THE COPPERBELT UNIVERSITY

SCHOOL OF MATHEMATICS AND NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

MA 110

MATHEMATICS METHODS

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2018

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INTRODUCTORY SET THEORY

1.1 THE CONCEPT OF SET THEORY

Definition 1.1.1

A *set* is a collection of distinct elements or objects.

We shall use capital letters for names of sets and small letters for elements of a set. For instance we write $x \in B$ if x is an element of B and if x is not an element of B, we may write $x \notin B$.

A set can be specified in one of the two ways:

(i) by listing the members of the collections or
(ii) by specifying what it is that the members of the collection have in common.

It is important to understand the following:

1. A set must be well defined-it must be possible to tell whether a given element belongs to a set either by checking it against the list of elements of the set or by deciding whether it satisfies or does not satisfy the rule governing membership for the set.
2. The elements in a set are distinct- if any object is listed as an element of a set it should not be listed a second time. For example, the set of letters needed to write $''teeth''$ contains only t, e, and h.
3. The order of the elements in a list is not significant- the set containing the element 1,2,3 is exactly the same as the set containing 2,3,1.

Set Notation

Several symbols are used for denoting sets. One of these is the trace { }. The set whose members or elements are 1,2,3,4 would be indicated by {1,2,3,4}. The same set can be described as $\{n|n$ is a counting number less than 5 }.

Examples 1.1.2

(a) Describe set $A = \{2,3,4,5,6\}$.

(b) List the elements of set $B = \{k \in S: k = 3n + 1, n = 0,1,2,3\}$

Solutions

(a) $A = \{n: n \in \mathbb{N} \text{ and } 1 < n < 7\}$
(b) $B = \{1,4,7,10\}$

Definition 1.1.2

Suppose A and B are sets. If every element of A is an element of B, then A is a subset of B. This is denoted $A \subset B$ does not rule out the possibility that $A = B$. The symbol $A \nsubseteq B$ means $A \subset B$ and $A \neq B$, then A is a proper subset of B.

Definition 1.1.3

Two sets A and B are equal $(A = B)$ if they contain same elements. That is $A = B$ if both $A \subset B$ and $B \subset A$.

Definition 1.1.4

The empty set or null set, denoted by $\emptyset$ or $\{\,\}$ is the set having no elements in it.

Note:

It has the property that the null set (empty set) is a subset of every other set.

Definition 1.1.5

The set containing the totality of elements for any particular discussion or situation is called the ***Universal set*** or the ***Universal set*** is a set which contains all the elements and is denoted by the symbols elements and is denoted by the symbols $\boldsymbol{U}$ or $\boldsymbol{E}$ and it contains every other set.

Definition 1.1.6

If A and B are sets, then $A - B$ is the set of all elements which belong to A but do not belong to B. That is $A \backslash B = A - B = \{\, x | x \in A$ and $x \notin B \,\}$

Example 1.1.3

Let $A = \{2,3,4,5\}$ and $B = \{3,4,5\}$. Find $A \backslash B = A - B$?

Solution

$A \backslash B = \{2\}$.

Definition 1.1.7

If U is the universal set and A is a set, then the complement of A denoted by

$$A' = U - A$$

Is a set, which contains all elements in the universal set that are not in the set A.

Definition 1.1.8

Let A and B be sets.

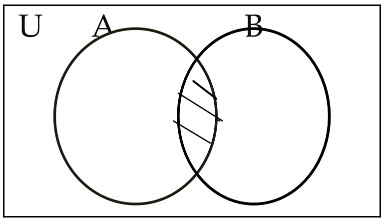
(a) $A \cup B = \{\, x: x \in A$ or $x \in B \,\}$. This is the union of A and B.
(b) $A \cap B = \{\, x: x \in A$ and $x \in B\}$. This is the intersection of A and B.

Venn Diagrams

1.2 Basic Operations and Properties of Sets

Intersection Sets

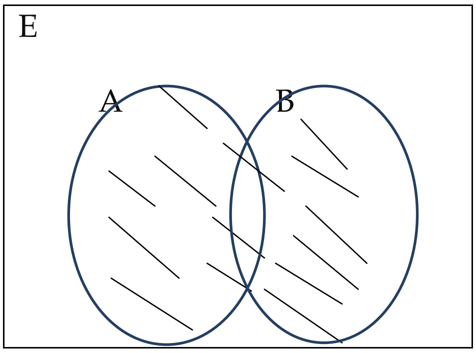
Let A and B be two sets, then the intersection of two sets A and B, denoted by $A \cap B$ is the set which contains elements which are common to both set A and set B. That is $A \cap B = \{ x |\ x \in A \text{ and } x \in B\}$.



Two elements are disjoint if their intersection is a null set. That is, A and B are disjoint implies that $A \cap B = \emptyset$.

Union Sets

Let A and B be any set, the union of A and B written as $A \cup B = \{ x | x \in A \text{ or } x \in B\}$



Example 1.2.1

Let $A = \{ 1,2,3,4,5,6,7,8\}$, $B = \{3,6,9,12,15\}$ and $C = \{2,5,7\}$

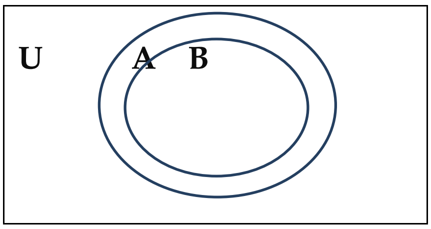
Find

(i) $A \cap B$
(ii) $B \cup C$
(iii) $B \cap C$

Solution

(i) $A \cap B = \{3,6\}$
(ii) $B \cup C = \{2,3,5,6,7,9,12,15\}$
(iii) $B \cap C = \emptyset$

If A and B are two sets such that B is a subset of A, this can be show below



Note that if $B \subset A$ then $A' \subset B'$.

Theorem 1.2.1

Let A, B and D be any set, then

a) $A \subset A$ (reflexive)
b) If $A \subset B$ and $B \subset A$, then $A = B$ (symmetric)
c) If $A \subset B$ and $B \subset D$, then $A \subset D$ (transitive)

Note

Every set has two subsets $\emptyset$ and itself which is called ***improper subsets***.

Theorem 1.2.3

Let A, B and C be sets. The following are immediate from the definitions.

1. $(A')' = A$

2. $A \cup A = A$; $A \cap A = A$; $A \backslash A = \emptyset$

3. Commutativity
   $A \cup B = B \cup A$
   $A \cap B = B \cap A$

4. Associativity
   $(A \cup B) \cup C = A \cup (B \cup C)$
   $(A \cap (B \cap C) = A \cap (B \cap C)$

5. Distributivity
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. $A \cap \emptyset = \emptyset$ $\quad A \cup \emptyset = A$
   $A \cap U = A$ $\quad A \cup E = E$
   $A \cup A' = E$ $\quad A \cap A' = \emptyset$
   $E' = \emptyset$
7. $A(B \cap C) = (A \backslash B) \cup (A \backslash C)$
8. De Morgan's laws state that for any sets A and B we have

   a) $(B \cap C)' = B' \cup C'$
   b) $(B \cup C)' = B' \cap C'$

Proof

1. $(A')' = A$
   We need to show that $(A')' \subset A$ and $A \subset (A')'$

(i) Let for all $x \in (A')' \Longrightarrow x \notin A' \Longrightarrow x \in A$. Therefore

$$\therefore (A')' \subset A.$$

(ii) Let for all $x \in A \Longrightarrow \forall\, x \notin A' \Longrightarrow \forall\, x \in (A')'$. Therefore

$$\therefore A \subset (A')'$$

From (i) and (ii) we have $(A')' = A$.

8. (b) $(B \cup C)' = B' \cap C'$

We need to show
(i) $(B \cup C)' \subset B' \cap C'$
(ii) $B' \cap C' \subset (B \cup C)'$

(i) Let for all $x \in (B \cup C)' \Rightarrow x \notin B \cup C$

$$\Rightarrow x \notin B \text{ or } x \notin C$$

$$\Rightarrow x \in B' \text{ and } x \in C'$$

$$\Rightarrow x \in B' \cap C'.$$

$$\therefore \ (B \cup C)' \subset B' \cap C'$$

(ii) Let for all $x \in B' \cap C' \Rightarrow x \in B'$ and $x \in C'$

$$\Rightarrow x \notin B \text{ or } x \notin C$$

$$\Rightarrow x \notin B \cup C$$

$$\Rightarrow x \in (B \cup C)'$$

From (i) and (ii) we conclude that

$$(B \cup C)' = B' \cap C'$$

(a) $(B \cap C)' = B' \cup C'$

We need to show

(i) $(B \cap C)' \subset B' \cup C'$

(ii) $B' \cup C' \subset (B \cap C)'$

(i) Let $x \in (B \cap C)' \Rightarrow x \notin B \cap C$
$\Rightarrow x \notin B$ and $x \notin C$
$\Rightarrow x \in B'$ or $x \in C'$
$\Rightarrow x \in B' \cup C'$

$$\therefore (B \cup C)' = B' \cap C'.$$

(ii) Let $x \in B' \cup C' \Rightarrow x \in B'$ or $x \in C'$
$\Rightarrow x \notin B$ and $x \notin C$
$\Rightarrow x \notin B \cap C$

$$\Longrightarrow x \in (B \cap C)'$$
$$\therefore B' \cup C' \subset (B \cap C)'$$

From (i) and (ii) we have $(B \cap C)' = B' \cup C'$.

Example 1.2.2

(a) Verify or show that $(A \cup B)' = A' \cap B'$, where $A = \{1,2,3,4,5\}, B = \{2,3,5,7\}$ and $U = \{1,2,3,4,5,6,7,8,9,10\}$.
(b) Using the associative and distributive properties of union and intersection of sets. Show that $A = (A \cap B) \cup (A \cap B')$

Solution

(a)

$$A \cup B = \{1,2,3,4,5,7\}$$

$$(A \cup B)' = \{6,8,9,10\}$$

Now $A' = \{6,7,8,9,10\}$ and $B' = \{1,4,6,8,9,10\}$

Thus $A' \cap B' = \{6,8,9,10\}$

(c) $(A \cap B) \cup (A \cap B') = A \cap (B \cup B')$ using distributive property.
$= A \cap U$
$= A.$

Set of Numbers

2. Set of Numbers

We use special symbols to denote sets of numbers.

1. $\mathbb{N} = N -$ natural numbers(positive integers)

   $N = \{1,2,3, \dots\}$.

2. $\mathbb{Z} = Z -$ set of integer

   $Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

3. $\mathbb{Q} = Q -$ is a set of rational numbers;

Rational numbers can be expressed in the form of $\frac{a}{b}$ where a and b are integers and $b \neq 0$

Example 2.1.1

Express each of the repeating decimals below in the form of ${}^{a}/_{b}$ where a and b are integers and $b \neq 0$

a) $0.\overline{3}$ b) $0.1\overline{3}$ c) $6.1\overline{56}$

Solutions

a) Let $x = 0.\overline{3}$ . . . (i)
$10x = 3.\overline{3}$ . . . (ii)

Subtract (i) from (ii)

$$9x = 3$$
$$x = {}^{3}/_{9}$$
$$x = {}^{1}/_{3}$$

b) Let $x = 0.1\overline{3}$ . . . (i)

$10x = 1.\overline{3}$ . . . (ii)

$100x = 13.\overline{3}$ . . . (iii)

Subtract (ii) from (iii) we get

$$90x = 12$$
$$x = {}^{12}/_{90}$$
$$x = {}^{2}/_{15}.$$

c) Let $x = 6.1\overline{56}$ . . . (i)

$10x = 61.\overline{56}$ . . . (ii)

$1000x = 6156.\overline{56}$ . . . (iii)

Subtract (ii) from (iii) we get

$$990x = 6095$$
$$x = \frac{6095}{990}$$
$$x = \frac{1219}{198}.$$

Irrational Numbers

An irrational number is defined to be a number that cannot be expressed in the form of ${}^{a}/_{b}$ when a and b are integers and $b \neq 0$. Is one whose decimal representation is none repeating and non-terminating. For example$\sqrt{3}$, $\sqrt{2}$, $\sqrt{3} + 5$ etc.

We shall introduce some mathematical proofs that will help us to show if the given number is irrational. For any given number, we may not tell if it is irrational unless we prove it. Consider the following examples.

Example 2.2.1

Prove that $\sqrt{2}$ is irrational number.

Solution

We shall prove this by method of contradictions. Suppose $\sqrt{2}$ is rational which can be written in the form of $^a/_b$ where a and b are integers and $b \neq 0$ with no common factor. Thus

$\sqrt{2} = {^a/_b}$ . . . (i)

Squaring both sides of (i) we get

$2 = a^2/b^2 \Rightarrow a^2 = 2b^2$ . . . (ii)

This implies that a^2 is an even number with a common factor 2. Hence a is an even number with the factor 2. Thus a can be written in form of $a = 2k$ where k is an integer. Substitute a in equation (ii) we get

$$(2k)^2 = 2b^2$$

$$4k^2 = 2b^2$$

$$2k^2 = b^2$$

Hence, b^2 is also an even number, consequently b is also an even number with the factor 2.Thus a and b have the common factor 2. This contradicts our earlier assumption that a and b has no common factor. Therefore, by the method of contradiction we have proved that $\sqrt{2}$ is an irrational number.

Example 2.2.2

Prove that $\sqrt{3}$ is irrational number.

Solution

We prove by the method of contradiction. Suppose that $\sqrt{3}$ is a rational number. Then it can be written in the form $^a/_b$ where $b \neq 0$, a and b has no common factor. That is

$$\sqrt{3} = {^a/_b} \ . \ . \ . \ \text{(i)}$$

Squaring (i) both sides we get

$$3 = \frac{a^2}{b^2}$$

$$a^2 = 3b^2 \ . \ . \ . \ \text{(ii)}$$

This implies that a^2 has a factor 3. Hence, a also has a factor 3 and can be expressed as

$a = 3k$ . . .(iii) where k is an integer

Substitute (iii) in (ii) we get $(3k)^2 = 3b^2$

$$9k^2 = 3b^2$$

$$3k^2 = b^2 \quad . \; . \; . \text{ (iv)}$$

Statement (iv) shows that b^2 has a common factor 3 and b has a common factor 3.Thus a and b has a common factor 3. This contradicts our earlier assumption that a and b has no common factor. Therefore, by method of contradiction, we have proved that $\sqrt{3}$ is an irrational number.

Example 2.2.3

Given that $\sqrt{2}$ is irrational, prove that $\sqrt{2}-1$ is irrational.

Solution

We prove by the method of contradiction. Suppose that $\sqrt{2}-1$ is rational, then it can be expressed as $\sqrt{2}-1 = {}^{a}/_{b}$ where a and b are integers, $b \neq 0$

$$\sqrt{2} = \frac{a}{b} + 1$$

$$\sqrt{2} = \frac{a+b}{b} \quad . \; . \; . \quad (i)$$

From statement (i) we see that $a+b$ is an integer, b also is and the fraction is rational. Since $\sqrt{2}$ is irrational then the statement is a contradiction that is, rational is not equal to irrational. Hence by the method of contradiction, $\sqrt{2}-1$ has been proved irrational number.

Review Exercise

1. Express the following in the form of ${}^{a}/_{b}$ where a and b are integers, $b \neq 0$.
   a) $0.\overline{16}$ b) $2.\overline{143}$ c) $1.17171717 \ldots$ d) $3.\overline{7}$
2. Show that the following are irrational numbers
   a) $1-\sqrt{3}$ b) $\sqrt{3}+5$ c) $\sqrt{5}-1$ d) $\sqrt{3-1}$

2.3 Binary Operations on Real Numbers

Definition 2.3.1

A binary operation denoted by ' $*$ ' on a non-empty set G is a rule that associates to each pair of elements a and b in G. We can denote a unique element a and b as $a * b$ of G, for example

a. **Addition** '$+$' is a binary operation on the set of natural numbers $\mathbb{N}$.
   Addition is a binary operation on the set of natural numbers $\mathbb{N}$ because if any numbers represented by a and b are members of natural numbers, then when we add those numbers they yield a sum that is also a natural number.
   This implies that if $a, b \in \mathbb{N}$ then $a+b \in \mathbb{N}$.
b. **Subtraction** '$-$' is not a binary operation on the set of natural numbers $\mathbb{N}$.

Subtraction is not a binary operation on the set of natural numbers because if any numbers represented by a and b are members of natural numbers, the when we subtract those numbers there is a possibility that their difference can be not a natural number.
The counter example is $5-7=-2$ where 5 and 7 are the members of natural numbers while -2 is a member of negative integers.

c. Both **addition** and **subtraction** are binary operation on a set of integers Z.

Properties of binary operation

The operation " $*$ " on any set G is said to be:

i. **Commutative** *:* if for every pair $a, b, c \in G$, we have $a * b = b * a$
ii. **Associative** *:* if for all $a, b \in G$ we have $(a * b) * c = a * (b * c)$

Example 2.3.2

Define an operation ' $*$ ' on a set of Real Numbers by $a * b = a + 2b$ for all $a, b \in R$. Is this operation commutative or associative?

Solution

For any two real numbers a and b then by the operation

$$a * b = a + 2b$$

$$b * a = b + 2a$$

If a and b are two distinct numbers then $a + 2b \neq b + 2a$ then, the operation defined by $a * b = a + 2b$ is not commutative. To show that the binary operation defined by $a * b = a + 2b$ is associative, we need to show that $(a * b) * c = a * (b * c)$

since $(a * b) * c = (a + 2b) * c = a + 2b + 2c\ \ .\ .\ .$ (i)

and $a * (b * c) = a * (b + 2c) = a + 2(b + 2c) = a + 2b + 4c\ \ .\ .\ .$ (ii)

The two statements are not equal which implies that the binary operation '*' defined by $a * b = a + 2b$ is not associative.

Example 2.3.3

Define an operation $*$ on the set of real numbers by $a * b = b^a$

i. Is $*$ a binary operation on the set of real numbers? Give reason for your answer.
ii. Is the operation commutative?
iii. Evaluate $(3 * 2) * -2$

Solution

i. The operation $*$ defined by $a * b = b^a$ is a binary operation on the set of real numbers since for all $a, b \in R$ under the operation $a * b$ yields $b^a \in R$.
ii. Since for any two distinct real numbers $a, b \in R$ such that $a * b = b^a$ and $b * a = a^b$ are not equal, then the operation in not commutative.
iii. By the operation defined $a * b = b^a$ then

$$\begin{aligned}(3 * 2) * -2 &= (2^3) * -2 \\ &= 8 * -2\end{aligned}$$

$$= (-2)^8$$
$$= 2^8$$

Example 2.3.4

Let‘ $*$’ be a binary operation on the set of real numbers defined by $a * b = -2^{a-b}$ where a and b are Members of real numbers.

i. Is $*$ commutative on real number? Justify your answer.
ii. Find $-1 * (4 * 9)$

Solutions

i. The operation $*$ is not commutative on real numbers since for any two real numbers a and b under the operation
$a * b = -2^{a-b}$ there exist a real number c such that
$$a * b = -2^c$$
or
$$a * b = -\frac{1}{2^c}$$

ii. Using the operation defined by $a * b = -2^{a-b}$ then
$$-1 * (4 * 9) = -1 * (-2^{4-9})$$
$$= -1 * (-2^{-5})$$
$$= -1 * (\frac{1}{-2^5})$$
$$= -1 * \frac{1}{-32}$$
$$= -1 * \left(-\frac{1}{32}\right)$$
$$= -2^{-1-\left(-\frac{1}{32}\right)}$$
$$= -2^{-\frac{31}{32}}$$
$$= \frac{32}{-2^{31}}\ .$$

2.4 Properties of real numbers

2.4.1 Algebraic Properties of Real Numbers.

If a, b and c are members of real numbers, that is $a, b, c \in R$, then

i. $a + b = b + a$ **commutative law of addition**
ii. $(a + b) + c = a + (b + c)$ **associative law of addition**
iii. There exists an element$0 \in \mathbb{R}$ such that $0 + a = a + 0 = a$ **adding identity** element.
iv. For all $a \in R$ there exist $-a \in \mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ **additive inverse**.
v. $a.b = b.a$ for all $a, b \in R$ **commutative law of multiplication**.
vi. $(a.b).c = a.(b.c)$ for all $a, b \in R$ **associative law of multiplication.**

vii. There exist an element $1 \in R$ such that for all $a \in \mathbb{R}$

$1. a = a$ and $a. 1 = a$ **multiplication identity**

viii. For each $a \neq 0 \in R$, there exist an element $1/a$ such that $a\left(\frac{1}{a}\right) = 1$

and $\left(\frac{1}{a}\right). a = 1$ **multiplication inverse**

ix. $a.(b + c) = (a.b) + (a.c)$ and $(b + c).a = (b.a) + (c.a)$

for all $a, b, c \in R$ **distributive law.**

Order of Relations

Let x and y be the members from real numbers, that is $x, y \in \mathbb{R}$, then;

i. $x > y \quad if \quad x - y > 0$

ii. $x < y \quad if \quad x - y < 0$

iii. The law of trichotomy holds for real numbers that is if $x, y \in \mathbb{R}$, then either

$x > y$ or $x < y$ or $x = y$

The Absolute Value property of Real Numbers

On a number line, the distance between a point x and zero is called the absolute value of x. The absolute value of x is denoted by $|x|$.

Definition 2.4.1

For any $a \in \mathbb{R}$, the absolute value is given as;

$$|a| = \begin{cases} a & if \quad a \geq 0 \\ -a & if \quad a < 0 \end{cases}$$

Example 2.4.2

Use the definition of absolute value, evaluate

i. $|7|$ ii. $|-4|$ iii. $|0|$ iv. $|-1/2|$

Solutions

i. $|7| = 7$ ii. $|-4| = -(-4) = 4$ iii. $|0| = 0$ iv. $|-1/2| = 1/2$

Properties of Absolute values

If $a, b \in \mathbb{R}$, then

a. $|a| \geq 0$ The absolute value of a real number is positive or zero (non-negative).
b. $|a| = |-a|$ The absolute value of real number is equal to the absolute of its opposite.
c. $|a - b| = |b - a|$ for all $a, b \in \mathbb{R}$.

Equations Involving Absolute Values property of real numbers

Example 2.4.3

Solve the following equations

a. $|x - 2| = 4$

b. $|2x-3|=7$
c. $|4x+5|=-1$

Solution

a. $|x-2|=4$
Since by the properties of the absolute value $|a|=|-a|$, then

$$x-2=4 \quad \text{or} \quad x-2=-4$$

It follows that

$$x=4+2 \quad \text{or} \quad x=-4+2$$

Hence

$$x=6 \quad \text{or} \quad x=-2$$

Therefore, the solution set is $(-2,6)$

b. $|2x-3|=7$

$$2x-3=7 \quad \text{or} \quad 2x-3=-7$$
$$2x=7+3 \quad \text{or} \quad 2x=-7+3$$
$$2x=10 \quad \text{or} \quad 2x=-4$$
$$x={}^{10}/_{2} \quad \text{or} \quad x={}^{-4}/_{2}$$
$$x=5 \quad \text{or} \quad x=-2$$

The solution set is $(-2\,,5)$

c. $|4x+5|=-1$
The absolute value of a real number can never be a negative. Hence the solution set is { }.

EXPONENTS, LAWS OF INDICES AND SURDS

Definition 3.1.1

If n is a positive integer and b is any real number, then

$$b^n=b.b\ldots b \qquad n-\text{times}$$

The number b is referred as a **base** and n is an **exponent.** The expression b^n can be read as b to the n^{th} power. The term squared and cubed are commonly associated with the exponents of two and three respectively. Example

$$2^3=2.2.2=8 \qquad 4^3=4.4.4=64 \qquad 1^5=1.1.1.1.1=1$$

Properties of exponents:

If a and b are real numbers, m and n are positive integers, then

1. $b^m.b^n=b^{m+n}$ product of two powers
2. $(b^m)^n=b^{mn}$ power of a power
3. $(ab)^m=a^m.b^m$ power of a product
4. $(a\backslash b)^m=\frac{a^m}{b^m}$ if $b\neq 0$ power of a quotient
5. $\frac{a^m}{b^n}=(\frac{a}{b})^{m-n}$ when $m>n$, $b\neq 0$ quotient of two powers

Example 3.1.2

Simplify the following

i. $(3x^2y)(4x^3y^2) = 3.4.x^{2+3}.y^{1+2} = 12x^5y^3$

ii. $(-2y^3)^5 = -2^5y^{15} = -32y^{15}$

iii. $(a^2\backslash b^4)^7 = \frac{(a^2)^7}{(b^4)^7} = \frac{a^{14}}{b^{28}}$

Zero and Negative Integers as an Exponent

A real number can be raised to a zero power or a negative integer power.

Definition 3.1.3

If b is non-zero real number, then $b^0 = 1$.

According to the definition, the following statements are true;

$5^0 = 1 \quad (xy)^0 = 1$ if $x \neq 0$ and $y \neq 0$.

Definition 3.1.4

If n is a positive integer and b is non zero real number, then

$$b^{-n} = \frac{1}{b^n}.$$

The following statements are true;

$$x^{-5} = \frac{1}{x^5} \quad , 2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

Example 3.1.5

Evaluate the following numerical expressions.

1. $(2^{-1}.3^2)^{-1}$ 2. $\left(\frac{2^{-3}}{3^{-2}}\right)^{-2}$

Solutions

1. $(2^{-1}.3^2)^{-1} = (2^{-1})^{-1}.(3^2)^{-1}$ power of a product
$= 2^1.3^{-2}$ power of a power
$= 2.(\frac{1}{3^2}) = \frac{2}{9}$

2. $\left(\frac{2^{-3}}{3^{-2}}\right)^{-2} = \frac{(2^{-3})^{-2}}{(3^{-2})^{-2}}$ power of a quotient

$= \frac{2^6}{3^4}$ power of a power

$= \frac{64}{81}$

Example 3.1.6

Simplify the following

a. $(3x^2y^{-4})(4x^{-3}y) = 3.4x^{2+(-3)}y^{-4+1}$ product of power

$= 12x^{-1}y^{-3} = \frac{12}{xy^3}$

b. $\frac{12a^2b^2}{-3a^{-1}b^5} = -4a^{3-(-1)}b^{2-5}$ quotient of power

$= -4a^4b^{-3} = -\frac{4a^4}{b^3}$

Example 3.1.7

a. Simplify
$2^{-3} + 3^{-1}$

Solution

$$2^{-3} + 3^{-1} = \frac{1}{2^3} + \frac{1}{3}$$

$$= \frac{1}{8} + \frac{1}{3}$$

$$= \frac{3}{24} + \frac{8}{24}$$

$$= \frac{11}{24}.$$

REVIEW EXERCISE

1. Evaluate each of the following expressions.

a. 2^{-3} b. $2^5 2^{-3}$ c. $3^{-2} - 2^3$ d. $\frac{1}{\left(\frac{4}{5}\right)^{-2}}$

2. Simplify each of the following; express final result without using zero or negative integer as an exponent.

a. $(x^3y^{-4})^{-1}$ b. $(a^3)^{-1}$ c. $(a^2b^{-2}c^{-1})^{-4}$ d. $\left(\frac{x^{-2}}{y^{-3}}\right)^{-2}$

d. $\frac{x^{-1}y^{-2}}{x^3y^{-1}}$ f. $\left(\frac{24x^5y^{-3}}{-8x^6y^{-1}}\right)^{-3}$

3.2 Radicals and surds of real numbers

1. Every positive real number has two square roots; one is positive and the other is negative.
2. Negative real numbers have no real number square roots because the square of any nonzero real number is positive.
3. The square root of zero is zero.

The following example illustrates the use of the square root notation.

$\sqrt{16} = 4$ Indicates the non negative or square root of 16

$-\sqrt{16} = -4$ Indicates the negative square root of 16

$\sqrt{0} = 0$ Zero has only one square root where$-\sqrt{0} = -0 = 0$

$\sqrt{-4}$ not a real number

$-\sqrt{-4}$ not a real number

$\sqrt[3]{8} = 2$ $\sqrt[3]{-8} = -2$ $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$ and $\sqrt[3]{-\frac{1}{27}} = -\frac{1}{3}$

The concept of root can be extended to fourth roots, fifth roots, and sixth root and so on.

Definition 3.2.1

$\sqrt[n]{b} = a$ if and only if $a^n = b$.

The following examples are applications of the definition

$\sqrt[4]{81} = 3$ because $3^4 = 81\sqrt[5]{32} = 2$ because $2^5 = 32$.

$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

Another property is that $(\sqrt[n]{b})^n = b$

$(\sqrt{4})^2 = 4$. If $b < 0$ and n is any positive integer greater than 1 or if $b < 0$ and n is an odd positive integer greater than 1, then $\sqrt[n]{b^n} = b$.

Simplest Radical Form

Let us use some examples to motivate another useful property of radicals.

$$\sqrt{25\ .\ 16} = \sqrt{400} = 20 \quad \text{and} \quad \sqrt{25}\,.\sqrt{16} = 5.4 = 20$$

$$\sqrt[3]{8\ .\ 27} = \sqrt[3]{216} = 6 \quad \text{and} \quad \sqrt[3]{8}\ .\sqrt[3]{27} = 2.3 = 6$$

In general, the following property can be stated.

Property 1

$\sqrt[n]{bc} = \sqrt[n]{b}\,.\sqrt[n]{c}$ if $\sqrt[n]{b}$ and $\sqrt[n]{c}$ are real numbers.

The definition of n^{th} root, along with Property1, provides the basis for changing radicals to simplest radical form. Consider the following examples of reductions to simplest radical form.

$$\sqrt{45} = \sqrt{9.5} = \sqrt{9}.\sqrt{5} = 3\sqrt{5}$$

$$\sqrt[3]{24} = \sqrt[3]{8.3} = \sqrt[3]{8}.\sqrt[3]{3} = 2\ \sqrt[3]{3}$$

$$\sqrt{80} = \sqrt{2^4.5} = \sqrt{2^4}.\sqrt{5} = 2^2\sqrt{5} = 4\sqrt{4}$$

Distributive property

$$3\sqrt{2} + 5\sqrt{2} = (3+5)\sqrt{2} = 8\sqrt{2}$$

$$7\sqrt[3]{5} - 3\sqrt[3]{5} = (7-3)\sqrt[3]{5} = 4\sqrt[3]{5}.$$

Property 2:

$\sqrt[n]{\frac{b}{c}} = \frac{\sqrt[n]{b}}{\sqrt[n]{c}}$ if $\sqrt[n]{b}$ and $\sqrt[n]{c}$ are real numbers and $c \neq 0$

For example: simplify by rationalizing the denominator

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}.$$

Example 3.2.2

Change the form $\frac{\sqrt{x+h} + \sqrt{x}}{h}$ by rationalizing the numerator.

Solution

$$\frac{\sqrt{x+h} + \sqrt{x}}{h} = \left(\frac{\sqrt{x+h} + \sqrt{x}}{h}\right) \cdot \left(\frac{\sqrt{x+h} - \sqrt{x}}{\sqrt{x+h} - \sqrt{x}}\right)$$

$$= \frac{x+h-x}{h\left(\sqrt{x+h} - \sqrt{x}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h} - \sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+h} - \sqrt{x}}.$$

REVIEW EXERCISE

1. Evaluate

   a. $\sqrt{81}$ *b*. $-\sqrt{49}$ *c*. $\sqrt[3]{125}$ *d*. $\sqrt[3]{\frac{64}{27}}$ *e*. $\sqrt[3]{-\frac{27}{8}}$

2. Express each of the following in the simplest radical form.
   a. $\sqrt{54}$
   b. $\sqrt{45xy^2}$
   c. $3\sqrt{32a^3}$
3. Rationalize the numerator

a. $\frac{\sqrt{2x+2h}-\sqrt{2x}}{h}$ b. $\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$ c. $\frac{2\sqrt{x+h}-2\sqrt{x}}{h}$

4: COMPLEX NUMBERS

Definition 4.1.1

A ***complex number*** is any number that can be expressed in the form $a + bi$ where a and b real numbers.

The form $a + bi$ is called the ***standard form*** of a complex number. The real number a is called the ***real part*** of the complex number, and b is called the ***imaginary part***. $7 + 8i$ is in form of $a + bi$. For example $11 - 3i$ can be written as $11 + (-3)i$ even though $11 - 3i$ is often used , $-8 + i\sqrt{3}$ can be written as $-8 + \sqrt{3}\, i$ and finally, $-9i$ can be written as $0 - 9i$ where $a = 0$ and it is called pure imaginary number.

Definition 4.1.2

Two complex numbers $a + bi$ and $c + bi$ are said to be ***equal*** if and only if $a = c$ and $b = d$. That is the real parts are equal and the imaginary parts are equal

Adding and Subtracting Complex Numbers
two or more complex numbers can be added as follows;

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Example 4.1.3

i. $(4 + 3i) + (5 + 9i) = (4 + 5) + (3 + 9)i = 9 + 12i$

ii. $(-6 + 4i) + (8 - 7i) = (-6 + 8) + (4 - 7)i = 2 - 3i$

iii. $\left(\frac{1}{2} + \frac{3}{4}\, i\right) + \left(\frac{3}{2} + \frac{3}{4} i\right) = \left(\frac{1}{2} + \frac{3}{2}\right) + \left(\frac{3}{4} + \frac{3}{4}\right) i = \left(\frac{7}{6} + \frac{19}{20} i\right)$

iv. $\left(3 + i\sqrt{2}\right) + \left(-4 + i\sqrt{2}\right) = (3 - 4) + \left(\sqrt{2} + \sqrt{2}\right)i = -1 + 2\sqrt{2}\, i$

The set of complex numbers is **closed with respect to addition**; that is, the sum of two complex numbers is a complex number. Furthermore, the commutative and associative properties of addition hold for all complex numbers. The additive identity element is $0 - 0i$, or just a real number 0. The additive inverse of $a + bi$ is $-a - bi$.

Multiplying and Dividing Complex Numbers
we define $i^2 = -1$, the number i is a square root of -1, so we write $i = \sqrt{-1}$.

$$(-i)^2 = (-i).(-i) = i^2 = -1.$$

Therefore, in the set of complex numbers, -1 has two square roots—namely, i and $-i$. This is symbolically expressed as $(i\sqrt{b})^2 = i^2(b) = -1(b) = -b$

Therefore, let us denote the **principal square root of $-b$** by $\sqrt{-b}$ and define it to be

$\sqrt{-b} = i\sqrt{b}$ where b is any positive real number. In other words, the principal square root of any negative real number can be represented as the product of a real number and the imaginary unit i.

$$\sqrt{-4} = i\sqrt{4} = 2i$$

$$\sqrt{-17} = i\sqrt{17}.$$

Example 4.1.3

Find the product of the two complex numbers $(2+3i)$ and $(4+5i)$

Solution

$$\begin{aligned}(2+3i)(4+5i) &= 2(4+5i)+3i(4+5i)\\ &= 8+10i+12i+15i^2\\ &= 8+22i+15(-1)\\ &= -15+8+22i\\ &= -7+22i.\end{aligned}$$

4.2 Quotient of complex numbers

We introduce a concept that helps us evaluate the quotient of any two complex numbers. This concept is called the ***conjugate.*** Two complex numbers $a+bi$ and $a-bi$ are called conjugates of each other if the product of a complex number $a+bi$ and its conjugate $a-bi$ is a real number. This concept is illustrated in the following example;

Example 4.2.1

a. Given a complex number $\frac{4-5i}{2i}$, expressed it in the form $a+bi$.

$$\frac{4-5i}{2i} = \frac{4-5i}{2i} \cdot \frac{-2i}{-2i}$$

$$= \frac{-2i(4-5i)}{2i(-2i)}$$

$$= \frac{-8i+10i^2}{-4i^2}$$

$$= \frac{-10-8i}{4}$$

$$= \frac{-10}{4} - \frac{8i}{4}$$

$$= -\frac{5}{2} - 2i.$$

The Absolute Value or the Magnitude of Complex Numbers

Example 4.2.3

Find the absolute value of

a. $Z = 4-3i$ b. $Z = -4i$ c. $Z = -4$

Solutions

$a.$ $|Z| = \sqrt{4^2+(-3)^2} = \sqrt{25} = 5$

$b.$ $|Z| = \sqrt{(-4)^2} = \sqrt{16} = 4$

$c.$ $|Z| = \sqrt{4^2} = 4$

Square Root of a Complex Number

Example 4.2.4

Evaluate $\sqrt{15+8i}$

Solution

By the property of the square root of complex numbers

$$\sqrt{15+8i} = a+bi$$

Our task is now to find the coefficients a and b. By squaring both sides we have

$$\left(\sqrt{15+8i}\right)^2 = (a+bi)^2$$

$$15+8i = a^2-b^2+2abi$$

Then by grouping the like terms

$$a^2-b^2 = 15 \quad . \; . \; . (i)$$

$$2abi = 8i \quad . \; . \; . (ii)$$

$$\therefore \qquad b = \frac{8}{2a}$$

We substitute (ii) into (i) we have

$$a^2 - \left(\frac{4}{a}\right)^2 = 15$$

$$a^2 - \frac{16}{a^2} = 15$$

Multiply both sides by a^2 gives

$$a^4 - 16 = 15a^2$$

$$a^4 - 15a^2 - 16 = 0$$

$$(a^2)^2 - 15a^2 - 16 = 0$$

We simplify by letting $a^2 = y$ then the equation becomes a quadratic equation

$$y^2 - 15y - 16 = 0.$$

We proceed by using factorization method of solving the quadratic equation

$$y^2 + y - 16y - 16 = 0$$

$$(y+1)(y-16) = 0$$

Since $y = a^2$ then we substitute back in the equation $(a^2+1)(a^2-16) = 0$

a should be a real number, then $a^2+1 \neq 0$ but $a^2-16=0$ then $a^2 = 16$ and $a = \pm 4$.

Therefore, we substitute back a in the equation $b = \frac{8}{2a}$ to solve for b.

$b = 1$ or $b = -1$.

Hence, the two possible square roots of $15 + 8i$ are $-4 + i$ or $-4 - i$.

REVIEW EXERCISE

1. Simplify the following
   a. $(5 + 2i)(8 + 6i)$ c. $(-7 - 3i) + (-4 + 4i)$ e. $(6 - 7i) - (7 - 6i)$

   b. $\left(\frac{1}{3} + \frac{2}{5}i\right) + \left(\frac{1}{2} + \frac{1}{4}i\right)$ d. $(4 + i\sqrt{3}) + (-6 - 2i\sqrt{3})$ f. $(-1 - 2i)^2$ g. $(-5i)(8i)$

2. Find each of the quotient and express the answer in the standard form.

   a. $\frac{4i}{3-2i}$ c. $\frac{4-10i}{-3+7i}$ e. $\frac{3+9i}{4-i}$

   b. $\frac{2+5i}{3+7i}$ d. $\frac{-1-i}{-2-3i}$ f. $\frac{-4+9i}{-3-6i}$
3. Find the absolute value and the square roots of the following
   a. $3 + 2i$ b. $15 + 8i$ c. $6 + 2i$.

5: EQUATIONS, INEQUALITIES, AND PROBLEM SOLVING

Linear equations

Solving an equation is the process of finding the number or numbers that makes an algebraic equation a true numerical statement. Such numbers are called the ***solutions*** or ***roots*** of the equation and are said to **satisfy the equation**. The set of all solutions of an equation is called its ***solution set***.

An equation that is satisfied by all numbers that can meaningfully replace the variable is called an **identity**. For example

$$3(x + 2) = 3x + 6\,, x^2 - 4 = (x + 2)(x - 2)$$

Equivalent equations are equations that have the same solution set. For example,

$7x - 1 = 20$, $7x = 21$ and $x = 3$ are all equivalent equations because $\{3\}$ is the solution set of each.

Properties of Equality

For all real numbers, a, b, and c
1. $a = a$. **Reflexive property**
2. If$a = b$, then$b = a$. **Symmetric property**
3. If $a = b$ and b = c, then $a = c$. **Transitive property**
4. If $a = b$, then a may be replaced by b, or b may be replaced by a, in any statement without changing the meaning of the statement. **Substitution property**
5. $a = b$ if and only if $a + c = b + c$. **Addition property**
6. $a = b$ if an only if $ac = bc$, where $c \neq 0$. **Multiplication property**

Now let us consider how these properties of equality can be used to solve a variety of linear equations.

Definition 5.1.1

A linear equation in the variable x is one that can be written in the form $ax + b = 0$ where a and b are real numbers and $a \neq 0$.

Example 5.1.2

Solve the equation $-4x - 3 = 2x + 9$

Solution

Solving this equation, we obtain

$$-4x - 3 = 2x + 9$$

$$-4x - 3 + (-2x) = 2x + 9 + (-2x)$$

$$-6x - 3 = 9$$

$$-6x = 12$$

$$x = -2.$$

Example 5.1.3

Solve $\frac{2y-3}{3} + \frac{y+1}{2} = 3$

Solution

$$\frac{2y-3}{3} + \frac{y+1}{2} = 3$$

$$6\left(\frac{2y-3}{3} + \frac{y+1}{2}\right) = 6(3)$$

$$6\left(\frac{2y-3}{3}\right) + 6\left(\frac{y+1}{2}\right) = 6(3)$$

$$2(2y-3) + 3(y+1) = 18$$

$$7y - 3 = 18$$

$$7y = 21$$

$$y = 3.$$

The solution set is $\{3\}$.

Example 5.1.4

If 2 is subtracted from five times a certain number n, the result is 28. Find the number n.

Solution

Let n represent the number to be found. The sentence If 2 is subtracted from five times a certain number, the result is 28 translates into the equation $5\text{n} - 2 = 28$.

Solving this equation, we obtain

$$5n - 2 = 28$$

$$5n = 30$$

$$n = 6$$

Example 5.1.5

Find three consecutive integers whose sum is -45.

Solution:

Let n represent the smallest integer; then $n + 1$ is the next integer and $n + 2$ is the largest of the three integers. Because the sum of the three consecutive integers is to be -45, we have the following equation.

$$n + (n + 1) + (n + 2) = -45$$

$$3n + 3 = -45$$

$$3n = -48$$

$$n = -16$$

If $n = -16$, $n + 1$ is -15 and $n + 2$ is -14.

Hence the three consecutive integers whose sum is -45 are $-16, -15$ and -14

Example 5.1.6

There are 51 students in a certain class. The number of females is 5 less than three times the number of males. Find the number of females and the number of males in the class.

Solution

Let m represent the number of males; then $3m - 5$ represents the number of females. The total number of students is 51, so the guideline is (number of males) and (number of females) equals 51. Thus, we can set up and solve the following equation $m + (3m - 5) = 51$

$$4m - 5 = 51$$

$$4m = 56$$

$$m = 14$$

Therefore, there are 14 males and $3(14) - 5 = 37$ females.

REVIEW EXERCISE

Solve each of the following problems

1. The sum of three consecutive integers is 21 larger than twice the smallest integer. Find the integers.
2. Find three consecutive even integers such that if the largest integer is subtracted from four times the smallest, the result is 6 more than twice the middle integer.
3. Find three consecutive odd integers such that three times the largest is 23 less than twice the sum of the two smallest integers.
4. Find two consecutive integers such that the difference of their squares is 37.

5. Find three consecutive integers such that the product of the two largest is 20 more than the square of the smallest integer.

6. Find four consecutive integers such that the product of the two largest is 46 more than the product of the two smallest integers.

7. Solve each equation

a. $9x - 3 = -21$

b. $5(2x - 1) = 13$

c. $3n - 2 = 2n + 5d. \; -2(y - 4) - (3y - 1) = -2 + 5(y + 1)$

d. $3(2t - 1) - 2(5t + 1) = 4(3t + 1)$

e. $\frac{-3x}{4} = \frac{9}{2}$

f. $\frac{2x}{3} - \frac{x}{5} = 7$

g. $\frac{2x+1}{14} - \frac{3x+4}{7} = \frac{x-1}{2}$

Equations with the denominator containing a variable

Now let us consider equations that contain the variable in one or more of the denominators. Our approach to solving such equations remains the same except that we must avoid any values of the variable that make a denominator zero.

Example 5.1.7

Solve $\frac{a}{a-2} + \frac{2}{3} = \frac{2}{a-2}$

Solution

$$\frac{a}{a-2} + \frac{2}{3} = \frac{2}{a-2}$$

$$3(a-2)\left(\frac{a}{a-2} + \frac{2}{3}\right) = 3(a-2)\left(\frac{2}{a-2}\right)$$

$$3a + 2(a-2) = 6$$

$$3a + 2a - 4 = 6$$

$$5a = 10$$

$$a = 2.$$

Example 5.1.7

Solve $\frac{x-2}{3} + \frac{x+2}{3} = 2$

Solution

$$\frac{x-2}{3} + \frac{4x+2}{3} = 2$$

$$3\left(\frac{x-2}{3} + \frac{4x+2}{3}\right) = 3(2)$$

$$x + 4x - 2 + 2 = 6$$

$$5x = 6$$

$$x = \frac{5}{6}.$$

REVIEW EXERCISE

1. Solve each equations

a. $\frac{x-2}{3} + \frac{x+1}{4} = \frac{1}{6}$ b. $\frac{5}{x} + \frac{1}{3} = \frac{8}{x}$ c. $\frac{n}{46-n} = 5 + \frac{4}{46-n}$ d. $\frac{a}{a+5} - 2 = \frac{3a}{a+5}$

e. $\frac{3}{x+3} + \frac{1}{x-1} = \frac{5}{2x+6}$ f. $\frac{4}{2y-3} - \frac{7}{3y-5} = 0$

5.2 Equations Involving Radicals

Example 5.2.1

Solve the following equations involving surds

1. $\sqrt{x+4} - \sqrt{x-1} = 1$
2. $\sqrt{x} + 6 = x$
3. $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$
4. $x^4 + 5x^2 - 36 = 0$
5. $15x^{-2} - 11x^{-1} - 12 = 0$

Solution

1.

$$\sqrt{x+4} = 1 + \sqrt{x-1}$$

$$\left(\sqrt{x+4}\right)^2 = \left(1 + \sqrt{x-1}\right)^2$$

$$x + 4 = x - 1 + 2\sqrt{x-1} + 1$$

$$\frac{4}{2} = \frac{2}{2}\sqrt{x-1}$$

$$(2)^2 = \left(\sqrt{x-1}\right)^2$$

$$4 = x - 1 \qquad x = 5$$

2.

$$\sqrt{x} = x - 6$$

$$\left(\sqrt{x}\right)^2 = (x-6)^2$$

$$x = x^2 - 12x + 36$$

$$0 = x^2 - 12x + 36$$

$$0 = (x-4)(x-9)$$

$$x = 4 \text{ or } x = 9$$

3.

$$x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$$

Let $u = x^{\frac{1}{3}}$ then $u^2 = x^{\frac{2}{3}}$

$$u^2 + u - 6 = 0$$

$$(u+3)(u-2) = 0$$

$$u = -3 \text{ or } u = 2$$

Thus,

$$u = x^{\frac{1}{3}} = -3 \text{ or } x^{\frac{1}{3}} = 2$$

$$\left(x^{\frac{1}{3}}\right)^3 = (-3)^3 \text{ or } \left(x^{\frac{1}{3}}\right)^3 = (2)^3$$

$$x = -27 \quad \text{or } x = 8$$

4.

$$15x^{-2} - 11x^{-1} - 12 = 0$$

Let $u = x^{-1}$ then $u^2 = x^{-2}$

$$15u^2 - 11u - 12 = 0$$

$$(5u+3)(3u-4) = 0$$

$$5u + 3 = 0 \text{ or } (3u-4) = 0$$

$$5u = -3 \text{ or } 3u = 4$$

$$u = -\tfrac{3}{5} \text{ or } u = \tfrac{4}{3}$$

Thus, $u = x^{-1} = -\frac{3}{5}$ or $x^{-1} = \frac{4}{3}$

$$\frac{1}{x} = -\frac{3}{5}$$

or

$$\frac{1}{x} = \frac{4}{3}$$

$$x = -\frac{5}{3}$$

or

$$x = \frac{3}{4}.$$

5.3 Equation Containing More than Two Radicals

Example 5.3.1

Solve $\sqrt{x+2}+\sqrt{2x}=\sqrt{18-x}$

In this case, it is impossible to isolate one radical on each sides of the equation, so we begin by squaring both sides as it is. Then proceed as follows

$$\left(\sqrt{x+2}+\sqrt{2x}\right)^2=\left(\sqrt{18-x}\right)^2$$

We use the algebraic property that $(a+b)^2=a^2+2ab+b^2$

$$\left(\sqrt{x+2}\right)^2+2\left(\sqrt{x+2}\right)\left(\sqrt{2x}\right)+\left(\sqrt{2x}\right)^2=18-x$$

$$x+2+2\left(\sqrt{x+2}\right)\left(\sqrt{2x}\right)+2x=18-x$$

$$2.\sqrt{2x}.\sqrt{x+2}=18-x-3x-2$$

$$\sqrt{2x}\,.\sqrt{x+2}=\frac{16-4x}{2}$$

$$\sqrt{2x}\ .\ \sqrt{x+2}=8-2x$$

Squaring both sides

$$\left(\sqrt{2x}\,.\sqrt{x+2}\right)^2=(8-2x)^2$$

$$2x(x+2)=64-32x+4x^2$$

$$x^2-18x+32=0$$

$$(x-16)(x-2)=0$$

$$x=16 \text{ or } x=2.$$

5.4 Two Systems of equations in two variables (Simultaneous equations)

The systems of equations in form of

$$ax+by=p$$

$$cx+dx=q$$

Where x and y are variables and a,b,c,d constants are called simultaneous equations

There are four possible methods that can be used to solve the variables x and y namely

1. **Graphical method**
2. **Substitution method**
3. **Elimination method**
4. **Matrix method**

We discuss two methods of solving the simultaneous equations that is substitution method and elmination method. The method of matrix is studied in the topic matrices.

Substitution method

Steps involved in substitution method

For any two simultaneous questions, we solve the variables x and y by following steps

I. Solve one of the equations for one variable in terms of the other. (making the subject of the formula
II. Substitute the expression obtained in step I into the other equation, producing an equation in one variable.
III. Solve the equation obtained in step II.
IV. Use the solution obtained in step III, along with the expression obtained in step I, to determine the solution of the system.

Example 5.4.1

Solve the system

$x - 3y = -25$

$4x + 5y = 19$

Solution

Solve the first equation for x in terms of y to produce

$x = 3y - 25$

Substitute $3y - 25$ for x in the second equation and solve for y.

$$4(3y - 25) + 5y = 19$$

$$12y - 100 + 5y = 19$$
$$12y + 5y = 100 + 19$$

$$17y = 119$$

$$y = 7$$

Next, substitute 7 for y in the equation $x = 3y - 25$ to obtain

$$x = 3(7) - 25$$

$$x = -4$$

Example 5.4.2

Solve

$5x + 9y = -2$

$2x + 4y = -1$

Solution

$$5x + 9y = -2$$

$$2x + 4y = -1$$

$$5x = -9y - 2$$
$$x = \frac{-9y - 2}{5}$$

We substitute the value of x in the second equation

$$2x + 4y = -1$$

$$2(\frac{-9y - 2}{5}) + 4y = -1$$

$$2(-9y - 2) + 20y = -5$$

$$-18y - 4 + 20y = -5$$

$$2y - 4 = -5$$

$$y = -\frac{1}{2}.$$

Substitute the value of y in the first equation

$$x = \frac{-9y - 2}{5}$$

$$x = \frac{-9(-\frac{1}{2}) - 2}{5} = \frac{1}{2}.$$

Elimination method

1. Any two equations of the system can be interchanged.
2. Any non-zero real number can multiply both sides of any equation of the system.
3. Any equation of the system can be replaced by the sum of that equation and a nonzero multiple of another equation.

Examples 5.4.2

Solve

a) $3x + 5y = -9$
   $-6y = -39$

b) $x + 2y = 8$
   $3x + y = 9$

c) $3x - 4y = 11$
   $7x - 5y = 4$

Non-Linear Simultaneous Equations

Example 9

Solve the simultaneous equations

a) $x^2 + y^2 = 13$
   $3x + y = 9$

b) $3y = 5 - x$
   $\frac{x}{y} + \frac{8y}{x} = 6$

QUADRATIC EQUATIONS

Definition 6.1.1

A quadratic equation of the variable x is defined as any equation that can be written in the form $ax^2 + bx + c = 0$, where $a, b\ c$ are real constants and x is an arbitrary variable. The form $ax^2 + bx + c = 0$ is called the **standard form** of a quadratic equation. The equation $x^2 + 2x - 15 = 0$ is an example of the quadratic equation.

Methods for solving the value of x

1. **Completing the square method.**
2. **Factorization method.**
3. **Quadratic formula method.**
4. **Sketching graph method (graphical method).**

Factorization method

For any quadratic equation $ax^2 + bx + c$ that is factorable, the variable x can be solved as follows

i) Find the product P such that; $P = ac$
ii) Find the sum S, that is $S = b$
iii) Find two real numbers r and t such that $rt = P$ and $r + t = S$
iv) Then express b in the equation $ax^2 + bx + c$ as a sum of r and t and factorize the like terms.

Example 6.1.2

Solve $x^2 + 2x - 15 = 0$

Solution

$$x^2 + 2x - 15 = 0$$

$$a = 1\ ,\quad b = 2$$

and

$$c = -15$$

$$P = ac = 1 \times (-15) = -15$$

$$S = b = 2$$

The two real numbers are $-3, 5$

$$x^2 - 3x + 5x - 15 = 0$$

$$(x + 5)(x - 3) = 0$$

$$x + 5 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -5 \quad \text{or} \quad x = 3$$

Example 6.1.3

Solve $x = -6x^2 + 12$

Solution

$$x = -6x^2 + 12$$

$$6x^2 + x - 12 = 0$$

$$a = 6 \ , \quad b = 1 \text{ and } c = -12$$

$$P = ac = 1 \times (-12) = -12$$

$$S = b = 1$$

The real numbers are $-3, \ 4$

$$6x^2 - 3x + 4x - 12 = 0$$

$$(3x - 4)(2x + 3) = 0$$

$$3x - 4 = 0 \qquad \text{or} \qquad 2x + 3 = 0$$

$$x = \tfrac{4}{3} \qquad \text{or} \quad x = -\tfrac{3}{2}$$

Now suppose that we want to solve $x^2 = k$, where k is any real number. We can proceed as follows.

$$x^2 = k$$

$$x^2 - k = 0$$

$$(x + \sqrt{k})(x - \sqrt{k}) = 0$$

$$x = -\sqrt{k} \quad \text{or} \quad x = \sqrt{k}$$

$$\text{Hence } \ x = \pm\sqrt{k}$$

Property 1
The solution set of $x^2 = k$ is $\left(-\sqrt{k}\, , \sqrt{k}\right)$, which can also be written
$\{\pm\sqrt{k}\}$.

Example 6.1.4

Solve each of the following

a. $x^2 = 72$ *b.* $(3n - 1)^2 = 26$

Solutions

a.
$$x^2 = 72$$
$$x = \pm\sqrt{72}$$
$$x = \pm 6\sqrt{2}.$$

b. $(3n - 1)^2 = 26$

$$3n - 1 = \pm\sqrt{26}$$
$$3n = 1 + \sqrt{26}$$
$$\text{or}$$
$$3n = 1 - \sqrt{26}$$

$$n = \frac{1+\sqrt{26}}{3}$$

or

$$n = \frac{1-\sqrt{26}}{3}.$$

Completing the Square method

Example 6.1.5

Solve $x^2 + 8x - 2 = 0$

Solution

$$x^2 + 8x - 2 = 0$$

$$x^2 + 8x = 2$$

Thus 16 has to be added to the left to make a perfect trinomial. Thus 16 has to be added to the right also.

$$x^2 + 8x + 16 = 2 + 16$$

$$(x+4)^2 = 18$$

$$x + 4 = \pm\sqrt{18}$$

$$x + 4 = \pm 3\sqrt{2}$$

$$x = -4 \pm 3\sqrt{2}.$$

Example 6.1.6

Solve $2x^2 + 6x - 3 = 0$

Solution

$$2x^2 + 6x - 3 = 0$$

$$2x^2 + 6x = 3$$

Multiply both sides by $\frac{1}{2}$

$$x^2 + 3x = \frac{3}{2}$$

add $\frac{9}{4}$ on both sides

$$x^2 + 3x + \frac{9}{4} = \frac{3}{2} + \frac{9}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{15}{4}$$

$$x + \frac{3}{2} = \pm\frac{\sqrt{15}}{2}$$

$$x = \frac{-3+\sqrt{15}}{2} \quad \text{or} \quad x = \frac{-3-\sqrt{15}}{2}$$

Quadratic Formula method

The process called **completing the square** can be used to solve *any* quadratic equation. If we use this process of completing the square to solve the general quadratic equation $ax^2 + bx + c = 0$, we obtain a formula known as the **quadratic formula**.

Definition 6.1.7

If $a \neq 0$, then the solutions (roots) of the equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How do we show that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We consider the quadratic equation $ax^2 + bx + c = 0$, and then proceed as follows

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

multiply both sides by $\frac{1}{a}$

$$\frac{1}{a}(ax^2 + bx) = (-c)\frac{1}{a}$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

complete the square by adding $\frac{b^2}{4a^2}$ on both sides

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

combine the right side in a single fraction

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$\sqrt{4a^2} = |2a|$$

but $2a$ can be used because of $\pm$.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

0r

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 6.1.8

Solve each of the following by using the quadratic formula.

a. $3x^2 - x - 5 = 0$
b. $25x^2 - 30x = -9$
c. $x^2 - 2x + 4 = 0$

Solutions

a. $3x^2 - x - 5 = 0$

We need to think of $3x^2 - x - 5 = 0$ as $3x^2 + (-x) + (-5) = 0$;
thus $a = 3$, $b = -1$ and $c = -5$. We then substitute these values into the quadratic formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)}$$

$$= \frac{1 \pm \sqrt{61}}{6}$$

$$x = \frac{1 - \sqrt{61}}{6}$$

or

$$x = \frac{1 + \sqrt{61}}{6}.$$

b. $25x^2 - 30x = -9$

$25x^2 - 30x + 9 = 0$.

$a = 25, b = -30$ and $c = 9$.

Now we use the formula.

$$x = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(25)(-9)}}{2(25)}$$

$$= \frac{30 \pm \sqrt{0}}{50}$$
$$= \frac{3}{5}$$

c. We substitute $a = 1, b = -2,$ and $c = 4$ into the quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2}$$
$$x = 1 \pm i\sqrt{3}.$$

Discriminant of the quadratic equation

The number $b^2 - 4ac$ in the quadratic equation $ax^2 + bx + c = 0$ is called the **discriminant** of the quadratic equation. It can be used to determine the **nature** of the **solutions** of the quadratic equation as follows.

Types of solutions of the quadratic equation

1. Real solutions (two real numbers)
2. Real solution (one real number)
3. Complex solution (two complex numbers).

Identifying the type of solution in a quadratic equation

Given any quadratic equation $ax^2 + bx + c = 0$

1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2. If $b^2 - 4ac = 0$, the equation has one real solution.
3. If $b^2 - 4ac < 0$, the equation has two complex but nonreal solutions.

The following examples illustrate each of these situations. (You may need to solve the equations completely to verify the conclusions.)

EQUATION	**DISCRIMINANT**	**NATURE OF SOLUTIONS**
$4x^2 - 7x - 1 = 0$	$b^2 - 4ac = 65$	two real solutions
$4x^2 + 12x + +9 = 0$	$b^2 - 4ac = 0$	$o$ne real solution
$5x^2 + 2x + 1 = 0$	$b^2 - 4ac = -16$	$t$wo complex solution

The Sum and the Product of Two Roots

Let

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The sum of the two roots are expressed as

$$x_1 + x_2 = \frac{-\mathrm{b} + \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}} + \frac{-\mathrm{b} - \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$

$$= \frac{-2\mathrm{b}}{2\mathrm{a}}$$

$$= -\frac{b}{a}$$

The product of the two roots are expressed as

$$(x_1)(x_2) = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} \quad = \frac{c}{a}.$$

6.6 Symmetric properties of the roots of quadratic equations

Let α and β be the two roots (solutions) of the quadratic equation $ax^2 + bx + c = 0$. Then the sum of the roots $\alpha + \beta = -\frac{b}{a}$ and the product of roots $\alpha\beta = \frac{c}{a}$

Proof

We have

$$ax^2 + bx + c = 0 \;\Rightarrow\; x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

If α and β are roots, then $(x - \alpha)(x - \beta) = 0$.

This implies that

$$x^2 - \beta x - \alpha x + \alpha\beta = 0 \Rightarrow x^2 - (\beta + \alpha)x + \alpha\beta = 0$$

Hence $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Example 6.6.1

The roots of the equation $3x^2 - 8x + 2 = 0$ are α and β. Find the values of

a) $\frac{1}{\alpha} + \frac{1}{\beta}$

b) $a^2 + b^2$

c) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

d) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

e) $\alpha^4+\beta^4$

Solutions

$3x^2-8x+2=0$

$$\alpha+\beta=-\frac{b}{a}=-\frac{-8}{3}=\frac{8}{3},$$

$$\alpha\beta=\frac{2}{3}.$$

a)

$$\begin{aligned}\frac{1}{\alpha}+\frac{1}{\beta}&=\frac{\beta+\alpha}{\alpha\beta}\\&=\frac{8}{3}\div\frac{2}{3}\\&=4.\end{aligned}$$

b) We have

$$(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$$

Then

$$\begin{aligned}(\alpha+\beta)^2&=\alpha^2+\beta^2+2\alpha\beta\\\alpha^2+\beta^2&=(\alpha+\beta)^2-2\alpha\beta\\\alpha^2+\beta^2&=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)\\&=\left(-\frac{8}{3}\right)^2-2\left(\frac{2}{3}\right)\\&=\frac{64}{9}-\frac{4}{3}\\&=5\frac{7}{9}\end{aligned}$$

c)

$$\begin{aligned}\frac{1}{\alpha^2}+\frac{1}{\beta^2}&=\frac{\beta^2+\alpha^2}{(\alpha\beta)^2}\\&=\frac{52}{9}\div\left(\frac{2}{3}\right)^2\\&=\frac{52}{9}\div\frac{4}{9}\\&=\frac{52}{9}\times\frac{9}{4}\\&=13.\end{aligned}$$

d)

$$\begin{aligned}\frac{\alpha}{\beta}+\frac{\beta}{\alpha}&=\frac{\alpha^2+\beta^2}{\alpha\beta}\\&=\frac{52}{9}\div\frac{2}{3}\\&=\frac{52}{9}\times\frac{3}{2}\end{aligned}$$

$$= \frac{26}{3}$$
$$= 8\frac{2}{3}$$

e)

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$$
$$= \left(\frac{52}{9}\right)^2 - 2\left(\frac{2}{3}\right)^2$$
$$= \frac{2704}{81} - \frac{8}{18}.$$

Equations of Quadratic Form

An equation such as $x^4 + 5x^2 - 36 = 0$ is not a quadratic equation. However, if we let

$u = x^2$ then , $u^2 = x^4$. We substitute in the equation we have $u^2 + 5u - 36 = 0$ which is a quadratic equation of the form $au^2 + bu + c = 0$ where $a \neq 0$ and u is an arbitrary variable of algebraic expression in x.

Example 6.6.2

Solve $x^{2/3} + x^{1/3} - 6 = 0$

Solution

$$x^{2/3} + x^{1/3} - 6 = 0$$

Let $u = x^{1/3}$ then $u^2 = x^{2/3}$ and the equation can be written as

$$u^2 + u - 6 = 0$$
$$(u + 3)(u - 2) = 0$$
$$u + 3 = 0 \quad \text{or} \quad u - 2 = 0$$
$$u = -3 \quad \text{or} \quad u = 2$$

Now substituting $x^{1/3}$ for u we have

$$x^{1/3} = -3 \quad \text{or} \quad x^{1/3} = 2$$

From which we obtain $x = (-3)^3$ or $x = 2^3$

$$x = -27 \quad \text{or} \quad x = 8$$

Example 6.6.3

Solve $x^4 + 5x^2 - 36 = 0$

Solution

$$x^4 + 5x^2 - 36 = 0$$
$$(x^2 + 9)(x^2 - 4) = 0$$
$$x^2 + 9 = 0 \quad \text{or} \quad x^2 - 4 = 0$$
$$x^2 = -9 \quad \text{or} \quad x^2 = 4$$
$$x = \pm 3i \quad \text{or} \quad x = \pm 2.$$

REVIEW EXERCISE

1. Solve each equations

a. $\sqrt{3x-2}=4$

b. $\sqrt{2t-1}+2=t$

c. $\sqrt{x+2}-1=\sqrt{x-3}$

d. $\sqrt{-2x-7}\ +\sqrt{x+9}=\sqrt{8-x}$

$e.$ $\sqrt{2x-3}=1$

f. $\sqrt{3x-2}=3x-2$

g. $\sqrt{3x+1}+\sqrt{2x+4}=3$

h. $\sqrt{x-2}\ +\sqrt{2x-11}=\sqrt{x-5}$

2. Solve each equation by factoring or by using the property, if $x^2=k$ then $x=\pm\sqrt{k}$.

a. $x^2-3x-28$ b. $2x^2-3x=0$ c. $9y^2=12$ d. $x^2-4x-12=0$

3. Use the method of completing the square to solve each equation.
   a. $x^2-10x+24=0$
   b. $n^2+10n-2=0$
   c. $y^2+5y=-2$
   d. $3n^2+5n-1=0$
   e. $x^2-6n+20=-1$
4. Use the quadratic formula to solve each equation
   a. $3x^2+16x=-5$
   b. $x^2+4=8x$
   c. $2a^2-6a+1=0$
   d. $x^2+24=0$
   e. $n^2-3n=-7$

5. Find the discriminant of each of the following quadratic equations and determine whether the equation has (1) two complex but non-real solutions, (2) one real solution, or (3) two unequal real solutions.
   a. $5x^2-2x-4=0$ b. $x^2+4x+7=0$ c. $16x^2=40x-25$
6. Solve each equation
   a. $x^4-5x^2+4=0$
   b. $6x^{2/3}-5x^{1/3}-6=0$
   c. $x^{2/3}+3x^{1/3}-10=0$
   d. $x^4-2x^2-35=0$

INEQUALITIES EQUATIONS

The following symbols are used $<$ less than, $>$greater that, $\leq$ less than or equal to , $\geq$ greater than or equal to

Properties

For any real numbers a and b, then

$a<b$	means a is less than b.
$a\ \leq\ b$	means a is less than or equal to b.
$a\ >b$	means a is greater than b.
$a\ \geq\ b$	means a is greater than or equal to b.

The following are examples are true

$8 + 7 > 10$ $\quad$ $4 - 1 < 6$ $\quad$ $5 + 4 \leq 9$ $\quad$ $9 - 3 \geq 6$

Linear inequalities

A linear inequality is an algebraic inequality contain one or more variables. The following are examples of algebraic inequalities.

$x + 3 > 8$, $3x + 2y \leq 4(x - 2)$, $(x + 4) \geq 0$, $x^2 + y^2 + z^2 \leq 16$

Properties in solving linear inequalities

1. For all real numbers a, b and $c, a > b$ if and only if $a + c > b + c$
2. For all real numbers a, b and c, with $c > 0, a > b$ if and only if $ac > bc$
3. For all real numbers a, b and c, with $c < 0, a < b$ if and only if $ac < bc$

Example 7.1.1

Solve $3(2x - 1) < 8x - 7$

Solution

$$3(2x - 1) < 8x - 7$$

$$6x - 3 < 8x - 7$$

$$-2x - 3 < -7$$

$$-2x < -4$$

Multiply both sides by $-\frac{1}{2}$ which reverses the inequality

$$x > 2\,.$$

Example 7.1.2

Solve $\frac{-3x+1}{2} > 4$

Solution

$$\frac{-3x + 1}{2} > 4$$

Multiply both sides by 2 $\quad \left(\frac{-3x+1}{2}\right) 2 > 4(2)$

$$-3x + 1 > 8$$

$$-3x > 8 - 1$$

$$-3x > 7$$

$$x < -\frac{7}{3}.$$

Example 7.1.3

Solve

$$\frac{x-4}{6}-\frac{x-2}{9}\leq\frac{5}{18}$$

Solution

$$\frac{x-4}{6}-\frac{x-2}{9}\leq\frac{5}{18}$$

$$18\left(\frac{x-4}{6}-\frac{x-2}{9}\right)\leq\left(\frac{5}{18}\right)18$$

$$3(x-4)-2(x-2)\leq 5$$

$$3x-12-2x+4\leq 5$$

$$x-8\leq 5$$

$$x\leq 13.$$

Compound Statements

we use the words **and**, **or** in mathematics to form compound statements.

1. $8+4=12$ and $-4<-3$ True
2. $-7<-2$ and $-6<-10$ True
3. $6>5$ and $-4<-8$ False
4. $4<2$ and $0<-10$ False
5. $-3+2=1$ and $5+4=8$ False

We call compound statements that use or disjunctions. The following are some examples of disjunctions that involve numerical statements.

6. $14>13$ or $35<37$ True
7. $\frac{3}{4}>\frac{1}{2}$ or $-4+(-3)=10$ true

Example 7.1.4

Solve $-2<\frac{3x+2}{2}<7$

Solution

$$-2<\frac{3x+2}{2}<7$$

$$-2(2)<2\left(\frac{3x+2}{2}\right)<2(7)$$

$$-4<3x+2<14$$

$$-4-2<3x<14-2$$

$$-6<3x<12$$

$$-2<x<4.$$

Definition 7.1.5

The **union** of two sets A and B (written as $A \cup B$) is the set of all elements that are in A or in B or in both. Using set-builder notation, we can write:

$$A \cup B = \{x | \ x \in A \ \text{ or } \ x \in B\}.$$

Example 7.1.6

Express $x < -1$ or $x > 2$ in interval notation

Solution

The solution set is $(-\infty, -1) \cup (2, \infty)$

7.2 Quadratic Inequalities

The equation $ax^2 + bx + c = 0$ is called the standard form of a quadratic equation in one variable. Similarly, the form $ax^2 + bx + c < 0$ is called **quadratic inequality**.

(The symbol $<$ can be replaced by $>, \leq$ or $\geq$ to produce other forms of quadratic inequalities.)
The number line can be used to help us solve quadratic inequalities where the quadratic polynomial is factorable.

Example 7.2.1

Solve $x^2 + x - 6 < 0$

Solution

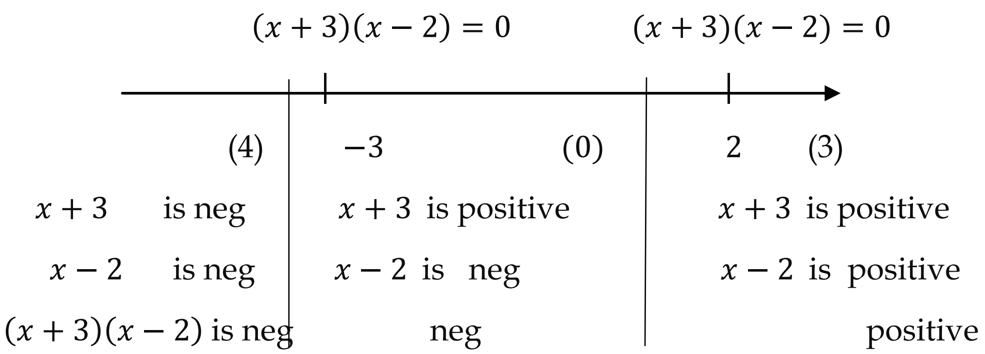
$$x^2 + x - 6 < 0$$

$$(x + 3)(x - 2) < 0$$

Let $(x + 3)(x - 2)$ is equal to zero. Then $x = -3$ and $x = 2$
The numbers -3 and 2 divide the number line into three interval.

We can choose a **test number** from each of these intervals and see how it affects the signs of the factors $x + 3$ and $x - 2$ and, consequently, the sign of the product of these factors.

Figure 1



Therefore, the given inequality $x^2 + x - 6 < 0$, is satisfied by the numbers between -3 and 2. That is, the solution set is the open interval $(-3, 2)$.

Definition 7.2.2

The numbers where the given polynomial or algebraic expression equals zero or is undefined, are referred to as **critical numbers**.

Example 7.2.3

Solve $6x^2 + 17x - 14 \geq 0$

Solution

$$6x^2 + 17x - 14 \geq 0$$

$$(2x + 7)(3x - 2) \geq 0$$

$$(2x + 7)(3x - 2) = 0$$

$$x = -\frac{7}{2}$$

and

$$x = \frac{2}{3}.$$

Now let us choose a test number from each of the three intervals and observe the sign behavior of the factors

$(2x + 7)(3x - 2) = 0$ $\quad$ $(2x + 7)(3x - 2) = 0$

(-4) $-\frac{7}{3}$ $\quad$ (0) $\quad$ $\frac{2}{3}$ $\quad$ (2)

$(2x + 7)$ is neg $\quad$ $(2x + 7)$is positive $\quad$ $(2x + 7)$is positive

$(3x - 2)$is neg $\quad$ $(3x - 2)$is neg $\quad$ $(3x - 2)$ is positive

p $\quad$ positive $\quad$ negative $\quad$ positive

Using the concept of set union, we can write the solution set$\left(\infty, \frac{7}{3}\right] \cup \left[\frac{2}{3}, \infty\right)$.

