CSCI E-82
Advanced Machine Learning, Data Mining & Artificial Intelligence
Lecture 4

Probability, Statistics, Regression Time Series Part I

Peter V. Henstock

Fall 2018

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Administrivia

- HW2 – how is it going?
- Topic Presentations
  - Accepting groups of 2 or 3
  - Sign up using the same Google spreadsheet
  - Welcome to keep your partner or switch
    - Please talk to your existing partner regardless
  - Goal: 15 minute presentation
    - Will schedule presentations shortly but you will have a week to prepare from them
  - Need to select a moderately narrow topic
    - I will review topics on the sheet.
    - First sign-ups for a given topic win

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1D Data Statistics

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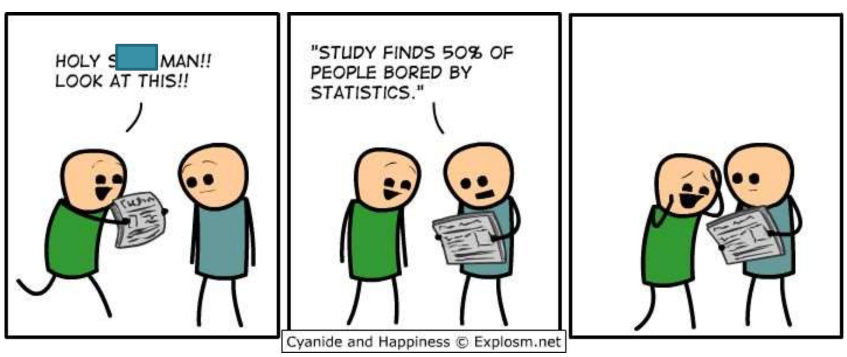
Let's Review Statistics

- Boringem.org



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Zoom Sign-in Times (min after hour) N=36

14	23	27
15	23	27
18	23	27
18	24	27
19	24	28
19	24	30
19	24	30
20	24	36
21	24	
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22	25	

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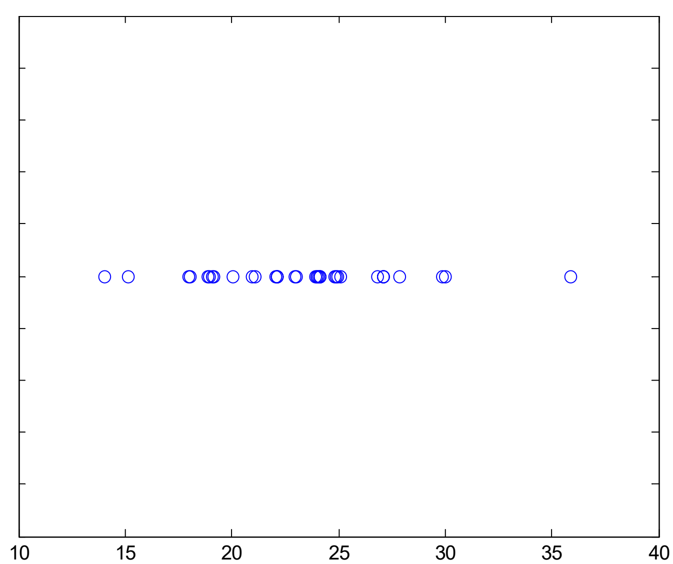
Plot (with jitter)



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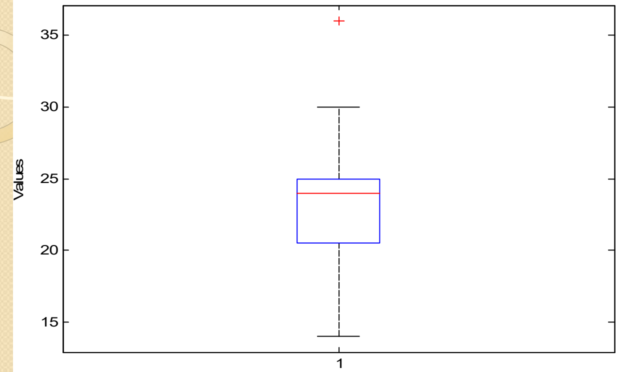
Box-Whisker Plot



- What are the components?

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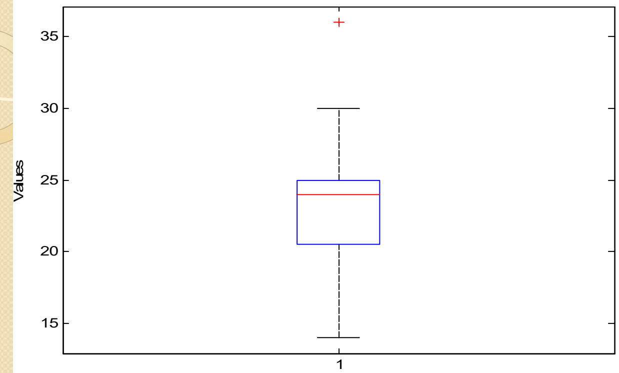
Box-Whisker Plot



- Red line = Q2 = median
- Box = Q1 to Q3
- Whiskers vary but might be:
  - Q1 - 1.5 IQR and Q3 + 1.5 IQR where
  - IQR = Inter Quartile Range = Q3-Q1

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Sample Statistics & Robustness

- Mean $\bar{\bar{x}} = \frac{1}{N}\sum_{i=0}^{N} x_i$
- Median =
  - x_i that is middle value of sorted X of odd N
  - $(x_i + x_{i+1})/2$ that is average of 2 central points
- Range = max(X) – min(X)
- Sample variance $s^2 = \frac{\sum_{i=1}^{N}(x_i - \bar{\bar{x}})^2}{N-1}$
- Robustness = immunity to outliers
- Which of these are robust?

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Sample Statistics?

- Mean $\bar{\bar{x}} = \frac{1}{N}\sum_{i=0}^{N} x_i$
- Median =
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- Range = max(X) – min(X)
- Sample variance $s^2 = \frac{\sum_{i=1}^{N}(x_i - \bar{\bar{x}})^2}{N-1}$
- Why are these “sample” statistics?

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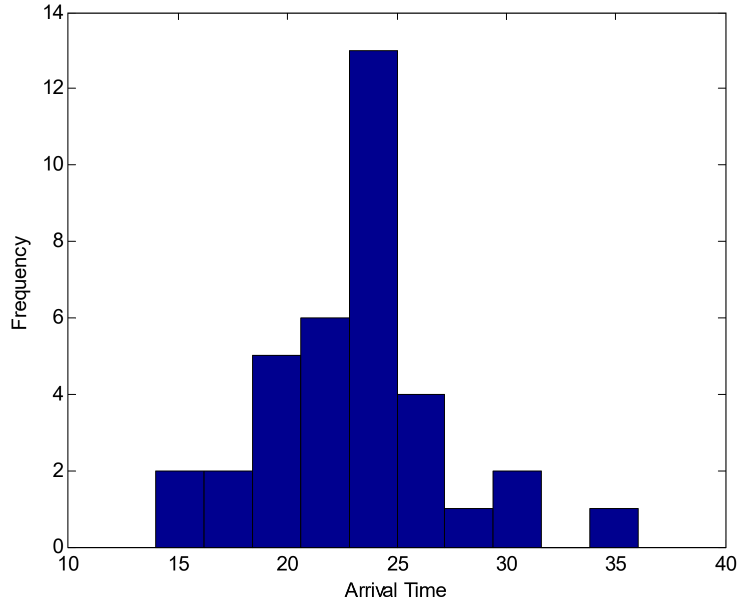
Histogram



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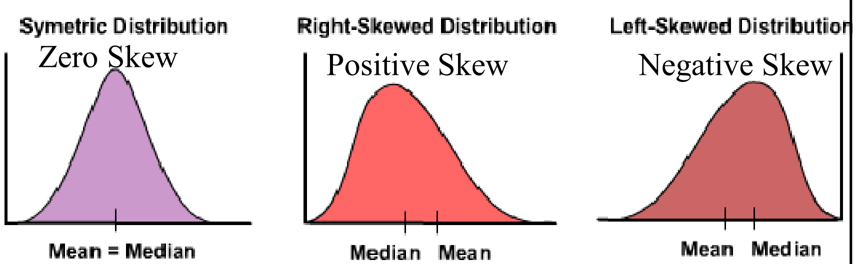
Skewness



- $Skewness = \frac{3(mean - media\ \ )}{s}$
- $Skewness = \frac{\sum (X_i - mean)^3}{(N-1)s^3}$
- N samples of X
- s = sample standard deviation
- https://onlinecourses.science.psu.edu/stat100/node/10

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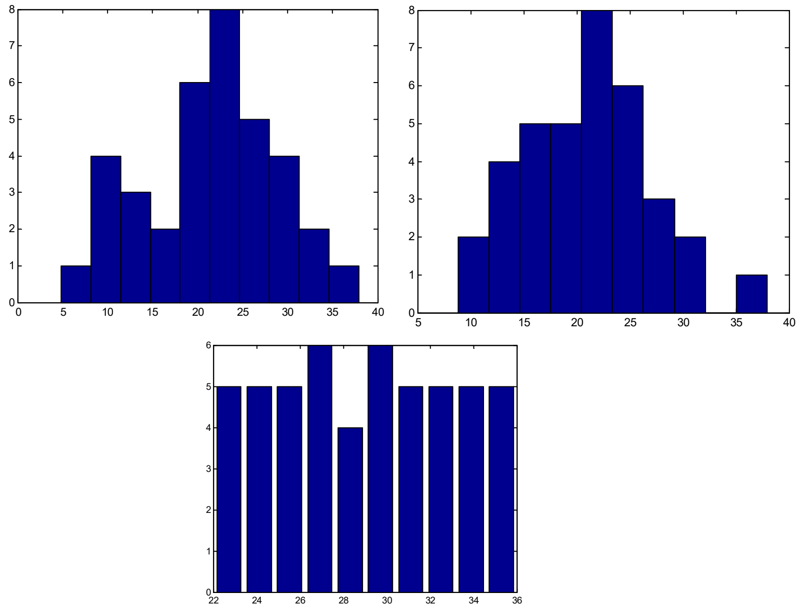
Modality ~ # of peaks



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Distributions

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Random Variable

- Variable that takes on a specific value
- Member of a group
- Group values are described according to a frequency distribution
- Types
  - Discrete
  - Continuous
- Frequently drawn from a distribution

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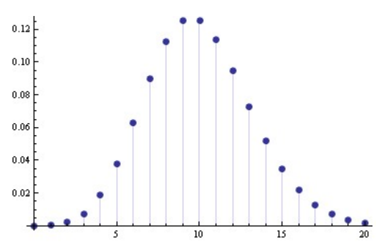
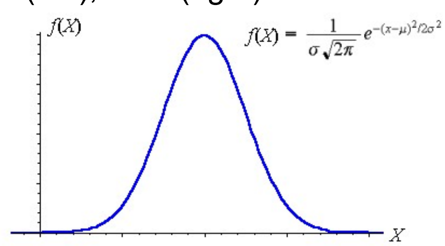
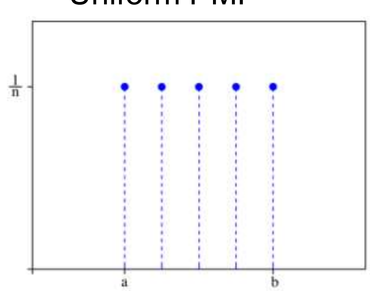
Probability Mass/Densities Function

Gaussian or Normal PMF (left), PDF (right)

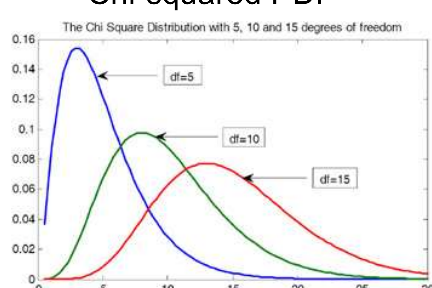




Uniform PMF



Chi-squared PDF



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Common density functions

- Uniform(a,b)
  - $f(x) = \frac{1}{b-a}$ for $x \in [a, b]$
- Bernoulli(p) for p in [0,1]
  - f(0) = 1-p = failure f(1) = p = success
- Binomial(n, p) = sum n Bernoulli trials
  - $f(x) = \binom{n}{x} p^x p^{n-x} \quad x = 0, \ldots n$
- Normal (Gaussian) distribution $\mathrm{N}(\mu, \sigma)$
  - $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

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Other Discrete/Continuous densities

