THE COPPERBELT UNIVERSITY

SCHOOL OF MATHEMATICS AND NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

TUTORIAL SHEET 3: **MA110-Mathematical Methods** 2022

BINARY OPERATIONS

1. Define an operation *on the set of real numbers by $a * b = b^a$
   i). Is $*$ a binary operation on the set of real numbers ? Give reason for your Answer.
   ii). Is the operation commutative?
   iii). Evaluate $(3 * 2) * -2$
2. Consider the binary operation $a * b = a + b - 2ab$, where $a\ and\ b$ are real numbers .
   i). Compute 1*(2*3) and (1*2)*3
   ii). Is * commutative? Justify your answer

3. Let $'*'$ be a binary operation on the set of real numbers defined by $a * b = -2^{b-a}$,
   Where a and b are real numbers .

   i). Is $*$ commutative on real number? Justify your answer.
   ii). Find $-1 * (4 * 9)$

4. Determine whether the following operations are binary on the set of Irrational numbers .
   a). Addition [+] (b). Multiplication [×] (c). Subtraction [−]

   Justify your answers.

5 a) Determine whether the binary operation $*$ defined is commutative and whether $*$ is associative
   i) $*$ defined on Z by letting $a * b = a - b$
   ii) $*$ defined on Q by letting $a * b = ab + 1$
   iii) $*$ defined on $Z^+$by letting $a * b = 2^{ab}$
   iv) . $*$ defined on $Z^+$by letting $a * b = a^b$

b) Determine whether the definition of $*$ does give a binary operation on the set and give a reason why.

i) On Z^+ define $*$ by letting $a*b = a-b$
ii) On Z^+ define $*$ by letting $a*b = a^b$
iii) On R define $*$ by letting $a*b = a-b$
iv) On Q, define $*$ by letting $a*b = a/b$
v) On Z^+ define $*$ by letting $a*b = a/b$

6. a) A binary operation $*$ is defined on the set of real numbers as follows:

$$a*b = 2^{-a} + b, \qquad a, b \in \mathbf{R}$$

(i) Is the operation $*$ commutative? If not give a counter example.
(ii) Find the value of $-1*(0*1)$ and $(-1*0)*1$, and state whether $*$ is associative.

b) State whether each of the following operation is a binary operation on **Z**, the set of integers, where a and b are integers:

(i) $a*b = 3a-b$ (ii) $a*b = (ab)^2$ (iii) $a*b = \sqrt{a-b}$

c) Each of the following operations in I, II, III, IV, V is a binary operation on **R**

I: $a*b = (a+b)(a-b)$ II: $a*b = ab$

III: $a*b = 2^{a-b}$ IV: $a*b = a+2b$ V: $a*b = a+b-ab$

a) Determine which ones are commutative and / or associative.
b) For each of the operations I, II, III, IV and V, evaluate $3*(7*4)$

d) Given the sets X = {0 , 1} and Y = {0 , 1 , 2},

(i) Determine whether each of the following operations $+, -, \times, \div$ is a binary operation on X and on Y.
(ii) Also check whether each of the operations is commutative or associative.

7. Let $\boldsymbol{A}$={1,2} and $\boldsymbol{B}$={3,4,5} be two set . List the element of $A \times B$ and $\times A$. State if

$A \times B \neq B \times A$?

SKETCHING

1 a) Sketch the following curves and indicate clearly the points of intersection with the axes:

i) $y = (x-3)(x-2)(x+1)$ ii) $y = (x+1)(1-x)(x+3)$
iii) $y = x(x+1)(x-1)$ iv) $y = x(2x-1)(x+3)$

b) Sketch the curves with the following equations:

i) $y = (x+1)^2(x-1)$ ii) $y = (x+2)(x-1)^2$

iii) $y = (x-1)^2 x$ iv) $y = x^2(x-2)$

c) Factories the following equations and then sketch the curves

i) $y = x^3 + x^2 - 2x$ ii) $y = x^3 + 5x^2 + 4x$

iii) $y = x - x^3$ iv) $y = 12x^3 - 3x$

d). Sketch the following curves and show their positions relative to the curve $y = x^3$

i) $y = (x-2)^3$ ii) $y = (2-x)^3$ (iii) $y = -(x+2)^3$

iv) $y = (x+2)^3$

e). Sketch the following and indicate the coordinates of the points where the curves cross the axes:

i) $y = (x+3)^3$ ii) $y = (1-x)^3$ iii) $y = -\left(x - \frac{1}{2}\right)^3$ iv) $y = (-x+2)^3$

f). Apply the following transformation to the curves with equations $y = f(x)$ where:

(i) $f(x) = x^2$ (ii) $f(x) = x^3$ (iii) $f(x) = \frac{1}{x}$

In each case state the coordinates of points where the curves cross the axes and in **(iii)**state the equations of any asymptotes.

a) $f(x+2)$ b) $f(x) + 2$ c) $f(x-1)$ d) $f(x) - 1$

g). Apply the following transformation to the curves with equations $y = f(x)$ where:

i) $f(x) = x^2$ ii) $f(x) = x^3$ iii) $f(x) = \frac{1}{x}$

In each case show both $f(x)$ and the transformation on the same diagram.