REVIEW EXERCISE

1. Solve each conjunction by using the compact form and express the solution sets in interval notation.
   a. $-17 \leq 3x - 2 \leq 10$ $\quad$ d. $-4 < \frac{x-1}{3} < 4$
   b. $4 > 3x + 1 > 1$ $\quad$ e. $-1 \leq \frac{x+2}{1} \leq 1$
2. Solve each inequality and express the solution sets in interval notation.
   a. $4x^2 - 4x + 1 > 0$ $\quad$ c. $9x^2 + 6x + 1 \leq 0$
   b. $15x^2 - 26x + 8 \leq 0$ $\quad$ d. $x^2 + 5x \leq 6$

7.3 Inequalities involving quotients and absolute value

Example 7.3.1

Solve $\quad \frac{x-2}{x+3} > 0$

Solution

$$\frac{x - 2}{x + 3} > 0$$

First we find that at $x = 2$, the quotient $\frac{x-2}{x+3}$ equals zero and that at $x = -3$, the quotient is undefined. The critical numbers -3 and 2 divide the number line into three intervals. Then, using a test number from each interval, we have

	-4	1	3
	-3	2	
	$x-2$ is neg	$x-2$ is neg	$x-2$ is positive
	$x+3$ is neg	$x+3$ is positive	$x+3$ is positive
	$\frac{x-2}{x+3}$ is positive	neg	positive

Therefore the solution set of $\frac{x-2}{x+3} > 0$ is of $(-\infty, 3) \cup (2, \infty)$ ■

Example 7.3.2

Solve $\frac{x+2}{x+4} \leq 3$

Solution

First, let us change the form of the given inequality.

$$\frac{x+2}{x+4} \leq 3$$

$$\frac{x+2}{x+4} - 3 \leq 0$$

$$\frac{x+2-3(x+4)}{x+4} \leq 0$$

$$\frac{-2x-10}{x+4} \leq 0.$$

If $x = -5$, then the quotient $\frac{-2x-10}{x+4}$ equals zero, and if $x = -4$, the quotient is undefined. Then, using test numbers such as $-6, -4\frac{1}{2}$ and-3, we are able to study the sign behavior of the quotient, as in the figure below

	-6	$-4\frac{1}{2}$	-3
$-2x-10$	Positive	neg	neg
$x+4$	neg	neg	positive
$\frac{-2x-10}{x+4}$	neg	positive	neg

The solution set of $\frac{x+2}{x+4} \leq 3$ is $(-\infty, -5] \cup (-4, \infty)$.

Absolute Value

we defined the ***absolute value*** of a real number by $|a| = \begin{cases} a, & if \quad a \geq 0 \\ -a, & if \quad a < 0 \end{cases}$

Property 1

For any real number $k > 0$, if $|x| = k$, then $x = k \ or \ \ x = -k$

Property 2

For any real number $k > 0$, if $|x| < k, then \ \ -k < x < k$.

Example 7.3.3

Solve $|2x + 1| < 5$

Solution

$$-5 < 2x + 1 < 5$$

$$-6 < 2x < 4$$

$$-3 < x < 2$$

The solution set is the interval $(-3,2)$.

Example 7.3.4

Solve $\left|\frac{x-2}{x+3}\right| < 4$

Solution

$-4 < \frac{x-2}{x+3} < 4$

Which can be written as

$$\frac{x-2}{x+3} > -4$$

and

$$\frac{x-2}{x+3} < 4$$

Each part of this and statement can be solved as we handled example earlier.

$$\frac{x-2}{x+3} + 4 > 0$$

and

$$\frac{x-2}{x+3} - 4 < 0$$

$$\frac{x - 2 + 4(x+3)}{x+3} > 0$$

and

$$\frac{x-2-4(x+3)}{x+3} - 4 < 0$$

$$\frac{x-2+4x+12}{x+3} > 0$$

and

$$\frac{x-2-4x+12}{x+3} < 0$$

$$\frac{5x+10}{x+3} > 0$$

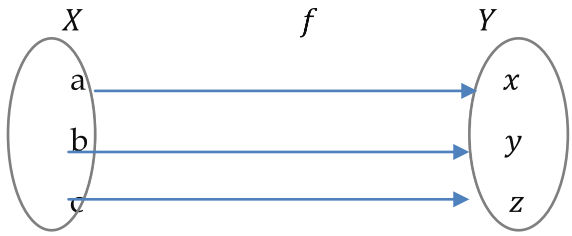
and

$$\frac{-3x-14}{x+3} < 0.$$

This solution set for $\left|\frac{x-2}{x+3}\right| < 4$ is

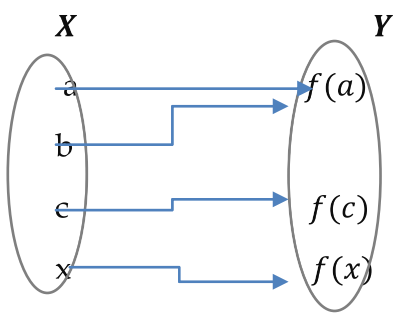
$$\left(-\infty, -\frac{14}{3}\right) \cup (-2, \infty).$$

FUNCTIONS



Definition 8.1.1

A ***function*** f is a correspondence between two sets X and Y that assigns to each element x of set X one and only one element y of set Y. The element y being assigned is called the **image** of x. The set X is called the **domain** of the function, and the set of all images is called the **range** of the function.



Definition 8.1.2

A ***relation*** is defined as a set of ordered pairs, and a function is defined as a relation in which no two ordered pairs have the same first element. For example, the equation $f(x) = 2x + 3$ indicates that to each value of x in the domain.

when $x = 1$ then $f(1) = 2(1) + 3 = 5$ gives the ordered pair (1,5)

when $x = 4$ then $f(4) = 2(4) + 3 = 11$ gives the ordered pair (4,11)

When $x = -2$ then $f(-2) = 2(-2) + 3 = -1$ gives the ordered pair $(2, -1)$

Example 8.1.3

If $f(x) = x^2 - x + 4$ and $g(x) = x^3 - x^2$, find $f(3), f(-2), g(4)$ and $g(-3)$

Solutions

$f(3) = 3^2 - 3 + 4 = 10$ and $g(4) = 4^3 - 4^2 = 48$

$f(-2) = (-2)^2 - (-2) + 4 = 10$ and $g(-3) = (-3)^3 - (-3)^3 = -36$.

Definition 8.1.4

A function f is called a **piecewise function** if it can be written in piecewise intervals. Let us consider an example of such a function.

Example 8.1.5

If $f(x) = \begin{cases} 2x + 1 & if \quad x \geq 0 \\ 3x - 1 & if \quad x < 0 \end{cases}$ find $f(2), f(4), f(-1)$ and $f(-3)$

Solutions

For $x \geq 0$, we use the assignment

$$f(x) = 2x + 1,$$

$$f(2) = 2(2) + 1 = 5,$$

,

$$f(4) = 2(4) + 1 = 9.$$

For $x < 0$, we use the assignment

$$f(x) = 3x - 1\,,$$

$$f(-1) = 3(-1) - 1 = -4 \quad ,$$

$$f(-3) = 3(-3) - 1 = -10.$$

8.2 Graphs of functions

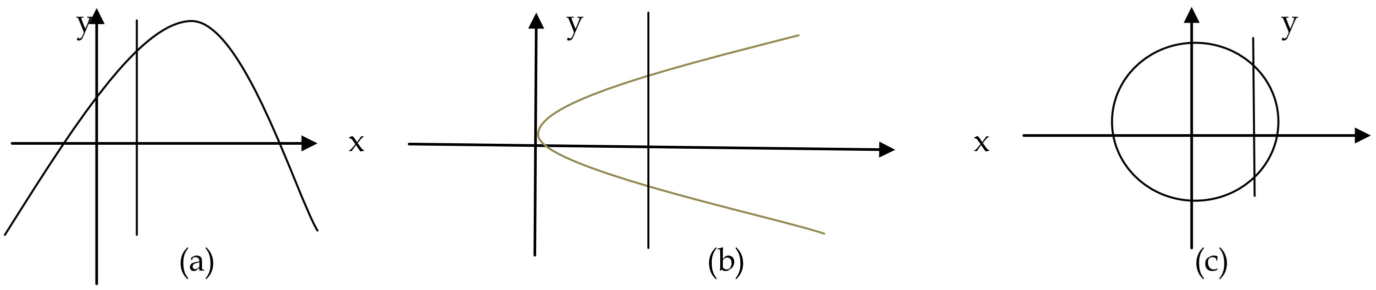
We can define the **graph** of a function f to be the set of all points in a plane of the form $(x,\ f\ (x))$, where x is from the domain of f. The graph of f is the same as the graph of the equation $y = f\ (x)$

Line test of the functions

We can know any given graph if it is a graph of a function or not by using the line test.

Vertical line test

A graph is said to be a graph of a function if the vertical line drawn at any point of the graph does not intersect the graph at more than one point.



Domain and Range of a Function

Example 8.2.1

For the function$f(x) = \sqrt{x-1}$, (a) specify the domain, (b) determine the range, and (c) evaluate $f\ (5), f(50)$, and $f\ (25)$.

Solutions

- ***a.*** The radicand must be nonnegative, so $x - 1 \geq 0$ and thus$x \geq 1$. Therefore, the domain (D) is $D = \{x|x \geq 1\}$
- ***b.*** The symbol $\sqrt{\ }$ indicates the nonnegative square root; thus the range ($\mathbb{R}$) is $R = \{\ f(x)|f(x) \geq 0\ \}$
- ***c.*** $f(5) = \sqrt{4} = 2$
  $f(50) = \sqrt{49} = 7$
  $f(25) = \sqrt{24} = 2\sqrt{6}$

Example 8.2.2

Find the domain for each of the following functions

a) $f(x) = \frac{3}{2x-5}$

b) $f(x) = \frac{1}{x^2-9}$

c) $f(x) = \sqrt{x^2 + 4x - 12}$

Solutions

a. We can replace x with any real number except $\frac{5}{2}$, because $\frac{5}{2}$ makes the denominator zero that makes the function undefined. Thus the domain is $D = \left\{x \,\middle|\, x \neq \frac{5}{2} \quad x \in \mathbb{R}\right\}$

b. We need to eliminate any values of x that will make the denominator zero. Therefore, let's solve the equation$x^2 - 9 = 0$.

$$x^2 - 9 = 0$$
$$x^2 = 9$$

$x = \pm 3$
The domain is thus the set
$D = \{x | x \neq 3 \; and \; x \neq -3 \quad x \in \mathbb{R}\}$

d. The radicand, $x^2 + 4x - 12$ must be nonnegative. Therefore, let's use a number line approach as we did in solving the inequality $x^2 + 4x - 12 \geq 0$

$(x + 6)(x - 2) \geq 0$
The critical values are $x = -6 \;\; and \;\; x = 2$

	-7	0	3
	-6		2
$(x + 6)$ is	negative	positive	positive
$(x - 2)$ is	negative	negative	positive
The product is	positive	negative	positive

The product $(x + 6)(x - 2)$ is nonnegative if $x \leq -6 \; or \; x \geq 2$. Using interval notation, we can express the domain as $(-\infty, -6] \cup [2, +\infty)$.

8.3 Even and Odd Functions

A function f having the property that $f(-x) = f(x)$ for every x in the domain of f is called an **even function**. A function f having the property that $f(-x) = -f(x)$ for every x in the domain of f is called an **odd function**.

Example 8.4
For each of the following, classify the function as even, odd, or neither even nor Odd.
a. $f(x) = 2x^3 - 4x$ b. $f(x) = x^4 - 7x^2$ c. $f(x) = x^2 + 2x - 3$

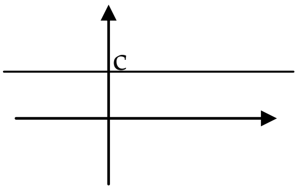
Solutions

a. The function $f(x) = 2x^3 - 4x$ is an odd function because
$f(-x) = 2(-x)^3 - 4(-x)$ which is $-2x^3 + 4x = -(2x^3 - 4x) = -f(x)$
b. The function $f(x) = x^4 - 7x^2$ is an even function since
$f(-x) = (-x)^4 - 7(-x)^2 = x^4 - 7x^2$ which is equal to $f(x)$.
c. The function $f(x) = x^2 + 2x - 3$ is neither even nor odd because
$f(-x) = (-x)^2 + 2(-x) - 3 = x^2 - 2x - 3$ Which is neither $f(x)$ nor $-f(x)$.

8.5 Sketching of some basic graphs of function

The constant function

$y = c \quad where\ c\ is\ a\ constant$

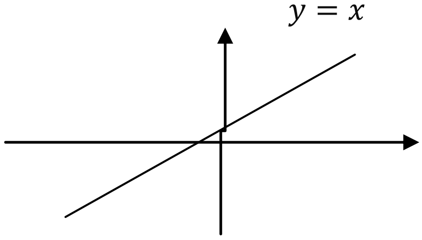


Domain $= \{x: x \in R\}$

Range$= \{y = c \ \ where\ c \in R\}$

identity function

$y = x$

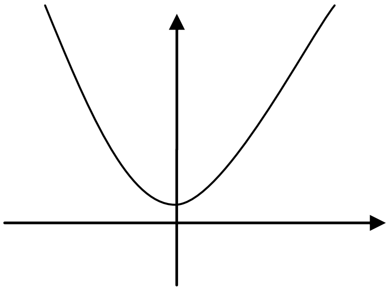


Domain$= \{x: x \in R\}$

Range$= \{y: y \in R\}$

Squaring function

$y = x^2$

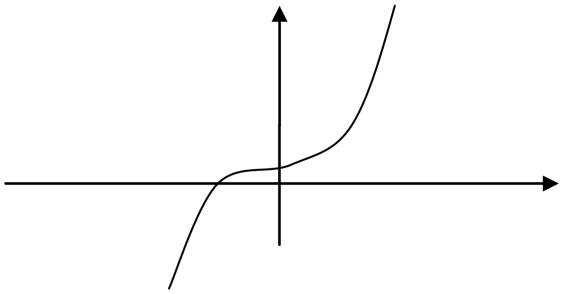


Domain$= \{x: x \in R\}$

Range$= \{y \in R: y \geq 0\}$

cubic function

$y = x^3$

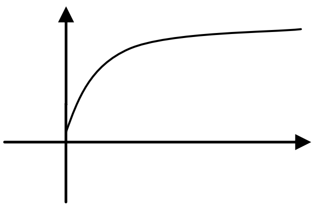


Domain$= \{x: x \in R\}$

Range$= \{y: y \in R\}$

Square root function

$y = \sqrt{x}$

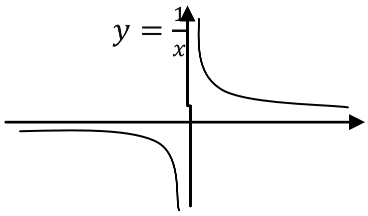


Domain$= \{x: x \in R, x \geq 0\}$

Range$= \{y: y \in R, y \geq 0\}$

Reciprocal function

$y = \frac{1}{x}$

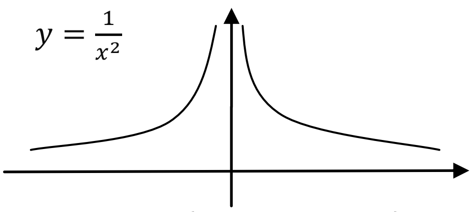


Domain$= \{x: x \in R, x \neq 0\}$

Range$= \{y: y \in R, y \neq 0\}$

Reciprocal square function

$y = \frac{1}{x^2}$

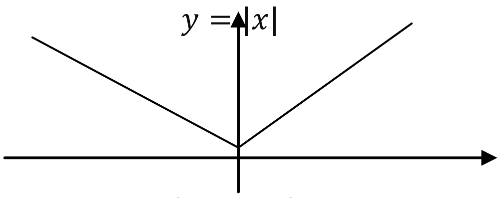


Domain$= \{x: x \in R, x \neq 0\}$

Range$= \{y \in R: y > 0\}$

Absolute value function

$y = |x|$



Domain$= \{x: x \in R\}$

Range$= \{y \in R: y \geq 0\}$

8.6 Transformation of Common Functions

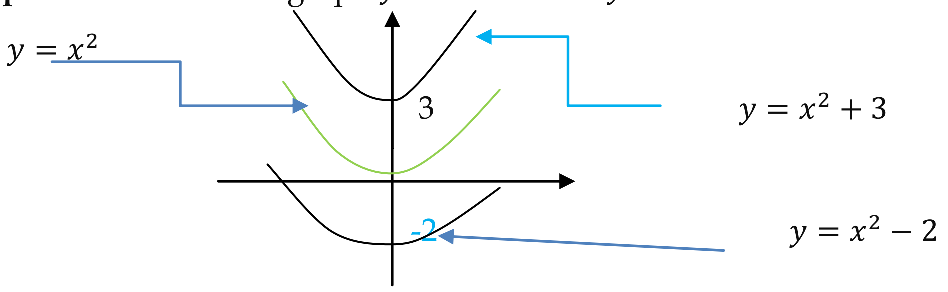
8.6.1 Translations of the Basic Curves

1. The graph of $f(x) = x^2 + 3$ is the graph of $f(x) = x^2$ moved up threeunits.
2. The graph of $f(x) = x^2 - 2$ is the graph of $f(x) = x^2$ moved down twounits.

Vertical Translation

The graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted k units upward if $k > 0$ or shifted $|k|$ units downward if $k < 0$.

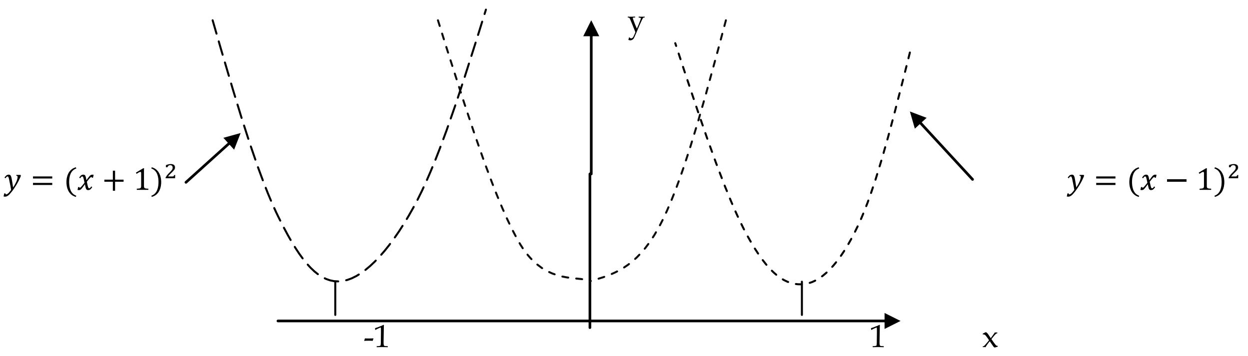
Example 8.6.1: sketch the graph $y = x^2 + 3$ and $y = x^2 - 2$ on the same axis



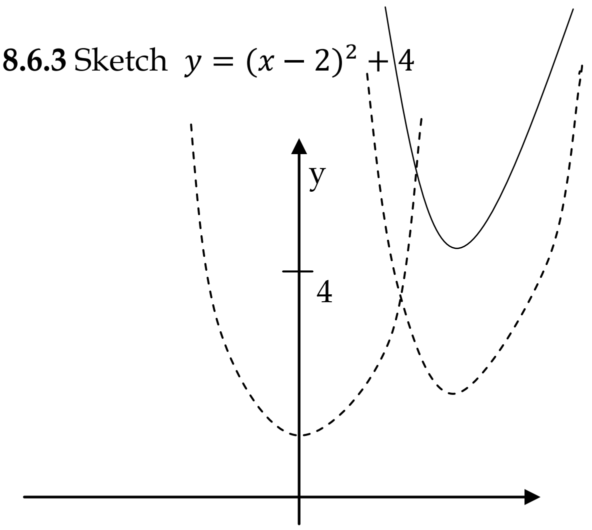
Horizontal Translation

The graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted h units to the right if $h > 0$ or shifted $|h|$ units to the left if $h < 0$.

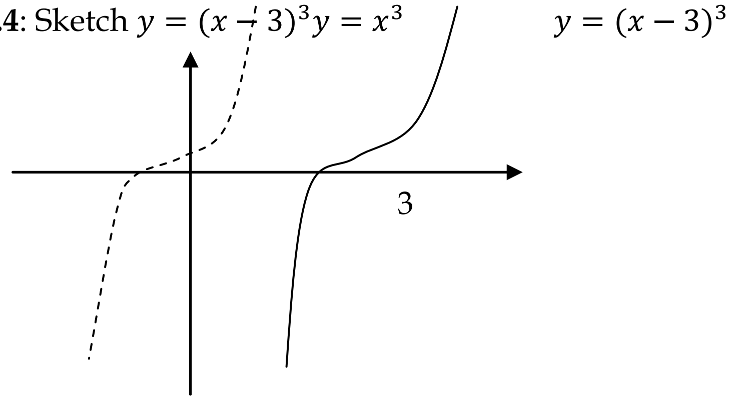
Example 8.6.2
Sketch $y = (x-1)^2$ and $y = (x+1)^2$ on the same axis.



Example 8.6.3 Sketch $y = (x-2)^2 + 4$



Example 8.6.4: Sketch $y = (x - 3)^3$

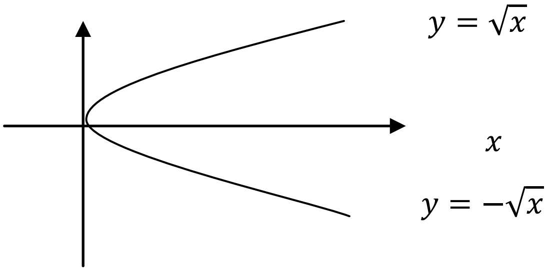


8.7 Reflections of the Basic Curves

x-axis Reflection

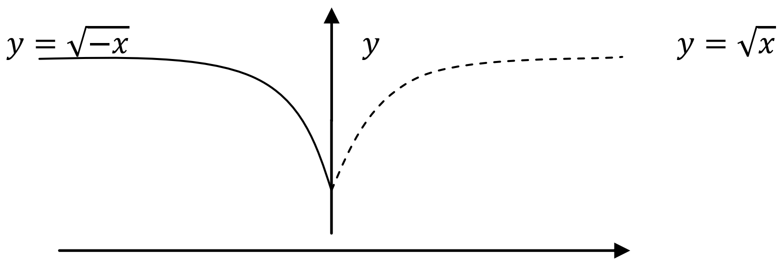
The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected through the x- axis.

The graph of $f(x) = -\sqrt{x}$ is obtained by reflecting the graph of $f(x) = \sqrt{x}$ through the x axis. Reflections are sometimes referred to as **mirror images**.



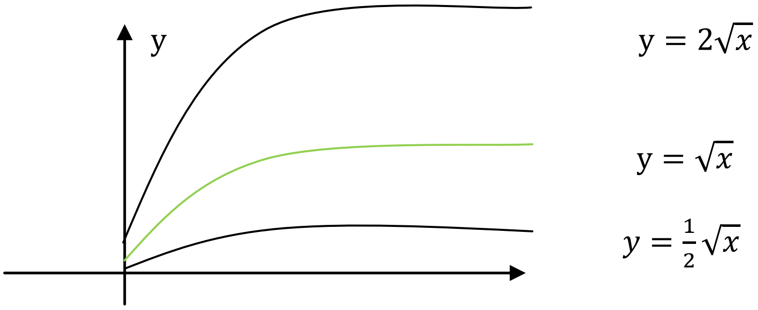
y-axis Reflection

The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected through the y −axis.
Now suppose that we want to do a $y - axis$ reflection of $f(x) = \sqrt{x}$. Because $f(x) = \sqrt{x}$ is defined for $x \geq 0$, the y −axis reflection $f(x) = \sqrt{-x}$ is defined for $-x \geq 0$, which is equivalent to $x \leq 0$. The figure shows the y −axis reflection of $f(x) = \sqrt{x}$.



Vertical Stretching and Shrinking

Translations and reflections are called **rigid transformations** because the basic shape of the curve being transformed is not changed. In other words, only the positions of the graphs are changed. The graph of $y = cf(x)$ is obtained from the graph of $y = f(x)$ by multiplying the y coordinates for $y = f(x)$ by c. If $c > 1$, the graph is said to be *stretched* by a factor of c, and if $0 < c < 1$, the graph is said to be *shrunk* by a factor of c.
The graph of $f(x) = 2\sqrt{x}$ is obtained by doubling the y- coordinates of points on the graph of $f(x) = \sqrt{x}$. Likewise, the graph of $f(x) = \frac{1}{2}\sqrt{x}$ is obtained by halving the y coordinates of points on the graph of $f(x) = \sqrt{x}$.

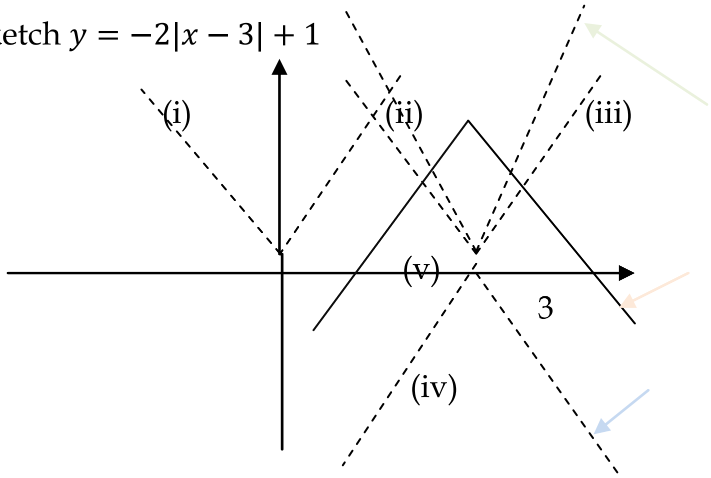


Successive Transformations

Some curves are the result of performing more than one transformation on a basic curve. That is the graph can involve stretching, a reflection, a horizontal translation, and a vertical translation of the basic absolute value function.

Example 8.7.1. Sketch $y = -2|x-3| + 1$

1



This is the basic absolute value curve stretched by a factor of 2, reflected through the x axis, shifted three units to the right, and shifted one unit upward. To sketch the graph, we locate the point (3, 1) and then determine a point on each of the rays.

REVIEW EXERCISE

Sketch each of the following functions, find the domain and the range from the graph.

a. $y = (x+2)^3$

b. $y = 2\sqrt{x-1}$

c. $y = \sqrt{2-x}$

d. $y = -2(x+1)^3 + 2$

$e.$ $y = 2x^3 + 3$

$f.$ $y = 2|x+1| - 4$

$g.$ $y = -\sqrt{x+2} + 2$

8.8 Combining functions

Functions are defined in terms of sums, differences, products, and quotients of simpler functions. In general, if f and g are functions and D is the intersection of their domains, then the following definitions can be made.

Sum

$$(f+g)(x) = f(x) + g(x)$$

Difference

$$(f-g)(x) = f(x) - g(x)$$

Product

$$(f.g)(x) = f(x).g(x)$$

Quotient

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} g(x) \neq 0.$$

Example 8.8.1
If $f = 3x - 1$ and $g(x) = x^2 - x - 2$. Find

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(f.g)(x)$
d. $\left(\frac{f}{g}\right)(x)$

and determine the domain of each.

Solutions

a. $(f+g)(x) = 3x - 1 + x^2 - x - 2 = x^2 + 2x - 3$

b. $(f-g)(x) = (3x-1) - (x^2 - x - 2) = -x^2 + 4x + 1$

c. $(f.g)(x) = (3x-1)(x^2 - x - 2) = 3x^3 - 4x^2 - 5x + 2$

d. $\left(\frac{f}{g}\right)(x) = \frac{3x-1}{x^2-x-2}$

The domain of both f and g is the set of all real numbers. Therefore, the domain of $f+g, f-g$, and $f.g$ is the set of all real numbers. For f/g, the denominator $x^2 - x - 2$ cannot equal zero. Solving $x^2 - x - 2 = 0$

$$(x-2)(x+2) = 0$$
$$(x-2) = 0 \text{ or } (x+2) = 0$$
$$x = 2 \text{ or } x = -1$$

Therefore, the domain for f/g is the set of all real numbers except 2 and -1.

8.9 Composition of Functions
Besides adding, subtracting, multiplying, and dividing functions, there is another important operation called ***composition***. The composition of two functions can be defined as follows.

Definition 8.9.1
The **composition** of functions f and g is defined by $(f \, o \, g)(x) = f(g(x))$ for all x in the domain of g such that $g(x)$ is in the domain of f.

Example 8.9.2

If $f(x) = x^2$ and $g(x) = 3x - 4$. Find $(f \ o \ g)(x), (g \ o \ f)(x)$ and determine its domain.

Solution

$$\begin{aligned}(f \ o \ g)(x) &= f\,(g(x)) \\ &= f(3x-4) \\ &= (3x-4)^2 \\ &= 9x^2 - 24x + 16.\end{aligned}$$

Since g and f are both defined for all real numbers, so is $f \ o \ g$.

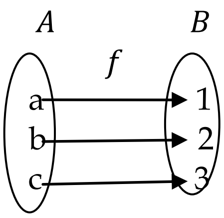
$$\begin{aligned}(g \ o \ f)(x) &= g\,(f(x)) \\ &= g(x^2) \\ &= 3x^2 - 4.\end{aligned}$$

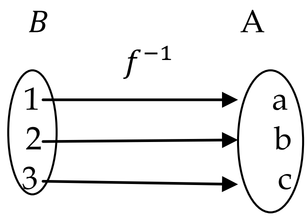
Since f and g are defined for all real numbers, so is $g \ o \ f$.

The composition of functions is not a commutative operation.
In other words, $f \ o \ g \ \neq g \ o \ f$ for all functions f and$g$. But there is a special class of functions for which $f \ o \ g \ = g \ o \ f$.

8.10 Inverse of Functions

Let f and g be two functions such that $f\big(g(x)\big) = x$ for every x in the domain of g and $g\big(f(x)\big) = x$ for every x in the domain of f under this condition, the function g is the inverse of the function$f$. The function g is usually denoted by f^{-1}.





Example 8.10.1
Show that the functions f and g are the inverses of each other where
$f(x) = x + 1$ and $g(x) = x - 1$

Solution
We show that $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$

$$f\big(g(x)\big) = (x-1) + 1 = x$$

$$g\big(f(x)\big) = (x+1) - 1 = x$$

Hence f and g are inverses of each other.

Example 8.10.2

Show that f and g are inverses of each other where $f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\frac{x+1}{2}}$

Solution

We show that $f(g(x)) = x$ and $g(f(x)) = x$

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$= 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$$

$$g(f(x)) = \sqrt[3]{\frac{(2x^3-1)+1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$$

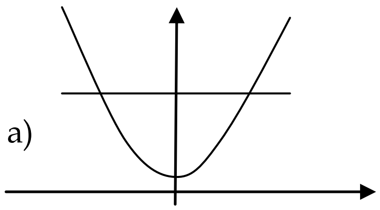
Hence f and g are inverses of each other.

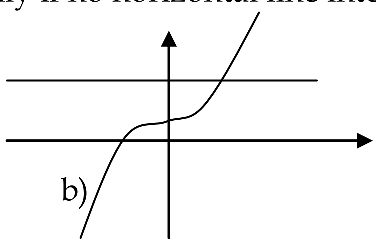
8.11 One –to – one function

A function f is one-to-one if each value of the dependent variable y corresponds to exactly one of the independent variable x. A function f has an inverse if and only if f is one-to-one. A one-to-one function is said to be ***injective***.

Horizontal line test for inverse function

A function f has an inverse function if and only if no horizontal line intersects the graph of f at more than one point.

a) 

b) 

a) The function $y = x^2$ has the horizontal line that cuts the graph at more than one point. Hence not one-to-one and has no inverse.
b) The function $y = x^3$ has the horizontal line that cuts the graph only at one point. Hence one-to-one and has the inverse.

Algebraic test for one-to-one functions

Let A and B be two sets, a function f from A to B is called one-to-one (injective) if

$$f(a) = f(b) \Rightarrow a = b$$

Example 8.11.1

Show if the functions $y = 3x - 2$ and $y = x^2$ are one-to-one or not.

Solution

$y = 3x - 2$

We show that $f(a) = f(b) \Rightarrow a = b$

$$f(a) = f(b)$$
$$3a - 2 = 3b - 2$$
$$3a = 3b$$
$$a = b \qquad \text{Hence one-to-one}$$

$y = x^2$
We show that $f(a) = f(b) \Rightarrow a = b$
$f(a) = f(b)$

$$a^2 = b^2$$
$$a^2 - b^2 = 0$$
$$(a - b)(a + b) = 0$$
$$a = b \quad \text{or} \quad a = -b$$

Hence f is not one – to – one

Finding the inverse of a function

i. Use the horizontal line test to decide whether the function f has an inverse.
ii. In the equation for *f(x)* replace *f(x)* by y.
iii. Interchange the roles of x and y, then solve for new y.
iv. Replace y by $f^{-1}(x)$ in the new equation.
v. Verify that f and f^{-1} are inverse of each other.

Example 8.11.2

Find the inverse of $f(x) = \frac{5-3x}{2}$

Solution

$$f(x) = \frac{5 - 3x}{2}$$
$$y = \frac{5 - 3x}{2}$$

$$x = \frac{5 - 3y}{2}$$
$$2x = 5 - 3y$$
$$3y = 5 - 2x$$

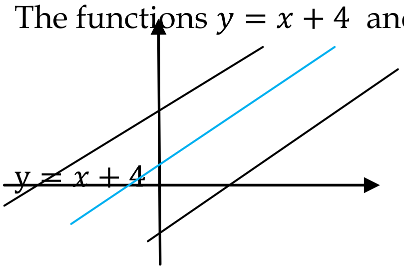
$$y = \frac{5 - 2x}{3}$$
$$f^{-1}(x) = \frac{5 - 2x}{3}$$

Relation between the Graph of the function and the Graph of its inverse.

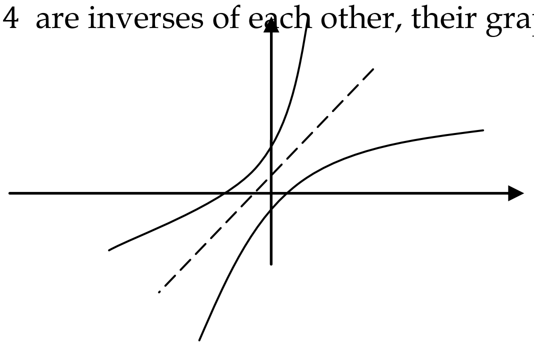
The graph $y = f^{-1}(x)$ is a reflection of the graph of the function $y = f(x)$ in the identity function $y = x$

Example 8.11.3

The functions $y = x + 4$ and $y = x - 4$ are inverses of each other, their graphs are related as



$y = x - 4$



REVIEW EXERCISE

1. Find $(fog)(x)$ *and* $(gof)(x)$ for the following
   i. $f(x) = 2x \quad , \quad g(x) = 3x - 1$
   ii. $f(x) = \frac{1}{x} \quad , \quad g(x) = 3x - 1$
   iii. $f(x) = \sqrt{x-2} \quad , \quad g(x) = 3x - 1$
   iv. $f(x) = \frac{4}{x+2} \quad , \quad g(x) = \frac{3}{2x}$
   v. If $f(x) = x^2 - 2 \quad and \quad g(x) = x + 4. Find \quad (fog)(-4)$

2. Show that $(fog)(x) = x \; and \; (gof)(x) = x$
   i. $f(x) = 2x \; , \; g(x) = \frac{1}{2}x$
   ii. $f(x) = 3x + 4 \; , \; g(x) = \frac{x-4}{3}$
   iii. $f(x) = 4x - 3 \; , \; g(x) = \frac{x+3}{4}$

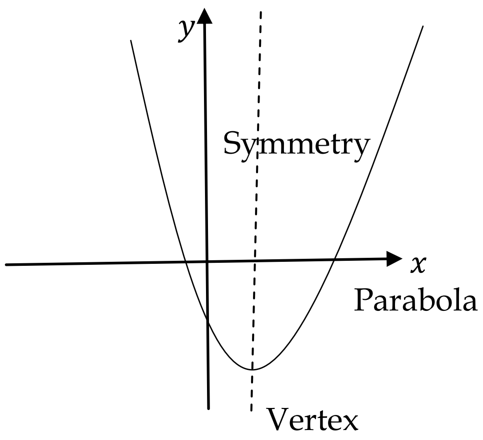
3. Determine whether the function is one-to-one, if it is, find the inverse and graph both the function and its inverse.
   i. $f(x) = x^3 - 2$
   ii. $f(x) = \frac{x}{\sqrt{x^2+4}}$
   iii. $f(x) = x^2 + 3$
   iv. $f(x) = x^3$
   v. $f(x) = \sqrt{x-1}$

QUADRATIC FUNCTIONS

The function $f(x) = ax^2 + bx + c$ where a, b and c are constants is called the quadratic function. The domain of a quadratic function is a set of all real numbers.

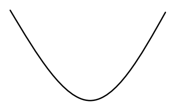
The graph of a quadratic function

The graph of a quadratic function is a special curve called **parabola.** All parabolas are symmetric with respect to a line called **axis of symmetry**. The point where the axis intersects the parabola is called the **vertex** of the parabola.

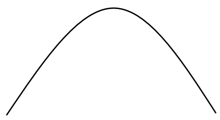


Properties of the parabola

i. If $a > 0$ in the function $f(x) = ax^2 + bx + c$.Then the graph of $f(x)$ opens up.



ii. If $a < 0$ in the function $f(x) = ax^2 + bx + c$, then the graph of $f(x)$ opens down.



iii. When $x = 0$ in the function $f(x) = ax^2 + bx + c$, then, $f(0) = c$. Hence, c is the y intercept. The graph of $f(x) = ax^2 + bx + c$ cuts y-axis at the point $(0, c)$.

iv. The vertex of a parabola is obtained by completing the square of the function

$$f(x) = ax^2 + bx + c$$
$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$
$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a^2}$$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Therefore, the parabola associated with the function $f(x) = ax^2 + bx + c$ has its vertex $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. And the equation of its axis of symmetry is $x = -\frac{b}{2a}$

Example 9.1.1
Graph $f(x) = 3x^2 - 6x + 5$

Solution

$$f(x) = 3x^2 - 6x + 5$$

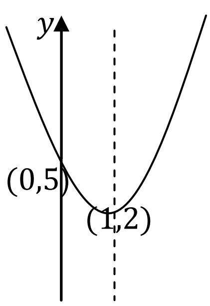
Step1 Because $a > 0$, the parabola opens upwards.
Step2

$$-\frac{b}{2a} = -\frac{-6}{2(3)} = 1$$

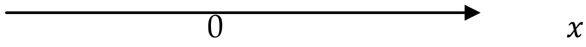
Step3

$$f\left(-\frac{b}{2a}\right) = f(1) = 3 - 6 + 5 = 2$$

thus, the vertex is (1,2)
Step 4 for $x = 0$, then $f(x) = 5$. The parabola cuts y −axis at (0,5)



$f(x) = 3x^2 - 6x + 5$



Example 9.1.2
Graph $f(x) = -x^2 - 4x - 7$

Solution

$$f(x) = -x^2 - 4x - 7$$

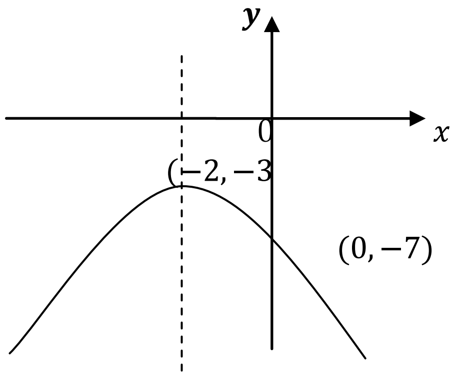
Step 1 since $a < 0$ the parabola opens downward
Step2

$$-\frac{b}{2a} = -\frac{-4}{2(-1)} = -2$$

Step3

$$f\left(-\frac{b}{2a}\right) = f(-2) = -(-2)^2 - 4(-2) - 7 = -3$$

then its vertex is $(-2, -3)$.
Step 4 $f(0) = -7$ then, the parabola intercept y −axis at $(0, -7)$

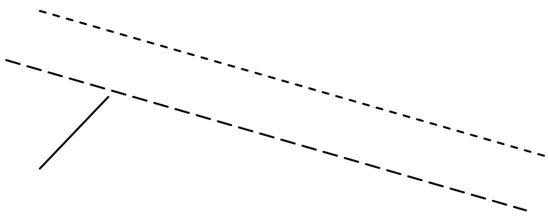


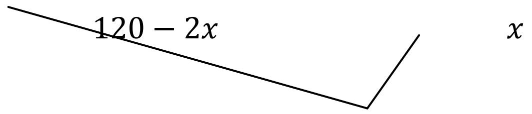
1. We can express the function in the form $f\ (x) = a(x-h)^2$and use the values of a, h, and k to determine the parabola.
2. We can express the function in the form $f(x) = ax^2 + bx + c$.

Applications of Quadratic functions in Problem Solving
As we have seen, the vertex of the graph of a quadratic function is either the lowest or the highest point on the graph. Thus, the **minimum value** or **maximum value** of a function is an application of the parabola.

Example 9.1.3
A farmer has 120 rods of fencing and wants to enclose a rectangular plot of land that requires fencing on only three sides because it is bounded by a river on one side. Find the length and width of the plot that will maximize the area.

Solution





The function $A(x) = x(120 - 2x)$ represents the area of the plot in terms of the width x. Because

$$A(x) = x(120 - 2x) \quad = 120x - 2x^2 \ = -2x^2 + 120x$$

We have a quadratic function with $a = -2$, $b = 120$, and $c = \ 0$. Therefore, the ***maximum*** value ($a \ < \ 0$ so the parabola opens downward) of the function is obtained where the x value is

$$-\frac{b}{2a} = -\frac{120}{2(-2)} = 30$$

If $x = 30$ then $120 - 2x = 120 - 2(30) = 60$. Thus the farmer should make the plot 30 rods wide and 60 rods long to maximize the area at (30)(60)= 1800 square rods.

Example 9.1.4
Find two numbers whose sum is 30, such that the sum of their squares is a minimum.

Solution
Let x represent one of the numbers; then $30 - x$ represents the other number. By expressing the sum of their squares as a function of x, we obtain

$$f(x) = x^2 + (30 - x)^2$$

which can be simplified to

$$f(x) = x^2 + 900 - 60x + x^2$$
$$f(x) = 2x^2 + 900 - 60x$$

This is a quadratic function with $a = \ 2, b = -60$, and $c = 900$. Therefore, the x value where the *minimum* occurs is $-\frac{b}{2a} = -\frac{-60}{2(2)} = 15$. If $x = 15$ then $30 - 15 = 15$ thus, the two numbers are both 15.

REVIEW EXERCISE

1. Determine the nature of the curve, find the turning point and sketch the following.

   i. $f(x) = -2 + 2x - x^2$ $\qquad$ $iv.$ $f(x) = 2x^2 - 3x - 4$

   ii. $f(x) = x^2$ $\qquad$ $v.$ $f(x) = x^2 + 2x - 3$

   iii. $f(x) = 5 - 2x - 4x^2$

2. Show that, the parabola associated with the function $f(x) = ax^2 + bx + c$ has its vertex $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

POLYNOMIAL FUNCTIONS AND EQUATIONS

The function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ is called the polynomial function in x of degree n when $a_n \neq 0$.The domain of any polynomial is a set of all real numbers.

Dividing polynomials by long division
Step 1 Use the conventional long division format and arrange both the dividend and the divisor in descending powers of the variable.
Step 2 Find the first term of the quotient by dividing the first term of the dividend by the first term of the divisor.

Step 3 Multiply the entire divisor by the quotient term in step 2 and place this product in position to be subtracted from the dividend.
Step 4 Subtract.
Step 5 Repeat steps 2, 3, and 4

Example 10.1.1
Divide $3x^3 - 5x^2 + 10x + 1$ by $3x + 1$

Solution

$$
\begin{array}{r|l}
 & x^2 - 2x + 4 \\
\hline
3x + 1 & 3x^3 - 5x^2 + 10x + 1 \\
 & -(3x^3 + x^2 \\
\hline
 & \quad -6x^2 + 10x + 1 \\
 & \quad -(-6x^2 - 2x \\
\hline
 & \qquad 12x + 1 \\
 & \qquad -(12x + 4 \\
\hline
 & \qquad\qquad -3
\end{array}
$$

Therefore, $3x^3 - 5x^2 + 10x + 1 = (3x + 1)(x^2 - 2x + 4) + (-3)$, which is of the familiar form

Dividend = (divisor)(quotient) + remainder

This result is commonly called the ***division algorithm for polynomials***, which can be stated in general terms as follows.

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$, then unique polynomials $q(x)$ and $r(x)$ exist such that

$f(x) = g(x)q(x) + r(x)$ where $f(x)$ is the dividend, $g(x)$ is the divisor, $q(x)$ is the quotient and $r(x)$ is the remainder.

Example 10.1.2

Divide $x^3 - 1$ by $x - 1$ and express in the form $f(x) = g(x)q(x) + r(x)$

Solution

$$
\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
x - 1 & x^3 + 0x^2 + 0x - 1 \\
 & -(x^3 - x^2 \\
\hline
 & \qquad x^2 + 0x
\end{array}
$$

$$
\begin{array}{r}
-(\,x^2 - x \\
\hline
x - 1 \\
-(x - 1 \\
\hline
0
\end{array}
$$

Then

$$x^3 - 1 \ = (x-1)(x^2 + x + 1) + 0.$$

Synthetic Division

If the divisor is of the form $x + c$ where c is a constant and $x + c$ is linear, then the typical long division algorithm can be simplified to a process called synthetic division just by working with the coefficients of the polynomial only. If $x + c$ is a factor of a polynomial $f(x)$, then c is one of its roots.

Example 10.1.3

Use synthetic division to divide $2x^4 + x^3 - 17x^2 + 13x + 2 \ by \ x - 2$ and express the answer in the form $f(x) = g(x)q(x) + r(x)$

Solution

$2x^4 + x^3 - 17x^2 + 13x + 2$ by $x - 2$
Let $f(x) = 2x^4 + x^3 - 17x^2 + 13x + 2$ and $g(x) = x - 2$

Then $x - 2 = 0$ implies that $x = 2$ and 2 is one of the roots of $f(x)$
We get the coefficient of $f(x)$ then we proceed by putting them in the synthetic table as follows;

$$
\begin{array}{c|ccccc|l}
2 & 2 & 1 & -17 & 13 & 2 & \text{(i)} \\
 & & 4 & 10 & -14 & -2 & \text{(ii)} \\
\hline
 & 2 & 5 & -7 & -1 & 0 & \text{(iii)}
\end{array}
$$

From part (iii)0 is the remainder and $2, 5, -7, -1$ are the coefficients of the quotient in the descending powers of x. Therefore,$q(x) = 2x^3 + 5x^2 - 7x - 1$

Hence

$$2x^4 + x^3 - 17x^2 + 13x + 2 \ = (x-2)(2x^3 + 5x^2 - 7x - 1) + 0$$

Example 10.1.4

Use synthetic division to divide $x^4 + 16$ by $x + 2$ and express the answer in the form $f(x) = g(x)q(x) + r(x)$

Solution

$f(x) = x^4 + 16 = x^4 + 0x^3 + 0x^2 + 0x + 16$
$g(x) = x + 2$

$x + 2 = 0,\ x = -2$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & 0 & 0 & 16 \\ & & -2 & 4 & -8 & 16 \\ \hline & 1 & -2 & 4 & -8 & 32 \end{array}$$

From the table of coefficients, 32 is the remainder and coefficients $1, -2, 4, -8$ are the coefficients of the quotient in the descending order of x. Therefore, $q(x) = x^3 - 2x^2 + 4x - 8$

$x^4 + 16 = (x + 2)(x^3 - 2x^2 + 4x - 8) + 32.$

Remainder and factor theorem

Let's consider the division algorithm (stated in the previous section) when the dividend, $f(x)$, is divided by a *linear polynomial* of the form $(x - c)$. Then the division algorithm,

$$f(x) = g(x)q(x) + r(x)$$

Becomes $f(x) = g(x - c)q(x) + r(x)$

Because the degree of the remainder, $r(x)$, must be less than the degree of the divisor, $(x - c)$, the remainder is a constant. Therefore, letting R represent the remainder, we have

$$f(x) = g(x - c)q(x) + R$$

If we evaluate f at c, we obtain

$$f(x) = g(c - c)q(c) + R$$

$$f(x) = g(0)q(c) + R$$

$$f(x) = 0. q(c) + R$$

$$f(x) = R$$

Remainder theorem.
If a polynomial $f(x)$ is divided by $(x - c)$, then the remainder is equal to $f(c)$.

Example 10.1.5
If $f(x) = x^3 + 2x^2 - 5x - 1$, find $f(2)$
(a) by using synthetic division and the remainder theorem and then
(b) by evaluating $f(2)$ directly.

Solution

a. $$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -1 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 5 \end{array}$$

$R = f(2) = 5$

b. $f(2) = 2^3 + 2(2)^2 - 5(2) - 1 = 8 + 8 - 10 - 1 = 5$.

Example 10.1.6

Find the remainder when $f(x) = x^3 + 3x^2 - 13x - 15$ is divided by $x + 1$

Solution

Let $f(x) = x^3 + 3x^2 - 13x - 15$ and write $x - (-1)$so that we can apply the remainder theorem.

$f(-1) = (-1)^3 + 3(-1)^2 - 13(-1) - 15 = 0$

Thus the remainder is zero and we say that $x + 1$ is a factor of $x^3 + 3x^2 - 13x - 15$.

Factor Theorem

A general *factor theorem* can be formulated by considering the equation$f(x) = g(x - c)q(x) + R$
If $x - c$ is a factor of $f(x)$, then the remainder$R$, which is also $f(c)$, then $f(c)$ must be zero.
Conversely, if $R = f(c) = 0$, then $f(x) = (x - c)q(x)$; in other words, $x - c$ is a factor of $f(x)$. The **factor theorem** can be stated as follows.

Factor Theorem

A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$.

Example 10.1.7

Is $x + 3$ a factor of $2x^3 + 5x^2 - 6x - 7$?

Solution

Let $f(x) = 2x^3 + 5x^2 - 6x - 7$

and $x + 3$ be written as $x - (-3)$.Then $f(-3) = 2(-3)^3 + 5(-3)^2 - 6(-3) - 7 = 2$

Since $f(-3) \neq 0$, by factor theorem, $x + 3$ is not a factor of

$2x^3 + 5x^2 - 6x - 7$.

Example 10.1.8

Is $x - 1$ a factor of $x^3 + 5x^2 - 2x - 8$?

Solution

Let $f(x) = x^3 + 5x^2 - 2x - 8$

and compute $f(1)$ we obtain

$$f(1) = (1)^3 + 5(1)^2 - 2(1) - 8 = 0$$

Therefore, by the factor theorem,

$x - 1$ is a factor of $x^3 + 5x^2 - 2x - 8$.

Example 10.1. 9
Show that $x-1$ is a factor of $x^3-2x^2-11x+12$ and find the other linear factors of the polynomial.

Solution

Let us use synthetic division to divide $x^3-2x^2-11x+12$ by $x-1$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & -12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

The last line indicates a quotient of x^2-x-12 and a remainder of zero. The zero remainder means that $x-1$ is a factor. Furthermore, we can write

$$x^3-2x^2-11x+12=(x-1)(x^2-x-12)$$

We can factor the quadratic polynomial x^2-x-12 as $(x-4)(x+3)$ by using our conventional factoring techniques. Thus we obtain

$$x^3-2x^2-11x+12=(x-1)(x-4)(x+3)$$

REVIEW EXERCISE

1. Use the long division to divide

   a. $x^3-2x^2-11x+12 \;\; by \;\; x-1$
   b. $2x^3+5x^2-6x-7 \;\; by \;\; x+3$
   c. $x^3+5x^2-2x-8 \;\; by \; x-1$

2. Find $f(c)$ (a) by using synthetic division and the remainder theorem and (b) by evaluating $f\ (c)$ directly.
   a. $f(x)=\ x^3+x^2-2x-4$ and $c=-1$
   b. $f(x)=\ 3x^3+4x^2-5x+3$ and $c=-4$
   c. $f(x)=x^4-2x^3-3x^2+5x-1$ and $c=-2$
   d. $f(x)=2x^4+x^3-4x^2+x+1$ and $c=2$

3. Use the factor theorem to help answer each question about factors.
   a. 1s $x-2$ a factor of $3x^2-4x-4$?
   b. 1s $x+3$ a factor of $6x^2+13x-15$?
   c. 1s $x-1$ a factor of $3x^3+5x^2-x-2$?
   d. 1s $x-3$ a factor of x^4-81?
   e. 1s $x+3$ a factor of x^4-81?

4. Use synthetic division to show that $g(x)$ is a factor of $f(x)$ and complete the factorization of $f(x)$.
   a. $g(x)=x+2$, $f(x)=x^3+7x^2+x-12$
   b. $g(x)=x-1$, $f(x)=3x^3+19x^2-38x+16$
   c. $g(x)=x-3$, $f(x)=6x^3-17x^2-5x+6$
   d. $g(x)=x+1$, $f(x)=x^3-2x^2-7x-4$
   e. $g(x)=x-5$, $f(x)=2x^3+x^2-61x+30$

10.2 POLYNOMIAL EQUATIONS

Linear and quadratic equations are special cases of a general class of equations we refer to as **polynomial equations**. The equation

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x^1+a_0=0$$

Where the coefficients $a_0, a_1, \dots, a_n$ are real numbers and n is a positive integer, is called a **polynomial equation of degree n**. The following are examples of polynomial equations.

$2x - 6 = 0$	degree of 1
$\frac{3}{4}x^2 - 5x + 1 = 0$	degree of 2
$3x^3 + 19x^2 - 38x + 16 = 0$	degree of 3
$x^4 - 81 = 0$	degree of 4

Remark: The most general polynomial equation allows complex numbers as coefficients. However, for our purposes in this text, we will restrict the coefficients to real numbers. We refer to such equations as **polynomial equations over the reals**.

Equation	**Solution Set**
$3x + 4 = 7$	$\{1\}$
$x^2 + x - 6 = 0$	$\{-3, 2\}$
$2x^3 - 3x^2 - 2x + 3 = 0$	$\left\{-1,\ 1,\ \frac{2}{3}\right\}$
$x^4 - 16 = 0$	$\{-2,\ 2, -2i,\ 2i\}$

Note that in each of these examples, the number of solutions corresponds to the degree of the equation.

Finding Rational Solutions

Rational Root Theorem

Consider the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$ where the coefficients $a_0, a_1, \dots, a_n$ are integers. If the rational number c/d, reduced to lowest terms, is a solution of the equation, then c is a factor of the constant term a_0, and d is a factor of the leading coefficient a_n.

Example 10.2.1

Find all rational solutions of $3x^3 + 8x^2 - 15x + 4 = 0$

Solution

If c/d is a rational solution, then c must be a factor of 4 and d must be a factor of 3.
Therefore, the possible values for c and d are as follows.

For c $\pm 1, \pm 2, \pm 4$
For *d* $\pm 1, \pm 3$
Thus the possible values for c/d are $\pm 1,\ \pm\frac{1}{3},\ \pm 2,\ \pm\frac{2}{3},\ \pm 4,\ \pm\frac{4}{3}$

By using synthetic division, we can test $x - 1$

$$\begin{array}{c|cccc} 1 & 3 & 8 & -15 & 4 \\ & & 3 & 11 & -4 \\ \hline & 3 & 11 & -4 & 0 \end{array}$$

This shows that $x - 1$ is a factor of the given polynomial; therefore, 1 is a rational solution of the equation.

$3x^3 + 8x^2 - 15x + 4 = 0$
$(x - 1)(3x^2 + 11x - 4) = 0$

The quadratic factor can be further factored by using techniques we are familiar with.

$(x - 1)(3x^2 + 11x - 4) = 0$
$(x - 1)(3x - 1)(x + 4) = 0$
$(x - 1) = 0$ or $(3x - 1) = 0$ or $(x + 4) = 0$
$x = 1$ or $x = \frac{1}{3}$ or $x = -4$

Thus the entire solution set consists of rational numbers and can be listed as

$\left\{-4, \frac{1}{3}, 1\right\}$.

REVIEW EXERCISE

1. Use the rational root theorem and the factor theorem to help solve each equation. Be sure that the number of solutions for each equation agrees with the degree of the polynomial.

   a. $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$

   b. $x^3 + x^2 - 4x - 4 = 0$

   c. $6x^3 + x^2 - 10x + 3 = 0$

   d. $x^3 - 2x^2 - 7x - 4 = 0$

   e. $x^3 + x^2 - 4x - 4 = 0$

   f. $x^4 + 4x^3 - x^2 - 16x - 12 = 0$

   g. $x^3 - 4x^2 + 8 = 0$

2. Find a polynomial equation with integral coefficients that has the given numbers as solutions and the indicated degree.

   a. $2, 4, -3$ degree 3

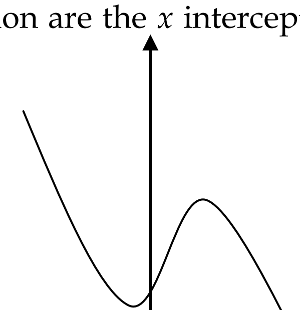
   b. $1, -1, 2, -4$ degree 4

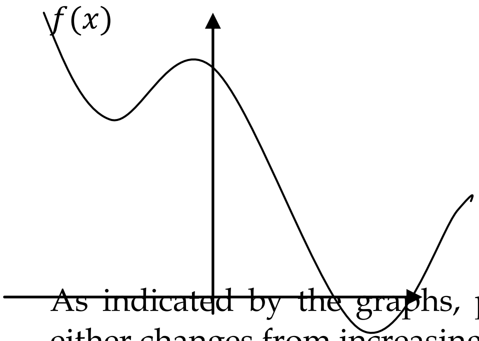
   c. $2, -1, 1$ degree 3

10.3 Graphing Polynomial Functions

Graphing Polynomial Functions in Factored Form

Every polynomial function of odd degree has at least ***one real zero***—that is, at least one real number c such that $f(c) = 0$. Geometrically, the zeros of the function are the x intercepts of the graph. The figure below shows some graphs of polynomial functions.





As indicated by the graphs, polynomial functions usually have **turning points** where the function either changes from increasing to decreasing or from decreasing to increasing.

Example 10.3.1

Graph $f(x) = (x+2)(x-1)(x-3)$

Solution

First, let us find the x intercepts (zeros of the function) by setting each factor equal to zero and solving for x.

$x + 2 = 0$ or $x - 1 = 0$ or $x - 3 = 0$

$x = -2$ or $x = 1$ or $x = 3$

Thus the points $(-2, 0), (1, 0)$, and $(3, 0)$ are on the graph. Second, the points associated with the x intercepts divide the x- axis into four intervals.

$x < -2$ | $-2 < x < 1$ | $1 < x < 3$ | $x > 3$

-2 $\quad$ 1 $\quad$ 3

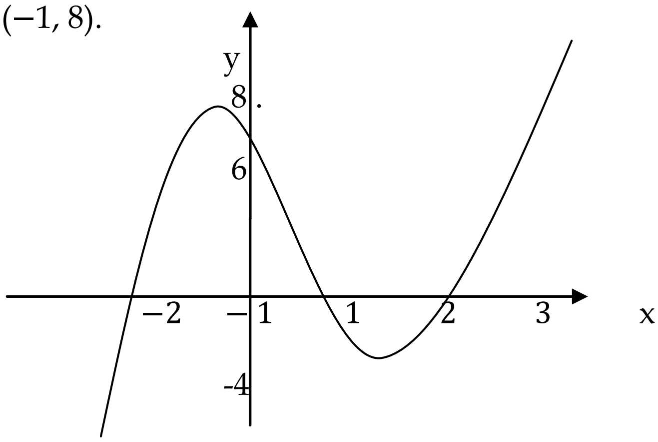
In each of these intervals, $f\ (x)$ is either always positive or always negative.

Interval	**Test value**	**Sign for $f(x)$**	**Location of graph**
$x < -2$	$f(-3) = -24$	Negative	Below $x -$ axis
$-2 < x < 1$	$f(0) = 0$	Positive	Above $x -$ axis
$1 < x < 3$	$f(2) = -4$	Negative	Below $x -$ axis
$x > 3$	$f(4) = 18$	Positive	Above $x -$ axis

Additional values: $f(-1) = 8$

$f(2) = -4.$

Making use of the x intercepts and the information in the table, we indicated turning points of the graph at (2, −4) and (−1, 8).



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Example10.3.2

Graph $f(x) = x^3 + 3x^2 - 4$

Solution

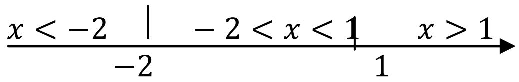
$$f(x) = x^3 + 3x^2 - 4$$

$$= (x-1)(x^2 + 4x + 4)$$
$$= (x-1)(x+2)^2$$

Now we can find the x intercepts.

$(x-1) = 0$ or $(x+2)^2 = 0$
$x = 1$ or $x = -2$

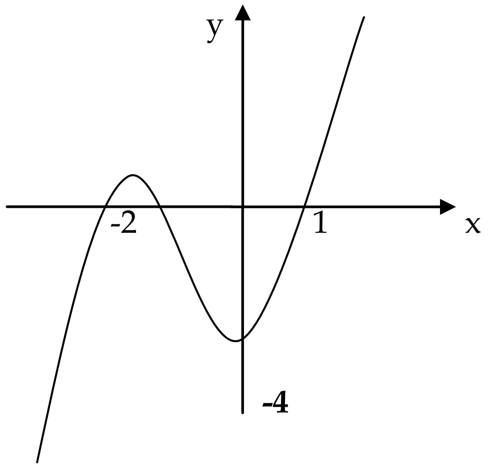
Thus the points (−2, 0) and (1, 0) are on the graph and divide the x axis into three intervals.



The following table determines some points and summarizes the sign behavior of $f(x)$.

interval	Test value	Sign of $f(x)$	Location of graph
$x < -2$	$f(-3) = -4$	Negative	Below $x -$ axis
$-2 < x < 1$	$f(0) = -4$	Negative	Below $x -$ axis
$x > 1$	$f(2) = 16$	Positive	

Additional values: $f(-1) = -2$ $\quad$ $f(-4) = -20$



REVIEW EXERCISE

1. graph each polynomial function.
   a. $f(x) = (x-2)(x+1)(x+3)$
   b. $f(x) = (x+3)(x+1)(x-1)(x-2)$
   c. $f(x) = -(x-2)^4$
   d. $f(x) = (x+4)(x+1)(x-1)$
   e. $f(x) = (x+2)^3(x-4)$

2. graph each polynomial function by first factoring the given polynomial. You may need to use some factoring techniques, as well as the rational root theorem and the factor theorem.
   a. $f(x) = x^3 + x^2 + 2x$
   b. $f(x) = x^3 + 2x^2 - x - 2$
   c. $f(x) = x^4 - 5x^2 + 4$
   d. $f(x) = x^3 - x^2 - 4x + 4$
   e. $f(x) = 2x^3 - 3x^2 - 3x + 2$

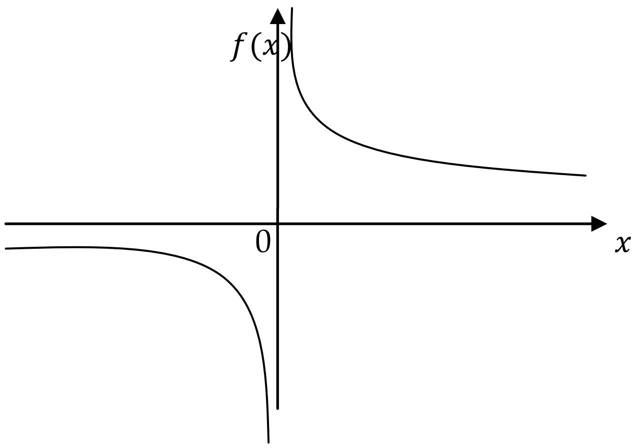
SKETCHING OF RATIONALAND PIECE-WISE FUNCTIONS

A function of the form $f(x) = \frac{p(x)}{q(x)}$ for $q(x) \neq 0$where $p(x)$ and $q(x)$ are both polynomial functions, is called a **rational function**. The following are examples of rational functions.

$$f(x) = \frac{x^2}{x^2 - x - 6} \quad , \quad f(x) = \frac{2}{x-1} \quad , \quad f(x) = \frac{x^8 - 8}{x+2}$$

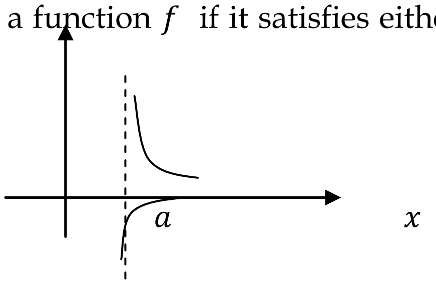
In each example, the domain of the rational function is the set of all real numbers except those that make the denominator zero.

Try to use the calculator with some test values an plot the graph of $f(x) = \frac{1}{x}$. You will see that the graph is

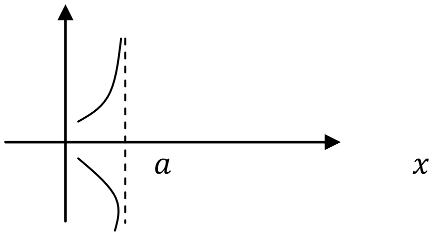


Vertical Asymptote
A line $x = a$ is a **vertical asymptote** for the graph of a function f if it satisfies either of the following two properties.

1. $f(x)$ either increases or decreases without bound as x approaches the number a from the right.

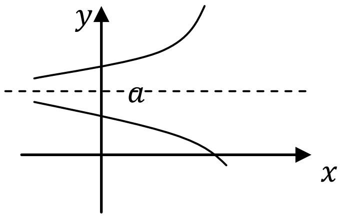


2. $f(x)$ either increases or decreases without bound as x approaches the number a from the left.

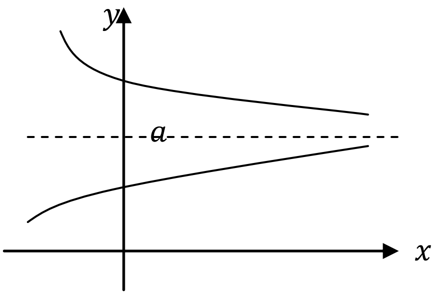


Horizontal Asymptote
A line $y = b$ [or $f(x) = b$] is a **horizontal asymptote**
for the graph of a function f if it satisfies either of the following two properties.

1. $f(x)$ approaches the number b from above or below as x gets infinitely small.



2. $f(x)$ approaches the number b from above or below as x gets infinitely large.



The following suggestions will help you graph rational functions of the type we are considering in this section

1. Check for y −axis symmetry and origin symmetry.
2. Find any vertical asymptote(s) by setting the denominator equal to zero and solving for x.
3. Find any horizontal asymptote(s) by studying the behavior of $f(x)$ as x gets infinitely large or as x gets infinitely small.
4. Study the behavior of the graph when it is close to the asymptotes.
5. Plot as many points as necessary to determine the shape of the graph. This may be affected by whether the graph has any symmetry.

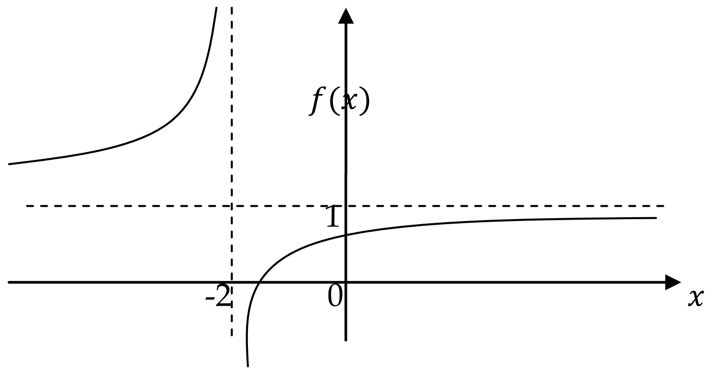
Example 11.1.1
Graph $f(x) = \frac{x}{x+2}$

Solution

Because $x = -2$ makes the denominator zero, the line $x = -2$ is a vertical asymptote.

$$f(x) = \frac{x}{x+2} = \frac{\frac{x}{x}}{\frac{x+2}{x}} = \frac{1}{1+\frac{2}{x}}$$

Now we can see that (i) as x gets larger and larger, the value of $f\,(x)$ approaches 1 from below, and (ii) as x gets smaller and smaller, the value of $f\,(x)$ approaches 1 from above. Thus, the line $f\,(x) = 1$ is a horizontal asymptote. Drawing the asymptotes (lines) and plotting a few points enable us to complete the graph shown



Example 11.1.2

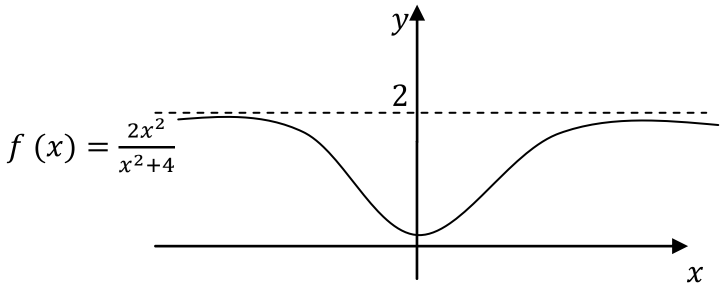
Sketch the graph of $f\,(x) = \frac{2x^2}{x^2+4}$

solution

First, note that $f\,(2x) = f\,(x)$; therefore, this graph is symmetric with respect to the $y\_$ axis. Second, the denominator $x^2 + 4$ cannot equal zero for any real number x. Thus there is no vertical asymptote. Third, dividing both the numerator and the denominator of the rational expression by x^2 produces

$$f\,(x) = \frac{2x^2}{x^2+4} = \frac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2}+\frac{4}{x^2}} = \frac{2}{1+\frac{4}{x^2}}$$

Now we can see that as x gets larger and larger, the value of $f\,(x)$ approaches 2 from below. Therefore, the line $f\,(x) = 2$ is a horizontal asymptote. We can plot a few points using positive values for x, sketch this part of the curve, and then reflect across the $f\,(x)$ axis to obtain the complete graph



Example11.1. 3

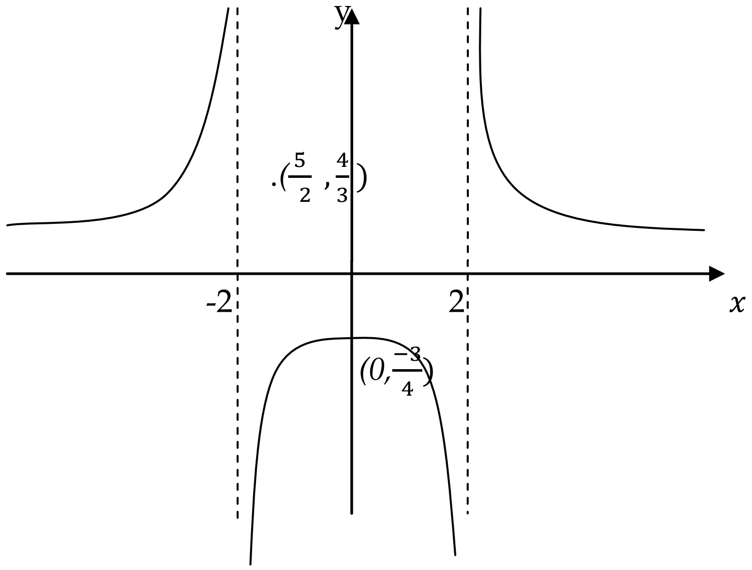
Sketch the graph of $f\,(x) = \frac{3}{x^2-4}$

Solution

First, note that $f\,(-x) = f\,(x)$; therefore, this graph is symmetric about the $f\,(x)$ axis. Second, by setting the denominator equal to zero and solving for x, we obtain

$$x^2 - 4 = 0$$
$$x^2 = 4 \qquad x = \pm 2.$$

The lines $x = 2$ and $x = -2$ are vertical asymptotes. Next, we can see that $\frac{3}{x^2-4}$ approaches zero from above as x gets larger and larger. Finally, we can plot a few points using positive values for x (not 2), sketch this part of the curve, and then reflect it across the $f\,(x)$ axis to obtain the complete graph



11.2 Oblique Asymptotes

Let us consider functions where the degree of the numerator is one greater than the degree of the denominator.

Example11.2.1

Graph $f\,(x) = \frac{x^2-1}{x-2}$

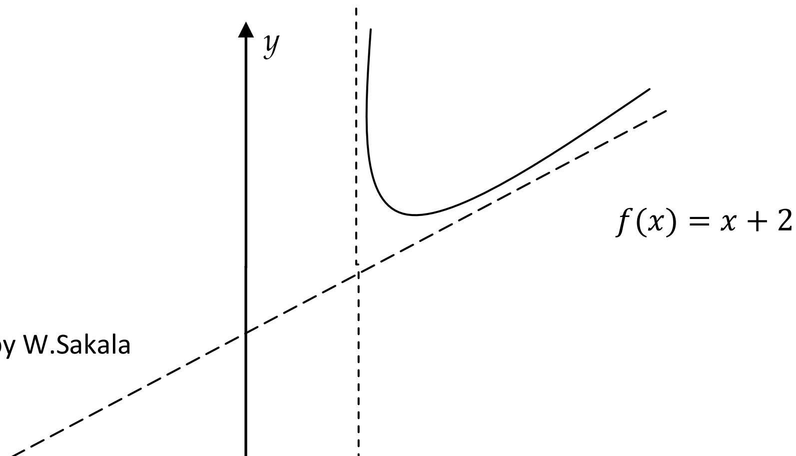
Solution

First, let us observe that $x = 2$ is a vertical asymptote. Second, because the degree of the numerator is greater than the degree of the denominator, we can change the form of the rational expression by division. We use synthetic division.

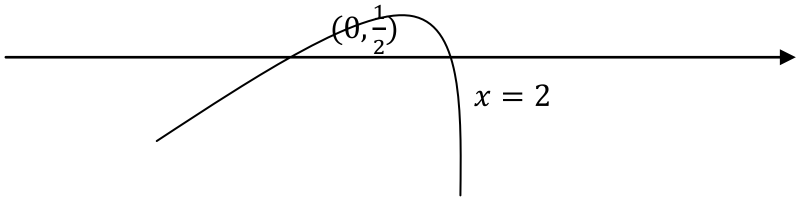
$$\begin{array}{r|rrr} 2 & 1 & 0 & -1 \\ & & 2 & 4 \\ \hline & 1 & 2 & 3 \end{array}$$

Therefore, the original function can be rewritten as $f\,(x) = \frac{x^2-1}{x-2} = x + 2 + \frac{3}{x-2}$

Now, for very large values of $|\,x|$, the fraction $\frac{3}{x-2}$. Therefore, as $|\,x|$ gets larger and larger, the graph of $f\,(x) = \frac{x^2-1}{x-2} = x + 2 + \frac{3}{x-2}$gets closer and closer to the line $f(x) = x + 2$. We call this line an **oblique asymptote.** Finally, because this is a new situation, it may be necessary to plot a large number of points on both sides of the vertical asymptote



$f(x) = \dfrac{x^2 - 1}{x - 2}$



If the degree of the numerator of a rational function is *exactly one more* than the degree of its denominator, then the graph of the function has an oblique asymptote.