- http://aleph0.clarku.edu/~djoyce/ma218/distributions.pdf

Distribution	Type	Mass/density function $f(x)$	Mean μ	Variance σ^2
Uniform(n)	D	$1/n$, for $x = 1, 2, \ldots, n$	$(n+1)/2$	$(n^2-1)/12$
Uniform(a, b)	C	$\frac{1}{b-a}$, for $x \in [a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Bernoulli(p)	D	$f(0) = 1-p, f(1) = p$	p	$p(1-p)$
Binomial(n, p)	D	$\binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, 1, \ldots, n$	np	npq
Geometric(p)	D	$q^{x-1}p$, for $x = 1, 2, \ldots$	$1/p$	$(1-p)/p^2$
NegativeBinomial(p,r)	D	$\binom{x-1}{r-1} p^r q^{x-r}$, for $x = r, r+1, \ldots$	r/p	$r(1-p)/p^2$
Hypergeometric(N,M,n)	D	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, for $x = 0, 1, \ldots, n$	np	$np(1-p)$
Poisson(λt)	D	$\frac{1}{x!}(\lambda t)^x e^{-\lambda t}$, for $x = 0, 1, \ldots$	λt	λt
Exponential(λ)	C	$\lambda e^{-\lambda x}$, for $x \in [0, \infty)$	$1/\lambda$	$1/\lambda^2$
Gamma(λ, r) Gamma(α, β)	C	$\frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}$ $= \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$, for $x \in [0, \infty)$	r/λ $= \alpha\beta$	r/λ^2 $= a\beta^2$

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Other Discrete/Continuous densities

- http://aleph0.clarku.edu/~djoyce/ma218/distributions.pdf

$\text{BETA}(\alpha, \beta)$	C	$\frac{1}{\text{B}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $0 \le x \le 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
$\text{NORMAL}(\mu, \sigma^2)$	C	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, for $x \in \mathbf{R}$	μ	σ^2
$\text{CHISQUARED}(\nu)$	C	$\frac{x^{\nu/2-1}e^{x/2}}{2^{\nu/2}\Gamma(\nu/2)}$, for $x \ge 0$	ν	2ν
$\text{T}(\nu)$	C	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\,\Gamma(\frac{\nu}{2})\,(1+x^2/\nu)^{(\nu+1)/2}}$ for $x \in \mathbf{R}$	0	$\nu/(\nu-2)$
$\text{F}(\nu_1, \nu_2)$	C	$\frac{1}{\text{B}(\frac{\nu_1}{2}, \frac{\nu_2}{2})} \frac{(\frac{\nu_1}{\nu_2})^{\nu_1/2} x^{\nu_1/2-1}}{(1+\frac{\nu_1}{\nu_2}x)^{(\nu_1+\nu_2)/2}}$ for >0	$\frac{\nu_2}{\nu_2-2}$	$\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$

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Properties of PMF/PDF

- prob(x) = p(x) = f(x) for discrete
- p(x $\in$ [a,b]) = $\sum_{i=a}^{b} f(x_i)$ for discrete
- p(x $\in$ [a,b]) = $\int_a^b f(x_i)\, dx$ for continuous
- $\sum_{-\infty}^{\infty} f(x_i)$ = 1 $\quad\quad$ $\int_{-\infty}^{\infty} f(x_i) dx$ = 1

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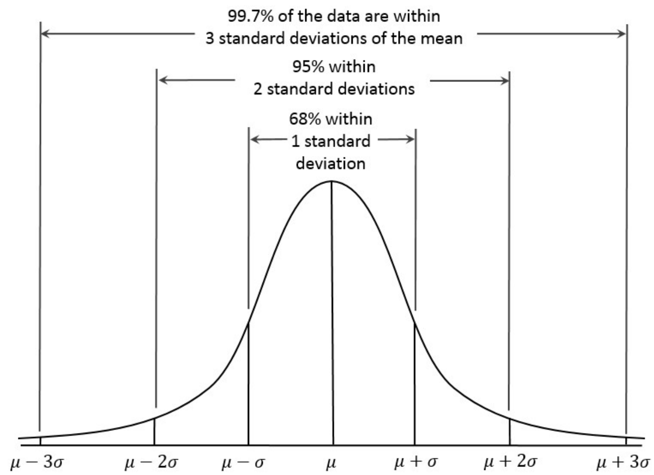
How to use this?

- Einstein had 160 IQ which is N(100,15)
- P(X >=160) = $\int_{160}^{\infty} f(x)dx$



https://en.wikipedia.org/wiki/68%E2%80%9395%E2%80%9399.7_rule#/media/File:Empirical_Rule.PNG

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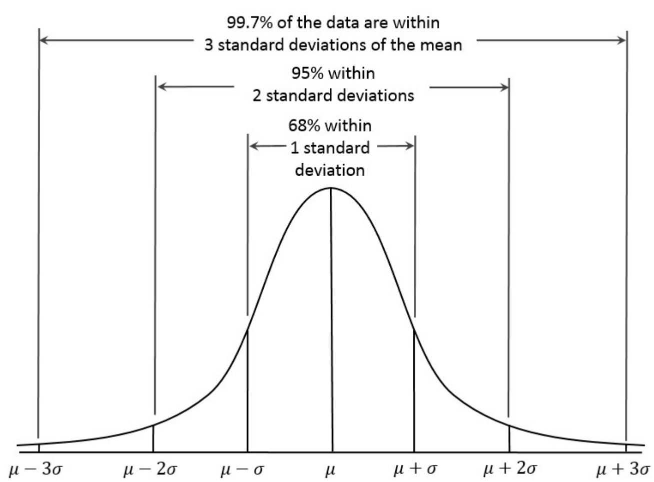
How to use this?

- What is probability of being within 1 stdev of “average”?



https://en.wikipedia.org/wiki/68%E2%80%9395%E2%80%9399.7_rule#/media/File:Empirical_Rule.PNG

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Statistics from PMF

- $Mean = \mu = \sum_{i=0}^{N} p_i x_i$
- $Var = \sigma^2 = \sum_{i=0}^{N} p_i (x_i - \mu)^2$

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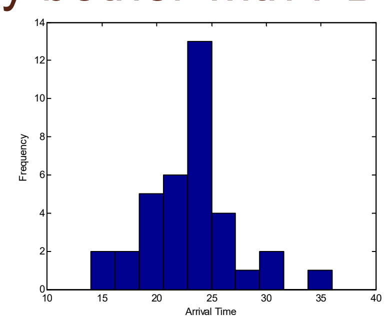
Why bother with PDF/PMFs?



- Provide an abstraction of actual sample histogram (normalized)
  - Avoids issues of outliers, etc.
- Data compression: N pts → parameters
- Enables statistical analyses

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Randomness

Characteristic	Pseudo-Random Number Generators	True Random Number Generators
Efficiency	Excellent	Poor
Determinism	Determinstic	Nondeterministic
Periodicity	Periodic	Aperiodic

- https://www.random.org/randomness/

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Normal Distribution

- If X and Y are random variables from a normal distribution, X+Y is also normal
- If you take many independent random variables from any single distribution, the sum approximates a normal distribution
  - Central Limit Theorem (with caveats)

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Probabilities

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Random Variable

- Variable that takes on a specific value
- Member of a group
- Group values are described according to a frequency distribution
- Types
  - Discrete
  - Continuous
- Frequently randomly drawn from a distribution

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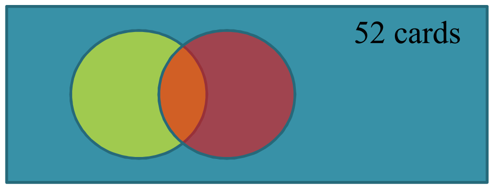
Probability Functions

- P(x) = probability of x occurring
  - Depends on the space of possibilities
- P(x) = 1 – P(~X)



- P(spades) = ?
- P(~spades) = ?
- P(Red “Royalty”)

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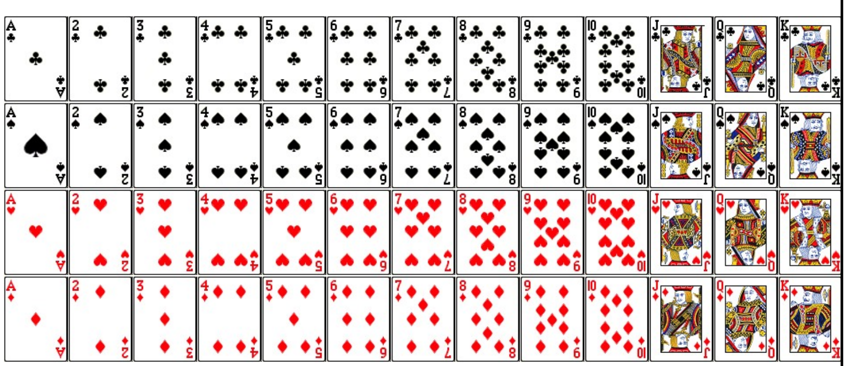
Deck of Cards



- http://www.milefoot.com/math/discrete/counting/cardfreq.htm

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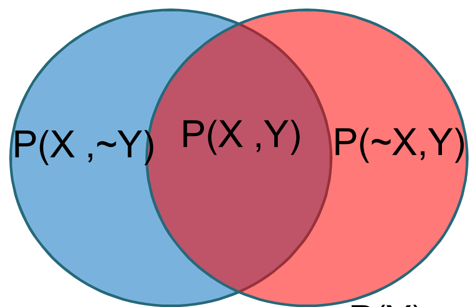
Joint Probabilities

- $P(X,Y) = P(X \text{ and } Y)$
- $P(X \text{ or } Y) = P(X) + P(Y) – P(X \text{ and } Y)$



P(X) = blue circle

P(Y) = red circle

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Joint Probabilities

- If X and Y are independent
  - $P(X,Y) = P(X)P(Y)$
- Example 1
  - X = probability of flipping coin
  - Y = probability of rolling a die (dice)
- Example 2
  - X = probability of grass is wet
  - Y = probability of rain

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What does this imply statistically?

A major airlines company received an anonymous bomb threat. To figure out ways of reducing the risk, a team was assembled. One of the members was a statistician. After careful thought and calculations, he handed a sealed bag to the airlines company and ordered them to put this bag in their plane during every journey. After a few journeys, the flight team were intrigued, and decided to open the bag. Inside, they found a bomb. They quickly confronted the statistician and asked for an explanation. He replied, "Well. Statistics show that there is a 1 in a 1,000,000 chance of a bomb being on a plane. But for 2 bombs to be on the same plane, the chances are only 1 in a 1,000,000,000,000."

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This is a joke. Neither the course staff nor Harvard endorses carrying bombs on planes

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Conditional Probabilities

- P(X|Y) = Probability of X "given" Y
- P(X|Y) = P(X,Y) / P(Y)

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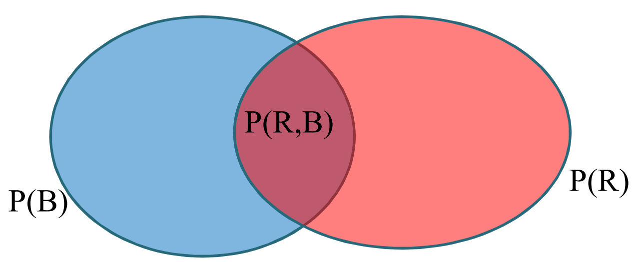
Conditional Probabilities

- P(X|Y) = Probability of X "given" Y
- P(X|Y) = P(X,Y) / P(Y)
- Probability in blue given it's in red?



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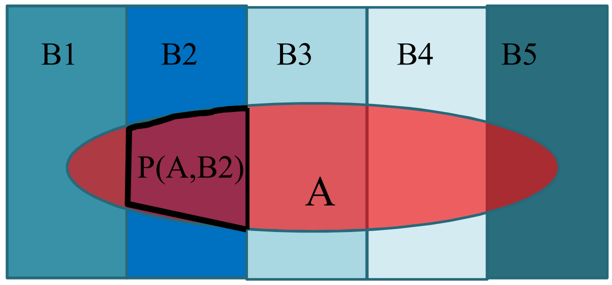
Visual Idea

- P(A) = $\sum_{i=0}^{n} P(A\,|B_i)P(B_i)$



- Region overlap: P(B2, A)
- P(B2,A) = P(A|B2)P(B2)
- If we add up all intersections over B's:
  - P(A) = $\sum_{i=0}^{n} P(A\,|B_i)P(B_i)$

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Basic Rules for Probabilities

- Sum Rule:
  - P(A U B) = P(A) + P(B) – P(A $\cap$ B)
- Product Rule:
  - P(A $\cap$ B) = P(A,B) = P(A|B)P(B) = P(B|A)P(A)
- Joint→Marginal
  - $\mathrm{P(A)} = \sum_{i=0}^{n} P(A, B_i)$
  - P(A) = $\sum_{i=0}^{n} P(A\,|B_i)P(B_i)$

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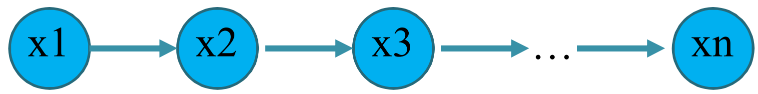
Chain Rule



What is the probability of this whole thing?

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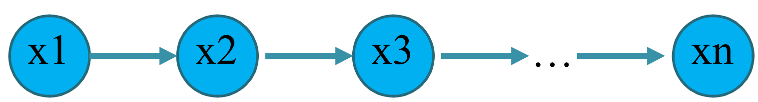
Chain Rule



What is the probability of this whole thing?

