MSM111 – Mathematical Methods I
SET THEORY

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Learning outcomes

This lecture develops work on sets. At the end of this session, students will be able to;

Outcomes

1. Discuss sets of numbers.
2. Define a set.
3. Carry out some set operations.
4. Apply set theory in solving some practical problems.

Definition (**Set**)

A set is a list or collection of well-defined objects. These objects are called elements or members of the set.

Sets are denoted by capital letters e.g, A, B, $\cdots$ and its elements are denoted by small letters e.g, $a, b, \cdots$.
If an element a is in a set A, we write

$$a \in A$$

and say that "a is a member (element) of A", or that "a belongs to A". If a is not in A, then we write

$$a \notin A.$$

Note: the elements are separated by commas and enclosed in braces $\{\}$.

Methods for Describing a Set

A set is often specified by:

(i) Roster/Listing Method : For example

$$A = \{a, e, i, o, u\}$$

denotes the set A whose members are vowels.

(ii) **Rule Method /Set-builder:** For example

$$B = \{x : x > 0 \quad , \quad x \in \mathbb{Z}\}$$

which reads "B is the set of x such that x is greater than zero and x is an integer".

Definition (**Finite and Infinite Sets**)

A set is said to be finite if it contains a finite number of elements or no elements otherwise it is infinite.

Example

(i) Let X be the set of the days of the week:- $X = \{$Monday, Tuesday, Wednesday, Thursday, Friday$\}$. Then X is a finite set.

(ii) Let $Y = \{2, 4, 6, 8, ...\}$. Then Y is infinite.

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Definition (**Empty/Null/Void -Set**)

A set which contains no elements is called an **empty** or **null set** and is denoted by $\emptyset$ or $\{\}$.

The empty set is considered to be finite and a subset of every other set, A**,**

$$\emptyset \subset A \subseteq U.$$

Example

For example:

(i) Let $A = \{x : x^2 = 4, x \text{ is odd}\}$. Then, A is an empty set, i.e. $A = \emptyset$.

(ii) Let B be the set of people in the world who are older than 2000 years. According to known statistics, B is an empty set.

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Definition (**Singleton Set**)

A singleton set is a set containing only one element.

Example

Let $A = \{2\}$, and let $B = \{1, 2, 3\}$. Then, A ia a singleton set but B is not.

Definition (**Subset and Superset**)

A set A is a subset of a set B or equivalently B is a superset of A, written

$$A \subseteq B \quad \textit{or} \quad B \supseteq A.$$

if every element of a set A also belongs to a set B. That is, $x \in A$ implies $x \in B$. We also say that A is contained in B or B contains A.

The negation of $A \subseteq B$ is written $A \nsubseteq B$ or $B \nsupseteq A$ and states that, there is $x \in A$ such that $x \notin B$.

Example

Consider the sets $A = \{1, 3, 5, 7, ...\}$ $B = \{5, 10, 15, 20, ...\}$ and $C = \{x : x > 2, x \text{ is prime}\} = \{3, 5, 7, 11, ...\}$.

Then $C \subseteq A$ since every prime number greater than 2 is old. On the other hand, $B \nsubseteq A$ since $10 \in B$ but $10 \notin A$

Definition (**Equal and Equivalent Sets**)

(i) Two sets A and B are said to be equal, if every element of A is a member of B and every element of B is a member of A and we write $A = B$. That is, A and B contain exactly the same elements.

Thus, to prove that the sets A and B are equal, we must show that

$$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

If it is not true that A and B are equal, then we write $A \neq B$.

(ii) Two sets A and B are said to be equivalent and we write $A \equiv B$ if A and B contain the same number of elements, not necessarily exact.

Example

(i) The set $E = \{2, 4, 6\}$ is a subset of the set $F = \{6, 2, 4\}$, since each element 2, 4 and 6 belongs to F. In fact, $E = F$. In the similar manner it can be shown that every set is a subset of itself.

(ii) For the sets $\mathbb{N}$ of natural numbers, $\mathbb{Z}$ of integers, $\mathbb{Q}$ of rational numbers, $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers, we have that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

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$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Definition (**Proper Subset**)

A set A is said to be a proper subset of a set B if $A \subseteq B$, but there is at least one element of B that is not in A, i.e., $A \neq B$. In this case we sometimes write

$$A \subset B.$$

Definition (Cardinality of a Set)

The number of elements in a set A is called the cardinality of A and is denoted $n(A)$ or $|A|$.

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Definition (Disjoint Sets)

Two sets A and B are called disjoint, if they do not have any common element, that is if

$$A \not\subseteq B \quad and \quad B \not\subseteq A.$$

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Definition (**Universal Set**)

A set consisting of all possible elements which occurs under consideration, is called a **universal set** and is denoted by U.

In any application of the theory of sets, all sets under investigation are regarded as subsets of a fixed set.

Definition (**Number of Subset of a Finite Set**)

The number of subsets of a given finite set (i.e., a set with n elements) A is defined by

$$2^n$$

Example

Consider the sets $A = \{1, 3\}$ and $B = \{a, b, c, d, e\}$. Then

$$\textbf{Number of subsets of A} = 2^2 = 4,$$

since the subsets of A are $\emptyset$, $\{1\}$, $\{3\}$ and $\{1, 3\}$. And

$$\textbf{Number of subsets of B} = 2^5 = 32,$$

Definition (**Power Set**)

The set formed by all the subsets of a given set A, is called the **power set** of A, denoted by $P(A)$ or 2^A.

Example

Consider the sets $A = \{1, 3\}$ and $B = \{a, b, c\}$. Then, the subsets of A are $\emptyset$, $\{1\}$, $\{3\}$ and $\{1, 3\}$. So,

$$P(A) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$$

and subsets of B are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$ $\{a, c\}$, $\{b, c\}$and $\{a, b, c\}$. So,

$$P(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Operations on Sets

We now define the methods of obtaining new sets from given ones.

Definition

Let A and B be nonempty sets and U be the universal set.