(a) $f(2x)$ (b) $f(-x)$ (c) $2f(x)$ (d) $4f(x)$ (e) $\frac{1}{4}f(x)$

2. Sketch the following rational functions:

i) $y = \frac{3}{x+1}$ ii) $f(x) = \frac{x^2+2}{x-1}$ iii) $f(x) = \frac{x^2}{x^2+x-3}$ iv) $f(x) = \frac{x+2}{x-2}$ v) $f(x) = \frac{x^2-5x+6}{x-2}$

vi) $f(x) = \frac{2x^4}{x^4+1}$ vii) $f(x) = \frac{2x^2}{x^2+4}$

3. A) Sketch the following and find the domains :

i) $f(x) = 1 - \sqrt{2-3x}$ ii) $f(x) = \sqrt{x+3}$ iii) $f(x) = 1 + \sqrt{\frac{x}{2}}$ iv) $f(x) = -\sqrt{-x+3}$

v) $f(x) = 2 + \sqrt{-3x+2}$ vi) $f(x) = -\sqrt{x}$ vii) $f(x) = 2 + 3\sqrt{-x+1}$ ix) $f(x) = 2\sqrt{-x+1}$

B) Graph each of the following:

i) $f(x) = \begin{cases} -x^2 & for\ x \geq 0 \\ 2x^2 & for\ x < 0 \end{cases}$ ii) $f(x) = \begin{cases} 2x+3 & if\ x < 0 \\ x^2 & if\ 0 \leq x < 2 \\ 1 & if\ x \geq 2 \end{cases}$

iii) $f(x) = \begin{cases} 2 & if\ x > 2 \\ 1 & if\ 0 < x \leq 2 \\ -1 & if\ x \leq 0 \end{cases}$ iv) $f(x) = \begin{cases} 2x+1 & if\ x \geq 0 \\ x^2 & if\ x < 0 \end{cases}$

Functions

1. Determine whether each of the given relation below is a function or not .

   (i) $y = x^2$ (ii) $y = \begin{cases} 2x+3\ , & x \leq 1 \\ 6 - x^2\ , & x \geq 1 \end{cases}$ (iii) $y = \begin{cases} x^3, & x < 2 \\ \frac{1}{x}, & x \geq 2 \end{cases}$ (iv) $y = 3x - 1$

2. For each of the following functions, find the image of -2, $-\frac{1}{2}$, 0 , 4 ,7

   (i) $f(x) = 1 - 6x$ (ii) $g(x) = (x-1)^2$ (iii) $h(x) = \frac{x-1}{x+1}$

3. Find the values of the constants in each of the following:

   (i) $f(x) = ax + b$, a and b are constants, $f(-2) = 7, f(1) = -1$

   (ii) $f(x) = ax^2 + bx + c$, a, b and c are constants, $f(0) = 7, f(1) = 6$and $f(-1) = 12$

4. Determine the domain of the following functions:

   i) $f(x) = 2x + \sqrt{x^2 + 4x - 12}$ ii) $h(x) = x + \sqrt{3x - 4}$

   iii) $k(x) = \sqrt{x} + \sqrt{x - 1}$ iv) $L(x) = -x^2 + 2x - 7$ v) $f(x) = x^2 - 2x + 2$

   vi) $f(x) = \sqrt{x^2 - 2x - 24} + 2$ vii)$f(x) = \sqrt[3]{x^2 - 4}$ viii) $f(x) = \sqrt[5]{x + 3}$

   ix) $k(x) = \sqrt[4]{2x + 1}$ x) $h(x) = \sqrt[8]{x^3 - 2}$ xi) $f(x) = x^2 + 4x - 1$

   xii) $f(x) = x^2 + \sqrt{x^2 + 4x - 12}$ xiii) $h(x) = x^2 + \sqrt{x - 1}$

NOT that for Questions 4 and 5 , determine the domain and the range of the function

5. The function f is defined by

$$f(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x & x \geq 0 \end{cases}$$

   a) Sketch $f(x)$.
   b) Find the value(s) of a such that $f(a) = 43$.
   c) Find the values of the domain that get mapped to themselves in the range.

6. a) The domain of the function $g(x) = \frac{2x-1}{x^2-1}$ is { 1,2,3,4}. Find the range of the function

b) The range of the function $g(x) = 1 - \frac{3}{x}$ is { -2,4, 5} , Find the domain of the function.