P(x1,x2,…xn)

How can we compute this using conditional probabilities?

P(X1)*

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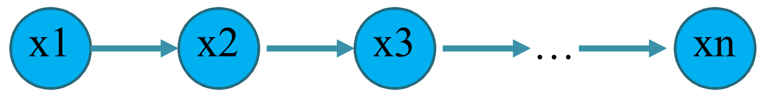
Chain Rule



What is the probability of this whole thing?

P(x1,x2,…xn)

How can we compute this using conditional probabilities?

P(X1)*P(X2|X1)

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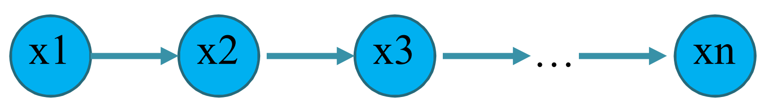
Chain Rule



What is the probability of this whole thing?

P(x1,x2,…xn)

How can we compute this using conditional probabilities?

P(X1)*P(X2|X1)*P(X3|X1,X2)…

…

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Chain Rule

$$x1 \rightarrow x2 \rightarrow x3 \rightarrow \dots \rightarrow xn$$

What is the probability of this whole thing?

P(x1,x2,…xn)

How can we compute this using conditional probabilities?

P(X1)*P(X2|X1)*P(X3|X1,X2)…

…P(Xn | X1..XN-1)

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Bayes Rule

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Bayes Rule

Thomas Bayes 1702-1761
Presbyterian minister in UK

Wikipedia

- $P(h|e) = \frac{P(e|h)P(h)}{P(e)}$
- Basis of classifiers
- Basis of network inferences
- Basis of structuring information or expert systems

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Bayes Rule

Thomas Bayes 1702-1761
Presbyterian minister in UK

Wikipedia

- $P(h|e) = \frac{P(e|h)P(h)}{P(e)}$
- $P(h|e) = \frac{P(h,e)}{P(e)}$ $\quad P(h,e) = P(e|h)P(h)$

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Components of Bayes Rule

- $P(hyp|data) = \frac{P(data|hyp)P(hyp)}{P(data)}$
- p(data|hyp) = p(data, hyp) / p(hyp)
- What is p(hyp|data) if hyp and data are independent?

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Conditional Independence

- P(X,Y|Z) = P(X|Z)P(Y|Z)
  - “X is independent of Y given Z”
  - X and Y are conditionally independent
- Clearly an extension of the general independence
  - P(A,B) = P(A)P(B)
- If X and Y are independent, can we say they are conditionally independent?

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Conditional Independence Examples

- https://www.quora.com/What-are-examples-of-events-that-are-independent-but-not-conditionally-independent-and-vice-versa

- Good examples to navigate:
  - [in]dependence
  - conditional [in]dependence

  - P(x|y) if independent p(x) →
  - P(x|y) = P(x,y) / P(y) = P(x)P(y)/P(y) → P(x)

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Components of Bayes Rule

- $P(hyp|data) = \frac{P(data|hyp)P(hyp)}{P(data)}$

- P(hyp) = "Prior"
  - Prior probability of a hypothesis
- P(hyp|data) = "posterior"
  - Probability of hypothesis given data
- P(data|hyp) = "likelihood"
  - Probability of data fitting a given hypothesis
- P(data) = normalizing constant
  - Data is same across all hypotheses

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2 Random Variables

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Two random variables

- Mean values are the same
- Standard deviations are the same

- Relationship:
  - Correlation Coefficient
  - Covariance

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Covariance

- Cov(X,Y) = σ(X,Y) = E[X-E(X)] E[Y-E(Y)]
- Cov(X,Y) $=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y})$
- Sample Covariance(X,Y)
  - $=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y})$

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Covariance

- Cov(X,Y) = σ(X,Y) = E[X-E(X)] E[Y-E(Y)]
- Cov(X,Y) $=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y})$
- Sample Covariance(X,Y)
  - $=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y})$
- What is the Cov(X,X)?

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Covariance

- Cov(X,Y) = σ(X,Y) = E[X-E(X)] E[Y-E(Y)]
- Cov(X,Y) $= \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})(Y_i - \bar{Y})$
- Sample Covariance(X,Y)
  - $= \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})(Y_i - \bar{Y})$
- What is the Cov(X,X)?
- If X and Y are independent, what is Cov(X,Y)?

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Covariance Matrix

- Let X = $[X_1 \ldots X_n]^T$ be a column vector
- Covariance matrix is written with Σ
- $\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = E[(X_i-\mu_i)(X_j-\mu_j)]$

$$\Sigma = \begin{bmatrix} \mathrm{E}[(X_1-\mu_1)(X_1-\mu_1)] & \mathrm{E}[(X_1-\mu_1)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_1-\mu_1)(X_n-\mu_n)] \\ \mathrm{E}[(X_2-\mu_2)(X_1-\mu_1)] & \mathrm{E}[(X_2-\mu_2)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_2-\mu_2)(X_n-\mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n-\mu_n)(X_1-\mu_1)] & \mathrm{E}[(X_n-\mu_n)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_n-\mu_n)(X_n-\mu_n)] \end{bmatrix}.$$

- $\Sigma = E(XX^T) - \mu\mu^T$

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Covariance Matrix

- Let X = $[X_1 \ldots X_n]^T$ be a column vector
- Covariance matrix is written with Σ
- $\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = E[(X_i-\mu_i)(X_j-\mu_j)]$

$$\Sigma = \begin{bmatrix} \mathrm{E}[(X_1-\mu_1)(X_1-\mu_1)] & \mathrm{E}[(X_1-\mu_1)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_1-\mu_1)(X_n-\mu_n)] \\ \mathrm{E}[(X_2-\mu_2)(X_1-\mu_1)] & \mathrm{E}[(X_2-\mu_2)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_2-\mu_2)(X_n-\mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n-\mu_n)(X_1-\mu_1)] & \mathrm{E}[(X_n-\mu_n)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_n-\mu_n)(X_n-\mu_n)] \end{bmatrix}.$$

- If X1…Xn are all independent, what does the covariance matrix look like?

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Sample Covariance Matrix example

X	Y
1	3
4	2
2	4
5	3
3	3
6	3

- Compute mean of X
- Compute mean of Y
- Sample cov of X, X
- Sample cov of Y, Y

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Sample Covariance Matrix example

X	Y
1	3
4	2
2	4
5	3
3	3
6	3

- Compute mean of X
  - Sum(X)/6 = 3.5
- Compute mean of Y
- Sample cov of X, X
- Sample cov of Y, Y
- Sample cov of X, Y

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Sample Covariance Matrix example

X	Y
1	3
4	2
2	4
5	3
3	3
6	3

- Compute mean of X
  - Sum(X)/6 = 3.5
- Compute mean of Y
  - Sum(Y)/6 = 3
- Sample cov of X, X
- Sample cov of Y, Y
- Sample cov of X, Y

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Sample Covariance Matrix example

X	Y
1	3
4	2
2	4
5	3
3	3
6	3

- Compute mean of X
  - Sum(X)/6 = 3.5
- Compute mean of Y
  - Sum(Y)/6 = 3
- Sample cov of X, X
  - $[\Sigma\ (x\text{-meanX})^2] / (N\text{-}1)$
- Sample cov of Y, Y
- Sample cov of X, Y

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Sample Covariance Matrix example

X	Y
1	3
4	2
2	4
5	3
3	3
6	3

- Compute mean of X
  - Sum(X)/6 = 3.5
- Compute mean of Y
  - Sum(Y)/6 = 3
- Sample cov of X, X
  - $[\Sigma\ (xi\text{-meanX})^2] / (N\text{-}1) = 3.5$
  - $[\Sigma\ (xi\text{-}3.5)^2] / (6\text{-}1) = 3.5$
- Sample cov of Y, Y
  - $[\Sigma\ (yi\text{-meanY})^2] / (N\text{-}1) = 0.4$
- Sample cov of X, Y

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Sample Covariance Matrix example

X	Y
1	3
4	2
2	4
5	3
3	3
6	3

- Compute mean of X
  - Sum(X)/6 = 3.5
- Compute mean of Y
  - Sum(Y)/6 = 3
- Sample cov of X, X
  - $[\Sigma\ (x\text{-meanX})^2]\ /\ (N\text{-}1) = 3.5$
  - $[\Sigma\ (x\text{-}3.5)^2]\ /\ (6\text{-}1) = 3.5$
- Sample cov of Y, Y
  - $[\Sigma\ (y\text{-meanY})^2]\ /\ (N\text{-}1) = 0.4$
- Sample cov of X, Y
  - $[\Sigma\ (x\text{-meanX})(y\text{-meanY})]\ /\ (N\text{-}1) = -0.4$

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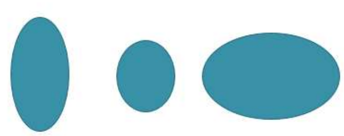
2D Normal Distributions

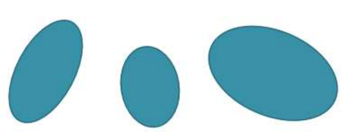
Normal Distribution

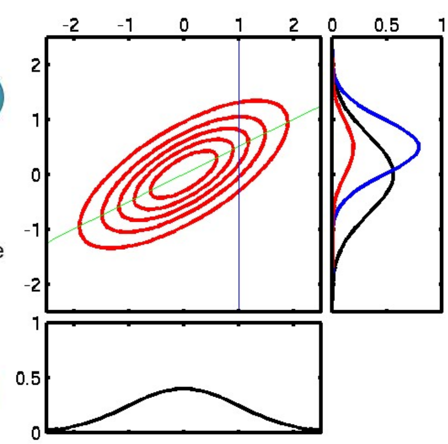
- What can we say about the covariance of these?



Normal Distribution

- What can we say about the covariance of these?





http://www.atmos.washington.edu/~hakim/591/code/

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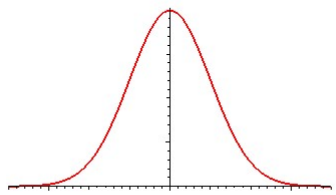
Gaussian or Normal Distribution

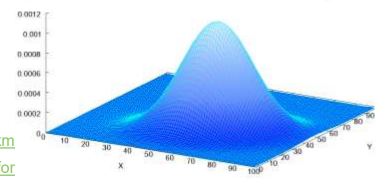


Univariate Gaussian

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(\mu-x)^2}{2\sigma^2}}$$

Multivariate Gaussian

$$f_{\mathbf{x}}(x_1,\ldots,x_k)=\frac{1}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma}|}}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$



- http://mydov.blogspot.com/2014/06/anomaly-detection.htm
- https://www.quora.com/What-is-an-intuitive-explanation-for multivariate-Gaussian-distribution-aka-multivariate-normal

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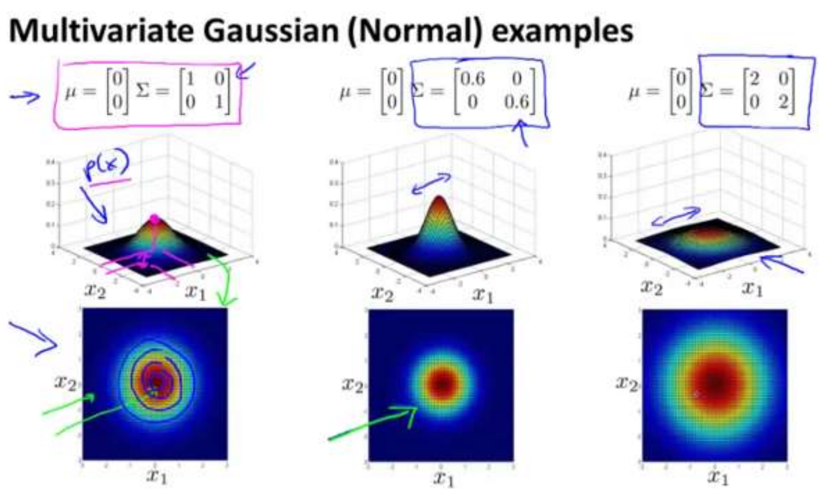
Zero Covariance Term



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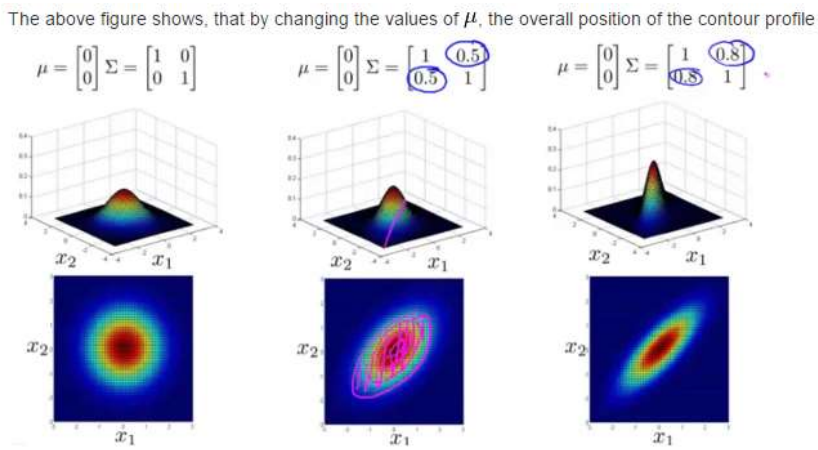
Non-zero Covariance Term