Example 11.2.2

Graph the rational function $f(x) = \frac{x^3 - 2x^2 - x - 1}{x^2 - 36}$

Solution

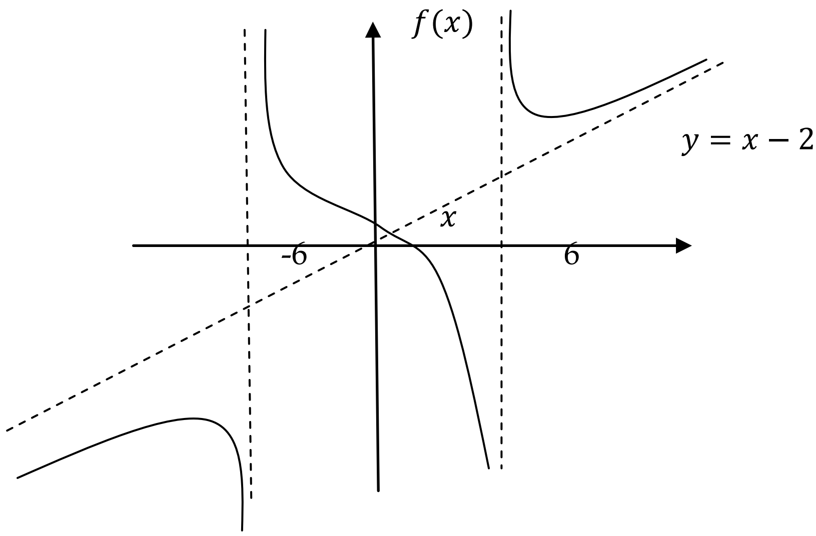
Let's analyze what we know about the graph.

1. Because $f(0) = \frac{1}{30}$, the point $(0, \frac{1}{30})$ is on the graph.
2. Because $f(-x) \neq f(x)$ and $f(x) \neq -f(x)$,there is no symmetry with respect to the origin or the y-axis.
3. The denominator is zero at $x = \pm 6$. Thus the lines $x = 6$ and $x = -6$ are vertical asymptotes.
4. Let's change the form of the rational expression by division.

$$
\begin{array}{r|l}
 & \quad\quad\quad x - 2 \\
x^2 - 36 & x^3 - 2x^2 - x - 1 \\
 & -(x^3 + 0x^2 - 36x \\
 & \quad\quad -2x^2 - 35x - 1 \\
 & \quad -(-2x^2 + 0x + 72 \\
 & \quad\quad\quad\quad 35x - 73
\end{array}
$$

Thus the original function can be rewritten as $f(x) = x - 2 + \frac{35x - 73}{x^2 - 36}$.
Therefore, the line $y = x - 2$ is an oblique asymptote.

Remember, you can use as many test values of *x as* you want to come up with nice shapes of the graph as shown below



REVIEW EXERCISE

1. Graph each rational function. Check first for symmetry and identify the asymptotes.

a. $f(x) = \frac{3}{x+1}$ d. $f(x) = \frac{x^2+2}{x-1}$

b. $f(x) = \frac{x^2-5x+6}{x-2}$ e. $f(x) = \frac{x^3+8}{x^2}$

c. $f(x) = \frac{2x^4}{x^4+1}$

2. Suppose $f(x) = \frac{1+x}{x^2-2x+1}$

   a. Find the domain of $f(x)$
   b. Find the vertical asymptotes if any
   c. Find the horizontal asymptote if any
   d. Sketch the graph of $f(x)$

11.3 GRAPHS OF PIECE WISE FUNCTIONS

The functions such as $f(x) = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases} \quad x \in R$ $\quad f(x) = \begin{cases} 2x+3 & if \;\; x < 0 \\ x^2 & if \;\; 0 \leq x \leq 2 \\ 1 & if \;\; x < 0 \end{cases}$

Are called ***piecewise*** functions. There graphs are sketched according to the defined function in a given interval.

Example 11.3.1

Sketch the graph of $f(x) = \begin{cases} 2x+3 & if \;\; x \leq 0 \\ x^2 & if \;\; 0 \leq x \leq 2 \\ 1 & if \;\; x \geq 2 \end{cases}$

Solution

We redefine the function as follows

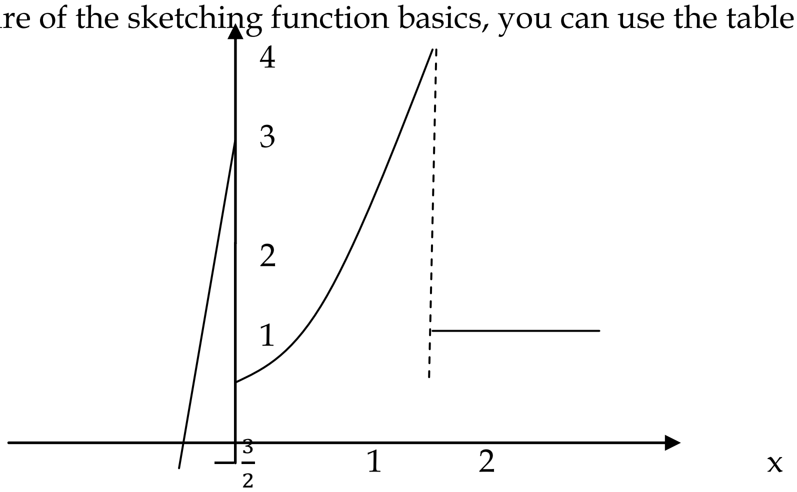
$$f(x) = 2x + 3 \quad \text{if } x < 0$$
$$f(x) = x^2 \quad \text{if } 0 \leq x \leq 2$$
$$f(x) = 1 \quad \text{if } x \geq 2$$

Sketching the functions on the same graph according to the given intervals we have

$2x + 3x^2$ | 1 |

0 $\quad$ 2

If you are not sure of the sketching function basics, you can use the table of values in each interval.



11.4 Sketching of absolute value functions

The function $f(x) = |x|$ can be redefined as

$$f(x) = \begin{cases} x & if \quad x \geq 0 \qquad x \in R \\ -x & if \quad x < 0 \end{cases}$$

Let's consider the following examples to show how to sketch the graphs of absolute value functions

Example 11.4.1

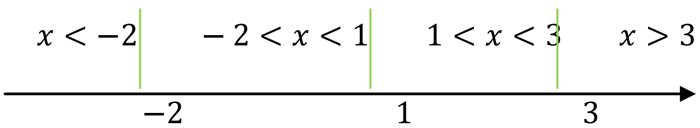
Sketch the graph of $f(x) = |x^3 - 2x^2 - 5x + 6|$

Solution

We consider the function $g(x) = x^3 - 2x^2 - 5x + 6$ first, the absolutes of the function $f(x) = |g(x)|$ implies that the graph is above x-axis. This means that, part of the graph that will be below x-axis shall be reflected in x-axis.The function $g(x) = x^3 - 2x^2 - 5x + 6$ can be simplified as

$$g(x) = (x + 2)(x - 1)(x - 3)$$
$$x + 2 = 0 \quad or \quad x - 1 = 0 \quad or \quad x - 3 = 0$$
$$x = -2 \quad or \quad x = 1 \quad or \quad x = 3$$

Thus the points $(-2, 0)$, $(1, 0)$, and $(3, 0)$ are on the graph. Second, the points associated with the x intercepts divide the x axis into four intervals.

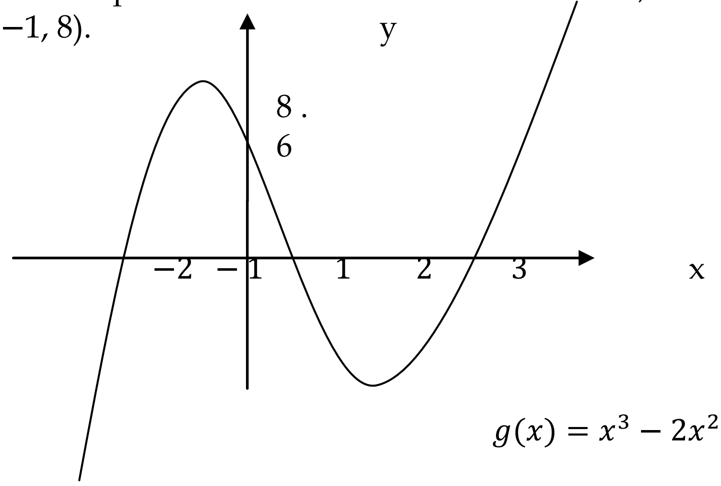


Using the test values of x

Interval	**Test value**	**Sign for *g(x)***	**Location of graph**
$x < -2$	$g(-3) = -24$	Negative	Below x axis
$-2 < x < 1$	$g(0) = 0$	Positive	Above x axis
$1 < x < 3$	$g(2) = -4$	Negative	Below x axis
$x > 3$	$g(4) = 18$	Positive	Above x axis

Additional values: $g(-1) = 8$ $\qquad$ $g(2) = -4$

Making use of the x intercepts and the information in the table, we indicated turning points of the graph at $(2, -4)$ and $(-1, 8)$.



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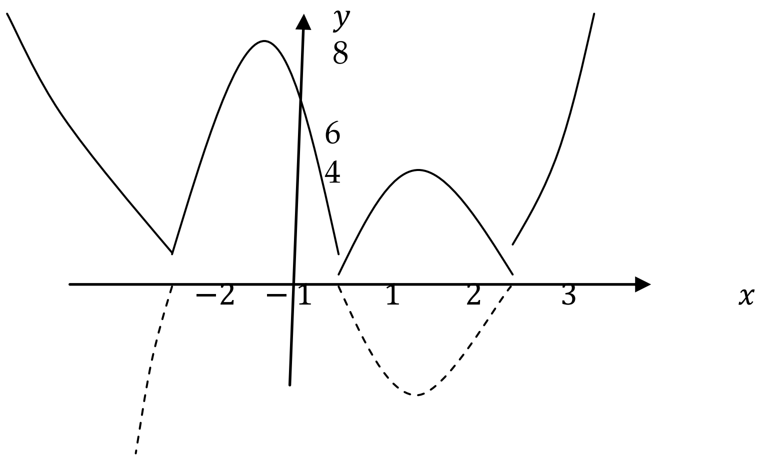
$g(x) = x^3 - 2x^2 - 5x + 6$

Now let us consider the graph of the function $f(x) = |g(x)| = |x^3 - 2x^2 - 5x + 6|$



Example11.4.2

Redefine the modulus function $f(x) = |3x + 2| - |2x - 1|$ by removing the modulus; hence sketch the graph of the function.

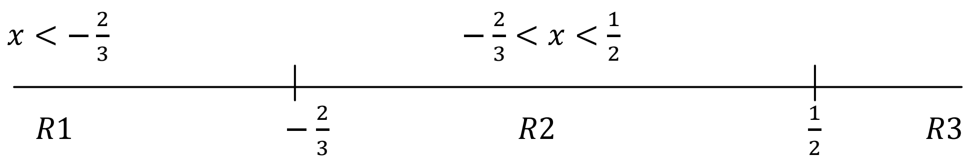
Solution

Let $g(x) = |3x + 2|$ and $h(x) = |2x - 1|$ then$f(x) = g(x) - h(x)$.

$$g(x) = \begin{cases} 3x + 2 \quad if \quad x \geq -\frac{2}{3} \\ -(3x + 2) \;\; if \quad x < -\frac{2}{3} \end{cases}$$

$$h(x) = \begin{cases} 2x - 1 \;\; if \;\; x \geq \frac{1}{2} \\ -(2x - 1) \;\; if \;\; x < \frac{1}{2} \end{cases}$$

The critical numbers are $-\frac{2}{3}$ and $\frac{1}{2}$ which can be divided into three regions



Now from piece wise functions *g and h* we get the functions defined in each region.

Region1 $x < -\frac{2}{3}$**:** $\quad g(x) = (-3x - 2)$ and $h(x) = (-2x + 1)$ then

$$f(x) = g(x) - h(x)$$

$$f(x) = (-3x - 2) - (-2x + 1)$$

$$f(x) = x - 3$$

Region 2 $\quad -\frac{2}{3} < x < \frac{1}{2}$

$g(x) = (3x + 2)$, $h(x) = (-2x + 1)$. Then,

$$f(x) = g(x) - h(x)$$

$$f(x) = (3x + 2) - (-2x + 1)$$

$$f(x) = 5x + 1$$

Region 3 $\frac{1}{2} < x$

$g(x) = (3x + 2)$, $h(x) = (2x - 1)$. Then

$$f(x) = g(x) - h(x)$$

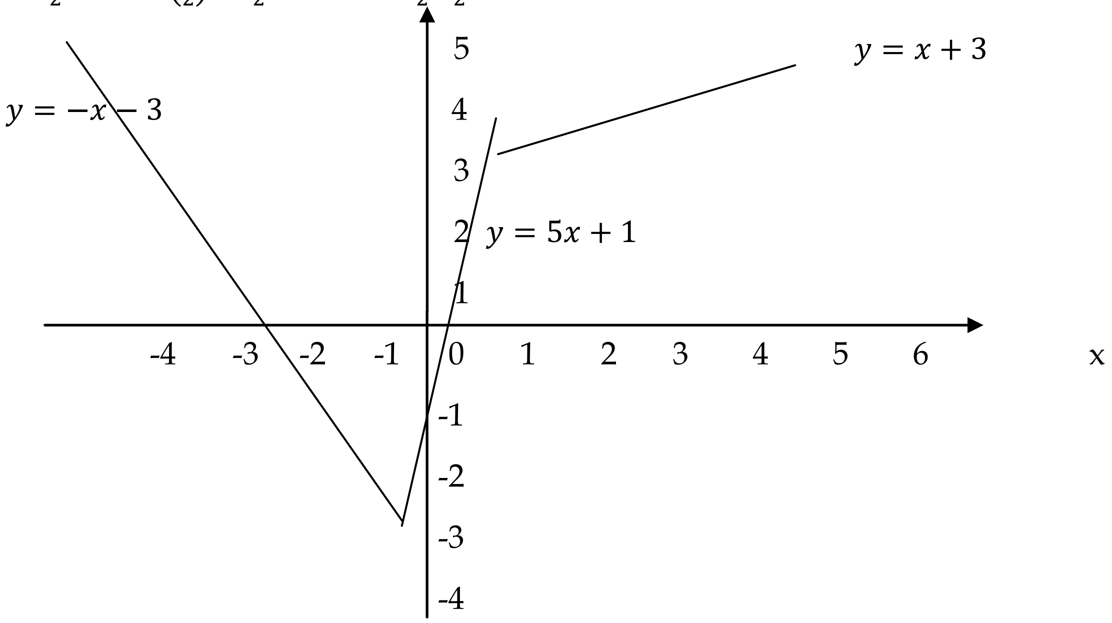
$$f(x) = (3x + 2) - (2x - 1) = x + 3$$

Now we redefine the function f according to the piecewise three regions

$$f(x) = \begin{cases} -x - 3 & if \;\; x \leq -\frac{2}{3} \\ 5x + 1 & if \;\; -\frac{2}{3} \leq x \leq \frac{1}{2} \\ x + 3 & if \;\; x \geq \frac{1}{2} \end{cases}$$

Then we can find the sketching co-ordinate of the function f in each region to make the shape of the graph. For $x = -\frac{2}{3}$, $f\left(-\frac{2}{3}\right) = -\frac{7}{3}$ hence $(-\frac{2}{3}, -\frac{7}{3})$ is the point

For $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = \frac{7}{2}$ hence $(\frac{1}{2}, \frac{7}{2})$ is the point.We can now sketch the piece wise function f



REVIEW EXERCISE

1. Sketch the following functions

   a. $f(x) = \begin{cases} -x^2 & for \;\; x \geq 0 \\ 2x^2 & for \;\; x < 0 \end{cases}$

   c. $f(x) = \begin{cases} 2x + 1 & if \;\; x \geq 0 \\ x^2 & if \;\; x < 0 \end{cases}$

   b. $f(x) = \begin{cases} -x - 3 & if \;\; x \leq -\frac{2}{3} \\ 5x + 1 & if \;\; -\frac{2}{3} \leq x \leq \frac{1}{2} \\ x + 4 & if \;\; x \geq \frac{1}{2} \end{cases}$

2. Sketch the graph of the following functions

   a. $f(x) = |x^3 - 2x^2 - 5x + 6| + 4$

   b. $f(x) = |x^2 + 5x + 4| - 2$

   c. $f(x) = |-3x^2 - 2x + 1| + 1$

3. Redefine the modulus function by removing the modulus; hence sketch the graph of the function.
   a. $f(x) = |3x + 1| + |2x - 3|$

   b. $f(x) = |2x - 1| - |x + 2|$

   c. $f(x) = |2x - 1| + |3x + 1|$

ARITHMETIC AND GEOMETRIC SEQUENCES

An **infinite sequence** is a function whose domain is the set of positive integers.
Example $f\ (n) = 5n + 1$ where the domain is the set of positive integers. If we substitute the numbers of the domain in order, starting with1. We can simply express the infinite sequence as
$6, 11, 16, 21, 26, \ldots$
The sequence is then expressed as $a_1, a_2, a_3, a_4, \ldots$ where a_1 is the **first term**, a_2 is the **second term**, a_3 is the **third term**, and so on.
The expression a_n, which defines the sequence, is called the **general term** of the sequence.

Example 12.1.1
Find the first five terms of the sequence where $a_n = 2n^2 - 3$; find the $20th$ term.

Solution
The first five terms are generated by replacing n with $1, 2, 3, 4,$ and 5.

$$\begin{aligned}
a1 &= 2(1)^2 - 3 = -1\\
a2 &= 2(2)^2 - 3 = 5\\
a3 &= 2(3)^2 - 3 = 15\\
a4 &= 2(4)^2 - 3 = 29\\
a5 &= 2(5)^2 - 3 = 47
\end{aligned}$$

The first five terms are thus $-1, 5, 15, 29,$ and 47. The $20th$ term is
$$a_{20} = 2(20)^2 - 3 = 797.$$

Arithmetic Sequences
An ***arithmetic sequence*** (also called an arithmetic progression AP) is a sequence that has a common difference between successive terms. The following are examples of arithmetic sequences.
$1, 8, 15, 22, 29, \ldots$
$4, 7, 10, 13, 16, \ldots$
$4, 1, -2, -5, 28, \ldots$
In a more general setting, we say that the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

A sequence is an arithmetic sequence if and only if there is a real number d such that

$a_{k+1} - a_k = d$ for every positive integer k. The number d is called the **common difference**.

We can generate an arithmetic sequence that has a common difference of d by starting with a first term a_1 and then simply adding d to each successive term.

First term: $\boldsymbol{a_1}$

Second term: $\boldsymbol{a_1 + d}$

Third term: $\boldsymbol{a_1 + 2d}$

The **general term** of an arithmetic sequence is given by

$$\boldsymbol{a_n = a_1 + (n-1)d}$$

where a_1 is the first term and d is the common difference.

Fourth term: $\boldsymbol{a_1 + 3d}$

.

Nth term: $\boldsymbol{a_1 + (n-1)d}$

Example 12.1.2
Find the general-term expression for the arithmetic sequence $6, 2, -2, -6, \ldots$

Solution
The common difference, d is $2 - 6 = -4$, and the first term, a_1 is 6. Substitute these values into $\boldsymbol{a_n} = a_1 + \boldsymbol{(n-1)d}$ and simplify we obtain

$\boldsymbol{a_n} = a_1 + (n-1)d$
$\boldsymbol{a_n} = 6 + (n-1)(-4)$
$\boldsymbol{a_n} = 6 - 4n + 4$
$\boldsymbol{a_n} = -4n + 2$

Example 12.1.3

Find the 40th term of the arithmetic sequence $1, 5, 9, 13, \ldots$

Solution

Using $\boldsymbol{a_n} = a_1 + (n-1)d$
we obtain

$$\begin{aligned} a_{40} &= 1 + (40-1)(4) \\ &= 1 + +(39)(4) \\ &= 157. \end{aligned}$$

Example 12.1.4
Find the first term of the arithmetic sequence where the fourth term is 26 and the ninth term is 61.
Solution
Using $\boldsymbol{a_n} = a_1 + (n-1)d$ with $a_4 = 26$ (the fourth term is 26) and $a_9 = 61$ (the ninth term is 61), we have $26 = a_1 + (4-1)d$

$61 = a_1 + (9-1)d$
Solving the system of equations
$26 = a_1 + 3d$
$61 = a_1 + 8d$

Yield $a_1 = 5$ and $d = 7$. Therefore 5 is the first term

Arithmetic Means

If numbers are inserted between two numbers a and b to form the arithmetic sequence, the inserted numbers are called arithmetic means between a and b. if a single number is inserted between a and b to form an arithmetic sequence, that number is called an arithmetic mean between a and b.

Example 12.1.5

Insert two arithmetic means between 6 and 27

Solution

From the sequence, we have $6, 6+d, 6+2d, 27$

Then $a_1 = 6$ and $a_4 = 27$. This implies that we have 4 elements and $n = 4$

Then

$$a_4 = a_1 + (n-1)d$$
$$27 = 6 + (4-1)d$$
$$27 = 6 + 3d$$
$$27 - 6 = 3d$$
$$d = 7$$

Since we have found the value for d, then we can put in the sequence to find the specific value

$$6 + d = 6 + 7 = 13$$
$$6 + 2d = 6 + 2(7) = 20$$

Hence, the two means are 13 and 20 making the sequence $6, 13, 20, 27$

Sums of Arithmetic Sequences

We often use sequences to solve problems, so we need to be able to find the sum of a certain number of terms of the sequence. Before we develop a general-sum formula for arithmetic sequences, let us consider an approach to a specific problem that we can then use in a general setting.

Example 12.1. 6

Find the sum of the first 100 positive integers.

Solution

We are being asked to find the sum of $1 + 2 + 3 + 4 + \cdots + 100$. Rather than adding in the usual way, let us find the sum in the following manner.

$$\begin{array}{cccccccccc} 1 & + & 2 & + & 3 & + & 4 & + \cdots & + 100 \\ 100 & + & 99 & + & 98 & + & 97 & + \cdots & + 1 \\ \hline 101 & & 101 & & 101 & & 101 & \cdots & 101 \end{array}$$

$$\frac{(101)100}{2} = 5050$$

Note that we simply wrote the indicated sum forward and backward, and then we added the results. In so doing, we produced 100 sums of 101, but half of them are repeats.For example, $100 + 1$ and $1 + 100$ are both counted in this process. Thus we divide the product $(100)(101)$ by 2, which yields the final result of 5050.

The ***forward–backward*** approach we used in Example 12.1.5 can be used to develop a formula for finding the sum of the first n terms of any arithmetic sequence. Consider an arithmetic sequence $a_1, a_2, a_3, \dots, a_n$ with a common difference of d. Use S_n to represent the sum of the first n terms and proceed as follows.

$$\boldsymbol{S_n} = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n$$
Now write this sum in reverse.
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$
Add the two equations to produce
$$2S_n = n(a_1 + a_n)$$
From which we obtain a **sum formula**:

$$\boldsymbol{S_n = \frac{n}{2}(a_1 + a_n)}$$

Example 12.1.7

Find the sum of the first 30 terms of the arithmetic sequence 3, 7, 11, 15, ...

Solution

Using $a_n = a_1 + (n-1)d$,we can find the $30th$ term.
$$a_{30} = 3 + (30 - 1)4$$
$$= 3 + 29(4) = 119$$
Now we can use the sum formula.
$$S_{30} = \frac{30\,(3 + 119)}{2} = 1830$$
Example12.1.8

Find the sum of
$$7 + 10 + 13 + \cdots + 157.$$

Solution

To use the sum formula, we need to know the number of terms. The *n*th-term formula will do that for us.
$$a_n = a_1 + (n-1)d$$
$$157 = 7 + (n-1)(3)$$
$$157 = 7 + 3n - 3$$
$$157 = 3n + 4$$
$$n = 51.$$
Now we can use the sum formula.

$$S_{51} = \frac{51\,(7 + 157)}{2} = 4182$$

Summation Notation

Sometimes a special notation is used to indicate the sum of a certain number of terms of a sequence. The capital Greek letter ***sigma*** $\sum$ is used as a **summation symbol**. For example, $\sum_{i=1}^{5} a_i$ represents the sum $a_1 + a_2 + a_3 + a_4 + a_5$. The letter i is frequently used as the **index of summation**; the letter i takes on all integer values from the lower limit to the upper limit, inclusive. Thus
$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n.$$
If $a_1, a_2, a_3, a_4, \ldots, a_n$, represents an arithmetic sequence, we can now write the sum formula
$$\sum_{n=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

Example 12.1.9

Compute the sum $\sum_{i=1}^{7} 2i^2$

Solution

The indicated sum means

$$\sum_{i=1}^{7} 2i^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 + 2(5)^2 + 2(6)^2 + 2(7)^2$$
$$= 8 + 18 + 32 + 50 + 72 + 98$$
$$= 278.$$

Example 12.1.10

Evaluate

$$\sum_{k=2}^{50} (2k+1)$$

Solution

$\sum_{k=2}^{50}(2k+1) = (2 \times 2 \ + 1) + (2 \times 3 + 1) + (2 \times 4 + 1) + \ldots + (2 \times 50 + 1)$

$= 5 + 7 + 9 + 11 + \cdots + 101$

We see that, the sequence has been made with $a_1 = 5, d = 2$ and $n = 49$

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{49} = \frac{49}{2}(5 + 101) = 49 \times 53 = 2597$$

REVIEW EXERCISE

1. Find the for the nth term for each of the following sequence below
   a. $1, -1, \ 1, -1, \ 1, \ldots$ b. 2,5,8,11,14,17, ….. c. 10,20,30, ….
2. Write down the indicated term in the following arithmetic sequence
   a. 3 , 11 , ……., 10th, ……….,19th, …….
   b. 4 ,7, …………11th ,…………,17th , ……….,20th , …
   c. 4 ,1, …………13th ,…………,15th , ……….,19th ,…
   d. 0, -6, …………,5th , ………,8th
3. Find the sum of the arithmetic sequence
   a. 4 ,10 ,………., 12th term
   b. 1, 2 ,3, 4,……………………..,200th term
4. Show that the sum of first n terms of the arithmetic progression with the first term a and the common difference d is $S_n = \frac{n}{2}(2a_1 \ + (n-1)d)$ or $S_n = \frac{n}{2}(a_1 \ + a_n)$
5. Show that the sum of the positive integers from 1 to n is $S_n = \frac{n}{2}(n+1)$
6. The fourth term of the arithmetic sequence is 18 and the common difference is -5, find the first term. And find the sum of sixteen terms
7. Fin the sum of odd numbers between 100 and 200
8. If the 6th term is of the arithmetic sequence is 12 and the 10th term is 16. Find the first term and the common difference.
9. Find the sum of the first 200 odd numbers
10. Find the sum of the first 40 term of the arithmetic sequence 2 ,6 ,10 ,14 ,18 ,….
11. Write down the first five term of the geometric sequence that has the following general terms
    a. $a_n = 3n^2 - 1$ c. $a_n = \begin{cases} 3n+1 & for\ n \le 3 \\ 4n-1 & for\ n > 3 \end{cases}$ d. $a_n = \frac{(-1)^{n+1}}{n(n+1)}$
    b. $d. \quad a_n = \begin{cases} 2n+1\ for\ n\ odd \\ 2n-1\ for\ n\ even \end{cases}$

12. Write each series in expanded form and find the sum

a. $\sum_{i=1}^{7} 2^i$ b. $\sum_{i=1}^{5} 3^{i-1}$ c. $\sum_{i=1}^{45}(5i+2)$ d. $\sum_{i=10}^{20} 4n$ e. $\sum_{i=4}^{8} \left(\frac{1}{2}\right)^i$

12.2 Geometric sequence

A **geometric sequence** or **geometric progression (GP)** is a sequence in which we obtain each term after the first by multiplying the preceding term by a common multiplier, called the **common ratio** of the sequence. We can find the common ratio of a geometric sequence by dividing any term (other than the first) by the preceding term.

Let us consider the sequence 1,3,9,27,81. We observe that, upon dividing a term by its preceding term we obtain a common ratio $r = 3$. This means that, to get any successive term, we must multiply the term with the common ratio.

Thus, the **general term** of a geometric sequence is given by

$\boldsymbol{a_n = a_1 r^{n-1}}$ where $a_1$the first term and r isthe common ratio.

Example12.2.1

A geometric sequence has the first term 2 and a common ratio 3.

a. Write down the first four terms.
b. Find the ninth term.

Solution

a. We have $a_1 = 2$ and $r = 3$. Then by the formula
 $a_n = a_1 r^{n-1}$ we have

$$a_1 = 2(3)^{2-1} = 2 \times 3 = 6$$

$$a_2 = 2(3)^{3-1} = 6 \times 3 = 18$$
$$a_3 = 2(3)^{4-1} = 18 \times 3 = 54$$

 Hence the sequence is $2, 6, 18, 54$

b. The ninth term

$$a_9 = 2(3)^{9-1} = 2(3)^8 = 13122$$

Example 12.2.2

The first three terms of the geometric sequence are 16 , 8 and 4. Find the seventh term.

Solution

From the formula we have

$$8 = 16\, r$$

$r = \frac{1}{2}$ Then, by the formula $a_n = a_1\, r^{n-1}$ we have

$$a_7 = 16(\frac{1}{2})^{7-1} = 16\left(\frac{1}{2}\right)^6 = 16$$

$$\left(\frac{1}{64}\right) = \frac{16}{64} = \frac{1}{4}$$

is the seventh term.

Geometric Means

If numbers are inserted between two numbers a and b to form a geometric sequence, the inserted numbers are called the geometric means between a and b

Example 12.2.3
Insert two geometric means between 2 and 16

Solution
We have $2, \_, \_, 16$. This implies that the first term is 2 and the fourth term is 16
By the formula $a_n = a_1 r^{n-1}$ we have $a_4 = 2r^{4-1}$

$$16 = 2r^{4-1}$$

$r = 2$ .Then we have the geometric sequence $2, a_1 r, a_1 r^2, 16$
2, 4, 8, 16 is the sequence.

The sum of the n[th] term of the geometric sequence

Let S_n be the sum of the geometric sequence, then $S_n = \frac{(a_1(1-(r)^n))}{1-r}$ where $r \neq 1$

Example12.2.4
Show that the sum of a geometric sequence is given by $S_n = \frac{a_1 - a_1(r)^n}{1-r}$ where $r \neq 1$
Solution
The geometric sequence is represented as $a_1, a_1 r, a_1 r^2, \ldots, a_1 r^{n-1}$
Then the sum of this sequence can be written as
$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$...(i)
Multiply (i) by r we have $S_n = a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + a_1 r^n$ ... (ii)
Subtract equation (ii) from (i) we have

$$S_n - rS_n = a_1 - a_1 r^n$$
$$S_n(1-r) = a_1 - a_1 r^n$$

$$S_n = \frac{a_1 - a_1(r)^n}{1-r}$$

where $r \neq 1$

Example12.2.5
Find the sum of the first six terms of the geometric progression 250, 50, 10
Solution
$a_1 = 250$, $r = 1/5$ and $n = 6$ .Then by the formula $S_n = \frac{a_1 - a_1(r)^n}{1-r}$ we have
S_6

$$S_6 = \frac{250 - 250\left(\frac{1}{5}\right)^6}{1 - \frac{1}{5}} = \frac{250\left(1 - \left(\frac{1}{5}\right)^6\right)}{1 - \frac{1}{5}} = \frac{250\left(1 - \frac{1}{5^6}\right)}{\frac{4}{5}}$$
$$= 25\,0(1 - \frac{1}{5^6}) \times \frac{5}{4} = \frac{625}{2}(1 - \frac{1}{5^6})$$

We can approximate $\frac{1}{5^6}$ as zero. $S_6 = \frac{625}{2}(1-0) \approx \frac{625}{2}$

Example 12.2.6
Evaluate $\sum_{k=1}^{100} 6(\frac{1}{2})^k$

Solution

$$\sum_{k=1}^{100} 6(\frac{1}{2})^{k} = 6\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^{2} + 6\left(\frac{1}{2}\right)^{3} + \cdots + 6\left(\frac{1}{2}\right)^{100}$$

$$= 3 + \frac{3}{2} + \frac{3}{4} + \cdots + 6\left(\frac{1}{2}\right)^{100}$$

This is the geometric progression with the common ratio $r = \frac{1}{2}$. Then $S_n = \frac{a_1 - a_1(r)^n}{1-r}$

$$S100 = \frac{3\left(1 - \left(\frac{1}{2}\right)^{100}\right)}{1 - \frac{1}{2}} = 6(1 - \frac{1}{2^{100}}) \approx 6$$

Example12.2.7

Find the sum of the first 5 terms of the geometric sequence where the nth term is given by

$$a_n = 3(2)^{n-1}$$

Solution

$$a_1 = 3(2)^{1-1} = 3$$
$$a_2 = 3(2)^{2-1} = 6$$
$$r = \frac{a2}{a1} = \frac{6}{3} = 2$$

Then for $n = 5$

$$S_5 = \frac{a_1 - a_1(r)^n}{1-r}$$
$$= \frac{3 - 3(2)^5}{1-2} = \frac{3(1-2^5)}{-1}$$
$$= \frac{3(1-32)}{-1} = \frac{3(-31)}{-1} = 93$$

Sum of an infinite geometric sequence

We consider the sum formula $S_n = \frac{a_1 - a_1(r)^n}{1-r}$

We rewrite this formula as

$$S_n = \frac{a_1}{1-r} - \frac{a_1(r\,)^n}{1-r}$$

Now if we observe the behavior of r^n for $|r| < 1$as $n \to \infty$, we see that $r^n \to 0$.

Hence $\frac{a_1(r\,)^n}{1-r} \to 0$ and ***sum*** $= \frac{a_1}{1-r}$ is the formula for the sum of an infinite geometric sequence

$$S_\infty = \frac{a_1}{1-r} \quad \text{for } |r| < 1.$$

Example 12.2. 8

Find the sum to infinite of the geometric sequence 64 , 32 , 16 , 8 , ...

Solution

$a_1 = 64$ $a_2 = 32$

$$r = \frac{a_2}{a1} = \frac{32}{64} = \frac{1}{2}$$

$$S_\infty = \frac{a_1}{1-r} = \frac{64}{1-\frac{1}{2}} = \frac{64}{\frac{1}{2}} = 128$$

Repeating decimals as a sum of infinite geometric sequence

We defined rational numbers to be numbers that have either a terminating or a repeating decimal representation. For example $0.\overline{3}$, $0.\overline{14}$, 0.23

Example 12.2.9

Change $0.\overline{14}$ to the form $\frac{a}{b}$ where a and b are integers, $b \neq 0$

Solution

The repeating decimal $0.\overline{14}$ can be written as the indicated sum of an infinite geometric sequence with first term 0.14 and common ratio 0.01.

$$0.14 + 0.0014 + 0.000014 + \cdots$$

$$S_\infty = \frac{a1}{1-r} = \frac{0.14}{1-0.01} = \frac{0.14}{0.99} = \frac{14}{99}$$

Thus,

$$0.\overline{14} = \frac{14}{99}$$

Example 12.2.10

Change $0.\overline{3}$ to the form $\frac{a}{b}$ where a and b are integers, $b \neq 0$

Solution

The repeating decimal $0.\overline{3}$ can be written as the indicated sum of an infinite geometric sequence with first term 0.3 and common ratio 0.1

$$0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

$$S_\infty = \frac{a1}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{3}{9} = \frac{1}{3}$$

Thus,

$$0.\overline{3} = \frac{1}{3}.$$

Example 12.2.11

Change $2.\overline{14}$ to the form $\frac{a}{b}$ where a and b are integers, $b \neq 0$

Solution

$2.\overline{14} = 2 + 0.\overline{14}$.

We have $0.\overline{14} = \frac{14}{99}$. Hence

$$2.\overline{14} = 2 + 0.\overline{14} = 2 + \frac{14}{99}$$

You should follow the procedure in example 9 when solving for $0.\overline{14} = \frac{14}{99}$

MATHEMATICAL INDUCTIONS

Mathematical induction is the general principle used in mathematics to proof mathematical statements, formulas, theorems that are true for the set of all natural numbers.

For example $2^n > n$ is true for all natural numbers. That is;

If $n = 1$, then $2n > n$ becomes $2^1 > 1$, a true statement.
I f$n = 2$, then $2^n > n$ becomes $2^2 > 2$, a true statement.
If $n = 3$, then $2^n > n$ becomes $2^3 > 3$, a true statement.

We can continue in this way as long as we want, but obviously we can never show in this manner that $2^n > n$ for *every* positive integer n. However, we do have a form of proof, called **proof by mathematical induction** that can be used to verify the truth of many mathematical statements involving positive integers. This form of proof is based on the following principle.

Step1: show that the statement is true for $n = 1$

Step 2: assume that the statement is true for $n = k$ and proof that the statement is true for $n = k + 1$ where k is any natural number.

Transitive property
If a, b and c natural numbers, then
- i) If $a \leq b$ and $b \leq c$, then $a \leq c$
- ii) If $a \geq b$ and $b \geq c$, then $a \geq c$

Example13.1.1
Prove that $2n > n$ for all positive integers of n.

Solution
We shall use the principle of mathematical induction to prove that $2^n > n$ for all positive integers of n.
Step1: If $n = 1$, then $2^n > n$ becomes $2^1 > 1$, a true statement.
Step 2: We assume that the statement is true for $n = k$, that is $2^k > k$ then we prove that the statement is true for $n = k + 1$, that is $2^{k+1} > k + 1$.
We start with the assumption to deduce the following statement.

$$2^k > k$$

Multiply by 2 on both side

$$2(2^k) > 2k$$
$$2^{k+1} > 2k$$

We have $2k > k$ and

$$2^{k+1} > 2k > k$$

By the transitive property

$$2^{k+1} > k$$

Since the statement is true for $n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

Example13.1. 2
Use mathematical induction to prove that $3^n \geq 2n + 1$.

Solution
Step1: If $n = 1$, then $3^n \geq 2n + 1$ becomes$3^1 \geq 2(1) + 1$ which is the true statement.
Step2: We assume that the statement is true for $n = k$, that is $3^k \geq 2k + 1$, and prove that the statement is true for $n = k + 1$,that is $3^{k+1} \geq 2(k + 1) + 1$.

We start with the assumption to deduce the following statement.

$$3^k \geq 2k+1$$

Multiply by 3 on both sides

$$(3^k \geq 2k+1)3$$
$$3^{k+1} \geq 6k+3$$

Now $6k+3 \geq 2(k+1)+1$ for any positive integer, this implies that

$$3^{k+1} \geq 6k+3 \geq 2(k+1)+1$$

By the transitive property

$$3^{k+1} \geq 2(k+1)+1$$

Since the statement is true for $n=1$, then true for $n=k$ and true for $n=k+1$, then the statement is true for all positive integers.

Example13.2. 3
Prove by mathematical induction that $4^n > n$ for all positive integers of n.

Solution
We shall use the principle of mathematical induction to prove that $4^n > n$ for all positive integers of n.
Step1: If $n = 1$, then $4^n > n$ becomes $4^1 > 1$, a true statement.
Step 2: We assume that the statement is true for $n = k$, that is $4^k > k$ then we prove that the statement is true for $n = k+1$, that is $4^{k+1} > k+1$.
We start with the assumption to deduce the following statement.

$$4k > k$$

Multiply by 4 on both side

$$4(4^k) > 4k$$
$$4^{k+1} > 4k$$

We have $4k > k$ and

$$4^{k+1} > 4k > k$$

By the transitive property

$$4^{k+1} > k$$

Since the statement is true for $n=1$, then true for $n=k$ and true for $n=k+1$, then the statement is true for all positive integers.

Example13.2. 4
Use mathematical induction to prove that $2^n \geq n+1$.

Solution
Step1: If $n=1$, then $2^n \geq n+1$ becomes$2^1 \geq 1+1$ which is the true statement.
Step2: We assume that the statement is true for $n=k$, that is $2^k \geq k+1$, and prove that the statement is true for $n=k+1$,that is $2^{k+1} \geq (k+1)+1$.
We start with the assumption to deduce the following statement.

$$2^k \geq k+1$$

Multiply by 2 on both sides

$$(2^k \geq (k+1)2$$
$$2^{k+1} \geq 2k+2$$

Now $2k+2 \geq (k+1)+1$ for any positive integer, this implies that

$$2^{k+1} \geq 2k+2 \geq (k+1)+1$$

By the transitive property

$$2^{k+1} \geq (k+1) + 1$$

Since the statement is true for $n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

Example13.2. 5

Use mathematical induction to show that $4^n - 1$ is divisible by 3.

Solution

Step1: For $n = 1$, then $4^n - 1 = 4^1 - 1 = 3$ which is divisible by 3

Step 2: We assume that the statement is true for $n = k$, that is $4^k - 1$ is divisible by 3 and prove that the statement is true for $n = k + 1$ that is $4^{k+1} - 1$ is divisible by 3.

We start with the assumption to deduce the following statement.

There exist an integer x such that

$$4^k - 1 = 3x$$

Multiply by 4 on both sides we have

$$(4^k - 1 = 3x)4$$
$$4(4^k - 1) = 12x$$
$$4^{k+1} - 4 = 12x$$
$$4^{k+1} - 1 - 3 = 12x$$
$$4^{k+1} - 1 = 12x + 3$$

$4^{k+1} - 1 = 3(4x + 1)$. The statement is divisible by 3

Since the statement is true for$n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

Example13.2. 6

Use mathematical induction to show that $6^n - 1$ is divisible by 5

Solution

Step1: For $n = 1$, then $6^n - 1 = 6^1 - 1 = 5$ which is divisible by 5

Step 2: We assume that the statement is true for $n = k$, that is $6^k - 1$ is divisible by 5 and prove that the statement is true for $n = k + 1$ that is $6^{k+1} - 1$ is divisible by 5.

We start with the assumption to deduce the following statement.

There exist an integer x such that

$$6^k - 1 = 5x$$

Multiply by 4 on both sides we have

$$(6^k - 1 = 5x)6$$
$$6(6^k - 1) = 30x$$
$$6^{k+1} - 6 = 30x$$
$$6^{k+1} - 1 - 5 = 30x$$
$$6^{k+1} - 1 = 30x + 5$$

$6^{k+1} - 1 = 5(6x + 1)$. the statement is divisible by 5

Since the statement is true for $n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

Example 13.2.7

Use mathematical induction to show that $9^n - 1$ is divisible by 4

Solution
Step1: For $n = 1$, then $9^n - 1 = 9^1 - 1 = 8$ which is divisible by 4
Step 2: We assume that the statement is true for $n = k$, that is $9^k - 1$ is divisible by 4 and prove that the statement is true for $n = k + 1$ that is $9^{k+1} - 1$ is divisible by 4.
We start with the assumption to deduce the following statement.
There exist an integer x such that

$$9^k - 1 = 4x$$

Multiply by 9 on both sides we have

$$(9^k - 1 = 4x)9$$
$$9(9^k - 1) = 36x$$
$$9^{k+1} - 9 = 36x$$
$$9^{k+1} - 1 - 8 = 36x$$
$$9^{k+1} - 1 = 36x + 8$$

$9^{k+1} - 1 = 4(9x + 2)$. the statement is divisible by 4

Since the statement is true for $n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

Example 13.2.8
Use mathematical induction, prove that the sum formula for the arithmetic sequence$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$ hold for all positive integer n. That is $S_n = \frac{n(n+1)}{2}$for $a_n = n$.

Solution

Step1: For $n = 1, 1 = \frac{1(1+1)}{2} = 1$ which is a true statement.
Step2: We assume that, the statement is true for $n = k$, that is $1 + 2 + 3 + 4 + \cdots + k = \frac{k(k+1)}{2}$ and prove that the statement is also true for $n = k + 1$, that is

$$1 + 2 + 3 + 4 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

We start with the assumption to deduce the following statement.

$$1 + 2 + 3 + 4 + \cdots + k = \frac{k(k + 1)}{2}$$

From the kth term, the next term is $(k + 1)th$ term. Then adding $(k + 1)th$ term on both sides we have

$$1 + 2 + 3 + 4 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$

$$= \frac{k(k + 1) + 2(k + 1)}{2}$$

$$= \frac{(k + 1)(k + 2)}{2}$$

$$1 + 2 + 3 + 4 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

Since the statement is true for $n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

Example13.2. 9
Use mathematical induction to prove that the sum formula for the sequence hold for all positive integer n. That is $S_n = 2(2^n - 1)$ for $a_n = 2^n$

Solution
We first introduce the sum of the sequence

$$2^1 + 2^2 + 2^3 + 2^4 \ldots + 2^n = 2(2^n - 1)$$

Step1*:* For $n = 1$, $2^1 = 2(2^n - 1) = 2(2^1 - 1) = 2$ which is a true statement.
Step2*:* We assume that, the statement is true for $n = k$,
That is $2^1 + 2^2 + 2^3 + 2^4 \ldots + 2^k = 2(2^k - 1)$and prove that the statement is also true for

$$n = k + 1,$$

That is $2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^k + 2^{k+1} = 2(2^{k+1} - 1)$

We start with the assumption to deduce the following statement.

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^k = 2(2^k - 1)$$

From the kth term, the next term is $(k+1)th$ term. Then adding $(k+1)th$ term on both sides we have

$$\begin{aligned}
2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^k + 2^{k+1} &= 2(2^k - 1) + 2^{k+1} \\
&= 2^{k+1} - 2 + 2^{k+1} \\
&= 2^{k+1} + 2^{k+1} - 2 \\
&= 2.2^{k+1} - 2 \\
&= 2(2^{k+1} - 1) \\
2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^k + 2^{k+1} &= 2(2^{k+1} - 1)
\end{aligned}$$

Since the statement is true for $n = 1$, then true for $n = k$ and true for $n = k + 1$, then the statement is true for all positive integers.

14: PARTIAL FRACTIONS

To understand the concept of partial fraction, let us first consider this example.

Example 14.1.1
Express $\frac{3}{x-2} + \frac{2}{x+3}$ as a single fraction in its lowest terms.

Solution

$$\frac{3}{x-2} + \frac{2}{x+3} = \frac{3(x+3) + 2(x-2)}{(x-2)(x+3)}$$

$$= \frac{3x + 9 + 2x - 4}{(x-2)(x+3)}$$

$$= \frac{3x + 2x + 9 - 4}{(x-2)(x+3)}$$

$$= \frac{5x + 5}{(x-2)(x+3)}$$

Now suppose that we want to reverse the process. That is, suppose we are given the rational expression $\frac{5x+5}{(x-2)(x+3)}$ and we want to express it as the sum of two simpler rational expressions called **partial fractions**. The process is called **partial fraction decomposition**.

Proper fractions and improper fractions

1. An algebraic fraction is said to be proper fraction when the degree of the numerator is less than the degree of the denominator.
2. An algebraic fraction is said to be an improper fraction when the degree of the denominator is less than the degree of the numerator.

Example14.1.2

Find the values A, B and C such that

$$5x+3 \equiv Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$

Solution

To find the values A, B and C we use the following methods

Method 1

Let $x=-3$

$$5x+3 = Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$
$$5(-3)+3 = A(-3)(-3+3)+B(-3)(-3-1)+C(-1)(-3+3)$$
$$-15+3 = -3B(-3-1)$$
$$-12 = 12B$$
$$B = -1$$

Let $x=1$

$$5x+3 = Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$
$$5(1)+3 = A(1)(1+3)+B(1)(1-1)+C(1)(1+3)$$
$$8 = 4A$$
$$A = 2$$

Let $x=0$

$$5x+3 = Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$
$$5(0)+3 = A(0)(0+3)+B(0)(0-1)+C(0-1)(0+3)$$
$$3 = 4-3C$$
$$C = -1$$

Method 2

$$5x+3 \equiv Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$
$$\equiv Ax^2+3Ax+Bx^2-Bx+Cx^2+2Cx-2C$$
$$\equiv (A+B+C)x^2+(3A-B+2C)x-3C$$

By comparing the coefficients

$$A+B+C=0$$
$$3A-B+2C=5$$
$$-3C=3$$

Solving the equations

$$C=-1\ ,A=2 \text{ and } B=1$$

Partial fraction decomposition with only linear factors denominator

Example14.1.3

Find the partial fraction decomposition of $\frac{8x-28}{x^2-6x+8}$

Solution

We know that $x^2 - 6x + 8$ can be factorized as $(x-2)(x-4)$. Then the faction becomes $\frac{8x-28}{(x-2)(x-4)}$

Now

$$\frac{8x-28}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} \quad \dots \quad \text{(i)}$$

The task is to find A and B

Multiply equation (i) on both sides by $(x-2)(x-4)$ we have

$$\left(\frac{8x-28}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}\right)(x-2)(x-4)$$

$$8x - 28 = A(x-4) + B(x-2)$$

Let $x = 4$

$$8(4) - 28 = A(4-4) + B(4-2)$$
$$4 = 2B$$
$$B = 2$$

Let $x = 2$

$$8(2) - 28 = A(2-4) + B(2-2)$$
$$-12 = -2A$$
$$A = 6$$

Hence

$$\frac{8x-28}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} = \frac{6}{x-2} + \frac{2}{x-4}$$

$$\frac{8x-28}{x^2-6x+8} = \frac{6}{x-2} + \frac{2}{x-4}$$

Example 14.1.4

Find the partial fraction decomposition of $\frac{x+3}{(x-2)(x+4)}$

Solution

$$\frac{x+3}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4} \quad \dots \quad \text{(i)}$$

The task is to find A and B

Multiply equation (i) on both sides by $(x-2)(x+4)$ we have

$$\left(\frac{x+3}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}\right)(x-2)(x+4)$$

$$x + 3 = A(x+4) + B(x-2)$$

Let $x = -4$

$$-4 + 3 = A(-4+4) + B(-4-2)$$
$$-1 = -6B$$

$$B = \frac{1}{6}$$

Let $x = 2$

$$2 + 3 = A(2 + 4) + B(2 - 2)$$
$$5 = 6A$$
$$A = \frac{5}{6}$$

Hence

$$\frac{x + 3}{(x - 2)(x + 4)} = \frac{A}{x - 2} + \frac{B}{x + 4} = \frac{\frac{5}{6}}{x - 2} + \frac{\frac{1}{6}}{x4}$$

$$\frac{x + 3}{(x - 2)(x + 4)} = \frac{5}{6(x - 2)} + \frac{1}{6(x + 4)} = \frac{5}{6x - 12} + \frac{1}{6x + 24)}$$

Denominator with an irreducible quadratic factor

Example 14.1.5

Express $\frac{3x+1}{(x-1)(x^2+1)}$ in partial fractions.

Solution

$$\frac{3x + 1}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}$$

$$\left(\frac{3x + 1}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}\right)(x - 1)(x^2 + 1)$$

$$3x + 1 = A(x^2 + 1) + Bx + C(x - 1)$$

Let $x = 1$

$$3(1) + 1 = A(1^2 + 1) + B(1) + C(1 - 1)$$
$$3 + 1 = 2A$$
$$A = 2$$

$$3x + 1 = A(x^2 + 1) + Bx + C(x - 1)$$
$$3x + 1 = Ax^2 + A + Bx^2 - Bx + Cx - C$$
$$3x + 1 = (A + B)x^2 + (-B + C)x + (A - C)$$

$$A + B = 0$$
$$2 + B = 0$$
$$B = -2$$

$$A - C = 1$$
$$2 - C = 1$$
$$C = 1$$

Hence

$$\frac{3x + 1}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1} = \frac{2}{x - 1} + \frac{-2x + 1}{x^2 + 1}$$

Example 14.1.6
Express $\frac{4}{(x+1)(2x^2+x+3)}$ in partial fractions.

Solution

$$\frac{4}{(x+1)(2x^2+x+3)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+x+3}$$

$$\left(\frac{4}{(x+1)(2x^2+x+3)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+x+3}\right)(x+1)(2x^2+x+3)$$

$$4 = A(2x^2+x+3) + (Bx+C)(x+1)$$

$$4 = 2Ax^2 + Ax + 3A + Bx^2 + Bx + Cx + C$$
$$4 = Ax^2 + A + Bx^2 - Bx + Cx - C$$
$$4 = (2A+B)x^2 + (A+B+C)x + (3A+C)$$

$$2A + B = 0$$
$$A + B + C = 0$$
$$3A + C = 4$$

Solving these three equations we have

$$A = 1\,, B = -2 \text{ and } \; C = 1$$

Hence

$$\frac{4}{(x+1)(2x^2+x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+3} = \frac{1}{x+1} + \frac{-2x+1}{x^2+x+3}$$

Denominator with the repeated linear factor

Example14.1.7
Express $\frac{1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{3}{(x-1)^3}$ as a single fraction.

Solution

$$\frac{1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{3}{(x-1)^3} = \frac{(x-1)^2 + 2(x-1) + 3}{(x-1)^3}$$

$$= \frac{x^2+2}{(x-1)^3}$$

Now, reversing this statement requires the concept of partial fraction decomposition of denominator with repeated linear factor.

Example14.1.8

Express $\frac{x^2+2}{(x-1)^3}$ in partial fractions

Solution

$$\frac{x^2+2}{(x-1)^3}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

Multiply on both sides by $(x-1)^3$

$$(\frac{x^2+2}{(x-1)^3}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3})(x-1)^3$$

$$x^2+2=A(x-1)^2+B(x-1)+C$$

$$=A(x^2-2x+1)+B(x-1)+C$$

$$=Ax^2-2Ax+A+Bx-B+C$$

$$=Ax^2+(-2A+B)x+(A-B+C)$$

Comparing the coefficients

$$A=1$$

$-2A+B=0$ then $B=2$

$A-B+C=2$ then $C=3$

Hence

$$\frac{x^2+2}{(x-1)^3}=\frac{1}{(x-1)}+\frac{2}{(x-1)^2}+\frac{3}{(x-1)^3}$$

Example14.1.9

Express $\frac{x-1}{(x+1)(x-2)^2}$ in partial fractions.

Solution

$$\frac{x-1}{(x+1)(x-2)^2}=\frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$$

$$\left(\frac{x-1}{(x+1)(x-2)^2}=\frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}\right)(x+1)(x-2)^2$$

$$x-1=A(x-2)^2+B(x-2)(x-1)+C(x-1)$$

$$=A(x^2-4x+4)+B(x-2)(x-1)+C(x-1)$$

$$= Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C$$

$$= (A+B)x^2 + (-4A - B + C)x + (-2B + 4A + C)$$

Comparing the coefficients

$$A + B = 0$$

$$-4A - B + C = 1$$

$$-2B + 4A + C = -1$$

Solving the equations

$$A = -\frac{2}{9}\ ,\quad B = \frac{2}{9}\quad ,\ C = \frac{1}{3}$$

Hence

$$\frac{x-1}{(x+1)(x-2)^2} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} = \frac{\frac{-2}{9}}{(x+1)} + \frac{\frac{2}{9}}{(x-2)} + \frac{\frac{1}{3}}{(x-2)^2}$$

$$\frac{x-1}{(x+1)(x-2)^2} = -\frac{2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}$$

Decomposition of improper fractions into partial fractions

An improper fraction whose numerator is of degree equal to or greater than the degree of the denominator. To deal with this fraction, we first divide the denominator into numerator to obtain a quotient and a proper fraction, and then split into partial fraction.

Example 14.1.10

Express $\frac{x^2-2}{(x+2)(x-1)}$ in partial fraction

Solution

$$\frac{x^2-2}{(x+2)(x-1)} = \frac{x^2-2}{x^2+2x-3}$$

We use long division

$(x^2 + 2x - 3 \quad 1$

$- \quad x2 \quad + 2x - 3x^2 + 0x - 2$

$-2x + 1$

$$\frac{x^2-2}{(x+2)(x-1)} = 1 + \frac{1-2x}{(x+3)(x-1)}$$

Now the partial fraction for a proper fraction $\frac{1-2x}{(x+3)(x-1)}$ can be obtained

$$\frac{1-2x}{(x+3)(x-1)} = \frac{A}{(x+3)} + \frac{B}{(x-1)}$$

$1-2x = A(x-1) + B(x+3)$
$A = -\frac{7}{4}$ and $B = -\frac{1}{4}$

Then

$$\frac{1-2x}{(x+3)(x-1)} = -\frac{7}{4(x+3)} - \frac{1}{4(x-1)}$$

Hence

$$\frac{x^2-2}{(x+2)(x-1)} = 1 + \frac{1-2x}{(x+3)(x-1)} = 1 - \frac{7}{4(x+3)} - \frac{1}{4(x-1)}$$

Example 14.1.11

Decompose $\frac{3x^2+7x-4}{(x^2+2)^2}$ into partial fractions.

Solutions

$$\frac{3x^2+7x-4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$$

$$\left(\frac{3x^2+7x-4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}\right)(x^2+2)^2$$

$$3x^2+7x-4 = (Ax+B)(x^2+2) + (Cx+D)$$

$$3x^2+7x-4 = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

Comparing the coefficients

$$A = 3B = 0$$

2A+$C = 7$ then $C = 1$

$2B + D = -4$ then $D = -4$

Hence

$$\frac{3x^2+7x-4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2} = \frac{3x}{(x^2+2)} + \frac{x-4}{(x^2+2)^2}$$

REVIEW EXERCISE

Find the partial fraction decomposition of the following

1. $\frac{11x-10}{(x-2)(x+1)}$
2. $\frac{-6x^2+19x+21}{x^2(x+3)}$
3. $\frac{-2x-8}{x^2-1}$
4. $\frac{3x^2+10x+9}{(x+2)^3}$
5. $\frac{x^2-18x+5}{(x-1)(x+2)(x-3)}$
6. $\frac{-2x^2-3x+10}{(x^2+1)(x-4)}$
7. $\frac{2x+1}{(x-2)^2}$
8. $\frac{4x^2+3x+14}{x^3-8}$

15: COMBINATION, PERMUTATIONAND BINOMIAL EXPANSIONS

Permutations and Combinations

Factorial notation becomes very useful. The notation $n!$ (Which is read n factorial) is used with positive integers as follows.

$$1! = 1$$
$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Note that the factorial notation refers to an *indicated product*. In general, we write

$$n! = n(n-1)(n-2) \times \dots \times 4 \times 3 \times 2 \times 1$$

We also define $0! = 1$ so that certain formulas will be true for all nonnegative integers.

Example15. 1.1

In how many ways can the three letters A, B, and C be arranged in a row?

Solution 1

Certainly one approach to the problem is simply to list and count the arrangements.

$ABC \quad ACB \quad BAC \quad BCA \quad CAB \quad CBA$

There are six arrangements of the three letters.

Solution 2

Another approach, one that can be generalized for more difficult problems, uses the fundamental principle of counting. Because there are three choices for the first letter of an arrangement, two choices for the second letter, and one choice for the third letter, there are $(3)(2)(1) = 6$ arrangements.

Ordered arrangements are called **permutations**. A permutation of a set of n elements is an ordered arrangement of the n elements denoted by $P(n, n)$

$$P(n, n) = n!$$

$$P(n, r) = \frac{n!}{(n-r)!} \text{ for } r < n$$

Example15.1.2

In how many ways can five students be seated in a row of five seats?

Solution

The problem is asking for the number of five-element permutations that can be formed from a set of five elements. Thus we can apply $P(n, n)$

$$P(n, n) = n!$$
$$P(5,5) = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Example 15.1.3

Evaluate $P(5,2)$

Solution

$$P(n, r) = \frac{n!}{(n-r)!}$$
$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{120}{6} = 20$$

Combination

A combination is an arrangement of objects in which the order is not important and denoted by $C(n,r)$ or $\binom{n}{r}$.

$$C(n,r) = \frac{n!}{(n-r)!r!} \quad \text{for } r < n$$

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \quad \text{for } r < n$$

Example 15.1.4
How many different five-card hands can be dealt from a deck of 52 playing cards?

Solution
Because the order in which the cards are dealt is not an issue, we are working with a combination (subset) problem. Thus, using the formula for $C(n,r\ )$, we obtain

$$C(52,5) = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{52\times 51\times 50\times 49\times 48\times 47!}{47!\ 5!}$$

$$= \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1}$$
$$= 2{,}598{,}960$$

There are 2,598,960 different five-card hands that can be dealt from a deck of 52 playing cards.

15.2 Binomial expansions
Let us consider the following expansions

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

First, note the pattern of the exponents for x and y on a term-by-term basis. The exponents of x begin with the exponent of the binomial and decrease by 1, term by term, until the last term has x^0, which is 1. The exponents of y begin with zero ($y^0 = 1$) and increase by 1, term by term, until the last term contains y to the power of the binomial.
Then when note the pattern of the coefficients, they are making a pattern of a triangle. This triangle is called ***Pascal's Triangle***.

```
                  1
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
```

Using this triangle, we can expand the expression $(x+y)^n$ when n is not very big.
Example 15.2.1
Use Pascal's triangle to expand $(x+y)^6$

Solution
From the triangle, when $n = 6$ we have the coefficients

1 6 15 20 15 6 1.

Following the pattern of exponents we have

$$1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$
$$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Now let's look for a pattern for the coefficients by examining specifically the expansion of $(x + y)^5$

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$C(5,0)$ $C(5,1)$ $C(5,2)$ $C(5,3)$ $C(5,4)$ $C(5,5)$

We can generalize from this pertains of the coefficients into a combination called binomial coefficients.

Binomial theorem
For any binomial $(x + y)$ and any natural number n

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 \dots + \binom{n}{n}y^n$$

Example 15.2.2
Expand $(x + y)^7$

Solution

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n$$

$$(x + y)^7 = x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7$$

$$(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Example15.2. 3
Expand $(x - y)^5$

Solution

$$(x - y)^5 = \big(x + (-y)\big)^5$$
$$\big(x + (-y)\big)^5 = x^5 + \binom{5}{1}x^4(-y) + \binom{5}{2}x^3(-y)^2 + \binom{5}{3}x^2(-y)^3 + \binom{5}{4}x(-y)^4 + \binom{5}{5}(-y)^5$$

$$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

Example 15.2.4
Expand $\left(x + \frac{1}{y}\right)^5$
Solution

$$\left(x + \frac{1}{y}\right)^5 = x^5 + \binom{5}{1}x^4\left(\frac{1}{y}\right) + \binom{5}{2}x^3\left(\frac{1}{y}\right)^2 + \binom{5}{3}x^2\left(\frac{1}{y}\right)^3 + \binom{5}{4}x\left(\frac{1}{y}\right)^4 + \binom{5}{5}\left(\frac{1}{y}\right)^5$$

$$= x^5 + 5x^4\frac{1}{y} + 10x^3\frac{1}{y^2} + 10x^2\frac{1}{y^3} + 5x\frac{1}{y^4} + \frac{1}{y^5}$$

$$= x^5 + \frac{5x^4}{y} + \frac{10x^3}{y^2} + \frac{10x^2}{y^3} + \frac{5x}{y^4} + \frac{1}{y^5}.$$

Finding the specific term

Let's consider the expansion

$$(x+y)^7 = x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7$$

Now if we can to find the sixth term without expanding the expression

We have $\binom{7}{5}x^2y^5$ has the sixth term.

We observe that

$$\binom{7}{5}x^2y^5 = \binom{7}{6-1}x^{n-(r-1)}y^{6-1} = \binom{n}{r-1}x^{n-r}y^{r-1}$$

In general, the r^{th} term in the expansion $(x+y)^n$ is given by

$$\boldsymbol{T_r = \binom{n}{r-1}x^{n-(r-1)}\,y^{r-1}}$$

Example 15.2.5

Find the $4th$ term in the binomial expansion $(x-2y^2)^6$

Solution

We have $n = 6$, $r = 4$ and $y = -2y^2$

$$T_r = \binom{n}{r-1}x^{n-(r-1)}y^{r-1}$$

$$\text{T4} = \binom{6}{4-1}x^{6-3}(-2y^2)^{4-1} = \binom{6}{3}x^3(-2y^2)^3 = 20x^3(-8)y^6 = -120x^3y^6$$

Example 15.2.6

Find the seventh term of $(x+y)^{11}$

Solution

We have $n = 11$, $r = 7$

$$Tr = \binom{n}{r-1}x^{n-(r-1)}y^{r-1}$$

$$\text{T4} = \binom{11}{7-1}(x)^{11-6}(y)^{7-1} = \binom{11}{6}x^5y^6 = 462x^3y^6.$$

Example15.2. 7

Find the fourth term of $(x+y)^8$

Solution

We have $n = 8$, $r = 4$

$$Tr = \binom{n}{r-1} x^{n-(r-1)} y^{r-1}$$

$$\text{T4} = \binom{8}{4-1} (x)^{8-3} (y)^{4-1} = \binom{8}{3} x^5 y^3 = 56x^3y^6$$

Example15.2. 8

Expand $\left(x + \frac{1}{x}\right)^6$

Solution

$$\left(x+\frac{1}{x}\right)^6 = x^6 + \binom{6}{1}x^5\left(\frac{1}{x}\right) + \binom{6}{2}x^4\left(\frac{1}{x}\right)^2 + \binom{6}{3}x^3\left(\frac{1}{x}\right)^3 + \binom{6}{4}x^2\left(\frac{1}{x}\right)^4 + \binom{6}{5}x\left(\frac{1}{x}\right)^5 + \binom{6}{6}\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^5\frac{1}{x} + 15x^4\frac{1}{x^2} + 20x^3\frac{1}{x^3} + 15x^2\frac{1}{x^4} + 6x\frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 + \frac{6x^5}{x} + \frac{15x^4}{x^2} + \frac{20x^3}{x^3} + \frac{15x^2}{x^4} + \frac{6x}{x^5} + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

From example 8, we see that there is a constant term 20 which is independent of the variable x.

Finding the constant term in the binomial expansion $\left(x + \frac{1}{x}\right)^n$ for n even.

The term independent of x or the constant term is given by the formula

$$\mathbf{T_r = \binom{n}{r-1} x^{n-(r-1)} y^{r-1}}$$

This formula is the same as with the one for finding the specific term, but the value of r will not be given. Let's consider the following example

Example 15.2.9

Without expanding, find the term independent of x in the binomial expansion $\left(x + \frac{1}{x}\right)^6$

Solution

$$n = 6$$

$$T_r = \binom{n}{r-1} x^{n-(r-1)} y^{r-1}$$

$$T_r = \binom{6}{r-1} x^{6-(r-1)} y^{r-1}$$

We have to know that at a constant term, the degree of the variable x and the degree of the variable y are equal. Then

$$n-(r-1)=r-1$$
$$6-(r-1)=r-1$$
$$6=r-1+r-1$$
$$6=2r-2$$
$$8=2r$$
$$r=4$$

$$T4=\binom{6}{4-1}x^{6-(4-1)}(\frac{1}{x})^{4-1}$$

$$T4=\binom{6}{3}x^{3}(\frac{1}{x})^{3}$$

$$T4=\binom{6}{3}x^{3}\frac{1}{x^{3}}$$

$$T4=\binom{6}{3}=20.$$

Binomial theorem for any index or any power

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\binom{n}{3}x^{n-3}y^3+\cdots+\binom{n}{n}y^n$$

Since

$$\binom{n}{0}=\binom{n}{n}=1$$

The theorem can be simplified as

$$(x+y)^n=x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\binom{n}{3}x^{n-3}y^3+\cdots+y^n$$

Then the theorem can be written as

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}y^3+\cdots+y^n$$

Now for the expression $(1+x)^n$ the expansion is given by

$$(1+x)^n=1^n+n1^{n-1}x+\frac{n(n-1)}{2!}1^{n-2}x^2+\frac{n(n-1)(n-2)}{3!}1^{n-3}x^3+\cdots+x^n$$

For any rational number n

$$\boldsymbol{(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots} \qquad \textit{Provided } |\boldsymbol{x}|<1$$

Example 15.2. 10
Expand $(1-2x)^3$

Solution
We have $n=3$

$$|-2x|<1$$
$$2|x|<1$$

$|x| < \frac{1}{2} < 1$
Hence valid

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

$$(1-2x)^3 = 1 + 3(-2x) + \frac{3(3-1)}{2!}(-2x)^2 + \frac{3(3-1)(3-2)}{3!}(-2x)^3 + \frac{3(3-1)(3-2)(3-3)}{4!}(-2x)^4$$

$$= 1 - 6x + 12x^2 - 8x^3$$

Example 15.2.11
Expand $\frac{1}{1+x}$

Solution

$$\frac{1}{1+x} = (1+x)^{-1}$$

We have $n = -1$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

$$(1+x)^{-1} = 1 + (-1)x + \frac{-1(-1-1)}{2!}x^2 + \frac{-1(-1-1)(-1-2)}{3!}x^3 + \cdots$$
$$= 1 - x + x^2 - x^3 + \cdots \text{Provided} |x| < 1.$$

Example 15.2.12

Expand $\sqrt{1-3x}$

Solution

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$

We have $n = \frac{1}{2}$

$$|-3x| < 1$$
$$3|x| < 1$$

$|x| < \frac{1}{3} < 1$
Hence valid

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$
$$(1-3x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-3x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-3x)^3 + \cdots$$

$$= 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{81}{6}x^3 + \cdots$$

15.3 Expanding partial fractions into binomial expressions

Example15.3.1

a. Express $\frac{4-5x}{(1+x)(2-x)}$ as partial fraction

b. Hence show that the binomial cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is

$$2-\frac{1}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$$

Solutions

a. We have

$$\frac{4-5\mathrm{x}}{(1+\mathrm{x})(2-\mathrm{x})}=\frac{A}{1+x}+\frac{B}{2-x}$$

$$\left(\frac{4-5\mathrm{x}}{(1+\mathrm{x})(2-\mathrm{x})}=\frac{A}{1+x}+\frac{B}{(2-x)}\right)(1+\mathrm{x})(2-\mathrm{x})$$

$$4-5x=A(2-x)+B(1+x)$$

For $x=2,\ B=-2$ and for $x=-1\,,\ A=3$

$$\frac{4-5\mathrm{x}}{(1+\mathrm{x})(2-\mathrm{x})}=\frac{3}{1+x}-\frac{2}{2-x}$$

b.

$$\frac{3}{1+x}-\frac{2}{2-x}=3\left(\frac{1}{1+x}\right)-2\left(\frac{1}{2-x}\right)$$

$$=3(1+x)^{-1}-2(2-x)^{-1}$$

$$\frac{1}{1+x}=(1+x)^{-1}\ \dots(i)$$

$$(1+x)^{-1}=1+(-1)x+\frac{-1(-1-1)}{2!}x^2+\frac{-1(-1-1)(-1-2)}{3!}x^3$$

$$=1-x+x^2-x^3$$

$$3(1+x)^{-1}=3-3x+3x^2-3x^3$$

$\frac{2}{2-x}=2(2-x)^{-1}$ ... (ii)

$$=2\left[\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}\right]=\left(1-\frac{x}{2}\right)^{-1}$$

$$\left(1-\frac{x}{2}\right)^{-1}=1+\left(-\frac{1}{2}x\right)+\frac{-1(-1-1)}{2!}(-\frac{x}{2})^2+\frac{-1(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3$$

$$=1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}$$

Combining (i) *and* (ii) we have

$$3\left(\frac{1}{1+x}\right)-2\left(\frac{1}{2-x}\right)=(3-3x+3x^2-3x^3)-(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8})$$

Subtracting the variables with the same degree

$$3\left(\frac{1}{1+x}\right)-2\left(\frac{1}{2-x}\right)=(3-3x+3x^2-3x^3)-\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}\right)$$

$$=2-\frac{1}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$$

Hence we have shown that the binomial cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is

$$2-\frac{1}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$$

Note: This approach is used to approximate the rational functions into the binomial expansion.

REVIEW EXERCISE

1. Evaluate
   a. $\frac{11!}{8!}$ b. $\frac{6!2!}{8!}$ c. $\frac{7!}{3!}$ d. $(3!)^2$ e. $5!\times 3!$
2. Express in factorial notation
   a. $5\times 4\times 3\times 2\times 1$ b. $9\times 8\times 7\times 6\times 5$ c. $\frac{4\times 3}{5\times 4\times 3}$
3. Evaluate
   a. $P(5,3)$b. $P(9,7)$ c. $P(7,1)$ d. $P(4,0)$
4. Evaluate
   a. $C(8,7)$ b. $C(5,2)$ c. $C(10,0)$ d. $C(7,7)$ e. $\binom{5}{3}$
5. A mixed net ball team containing 5 men and 6 women is to be chosen from 7 men and 9 women. In how many ways can this be done?
6. There are 10 possible players for the VI to represent the tennis club and of these the captain and the secretary must be in the team. In how many ways can the team be selected?
7. Use Pascal's triangle to find the expansion
   a. $(3x-y)^4$ b. $(x-\frac{1}{x})^5$ c. $(1-z)^4$ d. $(\frac{x}{2}+\frac{2}{x})^4$
8. Use the binomial theorem to expand the following
   a. $(x+y)^5$ b. $(2x+y)^4$ c. $(x-\frac{2}{y})^4$ d. $(\frac{1}{x}-\frac{1}{y})^3$
9. Write down the term indicated in the expansion of the following and simplify the answer
   a. $(x+2)^8$ term in x^5 b. $\left(2t-\frac{1}{2}\right)^{12}$ term in t^7 c. $(2x+y)^{11}$ term in x^3
10. Expand the following in ascending power of x, as far as the term in x^3 and state the values of x for which the expansions are valid
    a. $(1+\frac{x}{2})^{-3}$ b. $\frac{1}{\sqrt{(1+x^2)}}$ c. $\frac{1}{\sqrt{(2+x^2)}}$ d. $\frac{1}{3+x}$
11. Find the first four terms of the expansion in ascending power of x
    a. $\frac{1+x}{1-x}$ b. $\sqrt{\frac{(1-x)^2}{1+x}}$ c. $\frac{2}{(x+1)}+\frac{1}{(x+2)}$
12. Expand the following into partial fractions. Find three terms in ascending power of x in the expansion and state for what values of x are the expansion valid.
    a. $\frac{x+2}{x^2-1}$ b. $\frac{1}{(x+2)^2}$ c. $\frac{x+7}{(x+1)^2(x-2)}$

16: MATRICES

A matrix is a rectangular array of numbers in rows and columns. In plural it is called matrices.

