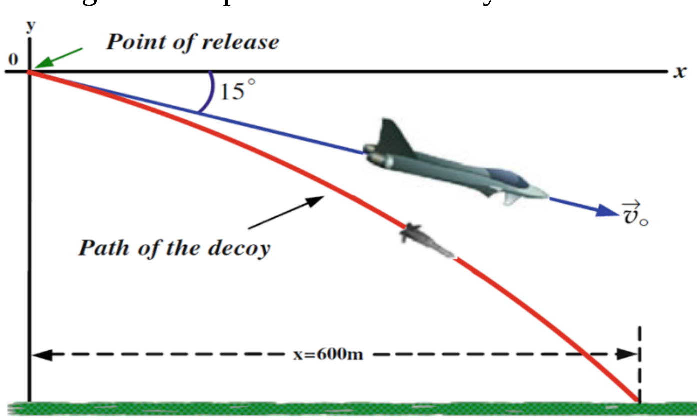

Figure 3.12 *See exercise 35*