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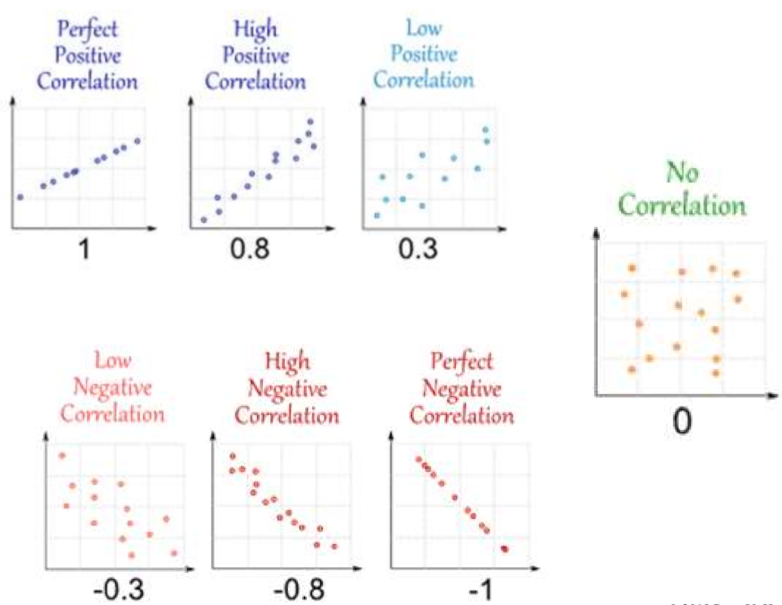
Correlation

https://www.mathsisfun.com/data/correlation.html



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Correlation coefficient

- r = CorrCoef(X,Y) = cov(X,Y) / ($s_X s_Y$)
  - Technically, the Pearson corr. Coef.
- $r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}}$
- Range of values for correlation is [-1, 1]
- No units

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Regression

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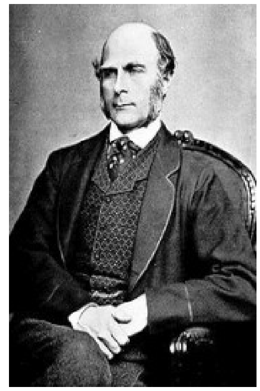
Sir Francis Galton



- 1822-1911 Victorian England
- Statistician & scientist
  - Regression
  - Standard deviation
  - Correlation
  - Psychometrics
  - Fingerprint classification
  - Weather map and scientific meteorology
- Coined terms in our lexicon:
  - Eugenics
  - Nature vs. Nurture

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Types of Learning

- Supervised
  - Provide output labels/values & input features
- Unsupervised
  - Provide only input features
- Semi-supervised learning
- Reinforcement learning

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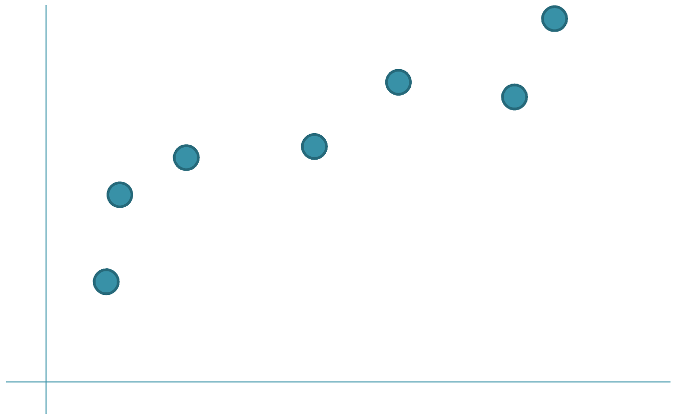
Let's model data



- What would be the simplest model for this type of result

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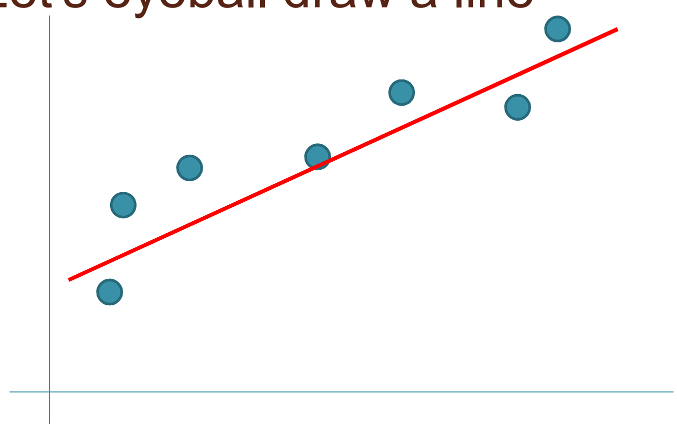
Let's eyeball draw a line



- How is this model?
- Does it fit the data well?

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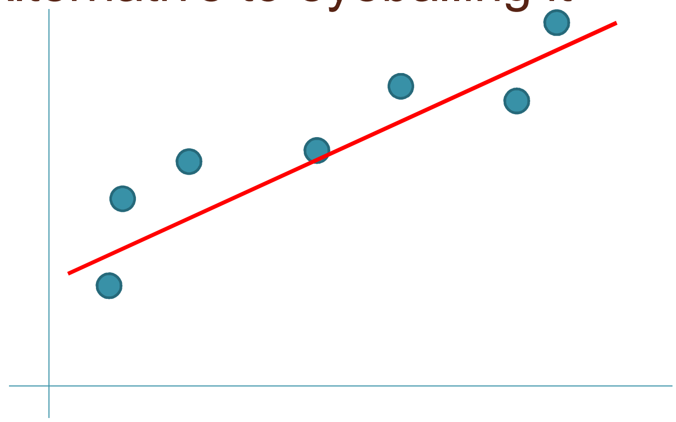
Alternative to eyeballing it



- How could we come up with a description of what a good linear fit would be for this set of blue dots?

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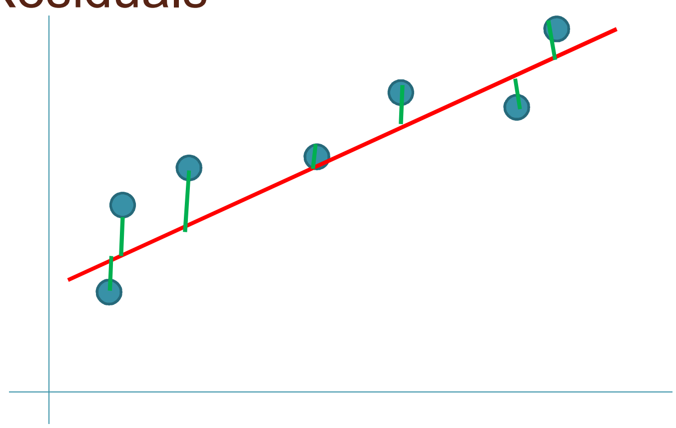
Residuals



- Residual = estimate on line – actual blue
- Residuals are drawn with green lines

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Regression vs. PCA

- Is this regression line fit the same one as the principal component?

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Generative Model

- What is it?

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Generative Model

- What does generative mean?
  - Could try to model the residuals and come up some useful ideas on model
  - Instead we are going to create a model that assumes the points were distributed by a generating model
  - Specifically, we start with points on a line and randomly shift the points according to a distribution
- $y_i = w_0 + w_1x_i + \varepsilon_i$
- $\varepsilon_i \sim N(\text{mean}=0, \text{stdev}=\sigma)$
- What does this have to do with residuals?

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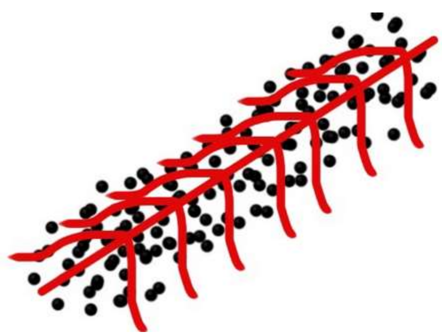
Residuals



- Residuals come from a normal distribution centered on the line
- http://stats.stackexchange.com/questions/148803/how-does-linear-regression-use-the-normal-distribution

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Residuals → Best Fit Model

- How can we use the residuals to come up with a best fit line?
- What would be a good function to optimize?

a) Σ residuals
b) Σ |residuals|
c) Σ [residuals]^2
d) Σ [yi - residuals]^2
e) Σ [residuals / yi]^2

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Simple case

- Optimize the least squares function
- $SumsqError = \sum_{i=1}^{N} \epsilon_i^2$
- But $y_i = w_0 + w_1 x_i + \epsilon_i$ but we don't know the true weights so we estimate
- so $\epsilon_i = y_i - \widehat{w_o} - \widehat{w_1}\ x_i$
- $SumsqError = \sum_{i=1}^{N} (y_i - \widehat{w_o} - \widehat{w_1}\ x_i)^2$
- How can we minimize the error?
- How can we find the best $\widehat{w_0}$ and $\widehat{w_1}$?

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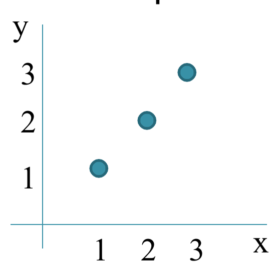
Sanity check example

- Sumsq error = cost function = J
- Goal is to minimize J
- J = sum-square distance between
  - predicted y values of line given parameters
  - the actual y values
  - at each x point



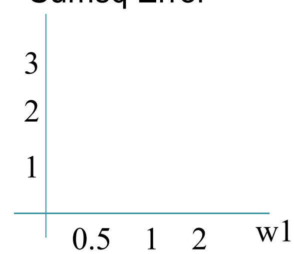


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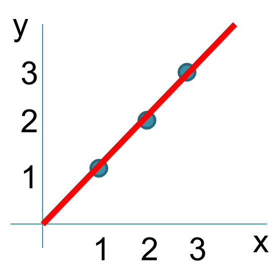
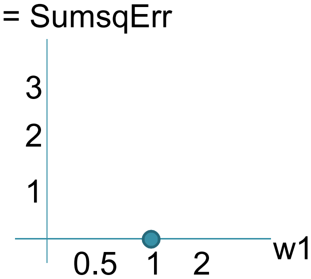
Figure out J for a few points





- Assume w0 = 0 (zero y-intercept)
- For w1 = 1:

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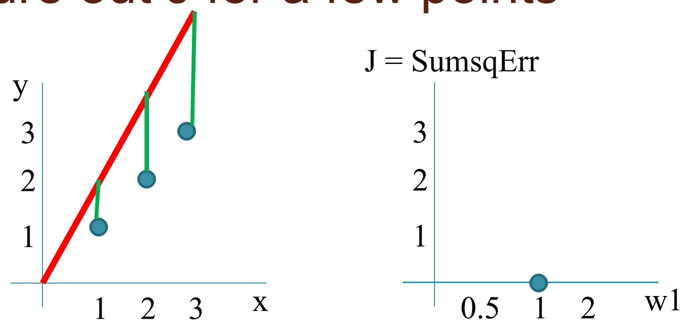
Figure out J for a few points



- Assume w0 = 0 (zero y-intercept)
- For w1 = 1: = 0
- For w1 = 2:

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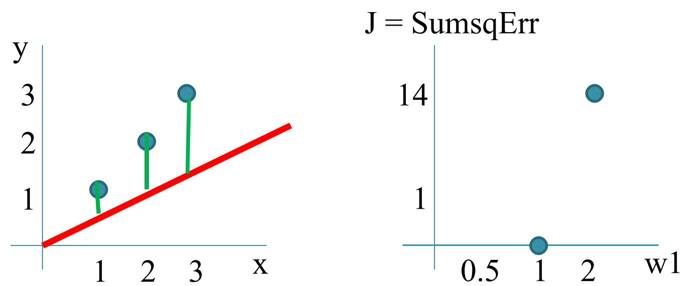
Figure out J for a few points



- Assume w0 = 0 (zero y-intercept)
- For w1 = 1: = 0
- For w1 = 2: $=(2-1)^2 + (4-2)^2 + (6-3)^2=14$
- For w1 = 0.5 =

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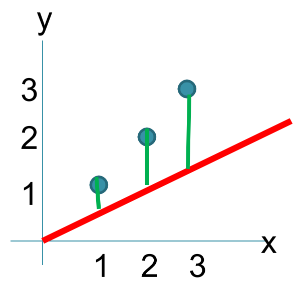
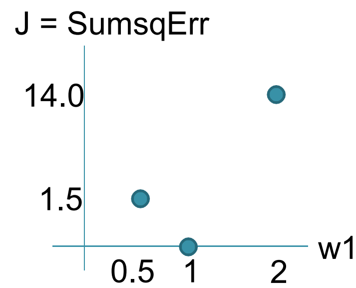
Figure out J for a few points





- Assume w0 = 0 (zero y-intercept)
- For w1 = 1: = 0
- For w1 = 2: $=(2-1)^2 + (4-2)^2 + (6-3)^2=14$
- For w1 = 0.5 = $(1-1/2)^2 + (2-1)^2 +(1.5-1)^2$ = 1.5

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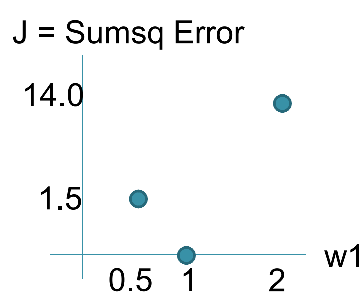
Modeling the error



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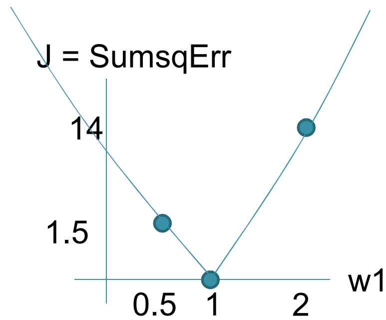
What shape is J vs. w1?