(i) The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to either A or B. In set-builder notation, this is written

$$A \cup B = \{x : x \in A \quad or \quad x \in B\}$$

(ii) The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of all elements which belong to both A and B. In set-builder notation, this is written

$$A \cap B = \{x : x \in A \quad and \quad x \in B\}$$

Hence, if $A \cap B = \emptyset$ then the sets A and B are said to be **disjoint** or **non-intersection**.

(iii) The **relative complement (or difference)** of a set B with respect to set A, denoted by $A \setminus B$, is the set of elements which belong to A but do not belong to B. In set-builder notation, this is written

$$A - B = A \setminus B = \{x : x \in A \quad and \quad x \notin B\}$$

Observe that $A \setminus B$ and B are disjoint: $(A \setminus B) \cap B = \emptyset$.

(iv) The **absolute complement**, or simply, **complement** of A denoted by A^c, is the set of elements which do not belong to A. In set-builder notation, this is written

$$A^c = \{x : x \in U, \quad x \notin A\}$$

That is A^c is the difference of the universal set and A: The set $U \setminus A = U \cap A^c$.

Example

Let $U = \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. Given that $A = \{0, 2, 4, 6, 10, 12\}$, $B = \{0, 3, 6, 12, 15\}$ and $C = \{1, 2, 3, 4, 5, 6, 7\}$, list

(i) $A \cup C$ (ii) $B \cap C$ (iii) $A \cap (B \cup C)$

(iv) $B \cup (A \cap C)$ (v) $A \cup B^c$ (vi) $(A \cap C)^c$

Solution.

(i)

$$\begin{aligned} A \cup C &= \{0, 2, 4, 6, 10, 12\} \cup \{1, 2, 3, 4, 5, 6, 7\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 10, 12\}. \end{aligned}$$

(ii) $B \cap C = \{0, 3, 6, 12, 15\} \cap \{1, 2, 3, 4, 5, 6, 7\} = \{3, 6\}$.

Solution Contn'd.

(iii) First determine $B \cup C = \{0, 3, 6, 12, 15\} \cup \{1, 2, 3, 4, 5, 6, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 12, 15\}$. Then

$$\begin{aligned} A \cap (B \cup C) &= \{0, 2, 4, 6, 10, 12\} \cap \{0, 1, 2, 3, 4, 5, 6, 7, 12, 15\} \\ &= \{0, 2, 4, 6, 12\} \end{aligned}$$

(iv) First determine $A \cap C = \{0, 2, 4, 6, 10, 12\} \cup \{1, 2, 3, 4, 5, 6, 7\} = \{2, 4, 6\}$. Then

$$B \cup (A \cap C) = \{0, 3, 6, 12, 15\} \cup \{2, 4, 6\} = \{0, 2, 3, 4, 6, 12, 15\}.$$

$\square$

Solution Contin'd.

(v) First determine $B^c = \{1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14\}$. Then

$$\begin{aligned} A \cup B^c &= \{0, 2, 4, 6, 10, 12\} \cup \{1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14\} \\ &= \{0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}. \end{aligned}$$

(vi) First determine
$A \cap C = \{0, 2, 4, 6, 10, 12\} \cap \{1, 2, 3, 4, 5, 6, 7\} = \{2, 4, 6\}$

$$\begin{aligned} (A \cap C)^c &= [\{2, 4, 6\}]^c \\ &= \{0, 1, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \end{aligned}$$

$\square$

(v) **Symmetric Difference of Two Sets:** Let A and B be sets. The symmetric difference of A and B, denoted by $A \Delta B$ is defined as

$$
\begin{aligned}
A \Delta B &= \{x : x \in A \text{ or } x \in B, \text{ but not both}\} \\
&= (A - B) \cup (B - A) = (A \cap B^c) \cup (B \cap A^c)
\end{aligned}
$$

Example: If $A = \{2, 1, 3, 5\}$ and $B = \{x, t, 7, 1\}$, then $A \cup B = \{1, 2, 3, 5, x, t, 7\}$ and $A \cap B = \{1\}$. Therefore,

$$A \Delta B = \{2, 3, 5, x, t, 7\}$$

(vi) **Cartesian Product of Sets:** Let A and B be sets. The cartesian product of A and B, denoted by $A \times B$ is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Example: If $A = \{0, 1, 2\}$ and $B = \{a, b\}$, then

$$A \times B = \{(0, a), (0, b), (1, a), (1, b), (2, 1), (2, b)\}$$

Sets under the above set operations satisfy various laws or identities which we shall next list.

Theorem (Laws of the Algebra of Set)

Sets satisfy the following algebraic laws

(i) **Idempotent Laws**

$$A \cup A = A \qquad A \cap A = A$$

(ii) **Commutative Laws**

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

(iii) **Associative Laws**

$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

Theorem 22 Contin'd.

(iv) **Distributive Laws**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(v) **Identity Laws**

$$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset, \quad A \cup U = U, \quad A \cap U = A$$

(vi) **Complement Laws**

$$A \cup A^c = U, \quad A \cap A^c = \emptyset, \quad (A^c)^c = A \quad U^c = \emptyset, \emptyset^c = U$$

(vii) **De-Morgan's Laws**

$$(A \cup B)^c = A^c \cap B^c \quad (A \cap B)^c = A^c \cup B^c$$

Venn Diagram

In a Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle or a closed geometrical figure inside the universal set as in the figure 1

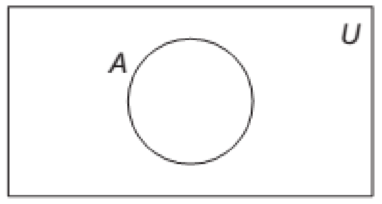

Figure: Venn Diagram

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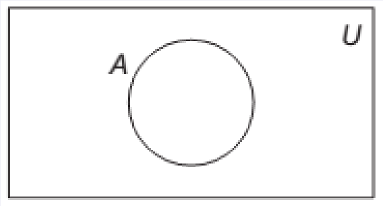

Figure: Venn Diagram

Example: Use of Venn diagrams to solve counting problems.