Order of matrix

The number of rows and the number columns of a matrix are called order of matrix. The number of rows are stated first.

Matrix	Order
$\begin{pmatrix} 3 & 6 \end{pmatrix}$	1×2
$\begin{pmatrix} 3 & 2 \\ 6 & 5 \end{pmatrix}$	2×2
$\begin{pmatrix} 2 & 7 & -5 \\ 3 & -1 & 4 \end{pmatrix}$	2×3

Notation of a Matrix

Matrix is defined by the capital letter and small letters are reserved for entries.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Types of Matrices

1. **Row and Column matrix**
   A row matrix is formed by a single row and a column matrix is formed by a single column. For example
   $$A = \begin{pmatrix} 6 & 1 & 3 \end{pmatrix} A = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$$
2. **Square matrix**
   A square matrix is formed by the same number of rows and columns. For example
   $$\begin{pmatrix} 1 & 5 & 4 \\ 3 & 2 & 0 \\ 7 & 8 & 6 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 5 & 1 \end{pmatrix}$$
3. **Zero matrix (Null matrix)**
   In a zero matrix, all entries are zero
   $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
4. **Diagonal matrix**
   This is a square matrix in which all elements above and below the main diagonal are zeros. For example
   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

5. **Lower triangular matrix**
   In a lower triangular matrix, the elements above the main diagonal are zeros. For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 7 & 8 & 6 \end{pmatrix}$$

6. **Identity matrix**
   An identity matrix is a diagonal matrix in which the main diagonal elements are each equal to 1. For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7. **Transpose matrix**
   Given a matrix A, the transpose of A usually denoted by A^T is a matrix where elements in rows of a matrix A becomes columns and the columns in a matrix A becomes rows. For example
   Given $A = \begin{pmatrix} 1 & 5 & 4 \\ 3 & 2 & 0 \\ 7 & 8 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 3 & 7 \\ 5 & 2 & 8 \\ 4 & 0 & 6 \end{pmatrix}$

   Properties of transpose matrix
   i) $(A^T)^T = A$
   ii) $(A + B)^T = A^T + B^T$
   iii) $(A.B)^T = B^T.A^T$

8. **Symmetric matrix**
   A matrix B is symmetric if $B = B^T$. For example

$$\begin{pmatrix} 6 & 2 & 5 \\ 2 & 1 & 3 \\ 5 & 3 & 7 \end{pmatrix}$$

9. **Equal matrices**
   Two matrices A and B are **equal** if and only if all elements in corresponding positions are equal.
   $A = \begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix}$ are equal

Example16.1.1
If the matrices A and B are equal, find the value of x given that

$$A = \begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -4 \\ x+6 & 7 \end{pmatrix}$$

Solution

$$A = B$$
$$\begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ x+6 & 7 \end{pmatrix}$$
$$5 = x + 6$$
$$x = 5 - 6$$
$$x = -1$$

Addition of matrices
To add two or more matrices, we add elements that appear in corresponding positions. Therefore the sum of two 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ u & z \end{pmatrix}$ is given by

$$A+B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\begin{pmatrix}x & y\\ u & z\end{pmatrix}=\begin{pmatrix}a+x & b+y\\ c+u & d+z\end{pmatrix}$$

Properties

i) $A+B=B+A$ *Commutative law*

ii) $(A+B)+C=A+(B+C)$ *Associative law*

Example 16.1.2

Given$A=\begin{pmatrix}2 & -3\\ 5 & 0\end{pmatrix}$ and $B=\begin{pmatrix}1 & 8\\ 7 & 6\end{pmatrix}$. Find $A+B$

Solution

$$A+B=\begin{pmatrix}2 & -3\\ 5 & 0\end{pmatrix}+\begin{pmatrix}1 & 8\\ 7 & 6\end{pmatrix}$$

$$=\begin{pmatrix}2+1 & -3+8\\ 5+7 & 0+6\end{pmatrix}$$

$$=\begin{pmatrix}3 & 5\\ 12 & 6\end{pmatrix}$$

Subtraction of matrices

To subtract two or more matrices, we subtract elements that appear in corresponding positions. Therefore the difference of two 2×2 matrices $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ *and* $B=\begin{pmatrix}x & y\\ u & z\end{pmatrix}$ is given by

$$A-B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}-\begin{pmatrix}x & y\\ u & z\end{pmatrix}=\begin{pmatrix}a-x & b-y\\ c-u & d-z\end{pmatrix}$$

Example16.1. 3

Given $A=\begin{pmatrix}2 & -3\\ 5 & 0\end{pmatrix}$ *and* $B=\begin{pmatrix}1 & 8\\ 7 & 6\end{pmatrix}$. Find $A+B$

Solution

$$A-B=\begin{pmatrix}2 & -3\\ 5 & 0\end{pmatrix}-\begin{pmatrix}1 & 8\\ 7 & 6\end{pmatrix}$$

$$=\begin{pmatrix}2-1 & -3-8\\ 5-7 & 0-6\end{pmatrix}$$

$$=\begin{pmatrix}1 & -11\\ -2 & -6\end{pmatrix}$$

Scalar multiplication

For any real number k and a matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, the scalar multiplication is defined as $kA=k\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}ka & kb\\ kc & kd\end{pmatrix}$. For example $2\begin{pmatrix}1 & -11\\ -2 & -6\end{pmatrix}=\begin{pmatrix}2 & -22\\ -4 & -12\end{pmatrix}$

Example 16.1.4

If $A=\begin{pmatrix}-4 & 3\\ 2 & -5\end{pmatrix}$ and $B=\begin{pmatrix}2 & -3\\ 7 & -6\end{pmatrix}$. Find $3A+2B$

Solution

$$3A+2B=3\begin{pmatrix}-4 & 3\\ 2 & -5\end{pmatrix}+2\begin{pmatrix}2 & -3\\ 7 & -6\end{pmatrix}$$

$$= \begin{pmatrix} -12 & 9 \\ 6 & -15 \end{pmatrix} + \begin{pmatrix} 4 & -6 \\ 14 & -12 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 3 \\ -6 & -27 \end{pmatrix}$$

Multiplication of two matrices
To multiply two matrices A and B, we use pertain called row-by-column pertain. We multiply the rows of A times the columns of B in a pair wise entry fashion, adding the results.

Let's consider two matrices$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ u & z \end{pmatrix}$, then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ u & z \end{pmatrix} = \begin{pmatrix} ax + bu & ay + bz \\ cx + du & cy + dz \end{pmatrix}$$

Example16.1. 5

If $A = \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ -1 & 7 \end{pmatrix}$. Find

a. AB
b. BA

Solutions

a. $AB = \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 7 \end{pmatrix}$

$$= \begin{pmatrix} (-2)(3) + (1)(-1) & (-2)(-2) + (1)(7) \\ (4)(3) + (5)(-1) & 4(-2) + (5)7 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 11 \\ 7 & 27 \end{pmatrix}$$

b. $BA = \begin{pmatrix} 3 & -2 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}$

$$= \begin{pmatrix} (3)(-2) + (-2)(4) & (3)(1) + (-2)(5) \\ (-1)(-2) + (7)(4) & (-1)(1) + (7)(7) \end{pmatrix}$$

$$= \begin{pmatrix} -14 & -7 \\ 30 & 34 \end{pmatrix}$$

Example16.1. 6
This example shows that **matrix multiplication is not a commutative operation**.
But the multiplications of matrices abide to the associative law and distributive law.

$$A(BC) = (AB)C$$
$$A(B + C) = AB + AC$$

Task
Use the matrix $A = \begin{pmatrix} 7 & -4 \\ 6 & 9 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 8 \\ -5 & 7 \end{pmatrix}$ *and* $C = \begin{pmatrix} 8 & -2 \\ 4 & -7 \end{pmatrix}$. Show that

a. $A(BC) = (AB)C$
b. $A(B + C) = AB + AC$

Note: Any two matrices can be multiplied if the number of columns of the first matrix is equal to the number of rows of the second matrix.

Determinant of a square matrix

Every square matrix is associated with a real number known as a determinant. The determinant of any matrix A is denoted by $\det(A\,)$ or $|A|$.
The determinant of a matrix is a real number (scalar) and not a matrix (vector).

Determinant of a 2×2 matrix

Let A be a 2×2 matrix, $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then the determinant of a matrix A is given by

$$\det(A) = |A| = ad - bc$$

Example 16.1. 7

If a matrix $A = \begin{pmatrix} 7 & -4 \\ 6 & 9 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 8 \\ -5 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 8 & -2 \\ 4 & -7 \end{pmatrix}$. Find

a. $\det\,(A)$
b. $|B|$
c. $\begin{vmatrix} 8 & -2 \\ 4 & -7 \end{vmatrix}$

Solutions

a. $\det(A) = ad - bc = (7)(9) - (6)(-4) = 63 + 24 = 87$
b. $|B| = ad - bc = (-3)(7) - (-5)(8) = -21 + 40 = 19$
c. $\begin{vmatrix} 8 & -2 \\ 4 & -7 \end{vmatrix} = ad - bc = (8)(-7) - (4)(-2) = -56 + 8 = -48$

If the determinant of a matrix is zero, then the square is called a **singular matrix**

Determinant of a 3×3 matrix

A 3×3 matrix is a matrix with 3 rows and 3 columns.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

To find the determinant of this matrix, we introduce another two concepts that will help in the evaluating the determinant, that is the minors and cofactors.

Minors

Let's consider a matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. For an element a, if we delete the row and a column that contains the entry a, we are left with a 2×2 matrix $\begin{pmatrix} e & f \\ h & i \end{pmatrix}$ whose determinant is $ei - hf$. This determinant is called the minor of entry a.

In general, $M_a = \begin{vmatrix} e & f \\ h & i \end{vmatrix}$ $\quad M_b = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$ $\quad M_i = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ and so on.

Cofactors

The cofactor of any entry x is equal to a minor of that entry if $(i + j)$ is an even integer and equal to a negative of the minor if $(i + j)$ is odd integer where $\boldsymbol{i}$is the position row number of the entry x and j is the position column number of the entry x.

Cofactor of $\boldsymbol{x} = (-\boldsymbol{1})^{\boldsymbol{i}+\boldsymbol{j}}$ M$_x$

Example 16.1.7

Given then matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Find

a. The minor of entry 1
b. The cofactor of entry 1
c. The cofactor of entry 6

Solutions

a. $M_1 = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (5 \times 9) - (6 \times 8) = 45 - 48 = -3$

b. Cofactor of $1 = (-1)^{i+j}\ M_1$

$$= (-1)^{1+1}(-3) = (1)(-3) = -3$$

c. M$_6$= $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 1 \times 8 - 7 \times 2 = -6$

Then

Cofactor of $6 = (-1)^{i+j}\ M_6 = (-1)^{2+3}(-6) = (-1)(-6) = 6$

We can find the cofactor of any 3×3 matrix A just by finding all the cofactors of entries in A, then replace each entry by its cofactor.

Note: for any 3×3 matrix, a cofactor is simply a signed minor according to the pertain

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Now for any 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the determinant is defined as

$\det(A) = a \times$ Cofactor of a $\quad + \quad b \times$ Cofactor of b $\quad + \quad c \times$Cofactor of c

$$\boldsymbol{detA} = (\boldsymbol{a} \times \boldsymbol{M_a}) - (\boldsymbol{b} \times \boldsymbol{M_b}) + (\boldsymbol{c} \times \boldsymbol{M_c})$$

Example16.1. 8

Find the determinant of a 3×3 matrix $A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 0 \end{pmatrix}$

Solution

$$|\boldsymbol{A}| = 1\begin{vmatrix} -1 & 4 \\ 1 & 0 \end{vmatrix} - 4\begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(0 - 4) - 4(0 - 8) + 2(3 + 2)$$
$$= -4 + 32 + 10 = 38$$

Example16.1. 9

Find the determinant of a 3×3 matrix $A = \begin{pmatrix} 1 & 3 & -6 \\ 3 & -1 & 4 \\ 2 & 1 & 0 \end{pmatrix}$

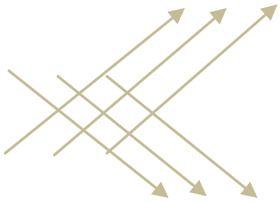
Solution

$$|\boldsymbol{A}| = \boldsymbol{1}\begin{vmatrix} -1 & 4 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix} + (-6)\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(0-4) - 3(0-8) - 6(3+2)$$
$$= -4 + 24 - 30$$
$$= -10.$$

There is another way of finding the determinant of a 3×3 matrix.

For any 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then



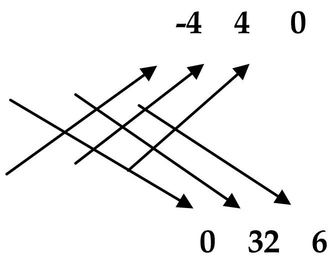
$$\det(A) = \left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

1. Multiply the number entries in each arrow.
2. In each arrow pointing up, multiply each answer with (-1)
3. Then add all the answers in all arrows

Example 16.1.10

Find the determinant of a 3×3 matrix $A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 0 \end{pmatrix}$

Solution



$$\det(A) = \left|\begin{matrix} 1 & 4 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 0 \end{matrix}\right| \begin{matrix} 1 & 4 \\ 3 & -1 \\ 2 & 1 \end{matrix}$$

$$\det(A) = -4 + 4 + 0 + 0 + 32 + 6$$
$$= 38$$

Multiplicative inverse of a square matrix

A non-singular square matrix A has a multiplicative inverse

$$BA = AB = I$$

Where I is the identity matrix

$$IA = AI = A$$

The inverse of a matrix A is denoted by A^{-1}

Hence $AA^{-1} = A^{-1}A = I$

A singular matrix has no inverse

Inverse of a non-singular 2×2 matrix

For any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ such that

$$AA^{-1} = I$$

$$AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Our task is to find the matrix $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ in the equation, in other words making A^{-1} a subject.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Two matrices are **equal** if and only if all elements in corresponding positions are equal.

$$\begin{aligned} aw + by &= 1 \\ cw + dy &= 0 \\ ax + bz &= 0 \\ cx + dz &= 1 \end{aligned}$$

We solve for the variables w, x, y, z taking a, b, c, d as constants

We solve for w

$$\begin{aligned} aw + by &= 1)d \\ cw + dy &= 0)b \end{aligned}$$

$$\begin{aligned} adw + bdy &= d \\ -(\ cbw + bdy &= 0 \end{aligned}$$

$$adw - cbw = d$$
$$w = \frac{d}{ad-bc} \quad \ldots \text{(i)}$$

We solve for y

$$\begin{aligned} aw + by &= 1)c \\ cw + dy &= 0)a \end{aligned}$$

$$\begin{aligned} acw + bcy &= c \\ -(\ caw + bay &= 0 \end{aligned}$$

$$bcy - ady = c$$

$y = \frac{c}{ad-bc}$ ... (ii)

We solve for x

$$ax + bz = 0$$
$$cx + dz = 1$$

$$ax + bz = 0)d$$
$$cx + dz = 1)b$$

$$adx + bdz = 0$$
$$\underline{-(cbx + dbz = b}$$
$$bdx - cbx = -b$$

$x = \frac{-b}{ad-bc}$ ...(iii)

We solve for z

$$ax + bz = 0$$
$$cx + dz = 1$$

$$ax + bz = 0)c$$
$$cx + dz = 1)a$$

$$acx + bcz = 0$$
$$\underline{-(cax + daz = a}$$
$$bcz - adx = -a$$

$z = \frac{a}{ad-bc}$ ...(iv)

Now replace these values in the matrix $A^{-1} = \begin{pmatrix} x & y \\ u & z \end{pmatrix}$ we have

$$A^{-1} = \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We know that $ad - bc$ is the determinant of A

$$\boldsymbol{A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Example 16.1.11

Find the inverse for a matrix , $B = \begin{pmatrix} -3 & 8 \\ -5 & 7 \end{pmatrix}$

Solution

$$B^{-1} = \frac{1}{|B|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$|B| = ad - bc = (-3)(7) - (-5)(8) = -21 + 40 = 19$$

$$B^{-1} = \frac{1}{19}\begin{pmatrix} 7 & -8 \\ 5 & -3 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 7/19 & -8/19 \\ 5/19 & -3/19 \end{pmatrix}$$

Inverse of 3 × 3 matrix

To find the inverse of a 3×3 matrix, we introduce another concept known as ad-joint of a matrix.

Ad-joint of a 3 × 3 matrix

Let A be any 3×3 matrix and a matrix B a cofactor of A. then an ad-joint of a matrix A is given by

$$adj\ A = B^T$$

Example 16.1.12

Find the ad-joint of a matrix $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ whose determinant is 45.

Solution

$A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$, we find a matrix B which is the cofactor of A

$$B = \begin{pmatrix} +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \\ +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \end{pmatrix}$$

$$B = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

$$adj\ A = B^T = \begin{pmatrix} -24 & 20 & 15 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Now observe this, when we multiply a matrix A with its ad-joint

$$A\,.\,adjA = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}\begin{pmatrix} -24 & 20 & 15 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} -48+18+75 & 40-15-25 & 26+24-50 \\ -96+6+90 & 80-5-30 & 52+8-60 \\ -24+24+0 & 20-20+0 & 13+32+0 \end{pmatrix}$$

$$= \begin{pmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{pmatrix}$$

$$= 45 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we see that,

$$A \,.\, adjA = |A|\; I$$

$$\frac{A.\, adjA}{|A|} = I \ldots (i)$$

But $AA^{-1} = I$, then substitute in equation (i) we have

$$\frac{A \,.\, adjA}{|A|} = AA^{-1}$$

Multiply on both sides by A^{-1}

$$\frac{A^{-1}A \,.\, adjA}{|A|} = A^{-1}AA^{-1}$$

$$\frac{I.\, adjA}{|A|} = I.\, A^{-1}$$

$$A^{-1} = \frac{1}{|A|}.\, adjA$$

For any 3×3 matrix A, the inverse is given by

$$\boldsymbol{A^{-1} = \frac{1}{|A|}.\, adjA}$$

Example 16.1.13

Find the inverse of a matrix $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ whose determinant is 45.

Solution

$A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$, we find a matrix B which is the cofactor of A

$$B = \begin{pmatrix} +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \\ +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \end{pmatrix}$$

$$B = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

$$adj\ A = B^T = \begin{pmatrix} -24 & 20 & 15 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Then

$$A^{-1} = \frac{1}{|A|}.\,adjA$$

$$A^{-1} = \frac{1}{45}\begin{pmatrix} -24 & 20 & 15 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix} = \begin{pmatrix} {}^{24}/_{45} & {}^{4}/_{9} & {}^{13}/_{45} \\ {}^{2}/_{15} & -{}^{1}/_{9} & {}^{8}/_{45} \\ {}^{1}/_{3} & -{}^{1}/_{9} & -{}^{2}/_{9} \end{pmatrix}$$

Note: if the determinant is not given, we have to use the approach of a determinant of 3×3 matrix to find it.

16.2 Solving systems of equations using matrix methods

Solving simultaneous equation using Cramer's Rule

Let the system of two equation with variables x and y be defined as

$$ax + by = p$$
$$cx + dy = q$$

This can be represented in a matrix form as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Now

$$ax + by = p)d$$
$$cx + dy = q)b$$

$$adx + bdy = pd$$
$$\underline{-(cbx + bdy = bq}$$
$$adx - cbx = pd - bq$$

$$x = \frac{pd - bq}{ad - cb}$$

$$ax + by = p)c$$
$$cx + dy = q)a$$

$$acx + bcy = pc$$
$$\underline{-(cax + bay = aq}$$
$$bcy - abyx = pc - aq$$

$$y = \frac{pc - aq}{ad - cb}$$

Now, from the values of x and y we observe that

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{Dx}{D}$$

$$y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{Dy}{D}$$

Example 16.2.1
Solve

$$2x + 5y = 9$$
$$3x - 2y = 4$$

Solution

$$2x + 5y = 9$$
$$3x - 2y = 4$$

$$\begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} 9 & 5 \\ 4 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix}} = \frac{-38}{-19} = 2$$

$$y = \frac{\begin{vmatrix} 2 & 9 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix}} = \frac{-19}{-19} = 1$$

Hence $x = 2,\ y = 1$
Example 16.2.2

Solve

$$y = -2x - 2$$
$$4x - 5y = 17$$

Solution

$$y = -2x - 2$$
$$4x - 5y = 17$$

This can be written as

$$2x + y = -2$$
$$4x - 5y = 17$$

$$\begin{pmatrix} 2 & 1 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 17 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} -2 & 1 \\ 17 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 4 & -5 \end{vmatrix}} = \frac{-7}{-14} = \frac{1}{2}$$

$$y = \frac{\begin{vmatrix} 2 & -2 \\ 4 & 17 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 4 & -5 \end{vmatrix}} = \frac{42}{-14} = -3$$

Hence $x = \frac{1}{2},\ y = -3$

Solving three systems of equations using Cramer's Rule

$$ax + by + cz = p$$
$$dx + ey + fz = q$$
$$gx + hy + iz = r$$

Where x, y, z are variables
This can be written as

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Then

$$x = \frac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{Dx}{D} y = \frac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{Dy}{D} z = \frac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{Dz}{D}$$

$$x = \frac{Dx}{D}, \quad y = \frac{Dy}{D}, \quad z = \frac{Dz}{D} \quad for \quad D \neq 0$$

Example 16.2.3
Use the Cramer's rule to solve the system

$$x - 2y + z = -4$$
$$2x + y - z = 5$$
$$3x + 2y + 4z = 3$$

Solution

$$x - 2y + z = -4$$
$$2x + y - z = 5 \qquad 3x + 2y + 4z = 3$$

This can be written as $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29 \qquad Dx = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29$$

$$Dy = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix} = 58 \qquad Dz = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{vmatrix} = -29$$

$$x = \frac{Dx}{D} = \frac{29}{29} = 1 \ , \qquad y = \frac{Dy}{D} = \frac{58}{29} = 2 \ , \qquad z = \frac{Dz}{D} = \frac{-29}{29} = -1$$

Hence $x = 1$, $y = 2$, $z = -1$

Solving system of equations using inverse matrix method
Let's consider the system of equations

$$x - y - z = -6$$
$$2x + y + z = 0$$
$$3x - 5y + 8z = 13$$

This can be written as $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 13 \end{pmatrix}$

Now, let $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{pmatrix}$ $\quad B = \begin{pmatrix} -6 \\ 0 \\ 13 \end{pmatrix}$

Then $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B$

To find the value of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we multiply on both sides by A^{-1}

$$A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$$

$$I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$$

Example 16.2.4
Use the inverse matrix method to solve the system

$$x - y - z = -6$$
$$2x + y + z = 0$$
$$3x - 5y + 8z = 13$$

Solution

$$x - y - z = -6$$
$$2x + y + z = 0$$
$$3x - 5y + 8z = 13$$

This can be written as $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 13 \end{pmatrix}$

Now, let $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} -6 \\ 0 \\ 13 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$$

We go step by step to find A^{-1}

$$\begin{array}{ccccccc} & & & 3 & 5 & 16 & \\ & \left|1\right. & -1 & -1 & \left|1\right. & -1 & \\ |A| = & \left|2\right. & 1 & 1 & \left|2\right. & 1 & \\ & \left|3\right. & -5 & 8 & \left|3\right. & -5 & \\ & & & 8 & -3 & 10 & \end{array}$$

$$|A| = 8 - 3 + 10 + 3 + 5 + 16$$
$$|A| = 39$$

Cofactor matrix C

$$C = \begin{vmatrix} +\begin{vmatrix} 1 & 1 \\ -5 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 8 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix} \\ -\begin{vmatrix} -1 & -1 \\ -5 & 8 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 3 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} \\ +\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{vmatrix}$$

$$C = \begin{pmatrix} 13 & -13 & -13 \\ 13 & 11 & 2 \\ 0 & -3 & 3 \end{pmatrix}$$

We find the ad-joint

$$adjA = C^T = \begin{pmatrix} 13 & 13 & 0 \\ -13 & 11 & -3 \\ -13 & 2 & 3 \end{pmatrix}$$

We find the inverse

$$A^{-1} = \frac{1}{|A|}.adjA$$

$$A^{-1} = \frac{1}{39}\begin{pmatrix} 13 & 13 & 0 \\ -13 & 11 & -3 \\ -13 & 2 & 3 \end{pmatrix}$$

Now we solve for the variables

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{39}\begin{pmatrix} 13 & 13 & 0 \\ -13 & 11 & -3 \\ -13 & 2 & 3 \end{pmatrix}\begin{pmatrix} -6 \\ 0 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{39}\begin{pmatrix} -78+0+0 \\ 78+0-39 \\ 78+0+39 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{39}\begin{pmatrix} -78 \\ 39 \\ 117 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

Hence $x = -2\ , y = 39\ \ and\ \ z = 3$

REVIEW EXERCISE

1. Given that $A = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 & 2 \\ 3 & 1 & 3 \end{pmatrix}$ evaluate
   a. $3A$ $\quad b.\ \ 2B$ $\quad c.\ \ 3A + 2B$ $\quad d.\ \ 3A - 2B$
2. Evaluate the matrix product
   a. $(2 \quad 3 \quad 1)\begin{pmatrix} 3 & 2 \\ 4 & 7 \\ 1 & 6 \end{pmatrix}$ $\quad b.\ \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 7 \\ 5 & -4 \end{pmatrix}$ $\quad c.\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 0 & 7 \end{pmatrix}$
3. Verify that if $M = \begin{pmatrix} -5 & 10 & 8 \\ 4 & -7 & -6 \\ -3 & 6 & 5 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 0 & 5 \end{pmatrix}$, then $MN = NM = I$ where $I\ is\ 3 \times 3$ matrix
4. Evaluate
   a. $\begin{vmatrix} 2 & -5 \\ 4 & 9 \end{vmatrix}$ $\quad b.\ \begin{vmatrix} 6 & -1 \\ 3 & -4 \end{vmatrix}$ $\quad c.\ \begin{vmatrix} 5 & 3 \\ -1 & 0 \end{vmatrix}$
5. Find the determinant
   a. $|\begin{pmatrix} 2 & -1 & 0 \\ 3 & 2 & 0 \\ 4 & 7 & 3 \end{pmatrix}$ $\quad b.\ \begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & 0 \\ 1 & -2 & 9 \end{pmatrix}$
6. Given a matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & -2 \\ 10 & 3 & 1 \end{pmatrix}$. Show that A is a singular matrix.
7. Let $= \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix}$ $\ B = \begin{pmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{pmatrix}$ $\ C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ $\ D = \begin{pmatrix} 0 & -3 \\ -2 & 1 \end{pmatrix}$. find where possible
   a. $A + 2D$ $\quad b.\ \ 3D - 2A$ $\quad c.\ \ AB$ $\quad d.\ \ A^3$ $\quad e.\ \ D + BC$ $\quad f.\ \ B^TB$ $\quad g.\ \ B - C^T$ $\quad h.\ \ B^TC^T - (CB)^T$
8. Find the matrix of the cofactors of matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$
9. Find the inverse of the matrix
   a. $A = \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix}$ $\quad b.\ \begin{pmatrix} 2 & 7 \\ -1 & 5 \end{pmatrix}$ $\quad c.\ \ D = \begin{pmatrix} 4 & 15 \\ 0 & 5 \end{pmatrix}$ $\quad d.\ \ B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$

10. Given $B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$, find B^{-1}

11. Use Cramer's rule to solve the system of equations

a. $x + y + 3z = 5$
$2x - y + 4z = 11$
$-y + z = 3$

$b.$ $2x + 3y - 5z = 4$
$2x + y + z = 2$
$x - 3y + z = 1$

12. Find the inverse of the matrix $A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$ and use the inverse to solve the system of equations$3x - y + 2z = 4$, $x + y + z = 2$, $2x + 2y - z = 3$

17: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

17.1.1 Exponent and exponential function

Definition 17.1.1

If n is a positive integer and b is any real number, then

$$b^n = b.b\dots b \quad n - times$$

Properties of exponents

if a and b are real numbers and m and n are positive integers, then

6. $b^m.b^n = b^{m+n}$ **product of two powers**
7. $(b^m)^n = b^{mn}$ **power of a power**
8. $(ab)^m = a^m.b^m$ **power of a product**
9. $(\frac{a}{b})^m = \frac{a^m}{b^m}$ $\quad if \ \ b \neq 0$ **power of a quotient**
10. $\frac{a^m}{b^n} = (\frac{a}{b})^{m-n}$ when $m > n, \quad b \neq 0$ **quotient of two powers**

Example 17.1.2

Simplify the following

iv. $(3x^2y)(4x^3y^2) = 3.4.x^{2+3}.y^{1+2}$
$= 12x^5y^3$

v. $(-2y^3)^5 = -2^5y^{15}$
$= -32y^{15}$

vi. $(\frac{a^2}{b^4})^7 = \frac{(a^2)^7}{(b^4)^7}$

$= \frac{a^{14}}{b^{28}}.$

Zero and Negative Integers as an Exponent

Now we can extend the concept of exponent to include the use of zero and negative integers. First let's consider zero as an exponent.

Definition 17.2.3

If b is non zero real number, then $b^0 = 1$

According to the definition, the following statements are true;

$5^0 = 1 \quad (xy)^0 = 1 \quad$ if $x \neq 0$ and $y \neq 0$

Definition 17.2.4

If n is a positive integer and b is non zero real number, then

$$b^{-n} = \frac{1}{b^n}$$

The following statements are true;

$$x^{-5} = \frac{1}{x^5} 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

Example 17.2.5

Evaluate the following numerical expressions.

1. $(2^{-1}.3^2)^{-1}$ 2. $\left(\frac{2^{-3}}{3^{-2}}\right)^{-2}$

Solutions

3. $(2^{-1}.3^2)^{-1} = (2^{-1})^{-1}.(3^2)^{-1}$ power of a product
$= 2^1.3^{-2}$ power of a power
$= 2.(\frac{1}{3^2})$
$= \frac{2}{9}$

4. $(\frac{2^{-3}}{3^{-2}})^{-2} = \frac{(2^{-3})^{-2}}{(3^{-2})^{-2}}$ power of a quotient

Exponential equations
If $b > 0, b \neq 1 \quad m, n \in R$ then $b^n = b^m$ if and only if $n = m$

Example17.2.8
Solve the following equations
a. $2^x = 32$ b. $9^x = 243$ c. $2^{3x} = 64^{-1}$ d. $(2^3)^{2x}.(2^2)^{2x-1} = 2^4$

Solutions

a. $2^x = 32$

$$2^x = 2^5$$
$$x = 5$$

b. $9^x = 243$

$$3^{2x} = 3^5$$
$$2x = 5$$
$$x = \frac{5}{2}$$

c. $2^{3x} = 64^{-1}$

$$2^{3x} = 2^{6(-1)}$$
$$2^{3x} = 2^{-6}$$
$$3x = -6$$
$$x = -2$$

d. $(2^3)^{2x}.(2^2)^{2x-1} = 2^4$

$$2^{6x}.2^{4x-2} = 2^4$$
$$6x + 4x - 2 = 4$$
$$6x + 4x = 6$$
$$x = \frac{3}{5}$$

Exponential functions

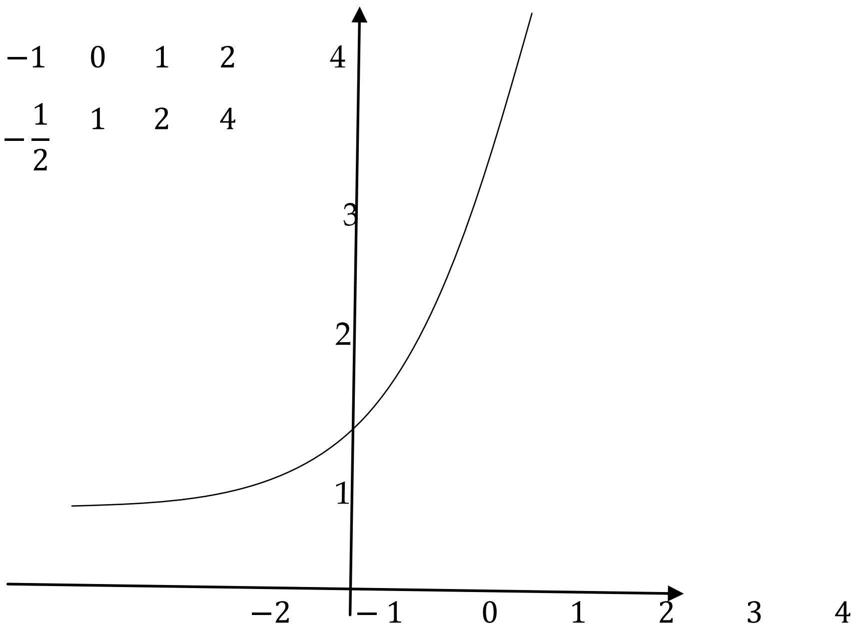
If $b > 0\,, b \neq 1$ then the function defined by $f(x) = b^x$ where b is a real number is called an exponential function with a base b.

The function $f(x) = 1^x = 1$ is a constant function and not an exponential function.

Graphs of exponential function

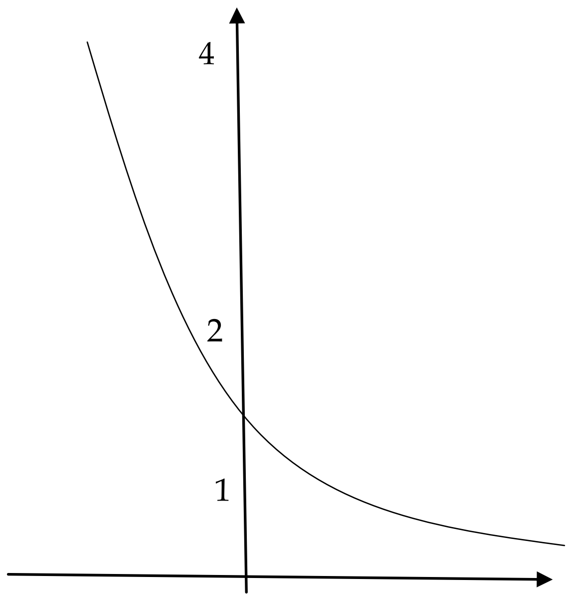
Consider the exponential function $f(x) = 2^x$

x	-2	-1	0	1	2
$f(x)$	$-\frac{1}{4}$	$-\frac{1}{2}$	1	2	4



Consider the exponential function $f(x) = (\frac{1}{2})^x$

x	-2	-1	0	1	2
$f(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



Properties of an exponential function

1. If $b > 1$, then the graph $f(x) = b^x$ goes up to the right and the function is called increasing function.
2. If $0 < b < 1$, then the graph $f(x) = b^x$ goes down to the right and the function is called decreasing function.
3. For $x = 0,\ b > 0$, then $b^x = b^0 = 1$then the graph $f(x) = b^x$ passes through the point (0,1)
4. The domain of the graph is $D = (-\infty, \infty)$ and the range is $R = (0, \infty)$
5. The graph of $f(x) = 2^x$ and $f(x) = (\frac{1}{2})^x$ both have $x - axis\ \ y = 0$ as horizontal asymptote.

Translations of exponential graphs

1. The graph of $f(x) = 2^x + 3$ is the graph of $f(x) = 2^x$ moves up three units
2. The graph of $f(x) = 2^{x-4}$ is the graph of $f(x) = 2^x$ moves to the right four units
3. The graph of $f(x) = \left(\frac{1}{2}\right)^x + 5$ is the graph of $f(x) = \left(\frac{1}{2}\right)^x$ moves up five units
4. The graph of $f(x) = \left(\frac{1}{2}\right)^{x+3} + 5$ is the graph of $f(x) = \left(\frac{1}{2}\right)^x$ moves to the left three units
5. The graph of $f(x) = -2^x$ is the graph of $f(x) = 2^x$ reflected across the x-axis
6. The graph of $f(x) = 2^x + 2^{-x}$ is symmetric with respect to the y- axis because $f(-x) = 2^{(-x)} + 2^{-(-x)} = f(x) = 2^x + 2^{-x}$

Applications of exponential functions

Many real-world situations that exhibit growth or decay can be represented by equations that describe exponential functions.

Example17.2.9

Suppose a farmer predict an animal inflation rate of 5% per year for the next 10 years. What will be the cost of an animal costing $K8$ today after;

a. One year.
b. two years,
c. n years.

d. After 10years

Solutions
This can be modeled as the exponential function
a. After one year
$$f(1) = K8(105\%)^1 = 8(1.05) = K8.40$$
b. After two years
$$f(2) = K8(105\%)^2 = 8(1.1025) = K8.82$$
c. After n years
$$f(n) = K8(105\%)^n = 8(1.05)^n = 8(1 + 0.5)^n$$
d. After 10 years
$$f(n) = K8(105\%)^n = 8(1.05)^{10} = 8(1 + 0.5)^{10} = K13.03$$

17.3 Compound interest
Compound interest provides another illustration of exponential growth. Suppose that $K500$ (called the **principal**) is invested at an interest rate of 8% **compounded annually**. The interest earned the first year is $K500\ (0.08) = K40$, and this amount is added to the original $K500$ to form a new principal of $K540$ for the second year. The interest earned during the second year is $K540\ (0.08) = K43.20$, and this amount is added to $K540$ to form a new principal of $K583.20$ for the third year. Each year a new principal is formed by reinvesting the interest earned during that year. In general, the compound interest is given by

$$\boldsymbol{A = P(1 + r)^t}$$

Where $\boldsymbol{A}$ is the compound interest $\boldsymbol{P}$ is the principle value initially invested, $\boldsymbol{r}$ is the rate of interest per year and $\boldsymbol{t}$ is time in years.

Example 17.3.2
A farmer invests the sum of $K2000$ into a bank account that earns compound interest of 5% per annum. Find the amount of money this sum accumulates to in 5- years.

Solution

$$A = P(1 + r)^t$$
$$A = 2000(1 + 0.05)^5$$
$$A = 2000(1.05)^5$$
$$A = K2{,}552.56$$
The compound interest of this investment is $K2,552.56$

Example 17.3.4
What rate of interest is needed for an investment of $K1,000$ to yield $K1,210$ in 2 years if the interest is compounded annually.

Solution
$$A = P(1 + r)^t$$
$$1210 = 1000(1 + r)^2$$

$$\frac{1210}{1000} = (1 + r)^2$$

$$1.21 = (1+r)^2$$
$$(1.21)^{\frac{1}{2}} = 1+r$$

$$1.1 = 1+r$$
$$r = 0.1 = 10\%$$

If n represents the number of compounding periods in a year, then the formula becomes
$$\boldsymbol{A = P\left(1+\frac{r}{n}\right)^{nt}}$$
Example 17.3.5
$K800$ invested for 5years at 4% interest per annum compounding semi-annually produces

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = K800\left(1+\frac{0.04}{2}\right)^{2(5)}$$
$$A = K800(1.02)^{10}$$
$K1,000$ invested for 10 −years at 6% interest per annum compounding quarterly produces

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = K1000\left(1+\frac{0.06}{4}\right)^{4(10)}$$
$$A = K1000(1.015)^{40}$$
$K20,000$ invested for 10-years at 6% interest per annum compounding quarterly produces

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = K20{,}000\left(1+\frac{0.06}{4}\right)^{4(10)}$$
$$A = K20{,}000(1.015)^{40}$$

The number e and natural exponential function
An interesting situation occurs if we consider the compound interest formula for

$$P = K1\,, \quad r = 100\% \quad \text{and} \quad t = 1\text{year}$$
$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 1\left(1+\frac{1}{n}\right)^{n}$$

$$A = \left(1+\frac{1}{n}\right)^{n}.$$

Now let's observe the behavior of the function $A = \left(1+\frac{1}{n}\right)^n$ as n increases.

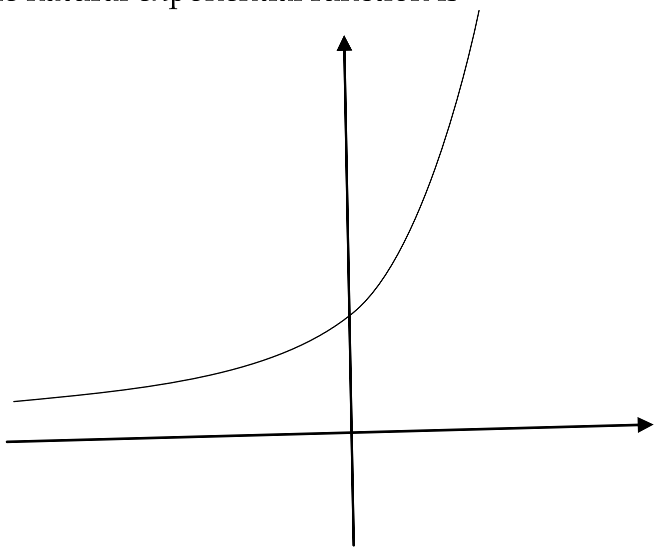
n	A
1	2.000
10	2.594
100	2.705
1000	2.717
10,000	2.718
10, 0000	2.718

As $n \to \infty, A \to 2.72$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.72 = e$$

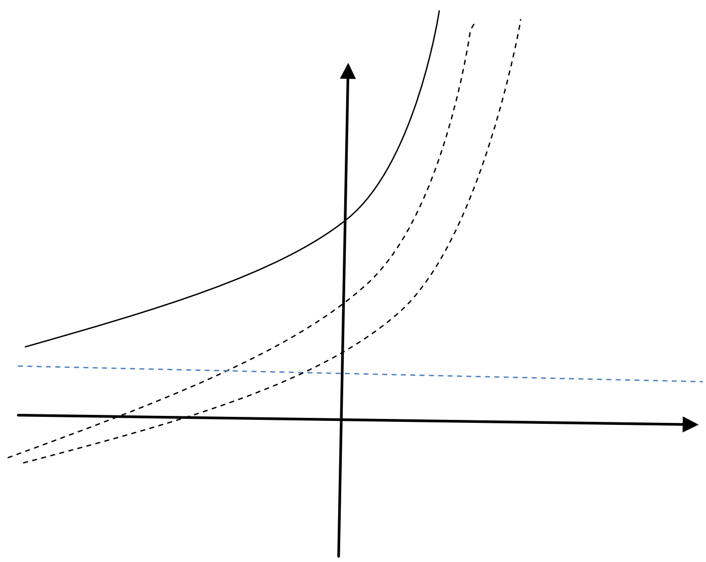
The number $\boldsymbol{e}$ is called natural base and $e = 2.72$
The function $f(x) = e^x$ is the natural exponential function
The graph of the natural exponential function is

2.72



Example 17.3.6
Sketch the graph of $f(x) = e^{x-1} + 2$

Solution



Exponential decay and half-life

Exponential decay is the approximated value in depreciation. Exponential decay is given by the formula

$$V = V0\ (1 - r)^t$$

Where $V_0$the initial amount invested, r is is the rate of depreciation and t is time in years

Example17.3. 7

Suppose that it is estimated that the value of a car depreciate 15% per year for the first 5years. How will a car that is costing $K95,000$ cost after 1 year and after 2 years.

Solutions

After 1 year

$V = V_0\,(1 - r)^t$

$$V = K95,000(1 - 0.15)^1$$

$$V = K95,000(0.85)$$

$$V = K80,750$$

After 2 years

$$V = V0\ (1 - r)^t$$

$$V = K95,000(1 - 0.15)^2$$

$$V = K95,000(0.85)^2$$

$$V = K95,000(0.7225)$$

$$V = K68,637.50$$

The rate of decay can be described exponentially and is based on the half-life of a substance. The **half-life** of a radio-active substance is the amount of time that it takes for one-half of an initial amount of the substance to disappear as the result of radio-active decay. The half-life of a substance is given by the formula

$$\boldsymbol{Q \ = \ Q0\left(\frac{1}{2}\right)^{t/h}}$$

Where Q_0 is the initial amount, h half-life time, t is the period of time and Q is the amount of substance remaining

Example 17.3.8

Barium-140 has a half-life of 13 days. If there are 500 milligrams of barium initially, how many milligrams remain after 26 days? After 100 days?

Solution

Using $Q_o = 500$ and $h = 13$, the half-life formula becomes

$$Q = Q_0\left(\frac{1}{2}\right)^{t/h}$$

$$Q = 500\left(\frac{1}{2}\right)^{t/13}$$

If $t = 26$ days

$$Q = 500\left(\frac{1}{2}\right)^{26/13}$$

$$Q = 500\left(\frac{1}{2}\right)^{2}$$

$$Q = 125 \text{ milligrams}$$

If $t = 100$ days

$$Q = 500\left(\frac{1}{2}\right)^{100/13}$$

$$Q = 500(0.5)^{100/13}$$

$$Q = 2.4 \text{ Milligrams}$$

Back to compound interest

If the number of compounding periods in a year is increased indefinitely, we arrive at the concept of **compounding continuously**. Mathematically, this can be accomplished by applying the limit concept to the expression

$$\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{nt}}$$

When $n \to \infty$
$\boldsymbol{A = P(e)^{nt}}$ when interest is compounded continually

Example 17.3.9

Find the accumulation of K800 invested for 5 years at 4% interest compounded continually.

Solution

$$A = P(e)^{nt}$$

$$A = 800(e)^{0.04\times 5}$$

$$A = 800(2.72)^{0.2}$$

$$A = 977.24$$

Law of exponential growth

The law of exponential growth is given by the function

$$Q(t) = Q0e^{kt}$$

Where$Q(t)$ is the substance amount at any time t, Q_0 *is* the initial substance amount, k is the growth constant.

Example 17.3.10

Suppose that in a certain culture, the equation $Q(t) = 15000e^{0.3t}$expresses the number of bacteria present as a function of the time *t*, where *t* is expressed in hours. Find

(a) the initial number of bacteria, and (b) the number of bacteria after 3 hours.

Solution

a. The initial number of bacteria is produced when $t = 0$.

$$Q(t) = 15000e^{0.3t}$$

$$Q(t) = 15000e^{0.3\times 0}$$

$$Q(t) = 15000e^{0}$$

$$Q(t) = 15000 \qquad e^0 = 1$$

b. $Q(t) = 15000e^{0.3t}$

$$Q(t) = 15000e^{0.3\times 3}$$

$$Q(t) = 15000e^{0.9}$$

$$Q(t) = 36,894 \qquad e^0 = 1$$

Example 17.3.11

Suppose the number of bacteria present in a certain culture after t minutes is given by the equation $Q(t) = Q_0e^{0.05t}$, where Q_0 represents the initial number of bacteria. If 5000 bacteria are present after 20 minutes, how many bacteria were present initially?

Solution

$$Q(t) = Q_0e^{0.05t}$$

$$5000 = Q_0e^{0.05\times 20}$$

$$5000 = Q_0e^{1}$$

$$\frac{5000}{e} = Q_0$$

$$Q_0 = 1839$$

Approximately 1839 bacteria's where present initially.

17.4 LOGARITHMS

Definition 17.4.1

If r is any positive real number, then the unique exponent t such that $b^t = r$ is called the **logarithm of *r* with base *b*** and is denoted by $log_b\ r$
Then $\log_b r = t$ is equivalent to $b^t = r$

$$\log_2 8 = 3 \text{ is equivalent to } 2^3 = 8$$
$$\log_3 81 = 4 \text{ is equivalent to } 3^4 = 81$$
$$\log_5 125 = 3 \text{ is equivalent to } 5^3 = 125$$
$$\left(\frac{1}{4}\right)^2 = \frac{1}{16} \text{ is equivalent to } \log_{\frac{1}{2}} {}^1\!/_{16} = 4$$

Logarithms can be determined by changing to exponential form and using the properties of exponents.

Example 17.4.2
Express the exponent in logarithmic form

a. $3^2 = 9$ $\quad$ $b.$ $10^1 = 10$

Solutions

a. $3^2 = 9$

$$\log_3 9 = 2$$

b. $10^1 = 10$

$$\log_{10} 10 = 1$$

Example 17.4.3
Evaluate

a. $\log_4 64$ $\quad$ b. $log_9\ 3$ $\quad$ c. $\log_5 5$

Solutions

a. $\log_4 64$

$$\text{let } m = \log_4 64$$
$$4^m = 64$$
$$4^m = 4^3$$
$$m = 3$$

Hence $\log_4 64 = m = 3$

b. $\log_9 3$

$$\text{let } m = \log_9 3$$
$$9^m = 3$$
$$3^{2m} = 3$$
$$2m = 1$$

$m = \frac{1}{2}$ hence $\log_9 3 = m = \frac{1}{2}$

c. $\log_5 5$

$$\text{let } m = \log_5 5$$
$$5^m = 5$$
$$5^m = 5^1$$

$$m = 1$$
$$\log_5 5 = 1$$

Example 17.4.4
Solve $\log_x 8 = 3$

Solution

$$\log_x 8 = 3$$
$$x^3 = 8$$
$$x^3 = 2^3$$
$$x = 2$$

Laws of logarithms

1. $\boldsymbol{\log_a x + \log_a y = \log_a(xy)}$

   Proof
   Let $\log_a x = m$ then $a^m = x$
   Let $\log_a y = n$ then $a^n = y$

   Now
   $$xy = a^m a^n = a^{m+n}$$

   Then
   $$\log_a x\,y = \log_a a^{m+n} = m + n$$

   $$\log_a x\,y = m + n = \log_a x + \log_a y$$

2. $\boldsymbol{\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)}$
   Proof
   Let $\log_a x = m$ then $a^m = x$
   Let $\log_a y = n$ then $a^n = y$

   Now
   $$\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}$$

   Then
   $$\log_a\left(\frac{x}{y}\right) = \log_a a^{m-n} = m - n$$

   $$\log_a\left(\frac{x}{y}\right) = m - n = \log_a x - \log_a y.$$

3. $\boldsymbol{\log_a x^n = n\log_a x}$

   Proof
   Let $\log_a x^n = z$ then $a^z = x^n$

$$a^{z\times\frac{1}{n}} = x^{n\times\frac{1}{n}}$$

$$a^{z\times\frac{1}{n}} = x$$

$$\log_a x = \frac{z}{n}$$

Multiply on both sides by n

$$\text{n}\log_a x = z$$
$$\log_a x^n = n\log_a x$$

4. For $b > 0\ \ and\ b \neq 1$

$$\mathbf{\log_b b = 1\quad and\quad \log_b 1 = 0}$$

Proof
Let $m = \log_b 1$

$$b^m = 1$$
$$b^m = 1^0$$
$$m = 0$$
$$\log_b 1 = 0$$

5. For $b > 0\ \ , b \neq 1$

$$\boldsymbol{b^{\log_b x} = x}$$

Example 17.4.5
Simplify

a. $log_a\ x + 3log_a\ y$ b. $2\log_a x - \frac{1}{2}\log_a y$

Solutions

a. $\log_a x + 3\log_a y$

$$\log_a x + \log_a y^3 = \log_a(xy^3)$$

b. $2\log_a x - \frac{1}{2}\log_a y$

$$\log_a x^2 - \log_a y^{\frac{1}{2}} = \log_a\left(\frac{x^2}{y^{1/2}}\right) = \log_a\left(\frac{x^2}{\sqrt{y}}\right) = \log_a\left(x^2\left(\sqrt{y}\right)^{-1}\right).$$

Example 17.4.6
Write the following in terms of $\log_a x$, $\log_a y$, $\log_a y$

a. $\log_a(x^2yz^2)$ b. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$ c. $\log_a\sqrt{\frac{x}{yz}}$

Solutions

a. $\log_a(x^2yz^2)$

$$\begin{aligned}\log_a(x^2yz^2) &= \log_a x^2 + \log_a y + \log_a z^2\\ &= 2\log_a x + \log_a y + 2\log_a z\end{aligned}$$

b. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

$$\log_a\left(\frac{x\sqrt{y}}{z}\right) = \log_a x + \log_a\sqrt{y} - \log_a z$$

$$= \log_a x + \log_a y^{\frac{1}{2}} - \log_a z$$

$$= \log_a x + \frac{1}{2}\log_a y - \log_a z$$

c. $\log_a \sqrt{\frac{x}{yz}} = \log_a \frac{\sqrt{x}}{\sqrt{yz}}$

$$= \log_a \sqrt{x} - \log_a \sqrt{yz}$$

$$= \log_a x^{\frac{1}{2}} - \log_a y^{\frac{1}{2}}z^{\frac{1}{2}}$$

$$= \frac{1}{2}\log_a x - \left(\frac{1}{2}\log_a y + \frac{1}{2}\log_a z\right)$$

$$= \frac{1}{2}\log_a x - \frac{1}{2}\log_a y - \frac{1}{2}\log_a z$$

Example 17.4.7

Express as a single logarithm $\log_5 x - 3$

Solution

$$\log_5 x - 3 = \log_5 x - 3\log_5 5$$

$$= \log_5 x - \log_5 5^3$$

$$= \log_5 x - \log_5 125 = \log_5\left(\frac{x}{125}\right)$$

Common base

Base 10 is known as the common base

If the logarithm $\log_a x$ is written as $\log x$. That is if the base of the logarithm is omitted then, it must be taken as the common base 10. For example $\log 2$, $\log 6$, $\log 9$

Change of base

Let's consider$\log_7 2$.

$$\text{let } m = \log_7 2$$

$$7^m = 2$$

It is not possible to find the value of m unless we change the base.

If a, b and c are positive numbers with $a \neq 1$, $b \neq 1$ then

$$\boldsymbol{\log_a c = \frac{\log_b c}{\log_b a}}$$

Proof

Let $m = \log_a c$

$$a^m = c$$

$$\log_b a^m = \log_b c$$

$$m\log_b a = \log_b c$$

$$\frac{m\log_b a}{\log_b a} = \frac{\log_b c}{\log_b a}$$

$$m = \frac{\log_b c}{\log_b a}$$

$$\log_a c = \frac{\log_b c}{\log_b a}$$

Example 17.4.8
Evaluate $\log_7 2$
Solution

$$let\ m = \log_7 2$$
$$7^m = 2$$
$$\log_{10} 7^m = \log_{10} 2$$

$$m \log_{10} 7 = \log_{10} 2$$

$$\frac{m \log_{10} 7}{\log_{10} 7} = \frac{\log_{10} 2}{\log_{10} 7}$$

$$m = \frac{\log_{10} 2}{\log_{10} 7} = \frac{\log 2}{\log 7}$$

Then $\log_7 2 = \frac{\log 2}{\log 7}$

Example 17.4.9
Change $\log_5 7$ to base 7

Solution

$$let\ m = \log_5 7$$
$$5^m = 7$$
$$\log_7 5^m = \log_7 7$$
$$m \log_7 5 = 1$$

$$\frac{m \log_7 5}{\log_7 5} = \frac{1}{\log_7 5}$$

$$m = \frac{1}{\log_7 5}$$
$$\log_5 7 = \frac{1}{\log_7 5}$$

17.5 Logarithmic functions

Definition 17.5.1
If $b > 0$ and $b \neq 1$ then the function defined by $f(x) = \log_b x$ where x is any positive real number is called the logarithmic function with the base b.

Sketching graphs of logarithmic functions
We employ the concept of inverse functions to develop the strategy of sketching the logarithmic functions.
Example 17.5.2

a. Find the inverse of $f(x) = 2^x$
b. Show that $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse of each other.

Solutions

$$f(x) = 2^x$$

a. Let $y = 2^x$

$$\log_2 y = \log_2 2^x$$

$$\log_2 y = x \log_2 2$$
$$\log_2 y = x$$

$$f^{-1}(x) = \log_2 x$$

b. To show that $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse of each other. We need to show that $(fog)(x) = (gof)(x)$

$$(fog)(x) = f[g(x)]$$
$$= f(\log_2 x)$$
$$= 2^{\log_2 x}$$
$$= x\,.$$

by property 5

$$(gof)(x) = g[f(x)]$$
$$= g(2^x)$$
$$= \log_2 2^x$$
$$= x \log_2 2$$
$$= x$$

Hence $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse of each other.

Remember that the graphs of a function and its inverse are reflections of each other through the line y $=x$. Thus the graph of a logarithmic function can be determined by reflecting the graph of its inverse exponential function through the line$y = x$.

Example 17.5.3

Sketch the graph of $f(x) = \log_2 x$

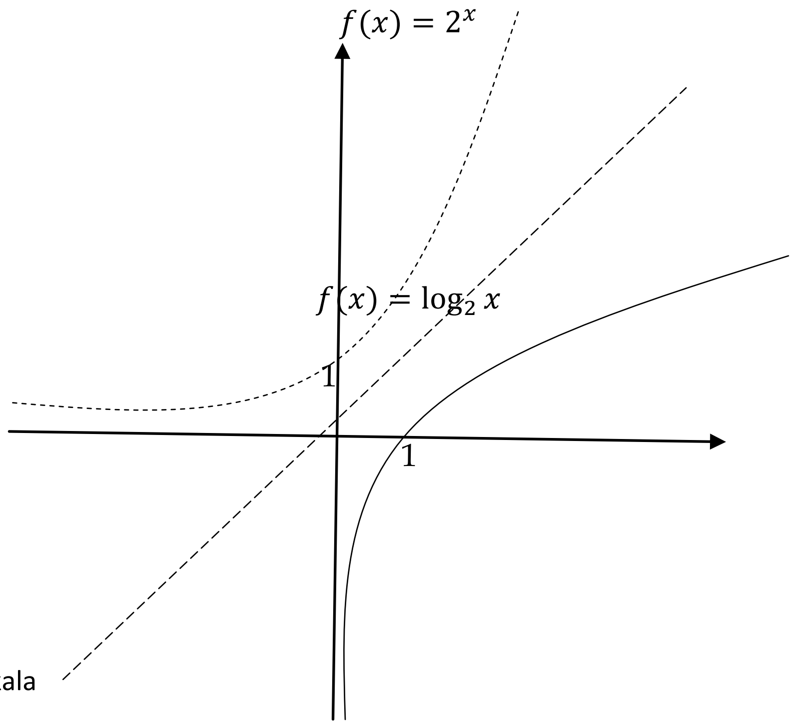
Solution

$$f(x) = \log_2 x$$

Let $y = \log_2 x$

$2^y = x$ then $f^{-1}(x) = 2^x$

We sketch the graph of $f^{-1}(x) = 2^x$ then we reflect in the line $y = x$.



From the graph, we can find the domain and the range of the logarithmic function

$$f(x) = \log_2 x$$
$$Domain = \{x|x > 0 \ \ for\ x \in R\}$$
$$Range = (-\infty, \infty)$$

Example 17.5.4

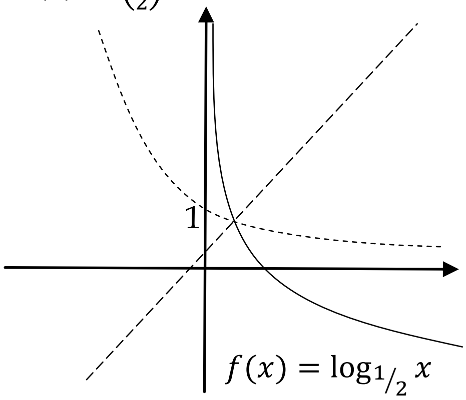
Sketch the graph of $f(x) = \log_{1/2} x$

Solution

$$f(x) = \log_{1/2} x$$

We have $f^{-1}(x) = \left(\frac{1}{2}\right)^x$

We sketch the graph of $f^{-1}(x) = \left(\frac{1}{2}\right)^x$ then we reflect in the line $y = x$.



Translations of graphs of the logarithmic functions

We regard graphs of the functions $f(x) = \log_2 x$ and $f(x) = \log_{1/2} x$ to be the parent graphs then apply the concepts of translations of functions.

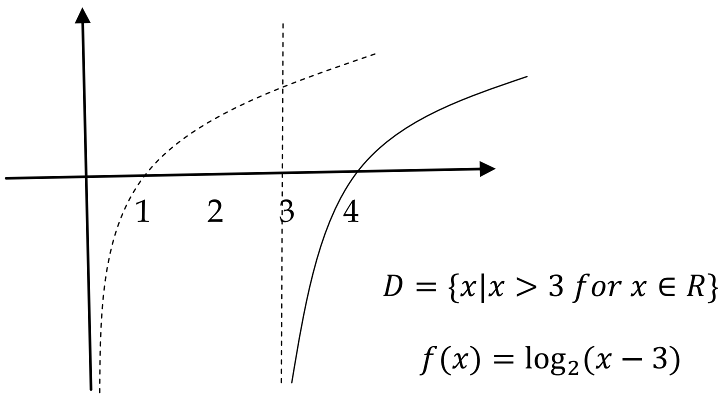
Example 17.5.5

Sketch the following logarithmic functions and find the domain

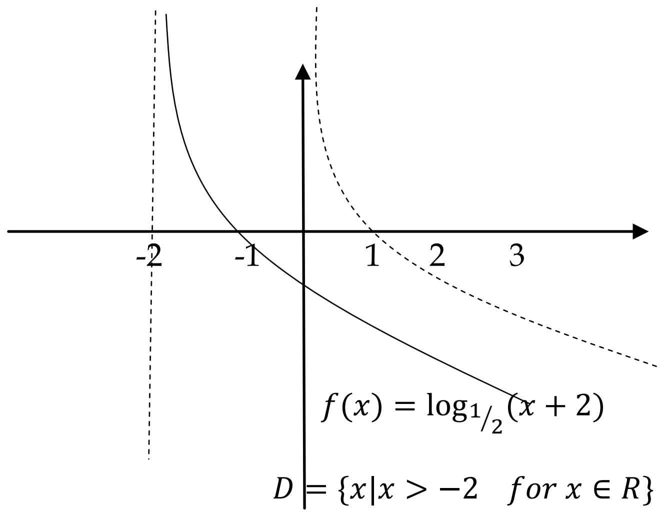
a. $f(x) = \log_2(x - 3)$ b. $f(x) = \log_{1/2}(x + 2)$

Solutions

a.



b.



Natural logarithm – base `e `

In many practical applications of logarithms, the number e (remember e ≈2.71828) is used as a base. Logarithms with a base of e are called **natural logarithms** and thesymbol ***lnx***is commonly used instead of $\log_e x$.

$$\mathbf{\log_e x = \ln x}$$

The **natural logarithmic function** is defined by the equation$f(x) = \ln x$. It is the inverse of the natural exponential function $f(x) = e^x$. Thus one way to graph $f(x) = \ln x$is to reflect the graph of $f(x) = e^x$across the line y =x.

Special Properties

1. $\log_e e = \ln e = 1$
2. $\log_e 1 = \ln 1 = 0$

Example 17.5.6

a. Express $\ln\left(\sqrt{\frac{x^3y^3}{z}}\right)$ as a sum or difference
b. Express as a single logarithm $\frac{1}{2}(3\ln x - 5\ln y + 16)$

Solutions

a. $\ln\sqrt{\frac{x^3y^2}{z}} = \ln\left(\frac{x^3y^2}{z}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{x^3y^2}{z}\right)$

$$= \frac{1}{2}(\ln x^3 + \ln y^2 - \ln z)$$
$$= \frac{1}{2}(3\ln x + 2\ln y - \ln z)$$

$$= \frac{3}{2}\ln x + \ln y - \frac{1}{2}\ln y$$

b. $\frac{1}{2}(3\ln x - 5\ln y + 16) = \frac{1}{2}(\ln x^3 - \ln y^5 + 16\ln e)$

$$= \frac{1}{2}(\ln x^3 - \ln y^5 + \ln e^{16})$$

$$= \frac{1}{2}\ln\left(\frac{x^3 e^{16}}{y^5}\right)$$

$$= \ln\sqrt{\frac{e^{16}x^3}{y^5}}.$$

Solving Equations evolving logarithms and exponential

If$x > 0\,, y > 0\,, b > 0 \; and \; b \neq 1$ then,$x = y$ if and only if $\log_b x = \log_b y$

Example 17.5.7

Solve the following equations

a. $5^x = 10$
b. $e^{x+1} = 5$
c. $2^{3x-2} = 3^{2x+1}$

Solutions

a. $5^x = 10$

$$\log_5 5^x = \log_5 10$$
$$x \log_5 5 = \log_5 10$$
$$x = \log_5 10$$

b. $e^{x+1} = 5$

$$\ln e^{x+1} = \ln 5$$
$$(x+1)\ln e = \ln 5$$
$$x + 1 = \ln 5$$
$$x = \ln 5 - 1$$

c. $2^{3x-2} = 3^{2x+1}$

$$\log 2^{3x-2} = \log 3^{2x+1}$$
$$(3x-2)\log 2 = (2x+1)\log 3$$
$$3x\log 2 - 2\log 2 = 2x\log 3 + \log 3$$

$$3x\log 2 - 2x\log 3 = 2\log 2 + \log 3$$
$$x(3\log 2 - 2\log 3) = 2\log 2 + \log 3$$

$$x = \frac{2\log 2 + \log 3}{3\log 2 - 2\log 3}$$

Example 17.5.8

Solve

a. $\log x + \log(x - 15) = 2$

b. $\ln(x+2) = \ln(x-4) + \ln 3$

Solution

a. $\log x + \log(x-15) = 2$
The domain $x > 15$

$$\log x(x-15) = 2\log 10$$
$$\log x(x-15) = \log 10^2$$
$$\log x(x-15) = \log 100$$

$$x(x-15) = 100$$
$$x^2 - 15x - 100 = 0$$
$$(x-20)(x+5) = 0$$
$$x = 20 \quad but\ x \neq -5$$

b. $\ln(x+2) = \ln(x-4) + \ln 3$

$$\ln(x+2) = \ln[3(x-4)]$$
$$x+2 = 3(x-4)$$
$$x+2 = 3x-12$$
$$2x = 14$$
$$x = 7$$

REVIEW EXERCISE

1. Assuming that the inflation is 7% per year, the equation $P = P_o(1.07)^t$ yields the prediction P of an item in *t* years if it presents the cost P_o. Find the predicted price of each of the following item for the indicated years ahead.
   a. K55 can of soup in 3 years
   b. K500 TV set in 7 years
   c. K350 of this book in 5 years
2. Suppose that it is estimated that the value of the car depreciated 20% per year for the first 5 years. The equation $A = P_o(1.07)^t$ yields the value (A) of a car after t years if the original price is P_0. Find the value of each of the following cars after the indicated time.
   a. K90,000 car after 4 years
   b. K5000 TV set after 7 years
3. Use the formula $A = P_o(1+\frac{r}{n})^{nt}$ to find the total amount of money accumulated at the end of the indicated time period for each of the following investments.
   a. K250 for 5 years compounded annually
   b. K300 for 6 years compounded semi-annually
   c. K750 for 15 years compounded quarterly
4. Use the formula $A = P_o\ e^{rt}$ to find the total amount of money accumulated at the end of the indicated time period compounded continuously. (use 2.718 as approximation for e
   a. K500 for 5years at 8%
   b. K800 for 10years at 10%
5. Sketch the following exponential graphs

$$i.\ \ f(x) = 2^{x-1} + 3 \qquad ii.\quad f(x) = \left(\frac{1}{2}\right)^{x+1} + 2 \qquad iii.\quad f(x) = 2^{-x}$$

$iv. \quad f(x) = e^{x+1} + 2 \qquad v. \quad f(x) = 3e^x + 1 \qquad vi. \quad f(x) = \left(-\frac{1}{4}\right)^{x-1} - 1$

6. Suppose that in a certain bacteria culture, the equation $Q(t) = 1000e^{0.04t}$ expresses the number of bacteria's present as a function of time, where t is in hours.
   How many bacteria's present at the end of 2hours, 3hours, 4hours and 6hours.
7. Write each of the following in the logarithmic form

   $a. \quad 3^2 = 9 \qquad b. \quad 2^5 = 32 \qquad c. \quad 2^{-4} = \frac{1}{16} \qquad d. \quad \left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$
8. Write each of the following in an exponential function

   $a. \quad \log_2 64 = 6 \qquad b. \quad \log_2\left(\frac{1}{16}\right) = -4 \qquad c. \quad \log_{10} 0.1 = -1$
9. Evaluate

   $a. \quad 5\log_5 13 \qquad b. \quad \log_5 \sqrt[3]{25} \qquad c. \quad \log_{1/2}\left(\frac{\sqrt[4]{8}}{2}\right)$
10. Solve each of the following equations

    $a. \quad \log_5 x = 2 \qquad b. \quad \log_4 m = \frac{3}{2} \qquad c. \quad \log_b 3 = \frac{1}{2} \qquad d. \quad \log_{10} x = 0$
11. Express each of the following as a sum or difference in a simpler logarithmic quantities.

    $a. \quad \log_b\left(\frac{x^2}{y}\right) \qquad b. \quad \log_b x^{2/3}y^{3/4} \qquad c. \quad \log_b \frac{x\sqrt{y}}{z}$
12. Express each of the following in simple logarithmic form

    $a. \quad 2\log_b x - 4\log_b y \qquad b. \quad \log_b x + \log_b y - \log_b z \qquad c. \quad \ln x - \ln y - 1$
13. Solve each of the following logarithmic equations

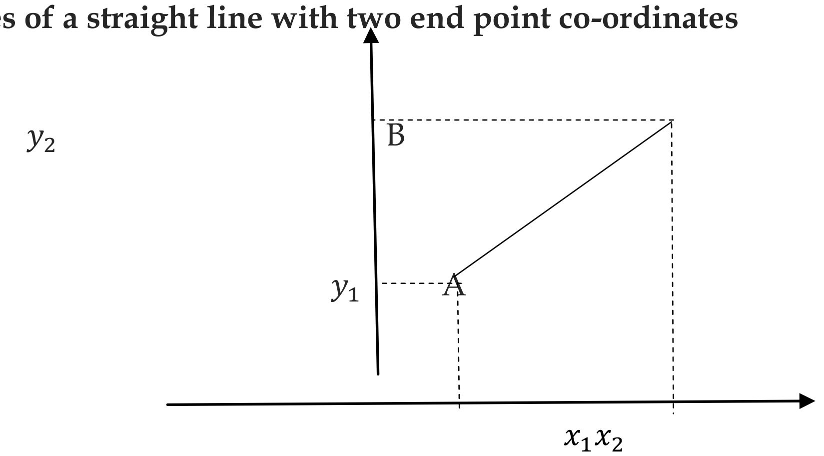
    $a. \quad \log_{10}(x-4) + \log_{10}(x-1) = 1 \qquad b. \quad \ln(x-2) = \ln(x-4) + \ln 3$

    $c. \quad \log_2 x = 8 + 9\log_x 2 \qquad d. \quad 9\log_x 5 = \log_5 x$
14. Graph each of the following logarithmic functions and determine its domain and the range.
    a. Graph of $f(x) = \log_3 x$ by changing in exponential form
    b. Graph of $f(x) = \log_{1/2} x$ by reflecting the graph $g(x) = \left(\frac{1}{2}\right)^x$ in the line $y = x$
    c. $f(x) = \log_{10}(x+2)$ $\quad d. \quad f(x) = \log_2(x-1) \quad e. \quad f(x) = 2 + \ln x$
15. Solve each of the following exponential equations

    $a. \quad 5^{3x+1} = 9$ b. $3^{2x+1} = 2^{3x+2}$ c. $3e^x - 1 = 17$ d. $2^{2x} + 3(2)^x - 4 = 0$

18:CO-ORDINATE GEOMETRY

Properties of a straight line with two end point co-ordinates



Mid - point of two points

The mid-point of $A(x_1 y_1,)$ and $B(\ x_2, y_2\ )$ is given by

Mid-point = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example 18.1.1

Find the midpoint of (3,4) and $(-5,2)$

Solution

Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$= \left(\frac{3-5}{2}, \frac{4+2}{2}\right)$$

$$= \left(\frac{-2}{2}, \frac{6}{2}\right)$$

$$= (-1, 3).$$

Example 18.1.2

Find the midpoint of (3,4) and $(-11,2)$

Solution

$$\text{Midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= \left(\frac{3-11}{2}, \frac{4+2}{2}\right)$$

$$= \left(\frac{-8}{2}, \frac{6}{2}\right)$$

$$= (-4, 3)$$

Example 18.1.3

If the mid-point of AB is $(-2,1)$ and $A(-3,2)$. Find the co-ordinate of B.

Solution

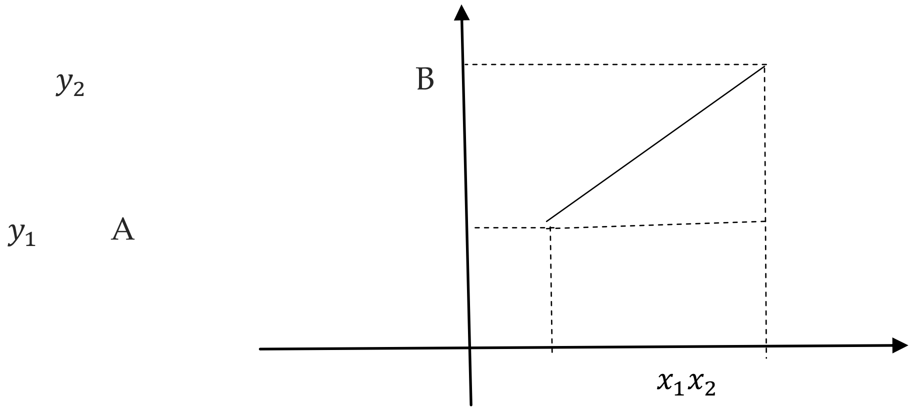
Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$(-2,1) = \left(\frac{-3+x_2}{2}, \frac{2+y_2}{2}\right)$$

$$\frac{-3+x2}{2} = -2 \quad , \quad \frac{2+y2}{2} = 1$$

$$x_2 = -1 \qquad y_2 = 0 \qquad B(-1,0)$$

Distance between two points



We have $A(x_1\,, x_2)$ and $B(y_1, y_2)$, using Pythagoras theorem

$$(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance between $(x_1\,, x_2)$ and (y_1, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 18.1.4

Find the distance between $A(2, -3)$and $B(-3,4)$living the answer in standard form.