- Parabola (badly drawn below)



- Min value will be 0 in this case—why?
- How could we optimize the J as a function of w1?

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Derivative Formulation

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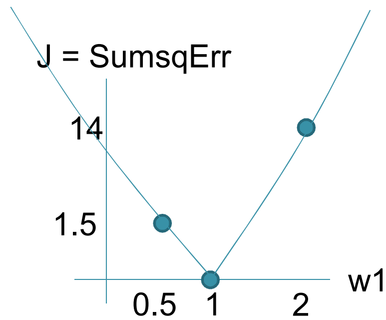
What shape is J vs. w1?

- Parabola (badly drawn below)



- Min value will be 0 in this case—why?
- How could we optimize the J as f(w1)?

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Take derivative and set to 0

- $\sum_{i=1}^{N}(y_i - \widehat{w_o} \;\; - \widehat{w_1} \;\; x_i)^2$
- Differentiate with respect to….?

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Take derivative and set to 0

- $\sum_{i=1}^{N}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{N}(y_i - \widehat{w_o} - \widehat{w_1}\, x_i)^2$
- Differentiate with respect to the w_o & w_1
- $\frac{\partial y}{\partial w} \sum_{i=1}^{N}(y_i - \widehat{w_o} - \widehat{w_1}\, x_i)^2 =$
- $2\sum_{i=1}^{N} y_i - 2N\widehat{w_o} - \widehat{w_1} \sum_{i=1}^{N} x_i)^2$
- $\widehat{w_o} = \frac{1}{N}[\sum_{i=1}^{N} y_i - \widehat{w_1} \sum_{i=1}^{N} x_i)]$
- $\widehat{w_1} = \dfrac{\sum_{i=1}^{N} y_i x_i - \dfrac{\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i}{N}}{\sum_{i=1}^{N} x_i^2 - \dfrac{\left(\sum_{i=1}^{N} x_i\right)^2}{N}}$

http://isites.harvard.edu/fs/docs/icb.topic515975.files/OLSDerivation.pdf

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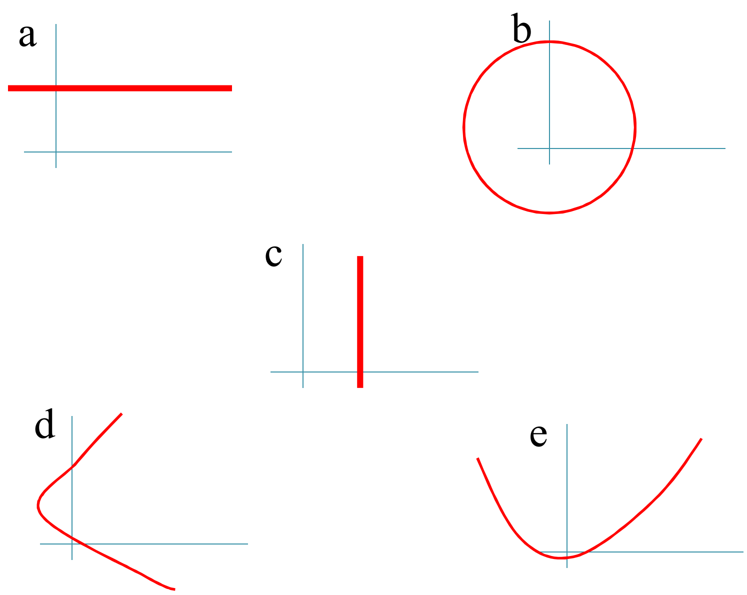
Which work for linear regression?



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Ordinary Least Squares vs. MLE

- Previous derived a generative model that was based on least-squares
- Reasonable approach

- MLE = maximum likelihood estimation
- Different framework for optimizing

- Likelihood = P(Data | hyp)
- MLE = arg max P(Data | hyp)
  - Select the parameters that maximize this quantity as our estimates

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MLE Estimate

- Assume P($data_i$ | hyp) are independent
- MLE = argmax $\prod_{i=1}^{N} P(data_i \mid hyp)$

- What is this P($data_i$ | hyp) using our previous assumptions?

- argmax $\prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(Y_i-\beta_0-\beta_1 X_i)^2}$

- What are we optimizing over in argmax?

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MLE Optimizing

- Argmax $\prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(Y_i - \beta_0 - \beta_1 X_i)^2}$
- To make the math easy, take a log
- To optimize, take derivative with respect to the betas
- Result in this case will be identical to the least squares approach
  - MLE estimates equal to OLS in this case

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MLE Estimate

- Assume P($data_i$ | hyp) are independent
- MLE = argmax $\prod_{i=1}^{N} P(data_i \mid hyp)$
- What is this P($data_i$ | hyp) using our previous assumptions?

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Multiple Linear Regression

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Multiple Linear Regression

- Previously had $y = w_0 + w_1x_1$
- What if we had 3 or 10 variables?
- We can still use regression
- Here is a simple case:
  $Y = w_0 + w_1x_1 + w_2x_2 + \ldots w_kx_k = \Sigma w_ix_i = w^Tx$

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Multiple Regression with Matrices

- $Y = XW + e$

- Y is Nx1 matrix
  - What do we usually call Y?

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Multiple Regression with Matrices

- $Y = XW + e$

- Y is N x 1 matrix
- X is N x (k+1)   include a column of 1s
- W is (k+1) x 1

- What do we usually call Y?

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Multiple Regression with Matrices

- Y = XW + e

- Y is N x 1 matrix
- X is N x (k+1)   include a column of 1s
- W is (k+1) x 1

- What do we usually call Y?
- What do we usually call X?

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Multiple Regression with Matrices

- Y = XW + e

- Y is N x 1 matrix (column vector)
- X is N x (k+1) include a column of 1s
- W is (k+1) x1 matrix
- e is N x 1 column vector

- What do we usually call Y?
- What do we usually call X?
- What do we usually call W?

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Multiple Regression with Matrices

- Y = XW + e

- Y is N x 1 matrix (column vector)
- X is N x (k+1) include a column of 1s
- W is (k+1) x 1 matrix
- e is N x 1 column vector

- What are the rows and column referring to for the X?

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Matrix Inverse

- For numbers:
  - Consider x * 1/x → 1
  - In a sense, 1/x is inverse of x
  - In a sense, x is inverse of 1/x

- For matrices
  - $A A^{-1} = I$ and $A^{-1}A = I$
  - Fairly easy for 2x2 matrices
  - Possible for 3x3 matrices by hand
  - Challenging for anything larger so use any linear algebra software or python

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Matrix Inverse

- Do all numbers have an inverse?
- Do all matrices have an inverse?

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Matrix Inverse

- Do all numbers have an inverse?
- Do all matrices have an inverse?
  - Officially, has to be a square matrix
  - (Note: there are pseudo inverses otherwise)
  - If it's square it also has to have "full rank"
    - Can't have rows that are products of each other
    - Can't have zero rows
    - Can't have rows that are weighted sums of any other rows
    - Determinant cannot be 0

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Least Squares Estimate

- SSE = e^Te
  - Is this an inner product or outer product?
- e = Y-XW
- SSE = $(Y-XW)^T(Y-XW)$
- SSE = $(Y^T-W^TX^T)(Y-XW)$
  - Note that $(AB)^T = B^TA^T$
- SSE = $(Y^TY – W^TX^TY + W^TX^TXW - Y^TXW)$
  - The 2nd and 4th term are equivalent
    - Not obvious from the matrix format
    - True when you multiply them out

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Least Squares Estimate

- SSE = e^Te
  - Is this an inner product or outer product?
- e = Y-XW
- SSE = $(Y-XW)^T(Y-XW)$
- SSE = $(Y^T-W^TX^T)(Y-XW)$
  - Note that $(AB)^T = B^TA^T$
- SSE=$(Y^TY – W^TX^TY + W^TX^TXW - Y^TXW)$
  - The 2nd and 4th term are equivalent
- SSE = $(Y^TY - 2Y^TXW + W^TX^TXW)$

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Least Squares Estimate

- SSE = $(Y^TY - 2Y^TXW + W^TX^TXW)$
- How to minimize this?
- Derivative wrt W and set to 0
- d/dW→ $0 – 2X^TY + 2X^TXW = 0$
  - Note that d/dW of $U^TVW = V^TU$
  - Last term with the W's on either side is a quadratic which yields 2 and removal of W^T
  - $X^TXW = X^TY$ → $W = (X^TX)^{-1}X^TY$
  - $(X^TX)^{-1}X^T$ is the Moore-Penrose Pseudoinverse of X
- Why can't we just do an inverse of X?

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Observation

- From before:
  - $Y = w_0 + w_1x_1 + w_2x_2 + \ldots w_kx_k = \Sigma w_ix_i$
- What does x_1 actually look like?
- If we modified it to be $3x_1$, what would happen to the overall equation?
- What if we used x_1^2 or cubed it to x_1^3 ?

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Observation

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  - $Y = w_0 + w_1x_1 + w_2x_2 + \dots w_kx_k = \Sigma w_ix_i$
- What does x_1 actually look like?
- If we modified it to be $3x_1$, what would happen to the overall equation?
- What if we used x_1^2 or cubed it to x_1^3 ?
  - Obtain polynomial regression

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Full multiple linear regression

- $Y = w_0 + w_1x_1 + w_2x_2 + \dots w_kx_k = \Sigma w_ix_i$
- $Y = w_0 + w_1x_1 + w_2x_2 + \dots w_kx_k + b_0x_1x_2 + b_1x_1x_3 + \dots bzx_{k-1}x_k + c_0x_1x_2x_3 + \dots$
- Y = sum of
  - Independent factors +
  - 2-way interactions +
  - 3-way interactions +
  - K-way interactions
- Can include squared, cubed, power terms

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Hierarchy Principles

- Only found this is statistics and never machine learning
- Need to include all lower terms if you include a higher order term
- If you have x_1^3
  - Need to include x_1^2 as well as x_1
- If you have $x_1^2x_2$
  - Need to include x_1^2, x_1x_2, x_1, x_2

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Further options

- $Y = \mathrm{sqrt}(X_1^5X_2X_3^3X_4)$
- Can multiple linear regression work for this kind of model?
- What does Y look like?
- What is log(ab)? → log(a) + log(b)
- log(sqrt(x))? → 1/2log(x)

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Big O notation

- Algorithms tend to be characterized by their approximate computational load
- Computation is a function of a parameter like n = size
- Sum of N numbers is O(N)
- Tree search is O(logN)
- We tend to round down so if the actual computation was $O(5N^3 + 4N + 17)$
  - Use just $O(N^3)$

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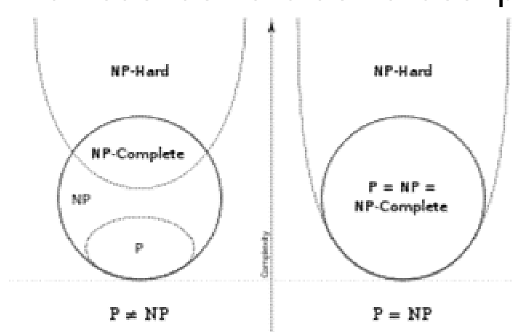
Complexity

- P = polynomial
- NP = nondeterministic polynomial time
- NP-complete = NP and NP-hard
- NP-hard =
  - at least as hard as hardest problem in NP



https://en.wikipedia.org/wiki/NP-hard#/media/File:P_np_np-complete_np-hard.svg

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Computational aspects

- Pseudoinverse
- Solution parameters = $(X^T X)^{-1} X^T y$

- What is the computational requirement?

- Matrix addition?
- Matrix multiplication?
- Matrix inverse?

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Gradient Descent

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Gradient Descent

- Problem:
  - You are placed on a hill in fog
- Goal:
  - Find the bottom of the hill
- Method
  - How would you do that?

