MAT1100 LECTURE NOTES

2.1 FUNCTIONS

Relations

Definition 2.1.1 Let A and B be two sets. Then the product (or Cartesian) product of A and B, written $A \times B$ and read "A cross B", is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. i.e.

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

Example 2.1.1 Let $A = \{1,2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

$$\mathrm{B}\times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

$$A \times A = A^2 = \{(1,1), (1,2), (2,1), (2,2)\}.$$

NOTE: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of ordered pairs of real numbers.

Definition 2.1.2 Let A and B be two sets. Then a binary relation or, simply a relation from A to B is a subset of $A \times B$. i.e. R is a relation from A to B if it is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. i.e.

$$R = \{(a, b): a \in A, b \in B\}$$

When $(a, b) \in R$ we say a is R-related to b and we write aRb.

Example 2.1.2 Let $A = \{1,2,3\}$ and $B = \{x, y, z\}$ and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since

$$R \subseteq A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\}.$$

The set of all the first components of the ordered pairs is called the **domain** of the relation and the set of all the second components of the ordered pairs is called the **range** of the relation.

The domain of R in example 2.2.2 is $\{1,3\}$ and the range is $\{y, z\}$.

NOTE: The domain of R is a subset of A and the range of R is the subset of B.

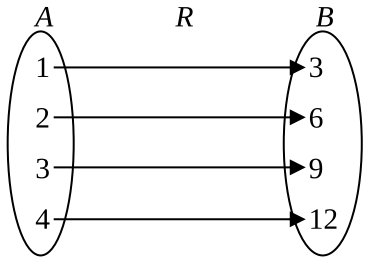
Relations can be defined by an equation or a rule or a table or an arrow diagram.

Example 2.1.3 Let the relation $R: A \to B$ be defined by

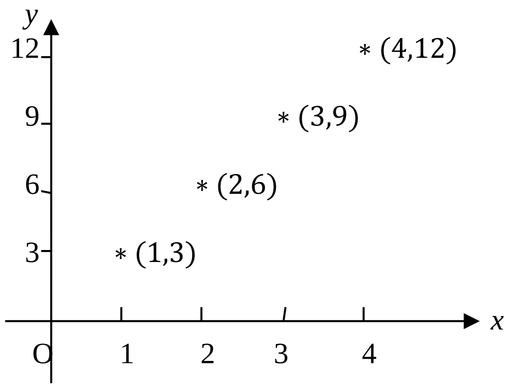
$$R = \{(1,3), (2,6), (3,9), (4,12)\}.$$

Then R can be defined by $y = 3x$ where $x \in A = \{1,2,3,4\}$ and $y \in B = \{3,6,9,12\}$.

It can also be defined using an arrow diagram



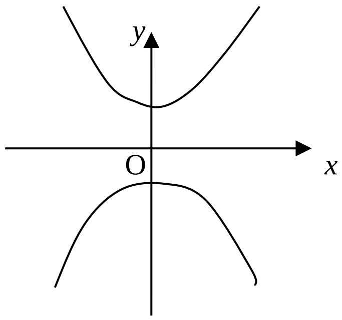
The relation can also be defined using the Cartesian coordinate system.

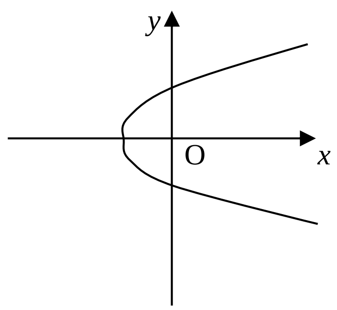


The domain of R is the set $\{1,2,3,4\}$ and its range is $\{3,6,9,12\}$.

Example 2.1.4 Find the domain and range of each relation whose defining rule and graph is given below:

(a) $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$ (b) $x = y^2 - 3$





Solution: (a) The domain of R is $\mathbb{R}$ and the range is $(-\infty, -4] \cup [4, \infty)$

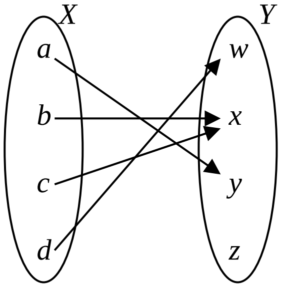
(b) The domain of $\mathbb{R}$ is $[-3, \infty)$ and the range is $\mathbb{R}$.

Functions

Definition 2.1.3 Let X and Y be two sets. Then a **function** f from X into Y is a rule that assigns each element $x \in X$ to unique (one and only one) element $y \in Y$. The notation for the function is $f: X \to Y$. This is read as f maps X into Y.

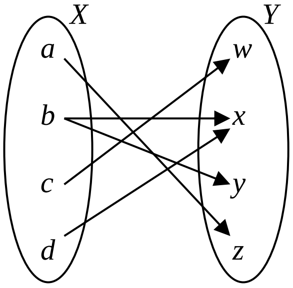
Example 2.5 Let $X = \{a, b, c, d\}$and $B = \{w, x, y, z\}$. Then the relation defined by the arrow diagram

(a)



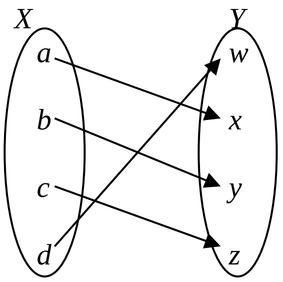
is a function since each element in X is related to only one element in Y

(b)



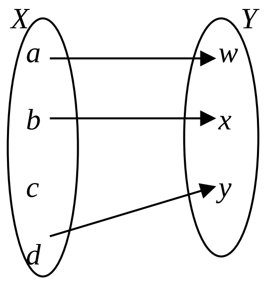
is not a function since there is an element b in X which is related to more than one element in Y.

(c)



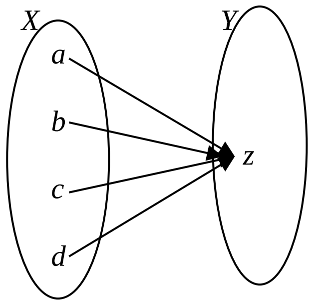
is a function since each element in X is related to only one element in Y

(d)



is not a function since there is an element c in X which is not related to any element in Y.

(e)



is a function since each element in X is related to only one element in Y.

For the function f the unique element $y \in Y$ related to $x \in X$ is called the **image** of x and it is written $f(x)$. The set of images is called the **range** of (or **image)** of f and is denoted by Ran(f) (or Im(f)). The **domain** of f is X. The elements of the domain corresponding to the images are called the **pre-images**. If X and Y are sets of real numbers, $f(x) \in \mathbb{R}$ and is the value of the function f at x.

NOTE: 1. To every function $f: X \to Y$ there corresponds the relation $\{(x, f(x)): x \in X\}$ in $A \times B$ i.e. $\{(x, f(x)): x \in X\} \subseteq A \times B$.

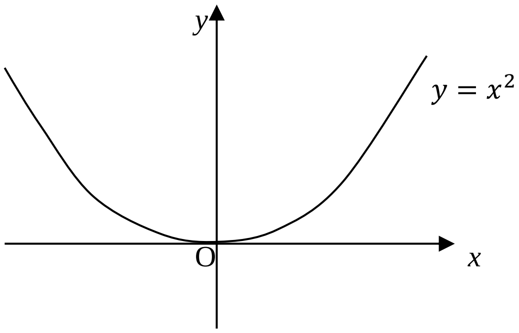
2. $f: X \to Y$ is a function if each $x \in X$ appears as the first coordinate in exactly one ordered pair (x, y) in f.

3. The range of f is denoted by $f(X)$ and is equal to $f(X) = \{f(x): x \in X\}$.

Example 2.1.6 Let $f: \mathbb{R} \to \mathbb{R}$ be a function which relates to each real number its square.

1. This function can be presented as an equation as: For each $\in \mathbb{R}, f(x) = x^2$. i.e. $\{(x, x^2): x \in \mathbb{R}\}$. It is said to be a real valued function.

2. The function $f(x) = x^2$ can also be represented as a graph as follows:

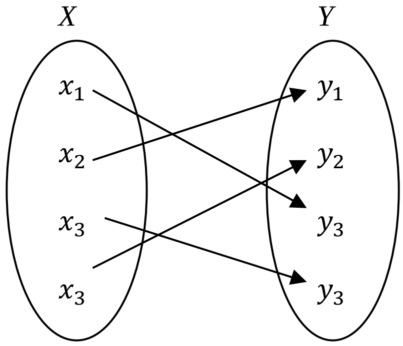


3. The domain of the function is $\mathbb{R}$ and its range is $f(\mathbb{R}) = \{x^2 : x \in \mathbb{R}\}$.

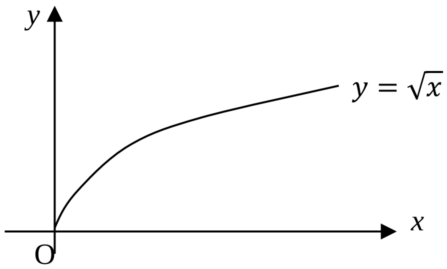
<u>**Definition**</u> **2.1.4** A function $f: X \to Y$ is said to be **one-to-one** (or one-one or 1-1) if each element in X corresponds to a distinct image in Y. i.e. f is one-to-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

<u>Example</u> 2.7 (a) The function $f: X \to Y$ defined by an arrow diagram shown below is one-one since there is a one to one correspondence between elements of set X and those of set Y.



(b) The function $f: [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ is one-one.



<u>Example</u> 2.1.8 Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is one to-to-one.

<u>Proof</u>: Let $x_1, x_2 \in \mathbb{R}$. We need to show that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Now,

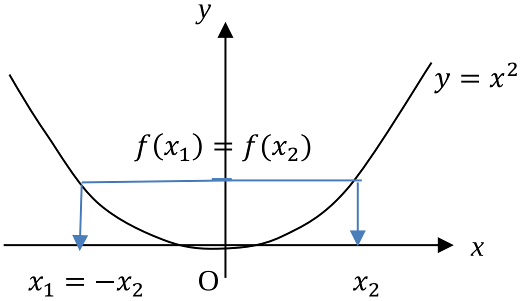
$$f(x_1) = f(x_2) \Rightarrow \sqrt{x_1} = \sqrt{x_2}$$

Squaring both sides we have

$$(\sqrt{x_1})^2 = (\sqrt{x_2})^2 \;\Rightarrow\; x_1 = x_2.$$

Hence, the function as defined is one-to-one.

Example 2.1.9 Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$



is not one-to-one.

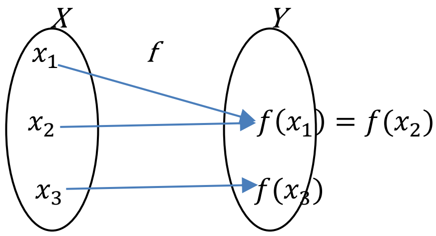
Proof: Let $x_1, x_2 \in \mathbb{R}$. Then

$$f(x_1) = f(x_2) \Rightarrow (x_1)^2 = (x_2)^2 \Rightarrow x_1 = \pm\sqrt{(x_2)^2} = \pm x_2.$$

i.e. $x_1 = +x_2$ and $x_1 = -x_2 \Rightarrow$ two different element in the domain are mapped to the same element in the range. Hence the function is not one-to-one.

Definition 2.1.5 A function $f: X \to Y$ is said to be many to one if there are at least two distinct elements $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$.

For example,



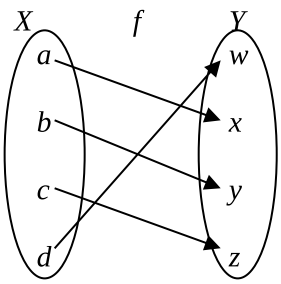
is a many to one function.

Definition 2.1.6 A function $f: X \to Y$ is said to be **onto** if every $y \in Y$ is the image of some $x \in X$, i.e. if $y \in Y \Rightarrow$ there exists $x \in X$ for which $f(x) = y$.

Note that if f is onto, then $f(X) = Y$.

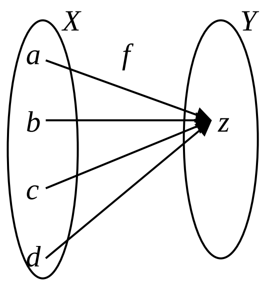
Example 2.1.10 The following functions as defined are onto functions:

(a)



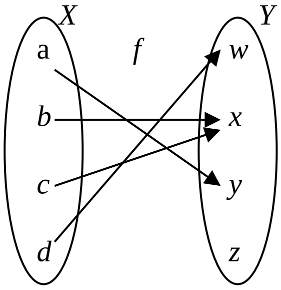
is an onto function since each element in Y is related to some element in X or $f(X) = Y$.

(b)



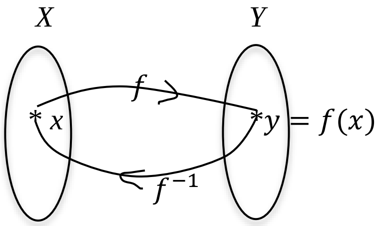
is an onto function since each element in Y is related to some element in X or $f(X) = Y$.

But the function defined below is not onto since there is an element in $z \in Y$ which is not related to any of the elements in *X*.



Inverse functions

The inverse of the function $f: X \to Y$ is the function which maps the elements of *Y* into the elements of *X* and it is denoted by $f^{-1}: Y \to X$, as shown in the arrow diagram below:



i.e. if $x \in X$, then $y = f(x) \in Y$, and for $y \in Y$, $f^{-1}(y) = f^{-1}\big(f(x)\big) = x \in X$.

NOTE: 1. A function f has an inverse function f^{-1} if and only if it is one-to-one and onto, and

2. The domain of the inverse function f^{-1} is the range of f and the range of f^{-1} is the domain of f.

To find the inverse of a given function $y = f(x)$, interchange *x* and *y* so that $x = f(y)$, and change the subject of the formula back to *y* and obtain $y = f^{-1}(x)$.