Solution

$A(x_1\,, x_2) = A(2, -3)B(y_1, y_2) = B(-3,4)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(2+3)^2 + (-3-4)^2} = \sqrt{(5)^2 + (7)^2}$$

$$= \sqrt{25 + 49} = \sqrt{74}.$$

Example 18.1.5

Find the distance between $A(4, -3)$ and $B(5,4)$ living the answer in standard form.

Solution

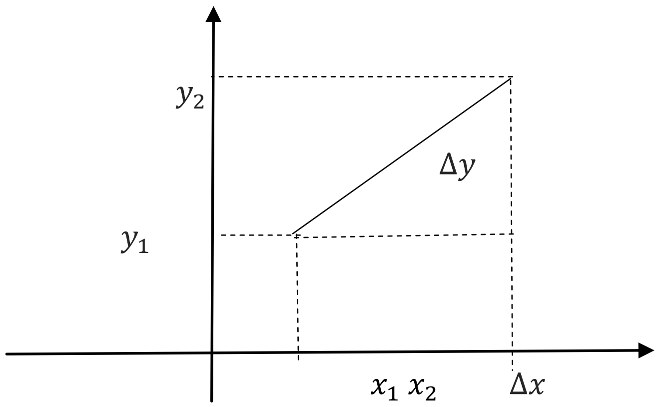
$$A(x_1\,, x_2) = A(4, -3)B(y_1, y_2) = B(5,4)$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4+3)^2 + (5-4)^2}$$

$$= \sqrt{(7)^2 + (1)^2}$$

$$= \sqrt{49 + 1} = \sqrt{50}$$

Gradient of a straight line



$$\textbf{Gradient} = \boldsymbol{m} = \frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$$

Example 18.1.6

Find the gradient of the line joining

a. $A(-2,7)$ and $B(4,5)$
b. C(2,−5) and $D(6,3)$

Solutions

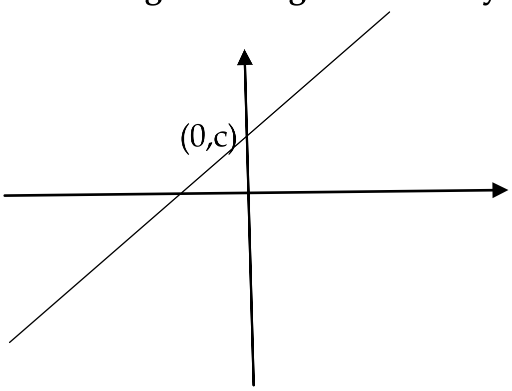
a. $A(-2,7)$ and $B(4,5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-7}{4+2} = \frac{-2}{6} = \frac{-1}{3}$$

b. $C(2,-5)$ and $D(6,3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3+5}{6-2} = \frac{8}{4} = 2$$

Equation of a straight line when given the gradient and y−intercept



The standard form of the equation of any given straight line is given by

$$\boldsymbol{y = mx + c}$$

Where m is the gradient, c is the y intercept, x and y are variables of the equation

The general form of the equation of a straight line is given by

$$\boldsymbol{ax + by + c = 0}$$

Where *a, b* and *c* are constants.

Example 18.1. 7

Write down the gradient of the following equations

a. $y = -3x + 5$
b. $4x - 2y + 5 = 0$

Solutions

a. $y = -3x + 5$

$$y = mx + c$$

$$m = -3$$

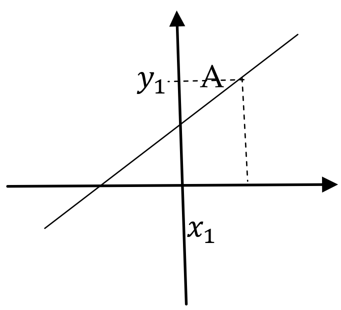
b. $4x - 2y + 5 = 0$

$$2y = 4x + 5$$
$$y = \frac{4}{2}x + \frac{5}{2}$$

$$y = 2x + \frac{5}{2}$$
$$y = mx + c$$
$$m = 2$$

Finding the equation of a straight line

i. **Given one point A$(\boldsymbol{x_1}, \boldsymbol{y_1})$ on the line and the gradient**



Given one point $A(x_1, y_1)$ with the gradient m, then the equation of a straight line is given by

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

Example 18.1.8

Find the equation of a straight line passing through $(3, -1)$whose gradient is $\frac{2}{3}$.

Solution

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = \frac{2}{3}(\,x - 3)$$

$$y + 1 = \frac{2}{3}(\,x - 5)$$

$$3(y + 1) = 2(x - 5)$$

$$3y + 3 = 2x - 6$$

$$2x - 3y - 9 = 0$$

Equation of a straight line given two points $\boldsymbol{A(x_1, y_1)}$ *and* $\boldsymbol{B(x_2, y_2)}$ on the line

Given two points $A(x_1, y_1)$and $B(x_2, y_2)$, the equation of a straight line joining the two points is given by$\boldsymbol{y - y_1 = m(x - x_1)}$ where m is the gradient

We first find m using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

Take any point either A or B, substitute the coordinates in the equation $y - y_1 = m(x - x_1)$ and substitute the gradient m then simplify.

$$y - y_1 = m(x - x_1)$$

$$\boldsymbol{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

Example18.1. 9

Find the equation of a straight line passing through (2, −3) and (−1,4).

Solution

Let $A(2, -3)$ and $B(-1,4)$

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-3)}{-1 - 2} = \frac{7}{-3} = -\frac{7}{3}$$

Taking the point $A(2, -3)$

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = -\frac{7}{3}(x - 2)$$

$$3(y + 3) = -7(x - 2)$$

$$3y + 9 = -7x + 14$$

$$3y + 7x - 5 = 0$$

Given the gradient and y intercept $(\mathbf{0}, \boldsymbol{c})$

We use the formula $y = mx + c$

Example 18.1. 10

Find the equation of a straight line passing through $(0, -7)$ whose gradient is 5.

Solution

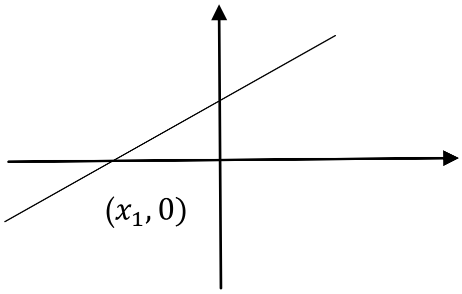
$$y = mx + c$$

$$y = 5x + (-7)$$

$$y = 5x - 7$$

ii. **Equation of a straight line when given two intercept**

$(0, y_1)$



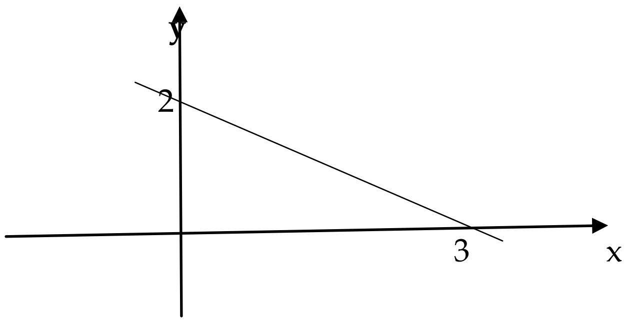
The equation of the straight line passing through $(x_1, 0)$ and $(0, y_1)$ is

$$\frac{x}{x_1} + \frac{y}{y_1} = 1$$

Example 18.1. 11

Find the equation of a straight line passing through (3,0) and (0,2)

Solution



$$\frac{x}{x_1} + \frac{y}{y_1} = 1$$

$$\frac{x}{3} + \frac{y}{2} = 1$$

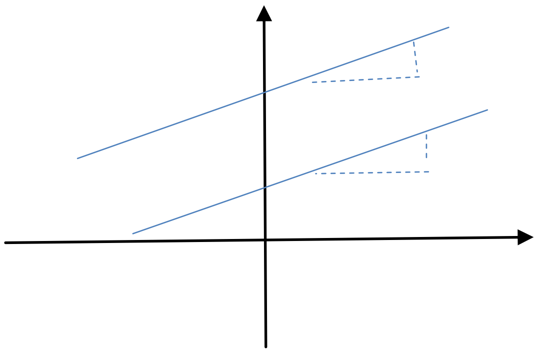
$$6\left(\frac{x}{3} + \frac{y}{2} = 1\right) \qquad 2x + 3y = 6$$

Parallel and perpendicular line

a. **Two Parallel lines**
Two lines are said to be parallel lines if they have equal gradients

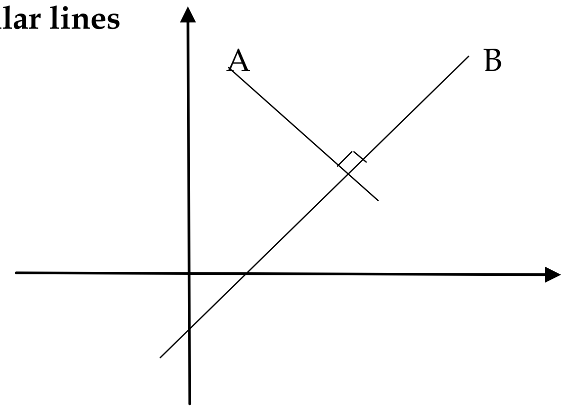
L_1

L_2



$$\boldsymbol{m_1 = m_2}$$

b. **Two perpendicular lines**



Two lines A and B are said to be perpendicular if the product of gradient of the line A and the gradient of the line B are equal to negative of one.

$$\boldsymbol{m_1 m_2 = -1}$$

Example18.1. 12

a. Find the line parallel to the line $y = -2x - 5$ passing through $(-2,5)$
b. Find the equation of the line passing through $(3, -1)$ and is perpendicular to the line $y = 2x - 4$

Solutions

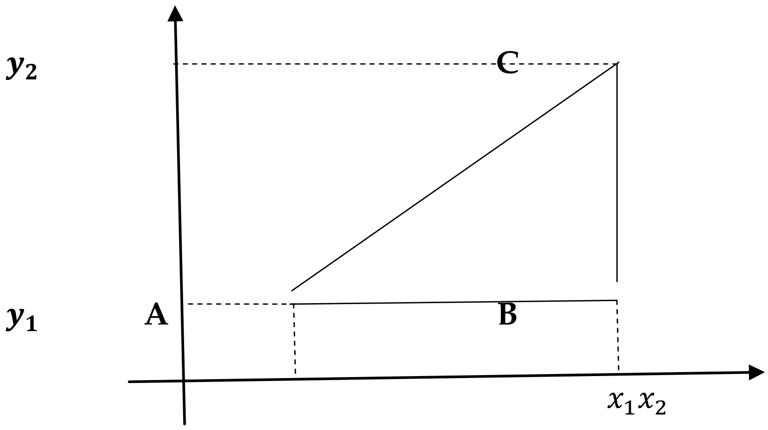
a. We have line1: L_1 $y = -2x - 5$ $m_1 = -2$
Line2: L_2 $m_1 = -2$, then $m_2 = m_1 = -2$

$$y - y_2 = m_2(x - x_2)$$
$$y - 5 = -2\big(x - (-2)\big)$$
$$y - 5 = -2x - 2$$

b. We have line1: L_1 $y = 2x - 4$ $m1 = 2$
Line2: L_2 $m_1 = 2$, then $m_2 = \frac{-1}{m_1} = \frac{-1}{2} = -{}^{1}/_{2}$

$$y - y_2 = m_2(x - x_2)$$
$$y - (-1) = -{}^{1}/_{2}\,(x - 3)$$
$$y + 1 = -{}^{1}/_{2}\,(x - 3)$$
$$2(y + 1) = -1(x - 3)$$
$$2y + 2 = -x + 3$$
$$2y + x - 1 = 0$$

Finding the area of a triangle using the determinant of matrix method



Area $= \frac{1}{2}\, bh \quad = \frac{1}{2}\,(AB) \times (BC)$

Now, given any triangle with vertices (x_1, y_1) , $B(x_2, y_2)$ and $C(x_3, y_3)$, the area of a triangle is given by

$$\boldsymbol{Area} = \frac{\mathbf{1}}{\mathbf{2}}\begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{y_1} & \mathbf{1} \\ \boldsymbol{x_2} & \boldsymbol{y_2} & \mathbf{1} \\ \boldsymbol{x_3} & \boldsymbol{y_3} & \mathbf{1} \end{vmatrix}$$

Example 13

Find the area of a triangle whose vertices are (1,1) , (3,1) and (3,3)

Solution

Let $A(1{,}1)$, $B(\,3{,}1)$ and $C(3{,}1)$ be the vertices

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix}\begin{matrix} 1 & 1 \\ 3 & 1 \\ 3 & 3 \end{matrix}$$

$$\begin{matrix} & -3 & & -3 & -3 \end{matrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix}\begin{matrix} 1 & 1 \\ 3 & 1 \\ 3 & 3 \end{matrix}$$

$$\begin{matrix} 1 & 3 & 9 \end{matrix}$$

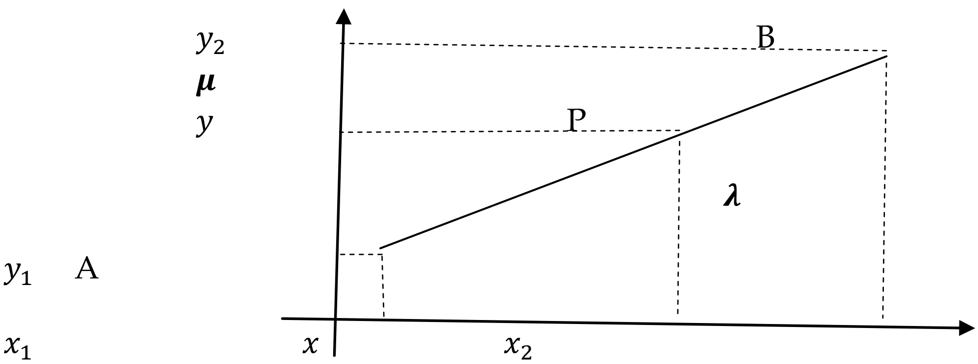
$= \frac{1}{2}(1+3+9-3-3-3) = \frac{1}{2}(4) = 2$ square units.

Further co-ordinate geometry

1. **Division of a straight line in given ratios(internally)**
   Suppose that a point $P(x,y)$ divides the line joining $A(x_1,y_1)$ to $B(x_2,y_2)$ in the ratio $\lambda:\mu$ internally then the co-ordinates of P is given by

$$x = \frac{\lambda\, x_2 + \mu\, x_1}{\lambda + \mu}$$

$$y = \frac{\lambda\ y_2 + \mu\, y_1}{\lambda + \mu}$$

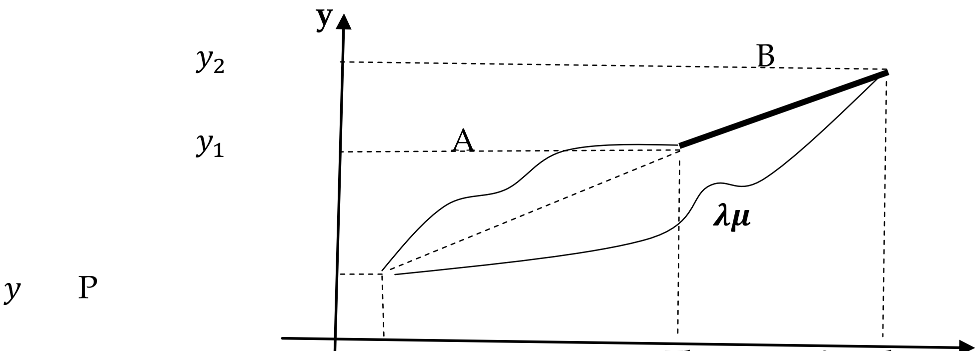


The terms $\boldsymbol{\lambda}$ and μ are being multiplied by the co-ordinates on the other side and are taken to be positives.

2. **Division of a straight line in given ratios(externally)**
   Suppose that a point $P(x,y)$ divides the line joining $A(\,x1,y1)$ to $B(x2,y2)$ in the ratio $\lambda:\mu$ externally then the co-ordinates of P is given by

$$x = \frac{-\lambda\, x_2 + \mu\, x_1}{-\lambda + \mu}$$

$$y = \frac{-\lambda\, y_2 + \mu\, y_1}{-\lambda + \mu}$$



The terms $\boldsymbol{\lambda}$ and μ are being multiplied by the co-ordinates on the other side and $\boldsymbol{\lambda}$ is taken to be negatives.

Example18.1. 14

Find the co-ordinates of the point $P(x,y)$ that divides the line joining $A(-2,5)$ and $B(4,2)$ in the ratio $2:1$

a. It is done internally
b. It is done externally

Solutions

a. $P(x,y)$ $\quad\quad A(-2,5)$ $\quad\quad B(4,2)\lambda{:}\,\mu\ = 2:1$

$$x = \frac{-\lambda\, x_2 + \mu\, x_1}{\lambda + \mu}$$
$$x = \frac{2(4) + 1(-2)}{-2-1} = \frac{-8+2}{-3} = 2$$

$$y = \frac{-\lambda\, y_2 + \mu\, y_1}{\lambda + \mu}$$

$$y = \frac{2(2)\ + 1(5)}{2+1} = \frac{4+5}{3} = 3$$
$$P(2,3)$$

b. $$x = \frac{-\lambda\, x_2 + \mu\, x_1}{\lambda + \mu}$$

$$x = \frac{-2(4) + 1(-2)}{-2+1} = \frac{-8-2}{-1} = 10$$

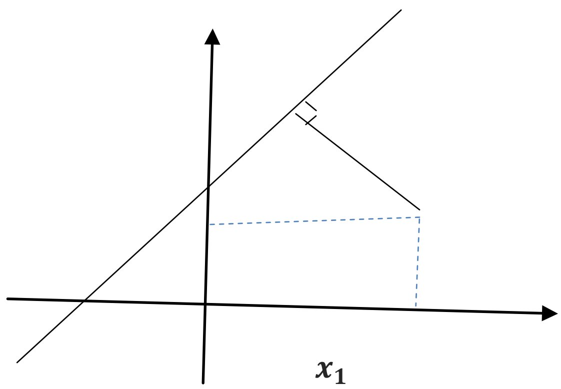
$$y = \frac{-\lambda\, y_2 + \mu\, y_1}{\lambda + \mu} = \frac{-2(2)\ + 1(5)}{-2+1} = \frac{-4+5}{-1} = -1$$
$$P(10,-1)$$

Further distance formulas

1. **The perpendicular distance of a point from a straight line**
   The distance of a point $A(x_1, y_1)$ from a straight line $ax + by + c = 0$ is given by

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

y_1 A



The line $ax + by + c = 0$ is a general form.

Example18.1. 15

Find the distance from $A(3,4)$ to the line $2x - y = 3$

Solution $\quad 2x - y = 3 \qquad 2x - y - 3 = 0$

$$a = 2\ , \qquad b = -1 \qquad c = -3$$

$$\boldsymbol{A}(x_1, y_1) = A(3,4)$$

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|2(3) + (-1)(4) + (-3)|}{\sqrt{2^2 + (-1)^2}} = \frac{|-1|}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$D = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

2. **Distance between two parallel lines**

$$ax + by + c_1 = 0$$
$$ax + by + c_2 = 0$$

Let's consider the following example to understand how the formula for the distance between these two lines that are parallel.

Example 18.1.1 16

Find the equation of a straight line L2 parallel to the line L1 $3x - 4y + 5 = 0$ and passing through $(1, -2)$.

Solution

L1 $3x - 4y + 5 = 0$

L2 $3x - 4y + c2 = 0$

To find $c2$ we substitute the point coordinates $(1, -2)$ in L2 : $3x - 4y + c2 = 0$

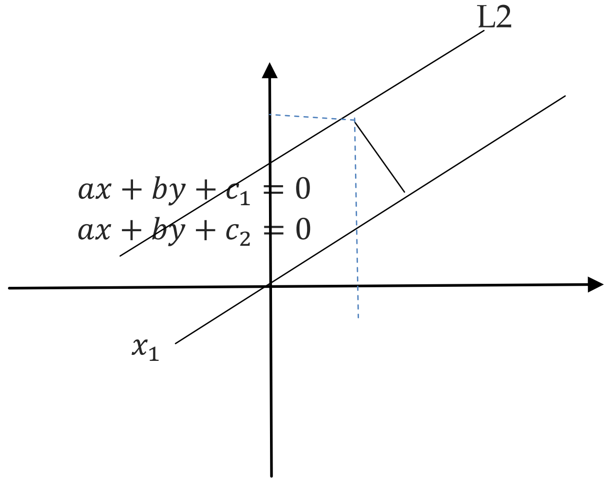
$$3(1) - 4(-2) + c2 = 0$$

$c_2 = -11$

L2 : $3x - 4y - 11 = 0$

y_1 L1

Now let's consider the equations



The distance between the two parallel lines L1 and L2 is the distance from a point (x_1, y_1) to the line L1

$$\boldsymbol{D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}}$$

Now   L2 $ax + by + c_2 = 0\ , where\ x = x_1 y = y_1 ax_1 + by_1 + c_2 = 0$
Solve for y_1 we have

$$\frac{by_1}{b} = \frac{-c_2 - ax_1}{b}$$
$$y1 = \frac{-c_2 - ax_1}{b}$$

Substitute in $D = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$we have

$$D = \frac{|ax_1 + b\left(\frac{-c_2 - ax_1}{b}\right) + c_1|}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|ax_1 - c_2 - ax_1 - c_1|}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$$

Hence the perpendicular distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

$$\boldsymbol{D = \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}}$$

Example 18.1.17
Find the perpendicular distance between $3x - 4y + 5 = 0$ and $3x - 4y + 7$

Solution
Let $a = 3\ , b = -4\ , c_1 = 5\ \ and\ \ c_2 = 7$

$$D = \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|5 - 7|}{\sqrt{3^2 + (-4)^2}}$$

$$D = \frac{|-2|}{\sqrt{9 + 16}}$$

$$D = \frac{2}{\sqrt{25}} \quad D = \frac{2}{5}.$$

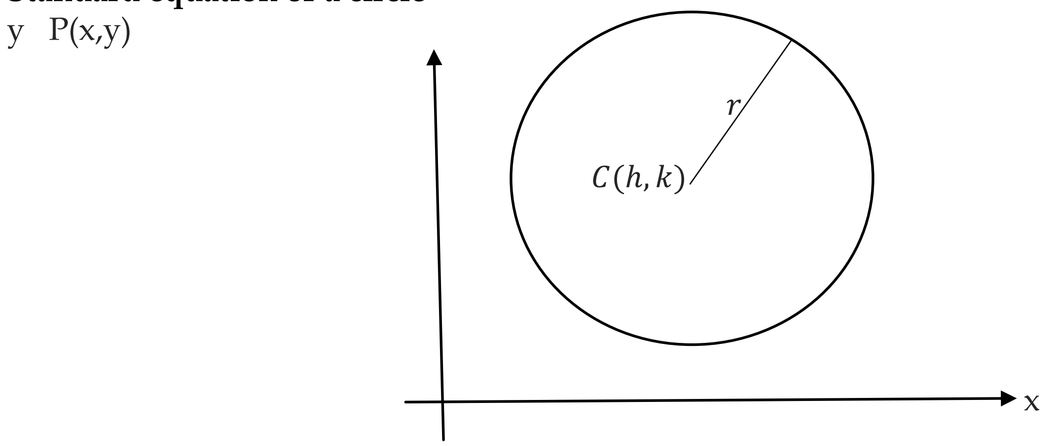
Coordinate geometry of a circle

The Circle

A **circle** is the set of all points in a plane equidistant from a given fixed point called the **center**. A line segment determined by the center and any point on the circle is called a **radius**.

Equations of a Circle

1. **Standard equation of a circle**



The distance between the center and a point p on the circumference is the radius given by

$$d = r = \sqrt{(x-h)^2 + (y-k)^2}$$

$$r^2 = (x-h)^2 + (y-k)^2$$

Hence the standard equation of a circle with the center at $C(h,k)$ and the radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 18.1.18

Find the equation of a circle that has its center at $(-3,5)$ of the length 4 units.

Solution

$$C(h,k) = C(-3,5)$$

$$h = -3,\ k = 5 \text{ and } r = 4$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-(-3))^2 + (y-5)^2 = 4^2$$

$$(x+3)^2 + (y-5)^2 = 16$$

2. **General equation of a circle**

   The general equation of a circle in x and $y$is given in the form of the quadratic

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

   Now we have to proof that this formula is true. That is it is equal to the standard formula.

   Proof

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$x^2 + y^2 + 2gx + 2fy = -c$$

By complete the square method

$$x^2 + y^2 + 2gx + 2fy = -c$$
$$x^2 + 2gx + y^2 + 2fy = -c$$

$$x^2 + 2gx + g^2 - g^2 + y^2 + 2fy + f^2 - f^2 = -c$$
$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = -c + g^2 + f^2$$
$$(x + g)^2 + (y + f)^2 = -c + g^2 + f^2$$
$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

With radius$r^2 = g^2 + f^2 - c$ $\qquad r = \sqrt{g^2 + f^2 - c}$

Center $C(h, k) = (-g, -f)$

$(x + g)^2 + (y + f)^2 = r^2$ which is similar to standerd equation

In the general equation, the coefficients of x^2 and y^2 are both ones.

Example 18.1.19

Find the Centre and the radius of the circle given by the equation

$$4x^2 + y^2 + 4x - 12y - 26 = 0$$

Solution

$$4x^2 + y^2 + 4x - 12y - 26 = 0$$

$$x^2 + y^2 + x - 3y - {}^{13}/_{2} = 0$$

Compare with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We have $2g = 1$ this implies that $g = {}^{1}/_{2}$

$2f = -3$Implies that $f = {}^{-3}/_{2}$

$$c = {}^{-13}/_{2}$$

Center: $C(-g, -f)$

$$C\left(-{}^{1}/_{2}, -({}^{-3}/_{2})\right)$$

$$C(-{}^{1}/_{2}, {}^{3}/_{2})$$

Radius: $r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-3}{2}\right)^2 - \left(\frac{-13}{2}\right)} = \sqrt{\frac{1}{4} + \frac{9}{4} - \left(\frac{13}{2}\right)} = \sqrt{\frac{1+9+26}{4}} = \sqrt{\frac{36}{4}} \;\; = \frac{6}{3} = 3$$

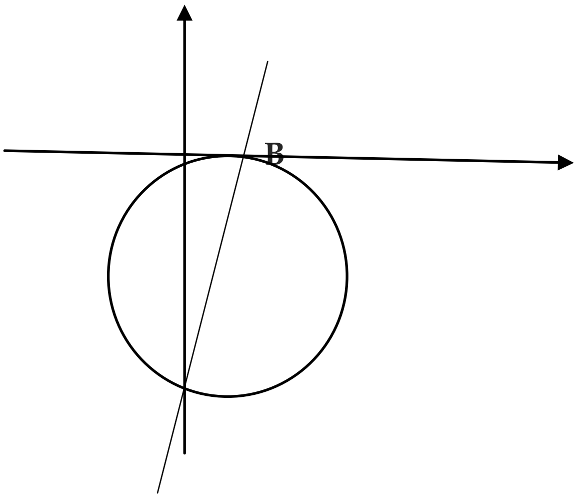
Example18.1. 20

Interpreting geometric questions on circles

a. The line $y = 2x - 8$ meets the co-ordinates axes at A and B. The line AB is the diameter of the circle. Find the equation of the circle
b. The circle with the center at (8,10)meets x-axis at (4,0) and (a,0)
   i) Find the radius of a circle
   ii) Find the value of a
   iii) Write the equation of a circle in the form $(x - h)^2 + (y - k)^2 = r^2$

Solutions

a. $y = 2x - 8y = 2x - 8$



A -8

The coordinates of B is calculated by $y = 2x - 8$ when $y = 0$, $x = 4$

B(4,0)

Then the center is the mid-point of $A(0, -8)$ and $B(4,0)$

Center$= \left(\frac{0+4}{2}, \frac{-8+0}{2}\right) = C(2, -4)$

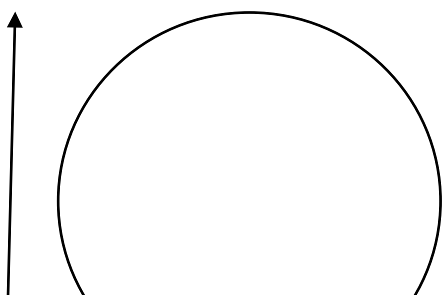
Radius: $r = \sqrt{(4-2)^2 + (0+4)^2} = \sqrt{20}$

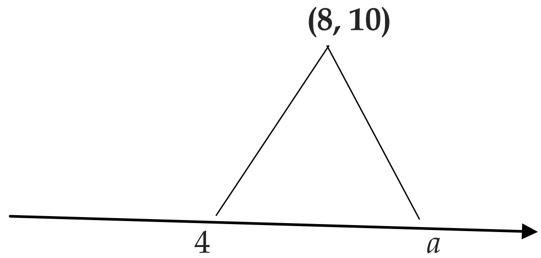
Equation: $(x - h)^2 + (y - k)^2 = r^2$

$$(x-2)^2 + \left(y - (4)\right)^2 = \sqrt{20}^2$$

$$(x-2)^2 + (y+4)^2 = 20$$

b. i) Center (8,10)
c.





$$r = \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$$

$$r = \sqrt{(8 - 4)^2 + (10 + 0)^2} = \sqrt{116}$$

iii) since same radius $r = \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$

$$\sqrt{(8 - a)^2 + (10 + 0)^2} = \sqrt{116}$$

$$(8 - a)^2 + (10 + 0)^2 = 116$$

$$(8 - a)^2 = 116 - 100$$

$$8 - a = \pm 4$$

$$a = 4 \quad or \quad a = 12$$

Then $a = 12$ *is valid*

REVIEW EXERCISE

1. A point A and B has the co-ordinates $(-4,6)$ *and* $(2,8)$respectively. A line p is drawn through B perpendicular to AB to meet y-axis at a point C.
   a. Find the equation of a line p
   b. Determine the co-ordinates of a point C
   c. Sketch the points and draw the lines joining them that makes a triangle.
   d. Find the area enclosed by this triangle ABC
2. The line l has equation $2x - y - 1 = 0$. The line m passes through the point $A(0,4)$ and is perpendicular to the line l.
   a. Find the gradient of the line m and its equation.
   b. Show that the line l and $m$intersect at a point P(2,3)
   c. The line n passes through the point B(3,0)and is parallel to the line m. Find the equation of the line n and find the point Q where the line n intersect l.
3. The points $(-1,-2), B(7,2)$ *and* $C(k,4)$ where k is a constant, are the vertices of ΔABC. Angle ABC is a right angle.
   a. Find the gradient of AB
   b. Calculate the value of k
   c. Find the equation of the line passing through B and C.
4. The straight line L_1 has the equation $4y + x = 0$. The straight line L_2 has the equation $y = 2x - 3$

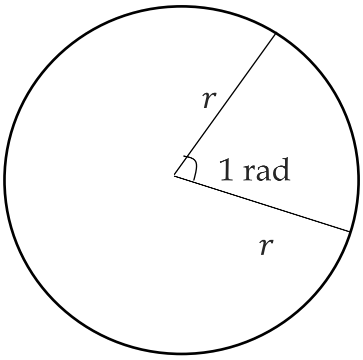
a. On the same axis, sketch the graph of L_1 and L_2. Show clearly the co-ordinates of all points at which the graph meets the co-ordinate axes.
b. The line L_1 and L_2 intersect at a point A. calculate the co-ordinates of A.

5. The points $A(4,5)$ and $B(7,-1)$ are two given points and a point C divides the line segment AB in the ratio 4:3. Find the co-ordinates of a point C if
   a. Divide internally
   b. Divide externally
6. Find the perpendicular distance from the given point to the given line
   a. $3x - 2y + 1 = 0$ $(2,0)$
   b. $5x + 12y - 7 = 0$ $(1,-2)$
   c. $y = 7$ $(2,-4)$
7. Find the center and the radius of each of the following circles and graph them
   a. $x^2 + y^2 - 6x - 10y + 30 = 0$
   b. $x^2 + y^2 - 10x = 0$
   c. $4x^2 + 4y^2 - 4x - 8y - 11 = 0$
   d. $x^2 + y^2 - x - 3y - 2 = 0$
8. Write the equation of the following circles in the form $x^2 + y^2 + Dx + Ey + F = 0$
   a. Center at $(-3,4)$, radius $r = 2$
   b. Center at $(2,3)$, radius $r = 5$
   c. Center at $(0,-4)$, radius $r = 6$

19: TRIGONOMETRY

Radian and Degree Measurements

A radian is the angle subs tended at the center of a circle by an arc whose length is equal to the radius of the circle.



We can find the number of radii that the complete circle has. We use the formula for the circumference $2\pi r = 360°$

So a complete revolution of a circle is 2π radian$= 360°$

Therefore $\pi\, rad = 180°$

$$\frac{\pi}{2}\, rad = 90°$$

$$\frac{\pi}{4}\ rad = 45°$$

In our calculations, we take $\pi\ rad = 180°$ as a conversion factor between radian measures and the degree measures.

Example 19.1.1

Change the following from degree to radians

a. 60° b. 270° c. 135°

Solutions

a. $\pi\ rad = 180°$

$$x = 60^{\circ}$$
$$60^{\circ}\ \pi\ rad = 180°\ x$$
$$x = \frac{60^{\circ}\ \pi\ rad}{180°}$$

$$x = \frac{\pi\ rad}{3} = \frac{\pi}{3}\ rad$$

b.

$$\pi\ rad = 180°$$
$$x = 270^{\circ}$$
$$270^{\circ}\ \pi\ rad = 180°\ x$$
$$x = \frac{270^{\circ}\ \pi\ rad}{180°}$$

$$x = \frac{3\ \pi\ rad}{2}$$

$$x = \frac{3\pi}{2}\ rad$$

c. $\pi\ rad = 180°$

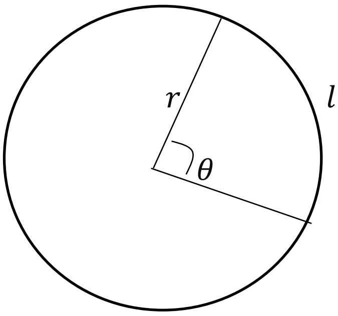
$$x = 135^{\circ}$$
$$135^{\circ}\ \pi\ rad = 180°\ x$$
$$x = \frac{135^{\circ}\ \pi\ rad}{180°}$$

$$x = \frac{3\pi\ rad}{4}$$

$$x = \frac{3\pi}{4}\ rad$$

The arc length of a circle

Let us consider the circle with the arc length l and the angle θ in radians.



From the radian measures and the circumference of a circle

$$C = 2\pi \quad \text{and} \quad C = 2\pi r$$

Then $$2\pi = 2\pi r$$

$$\theta = l$$

Then simplifying this two equations we get

$$2\pi l = 2\pi r\theta$$

$$l = \frac{2\pi r\theta}{2\pi}$$

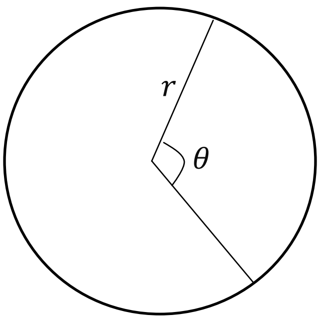
$$l = r\theta$$

For any given circle, the arc length is given by

$$\boldsymbol{l = r\theta}$$

Area of a sector of a circle

Taking the angle θ in radians, then the ratios hold



$$\frac{angle\ of\ an\ arc\ \theta}{2\pi} = \frac{area\ of\ an\ arc\ A}{\pi r^2}$$

$$2\pi A = \pi r^2\theta$$

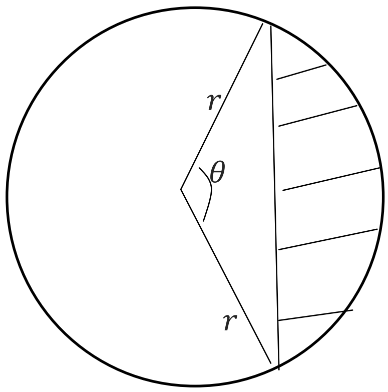
$$A = \frac{\pi r^2\theta}{2\pi}$$

$$A = \frac{1}{2}r^2\theta$$

The area of a sector with the angle θ in radians and the radius r is given by

$$\boldsymbol{A = \frac{1}{2}r^2\theta}.$$

The area of a segment of a circle



The shaded area is called the area of a segment.

$$A = \frac{1}{2}\theta r^2 - \frac{1}{2}r^2\sin\theta$$

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

Example 19.1. 2

Find the length of the arc of a circle of radius $5.2cm$, given that the arc subs tend an angle of 0.8 radians at the center of a circle.

$Solution \quad l = r\theta \qquad l = 0.8 \times 5.2 = 4.16\ cm.$

REVIEW EXERCISE

1. Each angle is expressed in radian. Change each angle to degrees without using the calculator.
   a) $\frac{2\pi}{5}$ $\quad b)$ $\frac{5\pi}{2}$ $\quad c)$ $\frac{7\pi}{12}$ $\quad d)$ $-\frac{7\pi}{6}$
2. Change the angle from degree to radian
   a) 120° $\quad b)$ 300° $\quad c)$ -330° $\quad d)$ -570°
3. a.) Find to the nearest tenth of each, the length of the arc length intercepted by a central angle of $\frac{2\pi}{3}$ radians given that the radius of the circle is 22cm.
4. Find to the nearest tenth of each, the length of the arc length intercepted by a central angle of 130° radians given that the radius of the circle is 8 meters.
5. What is the length of an arc which subtends an angle of 0.8 rad at the center of the circle, if the length of the arc is 3cm.
6. A cord AB subtends an angle of 60° at the center. What is the ratio of the cord to the arc length AB.

7. a) Show that the length of an arc is $S = r\theta$

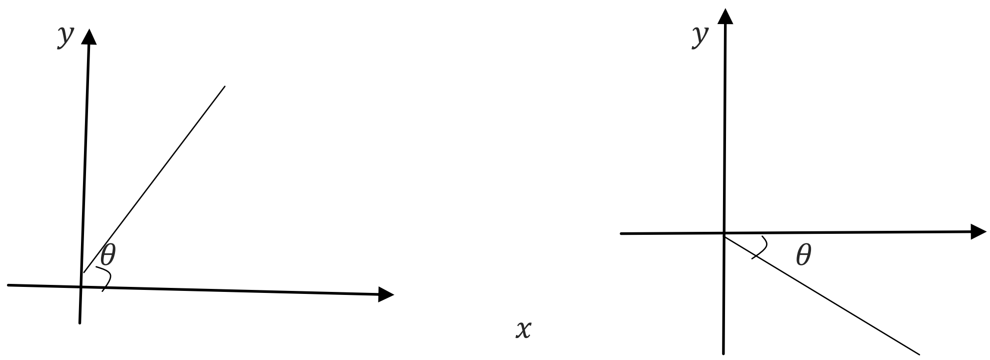
   b) Show that the area of the sector of a circle with radius r is $A = \frac{1}{2}r^2\theta$

   c) Show that the area of the segment of a circle with radius r is $A = \frac{1}{2}r^2(\theta - \sin\theta)$

   d) Show that the area of a triangle is $A = \frac{1}{2}\ ab\sin C$

The general terminal angles

1. **Positive and negative angles**
   An angle measured from the positive x-axis in an anticlockwise direction is considered to be positive angle. And an angle measured from the x-axis in a clockwise direction are taken to be negative angle.



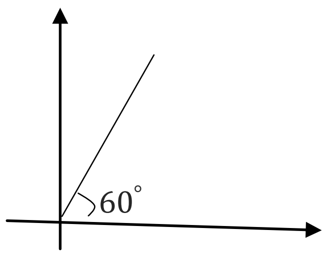
Positive angle　　　　　　　　　　　　negative angle

Example 19.1.5

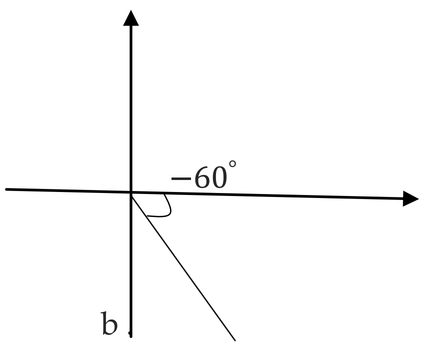
Illustrate the following angles on the ΔXOY plane

a. 60°　　b. -60°　　c. 300°　　d. -300°

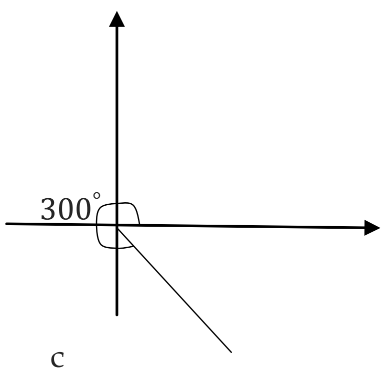
Solutions



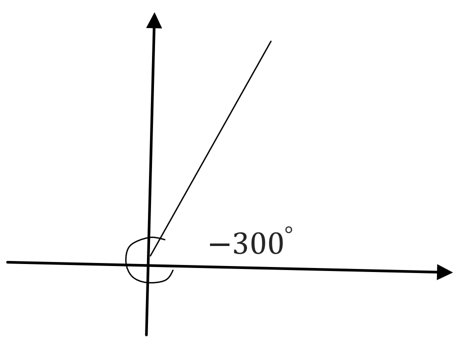
a.



b



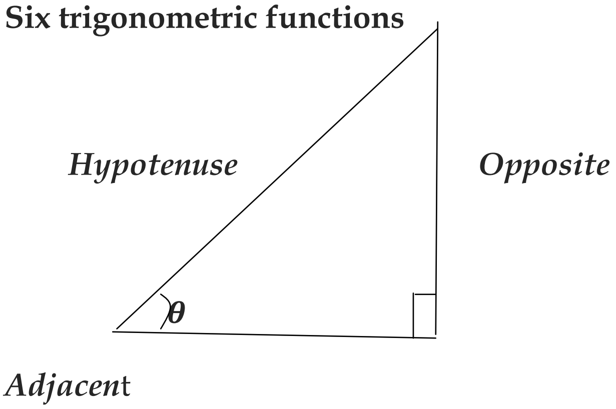
c



d

Trigonometry of a right angled triangle

Six trigonometric functions



Six ratios can be formed by using two lengths of the three sides of a right triangle. Each ratio defines a value of a trigonometric function of a given acute angle θ. The functions are **sine** (sin), **cosine** (cos), **tangent** (tan), **cotangent** (cot), **secant** (sec), and **cosecant** (csc).

Definition of trigonometric functions of an acute angle

Let θ be an acute angle of a right triangle. The values of the six trigonometric functions of are

$$\mathbf{sin\,\theta} = \frac{length\ of\ opposite\ side}{length\ of\ hyppotenus}$$

$$\mathbf{cos\,\theta} = \frac{\text{length of adjacent}}{\text{length of hypotenus}}$$

$$\mathbf{tan\,\theta} = \frac{\text{length of opposite side}}{\text{length of adjacent}}$$

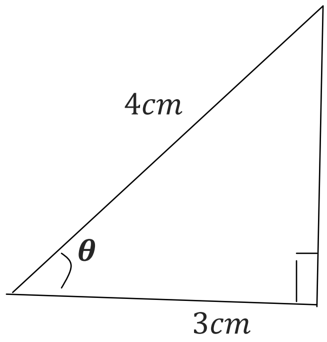
$$\mathbf{cot\,\theta} = \frac{\text{length of adjacent}}{\text{length of opposite}}$$

$$\mathbf{sec\,\theta} = \frac{\text{length of hypotenus}}{\text{length of adjacent}}$$

$$\mathbf{cosec\,\theta} = \frac{\text{length of hypotenus}}{\text{length of opposite}}$$

Example 19.1.6

Find the values of the six trigonometric functions of for the triangle given in the figure below



Solutions

By Pythagoras theorem, hyp= $\sqrt{a^2+b^2}=\sqrt{3^2+4^2}=\sqrt{25}=5$

$$\sin\theta=\frac{\text{length of opposite side}}{\text{length of hyppotenus}}=\frac{4}{5}$$

$$\cos\theta=\frac{\text{length of adjacent}}{\text{length of hypotenus}}=\frac{3}{5}$$

$$\tan\theta=\frac{\text{length of opposite side}}{\text{length of adjacent}}=\frac{4}{3}$$

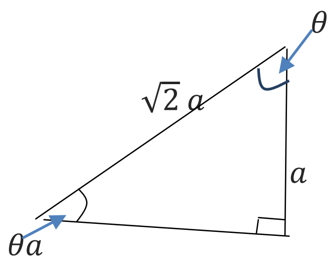
$$\cot\theta=\frac{\text{length of adjacent}}{\text{length of opposite}}=\frac{3}{4}$$

$$\sec\theta=\frac{\text{length of hypotenus}}{\text{length of adjacent}}=\frac{5}{3}$$

$$\operatorname{cosec}\theta=\frac{\text{length of hypotenus}}{\text{length of opposite}}=\frac{5}{4}$$

Trigonometric functions of special angles (30° , 45° *and* 60°)

We consider the right angled triangle with equal adjacent sides.



Where $\theta = 45^\circ$

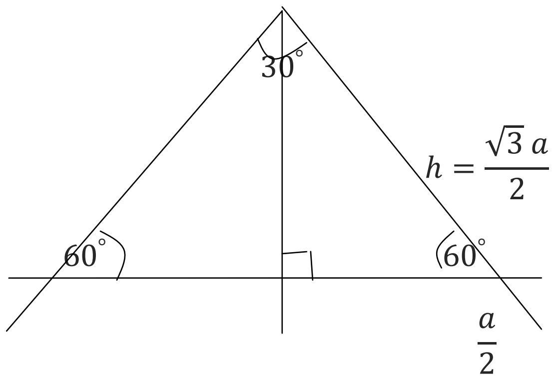
Then the six trigonometric functions of 45° are

$$\sin 45^\circ = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \cos 45^\circ = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan \theta = \frac{a}{a} = 1 \qquad \cot 45^\circ = \frac{a}{a} = 1$$

$$\sec 45^\circ = \frac{\sqrt{2}\,a}{a} = \sqrt{2} \operatorname{cosec} 45^\circ = \frac{\sqrt{2}\,a}{a} = \sqrt{2}$$

Now, we consider the isosceles triangle



$$a^2 = \left(\frac{a}{2}\right)^2 + h^2$$

$$a^2 = \frac{a^2}{4} + h^2$$

$$a^2 - \frac{a^2}{4} = h^2$$

$$\frac{4a^2 - a^2}{4} = h^2$$

$$\frac{3a^2}{4} = h^2$$

$$h = \frac{\sqrt{3}a}{2}$$

Then the value for six trigonometric functions for 30°

$$\sin 30^\circ = \frac{a/2}{a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}a/2}{a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{\frac{a}{2}}{\sqrt{3}a/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot 30^\circ = \frac{\sqrt{3}a/2}{a/2} = \sqrt{3}$$

$$\sec 30^\circ = \frac{a}{\sqrt{3}a/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\operatorname{cosec} 30^\circ = \frac{a}{a/2} = 2$$

The six trigonometric functions of 60°

$$\sin 60^\circ = \frac{\sqrt{3}a/2}{a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{a/2}{a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{\sqrt{3}a/2}{\frac{a}{2}} = \sqrt{3}$$

$$\cot 60^\circ = \frac{a/2}{\sqrt{3}a/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec 60^\circ = \frac{a}{a/2} = 2 \operatorname{cosec} 60^\circ = \frac{a}{\sqrt{3}a/2} = \frac{2\sqrt{3}}{3}$$

Summary of the six trigonometric functions of special angles

$\boldsymbol{\theta}$ $\boldsymbol{sin\ \theta}$ $\boldsymbol{cos\ \theta}$ $\boldsymbol{tan\ \theta}$ $\boldsymbol{cot\ \theta}$ $\boldsymbol{sec\ \theta}$ $\boldsymbol{csc\ \theta}$

$\mathbf{30^{\circ}\ ;\ \frac{\pi}{6}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\mathbf{45^{\circ}\ ;\ \frac{\pi}{4}}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\mathbf{60^{\circ}\ ;\ \frac{\pi}{3}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Reciprocal trigonometric functions

The sine and cosecant functions are called **reciprocal functions.** The cosine and secant are also reciprocal functions, as are the tangent and cotangent functions. These relationships hold for all values of θ for which both of the functions are defined.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\ \theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Trigonometric functions of any angle

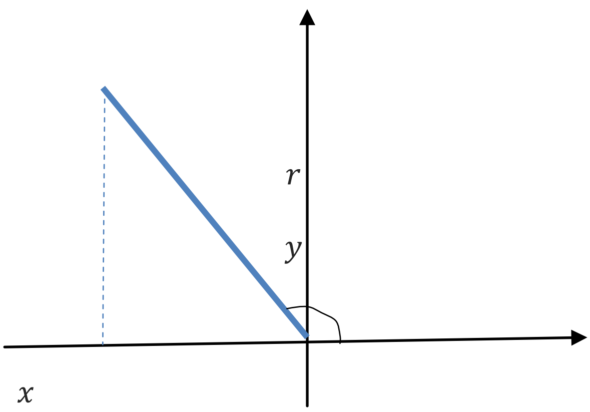
We consider the figure below,

y

$p(x, y)$

θ

x O



Definition of Trigonometric functions of any angle

Let$P(x, y)$ be any point, except the origin, on the terminal side of an angle in standard position. Let$r = d(O, P)$, the distance from the origin to P. The six trigonometric functions of θ are

$$\boldsymbol{\sin\theta = \frac{y}{r}\cos\theta = \frac{x}{r}\tan\theta = \frac{y}{x} \qquad for\ \ x \neq 0}$$

$$\csc\theta = \frac{r}{y} \quad y \neq 0 \qquad \sec\theta = \frac{r}{x} \quad x \neq 0 \qquad \cot\theta = \frac{x}{y} \quad y \neq 0$$

Where $r = \sqrt{x^2 + y^2}$

Example 19.1.7

Find the exact value of each of the six trigonometric functions of an angle in standard position whose terminal side contains the point $P(-3, -2)$

Solution

We have $x = -3$, $y = -2$ then $r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$

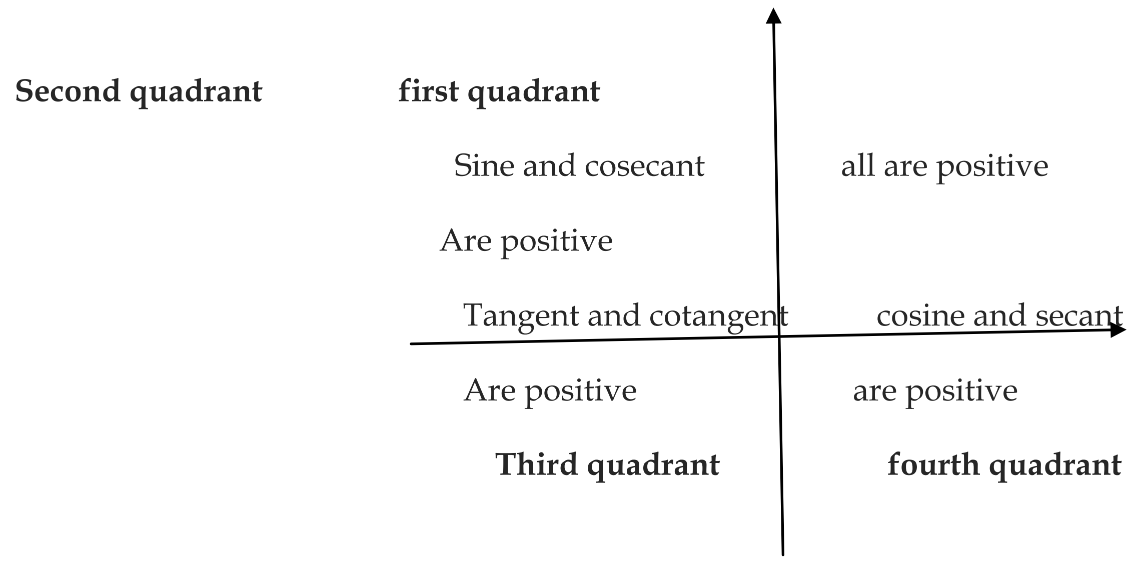
By the definition of the six trigonometric functions

$$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{13}} \cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} \tan\theta = \frac{y}{x} = \frac{-2}{-3} = \frac{2}{3}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{-2} \sec\theta = \frac{r}{x} = \frac{\sqrt{13}}{-3} \cot\theta = \frac{x}{y} = \frac{-3}{-2} = \frac{3}{2}$$

Quadrants and signs of trigonometric functions

The sign of a trigonometric function depends on the quadrant in which the terminal side of the angle lies.



Signs of the trigonometric functions

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin\theta$ *and* $\csc\theta$	Positive	Positive	Negative	Negative
$\cos\theta$ *and* $\sec\theta$	Positive	Negative	Negative	Positive

$\tan\theta$ *and* $\cot\theta$	Positive	Negative	Positive	Negative

Values of trigonometric functions of quadrant angles ($0^\circ, 90^\circ, 180^\circ$ *and* 270°)

The quadrant angle is an angle whose terminal side coincides with the x- axis or y-axis

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	1	0	$undefined$	1	$undefined$
90°	1	0	$undefined$	1	$undefined$	0
180°	0	-1	0	$undefined$	-1	$undefined$
270°	-1	0	$undefined$	-1	$undefined$	0

Example 19.1.8

Given $\tan\theta = -\frac{7}{5}$ and $\sin\theta < 0$. Find $\cos\theta$ and$\csc\theta$

Solution

The terminal side of angle θ must lie in Quadrant IV; that is the only quadrant in which sin and tan are both negative. Because

$$\tan\theta = -\frac{7}{5} = \frac{y}{x}$$

Then $y = -7, x = 5$

Now

$$r = \sqrt{x^2 + y^2} = \sqrt{5^2 + (-7)^2} = \sqrt{74}$$

Then

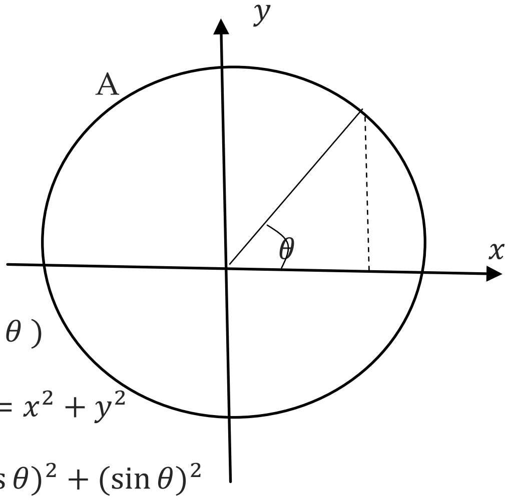
$$\cos\theta = \frac{x}{r} = \frac{5}{\sqrt{74}} = \frac{5\sqrt{74}}{74}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{74}}{-7} = -\frac{\sqrt{74}}{7}$$

Trigonometric Identities

Pythagorean trigonometric identities

Consider the triangle in a unity circle bellow.



By the Pythagoras theorem

$\sin\theta = \frac{y}{r}$, $\sin\theta = \frac{y}{1} \Rrightarrow \sin\theta = y$ 1

$\cos\theta = \frac{x}{r}$, $\cos\theta = \frac{x}{1} \Rrightarrow \cos\theta = x$

Then the point A is located at $(x, y) = (\cos\theta\ , \sin\theta\ )$

$$r^2 = x^2 + y^2$$

$$1^2 = (\cos\theta)^2 + (\sin\theta)^2$$

$$1 = cos^2\theta + sin^2\theta$$

$$cos^2\theta + sin^2\theta = 1$$

Reciprocal identities

1. $\sec\theta = \frac{1}{\cos\theta} \cos\theta = \frac{1}{\sec\theta}$
2. $\csc\theta = \frac{1}{\sin\theta} \sin\theta = \frac{1}{\csc\theta}$
3. $\cot\theta = \frac{1}{\tan\theta} \tan\theta = \frac{1}{\cot\theta}$

Quotient identities

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Now $1 = cos^2\theta + sin^2\theta$, then by division

$$\frac{1}{sin^2\theta} = \frac{cos^2\theta}{sin^2\theta} + \frac{sin^2\theta}{sin^2\theta} \Rrightarrow \quad \boldsymbol{csc^2\theta = 1 + cot^2\theta}$$

$$\frac{1}{cos^2\theta} = \frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta} \Rrightarrow \quad \boldsymbol{sec^2\theta = 1 + tan^2\theta}$$

Co-function identities

1. $\sin(\frac{1}{2} - \theta) = \cos\theta \cos(\frac{1}{2} - \theta) = \sin\theta$
2. $\tan(\frac{1}{2} - \theta) = \cot\theta \cot(\frac{1}{2} - \theta) = \tan\theta$
3. $\sec(\frac{1}{2} - \theta) = \csc\theta \csc(\frac{1}{2} - \theta) = \sec\theta$

Even and Odd identities

Even **Odd**

$$\cos(-\theta) = \cos\theta \sin(-\theta) = -\sin\theta$$

$$\sec(-\theta) = \sec\theta \csc(-\theta) = -\csc\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

Example 19.1.9

Simplify the following trigonometric expressions

a. $\sin\theta\cos^2\theta - \sin\theta$
b. $\sin\theta + \cot\theta\cos\theta$

Solutions

a. $\sin\theta\cos^2\theta - \sin\theta$

$$= \sin\theta\,(\cos^2\theta - 1)$$

$$= \sin\theta\sin^2\theta$$

$$= sin^3\theta$$

b. $\sin\theta + \cot\theta\cos\theta$

$$\frac{\sin\theta}{1} + \frac{\cos\theta\ \cos\theta}{\sin\theta} = \frac{sin^2\theta + cos^2\theta}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta$$

Example 19.1.10

Factorize the following

a. $sec^2\theta - 1$
b. $csc^2\theta - \cot\theta - 3$

Solutions

a. $sec^2\theta - 1$

$$sec^2\theta - 1^2$$

$$(\sec\theta + 1)(\sec\theta - 1)$$

b. $csc^2\theta - \cot\theta - 3$
We have the identity$csc^2\theta = 1 + cot^2\theta$, then

$$1 + cot^2\theta - \cot\theta - 3$$
$$cot^2\theta - \cot\theta - 2$$

Let $\cot\theta = x$

$$x^2 - x - 2 = 0$$
$$(x+1)(x-2)$$
$$(\cot\theta + 1)(\cot\theta - 2)$$

Example 19.1.11

Use the conjugate to express the following as not in fraction form

a. $\frac{1}{1+\sin\theta}$

b. $\frac{1}{1-\cos\theta}$

Solutions

a. $\frac{1}{1+\sin\theta}$

$$\frac{1}{1+\sin\theta} = \frac{1\,(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{1-\sin\theta}{1-sin^2\,\theta}$$

$$= \frac{1-\sin\theta}{cos^2\theta}$$

$$= \frac{1}{cos^2\theta} - \frac{\sin\theta}{cos\theta} \times \frac{1}{\cos\theta}$$

$$= sec^2\theta - \tan\theta\sec\theta$$

b. $\frac{1}{1-\cos\theta}$

$$\frac{1}{1-\cos\theta} = \frac{1\,(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1+\cos\theta}{1-cos^2\,\theta}$$

$$= \frac{1+\cos\theta}{sin^2\theta}$$

$$= \frac{1}{sin^2\theta} + \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta}$$

$$= csc^2\theta - \cot\theta\csc\theta$$

Verifying the identities.

1. Work with one side of the equation at a time.
2. Start with more complex side of the equation.
3. Factor or add fractions to reach at the required expression.

4. If the preceding guidelines are not helping, convert all terms to sine and cosine and simplify.

Example 19.1.12

Verify the following trigonometric identities

a. $\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\csc\theta$

b. $\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=2sec^2\theta$

c. $\frac{sec^2\theta-1}{sec^2\theta}=sin^2\theta$

d. $tan^2\theta-sin^2\theta=sin^4\theta sec^2\theta$

e. $\tan\theta+\cot\theta=\sec\theta\csc\theta$

f. $\sec x+\tan x=\frac{\cos x}{1-\sin x}$

g. $\frac{cot^2 y}{1+\csc y}=\frac{1-\sin y}{\sin y}$

Solutions

a. $\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\csc\theta$

LHS$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}$

$$\frac{sin^2\theta+\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)}=\frac{sin^2\theta+\cos\theta+cos^2\theta}{\sin\theta(1+\cos\theta)}=\frac{1+\cos\theta}{\sin\theta(1+\cos\theta)}=\frac{1}{\sin\theta}=\csc\theta$$

Hence $\boldsymbol{\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\csc\theta}$

b. $\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=2sec^2\theta$

LHS

$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}$$

$$\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)\,(1-\sin\theta)}=\frac{2}{1-\sin\theta)\,(1-\sin\theta)}=\frac{2}{1-sin^2\theta}=\frac{2}{cos^2\theta}=2sec^2\theta$$

Hence $\boldsymbol{\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=2sec^2\theta}$

c. $\frac{sec^2\theta-1}{sec^2\theta}=sin^2\theta$

$$LHS$$

$$\frac{sec^2\theta - 1}{sec^2\theta}$$

$$\frac{sec^2\theta}{sec^2\theta} - \frac{1}{sec^2\theta}$$

We have $sec^2\theta = 1 + tan^2\theta$

$$\frac{1 + tan^2\theta - 1}{sec^2\theta}$$

$$\frac{tan^2\theta}{sec^2\theta}$$

$$tan^2\theta \times \frac{1}{sec^2\theta}$$

$$\frac{sin^2\theta}{cos^2\theta} \times cos^2\theta = sin^2\theta$$

Hence $\frac{sec^2\theta - 1}{sec^2\theta} = sin^2\theta$

d. $tan^2\theta - sin^2\theta = sin^4\theta sec^2\theta$
LHS

$$tan^2\theta - sin^2\theta$$

$$\frac{sin^2\theta}{cos^2\theta} - sin^2\theta$$

$$\frac{sin^2\theta - sin^2\theta cos^2\theta}{cos^2\theta}$$

$$\frac{sin^2\theta(1 - cos^2\theta)}{cos^2\theta}$$

$$\frac{sin^2\theta sin^2\theta}{cos^2\theta}$$

$$\frac{sin^4\theta}{cos^2\theta}$$

$$sin^4\theta \times \frac{1}{cos^2\theta} = sin^4\theta\ sec^2\theta$$

Hence $tan^2\theta - sin^2\theta = sin^4\theta sec^2\theta$

e. $\tan\theta + \cot\theta = \sec\theta \csc\theta$

LHS

$$\tan\theta + \cot\theta$$

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\frac{sin^2\theta + cos^2\theta}{\sin\theta\ cos\theta}$$

$$\frac{1}{\sin\theta\cos\theta}$$

$$\frac{1}{\sin\theta} \times \frac{1}{\cos\theta} = \sec\theta\ csc\theta$$

Hence $\boldsymbol{\tan\theta + \cot\theta = \sec\theta\csc\theta}$

f. $\sec x + \tan x = \frac{\cos x}{1-\sin x}$
RHS

$$\frac{\cos x}{1-\sin x}$$

$$\frac{\cos x\,(1+\sin x)}{(1-\sin x)(1+\sin x)}$$

$$\frac{\cos x + \sin x\cos x}{(1-\sin x)(1+\sin x)}$$

$$\frac{\cos x + \sin x\cos x}{1 - sin^2x}$$

$$\frac{\cos x + \sin x\cos x}{cos^2x}$$

$$\frac{\cos x}{cos^2x} + \frac{\sin x cosx}{cos^2x}$$

$$\frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x$$

Hence $\boldsymbol{\sec x + \tan x = \frac{\cos x}{1-\sin x}}$

g. $\frac{cot^2\,y}{1+\csc y} = \frac{1-\sin y}{\sin y}$
LHS

$$\frac{cot^2\,y}{1+\csc y} = \frac{\frac{cos^2y}{sin^2y}}{1+\frac{1}{\sin y}} = \frac{\frac{cos^2y}{sin^2y}}{\frac{\sin y+1}{\sin y}} = \frac{cos^2y}{sin^2y} \times \frac{\sin y}{\sin y+1} = \frac{cos^2y}{\sin y(1+\sin y)}$$

$$\frac{1 - sin^2 y}{\sin y(1+\sin y)} = \frac{(1-\sin y)(1+\sin y)}{\sin y(1+\sin y)} = \frac{1-\sin y}{\sin y}$$

Hence

$$\boldsymbol{\frac{cot^2\, y}{1+\csc y} = \frac{1-\sin y}{\sin y}}.$$

Solving trigonometric equations

To solve trigonometric equations, we use algebraic techniques such as collecting like terms and factorization. Trigonometric functions are periodic, then most trigonometric equations has infinite many solutions.

Example 19.1.13

Solve for θ in the equation $2\cos\theta + 1 = 0$ for $0^{\circ} \leq \theta \leq 360^{\circ}$

Solution

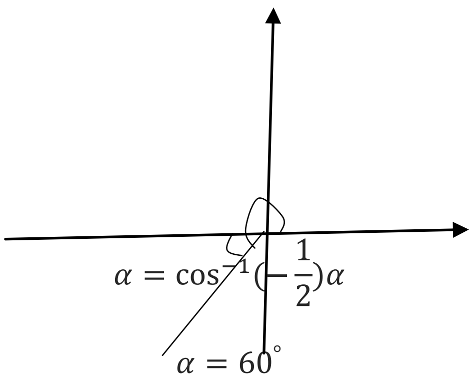
$2\cos\theta + 1 = 0$

$$2\cos\theta = -1$$

$$\frac{2\cos\theta}{2} = \frac{-1}{2}$$

$$\cos\theta = -\frac{1}{2}$$

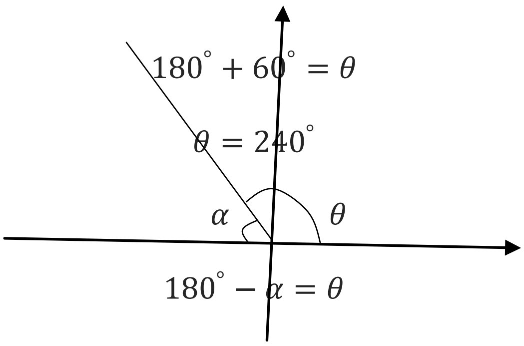
Let α be reference angle for $\theta\theta$



$$\alpha = \cos^{-1}(-\frac{1}{2})\alpha$$

$$\alpha = 60^{\circ}$$

Then $180^{\circ} + \alpha = \theta$



$$180^{\circ} + 60^{\circ} = \theta$$

$$\theta = 240^{\circ}$$

$$180^{\circ} - \alpha = \theta$$

$$180^{\circ} - 60^{\circ} = \theta$$

$$\theta = 120^{\circ}$$

Hence the solution set is $\{120^{\circ}, 240^{\circ}\}$

Note: when evaluating the trigonometric equations, consider the signs in the quadrants

Example 19.1.14

Solve$\sin x \cos x = 0$ for $0 \leq x \leq 2\pi$

Solution

$$\sin x \cos x = 0$$

Either $\sin x = 0$ or $\cos x = 0$

$$x = \{0. \pi\} \quad \text{or} \quad x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

Solution set is $\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$

Example 19.1.15

Solve $2sin^2 x + \sin x - 1 = 0$ for $0 \leq x \leq 2\pi$

Solution

$$2sin^2 x + \sin x - 1 = 0$$

Factorizing

$$(2 \sin x - 1)(\sin x + 1) = 0$$

$$2 \sin x - 1 = 0 \text{ or} \sin x + 1 = 0$$

$$\sin x = \frac{1}{2} \text{ or } \sin x = -1$$

$$x = \sin^{-1}(\frac{1}{2}) \quad \text{or} \quad x = \sin^{-1}(-1)$$

Solution set $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$

Example 19.1.16

Find the general solution of $2 \sin x - 1 = 0$

Solution

$$2\sin\theta - 1 = 0$$

$$2\sin\theta = 1$$

$$\frac{2\sin\theta}{2} = \frac{1}{2}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \sin^{-1}(\frac{1}{2})$$

$$x = \frac{\pi}{6} \quad , \qquad x = \frac{5\pi}{6}$$

Then the general solution is $x = \begin{cases} \frac{\pi}{6} + 2\pi n \\ \frac{5\pi}{6} + 2\pi n \qquad for\ n \in Z \end{cases}$

Example 19.1.17

Find the general solutions for $\sin x \tan x = \sin x$

Solution

$$\sin x \tan x = \sin x$$

$$\sin x \tan x - \sin x = 0$$

$$\sin x(\tan x - 1) = 0$$

$$\sin x = 0 \text{ or } \tan x - 1 = 0$$

$$x = \sin^{-1}(0) \text{ or } x = \tan^{-1}(1)$$

$$x = \pi \text{ or } x = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$$

Hence the general solution is

$$x = \begin{cases} \quad \pi n \\ \frac{\pi}{4} \qquad for\ n \in Z \\ \qquad \frac{5\pi}{4} \end{cases}$$

19.2 Further trigonometric identities

Sum and difference of angles.

i. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
ii. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
iii. $\cos(A+B) = \cos A \cos B - \sin A\ sinB$
iv. $\sin(A-B) = \cos A \cos B + \sin A \sin B$

v. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

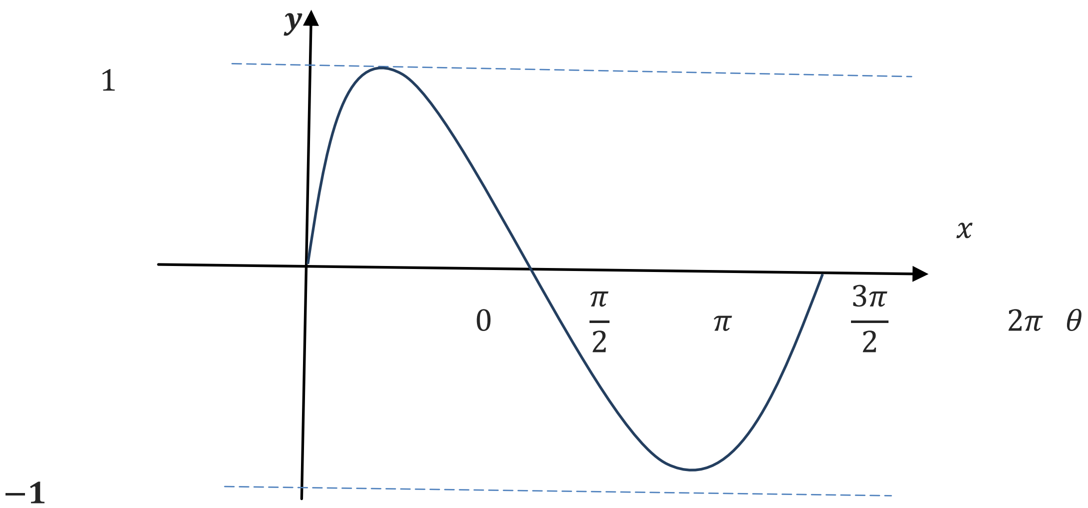
vi. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Graphs of trigonometric functions

The basic sine function is given by $y = \sin x$ and the cosine function as $y = \cos x$

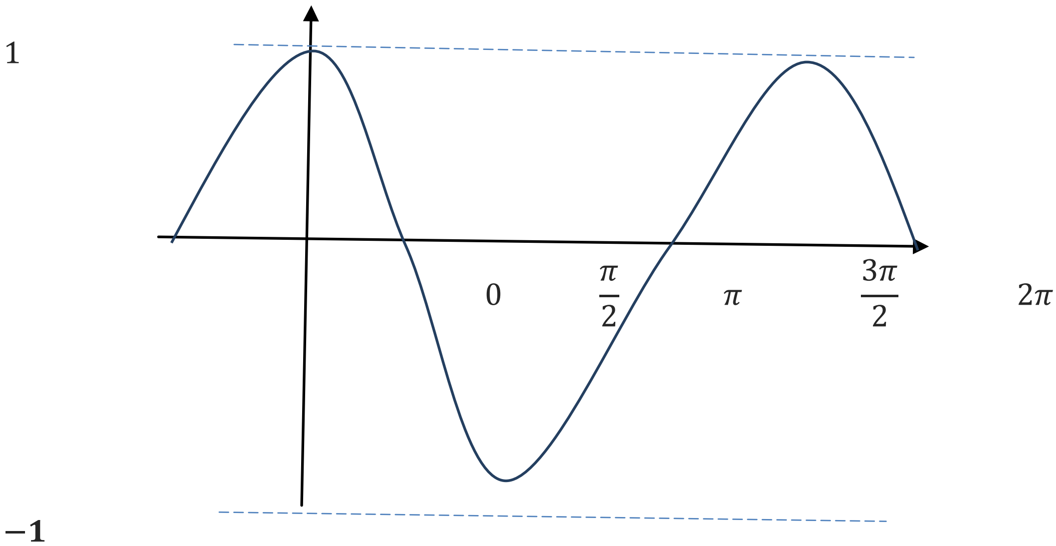
a) Graph of sine function

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0



b) Graph of Cosine function

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	-1	0	1



The general sine and cosine functions

i) $\boldsymbol{y = d + a\sin b(x - c)}$

ii) $\boldsymbol{y = d + a\cos b(x - c)}$ for a, b, c and d are constants

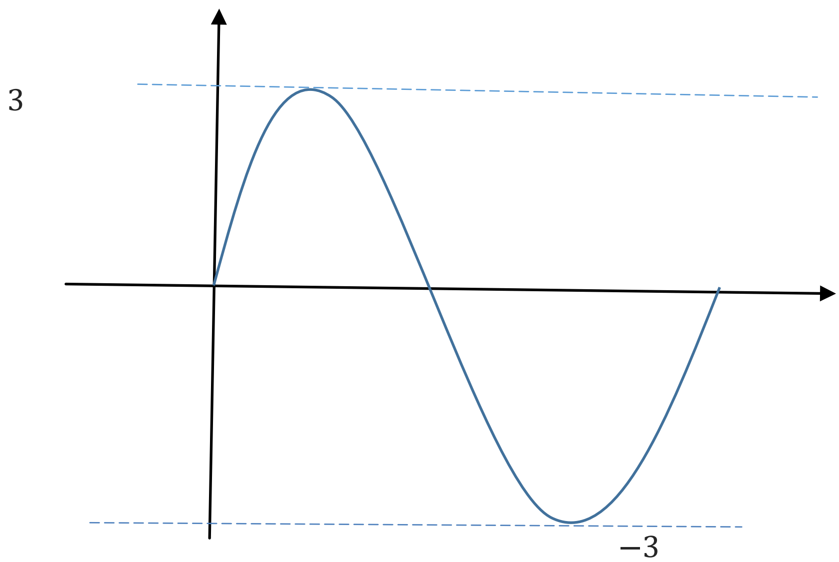
a. Amplitude

The largest value of y in the function $y = a\sin x$ *and* $y = a\cos x$ is called the amplitude.

$$\boldsymbol{Amplitude} = |\boldsymbol{a}|$$

Example 19.2.1

Sketch the graph of $y = 3\sin x$



b. Period of sine and cosine functions

Sine and cosine are periodic functions. 2π Makes a complete cycle of the graph of the sine and the cosine functions $y = \sin x$ *and* $y = \cos x$. Hence 2π is the period.