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Gradient Descent

- Idea: you are placed on a hill in fog
  - Find the bottom of the hill

do {
- Figure out which way is "down"
- Move a certain distance in that direction

} until at lowest spot

- What assumptions are we making?

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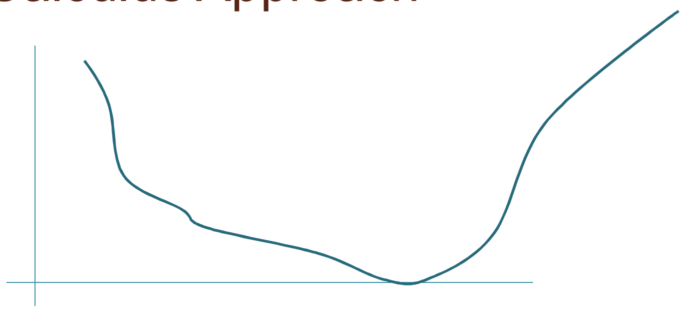
Calculus Approach



- Calculus tells us if that we have a function f then we need to do 2 things to find the minimum
  1)
  2)

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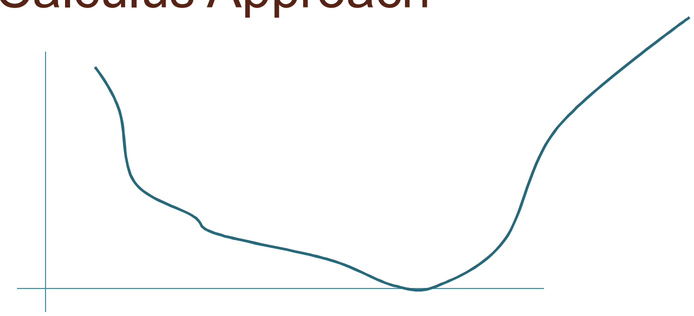
Calculus Approach



- Calculus tells us if that we have a function f then we need to do 2 things to find the minimum
  1) Compute the derivative
  2) Check the concavity using 2nd derivative

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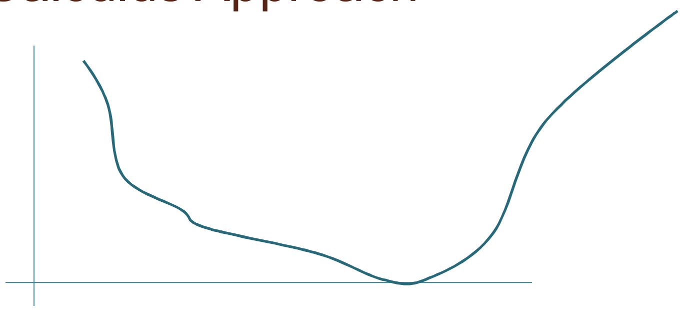
Calculus Approach



- Calculus tells us if that we have a function f then we need to do 2 things to find the minimum
  1) Compute the derivative
  2) Check the concavity using 2nd derivative

  How do we actually locate the minimum?

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General Gradient Descent

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \gamma_n \nabla F(\mathbf{x}_n),\ n \geq 0.$$

- For a partial derivative of 1 variable
  - $w_{n+1} = w_n - \alpha \frac{\partial}{\partial w_n} J(w_n)$
  - α is the learning rate
  - J() is the cost function
  - Iteratively adjusting w_n with better estimates using the gradient of the cost function as the direction.

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Diagnostics for Gradient Descent

- What should the gradient look like as a function of time (iterations)?
- Hint:
  - When should gradient descent end?

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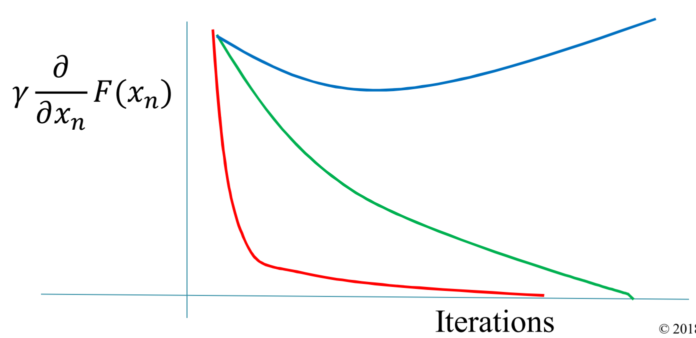
Diagnostics for Gradient Descent

- What should the gradient look like as a function of time (iterations)?
- Hint:
  - When should gradient descent end?



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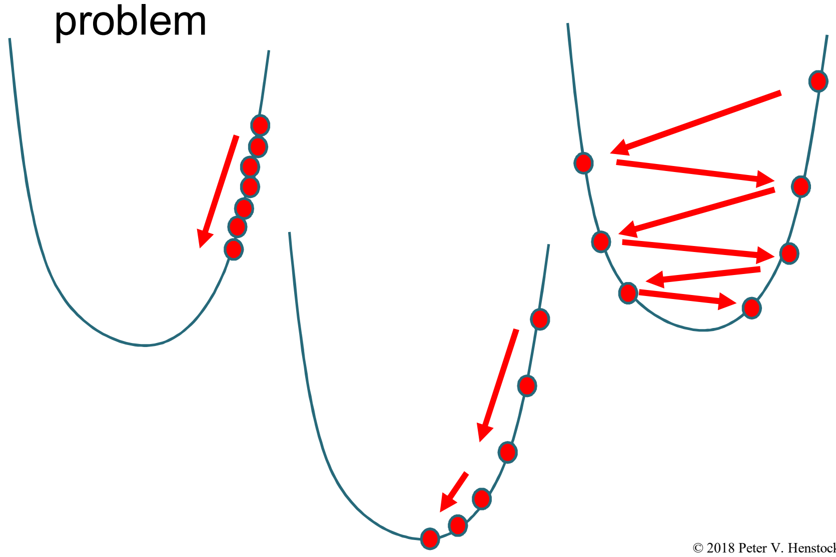
How large a learning rate?

- It depends…because it's a Goldilocks problem



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Gradient Descent

- Have an optimization function J we are trying to minimize
- Have 2 parameters w0 and w1
- Start with an initial guess of w0' and w1'
- Iteratively shift w0 and w1 in the direction to make J better until "done"

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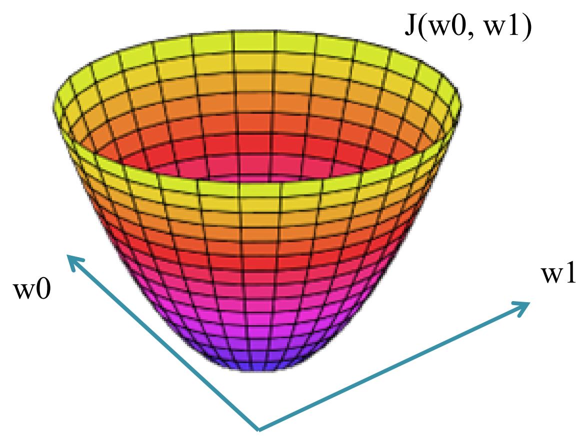
2D problem is same idea



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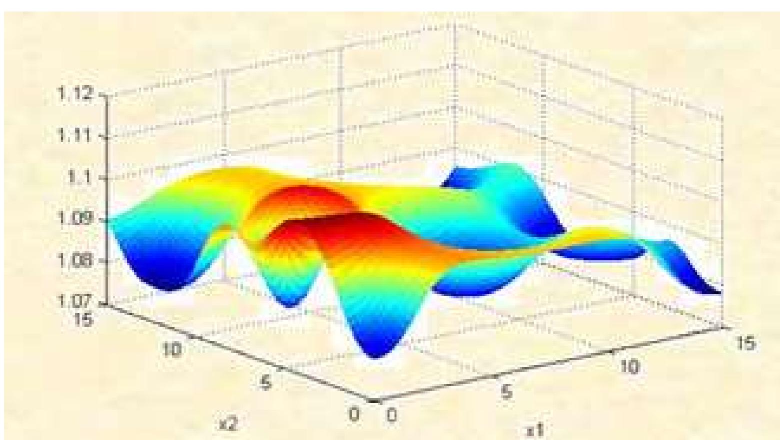
Will Gradient Descent Work?



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2D Gradient Descent

- For each loop
  - $w_1 \leftarrow w_1 - \alpha\, \partial J/\partial w_1$
  - $w_0 \leftarrow w_0 - \alpha\, \partial J/\partial w_0$
- Note: J is a function of w_0 and w_1
  - Be sure to update both new values using the older values
  - Do not update w_1 and use the updated w_1 to update w_0

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2D Gradient Descent

- For each loop
  - $w_1 \leftarrow w_1 - \alpha\, \partial J/\partial w_1$
  - $w_0 \leftarrow w_0 - \alpha\, \partial J/\partial w_0$
- $\partial J/\partial w_0 = 2\Sigma_i\, (w_1 x_i + w_0 - y_i)$
- $\partial J/\partial w_1 = 2\Sigma_i\, (w_1 x_i + w_0 - y_i) x_i$

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How to ensure achieved optimal?

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How to ensure achieved optimal?

- Start with multiple starting points
- Typically try a few different learning rates and plot the change of gradient to ensure things are going the right direction

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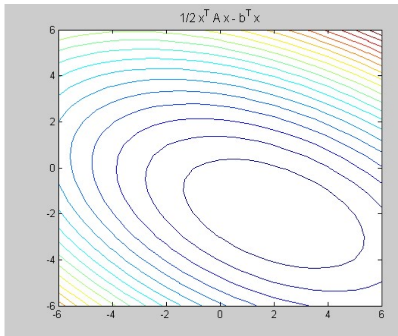
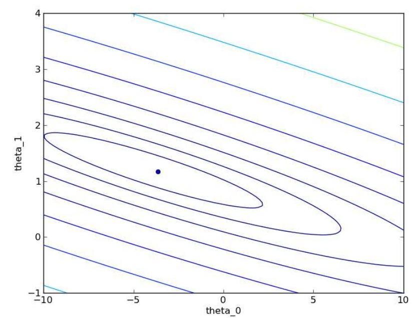
Which is easier for gradient descent?





- http://www.informatik.uni-konstanz.de/fileadmin/informatik/ag-saupe/Webpages/lehre/na_08/Lab1/10_CG/html/myCG.html
- http://aimotion.blogspot.com/2011/10/machine-learning-with-python-linear.html

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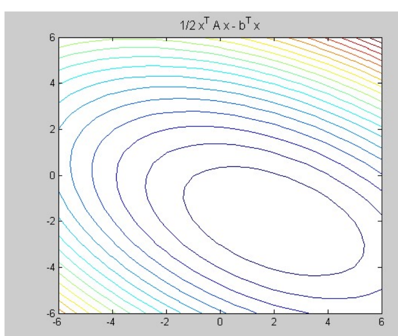
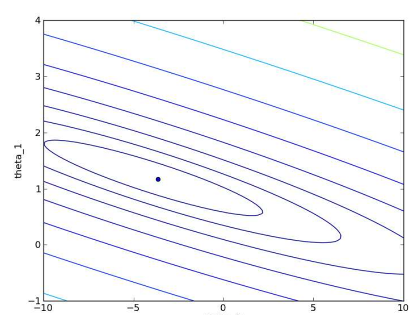
How to solve this problem?





- http://www.informatik.uni-konstanz.de/fileadmin/informatik/ag-saupe/Webpages/lehre/na_08/Lab1/10_CG/html/myCG.html
- http://aimotion.blogspot.com/2011/10/machine-learning-with-python-linear.html

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Regression Diagnostics

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Regression Process

- We load the data
- We perform a fit
- We're done!

- What could go wrong?

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Regression Process

- We load the data
- We perform a fit
- We're done!

- What could go wrong?
- Assumptions:
  - iid errors = independent & identically distributed
  - Common variance
  - N(0, variance)
  - Y and X are related

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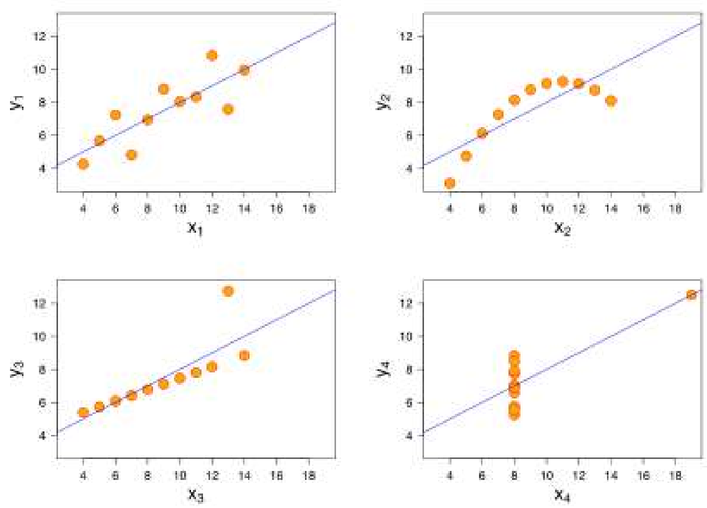
Classic Anscombe Example



https://en.wikipedia.org/wiki/Anscombe%27s_quartet 1973
Anscombe, F. J. 1973. Graphs in statistical analysis. The American Statistician, 27, pp 17-21.