<u>Example</u> 2.1.11 Find the inverse of the function

$$f(x) = \frac{2-x}{3x+2},\ x \neq -\frac{2}{3}.$$

<u>Solution</u>: Let $y = \frac{2-x}{3x+2}$. Then interchange *x* and *y* to obtain $x = \frac{2-y}{3y+2}$.

Make *y* the subject of the formula:

$$x(3y+2) = 2 - y$$
$$3xy + 2x = 2 - y$$
$$3xy + y = 2 - 2x$$
$$y(3x+1) = 2 - 2x$$
$$y = \frac{2-2x}{3x+1}$$

Therefore, $$f^{-1}(x) = \frac{2-2x}{3x+1},\ x \neq -\frac{1}{3}.$$

NOTE: The domain of the inverse function $f^{-1}$of f in Example 2.11 is

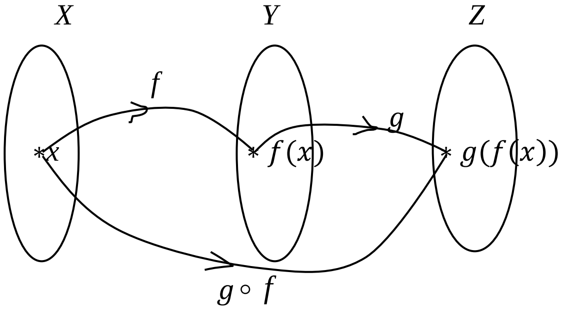
$$D_{f^{-1}} = \left\{x \in \mathbb{R}: x \neq -\frac{1}{3}\right\}.$$

Composite functions

<u>Definition</u> 2.7 Consider functions $f: X \to Y$ and $g: Y \to Z$ i.e. where the range of the of *f* is the domain of g. Pictorially is shown below:

$$X \xrightarrow{f} Y \xrightarrow{g} Z\,.$$

or



Let $x \in X$. Then the image of *x* under *f* is $f(x) \in Y$ (the domain of g). Accordingly, we can find the image of $f(x)$ under g, which is $g\big(f(x)\big) \in Z$. Thus the rule which assigns each element *x* in *X* an element $g(f(x))$ in *Z* is called **the composition function** of *f* and g, and it is denoted by $g \circ f$. Briefly, $g \circ f : X \to Z$ and it is defined by

$$(g \circ f)(x) = g[f(x)]$$

The function $f \circ g$ is defined by

$$(f \circ g)(x) = f[g(x)]$$

<u>Example</u> 2.1.12 Let the function *f* be defined by $f(x) = 3x - 5$ and the function g by $g(x) = x^2$. Find (a) $(g \circ f)(x)$ (b) $(g \circ f)(-2)$ (c) $(f \circ g)(x)$ (d) $(f \circ g)(-2)$.

<u>Solutions</u>:

(a) $(gof)(x) = g[f(x)] = g(3x - 5) = (3x - 5)^2$

(b) $(gof)(-2) = (3(-2) - 5)^2 = (-11)^2 = 121$

(c) $(fog)(x) = f[g(x)] = f(x^2) = 3(x^2) - 5 = 3x^2 - 5$

(d) $(fog)(-2) = 3(-2)^2 - 5 = 12 - 5 = 7$

Note that $(g \circ f)(x) \neq (f \circ g)(x)$ i.e. the composition of function is not commutative.

<u>Exercise</u> Let $f(x) = 3x - 4$. Show that

(a) $(f \circ f^{-1})(x) = x$

(b) $(f^{-1} \circ f)(x) = x$

In general for all functions *f*, $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x) = x$.

The composition of functions can be extended to a composite of more than two functions.

For example, if $f: W \to X$, $g: X \to Y$ and $h: Y \to Z$, then $h \circ (g \circ f)$ is defined by

$$h \circ (g \circ f)(x) = h\{g[f(x)]\}.$$

Example 2.1.13 Let the function f be defined by $f(x) = 5 - 3x$, the function g by $g(x) = x + 2$ and h by $h(x) = 2x^2$. Find (a) $[h \circ (g \circ f)](x)$ (b) $[(h \circ g) \circ f](x)$.

Solutions:

(a) $(gof)(x) = g[f(x)] = (5 - 3x) + 2 = 7 - 3x$

$[ho(gof)](x) = h[(gof)(x)] = h(7 - 3x) = 2(7 - 3x)^2$

(b) $(hog)(x) = h[g(x)] = h(x + 2) = 2(x + 2)^2$

$[(hog)of](x) = (hog)[f(x)] = (hog)[5 - 3x] = [2(5 - 3x + 2)^2]$

$= 2(7 - 3x)^2.$

Note that $h \circ (g \circ f)(x) = [(h \circ g) \circ f](x)$.

Domain of a composite function

Example 2.1.14 Let $f(x) = \frac{x-3}{x}$ and $g(x) = x + \frac{x-4}{x-1}$. Find the domain of the following composite functions:

(i) fog (ii) gof.

Solution: (i)

$$(fog)(x) = f[g(x)] = f\left(x + \frac{x-4}{x-1}\right) = \frac{\left(x + \frac{x-4}{x-1}\right) - 3}{x + \frac{x-4}{x-1}}$$

$$= \frac{x^2 - x + x - 4 - 3(x-1)}{x^2 - x + x - 4}$$

$$= \frac{x^2 - 3x - 1}{x^2 - 4}$$

$$= \frac{x^2 - 4x + 2}{(x+2)(x-2)}$$

Now, fog is not defined at $x = -2, x = 2$ and $g(x)$ is not defined at $x = 1$. This means that the domain of fog is $\{x \in \mathbb{R}: x \neq -2,1,2\}$.

(ii) $$(gof)(x) = g[f(x)] = g\left(\frac{x-3}{x}\right) = \frac{x-3}{x} + \frac{\left(\frac{x-3}{x}\right) - 4}{\left(\frac{x-3}{x}\right) - 1}$$

$$= \frac{x-3}{x} + \frac{\frac{x-3-4x}{x}}{\frac{x-3-x}{x}} = \frac{x-3}{x} + \frac{3x+3}{3}$$

$$= \frac{x^2+2x-3}{x}$$

Now, gof is not defined at $x = 0$, and $f(x)$ is also not defined at $x = 0$. Therefore, the domain of gof is $\{x \in \mathbb{R} : x \neq 0\}$.

<u>**Definition**</u> **2.1.8** A **peicewise** function is a function defined by at least two equations each of which applies to a different part of the domain.

Piecewise defined functions can take on a variety of forms. Their pieces may be all linear or a combination of functional forms (such as constant, linear, quadratic, cubic, square roots, cube roots etc.).

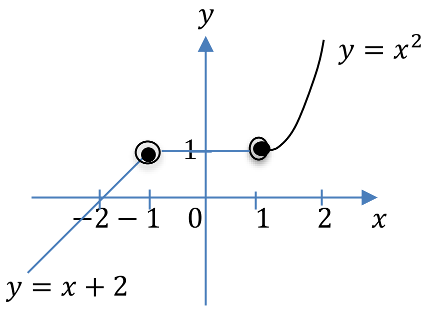
<u>Example</u> 2.1.15 For each of the following functions sketch and find

(a) $f(-1)$ (b) $f(0)$ (c) $f(3)$ (d) its domain and range:

1. $f(x) = \begin{cases} x+2, & x \leq -1 \\ 1, & -1 < x < 1 \\ x^2, & x \geq 1 \end{cases}$

2. $f(x) = \begin{cases} 0, & -5 \leq x < -2 \\ -x^2+4, & -2 \leq x \leq 1 \\ -x+3, & 2 < x \leq 5 \end{cases}$

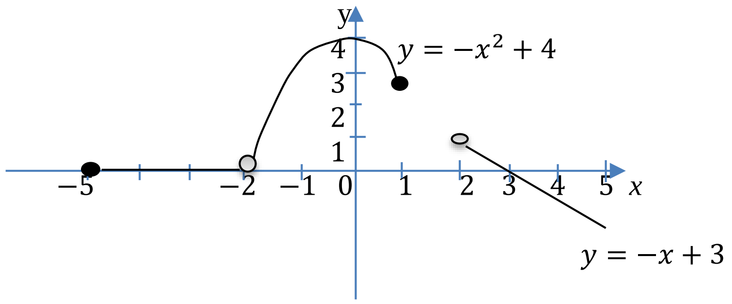
<u>Solution</u>: 1.



(a) $f(-1) = -1 + 2 = 1$ (b) $f(0) = 1$ (c) $f(3) = (3)^2 = 9$

(d) Domain of $f = \mathbb{R}$, Range of $f = \mathbb{R}$.

2.



(a) $f(-1) = -(-1)^2 + 4 = -1 + 4 = 3$

(b) $f(0) = -(0)^2 + 4 = 0 + 4 = 4$

(c) $f(3) = -3 + 3 = 0$

(d) Domain of $f = (-\infty, 1] \cup (2, \infty)$, Range of $f = (-\infty, 4]$.

<u>Definition</u> 2.1.9 Let $f: X \to Y$ be a function. Then *f* is said to be an **even** function if for each $x \in X$, $f(-x) = f(x)$.

<u>Example</u> 2.1.16 Show that the function $f(x) = 3x^2 - 4x^4$ is even.

<u>Solution</u>: $f(-x) = 3(-x)^2 - 4(-x)^4 = 3x^2 - 4x^4 = f(x)$. Since $f(-x) = f(x)$, *f* is an even function.

<u>Definition</u> 2.1.10 Let $f: X \to Y$ be a function. Then *f* is said to be an **odd** function if for each $x \in X$, $f(-x) = -f(x)$.

<u>Example</u> 2.1.17 Show that the function $f(x) = 6x^3 - 5x$ is odd.

<u>Solution</u>: $f(-x) = 6(-x)^3 - 5(-x) = -6x^3 + 5x = -(6x^3 - 5x) = -f(x)$. Since $f(-x) = -f(x)$, *f* is an odd function.

However, a function which is not even may not necessarily be odd and a function which is not odd may not necessarily be even. Some functions are neither even nor odd.

<u>Example</u> 2.18 Determine whether the function $f(x) = x^3 - 7x^2$ is even or odd or neither even nor odd.

<u>Solution</u>: $f(-x) = (-x)^3 - 7(-x)^2 = -x^3 + 7x^2 = -(x^3 - 7x^2) \neq -f(x)\ or\ f(x)$. Therefore, *f* is neither even nor odd.

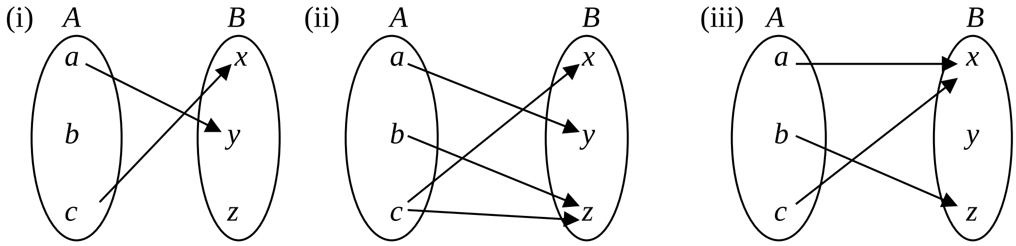
TUTORIAL SHEET 3

1. Which of the following sets of ordered pairs represent a functions:

   (a) $\{(1,4),(3,4),(7,3)\}$ (b) $\{(1,2),(1,3),(2,3)\}$ (c) $\{(4,3),(4,7),(3,4)\}$

   (d) $\{(1,2),(2,3),(3,4)\}$

2. State whether or not each of the diagrams defines a function from $A=\{a,b,c\}$ into $B=\{x,y,z\}$.



3. Let $X=\{1,2,3,4\}$ and $Y=\{1,2,3,4\}$. Illustrate each of the following in an arrow diagram and state whether or not each relation from X into Y is a function:

   (a) $f=\{(2,3),(1,4),(2,1),(3,2),(4,4)\}$ (b) $g=\{(3,1),(4,2),(1,1)\}$

   (c) $h=\{(2,1),(3,4),(1,4),(2,1),(4,4)\}$.

4. Verify that the two given functions are inverses of each other.

   (a) $f(x)=5x-9, \quad and \quad g(x)=\dfrac{x+9}{5}$

   (b) $f(x)=x^3+1 \quad and \quad g(x)=\sqrt[3]{x-1}$

   (c) $f(x)=\dfrac{1}{x-1}$ $for\ x>1 \quad and \quad g(x)=\dfrac{x+1}{x}$ $for\ x>0$.

5. Let f and g be two functions. Find $(f\circ g)(x)$ and $(g\circ f)(x)$. Also specify the domain for each.

   (a) $f(x)=3x+4,\ g(x)=x^2+1$ (b) $f(x)=2x^2-x-1,\ g(x)=x+4$

   (c) $f(x)=\sqrt{x-2},\ g(x)=3x-1$ (d) $f(x)=\dfrac{1}{x-1},\ g(x)=\dfrac{2}{x}$.

6. If $f(x)=\sqrt{x}, \quad g(x)=3x-1$, find $(f\circ g)(4)$ and $(g\circ f)(4)$.

7. For each given function, find f^{-1} and verify that $(f\circ f^{-1})(x)=x$ and $(f^{-1}\circ f)(x)=x$: (a) $f(x)=\dfrac{2}{x-1}$ for $x>1$ (b) $f(x)=\dfrac{1-x}{x}$

8. If $f(x)=2x+3$ and $g(x)=3x-5$, find

   (a) $(f\circ g)^{-1}(x)$ (b) $(f^{-1}\circ g^{-1})(x)$ (c) $(g^{-1}\circ g^{-1})(x)$.

9. Let the function be defined by $f(x)=\begin{cases} 2 & for \quad x<0 \\ x^2+1 & for\ 0\le x\le 4 \\ -1 & for \quad x>4 \end{cases}$.

Compute $f(3), f(6)$ and $f(-3)$ and sketch the graph of the function.