7. For each of the given functions find $\frac{f(a+h)-f(a)}{h}$.

(i) $f(x) = 4x + 5$ (ii) $f(x) = x^2 - 3x$ (iii) $f(x) = -x^2 + 4x - 2$

8. Determine whether the function f is even, odd or neither.

(i) $f(x) = x^2 + x$ (ii) $f(x) = \sqrt{2 - x^2}$ (iii) $f(x) = x^2$ (iv) $f(x) = \frac{1}{x}$

(v) $f(x) = 3x - 1$ (vi)$f(x) = x^5 + x^3 + x$ vii)$f(x) = x^4 + x^2 + 1$

(viii) $f(x) = x^3 + 1$ ix) $f(x) = x^2 + 1$ $f(x) = -x^3$

9. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ for the following and their domains

(i) $f(x) = 2x$, $g(x) = 3x - 1$. (ii) $f(x) = \frac{1}{x}$,$g(x) = 3x - 1$

(iii) $f(x) = \sqrt{x - 2}$, $g(x) = 3x - 1$ (iv) $f(x) = \frac{4}{x+2}$, $g(x) = \frac{3}{2x}$

10. Solve each of the following problems

(i) If $f(x) = x^2 - 2$ and $g(x) = x + 4$, find $(f \circ g)(2)$ and $(g \circ f)(-4)$

(ii) If $f(x) = \frac{1}{x}$ and $g(x) = 2x + 1$, find $(f \circ g)(1)$ and $(g \circ f)(2)$

(iii) If $f(x) = \sqrt{x + 1}$ and $g(x) = 3x - 1$, find $(f \circ g)(4)$ and $(g \circ f)(4)$

(iv) If $f(x) = x + 5$ and $g(x) = |x|$, find $(f \circ g)(-4)$ and $(g \circ f)(-4)$

11. Show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$

(i) $f(x) = 2x$,$g(x) = \frac{1}{2}x$ (ii) $f(x) = 3x + 4$, $g(x) = \frac{x-4}{3}$

(iii) $f(x) = 4x - 3$, $g(x) = \frac{x+3}{4}$

12. If $f(x) = 3x - 4$ and $g(x) = ax + b$, find the conditions on a and b that guarantee that

$$(f \circ g)(x) = (g \circ f)(x)$$

13. Let $f(x) = \frac{x}{x+2}$ and $g(x) = 2x - 1$.

(i) Find $(f \circ g)(x)$

(ii) Evaluate $(g \circ f)\left(\frac{3}{4}\right)$

(iii)Verify that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

14. Verify that the two given functions are inverses of each other

(i) $f(x) = -\frac{1}{2}x + \frac{5}{6}$ and $g(x) = -2x + \frac{5}{3}$

(ii) $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$

(iii) $f(x) = \sqrt{2x-4}$ for $x \geq 0$ and $g(x) = \frac{x^2+4}{2}$

(iv) $f(x) = x^2 - 4$ for $x \geq 0$ and $g(x) = \sqrt{x+4}$ for $x \geq -4$

15. Find f^{-1} and verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = (x)$

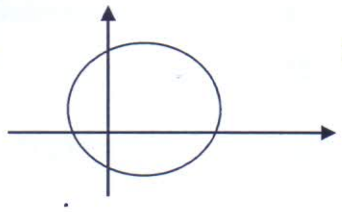
(i) $f(x) = \sqrt{x}$ for $x \geq 0$ (ii) $f(x) = \frac{1}{x}$ for $x \neq 0$ (iii) $f(x) = \frac{3}{4}x - \frac{5}{6}$

16. The function $f(x)$ is defined by $f(x) = x^2 - 3$ $\{x \in R,\ x > 0\}$

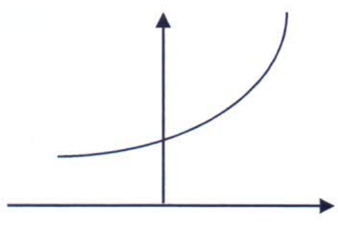
i) Find $f^{-1}(x)$ ii) Sketch $f^{-1}(x)$ iii) Find values of x such that $f(x) = f^{-1}(x)$

17. State whether or not the following graphs represent functions. Give reasons for your answers and describe the type of the function.

i)



ii)



18. The following mappings f and g are defined on all the real numbers by

$$f(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x \geq 4 \end{cases} \qquad g(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x > 4 \end{cases}$$

Explain why $f(x)$ is a function and $g(x)$ is not .