In a class of 50 college freshmen, 30 are studying C^{++}, 25 are studying Java and 10 are studying both languages.

1. How many freshmen are studying either computer language?
2. Determine $|A^c \cap B^c|$.

Solution

We let the universal set be
$U = \{\text{class of 50 freshmen}\}$,
$A = \{\text{students studying } C^{++}\}$,
$B = \{\text{students studying Java}\}$.

1. To answer the question we need $|AUB|$.
   $|A| = 30$, $|B| = 25$ and $|A \cap B| = 10$. From the Inclusion Exclusion Principle, we have that

$$|AUB| = |A| + |B| - |A \cap B| = 30 + 25 - 10 = 45$$

2. By De Morgan's Law $A^c \cap B^c = (A \cup B)^c$. Hence,

$$|A^c \cap B^c| = |(A \cup B)^c| = |U| - |AUB| = 50 - 45 = 5$$

Note: $|A^c \cap B^c|$ is the number of students who did not study any of the two languages.

SETS OF NUMBERS

There are certain sets of numbers that appear frequently in set theory and in many branches of mathematics. We begin this section by studying the decomposition of the real line into the following subsets:

(1) Natural Numbers $(\mathbb{N})$: are counting numbers i.e. $\mathbb{N} = 1, 2, 3, \cdots$.

(2) Integers $(\mathbb{Z})$: are natural numbers with their opposites (natural numbers with a negative sign) and includes 0 i.e. $\mathbb{Z} = \cdots, 0, 1, 2, 3, \cdots$.

(3) Rational Numbers $(\mathbb{Q})$: are numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. For example $\frac{1}{2}$, $46 = \frac{46}{1}$, $0.17 = \frac{17}{100}$. **Note:** The opposite of rational numbers are irrational numbers $(\mathbb{R})$. For example; $\sqrt{2}$, $\sqrt{3}$ are irrational numbers because they cannot be expressed as ratios of integers.
Note: Rational numbers can either be terminating or repeating decimal. e.g. 0.75(terminating decimal) and $0.\bar{3}$ (repeating decimal)

Example

Express the following rational numbers in the form $\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \in 0$ in their lowest terms:

1. $0.\overline{6}$
2. $0.3\overline{17}$

Example

Express the following rational numbers in the form $\frac{a}{b}$, $a, b \in \mathbb{Z}, b \in 0$ in their lowest terms:

1. $0.\overline{6}$
2. $0.3\overline{17}$

Solution

1. $$\text{Let } x = 0.\overline{6} \qquad (1)$$
   $$\text{Then, } 10x = 6.\overline{6} \qquad (2)$$

   Subtracting equation (1) from (2) we get,

   $$9x = 6 \implies x = \frac{1}{3}$$

Example

Prove that $\sqrt{2}$ is irrational.

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Solution

We prove by contradiction. Suppose that $\sqrt{2}$ is rational i.e. $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$. And that a and b have no common factor. Then,

$$\sqrt{2} = \frac{a}{b} \implies 2 = \left(\frac{a}{b}\right)^2$$
$$\implies 2 = \frac{a^2}{b^2} \implies a^2 = 2b^2$$

Thus, a^2 is divisible by 2 implying that a is also divisible by 2. Since a is divisible by 2. it can be writtem as $a = 2k$, $k \in \mathbb{Z}$.

Cont'

Then,

$$
\begin{aligned}
(2k)^2 &= 2b^2 \\
\implies 4k^2 &= 2b^2 \\
\implies b^2 &= 2k^2.
\end{aligned}
$$

Thus, b^2 is divisible by 2 implying that b is also divisible by 2. This means that a and b have a common factor 2. This is a contradiction. Hence, $\sqrt{2}$ is irrational.

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Thus, b^2 is divisible by 2 implying that b is also divisible by 2. This means that a and b have a common factor 2. This is a contradiction. Hence, $\sqrt{2}$ is irrational.

Task

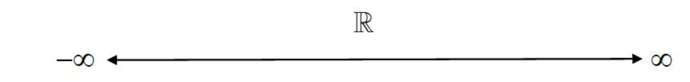
Prove that $\sqrt{3}$ is irrational.

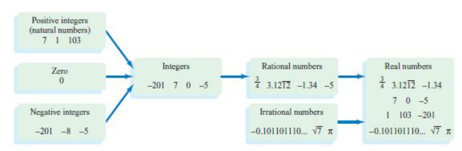
CONT'

(4) A Prime Number:is a positive integer greater than 1 that has no positive factors other than iteself and 1. For example, $2, 3, 5, 7, 7, 11, 13, 17, 19, 23, 29, \cdots$

(5) A composite number: is a positive integer greater than 1 that is not a prime number. For example, $4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \cdots$

(6) Real Numbers $(\mathbb{R})$: are a collection of rational and irrational numbers. Any number on the continous line is a real number.





Example

Determine which of the following numbers $-0.2, 0, 0.\overline{3}, 0.717717771777771\cdots, \pi, 6, 7, 41, 51$ are:

1. integers
2. rational numbers
3. irrational numbers
4. real numbers
5. prime numbers
6. composite numbers

Solution

1. $0, 6, 7, 41, 51$
2. $-0.2, 0, 0.\overline{3}, 6, 7, 41, 51$
3. $0.71771777177771\cdots, \pi$
4. $-0.2, 0, 0.\overline{3}, 0.71771777177771\cdots, \pi, 6, 7, 41, 51$
5. $7, 41$
6. $6, 51$

REVIEW OF PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$ $(b + c)a = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$ $(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we get if we multiply the number by each of the terms and then add the results.

REVIEW OF PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

REVIEW OF PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2+7}{5} = \frac{9}{5}$	When **adding fractions with the same denominator**, add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When **adding fractions with different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross-multiply**.

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7 Complex Numbers $(\mathbb{C})$:are numbers that can be expressed as $z = x + iy$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. x is referred to as the real part while y is referred to as the imaginary part of a complex number i.e. $Re(z) = x$ and $Im(z) = y$.