10. Determine which of the following function are even or odd or neither even nor odd.

(a) $f(x) = 4x - 7x^3$ (b) $f(x) = 3 + 5x - x^2$ (c) $f(x) = 5x^2 - 2x^4$

11. Prove that each of the following functions is one – to – one:

(a) $f(x) = 3 - 4x$ (b) $f(x) = \frac{x+2}{3x}$.

2.2 Linear and Quadratic Functions

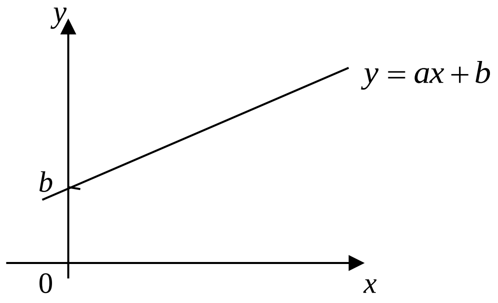
Linear Functions

<u>Definition</u> 2.2.1 A function is of the form

$$f(x) = ax + b$$

where *a* and *b* are constants is called a *linear function*.

The graph of a linear function is simply a straight line.



In the Cartesian plane, the constant *a* is the *gradient* or *slope* of the straight line and *b* is the *y*- intercept.

Quadratic Function

<u>Definition</u> 2.2.2 A quadratic function is of the form

$$f(x) = ax^2 + bx + c\ ,$$

where *a*, *b* and *c* are constants and $a \neq 0$.

Note that when $a = 0$, the function becomes a linear function.

A quadratic function can also be expressed in the form

$$f(x) = a(x + p)^2 + q\ ,$$

where *a*, *p* and *q* are constants. This is done by ***completing the square***.

<u>Example</u> 2.2.1 $f(x) = ax^2 + bx + c$

$$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$, by factoring out the coefficient of x^2.

Dividing the coefficient of *x* by 2 and squaring the result we write the expression in the form

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

Now the expression $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$, is a perfect square. Therefore

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

$$= a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a},$$

in which $p = \frac{b}{2a}$ and $q = \frac{4ac-b^2}{4a}$.

Example 2.2.2 Complete the square of each of the quadratic functions:

(a) $f(x) = 2x^2 - 4x + 5 = 2\left[x^2 - 2x + \frac{5}{2}\right] = 2\left[x^2 - 2x + (-1)^2 - (-1)^2 + \frac{5}{2}\right]$
$= 2\left[x^2 - 2x + 1 - 1 + \frac{5}{2}\right] = 2\left[(x-1)^2 - 1 + \frac{5}{2}\right] = 2\left[(x-1)^2 + \frac{3}{2}\right]$
$= 2(x-1)^2 + 3$

(b) $f(x) = 3 - 5x - x^2 = -x^2 - 5x + 3 = -[x^2 + 5x - 3] = -\left[x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 3\right]$

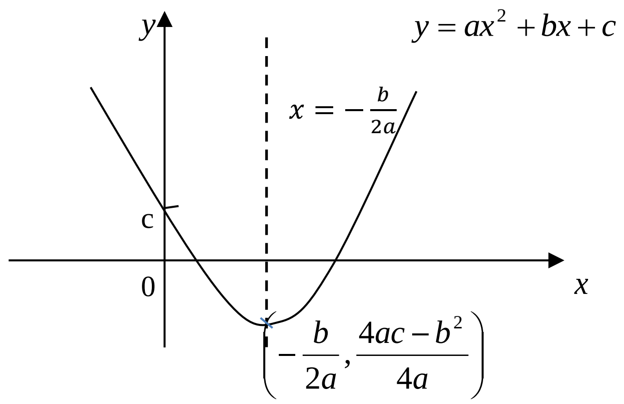
$$= -\left[\left(x+\frac{5}{2}\right)^2 - \frac{25}{4} - 3\right] = -\left[\left(x+\frac{5}{2}\right)^2 - \frac{37}{4}\right] = -\left(x+\frac{5}{2}\right)^2 + \frac{37}{4}$$

Graph of a Quadratic Function

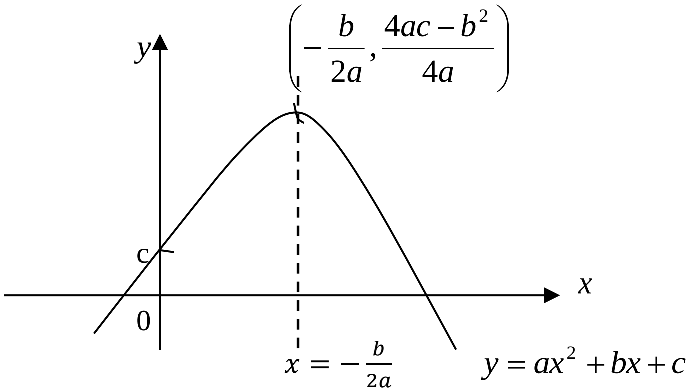
We consider an arbitrary function

$$f(x) = ax^2 + bx + c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}.$$

(a) If $a > 0$, the graph of the quadratic function opens upward and has a minimum turning point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.



(b) If $a < 0$, the graph of the quadratic function opens downward and has a maximum turning point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.



Note that in both cases, the turning point is given by $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ and the y - intercept is c.

(c) The equation of the line of symmetry of the graph of a quadratic function is $x = -\frac{b}{2a}$.

If the graph of the quadratic function cuts the x – axis, the x - intercepts are found by solving the quadratic equation $f(x) = 0$ i.e.

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} = 0\ .$$

Thus, $$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a}$$

$\Rightarrow$ $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$\Rightarrow$ $$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\ .$$

This is the ***quadratic formula*** used in finding the solutions or roots of a quadratic equation

$$ax^2 + bx + c = 0.$$

One x - intercept is $x = \frac{-b + \sqrt{b^2-4ac}}{2a}$ and the other is $x = \frac{-b - \sqrt{b^2-4ac}}{2a}$.

Example 2.2.3 Complete the square of each of the following quadratic functions. Hence sketch its graph indicating the turning point and the intercepts, and write down the equation of its line of symmetry.

1. $f(x) = 2x^2 + x - 10$ 2. $f(x) = 3 + 5x^2 - 2x^2$.

Solutions:

1. $f(x) = 2x^2 + x - 10 = 2\left(x^2 + \dfrac{1}{2}x - 5\right) = 2\left(x^2 + \dfrac{1}{2}x + \left(\dfrac{1}{4}\right)^2 - \left(\dfrac{1}{4}\right)^2 - 5\right)$

$$= 2\left(\left(x + \tfrac{1}{4}\right)^2 - \tfrac{1}{16} - 5\right) = 2\left(\left(x + \tfrac{1}{4}\right)^2 - \tfrac{81}{16}\right) = 2\left(x + \tfrac{1}{4}\right)^2 - \tfrac{81}{8}.$$

Since $a > 0$, the function has a *minimum turning point* and it occurs at point $\left(-\tfrac{1}{4}, -\tfrac{81}{8}\right)$.

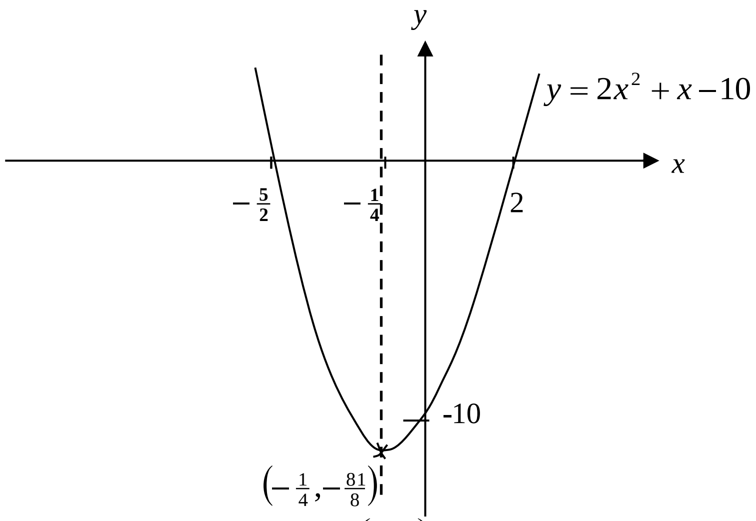
The *x*-intercepts are

$$x = \frac{-1 + \sqrt{1^2 - 4(2)(-10)}}{2(2)} \text{ and } x = \frac{-1 - \sqrt{1^2 - 4(2)(-10)}}{2(2)}$$

i.e. $x = \dfrac{-1 + \sqrt{81}}{4}$ and $x = \dfrac{-1 - \sqrt{81}}{4}$

i.e. $x = \dfrac{8}{4} = 2$ and $x = \dfrac{-10}{4} = \dfrac{-5}{2}$.

The *y* – intercept is the term independent of *x* in the quadratic equation, which in this case is -10.



The minimum value of the function is $f\left(-\dfrac{1}{4}\right) = -\dfrac{81}{8}$ and the line of symmetry is $x = -\tfrac{1}{4}$.

2. $f(x) = 3 + 5x - 2x^2 = -2\left(x^2 - \tfrac{5}{2}x - \tfrac{3}{2}\right) = -2\left(x^2 - \tfrac{5}{2}x + \left(-\tfrac{5}{4}\right)^2 - \left(-\tfrac{5}{4}\right)^2 - \tfrac{3}{2}\right)$

$$= -2\left(\left(x - \tfrac{5}{4}\right)^2 - \tfrac{25}{16} - \tfrac{3}{2}\right) = -2\left(\left(x - \tfrac{5}{4}\right)^2 - \tfrac{49}{16}\right) = -2\left(x - \tfrac{5}{4}\right)^2 + \tfrac{49}{8}.$$

The *x*-intercepts are

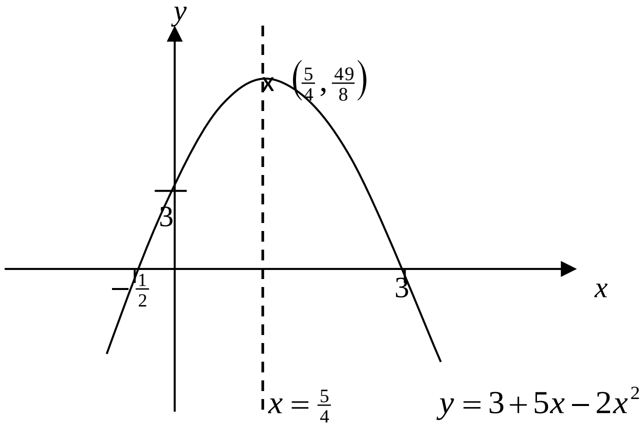
$$x = \frac{-5 + \sqrt{5^2 - 4(-2)(3)}}{2(-2)} \text{ and } x = \frac{-5 - \sqrt{1^2 - 4(-2)(3)}}{2(-2)}$$

i.e. $\qquad x = \dfrac{-5+\sqrt{49}}{-4}$ and $x = \dfrac{-5-\sqrt{49}}{-4}$

i.e. $\qquad x = \dfrac{2}{4} = -\dfrac{1}{2}$ and $x = \dfrac{-12}{4} = 3$.

Since $a < 0$, the function has a *maximum turning point* and it occurs at $\left(\frac{5}{4}, \frac{49}{8}\right)$.

The *y* – intercept is 3.



The maximum value of the function is $f\left(\dfrac{5}{4}\right) = \dfrac{49}{8}$ and the line of symmetry is $x = \frac{5}{4}$.

Applications of Quadratic Functions.

One method of solving a maximum or minimum problems which can be transformed into a quadratic function is the use of completion of the square.

<u>Example</u> 2.2.4. If the selling price *x* of an item is related to the profit *P* by the equation

$$P = 1000x - 25x^2$$

Determine the value of *x* that would yield maximum profit and state the maximum profit.

<u>Solution</u>: To find the value of *x* that that would yield maximum profit we have to use the method of completing the square.

$$\begin{aligned} P = 1000x - 25x^2 &= -25\left(x^2 - 40x\right) \\ &= -25\left(x^2 - 40x + (-20)^2 - (-20)^2\right) \\ &= -25\left(x^2 - 40x + (-20)^2 - 400\right) \\ &= -25\left((x-20)^2 - 400\right) \\ &= -25(x-20)^2 + 10000 \end{aligned}$$

The maximum profit is attained when $x = 20$ and the maximum profit is $10\,000$.

2. A farmer wishes to enclose a rectangular lot of maximum area with a fence 400 m long. Find the dimensions of the rectangle and state its maximum area.

Solution: Suppose the length of the rectangle is *x* and the width is *y*. Then the perimeter of the rectangle is

$$2x + 2y = 400$$

$$\Rightarrow \qquad x + y = 200 \ \Rightarrow y = 200 - x$$

The area of the rectangle is

$$A = xy$$

$$\Rightarrow \qquad A = x(200 - x) = 200x - x^2$$

This is a quadratic function

$$\begin{aligned} A(x) &= 200x - x^2 \\ &= -(x^2 - 200x) \\ &= -(x^2 - 200x + (-100)^2 - (-100)^2) \\ &= -((x-100)^2 - 10000) \\ &= -(x-100)^2 + 10000 \end{aligned}$$

This means that the maximum area of the rectangle is attained at $x = 100$.
Therefore, the dimensions of the rectangle are $length = 100\ m\ and\ width = 100\ m$ and hence, the maximum area is $10\,000\,m^2$.

2.3 Polynomail Functions

Let *n* be a nonnegative integer and let $a_0, a_1, a_2, \cdots, a_n$ be real numbers with $a_n \neq 0$, then the function defined by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a *polynomial function* of degree *n* (or simply a *polynomial*). The numbers $a_0, a_1, a_2, \cdots, a_n$ are called the *coefficients* and a_n the leading coefficient of the function p.

We have already encountered some special polynomials like the linear function $p(x) = a_1 x + a_0$, the quadratic function $p(x) = a_2 x^2 + a_1 x + a_0$. The constant function is defined by $p(x) = a_0$. A constant function is of the degree 0, a linear function is of degree 1, and a quadratic function is of degree 2.

Note that no degree is assigned to a zero function $p(x) = 0$.