For any function $y = \sin bx$ *or* $y = \cos bx$

$$\boldsymbol{period = \frac{2\pi}{b}}$$

Example 19.2.2

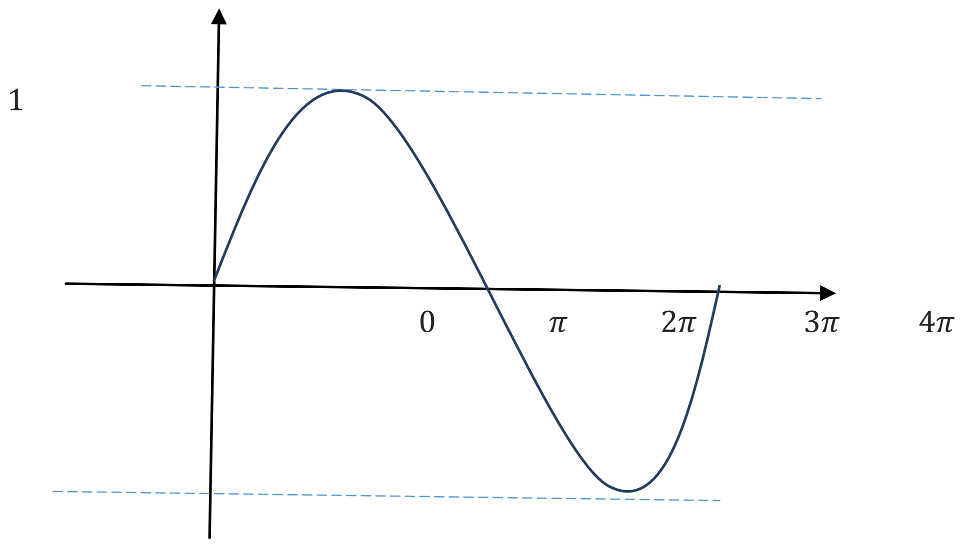
Find the period of the function$y = \sin\frac{x}{2}$. Hence sketch the graph of the function.

Solution

$$y = \sin\frac{x}{2}$$

$y = sin\ \frac{1}{2}x \Rightarrow b = \frac{1}{2}$

$$period = \frac{2\pi}{b}$$
$$period = \frac{2\pi}{\frac{1}{2}} = 4\pi$$



c. Phase shift

Sine and cosine functions can be translated or shifted its position either to the left or to the right just as any other functions.

$$y = \sin b(x - c)$$
$$y = \cos b(x - c)$$

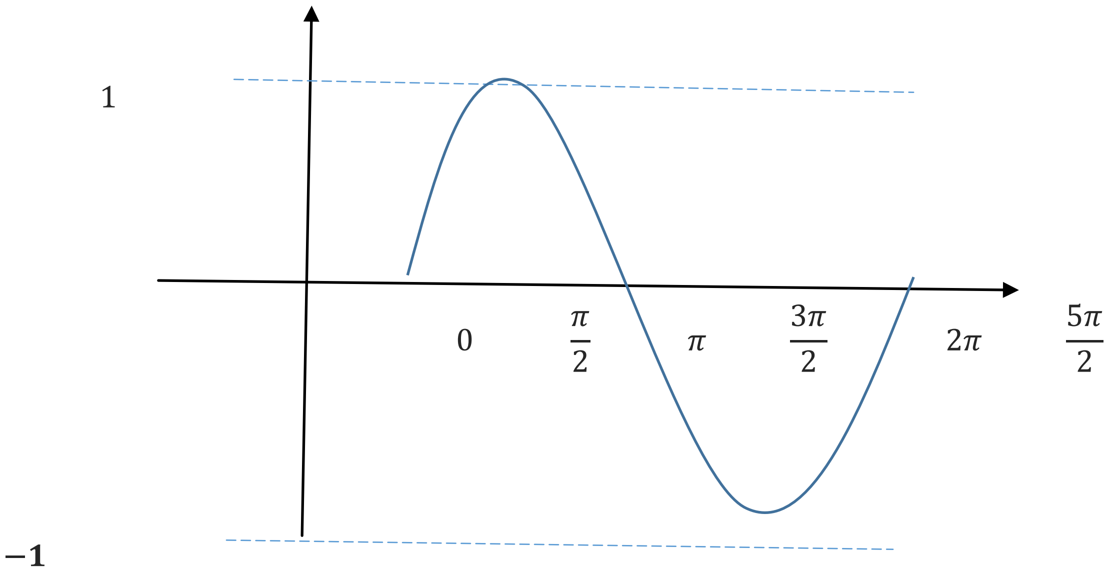
Then c is the phase shift of the function

Example 19.2.4

Sketch the graph $y = sin\left(x - \frac{\pi}{2}\right)$

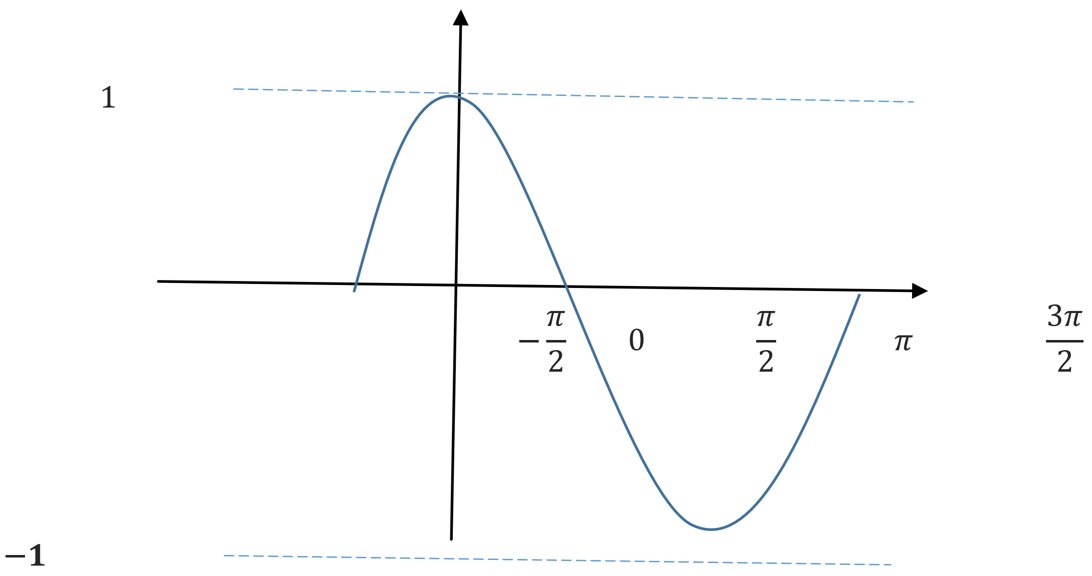
Solution

The graph of $y = \sin x$ moves $\frac{\pi}{2}$ units to the right



Example 19.2.5

Sketch the graph $y = \sin\left(x + \frac{\pi}{2}\right)$



d. Vertical shift

The translation caused by the constant d in the equation $y = d + a \sin b(x - c)$ and

$y = d + a\cos b(x - c)$. The graph moves d units up if d is positive and d units down if d is negative. Therefore, the graph oscillates about the horizontal $d - axis$ instead of x-axis

Example 19.2.6

Sketch the graph of $y = 2 + 3\cos 2\left(x - \frac{\pi}{2}\right)$

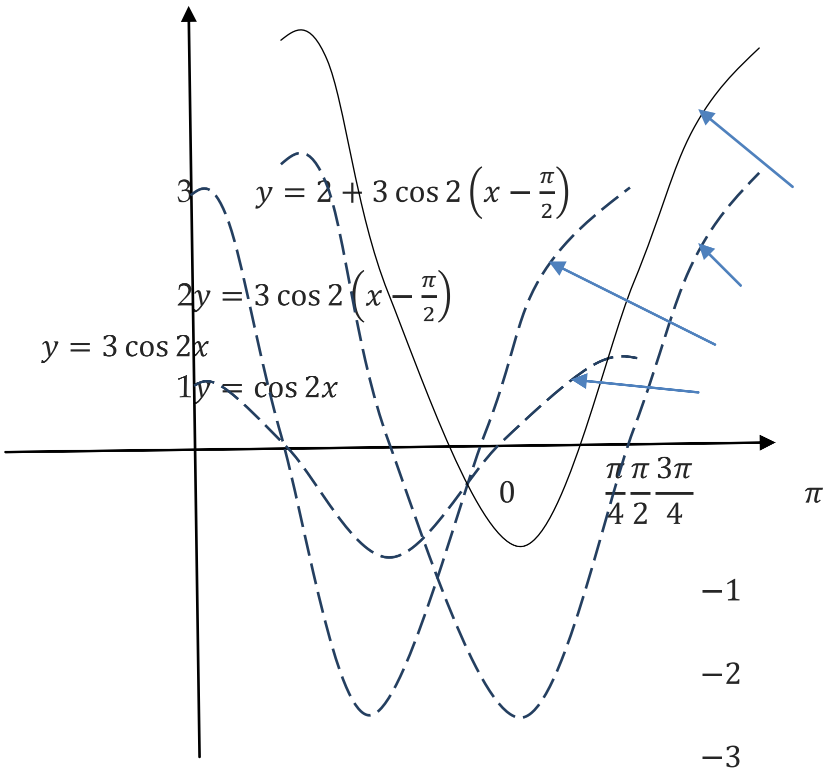
Solution

$$y = d + \mathrm{a}\cos b(x - c)$$

$$Amplitude = |a| = |3| = 3$$

$$period = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

phase shift $=$ c units to the right $= \frac{\pi}{2}$units right. Vertical shift $= d$ units up $= 2$ units up



Example 19.2.7

Sketch the graph of $y = -2 - 3\sin 2\left(x - \frac{\pi}{2}\right)$

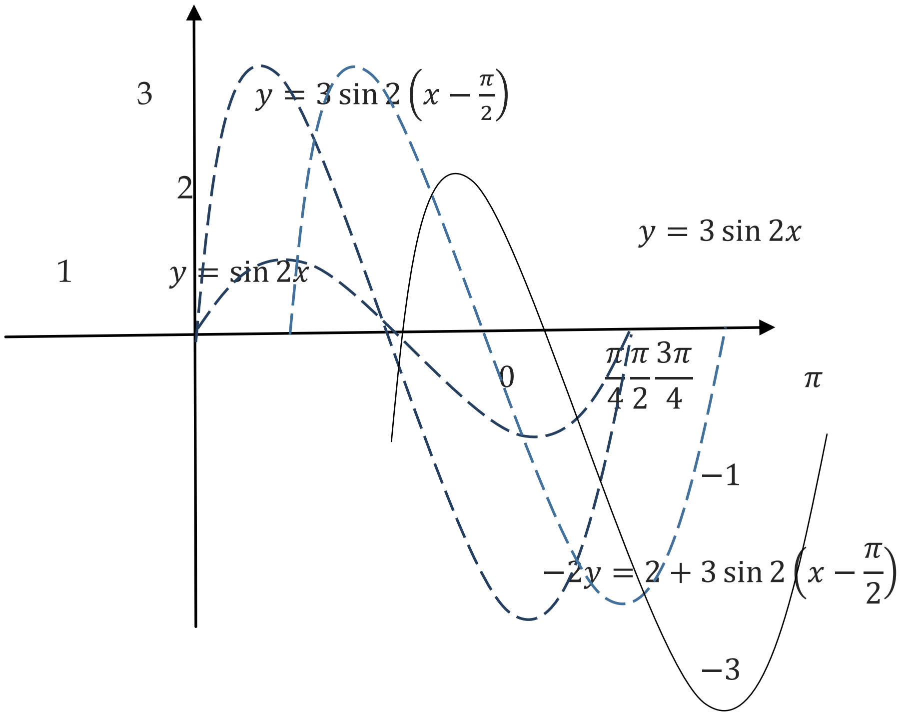
Solution

$$y = d + \mathrm{a}\sin b(x - c)$$

$$Amplitude = |a| = |-3| = 3period = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

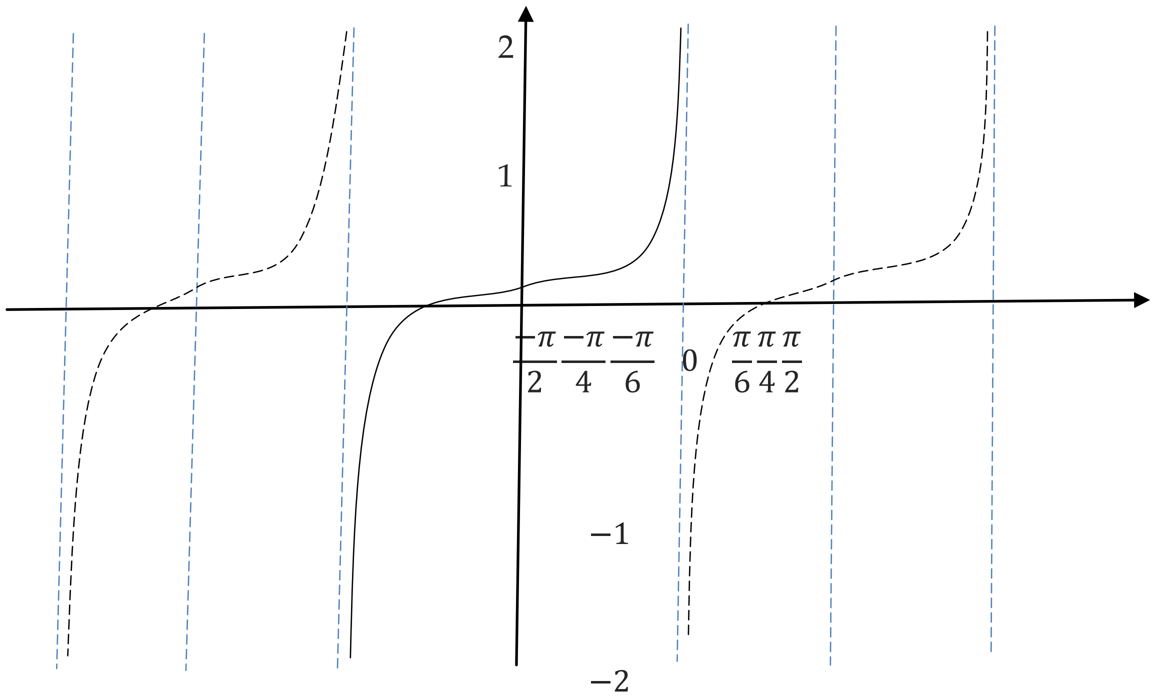
$$phase\ shift = c\ units\ to\ the\ right = \frac{\pi}{2}\ units\ right$$

$$vertical\ shift = d\ units\ down = 2\ units\ down$$



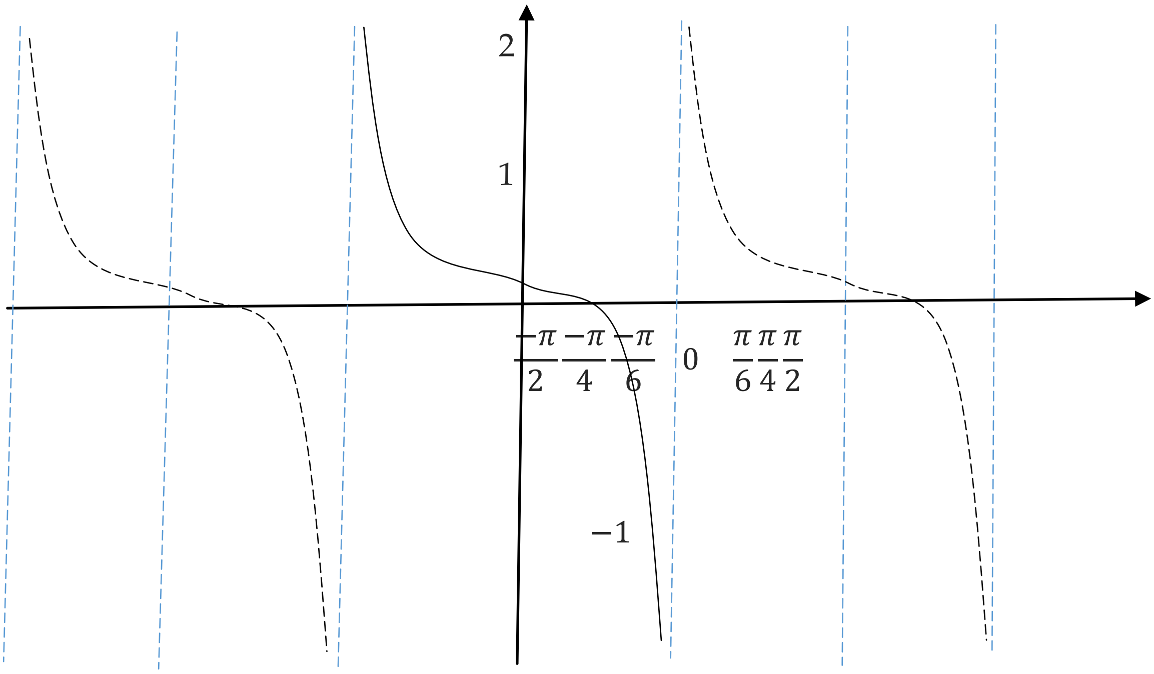
Graph of tangent function

x	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\tan x$	$un.defined$	-1	-0.58	0	0.58	1	$un.defined$



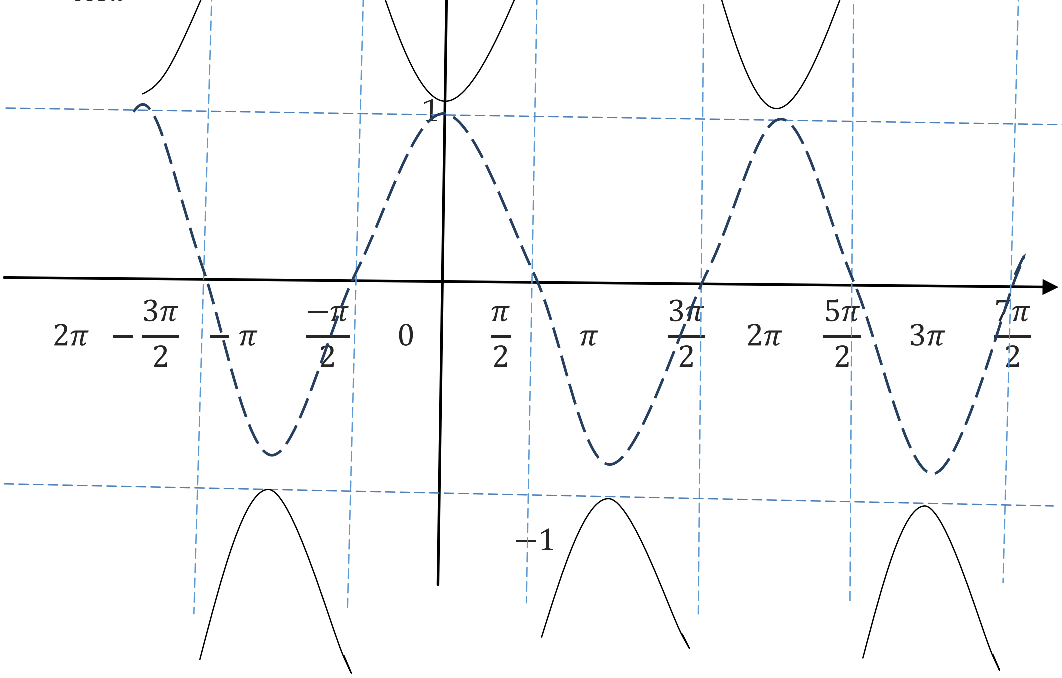
Graph of cotangent function

x	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\tan x$	$un.defined$	1	0.58	0	-0.58	-1	$un.defined$

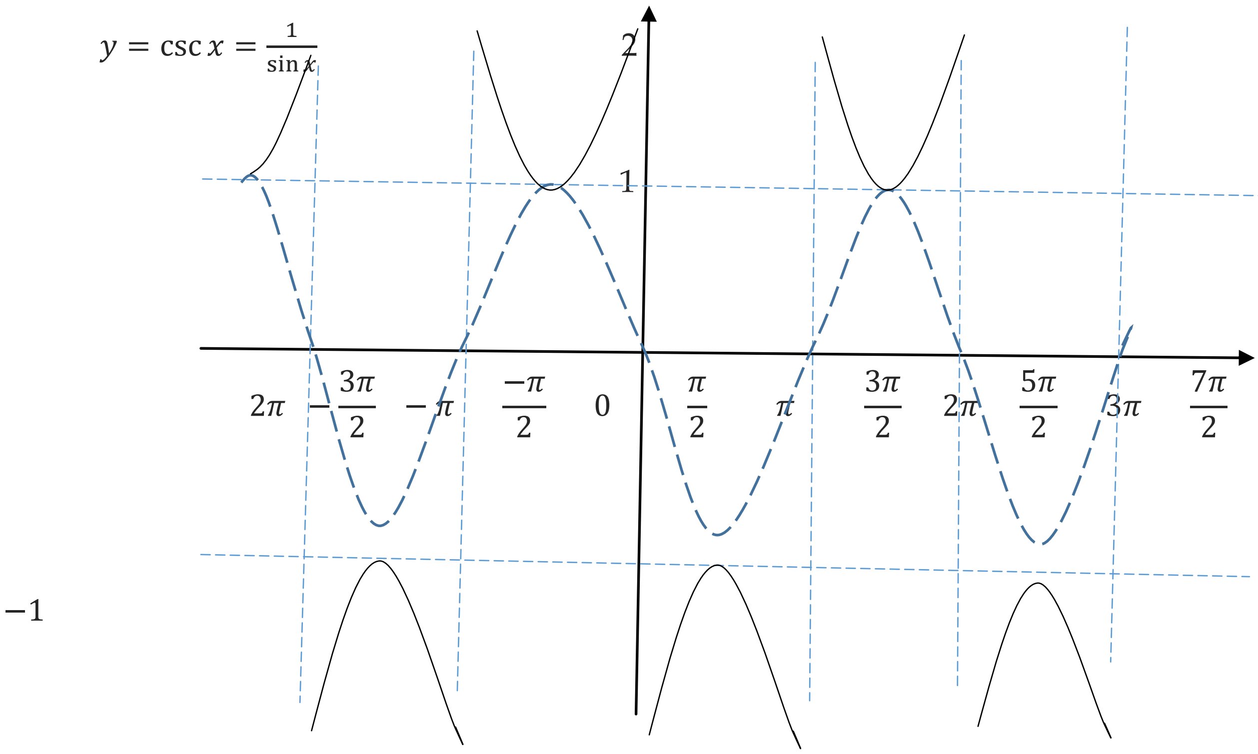


The graph of secant

$y = \sec x = \frac{1}{\cos x}$2



The graph of secant



Translations of tangent, secant and cosecant functions

Example 19.2.8

Sketch the graph of

a) $y = 2\sec 3x$
b) $y = 2\csc\left(x - \frac{\pi}{2}\right)$
c) $y = 2\sec\left(x + \frac{\pi}{2}\right)$

Double Angles

a.

$$\sin(A + A) = \sin A\cos A\ + \sin A\cos A$$

$$\sin 2A = \sin A\cos A\ + \sin A\cos A$$

$\boldsymbol{\sin 2A = 2\sin A\cos A}$ **and** $\boldsymbol{\sin A\ \cos A = \frac{1}{2}\ \sin 2A}$

b.

$$\cos(A + A) = \cos A\ \cos A - \sin A\sin A$$

$$\cos 2A = cos^2 A - sin^2 A$$

$$\cos 2A = cos^2 A - (1 - cos^2 A)$$

$$\cos 2A = cos^2 A - 1 + cos^2 A$$

$$\mathbf{\cos 2A = 2cos^2 A - 1}$$

$$\cos 2A = cos^2 A - 1 + cos^2 A$$

$$\cos 2A = 2cos^2 A - 1$$

$$\boldsymbol{cos^2 A = \frac{1 + \cos 2A}{2}}$$

c.

$$\cos(A + A) = \cos A \ \cos A - \sin A \sin A$$

$$\cos 2A = cos^2 A - sin^2 A$$

$$\cos 2A = (1 - sin^2 A) - sin^2 A)$$

$$\mathbf{\cos 2A = 1 - 2sin^2 A}$$

$$\boldsymbol{sin^2 A = \frac{1 - \cos 2A}{2}}$$

Example 19.2.9

a) Show that $\tan 2A = \frac{2 \tan A}{1 - tan^2 A}$

b) Verify that $\cos 2\theta = cos^4\theta - sin^2\theta$

c) Express $\sin 3\theta$ in terms of $\sin\theta$ ***Solutions***

a) $\tan 2A = \frac{2 \tan A}{1 - tan^2 A}$

LHS

$$\tan 2A = \frac{\sin 2A}{\cos 2A}$$

$$= \frac{2sinA \cos A}{cos^A - sin^2 A}$$

divide in $cos^2 A$

$$= \frac{\frac{2\sin A}{\cos A}}{1 - \frac{sin^2 A}{cos^2 A}}$$

$$= \frac{2\tan A}{1 - tan^2 A}$$

b) $\cos 2\theta = cos^4\theta - sin^2\theta$
RHS

$$cos^4\theta - sin^2\theta = (cos^2\theta)^2 - (sin^2\theta)^2$$

by different of two squares

$$= (cos^2\theta + sin^2\theta)(cos^2\theta - sin^2\theta)$$
$$= (cos^2\theta - sin^2\theta)$$
$$= \cos 2\theta$$

c) $\sin 3\theta = \sin(2\theta + \theta)$

$$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$
$$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2sin^2\theta)\sin\theta$$
$$= 2\sin\theta\, cos^2\theta + \sin\theta - 2sin^3\theta$$
$$= 2\sin\theta\,(1 - sin^2\theta) + \sin\theta - 2sin^3\theta$$
$$= 2\sin\theta - 2sin^3\theta + \sin\theta - 2sin^3\theta$$
$$= 3\sin\theta - 4sin^3\theta.$$

Half Angle Formula

i) $\boldsymbol{sin^2\left(\frac{\theta}{2}\right) = \frac{1-cos\,\theta}{2}}$

ii) $\boldsymbol{cos^2\left(\frac{\theta}{2}\right) = \frac{1+cos\theta}{2}}$

iii) $\boldsymbol{tan^2\left(\frac{\theta}{2}\right) = \frac{1-cos\,\theta}{1+cos\,\theta}}$

Example 19.2.10

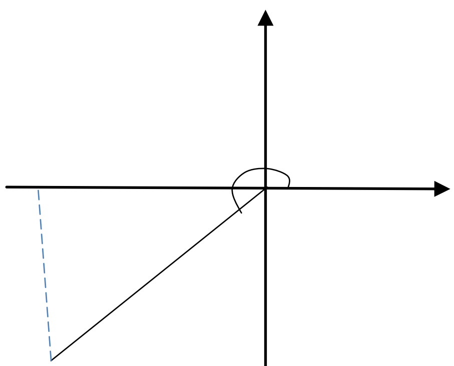
a) If $\cos\theta = \frac{-4}{5}$and θ is in the third quadrant, find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$ *and* $\tan\left(\frac{\theta}{2}\right)$
b) Evaluate tan 67.5 using half-angle formula

Solutions

a) $\cos\theta = \frac{-4}{5}$

-4

-3



$$cos\theta = \frac{-4}{5}, \quad \sin\theta = \frac{-3}{5}$$

Then $sin^2\left(\frac{\theta}{2}\right) = \frac{1-cos\,\theta}{2} = \frac{1-\left(\frac{-4}{5}\right)}{2} = \frac{9}{10} \Rightarrow \sin\left(\frac{\theta}{2}\right) = \frac{3}{\sqrt{10}}$

$$cos^2\left(\frac{\theta}{2}\right)=\frac{1+cos\theta}{2}=\frac{1+\left(\frac{-4}{5}\right)}{2}=\frac{1}{10}\quad \Rightarrow \quad \cos\left(\frac{\theta}{2}\right)=\frac{1}{\sqrt{10}}$$

$$tan^2\left(\frac{\theta}{2}\right)=\frac{1-cos\ \theta}{1+cos\ \theta}=\frac{1-\left(\frac{-4}{5}\right)}{1+\left(\frac{-4}{5}\right)}\quad \Rightarrow \tan\left(\frac{\theta}{2}\right)=-3$$

b) $let\ \frac{\theta}{2}=67.5^{\circ} \quad \Rightarrow \quad \theta=135^{\circ}$

The factor formulae

i) Product to Sum

a) $\sin A\cos B+\cos A\sin B=\sin(A+B)$

$$\sin A\cos B-\cos A\sin B=\sin(A-B)$$

Adding the two equations we get

$$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$

$$\mathbf{\sin \boldsymbol{A} \cos \boldsymbol{B}=\frac{1}{2}(\sin(\boldsymbol{A}+\boldsymbol{B})+\sin(\boldsymbol{A}-\boldsymbol{B}))}$$

b)

$$\sin A\cos B+\cos A\sin B=\sin(A+B)$$
$$\sin A\cos B-\cos A\sin B=\sin(A-B)$$

Subtracting the two equations we get

$$2\cos A\sin B=\sin(A+B)-\sin(A-B)$$

$$\boldsymbol{cosA\ sin\ B}=\mathbf{\frac{1}{2}(\sin(\boldsymbol{A}+\boldsymbol{B})-\sin(\boldsymbol{A}-\boldsymbol{B}))}$$

c) $\cos A\cos B+\sin A\sin B=\cos(A-B)$

$$\cos A\cos B-\sin A\sin B=\cos(A+B)$$

Adding the two equations we get

$$2\cos A\cos B=\cos(A-B)+\cos(A+B)$$

$$\mathbf{\cos \boldsymbol{A}\cos \boldsymbol{B}=\frac{1}{2}(\cos(\boldsymbol{A}-\boldsymbol{B})+\cos(\boldsymbol{A}+\boldsymbol{B}))}$$

d) $\cos A\cos B+\sin A\sin B=\cos(A-B)$

$$\cos A\cos B-\sin A\sin B=\cos(A+B)$$

Subtracting the two equations we get

$$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

$$\boldsymbol{sinA\ sinB}=\mathbf{\frac{1}{2}(\cos(\boldsymbol{A}-\boldsymbol{B})-\cos(\boldsymbol{A}+\boldsymbol{B}))}$$

Example 19.2.11

Express $\cos 5x\sin 4x$ as a product or different

Solution

$$cosA\ sin\ B = \frac{1}{2}(\ \sin(A+B) - \sin(A-B))$$

Let $A = 5x$ and $B = 4x$

$$cos5x\ sin\ 4x = \frac{1}{2}(\ \sin(5x+4x) - \sin(5x-4x))$$
$$cos5x\ sin\ 4x = \frac{1}{2}(\ \sin(9x) - \sin(x))$$
$$cos5x\ sin\ 4x = \frac{1}{2}(\ \sin 9x - \sin x)$$

ii) Sum to Product Formula

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$
$$let\ x = A+B$$
$$y = A-B$$

Adding the equations we get

$$x + y = 2B$$
$$B = \frac{x+y}{2}$$

subtracting the equations we get

$$x - y = 2A$$
$$A = \frac{x-y}{2}$$

Writing the equations in terms of $x\ and\ y$ we have

a) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$$\mathbf{\sin x + \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)}$$

b) $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$$\mathbf{\sin x - \sin y = 2\cos\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)}$$

c) $\cos(A-B) + \cos(A+B) = 2\cos A\cos B$

$$\mathbf{\cos y + \mathit{cos}\ x = 2\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)}$$

d) $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$$\mathbf{\cos y - \mathit{cos}\ x = 2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)}$$

Example 19.2.12

a. Express $\cos x + \cos 3x$ as a product
b. Find all the solutions of $\sin 5\theta + \sin 3\theta$ in the interval $(0, 2\pi]$

Solutions

a. $\cos x + \cos 3x$

$$\cos y + cos\ x = 2\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + cos\ 3x = 2\cos\left(\frac{3x - x}{2}\right)\cos\left(\frac{3x + x}{2}\right)$$
$$\cos x + cos\ 3x = 2\cos\left(\frac{2x}{2}\right)\cos\left(\frac{4x}{2}\right)$$
$$\cos x + cos\ 3x = 2\cos x\cos 2x$$

The Form a $\boldsymbol{\cos\theta + b\sin\theta}$

We can write the expression of the form $\boldsymbol{a\ \cos\theta + b\sin\theta}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants as

$$\boldsymbol{a\ \cos\theta + b\sin\theta = R\sin(\theta \pm \alpha)}$$

$$\boldsymbol{a\ \cos\theta + b\sin\theta = R\cos(\theta \pm \alpha)}$$

For $R > 0$ and $0 \leq \alpha \leq 90^{\circ}$ $\quad R = \sqrt{a^2 + b^2}$

Example 19.2.13

a) Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(\theta \pm \alpha)$
b) Show that $\cos\theta + \sin\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$
c) Show that $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin(2\theta - \alpha)$
d) Find the maximum value of the function $\sqrt{3}\cos\theta - \sin\theta$
e) Solve $\sqrt{3}\cos\theta + \sin\theta = 1$ in the interval $0 \leq \theta \leq 2\pi$

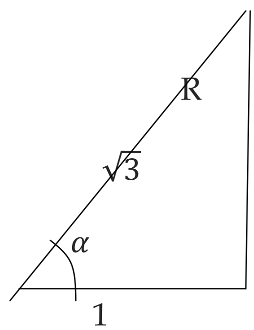
Solutions

a) $\sin x - \sqrt{3}\cos x \equiv R\sin(x - \alpha)$

$$\sin x - \sqrt{3}\cos x \equiv R(\sin x\cos\alpha - \cos x\sin\alpha)$$
$$\sin x - \sqrt{3}\cos x \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$$

Then

$$\sin x \equiv R\sin x\cos\alpha\ \sqrt{3}\cos x \equiv R\cos x\sin\alpha$$
$$1 = R\cos\alpha\ \sqrt{3} = R\sin\alpha$$
$$\cos\alpha = \frac{1}{R}\sin\alpha = \frac{\sqrt{3}}{R}$$



By Pythagoras theorem, $R = 2$

Then $\cos\alpha = \frac{1}{R} = \frac{1}{2}$

$$\alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\alpha = 60^{\circ}$$

We use the form $\sin x - \sqrt{3}\cos x \equiv R\sin(x - \alpha)$

$$\sin x - \sqrt{3}\cos x \equiv 2\sin(x - 60^{\circ})$$

b) $\cos\theta + \sin\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

By setting $\cos\theta + \sin\theta = R\sin(\theta + \alpha)$

$$R = \sqrt{a^2 + b^2}$$

$$R = \sqrt{1^2 + 1^2}$$

$$R = \sqrt{2}$$

Then $\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + \alpha)$

$$\cos\theta + \sin\theta = \sqrt{2}\sin\theta\cos\alpha - \sqrt{2}\cos\theta\sin\alpha$$

$$\cos\theta = \sqrt{2}\sin\theta\cos\alpha$$

$$1 = \sqrt{2}\sin\alpha$$

$$\sin\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = 45^{\circ}$$

Hence $\cos\theta + \sin\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

c) $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin(2\theta - \alpha)$

$$hints$$

By setting $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv R\sin(2\theta - \alpha)$

$$R = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$R = \sqrt{3 + 1}$$

$$R = 2$$

d) $\sqrt{3}\cos\theta - \sin\theta$

By setting $\sqrt{3}\cos\theta - \sin\theta \equiv R\cos(\theta - \alpha)$

$$R = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$R = \sqrt{3 + 1}$$

$$R = 2$$

Hence maximum value is 2

e) $\sqrt{3}\cos\theta + \sin\theta = 1$ for $0 \le \theta \le 2\pi$

By setting $\sqrt{3}\cos\theta - \sin\theta \equiv R\cos(\theta - \alpha)$

$$R = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$R = \sqrt{3 + 1}$$

$$R = 2$$
$$\sqrt{3}\cos\theta - \sin\theta \equiv 2\cos(\theta - \alpha)$$
$$\sqrt{3}\cos\theta - \sin\theta \equiv 2\cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\sqrt{3}\cos\theta = 2\cos\theta\cos\alpha$$
$$\cos\alpha = \frac{\sqrt{3}}{2}$$
$$\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$\alpha = \frac{\pi}{6}$$

Then

$$\sqrt{3}\cos\theta - \sin\theta \equiv 2\cos(\theta - \frac{\pi}{6})$$
$$2\cos(\theta - \frac{\pi}{6}) = 1$$

$$\cos(\theta - \frac{\pi}{6}) = \frac{1}{2}$$

Let $\varphi = \left(\theta - \frac{\pi}{6}\right)$

$$\cos\varphi = \frac{1}{2}$$
$$\varphi = \cos^{-1}\left(\frac{1}{2}\right)$$
$$\varphi = \frac{\pi}{3} \quad or \quad \varphi = \frac{5\pi}{3}$$

We substitute $\varphi = \left(\theta - \frac{\pi}{6}\right)$

$$\frac{\pi}{3} = \left(\theta - \frac{\pi}{6}\right) \quad or \quad \frac{5\pi}{3} = \left(\theta - \frac{\pi}{6}\right)$$

Solving for θ

$$\theta = \frac{\pi}{2} \ or \ \frac{11\pi}{6}$$

The solution set is $\{\frac{\pi}{2}, \frac{11\pi}{6}\}$

19.3 INVERSE TRIGONOMETRIC FUNCTIONS

Trigonometric functions are many to one functions. Hence in the open interval, trigonometric functions has no inverse. But by restricting the domain, they can be made one-to-one with the inverse.

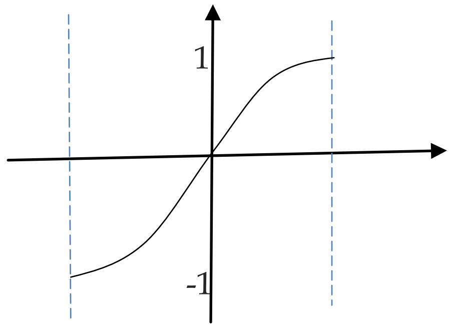
The inverse of sine function

The inverse of sine function is defined as $y = \sin^{-1} x$

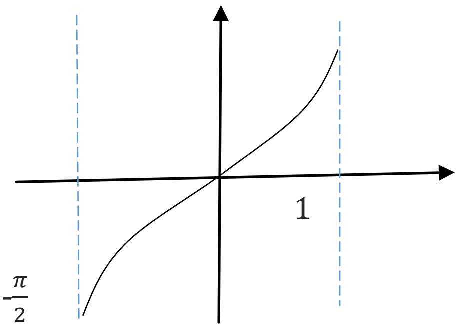
$y = \sin^{-1} x$ if and only if $\sin y = x$.

The invers function can also be written as $y = arc\sin x$

For the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ the graph of sine function satisfy the horizontal line test.



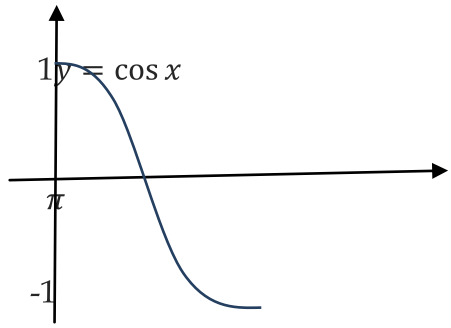
The graph of $y = \sin^{-1} x$ is defined for $-1 \leq x \leq 1$



Domain= $\{x: -1 \leq x \leq 1\}$
Range = $\{y: -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

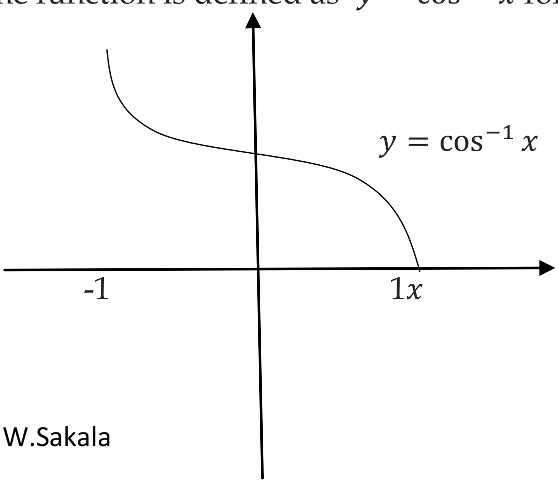
Inverse of cosine function

If the domain of $y = \cos x$ is restricted $0 \leq x \leq \pi$. The graph of $y = \cos x$ satisfies the horizontal line test and therefore the function becomes a one-to-one.



Domain = $\{x: 0 \leq x \leq \pi\}$ Range = $\{y: -1 \leq y \leq 1\}$

The inverse of cosine function is defined as $y = \cos^{-1} x$ for $-1 \leq x \leq 1$

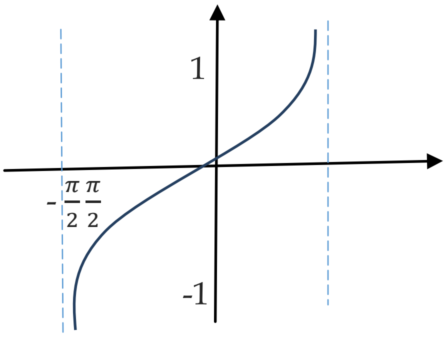


$$-\frac{\pi}{2}$$

Domain= $\{x| -1 \leq x \leq 1\}$ Range= $\{ y: 0 \leq y \leq \pi\}$

Inverse of the tangent function

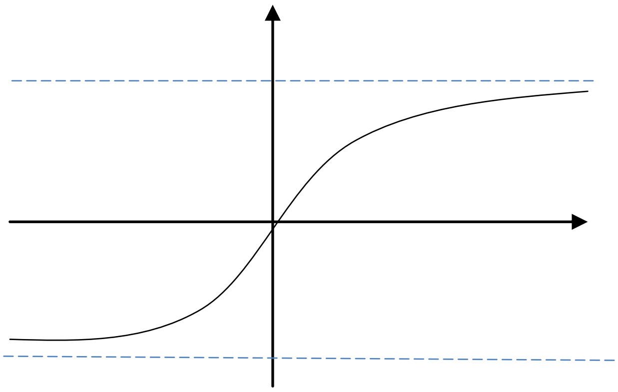
If the domain of $y = \tan x$ is restricted to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The graph of $y = \tan x$ satisfies the horizontal line test and therefore the function becomes a one-to-one.



The inverse of the tangent function is defined by $y = \tan^{-1} x$ or $y = arc\tan x$

The function $y = \tan^{-1} x$ is defined for $(-\infty, \infty)$

$\frac{\pi}{2}$



$-\frac{\pi}{2}$

Domain= $(-\infty, \infty\}$ Range= $\left\{y: -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

Expressions involving inverse trigonometric functions

Example 19.3.1

Evaluate

a) $arc\cos\left(\frac{\sqrt{2}}{2}\right)$ $b)$ $\sin^{-1}(-1)$ $c)$ $arc\tan(1)$ $d)$ $arc\sin\left(-\frac{\sqrt{3}}{2}\right)$

Solutions

a) $arc\cos\left(\frac{\sqrt{2}}{2}\right)$

$y = \cos^{-1} x$ for $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$

Then $arc\cos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

b) $\sin^{-1}(-1)$

$$y = \sin^{-1}(-1)$$
$$\sin y = -1$$
$$y = -\frac{\pi}{2}$$

c) $arc\tan(1)$

$y = \tan^{-1} x$ for the domain $= (-\infty, \infty)$ and Range$= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$y = \tan^{-1}(1)$$
$$\tan^{-1} y = 1$$
$$y = \frac{\pi}{4}$$

d) $arc\sin\left(-\frac{\sqrt{3}}{2}\right)$

$$y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$\sin^{-1} y = -\frac{\sqrt{3}}{2}$$

$\frac{\sqrt{3}}{2}$ is out of the domain $-1 \leq x \leq 1$. Hence the function is undefined.

Example 19.3.2

Evaluate

a) $\cos(arc\sin\frac{3}{5})$ b) $\sin[arc\cos\left(\frac{-2}{3}\right)]$

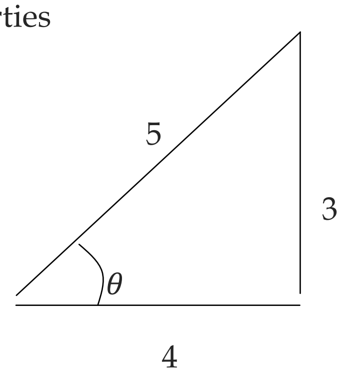
Solutions

a) $\cos(arc\sin\frac{3}{5})$

Let $\theta = arc\sin\frac{3}{5}$

$$\sin\theta = \frac{3}{5}$$

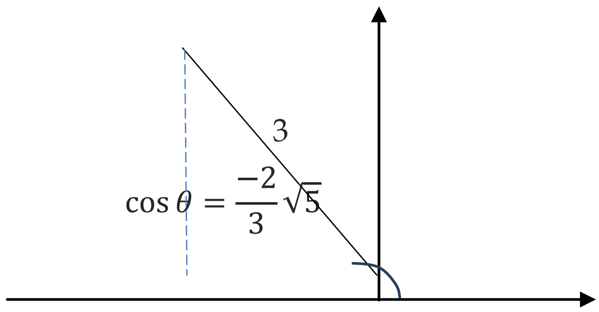
by the sine ratios properties



Then $\cos(arc\sin\frac{3}{5}) = \cos\theta = \frac{4}{5}$

b) $\sin[arc\cos\left(\frac{-2}{3}\right)]$

Let $\theta = \cos^{-1}\left(\frac{-2}{3}\right)$



θ

-2

Then$\sin[arc\cos\left(\frac{-2}{3}\right)] = \sin\theta = \frac{\sqrt{5}}{3}$

Example 19.3.2

Evaluate $\cos(arc\tan\frac{15}{5} - arc\sin\frac{7}{25})$

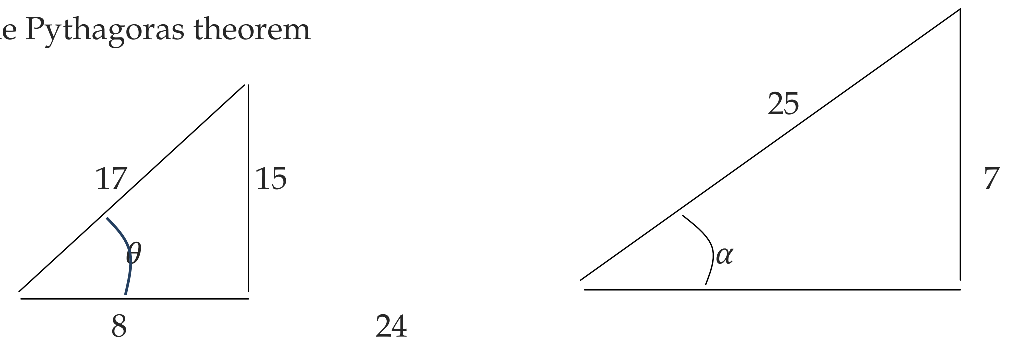
Solution

$\cos(arc\tan\frac{15}{5} - arc\sin\frac{7}{25})$

Let $\theta = arc\tan\frac{15}{5}$ and $\alpha = arc\sin\frac{7}{25}$

This implies that $\tan\theta = \frac{15}{5}$ and $\sin\alpha = \frac{7}{25}$

By the Pythagoras theorem



Then $\sin\theta = \frac{17}{15}$, $\cos\theta = \frac{8}{17}$, $\cos\alpha = \frac{24}{25}$, $\sin\alpha = \frac{7}{25}$

Now $\cos(arc\tan\frac{15}{5} - arc\sin\frac{7}{25}) = \cos(\theta - \alpha)$

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$\cos(\theta - \alpha) = \left(\frac{8}{17} \times \frac{24}{25}\right) + \left(\frac{17}{15} \times \frac{7}{25}\right) = \frac{297}{425}$$

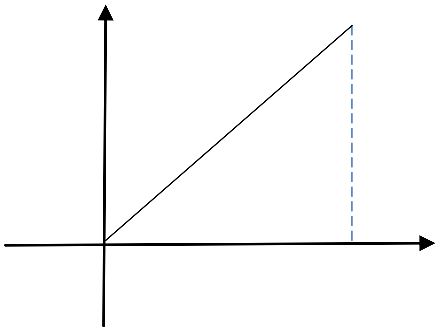
Hence $\cos(arc\tan\frac{15}{5} - arc\sin\frac{7}{25}$

20: FURTHER COMPLEX NUMBERS

Real numbers are graphed as points on the number line while complex numbers of the form $x + yi$are graphed in a coordinate plane called ***complex plane***.The complex number in form of $z = x + yi$ is called the ***Cartesian standard form*** or ***rectangular form***

Polar form of complex numbers

The modulus of the complex number $z = x + yi$ is given by $r = |z| = \sqrt{x^2 + y^2}$



Example 20.1.1

Find the modulus of the complex number

a) $z = -3 + 4i$
b) $z = -6i$

Solutions

a) $z = -3 + 4i$

$$x = -3 \quad , \quad y = 4$$
$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(-3)^2 + (4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$= 5$$

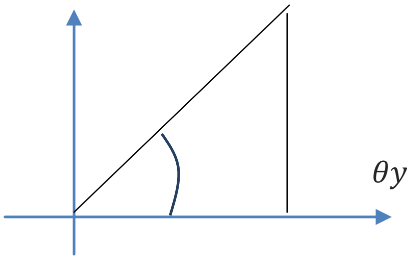
a) $z = -6i$

$$x = 0 \quad , \quad y = -6$$
$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(0)^2 + (-6)^2}$$
$$|z| = \sqrt{0 + 36}$$
$$|z| = \sqrt{36}$$
$$= 6$$

Trigonometric form of complex numbers

A complex number $z = x + yi$ can be written in form of trigonometric function.

r



$\theta y = r\sin\theta$

$x = r\cos\theta$

$$r = \sqrt{x^2 + y^2}$$

Then, the complex number $z = x + yi$ can be written in trigonometric form as

$$z = r\cos\theta \ + i\, r\sin\theta$$

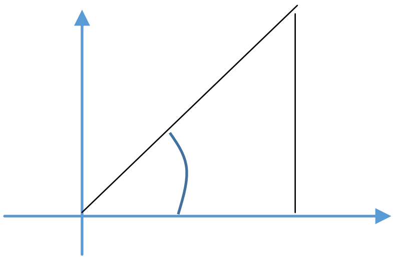
$$\boldsymbol{z = r(\cos\theta + i\sin\theta)}$$

$$\boldsymbol{\tan\theta = \frac{y}{x}}$$

Example 20.1.2

Write $2 + 2i$ in trigonometric form

Solution



$r = 2\sqrt{2}$

$\theta y = 2$

$$x = 2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{2^2 + 2^2}$$

$$r = \sqrt{8} = 2\sqrt{2}$$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{2}{2} \Rightarrow \tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) \Rightarrow \theta = 45^{\circ}$

The $\boldsymbol{z = r(\cos\theta + i\sin\theta)}$

$$\boldsymbol{z = 2\sqrt{2}(\cos 45^{\circ} + i\sin 45^{\circ})}.$$

Example 20.1.3

Write the complex in rectangular or Cartesian standard form.

a) $z = \sqrt{8}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin(-\frac{\pi}{3})\right)$
b) $z = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

Solutions

a) $z = \sqrt{8}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

From the even and odd trigonometric function properties $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$z = \sqrt{8}\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \sqrt{8}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \sqrt{2} - \sqrt{6}\,i$$

b) $z = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

by the reference angles $\frac{5\pi}{3} = -\frac{\pi}{3}$

$$z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\,i$$

Multiplicationof complex numbers in polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

Let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 \times z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$\boldsymbol{z_1 \times z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)}$$

Example 20.1.4

Find the product of $\left[2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right]\left[\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$

Solution

$$\left[2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right]\left[\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)\right]$$

$$= \boldsymbol{r_1 r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)}$$

$$= 2\sqrt{2}\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{2}\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$$

Division of complex numbers in polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

Let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{cos^2\theta_2 + sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\left[\frac{\cos(\theta_1-\theta_2) + isin\,(\theta_1-\theta_2)}{1}\right]$$

$$= \boldsymbol{\frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + isin\,(\theta_1-\theta_2))}.$$

Example 20.1.5

Express each of the following complex numbers in polar form, perform the operation and give the results in rectangular form / standard form.

a) $(-1+\sqrt{3}\,i)(\sqrt{3}+i)$

b) $\frac{4 \; - \; 4\sqrt{3}i}{-2\sqrt{3} \; + \; 2i}$

Solutions

a) $(-1+\sqrt{3}\,i)(\sqrt{3}+i)$

$$= [2(\cos 120^\circ + i\sin 120^\circ)][2(\cos 30^\circ + i\sin 30^\circ)]$$

$$= (2 \times 2)[\cos(120^\circ + 30^\circ) + i\sin(120^\circ + 30^\circ)]$$

$$= 4[\cos 150^\circ + i\sin 150^\circ)$$

$$= 4[-\cos 30^\circ + i\sin 30^\circ]$$

$$= 4\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$$

$$= -2\sqrt{3} + i$$

b) $\frac{4 \ - \ 4\sqrt{3}i}{-2\sqrt{3} \ + \ 2i}$

$$= \frac{8[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}]}{4[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}]}$$

$$= \frac{8}{4}\left[\cos\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right)\right]$$

$$= 2\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$

$$= 2\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= 2\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$$

$$= -\sqrt{3} + i$$

De Moivre`s Theorem

De Moivre`s Theorem is a procedure used for finding the powers and roots of complex numbers when expressed in trigonometric form. For example,$z^2 = z.z$.

It is not possible to evaluate the power of complex number when the power is very big but theDe Moivre`s Theorem provides a simpler way of evaluating it.

Let $z = r(\cos\theta + i\sin\theta)$be a non-zero complex number. Then,

$$z^2 = [r(\cos\theta + i\sin\theta)]\ [r(\cos\theta + i\sin\theta)]$$

$$= r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)]$$

$$= r^2[\cos(2\theta) + i\sin(2\theta)]$$

$$then \qquad z^3 = z.z^2 = [r(\cos\theta + i\sin\theta)]r^2[\cos(2\theta) + i\sin(2\theta)]$$

$$= r^3[\cos(\theta + 2\theta) + i\sin(\theta + 2\theta)]$$

$$= r^3[\cos(3\theta) + i\sin(3\theta)]$$

$$z^4 = z.z^3 = [r(\cos\theta + i\sin\theta)]r^3[\cos(3\theta) + i\sin(3\theta)]$$

$$= r^4[\cos(\theta + 3\theta) + i\sin(\theta + 3\theta)]$$

$$= r^4[\cos(4\theta) + i\sin(4\theta)]$$

When the power becomes very big, we have

$$\boldsymbol{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

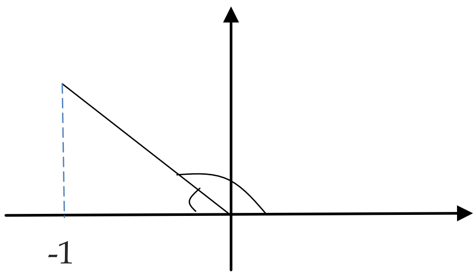
Example 20.1.6

Evaluate $\left(-1+\sqrt{3}\,i\right)^{12}$

Solution

$\sqrt{3}$

$\alpha\theta$



$$\tan\alpha = -\frac{\sqrt{3}}{1}$$

$\alpha = \tan^{-1}(\sqrt{3}),\qquad \alpha = \frac{\pi}{3}$. Then $\theta = \pi - \frac{\pi}{3},\qquad \theta = \frac{2\pi}{3}$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\left(-1+\sqrt{3}\,i\right)^{12} = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12}$$

$$= 2^{12}\left[\cos\left(12\times\frac{2\pi}{3}\right) + i\sin\left(12\times\frac{2\pi}{3}\right)\right]$$

$$= 2^{12}[\cos(8\pi) + i\sin(8\pi)]$$

$$= 2^{12}[\cos(2\pi) + i\sin(2\pi)]$$

$$= 2^{12}[\cos 0 + i\sin 0]$$

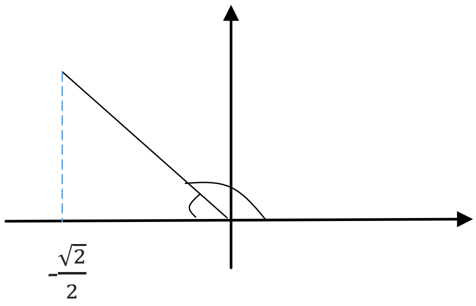
$$= 2^{12}[1+0]$$

$$= 2^{12}$$

Example 20.1.7

Use the De Moivre`s Theorem to evaluate $\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)^{10}$

Solution



$$\tan\alpha = \frac{\sqrt{2}}{2} \Big/ -\frac{\sqrt{2}}{2} = 1$$

$$\alpha = \frac{\pi}{4}$$

$$\theta = \pi - \frac{\pi}{4}$$

$$\theta = \frac{3\pi}{4}$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$$

$$r = \sqrt{\frac{2}{4} + \frac{2}{4}}$$

$$r = \sqrt{1}$$

$$r = 1$$

$$\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right)^{10} = \left[\,1\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^{10}$$

$$= 1^{10}\left[\cos\left(10 \times \frac{3\pi}{4}\right) + i\sin\left(10 \times \frac{3\pi}{4}\right)\right]$$

$$= \left[\cos\left(\frac{15\pi}{2}\right) + i\sin\left(\frac{15\pi}{2}\right)\right]$$

$$= \left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$$

$$= [0 + (-1)i]$$

$$= -i$$

Example 20.1.8

Use De Moivre`s Theorem to simplify $\frac{\cos 7\theta + i\sin 7\theta}{\cos 2\theta - i\sin 2\theta}$

Solution

$$\frac{\cos 7\theta + i\sin 7\theta}{\cos 2\theta - i\sin 2\theta}$$

$$= \frac{\cos 7\theta + i\sin 7\theta}{\cos(-2\theta) + i\sin(-2\theta)}$$

$$= \frac{(\cos\theta + i\sin\theta)^7}{(\cos\theta + i\sin\theta)^{-2}}$$

$$= (\cos\theta + i\sin\theta)^{7-(-2)}$$

$$= (\cos\theta + i\sin\theta)^9$$

$$= \cos 9\theta + i\sin 9\theta$$

Example 20.1.9

Use De Moivre`s Theorem to simplify $(\cos 2\theta + i\sin 2\theta)(\cos 5\theta + i\sin 5\theta)$

Solution$(\cos 2\theta + i\sin 2\theta)(\cos 5\theta + i\sin 5\theta)$

$$= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^5$$

$$= (\cos\theta + i\sin\theta)^{2+5}$$

$$= (\cos\theta + i\sin\theta)^7$$

$$= \cos 7\theta + i\sin 7\theta$$

Roots of complex numbers

The complex number $u = a + bi$ is an n^{th} root of the complex number $z = x + yi$ if

$z = u^n$ That is $x + yi = (a + bi)^n$

Example 20.1.10

Show that the complex numbers $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a sixthroot of 1

Solution

To show that $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a sixth root of 1, we evaluate $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^6$ and check if the answer is 1

$$\tan\alpha = \left(\frac{\sqrt{3}}{2}\Big/\frac{1}{2}\right) = 1$$

$$\alpha = \frac{\pi}{3}$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$r = 1$$

$$\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\,i\right)^6 = \left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^6$$

$$= 1^6\left[\cos\left(6 \times \frac{\pi}{3}\right) + i\sin\left(6 \times \frac{\pi}{3}\right)\right]$$

$$= [\cos(2\pi) + i\sin(2\pi)]$$

$$= [\cos(0) + i\sin(0)]$$

$$= [1 + 0]$$

$$= 1$$

Hence $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a sixth root of 1

n^{th} Roots of a complex number

If$z = r\,(\cos\theta\ + i\sin\theta)$ is a complex number, then there exist $n$distinct n^{th} roots of z given by

$$\boldsymbol{w_k} = \sqrt[n]{\boldsymbol{r}}\left[\mathbf{cos}\left(\frac{\boldsymbol{\theta} + 360^{\circ}\boldsymbol{k}}{\boldsymbol{n}}\right) + \boldsymbol{i}\,\mathbf{sin}\left(\frac{\boldsymbol{\theta} + 360^{\circ}\boldsymbol{k}}{\boldsymbol{n}}\right)\right] for\ k = 0,1,2,3,4,\dots,(n-1)$$

Example 20.1.11

a) Find all the cube roots of 8
b) Find the forth root of $-8 - 8\sqrt{3}\,i$

Solutions

a) Writing 8 in trigonometric form

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(8)^2 + (0)^2}$$

$$r = \sqrt{64 + 0}$$

$$r = 8$$

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{0}{8}$$

$$\theta = 0^{\circ}$$

$$8 = 8(\cos 0 + i\sin 0)$$

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^{\circ}k}{n}\right) + i\sin\left(\frac{\theta + 360^{\circ}k}{n}\right)\right] \quad for\ k = 0,1,2,3,4, \dots, (n-1)$$

The $n = 3$, $r = 8$, $\theta = 0$ and $k = 0,1,2$

$$w_k = \sqrt[3]{8}\left[\cos\left(\frac{0 + 360^{\circ}k}{3}\right) + i\sin\left(\frac{0 + 360^{\circ}k}{3}\right)\right]$$

$$w_k = \sqrt[3]{8}\left[\cos\left(\frac{360^{\circ}k}{3}\right) + i\sin\left(\frac{360^{\circ}k}{3}\right)\right]$$

$$for\ k = 0$$

$$w_0 = \sqrt[3]{8}\left[\cos\left(\frac{360^{\circ} \times 0}{3}\right) + i\sin\left(\frac{360^{\circ} \times 0}{3}\right)\right]$$

$$w_0 = \sqrt[3]{8}[\cos(0) + i\sin(0)]$$

$$w_0 = 2(1 + 0)$$

$$w_0 = 2$$

$$for\ k = 1$$

$$w_1 = \sqrt[3]{8}\left[\cos\left(\frac{360^{\circ} \times 1}{3}\right) + i\sin\left(\frac{360^{\circ} \times 1}{3}\right)\right]$$

$$w_1 = \sqrt[3]{8}[\cos(120^{\circ}) + i\sin(120^{\circ})]$$

$$w_1 = 2(1 + 0)$$

$$w_1 = 2$$

$$for\ k = 1$$

$$w_1 = \sqrt[3]{8}\left[\cos\left(\frac{360^{\circ} \times 1}{3}\right) + i\sin\left(\frac{360^{\circ} \times 1}{3}\right)\right]$$

$$w_1 = \sqrt[3]{8}[\cos(120^\circ) + i\sin(120^\circ)]$$

$$w_1 = 2(-\cos 60^\circ + i\sin 60^\circ)$$

$$w_1 = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\ i\right)$$

$$w_1 = -1 + \sqrt{3}i$$

$$for\ k = 2$$

$$w_2 = \sqrt[3]{8}\left[\cos\left(\frac{360^\circ \times 2}{3}\right) + i\sin\left(\frac{360^\circ \times 2}{3}\right)\right]$$

$$w_2 = \sqrt[3]{8}[\cos(240^\circ) + i\sin(240^\circ)]$$

$$w_2 = 2(-\cos 240^\circ - i\sin 240^\circ)$$

$$w_2 = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\ i\right)$$

$$w_2 = -1 - \sqrt{3}i$$

Hence the cube roots of 8 are $\{\ 2\ , -1 + \sqrt{3}i\ ,\ -1 - \sqrt{3}i\ \}$

b) $-8 - 8\sqrt{3}\ i$

$$x = -8 \quad and \quad y = -8\sqrt{3}$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-8)^2 + \left(-8\sqrt{3}\right)^2} = \sqrt{64 + 192} = \sqrt{256} = 16$$

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{-8\sqrt{3}}{-8}$$

$$\theta = \tan^{-1}\left(\sqrt{3}\right)$$

$$\theta = 240^\circ$$

$$-8 - 8\sqrt{3}i = 16(\cos 240^\circ + i\sin 240^\circ)$$

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right] \quad for\ k = 0,1,2,3,4,\ldots,(n-1)$$

The $n = 4\ , r = 16\ , \theta = 240^\circ$ and $k = 0,1,2,3$

$$w_k = \sqrt[4]{16}\left[\cos\left(\frac{240^\circ + 360^\circ k}{4}\right) + i\sin\left(\frac{240^\circ + 360^\circ k}{4}\right)\right]$$

$$w_k = 2\left[\cos\left(\frac{240^\circ + 360^\circ k}{4}\right) + i\sin\left(\frac{240^\circ + 360^\circ k}{4}\right)\right]$$

$$for\ k = 0$$

$$w_0 = 2\left[\cos\left(\frac{240^\circ + (360^\circ \times 0)}{4}\right) + i\sin\left(\frac{240^\circ + (360^\circ \times 0)}{4}\right)\right]$$

$$w_0 = 2\left[\cos\left(\frac{240^\circ}{4}\right) + i\sin\left(\frac{240^\circ}{4}\right)\right]$$

$$w_0 = 2\,[\cos 60^\circ + i\sin 60^\circ] = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = 1 + \sqrt{3}\,i$$

Continue calculations for k $= 1, 2, 3$

REVIEW EXERCISE

1. Express each of the following in trigonometric form where $0 \le \theta \le 2\pi$
   a) $-2 + 2i$ $b)$ $-4 - 4i$ $c)$ $-3i$ $d)$ -4 $e) - 4 - 4\sqrt{3}i$
2. Change the given complex from trigonometric form to $a + bi$ form
   a) $4(\cos 30^\circ + i\sin 30^\circ)$
   b) $5(\cos 120^\circ + i\sin 120^\circ)$
   c) $3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$
3. Find the product $z_1 z_2$ by using the trigonometric form of the numbers. Express the final result in $a + bi$.
   a) $z_1 = \sqrt{3} + i$ $z_2 = -2\sqrt{3} - 2i$
   b) $z_1 = 1 + \sqrt{3}i$ $z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
   c) $z_1 = 8 + 0i$ $z_2 = 0 - 3i$
   d) $z_1 = 5\sqrt{3} + 5i$ $z_2 = 6\sqrt{3} + 6i$
4. Use the De Moivre's to find the indicated powers. Express the result in $a + bi$
   a) $(1 + i)^{20}$
   b) $(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2})^{15}$
   c) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{10}$
5. Find the indicated roots. Express the roots in $a + bi$ form if they are exact. Otherwise, leave the answer in trigonometric form.
   a) The three cube roots of 3
   b) The four fourth roots of $-8 + 8\sqrt{3}i$
   c) The five fifth roots of $1 - i$
   d) The two square roots of $\frac{3}{2} + \frac{9\sqrt{3}}{2}i$
6. Use the De Moivre's theorem to simplify the following
   a) $(\cos 2\theta + i\sin 2\theta)(\cos 5\pi + i\sin 5\theta)$

b) $\frac{\cos 7\theta + i \sin 7\theta}{\cos 2\theta - i \sin 2\theta}$

c) $\frac{(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3})^5 (\cos\frac{2\pi}{3} - i \sin\frac{2\pi}{3})^4}{(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6})}$

7. Find the cube roots of $2 + 2i$
8. Express $\frac{2\sqrt{3}+2i}{\sqrt{3}-i}$ in the form $r\,(\cos\theta \; + i \sin\theta)$
9. Find the forth root of $\frac{2\sqrt{3}+2i}{\sqrt{3}-i}$

21: HYPERBOLIC FUNCTIONS

The hyperbolic functions is defined as $f(x) = \sinh x$, $f(x) = \cosh x$, $f(x) = \tanh x$

Definitions

The hyperbolic cosine of x is a function $\boldsymbol{\cosh x} = \frac{e^x + e^{-x}}{2}$

The hyperbolic sine of x is a function $\boldsymbol{\sinh x} = \frac{e^x - e^{-x}}{2}$

Identities for hyperbolic functions

Hyperbolic functions has the identities which are similar to the identities of trigonometric functions but they are not equal.

a) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

b) $\coth x = \frac{1}{\tanh x} = \frac{1}{\frac{e^x - e^{-x}}{e^x + e^{-x}}} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

c) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\frac{e^x + e^{-x}}{2}} = \frac{2}{e^x + e^{-x}}$

d) $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{1}{\frac{e^x - e^{-x}}{2}} = \frac{2}{e^x - e^{-x}}$

Example 21.1.1

Show that $\cosh^2 x - \sinh^2 x = 1$

Solution: We have $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}$$

Then by the deference of two squares $\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$

$$\cosh^2 x - \sinh^2 x = (e^x)(e^{-x}) = e^{x-x} = e^0 = 1$$

Hence

$$\boldsymbol{cosh^2x - sinh^2x = 1}$$

Example 21.1.2

Prove that $\sinh(2x) = 2\sinh x \cosh x$

Solution

From the left hand side

$$2\sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2} \times \frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{2x} + 1 - 1 - e^{-2x}}{2}$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$$= \sinh(2x).$$

Example 21.1.3

Prove that$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$

Solution

From the left hand side

$$\cosh A \cosh B + \sinh A \sinh B = \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) + \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$$

$$= \left(\frac{e^{A+B} + e^{A-B} + e^{B-A} + e^{-A-B}}{4}\right) + \left(\frac{e^{A+B} - e^{A-B} - e^{B-A} + e^{-A-B}}{4}\right)$$

$$= \frac{2e^{A+B} + 2e^{-A-B}}{4}$$

$$= \frac{2(e^{A+B} + e^{-A-B})}{4}$$

$$= \frac{\left(e^{A+B} + e^{-(A+B)}\right)}{2}$$

$$= \cosh(A + B).$$

Hence $\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$

The following identities hold

1. $\cos h^2 x - \sin h^2 x = 1$

2. $sech^2 x = 1 - tanh^2 x$
3. $\sinh 2x = 2\sinh x \cosh x$
4. $\cosh 2x = 1 + 2sinh^2 x$
5. $\sinh(A+B) = \sinh A \cosh B + \sinh B \cosh A$
6. $\sinh(A-B) = \sinh A \cosh B - \sinh B \cosh A$
7. $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$
8. $\cosh(A-B) = \cosh A \cosh B - \sinh A \sinh B$

9. $\tanh 2x = \frac{2\tanh x}{1+\tanh x}$

Equations of hyperbolic functions

Example 21.1.4

Solve for real values of x in each of the following hyperbolic functions.

a) $\sinh x + 4 = 4\cosh x$.
b) $\cosh 2x - 7\cosh x + 7 = 0$

Solutions

a) $\sinh x + 4 = 4\cosh x$

$$\frac{e^x - e^{-x}}{2} + 4 = 4\left(\frac{e^x + e^{-x}}{2}\right)$$
$$\frac{e^x - e^{-x}}{2} + 4 = 2(e^x + e^{-x})$$

$$e^x - e^{-x} + 8 = 4(e^x + e^{-x})$$
$$e^x - \frac{1}{e^x} + 8 = 4\left(e^x + \frac{1}{e^x}\right)$$

Let $y = e^x$, then

$$y - \frac{1}{y} + 8 = 4\left(y + \frac{1}{y}\right)$$
$$y^2 - 1 + 8y = 4y^2 + 4$$
$$y^2 - 4y^2 + 8y = 1 + 4$$
$$3y^2 - 8y + 5y = 0$$

By factorization method

$$(y-1)(3y-5)$$

Either $y = 1$ or $y = \frac{5}{3}$

Then $e^x = 1$

$$e^x = 1^0$$
$$x = 0$$

Or

$$e^x = \frac{5}{3}$$
$$\ln e^x = \ln\left(\frac{5}{3}\right)$$
$$x = \ln\left(\frac{5}{3}\right) = \ln 5 - \ln 3$$

b) $\cosh 2x - 7\cosh x + 7 = 0$

$$\cosh 2x = 2cosh^2 x - 1$$

Then $\quad \cosh 2x - 7\cosh x + 7 = 2cosh^2 x - 1 - 7\cosh x + 7 = 0$

$$2cosh^2 x - 1 - 7\cosh x + 7 = 0$$
$$2cosh^2 x - 7\cosh x + 6 = 0$$

Let $y = \cosh x$, then

$$2y^2 - 7y + 6 = 0$$
$$2y^2 - 4y - 3y + 6 = 0$$
$$(y-2)(2y-3) = 0$$
$$y = 2 \quad or \quad y = \frac{3}{2}$$

Then

$$\cosh x = 2 \quad or \quad \cosh x = \frac{3}{2}$$
$$\frac{e^x + e^{-x}}{2} = 2$$
$$e^x + e^{-x} = 4$$
$$e^x + \frac{1}{e^x} = 4$$

Let $m = e^x$

$$m + \frac{1}{m} = 4$$
$$m^2 - 4m + 1 = 0$$

Using the quadratic formula

$$m = 2 \pm \sqrt{3}$$
$$m = 2 + \sqrt{3} \quad or \quad m = 2 - \sqrt{3}$$
$$e^x = 2 + \sqrt{3}$$
$$\ln e^x = \ln\left(2 + \sqrt{3}\right)$$
$$x = \ln\left(2 + \sqrt{3}\right)$$

$$e^x = 2 - \sqrt{3}$$
$$\ln e^x = \ln\left(2 - \sqrt{3}\right)$$
$$x = \ln\left(2 - \sqrt{3}\right)$$
$$\cosh x = \frac{3}{2}$$
$$\frac{e^x + e^{-x}}{2} = \frac{3}{2}$$
$$e^x + e^{-x} = 3$$
$$e^x + \frac{1}{e^x} = 3$$

Let $m = e^x$

$$p + \frac{1}{m} = 3$$
$$p^2 - 3p + 1 = 0$$

Using the quadratic formula

$$p = \frac{3 \pm \sqrt{5}}{2}$$
$$m = \frac{3 + \sqrt{5}}{2} \quad or \quad m = \frac{3 - \sqrt{5}}{2}$$

$$e^x = \frac{3 \pm \sqrt{5}}{2}$$

$$\ln e^x = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right)$$

$$x = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right).$$

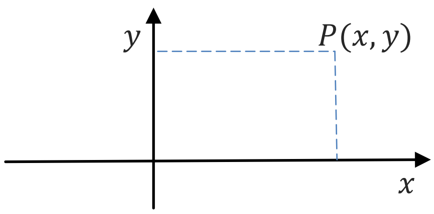
REVIEW EXERCISE

1. Prove the following identities
   a) $coth^2x - 1 = cosech^2x$
   b) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
   c) $\cosh 3x = 4\, cosh^3x - 3 \cosh x$
2. Solve for real values of $x$in each equation below.
   a) $\sinh x + 4 = 4 \cosh x$
   b) $4 \tanh x - \operatorname{sech} x = 1$
   c) $\cosh 2x - 7 \cosh x + 7 = 0$
   d) $sinh^2x - 3 \cosh x = 3$
   e) $sinh^2x - 5 \cosh x + 5 = 0$

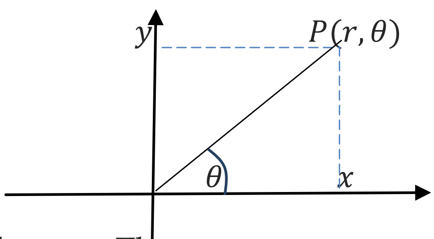
22: POLAR CO-ORDINATE SYSTEM

Position of a point in a plane can be represented in two ways. These are

a) Cartesian co-ordinate. That is (x, y)



b) Polar co-ordinate. That is (r, θ) where r the radius from a fixed point is and θ is an angle from a fixed point.



r

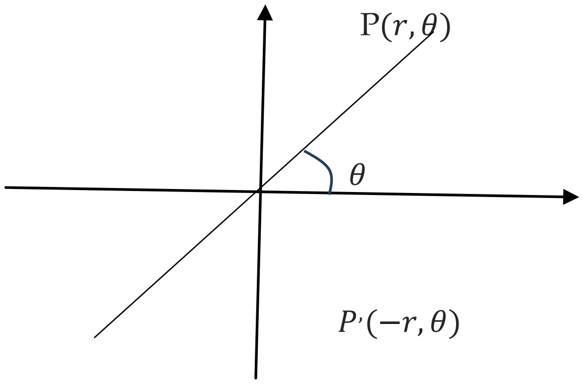
Using the Pythagoras Theorem,

$$\boldsymbol{r = \sqrt{x^2 + y^2}} \tan\theta = \frac{y}{x}$$

$$\boldsymbol{\theta = \tan^{-1}}\left(\frac{y}{x}\right)$$

i) r is the directed distance from the origin to a point P
ii) θ is the angle ant-clock wise from the polar axis to the line segment.