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How can we diagnose the issues?

- Residuals should have a common variance
- Residuals should have a N(0, variance)
- Residuals should be iid

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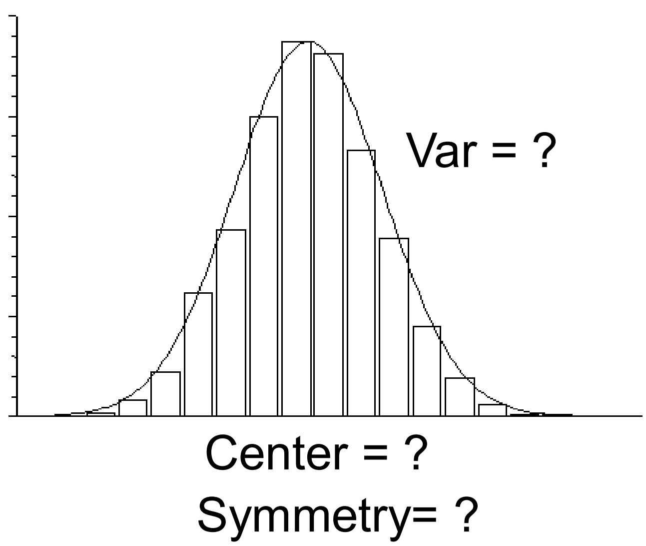
Histogram of Residuals



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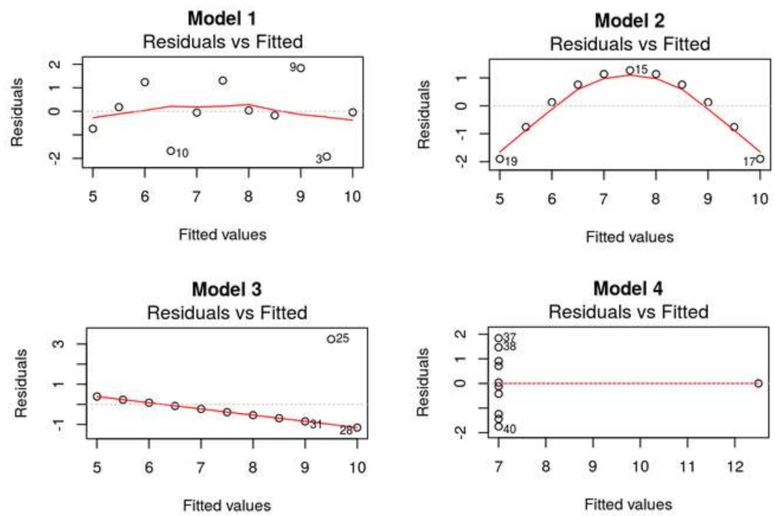
Residuals vs. X



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Heteroscedascity

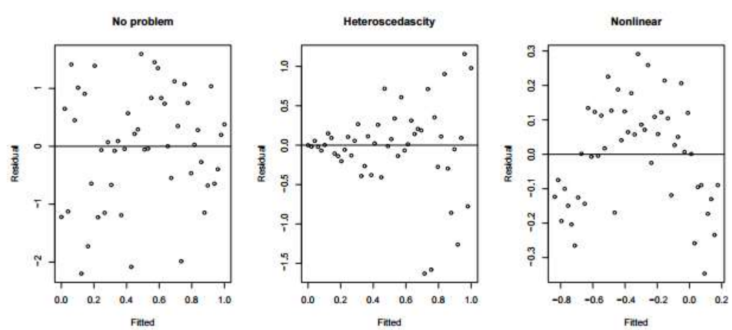

Figure 7.5: Residuals vs Fitted plots - the first suggests no change to the current model while the second shows non-constant variance and the third indicates some nonlinearity which should prompt some change in the structural form of the model

- Faraway: "Practical Regression and ANOVA using R" July 2002

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Data Transformations

- Common transformations include:
  - Log
  - Sqrt
  - Arcsin

- What do we apply these to?
  - Y?
  - X?

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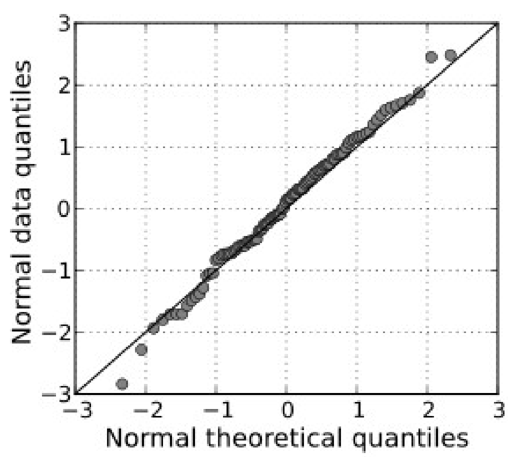
Quantile-Quantile (Q-Q) Plot



- Look for linear relationship
- Should have at least [-1 1] linear

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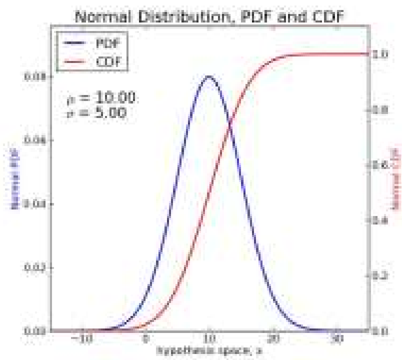
CMF = cumulative mass function

- Prob(a to b) = $\sum_a^b pmf$
- Cmf(x) = cumulative pmf = $\sum_{-\infty}^{x} pmf$
  - Maps from x → [0, 1] probability
- InvCmf(p) maps [0, 1] to an x-value



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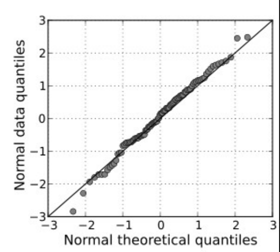
Quantile-Quantile (Q-Q) Plot



- Sort the residuals (y-axis)
  - Use Order Statistics
- Sample invnormal cdf evenly
  - Sample at probabilities 1/(n+1), 2/(n+1)…n/n+1
  - Convert the probabilities back to x-values for the x-axis
- Plot the sampled invcdf vs. residuals

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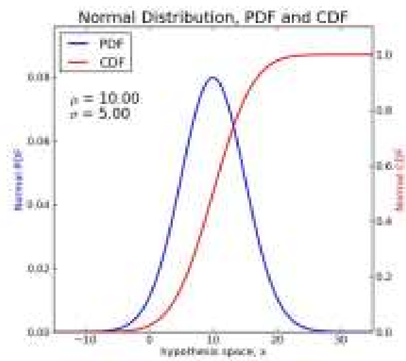
Q-Q plot intuition



- Central value of the distribution (50%)
  - Should match the mean
- Value mean ± stdev should line up
- Method works for other distributions too

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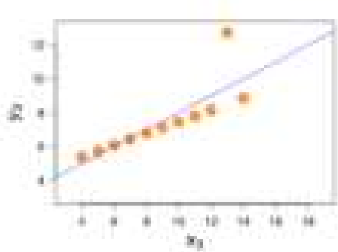
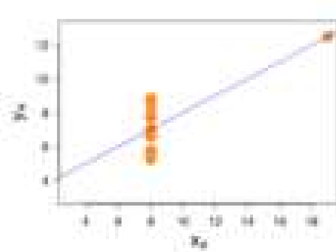
What about these cases?





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Leverage Points

- A single isolated point might have a lot of influence on the overall fit
- What characteristics might go into such a point?

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Leverage Points

- A single isolated point might have a lot of influence on the overall fit
- What characteristics might go into such a point?

1) Shifts the line
2) Isolated in x from other points

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Leverage Points

1) Shifts the line
2) Isolated in x from other points

- $h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{\bar{x}})^2}{\sum_{j=1}^{n} (x_j - \bar{\bar{x}})^2}$

- High leverage point if > 4/n

- Bad leverage point: distorts line (outlier)
- Good leverage point: consistent with line

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R Example for Regression

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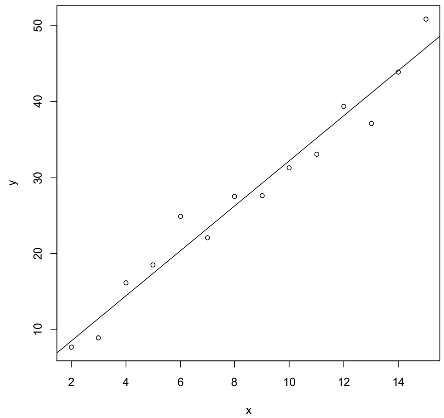
What does R do?

```
x <- 2:15
y <- 1:length(x)*3 + 5 +
rnorm(length(x))*2.5
plot(x,y)

fit <- lm(y ~ x)
summary(fit)
hist(resid(fit))
plot(fit)
```



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R call

```
Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max
-2.2088 -1.3598  0.3992  1.3213  1.6921

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.0393     0.9639   1.078    0.302
x             3.0456     0.1025  29.724 1.31e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.545 on 12 degrees of freedom
Multiple R-squared:  0.9866,    Adjusted R-squared:  0.9855
F-statistic: 883.5 on 1 and 12 DF,  p-value: 1.314e-12
```

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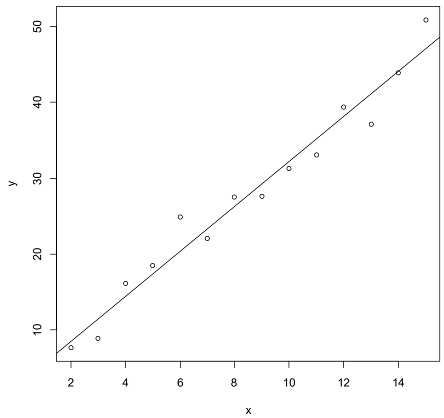
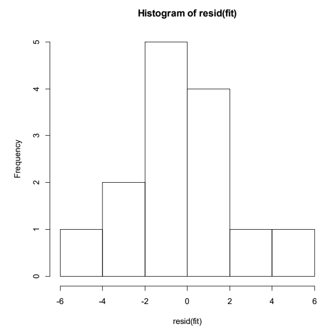
Original data with fit





- Checking fit and residual distribution

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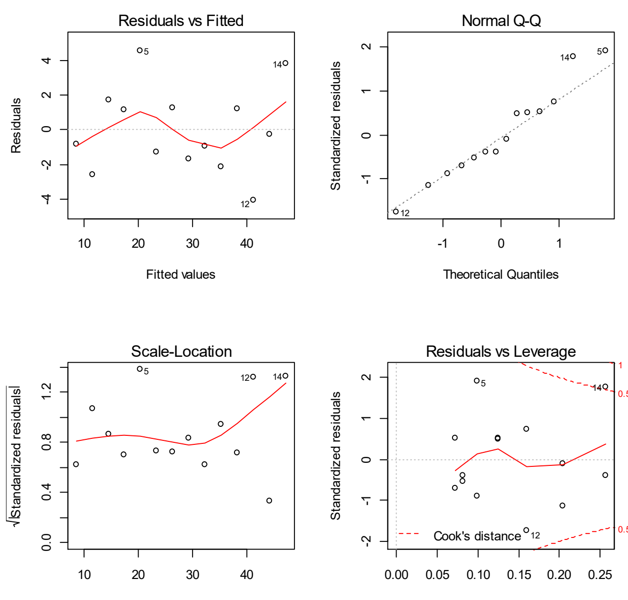
Regression diagnostics