Polynomials may be added, subtracted, multiplied, or divided. Thus if *p* and *q* are polynomials of degree *m* and *n* respectively, then

(i) $p \pm q$ is a polynomial of degree less than or equal to the maximum of *m* and *n*.

(ii) $p \cdot q$ is a polynomial of degree $m + n$.

Example 2.3.1 Let $p(x) = x^3 - 3x^2 + 5$ and $q(x) = x^3 + 2x^2 - x + 3$. Then

(i) $p(x) + q(x) = \left(x^3 - 3x^2 + 5\right) + \left(x^3 + 2x^2 - x + 3\right)$

$$= (x^3 + x^3) + (-3x^2 + 2x^2) + (-x) + (5 + 3)$$
$$= 2x^3 - x^2 - x + 8,$$
a polynomial of degree 3.

(ii) $p(x) - q(x) = (x^3 - 3x^2 + 5) - (x^3 + 2x^2 - x + 3)$
$$= (x^3 - x^3) + (-3x^2 - 2x^2) + (0x - (-x)) + (5 - 3)$$
$$= -5x^2 + x + 2,$$
a polynomial of degree 2.

(iii) $p(x) \cdot q(x) = \left(x^3 - 3x^2 + 5\right) \cdot \left(x^3 + 2x^2 - x + 3\right)$
$$= x^3\left(x^3 + 2x^2 - x + 3\right) - 3x^2\left(x^3 + 2x^2 - x + 3\right) + 5\left(x^3 + 2x^2 - x + 3\right)$$
$$= x^6 + 2x^5 - x^4 + 3x^3 - 3x^5 - 6x^4 + 3x^3 - 9x^2 + 5x^3 + 10x^2 - 5x + 15$$
$$= x^6 + (2x^5 - 3x^5) + (-x^4 - 6x^4) + (3x^3 + 5x^3) + (-9x^2 + 10x^2) + (-5x) + 15$$
$$= x^6 - x^5 - 7x^4 + 11x^3 + x^2 - 5x + 15,$$
a polynomial of degree 6.

The concept of division involving polynomials is quite similar to that of integers. Thus, if *p* and *h* are polynomials, then *p* is divisible by *h* if and only if there is a polynomial *q* such that

$$\frac{p}{h} = q.$$

i.e.
$$\frac{p(x)}{h(x)} = q(x)$$

or
$$p(x) = q(x)h(x).$$

<u>Example</u> 2.3.2 Let $p(x) = x^3 - 3x^2 + 5x - 6$ and $h(x) = x - 2$ be two polynomials. Then p is divisible by *h* if and only if there exist a polynomial $q(x) = x^2 - x + 3$ such that

$$\frac{p(x)}{h(x)} = q(x)$$

i.e.
$$\frac{x^3 - 3x^2 + 5x - 6}{x - 2} = x^2 - x + 3.$$

<u>**Theorem**</u> 2.3.1 If *p* and *h* are polynomials and *h* is of degree greater than zero, then there exists unique polynomials *q* and *r* such that

$$\frac{p(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)},$$

or
$$p(x) = q(x)h(x) + r(x),$$

where *r* is either a polynomial of degree less than the degree of *h* or the zero function.

The polynomial *p* is called the *dividend*, *h* is the *divisor*, *q* is the *quotient*, and *r* is the *remainder*.

Long Division of Polynomials

<u>Examples</u> 2.3.3

1. Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$

2. Divide $12x^3 - 6x^2 + 10$ by $2x + 1$

<u>Solutions</u>: 1.

$$
\begin{array}{r|l}
 & \quad 2x^2 \qquad\qquad +1 \\
\hline
x^2+2x-3 & 2x^4 + 4x^3 - 5x^2 + 3x - 2 \\
 & -(2x^4 + 4x^3 - 6x^2) \\
 & \qquad\qquad\quad x^2 + 3x - 2 \\
 & \qquad\qquad -(x^2 + 2x - 3) \\
 & \qquad\qquad\qquad\quad x + 1
\end{array}
$$

Therefore,

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x+1}{x^2 + 2x - 3}$$

The quotient $q(x) = 2x^2 + 1$ and the remainder $r(x) = x + 1$.

2.

$$
\begin{array}{r|l}
 & \quad 6x^2 - 6x + 3 \\
\hline
2x+1 & 12x^3 - 6x^2 + 0x + 10 \\
 & -(12x^3 + 6x^2) \\
 & \qquad -12x^2 + 0x \\
 & \quad -(-12x^2 - 6x) \\
 & \qquad\qquad\quad 6x + 10 \\
 & \qquad\qquad -(6x + 3) \\
 & \qquad\qquad\qquad\quad 7
\end{array}
$$

Therefore,

$$\frac{12x^3 - 6x^2 + 10}{2x+1} = 6x^2 - 6x + 3 + \frac{7}{2x+1}$$

The quotient $q(x) = 6x^2 - 6x + 3$ and the remainder $r(x) = 7$

Synthetic Division

There is a shortcut called synthetic division for long division of polynomials when dividing by divisors of the form $x - k$. The procedure is given below:

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following procedure

$$
\begin{array}{r|llll}
k & a & b & c & d \\
Add & & ka & kb + k^2a & kc + k^2b + k^3a \\
\hline
 & \underbrace{a \quad b + ka \quad c + kb + k^2a}_{} & & & d + ck + k^2b + k^3a = remainder
\end{array}
$$

coefficients of quotient $q(x)$ whose degree
is 1 less than that of that of the dividend.

Vertical pattern: Add terms in columns
Diagonal pattern: Multiply results by k

Hence

$$q(x) = ax^2 + (b + ka)x + (c + kb + k^2a)$$

and

$$r(x) = d + ck + k^2b + k^3a$$

Therefore,

$$\frac{ax^3 + bx^2 + cx + d}{x - k} = ax^2 + (b + ka)x + (c + kb + k^2a) + \frac{d + ck + k^2b + k^3a}{x - k}$$

Example 2.3.4 Use synthetic division to divide each of the following polynomials:

1. $2x^3 - 3x^2 + 4x + 5$ by $x - 2$
2. $x^4 + 3x^3 + x^2 - 2x - 6$ by $x + 3$

Solutions: 1.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & 5 \\ Add & \downarrow & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & 17 = r \end{array}$$

$$q(x) = 2x^2 + x + 6$$
$$r(x) = 17$$

Therefore,

$$\frac{2x^3 - 3x^2 + 4x + 5}{x - 2} = 2x^2 + x + 6 + \frac{17}{x - 2}$$

2.

$$\begin{array}{r|rrrrr} -3 & 1 & 3 & 1 & -2 & -6 \\ Add & & -3 & 0 & -3 & 15 \\ \hline & 1 & 0 & 1 & -5 & 9 = r \end{array}$$

Here the quotient is

$q(x) = x^3 + 0x^2 + x - 5$ i.e.

$$q(x) = x^3 + x - 5$$

$$r(x) = 9$$

Therefore,

$$\frac{x^4 + 3x^3 + x^2 - 2x - 6}{x + 3} = x^3 + x - 5 + \frac{9}{x + 3}$$

We have noted that when a polynomial $p(x)$ of degree n is divided by $(x - k)$ then there exists another polynomial $q(x)$ of degree $n - 1$ such that

$$p(x) = q(x)(x - k) + r,$$

for all x, where r is the remainder.

Now note that

$$p(k) = q(k)(k-k) + r = r \ ,$$

which is the remainder. This leads us to the remainder theorem.

<u>Theorem</u> 2.3.2 (Remainder theorem) If the polynomial $p(x)$ is divided by $(x-k)$ then the remainder is

$$p(k) = r \ .$$

<u>Example</u> 2.3.5 Use the remainder theorem to find the remainder when the polynomial $p(x)$ is divided by $(x-k)$:

1. $2x^3 - 3x^2 + 4x + 5$ by $x - 2$
2. $x^4 + 3x^3 + x^2 - 2x - 6$ by $x + 3$

<u>Solutions</u>: 1. Let $p(x) = 2x^3 - 3x^2 + 4x + 5$. Then, by the remainder theorem, the remainder is

$$p(2) = 2(2)^3 - 3(2)^2 + 4(2) + 5 = 16 - 12 + 8 + 5 = 17 = r \ ,$$

2. Let $p(x) = x^4 + 3x^3 + x^2 - 2x - 6$. Then the remainder is

$$\begin{aligned} p(-3) &= (-3)^4 + 3(-3)^3 + (-3)^2 - 2(-3) - 6 \\ &= 81 - 81 + 9 + 6 - 6 = 9 = r \ . \end{aligned}$$

When a polynomial $p(x)$ is divided by $(x-k)$ and the remainder is zero, i.e. $p(k) = 0$, we say that $p(x)$ is divisible by $(x-k)$ or $(x-k)$ is a factor of $p(x)$. This leads us to the Factor theorem:

<u>Theorem</u> 2.3.3 (Factor theorem) If $p(x)$ is a polynomial and *k* a real number such that $p(k) = 0$, then $(x-k)$ is a factor of $p(x)$.

Note: If $(x-k)$ is a factor of $p(x)$, then

$$p(x) = q(x)(x-k)$$

and the remainder $r = 0$.

<u>Example</u> 2.3.6 Show that $(x-k)$ is a factor of the given polynomial $p(x)$:

1. $p(x) = 2x^3 - x^2 - 4x + 3$; $x - 1$
2. $p(x) = x^4 + 2x^3 - x^2 - x + 2$; $x + 2$

<u>Solutions</u>: 1. Let $p(x) = 2x^3 - x^2 - 4x + 3$ .Then

$$p(1) = 2(1)^3 - (1)^2 + 4(1) + 3 = 2 - 1 - 4 + 3 = 0 \ .$$

By the factor theorem, $(x-1)$ if a factor of $2x^3 - x^2 - 4x + 3$.

Note: Dividing using synthetic division, we have

$$\begin{array}{r|rrrr} 1 & 2 & -1 & -4 & 3 \\ Add & & 2 & 1 & -3 \\ \hline & 2 & 1 & -3 & 0 = r \end{array}$$

$$2x^3 - x^2 - 4x + 3 = (x-1)(2x^2 + x - 3)$$

2. Let $p(x) = x^4 + 2x^3 - x^2 - x + 2$. Then

$$p(-2) = (-2)^4 + 2(-2)^3 - (-2)^2 - (-2) + 2$$
$$= 16 - 16 - 4 + 2 + 2 = 0$$

By the factor theorem, $(x+2)$ if a factor of $x^4 + 2x^3 - x^2 - x + 2$.

Dividing using synthetic division, we have

$$\begin{array}{r|rrrrr} -2 & 1 & 2 & -1 & -1 & 2 \\ Add & & -2 & 0 & 2 & -2 \\ \hline & 1 & 0 & -1 & 1 & 0 = r \end{array}$$

$$x^4 + 2x^3 - x^2 - x + 2 = (x+2)(x^3 - x + 1)$$

Zeros or Roots of a Polynomial

We have seen from the factor theorem that if *p* is a polynomial of degree $n \geq 1$ and *k* is a number, then $p(k) = 0$ implies that $x - k$ is a factor of *p*. The number *k* is called a zero (or root) of *p*. Geometrically, *k* represents the point where the graph of *p* intersects the *x* -axis.

Clearly, since a polynomial *p* of degree *n* cannot have more than *n* factors, then *p* has at most *n* zero.

For the rational zeros of a polynomial we have the following theorem:

<u>**Theorem**</u> 2.3.4 If $\frac{a}{b}$, a rational number in lowest terms, is a zero of the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the a_i's $(i = 0,1,2,\cdots,n)$ are integers and $a_n \neq 0$, the a is an integral factor of a_0 and b is an integral factor of a_n.

It must be noted that this theorem does not guarantee the existence of rational zeros of a polynomial. It merely enables us to identify the possible rational zeros. These are then checked using synthetic division or otherwise.

<u>Example</u> 2.3.7 Find all rational zeros of the polynomial

$$p(x) = 2x^3 + 5x^2 - 4x - 3.$$

<u>Solution</u>: If $\frac{a}{b}$ is a rational zero of *p*, then by the theorem, *a* must be an integral factor of 3 and *b* must be an integral factor of 2. i.e.

$$a \in \{-1,-3,1,3\} \text{ and } b \in \{-1,-2,1,2\}$$

and the set of possible rational zeros of *p* is

$$k = \frac{a}{b} \in \{-3, -\tfrac{3}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, 3\}$$

Using synthetic division we check each of the possible candidates starting with -3:

$$
\begin{array}{r|rrrr}
-3 & 2 & 5 & -4 & -3 \\
Add & & -6 & 3 & 3 \\
\hline
 & 2 & -1 & -1 & 0=r
\end{array}
$$

Thus, 3 is a zero of *p*.

Next we check $-\frac{3}{2}$:

$$
\begin{array}{r|rrrr}
-\frac{3}{2} & 2 & 5 & -4 & -3 \\
Add & & -3 & -3 & \frac{21}{2} \\
\hline
 & 2 & 2 & -7 & \frac{15}{2}=r\neq 0
\end{array}
$$

Thus, $-\frac{3}{2}$ is not a zero of *p*.

The remaining possible zeros can be checked the same way.

For this polynomial, the zeros are $-3,-\frac{1}{2}$ and 1.

Factoring a Polynomial

To factorize a polynomial we use the factor theorem sometimes combined with repeated division.

Example 2.3.8 Factorize the polynomial:

$$p(x)=2x^4-7x^3-2x^2+13x+6$$

Solution: Integral factors of 6 are $a\in\{-6,-3,-1,1,3,6\}$ and integral $p(x)$ factors of 2 are $b\in\{-2,-1,1,2\}$. The possible rational zeros of *p* are

$$k=\frac{a}{b}\in\{-6,-3,-\tfrac{3}{2},-1,-\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{3}{2},3,6\}.$$

We determine one of the actual zeros of *p* by using synthetic division:

$$
\begin{array}{r|rrrrr}
-1 & 2 & -7 & -2 & 13 & 6 \\
Add & & -2 & 9 & -7 & -6 \\
\hline
 & 2 & -9 & 7 & 6 & 0=r
\end{array}
$$

$\Rightarrow$ -1 is a zero of *p*, and by Factor theorem $(x+1)$ is a factor of $p(x)$. Thus

$$p(x)=(x+1)(2x^3-9x^2+7x+6).$$

Let $q(x)=2x^3-9x^2+7x+6$. Then again the integral factors of 6 are $a\in\{-6,-3,-1,1,3,6\}$ and the integral factors of 2 are $b\in\{-2,-1,1,2\}$.