INTERVAL NOTATION

The set of real numbers can be expressed in the form of an interval i.e. $\mathbb{R} = (-\infty, \infty)$.

Definition

In general, the interval notation

(i) (a, b) represents all real numbers between a and b not including a and b. This is an **open interval**. In set-builder notation, we write $\{x : a < x < b, x \in \mathbb{R}\}$.

(ii) $[a, b]$ represents all real numbers between a and b including a and b. This is a **closed interval**. In set-builder notation, we write $\{x : a \leq x \leq b, x \in \mathbb{R}\}$.

(iii) $(a, b]$ represents all real numbers between a and b not including a but including b. This is a **half-open interval**. In set-builder notation, we write $\{x : a < x \leq b, x \in \mathbb{R}\}$.

(iv) $[a, b)$ represents all real numbers between a and b including a but not including b. This is a **half-open interval**. In set-builder notation, we write $\{x : a \leq x < b, x \in \mathbb{R}\}$.

□

Note: Subsets of the real numbers whose graphs extend forever in one or both directions can be represented by interval notation using the infinity symbol ∞ or the negative infinity symbol $-\infty$.

Definition

The set

(i) $(-\infty, a)$ represents all real numbers less than a. In set-builder notation, we write $\{x : x < a, x \in \mathbb{R}\}$.

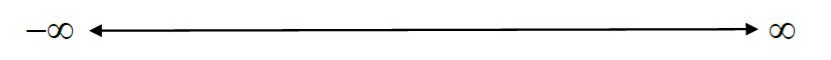
(ii) (a, ∞) represents all real numbers greater than a. In set-builder notation, we write $\{x : x > a, x \in \mathbb{R}\}$.

Cont'

(iii) $(-\infty, a]$ represents all real numbers less than or equal to a. In set-builder notation, we write $\{x : x \leq a, x \in \mathbb{R}\}$.

(iv) $[a, \infty)$ represents all real numbers greater than or equal to a. In set-builder notation, we write $\{x : x \geq a, x \in \mathbb{R}\}$.

(v) $(-\infty, \infty)$ represents all real numbers. In set-builder notation, we write $\{x : -\infty < x < \infty, x \in \mathbb{R}\}$.

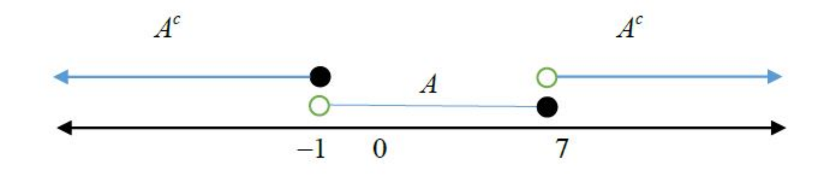


Example

Given that the universal set is $\mathbb{R}$ and that $A = (-1, 7]$, $B = [-5, 3]$ and $C = [-1, 10]$, find each of the following sets and display on the number line:

1. A^c
2. $B - A = B \setminus A$
3. $(B \cap A) \cap C^c$

Solution.

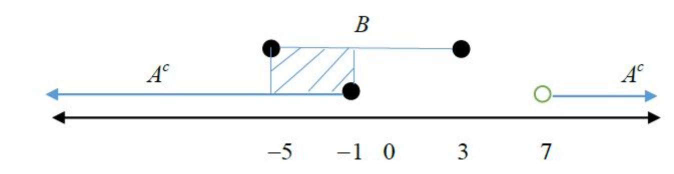
(1) 

$\therefore A^c = (-\infty, -1] \cup (7, \infty)$

□

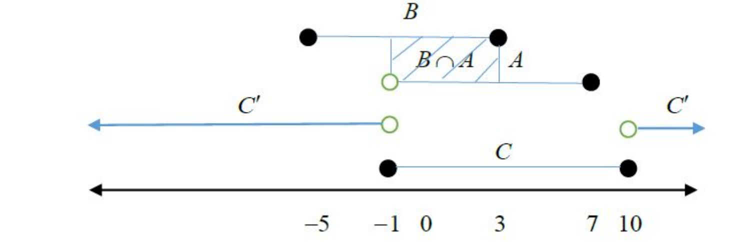
Solution Cont'

(2)



$\therefore B - A = B \cap A^c = [-5, -1]$

(3)



$B \cap A = (-1, 3]$ and $C' = C^c = (-\infty, -1) \cup (10, \infty)$
$\therefore B - A = B \cap A^c = [-5, -1]$

Example

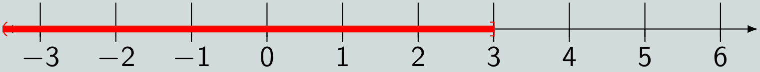
Graph $(-\infty, 3]$. Write the interval in set-builder notation.

Solution.

The set is of real numbers less than or equal to 3. In set-builder notation, this is the set

$$\{x : x \leq 3\}$$

Draw a square-right bracket at 3, and darken the number line to the left of 3, as shown:



Example

Write (i) and (ii) using interval notation. Write (iii) and (iv) using set-builder notation.

(i) $\{x : x \leq -1\} \cup \{x : x \geq 2\}$ (ii) $\{x : x \geq -1\} \cap \{x : x < 5\}$

(iii) $(-\infty, 0) \cup [1, 3]$ (iv) $[-1, 3] \cap (1, 5)$

Solution.

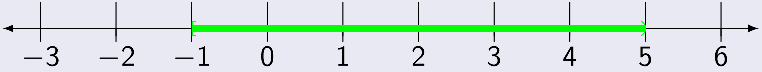
Here,

(i) $\{x : x \leq -1\} \cup \{x : x \geq 2\}$ is the set of real numbers that are either less than or equal to -1 or greater than or equal to 2. Hence, in interval form, we write

$$(-\infty, -1] \cup [2, \infty)$$

Solution Contin'd.