19. State if the following functions is one to one or many to one

i) $f(x) = 3x + 2$ for the domain $\{x > 0\}$

ii) $f(x) = x^2 + 5$ for the domain $\{x \geq 2\}$

iii) $f(x) = +\sqrt{x+2}$ for the domain $\{x \geq -2\}$

21. Find $f + g,\ f - g,\ f \cdot g,\ and\ \frac{f}{g}$ and determine their domain

i) $f(x) = \sqrt{x-1}\ ,\ g(x) = \sqrt{x}$ ii) $f(x) = \sqrt{x+2}\ ,\ g(x) = \sqrt{3x-1}$

iii) $f(x) = x^2 - 2x - 24\ ,\ g(x) = \sqrt{x}$ iv) $f(x) = -6x - 1,\ \ g(x) = -x - 1$

22) Find $(f°g)(x)\ \ and\ \ (g°f)(x)$ .Also specify the domain for each.

i) $f(x) = \frac{1}{x},\quad g(x) = 2x + 7$ ii) $f(x) = \sqrt{x-2}\,,\, g(x) = 3x - 1$

iii) $f(x) = \frac{1}{x-1},\quad g(x) = \frac{2}{x}$ iv) $f(x) = \frac{4}{x+2},\ g(x) = \frac{3}{2x}$

23). Determine whether the function is one-to-one . If it is,find the inverse and graph both the function and its inverse.

(i) $f(x) = x^3 - 2$ ii) $f(x) = \frac{x}{\sqrt{x^2+4}}$ iii) $f(x) = \sqrt{x^2+1}$ iv) $f(x) = \frac{x}{x+4}$

v) $f(x) = x^2 + 3$ vi) $f(x) = x^3$

24) . Determine the domain of the following fuctions

a). $f(x) = \frac{6}{\sqrt{6x-2}}$ (b). $g(y) = \sqrt{\frac{y}{y-8}}$ (c). $h(x) = \sqrt{x-7} + \sqrt{9-x}$

d). $f(x) = \sqrt{\frac{x+1}{x-1}}$ (e). $f(x) = \sqrt[3]{x}$

25) . Graph the function

$$\Bigg\{$$

$$f(x) = \begin{cases} 4 - x^2 , & \text{if } x \leq 1 \\ x - 3 , & \text{if } x > 1 \end{cases}$$

26). Given $f(x) = \frac{x}{|1-x|}$

(i). Find the domain of $f(x)$.
(ii). Sketch the graph of $f(x)$.
(iii). Find the range of $f(x)$

27). Suppose $f(x) = \frac{1+x}{x^2-2x+1}$

(i). Find the domain of $f(x)$
(ii). Find the vertical asymptotes if any
(iii). Find the horizontal asymptotes if any
(iv). Sketch the graph of $f(x)$.

28.) Sketch the graph of $f(x) = |2x+1|$. On the same diagram sketch also the graph of $g(x) = \sqrt{1-2x}$ and , hence, find the values such that $\sqrt{1-2x} > |2x+1|$

29). In each of the following determine whether f is even, odd or neither

i). $f(x) = x^4$ (ii). $f(x) = x + x^3$ (iii). $f(x) = x^3 - x$ (iv). $f(x) = 2 - x$ 30. Given the functions $f(x) = 3x + 1,\ h(x) = \frac{1}{x+1}$, $g(x) = x^3$ and $k(x) = \sqrt{x}$;
Find (i). $(fog)(x)$ (ii). $(gof)(x)$ (iii). $(kok)(x)$ (iv). $f[g(k(x))]$

31) . (i) If $f(x) = 3x - 4$ and $g(x) = ax + b$, find the conditions of a and b such that $(fog)(x) = (gof)(x)$.

(ii) If $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$. Find the domain and inverse of $(fog)(x)$ and $(gof)(x)$.

32). If $f(x) = 3 - x$ and $g(x) = \frac{3x}{x-3}$, $x \neq 3$. Show that this function is its own inverse. Is $(fog)(x)$ even, odd or neither.

33) . If $f(x) = 1 - x^2$ and $(x) = \sqrt{x}$. Find $(gof)(x)$ and $(fog)(x)$ and their domains

34). Show that the function f defined by $f(x) = x/\sqrt{x^2+1}$,$x \in R$, is a bijection on R on to $\{y: -1 < y < 1\}$

35) .Let $A = B = \{x \in R: -1 \le x \le 1\}$ and consider the subset $C = \{(x, y): x^2 + y^2 = 1\}$ of $A \times B$.Is this set a function ? Explain

36). a) If $f(x) = x^2$ and $g(x) = 3x - 4$,find $(gof)(x)$ and$(fog)(x)$and determine its domain

b) If $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$,find $(fog)(x)$ and determine its domain.

c) If $f(x) = \frac{2}{x-1}$ and $(x) = \frac{1}{x}$, find $(fog)(x)$ and determine its domain

d) If $f(x) = 1 - x^2$ and $g(x) = \sqrt{x}$, find $(gof)(x)$ and$(fog)(x)$ determine their domains