The possible rational zeros of *q* are

$$k=\frac{a}{b}\in\{-6,-3,-\tfrac{3}{2},-1,-\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{3}{2},3,6\}.$$

Again we determine one of the actual zeros of *q* by synthetic division:

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 7 & 6 \\ Add & & 4 & -10 & -6 \\ \hline & 2 & -5 & -3 & 0 = r \end{array}$$

$\Rightarrow$ 2 is a zero of *q* and by Factor theorem $(x-2)$ is a factor of $q(x)$. Thus

$$\begin{aligned} q(x) &= (x-2)(2x^2-5x-3) \\ &= (x-2)(2x+1)(x-3). \end{aligned}$$

Therefore,

$$p(x) = (x+1)(x-2)(2x+1)(x-3) .$$

Clearly note that the other zeros of *p* are $-\frac{1}{2}$ and 3.

Therefore when we solve the polynomial equation $p(x)=0$, i.e. say the equation

$$p(x) = (x+1)(x-2)(2x+1)(x-3) = 0 ,$$

we obtain $x=2,\ x=-1,\ x=-\frac{1}{2}$ and $x=3$, which are the zeros or roots of *p*. What this means is that the zeros or roots of a polynomial indicate where the value of the polynomial function is equal to zero, i.e. where the graph of the function cuts the *x* – axis. Using these *x*- intercepts and the *y* - intercept we can sketch the graph of polynomial.

Example 2.3.9 Sketch the graph of each of the following polynomial functions, indicating the points where the curve cuts the axes.

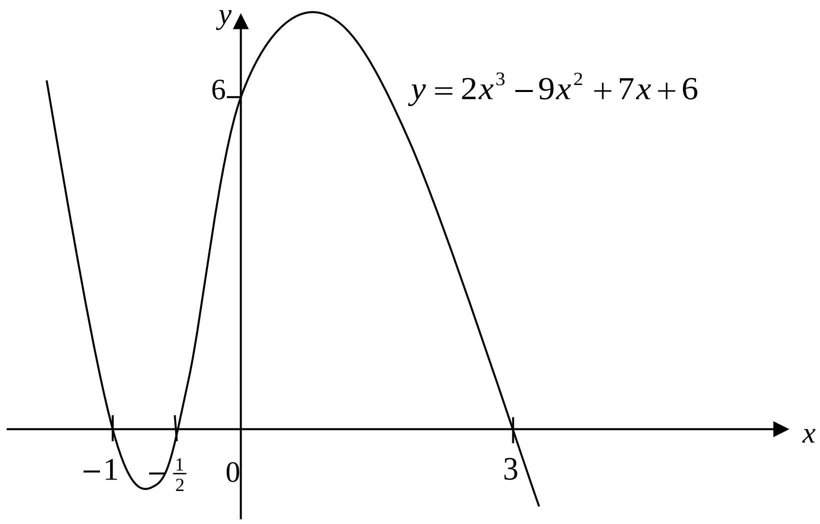
(a) $p(x) = 2x^3 - 9x^2 + 7x + 6$

(b) $p(x) = 2x^4 + 5x^3 - 5x^2 - 5x - 3$.

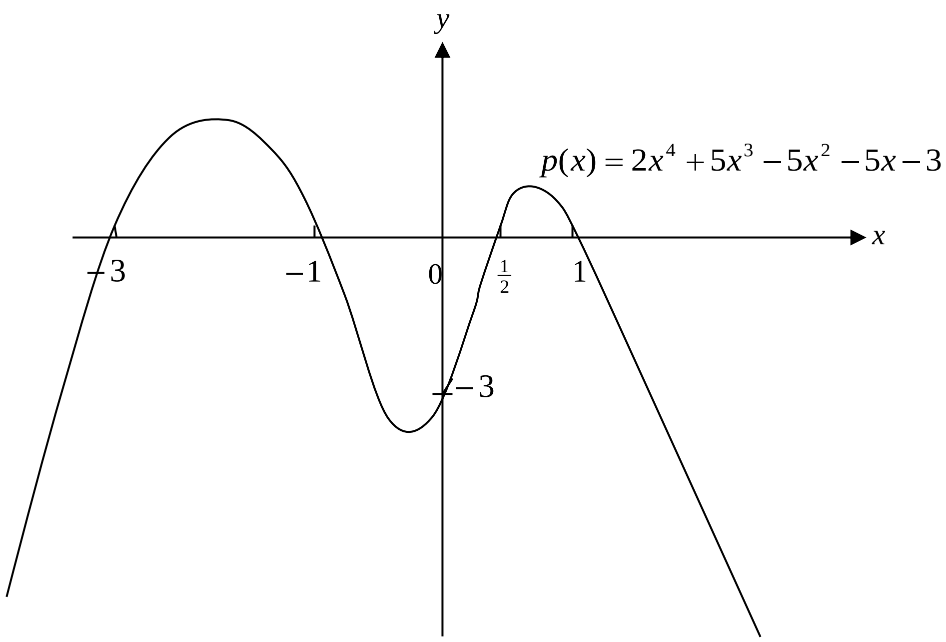
Solution: (a) The zeros of $p(x) = 2x^3 - 9x^2 + 7x + 6$ are $x=-1,\ x=-\frac{1}{2},\ x=3$. These are the *x* – intercepts the curve $y = 2x^3 - 9x^2 + 7x + 6$ and the *y* – intercept is 6.

Now note that a polynomial of degree 1 has no turning point, a polynomial of degree 2 has one turning point, a polynomial of degree 3 has 2 turning point, etc. This curve has 2 turning points.

Hence, we sketch the curve passing through the intercepts.



(b) The zeros of $p(x) = 2x^4 + 5x^3 - 5x^2 - 5x - 3$ are $-3, -1, \frac{1}{2}$ and 1, which are the x – intercepts of the curve. The y – intercept is $y = -3$.



We will only be able to find the exact turning points of a polynomial function of degree greater than 2 when we do differential calculus.

2.4 Rational Functions

A rational function is one that is written in the form of

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial.

We shall assume that $p(x)$ and $q(x)$ have no common factors.

Examples of rational functions are $\frac{1}{x}$, $\frac{x}{x^2-1}$, $\frac{3x^2-4x}{2x+1}$, etc.

The **domain** of a rational function of *x* includes all real numbers except *x* – values that make the denominator zero.

<u>Example</u> 2.4.1

1. The function $f(x)=\frac{1}{x}$ is not defined at $x=0$ and thus the domain of the function is the set $\{x\in\mathbf{R}:x\neq 0\}$.

2. The function $f(x)=\frac{x}{x^2-1}$ is not defined at $x=\pm 1$, and thus the domain of the function is the set $\{x\in\mathbf{R}:x\neq -1,1\}$.

3. The function $f(x)=\frac{3x^2-4x}{2x+1}$ is not defined at $x=-\frac{1}{2}$ and thus the domain of the function is the set $\{x\in\mathbf{R}:x\neq -\frac{1}{2}\}$.

Recall that the range of a function is the domain of its inverse. Thus, to find the range of a rational function $y=f(x)$, we need to find the domain of the inverse function $y=f^{-1}(x)$.

<u>Example</u> 2.4.1 Find the range of each of the following functions:

1. $f(x)=\frac{1}{x+2}$ 2. $f(x)=\frac{x+4}{2x-1}$

<u>Solutions</u>: 1. We first find the inverse of the function. Let $y=\frac{1}{x+2}$ and interchange *x* and y and obtain $x=\frac{1}{y+2}$. Then we make y the subject of the formula.

$y+2=\frac{1}{x}\Rightarrow y=\frac{1}{x}-2\ i.e.\ y=\frac{1-2x}{x}\Rightarrow f^{-1}(x)=\frac{1-2x}{x}, x\neq 0.$

Therefore, the range of the function is $\{x\in\mathbb{R}:x\neq 0\}$

2. $y=\frac{x+4}{2x-1}$ interchanging *x* and y we have $x=\frac{y+4}{2y-1}$. Making *y* the subject of the formula we have $x(2y-1)=y+4\Rightarrow 2xy-x=y+4\Rightarrow 2xy-y=x+4$

$\Rightarrow y(2x-1)=x+4\Rightarrow y=\frac{x+4}{2x-1}$. Thus $f^{-1}(x)=\frac{x+4}{2x-1}, x\neq\frac{1}{2}$.

Therefore, the range of the function is $\left\{x\in\mathbb{R}:x\neq\frac{1}{2}\right\}$

2.5 Modulus Function

A modulus function is a function of the form

$$f(x)=|g(x)|,$$

where $g(x)$ is a function.

Note: 1. When $g(x) \geq 0$, $|g(x)| = g(x)$.

2. When $g(x) < 0$, $|g(x)| = -g(x)$.

The **domain** of a modulus function is the same as that of the function $g(x)$ and its **range** is $\{y = f(x) \in \mathbf{R} : y = f(x) \geq 0\}$.

Graph of a Modulus Function

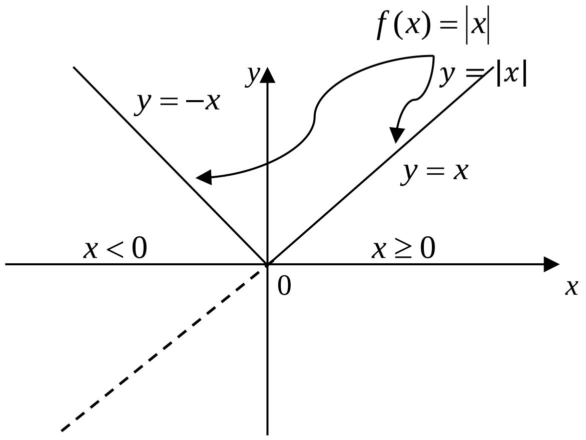
Example 2.5.1 Sketch the graph of each of the following modulus functions:

1. $f(x) = |x|$.

Solution: $y = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0. \end{cases}$

Step 1. Sketch the graph of $y = x$

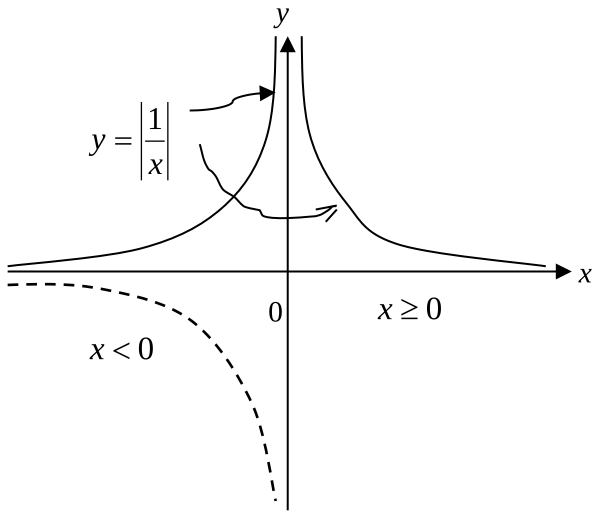
Step 2. For the part of the line below the x – axis (i.e. where $y < 0$), reflect the line in the x – axis.



Note: (a) For both $x < 0$ and $x \geq 0$, $y = |x| \geq 0$.

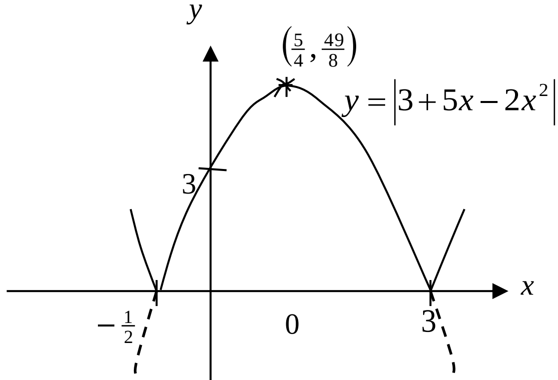
(b) Domain of $f(x) = |x|$ is $\mathbb{R}$ and its range is $\{y \in \mathbb{R}: y \geq 0\}$.

2. $f(x) = \left|\frac{1}{x}\right|$.



Domain of $f(x)=\left|\frac{1}{x}\right|$ is $\{x \in \mathbf{R} : x \neq 0\}$ and its range is $\{y \in \mathbf{R} : y > 0\}$.

3. $f(x)=\left|3+5x-2x^2\right|$



Domain of $f(x)=\left|3+5x-2x^2\right|$ is $\mathbf{R}$ and its range is $\{y \in \mathbf{R}: y \geq 0\}$

2.6 Radical Functions

Radical functions are functions involving roots (square roots, cube roots etc.)
For example,

$$f(x)=\sqrt{x}, \quad g(x)=\sqrt{2-x} \quad h(x)=\sqrt[3]{x+4} \text{ e.t.c.}$$

are all radical functions.

In this course we shall only consider radical functions involving the square root.

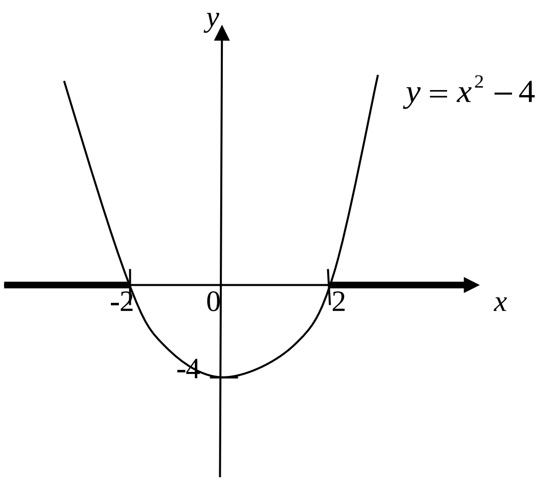
The **domain** of a radical function is the set of values of *x* for which the function is defined.