(ii) $\{x : x \geq -1\} \cap \{x : x < 5\}$ is the set of real numbers that are greater than or equal to -1 and less than 5. Hence, in interval form, we write $[-1, \infty) \cap (-\infty, 5) = [-1, 5)$



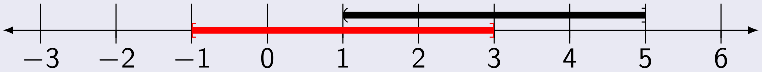
(iii) $(-\infty, 0) \cup [1, 3] \iff \{x : x < 0\} \cup \{x : 1 \leq x \leq 3\}$

□

Solution Contin's.

(iv) For $[-1,3]\cap(1,5)$, observe that $1\in[-1,3]$ but $1\notin(1,5)$. Therefore, 1 does not belong to the intersection of the sets. On the other hand $3\in[-1,3]$ and $3\in(1,5)$. Therefore, 3 belongs to the intersection of the sets. Thus we have the following

$$\begin{aligned}[-1,3]\cap(1,5) &= \{x:-1\le x\le 3\}\cap\{x:1<x<5\}\\ &= \{x:1<x\le 3\}.\end{aligned}$$



Note that the intersection of the sets occurs where the graphs intersect. □

Example

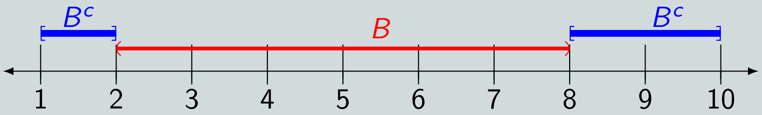
Let $X = [1, 10]$ be the universal st and $A = [1, 4]$, $B = (2, 8)$ and $C = [3, 6)$ be the subsets of X. Find each of the following sets and display them on the real line.

(i) B^c (ii) $A \cap B$ (iii) $A \cap (B - C)$

Solution.

(i) $B = (2, 8) = \{x : 2 < x < 8\}$ and so

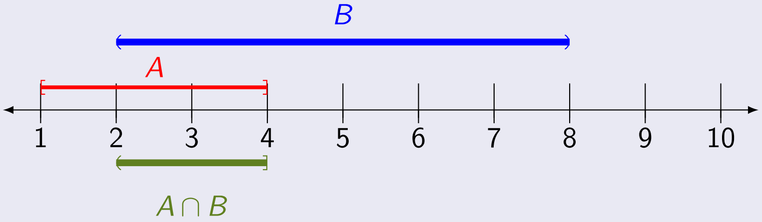
$$\begin{aligned} B^c &= \{x : 1 \leq x \leq 2 \quad or \quad 8 \leq x \leq 10\} \\ &= \{x : 1 \leq x \leq 2\} \cup \{x : \quad 8 \leq x \leq 10\} = [1, 2] \cup [8, 10] \end{aligned}$$



Solution Contn'd.

(ii) Here $A = [1,4] = \{x : 2 \leq x \leq 4\}$ and
$B = (2,8) = \{x : 1 < x < 8\}$ and so,

$$A \cap B = \{x : 1 \leq x \leq 4\} \cap \{x : 2 < x < 8\} = \{x : 2 < x \leq 4\} = (2,4]$$



□

Solution Contin'd.

(iii) $A \cap (B - C) = A \cap B \cap C^c$:- First determine $B - C = B \cap C^c$. Now, since $C = [3, 6)$, then $C^c = [1, 3) \cup [6, 10]$. Hence,

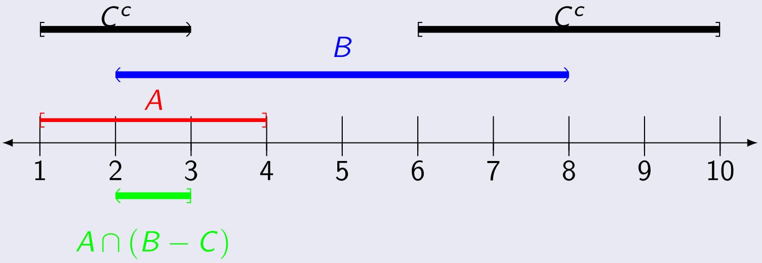
$$\begin{aligned} B - C = B \cap C^c &= (2, 8) \cap \{[1, 3) \cup [6, 10]\} \\ &= \{(2, 8) \cap (1, 3)\} \cup \{(2, 8) \cap [6, 10]\} \\ &= (2, 3) \cup [6, 8) \end{aligned}$$

and so,

$$\begin{aligned} A \cap (B - C) &= [1, 4] \cap \{(2, 3) \cup [6, 8)\} \\ &= \{[1, 4] \cap (2, 3)\} \cup \{[1, 4] \cap [6, 8)\} \quad \textbf{by distrubutive law} \\ &= (2, 3) \cup \varnothing = (2, 3) \end{aligned}$$

$\square$

Solution Contin'd.



Surds and Rationalisation of the Denominator

Definition

A surd is a square or n^{th} root that cannot be expressed as a rational number.

For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $\sqrt[3]{42}$, $\sqrt{110}$ are surds, they can not be evaluated directly. They are irrational numbers.

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Rules of Surds

(i) **Product Rule:** $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

(ii) **Quotient Rule:** $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

(iii) **Scalar Multiple and/or Sum Rule:** $c\sqrt{a} \pm d\sqrt{a} = (c \pm d)\sqrt{a}$.

It must carefully be noted that,

$$\sqrt{a} \pm \sqrt{b} \neq \sqrt{a \pm b}.$$

Example

Simplify the following as much as possible using some or all of the three rules stated above.

(i) $\sqrt{8} \times \sqrt{2}$

(ii) $\frac{\sqrt{78}}{\sqrt{6}}$

(iii) $11\sqrt{5} - 4\sqrt{5}$

(iv) $\frac{20(\sqrt{3}\times\sqrt{3}\times\sqrt{9})-16(\sqrt{9}\times\sqrt{9})}{\sqrt{8}\times\sqrt{2}}$

Solution.