We can extend the meaning of polar co-ordinate (r, θ) to the case where r is negative. The point $(-r, \theta)$ lies on the same line as (r, θ) but on the opposite side of the origin to P.

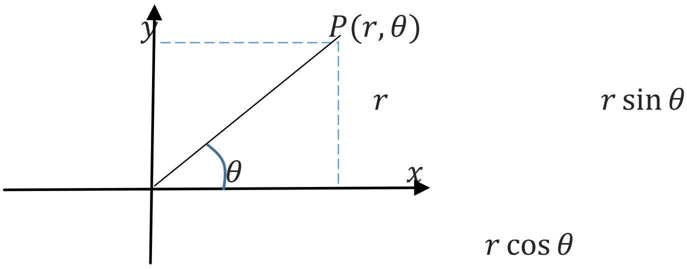


Example 22.1.1

Plot the points whose polar co-ordinates are

a) $\left(2, \frac{\pi}{4}\right)$ $\quad$ $b)$ $\left(-2, \frac{\pi}{4}\right)$ $\quad$ $b)$ $(3, 3\pi)$

Co-ordinate Conversion



The polar co-ordinates (r, θ) are related to the rectangular co-ordinates as follows

$$(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta) = r(\cos \theta, \sin \theta)$$

For $r = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$

Converting polar co-ordinate to rectangular co-ordinate.

Example 22.1.2

Change the following polar coordinates to rectangular co-ordinates.

a) $(2, \pi)$
$b)$ $\sqrt{3}, \frac{\pi}{6})$

Solutions

a) $(2\,,\pi)$

$$(2,\pi) = (r,\theta)$$
$$r = 2 \text{ and } \theta = \pi$$
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$
$$x = 2\cos\pi \quad \text{and} \quad y = 2\sin\pi$$
$$x = 2(-1) \quad \text{and} \quad y = 2(0)$$
$$x = -2 \quad \text{and} \quad y = 0$$

Then, the rectangular co-ordinate is $(x,y) = (-2\,,0)$

b) $\sqrt{3}\,,\,\frac{\pi}{6})$

$$(\sqrt{3}\,,\frac{\pi}{6}) = (r,\theta)$$
$$r = \sqrt{3} \text{ and } \theta = \frac{\pi}{6}$$
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$
$$x = \sqrt{3}\cos\frac{\pi}{6} \quad \text{and} \quad y = \sqrt{3}\sin\frac{\pi}{6}$$
$$x = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \quad \text{and} \quad y = \sqrt{3}\left(\frac{1}{2}\right)$$
$$x = \frac{3}{2} \quad \text{and} \quad y = \frac{\sqrt{3}}{2}$$

Then, the rectangular co-ordinate is $(x,y) = \left(\frac{3}{2}\,,\,\frac{\sqrt{3}}{2}\right)$

Converting rectangular co-ordinates to polar co-ordinates

Example 22.1.3

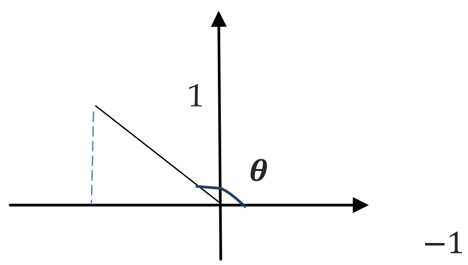
Convert the following Cartesian co-ordinates to polar co-ordinates.

a) $(-1{,}1)$ b) $(0{,}2)$

Solutions

a) $(-1{,}1)$

$$x = -1 \quad , \quad y = 1$$



$$r = \sqrt{2}$$
$$\tan\theta = \frac{1}{-1}$$
$$\theta = \tan^{-1}(-1)$$

$$\theta = \frac{3\pi}{4}$$

Then, the polar co-ordinate is $(r, \theta) = \left(\sqrt{2}, \frac{3\pi}{4}\right)$

Polar equations

We have $y = r \sin\theta$ and $x = r\cos\theta$, then

$$y^2 = r^2 sin^2\theta \quad \text{and} \quad x = r^2 cos^2\theta$$

We can convert a rectangular equation to polar form by simply replacing x by $r\cos\theta$ and y by $r\sin\theta$.

Example 22.1.4

Convert the following equations to polar form.

a) $x^2 + y^2 = 9$
b) $xy = 4$

Solutions

a) $x^2 + y^2 = 9$

$$\begin{aligned} r^2 cos^2\theta + r^2 sin^2\theta &= 9 \\ r^2(cos^2\theta + sin^2\theta) &= 9 \\ r^2 &= 9 \\ r &= 3 \end{aligned}$$

b) $xy = 4$

$$\begin{aligned} (r\cos\theta)(r\sin\theta) &= 4 \\ r^2 \cos\theta \sin\theta &= 4 \\ r^2 \frac{1}{2}\sin 2\theta &= 4 \\ r^2 \sin 2\theta &= 8 \end{aligned}$$

Example 22.1.5

Convert the following to rectangular form

a) $r = 2$
b) $r = \sec\theta$
c) $\theta = \frac{\pi}{4}$
d) $r = \frac{6}{2\cos\theta + 3\sin\theta}$

Solutions

a) $r = 2$

$$\sqrt{x^2 + y^2} = 2$$
$$x^2 + y^2 = 4$$

b) $r = \sec\theta$

$$r = \frac{1}{\cos\theta}$$
$$r\cos\theta = 1$$
$$x = 1$$

c) $\theta = \frac{\pi}{4}$

$$\tan\theta = tan\frac{\pi}{4}$$
$$\frac{y}{x} = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1$$
$$y = x$$

d) $r = \frac{6}{2\cos\theta + 3\sin\theta}$

$$r(2\cos\theta + 3\sin\theta) = 6$$
$$2\,\mathrm{r}\cos\theta + 3r\sin\theta = 6$$
$$2x + 3y = 6$$

Graphs of polar equations

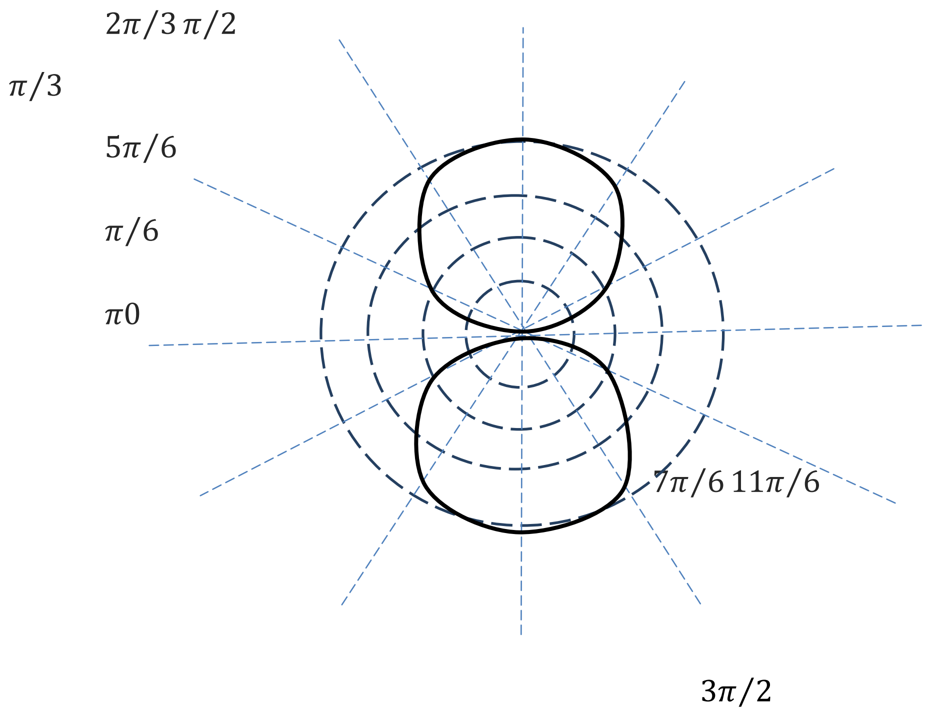
Example 22.1. 6

Sketch the graph of $r = 4\sin\theta$

Solution

To sketch the graph of$r = 4\sin\theta$, we get the values for $0 \le \theta \le 2\pi$ and take the approximation for $\sqrt{2} = 1.41$ $\quad\sqrt{3} = 1.73$

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$\boldsymbol{r}$	0	2	3.46	4	3.76	2	0	-2	-4	-2	0



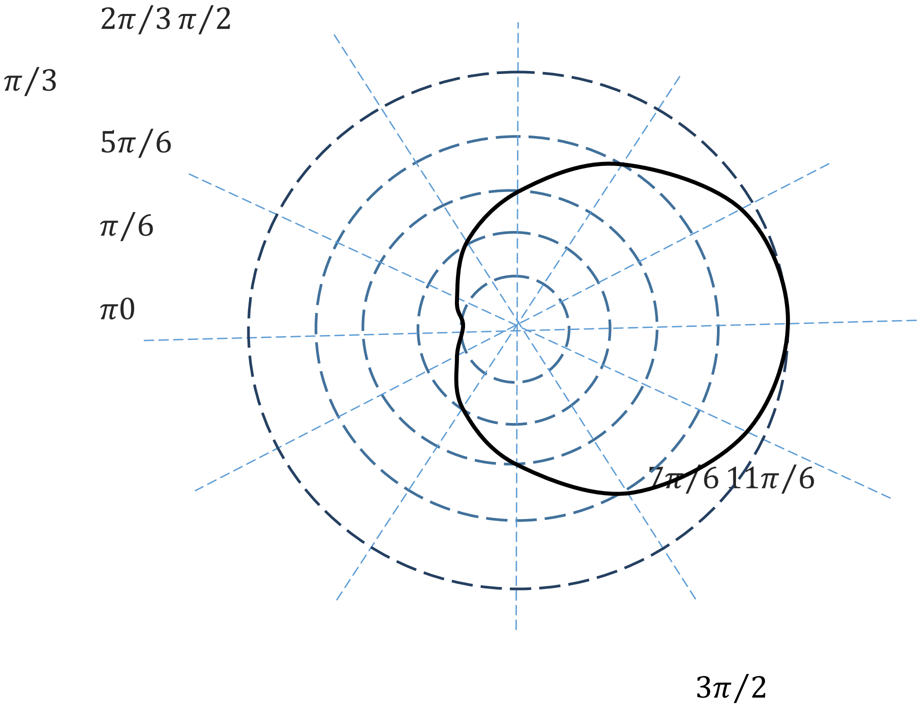
Example 22.1.7

Sketch $r = 3 + 2\cos\theta$

Solution

To sketch the graph of $r = 3 + 2\cos\theta$, we get the values for $0 \le \theta \le 2\pi$ and take the approximation for $\sqrt{2} = 1.41$ $\quad \sqrt{3} = 1.73$

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$2\cos\theta$	2	1.73	1	0	-1	-1.73	-2				
$r = 3 + 2\cos\theta$	5	4.73	4	3	2	-1.27	1				



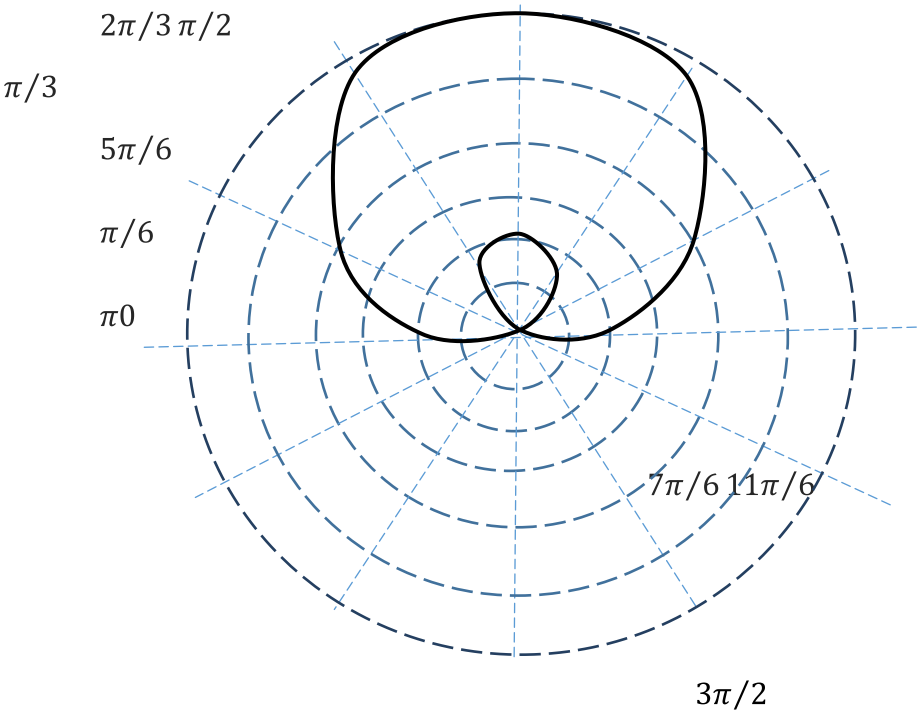
Example 22.1.8

Sketch $r = 2 + 4\sin\theta$

Solution

To sketch the graph of $\boldsymbol{r = 2 + 4\sin\theta}$, we get the values for $0 \leq \theta \leq 2\pi$ and take the approximation for $\sqrt{2} = 1.41$ $\quad \sqrt{3} = 1.73$

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$4\sin\theta$	0	2	3.46	4	-2	-3.46	-4				
$r = 2 + 4\sin\theta$	2	4	5.46	6	0	-1.46	-2				



REVIEW EXERCISE

a) Plot the following points (given in polar form). Then find the all the polar coordinates of each point.

   i) $\left(2, {}^{\pi}/_{2}\right)$ ii) $\left(-2, {}^{\pi}/_{2}\right)$ iii) $\left(4, {}^{\pi}/_{3}\right)$

b) Change each of the following to polar form

   i) $5x + 4y = 10$ $ii)$ $x^2 + y^2 = \sqrt{x^2 + y^2}$ $iii)$ $x^2 + y^2 + 6y = 0$

c) Change each polar equation to rectangular form

   i) $r = 2\sin\theta$ $ii)$ $r = 2\cos\theta + 3\sin\theta$ $iii)$ $r = \frac{4}{2+\cos\theta}$ $iv)$ $r = \frac{5}{2-2\sin\theta}$

d) Sketch the graph of each of the polar equation.

   i) $r = 1 + 2\sin\theta$
   ii) $r = 2 - 2\cos\theta$
   iii) $r^2 = \cos\theta$
   iv) $r^2 = 4\cos 2\theta$
   v) $r^2 = -\sin 2\theta$

CALCULUS

Calculus isconcerned with the study of concepts as the rate of change of one variable quantity with respect to another, the slope of a curve at a prescribed point, finding the maximum and minimumpoint of functions, finding the area bounded by curves and the volume of revolution of the curve.

23: LIMITS

We study different functions, how their graphs are and how they behave at a given point or as the variable approaches a certain constant c.

The foundation of Calculus is based on the concept of limits.

Definition of limits

L et $f(x)$ be defined on an open interval (c, d) except possiblyat $a \in (c, d)$. If $f(x)$is arbitrary close for all x sufficiently close to a, we say that f approaches the limit L as x approaches a and we write

$$\lim_{x \to a} f(x) = L.$$

Let us examine the following function at a given point

Example23.1.1

Consider the function $f(x) = \frac{x^2-4}{x-2}$ at$x = 2$, the function is undefined. What is the behavior of the function as x values approaches 2 from the left and from the right?

Solution

x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x)$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

By using the calculator, we observe that as the values of x approaches 2 from the left, the function goes to 4 and as the value of x approaches 2 from the right, the function goes to 4.

$$\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = 4 \qquad \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = 4$$

The limit of a function f exist if and only if both corresponding sided limits exist and are equal that is $\lim_{x \to a-} f(x) = L$ if and only iff $\lim_{x \to a_+} f(x) = L$. That is the left hand sided limit and the right hand sided limit.

Example 23.1.2

Consider the function $f(x) = \frac{x^2-5}{x-2}$ at $x = 2$, the function is undefined. What is the behavior of the function as x values approaches 2 from the left and from the right?

Solution

x	$f(x)$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	1000.9999

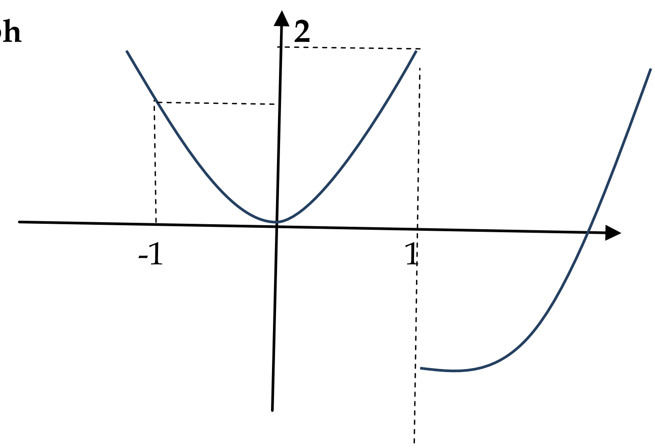
x	$f(x)$
2.1	-5.9
2.01	-95.99
2.001	995.999
2.0001	-9995.999

By using the calculator, we observe that as the values of x approaches 2 from the left, the function goes to infinity and as the value of x approaches 2 from the right, the function goes to negative infinity.

$\lim_{x \to 2} \frac{x^2-4}{x-2}$ the limit does not exist

Examining the limits from the graph

Example 23.1.3



Find

a. $\lim_{x \to -1} f(x)$
b. The limit at $x = 1$

Solutions

a. $\lim_{x \to -1} f(x)$

$$\lim_{x \to -1^-} f(x) = 1 \qquad \lim_{x \to -1^+} f(x) = 1 \quad \text{hence } \lim_{x \to -1} f(x) = 1$$

b. $\lim_{x \to 1^-} f(x) = 1.7$

$$\lim_{x \to 1^+} f(x) = -1 \qquad \text{hence } \lim_{x \to 1} f(x) \text{ does not exist}$$

Computing of limits

Limits are computed by following the rules

Rules for computing of limits of Limits

Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ and let c be a constant. Then the following apply;

i. **Constant multiple rule**
$\lim_{x \to a} c.f(x) =$ c.$\lim_{x \to a} f(x)$=c.L

ii. **Sum rule**
$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$=L±M

iii. **Product rule**

$\lim_{x \to a}[f(x).g(x)] = \lim_{x \to a} f(x).\lim_{x \to a} f(x)$=L.M

iv. **Quotient rule**
$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right]$=[$\lim_{x \to a} f(x) / \lim_{x \to a} g(x)$]=$\frac{L}{M}$ where M≠ 0

v. **Power rule**

$\lim_{x \to a}[f(x)]^n$= $[\lim_{x \to a} f(x)]^n = L^n$

vi. **Constant rule**
$\lim_{x \to a} c = c$ where c is a constant

Example 23.1.4

Evaluate the following limits

a) $\lim_{x\to 2} 7$

Solution

By using the rule of a constant,

$$\lim_{x\to 2} 7 = 7$$

Example 23.1.5

Evaluate $\lim_{x\to 2} 5x + 2$

Solution

By sum rule $\lim_{x\to 2} 5x + 2 = 12$

Example 23.1.6

Evaluate$\lim_{x\to 3} \frac{5x+4}{x-1}$

Solution

By quotient rule$\lim_{x\to 3} \frac{5x+4}{x-1} = \frac{19}{2}$

Limits of a Polynomials

A function of the form of $f(x) = a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_1 \pm a_0$where n is a member of Natural Numbers, then the limit of any polynomial f(x) is

$$\lim_{x\to a} f(x) = f(a)$$

Example 23.1.7

Evaluate $\lim_{x\to 2} 2x^3 - 4x^2 + 5$

Solution

$$\lim_{x\to 2} f(x) = 5$$

Factorization Method of Evaluating Limits

Some rational functions are undefined at a constant c that is f (c) is undefined. Remember for any function f, then$\frac{f(x)}{0}$ is undefined. Even though f(c) may be undefined, the limit as x→ c may or may not exist. We employ the method of factorization to evaluate such limits. Before start evaluating the limit of any rational function check the following:

i. Check if the function is defined, if so then proceed with quotient rule. If not then,
ii. Check if the numerator or denominator is factorable. If so proceed by factorizing and eliminate what is common then apply the limits.

Example 23.1.8

Evaluate the limit of $f(x) = \frac{x^2-4}{x-2} \quad as\ x \to 2$

Solution

We have $f(2)$ undefined. But the numerator has a factorable function.
Then we proceed with factorization method.

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{x^2-2^2}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

Therefore we see that the function f may not be defined at a point but has a limit as the variable in the function approaches that point.

The limit of the function does not exist only if the left hand limit and the right hand limit are not equal. Meaning as we approach a point c from the left, f approaches L and as we approach f from the right f approaches M

The limit of a function exist if and only if $L = M$

Example23.1. 9

Show that $\lim_{x\to 2}\frac{|x|}{x}$ does not exist.

Solution

$$|x| = f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Therefore, for $(x < 0)$

$$\lim_{x\to 2}\frac{-x}{x} = \lim_{x\to 2} -1 = -1$$

And, for $(x \geq 0)$

$$\lim_{x\to 2}\frac{x}{x} = \lim_{x\to 2} 1 = 1$$

Hence, since the left and the right hand limit are not equal, then the limit of a function f does not exist.

Example 23.1.10

Evaluate, $\lim_{x \to -1} f(x) \quad where\ f(x) = \begin{cases} 2x+1 & if\ x < -1 \\ 3 & if - 1 < x < 1 \\ 2x+1 & if\ x > 1 \end{cases}$

Solution

This is what we call a piece wise function. So we observe the left and the right hand limits.

$$\lim_{x \to -1} f(x) = -1 \qquad for \qquad x < -1$$
$$\lim_{x \to -1} f(x) = 3 \qquad for \qquad x > -1$$

Therefore, since the left and the right hand limit are not equal, then the limit of $f(x)$ does not exist.

Limits by Rationalization

Some Rational functions contains the denominator or the numerator that are in form of $\sqrt{f(x)}$ or both are in $\sqrt{f(x)}$ to evaluate the limits of such functions we rationalize either the denominator or the numerator.

Example 23.1.11

Evaluate $\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

Solution

$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \to 0} \frac{(\sqrt{x+2})^2 - (\sqrt{2})^2}{x(\sqrt{x+2} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \to 0} \frac{x}{x(\sqrt{x+2} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{x+2} + \sqrt{2})} = \frac{1}{(\sqrt{2} + \sqrt{2})} = \frac{1}{2\sqrt{2}}.$$

Limits of Radicals

Suppose that $\lim_{x \to c} f(x) = L$ and n is a positive integer then,

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$$

Example 23.1.12

Evaluate $\lim_{x \to 10} \sqrt[3]{x-2}$

Solution $\sqrt[3]{(lim_{x \to 10}\, x - 2\, x\,)} = \sqrt[3]{8} = 2.$

REVIEW EXERCISE

1. Evaluate the following limits
   a) $\lim_{x\to 2}(2x^2 - x + 7)$ b) $\lim_{x\to -2}\frac{2x^2+5x+2}{x^2+9x+14}$ c) $\lim_{x\to}\frac{1}{x-1}$

   b) $\lim_{x\to 0}\frac{x}{|x|}$ e) $\lim_{x\to -3}\frac{x^2-9}{x+3}$
2. Determine the following limits
   i. $\lim_{x\to 0}\frac{x}{5x^3+2x^2}$ iii. $\lim_{x\to 2}\sqrt[3]{2x+1}$

   ii. $\lim_{x\to 0}\frac{\sqrt{x^2+x+4}-2}{x^2+x}$ iv. $\lim_{x\to 0}\frac{x^2}{2-x^2}$
3. Find the limits of the following ;
   i. $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$ ii.$\lim_{x\to 0} 2x$

Find $\lim_{x\to 2} f(x)$, $f(x) = \begin{cases} x^2 & if\ x \neq 2 \\ 1 & if\ x = 1 \end{cases}$

Properties of limits of trigonometric functions

For any real number c we have;

i. $\lim_{x\to c} sinx = \sin c$
ii. $\lim_{x\to c} cosx = \sin c$
iii. $\lim_{x\to c} e^x = e^c$
iv. $\lim_{x\to c} Inx = In\ c$ $for\ \ c > 0$
v. $\lim_{x\to c} sin^{-1} x = sin^{-1}c$ $for -1 < c < 1$
vi. $\lim_{x\to c} cos^{-1}x = cos^{-1} c$ $for -1 < c < 1$
vii. $\lim_{x\to c} tan^{-1} x = tan^{-1}c$ $for\ \ -\infty < c < \infty$

Example 23.1.13

Evaluate $\lim_{x\to 0} sin^{-1}(\frac{x+1}{2})$

Solution
Since$-1 < c = 0 < 1$, then
$\lim_{x\to 0} sin^{-1}(\frac{x+1}{2}) = {}^{\pi}/_{6}$. Note: we use the radian measure for the angles

Property
a) $\lim_{X\to 0}\frac{\sin x}{x} = 1$ b) $\lim_{x\to 0}\frac{cosx-1}{x} = 0$

Example 23.1.14
Evaluate $\lim_{x\to 0} x \cot x$

Solution
We know that $cotx = \frac{cosx}{sinx}$

Then $\lim_{x\to 0}(x \ .\frac{cosx}{sinx})$ $= \lim_{x\to 0}(cosx. \ \frac{x}{sinx})$ $= \lim_{x\to 0}(\frac{cosx}{sinx/_{x}})$

From the theorem
$\lim_{x\to 0}\frac{sinx}{x} = 1$ and $\lim_{x\to 0} cosx = 1$

Hence $\lim_{x\to 0} x. cotx = 1$

Squeeze or Sandwich Theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in the interval (c,d), except possibly at $a \in (c,d)$ then if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$

Example 23.1.15
Find the limit of $y = x^2. cos\ (1\ /\ x)$

Solution
We know that $-1 \leq cosx \leq 1$
Then $-1 \leq cos\ (^1/_x) \leq 1$

Multiplying by x^2 gives

$$-x^2 \leq \cos\ (^1/_x) \leq x^2$$

Let $f(x) = -x^2$, $h(x) = x^2$ since $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$ hence by the Sandwich theorem

$$\lim_{x\to 0} x^2. \cos\ (^1/_x) = 0.$$

Limit at infinity

limit at $-\infty$ *or* ∞

1. We say $f(x)$ has a limit L as x approaches infinity and we write
$$\lim_{x\to\infty} f(x) = L$$

Note:

i. $x \to \infty$ means, as x is approaching a very big number

Property

a)$\lim_{x\to\pm\infty} c = c$ b) $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ where n is a positive number

Example 23.1.16
Evaluate $\lim_{x\to\infty} \frac{1}{x^4}$

Solution
Since $n = 4 > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^4} = 0$$

Limits at infinity of Rational Functions

To determine the limit of a rational function as x→ $\pm\infty$ we can divide the numerator and the denominator by the highest power of x in in the denominator.

Example 23.1. 17
Evaluate

$$\lim_{x\to\infty} \frac{5x+8}{7x+3}$$

Solution
Since the function has x has the highest power of the polynomial in the denominator. We divide the numerator and the denominator by x, then we apply the limit at infinity that yield,

$$\lim_{x\to\infty} \frac{5 + {}^{8}/_{x}}{7 + {}^{3}/_{x}} = \frac{5}{7}$$

EXERCISE

1. Use the Squeeze theorem to find the limits of the following
   i. $\lim_{x\to 0} x^3 sin({}^{1}/_{2x})$ ii. $\lim_{x\to 0} x^3 cos(\frac{1}{x})$

2. Determine the following limits

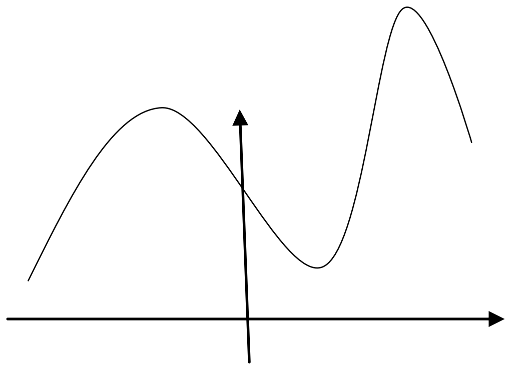
24: CONTINUITY OF FUNCTIONS

When we plot the results collected from the field, we often connect the plotted points with an unbroken curve to show that the functions values are likely to have been at the times we did not measure. In doing so we are assuming that we are working with a continuous function, so its output vary continuously with the input and do not jump from one value to another without taking on the values in between. The limit of the function as x is approaching a constant c can be found just by calculating the value of a function at a constant c. (we found this to be true by the polynomials).

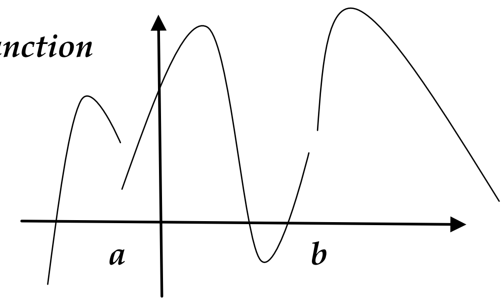
Any function $y = f(x)$ whose graph can be sketched over its domain in one continuous motion without lifting a pencil is an example of a continuous function. In this section we investigate precisely what it mean for a function to be continuous or discontinuous.

We also study the properties of the continuous function.

Graph of a continuous function



Graph of a discontinuous function



Continuity of a function at a Point

We investigate the continuity of a function graphically.

DEFINITION Continuity at a Point

Interior point: A function $y = f(x)$ is continuous at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

End point: A function $y = f(x)$ is continuous at left end point aor right end point b of its domain if

$\lim_{x \to a_+} f(x) = f(a)$ $\lim_{x \to b-} f(x) = f(b)$ respectively

If a function f is not continuous at a point c, we say that the **function f is discontinuous at c** and **c is a point of discontinuity** of f.Note that c has to be in the domain of f.

Example 24.1.1

A function $f(x) = \sqrt{4 - x^2}$ is continuous at every point in its domain $[-2, 2]$, including $x = -2$ where f is right continuous and $x = 2$ where f is left continuous.

Continuity Test

A function f is continuous at $x = c$ if and only if it meets the following three steps.

1. $f(c)$ *exists* (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exist (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example 24.1.2

Discuss the continuity of the function $y = \frac{x^2-4}{x-2}$ at $x = 2$

Solution.

From the continuity test,$f(2)$ is undefined. Hence the function $f(x)$ given by

$y = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$

Note: when discussing the continuity of a function, try all the three conditions for continuity and if all the conditions are satisfied then conclude that the function is continuous at a given point. But if any condition fails to be satisfied then conclude that the function is discontinuous at a given point.

Example 24.1.3

Discus the continuity of the function $f(x) = \begin{cases} 4 - x^2 & , x \neq 1 \\ 3 & , x = 1 \end{cases}$ at $x = 1$

Solution

I. $f(c) = f(1) = 3$
II. $\lim_{x\to 1} f(x) = \lim_{x\to 1} 4 - x^2 = 3$
III. $\lim_{x\to 1} f(x) = f(1) = 3$

Since all the three conditions for continuity test satisfied. Hence f is continuous at $x = 1$

Example 24.1. 4

Explain why the function $f(x) = \sin(\frac{1}{x})$ is discontinuous at $x = 0$

Solution

Since, $f(0) = \sin\frac{1}{0}$ is undefined. Hence f is discontinuous

Removable Discontinuity

1. A function f has a **removable discontinuity** at $x = c$ if $\lim_{x\to c} f(x)$ exist and either $f(c)$ is undefined or $lim_{x\to c} f(x) \neq f(c)$.
2. If $\lim_{x\to c} f(x)$does not exist, then the function f has a **non-removable discontinuity**.

Example 24.1. 5

a. Determine whether $f(x) = \frac{x^2+2x-3}{x-1}$ *is continuous at* $x = 1$
b. If not is it possible to remove the discontinuity. If possible, then
c. Redefine a new function $g(x)$that is continuous at $x = 1$

Solution

a. Since $f(1)$ is undefined, then the function f is discontinuous.
b. $\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$

$$\lim_{x\to 1} ((x+3)(x-1))/((x-1))$$
$$\lim_{x\to 1}(x+3) = 4$$

since f has a limit then f has a removable discontinuity.

c. We redefine f has g where

$$g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & , \quad x \neq 1 \\ 4 & , \quad x = 1 \end{cases}$$

at $x = 1$

Example 24.1. 6

Find all discontinuity of $f(x) = \frac{1}{x^2}$ at $x = 0$

Solution

Since $f(0)$ is undefined then the function f is discontinuous.

Since $\lim_{x \to o} \frac{1}{x^2}$ does not exist, then the function f has a non-removable discontinuity

Properties

1. All polynomials are continuous for all values of x.
2. $sin\ x, cos\ x,\ \ tan^{-1}x$ and $e^x$are continuous everywhere.
3. Rational functions $\frac{f(x)}{g(x)}$ are continuous except at $g(x) = 0$
4. $\sqrt[n]{x}$ i) is continuous for all x when n is odd. ii. Is continuous for $x \geq 0$ when n is a positive positive integer
5. lnx is continuous when $x > 0$ and $sin^{-1}\ x, cos^{-1}\ x$ are continuous for $-1 < x < 1$

Example 24.1.7
Determine the interval on which each function is continuous

a) $f(x) = x^2 - 2x + 1$
b) $\sqrt[3]{x^2 + 2}$
c) $\sqrt{4 - x^2}$

Solution

a) The function f given by $f(x) = x^2 - 2x + 1$ is a polynomial. Hence by the theorem f is continuous on $(-\infty, \infty)$
b) The function f is in form of $\sqrt[n]{x}$ with $n = 3$ is odd. Then by the theorem, f is continuous for all values of x.
c) $f(x) = \sqrt{4 - x^2}$ is continuous when $4 - x^2 > 0$ then

Continuous Functions
A function is ***continuous*** on an interval if and only if it is continuous at every point of the interval. A ***continuous function*** is one that is continuous at every point of its domain.

Example24.1. 8

a) The function $f(x) = {}^{1}/_{x}$ is continuous at every point in the domain $(-\infty, 0) \cup (0, \infty)$. It has a point of discontinuity at $x = 0$ because it is not defined there.
b) The identity function $f(x) = x$ and constant function $f(x) = c$ are continuous everywhere.

Properties of Continuous Functions
If the functions f and g are continuous at $x = c$, the following combinations are continuous at $x = c$

a) **Sums:** $f \pm g$
b) **Product**: $f.g$
c) **Constant multiples**: $k.f$ for any number k
d) **Quotient:** ${}^{f}/_{g}$ provided $g(c) \neq 0$
e) **Power:**f^n provided n in defined in the interval and n is a positive interger

Note: Every polynomial functions $f(x)$ are continuous everywhere.

Composite functions

All composite function are continuous. The idea is that $f(x)$is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$,then $g\ o\ f$ continuous at $x = c$.

THEOREM (Composite of Continuous Function)

If f is continuous at c and g is continuous at $f(c)$, then the composite gof is continuous at c

Continuity of Absolute value function

The function $f(x) = |x|$ is continuous at every value of x since

$f(x) = \begin{cases} x\ , & x > 0 \\ -x\ , & x < 0 \end{cases}$ and $\lim_{x\to 0} |x| = 0$

Example 24.1.9

Show that the following functions are continuous everywhere on the respective domains

a) $y = \sqrt{x^2 - 2x - 5}$

b) $y = |\frac{xsinx}{x^2+2}|$

Solution

a) The square root function is continuous on $[0,\infty)$ because it is a rational power of identity functions$f(x) = x$. The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $(t) = \sqrt{t}$. Hence the function f is continuous on its interval.

b) Since $\sin x$ is continuous everywhere, the numerator term is the product of continuous functions and denominator term $x^2 + 2$ is everywhere-positive polynomial. The function is a composite of a quotient of continuous function with an absolute value function. Hence f is continuous on its interval.

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that makes them useful in Mathematics and its application. One of these is the Intermediate value property. A function is said to have the Intermediate value property if whenever it takes on two values it also takes all the values in between.

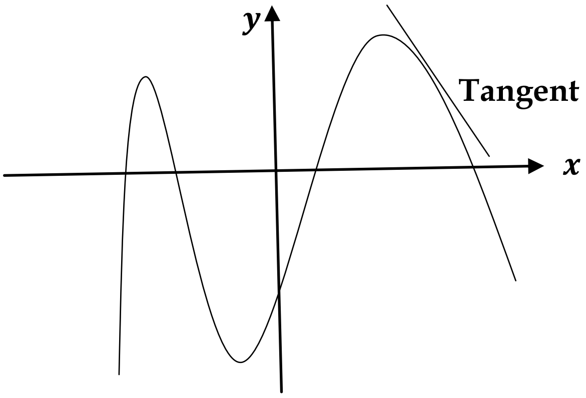
THEOREM Intermediate value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval [a, b] takes on every values in between$f(a)$ and $f(b)$. In other words if y_1 is any value in between$f(a)$and $f(b)$. Then $y_1 = f(c)$where $c \in [a, b]$

25: DEFFERENTIATION

TANGENT LINE TO THE FUNCTION

For any given function, the tangent line to its curve is given by $y - y_1 = m(x - x_1)$ where m is the gradient or the slop of the curve, y_1 and x_1 are the coordinate points.



Definition (Gradient, Tangent Line)

The **gradient to the curve** $y = f(x)$at a point $P(c, f(c))$ is a number

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

(provided the limit exist)

The **tangent line to the curve** at a point P is the line through P with this slope

Finding the Tangent of $y = f(x)$ at (x_1, y_1)

i. Calculate $f(c)$ and $f(c+h)$
ii. Calculate the slop

$$m = \lim_{h \to o} \frac{f(c+h) - f(c)}{h}$$

iii. If the limit exist find the tangent line as

$$y - y_1 = m(x - x_1)$$

NOTE: The expression $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ is called the **derivative at a point.**

The Derivative as a Function

Definition (Derivative Function)

The derivative of the function f with respect to the variable x is the function

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Provided the limit exist.

We use the notation $f(x)$ rather than f in the definition to emphasize the independent variable x, which we are differentiating with respect to. The domain of $\frac{dy}{dx}$ is the set of points in the domain f for which the limit exist, and the domain may be the same or smaller than the domain in f. If $\frac{dy}{dx}$ exist at a particular x, we say that f is **differentiable at x**.

If $\frac{dy}{dx}$ exist at every point in the domain of f, we say f is **differentiable.**

The process of calculating the derivative is called **differentiation.**

Example 25.1.1

Compute the derivative of $f(x) = 3x^2 - 4x + 1$ at $x = 1$ using the definition of the derivative at a point.

Solution

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Let $f(c) = 3c^2 - 4c + 1$ and $f(c+h) = 3(c+h)^2 - 4(c+h) + 1$

Substitute in the formula with $x = c = 1$,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[3(1+h)^2 - 4(1+h) + 1] - [3(1)^2 - 4(1) + 1]}{h}$$

Expand then simplify

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{3 + 6h + 3h^2 - 3 - 4h}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2h + 3h^2}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{h(2 + 3h)}{h} = \lim_{h \to 0}(2 + 3h) = 2.$$

Hence the derivative of $f(x)$ at $x = 1$ is 2.

The method of finding the derivative using the definition is called **differentiating from the first principle.**

Example25.1. 2

Differentiate from the first principle $f(x) = \sqrt{x}$

Solution

Let $f(x) = \sqrt{x}$ and $f(x+h) = \sqrt{x+h}$

Then

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalize the numerator

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sqrt{x+h}^{2}-\sqrt{x}^{2}}{h(\sqrt{x+h}+\sqrt{x}}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

When Does a Function Not Have a Derivative at a given Point c.

1) When the limit of the function $f(x)$ does not exist at the given point c.
2) When the function $f(x)$ is discontinuous at a point c.

Non Differentiable function.

The function is non differentiable if.

1. The limit of the function does not exist
2. The function is discontinuous.

Differentiable Functions Are Continuous

A function is continuous at every point where it is differentiable.

Differentiability Implies Continuity.

If f has a derivative at $x = c$, then f is continuous at $x = c$.

THEOREM (Intermediate Value Property of Derivatives)

If **a** and **b** are any two points in the interval on which f is diferentiable,then $\frac{dy}{dx}$ takes every point between the derivative at **a** and at **b**.

DIFFERENTIATION RULES

We can differentiate functions without using the definition but by just applying the following rules;

RULE 1. Derivative of a Constant

If f has a constant value $f(x) = c$, then

$$\frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

Example 25.1.3

If f has a constant value $f(x) = 8$, then $\frac{dy}{dx} = \frac{d}{dx}(8) = 0$.

Similarly if $f(x) = \frac{\pi}{2}$ then

$$\frac{dy}{dx} = \frac{d}{dx}(\frac{\pi}{2}) = 0.$$

Proof

We apply the definition of the derivative to $f(x) = c$ whose output is c.

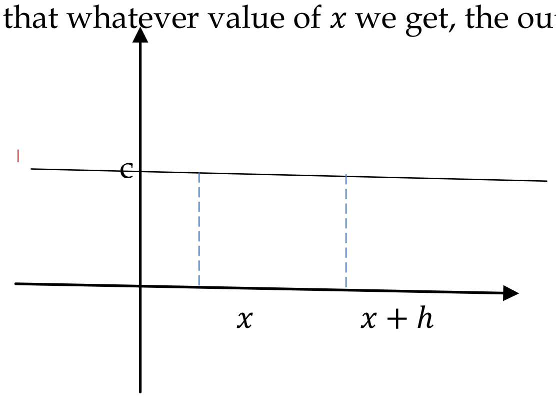
At every value of x we find that

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

Hence

$$\frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

We observe from the graph that whatever value of x we get, the output is c



Rule 2. Power Rule

If n is an integer, then

$$\frac{dy}{dx} = \frac{d}{dx}x^n = nx^{n-1}$$

Example 25.1.4

f	x	x^2	x^3	x^4
$\frac{dy}{dx}$	1	$2x$	$3x^2$	$4x^3$

Example 25.1.5

Differentiate the following functions

a) $f(x) = 4x^4 - 5x^2 + 3x - 7$
b) $f(x) = x^3 + x - 8$

Solution

a) $\frac{dy}{dx} = 16x^3 - 10x + 3$

b) $\frac{dy}{dx} = 3x^2 + 1$

Rule 3. Constant Multiple Rule

If $\boldsymbol{y}$ is a differentiable function on $\boldsymbol{x}$, and $\boldsymbol{c}$ is a constant, then $\frac{d}{dx}(c.y) = c.\frac{dy}{dx}$

Example 25.1.6

Differentiate $y = \pi x^2$

Solution

$$\frac{dy}{dx} = 2\pi x$$

Rule 4. Derivative of Sum Rule

If u and v are differential functions in x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At that point,

$$\frac{dy}{dx} = \frac{d}{dx}\,(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof

We apply the definition of derivative to $f(x) = u(x) + v(x)$

$$\frac{d}{dx}[\,u(x) + v(x)] = \lim_{h\to 0}\frac{[\,u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h) - u(x)}{h} + \lim_{h\to 0}\frac{v(x+h) - v(x)}{h}$$

$$= \frac{du}{dx} + \frac{dv}{dx}.$$

Example 25.1.7 (Derivative of Polynomial)

Differentiate $y = x^3 - 5x + 1$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx(5x)} + \frac{d}{dx(1)} = 3x^2 - 5 + 0 = 3x^2 - 5.$$

RULE 5. Product Rule

If u and v are differentiable at x, then so is their product uv and

$$\frac{d}{dx}(u.\,v) = u\,.\frac{dv}{dx} + v.\frac{du}{dx}$$

Example 25.1.8

Differentiate $\quad y = (2x^5)(\pi x^2)$

Solution

Let $u = 2x^5$, then $\frac{du}{dx} = 10x^4$ and let $v = \pi x^2$, then $\frac{dv}{dx} = 2\pi x$

Applying product rule

$$\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

$$= (2x^5)(2\pi x) + (2x^2)(10x^2) = 4\pi x^6 + 20x^4$$

RULE 6. Derivative of Quotient Rule

If u and v are differentiable on x and $v(x) \neq 0$,

then the quotient $^u/_v$is differentiable at x, and

$$\frac{d}{dx}(^u/_v) = \frac{v.\frac{du}{dx} \quad - \quad u.\frac{dv}{dx}}{v^2}.$$

Example 25.1.9

Differentiate $y = \frac{t^2-1}{t^2+1}$

Solution

Apply quotient rule to $u = t^2 - 1$ and $v = t^2 + 1$

$$\frac{dy}{dt} = \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

Rule 7. Chain Rule and Composite Functions

If u is differentiable at x and f is differentiable at $u(x)$,

then their composite $f[u](x)$ or fou is also differentiable at x and

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)](\boldsymbol{x})] = \frac{\boldsymbol{dy}}{\boldsymbol{du}}.\frac{\boldsymbol{du}}{\boldsymbol{dx}}.$$

Example 25.1.10

Find the derivative of $y = (15x^2 + 1)^7$

Solution

$$y = (15x^2 + 1)^7$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 7(15x^2+1)^6 \cdot \frac{dy}{dx}(15x^2+1)$$

$$\frac{dy}{dx} = 7(15x^2+1)^6 . 30x$$

$$\frac{dy}{dx} = 210(15x^2+1)^6$$

Second and Higher Order Derivatives

If f is differentiable at x, then its derivative $\frac{dy}{dx}$ is also differentiable at x

and we call the **second derivative** denoted by $\frac{d^2y}{dx^2}$. Then the second derivative is also

differentiable at x and we call the **third derivative** denoted by $\frac{d^3y}{dx^3}$ and so on. Then the $\boldsymbol{n^{th}}$ **derivative denoted** by y^n or $\frac{d^ny}{dx^n}$ is also differentiable at x.

Example 25.1.11

Find the first four derivatives of $y = x^3 - 3x^2 + 2$

Solution

$\frac{dy}{dx} = 3x^2 - 6x$ First derivative

$\frac{d^2y}{dx^2} = 6x$ Second derivative

$\frac{d^3y}{dx^3} = 6$ Third derivative

$\frac{d^4y}{dx^4} = 0$ Fourth derivative

Note some functions are finitely differentiable, they become zero at some point as we continue differentiating it. While some are infinitely differentiable.

RWVIEW EXERCISE

1. differentiate using the first principal:
a) $y = \sqrt{2x}$

b) $y = \frac{1}{x+1}$

2. Find the first and the second derivatives:
   a) $y = -x^2 + 3$

b) $y = x^2 + x + 8$

3. Use the appropriate rule to differentiate the following:
   a) $y = \frac{x+1}{x}$

   b) $y = (x-5)^5$

   c) $y = 280$

   d. $y = (x+1)(x^3-1)$

Derivatives of Trigonometric Functions

Derivative of sine function is cosine function;

$$\frac{d}{dx}(sinx) = cosx$$

Proof

Let $f(x) = sinx$ and let $f(x+h) = \sin(x+h)$

By the derivative definition

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \quad = \lim_{h\to 0}\frac{\sin(x+h) - sinx}{h}$$

By sum angle trigonometric formula $\sin(x+h) = \sin x \cos h + \cos x \sin h$

Then

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{sinx\ cosh + cosx\ sinx - sinx}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{sinx\ cosh - sinx + cosx\ sinh}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{sinx\ (cosh - 1) + cosx\ sinh}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{sinx(cosx - 1)}{h} + \lim_{h\to 0}\frac{cosxsinh}{h}$$

$$\frac{dy}{dx} = 0.sinx + 1.cosx$$

$$\frac{dy}{dx} = cosx$$

Hence

$$\frac{d}{dx}(sinx) = cosx$$

Example 25.1.12

Differentiate

a) $y = x^2 - sinx$

Solution

By sum rule

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(sinx)$$

$$\frac{dy}{dx} = 2x - cosx$$

b) $y = \frac{sinx}{x}$

Solution

By quotient rule

$$\frac{dy}{dx} = \frac{x.cosx - sinx}{x^2}$$

Derivative of cosine function is negative of sine function

$$\frac{d}{dx}(cosx) = -sinx$$

Proof

Let $f(x) = cosx$ and $f(x+h) = \cos(x+h)$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\cos(x+h) - cosx}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{cosx\ cosh - sinx\ sinh - cosx}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{cosx\ cosh - cosx - sinx\ sinx}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{cosx(\cosh - 1) - sinx\ sinh}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} six.\frac{(cosh - 1)}{h} - \lim_{h \to 0} sinx.\frac{sinh}{h}$$

$$\frac{dy}{dx} = sinx.0 - sinx.1 = -sinx$$

$$\boldsymbol{\frac{d}{dx}(cosx) = -sinx.}$$

Example 25.1.13

1. Differentiate the following
   a) $y = 5x + \cos x$
   b) $y = \sin x \cos x$

Solutions

a) By sum rule

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$$
$$\frac{dy}{dx} = 5 + (-\sin x)$$
$$\frac{dy}{dx} = 5 - \sin x.$$

b) By product rule

$$\frac{dy}{dx} = \sin x\,.\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \sin x\,.(-\sin x) + \cos x \cos x$$

$$\frac{dy}{dx} = -sin^2x + cos^2x.$$

Derivative of tan function is the square of secant function $\boldsymbol{\frac{d}{dx}(tanx) = sec^2x}$

Proof

Let $y = tanx = \frac{sinx}{cosx}$

By quotient rule

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x\,(-\sin x)}{cos^2x} = \frac{cos^2x + sin^2x}{cos^2x} = \frac{1}{cos^2x} = sec^2x.$$

Hence shown

SUMMARY

$$\frac{d}{dx}(sinx) = cosx$$

$$\frac{d}{dx}(cosx) = -sinx$$

$$\frac{d}{dx}(tanx) = sec^2x$$

$$\frac{d}{dx}(cotx) = -csec^2x$$

$$\frac{d}{dx}(secx) = secx\,tanx$$

$$\frac{d}{dx}(csecx) = -csecx\,cotx.$$

Example 25.1.14

Find the second derivative of $y = secx$

Solution

$$y = secx$$

$$\frac{dy}{dx} = secx\ tanx$$

By product rule

$$\frac{d^2x}{dx^2} = secx\ sec^2x + tanx.secx\ tanx$$

$$\frac{d^2y}{dx^2} = sec^3x + secxtan^2x.$$

Parametric Equations

Definition: (Parametric Curve)

If x and y are given as functions

$$x = f(t) \quad , y = g(t)$$

Over an interval of t-values, then the set of points

$(x, y) = f(t), g(t)$defined by these equations is **a parametric curve**.

The equation are **parametric equations** for the curve.

The value t is the **parameter** for the curve, and its domain I is **parameter interval**. If I is a closed interval $a \leq \mathrm{t} \leq \mathrm{b}$, the points $f(a), g(a)$are called the **initial point** of the curve. The point $f(b), g(b)$ is called the **terminal point**.

When we give the parametric equation and a parameter interval for a curve,

we say we have **parametized** the curve. The equation and the parameter interval together constitute a **parametrization** of a curve.

Parametric Formula for $\boldsymbol{dy}/\boldsymbol{dx}$

The parametized curve $x = f(t)$ and $y = g(t)$ are differentiable at t,

if f and g are differentiable at t and $\frac{dx}{dt} \neq 0$

then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Example 25.1.15

If $x = 2t + 3$, $y = t^2 - 1$, find the value of $\frac{dy}{dx}$ at $t = 6$

Solution

$$\frac{dx}{dt} = 2 \quad , \quad \frac{dy}{dt} = 2t$$

Then at

$$\frac{dy}{dt} = 12$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12}{2} = 6.$$

Example 25.1.16

Find the equation of the tangent to the curve with parametric equations

a) $x = 3t - 2sint$, $y = t^2 + t\cos t$, at the point p, where $t = \frac{\pi}{2}$
b) $x = e^t$, $y = e^t + e^{-t}$, at the point p,where $t = 0$

Solutions

a) $x = 3t - 2sint$, $y = t^2 + tcos$
at a point p, where $t = \frac{\pi}{2}$ the coordinates are

$$x = 3\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} - 2$$

$$y = (\frac{\pi}{2})^2 + \frac{\pi}{2} cos\frac{\pi}{2} = \frac{\pi^2}{4}$$

$$(x, y) = (\frac{3\pi}{2} - 2 \ , \frac{\pi^2}{4})$$

The gradient

$$\frac{dx}{dt} = 3 - 2cost \qquad \frac{dy}{dt} = 2t + t(sint) + 1. cost$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - tsint + cost}{3 - 2cost}$$

At $t = \frac{\pi}{2}$ then

$$\frac{dy}{dx} = \frac{\pi - \frac{\pi}{2}}{3}$$

Hence gradient= $m = \frac{\pi}{6}$

The equation of tangent

$y - y_1 = m(x - x_1)$

$$y - \frac{\pi^2}{4} = \frac{\pi}{6}\left(x - \left(\frac{3\pi}{2} - 2\right)\right)$$

$$y = \frac{\pi}{6}x + \frac{\pi}{3}$$

a) $x = e^t$, $y = e^t + e^{-t}$

At $t = 0$ $x = e^0 = 1$, $y = e^0 + e^0 = 2$

$$P(x, y) = P(1,2)$$

The gradient

$$\frac{dx}{dt} = e^t \quad , \quad \frac{dy}{dt} = e^t - e^{-t}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t - e^{-t}}{e^t}$$

at $t = 0$

$$m = 0$$

Then tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 0(x - 1)$$
$$y = 2$$

Note: to find the equation of normal line to the tangent we use the property

$$m_1 m_2 = -1.$$

Implicit Differentiation

Many of the functions we have dealt with so far have been described by the equation of the form of $y = f(x)$that expresses y **explicitly** in terms of the variable x. We have learned the rules for differentiating functions defined in this way. We also learnt how to differentiate the **parametric** equation in form of $x = f(t), \quad y = g(t)$. The third situation occur when we encounter equations like

$$y^2 + x^2 - xy = 25$$

Where y cannot be solved **explicitly** or a **subject of the formula**. Then we use **implicit differentiation method** to find the derivative of such functions.

Implicit Differentiation Rule

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Write $\frac{dy}{dx}$ after every differentiation involving function in y
3. Collect the terms with $\frac{dy}{dx}$ on one side of the equation.
4. Factor out $\frac{dy}{dx}$, and Solve for $\frac{dy}{dx}$

Example 25.1.17

Differentiate

a) $x^2 + y^2 = 5$
b) $3x + y^3 - 4y = 10x^2$
c) $cosy - y^2 = 8$
d) $x^2 + 2xy + y^2 = 8$

Solutions

a) $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5)$

$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

b) $\frac{d}{dx}(3x + y^3 - 4y) = \frac{d}{dx}(10x^2)$

$$3 + 3y^2\frac{dy}{dx} - 4\frac{dy}{dx} = 20x$$
$$3y^2\frac{dy}{dx} - 4\frac{dy}{dx} = 20x - 3$$
$$\frac{dy}{dx}(3y^2 - 4) = 20x - 3$$
$$\frac{dy}{dx} = \frac{20x - 3}{3y^2 - 4}.$$

c)

$$\frac{d}{dx}(cosy - y^2) = \frac{d}{dx}(8)$$
$$-siny\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(-siny - 2y) = 0$$
$$\frac{dy}{dx} = \frac{0}{-siny - 2y}$$
$$\frac{dy}{dx} = 0.$$

c) $x^2 + 2xy + y^2 = 8$

$$2x + 2\left(x.\frac{dy}{dx} + y\right) + 2y.\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx}(2x + 2y) = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-(2x + 2y)}{2x + 2y}$$

$$\frac{dy}{dx} = -1.$$

REVIEW EXERCISE

1. Differentiate the following
   a) $y = \cot(4x)$
   b) $y = -10x + 3cosx$
   c) $y = sinx^5$

3) Find the equation of the tangent to the curve and the equation of the normal to the tangent line with parametric equation
   a) $x = 9 - t^2, \quad y = t^2 + 6t,$ at the point P, where $t = 2$

4) Differentiate the following
   a) $3x^2 - 7y^2 + 4xy - 8x = 0$
   b) $x^2y^2 + 3y = 4x$

Derivative of Inverse Trigonometric Functions

If f is any one –to- one differentiable function, it can be proved that its inverse function$f^{-1}$is also differentiable, except where its tangent are vertical. This is possible because differentiable functions are continuous on its interval. If we reflect the function at $y = x$ then the inverse is also continuous implies differentiable. We use implicit differentiation to find the derivatives of inverse trigonometric function assuming it is differentiable.

Example 25.1.18

Find the derivative of $y = \sin^{-1} x$

Solution

Recall $\quad y = si\,n^{-1} x \qquad$ means $\quad sin\ y = x \qquad$ for $\quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Differentiate $\sin^{-1} y = x$ implicitly, we obtain

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

From trigonometric identities $\quad cos^2y + sin^2y = 1 \quad$ implies that $\cos y = \sqrt{1 - sin^2y}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - sin^2y}}$$

But $sin\ y = x \qquad$ implies $\qquad sin^2\ y = x^2$ substitute in $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Example 25.1.19

Differentiate $y = \cos^{-1} x$

Solution

$cos\ y = x$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Differentiate implicitly

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{-\sin y} \qquad ,$$

But $\sin y = \sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{1}{-\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$$

Example 25.1.20

Differentiate $y = \tan^{-1} x$

Solution

$$\tan y = x$$

Differentiate implicitly

$$sec^2 y \frac{dy}{dx} = 1$$

Implies that

$$\frac{dy}{dx} = \frac{1}{sec^2 y}$$

From trigonometric identity

$$sec^2 y = 1 + tan^2 y = 1 + x^2$$

Then

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

SAMMARY

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad -1 \le x \le 1$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad -1 \le x \le 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad (-\infty, \infty)$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2} \qquad (-\infty, \infty)$$

$$\frac{d}{dx}(sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \text{for } |x| > 1$$

$$\frac{d}{dx}(cs\,c^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

for $|x| > 1$

Example 25.1.21

Differentiate

a) $y = \cos^{-1}(3x)$
b) $y = \tan^{-1}(x^3)$
c) $y = \csc^{-1}(2x+1)$

Solutions

a) $y = \cos^{-1} 3x$

Let $u = 3x$ then

$$\frac{du}{dx} = 3$$

$$y = \cos^{-1} u$$

apply chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \times 3$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-(3x)^2}}$$
$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-9x^2}}.$$

b) $y = \tan^{-1}(x^3)$, let

$$u = x^3 \frac{du}{dx} = 3x^2$$
$$y = \tan^{-1} u$$

apply chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{1+u^2} \times 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{1+u^2}$$

$$\frac{dy}{dx} = \frac{3x^2}{1+(x^3)^2} = \frac{3x^2}{1+x^6}$$

c) $y = \csc^{-1}(2x+1)$
Let

$$u = 2x+1 \qquad \frac{du}{dx} = 2$$
$$y = \csc^{-1} u$$

apply chain rule

$$\frac{dy}{dx} = \frac{-1}{|u|\sqrt{u^2-1}} \times 2 \quad = \frac{-2}{|2x+1|\sqrt{(2x+1)^2-1}}.$$

Derivatives of Logarithmic Functions

We use implicit differentiation to find the derivative of logarithmic functions

$y = \log_a x$. In particular, the natural logarithm function $y = \ln x$ can be proved that logarithmic functions are differentiable.

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Proof

$let\ \ y = \log_a x$ Then $a^y = x$

differentiate this function implicitly, we get

$$a^y ln\ \ a\, dy/dx\ = 1$$

then

$$\frac{dy}{dx} = \frac{1}{a^y \, lna} \quad = \frac{1}{xl\,n\,x}.$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Example 25.1.22

Differentiate

a) $y = \ln(x^2 + 1)$
b) $y = \ln(cosx)$
c) $y = \log_3(7x^3 + 2x - 1)$

Solutions

a) $y = \ln(x^2 + 1)$
Let $\quad u = x^2 + 1 \quad , \quad \frac{du}{dx} = 2x$

$$y = \ln u$$

Applying chain rule

$$\frac{dy}{dx} = \frac{2x}{u} = \frac{2x}{(x^2 + 1)}$$

b) $y = \ln(cosx)$
Let

$$u = \cos x \quad , \quad \frac{du}{dx} = -sinx$$
$$y = \ln u$$

Applying chain rule

$$\frac{dy}{dx} = \frac{1}{u} \times (-sinx) = \frac{-sinx}{cosx}$$

c) $y = \log_3(7x^3 + 2x - 1)$
Let

$$u = 7x^3 + 2x - 1 \quad , \quad \frac{du}{dx} = 21x^2 + 2$$

Applying chain rule

$$\frac{dy}{dx} = \frac{1}{u\,ln3} \times (21x^2 + 2)$$
$$= \frac{1}{(7x^3 + 2x - 1)ln3} \times (21x^2 + 2)$$
$$= \frac{21x^2 + 2}{(7x^3 + 2x - 1)ln3}.$$

In general

$$\frac{d}{dx}\ln\left(u(x)\right) = \frac{du/dx}{u}$$

$$\frac{d}{dx}\log_a u(x) = \frac{du/dx}{u\, lna}$$

Exponential and logarithmic derivative

The exponential functions in form of $y = a^x$ and $y = x^x$ $y = e^x$

$$\frac{d}{dx}(e^x) = e^x$$

in general

$$\frac{d}{dx}(e^u) = \frac{du}{dx} \times e^u$$

$$\frac{d}{dx}(a^x) = a^x lna$$

in general

$$\frac{d}{dx}a^u = \frac{du}{dx} \times a^u lna.$$

Steps to follow when evaluating the derivative of exponential functions.

1. Take natural logarithm on both sides of the equation $y = f(x)$ and use the laws of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting result for $\frac{dy}{dx}$.

Example 25.1.23

Differentiate the following

a) $y = 3^x$ b) $y = 4^{-3x+1}$ d) $y = 4e^{3x}$

Solutions

a) $y = 3^x \frac{dy}{dx} = 3^x ln3$

b) $y = 4^{-3x+1}$

$$let\ u = -3x + 1 \quad , \quad \frac{du}{dx} = -3$$

Applying chain rule

$$\frac{dy}{dx} = \frac{du}{dx} \times a^u lna$$

$$\frac{dy}{dx} = -3.4^{-3x+1} ln4$$

c) $y = 4e^{3x}$

Let $u = 3x$, $\frac{du}{dx} = 3$

$$y = 4e^u$$

Applying chain rule

$$\frac{dy}{dx} = 4\frac{du}{dx} \times e^u$$

$$\frac{dy}{dx} = 4.3. e^u = 12e^{3x}$$

Example 25.1.24

Differentiate the following

a) $y = x^x$ $\quad$ $b)$ $y = x^{sinx}$

Solutions

We apply the three steps of derivative of exponential

a) $y = x^x$

$$\ln y = \ln x^x$$

$lny = x\ lnx$ Apply product and implicit

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x\ .1$$

$$\frac{dy}{dx} = y(1 + lnx)$$

replace y

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

b) $y = x^{sinx}$

$lny = sinx\ \ lnx$ apply product and implicit

$$\frac{1}{y}\frac{dy}{dx} = sinx.\frac{1}{x} + lnx. cosx$$

$$\frac{dy}{dx} = y(\frac{sinx}{x} + \ln x \cos x)$$

replace y

$$\frac{dy}{dx} = x^{sinx}(\frac{sinx}{x} + \ln x \cos x).$$

REVIEW EXERCISE

1. Find the derivatives of the given functions
   a) $y = 4\sec^{-1}(x^{4)}$
   b) $y = \tan^{-1}\sqrt{x}$
   c) $y = \frac{x^2}{\cot^{-1} x}$

2. Find the derivative of the functions
   a) $y = 2^x$
   b) $y = x^{cosx}$
   c) $y = (cosx)^x$
   e. $y = e^{2x}$

Derivative of Hyperbolic Function

Certain even and odd combinations of the exponential $\boldsymbol{e^x}$ and $\boldsymbol{e^{-x}}$ arise so frequently in Mathematics and its applications that they deserve to be given a special name.
In many ways, they are similar to the trigonometric functions and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason, they are collectively called **hyperbolic functions** and individually called **hyperbolic sine, hyperbolic cosine,** and so on

Definition of hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad , \quad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad , \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

Derivatives of the hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(cos\,h\,x) = sinhx$$
$$\frac{d}{dx}(sechx) = sechx\ tanh\ x$$
$$\frac{d}{dx}(\tanh x) = sech^2\ x$$
$$\frac{d}{dx}(cothx) = csch^2 x$$

Example 25.1.25 Find the derivative of $y = \cosh\sqrt{x}$

Solution Any of these differentiation rules can apply

$$\frac{d}{dx}\left(\cosh\sqrt{x}\right) = \sinh\sqrt{x}\,.\frac{d}{dx}\sqrt{x} = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

Derivatives of the inverse hyperbolic functions

$$\frac{d}{dx}(sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}\quad \frac{d}{dx}(csch^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(cosh^{-1}x) = -\frac{1}{\sqrt{x^2+1}}\quad \frac{d}{dx}(sech^{-1}) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(tanh^{-1}x) = \frac{1}{1-x^2}\quad \frac{d}{dx}(coth^{-1}x) = \frac{1}{1-x^2}.$$

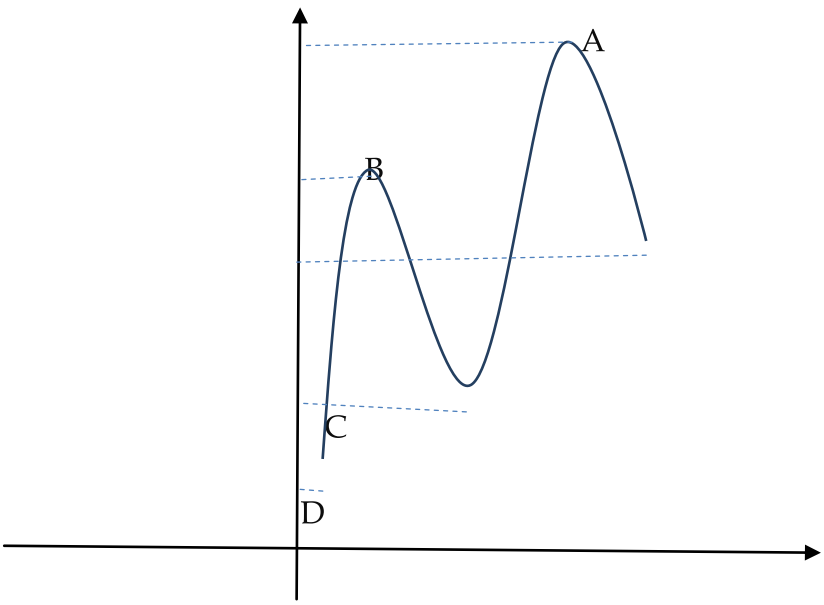
26: APPLICATIONS OF DIFFERENTIATION TO FUNCTIONS

Maximum and minimum values
This section shows how to locate and identify extreme values of a continuous function from its derivative. Once we do this, we can solve and sketch a variety of optimization problems in which we find the optional (best) way to do something in given situation.

Definitions (Absolute Maximum and Absolute Minimum)
Let f be a function with domain D. Then f has an**Absolute maximum value** on D at a point c if

$f(x) \le f(c)$f or all x in D and an **absolute minimum value** on D at c if

$f(x) \ge f(c)$ for all x in D.



A: absolute maximum **B:** local maximum **C:** local minimum **D:** absolute minimum

Absolute maximum and absolute minimum values are called **extrema**, plural **extremum.** Absolute extrema are also called **global extrema**.
Functions with the same defining rules can have different extrema, depending on the domain.

Definition (Local Maximum and Local Minimum)
A function f has a **local maximum value** at an interior point c of its domain if $f(x) \leq (x)$ for all x in some open interval containing c. And f has a **local minimum value** at point c of its domain if $f(x) \geq f(c)$ for all x in some open interval containing c.

We can extend the definition of local extrema to the end points of the intervals by defining f to have a **local maximum** or **local minimum** value at an end point c if the appropriate equality hold for all x in some half- open in its domain containing$c$. In the figure, the function f has local minima at a, e, b.
Local extrema are also called **relative extrema.**
An absolute maximum is also a local maximum, being the largest value overall. It is also the largest value in its immediate neighborhood, hence a list of all local maxima will be include the absolute maximum value if there is one. Similarly, a list of all local minima will include the absolute minima if there is one.

THEOREM (The First Derivative Theorem for Local Extreme Values)
If f has a local maximum or local minimum value at an interior point c of its domain, and if $\frac{dy}{dx}$ is defined at c, then

$$\boldsymbol{\frac{d}{dx} f(c) = 0}.$$

The theorem says that a functions first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only place where a function f can possibly have an extreme value (local or global) are

1. Interior point where $\frac{dy}{dx} = 0$
2. Interior point where $\frac{dy}{dx}$ is un defined
3. End point of the domain of f

Definition (Critical Point)
An interior point of the domain of a function f where $\frac{dy}{dx}$ is zero or undefined is a **critical point** of f and the value of x at that point is a **critical number**

The only domain points where a function can assume extreme value are critical points and end points. But the converse is not true.

A differentiable function may have a critical point at $x = c$ without having a local extreme value there. For instance, the function $f(x) = x^3$ has a critical point at zero but has no extreme value. Instead, it has a **point of inflection** there.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and end points
2. Take the largest and the smallest of these values.

Example 26.1.1

Find the critical points of

a) $y = x^3 - 6x^2 + 9x + 1$
b) $y = x^{2/3}$

Solutions

a) $y = x^3 - 6x^2 + 9x + 1$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

Set $\frac{dy}{dx} = 0$ then $3x^2 - 12x + 9 = 0$

By factorization $3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$

b) $y = x^{2/3}$

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

The function is undefined at $x = 0$

Hence $f(0) = 0$ and the critical point is (0,0)

Monotonic Functions and First Derivative Test

Sketching of Differentiable Functions

In sketching the graph of a differentiable function it is useful to know where the graph is increasing (rises from left to right) and where it decreases (fall from left to right) over an interval.

This section defines precisely what it means for a function to be increasing or decreasing over an interval, and gives a test to determine where it is increases and decreases. We also show how to test critical points of a function for the presence local extreme value.