Example 2.6.1 The domain of

1. $f(x)=\sqrt{x}$ is the set $\{x \in \mathbf{R} : x \geq 0\}$.

2. $f(x)=\sqrt{2-x}$ is the set $\{x \in \mathbf{R} : x \leq 2\}$ since for $x \leq 2$, $2-x \geq 0$.

3. $f(x)=\sqrt{x^2-4}$ is the set $\{x \in \mathbf{R} : x \leq -2 \text{ or } x \geq 2\}$ since the function is defined for values of *x* for which $x^2-4 \geq 0$ i.e. $(x+2)(x-2) \geq 0$ i.e. $x \leq -2 \text{ or } x \geq 2$.



The **range** of a radical function $y = f(x)$ is the set of values *y* takes for all values of *x* within the domain of *f*.

Example 2.6.1 The range of

1. $f(x) = \sqrt{x}$ is the set $\{y = f(x) \in \mathbf{R} : x \geq 0\} = [0, \infty)$.
2. $f(x) = -\sqrt{2-x}$ is the set $\{y = f(x) \in \mathbf{R} : x \leq 2\} = (-\infty, 0]$.
3. $f(x) = \sqrt{x^2 - 4}$ is the set $\{y = f(x) \in \mathbf{R} : x \leq -2 \text{ or } x \geq 2\} = [0, \infty)$ since the value of the function is greater or equal to zero for all $x \leq -2 \text{ or } x \geq 2$.

Graphs of a Radical Functions

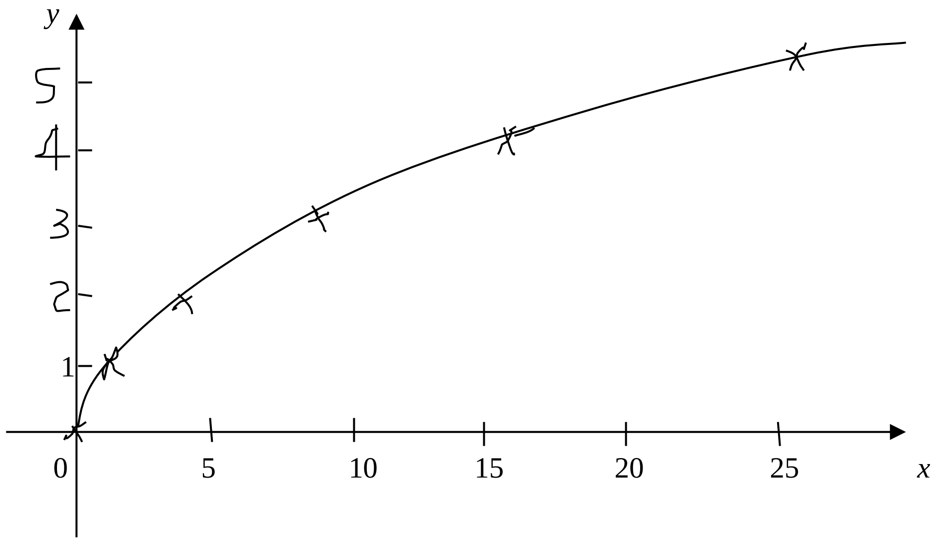
Example 2.6.2 Sketch the graph of each of the following functions:

1. $f(x) = \sqrt{x}$

Solution: Step 1: Plot the points.

x	0	1	4	9	16	25
$f(x)$	0	1	2	3	4	5

Step 2: Sketch the curve of the function passing through the plotted points.

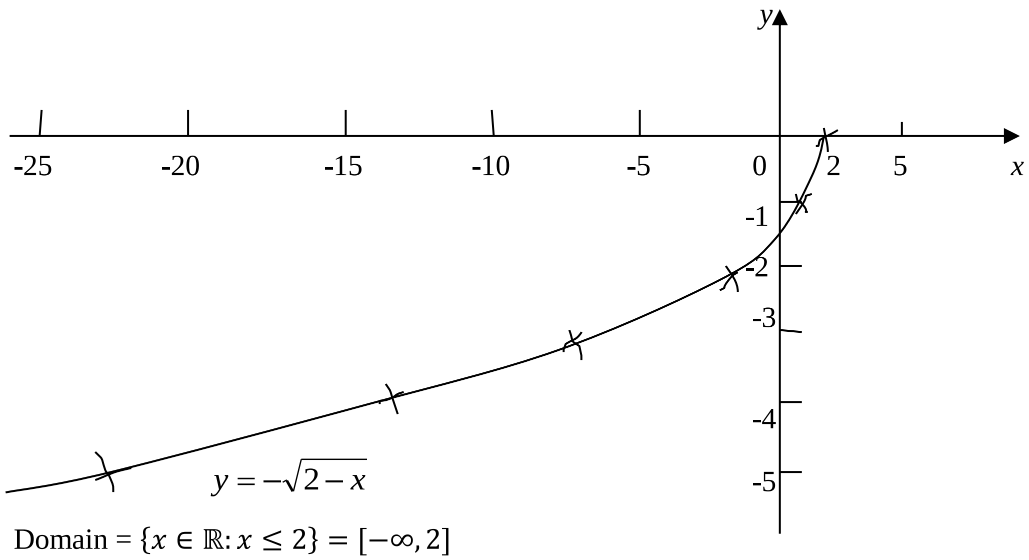


2. $f(x) = -\sqrt{2-x}$

Solution: Step 1: Plot the points in the table.

x	-23	-14	-7	-2	1	2
$f(x)$	-5	-4	-3	-2	-1	0

Step 2: Sketch the curve of the function passing through the plotted points.



Domain = $\{x \in \mathbb{R}: x \leq 2\} = [-\infty, 2]$

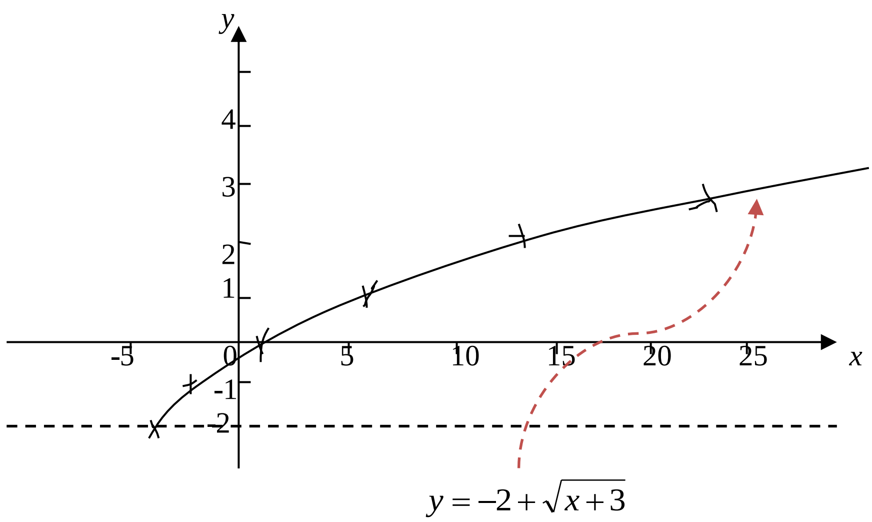
Range = $\{y \in \mathbb{R}: y \leq 0\} = [-\infty, 0]$

3. $f(x) = -2 + \sqrt{x+3}$

<u>Solution</u>: Step 1: Plot the points in the table.

x	-3	-2	1	6	13	22
$f(x)$	-2	-1	0	1	2	3

Step 2: Sketch the curve of the function passing through the plotted points.



Domain = $\{x \in \mathbb{R}: x \geq -3\} = [-3, \infty)$

Range = $\{y \in \mathbb{R}: y \geq -2\} = [-2, \infty)$

TUTORIAL SHEET 4

1. Complete the square of each of the following quadratic functions. Hence sketch the graph of the function, showing clearly the x – and y – intercepts and the turning point. State

   (i) the line of symmetry, and

   (ii) the maximum or minimum value of the function.

   (a) $f(x) = 2x^2 - 4x + 5$ (b) $f(x) = x^2 + 2x - 5$ (c) $f(x) = 4 - 3x^2$

   (d) $f(x) = 3 - 7x - 3x^2$.

2. What are the dimension of the largest rectangular field which can be enclosed by 1200 m of fencing?

3. If the profit p in the manufacture and sale of x units of a product is given by

$$p(x) = 200x - 0.001x^2,$$

   (a) Find the number x that yields the maximum profit.

   (b) Find the maximum profit if each item is sold at K2.50.

   (c) Sketch the graph of the function p.

4. A window is to be constructed in the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 540 cm, find its dimensions for maximum area.

5. Let $p(x) = 6^3 + 3x^2 - 2x - 7$ and $q(x) = x^3 + 3x^2 + x + 5$. Find

   (a) $p(x) + qx)$ (b) $p(x) - qx)$ (c) $p(x) - qx)$ (d) $p(x) \cdot qx)$

6. Use long division to divide:

Dividend	Divisor
(i) $x^3 + 8x^2 - 5x - 1$	$x - 2$
(ii) $2x^3 + 6x^2 - x + 5$	$2x^2 - 1$
(iii) $x^4 - 4x^2 + 3$	$3 + 2x - x^2$

7. Use synthetic division to divide the polynomials and write the function in the form $p(x) = (x - k)q(x) + r$, where $q(x)$ is the quotient and r is the remainder:

Dividend	Divisor
(i) $x^3 - 10x^2 + 31x - 30$	$x + 3$
(ii) $x^3 + 15x^2 + 68x + 96$	$x - 2$
(iii) $6x^3 + x^2 - 21x - 10$	$2x - 1$

8. Write the function in the form $p(x) = (x - k)q(x) + r$, where $q(x)$ is the quotient and r is the remainder:

   (i) $p(x) = x^3 + x^2 - 12x + 20, \quad k = 2$

   (ii) $p(x) = x^3 - 2x^2 - 15x + 7, \quad k = -4$

   (iii) $p(x) = x^3 + 2x^2 - 3x - 12, \quad k = \sqrt{3}$

(iv) $p(x) = 3x^3 - 19x^2 + 27x - 7, \quad k = 3 - \sqrt{2}$

9. Factorize the polynomial completely:

(i) $p(x) = x^3 - 12x - 16$ (ii) $p(x) = 3x^3 + 10x^2 - 27x - 10$

(iii) $p(x) = x^3 + 2x^2 - 3x - 6$ (iv) $p(x) = x^3 + 2x^2 - 2x - 4$

10. Given that $(x-1)$ and $(x+1)$ are factors of $px^3 + qx^2 - 3x - 7$, find the value of *p* and *q*.

11. The expression $2x^3 - ax^2 + bx + 3$ gives a remainder -15 when divided by $(x+1)$ and a remainder -46 when divided by $(x-3)$. Find the value of *a* and of *b*.

12. Find the zeros of each of the following polynomial functions. Hence sketch its graph indicating the *x* – and *y* - intercepts:

(i) $p(x) = x^3 - 2x - 7x + 12$ (ii) $p(x) = -x^3 + x^2 + 5x - 2$

(iii) $p(x) = 15 + 5x - 3x^2 - x^3$ (iv) $p(x) = x^3 + 5x^2 + 6x + 2$

13. (a) Show that $(x-2)$ is a factor of $p(x) = x^3 + x^2 - 5x - 2$.

(b) Hence, or otherwise, find the exact solutions of the equation $p(x) = 0$.

14. Sketch the graph of each of the following modulus functions:

(a) $f(x) = -|x| + 3$ (b) $f(x) = |x^3|$ (c) $f(x) = |(x+1)(2-x)|$

(d) $f(x) = |2x^2 - 7x + 3|$.

15. Sketch the graphs of the following functions and determine its domain and range:

(a) $f(x) = \sqrt{x-2}$ (b) $g(x) = -4 + \sqrt{x+3}$ (c) $h(x) = 1 + \sqrt{-x-2}$

(d) $y = -\sqrt{3x+1}$.

16. The description of body-heat loss due to convection involves a coefficient of convection K_c, which depends on wind speed *v* according to the equation:

$K_c = 4\sqrt{4v+1}$.

(a) What is the domain?

(b) What restrictions do nature and common sense put on *v*?

2.7 Equations

Quadratic Equations

Any equation of the form

$$ax^2 + bx + c = 0$$

is called a quadratic equation.

<u>Nature of Roots of a Quadratic Equation</u>

By completing the square of the quadratic function $f(x) = ax^2 + bx + c$ and equating to zero we have the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

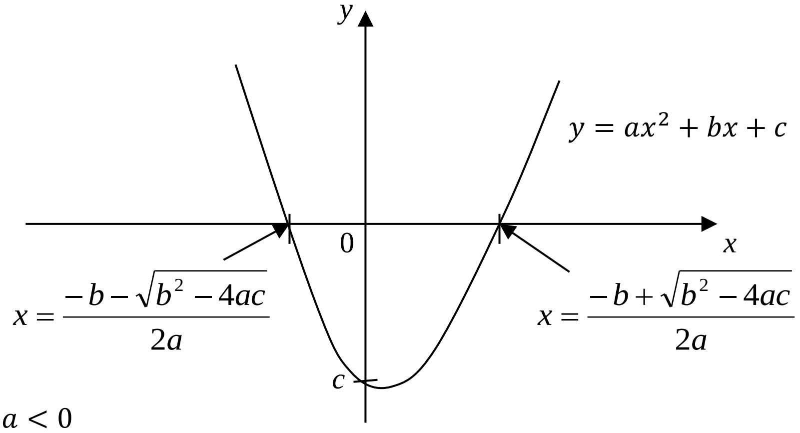
used to obtain the solutions (or the roots) of the quadratic equation.

The expression $b^2 - 4ac$, under the square root sign, is called the **discriminant,** and it determines the nature of the roots of the quadratic equation.