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Actual Fit Values

```
Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max
-2.2088 -1.3598  0.3992  1.3213  1.6921

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.0393     0.9639   1.078    0.302
x             3.0456     0.1025  29.724 1.31e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

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```

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Time Series Modeling

http://www.artnet.com/artists/ert%C3%A9/complete-numbers-suite-set-of-10-HLI_PTOIsn_8aIdkjAWWWw2

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Why Do Time Series Analysis?

- Model of data

- Interpretation
- Forecasting
- Control
- Hypothesis testing
- Simulation

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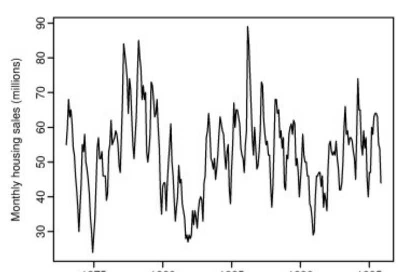
Time Series Modeling

- Previously with regression:
  - Y ~ f(X)
  - X samples are i.i.d.
  - Generative model: Y is f(x) + N(0,sigma)

- Why don't just use Y ~ f(t) ?



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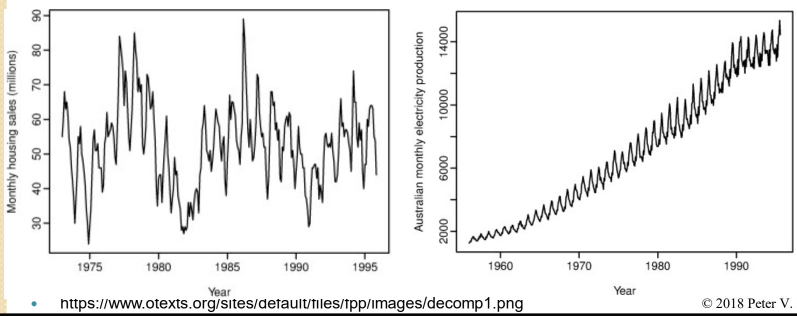
Signal vs. Noise Issue

- Regression model would probably smooth the data to provide a good trend
- Decomposition:
  - Could extract the trend and seasonality
  - Are they additive or multiplicative?
  - Then model the remainder which is what?



- https://www.otexts.org/sites/default/files/fpp/images/decomp1.png

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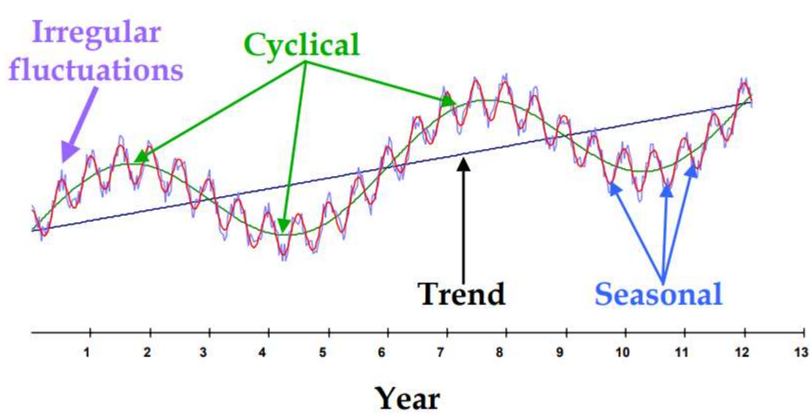
Time Series Components



- http://www.statoek.wiso.uni-goettingen.de/veranstaltungen/graduateseminar/SmoothingMethods_Narodzonek-Karpowska.pdf

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Smoothing

- Moving Average: average over last m
  - $x(n) = \frac{1}{m}\sum_{k=0}^{m-1} x(n-k)$
- Weighted Moving Average: avg over last m
  - $x(n) = \frac{1}{\sum_{k=0}^{m} w_k}\sum_{k=0}^{m} w_k x(n-k)$
  - Weights usually sum to 1
  - Use this to heavily weight more recent
- Exponential Smoothing
  - Single: s(n) = αx(n) + (1-α)s(n-1)
  - Double: s(n) = αx(n) + (1-α)[s(n-1) + b(n-1)]
    - b(n) = β[s(n)-s(n-1)] + (1-β) b(n-1)

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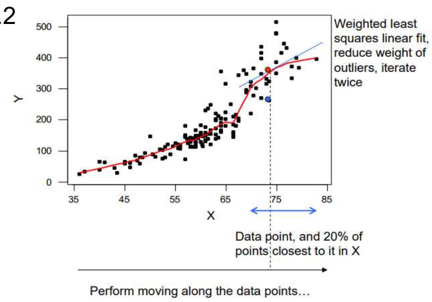
Lowess smoothing

- Lowest (weighted) regression
- Non-parametric model
- http://sites.stat.psu.edu/~fxc11/Stat462_STABLE/Lect12_lowess.pdf

- Smoothing param 0.2 = fraction of points
- Degree 1 = linear



Weighted least squares linear fit

$$\min_{\beta_0,\beta_1} \sum_{j\in N_i} w_j (y_j - (\beta_0 + \beta_1 x_j))^2$$

Weight function

$$w_i = \left(1 - \left(\frac{d(x_j, x_i)}{\max_{\ell \in N_i} d(x_\ell, x_i)}\right)^3\right)^3$$

F. Chiaromonte 3

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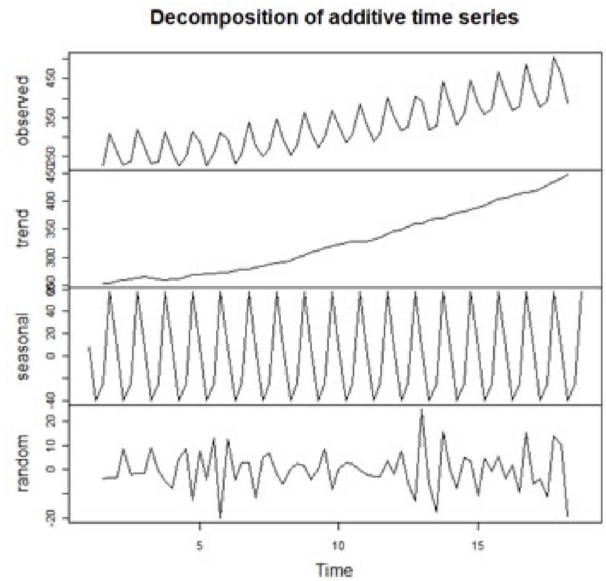
Signal Decomposition



https://onlinecourses.science.psu.edu/stat510/node/69

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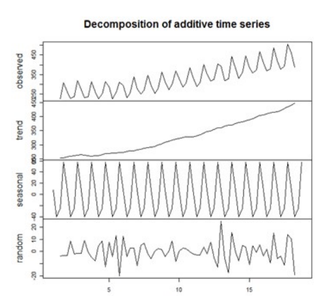
In this lecture and next

- Remove the trend
- Figure out seasonality*
- Figure out frequency*
- Model the “random”

- * = next lecture



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Time Series Framework

- Need a model to explain random signals
- Existing models may not be explainable by the Y~f(t) + noise
- Existing models may not be predictable
- But, we know something about Y
- Framework: random signals
  - Y(t) characterized by a distribution at each t

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Framework is Random Sequence

- $x_1, x_2, \ldots x_n$ (Changed notation to x)
- Need to exploit the order of sequence
- As random, it cannot be predicted
- Deterministic process: known / predictable
- Stochastic or Random process:
  - System that generates all possible random sequences of which we have a realization $x_1 \ldots x_n$ with the specific random values
  - Goal is to model the process
  - Model process as random since it's either random or we do not have a better model

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Time Series Review

- Random Process
  - w(k) = 0, 2, 0, -1, 2, -1, 2, 1, 0, 0, 1, 2, -1, -2
- AR Model
  - $x(k) = \phi_1 x(k-1) + \phi_2 x(k-2) + \ldots \phi_p x(k-p) + w(k)$
  - Let's use a 1st order model $\phi_1 = 0.9$
  - x(0) = 0
  - x(1) = 0.9*0 + 2 = 2
  - x(2) = 0.9*2 + 0 = 1.8
  - x(3) = 0.9*1.8 + -1 = 0.62

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Time Series Review

- Random Process
  - w() = 0, 2, 0, -1, -2, -1, 2, 1, 0, 0, 1, 2, -1, -2
- MA Model (clearer notation)
  - $x(k) = w(k) + \theta_1 w(k-1) + \theta_2 w(k-2) + \ldots \theta_p w(k-p)$
  - Let's use a 2nd order model $\theta_1 = 0.7$, $\theta_2 = 0.3$
  - Assume x(-1) = 0
  - x(0) = 0
  - x(1) = 2 + 0.7(0) = 2
  - x(2) = 0 + 0.7(2) + 0.3(0) = 1.4
  - x(3) = -1 + 0.7(0) + 0.3(2) = -0.4

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Ergodic Process

- Time estimate converges to the true estimate as N → infinity
  - Ergodic mean
    - True if $1/N \Sigma x(k) = E[x(k)] = \mu$ as N→infinity
- If we take a reasonably long sample,
  - Then we can infer the statistical properties of the whole sequence

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Strict Stationary

- If joint distribution of N observations is invariant to time shifts
- $F_{x(k1)}(x1) = F_{x(k1+T)}(x1)$
  - All have same distribution regardless of T

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(Wide Sense or Weak) Stationarity

- Few real systems are strictly stationary
  - Joint distribution of all vectors of d dimensions remains the same for any fixed d
- Weaker definition is WSS
  - First order stationary: E[X(t)] is same for all t
  - Second order:
    1) First order stationary and
    2) Cov[X(t),X(t-lag)] is function of only lag

Note: Gaussian processes described by mean & cov

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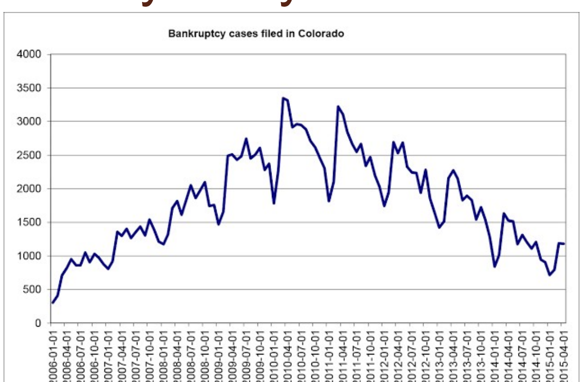
Exploratory analysis



http://www.ryanmcmaken.com/2015/05/chart-of-day-colorado-bankruptcy.html

Constant mean or trend?
Constant variance?
Seasonality?
Outliers?
Cycles?

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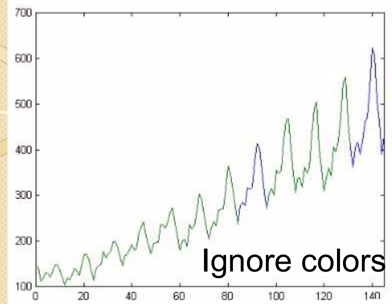
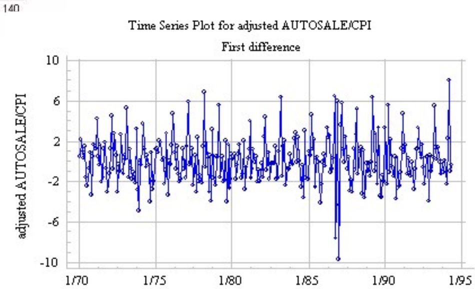
Which is stationary? Which order?





http://www.es.lancs.ac.uk/cres/captain/dhrdemo.html
https://people.duke.edu/~rnau/411diff.htm

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Mathematical idea of stationary

x(k) = Acos(wk + u) where u is $U(0,\pi)$

Is this stationary?

How would we determine the answer?

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Mathematical idea of stationary

x(k) = Acos(wk + u) where u is U(0,π)
Is this stationary?
How would we determine the answer?

E(x|k) = ?

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Mathematical idea of stationary

x(k) = Acos(wk + u) where u is U(0,π)
Is this stationary?
How would we determine the answer?

E(x|k) = $\int_0^{\pi} [?]\, f(u) du$ where f(u) = ?

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Mathematical idea of stationary

x(k) = Acos(wk + u) where u is U(0,π)
Is this stationary?
How would we determine the answer?

E(x|k) = $\int_0^{\pi}$ Acos(wk + u)$f(u)du$
E(x|k) =$A\int_0^{\pi}$ cos(wk + u)$\frac{1}{\pi}du = \frac{-2}{\pi}\sin(wk)$

So what can you conclude?

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Mathematical idea of stationary

x(k) = Acos(wk + u) where u is U(0,π)
Is this stationary?
How would we determine the answer?

E(x|k) = $\int_0^{\pi}$ Acos(wk + u)$f(u)du$
E(x|k) = $\frac{A}{\pi}\int_0^{\pi}$ cos(wk + u)$1du = \frac{-2A}{\pi}\sin(wk)$
E(x|k) depends on k so not stationary

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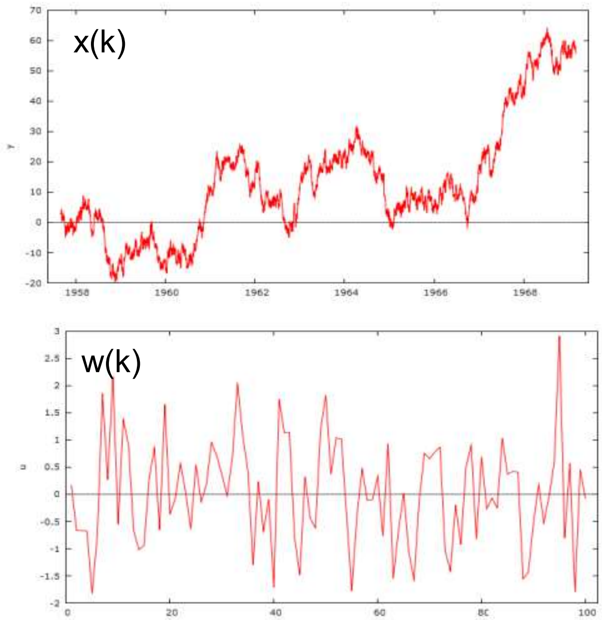
Random Walk



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