(i)

$$\sqrt{8} \times \sqrt{2} = \sqrt{8 \times 2} = \sqrt{16} = 4.$$

(ii)

$$\frac{\sqrt{78}}{\sqrt{6}} = \sqrt{\frac{78}{6}} = \sqrt{13}.$$

(iii)

$$11\sqrt{5} - 4\sqrt{5} = (11 - 4)\sqrt{5} = 7\sqrt{5}.$$

Solution Contin'd.

(iv)

$$
\begin{aligned}
\frac{20(\sqrt{3}\times\sqrt{3}\times\sqrt{9})-16(\sqrt{9}\times\sqrt{9})}{\sqrt{8}\times\sqrt{2}} &= \frac{20\sqrt{3\times 3\times 9}-16\sqrt{9\times 9}}{\sqrt{8\times 2}}\\
&= \frac{20\sqrt{81}-16\sqrt{81}}{\sqrt{16}}\\
&= \frac{20\sqrt{81}-16\sqrt{81}}{\sqrt{16}}\\
&= \frac{(20-16)\sqrt{81}}{\sqrt{16}}\\
&= \frac{4\sqrt{81}}{\sqrt{16}}\\
&= \frac{4\times 9}{4}\\
&= 9
\end{aligned}
$$

Example

Simplify $\sqrt{27}$.

Solution.

Here,

$$\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$$

$\square$

Example

Simplify $\sqrt{27}$.

Solution.

Here,

$$\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$$

□

Multiplying a Surd by a Real Number

Any real number, like 3, 7, 49 etc. can be expressed as a root by remembering that $a = \sqrt{a^2}$. For example, we can write 7 as

$$7 = \sqrt{7^2} = \sqrt{49}.$$

Note: Do not get confused and write an incorrect statement, like $\sqrt{7} \times \sqrt{7} = 49$.

Example

Simplify the following as much as possible.

(i) $\sqrt{512} - 15\sqrt{2}$ (ii) $5\sqrt{75} + 2\sqrt{3}$

Solution.

(i)

$$\begin{aligned}\sqrt{512} - 15\sqrt{2} = \sqrt{2 \times 256} - 15\sqrt{2} &= \sqrt{256} \times \sqrt{2} - 15\sqrt{2}\\ &= 16\sqrt{2} - 15\sqrt{2}\\ &= \sqrt{2}\end{aligned}$$

(ii)

$$\begin{aligned}5\sqrt{75} + 2\sqrt{3} = 5\sqrt{25 \times 3} + 2\sqrt{3} &= 5(\sqrt{25} \times \sqrt{3}) + 2\sqrt{3}\\ &= 5(5\sqrt{3}) + 2\sqrt{3}\\ &= 25\sqrt{3} + 2\sqrt{3}\\ &= 27\sqrt{3}\end{aligned}$$

Rationalising the Denominator

When dealing with expressions where surds appear in the denominator, it is normal practice to eliminate all surds in the denominator where possible. This is called **rationalizing the denominator**. We have two types of expression, namely

(i) **Expression of type** $\frac{1}{\sqrt{a}}$: Here, multiply the numerator and denominator by $\sqrt{a}$, giving

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}.$$

(ii) **Expression of type** $\frac{1}{b \pm \sqrt{a}}$: Here, if the expression is

(a) $\frac{1}{b+\sqrt{a}}$, multiply the numerator and denominator by $b - \sqrt{a}$, giving

$$\frac{1}{b+\sqrt{a}} = \frac{1}{b+\sqrt{a}} \times \frac{b-\sqrt{a}}{b-\sqrt{a}} = \frac{b-\sqrt{a}}{b^2+a}.$$

(b) $\frac{1}{b-\sqrt{a}}$, multiply the numerator and denominator by $b+\sqrt{a}$, giving

$$\frac{1}{b-\sqrt{a}} = \frac{1}{b-\sqrt{a}} \times \frac{b+\sqrt{a}}{b+\sqrt{a}} = \frac{b+\sqrt{a}}{b^2+a}.$$

In general, to rationalize the denominator of

$$\frac{1}{a\sqrt{b} \pm c\sqrt{d}},$$

we multiply numerator and denominator by

$$a\sqrt{b} \mp a\sqrt{d}.$$

Example

Rationalise the denominator in the following expression:

(i) $\frac{3\sqrt{5}}{2\sqrt{6}}$ (ii) $\frac{10}{\sqrt{3}-1}$ (iii) $\frac{2\sqrt{3}+\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$

Solution.

(i)

$$\frac{3\sqrt{5}}{2\sqrt{6}} = \frac{3\sqrt{5}}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{3\sqrt{5} \times \sqrt{6}}{2 \times 6} = \frac{3\sqrt{30}}{12}$$
$$= \frac{\sqrt{30}}{4}.$$

(ii)

$$\frac{10}{\sqrt{3}-1} = \frac{10(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{10(\sqrt{3}+1)}{3-1}$$
$$= 5(\sqrt{3}+1)$$

Solution.

(iii) For the case, $\frac{2\sqrt{3}+\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$, we can rationalise the denominator by multiplying both the numerator and denominator by $3\sqrt{2}-\sqrt{3}$:

$$\begin{aligned}
\frac{2\sqrt{3}+\sqrt{2}}{3\sqrt{2}+\sqrt{3}} &= \frac{(2\sqrt{3}+\sqrt{2})(3\sqrt{2}-\sqrt{3})}{(3\sqrt{2}+\sqrt{3})(3\sqrt{2}-\sqrt{3})} \\
&= \frac{6\sqrt{6}-2\sqrt{9}+3\sqrt{4}-\sqrt{6}}{9\sqrt{4}-3} \\
&= \frac{6\sqrt{6}-2(3)+3(2)-\sqrt{6}}{9\sqrt{4}-3} \\
&= \frac{5\sqrt{6}-6+6}{15} \\
&= \frac{\sqrt{6}}{3}.
\end{aligned}$$

□

Example

Express $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ in the form $a\sqrt{2}+b$ where a and b are integers and hence, state the values of a and b.