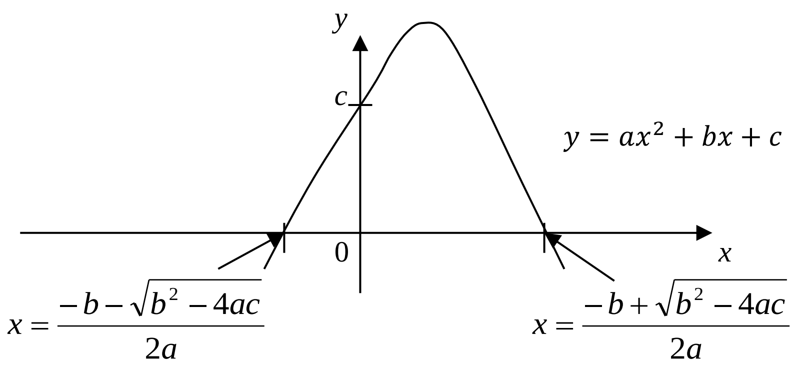
1. If $b^2 - 4ac > 0$, the equation has two and ***two distinct real*** roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

<u>Case</u> 1. $a > 0$



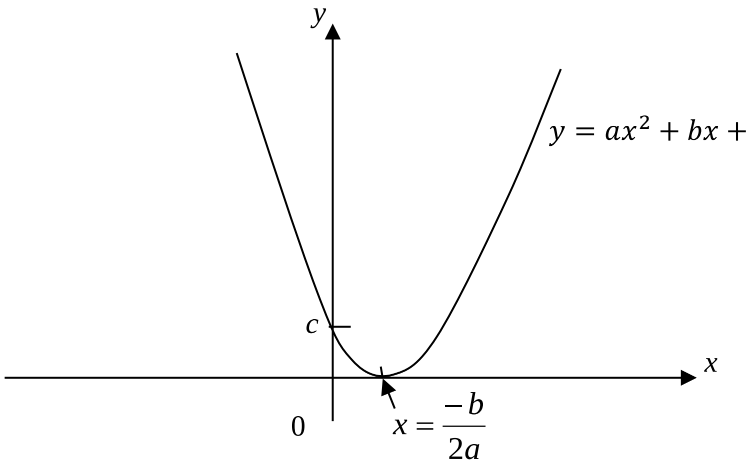
<u>Case 2</u>. $a < 0$



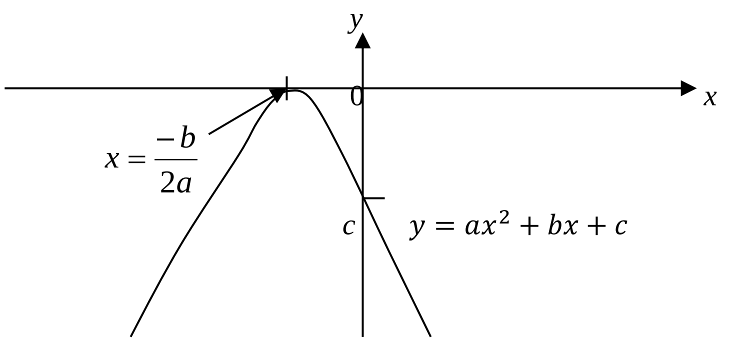
Note that when the quadratic equation has two distinct real roots the graph of the curve $y = ax^2 + bx + c$ cuts the x – axis at two distinct points.

2. If $b^2 - 4ac = 0$, the equation has ***two equal real*** roots $x = \dfrac{-b}{2a}$.

<u>Case</u> 1. $a > 0$



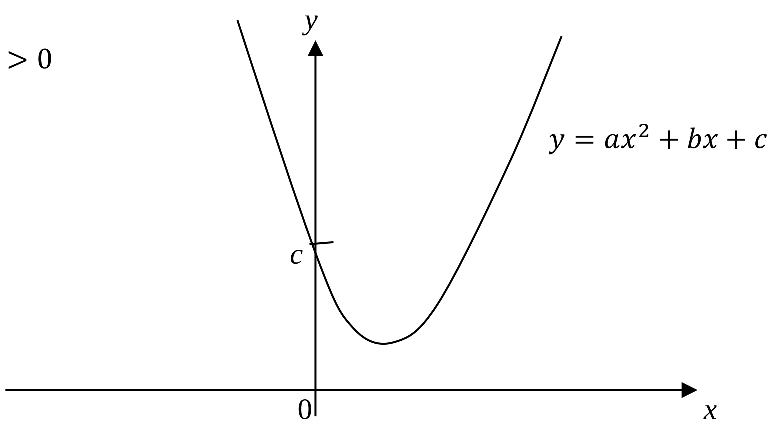
<u>Case</u> 2. $a > 0$



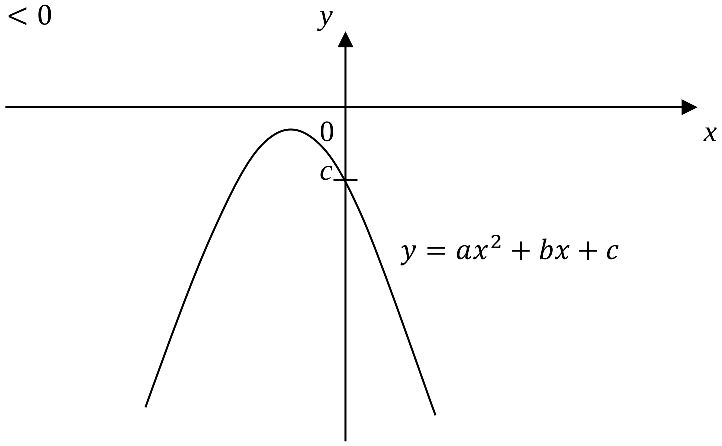
Note that when the quadratic equation has two equal real roots the graph of the curve $y = ax^2 + bx + c$ touches the x – axis at exactly one point.

3. If $b^2 - 4ac < 0$, the equation has no real roots. It has **two complex roots** which are conjugates of each other.

<u>Case</u> 1. $a > 0$



Case 2. $a < 0$



Note that when the quadratic equation has complex roots the graph of the curve $y = ax^2 + bx + c$ does not cut or touch the x – axis.

Example 2.7.1 Determine the nature of the roots of each of the following quadratic equations:

1. $x^2 - 6x + 9 = 0$

Solution: $a = 1, b = -6,$ and $c = 9$.

Using the discriminant, we have

$$b^2 - 4ac = (-6)^2 - 4(1)(9) = 36 + 36 = 0.$$

⇒ the equation has two equal real roots.

2. $x^2 + 4x - 8 = 0$

Solution: $a = 1, b = 4,$ and $c = -8$.

Using the discriminant, we have

$$b^2 - 4ac = (4)^2 - 4(1)(-8) = 16 + 32 = 48 > 0.$$

⇒ the equation has two distinct real roots.

3. $3x^2 + 4x + 2 = 0$

Solution: $a = 3, b = 4,$ and $c = 2$.

Using the discriminant, we have

$$b^2 - 4ac = (4)^2 - 4(3)(2) = 16 - 24 = -8 < 0.$$

⇒ the equation has two complex roots.

4. Prove that $kx^2 + 2x - (k-2) = 0$ has real roots for any value of k.

Proof: If the equation has real roots then $b^2 - 4ac \geq 0$.

Now, $b^2 - 4ac = 2^2 - 4(k)\big(-(k-2)\big) = 4 + 4k^2 - 8k$

$= 4\left(k^2 - 2k + 1\right) = 4(k-1)^2 \geq 0$ for any value of k.

Relationships between the Roots and Coefficients of a Quadratic Equation

Let α and β be the roots of a quadratic equation

$$ax^2 + bx + c = 0.$$

Then the equations

$$(x-\alpha)(x-\beta) = 0 \qquad \text{(I)}$$

and

$$ax^2 + bx + c = 0 \qquad \text{(II)}$$

have the same roots.

But from (I)

$$(x-\alpha)(x-\beta) = x^2(\alpha+\beta)x + \alpha\beta = 0$$

i.e.

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0 \qquad \text{(III)}$$

Dividing (II) by a we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(IV)}$$

Now, (II) and (IV) have the same roots.

Comparing the coefficients of (III) and (IV) we have

$$\boxed{\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.}$$

Example 2.7.2 If the equation

$$2x^2 - 3x + 6 = 0$$

has roots α and β, then the sum of roots

$$\alpha+\beta = -\frac{b}{a} = -\frac{-3}{2} = \frac{3}{2}$$

and the product of roots

$$\alpha\beta = \frac{c}{a} = \frac{6}{2} = 3.$$

Example 2.7.3 The roots of the equation

$$2x^2 + x - 7 = 0$$

are α and β. Find the values of $\frac{1}{\alpha} + \frac{1}{\beta}$ and $\frac{1}{\alpha\beta}$.

Solution: sum of roots = $\alpha+\beta = -\frac{b}{a} = -\frac{1}{2}$

Product of roots = $\alpha\beta = \frac{c}{a} = \frac{-7}{2}$.

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}=\frac{-\frac{1}{2}}{-\frac{7}{2}}=\frac{1}{7} \text{ and } \frac{1}{\alpha\beta}=\frac{1}{-\frac{7}{2}}=-\frac{2}{7}.$$

<u>Example</u> 2.7.4 If α and β are the roots of the function $x(x-3)=x+4$, find the values of $\alpha^3+\beta^3$ and $\alpha^3\beta^3$.

<u>Solution</u>: $x(x-3)=x+4 \Rightarrow x^2-4x-4=0$ Thus $a=1, b=-4$ and $c=-4$.

sum of roots = $\alpha+\beta=-\frac{b}{a}=-\frac{-4}{1}=4$

Product of roots = $\alpha\beta=\frac{c}{a}=\frac{-4}{1}=-4$.

But

$$\begin{aligned}(\alpha+\beta)^3 &= (\alpha+\beta)(\alpha+\beta)^2=(\alpha+\beta)(\alpha^2+2\alpha\beta+\beta^2)\\ &= \alpha^3+3\alpha\beta^2+3\alpha\beta^2+\beta^3\end{aligned}$$

$\Rightarrow \alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(4)^3-3(-4)(4)=64+48=112$

and $\quad \alpha^3\beta^3=(\alpha\beta)^3=(-4)^3=-64.$

Note from equations (III) and (IV) that a quadratic equation can be written as

$$\boxed{x^2-(sum\ of\ roots)x+product\ of\ roots=0.}$$

Using this we consider the following Example:

<u>Example</u> 2.7.5 Write down the quadratic equation whose roots are $\frac{1}{3}$ and $-\frac{2}{3}$.

<u>Solution</u>: $\quad x^2-(\alpha+\beta)x+\alpha\beta=0$

$$\alpha+\beta=\frac{1}{3}+\frac{-2}{3}=-\frac{1}{3} \text{ and } \alpha\beta=\frac{1}{3}\times\frac{-2}{3}=-\frac{2}{9}.$$

Therefore, the equation is

$$x^2+\frac{1}{3}x-\frac{2}{9}=0$$

or
$$9x^2+3x-2=0.$$

2.8 <u>Polynomial Equations</u>

Polynomial equations are of the form

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0=0,$$

where $a_0, a_1, a_2, \cdots, a_n$ are real numbers and $a_n\neq 0$.

We solve the polynomial equations the same way we find roots or zeros of a polynomial function.

<u>Example</u> 2.8.1 Solve the polynomial equation

$$2x^3 - 5x^2 + x + 2 = 0 .$$

Solution: Find a zero of $f(x) = 2x^3 - 5x^2 + x + 2$

$$f(1) = 2(1)^3 - 5(1)^2 + (1) + 2 = 2 - 5 + 1 + 2 = 0$$

$\Rightarrow$ 1 is a zero of $f(x)$.

$\Rightarrow$ $(x-1)$ is factor of $f(x)$.

$$\begin{array}{r|rrrr} 1 & 2 & -5 & 1 & 2 \\ Add & & 2 & -3 & -2 \\ \hline & 2 & -3 & -2 & 0 = r \end{array}$$

$$\Rightarrow \; f(x) = (x-1)(2x^2 - 3x - 2) = (x-1)(2x+1)(x-2)$$

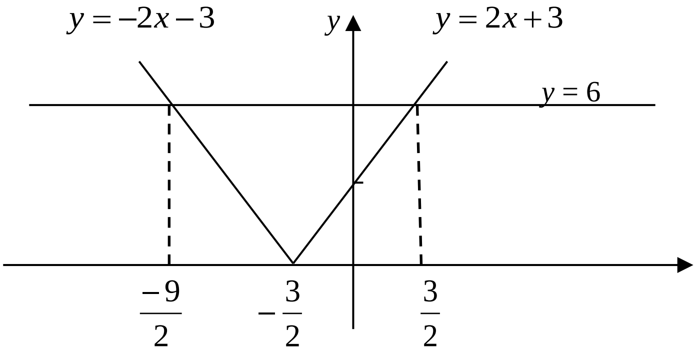
$$(x-1)(2x+1)(x-2) = 0 \quad \Rightarrow x = 1, \quad x = -\tfrac{1}{2}, \quad x = 2 .$$

2.9 Equations Involving the Absolute Value

Example 2.9.1 Solve each of the following equations:

1. $|2x+3| = 6$

Solution: Method 1: $|2x+3| = \begin{cases} 2x+3 \text{ if } x \geq -\frac{3}{2} \\ -(2x+3) \text{ if } x < -\frac{3}{2} \end{cases}$



$$\Rightarrow \quad 2x + 3 = 6 \quad \Rightarrow x = \frac{3}{2} \text{ and } -(2x+3) = 6 \quad \Rightarrow x = \frac{-9}{2} .$$