Increasing Functions and Decreasing Functions

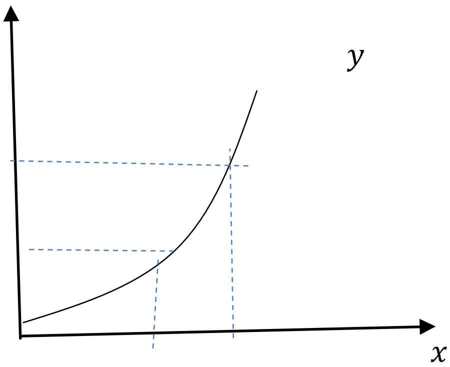
What kind of functions have positive derivatives or negative derivatives? Mean value theorem gives answer to this question. The only function with positive derivative are increasing functions and the only functions with negative derivatives are decreasing functions.

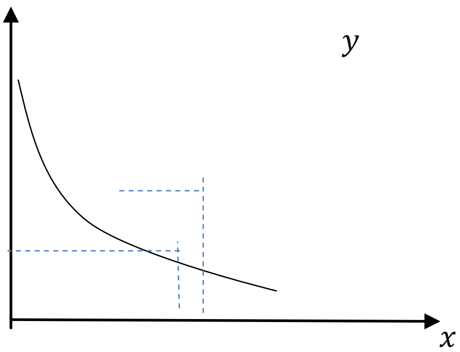
Definitions (Increasing, Decreasing Function)

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in$I$.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ then f is said to be **increasing** on I.
2. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ then f is said to be **decreasing** on I.

A function that is either increasing or decreasing on I is called a ***Monotonic function.***





First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b)

If $\frac{d}{dx}f(x) > 0$ at each point $x \in (a, b)$, then f is said to be increasing on $[a, b]$

If $\frac{d}{dx} f(x) < 0$ at each point $x \in (a, b)$, then f is said to be increasing on$[a, b]$.

Example 26.1.2

Find where the function $y = 3x^4 - 4x^3 - 12x^2 + 5$ increasing and where it is decreasing.

Solution

$$y = 3x^4 - 4x^3 - 12x^2 + 5$$

$$\frac{dy}{dx} = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 12)$$

$$= 12x(x - 2)(x + 1)$$

Set $\frac{dy}{dx} = 0$ implies $12x(x - 2)(x + 1) = 0$

then critical numbers are $x = -1$, $x = 0,$ $x = 2.$

We use a table interval to test where the function is increasing or decreasing

	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$2 < x$
$12x$	$-$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$
$(x+1)$	$-$	$+$	$+$	$+$
$\frac{dy}{dx}$	$-$	$+$	$-$	$+$
$y = f(x)$	decreasing	increasing	decreasing	increasing
interval	$(-\infty, -1]$	$[-1,0]$	$[0,2]$	$[2, \infty)$

Note: in a previous example the interval is not given, in this case we observe the behavior of the graph taking the interval from negative infinity to positive infinity. In the next example, we will look on when the interval is given and observe what it means when the graph has the end points.

First Derivative Test for Local Extrema (maximum or minimum)

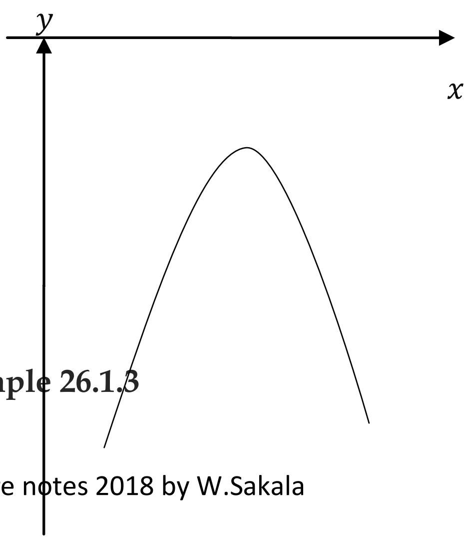
Suppose that c is a critical point of a continuous function $y = f(x)$ and that f is differentiable at every in some interval containing c except possibly at c itself.

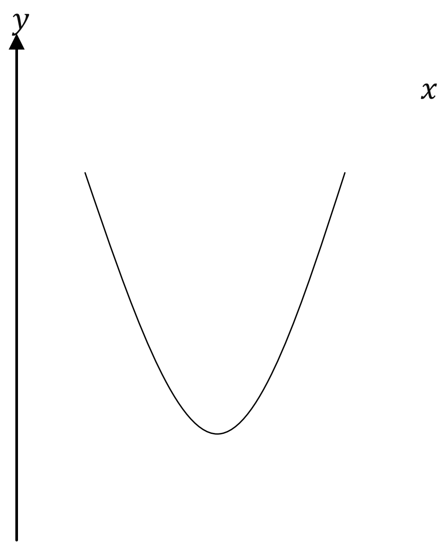
Moving from left to right

1. If $\frac{dy}{dx}$ changes from negative to positive at a point c, then f has a local minimum at c.

2. If $\frac{dy}{dx}$ changes from positive to negative at a point c, then f has a local maximum at c.

3. If $\frac{dy}{dx}$ does not change the sign at c then f has no local extremum at c, $e.g \quad y = x^3$





Example 26.1.3

Find all the critical numbers and use the first derivative test to classify each as the location of a local maximum, local minimum or neither. Hence, sketch the graph.

a) $y = x^3 - 9x^2 + 24x - 10 \quad , \quad 0 \le x \le 4$

Solution

$$y = x^3 - 9x^2 + 24x - 10$$

$$\frac{dy}{dx} = 3x^2 - 18x + 24$$

Set $\frac{dy}{dx} = 0$ implies $3x^2 - 18x + 24 = 0$

By factorization, we solve for x to find critical numbers

$$3(x^2 - 6x + 8) = 3(x-2)(x-4)$$

$$x = 0, x = 2 \ , x = 4$$

We are including $x = 0$ since it is the critical endpoint.

	$0 < x < 2$	$2 < x < 4$
	$[0,2]$	$[2,4]$
$(x-2)$	$-$	$+$
$(x-4)$	$-$	$-$
$\frac{dy}{dx}$	$+$	$-$
$y = f(x)$ behavior	positive function increasing	negative function decreasing

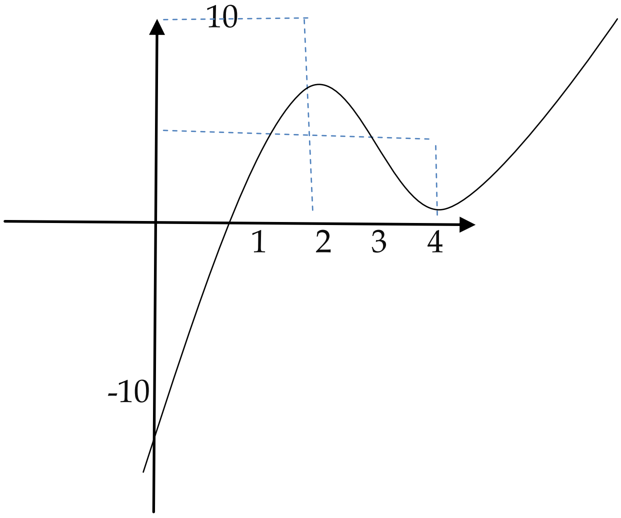
Coordinates Substitute the critical value x in $y = f(x)$ and $s$olve for y.

at $x = 0 \quad y = -10 \qquad (0, -10) \qquad$ local minimum

at $x = 2 \quad y = 10 \qquad (2,10) \qquad$ local maximum

at $x = 4 \quad y = 6 \qquad (4,6) \qquad$ absolute minimum

Graph



REVIEW EXERCISE

1. Find the interval where the function is increasing and decreasing
   a) $y = x^3 - 3x^2 - 9x + 1$ $\quad b)$ $y = x^4 - 8x^2 + 1$ $\quad c)$ $y = (x+1)^{2/3}$
2. Let $y = 1 + sin^2\, x + sin\; x$ be a function defined on the interval $0 < x < 2\pi$.
   (i) Find all the critical points of the function on $0 < x < 2\pi$
   (ii) Find the tangent to the graph of the function at a point where $x = \pi$
3. Find all the critical numbers and use the first derivative test to classify each as the location of a local maximum, local minimum or neither. Hence, sketch the graph.
   a) $y = x^3 - 3x + 6x$ $\quad c)$ $y = x^4 + 4x^3 - 2$
   b) $y = \frac{x}{1+x^2}$ $\quad d)$ $y = \tan^{-1}(x^2)$

Concavity, Curve Sketching and Second Derivative Test

Concavity

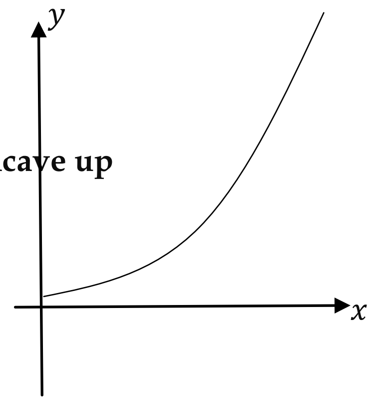
The graph of the curve $y = x^3$ rises as x increases, but the portions defines on the intervals $(-\infty, 0)$and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangent. The slop of the tangents decreases on the interval$(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises up above its tangent. The slop of the tangent are increasing on the interval$(0, \infty)$. This bending or turning behavior defines the **concavity** of the curve.

Definition (Concave Up, Concave Down)

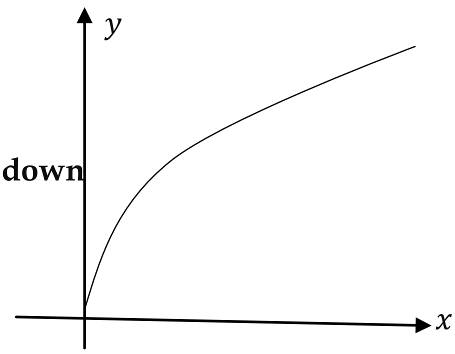
The graph of a differentiable function $y = f(x)\ is$

1. **Concave up**on an open interval I if $\frac{dy}{dx}$ is increasing on I.
2. **Concave down** on an open interval I if $\frac{dy}{dx}$ is decreasing on I.

Concave up



concave down



Second Derivative Test for Concavity.

Let $y = f(x)$ be twice- differentiable on an interval I.

1. If $\frac{d^2y}{dx^2} > 0$ on I, the graph of f concave up on I.
2. If $\frac{d^2y}{dx^2} < 0$ $on\ I$, the graph of f concave down on I.

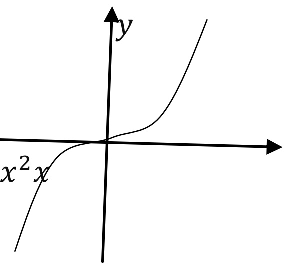
Example 26.1.4

Determine the interval where the graph of a given function is concave up and concave down.

a) $y = x^3$ $\qquad$ $b)$ $\quad y = 2x^3 + 9x^2 - 24x - 10$

Solutions

a) $y = x^3$



$$\frac{dy}{dx} = 3x^2$$

$$\frac{d^2y}{dx^2} = 6x$$

The graph concave up when $\quad \frac{d^2y}{dx^2} > 0 \quad then$

$f(x) = x^3$is concave up in the interval $(0, \infty)$

The graph of y is concave down when$\frac{d^2y}{dx^2} < 0 \quad then$

$y = x^3$is concave down in the interval $(-\infty, 0)$

b) $y = 2x^3 + 9x^2 - 24x - 10$

$$\frac{dy}{dx} = 6x^2 + 18x - 24$$

$$\frac{d^2y}{dx^2} = 12x + 18$$

For concave up $\frac{d^2y}{dx^2} > 0$ then

$$12x + 18 > 0$$

implies

$$x > -\frac{3}{2}$$

the graph of y is concave up on $(-\frac{3}{2}, \infty)$

For concave down $\frac{d^2y}{dx^2} < 0$ $\qquad$ the

$$12x + 18 < 0$$

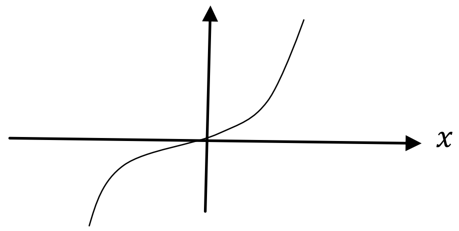
implies

$$x < -\frac{3}{2}$$

the graph of y is concave down on the interval $(-\infty, -\frac{3}{2})$

Inflection Point

The graph of the function $y = x^3$ canges concavity at the point$(0,0)$, wecall the point $(0, 0)$ **point of inflection** of the curve.



Definition (Point of Inflection)

A point where the graph of the function has a tangent line and where the concavity changes is a point of inflection.

A point on a curve where $\frac{d^2y}{dx^2}$ is positive on one side and negative on the other side is a point of inflection. At that point, $\frac{d^2y}{dx^2}$ is either zero or undefined. If the function y is twice differentiable function, then $\frac{d^2y}{dx^2} = 0$ at a point of inflection.

Example 26.1.5

Find the inflection point of $y = x^3 - 6x^2 + 9x + 1$

Solution

$$y = x^3 - 6x^2 + 9x + 1$$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

Set $\frac{d^2y}{dx^2} = 0$ then $6x - 12 = 0$ implies $x = 2$

When $x = 2$ $f(2) = 2^3 - 6 \times 2^2 + 9 \times 2 + 1 = 35$

$\therefore$ The point of inflection is at $(2,35)$

Second derivative Test for Local Extrema.

Instead of looking for sign change in $\frac{dy}{dx}$ at critical points, we can sometimes use the following test to determine the presence and character of local extrema.

THEOREM (Second Derivative Test for Local Extrema)
Suppose $\frac{d^2y}{dx^2}$ is continuous on an open interval that contains $x = c$.

1. If $\frac{d}{dx}f(c) = 0$ and $\frac{d^2}{dx^2}f(c) < 0$ then f has a local maximum at $x = c$
2. If $\frac{d}{dx}f(c) = 0$ and $\frac{d^2}{dx^2}f(c) > 0$, then f has a local minimum at $x = c$.

3. If $\frac{d}{dx}f(c) = 0$ and $\frac{d^2}{dx^2}f(c) = 0$, then test fail. the function f may have a local maximum, local minimum or neither.

Asymptotes and Review of Limits at Infinity

Limit at Infinity

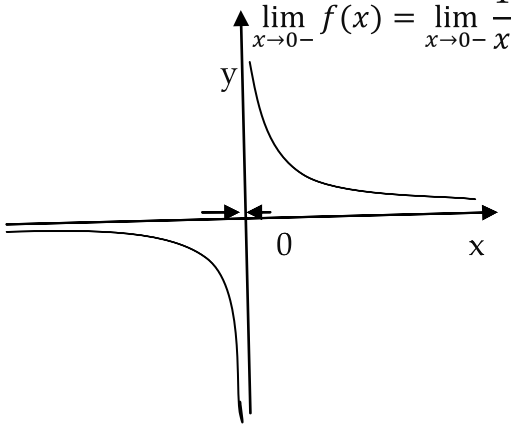
Let us look again at the function $f(x) = \frac{1}{x}$. As $x \to 0^+$, the value of f grows without bound, eventually reaching and surpassing every positive real number. It is nevertheless convenience to describe the behavior f by $f(x) \to \infty$ *as* $x \to 0^+$, we write

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{1}{x} = \infty$$

We are not saying that the limit exist nor are we saying that there is a real number ∞, for there is no such number. But we say the limit does not exist because $\frac{1}{x}$ becomes arbitrary large as $x \to 0^+$

as $x \to 0^-$, the value of $f(x) = \frac{1}{x}$ become arbitrary large and negative, we write

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{1}{x} = -\infty$$



Definition (Horizontal Asymptote)

A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if

$$\lim_{x\to\pm\infty} f(x) = b$$

We observe that the horizontal asymptote of the curve $y = \frac{1}{x}$ is 0

Example 26.1.6

Find the horizontal asymptote of $y = \frac{5x^2+8x-3}{3x^2+2}$

Solution

Since $\lim_{x\to\infty} f(x) = \frac{5}{3}$ and $\lim_{x\to-\infty} f(x) = \frac{5}{3}$

Then the horizontal asymptote is $\frac{5}{3}$

Vertical Asymptotes of Rational Functions

We say the line $x = 0$ (the $y -$ $axis$) is a vertical asymptote of the graph $y = \frac{1}{x}$. Observe that the denominator is zero at $x = 0$ and the function is

Undefined there.

Definition (Vertical Asymptote)
A line $x = a^{+}$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$\lim_{x \to a+} f(x) = \pm\infty$ or $\lim_{x \to a-} f(x) = \pm\infty$

Example 26.1.7

Find the vertical and the horizontal asymptote of the curve $y = \frac{x+3}{x+2}$

Solution

We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$ where the function is undefined since the denominator is zero.

The asymptote are quickly revealed if we recast the rational function as a polynomial with a remainder by dividing $(x + 2)$ $into$ $(x + 3)$ by using long division.

The result enables us to write

$$y = 1 + \frac{1}{x + 2}$$

We now see that the curve in the question is the graph of $y = \frac{1}{x}$ shifted 1 unit up and two units left. The asymptotes, instead of being the coordinate axis are now the line $y = 1$ and $x = -2$ as horizontal and vertical asymptote respectively.

Oblique Asymptotes

A rational function with the degree of the numerator greater than the degree of the denominator has at least one special asymptote that its line is not vertical or horizontal asymptote. We call this special line called **oblique asymptote**.))

Example 26.1.8

Find the asymptotes of the graph $y = \frac{x^2-3}{2x-4}$

Solution

We are interested in the behavior as $x \to \pm\infty$ and $as\ x \to 2$ where the denominator is zero. Divide $(2x-4)$into (x^2-3) by long division

$$f(x) = \frac{x^2-3}{2x-4} = \frac{x}{2} + 1 + \frac{1}{2x-4}$$

Since $\lim_{x\to 2+} f(x) = \infty$ and $lim_{x\to 2-} f(x) = -\infty$ the line $x = 2$ is a two sided Vertical asymptote. As $x \to \pm\infty$ the remainder aproaches 0 and $f(x) \to \left({}^{x}/_{2}\right) + 1$, the line$y = \frac{x}{2} + 1$ is an oblique asymptote both to the Left and to the right.

Strategy for Graphing/ Sketching the Function $y = f(x)$

1. Identify the domain of the function $y = f(x)$
2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
3. Find the critical points of f and identify the behavior at each point.
4. Find where the function is increasing and where is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and points in steps 3-5 then sketch the graph.

REVIEW EXERCISE

1. Determine all significant features and sketch the graph
   a) $y = \frac{x}{x+2}$ $d)$ $y = \frac{x^2}{x^2-9}$

   b) $y = x^3 - 3x^2 + 3x$ $e)$ $y = \frac{x^2+1}{3x^2-1}$

   c) $y = x^2 \ln x$

2. Determine the interval where the graph of a given function is concave up and concave down.
   a) $y = x^4 - 6x^2 + 2x + 3$
   b) $y = \sin x - \cos x$

3. Find all critical numbers and use the second derivative test to determine al local extrema.
   a) $y = x^4 + 4x^3 - 1$ $b)$ $y = \frac{x^2-5x+4}{x}$ $c)$ $y = xe^{-x}$

27: APPLICATION OF DIFFERENTIATION IN INDUSTRY

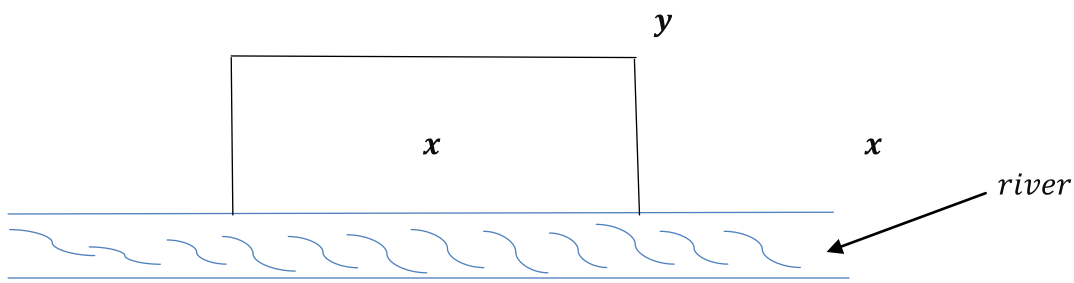
Steps in Solving Optimization Problems

1. Understand the Problem: the first step is to read the problem carefully until it is clearly understood. What are the given conditions?
2. Draw a Diagram: in most problems, it is useful to draw a diagram and identify the given and the required quantities on the diagram.
3. Introduce Notations: assign a symbol to the quantity that is to be maximized or minimized (let us call it Q). Also, select symbols $(a, b, c, \dots, x, y)$for other unknown quantities and label the diagram with these symbols. Example h for height, A for area, t for time and V for volume.
4. Express Q in terms of some other symbols from step3.
5. If Q has been expressed as a function of more than one variable in step 4, use the given information to find relationships (in form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of one variable x, say, $Q = f(x)$. Write the domain of this function.
6. Use the methods of finding the absolute maximum or minimum values of f

Example 26.1.9

A farmer has 2400m of fencing and wants to fence off a rectangular field that borders a straight river. He needs to no fence along the river. What dimensions of the field has the largest area?

Solution



We want to maximize the area of the field, let y and x be the length and breadth respectively. Then we express A in terms of x and y

$$A = xy$$

But total length of the fence $x + x + y = 2400$

$2x + y = 2400$ implies $y = (2400 - 2x)$

Substitute y in the equation of the Area

$$A = x(2400 - 2x) = 2400x - 2x^2$$

to maximize the area we find the derivative of the area

$$\frac{dA}{dx} = 2400 - 4x$$

Set the derivative equal to zero

$2400 - 4x = 0$ then

$$4x = 2400 \qquad x = 600$$

We now solve for y, $y = 2400 - 2x = 2400 - 2(600) = 120$

∴ Maximum area dimension

$$A = xy$$

$$A = 1200 \times 600 \qquad A = 720{,}000m^2$$

Rate of Change

We look at problems that ask the rate at which some variables changes. In each case the rate is the derivative that has to be computed from the rate at which some other variables is known to be change. To find it, we write the equation that relates the variables involves and differentiate it to get the equation that relate to the rate of change we seek to the rate of change we know. The problem of finding the rate you cannot measure easily from other rates you can is called related rate problem.

Related Rates Equation

Suppose that we are pumping air into a spherical balloon. Both the volume V and the radius r is increasing overtime, then since we know the equation of finding the volume of a spherical balloon, we can relate the rate of change to it.

$$V = \frac{4}{3}\pi r^3$$

Using chain rule, we differentiate to find the related rate equation

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = 4\pi r^2$$

Therefore, if we know the radius of the balloon and the rate ${dv}/{dt}$ at which the volume is increase at an instant time, then we can calculate the last equation ${dr}/{dt}$ to find how fast the radius is increasing at the instant rate. The related equation allows us to calculate ${dr}/{dt}$ from ${dv}/{dt}$ which the rate of increase is if the answer is positive and rate decrease if the answer is negative. On the other hand, we can calculate the rate of change of volume if we have been given the rate at which the radius is changing.

Example 26.1.10 (Filling a Conical tank)

Mopani Copper Mine Plc has a water-reserving tank that is to be used when there is an argent need. The tank is in a conical shape. Water runs into the tank at a rate of $9m^3/\text{min}$. The tank stands point down and has a height of 10m and a base of radius 5m. How fast is the water level rising when the water level is at 6m?

Example 26.1.11

a) $y = (2r^2 - r + 1)^2$ and $x = 4r$. At what rate is y changing with respect to x when $r = 0.5$?
b) A closed cylinder of a fixed length $10cm$ but its radius is increasing at the rate of $1.5cms^{-1}$. Find the rate of increase of its total surface area when the radius is $4cm$. (Leave the answer in terms of π)
c) Show that the rectangle has a maximum area for a given is a square.

Solutions $y = (2r^2 - r + 1)^2$, $x = 4r$

We find the derivative. Recall the parametric derivatives

$$\frac{dy}{dx} = \frac{dy/dr}{dx/dr}$$

$$\frac{dy}{dr} = 2(2r^2 - r + 1).4r - 1$$
$$= (8r - 2)(2r^2 - r + 1)$$

$$\frac{dx}{dr} = 4$$

Then

$$\frac{dy}{dx} = \frac{(8r - 2)(2r^2 - r + 1)}{4}$$

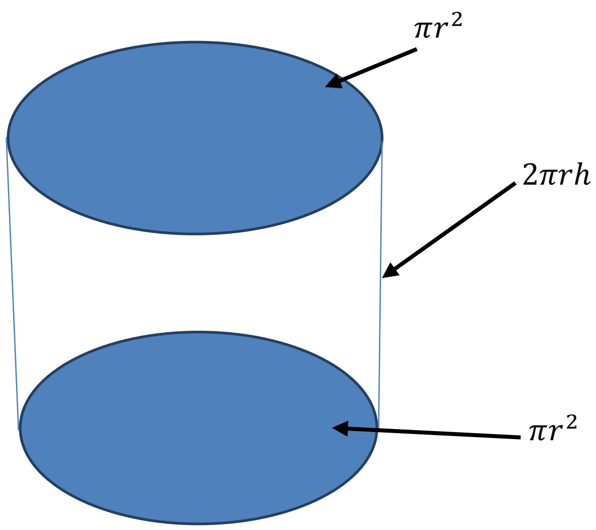
When $r = 0.5$ $\frac{dy}{dx} = \frac{3}{4}$ as rate of change in y with respect to x.

a) **Data**

$$h = 10cm \quad ,$$

$$\frac{dr}{dt} = 1.5cms^{-1}$$

Rate of increase in r



We express the Area of a cylinder in terms of r and h

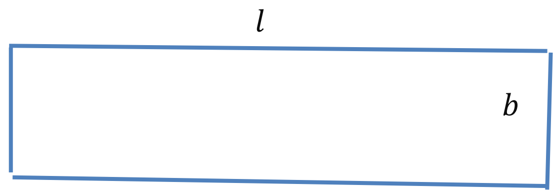
$$A = 2\pi rh + 2\pi r^2$$
$$\frac{dA}{dr} = 2\pi h + 4\pi r$$

$at\ r = 4\ and \quad h = 10$ $\qquad \frac{dA}{dr} = 2\pi(10) + 4\pi(4) = 36\pi\ m$

We find the rate of increase of the total surface area by chain rule

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 36\pi\ m \times 1.5cms^{-1} = 54\pi\ cm^2s^{-1}$$

b) The rectangle



Perimeter $\quad p = 2(l + b)\ ,\qquad b = \frac{p-2l}{2}$

Maximum area is found when $l = b$

$$A = l \times b$$

$$A = l \times \left(\frac{p - 2l}{2}\right) = \frac{pl}{2} - l^2$$

We maximize by finding the derivative

$$\frac{dA}{dl} = \frac{p}{2} - 2l$$

Set the derivative equal to zero

$$\frac{p}{2} - 2l = 0$$

$p = 4l$ which is the perimeter of a square.

REVIEW EXERCISE

1. A closed cylinder has a total surface area equal to 600π. Show that the volume,$V\ cm^3$, of this cylinder is given by the formula $V = 300\pi r - \pi r^3 \quad where\ r\ cm$ is the radius of the cylinder. Find the maximum value of such cylinder.

28: INTEGRATION

ANTIDERIVATIVES

We have studied how to find the derivatives of a function. However many problems require a function from its known derivative. More generally, we want to find a function F from its derivative f. If such function exists, it is called ***anti derivative*** of f.

Definition (Anti-derivative)

A function F is an anti- derivative of f on an interval I if $\frac{d}{dx}F(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative f is called ant-defferentiation. We use capital letters such as F to represent an ant-derivative of f, G to represent an ant-derivative of g, and so forth.

Example 28.1.1

Find the ant-derivative of the following functions

a) $f(x) = 2x$
b) $g(x) = \cos x$
c) $h(x) = 2x + \cos x$

Solutions

a) $F(x) = x^2$
b) $G(x) = \sin x$
c) $H(x) = x^2 + \sin x$
Each answer can be checked by differentiating. The derivative of $F(x) = x^2$is $2x$.

Arbitrary Constant.

The function $F(x) = x^2$ is not the only function with the derivative $2x$. The function $x^2 + 1$ has the same derivative. So does$x^2 + c$ for any constant. Where c is an **arbitrary constant**. More generally, we have the following results

If f is an ant derivative of on the interval I, then the general ant derivative of f on I is $F(x)$+c Where c is an arbitrary constant.

Ant-derivative Formulas

Function Ant-derivative

1. $\boldsymbol{x^n \frac{x^{n+1}}{n+1} + c \qquad n \neq -1, n\, rational}$
2. $\boldsymbol{\sin kx \qquad -\frac{\cos kx}{k} + c, \quad k \;\; a\, constant, k \neq 0}$
3. $\boldsymbol{\cos kx \qquad \frac{\sin kx}{k} + c, \quad k \;\; a\, constant, k \neq 0}$
4. $\boldsymbol{sec^2 x \qquad \tan x + c}$
5. $\boldsymbol{csc^2 x \qquad -\cot x + c}$
6. $\boldsymbol{\sec x \tan x \qquad \sec x + c}$
7. $\boldsymbol{\csc x \cot x \qquad -\csc x + c}$

Example 28.1. 2

Find the general ant derivative of each of the following

a) $f(x) = x^5$ $\quad b)$ $f(x) = \sin 2x$ $\quad c)$ $f(x) = \cos\left(\frac{x}{2}\right)$

Solution

a) $F(x) = \frac{x^6}{6} + c$ $\quad b)$ $F(x) = \frac{-\cos 2x}{2} + c$ $\quad c)$ $F(x) = \frac{\sin x/2}{1/2} + c$

Ant derivative Linearity Rules

<u>Function General Ant-derivative</u>

1. **constant multiple rule** $\quad kf(x) \quad kF(x) + c \quad$ k constant
2. **Negative Rule** $\quad -f(x) \quad -F(x) + c$
3. **Sum or difference Rule** $\quad f(x) \pm g(x) \quad F(x) \pm G(x) + c$

Indefinite Integrals

The special symbol used to denote the collection of all ant-derivatives of a function f.

Definition(Indefinite Integrals, Integrand)

The set of all ant-derivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int \boldsymbol{f(x)dx}$$

The symbol $\int$ is an **integral sign**, the function f is the **integrand** of the integral and x is the **variable of integration.**

The process of evaluating anti- derivatives using an integral is called **Integration.**

Using this notation, we restate the solutions as

$$\int 2x\,dx = x^2 + c \int \cos x\,dx = \sin x + c$$

Integrating a Constant

Remember $\frac{d}{dx}(ax) = a$ where a is any constant, then

$$\int \boldsymbol{a\,dx = ax + c}$$

Example 28.1.3

The integral of

a) $\int 2\,dx = 2x + c$
b) $\int \pi\,dx = \pi x + c$
c) $\int dx = x + c$

Integration Involving Inverse Trigonometric Functions

Remember

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1} x + C$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \Rightarrow \quad \int \frac{1}{1+x^2}dx = \tan^{-1} x + C$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \Rightarrow \quad \int \frac{1}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$$

Example28.1. 4

Find

$$\int (4\ \sin x - \frac{2}{x^2+1})dx$$

Solution

$$4\int \sin x\, dx - 2\int \frac{1}{x^2+1}dx = 4(-\cos x)\ - 2(\tan^{-1} x) + C$$

$$= -4\cos x - 2\tan^{-1} x + C$$

Integrating $\boldsymbol{y = \frac{1}{x}}$

The power rule fails because $y = x^{-1}$, $\int x^{-1}\, dx = \frac{x^{-1+1}}{-1+1} + C$ which is undefined

Corollary 1

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

Remember

$$\frac{d}{dx}\ln f(x) = \frac{\frac{d}{dx}f(x)}{f(x)}$$

More generally

$$\int \frac{\frac{d}{dx}f(x)}{f(x)}\ dx = \ln|f(x)| + C$$

Example 28.1.5

Find

a) $\int \cot x\, dx$ $b)$ $\int \tan x\, dx$ $c) \int \sec x\, dx$

Solutions

a) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\ dx$
Let

$$f(x) = \sin x\ ,\quad \frac{d}{dx}\sin x = \cos x$$

apply the corollary

$$\int \cot x = \ln|\sin x| + C$$

b) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\ dx = -\int \frac{-\sin x}{\cos x}\ dx$

$$= -\ln|\cos x| + C = \ln(\cos x)^{-1} + C$$
$$= \ln\frac{1}{\cos x} + C = \ln\sec x + C$$

c) $\int \sec x \, dx = \int \sec x \, \frac{(\sec x + \tan x)}{(\sec x + \tan x)} \, dx$

$$\int \frac{sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + C$$

Integration of Exponential functions

Remember $\frac{d}{dx}(e^{kx}) = ke^{kx}$ $\quad then \quad \int e^{kx} \, dx = \frac{e^{kx}}{k} + C$

$$\frac{d}{dx}a^x = a^x \ln a \, ,$$

then

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

Example 28.1.6

Consider the following

a) $\int e^x \, dx = e^x + c$

b) $\int e^{3x} \, dx = \frac{e^{3x}}{3} + C$

c) $\int 2^x \, dx = \frac{2^x}{\ln 2} + C$

d) $\int 3^x \ln 3 \, dx = \ln 3 \int 3^x \, dx$

$$= \ln 3 \times \frac{3^x}{\ln 3} + C = 3^x + C$$

REVIEW EXERCISE

. Evaluate the indefinite integrals

a) $\int (x^2 - 2x + 1)dx$

b) $\int (1 - x^2 - 3x^5)\, dx$

c) $\int (2\sin x + \cos x)\, dx$

d) $\int (4x - 2e^x)\, dx$

e) $\int (2t^2 - 3t^{-3/2} + 1)\, dx$

f) $\int (2\cos x - \sqrt{e^{2x}})\, dx$

. Find the most general ant-derivative and check your answer by differentiation

a) $\int (5 - 6x)\, dx$

b) $\int (-2\cos t$

c) $\int \frac{1}{\sqrt{x}}\, dx$

d) $\int (x+3)^3\ dx$

Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)dx$$

Where a is the lower limit, b is the upper limit, $f(x)$ is the integrand and dx is the variable of the integration.

THEOREM (The Existence of Definite Integral)

A continuous function is integrable. That is, if a function f is continuous on an interval$[a, b]$, then its definite integral over $[a, b]$ exist. Then $\int_a^b f(x)dx = F(b) - F(a)$

Example 28.1.7

Find the definite integral $\int_0^3 (x^2 - 4)dx$

Solution $\int_0^3 (x^2 - 4)dx = \frac{x^3}{3} - 4x|\,_0^3$

$$= \left(\frac{3^3}{3} - 4(3)\right) - \left(\frac{0^3}{3} - 4(0)\right)$$

$$= \frac{27}{3} - 12$$

$$= 9 - 12 = -3$$

Rules of Definite Integral

If f and g are integrable, the definite integral satisfy the following

1. order of integration: $\int_b^a f(x)\,dx = -\int_a^b f(x)dx$
2. zero with interval: $\int_a^a f(x)dx = 0$
3. constant multiple: $\int_a^b k\,f(x)dx = k\int_a^b f(x)dx$
4. sum and difference: $\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. adidivity : $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
6. domination: $f(x) \geq g(x)\ \ on\ \ [a,b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)\,dx$

Indefinite Integral and Substitution Rule

Recall that the set of all anti-derivatives of the function f is called **indefinite integral** with respect to x and is symbolized by

$$\int f(x)dx$$

When finding indefinite integral of a function f, remember that it always includes an arbitrary constant C.

The Power Rule I Integral form

If u is a differentiable function of x and n is a rational number different from -1, the chain rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$$

Therefore,

$$\int (u^n \frac{du}{dx})\,dx = \frac{u^{n+1}}{n+1} + C$$

The integral on the left hand side is written in simpler form as

$$\int u^n\,du,$$

Example Using Power Rule $\int(\sqrt{1+x^2}\,.\,2x) = \int \sqrt{u}\left(\frac{du}{dx}\right)dx$ $\quad let\ u = 1 + x^2,\ \ \frac{du}{dx} = 2x$

$$= \int u^{1/2}\ du \qquad\qquad = \frac{u^{(1/2)+1}}{1/2+1} + C$$

THEOREM (The Substitution Rule)

If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x)).\frac{d}{dx}g(x) = \int f(u).\ du$$

Proof

This rule is true because, by chain rule, $F(g(x))$ is an ant derivative $f(g(x).\frac{d}{dx}g(x)$ where F is an antiderivative of f.

$$\frac{d}{dx}F(g(x) = \frac{d}{dx}F(g(x)).\frac{d}{dx}g(x)$$

$$= f(g(x)).\frac{d}{dx}g(x)$$

If we substitute $u = g(x)$, then

$$\int f(g(x)).\frac{d}{dx}g(x) = \int \frac{d}{dx}F(g(x))dx$$

$$= F(g(x)) + C$$

$$= F(u) + C \qquad\qquad u = g(x)$$

$$= \int \frac{d}{du}F(u)du$$

$$= \int f(u) + C\ \frac{d}{du}F = f$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x).\frac{d}{dx}g(x)\ dx$$

When f and $\frac{d}{dx}g(x)$ are continuous,

1. Substitute $u = g(x)$ and $du = \frac{d}{dx}g(x).dx$ to obtain the integral
$$\int f(u)du$$
2. Integrate with respect to u
3. Replace u by $g(x)$ in the result

Example 28.1.9

Evaluate the indefinite integrals by substitution

a) $\int x^2\sqrt{x^3+2}\ dx$

b) $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}\ dx$

c) $\int \frac{4}{x(\ln x)^2}\ dx$

d) $\int \frac{x^3}{\sqrt{1-x^2}}\ dx$

e) $\int x^2\, sec^2 x^3\ \ dx$

f) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$

g) $\int \frac{(\tan^{-1}x)^2}{1+x^2}\ dx$

h) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

i) $\int \frac{sec^2 x}{\tan x}\ dx$

j) $\int x\cos x^2\ \ dx$

Solutions $\int x^2\sqrt{x^3+2}\ dx$ $\int x^2\,(x^3+2)^{1/2}\ dx$

$$let\ u = x^3+2$$
$$\frac{du}{dx} = 3x^2$$
$$dx = \frac{du}{3x^2}$$
$$Replace\ in\ \ u \quad and\ dx$$
$$\therefore \int x^2\,u^{1/2}\,\frac{du}{3x^2}$$
$$\frac{1}{3}\int u^{1/2}\ du$$
$$\frac{1}{3}\left(\frac{u^{3/2}}{3/2}\right)+C$$
$$=\frac{1}{9}(u)^{3/2}\ +C$$
$$=\frac{1}{9}(x^3+2)^{3/2}\ +C$$

a) $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}\ dx$ $let\ \ u = \sqrt{x}+2$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ $\quad dx = 2\sqrt{x}\,du$ $\quad$ substitute and cancel what is comon

$$= 2\int u^3\,du$$

$$= 2\left(\frac{u^4}{4}\right) + C$$

$$replace\ u$$

$$= \frac{1}{2}(\sqrt{x} + 2)^4 + C$$

b) $\int \frac{4}{x(\ln x)^2}\,dx$

Let $u = \ln(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

$$dx = x\,du$$

$$= \int \frac{4}{xu^2} x\,du = 4\int u^{-2}\ du = -8u^{-3} + C = -8(\ln x)^{-3} + C$$

c) $\int \frac{x^3}{\sqrt{1-x^4}}\,dx$

$$\int (1 - x^4)^{-1/2}\,x^3\,dx$$

$$let\ \ u = 1 - x^4$$

$$\frac{du}{dx} = -4x^3$$

$$dx = \frac{du}{-4x^3}$$

$$\int x^3 u^{-1}\frac{du}{-4x^3}$$

$$-\frac{1}{4}\int u^{-1/2}\,du = -\frac{1}{4}.\frac{u^{1/2}}{1/2} + C = -\frac{1}{2}\sqrt{1 - x^4}\ + C$$

d) $\int x^2 sec^2 x^3\,dx$

$$\int x^2\,(\sec x^3)^2\ dx$$

$$let\ u = x^3$$

$$\frac{du}{dx} = 3x^2$$

$$dx = \frac{du}{3x^2}$$

$$\frac{1}{3}\int sec^2 u\ \ du = \frac{1}{3}\tan u + C$$

$$\frac{1}{3}\tan x^3\ + C.$$

e) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

$$let\ u = \sqrt{x}$$
$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$
$$dx = 2\sqrt{x}\, du$$
$$2\int \sin u\, du = -2\cos u + C = -2\cos\sqrt{x} + C$$

f) $\int \frac{(\tan^{-1}x)^2}{1+x^2}\, dx$

$$let\ \ u = \tan^{-1}x$$
$$dx = (1+x^2)du$$
$$\int \frac{u^2}{1+x^2}(1+x^2)du$$
$$\int u^2\ du$$
$$= \frac{u^3}{3} + C.$$
$$= \frac{(\tan^{-1}x)^3}{3} + C$$

g) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$ $\qquad let\ u = \sqrt{x}$

$$dx = \frac{1}{2\sqrt{x}}$$
$$2\int e^u\ du = 2e^u + C = 2e^{\sqrt{x}} + C$$

h) $\int \frac{sec^2x}{\tan x}\, dx$ $\qquad let\ \ u = \tan x$

$$\frac{du}{dx} = sec^2x$$
$$dx = \frac{du}{sec^2x}$$
$$\int \frac{1}{u}\ du = \ln|u| + C = \ln|\tan x| + C$$

i) $\int x\cos x^2\ dx$

Let

$$u = x^2$$
$$\frac{du}{dx} = 2x$$
$$dx = \frac{du}{2x}$$
$$\frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin x^2 + C.$$

REVIEW EXERCISE

1. Evaluate the indefinite integrals by substitution
   a) $\int 2\sin x\cos x\, dx$

   b) $\int sec^2x\tan x\, dx$

   c) $\int 3x\sqrt{7-3x^2}\ dx$

   d) $\int \frac{}{\sqrt{5x+8}}\, dx$

   e) $\int \sqrt{\cot x}\, csc^2x\, dx$

   f) $\int (3\sin x+4)^6\cos x\, dx$

Integration by Part

Since

$$\int x\, dx = \frac{1}{2}x^2 + C$$
$$\int x^2\, dx = \frac{1}{3}x^3 + C$$

It is apparently that

$$\int x.x\, dx \neq \int xdx.\ \int x\, dx$$

Integration by part helps simplifying integrals of the form

$$\int f(x).g(x)dx$$

It is useful that f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int x\, e^x\, dx$$

Is such an integral because $f(x) = x$ is differentiated twice, become zero and $g(x) = e^x$ can be integrated repeatedly without difficult.

Product Rule in Integral Form, integration by Part formula

If f and g are differentiable functions of x, the product rule says

$$\frac{d}{dx}[f(x).g(x)] = \frac{d}{dx}f(x).g(x) + \frac{d}{dx}g(x).f(x)$$

The terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x).g(x)]dx = \int [\frac{d}{dx}f(x).g(x) + \frac{d}{dx}g(x).f(x)]\ dx$$

or

$$\int \frac{d}{dx}[f(x).g(x)dx = \int \frac{d}{dx}f(x).g(x)\ dx + \int \frac{d}{dx}\ g(x).f(x)dx$$

Rearranging the last equation

$$\int f(x).\ \frac{d}{dx}g(x)dx = \int \frac{d}{dx}[f(x)g(x)]dx - \int \frac{d}{dx}f(x).g(x)dx$$

$$\int f(x).\frac{d}{dx}g(x)dx = f(x).g(x) - \int \frac{d}{dx}f(x).g(x)dx$$

Let $u = f(x)$ and $v = g(x)$

$$\int u\,.dv = u.v - \int v\ du$$

Integration by Parts Formula

$$\int u.dv\ = uv - \int v\,du$$

Choosing of u and dv when using reduction formula

Remember

We said that, $u = f$ has to be chosen such that it is differentiable repeatedly, and dv that is integrated without any difficult.

Now there is one simple criteria we can use to choose u when two different functions being given. From this, choose the function that comes first in this order as u.

L I A T E

Logarithm, Inverse trigonometry, Algebra, Trigonometric, Exponential

Example 28.1.10

Evaluate the indefinite integrals by parts

a) $\int x\cos x\,dx$

b) $\int x\sin 4x\,dx$

c) $\int \ln x\,dx$

d) $\int x\ln x\,dx$

e) $\int x\,sec^2\,x\,dx$

Solutions

a) $\int x\cos x\,dx$ $\qquad$ *Let* $u = x$

$$du = dx$$

$$dv = \cos x$$

$$\int dv = v = \int \cos x = \sin x$$

$$\int u\,dv = uv - \int v\,du$$

$$= x.\sin x - \int \sin x.\ dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

$$\therefore \int x\cos x\,dx = x\sin x + \cos x + C$$

b) $\int x \sin 4x \, dx$

Let

$$u = x$$
$$du = dx$$

Let $dv = \sin 4x \, dx$

$$\int dv = v = \int \sin 4x \, dx$$
$$v = \frac{-\cos 4x}{4}$$
$$\int u \, dv = uv - \int v \, du$$
$$= x.\frac{-\cos 4x}{4} - \int -\frac{\cos 4x}{4} \, dx$$
$$= -\frac{1}{4} x \cos 4x \; + \frac{1}{4}\int \cos 4x \, dx$$
$$= -\frac{1}{4} x \cos 4x \; + \frac{1}{4}(\frac{1}{4}\sin 4x) + C$$
$$= -\frac{1}{4} x \cos 4x + \frac{1}{16}\sin 4x + C$$

$$\therefore \int x \sin 4x \, dx = -\frac{1}{4} x \cos 4x \; + \frac{1}{16}\sin 4x + C$$

c) $\int \ln x \, dx$

Let

$$u = \ln x$$
$$du = \frac{1}{x} \, dx$$

Let

$$dv = dx$$
$$\int dv = v = \int dx$$
$$v = x$$
$$\int u \, dv = uv - \int v \, du$$
$$= \ln x . x - \int x \, \frac{1}{x} \, dx$$
$$= x \ln x - \int dx$$
$$= x \ln x - x + C$$
$$\therefore \int \ln x \, dx = x \ln x - x + C$$

d) $\int x \ln x \, dx$

Let

$$u = \ln x$$
$$du = \frac{1}{x} dx$$

Let

$$dv = xdx$$

$$\int dv = v = \int x\, dx$$

$$v = \frac{x^2}{2}$$

$$\int u\, dv = uv - \int v\, du$$

$$= \ln x . \frac{x^2}{2} - \int \frac{x^2}{2} . \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\, dx$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} (\frac{x^2}{2}) + C$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$$

$$\therefore \quad \int x \ln x\, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$$

e) $\int x\, sec^2\, x\, dx$

Let

$$u = x$$

$$du = dx$$

Let

$$dv = sec^2 x\, dx$$

$$\int dv = v = \int sec^2\, x\, dx$$

$$v = \tan x$$

$$\int u\, dv = uv - \int v\, du$$

$$= x \tan x - \int \tan x\, dx$$

$$= x \tan x - \int \frac{\sin x}{\cos x}\, dx$$

$$= x \tan x + \ln|\cos x| + C$$

$$\therefore \quad \int x\, sec^2 x = x \tan x \ + \ln|\cos x| + C$$

Repeated Use of Integration by Parts

Example 28.1.11

Evaluate

$$\int x^2\, e^x\, dx$$

Solution

Let $u = x^2$, $du = 2x$ and

Let $dv = e^x$, $\int dv = v = \int e^x = e^x$

$$\int u\, dv = uv - \int v\, du$$

$$\int u\,dv = x^2\,e^x - \int e^x\;2x\,dx$$

$$\int u\,dv = x^2\;e^x - 2\int xe^x\;dx \qquad \text{(i)}$$

The new integral is now less complicated that can be evaluated using integration by part again.

$$\int xe^x dx$$

Let $u = x,\; du = dx$

Let $dv = e^x dx, \quad \int dv = v = \int e^x\,dx = e^x$

$$\int u\,dv = xe^x - \int e^x dx$$

$$\int u\,dv = xe^x - e^x \qquad \text{(ii)}$$

Substitute (ii) in (i)

$$\int u\,dv = x^2\,e^x - 2(xe^x - e^x) + C$$

$$\int u\,dv = x^2 e^x + 2xe^x + e^x + C$$

Example 28.1.12

Evaluate

$$\int e^x \cos x\,dx$$

Solution

Let $u = e^x\,,\; du = e^x dx$

Let $dv = \cos x\,dx \quad , \quad \int dv = v = \int \cos x\,dx = \sin x$

$$\int e^x \cos x\,dx = e^x \sin x - \int \sin x . e^x\;dx$$

Integrate the second integral by part

$$u = e^x\;,\;\; du = e^x\,dx \quad , \quad dv = \sin x\,dx\;,\;\; v = -\cos x$$

$\int e^x \cos x\,dx = e^x \sin x - (-e^x . \cos x - \int -\cos x\,e^x\;dx)$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

The unknown integral now appears on both sides of the equation

Rearranging them gives,

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\therefore \quad \int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

REVIEW EXERCISE

1. Evaluate the indefinite integrals by parts
   a) $\int x \sin\frac{x}{2}\;dx$
   b) $\int 4x\;sec^2\,x\;dx$
   c) $\int xe^{2x}\;dx$ d) $\int e^x \sin 4x$

Integration by Reduction Formula

Given $\int cos^n\, x\, dx$ when n is small, it is easy to use the cosine and sine identities, but when n is large, it becomes hard to use the identities. The reduction method enables us to handle such integrals.

Example 28.1.13

Obtain a reduction formula that expresses the integral

$$\int cos^n x\, dx$$

In terms of an integral of lower power of $\cos x$

Solution

We may think of $cos^n x$ as $cos^{n-1}x.\cos x$. Then we let

$$u = cos^{n-1}x \quad , \quad dv = \cos x\, dx$$

So that
$du = (n-1)cos^{n-2}\, x sinx\, dx$ and $v = sin\; x$
Hence

$$\int cos^n\, x\, dx = cos^{n-1}x \sin x + \int \sin x\,(n-1)cos^{n-2}x \sin x\, dx$$

$$= cos^{n-1}x \sin x + (n-1)\int sin^2 x\, cos^{n-2}x\, dx$$

$$= cos^{n-1}x \sin x + (n-1)\int (1 - cos^2 x)cos^{n-2}\, x\, dx$$

$$= cos^{n-1}x \sin x + (n-1)\int cos^{n-2}x - cos^n\, x\, dx$$

$$= cos^{n-1}x \sin x + (n-1)\int cos^{n-2}dx - (n-1)\int cos^n\, x\, dx$$

If we add

$$(n-1)\int cos^n x\, dx$$

On both sides, we obtain

$$n\int cos^n x\, dx = cos^{n-1}x \sin x + (n-1)\int cos^{n-2}\, x\, dx$$

Divide both sides by n, we get

$$\int cos^n\, x\, dx = \frac{cos^{n-1}x \sin x}{n} + \frac{(n-1)}{n}\int cos^{n-2}\, x\, dx$$

This allows us to reduce the exponent on $\cos x$ by 2 and it is very useful formula. When n is an integer, we apply the formula repeatedly until the remaining integral is either
$\int cosx\, dx = sinx + C$ or $\int cos^0\, x\, dx = \int dx = x + C$

Evaluating the indefinite integral of

$$\int sin^n x = \frac{sin^{n-1}x \cos x}{n} - \frac{(n-1)}{n}\int sin^{n-2}x\, dx$$

Now let $I_n = \int sin^n\, x\, dx$ then

$$I_n = \frac{cos^{n-1}x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$$

$$I_n = \frac{sin^{n-1}x \cos x}{n} - \frac{(n-1)}{n} I_{n-2}$$

Example 28.1.14

Evaluate the indefinite integral by reduction formula $\int cos^3 x\, dx$

Solution From the formula $\int cos^3 x\, dx = \frac{cos^2 x\, sinx}{3} + \frac{2}{3}\int \cos x\, dx$

$$= \frac{cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C$$

Example 28.1.15

Evaluate the indefinite integral by reduction formula $\int cos^5 x\, dx$

Solution Since $n = 5$ then $I_5 = \int cos^5\, x\, dx$

$$I_5 = \frac{cos^4\, xsi\,n\,x}{5} + \frac{4}{5}\, I_3$$

$$I_3 = \frac{cos^2\, xsi\,n\,x}{3} + \frac{2}{3}\, I_1$$

$$I_1 = \int cosx = -sinx.$$

Then we do a backward substitution

$$I_3 = \frac{cos^2 x\, sin\, x}{3} - \frac{2}{3} sin\, x$$

$$I_5 = \frac{cos^4 x\, sin\, x}{5} + \frac{4}{5}(\frac{cos^2 x\, sin\, x}{3} - \frac{2}{3} sin\, x)$$

Hence since $I_5 = \int cos^5\, x\, dx$

$$\therefore \int cos^5 x\, dx = \frac{cos^4 x \sin x}{5} + \frac{4}{15} cos^2 x \sin x - \frac{8}{15}\sin x + C$$

Example 28.1.16

Evaluate the indefinite integral using reduction formula $\int sin^4 x\, dx$

Solution $n = 4$ then $I_4 = \int sin^4\, x\, dx$

$$I_4 = \frac{sin^3\,xco\,s\,x}{4} - \frac{3}{4}\ I_2$$

$$I_2 = \quad sinx\frac{co\,s\,x}{2} - \frac{1}{2}\ I_0$$

$$I_0 = \int dx = x$$

Back ward substitution

$$I_2 = \quad sinx\frac{co\,s\,x}{2} - \frac{1}{2}\ x$$

$$I_4 = \frac{sin^3x\,cos\,x}{4} - \frac{3}{4}\left(\frac{sin\,x\,cos\,x}{2} - \frac{1}{2}\ x\right) + C$$

Hence

$$\int \boldsymbol{sin^4x\,dx} = \frac{\boldsymbol{sin^3x\cos x}}{\mathbf{4}} - \frac{\mathbf{3}}{\mathbf{8}}\mathbf{sin}\,\boldsymbol{x}\,\mathbf{cos}\,\boldsymbol{x} + \frac{\mathbf{3}}{\mathbf{8}}\ \boldsymbol{x} + \boldsymbol{C}$$

Special reduction formula

Integration of Rational Functions by Partial Fractions

Rational function $\frac{(5x-3)}{(x^2-2x-3)}$ can be written as $\frac{(5x-3)}{(x^2-2x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$

Which can be verified algebraically by placing a fraction on the right side over a common denominator $(x+1)(x-3)$.

We simply sum the integral of the fraction on the right side.

$$\int \frac{(5x-3)}{(x^2-2x-3)}\ dx = \int \frac{2}{x+1}\ dx + \int \frac{3}{x-3}\ dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

The method for writing rational functions as a sum of simpler fractions is called the method of partial fractions.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things;

1. The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be a proper fraction.
2. We must know the factors of $g(x)$

Example 28.1.17

Distinct Linear Factors

Evaluate

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx \quad \text{using partial fractions}$$

Solution

The partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$$

We find the undetermined coefficient A, B and C, we clear the fraction and get

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C)$$

The polynomials on both sides of the above equations are identical, so we equate coefficients of like powers of x obtaining

Coefficient of

$$x^2: \qquad A+B+C=1$$

Coefficient of

$$x: \qquad 4A+2B \quad =4$$

Coefficient of

$$x^0: \qquad 3A-3B-C=1$$

There are many ways for solving such a system of linear equations for the unknowns A, B and C including elimination.

The solutions are $A=\frac{3}{4,}$, $B=\frac{1}{2}$ and $C=\frac{-1}{4}$

Then

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\, dx = \int [\frac{3}{4}\frac{1}{(x-1)} + \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{4}\frac{1}{(x+3)}]\, dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

Example 28.1.19 Repeating Linear Factors

Evaluate $\int \frac{6x+7}{(x+2)^2}\, dx$

Solution

First, we express the integrand as sum of partial fractions with coefficients

$$\frac{(6x+7)}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

Multiply both sides by $(x+2)^2$, we get

$$6x+2 = A(x+2) + B$$

$$= Ax + (2A + B)$$

Equating the coefficients of correspondence powers of x gives

$$A = 6, \;\; B = -5$$

Therefore

$$\int \frac{6x+7}{(x+2)^2}\, dx = \int (\frac{6}{(x+2)} - \frac{5}{(x+2)^2})\, dx = 6\int \frac{1}{(x+2)}\, dx - 5\int (x+2)^{-2}\, dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$

Example 28.1.20 (Integrating Improper Fraction)

Evaluate

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\, dx$$

Solution

First, we divide the denominator into the numerator by long division to get a polynomial plus a proper fraction.

$$\frac{2x^2-4x^2-x-3}{x^2-2x-3}\, dx = 2x + \frac{5x-3}{x^2-2x-3}$$

$$= \int 2x\, dx + \int \frac{2}{(x+1)}\, dx + \int \frac{3}{(x-3)}\, dx$$

$$= x^2 + 2\ln|\, x+1| + 3\ln|\, x-3| + C$$

A quadratic polynomial is **irreducible** if it cannot be written as a product of two linear factors with real coefficients.

Example 28.1.21 (Integrate the irreducible factor)

Evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx$

Solution

The denominator has a an irreducible quadratic factor as well as the repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Clearing the equation gives

$$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+)$$

$$= (A+C)x^3 + (-2A+B-C)x^2 + (A-2B+C)x + (B-C+D)$$

Equating the coefficients of like terms gives

coeff of

$$x^3: \qquad 0 = A + C$$

$$x^2: \qquad 0 = -2A + B - C$$

$$x^1: \qquad -2 = A - 2B + C$$

$$x^0: \qquad 4 = B - C + D$$

We solve these equations simultaneously $A = 2$, $B = 1$, $C = -2$ and $D = 1$

We substitute these values

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

Finally, using the expansion above we integrate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx = \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx$$

$$= \ln|x^2+1| + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C.$$

For any given partial fraction, apply the rules of decomposition of partial fractions then integrate the partial sums.

REVIEW EXERCISE

Find the partial decomposition and an ant derivative of the following

a) $\frac{x-5}{x^2-1}$

b) $\frac{3x}{x^2-3x-4}$

c) $\frac{x^3-4}{x^3+2x^2+2x}$

d) $\frac{2x^3-4x^2-15x+15}{x^2-2x-8}$

e) $\frac{5x-13}{(x-3)(x-2)}$

f) $\frac{x+4}{(x+1)^2}$

g) $\frac{x+1}{x^2(x-1)}$

29: APPLICATIONS OF INTEGRATION

Area under the Curve

We now make precise the notation of the area of a region with curved boundary. The area under the graph of a non-negative continuous function is defined by the definite integral.

Definition (Area under a curve as a definite integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$over $[a, b]$ is the integral of f from a to b

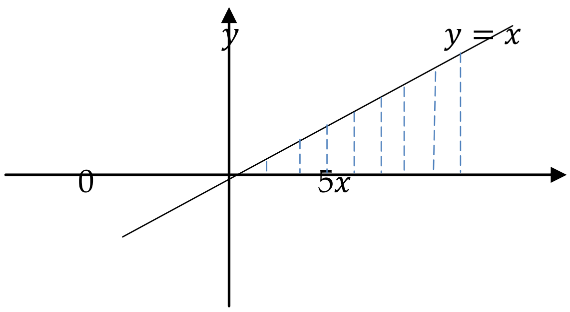
$$A = \int_a^b f(x)dx$$

Example 29.1.1

Find the area bounded by the straight line $y = x$ in the interval [0,5]

Solution

We first draw the shape of the region required



We see that the area under the curve gives the area of a triangle but we cannot use the formula for the area of a triangle since we have not been given the height to satisfy the formula. Now we use integration to evaluate the area bounded by this line $y = x$ on the interval [0,5]

By the formula $\int_a^b f(x)dx$, $a = 0,\ b = 5$ and $f(x) = x$ then,

$$\int_0^5 x\ dx = [\ \frac{x^2}{2}\ ]_0^5$$

$$= \frac{5^2}{2} - \frac{0^2}{2} = \frac{25}{2}\ m$$

2

Example 29.1.2

Find the area bounded by the curve$y = x^2 + 1$, the $x -$ axis and the coordinates $x = 0$ and $x = 2$

Solution

Part of the curve $y = x^2 + 1$ is shown below.

$$A = \int_a^b f(x)dx$$

$$= \int_0^2 (x^2 + 1)\ dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^2$$

$$= \left(\frac{2^3}{3} + 2\right) - (0 + 0)$$

$$= 4\frac{2}{3} units^2.$$

Example 29.1.24

Find the equation of the area $y = x^2 - 4x$ on the interval $[0, p]$

Solution By the formula $\int_a^b f(x)dx$

We have $A = \int_0^p (x^2 - 4x)dx$

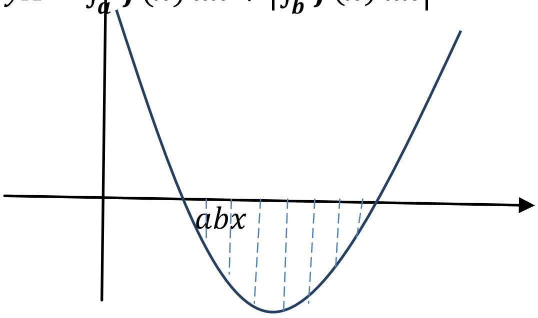
$$= \left(\frac{x^3}{3} - 4.\frac{x^2}{2}\right)_0^p$$

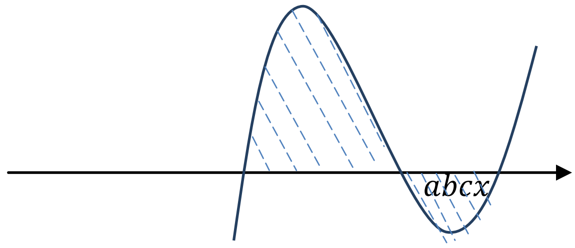
$$A = \frac{p^3}{3} - 2p^2.$$

Area below x −axis If area is below the x −axis, then

$$A = \left|\int_a^b \boldsymbol{f(x)dx}\right| y$$

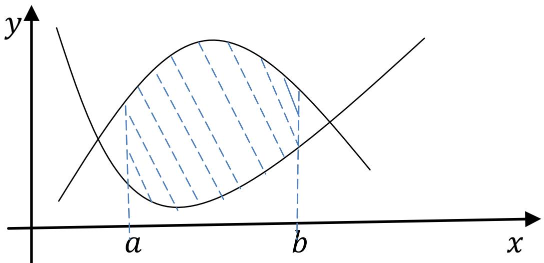
$y\boldsymbol{A} = \int_a^b \boldsymbol{f(x)\,dx} + \left|\int_b^c \boldsymbol{f(x)\,dx}\right|$





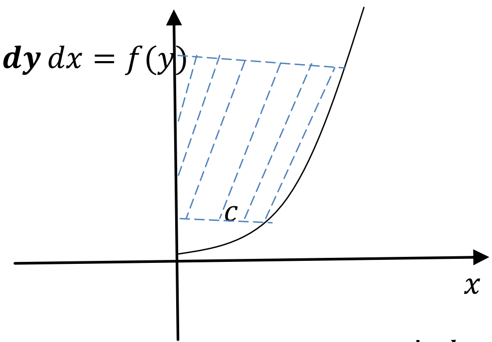
Area Enclosed By Two Curves : If the area is between the two curves $y = f(x)$ and $y = g(x)$, then

$$\boldsymbol{A} = \int_a^b \boldsymbol{f(x)dx} - \int_a^b \boldsymbol{g(x)dx}$$



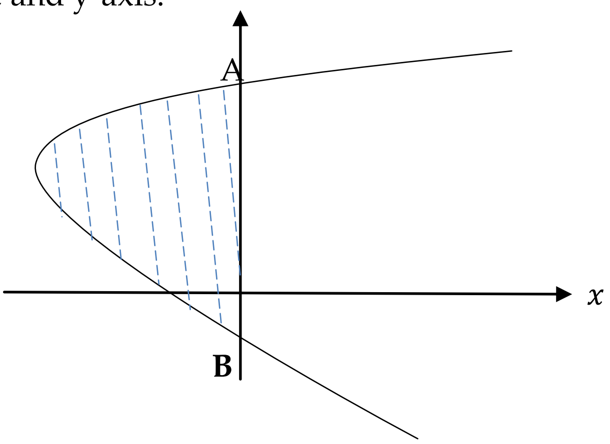
If the area is between the curve and $y - axis$, then

$\boldsymbol{A} = \int_c^d \boldsymbol{f(y)dy}$or $\boldsymbol{A} = \int_c^d \boldsymbol{x\,dy}$ $dx = f(y)$



If the region is on the left side of y –axis, then $A = \boldsymbol{A} = \left|\int_c^d \boldsymbol{x\,dy}\right|$

Example 29.1.3 The curve $(y-1)^2 = x + 4$ meets y-axis at A and B. calculate the area of the region bounded by the curve and y-axis.



Solution

$$(y-1)^2 = x+4$$

$$(y-1)^2 - 4 = x$$

At $x = 0$

$$(y-1)^2 - 4 = 0$$

$$(y-1)^2 = 4$$

$$y = 3 \; or \; y = -1$$

Therefore $\quad$ A$(0,3)$ and $B(0,-1)$

$$\boldsymbol{A} = \left|\int_{c}^{d} \boldsymbol{x\,dy}\right|$$

$$= \left|\int_{-1}^{3} (y-1)^2 - 4\,dy\right|$$

$$= \left|\int_{-1}^{3} (y^2 - 2y - 34\,)dy\right|$$

$$= \left|\left[\frac{y^3}{3} - y^2 - 3y\right]_{-1}^{3}\right|$$

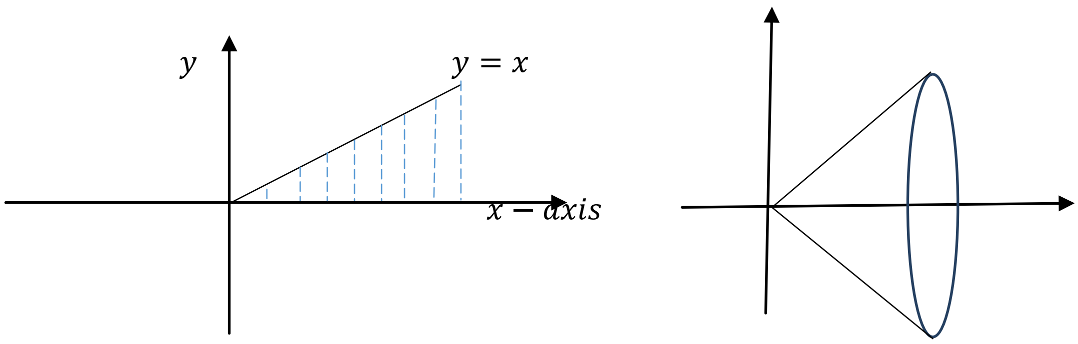
$$= \left|-\frac{32}{3}\right|$$

$$= 10\frac{2}{3}\; units.$$

30: APPLICATION OF INTEGRATION: VOLUMES OF REVOLUTION

Volume is obtained by rotating the area about the axis

a) **Rotation about the x-axis**$(\mathbf{360}^{\circ})$

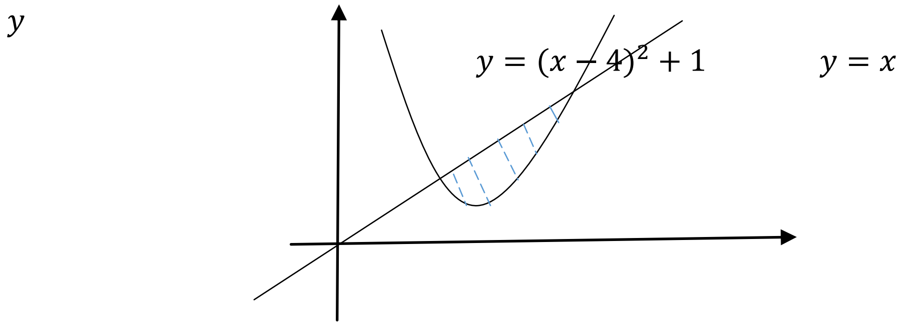


$$V = \pi \int_0^a [f(x)]^2 \, dx = \pi \int_0^a y^2 \, dx$$

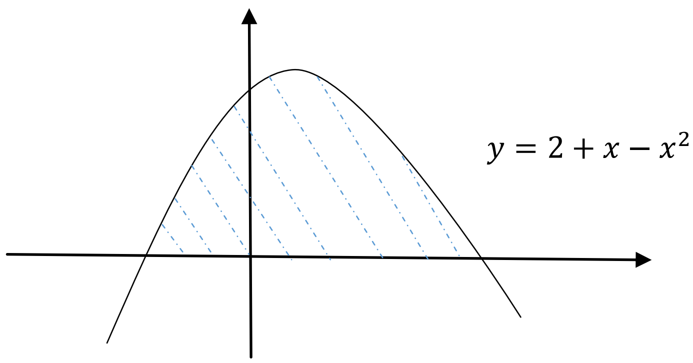
b) About y-axis(360°)

REVIEW EXERCISE

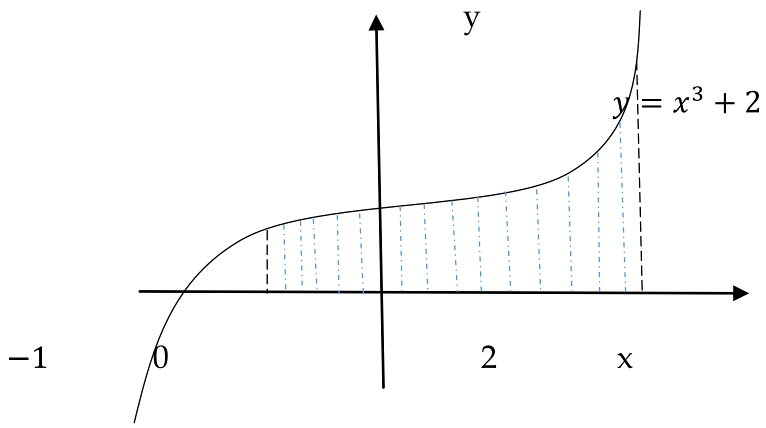
1. Find the indefinite integral $\int x^2\sqrt{x^3+2}\ dx$
2. Fin the area enclosed by the curve $y = x^2 + 4$ with x −axis in the interval $0 \le x \le 8$
3. Fine the area enclosed the two equations



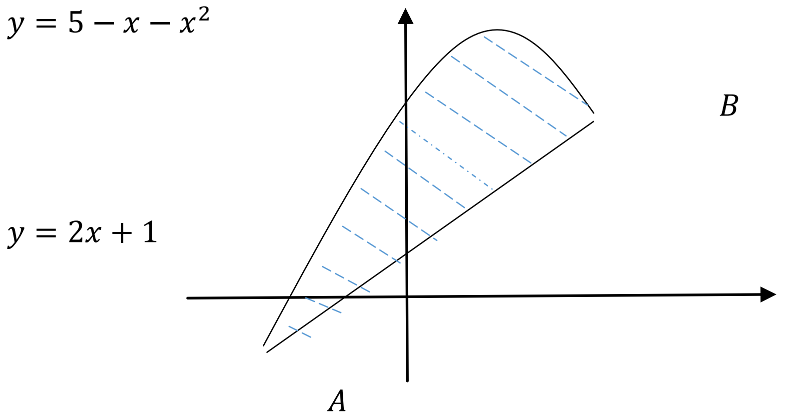
4. Find the area under the curve



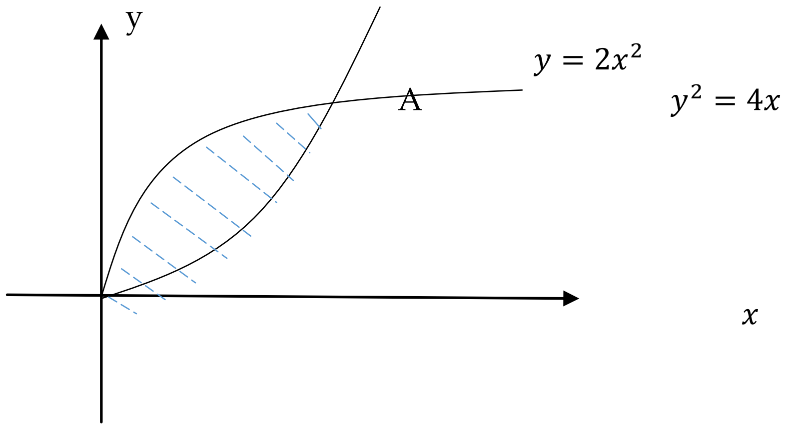
5. Find the area under the curve $y = x^3 + 2$



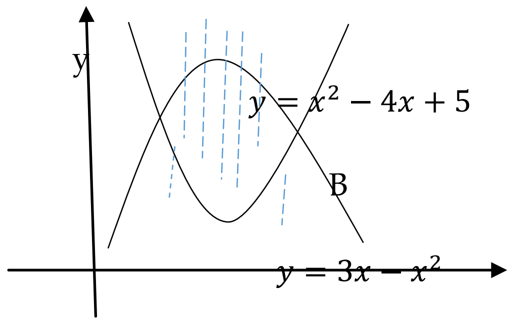
6. The curve $y = 5 - x - x^2$ meets the line $y = 2x + 1$ at A and B as shown in the diagram
   a) Find the coordinates of A and B
   b) The area of the shaded region



7. The curve $y = x^2$ and $y = 2 - x^2$ for $x > 0$ intersect at A. Find the coordinate of A. The region bounded by the curves and y-axis is rotated about x axis through 360°. Find the volume of the revolution.
8. The curves $y^2 = 4x$ and $y = 2x^2$ intersects at O and A.



   a) Find the coordinates of A.
   b) Find the area enclosed by the two curves.
   c) If the region bounded by the two curves is rotated about (i) the x-axis , (ii) the y-axis. Find the ratio of the two volumes created.
9. The diagram bellow shows the curves $y = 3x - x^2$ and $y = x^2 - 4x + 5$ intersect at A and B



   a) Find the coordinates of A and B
   b) Calculate the area of the shaded region.
   c) The shaded region is rotated about 360° about the x-axis to form a solid of revolution. Calculate the volume generated.

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