Solution.

The obvious thing to do here is to rationalise the denominator:

$$\begin{aligned}\frac{\sqrt{2}+1}{\sqrt{2}-1} &= \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \\ &= \frac{2+\sqrt{2}+\sqrt{2}+1}{2-1} \\ &= 3+2\sqrt{2}.\end{aligned}$$

So we see that $a=2$ and $b=3$. □

Binary Operations

Definition

Let A be a set. A binary operation $*$ on A is a well defined function $* : A \times A \longrightarrow A$ on the Cartesian product of A onto itself whose image is in A

Generally, a binary operation on a set is a function that acts on two elements of the set to produces a third element which is unique in the set.

Binary operations are usually denoted by special symbols such as: $+$, $-$, $/$, $\times$, $*$, $\cup$, $\cap$, or, and.

If $a, b \in A$, we usually write $a * b$ instead of $*(a, b)$.

Definition (**Closure Property**)

An operation $*$ on a non-empty set A is said to satisfy the closure property, if

$$a \in A, \quad b \in A \quad \Longrightarrow \quad a * b \in A.$$

An operation $*$, satisfying the closure property is known as a binary operation. In this case, we say that A is closed under the operation $*$.

A set has closure under an operation if the result of performing the operation with two elements of the set is also also an element of the set.

Theorem (Algebraic Laws of Binary Operations)

Let $*$ and $\circ$ be two binary operations on a non-empty set A. Then, we have the following;

(i) **Associative Law:-** $*$ *is said to be associative, if*

$$(a * b) * c = a * (b * c), \quad \forall a, b, c \in A.$$

On the set of real numbers $\mathbb{R}$, addition and multiplication satisfies the associative law, that is they are associative binary operations.

(ii) **Commutative Law:-** $*$ *is said to be commutative, if*

$$a * b = b * a, \quad \textit{forall} \quad a, b \in A$$

Addition and multiplication are commutative binary operations on $\mathbb{Z}$ but subtraction is not a commutative binary operation, since

Theorem 37 Contin'd.

Union and intersection are commutative binary operations on the power set $P(A) = 2^A$ of all subsets of set A. But difference of sets is not a commutative binary operation on 2^A.

(iii) **Distributive Law:-** we say that $*$ is distributed over $\circ$, if

$$a * (b \circ c) = (a * b) \circ (a * c), \quad forall \quad a, b, c \in A$$

This is also called **left distribution** and

$$(b \circ c) * a = (b * a) \circ (c * a), \quad for \;\; all \;\; a, b, c \in A$$

and is also called **right distribution**.

Let $\mathbb{R}$ be the set of all real numbers, then multiplication distributes addition on $\mathbb{R}$. Since $\forall a, b, c \in \mathbb{R}$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

Theorem 37 Contin'd.

(iv) **Identity Element:** an element $e \in A$, if it exist such that

$$a * e = e * a = a, \ \ for \ \ all \ \ a \in A.$$

is called an identity element of A, with respect to $*$.

For addition on $\mathbb{R}$, 0 is the identity element in $\mathbb{R}$, since $\forall a \in \mathbb{R}$,

$$a + 0 = 0 + a = a.$$

For multiplication on $\mathbb{R}$, 1 is the identity element in $\mathbb{R}$, since $\forall a \in \mathbb{R}$,

$$a \cdot 1 = 1 \cdot a = a.$$

For addition on $\mathbb{N}$ the identity element does not exist. But for multiplication on $\mathbb{N}$ the identity element is 1.

□

Theorem 37 Contin'd.

Let $P(A)$ be the power set of a non-empty set A. Then, $\emptyset$ is the identity element for union on $P(A)$ as

$$X \cup \emptyset = \emptyset \cup X = X, \ \ for \ \ all \ \ X \in P(A).$$

Also, A is the identity element for intersection on $P(A)$ since,

$$X \cap A = A \cap X = X, \ \ for \ \ all \ \ X \in P(A)$$

$\square$

Theorem 37 Contin'd.

(v) **Inverse of an Element:-** Let $e \in A$ be the identity element and $a \in A$. We say that a is invertible, if there exists an element $b \in A$ such that

$$a * b = b * a = e$$

In this case, b is called the **inverse** of a and we write, $a^{-1} = b$.

Addition on $\mathbb{N}$ has no identity element and accordingly $\mathbb{N}$ has no invertible element.

Multiplication on $\mathbb{N}$ has 1 as the identity element and no element other than 1 is invertible. Similarly on $\mathbb{Z}$.

$\square$

Example

Define $*$ on $\mathbb{Q}$ by

$$x * y = \frac{xy}{5}.$$

Test for above laws.

Solution.

(i) **Closure Law:** Since $\mathbb{Q}$ is closed with respect to multiplication then $\frac{xy}{5} = x * y \in \mathbb{Q}$. Hence, closure holds in $\mathbb{Q}$ with respect to $*$.

(ii) **Associative law:** For all $x, y, z \in \mathbb{Q}$, we have

$$\begin{aligned} x * (y * z) = x * \left(\frac{yz}{5}\right) = \frac{1}{5}\frac{x(yz)}{5} = \frac{(xy)z}{5}\frac{1}{5} &= \left(\frac{xy}{5}\right) * z \\ &= (x * y) * z \end{aligned}$$

since, multiplication is associative. Hence, associativity law holds in $\mathbb{Q}$ with respect to $*$.

Solution Contin'd.

(iii) **Commutative law:** For $x, y \in \mathbb{Q}$, we obtain

$$x * y = \frac{xy}{5} = \frac{yx}{5} = y * x$$

since multiplication is commutative. Hence, commutativity law holds in $\mathbb{Q}$ with respect to $*$.

(iv) **Identity law:** Let $e \in \mathbb{Q}$ be the identity. Then for any $a \in \mathbb{Q}$,

$$a = e * a = \frac{ea}{5}$$

and so, solving for e in $a = \frac{ea}{5}$, we obtain,

$$e = 5 \in \mathbb{Q}.$$

Hence identity law holds in $\mathbb{Q}$ with respect to $*$.