Method 2: It must be noted that $|x|$ can also be defined as

$$|x| = \sqrt{x^2} .$$

Thus, $|2x+3| = 6 \quad \Rightarrow \quad \sqrt{(2x+3)^2} = 6 \quad \Rightarrow \quad \left(\sqrt{(2x+3)^2}\right)^2 = 6^2$

$$\Rightarrow \quad (2x+3)^2 = 36 \quad \Rightarrow \quad 4x^2 + 12x + 9 = 36$$

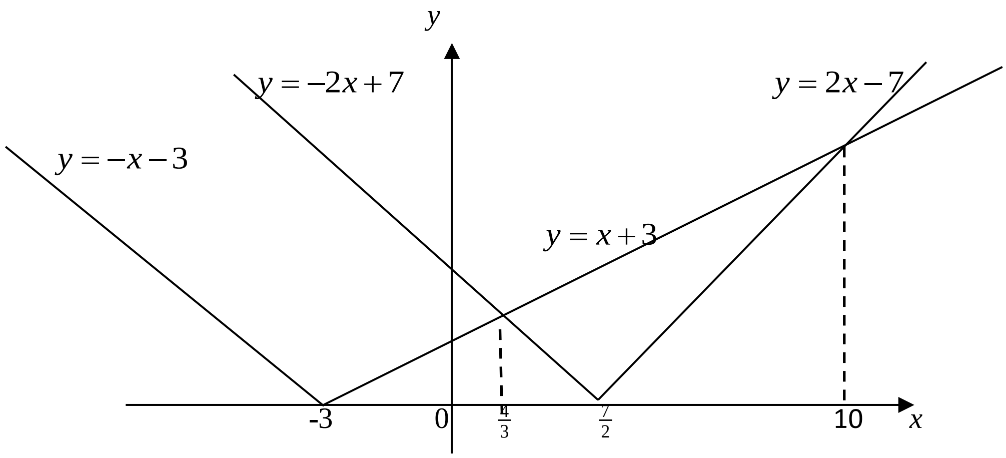
$$\Rightarrow \quad 4x^2+12x-27=0 \quad \Rightarrow \quad (2x-3)(2x+9)=0$$

$$\Rightarrow \quad x=\frac{3}{2} \quad \text{or} \quad x=-\frac{9}{2}.$$

2. $|2x-7|=|x+3|$

Solution: Method 1: $|2x-7|=\begin{cases} 2x-7 \text{ } if \text{ } x\ge \frac{7}{2} \\ -(2x-7) \text{ } if \text{ } x<\frac{7}{2} \end{cases}$

and $|x+3|=\begin{cases} x+3 \text{ } if \text{ } x\ge -3 \\ -(x+3) \text{ } if \text{ } x<-3 \end{cases}$



$$\Rightarrow \quad 2x-7=x+3 \quad \Rightarrow x=10 \text{ and } -2x+7=x+3 \quad \Rightarrow x=\frac{4}{3}$$

Note that $2x-7$ is only defined for $x\ge \frac{7}{2}$ and $-(x+3)$ is only defined for $x<-3$, thus $2x-7\ne -(x+3)$ for $x<-3$ and $x\ge \frac{7}{2}$.

Method 2: $|2x-7|=|x+3| \quad \Rightarrow \quad \left(\sqrt{(2x-7)^2}\right)^2=\left(\sqrt{(x+3)^2}\right)^2$

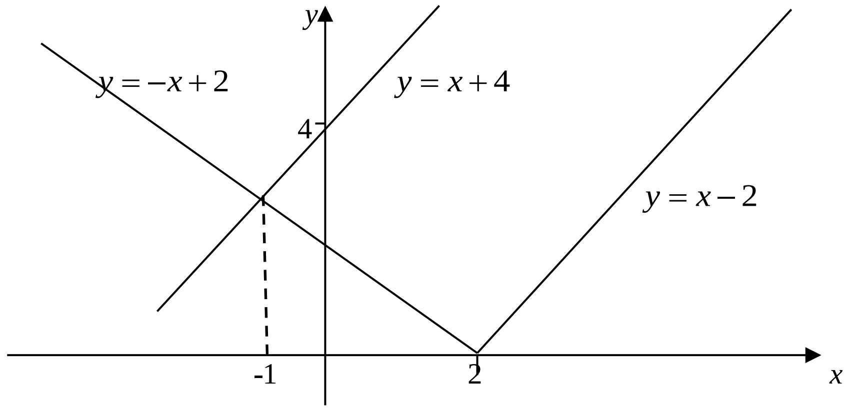
$$\Rightarrow \quad (2x-7)^2=(x+3)^2 \quad \Rightarrow \quad 4x^2-28x+49=x^2+6x+9$$

$$\Rightarrow \quad 3x^2-34x+40=0 \quad \Rightarrow \quad (3x-4)(x-10)=0$$

$$\Rightarrow \quad x=\frac{4}{3} \quad \text{or} \quad x=10.$$

3. $|x-2|=x+4$.

Solution: Method 1: $|x-2|=\begin{cases} x-2 \text{ } if \text{ } x\ge 2 \\ -(x-2) \text{ } if \text{ } x<2 \end{cases}$.



$$-(x-2)=x+4 \quad \Rightarrow -2x=2 \quad \Rightarrow x=-1$$

Note that $x-2=x+4$ has no solution because the lines $y=x-2$ and $y=x+4$ do not intersect since they have the same gradient and they are parallel.

Method 2 $|x-2|=x+4 \quad \Rightarrow \sqrt{(x-2)^2}=x+4 \quad \Rightarrow \left(\sqrt{(x-2)^2}\right)^2=(x+4)^2$

$\Rightarrow \quad (x-2)^2=(x+4)^2 \quad \Rightarrow x^2-4x+4=x^2+8x+16 \quad \Rightarrow 12x=-12 \quad \Rightarrow x=-1$

2.10 Equations Involving Radicals

Example 2.10.1 Solve each of the following equations:

1. $\sqrt{3x-8}-\sqrt{x-2}=0$
2. $\sqrt{3x+1}+\sqrt{2x+4}=3$.
3. $\sqrt{x-2}-\sqrt{2x-11}=\sqrt{x-5}$

Solution: 1. $\sqrt{3x-8}=\sqrt{x-2}$

$$\left(\sqrt{3x-8}\right)^2=\left(\sqrt{x-2}\right)^2$$

$$\Rightarrow \quad 3x-8=x-2 \quad \Rightarrow 2x=6 \quad \Rightarrow x=3$$

Test the root: When $x=3$, LHS = $\sqrt{3(3)-8}-\sqrt{3-2}=1-1=0=$ RHS

$\Rightarrow x=3$ is a root of the given equation.

2. $\sqrt{3x+1}+\sqrt{2x+4}=3$.

Solution: $\sqrt{3x+1}=3-\sqrt{2x+4}$

$$\Rightarrow \quad \left(\sqrt{3x+1}\right)^2=\left(3-\sqrt{2x+4}\right)^2$$

$$\Rightarrow \quad 3x+1=9-6\sqrt{2x+4}+2x+4$$

$$\Rightarrow \quad x-12=-6\sqrt{2x+4}$$

$$\Rightarrow \quad (x-12)^2=\left(-6\sqrt{2x+4}\right)^2$$

$$\Rightarrow \quad x^2-24x+144=36(2x+4)=72x+144$$

$$\Rightarrow \quad x^2-96x=0 \quad \Rightarrow x(x-96)=0 \quad \Rightarrow \quad x=0 \text{ or } x=96$$

Test the roots: When $x=0$, LHS = $\sqrt{3(0)+1}+\sqrt{2(0)+4}=1+2=3=$ RHS

$\Rightarrow x=0$ is a root of the given equation.

When $x=96$, LHS = $\sqrt{3(96)+1}+\sqrt{2(96)+4}=17+14\neq 3=$ RHS

$\Rightarrow x=96$ is not a root of the given equation.

Therefore, the equation on has one root $x=0$.

3. $\sqrt{x-2}-\sqrt{2x-11}=\sqrt{x-5}$

Solution: $\sqrt{x-2}=\sqrt{2x-11}+\sqrt{x-5}$

$\Rightarrow \quad \left(\sqrt{x-2}\right)^2=\left(\sqrt{2x-11}+\sqrt{x-5}\right)^2$

$\Rightarrow \quad x-2=2x-11+2\sqrt{2x-11}\cdot\sqrt{x-5}+x-5$

$\Rightarrow \quad 2x-14=2\sqrt{2x-11}\cdot\sqrt{x-5}$

$\Rightarrow \quad (2x-14)^2=\left(2\sqrt{2x-11}\cdot\sqrt{x-5}\right)^2$

$\Rightarrow \quad 4x^2-56x+196=4(2x-11)(x-5)=8x^2-84x+220$

$\Rightarrow \quad 4x^2-28x+24=0 \quad \Rightarrow x^2-7x+6=0 \quad \Rightarrow (x-1)(x-6)=0$

$\Rightarrow \quad x=1 \; or \; x=6$

Test the roots: When $x=1$, LHS = $\sqrt{1-2}+\sqrt{2(1)+4}$ is not defined.

$\Rightarrow x=1$ is not a root of the given equation.

When $x=6$, LHS = $\sqrt{6-2}-\sqrt{2(6)-11}=2-1=1$

RHS $=\sqrt{6-5}=1=$ RHS

$\Rightarrow x=6$ is a root of the given equation.

Therefore, the equation on has one root $x=6$.

2.11 System of Equations in Two unknowns

Elimination Method

Example 2.11.1 Solve the system of equations by elimination method:

1. $3x+2y=1;\ 5x-2y=23$

Solution: $3x+2y=1$

$+\ (5x-2y=23)$

$8x \qquad =24 \quad \Rightarrow x=3$ and $3(3)+2y=1 \quad \Rightarrow y=-4$.

Substitution Method

Example 3.9 Solve the system of equations by substitution method:

1. $3x+2y=1;\ 5x-2y=23$

2. $y=2x; \quad y=x^2-1$

3. $3x-7y+6=0; \quad x^2-y^2=4$

<u>Solutions</u>:

1. $3x+2y=1; 5x-2y=23$

We make either *x* or *y* the subject of the formula of one equation and substitute in the other equation:

$x=\frac{1-2y}{3} \Rightarrow 5\left(\frac{1-2y}{3}\right)-2y=23 \Rightarrow 5-10y-6y=69$

$\Rightarrow y=-4$. Replacing this in the first equation we have $x=\frac{1-2(-4)}{3}=3$.

Therefore the solution set is $\{3,-4\}$.

2. $y=2x; \quad y=x^2-1$

Replacing $y=2x$ in the other equation we have

$2x=x^2-1$ or $x^2-2x-1=0 \Rightarrow x=1\pm\sqrt{2}$.

When $x=1\pm\sqrt{2}$, $y=2\pm2\sqrt{2}$. Therefore the solution set is

$\{(1+\sqrt{2},2+2\sqrt{2}),(1-\sqrt{2},2-2\sqrt{2})\}$.

3. $3x-7y+6=0;\ x^2-y^2=4$

From the first equation we have

$x=\frac{7y-6}{3}$. Replacing this in the second equation we have

$\left(\frac{7y-6}{3}\right)^2-y^2=4$ or $(7y-6)^2-9y^2=36$

$49y^2-84y+36-9y^2=36$ or $40y^2-84y=0$

$\Rightarrow 4y(y-21)=0 \ \Rightarrow y=0\ or\ y=21$

When $y=0, x=-2$ and when $y=21, x=47$.

Therefore, the solution set is $\{(-2,0),(47,21)\}$.

TUTORIAL SHEET 5

1. Without solving the equations determine the nature of the roots of each of the following quadratic equations.

   (a) $3x^2 + 13x - 10 = 0$ (b) $2x^2 + 3x + 2 = 0$ (c) $4x^2 - 12x + 9 = 0$

2. If the roots of the quadratic equation $kx^2 + 30x + 25 = 0$ are equal, find the value of *k*.

3. Prove that $kx^2 + 2x - (k-2) = 0$ has real roots for any value of *k*.

4. Find a relationship between *p* and *q* if the roots of $px^2 + qx + 1 = 0$ are equal.

5. Without solving write down the sums and products of the roots of the following equations:

   (a) $4x^2 + 7x - 3 = 0$ (b) $\frac{x-1}{2} = \frac{3}{x+2}$ (c) $ax^2 - x(a+2) - a = 0$.

6. The roots of the quadratic equation $3x^2 + 13x - 10 = 0$ are α and β. Find the value of :

   (a) $\alpha^2 + \beta^2$ (b) $\frac{1}{\alpha^2+1} + \frac{1}{\beta^2+1}$ (d) $(\alpha - \beta)^2$.

7. The roots $x^2 + 3x - 10 = 0$ are α and β. Without finding the values of α and β, find the equations whose roots are:

   (a) $\alpha + 2, \beta + 2$ (b) $\frac{1}{\alpha}, \frac{1}{\beta}$ (c) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (d) $\alpha - \beta, \beta - \alpha$.

8. (a) On the same diagram, sketch the graphs of $y = |x|$ and $y = |2x - 1|$.

   (b) Solve the equation $|x| = |2x - 1|$.

9. Solve each of the following equations:

   (a) $|2x - 5| = 7$ (b) $|2x+1| = |4x-3|$ (c) $\left|\frac{k-2}{k-1}\right| = 3$.

10. On the same diagram, sketch the graphs of $y = 24 + 2x - x^2$ and $y = |5x - 4|$.

    Solve the equation $24 + 2x - x^2 = |5x - 4|$.

11. Solve each of the following equations:

    (a) $\sqrt{2t-1} + 2 = t$ (b) $\sqrt{2x-1} - \sqrt{x+3} = 1$ (c) $\sqrt{2x-1} - \sqrt{x+3} = 1$

    (d) $\sqrt{x-2} - \sqrt{2x-11} = \sqrt{x-5}$ (e) $\sqrt{1+2\sqrt{x}} = \sqrt{x+1}$.

12. Solve each of following system of equations by the substitution method:

    (a) $\begin{cases} x + 2y = 3 \\ x - 2y = 1 \end{cases}$ (b) $\begin{cases} 3x - 5y = 2 \\ 2x + 5yy = 13 \end{cases}$.

13. Solve each of following simultaneous equations:

    (a) $\begin{cases} 2x - y + 3 = 0 \\ x^2 - 2x - y = 3 \end{cases}$ (b)) $\begin{cases} x - y = 2 \\ x^2 - y^2 = 8 \end{cases}$