$\square$

Proof Contin'd.

(v) **Inverse law**: Suppose $a, b \in \mathbb{Q}$ are such that

$$a * b = e = 5.$$

Then $\frac{ab}{5} = 5$, and solving for b, we get that

$$b = \frac{25}{a}.$$

Since

$$\frac{25}{a} \in \mathbb{Q},$$

we conclude that $\frac{25}{a}$ is the inverse of a. Hence, inverse law holds in $\mathbb{Q}$ with respect to $*$.

$\square$

Example

Define $*$ on $\mathbb{Z}$ by

$$x * y = x + y - 1.$$

Do closure, associative, commutative, identity and inverse laws hold in $\mathbb{Z}$ with respect to (w.r.t) $*$?

Solution.

(i) **Closure holds** since $\mathbb{Z}$ is closed with respect to $+$ and $-$, so

$$x + y - 1 = x * y \in \mathbb{Z}.$$

Hence, closure law hold in $\mathbb{Z}$ with respect to $*$.

□

Solution Contin'd.

(ii) For **associative law** we have,

$$\begin{aligned} x*(y*z) &= x+(y*z)-1 = x+[y+z-1]-1 \\ &= x+y+z-1-1 \\ &= (x+y-1)+z-1 \\ &= (x*y)+z-1 \\ &= (x*y)*z. \end{aligned}$$

Hence, associative law hold in $\mathbb{Z}$ with respect to $*$.

(ii) For **commutative law** we have,

$$x*y = x+y-1 = y+x-1 = y*x.$$

Hence, commutative law hold in $\mathbb{Z}$ with respect to $*$.

Solution Contin'd.

(ii) For **Identity law**, let $e \in \mathbb{Z}$ be the identity. Then for every $a \in \mathbb{Z}$, we have

$$a = a * e = a + e - 1$$

and so, $a = a + e - 1$. Solving for e, we get

$$e = a - a + 1 \implies e = 1.$$

Hence, identity law hold in $\mathbb{Z}$ with respect to $*$.

(iv) **Inverse Law**: Suppose $a, b \in \mathbb{Z}$ are such that $a * b = e = 1$, that is,

$$a + b - 1 = 1,$$

then $b = 2 - a$ is the inverse of a in $\mathbb{Z}$. Hence, inverse law hold in $\mathbb{Z}$ with respect to $*$.

□

Example

Let $*$ be a binary operation on $\mathbb{Z}$ defined by $x * y = xy + 5$ for all $x, y \in \mathbb{Z}$.

(i) Determine $(4 * -1) * 6$.

(i) Is $*$ an associative binary operation?

(iii) Does $\mathbb{Z}$ have an identity element with respect to $*$?

Solution.

(i) By the definition of $*$, we have

$$\begin{aligned}(4 * -1) * 6 &= [4(-1) + 5] * 6 \\ &= [-4 + 5] * 6 \\ &= 1 * 6 \\ &= 1 \times 6 + 5\end{aligned}$$

Solution Contn'd.

(ii) **Associative law:** For $x, y \in \mathbb{Z}$, we want to check if $x * (y * z) = (x * y) * z$? Now,

$$
\begin{aligned}
L.H.S = x * (y * z) &= x(y * z) + 5 \\
&= x(yz + 5) + 5 \\
&= xyz + 5x + 5
\end{aligned}
$$

Also,

$$
\begin{aligned}
R.H.S = (x * y) * z &= (xy + 5) * z \\
&= (xy + 5)z + 5 \\
&= xyz + 5z + 5
\end{aligned}
$$

Since $x * (y * z) = xyz + 5x + 5 \neq xyz + 5z + 5 = (x * y) * z$, we conclude that the associative law does not hold.

Proof Contin'd.

(ii) **Identity law:** Let $e \in \mathbb{Z}$ be the identity, then for every $a \in \mathbb{Z}$, we have

$$a = a * e = ae + 5$$

and so,

$$\begin{aligned} \implies \quad & a = ae + 5 \\ \implies \quad & ae = 5 - a \\ \implies \quad & e = \frac{5-a}{a} = \frac{5}{a} - 1. \end{aligned}$$

But $\frac{5-a}{a} \notin \mathbb{Z}$, which is a contradiction that $e \in \mathbb{Z}$. Hence, e does not exist and so, the identity law does not hold.

□

Example

Let $*$ and $\circ$ be a binary operations on $\mathbb{Z}$ defined by $x * y = \frac{xy}{5}$ and $x \circ y = xy + 5$ for all $x, y \in \mathbb{Z}$.

(i) Is $*$ distributive over $\circ$?

(ii) Is $\circ$ distributive over $*$?

Solution.

(i) We check if $x * (y \circ z) = (x * y) \circ (x * z)$ holds. Now,

$$L.H.S = x * (y \circ z) = x * (yz + 5) = \frac{x(yz+5)}{5} = \frac{xyz + 5x}{5}$$

$$Also,$$

$$R.H.S = (x * y) \circ (x * z) = \frac{xy}{5} \circ \frac{xz}{5} = \frac{xy}{5}\frac{xz}{5} + 5 = \frac{x^2yz}{25} + 5$$

Since $L.H.S \neq R.H.S$, $*$ is not distributive over $\circ$.

Solution Contin'd.

(ii) We check $x \circ (y * z) = (x \circ y) * (x \circ z)$

$$\begin{aligned} L.H.S = x \circ (y * z) &= x \circ \frac{yz}{5} \\ &= \frac{xyz}{5} + 5 \end{aligned}$$

Also,

$$\begin{aligned} R.H.S = (x \circ y) * (x \circ z) &= (xy + 5) * (xz + 5) \\ &= \frac{(xy + 5)(xz + 5)}{5} \\ &= \frac{x^2yz + 5xy + 5xz + 25}{5} \end{aligned}$$

Since $L.H.S \neq R.H.S$, $\circ$ is not distributive over $*$. $\square$


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